Studies on algebraic structure of dynamical Yang-Baxter maps

ダイナミカル・ヤン・バクスター写像の代数構造の研究

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Contents

Chapter 1

Introduction

In this doctoral thesis, we discuss the algebraic structure of dynamical Yang-Baxter maps under some conditions. A dynamical Yang-Baxter map, which was proposed by Shibukawa [33], is a set-theoretical solution of the dynamical Yang-Baxter equation that is a dynamical analogue of the quantum Yang-Baxter equation. This thesis is based on the articles [27, 28] which were already published. First of all, we mention some history, from quantum Yang-Baxter equation to dynamical Yang-Baxter map.

The quantum Yang-Baxter equation

The definition of the quantum Yang-Baxter equation is as follows.

Definition 1. Let V be a vector space, and R be a linear operator on $V \otimes V$. The following equation on $V \otimes V \otimes V$ is called the quantum Yang-Baxter equation

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}. \tag{1.0.1}$$

Here, R_{ij} denotes the action of the linear operator $R : V \otimes V \to V \otimes V$ on the i-th and the j-th components of $V \otimes V \otimes V$.

The quantum Yang-Baxter equation first appeared manifestly in the work of McGuire [29] in 1964 and Yang [42] in 1967. In these articles they used the quantum Yang-Baxter equation to solve a one-dimensional quantum many-body problem, and in [2, 3] Baxter showed the importance of the quantum Yang-Baxter equation by solving the eight-vertex lattice model. Today the quantum Yang-Baxter equation has turned out to be one of the fundamental equations in the theory of integrable systems.

As an important event in the study of the quantum Yang-Baxter equation, we mention quantum groups briefly. In the beginning of 80's, the study of the quantum Yang-Baxter equation has been performed actively in Russia [22, 23]. This study led to the idea of a quantum group. Through these studies Drinfel'd [7] and Jimbo [17] introduced a quantum group as a deformation of group or a Lie algebra, which has a non commutative and a non co-commutative Hopf algebra structure. Using the quantum group Drinfel'd and Jimbo construct the solutions of the quantum Yang-Baxter equation systematically.

The Yang-Baxter map

In the 90's, Drinfel'd [8] suggested to study set-theoretical solutions of the quantum Yang-Baxter equation, which are called Yang-Baxter maps [41], and defined them as follows.

Definition 2. Let X be a non-empty set. The Yang-Baxter map is a map $R : X \times X \to X \times X$ which satisfies the following equation on $X \times X \times X$,

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}. \tag{1.0.2}$$

Here $R_{12}, R_{23}, \cdots$ are maps from $X \times X \times X$ to $X \times X \times X$ defined as follows:

$$\begin{aligned} &R_{12}(a, b, c) = (R(a, b), c), \\ &R_{23}(a, b, c) = (a, R(b, c)), \cdots (a, b, c \in X). \end{aligned}$$

The Yang-Baxter map has relations with many areas [1, 10, 14, 16, 24, 31]. In [24], Lu-Yan-Zhu construct invertible Yang-Baxter maps satisfying compatibility conditions by means of bijective 1-cocycles. In [10], Etingof, Schedler, and Soloviev gave a classification of the invertible Yang-Baxter maps satisfying non-degenerate and unitary conditions, and they discuss the geometric and algebraic aspects of these Yang-Baxter maps.

The quantum dynamical Yang-Baxter equation

Gervais and Neveu introduced a quantum dynamical Yang-Baxter equation as a generalization of the quantum Yang-Baxter equation in a physics paper [15], and the study of mathematical aspect was started by Felder in [12]. Therein he proposed the quantum dynamical Yang-Baxter equation as a quantization of the classical dynamical Yang-Baxter equation, and explained a relation with conformal field theory and statistical mechanics.

As a generalization of the quantum Yang-Baxter equation the quantum dynamical Yang-Baxter equation is defined as follows [11].

Definition 3. Let $\mathfrak{h}$ be a finite dimensional commutative Lie algebra over $\mathbb{C}$, $\mathfrak{h}^*$ a dual space of $\mathfrak{h}$, and V a semisimple $\mathfrak{h}$-module. Then the following equation with respect to (meromorphic) functions $R : \mathfrak{h}^* \to \mathrm{End}_{\mathfrak{h}} V \otimes V$ is called the quantum dynamical Yang-Baxter equation

$$R_{23}(\lambda)R_{13}(\lambda - h^{(2)})R_{12}(\lambda) = R_{12}(\lambda - h^{(3)})R_{13}(\lambda)R_{23}(\lambda - h^{(1)}) \ (\forall\lambda \in \mathfrak{h}^*). \tag{1.0.3}$$

Here $R_{12}(\lambda), R_{12}(\lambda - h^{(3)})), \cdots$ are linear transformation on $V \otimes V \otimes V$ defined as follows:

$$\begin{aligned}
&R_{12}(\lambda)(u \otimes v \otimes w) = (R(\lambda)(u \otimes v) \otimes w),\\
&R_{12}(\lambda - h^{(3)})(u \otimes v \otimes w) = (R(\lambda - wt(w))(u \otimes v) \otimes w), \cdots \ (u, v, w \in V),
\end{aligned}$$

$wt(w)$ means a weight of w under $\mathfrak{h}$.

In this definition $\lambda \in \mathfrak{h}$ means a dynamical parameter, which differs from spectral parameter. If $\mathfrak{h} = 0$, the quantum dynamical Yang-Baxter equation turns into the quantum Yang-Baxter equation. As an important generalization, we can define the quantum dynamical Yang-Baxter equation with spectral parameter as follows [11].

Definition 4. Let $\mathfrak{h}$ be a finite dimensional commutative Lie algebra over $\mathbb{C}$, $\mathfrak{h}^*$ a dual space of $\mathfrak{h}$, and V a semisimple $\mathfrak{h}$-module. Then the following equation with respect to (meromorphic) functions $R : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}_{\mathfrak{h}} V \otimes V$ is called the quantum dynamical Yang-Baxter equation

$$\begin{aligned}
&R_{23}(u_{23}, \lambda)R_{13}(u_{13}, \lambda - h^{(2)})R_{12}(u_{12}, \lambda)\\
&\quad = R_{12}(u_{12}, \lambda - h^{(3)})R_{13}(u_{13}, \lambda)R_{23}(u_{23}\lambda - h^{(1)}) \ (\forall\lambda \in \mathfrak{h}^*).
\end{aligned} \tag{1.0.4}$$

Here $u_{ij} = u_i - u_j$.

As in the case of the quantum groups, many people tried to define an algebraic system from the quantum dynamical Yang-Baxter equation [9, 12, 13, 18]. There are two types of this algebraic system. In contrast with the quantum groups these algebraic systems are not Hopf algebras, which have a generalized Hopf algebra structure called quasi-Hopf algebra or $\mathfrak{h}$-Hopf algebroid.

The dynamical Yang-Baxter map

A dynamical Yang-Baxter map is a set-theoretical solution of the quantum dynamical Yang-Baxter equation, which was proposed by Shibukawa [33] in 2005 as follows.

Definition 5. Let H and X be non-empty sets, $\phi : H \times X \to H$. The dynamical Yang-Baxter map associated with H, X, ϕ is a map $R(\lambda) : X \times X \to X \times X (\lambda \in H)$ which satisfies the following equation on $X \times X \times X$,

$$R_{23}(\lambda)R_{13}(\phi(\lambda, X^{(2)}))R_{12}(\lambda) = R_{12}(\phi(\lambda, X^{(3)}))R_{13}(\lambda)R_{23}(\phi(\lambda, X^{(1)})). \tag{1.0.5}$$

Here $R_{12}(\lambda), R_{12}(\phi(\lambda, X^{(3)})), \cdots$ are maps from $X \times X \times X$ to $X \times X \times X$ defined as follows:

$$R_{12}(\lambda)(a, b, c) = (R(\lambda)(a, b), c),$$
$$R_{12}(\phi(\lambda, X^{(3)}))(a, b, c) = (R(\phi(\lambda, c))(a, b), c), \cdots \ (a, b, c \in X).$$

As a special case the dynamical Yang-Baxter map includes the Yang-Baxter map.

In [34], Shibukawa gave a characterization of the dynamical Yang-Baxter maps satisfying invariance conditions by using left quasigroups and ternary operations. The dynamical Yang-Baxter map yields bialgebroids [37] and discrete integrable systems through 3D compatible ternary systems [21]. Furthermore, suitable homogeneous pre-systems [19], related to reductive homogeneous spaces, can produce the dynamical Yang-Baxter map. Until now, there are many interesting results [33, 34, 35, 36]. The dynamical Yang-Baxter map are expected to relate with many areas like the ultra discrete integrable systems and discrete geometries.

This paper consists of two parts, based on the articles [27] and [28]. Here we explain about these articles. For details see Chapter 2 and Chapter 3.

Dynamical braces and dynamical Yang-Baxter maps

In the first part, which is based on [27], we discuss right non-degenerate dynamical Yang-Baxter maps with unitary condition, and study these algebraic and combinatorial structures.

First, we propose an algebraic system called a dynamical brace. The dynamical brace is a generalization of the brace that was proposed by Rump in [32] as a generalization of the radical ring. The radical ring means a ring $(R, +, \cdot)$, which has a group structure with respect to multiplication $a * b := a \cdot b + a + b$ $(\forall a, b \in R)$. For examples of the radical ring, consider the Jacobson radical. In [32] Rump shows a relation between brace and non-degenerate Yang-Baxter map with unitary condition. The dynamical brace is an algebraic system with a family of multiplications that is defined as follows.

Definition 6. Let H be a non-empty set, $(A, +)$ an abelian group with the family of multiplications $\{\cdot_\lambda : A \times A \to A\}_{\lambda \in H}$ and $\phi : H \times A \to H$. We call $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda \in H})$ a dynamical brace if the following conditions are satisfied for all $(\lambda, a, b, c) \in H \times A \times A \times A$:

(1) $(a + b) \cdot_\lambda c = a \cdot_\lambda c + b \cdot_\lambda c$,

(2) $a \cdot_\lambda (b \cdot_\lambda c + b + c) = (a \cdot_{\phi(\lambda,c)} b) \cdot_\lambda c + a \cdot_{\phi(\lambda,c)} b + a \cdot_\lambda c$,

(3) The map $\gamma_\lambda(b) : a \mapsto a \cdot_\lambda b + a$ is bijective.

Using a dynamical brace we can construct a dynamical Yang-Baxter map as follows. Let $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda \in H})$ be a dynamical brace, then $R(\lambda) : A \times A \to A \times A (\lambda \in H)$ defined by

$$R(\lambda)(a, b) = (\mathfrak{R}_b^\lambda(a), \mathfrak{L}_a^\lambda(b)) := (\gamma_\lambda(\gamma_\lambda(a)(b))^{-1}(a), \gamma_\lambda(a)(b)) \tag{1.0.6}$$

is a right non-degenerate dynamical Yang-Baxter map associated with A, H, ϕ, which satisfies the unitary condition

$$PR(\lambda)PR(\lambda) = \mathrm{id}_{A \times A}, \ (\forall \lambda \in H).$$

Here P is a map defined as follows

$$P : A \times A \to A \times A, (a, b) \mapsto (b, a).$$

This result is obtained as a corollary of Theorem 6 in Chapter 2. In Theorem 6, we give a characterization of the dynamical Yang-Baxter map, which corresponds to the dynamical brace.

Like the brace, the dynamical brace satisfies the next relation with respect to multiplications $a *_\lambda b := a \cdot_\lambda b + a + b$ $(\forall a, b \in A, \lambda \in H)$,

$$(a *_{\phi(\lambda,c)} b) *_\lambda c = a *_\lambda (b *_\lambda c).$$

This relation can be considered as an associative law of a dynamical algebraic system.

In the latter part of Chapter 2, we describe the combinatorial aspects of the dynamical brace. For the dynamical brace $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda \in H})$, we identify the map

$$R_\lambda(a) : A \to A, b \mapsto b *_\lambda a = b \cdot_\lambda a + b + a = \gamma_\lambda(a)(b) + a$$

with the action of the element $(a, \gamma_\lambda(a))$ of semidirect product $A \rtimes \mathrm{Aut}(A)$ on A. By using this identification we regard $S_\lambda = \{R_\lambda(a) | \lambda \in H, a \in A\}$ as a subset of $A \rtimes \mathrm{Aut}(A)$,

$$\{(a, \gamma_\lambda(a)) | a \in A\}$$

for all $\lambda \in A$, and we characterize the dynamical brace in a combinatorial way as follows (See Theorem 8).

Theorem 1. Let $(A, +)$ be an abelian group and H a non-empty set.

(1) Let $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda \in H})$ be a dynamical brace. We set a family of subsets $\{S_\lambda\}_{\lambda \in H}$ as follows. $S_\lambda := \{R_\lambda(a) : A \to A, b \mapsto b *_\lambda a | a \in A\} \subset A \rtimes \mathrm{Aut}(A)$. Then, $\{S_\lambda\}_{\lambda \in H}$ satisfies the following conditions:

  (a) $\forall a \in A$, $\exists! f \in \mathrm{Aut}(A)$ s.t., $(a, f) \in S_\lambda$,

  (b) $\forall (a, f) \in S_\lambda$, $\exists! \mu \in H$ s.t., $(a, f)^{-1} S_\lambda = \{(a, f)^{-1}(b, g) | (b, g) \in S_\lambda\} = S_\mu$.

  We denote the unique $f \in \mathrm{Aut}(A)$ of condition (a) by $f_\lambda(a)$.

(2) Let $\{S_\lambda\}_{\lambda \in H}$ be a family of subsets of $A \rtimes \mathrm{Aut}(A)$ and suppose that $\{S_\lambda\}_{\lambda \in H}$ satisfies the above conditions (a) and (b). Define multiplications $\{\cdot_\lambda\}_{\lambda \in H}$ on A by $a \cdot_\lambda b := f_\lambda(b)(a) - a$, and a map ϕ from $H \times A$ to H by $\phi(\lambda, a) = \mu$, which is determined uniquely by condition (b). Then $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda \in H})$ is a dynamical brace.

(3) The correspondence between (1) and (2) is one-to-one.

As a special case, when $\#(H) = 1$, $\{S_\lambda\}_{\lambda \in H}$ corresponds to a regular subgroup of $A \rtimes \mathrm{Aut}(A)$ [5, 6]. A subgroup S of $A \rtimes \mathrm{Aut}(A)$ is said to be regular if, given any $a \in A$, then for each $b \in A$ there exists a unique $x \in S$ such that $x.a = b$. Here . denotes an action of S on A.

Through this combinatorial expression, we obtain a way to construct dynamical braces, and we exhibit some examples associated with abelian groups $\mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2$.

Quantum Yang-Baxter equation, braided semigroups, and dynamical Yang-Baxter maps

In the second part, which is based on [28], we start from the following simple examples of idempotent Yang-Baxter maps [4, 34].

Let G be a group, and let e_G denote unit element of G. Then the maps $\sigma_i : G \times G \to G \times G$ $(i = 1, 2)$,

$$\sigma_1(a, b) = (e_G, ab) \text{ and } \sigma_2(a, b) = (ab, e_G) \ (a, b \in G), \tag{1.0.7}$$

satisfy the idempotent condition

$$\sigma_i^2 = \sigma_i, \ (i = 1, 2),$$

and the quantum Yang-Baxter equation

$$\sigma_i \times \mathrm{id}_G \circ \mathrm{id}_G \times \sigma_i \circ \sigma_i \times \mathrm{id}_G = \mathrm{id}_G \times \sigma_i \circ \sigma_i \times \mathrm{id}_G \circ \mathrm{id}_G \times \sigma_i \quad (i = 1, 2),$$

This equation is equivalent to the quantum Yang-Baxter equation of the form (1.0.2).

The aim of this part is to generalize the above examples from the viewpoint of category theory. In this generalization braided semigroups play an important role. The braided semigroup is a generalization of the braided group [36, 40], which is a useful concept in the construction of the Yang-Baxter map [24]. To define a braided semigroup, we use a tensor category. A tensor category is a category C with the following data,

(1) a functor $\otimes : C \times C \to C$, which is called tensor product,

(2) a unit object I,

(3) a natural isomorphism $a : \otimes \circ (\otimes \times \mathrm{id}) \to \otimes \circ (\mathrm{id} \times \otimes)$, which is called an associativity constraint,

(4) natural isomorphisms $l : \otimes(I \times \mathrm{id}) \to \mathrm{id}$, $r : \otimes(\mathrm{id} \times I) \to \mathrm{id}$, which are called left and right unit constraints with respect to I,

satisfying the pentagon axiom and the triangle axiom. We denote by $1_X : X \to X$ the identity morphism of an object X.

By using the tensor category, the quantum Yang-Baxter equation is defined as follows,

Definition 7. Let C be a tensor category, X an object of C and $\sigma_{XX} : X \otimes X \to X \otimes X$ a morphism of C. Then the following relation is called a quantum Yang-Baxter equation in C,

$$a \circ \sigma_{XX} \otimes 1_X \circ a^{-1} \circ 1_X \otimes \sigma_{XX} \circ a \circ \sigma_{XX} \otimes 1_X = 1_X \otimes \sigma_{XX} \circ a \circ \sigma_{XX} \otimes 1_X \circ a^{-1} \circ 1_X \otimes \sigma_{XX} \circ a. \tag{1.0.8}$$

Here, $a = a_{X,X,X}$.

As a generalization of the braided group, we define a braided semigroup by using the tensor category as follows.

Definition 8. Let $\sigma_{XY} : X \otimes Y \to Y \otimes X$ be a morphism of the tensor category C.

(1) A pair (X, m_X) of an object X and a morphism $m_X : X \otimes X \to X$ is a semigroup, if and only if m_X satisfies

$$m_X \circ (m_X \otimes 1_X) = m_X \circ (1_X \otimes m_X) \circ a_{X,X,X}. \tag{1.0.9}$$

This morphism m_X is called a multiplication.

(2) A pair (X, Δ_X) of an object X and a morphism $\Delta_X : X \to X \otimes X$ is a co-semigroup, the dual concept of the semigroup, if and only if Δ_X satisfies

$$a_{X,X,X} \circ (\Delta_X \otimes 1_X) \circ \Delta_X = (1_X \otimes \Delta_X) \circ \Delta_X. \tag{1.0.10}$$

The morphism Δ_X is said to be a comultiplication.

(3) A matched pair of semigroups $X = (X, m_X)$ and $Y = (Y, m_Y)$ (Cf. [26, 36, 39, 40]) is a triple (X, Y, σ_{XY}) satisfying:

$$\begin{aligned} &(1_Y \otimes m_X) \circ a_{Y,X,X} \circ (\sigma_{XY} \otimes 1_X) \circ a^{-1}_{X,Y,X} \circ (1_X \otimes \sigma_{XY}) \\ =&\sigma_{XY} \circ (m_X \otimes 1_Y) \circ a^{-1}_{X,X,Y}; \qquad (1.0.11) \\ &(m_Y \otimes 1_X) \circ a^{-1}_{Y,Y,X} \circ (1_Y \otimes \sigma_{XY}) \circ a_{Y,X,Y} \circ (\sigma_{XY} \otimes 1_Y) \\ =&\sigma_{XY} \circ (1_X \otimes m_Y) \circ a_{X,Y,Y}. \qquad (1.0.12) \end{aligned}$$

A pair (X, σ_{XX}) of a semigroup X and a morphism $\sigma_{XX} : X \otimes X \to X \otimes X$ is called a braided semigroup, if and only if the triple (X, X, σ_{XX}) is a matched pair of semigroups.

We obtain the following results as our main theorem.

Theorem 2. Let $X = (X, m_X)$ be a semigroup with a comultiplication $\Delta_X : X \to X \otimes X$ on the tensor category C. If the pair $(X, \sigma_{XX} := \Delta_X \circ m_X)$ is a braided semigroup, then σ satisfies the quantum Yang-Baxter equation in the tensor category C.

Theorem 2 show that the braided semigroup plays an important role in a construction of a solution of the quantum Yang-Baxter equation.

In section 3.3, we construct the braided semigroup and comultiplication by means of semigroup with a left or right unit. A left unit η_l (resp. a right unit η_r) of a semigroup (S, m_S) is a morphism $\eta_l : I \to S$ (resp. $\eta_r : I \to S$) satisfying $m_S \circ \eta_l \otimes 1_S = l_S$ (resp. $m_S \circ 1_S \otimes \eta_r = r_S$). Define comultiplication as follows:

$$\Delta_1 := (\eta_l \otimes 1_S) \circ l_S^{-1} \text{ and } \Delta_2 := (1_S \otimes \eta_r) \circ r_s^{-1}.$$

Then $(S, \sigma_i := \Delta_i \circ m_S)$ $(i = 1, 2)$ is a braided semigroup. In this construction the multiplication m_S and the comultiplications $\Delta_i (i = 1, 2)$ satisfy the following relation

$$m_S \circ \Delta_i = \mathrm{id}_S, \ (i = 1, 2).$$

This relation $\sigma_i (i = 1, 2)$ satisfies the idempotent condition.

We introduce a tensor category $\mathbf{Set}_H$, which is associated with a non-empty set H, to construct the dynamical Yang-Baxter map.

Definition 9. Let H be a non-empty set. $\mathbf{Set}_H$ denotes the following category:

(1) an object is a pair $(X, \cdot_X)$ of a set X and a map $\cdot_X : H \times X \to H, (\lambda, x) \mapsto \lambda \cdot_X x$,

(2) a morphism $f : (X, \cdot_X) \to (Y, \cdot_Y)$ is a map $f : H \to \mathrm{Map}(X, Y)$ satisfying

$$\lambda \cdot_Y f(\lambda)(x) = \lambda \cdot_X x, \ (\forall \lambda \in H, \forall x \in X),$$

(3) the identity 1 and the composition $\circ$ are defined by

$$1_X(\lambda)(x) = x \ (\lambda \in H, x \in X) \text{ and } (g \circ f)(\lambda) = g(\lambda) \circ f(\lambda) \ (\lambda \in H),$$

for objects X, Y, Z and morphisms $f : X \to Y, g : Y \to Z$.

The $\mathbf{Set}_H$ has a tensor category structure as follows:

(1) the tensor product $X \otimes Y$ of the objects $(X, \cdot_X)$ and $(Y, \cdot_Y)$ is a pair $(X \times Y, \cdot)$ of the Cartesian product $X \times Y$ and the following map $\cdot : H \times (X \times Y) \to H$,

$$\lambda \cdot (x, y) = (\lambda \cdot_X x) \cdot_Y y, \ (\lambda \in H, (x, y) \in X \times Y).$$

(2) the tensor product of the morphisms $f : X \to X'$ and $g : Y \to Y' is$ is defined by

$$(f \otimes g)(\lambda)(x, y) = (f(\lambda)(x), g(\lambda \cdot_X x)(y)), \ (\lambda \in H, (x, y) \in X \times Y).$$

(3) the associativity constraint a, the unit I, and the left and the right unit constraints l, r are as follows,

(a) $a_{XYZ}(\lambda)((x, y), z) = (x, (y, z))$,

(b) $I = (\{e\}, \cdot_I)$, a pair of the set $\{e\}$ of one element and the map $\cdot_I$ defined by $\lambda \cdot_I e = \lambda$,

(c) $l_X(\lambda)(e, x) = x = r_X(\lambda)(x, e)$.

The tensor category $\mathbf{Set}_H$ is a generalization of the tensor category $\mathbf{Set}$.

Definition 10. A morphism $\sigma : X \times X \to X \times X$ of $\mathbf{Set}_H$ is a dynamical Yang-Baxter map if and only if σ satisfies the quantum Yang-Baxter equation in $\mathbf{Set}_H$.

As an application, we construct the braided semigroup with left or right unit by means of left quasigroups [30, 38].

Definition 11. A left quasigroup Q is a non-empty set, together with a binary operation $\cdot : Q \times Q \to Q$ such that the left translation map $L(a) : Q \ni b \mapsto a \cdot b \in Q$ is bijective for all $a \in Q$.

For simplicity of notation, we write ab $(a, b \in Q)$ instead of $a \cdot b$, and denote $L(a)^{-1}(b)(\in Q)$ by $a \backslash b$. Here, $L(a)^{-1} : Q \to Q$ is the inverse of $L(a)$. A left quasigroup is a generalization of a group, which is not always associative For examples of left quasigroups see Example 7 of Chapter 3.

For a left quasigroup $(Q, \cdot)$ and $\lambda \in Q$, we define the binary operation $\cdot_\lambda$ on Q, and equivalence relation $\sim$ on Q by

$$a \cdot_\lambda b = \lambda \backslash ((\lambda a) b) \ (a, b \in Q),$$

$$\lambda \sim \mu \iff a \cdot_\lambda b = a \cdot_\mu b \ (\forall a, b \in Q).$$

We write $H := Q/\sim$. Let $s : H \to Q$ be a right inverse of the projection $Q \ni \lambda \mapsto [\lambda] \in H$; that is, $s : H \to Q$ is a map satisfying $s([\lambda]) \sim \lambda$ for all $\lambda \in Q$, and we define a map $\cdot_Q : H \times Q \to H$ by $[\lambda] \cdot_Q a := [\lambda a]$ $(\lambda, a \in Q)$. This $Q = (Q, \cdot_Q)$ is an object of $\mathbf{Set}_H$.

Theorem 3. The maps $\sigma_1([\lambda]), \sigma_2([\lambda]) : Q \times Q \to Q \times Q,\ (\lambda \in Q)$ defined by:

$$\begin{aligned} &\sigma_1([\lambda])(a, b) = (s([\lambda]) \backslash s([\lambda]), \lambda \backslash ((\lambda a) b)) \\ &\sigma_2([\lambda])(a, b) = (\lambda \backslash ((\lambda a) b), s([(\lambda a) b]) \backslash s([(\lambda a) b])) \quad (a, b \in Q) \end{aligned} \tag{1.0.13}$$

are idempotent dynamical Yang-Baxter maps.

In this construction the set of dynamical parameters H has a relation with the left nucleus

$$N_l(Q) = \{a \in Q | (a \cdot x) \cdot y = a \cdot (x \cdot y) \ (\forall x, y \in Q)\},$$

of the left quasigroup $(Q, \cdot)$ (See Remark 6). If Q is a group, which is an example of left quasigroup, both σ_1 and σ_2 are the same as the Yang-Baxter map in 1.0.7 for any right inverse s.

Chapter 2

Dynamical braces and dynamical Yang-Baxter maps

2.1 Dynamical Yang-Baxter maps

Let X, H be non-empty sets and ϕ a map from $H \times X$ to H. We call elements of H dynamical parameters.

Definition 12. A map $R(\lambda) : X \times X \to X \times X$ ($\lambda \in H$) is a dynamical Yang-Baxter map (DYB map) associated with X, H, ϕ if $R(\lambda)$ satisfies the following equation on $X \times X \times X$ for all $\lambda \in H$:

$$R_{23}(\lambda)R_{13}(\phi(\lambda, X^{(2)}))R_{12}(\lambda) = R_{12}(\phi(\lambda, X^{(3)}))R_{13}(\lambda)R_{23}(\phi(\lambda, X^{(1)})). \tag{2.1.1}$$

Here $R_{12}(\lambda), R_{12}(\phi(\lambda, X^{(3)})), \cdots$ are maps from $X \times X \times X$ to $X \times X \times X$ defined by

$$R_{12}(\lambda)(a, b, c) = (R(\lambda)(a, b), c),$$
$$R_{12}(\phi(\lambda, X^{(3)}))(a, b, c) = (R(\phi(\lambda, c))(a, b), c) \quad (a, b, c \in X).$$

As a special case of DYB maps, we can define Yang-Baxter maps as follows.

Definition 13. A map $R : X \times X \to X \times X$ is a Yang-Baxter map(YB map) if R satisfies the following equation on $X \times X \times X$:

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}. \tag{2.1.2}$$

Here R_{ij} are defined in the same way as in the definition above.

As can be seen from the definitions above, a YB map is just a DYB map which is independent of the dynamical parameter.

We represent a map $R(\lambda) : X \times X \to X \times X$ $(\lambda \in H)$ by

$$R(\lambda)(a,b) = (\mathfrak{R}_b^\lambda(a), \mathfrak{L}_a^\lambda(b)) \quad (\lambda, a, b) \in H \times X \times X. \tag{2.1.3}$$

For $(a, \lambda) \in X \times H$, we define maps $\mathfrak{L}_a^\lambda : X \to X, \mathfrak{R}_a^\lambda : X \to X$ by

$$\mathfrak{L}_a^\lambda : b \mapsto \mathfrak{L}_a^\lambda(b), \mathfrak{R}_a^\lambda : b \mapsto \mathfrak{R}_a^\lambda(b). \tag{2.1.4}$$

For $\lambda \in H$, we set $\mathfrak{L}^\lambda : X \times X \to X, \mathfrak{R}^\lambda : X \times X \to X$ by

$$\mathfrak{L}^\lambda : (a,b) \mapsto \mathfrak{L}_a^\lambda(b), \mathfrak{R}^\lambda : (a,b) \mapsto \mathfrak{R}_b^\lambda(a). \tag{2.1.5}$$

Let $\mathfrak{L}$ be a map $\lambda \mapsto \mathfrak{L}^\lambda$ and $\mathfrak{R}$ a map $\lambda \mapsto \mathfrak{R}^\lambda$. By rewriting the definition of the DYB map we obtain the next lemma.

Lemma 1. A map $R(\lambda) : X \times X \to X \times X$ $(\lambda \in H)$ associated with X, H, ϕ is a DYB map if and only if $\mathfrak{L}, \mathfrak{R}$ satisfies the next three relations for all $(\lambda, a, b, c) \in H \times X \times X \times X$:

$$\mathfrak{L}_a^\lambda \cdot \mathfrak{L}_b^{\phi(\lambda,a)} = \mathfrak{L}_{\mathfrak{L}_a^\lambda(b)}^\lambda \cdot \mathfrak{L}_{\mathfrak{R}_b^\lambda(a)}^{\phi(\lambda,\mathfrak{L}_a^\lambda(b))}, \tag{2.1.6}$$

$$\mathfrak{R}^\lambda_{(\mathfrak{L}^{\phi(\lambda,\mathfrak{L}_a^\lambda(b))}_{\mathfrak{R}_b^\lambda(a)}(c))} \cdot \mathfrak{L}_a^\lambda(b) = \mathfrak{L}^{\phi(\lambda,\mathfrak{L}_a^\lambda \mathfrak{L}_b^{\phi(\lambda,a)}(c))}_{(\mathfrak{R}^\lambda_{\mathfrak{L}_b^{\phi(\lambda,a)}(c)}(a))} \cdot \mathfrak{R}_c^{\phi(\lambda,a)}(b), \tag{2.1.7}$$

$$\mathfrak{R}_c^{\phi(\lambda,\mathfrak{L}_a^\lambda(b))} \cdot \mathfrak{R}_b^\lambda(a) = \mathfrak{R}^{\phi(\lambda,\mathfrak{L}_a^\lambda \mathfrak{L}_b^{\phi(\lambda,a)}(c))}_{(\mathfrak{R}_c^{\phi(\lambda,a)}(b))} \cdot \mathfrak{R}^\lambda_{\mathfrak{L}_b^{\phi(\lambda,a)}(c)}(a). \tag{2.1.8}$$

Proof. The proof is straightforward. □

Definition 14. Let $R(\lambda)$ be a DYB map associated with X, H, ϕ.

(1) We say that $R(\lambda)$ is left non-degenerate if the map $\mathfrak{R}_a^\lambda$ is bijection, and $R(\lambda)$ is called right non-degenerate if the map $\mathfrak{L}_b^\lambda$ is bijection for all $(\lambda, a, b) \in H \times X \times X$. When $R(\lambda)$ is left and right non-degenerate we call it non-degenerate.

(2) Let P be a map from $X \times X$ to $X \times X$ defined by $P(a,b) = (b,a)$. We say that $R(\lambda)$ satisfies the unitary condition if $R(\lambda)$ satisfies $PR(\lambda)PR(\lambda)$ $= \mathrm{id}_{X \times X}$ for all $\lambda \in H$. When a DYB map satisfies the unitary condition we call it a unitary DYB map.

(3) We call the next condition about a map $\phi : H\times X \to H$ the weight-zero condition:
$$\phi(\phi(\lambda,a),b) = \phi(\phi(\lambda,\mathfrak{L}_a^\lambda(b)),\mathfrak{R}_b^\lambda(a)),$$
for all $(\lambda,a,b) \in H \times X \times X$.

Lemma 2. A DYB map $R(\lambda) : X \times X \to X \times X (\lambda \in H)$ associated with X, H, ϕ satisfies the unitary condition if and only if $\mathfrak{L}, \mathfrak{R}$ satisfy the next relations for all $(\lambda,a,b) \in H \times X \times X$:

$$\mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)} \cdot \mathfrak{R}_b^\lambda(a) = a, \tag{2.1.9}$$

$$\mathfrak{R}^\lambda_{\mathfrak{R}_b^\lambda(a)} \cdot \mathfrak{L}_a^\lambda(b) = b. \tag{2.1.10}$$

Proof. The proof is straightforward. □

Example 1. (1) Let X be a non-empty set and $\mathrm{id}_{X\times X}$ the identity map. Then $(X, \mathrm{id}_{X\times X})$ is a unitary YB map. We call this YB map the trivial solution.

(2) (Lyubashenko, see [10]) Let X be a non-empty set. $r : X \times X \to X \times X, (a,b) \mapsto (\mathfrak{R}(a), \mathfrak{L}(b))$. Here $\mathfrak{L}, \mathfrak{R}$ are maps from X to X. Suppose that $\mathfrak{L}$ and $\mathfrak{R}$ are bijections. Then (X, r) is a YB map if and only if $\mathfrak{L}\mathfrak{R} = \mathfrak{R}\mathfrak{L}$. Moreover (X, r) satisfies the unitary condition if and only if $\mathfrak{R} = \mathfrak{L}^{-1}$. We call this solution (X, r) a permutation solution.

The following proposition give relations between two DYB maps associated with distinct spaces.

Proposition 1. [33, Y. Shibukawa]

(1) Let H be a non-empty set and $R'(\lambda)$ a DYB map associated with X, H', ϕ. If there exist maps $\psi : H \to H'$, $\rho : H' \to H(\psi\rho = \mathrm{id}_{H'})$, then the map $R(\lambda) : X \times X \to X \times X$ $(\lambda \in H)$, $R(\lambda) = R'(\psi(\lambda))$ is a DYB map associated with $X, H, \rho\phi(\psi \times \mathrm{id}_X)$.

(2) Let X be a non-empty set and $R'(\lambda)$ a DYB map associated with X', H, ϕ. If there exist maps $\rho : X' \to X$, $\psi : X \to X'$ such that $(\psi\rho = \mathrm{id}_{X'})$, then the map $R(\lambda) : X \times X \to X \times X$ $(\lambda \in H), R(\lambda) = (\rho \times \rho)R'(\lambda)(\psi \times \psi)$ is a DYB map associated with $X, H, \phi(\mathrm{id}_X \times\psi)$.

Definition 15. Let $R(\lambda)$ be a DYB map associated with X, H, ϕ and $R^{'}(\lambda^{'})$ a DYB map associated with $X^{'}, H^{'}, \phi^{'}$. $R(\lambda)$ is equivalent to $R^{'}(\lambda^{'})$ if and only if there exist two bijections $F : X \to X^{'}$, $p : H \to H^{'}$ such that

(1) $p\phi = \phi^{'}(p \times F)$,

(2) $(F \times F)R(\lambda) = R^{'}(p(\lambda))(F \times F)$,

for all $\lambda \in H$.

The next theorem show us that the right non-degeneracy condition and the unitary condition are suitable conditions to simplify DYB maps. In [31], Rump showed it in the case of the YB maps.

Theorem 4. Let $\mathfrak{L}_a^\lambda : X \to X$ be bijections for all $(a, \lambda) \in X \times H$, and $\mathfrak{R}_b^\lambda(a) := (\mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)})^{-1}(a)$. Suppose that the maps $\mathfrak{L}_a^\lambda, \mathfrak{R}_b^\lambda$ satisfy the relation (2.1.6) of Lemma.1. Then a map $R(\lambda) : X \times X \to X \times X$ defined by

$$R(\lambda)(a, b) := (\mathfrak{R}_b^\lambda(a), \mathfrak{L}_a^\lambda(b)) = ((\mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)})^{-1}(a), \mathfrak{L}_a^\lambda(b))$$

is a right non-degenerate unitary DYB map associated with X, H, ϕ.

Proof. First, we show that the relation (2.1.7) follows from the relation (2.1.6).
Put $A = \mathfrak{L}^{\phi(\lambda, \mathfrak{L}_a^\lambda(b))}_{\mathfrak{R}_b^\lambda(a)}(c), B = \mathfrak{L}_a^\lambda \mathfrak{L}_b^{\phi(\lambda,a)}(c)$. Then

$$\begin{aligned}
\text{LHS of (2.1.7)} &= \mathfrak{R}_A^\lambda \mathfrak{L}_a^\lambda(b) \\
&= (\mathfrak{L}^\lambda_{\mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)}(A)})^{-1}\mathfrak{L}_a^\lambda(b) \\
&= (\mathfrak{L}_B^\lambda)^{-1}\mathfrak{L}_a^\lambda(b), \\
\text{RHS of (2.1.7)} &= \mathfrak{L}^{\phi(\lambda,B)}_{(\mathfrak{R}^\lambda_{\mathfrak{L}_b^{\phi(\lambda,a)}(c)}(a))}\mathfrak{R}_c^{\phi(\lambda,a)}(b) \\
&= \mathfrak{L}^{\phi(\lambda,B)}_{(\mathfrak{L}_B^\lambda)^{-1}(a)}(\mathfrak{L}^{\phi(\lambda,a)}_{\mathfrak{L}_b^{\phi(\lambda,a)}(c)})^{-1}(b).
\end{aligned}$$

Thus, we must prove $(\mathfrak{L}_B^\lambda)^{-1}\mathfrak{L}_a^\lambda(b) = \mathfrak{L}^{\phi(\lambda,B)}_{(\mathfrak{L}_B^\lambda)^{-1}(a)}(\mathfrak{L}^{\phi(\lambda,a)}_{\mathfrak{L}_b^{\phi(\lambda,a)}(c)})^{-1}(b)$. We have

$$\begin{aligned}
(\mathfrak{L}_a^\lambda)^{-1}\mathfrak{L}_B^\lambda\mathfrak{L}^{\phi(\lambda,B)}_{(\mathfrak{L}_B^\lambda)^{-1}(a)}(\mathfrak{L}^{\phi(\lambda,a)}_{\mathfrak{L}_b^{\phi(\lambda,a)}(c)})^{-1}(b) &= (\mathfrak{L}_a^\lambda)^{-1}(\mathfrak{L}_a^\lambda\mathfrak{L}^{\phi(\lambda,a)}_{\mathfrak{R}^\lambda_{(\mathfrak{L}_B^\lambda)^{-1}(a)}(B)})(\mathfrak{L}^{\phi(\lambda,a)}_{\mathfrak{L}_b^{\phi(\lambda,a)}(c)})^{-1}(b) \\
&= \mathfrak{L}^{\phi(\lambda,a)}_{\mathfrak{R}^\lambda_{(\mathfrak{L}_B^\lambda)^{-1}(a)}(B)}(\mathfrak{L}^{\phi(\lambda,a)}_{\mathfrak{L}_b^{\phi(\lambda,a)}(c)})^{-1}(b) \\
&= b.
\end{aligned}$$

Next, we show that the relation (2.1.8) follows from the relation (2.1.6). Put $X = \mathfrak{L}_a^\lambda(b), Y = \mathfrak{L}_{\mathfrak{R}_b^\lambda(a)}^{\phi(\lambda, \mathfrak{L}_a^\lambda(b))}(c), Z = \mathfrak{L}_X^\lambda(Y)$. Then $\mathfrak{R}_Y^\lambda(X)$ =LHS of (2.1.7) and

$$
\begin{aligned}
\text{LHS of (2.1.8)} &= (\mathfrak{L}^{\phi(\lambda, \mathfrak{L}_a^\lambda(b))}_{\mathfrak{L}^{\phi(\lambda, \mathfrak{L}_a^\lambda(b))}_{\mathfrak{R}_b^\lambda(a)}(c)})^{-1}(\mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)})^{-1}(a) \\
&= (\mathfrak{L}_X^\lambda \mathfrak{L}_Y^{\phi(\lambda,X)})^{-1}(a) \\
&= (\mathfrak{L}_Z^\lambda \mathfrak{L}^{\phi(\lambda,Z)}_{\mathfrak{R}_Y^\lambda(X)})^{-1}(a) \\
&= (\mathfrak{L}^{\phi(\lambda,Z)}_{\mathfrak{R}_Y^\lambda(X)})^{-1}(\mathfrak{L}_Z^\lambda)^{-1}(a) \\
&= (\mathfrak{L}^{\phi(\lambda,Z)}_{\mathfrak{L}^{\phi(\lambda,Z)}_{\mathfrak{R}^\lambda_{\mathfrak{L}_b^{\phi(\lambda,a)}(c)}(a)} \mathfrak{R}_c^{\phi(\lambda,a)}(b)})^{-1}(\mathfrak{L}_Z^\lambda)^{-1}(a) \\
&= \mathfrak{R}^{\phi(\lambda,Z)}_{\mathfrak{R}_c^{\phi(\lambda,a)}(b)} \mathfrak{R}^\lambda_{\mathfrak{L}_b^{\phi(\lambda,a)}(c)}(a) \\
&= \text{RHS of (2.1.8)}.
\end{aligned}
$$

$\square$

Next, we consider a non-empty set X with a commutative binary operator $+ : X \times X \to X, (x, y) \mapsto x + y$. (The associativity of $+$ is not assumed here.)

Corollary 1. Let $X = (X, +)$ be a non-empty set with a commutative binary operator $+$, and bijections $\mathfrak{L}_a^\lambda : X \to X$ satisfying

$$\mathfrak{L}_a^\lambda \cdot \mathfrak{L}_b^{\phi(\lambda,a)} = \mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)+a}, \tag{2.1.11}$$

for all $(\lambda, a, b) \in H \times X \times X$. Then a map $R(\lambda) : X \times X \to X \times X$

$$R(\lambda)(a, b) = (\mathfrak{R}_b^\lambda(a), \mathfrak{L}_a^\lambda(b)) := ((\mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)})^{-1}(a), \mathfrak{L}_a^\lambda(b)),$$

gives a right non-degenerate unitary DYB map associated with X, H, ϕ.

Proof. We show that the relation (2.1.6) follows from the relation (2.1.11).

$$\text{RHS of (2.1.6)} = \mathfrak{L}^\lambda_{\mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)}(\mathfrak{R}_b^\lambda(a)) + \mathfrak{L}_a^\lambda(b)} = \mathfrak{L}^\lambda_{a+\mathfrak{L}_a^\lambda(b)} = \text{LHS of (2.1.6)}.$$

$\square$

2.2 Braces and dynamical braces

In this section, we begin with an introduction of a relation between the brace and the YB map. This relation was proved by Rump in [32].

Definition 16. [32, W. Rump] Let $A = (A, +)$ be an abelian group with a multiplication $\cdot : A \times A \to A$. We call $(A, +, \cdot)$ a brace if the following conditions are satisfied for all $a, b, c \in A$:

(1) $(a + b) \cdot c = a \cdot c + b \cdot c$ (Right distributive law),

(2) $a \cdot (b \cdot c + b + c) = (a \cdot b) \cdot c + a \cdot b + a \cdot c$,

(3) The map $\gamma(b) : a \mapsto a \cdot b + a$ is bijective.

Proposition 2. [32, W. Rump] An abelian group $A = (A, +)$ with a right distributive multiplication is a brace if and only if A is a group with respect to the operation $a * b := a \cdot b + a + b$, $(a, b \in A)$.

Proposition 3. Let $(A, +, \cdot)$ be a brace and 0 the unit of the abelian group $(A, +)$. Then $(A, +, \cdot)$ satisfies the next relation for all $a \in A$,

$$0 \cdot a = a \cdot 0 = 0.$$

Proof. 1. $0 \cdot a = 0$ is trivial.
2. $a \cdot 0 = a \cdot (0 \cdot 0 + 0 + 0) = (a \cdot 0) \cdot 0 + a \cdot 0 + a \cdot 0$, hence $\gamma(0)(a \cdot 0) = (a \cdot 0) \cdot 0 + a \cdot 0 = 0 = 0 \cdot 0 + 0 = \gamma(0)(0)$. Therefore we obtain $a \cdot 0 = 0$ by using bijectivity of $\gamma(0)$. □

Example 2. [31, W. Rump] 1. Abelian group $(A, +)$ with a multiplication $a \cdot b = 0$ is a brace $(a, b \in A)$. We call this $(A, +, \cdot)$ trivial brace.
2. Let $R = (R, +, \cdot)$ be a ring and $\mathrm{Jac}(R)$ a Jacobson radical of R. Then $\mathrm{Jac}(R)$ has a group structure with respect to the operation $a*b = a \cdot b + a + b$ $(a, b \in \mathrm{Jac}(R))$. Therefore $(\mathrm{Jac}(R), +, \cdot)$ is a brace. In general, a ring $R = (R, +, \cdot)$ having a group structure with a multiplication $a * b = a \cdot b + a + b$ is called radical ring. On account of this, the brace is a generalization of the radical ring.

Theorem 5. [32, W. Rump] Let $(A, +, \cdot)$ be a brace. Then a map $R : A \times A \to A \times A$ defined by

$$R(a, b) := (\gamma(\gamma(a)(b))^{-1}(a), \gamma(a)(b)) \quad (a, b \in A),$$

is a non-degenerate unitary YB map.

Next we introduce the dynamical brace as a generalization of the brace.

Definition 17. Let H be a non-empty set, $A = (A, +)$ an abelian group with the family of multiplications $\{\cdot_\lambda : A \times A \to A\}_{\lambda \in H}$ and ϕ a map from $H \times A$ to H. We call $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda \in H})$ a dynamical brace (d-brace) if the following conditions are satisfied for all $(\lambda, a, b, c) \in H \times A \times A \times A$:

(1) $(a + b) \cdot_\lambda c = a \cdot_\lambda c + b \cdot_\lambda c$ (Right distributive law),

(2) $a \cdot_\lambda (b \cdot_\lambda c + b + c) = (a \cdot_{\phi(\lambda,c)} b) \cdot_\lambda c + a \cdot_{\phi(\lambda,c)} b + a \cdot_\lambda c$,

(3) The map $\gamma_\lambda(b) : a \mapsto a \cdot_\lambda b + a$ is bijective.

Definition 18. $(Q, \cdot)$ is a right quasigroup if and only if Q is a non-empty set with a binary operation $(\cdot)$ having the property below:

$$R(a) : Q \to Q, b \mapsto b \cdot a \quad \text{is bijective for all } a \in Q.$$

A left quasigroup are similarly defined, and a non-empty set Q with left and right quasigroup structure is called a quasigroup [30].

We can extend Proposition 2 to the d-brace as follows.

Proposition 4. Let H be a non-empty set, $A = (A, +)$ an abelian group with a family of right distributive multiplications $\{\cdot_\lambda : A \times A \to A\}_{\lambda \in H}$ and ϕ a map from $H \times A$ to H. Then $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda \in H})$ is a d-brace if and only if A is a right quasigroup with respect to operations

$$a *_\lambda b := a \cdot_\lambda b + a + b, \tag{2.2.1}$$

and satisfies the next relation for all $(\lambda, a, b, c) \in H \times A \times A \times A$,

$$(a *_{\phi(\lambda,c)} b) *_\lambda c = a *_\lambda (b *_\lambda c). \tag{2.2.2}$$

Proof. 1. Let $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda \in H})$ be a d-brace. Consider maps $R^*_\lambda(b) : a \mapsto a *_\lambda b = a \cdot_\lambda b + a + b = \gamma_\lambda(b)(a) + b$ $(b \in A)$. Because of bijectivity of $\gamma_\lambda(b)$, $R^*_\lambda(b)$ is bijection. Hence $(A, *_\lambda)$ is a right quasigroup. The relation (2.2.2) follows from conditions (1) and (2) of d-brace.
2. Suppose that A satisfies the conditions of proposition. Then the relation (2.2.2) implies condition (2) of Definition 17, and bijectivity of $\gamma_\lambda(b)$ follows from a right quasigroup structure of $(A, *_\lambda)$. $\square$

Note that

$$a *_\lambda b = a *_\mu b \iff a \cdot_\lambda b = a \cdot_\mu b, \tag{2.2.3}$$

for all $(\lambda, \mu, a, b) \in H \times H \times A \times A$.

Proposition 5. Let $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda \in H})$ be a d-brace. Then

$$\cdot_{\phi(\phi(\lambda,a),b)} = \cdot_{\phi(\lambda, b *_\lambda a)}, \tag{2.2.4}$$

as a map from $A \times A$ to A, for all $(\lambda, a, b) \in H \times A \times A$.

Proof. It follows from the next calculation:

$$\begin{aligned} (d *_{\phi(\phi(\lambda,a),b)} c) *_\lambda (b *_\lambda a) &= \{(d *_{\phi(\phi(\lambda,a),b)} c) *_{\phi(\lambda,a)} b\} *_\lambda a \\ &= \{d *_{\phi(\lambda,a)} (c *_{\phi(\lambda,a)} b)\} *_\lambda a \\ &= d *_\lambda \{c *_\lambda (b *_\lambda a)\} \\ &= (d *_{\phi(\lambda, b *_\lambda a)} c) *_\lambda (b_{*_\lambda} a). \end{aligned}$$

Therefore we obtain $d *_{\phi(\phi(\lambda,a),b)} c = d *_{\phi(\lambda, b *_\lambda a)} c$, for all $c, d \in A$. □

Proposition 6. Let $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda \in H})$ be a d-brace and 0 the unit of the abelian group $(A, +)$. Then

(1) $0 \cdot_\lambda a = 0$,

(2) $a \cdot_{\phi(\lambda,0)} 0 = 0$,

for all $(\lambda, a) \in H \times A$.

Proof. 1. $0 \cdot_\lambda a = 0$ is trivial.
2. $a \cdot_\lambda 0 = a \cdot_\lambda (0 \cdot_\lambda 0 + 0 + 0) = (a \cdot_{\phi(\lambda,0)} 0) \cdot_\lambda 0 + a \cdot_{\phi(\lambda,0)} 0 + a \cdot_\lambda 0$, hence $\gamma_\lambda(0)(a \cdot_{\phi(\lambda,0)} 0) = (a \cdot_{\phi(\lambda,0)} 0) \cdot_\lambda 0 + a \cdot_{\phi(\lambda,0)} 0 = 0 = 0 \cdot_\lambda 0 + 0 = \gamma_\lambda(0)(0)$. Therefore we obtain $a \cdot_{\phi(\lambda,0)} 0 = 0$ by using the bijectivity of $\gamma_\lambda(0)$. □

Definition 19. (1) Let $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda \in H})$ be a d-brace. If a multiplication $\cdot_\lambda$ satisfies $a \cdot_\lambda 0 = 0 \cdot_\lambda a = 0$ for all $a \in A$, we call $\cdot_\lambda$ zero-symmetric. We call the d-brace zero-symmetric if all multiplications of the d-brace are zero-symmetric.

(2) Let $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda \in H})$ be a d-brace and K a subset of H.
If $(A, K, \phi|_{K \times A}; +, \{\cdot_\lambda\}_{\lambda \in K})$ is again a d-brace, we call it a restricted d-brace.

(3) Two d-braces $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda\in H})$ and $(A^{'}, H^{'}, \phi^{'}; +^{'}, \{*_{\lambda^{'}}\}_{\lambda^{'}\in H^{'}})$ are isomorphic if and only if there are bijections $F : A \to A^{'}$, $p : H \to H^{'}$ such that

(a) $F(a + b) = F(a) +^{'} F(b)$,

(b) $F(a \cdot_\lambda b) = F(a) *_{p(\lambda)} F(b)$,

(c) $p\phi = \phi^{'}(p \times F)$,

for all $(\lambda, a, b) \in H \times A \times A$.

In general, the d-brace is not zero-symmetric (see Example 5).

Let us reconsider Corollary 1 stated in the section 2.1. Suppose that $A = (A, +)$ is an abelian group, H a non-empty set and ϕ a map from $H \times A$ to H. To obtain a DYB map associated with A, H, ϕ, we need to construct maps $\mathfrak{L}^\lambda_a : A \to A$ that satisfy $\mathfrak{L}^\lambda_a \cdot \mathfrak{L}^{\phi(\lambda,a)}_b = \mathfrak{L}^\lambda_{\mathfrak{L}^\lambda_a(b)+a}$, for all $(\lambda, a, b) \in H \times A \times A$. The next theorem states a relation between d-braces and DYB maps. This theorem is a generalization of Theorem 5 to the case of the DYB maps.

Theorem 6. Let $A = (A, +)$ be an abelian group, H a non-empty set and ϕ a map from $H \times A$ to H.

(1) Let $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda\in H})$ be a d-brace. Then $\{\mathfrak{L}^\lambda_a := \gamma_\lambda(a) : A \to A\}_{(\lambda,a)\in H\times A}$ is a family of automorphisms of the abelian group $(A, +)$ that satisfies $\mathfrak{L}^\lambda_a \cdot \mathfrak{L}^{\phi(\lambda,a)}_b = \mathfrak{L}^\lambda_{\mathfrak{L}^\lambda_a(b)+a}$, for all $(\lambda, a, b) \in H \times A \times A$.

(2) Let $\{\mathfrak{L}^\lambda_a : A \to A\}_{(\lambda,a)\in H\times A}$ be a family of automorphisms of the abelian group $(A, +)$ that satisfies $\mathfrak{L}^\lambda_a \cdot \mathfrak{L}^{\phi(\lambda,a)}_b = \mathfrak{L}^\lambda_{\mathfrak{L}^\lambda_a(b)+a}$. Define multiplications on A by $a \cdot_\lambda b := \mathfrak{L}^\lambda_b(a) - a$, for all $(\lambda, a, b) \in H \times A \times A$. Then $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda\in H})$ is a d-brace.

(3) The correspondence between (1) and (2) is one-to-one.

Proof. 1. We prove that $\{\mathfrak{L}^\lambda_a := \gamma_\lambda(a) : A \to A\}_{(\lambda,a)\in H\times A}$ is a family of automorphisms of the abelian group A and satisfies $\mathfrak{L}^\lambda_a \cdot \mathfrak{L}^{\phi(\lambda,a)}_b = \mathfrak{L}^\lambda_{\mathfrak{L}^\lambda_a(b)+a}$, for all $(\lambda, a, b) \in H \times A \times A$.

(i) As a result of the definition of d-brace, $\mathfrak{L}_a^\lambda$ is automorphism.
(ii) The relation $\mathfrak{L}_a^\lambda \cdot \mathfrak{L}_b^{\phi(\lambda,a)} = \mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)+a}$ is proved as follows.

$$\begin{aligned}
\text{LHS} &= \gamma_\lambda(a)\gamma_{\phi(\lambda,a)}(b)(c) \\
&= (c \cdot_{\phi(\lambda,a)} b + c) \cdot_\lambda a + c \cdot_{\phi(\lambda,a)} b + c \\
&= (c \cdot_{\phi(\lambda,a)} b) \cdot_\lambda a + c \cdot_\lambda a + c \cdot_{\phi(\lambda,a)} b + c \\
&= c \cdot_\lambda (b \cdot_\lambda a + b + a) + c \\
&= c \cdot_\lambda (\gamma_\lambda(a)(b) + a) + c \\
&= \gamma_\lambda(\gamma_\lambda(a)(b) + a)(c) \\
&= \text{RHS}.
\end{aligned}$$

2. We prove that $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda\in H})$ is a d-brace.
(i) By the definition of $\mathfrak{L}_b^\lambda$, multiplication $\cdot_\lambda$ satisfies the right distributive law.
(ii) As a consequence of $\gamma_\lambda(b)(a) = a \cdot_\lambda b + a = \mathfrak{L}_b^\lambda(a)$, $\gamma_\lambda(b)$ is a bijection.
(iii) The relation $a \cdot_\lambda (b \cdot_\lambda c + b + c) = (a \cdot_{\phi(\lambda,c)} b) \cdot_\lambda c + a \cdot_{\phi(\lambda,c)} b + a \cdot_\lambda c$ is proved as follows.

$$\begin{aligned}
\text{LHS} &= a \cdot_\lambda (\mathfrak{L}_c^\lambda(b) + c) \\
&= \mathfrak{L}^\lambda_{\mathfrak{L}_c^\lambda(b)+c}(a) - a \\
&= \mathfrak{L}_c^\lambda \mathfrak{L}_b^{\phi(\lambda,c)}(a) - a \\
&= \mathfrak{L}_c^\lambda(a \cdot_{\phi(\lambda,c)} b + a) - a \\
&= \mathfrak{L}_c^\lambda(a \cdot_{\phi(\lambda,c)} b) + \mathfrak{L}_c^\lambda(a) - a \\
&= \text{RHS}.
\end{aligned}$$

3. The proof is straightforward. □

Corollary 2. Let $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda\in H})$ be a d-brace. Then the map $R(\lambda) : A \times A \to A \times A$ $(\lambda \in H)$ defined by

$$R(\lambda)(a, b) = (\mathfrak{R}_b^\lambda(a), \mathfrak{L}_a^\lambda(b)) := (\gamma_\lambda(\gamma_\lambda(a)(b))^{-1}(a), \gamma_\lambda(a)(b)), \tag{2.2.5}$$

is a right non-degenerate unitary DYB map associated with A, H, ϕ.

Proposition 7. Let $\{\mathfrak{L}_a^\lambda : A \to A\}_{(\lambda,a)\in H\times A}$ be a family of automorphisms of the abelian group A and satisfies,

$$\mathfrak{L}_a^\lambda \cdot \mathfrak{L}_b^{\phi(\lambda,a)} = \mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)+a} \text{ (for all } (\lambda, a, b) \in H \times A \times A).$$

Then $\{\mathfrak{L}_a^\lambda : A \to A\}_{(\lambda,a)\in H\times A}$ satisfies

$$\mathfrak{L}_c^{\phi(\phi(\lambda,a),b)} = \mathfrak{L}_c^{\phi(\lambda,\mathfrak{L}_a^\lambda(b)+a)},$$

for all $(\lambda, a, b, c) \in H \times A \times A \times A$.

Proof.

$$\begin{aligned}
\mathfrak{L}_a^\lambda(\mathfrak{L}_b^{\phi(\lambda,a)}\mathfrak{L}_c^{\phi(\phi(\lambda,a),b)}) &= \mathfrak{L}_a^\lambda\mathfrak{L}^{\phi(\lambda,a)}_{\mathfrak{L}_b^{\phi(\lambda,a)}(c)+b} \\
&= \mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(\mathfrak{L}_b^{\phi(\lambda,a)}(c)+b)+a} \\
&= \mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda\mathfrak{L}_b^{\phi(\lambda,a)}(c)+\mathfrak{L}_a^\lambda(b)+a} \\
&= \mathfrak{L}^\lambda_{\mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)+a}(c)+\mathfrak{L}_a^\lambda(b)+a} \\
&= \mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)+a}\mathfrak{L}_c^{\phi(\lambda,\mathfrak{L}_a^\lambda(b)+a)} \\
&= \mathfrak{L}_a^\lambda(\mathfrak{L}_b^{\phi(\lambda,a)}\mathfrak{L}_c^{\phi(\lambda,\mathfrak{L}_a^\lambda(b)+a)}).
\end{aligned}$$

□

As a consequence of this corollary, if the map $\mathfrak{L} : H \to \mathrm{Map}(A \times A, A)$ is an injection, ϕ satisfies the weight-zero condition. For $\mathfrak{R}_a^\lambda(b) := (\mathfrak{L}^\lambda_{\mathfrak{L}_a^\lambda(b)})^{-1}(a)$.

Remark 1. In general, it seems to be natural to assume that the d-brace $(A, H, \phi; +, \{\cdot_\lambda\}_{\lambda\in H})$ satisfies the condition $\cdot_\lambda = \cdot_\mu \iff \lambda = \mu$, $(\lambda, \mu \in H)$. For this reason, the injectivity of $\mathfrak{L}$ with respect to parameter H seems to be natural. Therefore the weight-zero condition also seems to be natural.

The next theorem gives a relation between brace structures and d-brace structures over modules. (i.e., a relation between some YB maps and DYB maps).

Theorem 7. Let G be a group, $A = (A, +)$ a G-module and $(A, +, \cdot)$ a brace. We denote an action of $\lambda \in G$ by f_λ. Suppose ϕ be a map from $G \times A$ to G, and define multiplications $\cdot_\lambda$ $(\lambda \in G)$ over A by

$$a \cdot_\lambda b := f_\lambda^{-1}(f_{\phi(\lambda,b)}(a) \cdot f_\lambda(b) + f_{\phi(\lambda,b)}(a)) - a, \tag{2.2.6}$$

for all $a, b \in A$. Then $(A, G, \phi; +, \{\cdot_\lambda\}_{\lambda\in G})$ is a d-brace if and only if the map $\phi : G \times A \to G$ satisfies

$$f_{\phi(\lambda,b*_\lambda a)} = f_{\phi(\phi(\lambda,a),b)},$$

for all $(\lambda, a, b) \in G \times A \times A$. Here multiplications $*_\lambda$ are defined by $a *_\lambda b := a \cdot_\lambda b + a + b$.

Proof. Using $a \cdot_\lambda b = f_\lambda^{-1}(f_{\phi(\lambda,b)}(a) \cdot f_\lambda(b) + f_{\phi(\lambda,b)}(a)) - a$ we can express operations $*_\lambda$ as follows

$$a *_\lambda b = f_\lambda^{-1}(f_{\phi(\lambda,b)}(a) * f_\lambda(b)),$$

this multiplication satisfies the right distributivity, and $(A, *_\lambda)$ is a right quasigroup for all $\lambda \in H$. To obtain the theorem we need to see $(a *_{\phi(\lambda,c)} b) *_\lambda c = a *_\lambda (b *_\lambda c)$.

$$\begin{aligned} \text{LHS} &= f_\lambda^{-1}(f_{\phi(\lambda,c)}(a *_{\phi(\lambda,c)} b) * f_\lambda(c)) \\ &= f_\lambda^{-1}((f_{\phi(\phi(\lambda,c),b)}(a) * f_{\phi(\lambda,c)}(b)) * f_\lambda(c)), \end{aligned}$$

and

$$\begin{aligned} \text{RHS} &= f_\lambda^{-1}(f_{\phi(\lambda,b*_\lambda c)}(a) * f_\lambda(b *_\lambda c)) \\ &= f_\lambda^{-1}(f_{\phi(\lambda,b*_\lambda c)}(a) * (f_{\phi(\lambda,c)}(b) * f_\lambda(c))), \end{aligned}$$

hence we obtain the theorem by comparison of LHS and RHS. □

Remark 2. If an action of group G is faithful, a map ϕ satisfies

$$\phi(\lambda, b *_\lambda a) = \phi(\phi(\lambda, a), b) \quad ((\lambda, a, b) \in G \times A \times A). \tag{2.2.7}$$

This relation corresponds to the weight-zero condition in the DYB map.

Example 3. Let $(F, +, \times)$ be any field with a trivial brace structure $\cdot$. Define an action of $a \in F$ by $f_a(b) := a^2 b$ and define $\phi : F \times F \to F$ by $\phi(a, b) := f_a(b) + a = a(ab + 1)$. From Theorem 7 we obtain $a \cdot_b c = \{(bc+1)^2 - 1\}a$. Then ϕ satisfies $\phi(a, b *_a c) = \phi(\phi(a, c), b)$ (i.e., weight-zero condition). Hence $(F, F, \phi; +, \{\cdot_a\}_{a \in F})$ is a d-brace.

The DYB map $R(a)$ $(a \in F)$ associated with F, F, ϕ which corresponds to this d-brace is as follows.

$$R(a)(b, c) = (\{a(ab + 1)^2 c + 1\}^{-1} b, (ab + 1)^2 c),$$

for all $a, b, c \in F$.

2.3 Combinatorial aspects of dynamical braces

In this section, we give the combinatorial aspects of the d-brace. From these aspects we obtain a way to describe the d-brace as some family of subsets.

Theorem 8. Let $(A,+)$ be an abelian group and H a non-empty set.

(1) Let $(A,H,\phi;+,\{\cdot_\lambda\}_{\lambda\in H})$ be a d-brace. We set a family of subsets $\{S_\lambda\}_{\lambda\in H}$ as follows. $S_\lambda := \{R_\lambda(a) : A \to A, b \mapsto b *_\lambda a | a \in A\} \subset A \rtimes \mathrm{Aut}(A)$. Then, $\{S_\lambda\}_{\lambda\in H}$ satisfy the following conditions:

  (a) $\forall a \in A$, $\exists! f \in \mathrm{Aut}(A)$ s.t., $(a,f) \in S_\lambda$,

  (b) $\forall (a,f) \in S_\lambda$, $\exists! \mu \in H$ s.t., $(a,f)^{-1}S_\lambda = \{(a,f)^{-1}(b,g) | (b,g) \in S_\lambda\} = S_\mu$.

  We denote the unique $f \in \mathrm{Aut}(A)$ of condition (a) by $f_\lambda(a)$.

(2) Let $\{S_\lambda\}_{\lambda\in H}$ be a family of subsets of $A \rtimes \mathrm{Aut}(A)$ and suppose that $\{S_\lambda\}_{\lambda\in H}$ satisfy the above conditions (a) and (b). Define multiplications $\{\cdot_\lambda\}_{\lambda\in H}$ on A by $a \cdot_\lambda b := f_\lambda(b)(a) - a$, and define a map ϕ from $H \times A$ to H by $\phi(\lambda,a) = \mu$, which determines uniquely in the condition (b). Then $(A,H,\phi;+,\{\cdot_\lambda\}_{\lambda\in H})$ is a d-brace.

(3) The correspondence between (1) and (2) is one-to-one.

Proof. 1. Because of $R_\lambda(a)(b) = b *_\lambda a = b \cdot_\lambda a + b + a = \gamma_\lambda(a)(b) + a$ and $\gamma_\lambda(a) \in \mathrm{Aut}(A)$, we can regard $R_\lambda(a)$ as an action of $(a, \gamma_\lambda(a))$. Therefore $S_\lambda \simeq \{(a,\gamma_\lambda(a)) | a \in A\} \subset A \rtimes \mathrm{Aut}(A)$. Next we prove that $\{S_\lambda\}_{\lambda\in H}$ satisfies conditions (a) and (b).
(i) Condition (a) follows from the definition of S_λ.
(ii) For $R_{\phi(\lambda,a)}(b) \in S_{\phi(\lambda,a)}$, we obtain $R_\lambda(a)R_{\phi(\lambda,a)}(b)(c) = (c *_{\phi(\lambda.a)} b) *_\lambda a = c *_\lambda (b *_\lambda a) = R_\lambda(R_\lambda(a)(b))(c)$ for all $b \in A$. Because $R_\lambda(a)$ is a bijection, we obtain the following equality

$$\begin{aligned} R_\lambda(a)S_{\phi(\lambda,a)} &= \{R_\lambda(a)R_{\phi(\lambda,a)}(b) | b \in A\} \\ &= \{R_\lambda(R_\lambda(a)(b)) | b \in A\} \\ &= S_\lambda. \end{aligned}$$

2. We prove that $(A, *_\lambda)$ is right quasigroup, and satisfies $(a *_{\phi(\lambda,c)} b) *_\lambda c = a *_\lambda (b *_\lambda c)$ for all $(\lambda,a,b,c) \in H \times A \times A \times A$.

(i) By definition of multiplications, we obtain $a *_\lambda b = f_\lambda(b)(a) + b$, therefore $*_\lambda$ is an action of $(b, f_\lambda(b))$. Hence $(A, *_\lambda)$ is right quasigroup.
(ii) We prove that $(A, *_\lambda)$ satisfies the relation (2.2.2). Take $(b, f_{\phi(\lambda,c)(b)}) \in S_{\phi(\lambda,c)}$, $(c, f_\lambda(c)) \in S_\lambda$, by definition of ϕ

$$(c, f_\lambda(c))(b, f_{\phi(\lambda,c)}(b)) = (c + f_\lambda(c)(b), f_\lambda(c)f_{\phi(\lambda,c)}(b)) \in S_\lambda.$$

Therefore $(c + f_\lambda(c)(b), f_\lambda(c)f_{\phi(\lambda,c)}(b)) = (c + f_\lambda(c)(b), f_\lambda(c + f_\lambda(c)(b)))$ by condition (a). From this we obtain the relation (2.2.2) as follows

$$\begin{aligned} (a *_{\phi(\lambda,c)} b) *_\lambda c &= f_\lambda(c)f_{\phi(\lambda,c)}(b)(a) + f_\lambda(c)(b) + c \\ &= f_\lambda(f_\lambda(c)(b) + c)(a) + f_\lambda(c)(b) + c \\ &= a *_\lambda (b *_\lambda c). \end{aligned}$$

3. The proof is straightforward. □

A subgroup S of $A \rtimes \mathrm{Aut}(A)$ is said to be regular if, given any $a \in A$, then for each $b \in A$ there exists a unique $x \in S$ such that $x.a = b$. Here . denotes an action of S. From this, we express regular subgroup as $S = \{(a, f(a)) | a \in A\}$.

Corollary 3. Let $A = (A, +)$ be an abelian group.

(1) Let $(A, +, \cdot)$ be a brace. Then $\{R(a) : A \to A, b \mapsto b * a | a \in A\}$ is a regular subgroup of $A \rtimes \mathrm{Aut}(A)$.

(2) Let S be a regular subgroup of $A \rtimes \mathrm{Aut}(A)$. Define a multiplication on A by $a \cdot b := f(b)(a) - a$. Then $(A, +, \cdot)$ is a brace.

(3) The correspondence between (1) and (2) is one-to-one.

Proof. A case of $\#(H) = 1$. □

Remark 3. F.Catino and R.Frizz have obtained a similar result in [5, 6]. In [6] they called an algebra with brace structure a radical circle algebra.

From Theorem 8, we obtain a way to construct d-braces. i.e., start from a subset X with condition (a), and consider all subsets $\{(a, f)^{-1}X\}_{(a,f)\in X}$. Continue this operations to all subsets $\{(a, f)^{-1}X\}_{(a,f)\in X}$ until it closes.

The next proposition corresponds to Proposition 5 and Proposition 7.

Proposition 8. Let $\{S_\lambda\}_{\lambda\in H}$ be a family of subsets of $A \rtimes \mathrm{Aut}(A)$, and suppose that $\{S_\lambda\}_{\lambda\in H}$ satisfy the conditions of Theorem 8. Then $\{S_\lambda\}_{\lambda\in H}$ satisfy

$$S_{\phi(\phi(\lambda,a),b)} = S_{\phi(\lambda,f_\lambda(a)(b)+a)},$$

for all $(\lambda, a, b) \in H \times A \times A$.

Proof. Because of condition (b) of Theorem 8, we obtain
$(a, f_\lambda(a))^{-1}S_\lambda = S_{\phi(\lambda,a)}$. Therefore

$$\begin{aligned} S_{\phi(\phi(\lambda,a),b)} &= (b, f_{\phi(\lambda,a)}(b))^{-1}S_{\phi(\lambda,a)} \\ &= (b, f_{\phi(\lambda,a)}(b))^{-1}\{(a, f_\lambda(a))^{-1}S_\lambda\} \\ &= \{(b, f_{\phi(\lambda,a)}(b))^{-1}(a, f_\lambda(a))^{-1}\}S_\lambda \\ &= (f_\lambda(a)(b) + a, f_\lambda(a)f_{\phi(\lambda,a)}(b))^{-1}S_\lambda \\ &= S_{\phi(\lambda,f_\lambda(a)(b)+a)}. \end{aligned}$$

□

2.4 Graphs of dynamical braces and their properties

Let $(A, +)$ be an abelian group, H a non-empty set and $\{S_\lambda\}_{\lambda\in H}$ a family of subsets of $A \rtimes \mathrm{Aut}(A)$. Set $S_\lambda = \{(a, f_\lambda(a))|a \in A\}$ and suppose that $\{S_\lambda\}_{\lambda\in H}$ satisfies the conditions (a) and (b) of Theorem 8. Then, by the condition (b) we obtain a directed graph $G(A)$ that consists of

$$V(A) = \{S_\lambda|\lambda \in H\} \quad \text{(vertex set)},$$

$$E(A) = \{(S_\lambda, S_{\phi(\lambda,a)})|\lambda \in H, a \in A\} \quad \text{(edge set)}.$$

We call this graph $G(A)$ associated with $(A, +, \{S_\lambda\}_{\lambda\in H})$ a graph of d-brace.

As a consequence of this, vertex S_λ corresponds to multiplication $\cdot_\lambda$ (i.e., dynamical parameters correspond to vertices of graphs), map ϕ means a connection of edges, and $\#(A)$ is a degree of graph. This graph has the following properties.

Proposition 9. (1) Each vertex $S_{\phi(\lambda,a)}$ has a loop. Namely $(S_{\phi(\lambda,a)}, S_{\phi(\lambda,a)}) \in E(A)$ for all $(\lambda, a) \in H \times A$.

(2) The edge $(S_{\phi(\lambda,a)}, S_{\phi(\phi(\lambda,a),b)}) \in E(A)$ has an inverse edge. Namely $(S_{\phi(\phi(\lambda,a),b)}, S_{\phi(\lambda,a)}) \in E(A)$ for all $(\lambda, a, b) \in H \times A \times A$.

(3) For the edge $(S_{\phi(\lambda,a)}, S_{\phi(\lambda,a)}) \in E(A)$, the corresponding multiplication $\cdot_{\phi(\lambda,a)}$ is zero-symmetric. Hence all d-braces include a zero-symmetric restricted d-brace.

(4) Two isomorphic d-braces give the same underlying graph.

Proof. 1. By Proposition 8.

$$(S_{\phi(\lambda,a)}, S_{\phi(\lambda,a)}) = (S_{\phi(\lambda,a)}, S_{\phi(\phi(\lambda,a),0))}) \in E(A).$$

2. By definition $(S_{\phi(\phi(\lambda,a),b)}, S_{\phi(\phi(\phi(\lambda,a),b),f_{\phi(\lambda,a)}(b)^{-1}(-b)})) \in E(A)$, and

$$S_{\phi(\phi(\phi(\lambda,a),b),f_{\phi(\lambda,a)}(b)^{-1}(-b))} = S_{\phi(\phi(\lambda,a),0)} = S_{\phi(\lambda,a)},$$

follows from Proposition 8. Therefore $(S_{\phi(\phi(\lambda,a),b)}, S_{\phi(\lambda,a)}) \in E(A)$.
3. This follows from the definition of $\cdot_\lambda$ and Proposition 6. For the latter part, we restrict the set of dynamical parameters to $\mathrm{Im}\phi$. Elements of $\mathrm{Im}\phi$ correspond to vertices with loops.
4. Let $(A, H, \phi; +_A, \{\cdot_\lambda\}_{\lambda\in H})$ and $(B, I, \psi; +_B, \{\cdot_\mu\}_{\mu\in I})$ be two isomorphic d-braces. By definition of isomorphism, there are maps $F : A \to A'$, $p : H \to H'$ such that F, p satisfy the conditions of Definition 19. If $(S_\lambda, S_{\phi(\lambda,a)}) \in E(A)$, then $(S_{p(\lambda)}, S_{p\phi(\lambda,a)}) = (S_{p(\lambda)}, S_{\psi(p(\lambda),F(a))}) \in E(B)$. Therefore we obtain a bijection between two graphs. $\square$

Lastly, we give some examples of graphs, and corresponding multiplication tables of d-braces. In the following graphs, we ignore the degree of loops, i.e., we denote a loop by a single edge.

Example 4. Let A be an abelian group. Then A itself is a regular subgroup of $A \rtimes \mathrm{Aut}(A)$. This regular subgroup corresponds to the trivial brace structure on A. (See Example 2).

Example 5. Set $A = \{0, 1, 2\} = \mathbb{Z}_3$, $\mathrm{Aut}(A) = \{\mathrm{id}_A, \tau\}, \tau : (0, 1, 2) \mapsto (0, 2, 1)$ and

$$A \rtimes \mathrm{Aut}(A) = \{I = (0, \mathrm{id}_A), (0, \tau), (1, \mathrm{id}_A), (1, \tau), (2, \mathrm{id}_A), (2, \tau)\}.$$

Then following families of subsets of $A \rtimes \mathrm{Aut}(A)$ satisfy conditions (a) and (b): $(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}, S_{\lambda_4})$, $(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}, S_{\lambda_5})$, $(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}, S_{\lambda_6})$. Here

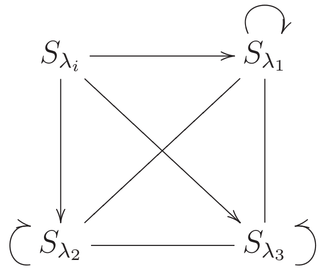
$$S_{\lambda_1} := \{I, (1,\tau), (2,\tau)\}, \quad S_{\lambda_2} := \{I, (1, \mathrm{id}_A), (2,\tau)\},$$

$$S_{\lambda_3} := \{I, (1,\tau), (2, \mathrm{id}_A)\}, \quad S_{\lambda_4} := \{(0,\tau), (1, \mathrm{id}_A), (2, \mathrm{id}_A)\},$$

$$S_{\lambda_5} := \{(0,\tau), (1,\tau), (2, \mathrm{id}_A)\}, \quad S_{\lambda_6} := \{(0,\tau), (1, \mathrm{id}_A), (2,\tau)\}.$$

In this case the set of dynamical parameters is $H = \{\lambda_1, \lambda_2, \lambda_3, \lambda_i\}$ $(i = 4, 5, 6)$.

Graphs of $(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}, S_{\lambda_i})$, and corresponding multiplication tables of $S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}, S_{\lambda_4}, S_{\lambda_5}, S_{\lambda_6}$ are as follows: (the three graphs are same).



$\cdot_{\lambda_1}$	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

$\cdot_{\lambda_2}$	0	1	2
0	0	0	0
1	0	0	1
2	0	0	2

$\cdot_{\lambda_3}$	0	1	2
0	0	0	0
1	0	1	0
2	0	2	0

$\cdot_{\lambda_4}$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

$\cdot_{\lambda_5}$	0	1	2
0	0	0	0
1	1	1	0
2	2	2	0

$\cdot_{\lambda_6}$	0	1	2
0	0	0	0
1	1	0	1
2	2	0	2

Therefore d-braces corresponding to $(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}, S_{\lambda_4})$, $(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}, S_{\lambda_5})$ and $(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}, S_{\lambda_6})$ are not isomorphic. Hence the inverse of Proposition 9. (4) is not true.

Moreover in this example, the triple $(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3})$ again satisfies conditions (a) and (b). From this we obtain a subgraph of the above graph as follows.

$$\begin{tikzcd}
 & S_{\lambda_1} \arrow[loop above] \arrow[dl, dash] \arrow[dr, dash] & \\
S_{\lambda_2} \arrow[loop left] \arrow[rr, dash] & & S_{\lambda_3} \arrow[loop right]
\end{tikzcd}$$

It means that the d-brace of $(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3})$ is a restricted d-brace of d-braces of $(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}, S_{\lambda_i})$, $i = 4, 5, 6$.

Example 6. We give three examples over $A = \{(0,0), (0,1), (1,0), (1,1)\} = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let τ, π and σ be automorphisms of A defined by

$$\tau : ((0,1), (1,0), (1,1)) \mapsto ((0,1), (1,1), (1,0)),$$

$$\pi : ((0,1), (1,0), (1,1)) \mapsto ((1,0), (0,1), (1,1)),$$

$$\sigma : ((0,1), (1,0), (1,1)) \mapsto ((1,0), (1,1), (0,1)).$$

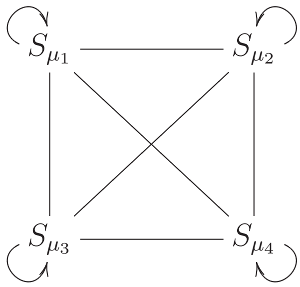
Then the pairs (S_{λ_0}), $(S_{\lambda_1}, S_{\lambda_2})$, $(S_{\mu_1}, S_{\mu_2}, S_{\mu_3}, S_{\mu_4})$

$$\begin{aligned}
S_{\lambda_0} &= \{I = ((0,0), \mathrm{id}_A), ((0,1), \pi), ((1,0), \pi), ((1,1), \mathrm{id}_A)\}, \\
S_{\lambda_1} &= \{I, ((0,1), \tau), ((1,0), \tau), ((1,1), \mathrm{id}_A)\}, \\
S_{\lambda_2} &= \{I, ((0,1), \tau), ((1,0), \mathrm{id}_A), ((1,1), \tau)\}, \\
S_{\mu_1} &= \{I, ((0,1), \tau), ((1,0), \sigma), ((1,1), \mathrm{id}_A)\}, \\
S_{\mu_2} &= \{I, ((0,1), \tau), ((1,0), \tau\sigma), ((1,1), \tau)\}, \\
S_{\mu_3} &= \{I, ((0,1), \sigma^{-1}), ((1,0), \tau\sigma), ((1,1), \sigma^{-1})\}, \\
S_{\mu_4} &= \{I, ((0,1), \sigma), ((1,0), \tau), ((1,1), \mathrm{id}_A)\},
\end{aligned}$$

satisfy the conditions (a) and (b). The graphs of (S_{λ_0}), $(S_{\lambda_1}, S_{\lambda_2})$ and $(S_{\mu_1}, S_{\mu_2}, S_{\mu_3}, S_{\mu_4})$ are as follows. Because $S_{\lambda_0} \not\simeq A$, S_{λ_0} corresponds to a non-trivial brace.

$$\begin{tikzcd}
S_{\lambda_0} \arrow[loop above]
\end{tikzcd}$$

$$\begin{tikzcd}
S_{\lambda_1} \arrow[loop left] \arrow[r, equal] & S_{\lambda_2} \arrow[loop right]
\end{tikzcd}$$



The multiplication tables of $\cdot_{\lambda_i}$, $\cdot_{\mu_j}$ that correspond to S_{λ_i}, S_{μ_j} are as follows: $i = 0, 1, 2$ and $j = 1, 2, 3, 4$.

$\cdot_{\lambda_0}$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(0,1)	(0,0)	(1,1)	(1,1)	(0,0)
(1,0)	(0,0)	(1,1)	(1,1)	(0,0)
(1,1)	(0,0)	(0,0)	(0,0)	(0,0)

$\cdot_{\lambda_1}$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(0,1)	(0,0)	(0,0)	(0,0)	(0,0)
(1,0)	(0,0)	(0,1)	(0,1)	(0,0)
(1,1)	(0,0)	(0,1)	(0,1)	(0,0)

$\cdot_{\lambda_2}$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(0,1)	(0,0)	(0,0)	(0,0)	(0,0)
(1,0)	(0,0)	(0,1)	(0,0)	(0,1)
(1,1)	(0,0)	(0,1)	(0,0)	(0,1)

$\cdot_{\mu_1}$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(0,1)	(0,0)	(0,0)	(1,1)	(0,0)
(1,0)	(0,0)	(0,1)	(0,1)	(0,0)
(1,1)	(0,0)	(0,1)	(1,0)	(0,0)

$\cdot_{\mu_2}$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(0,1)	(0,0)	(0,0)	(1,0)	(0,0)
(1,0)	(0,0)	(0,1)	(0,0)	(0,1)
(1,1)	(0,0)	(0,1)	(1,0)	(0,1)

$\cdot_{\mu_3}$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(0,1)	(0,0)	(1,0)	(1,0)	(1,0)
(1,0)	(0,0)	(1,1)	(0,0)	(1,1)
(1,1)	(0,0)	(0,1)	(1,0)	(0,1)

$\cdot_{\mu_4}$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(0,1)	(0,0)	(1,1)	(0,0)	(0,0)
(1,0)	(0,0)	(0,1)	(0,1)	(0,0)
(1,1)	(0,0)	(1,0)	(1,0)	(0,0)

Chapter 3

Quantum Yang-Baxter equation, braided semigroups, and dynamical Yang-Baxter maps

3.1 Tensor category $\mathbf{Set}_H$ and dynamical Yang-Baxter maps

In this section, we construct dynamical Yang-Baxter maps, which generalize (1.0.7), after a brief review of the tensor category $\mathbf{Set}_H$ [35, 36]. For category theory, see [20, 25].

Let H be a non-empty set. We denote by $\mathbf{Set}_H$ the following category: its objects are pairs $(X, \cdot_X)$ of a set X and a map $\cdot_X : H \times X \ni (\lambda, x) \mapsto \lambda \cdot_X x \in H$; its morphisms $f : (X, \cdot_X) \to (Y, \cdot_Y)$ are maps $f : H \to \mathrm{Map}(X, Y)$ satisfying $\lambda \cdot_Y f(\lambda)(x) = \lambda \cdot_X x \quad (\forall \lambda \in H, \forall x \in X)$; the identity 1 and the composition $\circ$ are defined by

$$1_X(\lambda)(x) = x \quad (\lambda \in H, x \in X) \text{ and } (g \circ f)(\lambda) = g(\lambda) \circ f(\lambda) \quad (\lambda \in H)$$

for objects X, Y, Z and morphisms $f : X \to Y$, $g : Y \to Z$. We will often write $\lambda \cdot_X x$ simply by λx.

This $\mathbf{Set}_H$ is a tensor category. In fact, the tensor product $X \otimes Y$ of the objects $X = (X, \cdot_X)$ and $Y = (Y, \cdot_Y)$ is a pair $(X \times Y, \cdot)$ consisting of the

Cartesian product $X \times Y$ and the following map $\cdot : H \times (X \times Y) \to H$.

$$\lambda \cdot (x, y) = (\lambda \cdot_X x) \cdot_Y y \quad (\lambda \in H, (x, y) \in X \times Y).$$

The tensor product of the morphisms $f : X \to X'$ and $g : Y \to Y'$ is defined by $(f \otimes g)(\lambda)(x, y) = (f(\lambda)(x), g(\lambda x)(y))$ $(\lambda \in H, (x, y) \in X \times Y)$. The definitions of the associativity constraint a, the unit I, and the left and the right unit constraints l, r are as follows: $a_{XYZ}(\lambda)((x, y), z) = (x, (y, z))$; $I = (\{e\}, \cdot_I)$, a pair of the set $\{e\}$ of one element and the map $\cdot_I$ defined by $\lambda \cdot_I e = \lambda$; $l_X(\lambda)(e, x) = x = r_X(\lambda)(x, e)$.

Definition 20. A morphism $\sigma : X \otimes X \to X \otimes X$ of $\mathbf{Set}_H$ is a dynamical Yang-Baxter map [27, 33, 35, 36], iff σ satisfies the QYBE (3.1.1) in $\mathbf{Set}_H$.

$$a \circ \sigma \otimes 1_X \circ a^{-1} \circ 1_X \otimes \sigma \circ a \circ \sigma \otimes 1_X = 1_X \otimes \sigma \circ a \circ \sigma \otimes 1_X \circ a^{-1} \circ 1_X \otimes \sigma \circ a. \quad (3.1.1)$$

Here, $a = a_{X,X,X}$.

If this dynamical Yang-Baxter map is an automorphism in $\mathbf{Set}_H$, then it is exactly a Yang-Baxter operator [20, Definition XIII.3.1].

Remark 4. (1) The dynamical Yang-Baxter map in the above definition is called a dynamical braiding map satisfying an invariance condition [34, Definition 2.8].

(2) If H is a set of one element, then the tensor category $\mathbf{Set}_H$ is the tensor category $\mathbf{Set}$ consisting of sets, and the dynamical Yang-Baxter map is exactly a Yang-Baxter map.

Every left quasigroup [30, 38] can produce dynamical Yang-Baxter maps.

Definition 21. A left quasigroup Q is a non-empty set, together with a binary operation $(\cdot)$ on Q such that the left translation map $L(a) : Q \ni b \mapsto a \cdot b \in Q$ is bijective for all $a \in Q$.

For simplicity of notation, we write ab $(a, b \in Q)$ instead of $a \cdot b$, and denote $L(a)^{-1}(b) (\in Q)$ by $a \backslash b$. Here, $L(a)^{-1} : Q \to Q$ is the inverse of $L(a)$. We note that the binary operation of the left quasigroup is not always associative.

Example 7. (1) The group G is a left quasigroup.

(2) The set $\mathbb{Z}$ of integers, together with a binary operation $a \cdot b := b - a$ ($a, b \in \mathbb{Z}$), is a left quasigroup. This binary operation is not associative.

(3) Let V denote the vector space $(\mathbb{Z}/3\mathbb{Z})^4$ over the finite field $\mathbb{Z}/3\mathbb{Z}$. We define the binary operation $(\cdot)$ on V by

$$a \cdot b = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4 + (a_3 - b_3)(a_1 b_2 - a_2 b_1))$$

for $a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4) \in V$. The pair $(V, \cdot)$ is a left quasigroup; moreover, this is a smallest commutative Moufang loop that is not a group [30, Example IV.5.1].

For a left quasigroup Q and $\lambda \in Q$, we define the binary operation $(\cdot_\lambda)$ on Q by

$$a \cdot_\lambda b = \lambda \backslash ((\lambda a) b) \quad (a, b \in Q). \tag{3.1.2}$$

We denote by Q_λ the set Q with the above binary operation $(\cdot_\lambda)$ (3.1.2).

Proposition 10. Q_λ is a left quasigroup for any $\lambda \in Q$.

Remark 5. The binary operation $(\cdot_\lambda)$ is called a left derivative of $(\cdot)$ with respect to $\lambda \in Q$ [30, Section III.5].

A relation $\sim$ on Q defined by

$$\lambda \sim \mu \;\Leftrightarrow\; a \cdot_\lambda b = a \cdot_\mu b \quad (\forall a, b \in Q) \tag{3.1.3}$$

is an equivalence relation on Q. We write $H := Q/\sim$.

Proposition 11. If $\lambda \sim \mu$, then $\lambda a \sim \mu a$ for any $a \in Q$.

Proof. By the definition (3.1.2), $a \cdot_\lambda (x \cdot_{\lambda a} y) = (a \cdot_\lambda x) \cdot_\lambda y$ for $x, y \in Q$, and consequently $a \cdot_\lambda (x \cdot_{\lambda a} y) = a \cdot_\mu (x \cdot_{\mu a} y) = a \cdot_\lambda (x \cdot_{\mu a} y)$, since $\lambda \sim \mu$. On account of Proposition 10, $x \cdot_{\lambda a} y = x \cdot_{\mu a} y$ for $x, y \in Q$, which is the desired result. $\square$

Remark 6. Each equivalence class $[\lambda] \in H$ containing $\lambda \in Q$ is the set $\lambda \cdot N_l(Q_\lambda)$. Here, $N_l(Q_\lambda)$ is a left nucleus [30, Definition I.3.2] of the left quasigroup Q_λ: $N_l(Q_\lambda) = \{a \in Q_\lambda \mid (a \cdot_\lambda x) \cdot_\lambda y = a \cdot_\lambda (x \cdot_\lambda y) \quad (\forall x, y \in Q_\lambda)\}$.

We define a map $(\cdot) : H \times Q \to H$ by $[\lambda] \cdot a := [\lambda a]$ ($\lambda, a \in Q$). Because of Proposition 11, this definition makes sense.

Proposition 12. $Q = (Q, \cdot)$ is an object of $\mathbf{Set}_H$.

Let $s : H \to Q$ be a right inverse of the projection $Q \ni \lambda \mapsto [\lambda] \in H$; that is, $s : H \to Q$ is a map satisfying $s([\lambda]) \sim \lambda$ for all $\lambda \in Q$. We denote by $\sigma_1([\lambda])$ and $\sigma_2([\lambda])$ ($\lambda \in Q$) the maps on $Q \times Q$ defined by:

$$\begin{aligned}
&\sigma_1([\lambda])(a, b) = (s([\lambda])\backslash s([\lambda]), \lambda\backslash((\lambda a)b));\\
&\sigma_2([\lambda])(a, b) = (\lambda\backslash((\lambda a)b), s([(\lambda a)b])\backslash s([(\lambda a)b])) \quad (a, b \in Q).
\end{aligned} \tag{3.1.4}$$

Theorem 9. Both σ_1 and σ_2 are dynamical Yang-Baxter maps.

We will give a proof of this theorem in Section 3.4 after clarifying the structure of σ_i ($i = 1, 2$) in Sections 3.2 and 3.3 from the viewpoint of category theory.

Example 8. (1) If the left quasigroup Q is a group G (See Example 7 (1)), then the set H has only one element, and the maps $\sigma_i := \sigma_i([\lambda])$ ($i = 1, 2, \lambda \in G$) are the same as those in (1.0.7) for any right inverse s (See also Remark 4 (2)).

(2) If the left quasigroup Q is $\mathbb{Z}$ in Example 7 (2), then $\lambda \sim \mu \Leftrightarrow \lambda = \mu$ ($\lambda, \mu \in \mathbb{Z}$). As a result, the set H is isomorphic to $\mathbb{Z}$ as sets, and every right inverse s satisfies $s([\lambda]) = \lambda$ ($\lambda \in \mathbb{Z}$). The maps $\sigma_i([\lambda])$ ($i = 1, 2, \lambda \in \mathbb{Z}$) are as follows:

$$\begin{aligned}
&\sigma_1([\lambda])(a, b) = (2\lambda, b - a + 2\lambda);\\
&\sigma_2([\lambda])(a, b) = (b - a + 2\lambda, 2b - 2a + 2\lambda) \quad (\lambda, a, b \in \mathbb{Z}).
\end{aligned}$$

(3) If the left quasigroup Q is V in Example 7 (3), then $1 < \#(H) < \#(V)(= 81)$. The element $1_V := (0, 0, 0, 0)$ is the unit element of $(V, \cdot)$, and the inverse a^{-1} of $a (\in V)$ is $-a$. Because $(V, \cdot)$ is a Moufang loop, $(ba^{-1})a = b$ for any $a, b \in V$ [30, Section IV.1]. By virtue of Proposition 11,

$$a \sim b \Rightarrow aa^{-1} \sim ba^{-1} \Leftrightarrow 1_V \sim ba^{-1} \Rightarrow 1_V a \sim (ba^{-1})a \Leftrightarrow a \sim b;$$

that is, $a \sim b \Leftrightarrow ba^{-1} \sim 1_V$. A straightforward computation shows that $a \sim 1_V \Leftrightarrow a = (0, 0, 0, a_4)$ ($a_4 \in \mathbb{Z}/3\mathbb{Z}$), and the relation $a \sim b$ is consequently equivalent to that $b = a + (0, 0, 0, r)$ ($\exists r \in \mathbb{Z}/3\mathbb{Z}$). $H (= V/\sim)$ is thus a set of order 27. Since the orders of the sets H

and V are different, the method in [34] does not produce this example directly. Finally, the maps $\sigma_i([\lambda])$ $(i = 1, 2, \lambda \in V)$ for any right inverse s are as follows:

$$\sigma_1([\lambda])(a, b) = (1_V, x); \sigma_2([\lambda])(a, b) = (x, 1_V) \quad (\lambda, a, b \in V).$$

Here, the element $x \in V$ is defined by

$$\begin{aligned} x = &(a_1 + b_1, a_2 + b_2, a_3 + b_3, \\ &a_4 + b_4 + \lambda_1(a_2b_3 - a_3b_2) + \lambda_2(a_3b_1 - a_1b_3) + (\lambda_3 + a_3 - b_3)(a_1b_2 - a_2b_1)) \end{aligned}$$

for $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4), a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4) \in V$.

3.2 QYBE and braided semigroups

This section establishes a relation between the QYBE and braided semigroups in tensor categories, which play an essential role in the proof of Theorem 9.

Let $C = (C, \otimes, a, I, l, r)$ be a tensor category. That is to say, C is a category with a tensor product $\otimes : C \times C \to C$, an associativity constraint $a : \otimes \circ (\otimes \times \mathrm{id}) \to \otimes \circ (\mathrm{id} \times \otimes)$, a unit object I, and left and right unit constraints $l : \otimes(I \times \mathrm{id}) \to \mathrm{id}$, $r : \otimes(\mathrm{id} \times I) \to \mathrm{id}$ with respect to I, satisfying the pentagon axiom and the triangle axiom. We denote by $1_X : X \to X$ the identity morphism of an object X.

A pair (X, m_X) of an object X and a morphism $m_X : X \otimes X \to X$ is a semigroup, iff m_X satisfies

$$m_X \circ (m_X \otimes 1_X) = m_X \circ (1_X \otimes m_X) \circ a_{X,X,X}. \tag{3.2.1}$$

This morphism m_X is called a multiplication. A pair (X, Δ_X) of an object X and a morphism $\Delta_X : X \to X \otimes X$ is a co-semigroup, the dual concept of the semigroup, iff Δ_X satisfies

$$a_{X,X,X} \circ (\Delta_X \otimes 1_X) \circ \Delta_X = (1_X \otimes \Delta_X) \circ \Delta_X. \tag{3.2.2}$$

The morphism Δ_X is said to be a comultiplication.

Let $\sigma_{XY} : X \otimes Y \to Y \otimes X$ be a morphism of the tensor category C.

Definition 22. A matched pair of semigroups $X = (X, m_X)$ and $Y = (Y, m_Y)$ (Cf. [26, 36, 39, 40]) is a triple (X, Y, σ_{XY}) satisfying:

$$\begin{aligned}&(1_Y \otimes m_X) \circ a_{Y,X,X} \circ (\sigma_{XY} \otimes 1_X) \circ a^{-1}_{X,Y,X} \circ (1_X \otimes \sigma_{XY})\\ =&\sigma_{XY} \circ (m_X \otimes 1_Y) \circ a^{-1}_{X,X,Y}; \qquad (3.2.3)\\ &(m_Y \otimes 1_X) \circ a^{-1}_{Y,Y,X} \circ (1_Y \otimes \sigma_{XY}) \circ a_{Y,X,Y} \circ (\sigma_{XY} \otimes 1_Y)\\ =&\sigma_{XY} \circ (1_X \otimes m_Y) \circ a_{X,Y,Y}. \qquad (3.2.4)\end{aligned}$$

A pair (X, σ_X) of a semigroup X and a morphism $\sigma_X : X \otimes X \to X \otimes X$ is called a braided semigroup, iff the triple (X, X, σ_X) is a matched pair of semigroups.

Remark 7. The matched pair (X, Y, σ_{XY}) of semigroups gives birth to a semigroup. In fact, $(Y \otimes X, m_{Y \otimes X})$ is a semigroup with the morphism $m_{Y\otimes X}$: $(Y \otimes X) \otimes (Y \otimes X) \to Y \otimes X$ defined by

$$m_{Y\otimes X} = m_Y \otimes m_X \circ a_{Y\otimes Y,X,X} \circ (a^{-1}_{Y,Y,X} \circ 1_Y \otimes \sigma_{XY} \circ a_{Y,X,Y}) \otimes 1_X \circ a^{-1}_{Y\otimes X,Y,X}. \qquad (3.2.5)$$

$(X \otimes X, m_{X\otimes X})$ is hence a semigroup, if (X, σ_X) is a braided semigroup.

Let $X = (X, m_X)$ be a semigroup with a comultiplication $\Delta_X : X \to X \otimes X$. We write $\sigma := \Delta_X \circ m_X : X \otimes X \to X \otimes X$.

Theorem 10. If the pair (X, σ) is a braided semigroup, then σ satisfies the QYBE (3.1.1) in the tensor category C.

Proof. Because $\sigma = \Delta_X \circ m_X$,

(Right-hand-side of (3.1.1)) $= (1_X\otimes\Delta_X)\circ(1_X\otimes m_X)\circ a\circ(\sigma\otimes 1_X)\circ a^{-1}\circ(1_X\otimes\sigma)\circ a$.

Here, $a = a_{X,X,X}$. On account of (3.2.3), the right-hand-side of the above equation is $(1_X \otimes \Delta_X) \circ \sigma \circ (m_X \otimes 1_X)$. By making use of $\sigma = \Delta_X \circ m_X$ again, $(1_X \otimes \Delta_X) \circ \sigma \circ (m_X \otimes 1_X) = (1_X \otimes \Delta_X) \circ \Delta_X \circ m_X \circ (m_X \otimes 1_X)$. From (3.2.4), a similar argument induces that

(Left-hand-side of (3.1.1)) $= a \circ (\Delta_X \otimes 1_X) \circ \Delta_X \circ m_X \circ (1_X \otimes m_X) \circ a$.

This completes the proof in view of (3.2.1) and (3.2.2). $\square$

Remark 8. (1) The dual concept of the braided semigroup is a braided co-semigroup. Let $\sigma_{XY} : X \otimes Y \to Y \otimes X$ be a morphism of a tensor category C. A matched pair of co-semigroups (X, Δ_X) and (Y, Δ_Y) is a triple (X, Y, σ_{XY}) satisfying:

$$\sigma_{XY} \otimes 1_X \circ a^{-1}_{X,Y,X} \circ 1_X \otimes \sigma_{XY} \circ a_{X,X,Y} \circ \Delta_X \otimes 1_Y = a^{-1}_{Y,X,X} \circ 1_Y \otimes \Delta_X \circ \sigma_{XY};$$
$$1_Y \otimes \sigma_{XY} \circ a_{Y,X,Y} \circ \sigma_{XY} \otimes 1_Y \circ a^{-1}_{X,Y,Y} \circ 1_X \otimes \Delta_Y = a_{Y,Y,X} \circ \Delta_Y \otimes 1_X \circ \sigma_{XY}.$$

A pair (X, σ_X) of a co-semigroup X and a morphism $\sigma_X : X \otimes X \to X \otimes X$ is a braided co-semigroup, iff the triple (X, X, σ_X) is a matched pair of co-semigroups.

(2) The matched pair (X, Y, σ_{XY}) of co-semigroups defines a co-semigroup $(X \otimes Y, \Delta_{X \otimes Y})$. Here,

$$\Delta_{X\otimes Y} := a_{X\otimes Y,X,Y} \circ (a^{-1}_{X,Y,X} \circ 1_X \otimes \sigma_{XY} \circ a_{X,X,Y}) \otimes 1_Y \circ a^{-1}_{X\otimes X,Y,Y} \circ \Delta_X \otimes \Delta_Y. \tag{3.2.6}$$

From this fact, $(X \otimes X, \Delta_{X\otimes X})$ is a co-semigroup, if (X, σ_X) is a braided co-semigroup.

(3) A dual of Theorem 10 is also true. Let (X, Δ_X) be a co-semigroup with a multiplication m_X. We set $\sigma := \Delta_X \circ m_X$. If (X, σ) is a braided co-semigroup, then σ satisfies the QYBE. The proof is similar to that of Theorem 10.

3.3 Semigroups with left or right unit

In this section, we construct the braided semigroups in Theorem 10 by means of semigroups with a left or right unit.

Let $C = (C, \otimes, a, I, l, r)$ be a tensor category, S an object of C, and $\eta : I \to S$ a morphism of C. We define the morphisms $\Delta_i : S \to S \otimes S$ $(i = 1, 2)$ by

$$\Delta_1 = (\eta \otimes 1_S) \circ l^{-1}_S \text{ and } \Delta_2 = (1_S \otimes \eta) \circ r^{-1}_S. \tag{3.3.1}$$

Proposition 13. Both (S, Δ_1) and (S, Δ_2) are co-semigroups.

Proof. From (3.3.1), $\Delta_1 \circ \eta = \eta \otimes \eta \circ l^{-1}_I = \eta \otimes \eta \circ r^{-1}_I = \Delta_2 \circ \eta$, which induces

$$\Delta_1 \otimes 1_S \circ \Delta_1 = \Delta_2 \otimes 1_S \circ \Delta_1 \text{ and } 1_S \otimes \Delta_2 \circ \Delta_2 = 1_S \otimes \Delta_1 \circ \Delta_2. \tag{3.3.2}$$

By virtue of the triangle axiom,

$$a_{S,S,S} \circ \Delta_2 \otimes 1_S = 1_S \otimes \Delta_1. \tag{3.3.3}$$

It follows immediately from (3.3.2) and (3.3.3) that Δ_1 and Δ_2 satisfy (3.2.2). This proves the proposition. $\square$

A morphism $\eta : I \to S$ is called a left unit (resp. a right unit) of a semigroup (S, m_S), iff η satisfies $m_S \circ \eta \otimes 1_S = l_S$ (resp. $m_S \circ 1_S \otimes \eta = r_S$).

Let (S, m_S) be a semigroup with a left or right unit η. With the aid of the above proposition, the morphisms Δ_1 and Δ_2 are comultiplications of S. We define the morphisms σ_i $(i = 1, 2)$ by $\sigma_i := \Delta_i \circ m_S$.

Proposition 14. For $i = 1, 2$, (S, σ_i) is a braided semigroup.

Proof. The following lemma and (3.2.1) immediately establish (3.2.3) and (3.2.4) for the case $i = 1$.

Lemma 3. Δ_1 satisfies:

$$1_S \otimes m_S \circ a_{S,S,S} \circ \Delta_1 \otimes 1_S = \Delta_1 \circ m_S; \tag{3.3.4}$$

$$m_S \circ m_S \otimes 1_S \circ a^{-1}_{S,S,S} \circ 1_S \otimes \Delta_1 = m_S; \tag{3.3.5}$$

$$m_S \otimes 1_S \circ a^{-1}_{S,S,S} \circ 1_S \otimes \Delta_1 \circ \Delta_1 = \Delta_1. \tag{3.3.6}$$

For the proof of the case $i = 2$, we use:

$$m_S \otimes 1_S \circ a^{-1}_{S,S,S} \circ 1_S \otimes \Delta_2 = \Delta_2 \circ m_S; \tag{3.3.7}$$

$$m_S \circ 1_S \otimes m_S \circ a_{S,S,S} \circ \Delta_2 \otimes 1_S = m_S; \tag{3.3.8}$$

$$1_S \otimes m_S \circ a_{S,S,S} \circ \Delta_2 \otimes 1_S \circ \Delta_2 = \Delta_2. \tag{3.3.9}$$

This completes the proof. $\square$

Proof of Lemma 3. The naturality of the left unit constraint l, together with the fact that $l_{S \otimes S} \circ a_{I,S,S} = l_S \otimes 1_S$, implies (3.3.4).

If η is a left unit of the semigroup (S, m_S), then

$$m_S \circ \Delta_1 = 1_S, \tag{3.3.10}$$

which induces (3.3.5) by virtue of the associativity (3.2.1) of m_S.

If η is a right unit of (S, m_S), then

$$m_S \otimes 1_S \circ a^{-1}_{S,S,S} \circ 1_S \otimes \Delta_1 = 1_{S \otimes S}. \tag{3.3.11}$$

In fact, the left-hand-side of (3.3.11) is $(m_S \circ 1_S \otimes \eta) \otimes 1_S \circ a_{S,I,S}^{-1} \circ 1_S \otimes l_S^{-1}$. The triangle axiom, together with the fact that η is a right unit, gives (3.3.11), which consequently yields (3.3.5).

The proof of (3.3.6) is immediate by taking (3.2.2), (3.3.10), and (3.3.11) into account. □

Some σ_i are idempotent.

Proposition 15. If η is a left unit, then $\sigma_1^2 = \sigma_1$; and, if η is a right unit, then $\sigma_2^2 = \sigma_2$.

Proof. From (3.3.10) and the fact that $\sigma_1 = \Delta_1 \circ m_S$, $\sigma_1^2 = \sigma_1$, if η is a left unit. A similar argument induces that $\sigma_2^2 = \sigma_2$, if η is a right unit. □

Remark 9. (1) (S, σ_i) $(i = 1, 2)$ in Proposition 14 are also braided co-semigroups because of (3.2.2) and (3.3.4)–(3.3.9). As a result, $S \otimes S$ is a semigroup with respect to $m_{S\otimes S}$, but also a co-semigroup with respect to $(\Delta_i)_{S\otimes S}$. Here, $m_{S\otimes S}$ is the morphism (3.2.5) for $X = Y = S$, and $(\Delta_i)_{S\otimes S}$ is the morphism (3.2.6) for $X = Y = S$ and $\Delta_X = \Delta_Y = \Delta_i$.

(2) The quartet $(S, m_S, \Delta_i, \sigma_i)$ $(i = 1, 2)$ is a "bi-semigroup." In fact, it follows from (3.3.4)–(3.3.9) that

$$m_{S\otimes S} \circ \Delta_i \otimes \Delta_i = \Delta_i \circ m_S. \tag{3.3.12}$$

The morphism $\Delta_i : S \to S \otimes S$ hence respects the semigroup structures (See (1)). On the other hand, from (3.2.5) and (3.2.6), (3.3.12) is exactly the same as

$$m_S \otimes m_S \circ (\Delta_i)_{S\otimes S} = \Delta_i \circ m_S,$$

which means that $m_S : S \otimes S \to S$ respects the co-semigroup structures. Therefore, we can regard S as a bi-semigroup.

3.4 Proof of Theorem 9

This section is devoted to a proof of Theorem 9. We follow the notation used in Section 3.1.

Let $Q = (Q, \cdot)$ be a left quasigroup (see Definition 21). We denote by H the set of all equivalence classes of the relation (3.1.3) on Q.

For any $[\lambda] \in H$ $(\lambda \in Q)$, $m_Q([\lambda])$ is the map from $Q \times Q$ to Q defined by $m_Q([\lambda])(a,b) = a \cdot_\lambda b$ for $a, b \in Q$ (Cf. [35, (3.3)]). For $(\cdot_\lambda)$, see (3.1.2). On account of (3.1.3), this definition is unambiguous.

Proposition 16. $m_Q : Q \otimes Q \to Q$ is a morphism of $\mathbf{Set}_H$. Moreover, $Q = (Q, m_Q)$ is a semigroup in $\mathbf{Set}_H$.

Let $s : H \to Q$ be a right inverse of the projection $Q \ni \lambda \mapsto [\lambda] \in H$. For any $[\lambda] \in H$ $(\lambda \in Q)$, $\eta_Q^{(s)}([\lambda])$ is the map from $I = \{e\}$ to Q defined by $\eta_Q^{(s)}([\lambda])(e) = s([\lambda]) \backslash s([\lambda])$.

Proposition 17. (1) $\eta_Q^{(s)} : I \to Q$ is a morphism of $\mathbf{Set}_H$.
(2) $\eta_Q^{(s)}$ is a left unit of the semigroup (Q, m_Q) (Cf. [35, (3.5)]).

Proof. We give only the proof of (1). For the proof, it is sufficient to show that $[\lambda]\eta_Q^{(s)}([\lambda])(e) = [\lambda]e$ $(\lambda \in Q, I = \{e\})$. Because $s([\lambda]) \sim \lambda$, $[s([\lambda])] = [\lambda]$. Hence,

$$[\lambda]\eta_Q^{(s)}([\lambda])(e) = [s([\lambda])](s([\lambda]) \backslash s([\lambda])) = [s([\lambda])] = [\lambda] = [\lambda]e.$$

This completes the proof. $\square$

We set $\Delta_1 = (\eta_Q^{(s)} \otimes 1_Q) \circ l_Q^{-1}$, $\Delta_2 = (1_Q \otimes \eta_Q^{(s)}) \circ r_Q^{-1}$, and $\sigma_i = \Delta_i \circ m_Q$ $(i = 1, 2)$. It follows from Propositions 13, 14, 16, and 17 that each (Q, σ_i) $(i = 1, 2)$ is a braided semigroup in $\mathbf{Set}_H$ with the comultiplication Δ_i, and a straightforward calculation shows that the morphisms σ_1 and σ_2 are exactly the same as those in (3.1.4). Theorem 10 thus proves Theorem 9.

Remark 10. (1) Remarks 8 (3) and 9 (1) with Propositions 16 and 17 also imply Theorem 9.

(2) The construction [27] of dynamical Yang-Baxter maps using dynamical braces cannot produce the above dynamical Yang-Baxter map σ_1, if $\#(Q) > 1$. Suppose, contrary to our claim, that there exists a dynamical brace that gives birth to σ_1 by the method in [27, Corollary 3.2.1]. Then this dynamical Yang-Baxter map satisfies the unitary condition $\sigma_1^2 = 1_{Q \otimes Q}$. As a result, $\sigma_1 = \sigma_1^2 = 1_{Q \otimes Q}$ from Proposition 15. This contradicts the condition that $\#(Q) > 1$.

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