Markov-like set-valued functions on finite
graphs and their inverse limits

有限グラフ上のマルコフ型集合値写像
およびその射影系と極限

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Introduction

Inverse sequences of continuous functions on compacta (= compact metric spaces) and inverse limits are fundamental tools of describing complicated continua. For example, using a technical inverse limit, Anderson and Chouquet [AC] construct a continuum such that any two mutually distinct non-degenerate subcontinua are not homeomorphic.

On the other hand, inverse limits are also important tools for investigating dynamical systems of continuous functions. Smale horseshoe map is known as a fundamental example in dynamical systems. In continuum theory, the attracting set for the horseshoe map is homeomorphic to a well-known continuum as a Knaster continuum which is an inverse limit of the tent map [B, W]. In fact, an inverse limit of the tent map is a complicated continuum which is indecomposable (see [IM2, Example 22]).

In order to study continua from a slightly different viewpoint, in 2004, Mahavier [M] introduced generalized inverse limits with set-valued functions on intervals. Later Ingram and Mahavier [IM1] generalized the notation to set-valued functions on compact metric spaces as follows.

Definition 0.0.1. For any $i \in \mathbb{N}$, let X_i be a compactum and let 2^{X_i} be the space of all nonempty closed sets of X_i with the Hausdorff metric. Let $f_i : X_{i+1} \rightarrow 2^{X_i}$. A *generalized inverse sequence* is defined as a sequence of pairs X_i and f_i , which is denoted by $\{X_i, f_i\}_{i \in \mathbb{N}}$. The *generalized inverse limit* $\varprojlim \{X_i, f_i\}$ of the inverse sequence $\{X_i, f_i\}_{i \in \mathbb{N}}$ is defined by

$$\varprojlim \{X_i, f_i\} := \left\{ (x_1, x_2, \dots) \in \prod_{i=1}^{\infty} X_i \mid x_i \in f_i(x_{i+1}) \text{ for any } i \in \mathbb{N} \right\}.$$

$\varprojlim \{X_i, f_i\}$ is a subspace of $\prod_{i=1}^{\infty} X_i$ endowed with the product topology. Thus we can assume that $\varprojlim \{X_i, f_i\}$ is a metric space with a metric function d defined by

$$d(\mathbf{x}, \mathbf{x}') = \sum_{i=1}^{\infty} 2^{-i} d_{X_i}(x_i, x'_i).$$

Then each f_i is called as a bonding function of the inverse sequence. In the case where $X_i = X$ and $f_i = f$ for every $i \in \mathbb{N}$, we write the inverse limit by $\varprojlim \{X, f\}$.

In case of usual inverse sequence $\{X_i, f_i\}_{i \in \mathbb{N}}$, the inverse limit keeps good topological properties of factor spaces. For example, $\dim \left(\varprojlim \{X_i, f_i\} \right) \leq \max_{i \in \mathbb{N}} \{\dim(X_i)\}$. If all X_i are continua, $\varprojlim \{X_i, f_i\}$ is also a continuum. If X_i is a closed interval $\mathbb{I}$ for each $i \in \mathbb{N}$, then $\varprojlim \{X_i, f_i\}$ is always embeddable into the plane, and do not contain any triod.

In general, it is not easy to analyze topological properties of continua described by generalized inverse limits. In fact, generalized inverse limits do not have the useful properties as above. There is a generalized inverse limits of closed intervals whose dimension is equal to n for any $n = 1, 2, \dots, \infty$, and there is also a generalized inverse limit of closed intervals which is not a continuum (see [I, Section 2.2, Section 5.3]). Moreover, let $\{X_i, f_i\}_{i \in \mathbb{N}}$ be a usual inverse sequence of continuous maps, and let $\{n_i\}_{i \in \mathbb{N}}$ be an increasing sequence of natural numbers where $n_1 = 1$. Then the limit of a usual inverse sequence $\{X_{n_i}, f_{n_i, n_{i+1}}\}_{i \in \mathbb{N}}$ where $f_{n_i, n_{i+1}} := f_{n_i} \circ \dots \circ f_{n_{i+1}-1}$ is homeomorphic to $\varprojlim \{X_i, f_i\}$. We call such a fundamental result “the subsequence theorem”. However, the theorem can fail to hold in generalized inverse limits (see [GY], and see [BCMM3, N, N2] for other fundamental properties). Then the problem “What are conditions for generalized inverse sequences to have analogs of properties of usual inverse limits ?” may be important.

In this paper, we focus an analog of the property of Markov and the same pattern between two Markov maps on intervals. Markov maps on intervals are introduced as a special case of piecewise monotonic maps with the set of their critical points is an invariant set, e.g., tent maps and quadratic maps. In 2002 S. E. Holte [H2] introduced the notation of the same pattern between Markov maps, and showed that any two inverse limits of inverse sequences whose bonding maps are Markov maps with the same pattern are homeomorphic. There are many generalizations of Markov interval maps [AK, BL, BC, CL, R]. In 2013 I. Banič and T. Lunder [BL] extended the notation of Markov continuous maps on intervals to of upper semi-continuous set-valued functions having interval valued images on finitely many points of their domain and called generalized Markov interval functions. They also defined the notation of the same pattern between generalized Markov-interval functions and showed the corresponding theorem for generalized inverse limits to Holte’s theorem.

In Part I, we study a generalization of Markov maps on intervals. In Chapter 1, we will introduce Markov-like functions on intervals and the same pattern between Markov-like functions, and show the corresponding theorem to Holte’s theorem. Moreover, we will show that every generalized inverse limit with Markov-like functions on intervals is one-dimensional. In Section 1.6, we will give examples of Markov-like functions such that they do not have the same pattern but their generalized inverse limits are homeomorphic.

In Chapter 2, we will generalize the notation of Markov-like functions on intervals to finite graphs and define the notation of the same pattern among those Markov-like functions, and show that any two generalized inverse limits with Markov-like bonding

functions on finite graphs having the same pattern are homeomorphic. Moreover, in Section 2.2, we will show that every generalized inverse limit with Markov-like functions on finite graphs is one-dimensional. In Section 2.6 we will give examples of Markov-like functions and apply Theorem 2.2.5.

Markov-like functions on intervals have an interesting example. The Markov-like function f in Example 1.5.4 is given by slightly deforming the tent map (see [BCMM2]). Every inverse limits of continuous functions on intervals can be embeddable into the plane. However, the generalized inverse limit of f is a non-planar continuum.

In Part II, we study non-planar one-dimensional generalized inverse limits of intervals. In Section 3.2, we introduce a sufficient condition in order to generalized inverse limits with upper semi-continuous functions on intervals is a non-planar one-dimensional continuum. In Section 3.3, we prove that the condition induces non-planar one-dimensional continua as generalized inverse limit spaces. In Section 3.4, we apply Theorem 3.2.4 to Example 1.5.4, and in Section 3.5, we show that the condition in Theorem 3.2.4 is essential to construct non-planar continua.

Preliminaries

In this paper, we may assume that all spaces are compacta and simplicial complexes are finite and use the following notation:

- For any continuous map $f : X \rightarrow Y$, let $G(f) := \{(x, y) \in X \times Y \mid y = f(x)\}$ be the graph of f .
- For any set-valued function $g : X \rightarrow 2^Y$, let $G(g) := \{(x, y) \in X \times Y \mid y \in g(x)\}$ be the graph of g .
- A set-valued function $g : X \rightarrow 2^Y$ is said to have a surjective graph (or, simply, to be surjective) if for any $y \in Y$ there exists $x \in X$ such that $y \in g(x)$.
- For any continuous map $f : X \rightarrow Y$, $\tilde{f} : 2^X \rightarrow 2^Y$ is the induced map of f given by $\tilde{f}(A) = f(A)$.

We will need some background theorems about generalized inverse limits of upper semi-continuous functions proved by Ingram and Mahavier [IM1]. We remark them here:

Theorem 0.0.2. ([IM1, Theorem 2.1]) *Suppose each of X and Y is a compact Hausdorff space and M is a subset of $X \times Y$ such that if x is in X then there is a point y in Y such that (x, y) is in M . Then M is closed if and only if there is an upper-semi continuous function $f : X \rightarrow 2^Y$ such that $M = G(f)$.*

Theorem 0.0.3. ([IM1, Theorem 3.2]) Suppose that $\{X_i\}_{i \in \mathbb{N}}$ is a sequence of compact Hausdorff spaces and $\{f_i : X_{i+1} \rightarrow 2^{X_i}\}_{i \in \mathbb{N}}$ is a sequence of upper semi-continuous functions. Then $\varprojlim \{X_i, f_i\}$ is non-empty and compact.

Theorem 0.0.4. ([IM1, Theorem 4.7]) Suppose $\{X_i\}_{i \in \mathbb{N}}$ is a sequence of Hausdorff continua and $\{f_i : X_{i+1} \rightarrow 2^{X_i}\}_{i \in \mathbb{N}}$ is a sequence of upper semi-continuous functions such that for each $i \in \mathbb{N}$ and any $x \in X_{i+1}$, $f_i(x)$ is connected. Then $\varprojlim \{X_i, f_i\}$ is a Hausdorff continuum.

Part I

Markov-like set-valued functions and their inverse limits

Chapter 1

Markov-like set-valued functions on intervals and their inverse limits

1.1 Introduction

Markov maps on intervals introduced as a generalization of piecewise monotonic maps. In one-dimensional dynamical systems, Markov maps on intervals are known as an important class of maps whose much of dynamics can be obtained, e.g., topological entropy (see [BG], [BB]). In continuum theory, as a generalization of the tent maps, Markov maps on intervals are used to give a topological classification of the attractor of generalized horseshoe maps (see [B] and [H1]).

Definition 1.1.1. ([H2]) Fix $m \in \mathbb{N}_{\geq 2}$. A map $f : [a_1, a_m] \rightarrow [a_1, a_m]$ is Markov if there exists a finite set of points, $A = \{a_1 < \cdots < a_m\}$ such that $f(A) \subseteq A$, and the restriction of f to each component of $[a_1, a_m] \setminus A$ is one-to-one.

In 2002, Holte introduced the same pattern between two Markov maps on intervals [H2].

Definition 1.1.2. ([H2]) Let $f : [a_1, a_m] \rightarrow [a_1, a_m]$ be a Markov with respect to $A = \{a_1 < \cdots < a_m\}$, and $g : [b_1, b_m] \rightarrow [b_1, b_m]$ be Markov with respect to $B = \{b_1 < \cdots < b_m\}$, respectively. We say that f and g are Markov with same pattern if

$$f(a_j) = a_k \text{ if and only if } g(b_j) = b_k.$$

Moreover, she showed the same pattern between continuous Markov maps induce homeomorphic inverse limit spaces.

Theorem 1.1.3. ([H2, Theorem 1]) Let $\{f_i\}_{i \in \mathbb{N}}$ be a sequence of continuous surjective self maps of $[a_1, a_m]$ which are all Markov with respect to $A = \{a_1 < \cdots < a_m\}$

Figure 1.1: Markov maps f and g having the same pattern by Holte

and $\{g_i\}_{i \in \mathbb{N}}$ be a sequence of continuous surjective self maps of $[b_1, b_m]$ which are all Markov with respect to $B = \{b_1 < \dots < b_m\}$. If f_i and g_i are Markov with the same pattern for each $i \in \mathbb{N}$, both inverse limits $\varprojlim \{[a_1, a_m], f_i\}$ and $\varprojlim \{[b_1, b_m], g_i\}$ are homeomorphic.

In 2013, I. Banič and T. Lunder [BL] extended the notation of Markov continuous maps on intervals to of upper semi-continuous set-valued functions having interval valued images on finitely many points of their domain and called generalized Markov interval functions.

Definition 1.1.4. [BL, Definition3.1] Let $a, b \in \mathbb{R}, a < b$, and $m \in \mathbb{N}_{\geq 2}$. We say that an upper-semi continuous function $f : [a, b] \rightarrow 2^{[a, b]}$ is a generalized Markov interval function with respect to A , where $A = \{a_1, \dots, a_m\} \subseteq [a, b]$, if

1. $a = a_1 < \dots < a_m = b$,
2. the restriction of f on every component of $[a, b] \setminus A$ is an injective single valued function,
3. for each $j \in \{1, \dots, m\}$, the image $f(a_j)$ is an interval (possibly degenerate) $[a_{r_1(j)}, a_{r_2(j)}]$, where $a_{r_1(j)}, a_{r_2(j)} \in A$ ($a_{r_1(j)} < a_{r_2(j)}$),
4. for each $j \in \{1, \dots, m-1\}$, $\lim_{x \uparrow a_{j+1}} f(x), \lim_{x \downarrow a_j} f(x) \in A$.

They also defined the notation of the same pattern between generalized Markov-interval functions and showed the corresponding theorem for generalized inverse limits to Holte's theorem.

Definition 1.1.5. ([BL, Definition3.3]) Let $f : [a_1, a_m] \rightarrow 2^{[a_1, a_m]}$ be a generalized Markov interval function with respect to $A = \{a_1 < \dots < a_m\}$ and let $g : [b_1, b_m] \rightarrow 2^{[b_1, b_m]}$ be a generalized Markov interval function with respect to $B = \{b_1 < \dots < b_m\}$. We say that f and g are generalized Markov interval function with the same pattern if i) and ii) hold true:

i) for every $j \in \{1, \dots, m\}$,

$$f(a_j) = [a_{r_1(j)}, a_{r_2(j)}] \text{ if and only if } g(b_j) = [b_{r_1(j)}, b_{r_2(j)}],$$

ii) for every $j \in \{1, \dots, m-1\}$,

$$\begin{aligned} \lim_{x \uparrow a_{j+1}} f(x) = a_k & \text{ if and only if } \lim_{y \uparrow b_{j+1}} g(y) = b_k, \text{ and} \\ \lim_{x \downarrow a_j} f(x) = a_l & \text{ if and only if } \lim_{y \downarrow b_j} g(y) = b_l. \end{aligned}$$

Figure 1.2: Markov maps f and g having the same pattern by Banič and Lunder

Theorem 1.1.6. ([BL, Theorem 1.2]) Let $\{f_i\}_{i \in \mathbb{N}}$ be a sequence of upper semi-continuous functions from $[a_1, a_m]$ to $2^{[a_1, a_m]}$ with surjective graphs, which are all generalized Markov interval functions with respect to $A = \{a_1, \dots, a_m\}$ and let $\{g_i\}_{i \in \mathbb{N}}$ be a sequence of upper semi-continuous functions from $[b_1, b_m]$ to $2^{[b_1, b_m]}$ with surjective graphs, which are all generalized Markov interval functions with respect to $B = \{b_1, \dots, b_m\}$. If for each $i \in \mathbb{N}$, f_i and g_i are generalized Markov interval functions with the same pattern, then $\lim_{\leftarrow} \{[a_1, a_m], f_i\}$ is homeomorphic to $\lim_{\leftarrow} \{[b_1, b_m], g_i\}$.

In this chapter, we will introduce Markov-like functions as a generalization of generalized Markov interval functions and define the notation of the same pattern between Markov-like functions. A Markov-like function allows to have finite values on each component of the complement of finitely many points, and its graph may be disconnected. Generalized inverse sequences of Markov-like functions yield interesting examples. For example, the function f in [I, Example 2.11] is a Markov-like function and the generalized inverse limit M is the Hurewicz continuum. This example is studied in [EK], [KK] and [KT]. In Section 1.2.3, we will prove that any two generalized inverse limits with Markov-like bonding functions having the same pattern are homeomorphic. Consequently we can have a generalization of [BL] and [H2].

1.2 Markov-like functions on intervals

In this section, we will define Markov-like functions on intervals, and the same pattern between two Markov-like functions.

Definition 1.2.1. Fix $m \in \mathbb{N}_{\geq 2}$. Let $\mathbb{I} = [a_1, a_m]$ be a closed interval. Let $A : a_1 < a_2 < \dots < a_m$ be a finite partition of $\mathbb{I}$. A set-valued function $f : \mathbb{I} \rightarrow 2^{\mathbb{I}}$ having a surjective graph is *Markov-like* with respect to A if the following conditions are satisfied.

- (1) For each $j \in \{1, \dots, m\}$, there exist $\frac{s_j}{2}$ mutually disjoint closed intervals (they can be degenerate) $[a_{r_1(j)}, a_{r_2(j)}], \dots, [a_{r_{s_j-1}(j)}, a_{r_{s_j}(j)}]$ such that

$$f(a_j) = \bigcup_{k=1}^{\frac{s_j}{2}} [a_{r_{2k-1}(j)}, a_{r_{2k}(j)}], \text{ and}$$

$$a_{r_l(j)} \in A \text{ for each } l \in \{1, \dots, s_j\}.$$

- (2) For each $j \in \{1, \dots, m-1\}$, there exist $n_f(j)$ strictly monotone continuous functions $f_j^1, f_j^2, \dots, f_j^{n_f(j)}$ defined on $[a_j, a_{j+1}]$ such that for each $j = 1, 2, \dots, m-1$,

$$G(f|_{\text{Int}([a_j, a_{j+1}])}) = \bigcup_{k=1}^{n_f(j)} G(f_j^k|_{\text{Int}([a_j, a_{j+1}])}),$$

and for each $1 \leq k \leq n_f(j)$,

$$f_j^k(a_j) \in f(a_j) \cap A \text{ and } f_j^k(a_{j+1}) \in f(a_{j+1}) \cap A.$$

Moreover, if $k_1 \neq k_2$ then

$$G(f_j^{k_1}|_{\text{Int}([a_j, a_{j+1}])}) \cap G(f_j^{k_2}|_{\text{Int}([a_j, a_{j+1}])}) = \emptyset.$$

The graph of each Markov-like function is the union of finite arcs. Hence each Markov-like function is upper semi-continuous.

Definition 1.2.2. Let $\mathbb{I} = [a_1, a_m]$ and $\mathbb{J} = [b_1, b_m]$ be closed intervals and $A : a_1 < a_2 < \dots < a_m$ and $B : b_1 < b_2 < \dots < b_m$ be partitions of $\mathbb{I}$ and $\mathbb{J}$ respectively. A Markov-like function $f : \mathbb{I} \rightarrow 2^{\mathbb{I}}$ with respect to A and a Markov-like function $g : \mathbb{J} \rightarrow 2^{\mathbb{J}}$ with respect to B have *the same pattern* if the following conditions are satisfied.

- (3) For each $j \in \{1, \dots, m\}$,

$$f(a_j) \supseteq [a_{r_1(j)}, a_{r_2(j)}] \text{ if and only if } g(b_j) \supseteq [b_{r_1(j)}, b_{r_2(j)}].$$

- (4) For each $j \in \{1, \dots, m-1\}$, $n_f(j) = n_g(j)$ and there exists a bijection $\phi_j : \{1, \dots, n_f(j)\} \rightarrow \{1, \dots, n_g(j)\}$ such that

$$f_j^k(a_j) = a_{l_1(j)} \text{ if and only if } g_j^{\phi_j(k)}(b_j) = b_{l_1(j)},$$

$$f_j^k(a_{j+1}) = a_{l_2(j)} \text{ if and only if } g_j^{\phi_j(k)}(b_{j+1}) = b_{l_2(j)}.$$

Suppose $\phi_1 : \{1\} \rightarrow \{1\}$, $\phi_2 : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$, $\phi_3 : \{1, 2\} \rightarrow \{1, 2\}$, $\phi_4 : \{1\} \rightarrow \{1\}$ are the identity mappings. Figure 1.3 is an example of the graphs of Markov-like bonding functions f and g having the same pattern with ϕ_1, ϕ_2, ϕ_3 , and ϕ_4 .

Figure 1.3: Markov-like functions f and g having the same pattern

In this chapter, our main result is Theorem 1.2.3.

Theorem 1.2.3. Fix $m \in \mathbb{N}_{\geq 2}$. Let $\mathbb{I} = [a_1, a_m]$ and $\mathbb{J} = [b_1, b_m]$ be closed intervals and $A : a_1 < a_2 < \dots < a_m$ and $B : b_1 < b_2 < \dots < b_m$ be partitions of $\mathbb{I}$ and $\mathbb{J}$, respectively. Let $\{f_i\}_{i \in \mathbb{N}}$ and $\{g_i\}_{i \in \mathbb{N}}$ be sequences of Markov-like functions with respect to A and B respectively. If f_i and g_i have the same pattern for every $i \in \mathbb{N}$, two generalized inverse limits $\varprojlim \{\mathbb{I}, f_i\}$ and $\varprojlim \{\mathbb{J}, g_i\}$ are homeomorphic.

1.3 Preliminaries

We will need several preliminary results to prove Theorem 1.2.3. Throughout this section, we assume all hypotheses of Theorem 1.2.3 hold. That is, $\mathbb{I} = [a_1, a_m]$ and $\mathbb{J} = [b_1, b_m]$ are closed intervals, $A : a_1 < a_2 < \dots < a_m$ and $B : b_1 < b_2 < \dots < b_m$ are partitions of $\mathbb{I}$ and $\mathbb{J}$, respectively, $\{f_i\}_{i \in \mathbb{N}}$ and $\{g_i\}_{i \in \mathbb{N}}$ are sequences of Markov-like functions with respect to A and B respectively, and f_i and g_i have the same pattern for every $i \in \mathbb{N}$.

1.3.1 Topological framework

From Theorem 1.3.3, we can assume both $\mathbb{I}$ and $\mathbb{J}$ are the unit interval $[0, 1]$. To show that we need Lemma 1.3.1 and 1.3.2.

Lemma 1.3.1. Let $\mathbb{I} = [a_1, a_m]$ be a closed interval and $A : a_1 < \dots < a_m$ be a partition of $\mathbb{I}$, where $m \geq 2$. Let $\mathbb{J} = [0, 1]$. Let $f : \mathbb{I} \rightarrow 2^{\mathbb{I}}$ be a Markov-like function with respect to A . Put a piecewise linear homeomorphism $h : \mathbb{I} \rightarrow \mathbb{J}$ such that $h(a_1) = 0, h(a_m) = 1$. Define $b_j = h(a_j)$ for each $j = 1, \dots, m$ and take a partition $B : 0 = b_1 < \dots < b_m = 1$ of $\mathbb{J}$. Then there is a Markov-like function $g : \mathbb{J} \rightarrow 2^{\mathbb{J}}$ with respect to B such that

$$\tilde{h} \circ f = g \circ h, \text{ and}$$

f and g have the same pattern.

Proof. Let α be the inverse map of h . Let $g : \mathbb{J} \rightarrow 2^{\mathbb{J}}$ be defined by

$$g(x) := \tilde{h} \circ f \circ \alpha(x) \quad \text{for } x \in \mathbb{J}.$$

We have

$$\begin{aligned} g \circ h(x) &= \tilde{h} \circ f \circ \alpha \circ h(x) \\ &= \tilde{h} \circ f(x). \end{aligned}$$

We will show that g is Markov-like with respect to B . Since f is Markov-like with respect to A , for each $j \in \{1, \dots, m\}$, there exist $\frac{s_j}{2}$ mutually disjoint closed intervals $[a_{r_1(j)}, a_{r_2(j)}], \dots, [a_{r_{s_j-1}(j)}, a_{r_{s_j}(j)}]$ such that

$$f(a_j) = \bigcup_{k=1}^{\frac{s_j}{2}} [a_{r_{2k-1}(j)}, a_{r_{2k}(j)}],$$

where $a_{r_l(j)} \in A$ for each $l \in \{1, \dots, s_j\}$.

By the definition of h , for each $a_{r_1}, a_{r_2} \in A$,

$$\begin{aligned}\tilde{h}([a_{r_1}, a_{r_2}]) &= \{h(x) \mid x \in [a_{r_1}, a_{r_2}]\} \\ &= [b_{r_1}, b_{r_2}].\end{aligned}$$

Thus,

$$\begin{aligned}g(b_j) &= \tilde{h} \circ f \circ \alpha(b_j) \\ &= \tilde{h} \circ f(a_j) \\ &= \bigcup_{k=1}^{\frac{s_j}{2}} [b_{r_{2k-1}(j)}, b_{r_{2k}(j)}] \quad (\dagger).\end{aligned}$$

Fix $j \in \{1, \dots, m-1\}$. Since $\alpha(\text{Int}([b_j, b_{j+1}])) = \text{Int}([a_j, a_{j+1}])$,

$$\begin{aligned}G(g|_{\text{Int}([b_j, b_{j+1}])}) &= \left\{ (x, y) \in G \left(\tilde{h} \circ f \circ \alpha \right) \mid x \in \text{Int}([b_j, b_{j+1}]) \right\} \\ &= \bigcup_{k=1}^{n_f(j)} \left\{ (x, y) \in G(h \circ f_j^k \circ \alpha) \mid x \in \text{Int}([b_j, b_{j+1}]) \right\}.\end{aligned}$$

Put $g_j^k := h \circ f_j^k \circ \alpha$ for each $k \in \{1, \dots, n_f(j)\}$. Then g_j^k is a strictly monotone continuous function for each $k \in \{1, \dots, n_f(j)\}$. Put $k_1, k_2 \in \{1, \dots, n_f(j)\}$ with $k_1 \neq k_2$. Then $G(g_j^{k_1}|_{\text{Int}([b_j, b_{j+1}])}) \cap G(g_j^{k_2}|_{\text{Int}([b_j, b_{j+1}])}) = \emptyset$ since $f_j^{k_1}, f_j^{k_2}$, and h are injections.

For each $k \in \{1, \dots, n_f(j)\}$,

$$\begin{aligned}g_j^k(b_j) &= h \circ f_j^k \circ \alpha(b_j) \\ &\in \tilde{h}(f(a_j) \cap A) \\ &= \tilde{h} \circ f(a_j) \cap \tilde{h}(A) \\ &= g(b_j) \cap B.\end{aligned}$$

Similarly, $g_j^k(b_{j+1}) \in g(b_{j+1}) \cap B$. Thus g is a Markov-like function with respect to B .

We will show that f and g have same pattern. From $(\dagger)$, f and g satisfy the property (3) in Definition 1.2.2. Thus we will verify that the property (4) is satisfied. For each $j \in \{1, \dots, m-1\}$, $n_g(j) = n_f(j)$ by the definition of g . Let us define a bijection $\phi_j : \{1, \dots, n_f(j)\} \rightarrow \{1, \dots, n_g(j)\}$ by $\phi_j(k) = k$. Then for each $k \in \{1, \dots, n_f(j)\}$,

$$\begin{aligned}f_j^k(a_j) = a_{r_1(j)} &\text{ if and only if } h \circ f_j^k(a_j) = b_{r_1(j)} \\ &\text{ if and only if } h \circ f_j^k \circ \alpha(b_j) = b_{r_1(j)} \\ &\text{ if and only if } g_j^{\phi_j(k)}(b_j) = b_{r_1(j)}.\end{aligned}$$

Similarly $f_j^k(a_{j+1}) = a_{r_2(j)}$ if and only if $g_j^{\phi_j(k)}(b_j) = b_{r_2(j)}$. Thus the property (4) is satisfied. Therefore f and g have the same pattern. $\square$

Lemma 1.3.2. *Let $\mathbb{I}$ and $\mathbb{J}$ be closed intervals. Let $\{f_i : \mathbb{I} \rightarrow 2^{\mathbb{I}}\}_{i \in \mathbb{N}}$ and $\{g_i : \mathbb{J} \rightarrow 2^{\mathbb{J}}\}_{i \in \mathbb{N}}$ be sequences of set-valued functions and let $\{h_i : \mathbb{I} \rightarrow \mathbb{J}\}_{i \in \mathbb{N}}$ be a sequence of homeomorphisms such that*

$$\tilde{h}_i \circ f_i = g_i \circ h_{i+1} \quad \text{for every } i \in \mathbb{N},$$

where $\tilde{h}_i : 2^{\mathbb{I}} \rightarrow 2^{\mathbb{J}}$ is the induced homeomorphism by h_i . Then the generalized inverse limits $\varprojlim \{\mathbb{I}, f_i\}$ and $\varprojlim \{\mathbb{J}, g_i\}$ are homeomorphic.

Lemma 1.3.2 can be obtained in a similar way as [I, Theorem 2.9].

Theorem 1.3.3. *Let $\mathbb{I} = [a_1, a_m]$ be a closed interval and $A : a_1 < \dots < a_m$ be a partition of $\mathbb{I}$, where $m \geq 2$. Let $\{f_i : \mathbb{I} \rightarrow 2^{\mathbb{I}}\}_{i \in \mathbb{N}}$ be a sequence of Markov-like functions with respect to A . Suppose that $h : \mathbb{I} \rightarrow \mathbb{J} = [0, 1]$ is a piecewise linear homeomorphism such that $h(a_1) = 0, h(a_m) = 1$. Define $b_j = h(a_j)$ for each $j = 1, \dots, m$ and take a partition $B : 0 = b_1 < \dots < b_m = 1$ of $\mathbb{J}$. Then there exists a sequence $\{g_i : \mathbb{J} \rightarrow 2^{\mathbb{J}}\}_{i \in \mathbb{N}}$ of Markov-like functions with respect to B such that the generalized inverse limits $\varprojlim \{\mathbb{I}, f_i\}$ and $\varprojlim \{\mathbb{J}, g_i\}$ are homeomorphic.*

Proof. Let α is the inverse mapping of h . For each $i \in \mathbb{N}$, let $g_i : \mathbb{J} \rightarrow 2^{\mathbb{J}}$ be defined by $g_i := \tilde{h} \circ f_i \circ \alpha$. From Lemma 1.3.1, f_i and g_i have same pattern. Moreover $\tilde{h} \circ f_i = g_i \circ h$ for each $i \in \mathbb{N}$. Therefore, by Lemma 1.3.2, $\varprojlim \{\mathbb{I}, f_i\}$ and $\varprojlim \{\mathbb{J}, g_i\}$ are homeomorphic. $\square$

1.3.2 A construction of a map $H : \varprojlim \{\mathbb{I}, f_i\} \rightarrow \varprojlim \{\mathbb{J}, g_i\}$

From Theorem 1.3.3, we can assume both $\mathbb{I}$ and $\mathbb{J}$ are the unit interval $[0, 1]$. We now construct a map H from $\varprojlim \{\mathbb{I}, f_i\}$ to $\varprojlim \{\mathbb{J}, g_i\}$.

Proposition 1.3.4. *Let $h : \mathbb{I} \rightarrow \mathbb{J}$ be a piecewise linear homeomorphism such that $h([a_j, a_{j+1}]) = [b_j, b_{j+1}]$ for all $j = 1, 2, \dots, m-1$. For any point $\mathbf{x} = (x_1, x_2, \dots) \in \varprojlim \{\mathbb{I}, f_i\}$, there exists exactly one point $\mathbf{y} = (y_1, y_2, \dots) \in \varprojlim \{\mathbb{J}, g_i\}$ with $y_1 = h(x_1)$ and satisfying the following properties for each $i \in \mathbb{N}$:*

- (1)-(i) $x_i \in \text{Int}([a_j, a_{j+1}])$ if and only if $y_i \in \text{Int}([b_j, b_{j+1}])$,
- (2)-(i) $x_i = a_j$ if and only if $y_i = b_j$,
- (3)-(i) In the case of $i \geq 2$, $x_{i-1} = f_{i-1}^k(x_i)$ if and only if $y_{i-1} = g_{i-1}^{\phi_{i-1}^j(k)}(y_i)$.

Proof. To prove the existence of $\mathbf{y}$, we show that there is a decreasing sequence of closed sets $\{K_n\}_{n \in \mathbb{N}}$ of the Hilbert cube such that for each $n \in \mathbb{N}$, every points of K_n satisfy the properties (1)-(i), (2)-(i), and (3)-(i) for $i = 1, \dots, n$.

Let $y_1 = h(x_1)$ and $K_1 := \{h(x_1)\} \times \prod_{j=2}^{\infty} \mathbb{J}$. By the definition of h , the conditions (1)-(1), (2)-(1), and (3)-(1) are satisfied for each point of K_1 . We note that K_1 is non-empty and closed. Suppose we defined non-empty closed sets $K_1 \supseteq K_2 \supseteq \dots \supseteq K_n$ such that for each $i = 1, \dots, n$, every point of K_i satisfies the properties (1)-(j), (2)-(j), and (3)-(j) for each $j \leq i$. We will define K_{n+1} . Let $\pi_{<1,n>} : \prod_{j=1}^{\infty} \mathbb{J} \rightarrow \prod_{j=1}^n \mathbb{J}$ be the projection mapping to the first n -coordinates. Fix $(y_1, y_2, \dots, y_n) \in \pi_{<1,n>}(K_n)$. We define y_{n+1} by the following way.

Case 1. Suppose $x_{n+1} \in \text{Int}([a_j, a_{j+1}])$ for some $j \in \{1, \dots, m-1\}$. By the definition of f_n , there exists $k \in \{1, 2, \dots, n_f(j)\}$ such that

$$x_n = f_{n,j}^k(x_{n+1}) \in f_{n,j}^k(\text{Int}([a_j, a_{j+1}])).$$

Then we can choose $a_{l_1}, a_{l_2} \in A$ with

$$(a_{l_1}, a_{l_2}) = f_{n,j}^k(\text{Int}([a_j, a_{j+1}])).$$

Hence $y_n \in (b_{l_1}, b_{l_2})$ by the assumption. Since f_n and g_n have the same pattern,

$$(b_{l_1}, b_{l_2}) = g_{n,j}^{\phi_{n,j}(k)}(\text{Int}([b_j, b_{j+1}])).$$

Therefore, there exists a $y_{n+1} \in \text{Int}([b_j, b_{j+1}]))$ such that

$$y_n = g_{n,j}^{\phi_{n,j}(k)}(y_{n+1}).$$

Case 2. Suppose $x_{n+1} \in A$. Choose $a_j \in A$ with $x_{n+1} = a_j$. Let $y_{n+1} = b_j$. By the definition of f_n , there are $a_{r_1(j)}, a_{r_2(j)} \in A$ such that

$$x_n \in [a_{r_1(j)}, a_{r_2(j)}] \subseteq f_n(a_j).$$

By the assumption, $y_n \in [b_{r_1(j)}, b_{r_2(j)}]$. Since f_n and g_n have the same pattern,

$$[b_{r_1(j)}, b_{r_2(j)}] \subseteq g_n(b_j) = g_n(y_{n+1}).$$

Let $K_{n+1} := K_n \cap (\{(y_1, y_2, \dots, y_{n+1})\} \times \prod_{i=n+2}^{\infty} \mathbb{J})$. Then K_{n+1} is a non-empty closed set and each point of K_{n+1} satisfies the properties (1)-(i), (2)-(i), and (3)-(i) for $i = 1, \dots, n+1$. Thus, we can have a desired decreasing sequence of closed sets $\{K_n\}_{n \in \mathbb{N}}$. Then there exists $\mathbf{y} \in \bigcap_{n=1}^{\infty} K_n$. By the construction of $\mathbf{y}$, $\mathbf{y} \in \varprojlim \{\mathbb{J}, g_i\}$.

We will show that $\mathbf{y}$ is uniquely determined. Suppose that for each $\mathbf{x} \in \{\mathbb{I}, f_i\}$, there are mutually distinct points $\mathbf{y}, \mathbf{y}' \in \varprojlim \{\mathbb{J}, g_i\}$ such that

$$y_1 = h(x_1) = y'_1$$

and both points satisfy the properties (1)-(i), (2)-(i), and (3)-(i) for each $i \in \mathbb{N}$. We may assume that there exists $i_0 \in \mathbb{N}_{\geq 2}$ such that $y_{i_0} \neq y'_{i_0}$ and, $y_i = y'_i$ if $i < i_0$.

Case 1. Suppose that $x_{i_0} \in \text{Int}([a_j, a_{j+1}])$ for some $j \in \{1, \dots, m-1\}$. Since the property (1)-(i₀) is satisfied, $y_{i_0}, y'_{i_0} \in \text{Int}([b_j, b_{j+1}])$. By the definition of f_{i_0-1} , there exists $k \in \{1, \dots, n_{f_{i_0-1}}(j)\}$ such that

$$x_{i_0-1} = f_{i_0-1}^k(x_{i_0}).$$

From the property (3)-(i₀),

$$\begin{aligned} y_{i_0-1} &= g_{i_0-1}^{\phi_{i_0-1} j(k)}(y_{i_0}), \\ y'_{i_0-1} &= g_{i_0-1}^{\phi_{i_0-1} j(k)}(y'_{i_0}). \end{aligned}$$

Since $g_{i_0-1}^{\phi_{i_0-1} j(k)}$ is a injective map, $y_{i_0-1} \neq y'_{i_0-1}$. It is a contradiction.

Case 2. Suppose that $x_{i_0} = a_j \in A$ for some $j \in \{1, \dots, m\}$. From (2)-(i₀), $y_{i_0} = b_j = y'_{i_0}$. It is a contradiction. Thus such a case does not occur.

Since we have a contradiction in each case, $\mathbf{y}$ is uniquely determined by $\mathbf{x}$. $\square$

For any $\mathbf{x} \in \varprojlim \{\mathbb{I}, f_i\}$, choosing the point $\mathbf{y} \in \varprojlim \{\mathbb{J}, g_i\}$ as in Proposition 1.3.4, we can define the function

$$H : \varprojlim \{\mathbb{I}, f_i\} \rightarrow \varprojlim \{\mathbb{J}, g_i\}.$$

1.3.3 Some properties of H

We provide some lemmas to show that H is a homeomorphism.

Lemma 1.3.5. *Fix $\mathbf{x} \in \varprojlim \{\mathbb{I}, f_i\}$. For each $i \in \mathbb{N}$, there exists $\delta_i > 0$ such that for each $\mathbf{x}' \in \varprojlim \{\mathbb{I}, f_i\}$ with $d(\mathbf{x}, \mathbf{x}') < \delta_i$, at least one of the following statements hold:*

- (1) $x_{i+1} = x'_{i+1}$, $x_{i+1}, x'_{i+1} \in A$,
- (2) There exist $j \in \{1, \dots, m\}$ and $k \in \{1, \dots, n_{f_i}(j)\}$ such that

$$(x_{i+1}, x_i), (x'_{i+1}, x'_i) \in G(f_i^k_j).$$

Proof. Fix $\mathbf{x} \in \varprojlim \{\mathbb{I}, f_n\}$ and $i \in \mathbb{N}$. Let

$$K := \left\{ (k', j') \mid (x_{i+1}, x_i) \in G(f_i^{k'}_{j'}) \right\}$$

and

$$L := G(f_i) \setminus \left(G(f_i|_{\{x_{i+1}\}}) \cup \bigcup_{(k', j') \in K} G(f_i^{k'}|_{j'}) \right).$$

We will show that $(x_{i+1}, x_i) \notin \text{Cl}(L)$.

Assume that $(x_{i+1}, x_i) \in \text{Cl}(L)$. Then there is a sequence $\{(z_{i+1}^n, z_i^n)\}_{n \in \mathbb{N}}$ in L such that

$$\lim_{n \rightarrow \infty} |x_i - z_i^n| = 0, \lim_{n \rightarrow \infty} |x_{i+1} - z_{i+1}^n| = 0.$$

Then we can assume that $z_{i+1}^n \notin A$ for every $n \in \mathbb{N}$ regardless whether x_{i+1} is in A or not. Thus, for each $n \in \mathbb{N}$, there exist $j' \in \{1, \dots, m-1\}$ and $k' \in \{1, \dots, n_{f_i}(j')\}$ such that

$$(z_{i+1}^n, z_i^n) \in G(f_i^{k'}|_{j'}).$$

Let

$$K_L := \left\{ (k', j') \mid (z_{i+1}^n, z_i^n) \in G(f_i^{k'}|_{j'}) \text{ for some } n \in \mathbb{N} \right\}.$$

Since K_L is finite, there exists $(k, j) \in K_L$ and an infinite subset M of $\mathbb{N}$ such that

$$(z_{i+1}^m, z_i^m) \in G(f_i^k|_j) \text{ for each } m \in M.$$

Then $(x_{i+1}, x_i) \in G(f_i^k|_j)$. It is a contradiction. Thus there exists $\delta > 0$ such that

$$|z_i - x_i|^2 + |z_{i+1} - x_{i+1}|^2 \geq \delta^2 \text{ for any } (z_{i+1}, z_i) \in L.$$

Next we define $\delta_i = 2^{-(i+2)}\delta$ and show that δ_i satisfy the condition. In fact, if

$$d(\mathbf{x}, \mathbf{x}') = \sum_{j=1}^{\infty} 2^{-j} |x_j - x'_j| < \delta_i,$$

then

$$2^{-i} |x_i - x'_i| < 2^{-(i+2)}\delta \text{ and } 2^{-(i+1)} |x_{i+1} - x'_{i+1}| < 2^{-(i+2)}\delta.$$

It follows that

$$|x_i - x'_i|^2 + |x_{i+1} - x'_{i+1}|^2 < \frac{5}{16}\delta^2.$$

Therefore $(x_{i+1}, x_i) \notin L$. □

Lemma 1.3.6. Choose $\mathbf{x}, \mathbf{x}' \in \varprojlim \{\mathbb{I}, f_i\}$ and let $\mathbf{y} = H(\mathbf{x})$, $\mathbf{y}' = H(\mathbf{x}')$. Then we have the following:

(1) If $x_{i+1} = x'_{i+1}$ and $x_{i+1}, x'_{i+1} \in A$, then $y_{i+1} = y'_{i+1}$,

(2) If there exist $j \in \{1, \dots, m-1\}$ and $k \in \{1, \dots, n_{f_i}(j)\}$ such that

$$(x_{i+1}, x_i), (x'_{i+1}, x'_i) \in G(f_{i,j}^k),$$

then

$$(y_{i+1}, y_i), (y'_{i+1}, y'_i) \in G(g_{i,j}^{\phi_{i,j}(k)}).$$

Proof. This lemma is obtained directly from the definition of H . $\square$

Definition 1.3.7. Fix $i \in \mathbb{N}$. For any $(y_{i+1}, y_i) \in G(g_i)$, $G_{(y_{i+1}, y_i)}(g_i)$ is the subset of $G(g_i)$ consisting of points $(y'_{i+1}, y'_i) \in G(g_i)$ which satisfy at least one of the following conditions:

1. $y_{i+1} = y'_{i+1}$, or
2. there exist $j \in \{1, \dots, m-1\}$ and $k \in \{1, \dots, n_{g_i}(j)\}$ such that

$$(y_{i+1}, y_i), (y'_{i+1}, y'_i) \in G(g_{i,j}^k).$$

To show the continuity of H , we need Lemma 1.3.9. We can easily prove Lemma 1.3.9 with Lemma 1.3.8.

Lemma 1.3.8. Fix $i \in \mathbb{N}$. For any $(y_{i+1}, y_i) \in G(g_i)$ and $\epsilon > 0$, there exists $\delta > 0$ such that if

$$(y'_{i+1}, y'_i) \in G_{(y_{i+1}, y_i)}(g_i), \quad |y_i - y'_i| < \delta,$$

then

$$|y_{i+1} - y'_{i+1}| < \epsilon.$$

Proof. Fix $i \in \mathbb{N}$ and $(y_{i+1}, y_i) \in G(g_i)$. For each $j \in \{1, \dots, m-1\}$ and $k \in \{1, \dots, n_{g_i}(j)\}$, the inverse map of $g_{i,j}^k$ is uniformly continuous. Thus the statement is easily seen by the definition of $G_{(y_{i+1}, y_i)}(g_i)$. $\square$

Lemma 1.3.9. Fix $n \in \mathbb{N}$ and $(y_1, y_2, \dots) \in \mathbb{I}^{\mathbb{N}}$ with $(y_{i+1}, y_i) \in G(g_i)$ for $i = 1, \dots, n-1$. For any $\epsilon > 0$, there exists $\delta_n > 0$ such that for any $(y'_1, \dots, y'_n, \dots) \in \mathbb{I}^{\mathbb{N}}$ with $(y'_{i+1}, y'_i) \in G_{(y_{i+1}, y_i)}(g_i)$ for $i = 1, \dots, n-1$, if $|y_1 - y'_1| < \delta_n$, then

$$|y_{i+1} - y'_{i+1}| < \epsilon \quad \text{for } 1 \leq i \leq n-1.$$

1.4 Proof of Theorem 1.2.3

We now prove Theorem 1.2.3.

Proposition 1.4.1. $H : \varprojlim \{\mathbb{I}, f_i\} \rightarrow \varprojlim \{\mathbb{J}, g_i\}$ is a homeomorphism.

Proof. Fix $\mathbf{x} \in \varprojlim \{\mathbb{I}, f_i\}$ and let $\mathbf{y} = H(\mathbf{x})$. Fix any $\epsilon > 0$ and choose $n_\epsilon \in \mathbb{N}$ with $\sum_{i=n_\epsilon+1}^{\infty} 2^{-i} < \frac{\epsilon}{2}$. From Lemma 2.3.3 and Lemma 2.3.4, there exists $\delta_{n_\epsilon} > 0$ such that if $d(\mathbf{x}, \mathbf{x}') < \delta_{n_\epsilon}$ then

$$(\pi_i \circ H(\mathbf{x}'), \pi_{i+1} \circ H(\mathbf{x}')) \in G_{(y_{i+1}, y_i)}(g_i) \quad \text{for } 1 \leq i \leq n_\epsilon - 1.$$

Moreover, from Lemma 2.3.6, there exists $\eta_{n_\epsilon} > 0$ such that for any $(y'_1, \dots, y'_{n_\epsilon}, \dots) \in \mathbb{I}^\mathbb{N}$ with $(y'_{i+1}, y'_i) \in G_{(y_{i+1}, y_i)}(g_i)$ for $1 \leq i \leq n_\epsilon - 1$, if $|y_1 - y'_1| < \eta_{n_\epsilon}$ then

$$|y_{i+1} - y'_{i+1}| < \frac{\epsilon}{2n_\epsilon} \quad \text{for } 1 \leq i \leq n_\epsilon - 1.$$

Since $h : \mathbb{I} \rightarrow \mathbb{I}$ is continuous, there exists $\delta'_{n_\epsilon} > 0$ such that if $d(\mathbf{x}, \mathbf{x}') < \delta'_{n_\epsilon}$ then

$$|y_1 - \pi_1 \circ H(\mathbf{x}')| = |h(x_1) - h(x'_1)| < \min \left\{ \frac{\epsilon}{2n_\epsilon}, \eta_{n_\epsilon} \right\}.$$

Thus put $\delta := \min \{\delta_{n_\epsilon}, \delta'_{n_\epsilon}\}$. If $d(\mathbf{x}, \mathbf{x}') < \delta$ then

$$|y_i - \pi_i \circ H(\mathbf{x}')| < \frac{\epsilon}{2n_\epsilon} \quad \text{for } 1 \leq i \leq n_\epsilon.$$

Therefore

$$\begin{aligned} d(\mathbf{y}, H(\mathbf{x}')) &= \sum_{i=1}^{\infty} 2^{-i} |y_i - \pi_i \circ H(\mathbf{x}')| \\ &= \sum_{i=1}^{n_\epsilon} 2^{-i} |y_i - \pi_i \circ H(\mathbf{x}')| + \sum_{i=n_\epsilon+1}^{\infty} 2^{-i} |y_i - \pi_i \circ H(\mathbf{x}')| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, H is continuous. The same proof can be applied to the inverse map of H . Therefore we have that H is a homeomorphism. $\square$

From Proposition 1.3.4 and 1.4.1, we complete the proof of Theorem 1.2.3.

1.5 Dimension of inverse limits of Markov-like functions on intervals

Dimension of generalized inverse limits of set-valued functions even on $[0, 1]$ may be greater than one (see [I, Chapter 5]). Recently H. Kato [K] gave an interesting criterion for evaluating dimension of some generalized inverse limits of set-valued functions. Using the criterion, we will show that generalized inverse limits on an interval $\mathbb{I}$ with Markov-like bonding functions is one-dimensional.

Definition 1.5.1. Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of compact metric space. Let $f_i : X_{i+1} \rightarrow 2^{X_i}$ be a upper-semi continuous function for each $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, define $D_1(f_i)$ and $D_1(f_i^{-1})$ as

$$D_1(f_i) := \{x \in X_{i+1} \mid \dim f_i(x) \geq 1\},$$

$$D_1(f_i^{-1}) := \{y \in X_i \mid \dim f_i^{-1}(y) \geq 1\}.$$

The following theorem is a direct consequence of [K, Corollary 3.7] and its proof.

Theorem 1.5.2. ([K, Corollary 3.7]) Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of 1-dimensional compact metric space. Let $f_i : X_{i+1} \rightarrow 2^{X_i}$ be a upper-semi continuous function for each $i \in \mathbb{N}$. Suppose $\dim D_1(f_i) \leq 0$ and $D_1(f_i^{-1}) = \emptyset$ for each $i \in \mathbb{N}$. Then $\dim \varprojlim \{X_i, f_i\} \leq 1$.

Theorem 1.5.3. Let $\mathbb{I} = [a_1, a_m]$ be a closed intervals and $A : a_1 < a_2 < \dots < a_m$ be a partitions of $\mathbb{I}$. Let $f_i : \mathbb{I} \rightarrow 2^{\mathbb{I}}$ be a Markov-like function with respect to A for each $i \in \mathbb{N}$. Then $\dim \varprojlim \{X_i, f_i\} = 1$.

Proof. Fix $i \in \mathbb{N}$. From the definition of Markov-like functions, if $x \in D_1(f_i)$, then $x \in A$. Hence $\dim D_1(f_i) \leq \dim A = 0$. Moreover, since the graph of f_i cannot contain a graph of any constant function, $D_1(f_i^{-1}) = \emptyset$. Therefore, from Theorem 1.5.2, $\dim \varprojlim \{X_i, f_i\} \leq 1$.

We next show that $\dim \varprojlim \{X_i, f_i\} \geq 1$. Fix $i \in \mathbb{N}$ and put an non-degenerate closed interval $\mathbb{J}_i \subseteq \mathbb{I}$ with $\mathbb{J}_i \cap A = \emptyset$. From the definition of Markov-like functions, the one of the followings hold.

- (1) There exists non-degenerate closed interval $\mathbb{J}_{i+1} \subseteq \mathbb{I}$ such that $\mathbb{J}_{i+1} \subseteq \text{Int}([a_j, a_{j+1}])$ and $\mathbb{J}_i = f_{i,j}^k(\mathbb{J}_{i+1})$ for some $j \in \{1, \dots, m-1\}$ and $k \in \{1, \dots, n_{f_i(j)}\}$.
- (2) There exists degenerate interval $\mathbb{J}_{i+1} = \{a_j\} \subseteq A$ such that $\mathbb{J}_i \subseteq f_i(a_j)$.

Hence we can construct a inverse sequence $\{\mathbb{J}_i, g_i\}$ such that $\mathbb{J}_1$ is non-degenerate closed interval with $\mathbb{J}_1 \cap A = \emptyset$ and for each $i \in \mathbb{N}$, if $\mathbb{J}_{i+1}$ is non-degenerate, there exist j and k such that

$$\mathbb{J}_i = f_{i,j}^k(\mathbb{J}_{i+1}) \text{ and } G(g_i) = G(f_{i,j}^k|_{\mathbb{J}_{i+1}}),$$

if $\mathbb{J}_{i+1}$ is degenerate,

$$\mathbb{J}_i \subseteq f_i(\mathbb{J}_{i+1}) \text{ and } G(g_i) = G(f_i) \cap (\mathbb{J}_{i+1} \times \mathbb{J}_i).$$

Then $\varprojlim \{\mathbb{J}_i, g_i\} \subset \varprojlim \{X_i, f_i\}$ is an arc. Therefore $\dim \varprojlim \{X_i, f_i\} \geq 1$. $\square$

Theorem 1.5.3 says that any generalized inverse limit of Markov-like functions is one-dimensional. Thereby, as any chainable continuum is planar, a reader may suppose that any generalized inverse limit of Markov-like functions is planar. However there are examples of nonplanar generalized inverse limits with Markov-like functions in [I, Example 2.11 and 2.15]. The function in [I, Example 2.15] is also studied in [BCMM1], [BCMM2], [C] and [KN].

Example 1.5.4. ([I, Example 2.15]) Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f(0) = [0, 1]$ and $f(x) = \{1 - x\}$ for $0 < x \leq 1$. See Figure 1.4. Then $\varprojlim \{[0, 1], f\}$ is a non-planar one-dimensional continuum.

We say $T : [0, 1] \rightarrow [0, 1]$ is the tent map if the graph of T is the union of straight lines connecting $(0, 0)$ and $(c, 1)$, and $(c, 1)$ and $(1, 1)$ in $[0, 1]^2$, where $0 < c < 1$. The inverse limit of the tent map is a Knaster continuum which is planar and one-dimensional. The function f given by shifting the critical point c of the tent map to the point 0. See Figure 1.4. Although the generalized inverse limit of f is one-dimensional, it can not be embeddable into the plane.

In Section 3.4, we will prove that the generalized inverse limits of the function f in Example 1.5.4 is a non-planar one-dimensional continuum.

Figure 1.4: Graphs of f from Example 1.5.4 and the tent map T

1.6 Examples of Markov-like functions on intervals

In this section we show that there are Markov-like functions which generate homeomorphic generalized inverse limits without the same pattern. To see that, we refer to the following examples 1.6.1, 1.6.2. Here we suppose that $\mathbb{I}$ means the unit interval $[0, 1]$.

Example 1.6.1. Let $\{f_i : \mathbb{I} \rightarrow 2^{\mathbb{I}} \mid i \in \mathbb{N}\}$ be an sequence of upper semicontinuous functions. Suppose that there is a sequence of strictly monotone continuous functions $\{g_i : \mathbb{I} \rightarrow \mathbb{I} \mid i \in \mathbb{N}\}$ such that $(0, 0), (1, 1) \in G(g_i)$ and $G(f_i) = G(g_i)$ for each $i \in \mathbb{N}$. Then $\varprojlim \{\mathbb{I}, f\}$ is an arc with endpoints $(0, 0, \dots)$, and $(1, 1, \dots)$.

Example 1.6.2. Fix $n \in \mathbb{N}_{\geq 2}$. Suppose that $f_1, \dots, f_n : \mathbb{I} \rightarrow \mathbb{I}$ are strictly monotone continuous functions such that

$$i \neq j \Rightarrow G(f_i) \cap G(f_j) = \{(0, 0), (1, 1)\}.$$

Let $f : \mathbb{I} \rightarrow 2^{\mathbb{I}}$ be the upper semi-continuous function defined by

$$G(f) = \bigcup_{i=1}^n G(f_i).$$

Then $\lim_{\leftarrow} \{\mathbb{I}, f\}$ is homeomorphic to the suspension of a Cantor set.

Figure 1.5: Graph of f where $n = 2$ from Example 1.6.2

Proof. Let C be a Cantor set and let $S(C)$ the suspension of C . Here the suspension $S(C)$ is the quotient space obtained from the product space $\mathbb{I} \times C$ by shrinking the top set $\{1\} \times C$ and the bottom set $\{0\} \times C$ to the points N and S , respectively. Let $p : \mathbb{I} \times C \rightarrow S(C)$ be the natural projection and for each $(t, z) \in \mathbb{I} \times C$ the equivalence class of (t, z) we write $[t, z] = p(t, z)$. Let $\Lambda_n := \prod_{k \in \mathbb{N}} \{1, \dots, n\}$ be the product space of countable copies of the n -point space $\{1, \dots, n\}$. Then there exists a homeomorphism $\varphi : C \rightarrow \Lambda_n$. We define $h : S(C) \rightarrow \lim_{\leftarrow} \{\mathbb{I}, f\}$ as follows: for any $[t, z] \in S(C)$,

$$h([t, z]) := (x_1, x_2, \dots) \text{ with } x_1 = t \text{ and } (x_{k+1}, x_k) \in G(f_{s_k}) \text{ for each } k \in \mathbb{N},$$

$$\text{where } s = (s_1, s_2, \dots) = \varphi(z).$$

In the case of $t \in (0, 1)$, $\mathbf{x} = h([t, z]) \in \lim_{\leftarrow} \{\mathbb{I}, f\}$ is uniquely determined. Moreover

$$h([0, z]) = (0, 0, \dots), \quad h([1, z]) = (1, 1, \dots) \text{ for any } z \in C.$$

Thus, $h(N) = (0, 0, \dots)$ and $h(S) = (1, 1, \dots)$. Hence h is well-defined. We note that h is a bijection. In fact, for each $s = (s_1, s_2, \dots) \in \Lambda_n$, let denote

$$L_s := \left\{ \mathbf{x} \in \prod_{k=1}^{\infty} \mathbb{I} \mid (x_{k+1}, x_k) \in G(f_{s_k}) \text{ for each } k \in \mathbb{N} \right\}.$$

Then

$$\lim_{\leftarrow} \{\mathbb{I}, f\} = \bigcup_{s \in \Lambda_n} L_s, \text{ and } L_s \cap L_{s'} = \{(0, 0, \dots), (1, 1, \dots)\} \text{ if } s \neq s'.$$

Hence we can easily see that h is bijective. Therefore to prove that h is a homeomorphism we will show that h is continuous.

Let take a point $[t_0, z_0] \in S(C)$ and put $\mathbf{x} = (x_1, x_2, \dots) = h([t_0, z_0])$. For an arbitrary $i \in \mathbb{N}$, take any open subset $U \subseteq \mathbb{I}$ with $x_i \in U$. We will show that there exists an open set $O_i \subset S(C)$ with $[t_0, z_0] \in O_i$ such that $\pi_i(h(O_i)) \subset U$.

In the case of $i = 1$, since $x_1 = t_0$,

$$O_1 = \{[t, c] \in S(C) \mid t \in U\}$$

is an open subset of $S(C)$, $[t_0, z_0] \in O_1$ and $\pi_1(h(O_1)) \subset U$.

In general case $i > 1$, let the subset $\tilde{\Lambda}$ of $\prod_{k=1}^{i-1} \{1, \dots, n\}$ consisting of points $(u_1, \dots, u_{i-1})$ which satisfy the condition:

$$t_0 = x_1 \in f_{u_1} \circ \dots \circ f_{u_{i-1}}(U).$$

By the definition of h , $(s_1, \dots, s_{i-1}) \in \tilde{\Lambda}$ where $\varphi(z_0) = (s_1, \dots, s_{i-1}, s_i, \dots) \in \Lambda_n$. Hence

$$V := \tilde{\Lambda} \times \prod_{k=i}^{\infty} \{1, \dots, n\}$$

is an open subset of Λ_n and $\varphi(z_0) \in V$. Note that, if $x_i = 0$, for any $(u_1, \dots, u_{i-1}) \in \prod_{k=1}^{i-1} \{1, \dots, n\}$, $f_{u_1} \circ \dots \circ f_{u_{i-1}}(x_i) = 0 = x_1$. Hence $\tilde{\Lambda} = \prod_{k=1}^{i-1} \{1, \dots, n\}$ and $V = \Lambda_n$. If $x_i = 1$, we similarly have that $V = \Lambda_n$. In any case

$$W := \bigcap_{s \in V} f_{s_1} \circ \dots \circ f_{s_{i-1}}(U) = \bigcap_{(s_1, \dots, s_{i-1}) \in \tilde{\Lambda}} f_{s_1} \circ \dots \circ f_{s_{i-1}}(U)$$

is an open set in $\mathbb{I}$ with $t_0 = x_1 \in W$ having the property: for any finite sequence $(y_1, \dots, y_i) \in \mathbb{I} \times \dots \times \mathbb{I}$ and any $(s_1, \dots, s_{i-1}) \in \tilde{\Lambda}$,

$$y_1 \in W \text{ and } y_k = f_{s_k}(y_{k+1}) \ (k = 1, \dots, i-1) \implies y_i \in U.$$

Then

$$O_i = p(W \times \varphi^{-1}(V)) = \{[t, z] \in S(C) \mid t \in W, \varphi(z) \in V\}$$

is an open set in $S(C)$. If $t_0 \in (0, 1)$, then we may assume that $W \subset (0, 1)$. Note that if $t_0 = 0$ or 1 , $\varphi^{-1}(V) = C$. Hence in these cases we can take W a half open interval of the form $[0, r)$ or $(r, 1]$, respectively. Thus, O_i is an open subset of $S(C)$ such that $[t_0, z_0] \in O_i$ and $\pi_i(h(O_i)) \subset U$. Therefore h is continuous. $\square$

Example 1.6.2 induces an example of two Markov-like functions f and g such that they do not have the same pattern but their generalized inverse limits are homeomorphic.

Example 1.6.3. Let $l, m \in \mathbb{N}_{\geq 2}$ with $l \neq m$. Suppose that $f_1, \dots, f_l, g_1, \dots, g_m : \mathbb{I} \rightarrow \mathbb{I}$ are strictly monotone continuous functions such that

$$i \neq j \Rightarrow G(f_i) \cap G(f_j) = \{(0, 0), (1, 1)\},$$

$$i' \neq j' \Rightarrow G(g_{i'}) \cap G(g_{j'}) = \{(0, 0), (1, 1)\}.$$

Define set-valued functions $f, g : \mathbb{I} \rightarrow 2^{\mathbb{I}}$ by

$$G(f) = \bigcup_{i=1}^l G(f_i), \quad G(g) = \bigcup_{j=1}^m G(g_j).$$

Then f and g do not have the same pattern but their generalized inverse limits $\varprojlim \{\mathbb{I}, f\}$ and $\varprojlim \{\mathbb{I}, g\}$ are homeomorphic.

From the set-valued functions f and g in Example 1.6.3 we will give an example of a pair (φ, ψ) of Markov-like functions whose graphs coincide with $G(f) \cup \{1\} \times \mathbb{I}$ and $G(g) \cup \{1\} \times \mathbb{I}$, respectively but they induce non-homeomorphic generalized inverse limits $\varprojlim \{\mathbb{I}, \varphi\}$ and $\varprojlim \{\mathbb{I}, \psi\}$.

Example 1.6.4. Let $f, g : \mathbb{I} \rightarrow 2^{\mathbb{I}}$ be the set-valued functions $f, g : \mathbb{I} \rightarrow 2^{\mathbb{I}}$ defined in Example 1.6.3 as in the case of $l = 2$ and $m = 3$. Let $\varphi, \psi : \mathbb{I} \rightarrow 2^{\mathbb{I}}$ be defined by

$$\varphi(x) = \begin{cases} \{f(x)\} & \text{if } x \in [0, 1) \\ [0, 1] & \text{if } x = 1. \end{cases}$$

$$\psi(x) = \begin{cases} \{g(x)\} & \text{if } x \in [0, 1) \\ [0, 1] & \text{if } x = 1. \end{cases}$$

Then $\varprojlim \{\mathbb{I}, \varphi\}$ and $\varprojlim \{\mathbb{I}, \psi\}$ are not homeomorphic.

Proof. Let

$$A_0 := \left\{ \mathbf{x} \in \prod_{j \in \mathbb{N}} \mathbb{I} \mid (x_{j+1}, x_j) \in G(f) \text{ for each } j \in \mathbb{N} \right\},$$

$$A_1 := \left\{ \mathbf{x} \in \prod_{j \in \mathbb{N}} \mathbb{I} \mid x_2 = 1, \text{ and } x_j \in \varphi(x_{j+1}) \text{ for any } j \geq 2 \right\}$$

Figure 1.6: Graphs of φ and ψ from Example 1.6.4

$$B_0 := \left\{ \mathbf{x} \in \prod_{j \in \mathbb{N}} \mathbb{I} \mid (x_{j+1}, x_j) \in G(g) \text{ for each } j \in \mathbb{N} \right\},$$

$$B_1 := \left\{ \mathbf{x} \in \prod_{j \in \mathbb{N}} \mathbb{I} \mid x_2 = 1, \text{ and } x_j \in \psi(x_{j+1}) \text{ for any } j \geq 2 \right\}.$$

For each $i \in \mathbb{N}_{\geq 2}$, let

$$A_i := \left\{ \mathbf{x} \in \prod_{j \in \mathbb{N}} \mathbb{I} \mid x_{i+1} = 1, (x_{j+1}, x_j) \in G(f) \ (j < i), x_j \in \varphi(x_{j+1}) \ (j \geq i+1) \right\},$$

$$B_i := \left\{ \mathbf{x} \in \prod_{j \in \mathbb{N}} \mathbb{I} \mid x_{i+1} = 1, (x_{j+1}, x_j) \in G(g) \ (j < i), x_j \in \psi(x_{j+1}) \ (j \geq i+1) \right\}.$$

Then we can easily see that

$$\lim_{\leftarrow} \{\mathbb{I}, \varphi\} = \bigcup_{i=0}^{\infty} A_i, \quad \lim_{\leftarrow} \{\mathbb{I}, \psi\} = \bigcup_{i=0}^{\infty} B_i,$$

and

$$A_{i_1} \cap A_{i_2} = (1, 1, \dots), \quad B_{i_1} \cap B_{i_2} = (1, 1, \dots) \quad \text{if } i_1 \neq i_2.$$

We will consider A_i and B_i for each $i \in \mathbb{Z}_{\geq 0}$. By Example 1.6.2, A_0 and B_0 are the union of uncountable arcs with endpoints $(0, 0, \dots)$ and $(1, 1, \dots)$, respectively.

For each $i \geq 1$, since $\varphi^{-1}(\{1\}) = \{1\}$, $\pi_{<1,i+1>} : A_i \rightarrow \pi_{<1,i+1>}(A_i)$ is a homeomorphism. Moreover,

$$\pi_{<1,i+1>}(A) = \left\{ \mathbf{x} \in \prod_{j=1}^{i+1} \mathbb{I} \mid x_{i+1} = 1, (x_{j+1}, x_j) \in G(f) \ (j < i) \right\}$$

is the union of 2^{i-1} many arcs. These arcs have same endpoints $u_i = (0, 0, \dots, 0, 1)$, $v_i = (1, 1, \dots, 1)$ and they are mutually disjoint on each point without their endpoints. Thus A_i is the union of 2^{i-1} many arcs with same endpoints p_i, q_i such that j -th coordinate of p_i and q_i are equal to u_i and v_i respectively if $j \leq i + 1$ and, j -th coordinate of p_i and q_i are equal to 1 respectively if $j > i + 1$. These arcs are pairwise disjoint on each point without their endpoints. By similar way, it is also seen that B_i is the union of 3^{i-1} many arcs such that they have same endpoints and they are mutually disjoint on each point without their endpoints.

The point $p = (0, 1, 1, \dots) \in \varprojlim \{\mathbb{I}, \psi\}$ is an endpoint of B_2 . Then the maximum number of arcs such that they contain p as common endpoint and they only intersect on p is 3. However for each point $q \in \varprojlim \{\mathbb{I}, \varphi\}$, the maximum number of such arcs is equal to 2^n for some $n \in \mathbb{Z}_{\geq 0}$, or countably infinite. Therefore $\varprojlim \{[0, 1], \varphi\}$ and $\varprojlim \{[0, 1], \psi\}$ are not homeomorphic. $\square$

Chapter 2

Markov-like set-valued functions on finite graphs and their inverse limits

2.1 Introduction

In this chapter, we will generalize the notation of Markov-like functions on intervals to finite graphs and define the notation of the same pattern among those Markov-like functions. From Definition 2.2.3, readers may notice that the same pattern between two Markov-like functions is a generalization of topological conjugacy.

In Section 2.4, we will prove that any two generalized inverse limits with Markov-like bonding functions on finite graphs having the same pattern are homeomorphic. Consequently, we can have a generalization of Chapter 1.

Moreover, in Section 2.5, we will show that every generalized inverse limits with surjective Markov-like functions on finite graphs is one-dimensional. In Section 2.6 we will give examples of Markov-like functions and apply Theorem 2.2.5.

2.2 Markov-like functions on finite graphs

We begin the process of defining Markov-like function by defining finite graphs.

Definition 2.2.1. A continuum T is a *finite graph* if there exist a one-dimensional finite connected simplicial complex $\mathcal{T}$ and a homeomorphism from the polyhedron $|\mathcal{T}|$ of $\mathcal{T}$ onto T . Then we say T is triangulated by $\mathcal{T}$.

We usually identify T with $|\mathcal{T}|$ without writing a homeomorphism between them when we do not need to specify it (see Munkres [Mu, §2]). By $\mathcal{V}(\mathcal{T})$ and $\mathcal{E}(\mathcal{T})$ we denote the sets of vertices (= 0-simplices) and edges (= 1-simplices) of $\mathcal{T}$, respectively.

Definition 2.2.2. Let T be a finite graph triangulated by $\mathcal{T}$. A function $f : T \rightarrow 2^T$ is said to be *Markov-like with respect to $\mathcal{T}$* if the following statements are satisfied.

- (1) For each $v \in \mathcal{V}(\mathcal{T})$ there exists a subcomplex $\mathcal{T}_v$ of $\mathcal{T}$ such that

$$f(v) = |\mathcal{T}_v|.$$

- (2) For each $e \in \mathcal{E}(\mathcal{T})$ there exist $n_f(e) \in \mathbb{N}$ and injective continuous functions $f_e^1, \dots, f_e^{n_f(e)} : e \rightarrow T$ such that

$$f_e^k(v) \in \mathcal{V}(\mathcal{T}_v) \text{ for any } 1 \leq k \leq n_f(e) \text{ and } v \in \partial(e),$$

$$G(f|_{e \setminus \partial(e)}) = \bigcup_{k=1}^{n_f(e)} G(f_e^k|_{e \setminus \partial(e)}), \text{ and}$$

$$G(f_e^{k_1}|_{e \setminus \partial(e)}) \cap G(f_e^{k_2}|_{e \setminus \partial(e)}) = \emptyset \quad \text{if } k_1 \neq k_2,$$

where $\partial(e)$ is the set of vertices of e .

The graph of a Markov-like function is a finite union of arcs. Hence each Markov-like function is upper semi-continuous.

In the following definition we introduce the same pattern between two Markov-like functions.

Definition 2.2.3. Let T and S be finite graphs with triangulations $\mathcal{T}$ and $\mathcal{S}$ respectively. Let $f : T \rightarrow 2^T$ and $g : S \rightarrow 2^S$ be Markov-like functions with respect to $\mathcal{T}$ and $\mathcal{S}$ respectively. Let $h, h' : (T, \mathcal{T}) \rightarrow (S, \mathcal{S})$ be simplicial homeomorphisms. We say that f and g *have the same pattern* with respect to the pair (h, h') if the following conditions are satisfied:

- (3) For each $v \in \mathcal{V}(\mathcal{T})$

$$\tilde{h}(f(v)) = g(h'(v)).$$

- (4) For each $e \in \mathcal{E}(\mathcal{T})$ there exists a bijection $\phi_e : \{1, \dots, n_f(e)\} \rightarrow \{1, \dots, n_g(\tilde{h}'(e))\}$ such that for each $v \in \partial(e)$,

$$h(f_e^k(v)) = g_{\tilde{h}'(e)}^{\phi_e(k)}(h'(v)) \text{ and}$$

$$\tilde{h}(f_e^k(e)) = \tilde{g}_{\tilde{h}'(e)}^{\phi_e(k)}(\tilde{h}'(e)) \text{ for each } 1 \leq k \leq n_f(e).$$

Here we give an example of Markov-like functions which have the same pattern.

Example 2.2.4. Let T and S be finite graphs homeomorphic to $\mathbb{S}^1$ with triangulations $\mathcal{T}$ and $\mathcal{S}$ such that

$$\begin{aligned}\mathcal{V}(\mathcal{T}) &= \{v_1, v_2, v_3, v_4\}, \quad \mathcal{E}(\mathcal{T}) = \{e_1, e_2, e_3, e_4\}; \\ \mathcal{V}(\mathcal{S}) &= \{u_1, u_2, u_3, u_4\}, \quad \mathcal{E}(\mathcal{S}) = \{w_1, w_2, w_3, w_4\}\end{aligned}$$

with

$$\begin{aligned}\partial(e_i) &= \{v_i, v_{i+1}\}, \quad \partial(w_i) = \{u_i, u_{i+1}\} \quad \text{for } 1 \leq i \leq 3, \quad \text{and} \\ \partial(e_4) &= \{v_4, v_1\}, \quad \partial(w_4) = \{u_4, u_1\}.\end{aligned}$$

Let $h : (T, \mathcal{T}) \rightarrow (S, \mathcal{S})$ be a simplicial homeomorphism such that

$$h(v_i) = u_i \quad \text{for } 1 \leq i \leq 4.$$

Let graphs of $f : T \rightarrow 2^T$ and $g : S \rightarrow 2^S$ be defined by Figure 2.1 respectively. Bijective maps $\phi_{e_1} : \{1\} \rightarrow \{1\}, \phi_{e_2} : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}, \phi_{e_3} : \{1, 2\} \rightarrow \{1, 2\}, \phi_{e_4} : \{1\} \rightarrow \{1\}$ are the identity mappings. Then f and g have the same pattern with respect to (h, h) .

Figure 2.1: Graphs of f and g from Example 2.2.4

The notations of generalized Markov interval maps and having the same pattern were introduced by Banič and Lunder [BL] and the author [1] gave further generalization. In both [BL] and [1] they considered only interval maps and defined the notation of having the same pattern by defining correspondence from a Markov partition to the other Markov partition in accordance with the order of real numbers. However, finite graphs do not have an appropriate order. Thereby, in order to introduce the notation of having the same pattern, we use simplicial homeomorphisms.

Hence we can give a unified definition including one by [BL] and [1]. In particular, in both [BL] and [1], they considered the same pattern with respect to a pair (h, h) of a piecewise linear homeomorphism h as the above example. In Section 2.6, we will give examples of Markov-like functions having the same pattern in the new sense.

The goal of the section is to prove Theorem 2.2.5.

Theorem 2.2.5. *Let T and S be finite graphs triangulated by $\mathcal{T}$ and $\mathcal{S}$ respectively. Let $\{f_i : T \rightarrow 2^T\}_{i \in \mathbb{N}}$ and $\{g_i : S \rightarrow 2^S\}_{i \in \mathbb{N}}$ be sequences of Markov-like functions with respect to $\mathcal{T}$ and $\mathcal{S}$ respectively. Let $\{h_i : T \rightarrow S\}_{i \in \mathbb{N}}$ be a sequence of simplicial homeomorphisms. If f_i and g_i have the same pattern with respect to (h_i, h_{i+1}) for each $i \in \mathbb{N}$, then $\varprojlim \{T, f_i\}$ and $\varprojlim \{S, g_i\}$ are homeomorphic.*

2.3 Preliminaries

We will need several preliminary results to prove Theorem 2.2.5. Throughout this section, we assume all hypotheses of Theorem 2.2.5 hold. That is, T and S are finite graphs triangulated by $\mathcal{T}$ and $\mathcal{S}$, $\{f_i\}_{i \in \mathbb{N}}$ and $\{g_i\}_{i \in \mathbb{N}}$ are Markov-like functions, $\{h_i\}_{i \in \mathbb{N}}$ are simplicial homeomorphisms, and f_i and g_i have the same pattern with respect to (h_i, h_{i+1}) for all $i \in \mathbb{N}$. From the definition, we use the following terminology: For $i \in \mathbb{N}$ and edges $e \in \mathcal{E}(\mathcal{T})$, $w \in \mathcal{E}(\mathcal{S})$, $f_{i,e}^k$ and $g_{i,w}^l$ are the k -th continuous injection defined on e with f_i and the l -th continuous injection defined on w with g_i , respectively. $\phi_{i,e} : \{1, \dots, n_{f_i}(e)\} \rightarrow \{1, \dots, n_{g_i}(\tilde{h}_{i+1}(e))\}$ is the bijection.

2.3.1 A construction of a map $H : \varprojlim \{T, f_i\} \rightarrow \varprojlim \{S, g_i\}$

We will construct a map $h : \varprojlim \{T, f_i\} \rightarrow \varprojlim \{S, g_i\}$.

Proposition 2.3.1. *For any point $\mathbf{x} = (x_1, x_2, \dots) \in \varprojlim \{T, f_i\}$ there exists exactly one point $\mathbf{y} = (y_1, y_2, \dots) \in \varprojlim \{S, g_i\}$ satisfying the following properties: for each $i \in \mathbb{N}$,*

- (1)-(i) *If $x_i \in \mathcal{V}(\mathcal{T})$, then $y_i = h_i(x_i)$.*
- (2)-(i) *If $x_i \in e$ for some $e \in \mathcal{E}(\mathcal{T})$, then $y_i \in \tilde{h}_i(e)$.*
- (3)-(i) *If $i \geq 2$ and $(x_i, x_{i-1}) \in G(f_{i-1}^k e)$, then $(y_i, y_{i-1}) \in G(g_{i-1}^{\phi_{i-1,e}(k)} \tilde{h}_i(e))$.*

Proof. Let $\pi_{<1, \dots, m>} : \prod_{i=1}^n S \rightarrow \prod_{i=1}^m S$ be the projection for $m \in \mathbb{N}$ with $m < n$.

To prove the existence of $\mathbf{y}$, we show that there is a sequence $\{L_n\}_{n \in \mathbb{N}}$ such that

- L_n is a closed subset of $\prod_{j=1}^n S$ for each $n \in \mathbb{N}$,
- every point of L_n satisfies the properties (1)-(i), (2)-(i), and (3)-(i)

for $i = 1, \dots, n$, and

• if $n \geq 2$ then $\pi_{<1, n-1>}(L_n) \subseteq L_{n-1}$.

Let $y_1 = h_1(x_1)$ and $L_1 := \{y_1\}$. By the definition of h_1 , the conditions (1)-(1), (2)-(1), and (3)-(1) are satisfied for each point of L_1 . Suppose we define closed sets $L_1, L_2, \dots, L_n$ such that for each $i = 1, \dots, n$, every point of L_i satisfies the properties (1)-(j), (2)-(j), and (3)-(j) for each $j \leq i$, and if $i \geq 2$, then $\pi_{<1, i-1>}(L_i) \subseteq L_{i-1}$. We will define L_{n+1} . Fix $(y_1, y_2, \dots, y_n) \in L_n$. We define y_{n+1} in the following way.

In the case where $x_{n+1} \notin \mathcal{V}(\mathcal{T})$, there uniquely exists $e \in \mathcal{E}(\mathcal{T})$ and $k \in \{1, \dots, n_{f_n}(e)\}$ such that $(x_{n+1}, x_n) \in G(f_n^k e)$. Since f_n and g_n have the same pattern, and $g_n^{\phi_n e(k)} \tilde{h}_{n+1}(e)$ is injective, we can put y_{n+1} as $\left(g_n^{\phi_n e(k)} \tilde{h}_{n+1}(e)\right)^{-1}(y_n)$. In the other case, $x_{n+1} \in \mathcal{V}(\mathcal{T})$, then put $y_{n+1} = h_{n+1}(x_{n+1})$.

Let $L_{n+1} := \{(y_1, \dots, y_{n+1})\}$. Then L_{n+1} is a non-empty closed set of $\prod_{j=1}^{n+1} S$, and $(y_1, \dots, y_{n+1})$ satisfies the properties (1)-(i), (2)-(i), and (3)-(i) for $1 \leq i \leq n+1$. Thus, we can have a desired sequence $\{L_n\}_{n \in \mathbb{N}}$. Then define $\{K_n\}_{n \in \mathbb{N}}$ by for each $n \in \mathbb{N}$

$$K_n := L_n \times \prod_{j=n+1}^{\infty} S.$$

Since $\{K_n\}$ is a decreasing sequence of closed sets, there exists $\mathbf{y} \in \bigcap_{n=1}^{\infty} K_n$. By the construction of $\mathbf{y}$, $\mathbf{y} \in \varprojlim \{S, g_i\}$ and $\mathbf{y}$ satisfies the properties (1)-(i), (2)-(i), and (3)-(i) for each $i \in \mathbb{N}$.

We will show that $\mathbf{y}$ is uniquely determined. Suppose that for each $\mathbf{x} \in \varprojlim \{T, f_i\}$, there are mutually distinct points $\mathbf{y}, \mathbf{y}' \in \varprojlim \{S, g_i\}$ such that $\mathbf{y}$ and $\mathbf{y}'$ satisfy the properties (1)-(i), (2)-(i) and (3)-(i) for each $i \in \mathbb{N}$. We may assume that there exists $i_0 \in \mathbb{N}_{\geq 2}$ such that $y_{i_0} \neq y'_{i_0}$, and $y_i = y'_i$ for $i < i_0$.

By the definition we see that $x_{i_0} \notin \mathcal{V}(\mathcal{T})$. Then there uniquely exists $e \in \mathcal{E}(\mathcal{T})$ and $k \in \{1, \dots, n_{f_{i_0-1}}(e)\}$ such that $(x_{i_0}, x_{i_0-1}) \in G(f_{i_0-1}^k e)$. From the construction of $\mathbf{y}$ and $\mathbf{y}'$, $(y_{i_0-1}, y_{i_0}), (y'_{i_0-1}, y'_{i_0}) \in G(g_{i_0-1}^{\phi_{i_0-1} e(k)} \tilde{h}_{i_0}(e))$. Since $g_{i_0-1}^{\phi_{i_0-1} e(k)} \tilde{h}_{i_0}(e)$ is injective, it contradicts the definition of i_0 . Thus $\mathbf{y}$ is uniquely determined. $\square$

For any $\mathbf{x} \in \varprojlim \{T, f_i\}$, choosing the point $\mathbf{y} \in \varprojlim \{S, g_i\}$ of Proposition 2.3.1, we can define the function

$$H : \varprojlim \{T, f_i\} \rightarrow \varprojlim \{S, g_i\}.$$

2.3.2 Some properties of H

We will provide some propositions to show that H is a homeomorphism.

Proposition 2.3.2. *For any $\mathbf{x} \in \lim_{\leftarrow} \{T, f_i\}$ and $l \in \mathbb{N}$, there exists $\delta > 0$ such that for each $\mathbf{x}' \in \lim_{\leftarrow} \{T, f_i\}$ with $d(\mathbf{x}, \mathbf{x}') < \delta$, at least one of the following statements hold for each $i \in \{1, \dots, l\}$:*

(1)-(i) $y_{i+1} = y'_{i+1}$,

(2)-(i) $(y_{i+1}, y_i), (y'_{i+1}, y'_i) \in G(g_{i \ e}^k)$ for some $e \in \mathcal{E}(\mathcal{S})$ and $k \in \{1, \dots, n_{g_i}(e)\}$

where $\mathbf{y} = H(\mathbf{x})$, $\mathbf{y}' = H(\mathbf{x}')$.

Proposition 2.3.2 is an easy consequence of Lemma 2.3.3 and Lemma 2.3.4.

Lemma 2.3.3. *For any $\mathbf{x} \in \lim_{\leftarrow} \{T, f_i\}$ and $l \in \mathbb{N}$, there exists $\delta > 0$ such that for any $\mathbf{x}' \in \lim_{\leftarrow} \{T, f_i\}$ with $d(\mathbf{x}, \mathbf{x}') < \delta$, at least one of the following statements hold for each $i \in \{1, \dots, l\}$:*

(1)-(i) $x_{i+1} = x'_{i+1}$, $x_{i+1}, x'_{i+1} \in \mathcal{V}(\mathcal{T})$,

(2)-(i) $(x_{i+1}, x_i), (x'_{i+1}, x'_i) \in G(f_{i \ e}^k)$ for some $e \in \mathcal{E}(\mathcal{T})$ and $k \in \{1, \dots, n_{f_i}(e)\}$.

Proof. Fix $\mathbf{x} \in \lim_{\leftarrow} \{T, f_i\}$ and $l \in \mathbb{N}$. Put an arbitrary $i \in \{1, \dots, l\}$. Let

$$K := \left\{ (e', k') \mid (x_{i+1}, x_i) \in G(f_{i \ e'}^{k'}) \right\}$$

and

$$L := G(f_i) \setminus \left(G(f_i|_{\{x_{i+1}\}}) \cup \bigcup_{(e', k') \in K} G(f_{i \ e'}^{k'}) \right)$$

We will show that $(x_{i+1}, x_i) \notin \text{Cl}(L)$.

Assume that $(x_{i+1}, x_i) \in \text{Cl}(L)$. Then there is a sequence $\{(z_{i+1}^n, z_i^n)\}_{n \in \mathbb{N}}$ in L such that

$$\lim_{n \rightarrow \infty} d_T(z_{i+1}^n, x_{i+1}) = 0, \quad \lim_{n \rightarrow \infty} d_T(z_i^n, x_i) = 0.$$

Then we can assume that $z_{i+1}^n \notin \mathcal{V}(\mathcal{T})$ for every $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, there exist $e' \in \mathcal{E}(\mathcal{T})$ and $k' \in \{1, \dots, n_{f_i}(e')\}$ such that

$$(z_{i+1}^n, z_i^n) \in G(f_{i \ e'}^{k'}).$$

Let

$$K_L := \left\{ (e', k') \mid (z_{i+1}^n, z_i^n) \in G(f_{i \ e'}^{k'}) \text{ for some } n \in \mathbb{N} \right\}.$$

Since K_L is finite, there exist $(e, k) \in K_L$ and an infinite subset M of $\mathbb{N}$ such that

$$(z_{i+1}^m, z_i^m) \in G(f_{i \ e}^k) \text{ for each } m \in M.$$

Then $(x_{i+1}, x_i) \in G(f_{i \ e}^k)$. It is a contradiction. Thus there exists $\eta > 0$ such that

$$d_T(x_i, z_i)^2 + d_T(x_{i+1}, z_{i+1})^2 \geq \eta^2 \text{ for any } (z_{i+1}, z_i) \in L.$$

Then $\delta_i = 2^{-(i+2)}\eta$ satisfies the required property. In fact, if

$$d(\mathbf{x}, \mathbf{x}') = \sum_{j=1}^{\infty} 2^{-j} d_T(x_j, x'_j) < \delta_i,$$

then

$$2^{-i} d_T(x_i, x'_i) < 2^{-(i+2)} \eta, \quad 2^{-(i+1)} d_T(x_{i+1}, x'_{i+1}) < 2^{-(i+2)} \eta.$$

It follows that

$$d_T(x_i, x'_i)^2 + d_T(x_{i+1}, x'_{i+1})^2 < \frac{5}{16} \eta^2.$$

Thus $(x'_{i+1}, x'_i) \notin L$. Therefore put $\delta := \min \{\delta_i \mid 1 \leq i \leq l\}$. □

Lemma 2.3.4. Choose $\mathbf{x}, \mathbf{x}' \in \lim_{\leftarrow} \{T, f_i\}$ and let $\mathbf{y} = H(\mathbf{x})$, $\mathbf{y}' = H(\mathbf{x}')$. Then, for each $i \in \mathbb{N}$, we have the following:

- (1) If $x_{i+1} = x'_{i+1}$ and $x_{i+1}, x'_{i+1} \in \mathcal{V}(\mathcal{T})$, then $y_{i+1} = y'_{i+1}$,
- (2) If $(x_{i+1}, x_i), (x'_{i+1}, x'_i) \in G(f_{i+1}^k)$, for some $e \in \mathcal{E}(\mathcal{T})$ and $k \in \{1, \dots, n_{f_i}(e)\}$, then

$$(y_{i+1}, y_i), (y'_{i+1}, y'_i) \in G\left(g_i^{\phi_i \ e(k)}\right).$$

Proof. This lemma is directly given from the definition of H . □

In order to prove H is continuous, we will need Proposition 2.3.6, which we can easily prove with Lemma 2.3.5.

Lemma 2.3.5. Fix $i \in \mathbb{N}$. For each $\epsilon > 0$, there exists $\delta > 0$ such that for each $(y_{i+1}, y_i), (y'_{i+1}, y'_i) \in G(g_i)$ having at least one of the following properties:

- (1) $y_{i+1} = y'_{i+1}$,
- (2) $(y_{i+1}, y_i), (y'_{i+1}, y'_i) \in G(g_i^k)$ for some $e \in \mathcal{E}(\mathcal{S})$ and $k \in \{1, \dots, n_{g_i}(e)\}$,

if $d_S(y_i, y'_i) < \delta$ then $d_S(y_{i+1}, y'_{i+1}) < \epsilon$.

Proof. Fix $i \in \mathbb{N}$. For each $e \in \mathcal{E}(\mathcal{S})$ and $k \in \{1, \dots, n_{g_i}(e)\}$, the inverse map of g_i^k is uniformly continuous. Since there are only finitely many e and k , the result is directly seen. □

Proposition 2.3.6. Fix $l \in \mathbb{N}$. For each $\epsilon > 0$, there exists $\delta > 0$ such that for each $\mathbf{y}, \mathbf{y}' \in \lim_{\leftarrow} \{S, g_i\}$ having at least one of the following properties for each $1 \leq i \leq l$:

- (1)-(i) $y_{i+1} = y'_{i+1}$,
 - (2)-(i) $(y_{i+1}, y_i), (y'_{i+1}, y'_i) \in G(g_i^k)$ for some $e \in \mathcal{E}(\mathcal{S})$ and $k \in \{1, \dots, n_{g_i}(e)\}$,
- if $d_S(y_1, y'_1) < \delta$, then $d_S(y_{i+1}, y'_{i+1}) < \epsilon$ for each $1 \leq i \leq l$.

2.4 Proof of Theorem 2.2.5

We are ready to prove that H is a homeomorphism.

Proposition 2.4.1. *H is a homeomorphism.*

Proof. Fix $\mathbf{x} \in \varprojlim \{T, f_i\}$ and let $\mathbf{y} = H(\mathbf{x})$. Fix any $\epsilon > 0$ and choose $n_\epsilon \in \mathbb{N}$ with $\sum_{i=n_\epsilon+1}^{\infty} 2^{-i} \text{diam}(S) < \frac{\epsilon}{2}$. From Proposition 2.3.2 and Proposition 2.3.6, there exists $\delta_{n_\epsilon} > 0$ such that if $d(\mathbf{x}, \mathbf{x}') < \delta_{n_\epsilon}$ then

$$\begin{aligned} &\text{there exists } \eta_{n_\epsilon} > 0 \text{ such that if } d_S(y_1, y'_1) < \eta_{n_\epsilon} \text{ then} \\ &d_S(y_i, y'_i) < \frac{\epsilon}{2n_\epsilon} \text{ for each } 2 \leq i \leq n_\epsilon, \end{aligned}$$

where $\mathbf{y}' = H(\mathbf{x}')$.

Since $h_1 : T \rightarrow S$ is continuous, there exists $\delta'_{n_\epsilon} > 0$ such that if $d(\mathbf{x}, \mathbf{x}') < \delta'_{n_\epsilon}$ then

$$d_S(y_1, y'_1) = d_S(h_1(x_1), h_1(x'_1)) < \min \left\{ \frac{\epsilon}{2n_\epsilon}, \eta_{n_\epsilon} \right\}.$$

Thus put $\delta := \min \{\delta_{n_\epsilon}, \delta'_{n_\epsilon}\}$. If $d_T(\mathbf{x}, \mathbf{x}') < \delta$ then

$$d_S(y_i, y'_i) < \frac{\epsilon}{2n_\epsilon} \text{ for } 1 \leq i \leq n_\epsilon.$$

Therefore

$$\begin{aligned} d(\mathbf{y}, \mathbf{y}') &= \sum_{i=1}^{\infty} 2^{-i} d_S(y_i, y'_i) \\ &= \sum_{i=1}^{n_\epsilon} 2^{-i} d_S(y_i, y'_i) + \sum_{i=n_\epsilon+1}^{\infty} 2^{-i} d_S(y_i, y'_i) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, H is continuous. The same proof can be applied to the inverse map of H . Therefore we have that H is a homeomorphism. $\square$

Theorem 2.2.5 follows from Proposition 2.3.1 and 2.4.1.

2.5 Dimension of inverse limits of Markov-like functions on finite graphs

In Section 1.5, we showed that every generalized inverse limit on intervals with Markov-like bonding functions is one-dimensional. In similar way as Section 1.5, we can show that any generalized inverse limit on finite graphs T with surjective Markov-like bonding functions is one-dimensional. We now remark the notations to show Theorem 2.5.2.

Remark 2.5.1. Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of compacta. Let $f_i : X_{i+1} \rightarrow 2^{X_i}$ be an upper-semi continuous function for each $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, define $D_1(f_i)$ and $D_1(f_i^{-1})$ as

$$D_1(f_i) := \{x \in X_{i+1} \mid \dim f_i(x) \geq 1\},$$

$$D_1(f_i^{-1}) := \{y \in X_i \mid \dim f_i^{-1}(y) \geq 1\}.$$

Theorem 2.5.2. Let T be a finite graph with a triangulation $\mathcal{T}$. Let $f_i : T \rightarrow 2^T$ be a surjective Markov-like function with respect to $\mathcal{T}$ for each $i \in \mathbb{N}$. Then $\dim \lim_{\leftarrow} \{T, f_i\} = 1$.

Proof. Fix $i \in \mathbb{N}$. From the definition of Markov-like functions, if $x \in D_1(f_i)$, then $x \in \mathcal{V}(\mathcal{T})$. Hence $\dim D_1(f_i) \leq \dim \mathcal{V}(\mathcal{T}) = 0$. Moreover, since the graph of f_i cannot contain a graph of any constant function, $D_1(f_i^{-1}) = \emptyset$. Therefore, from Theorem 1.5.2, $\dim \lim_{\leftarrow} \{T, f_i\} \leq 1$.

We next show that $\dim \lim_{\leftarrow} \{T, f_i\} \geq 1$. Fix $i \in \mathbb{N}$. Put an edge $e_i \in \mathcal{E}(\mathcal{T})$ and put an arc $\mathbb{J}_i \subseteq e_i$ with $\mathbb{J}_i \cap \mathcal{V}(\mathcal{T}) = \emptyset$. Since f_i is a surjective Markov-like function, one of the followings hold:

- (1) There is an arc $\mathbb{J}_{i+1} \subseteq T$ such that $\mathbb{J}_{i+1} \subseteq e_{i+1} \setminus \partial(e_{i+1})$ and $\mathbb{J}_i = f_{i, e_{i+1}}^k(\mathbb{J}_{i+1})$ for some $e_{i+1} \in \mathcal{E}(\mathcal{T})$ and $k \in \{1, \dots, n_{f_i}(e_{i+1})\}$.
- (2) There exists singleton $\mathbb{J}_{i+1} = v_{i+1}$ for some $v_{i+1} \in \mathcal{V}(\mathcal{T})$ such that $\mathbb{J}_i \subseteq f(\mathbb{J}_{i+1})$.

We note that for any $x \in T$, $f_i^{-1}(x)$ is finite. Hence we can construct a generalized inverse sequence $\{\mathbb{J}_i, g_i\}_{i \in \mathbb{N}}$ such that $\mathbb{J}_1 \subseteq T$ is an arc with $\mathbb{J}_1 \cap \mathcal{V}(\mathcal{T}) = \emptyset$, and for each $i \in \mathbb{N}$,

if $\mathbb{J}_{i+1}$ is non-degenerate, there exists $e_{i+1} \in \mathcal{E}(\mathcal{T})$ and $k \in \{1, \dots, n_{f_i}(e_{i+1})\}$ such that

$$\mathbb{J}_{i+1} \subseteq e_{i+1}, \mathbb{J}_i = f_{i, e_{i+1}}^k(\mathbb{J}_{i+1}) \text{ and } G(g_i) = G\left(f_{i, e_{i+1}}^k|_{\mathbb{J}_{i+1}}\right),$$

if $\mathbb{J}_{i+1}$ is degenerate,

$$\mathbb{J}_i \subseteq f_i(\mathbb{J}_{i+1}) \text{ and } G(g_i) = G(f_i) \cap (\mathbb{J}_{i+1} \times \mathbb{J}_i).$$

Then $\lim_{\leftarrow} \{\mathbb{J}_i, g_i\} \subseteq \lim_{\leftarrow} \{T, f_i\}$ is an arc. Therefore $\dim \lim_{\leftarrow} \{T, f_i\} \geq 1$. $\square$

2.6 Examples of Markov-like functions on finite graphs

In this section, we show that there exist Markov-like functions f and g on a simple closed circles which do not have the same pattern with respect to a pair (h, h) of any simplicial homeomorphism h but have the same pattern with respect to some pair (h, h') of distinct simplicial homeomorphisms h and h' .

Example 2.6.1. Let T be the boundary of the square $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$, where we employ a metric d on $\mathbb{R}^2$ given by $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. We give a triangulation $\mathcal{T}$ of the finite graph T as follows:

$$\mathcal{V}(\mathcal{T}) = \{v_1 = (0, 0), v_2 = (0, 1), v_3 = (1, 1), v_4 = (1, 0)\};$$

$$\mathcal{E}(\mathcal{T}) = \{e_1 = \{0\} \times [0, 1], e_2 = [0, 1] \times \{1\}, e_3 = \{1\} \times [0, 1], e_4 = [0, 1] \times \{0\}\}.$$

Thus, each edge e_i is the segment joining v_i and v_{i+1} , where $v_5 = v_1$.

We consider functions $f, g : T \rightarrow 2^T$ defined by the following: For any $t \in [0, 1]$,

$$\begin{aligned} f(0, t) &= \{f_{e_1}^1(0, t), f_{e_1}^2(0, t)\} = \{(0, t), (t, 0)\}, \\ f(t, 1) &= \{f_{e_2}^1(t, 1), f_{e_2}^2(t, 1)\} = \{(t, 1), (1, t)\}, \\ f(1, t) &= \{f_{e_3}^1(1, t), f_{e_3}^2(1, t)\} = \{(t, 1), (1, t)\}, \\ f(t, 0) &= \{f_{e_4}^1(t, 0), f_{e_4}^2(t, 0)\} = \{(0, t), (t, 0)\}, \text{ and} \\ g(0, t) &= \{g_{e_1}^1(0, t), g_{e_1}^2(0, t)\} = \{(1-t, 1), (1, 1-t)\}, \\ g(t, 1) &= \{g_{e_2}^1(t, 1), g_{e_2}^2(t, 1)\} = \{(0, 1-t), (1-t, 0)\}, \\ g(1, t) &= \{g_{e_3}^1(1, t), g_{e_3}^2(1, t)\} = \{(0, 1-t), (1-t, 0)\}, \\ g(t, 0) &= \{g_{e_4}^1(t, 0), g_{e_4}^2(t, 0)\} = \{(1-t, 1), (1, 1-t)\}. \end{aligned}$$

Then f and g are Markov-like functions with respect to $\mathcal{T}$. See Figure 2.2.

Figure 2.2: Graphs of f and g from Example 2.6.1

First, we show that f and g do not have the same pattern with respect to (h, h) for any simplicial homeomorphism $h : T \rightarrow T$. We note that $v_i \in f(v_i)$ for all i but $v_i \notin g(v_i)$ if $i = 1, 3$. Assume that f and g have the same pattern with respect to

(h, h) for some simplicial homeomorphism $h : T \rightarrow T$. Then $v_3 = h(v_{i(3)})$ for some $i(3) \in \{1, 2, 3, 4\}$. However

$$v_3 = h(v_{i(3)}) \in \tilde{h}(f(v_{i(3)})) = g(v_3).$$

This is a contradiction; hence f and g do not have the same pattern with respect to (h, h) .

On the other hand, we define a homeomorphism $h : T \rightarrow T$ by

$$\begin{aligned} h(0, t) &= (1, 1 - t), \\ h(t, 1) &= (1 - t, 0), \\ h(1, t) &= (0, 1 - t), \\ h(t, 0) &= (1 - t, 1) \quad \text{for any } t \in [0, 1], \text{ and} \end{aligned}$$

let $h' : T \rightarrow T$ be the identity mapping. Then we can easily see that f and g have the same pattern with respect to (h, h') with bijective maps $\phi_{e_1}, \phi_{e_2}, \phi_{e_3}, \phi_{e_4} : \{1, 2\} \rightarrow \{1, 2\}$ defined by

$$\phi_{e_i}(1) = 2, \quad \phi_{e_i}(2) = 1 \quad \text{for } 1 \leq i \leq 4.$$

Moreover f and g have the same pattern with respect to (h', h) with bijective maps $\phi_{e_1}, \phi_{e_2}, \phi_{e_3}, \phi_{e_4} : \{1, 2\} \rightarrow \{1, 2\}$ defined by

$$\phi_{e_i}(x) = x \quad \text{for } 1 \leq i \leq 4.$$

Hence, by Theorem 2.2.5, $\varprojlim \{T, f\}$ and $\varprojlim \{T, g\}$ are homeomorphic with the sequence $\{h_i\}_{i \in \mathbb{N}}$ defined by $h_i = h$ if i is odd, and $h_i = h'$ if i is even.

In fact, $\varprojlim \{T, f\}$ is homeomorphic to the suspension of a Cantor set. We will now show this fact. Let $E_0 = e_1 \cup e_2$ and $E_1 = e_3 \cup e_4$. Define maps $f_1 : E_0 \rightarrow E_0$, $f_2 : E_0 \rightarrow E_1$, $f_3 : E_1 \rightarrow E_1$, $f_4 : E_1 \rightarrow E_0$, $F_0 : T \rightarrow T$, and $F_1 : T \rightarrow T$ as follows:

$$f_1(x, y) = (x, y), \quad f_2(x, y) = (y, x), \quad f_3(x, y) = (x, y), \quad f_4(x, y) = (y, x), \quad \text{and}$$

$$G(F_0) = G(f_1) \cup G(f_3), \quad G(F_1) = G(f_2) \cup G(f_4).$$

Let C be a Cantor set. Here the *suspension* $S(C)$ of C is the quotient space obtained from the product space $[0, 2] \times C$ by shrinking the top set $\{2\} \times C$ and the bottom set $\{0\} \times C$ to the points N and S , respectively. Let $p : [0, 2] \times C \rightarrow S(C)$ be the natural projection and for each $(t, z) \in [0, 2] \times C$ the equivalence class of (t, z) is written by $[t, z] = p(t, z)$.

Let

$$\Lambda = \left\{ \mathbf{s} = (s_j) \in \prod_{j \in \mathbb{N}} \{1, 2, 3, 4\} \mid \text{Dom } f_{s_j} = \text{Im } f_{s_{j+1}} \quad \text{for all } j \in \mathbb{N} \right\}.$$

Λ is clearly a Cantor set and therefore there exists a homeomorphism $\varphi : C \rightarrow \Lambda$. Then we define $h : S(C) \rightarrow \varprojlim \{T, f\}$ as follows: for any $[t, z] \in S(C)$ we write $\varphi(z) = \mathbf{s} = (s_j) \in \Lambda$ and define

$$h([t, z]) = (x_1, x_2, \dots),$$

where $x_1 \in \text{Im}(f_{s_1})$ is the unique point such that $d(v_3, x_1) = t$ and $(x_{j+1}, x_j) \in G(f_{s_j})$ for every $j \in \mathbb{N}$. If $t = 0$, then $x_1 = v_3$ and $f_i(v_3) = v_3$ for all $i = 1, \dots, 4$. Hence $h(N) = h([0, z]) = (v_3, v_3, \dots)$. Similarly $h(S) = (v_1, v_1, \dots)$. Thus, h is well-defined.

We show that h is bijective. For each $\mathbf{s} = (s_j) \in \Lambda$, let

$$L_{\mathbf{s}} = \left\{ \mathbf{x} \in \prod_{j \in \mathbb{N}} T \mid (x_{j+1}, x_j) \in G(f_{s_j}) \text{ for each } j \in \mathbb{N} \right\}.$$

Then $\varprojlim \{T, f\} = \bigcup_{\mathbf{s} \in \Lambda} L_{\mathbf{s}}$. Hence, for each $\mathbf{x} \in \varprojlim \{T, f\}$, there exists $\mathbf{s} \in \Lambda$ such that $\mathbf{x} \in L_{\mathbf{s}}$. Then $\mathbf{x} = h([t, z])$, where $t = d(x_1, v_3)$ and $z = \varphi^{-1}(\mathbf{s})$. Hence h is surjective.

Let us take arbitrary distinguished points $[t_1, z_1], [t_2, z_2] \in S(C)$ and put $h([t_1, z_1]) = \mathbf{x}$ and $h([t_2, z_2]) = \mathbf{y}$. Suppose that $t_1, t_2 \in (0, 2)$. If $t_1 \neq t_2$, then $x_1 \neq y_1$. Hence $\mathbf{x} \neq \mathbf{y}$. If $t_1 = t_2$, then $z_1 \neq z_2$ and $\varphi(z_1) = (s_j^1) \neq (s_j^2) = \varphi(z_2)$. Hence $s_j^1 \neq s_j^2$ for some $j \geq 1$. Then $(x_{j+1}, x_j) \neq (y_{j+1}, y_j)$ because $G(f_{s_j^1}) \cap G(f_{s_j^2}) = \{(v_1, v_1), (v_3, v_3)\}$. Thus $\mathbf{x} \neq \mathbf{y}$. In the other cases we can easily see that $\mathbf{x} \neq \mathbf{y}$. Thus, h is injective.

Next we show that h is continuous. Take a point $[t_0, z_0] \in S(C)$ and write $h([t_0, z_0]) = \mathbf{x}$. For any $j \in \mathbb{N}$ and any open subset $U \subseteq T$ with $x_j \in U$ we can find an open neighborhood O_j of $[t_0, z_0]$ in $S(C)$ such that $\pi_j(h(O_j)) \subset U$. We may assume that $U \subseteq \text{Int } E_i$ for some $i \in \{0, 1\}$ if $x_j \notin \{v_1, v_3\}$. First we consider the case of $j \geq 2$. Define the subset $\tilde{\Lambda}$ of $\prod_{k=1}^{j-1} \{1, 2, 3, 4\}$ consisting of sequences $(u_1, \dots, u_{j-1})$ which satisfy the conditions:

$$x_1 \in f_{u_1} \circ \dots \circ f_{u_{j-1}}(U), \text{ and}$$

$$\text{Dom } f_{u_k} = \text{Im } f_{u_{k+1}} \text{ for all } k = 1, \dots, j-2,$$

and

$$V = \left(\tilde{\Lambda} \times \prod_{k=j}^{\infty} \{1, 2, 3, 4\} \right) \cap \Lambda.$$

Then V is open in Λ . Note that $V = \Lambda$ if $t_0 = 0$ or 2 . Furthermore we define

$$W = \bigcap_{\mathbf{s} \in V} f_{s_1} \circ \dots \circ f_{s_{j-1}}(U) = \bigcap_{\mathbf{s} \in \tilde{\Lambda}} f_{s_1} \circ \dots \circ f_{s_{j-1}}(U) \text{ if } t_0 \in (0, 2),$$

$$W = \bigcap_{\mathbf{u} \in \prod_{k=1}^{j-1} \{0, 1\}} F_{u_1} \circ \dots \circ F_{u_{j-1}}(U) \text{ if } t_0 \in \{0, 2\}.$$

Then W is an open subset in T containing x_1 with the following property: for $(y_1, \dots, y_j) \in \prod_{k=1}^j T$ and $(s_1, \dots, s_{j-1}) \in \tilde{\Lambda}$, if $y_1 \in W$ and $y_k = f_{s_k}(y_{k+1})$ for every $k = 1, \dots, j-1$, then $y_j \in U$. Taking $\delta > 0$ such that $B_d(x_1; \delta) \subseteq W$, we define the open subset $O_j \subset S(C)$ by

$$O_j = \{[t, z] \in S(C) \mid |t_0 - t| < \delta, \quad z \in \varphi^{-1}(V)\}.$$

Then we can easily check that $[t_0, z_0] \in O_j$ and $\pi_j(h(O_j)) \subseteq U$. In the case of $j = 1$, we can also construct O_1 putting $W = U$, $V = \{\mathbf{s} \in \Lambda \mid x_1 \in \text{Im } f_{s_1}\}$ if $t_0 \in (0, 2)$ and $V = \Lambda$ if $t_0 = 0$ or 2 . Therefore h is a homeomorphism.

Since the graph of g is equal to the graph of f shifted by two edges, a reader may suppose that if the upper semi-continuous function from T to 2^T having the graph which is equal to the graph of f shifted by one or three edges then both generalized inverse limit and $\varprojlim \{T, f\}$ are homeomorphic. However, it is not true. In Example 2.6.2, we show that generalized inverse limits of the function with the graph which is equal to the graph of f shifted by one or three edges are not homeomorphic to $\varprojlim \{T, f\}$.

Example 2.6.2. We consider functions $\varphi, \psi : T \rightarrow 2^T$ defined by the following: For any $t \in [0, 1]$,

$$\begin{aligned} \varphi(0, t) &= \{\varphi_{e_1}^1(0, t), \varphi_{e_1}^2(0, t)\} = \{(t, 1), (1, t)\}, \\ \varphi(t, 1) &= \{\varphi_{e_2}^1(t, 1), \varphi_{e_2}^2(t, 1)\} = \{(1-t, 1), (1, 1-t)\}, \\ \varphi(1, t) &= \{\varphi_{e_3}^1(1, t), \varphi_{e_3}^2(1, t)\} = \{(0, t), (t, 0)\}, \\ \varphi(t, 0) &= \{\varphi_{e_4}^1(t, 0), \varphi_{e_4}^2(t, 0)\} = \{(0, 1-t), (1-t, 0)\}, \text{ and} \\ \psi(0, t) &= \{\psi_{e_1}^1(0, t), \psi_{e_1}^2(0, t)\} = \{(0, 1-t), (1-t, 0)\}, \\ \psi(t, 1) &= \{\psi_{e_2}^1(t, 1), \psi_{e_2}^2(t, 1)\} = \{(0, t), (t, 0)\}, \\ \psi(1, t) &= \{\psi_{e_3}^1(1, t), \psi_{e_3}^2(1, t)\} = \{(1-t, 1), (1, 1-t)\}, \\ \psi(t, 0) &= \{\psi_{e_4}^1(t, 0), \psi_{e_4}^2(t, 0)\} = \{(t, 1), (1, t)\}. \end{aligned}$$

φ and ψ are Markov-like functions with respect to $\mathcal{T}$, see Figure 2.3.

Let $f : T \rightarrow 2^T$ be the Markov-like function as in Example 2.6.1. Then, $\varprojlim \{T, \varphi\}$ and $\varprojlim \{T, \psi\}$ are not homeomorphic to $\varprojlim \{T, f\}$.

Let $h : T \rightarrow T$ be the homeomorphism as in Example 2.6.1 and let $h' : T \rightarrow T$ be the identity mapping, respectively. Then we can easily see that φ and ψ have the same pattern with respect to (h, h') and (h', h) respectively. Thus both $\varprojlim \{T, \varphi\}$ and $\varprojlim \{T, \psi\}$ are homeomorphic. Hence, we show that $\varprojlim \{T, \varphi\}$ is not homeomorphic to $\varprojlim \{T, f\}$.

We begin the process of proving this by labelling the functions on each edges by the following: if $G(\varphi) \cap (e_i \times e_j) \neq \emptyset$, then let $\varphi_{j,i} : e_i \rightarrow e_j$ is the map having the graph $G(\varphi) \cap G(\varphi) \cap (e_i \times e_j)$. See Figure 2.4.

Figure 2.3: Graphs of φ and ψ from Example 2.6.2

Figure 2.4: The new label of the graph of φ

Moreover we define

$$\begin{aligned} \mathbf{x} &:= (v_2, v_3, \dots), \\ V &:= \{(v_2, v_1, v_4, v_3, v_2), (v_2, v_1, v_4, v_1, v_4), (v_4, v_1, v_4, v_1, v_4), (v_4, v_3, v_2, v_3, v_2)\}, \\ W &:= \left\{ \mathbf{w} \in \prod_{j=1}^{\infty} T \mid \pi_{\langle 4k+1, 4(k+1)+1 \rangle}(\mathbf{w}) \in V \text{ for each } k \in \mathbb{N} \right\}. \end{aligned}$$

We note that W is a Cantor set in $\varprojlim \{T, \varphi\}$. Then we have Proposition 2.6.3.

Proposition 2.6.3. *For any $\mathbf{w} \in W$, there exists an arc $L_{\mathbf{w}} \subseteq \varprojlim \{T, \varphi\}$ whose endpoints are $\mathbf{w}$ and $\mathbf{x}$. Moreover, if $\mathbf{w} \neq \mathbf{w}'$ then $L_{\mathbf{w}} \cap L_{\mathbf{w}'} = \{\mathbf{x}\}$. Hence, $\mathbf{x}$ is embedded in $\varprojlim \{T, \varphi\}$ as the centred point of a Cantor fan.*

Proof. For any $\mathbf{w} \in W$, we construct an arc $L_{\mathbf{w}}$ as an inverse limit of homeomorphisms $\{\eta_i\}_{i \in \mathbb{N}}$ between edges inductively.

We consider the case where $\pi_{<1,5>}(\mathbf{w}) = (v_2, v_1, v_4, v_3, v_2)$. Then define

$$\eta_1 := \varphi_{2,1}, \eta_2 := \varphi_{1,3}, \eta_3 := \varphi_{3,2}, \eta_4 := \varphi_{2,2}.$$

We note that both $\pi_{<1,5>}(\mathbf{x})$ and $\pi_{<1,5>}(\mathbf{w})$ are the endpoints of the arc

$$\left\{ (y_1, \dots, y_5) \in \prod_{j=1}^5 T \mid y_j \in \eta_j(y_{j+1}) \ (1 \leq j \leq 4) \right\}.$$

Since $\pi_5(\mathbf{w}) = v_2$, $\pi_{<5,9>}(\mathbf{w}) \in \{(v_2, v_1, v_4, v_1, v_4), (v_2, v_1, v_4, v_3, v_2)\}$.

If $\pi_{<5,9>}(\mathbf{w}) = (v_2, v_1, v_4, v_1, v_4)$, define

$$\eta_5 := \varphi_{2,1}, \eta_6 := \varphi_{1,3}, \eta_7 := \varphi_{3,1}, \eta_8 := \varphi_{1,3}.$$

If $\pi_{<5,9>}(\mathbf{w}) = (v_2, v_1, v_4, v_3, v_2)$, define

$$\eta_5 := \varphi_{2,1}, \eta_6 := \varphi_{1,3}, \eta_7 := \varphi_{3,2}, \eta_8 := \varphi_{2,2}.$$

Then we note that both $\pi_{<5,9>}(\mathbf{x})$ and $\pi_{<5,9>}(\mathbf{w})$ are the endpoints of the arc

$$\left\{ (y_5, \dots, y_9) \in T \mid y_j \in \eta_j(y_{j+1}) \ (5 \leq j \leq 9) \right\}.$$

Since $\pi_5(\mathbf{w}) = v_2$, $\pi_{<5,9>}(\mathbf{w}) \in \{(v_2, v_1, v_4, v_1, v_4), (v_2, v_1, v_4, v_3, v_2)\}$.

To continue to this process, we can construct a sequence $\{\eta_i\}_{i \in \mathbb{N}}$ of maps. Since η_i is a homeomorphism between arcs and $\text{Dom}(\eta_i) = \text{Im}(\eta_{i+1})$ have the endpoints x_{i+1}, w_{i+1} for each $i \in \mathbb{N}$, the inverse limit of $\{\eta_i\}_{i \in \mathbb{N}}$ is an arc with the endpoints $\mathbf{x}$ and $\mathbf{w}$, the required $L_{\mathbf{w}}$. Similarly, we can construct $L_{\mathbf{w}}$ in the cases where $\pi_{<1,5>}(\mathbf{w}) \in \{(v_2, v_1, v_4, v_1, v_4), (v_4, v_1, v_4, v_1, v_4), (v_4, v_3, v_2, v_3, v_2)\}$.

We show that if $\mathbf{w} \neq \mathbf{w}'$ then $L_{\mathbf{w}} \cap L_{\mathbf{w}'} = \{\mathbf{x}\}$. We assume $\mathbf{w} \neq \mathbf{w}'$ and $\{\eta_i\}_{i \in \mathbb{N}}$ and $\{\eta'_i\}_{i \in \mathbb{N}}$ are the sequences of bonding functions correspond with $L_{\mathbf{w}}$ and $L_{\mathbf{w}'}$ respectively. Then, there exists $k \in \mathbb{N}$ such that

$$\pi_{<4k+1,4(k+1)+1>}(\mathbf{w}) \neq \pi_{<4k+1,4(k+1)+1>}(\mathbf{w}').$$

From the definitions of V and W , one of the following is satisfied:

$$\eta_{4k+1} \neq \eta'_{4k+1} \text{ or } \eta_{4k+3} \neq \eta'_{4k+3}.$$

In the case where $\eta_{4k+1} \neq \eta'_{4k+1}$, $G(\eta_{4k+1}) \cap G(\eta'_{4k+1}) = \{(v_3, v_2)\}$. Since η_i is bijective, $L_{\mathbf{w}} \cap L_{\mathbf{w}'} = \{\mathbf{x}\}$. On the other case, we can see that by the same way. Since W is a Cantor set, $\mathbf{x}$ is embedded into $\varprojlim \{T, \varphi\}$ as the centred point of a Cantor fan $\bigcup_{\mathbf{w} \in W} L_{\mathbf{w}}$. $\square$

We return to prove that $\varprojlim \{T, \varphi\}$ is not homeomorphic to $\varprojlim \{T, f\}$
We define a homeomorphism $\theta : T \rightarrow T$ by

$$G(\theta) = G(\varphi_{2,1}) \cup G(\varphi_{3,2}) \cup G(\varphi_{4,3}) \cup G(\varphi_{1,4}).$$

Moreover define $\Theta : \varprojlim \{T, \varphi\} \rightarrow \varprojlim \{T, \varphi\}$ by

$$\Theta(\mathbf{y}) := (\theta(y_1), y_1, y_2, \dots) \text{ for } \mathbf{y} \in \varprojlim \{T, \varphi\}.$$

Then Θ is continuous injective map. We now consider the point $\mathbf{x}$ as in Proposition 2.6.3. Then

$$\Theta(\mathbf{x}) = (v_4, v_3, v_2, v_3, v_2, \dots), \Theta^2(\mathbf{x}) = (v_1, v_4, v_3, v_2, v_3, \dots).$$

Thus, $\mathbf{x} \neq \Theta(\mathbf{x})$, $\mathbf{x} \neq \Theta^2(\mathbf{x})$, and $\Theta(\mathbf{x}) \neq \Theta^2(\mathbf{x})$. Since Θ is an embedding and $\mathbf{x}$ is the centred point of a Cantor fan, $\varprojlim \{T, \varphi\}$ have atleast three centred points $\mathbf{x}$, $\Theta(\mathbf{x})$, and $\Theta^2(\mathbf{x})$ of Cantor fans. However, since $\varprojlim \{T, f\}$ is the suspension of a Cantor set, $\varprojlim \{T, f\}$ have only two centred points of Cantor fans. Therefore $\varprojlim \{T, \varphi\}$ is not homeomorphic to $\varprojlim \{T, f\}$.

Part II

A simple construction of non-planar one-dimensional continua as generalized inverse limits

Chapter 3

A simple construction of non-planar one-dimensional continua

3.1 Introduction

The topological structure of generalized inverse limits is much more complicated than one of (usual) inverse limits. For example, non-degenerate chainable continua (= inverse limits of intervals) are always one-dimensional and embeddable in the plane, but for any $n = 1, 2, \dots, \infty$, there exists a generalized inverse limit of intervals whose dimension is equal to n . Thus, we have non-planar examples of generalized inverse limits of intervals.

The Markov-like function f in Example 1.5.4 is studied in [BCMM1, BCMM2, C, KN]. The function f given by shifting the critical point of the tent map to the minimum value of their domain. Although the generalized inverse limit of f is one-dimensional, it can not be embeddable into the plane.

In this chapter, we will give a simple construction of generalized inverse sequence with single bonding function on an interval whose limit is one-dimensional but not embeddable into the plane. This construction is a generalization of Example 1.5.4.

3.2 A simple construction of non-planar continua

Our main result is Theorem 3.2.4. To prove that we remark Proposition 3.2.1 and 3.2.3.

Proposition 3.2.1. (*[N, Theorem 5.4]*) *If X_1 is a continuum such that every non-degenerate subcontinuum K of X_1 contains a countable set that separates K , and for each i , X_i is compact and $f_i : X_{i+1} \rightarrow 2^{X_i}$ is upper-semi continuous, and for each $y \in X_i$, $\dim(\{x \in X_{i+1} \mid y \in f_i(x)\}) = 0$, then $\dim\left(\varprojlim \{X_i, f_i\}\right) \leq 1$.*

Definition 3.2.2. A continuum M will be said to be *triodic* if there exist subcontinua $M_1, M_2, M_3 \subseteq M$ such that $\bigcap_{i=1}^3 M_i$ is a non-empty continuum, and for any $j, k \in \{1, 2, 3\}$ with $j \neq k$, $\bigcap_{i=1}^3 M_i = M_j \cap M_k$.

Proposition 3.2.3. ([Mo]) *If, in the plane, $\mathcal{G}$ is an uncountable family of bounded triodic continua, there exists an uncountable subfamily $\mathcal{H}$ of $\mathcal{G}$ such that every two continua of the set $\mathcal{H}$ have a point in common.*

Theorem 3.2.4. *Fix $n \in \mathbb{N}_{\geq 2}$. Let $g : \mathbb{I} \rightarrow \mathbb{I}$ be a surjective piecewise monotonic continuous function. Suppose that g has a periodic point of prime period n and L is a closed interval such that $p, g(p) \in L$. If $f : \mathbb{I} \rightarrow 2^{\mathbb{I}}$ satisfies that*

$$G(g) \cup (\{p\} \times L) \subseteq G(f),$$

then $\varprojlim \{\mathbb{I}, f\}$ is non-planar. Moreover, if $G(g) \cup (\{p\} \times L) = G(f)$ then $\varprojlim \{\mathbb{I}, f\}$ is a one-dimensional continuum.

3.3 Proof of Theorem 3.2.4

We can assume that $G(f) = G(g) \cup (\{p\} \times L)$ and $p < g(p)$. We prove that $\varprojlim \{\mathbb{I}, f\}$ is non-planar. Let Q, Q_1, Q_2, Q_3 be the points of $\prod_{j=1}^{3n} \mathbb{I}$ defined by the following:

$$\begin{aligned} Q &:= (g^{n-1}(p), \dots, g^2(p), g(p), p, g^{n-1}(p), \dots, g^2(p), g(p), p, g^{n-1}(p), \dots, g^2(p), g(p), p), \\ Q_1 &:= (g^{n-2}(p), \dots, g(p), p, p, g^{n-1}(p), \dots, g^2(p), g(p), p, g^{n-1}(p), \dots, g^2(p), g(p), p), \\ Q_2 &:= (g^{n-2}(p), \dots, g(p), p, g^{n-1}(p), g^{n-2}(p), \dots, g(p), p, p, g^{n-1}(p), \dots, g^2(p), g(p), p), \\ Q_3 &:= (g^{n-2}(p), \dots, g(p), p, g^{n-1}(p), g^{n-2}(p), \dots, g(p), p, g^{n-1}(p), g^{n-2}(p), \dots, g(p), p, p). \end{aligned}$$

Moreover let A_1, A_2, A_3 be the subsets of $\prod_{j=1}^{3n} \mathbb{I}$ defined by

$$\begin{aligned} A_1 &:= \left\{ \mathbf{x} \in \prod_{j=1}^{n-1} \mathbb{I} \mid x_j = g(x_{j+1}) (j \leq n-2), x_{n-1} \in [p, g(p)] \right\} \\ &\quad \times \{ (p, g^{n-1}(p), \dots, g(p), p, g^{n-1}(p), \dots, g(p), p) \}, \\ A_2 &:= \left\{ \mathbf{x} \in \prod_{j=1}^{2n-1} \mathbb{I} \mid x_j = g(x_{j+1}) (j \leq 2n-2), x_{2n-1} \in [p, g(p)] \right\} \\ &\quad \times \{ (p, g^{n-1}(p), \dots, g(p), p) \}, \\ A_3 &:= \left\{ \mathbf{x} \in \prod_{j=1}^{3n} \mathbb{I} \mid x_j = g(x_{j+1}) (j \leq 3n-2), x_{3n-1} \in [p, g(p)], x_{3n} = p \right\}. \end{aligned}$$

Then each A_i is an arc having end points Q and Q_i . If $i \neq j$, $A_i \cap A_j = \{Q\}$. Thus $A = \bigcup_{i=1}^3 A_i$ is a triodic continuum in $\prod_{j=1}^{3n} \mathbb{I}$.

For each $k \in \mathbb{N}$, we define $J_{k,1}$ and $J_{k,2}$ by

$$J_{k,1} := L \times \{p\} \times \{g^{n-1}(p)\} \times \cdots \times \{g(p)\} \subseteq \prod_{i=3n+(k-1)(n+1)}^{3n+k(n+1)} \mathbb{I},$$

$$J_{k,2} := L \times \{p\} \times \cdots \times \{p\} \times \{p\} \subseteq \prod_{i=3n+(k-1)(n+1)}^{3n+k(n+1)} \mathbb{I}.$$

For each $s = (s_1, s_2, \dots) \in \prod_{j=1}^{\infty} \{1, 2\}$, let us define

$$B_s := \left\{ \mathbf{x} \in \prod_{j=1}^{\infty} \mathbb{I} \mid \pi_{\langle 1, 3n \rangle}(\mathbf{x}) \in A, \pi_{\langle 3n+(k-1)(n+1), 3n+k(n+1) \rangle}(\mathbf{x}) \in J_{k,s_k} \text{ for each } k \in \mathbb{N} \right\}.$$

Then B_s is a triodic continuum in $\varprojlim \{\mathbb{I}, f\}$. Moreover we can easily see that $B_s \cap B_{s'} = \emptyset$ if $s \neq s'$. Thus $\varprojlim \{\mathbb{I}, f\}$ has uncountably many mutually disjoint triodic subcontinua. Therefore $\varprojlim \{\mathbb{I}, f\}$ is non-planar by Proposition 3.2.3.

Since f has continuum value on each point on $\mathbb{I}$, $\varprojlim \{\mathbb{I}, f\}$ is a continuum from Theorem 0.0.4. Moreover any non-degenerate subcontinuum K of $\mathbb{I}$ is a closed interval and any point p of $\text{Int}(K)$ separates K . Thus $\dim \left(\varprojlim \{\mathbb{I}, f\} \right) \leq 1$ from Proposition 3.2.1. Since $\varprojlim \{\mathbb{I}, g\} \subseteq \varprojlim \{\mathbb{I}, f\}$, $\dim \left(\varprojlim \{\mathbb{I}, f\} \right) = 1$.

3.4 Proof of Example 1.5.4

We prove that the generalized inverse limit of the function f in Example 1.5.4 is a non-planar one-dimensional continuum with applying Theorem 3.2.4.

We remark the definition of f .

Remark 3.4.1. (Example 1.5.4) $f : [0, 1] \rightarrow 2^{[0,1]}$ is given by $f(0) = [0, 1]$ and $f(x) = \{1 - x\}$ for $0 < x \leq 1$.

Proof of Example 1.5.4 Let $g : [0, 1] \rightarrow [0, 1]$ be defined by $g(x) = 1 - x$. Then g is strictly monotonic and $p = 0$ is the periodic point of this function. Then $f(p) = [p, g(p)] = [0, 1]$. Therefore, by Theorem 3.2.4, $\varprojlim \{[0, 1], f\}$ is a non-planar one-dimensional continuum.

3.5 On the condition in Theorem 3.2.4

We note that the condition in Theorem 3.2.4, $[p, g(p)] \subseteq f(p)$, is essential.

Example 3.5.1. ([I, Example 2.19]) Let $b \in (0, 1)$ and let $f : [0, 1] \rightarrow 2^{[0,1]}$ be defined by $f(0) = [b, 1]$ and $f(x) = \{1 - x\}$ for $0 < x \leq 1$. See Figure 3.1. Then $\varprojlim \{[0, 1], f\}$ is embeddable in the plane.

Figure 3.1: Graphs of f from Example 3.5.1

Proof. Let

$$A_0 = \left\{ \mathbf{x} \in \prod_{j=1}^{\infty} \mathbb{I} \mid x_j = 1 - x_{j+1} \text{ for any } j \in \mathbb{N} \right\}$$

and for each $n \in \mathbb{N}$, let

$$A_{2n-1} := \left\{ \mathbf{x} \in \prod_{j=1}^{\infty} \mathbb{I} \mid x_{2n-1} \in [b, 1], x_{2n} = 0, x_j = 1 - x_{j+1} \text{ for } j \neq 2n-1 \right\},$$

$$A_{2n} := \left\{ \mathbf{x} \in \prod_{j=1}^{\infty} \mathbb{I} \mid x_{2n} \in [b, 1], x_{2n+1} = 0, x_j = 1 - x_{j+1} \text{ for } j \neq 2n \right\}.$$

Then we can easily see that

$$\lim_{\leftarrow} \{[0, 1], f\} = \bigcup_{n=0}^{\infty} A_n.$$

Let O be the origin $(0, 0)$ and let O_1 be the point $(0, 1)$ in the plane $\mathbb{R}^2$. For each $n \in \mathbb{N}$ and $s \in [0, 1]$, let K_n^s and L_n^s be the points $(1/n, s)$ and $(-1/n, s)$ in $\mathbb{R}^2$ respectively. For each $b \in (0, 1)$, define

$$X_b := \left(\bigcup_{n \in \mathbb{N}} K_n^0 K_n^{1-b} \right) \cup OO_1 \cup \left(\bigcup_{n \in \mathbb{N}} L_n^1 L_n^b \right),$$

here for points $P, Q \in \mathbb{R}^2$ the segment joining P and Q is denoted by PQ . Then we define the quotient space Y_b of X_b identifying the sets $\{(1, 0), (1/2, 0), \dots, (0, 0)\}$

and $\{(-1, 1), (-1/2, 1), \dots, (0, 1)\}$ to the points S and N respectively. Clearly Y_b is a planar continuum. Put

$$Y_b^+ := p \left(\left(\bigcup_{n \in \mathbb{N}} K_n^0 K_n^{1-b} \right) \cup OO_1 \right) \quad \text{and} \quad Y_b^- := p \left(OO_1 \cup \left(\bigcup_{n \in \mathbb{N}} L_n^1 L_n^b \right) \right),$$

where $p : X_b \rightarrow Y_b$ is the natural projection. For each $(z, t) \in X_b$ the point $p(z, t)$ is denoted by $[z, t]$.

We define $h_1 : Y_b^+ \rightarrow A_0 \cup \bigcup_{n \in \mathbb{N}} A_{2n}$ as follows: for $(z, t) \in \left(\bigcup_{n \in \mathbb{N}} K_n^0 K_n^{1-b} \right) \cup OO_1$, put $h_1([z, t]) = \mathbf{x} \in \prod_{j=1}^{\infty} \mathbb{I}$ such that

(i) If $(z, t) \in K_n^0 K_n^{1-b}$, then

$$x_1 = t, \quad x_{2n+1} = 0 \quad \text{and} \quad x_j = 1 - x_{j+1} \quad \text{for } j \neq 2n.$$

(ii) If $(z, t) \in OO_1$, then

$$x_1 = t, \quad \text{and} \quad x_j = 1 - x_{j+1} \quad \text{for each } j \geq 1.$$

$$\text{i.e., } h_1([1/n, t]) = \mathbf{x} = (t, 1 - t, t, \dots, 1 - t, \overset{2n+1}{0}, 1, 0, 1, \dots) \in A_{2n}.$$

$$h_1([0, t]) = \mathbf{x} = (t, 1 - t, t, \dots, t, 1 - t, \dots) \in A_0.$$

Since $h_1([z, 0]) = (0, 1, 0, \dots)$ for any $z \in Q$, h_1 is well-defined. Moreover, since $\lim_{\substack{n \rightarrow \infty \\ s \rightarrow t}} h_1([1/n, s]) = h_1([0, t])$ for any $t \in [0, 1 - b]$, we can see that h_1 is continuous.

Thus, h_1 is a homeomorphism.

Similarly we have a homeomorphism $h_2 : Y_b^- \rightarrow A_0 \cup \bigcup_{n \in \mathbb{N}} A_{2n-1}$ such that

$$h_2([0, t]) = (t, 1 - t, t, \dots, t, 1 - t, \dots) \in A_0 \quad \text{for all } t \in [0, 1].$$

Hence we can have a homeomorphism $h : Y_b = Y_b^+ \cup Y_b^- \rightarrow \bigcup_{n=0}^{\infty} A_n$ by $h|_{Y_b^+} = h_1$ and $h|_{Y_b^-} = h_2$. Therefore, $\lim_{\leftarrow} \{[0, 1], f\}$ is embeddable in the plane. □

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- [1] H. Imamura, Markov-like set-valued functions on interval and their inverse limits, to appear in Glas. Mat. Ser. III.
- [2] H. Imamura, Markov-like set-valued functions on finite graphs and their inverse limits, to appear in Topology Appl.

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学会発表 (国内)	Markov-like generalized inverse systems and their limits	RIMS 共同研究（公開型）集合論的・幾何学的トポロジーの動向と諸分野との連携	2017.6.14	申請者
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