A Study on Renormalization Theory of Weak Solutions for Nonlinear Partial Differential Equations

非線形偏微分方程式における弱解の再正規化理論についての研究

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Contents

Chapter 1

Introduction

The objective of this dissertation is to extend the notion of weak solutions for nonlinear partial differential equations by renormalization theory, and moreover to characterize the solutions for the Cauchy problem of nonlinear degenerate partial differential equations given general L^1 data.

A partial differential equation is an equation involving an unknown function of two or more variables and some of its partial derivatives, and describes various phenomena in physics, biology, chemistry, engineering, economics, and so on. For instance, a second order partial differential equation has the form

$$f(D^2u, Du, u, x) \;=\; 0, \tag{1.1}$$

where $x \in \mathbf{R}^N$, $u : \mathbf{R}^N \to \mathbf{R}$ is an unknown function, $f : \mathbf{R}^{N^2} \times \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ is a given function, and $D^k u$ denotes all k-th order derivatives for $k = 1, 2$, that is,

$$D^k u \;:=\; \frac{\partial^k u}{\partial x_1^{k_1} \cdots \partial x_N^{k_N}}$$

with $\sum_{i=1}^N k_i = k$. Furthermore, we classify the partial differential equation (1.1) into the following four categories. Let a, a_0, a_1 and a_2 be given functions.

(C1) The partial differential equation (1.1) is called *linear* provided it has the form

$$a_2(x)\, D^2u + a_1(x)\, Du + a_0(x)\, u + a(x) \;=\; 0.$$

(C2) The partial differential equation (1.1) is called *semilinear* provided it has the form

$$a_2(x)\,D^2u + a(Du,u,x) \;=\; 0.$$

(C3) The partial differential equation (1.1) is called *quasilinear* provided it has the form

$$a_2(Du,u,x)\,D^2u + a(Du,u,x) \;=\; 0.$$

(C4) The partial differential equation (1.1) is called *fully nonlinear* provided it depends on nonlinearly the second order derivatives.

We usually say the partial differential equation (1.1) is *nonlinear* if (1.1) is not linear, namely, we sometimes regard semilinear and quasilinear as nonlinear.

In this dissertation, we mainly deal with the following nonlinear, quasilinear formally, degenerate parabolic-hyperbolic equation

$$\frac{\partial u}{\partial t} + \operatorname{div}\mathbf{F}(u) \;=\; \operatorname{div}\left(A(u)\nabla u\right) + f \qquad \text{in } \; Q := (0,T)\times\mathbf{R}^N, \tag{1.2}$$

where $u = u(t,x)$ is the unknown, $t \in (0,T)$, $T > 0$, $x \in \mathbf{R}^N$ and $N \geq 1$. The flux $\mathbf{F} : \mathbf{R} \to \mathbf{R}^N$, the diffusion matrix $A = (a_{ij})$ and external forces f are given. We employ

$$\nabla u \;:=\; \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N}\right) \qquad \text{and} \qquad \operatorname{div}\mathbf{F}(u) \;:=\; \sum_{i=1}^{N} \frac{\partial F_i(u)}{\partial x_i}$$

for $\mathbf{F}(u) := (F_1(u), \cdots, F_N(u))$ as usual. Afterward, for simplicity, we write u_{x_i} for $\frac{\partial u}{\partial x_i}$. The diffusion term $\operatorname{div}(A(u)\nabla u)$ is sometimes written as

$$\sum_{i,j=1}^{N} A_{ij}(u)_{x_i x_j}$$

with $A_{ij}(r) := \int_0^r a_{ij}(\xi)\,d\xi$, $A_{ij}(0) = 0$ for $r \in \mathbf{R}$. This type of equation typically appears in biology, for example, it describes the evolution of a biological species in porous medium.

If $A \equiv O$, that is to say the diffusion term degenerates, this equation becomes the so-called conservation law

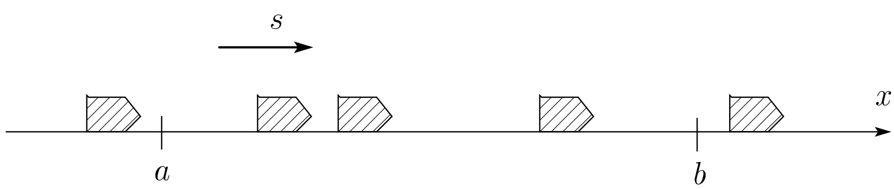
$$u_t + \operatorname{div} \mathbf{F}(u) = f \qquad \text{in} \quad Q, \tag{1.3}$$

which can be applied to traffic flows or gas dynamics. Moreover, we handle this partial differential equation with an initial condition. This means that we consider the following initial Cauchy problem with appropriate data f and u_0,

$$\begin{cases} u_t + \operatorname{div} \mathbf{F}(u) = f & \text{in} \quad Q, \\ u(0, \cdot) = u_0 & \text{in} \quad \mathbf{R}^N. \end{cases} \tag{1.4}$$

As a typical example of appearance of a conservation law in the real world, we now consider the traffic flow on an expressway.

Example 1: Let $u(t, x)$ be the density on an expressway at time t and point x. We assume, for simplicity, that u is continuous in t and x, and the speed s of the cars depends only upon their density, which means that $s = s(u)$ and $s' < 0$.



The traffic flow on the expressway

For any two points a, b on the expressway, the number of cars between a and b

depends upon the inflow at $x = a$ and outflow at $x = b$, namely,

$$\begin{aligned} \frac{d}{dt}\int_a^b u(t,x)\,dx &= s(u(t,a))\,u(t,a) - s(u(t,b))\,u(t,b) \\ &= -\Big[s(u(t,x))\,u(t,x)\Big]_{x=a}^{x=b} \\ &= -\int_a^b \Big(s(u(t,x))\,u(t,x)\Big)_x\,dx \end{aligned}$$

holds for any a, b. Since u is continuous and a, b are arbitrary, we have the conservation law

$$u_t + F(u)_x = 0$$

with the flux $F(u) := s(u)\,u$. Statistically, one usually takes

$$s(u) = C_1 \log \frac{C_2}{u}$$

with $0 < u \le C_2$ for appropriate positive constants C_1, C_2.

We refer to Evans [E1998] for the theory and applications of various partial differential equations.

1.1 Weak solutions

We are interested in finding all solutions of a partial differential equation and furthermore investigating the existence, uniqueness, asymptotic behavior and other properties of solutions for general data. Most partial differential equations, however, are not expected to have smooth solutions.

Example 2: Consider the Cauchy problem of the inviscid Burgers' equation

$$u_t + uu_x = 0, \qquad u(0,x) = \frac{1}{1+x^2}. \tag{1.5}$$

It is known that a smooth solution for (1.5) blows up in finite time, namely, the discontinuity of the smooth solution appears in finite time even if the initial datum is sufficiently smooth. This corresponds a shock wave in gas dynamics. In this problem, we see that the smooth solution for (1.5) blows up at $t = 8/\sqrt{27}$ by the implicit function theorem. We now investigate the phenomena at

$$t = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ \frac{8}{\sqrt{27}} - \frac{1}{1000000},\ \frac{8}{\sqrt{27}},\ 2$$

using "Maple".

```
>   with(plots):
>   with(PDEtools):
>   pde:=diff(u(t,x),t)+u(t,x)*diff(u(t,x),x)=0:
>   ini:=[0,s,1/(1+s^2)]:
>   sol:=pdsolve(pde,u(t,x)):
>   solve(x=y+t/(1+y^2),y):
>   v:=(t,x)->(x,y):
>   plot3d(v(t,x),t=0..3,x=-2..3):
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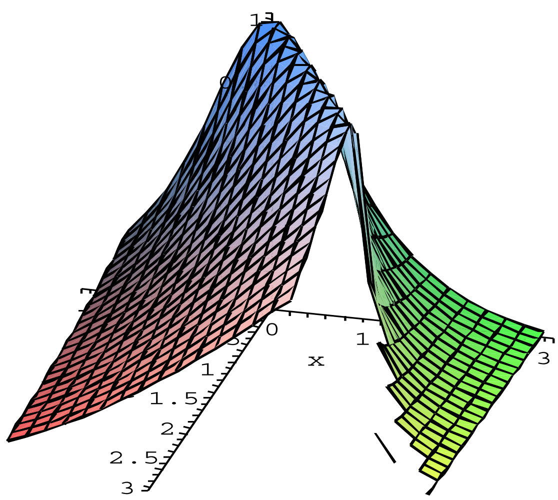
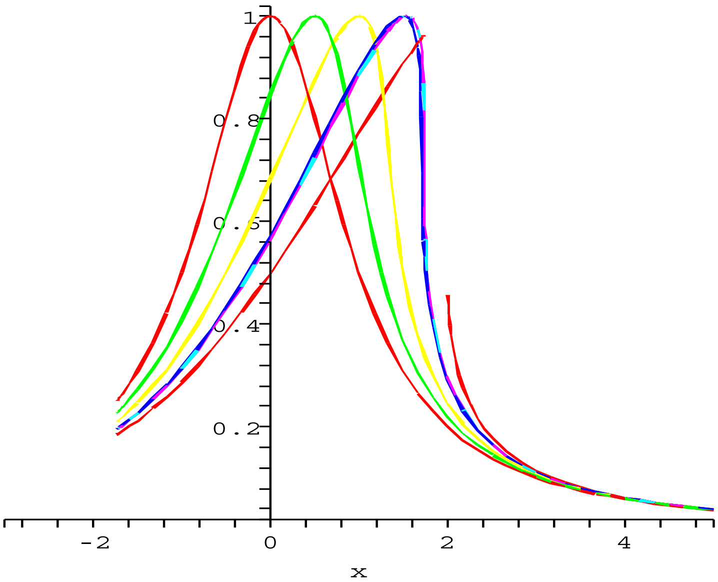


The graph of $u(t, x)$

```
> f1:=v(0,x):
> f2:=v(1/2,x):
> f3:=v(1,x):
> f4:=v(3/2,x):
> f5:=v(8/sqrt(27)-1/1000000,x):
> f6:=v(8/sqrt(27),x):
> f7:=v(2,x):
> plot([f1,f2,f3,f4,f5,f6,f7],x=-3..5);
```



The graphs of f1, $\cdots$, f7

As we saw before, we need to extend the notion of solutions to nonsmooth solutions including discontinuous solutions interpreted in the sense of distributions, which is the so-called weak solution. Thanks to the notion of weak solutions, we can deal with a lot of partial differential equations which have not been handled in the classical sense up to then. We now state the definition of a weak solution for the Cauchy problem of a conservation law (1.4).

Definition 1.1. *We say $u \in L^\infty(Q)$ is a* weak solution *for the Cauchy problem* (1.4) *provided*

$$\iint_Q \big(u\,\phi_t + \mathbf{F}(u)\cdot\nabla\phi + f\,\phi\big)\,dxdt \;=\; 0$$

holds for any $\phi \in C_0^1(Q)$, and $u(t,\cdot) \to u_0$ in $L^1_{loc}(\mathbf{R}^N)$ as $t \downarrow 0$ essentially.

Nevertheless, it is known that there exist many weak solutions in general for the Cauchy problem of nonlinear degenerate parabolic-hyperbolic equations including conservation laws.

Example 3: Consider the Cauchy problem of the inviscid Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x \;=\; 0, \qquad u(0,x) \;=\; \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{1.6}$$

For every $r \in (0,1)$, we define the piecewise constant function $u_r : [0,\infty)\times\mathbf{R} \to \mathbf{R}$ as

$$u_r(t,x) \;:=\; \begin{cases} 0 & \text{if } x < \dfrac{r\,t}{2} \\ r & \text{if } \dfrac{r\,t}{2} \le x < \dfrac{(1+r)\,t}{2} \\ 1 & \text{if } x \ge \dfrac{(1+r)\,t}{2}. \end{cases}$$

Then, each u_r is a solution for the problem (1.6) since it satisfies the equation almost everywhere and Rankine-Hugoniot conditions hold along the two lines of discontinuity

$$\ell_1(t) \;=\; \frac{r\,t}{2} \qquad \text{and} \qquad \ell_2(t) \;=\; \frac{(1+r)\,t}{2}.$$

Thereupon, Kružkov [Kr1970] introduced a new notion of an entropy solution which is a weak solution satisfying an entropy inequality, and proved the uniqueness of an entropy solution for a conservation law. This 'entropy' comes, roughly speaking, from the thermodynamic principle that physical entropy can not decrease as time goes forward. The entropy inequality is a suitable criterion to extract accurately the exact one weak solution according as physical demands, and ensure the uniqueness of weak solutions. We here refer to the definition of an entropy solution for the Cauchy problem of a scalar conservation law (1.4).

Definition 1.2. *Let $\eta \in C^1(\mathbf{R})$ be a convex function. If there exist functions $q_i \in C^1(\mathbf{R})$, $i = 1, \cdots, N$, such that for any $r \in \mathbf{R}$*

$$\eta'(r)\, F_i'(r) \;=\; q_i'(r) \qquad i = 1, \cdots, N,$$

then $(\eta, \mathbf{q})$ is called an entropy-entropy flux pair *of the conservation law* (1.3).

Definition 1.3. *We say $u \in L^\infty(Q)$ is an* entropy solution *for the Cauchy problem* (1.4) *provided for every entropy-entropy flux pair $(\eta, \mathbf{q})$ of the conservation law* (1.3)*, the so-called entropy inequality*

$$\eta(u)_t + \operatorname{div} \mathbf{q}(u) \;\leq\; f$$

holds in the sense of distributions, namely,

$$\iint_Q \big(\eta(u)\, \phi_t + \mathbf{q}(u)\cdot\nabla\phi + f\,\phi\big)\, dxdt \;\geq\; 0$$

is fulfilled for any $\phi \in C_0^1(Q)^+$, and $u(t,\cdot) \to u_0$ in $L^1_{loc}(\mathbf{R}^N)$ as $t \downarrow 0$ essentially. Here $C_0^1(Q)^+$ denotes the space of all nonnegative functions belong to $C_0^1(Q)$.

Denote by $S_0(r)$ the sign function taking 1 if $r > 0$, 0 if $r = 0$ or -1 if $r < 0$. Note that we may consider

$$\begin{aligned} \eta(u) \;&=\; \big|u - k\big|, \\ q_i(u) \;&=\; S_0(u-k)\,\big(F_i(u) - F_i(k)\big), \qquad i = 1, \cdots, N \end{aligned}$$

for $k \in \mathbf{R}$ as an entropy-entropy flux pair even if they are not smooth enough by taking account of an appropriate smoothing function. To be sure, we consider a function $G \in C^\infty(\mathbf{R})$ with $G(x) = |x|$ for all $|x| \geq 1$, $G'(0) = 0$ and $G'' \geq 0$. For fixed $k \in \mathbf{R}$, we set

$$G_\varepsilon := \varepsilon\, G((x-k)/\varepsilon).$$

Then we see that $G_\varepsilon \to |x-k|$ as $\varepsilon \downarrow 0$. This formulation makes it possible to define an entropy solution in the following way:

Definition 1.4. *We say $u \in L^\infty(Q)$ is an* entropy solution *for the Cauchy problem* (1.4) *provided*

$$\iint_Q \Big(|u-k|\, \phi_t + S_0(u-k)\, \big(\mathbf{F}(u) - \mathbf{F}(k)\big) \cdot \nabla\phi + S_0(u-k)\, f\, \phi \Big)\, dxdt \;\geq\; 0$$

holds for any $k \in \mathbf{R}$ and any $\phi \in C_0^1(Q)^+$, and $u(t,\cdot) \to u_0$ in $L^1_{loc}(\mathbf{R}^N)$ as $t \downarrow 0$ essentially.

We can easily check that an entropy solution is a weak solution. Indeed, let u be an entropy solution for the Cauchy problem (1.4). If u is bounded, choosing $k < -\|u\|_{L^\infty(Q)}$ we have

$$\begin{aligned} 0 \;&\leq\; \iint_Q \Big((u-k)\, \phi_t + \big(\mathbf{F}(u) - \mathbf{F}(k)\big) \cdot \nabla\phi + f\, \phi \Big)\, dxdt \\ &=\; \iint_Q \Big(u\, \phi_t + \mathbf{F}(u) \cdot \nabla\phi + f\, \phi \Big)\, dxdt. \end{aligned}$$

On the other hand, choosing $k > \|u\|_{L^\infty(Q)}$ we have

$$\begin{aligned} 0 \;&\leq\; \iint_Q \Big((k-u)\, \phi_t + \big(\mathbf{F}(k) - \mathbf{F}(u)\big) \cdot \nabla\phi - f\, \phi \Big)\, dxdt \\ &=\; -\iint_Q \Big(u\, \phi_t + \mathbf{F}(u) \cdot \nabla\phi + f\, \phi \Big)\, dxdt. \end{aligned}$$

Combining these estimates, we deduce that

$$\iint_Q \Big(u\, \phi_t + \mathbf{F}(u) \cdot \nabla\phi + f\, \phi \Big)\, dxdt \;=\; 0,$$

which means u is a weak solution for the Cauchy problem (1.4).

Since the notion of entropy solutions was introduced, many researchers have studied the Cauchy problems and initial-boundary value problems for nonlinear degenerate equations as well as conservation laws.

On the other hand, Portilheiro proved that the equivalence of an entropy solution and a dissipative solution of a conservation law (1.3). The notion of dissipative solutions was introduced first by Evans, and established afterward by Portilheiro for conservation laws. The original definition of dissipative solutions is as follows:

Definition 1.5. *Let X be a certain Banach space. We say $A : D(A) \to 2^X$ is an* accretive operator *if*

$$\|u - v\| \le \|(u - v) + \lambda (Au - Av)\|$$

holds for any $u, v \in D(A)$ and $\lambda > 0$, where $\|\cdot\|$ denotes the norm in X.

Definition 1.6. *We say u is a* dissipative solution *of the equation*

$$Au = f$$

with possibly multivalued accretive operator $A : D(A) \to 2^X$ defined as a subset of some Banach space X if

$$[u - \phi, f - A\phi]_+ \ge 0$$

holds for every 'nice' function ϕ, where $[\cdot,\cdot]_+$ denotes the Kato bracket defined as

$$[u, v]_+ := \lim_{\lambda \downarrow 0} \frac{\|u + \lambda v\| - \|u\|}{\lambda}.$$

We note that A is accretive if and only if

$$[u - v, Au - Av]_+ \ge 0$$

holds for any $u, v \in D(A)$. We also note that if $X = L^1(Q)$ particularly, then the Kato bracket is given by

$$[f, g]_+ := \iint_{f\neq 0} S_0(f)\, g\, dxdt + \iint_{f=0} \big|g\big|\, dxdt$$

for any $f, g \in L^1(Q)$.

The definition of dissipative solutions of a scalar conservation law (1.3) with globally Lipschitz-continuous flux $\mathbf{F}$ is given as follows:

Definition 1.7. *We say $u \in L^1(Q)$ is a* dissipative solution *of a conservation law* (1.3) *with globally Lipschitz-continuous flux* $\mathbf{F}$ *provided*

$$\iint_Q S_0(u-\phi)\,\big(f - \phi_t - \operatorname{div}\mathbf{F}(\phi)\big)\, dxdt \;\geq\; 0$$

holds for any $\phi \in C_0^1(Q)$ such that $\phi(t,x) \equiv k$ for large x, and $u(t,\cdot) \to u_0$ in $L^1_{loc}(\mathbf{R}^N)$ as $t \downarrow 0$ essentially.

Direct proofs of existence and uniqueness of dissipative solutions have not been obtained yet, but the notion of dissipative solutions is flexible and suitable to handle relaxation systems than the entropy framework.

As we mentioned before, the notion of entropy and dissipative solutions is useful and important to resolve the mechanism of phenomena describing as partial differential equations. On the other hand, it is known from Crandall [C1972] that a weak solution for (1.4) has been constructed for any L^1 data using nonlinear semigroup theory (see Crandall and Liggett [CL1971] for example). In this case, however, if the flux has no growth, then it is impossible to construct a solution even in the sense of distributions since the flux function may fail to be locally integrable. To resolve the difficulty, DiPerna and Lions introduced the following renormalization theory.

1.2 Renormalization

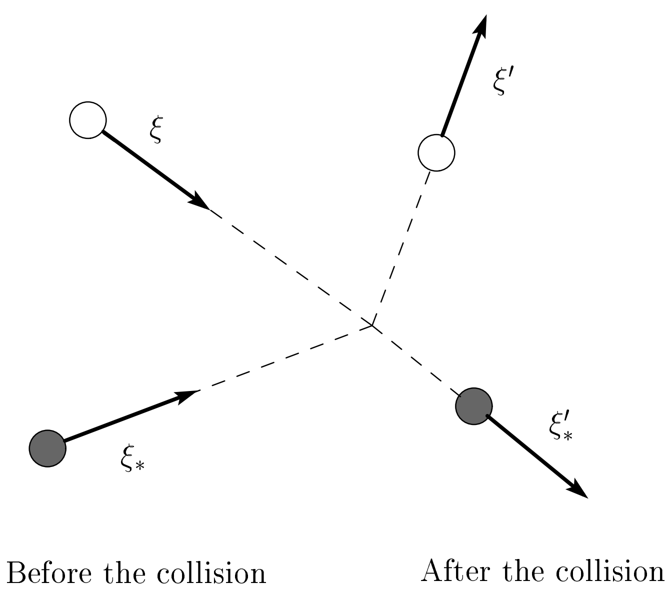
DiPerna and Lions [DPL1989] studied the Cauchy problem for the Boltzmann equation

$$f_t + \xi\cdot\nabla_x f \;=\; Q(f,f) \qquad \text{in} \;\; (0,\infty)\times\mathbf{R}^N\times\mathbf{R}^N \tag{1.7}$$

where $f = f(t,x,\xi)$, $t>0$, $x\in\mathbf{R}^N$, $\xi\in\mathbf{R}^N$, $N\geq 1$ and $Q(f,f)$ is a collision operator defined by

$$Q(h,h) \;:=\; \int_{\mathbf{R}^N}\int_{S^{N-1}} \Big(h(\xi')\,h(\xi'_*) - h(\xi)\,h(\xi_*)\Big) B(\xi-\xi_*,w)\,dwd\xi_*$$

for $h\in C_0^\infty(\mathbf{R}^N)$. In fact, $Q(f,f)$ means $Q(f(t,x,\cdot),f(t,x,\cdot))$ in (1.7). Here, $\xi' = \xi-(\xi-\xi_*,w)w$, $\xi'_* = \xi_*+(\xi-\xi_*,w)w$ with inner product $(\cdot,\cdot)$, and $B(z,w)\geq 0$ called the collision kernel is a given function of $|z|$ and $|(z,w)|$ only.



The physical interpretation of ξ, ξ_*, ξ', ξ'_* is as follows: ξ, ξ_* are the velocities of two colliding molecules immediately before collision while ξ', ξ'_* are the velocities immediately after the collision. Furthermore, local conservation laws of momentum and kinetic energy

$$\xi + \xi_* = \xi' + \xi'_* \qquad \text{and} \qquad |\xi|^2 + |\xi_*|^2 = |\xi'|^2 + |\xi'_*|^2$$

hold for binary interaction.

DiPerna and Lions proved that sequences of classical solutions of (1.7) with uniform a priori bounds obtained from the standard physical identities associated with (1.7) converge weakly in L^1 to a renormalized solution of (1.7) defined below, and also deduced from this stability result the existence of a global renormalized solution of (1.7) with an initial condition. Due to the definition of the collision operator $Q(f,f)$, it is reasonable to ask for an estimate of the following form

$$f \in L^2_{loc}((0,\infty) \times \mathbf{R}^N_x\, ;\, L^1(\mathbf{R}^N_\xi))$$

and such an estimate does not seem to be available in general. This lack of the estimate has been the major obstruction to a complete understanding of the Cauchy problem for (1.7). To overcome this difficulty, the notion of renormalized solutions was introduced. The definition of renormalized solutions is as follows:

Definition 1.8. *A nonnegative function f is a* renormalized solution *of* (1.7) *if* $(1+f)^{-1}Q(f,f) \in L^1_{loc}$ *and* $g := \log(1+f)$ *solves a renormalized Boltzmann equation*

$$g_t + \xi\cdot\nabla_x g = \frac{1}{1+f}\, Q(f,f) \qquad \text{in} \quad (0,\infty) \times \mathbf{R}^N \times \mathbf{R}^N \tag{1.8}$$

in the sense of distributions.

Similar ideas for nonlinear elliptic equations also appear in Bénilan et al. [BBGGPV1995] and Boccardo et al. [BGDM1993].

As we mentioned above, for the Cauchy problem of degenerate parabolic-hyperbolic equations, it is known that if initial data and external forces are unbounded then the solution constructed by nonlinear semigroup theory is also unbounded in general. Furthermore, if no growth conditions are assumed on the flux, the flux function may fail to be locally integrable, and therefore the Cauchy problem does not possess a solution even in the sense of distributions. In order to conquer this difficulty, Bénilan et al. [BCW2000] introduced a new notion of renormalized entropy solutions and obtained existence and uniqueness results of renormalized entropy solutions for the Cauchy problem of conservation laws with general L^1 data. As to the definition of renormalized entropy solutions, see Definition 4.1 in Chapter 4 later.

As related problems, initial-boundary value problems of nonlinear degenerate parabolic-hyperbolic equations or uniqueness of solutions for systems of conservation laws also rouse our interest.

1.3 Overview

This dissertation is organized as follows: We begin in Chapter 2 with a study of construction of local C^k center unstable manifolds for time dependent evolution equations of parabolic type in Banach spaces. In the case of unbounded domain, there is no well-defined spectral gap due to the appearance of continuous spectrum of the linearized equation. In order to apply the partial differential equations on unbounded domains, we shall present a useful theorem to formulate a local center unstable manifolds for the evolution equations in Banach spaces. Furthermore, our result is used in Kobayasi [Ko2002] to construct a local invariant manifold for a nonlinear parabolic equation on the whole space $\mathbf{R}^N$. Contents of this chapter is based on the paper [KT2003] which is a joint work with Professor Kazuo Kobayasi.

Chapter 3 is concerned with renormalized solutions for degenerate quasilinear elliptic equations with no growth convection term on unbounded domain. The existence result of renormalized solutions for this problem was obtained by

Kobayasi [Ko1998]. In terms of this, we focus on the uniqueness of renormalized solutions, and apply our theory to the stationary problem of p-Laplace equations. Contents of this chapter is based on the paper [KTU2000] jointly with Professor Kazuo Kobayasi and Mr. Takeshi Uehara.

Chapter 4 is devoted to the relation of specific weak solutions for the Cauchy problem of a scalar conservation law with locally Lipschitz-continuous flux. In case of globally Lipschitz-continuous flux, Portilheiro [P2003] introduced a notion of dissipative solutions and proved the equivalence of such solutions and entropy solutions. The structure of dissipative solutions is flexible and suitable to deal with relaxation systems than entropy scheme. In this chapter, we shall extend some results obtained by Portilheiro to the case of locally Lipschitz-continuous flux. We introduce a new notion of renormalized dissipative solutions which is a generalization of dissipative solutions in the sense of Portilheiro for the Cauchy problem of a scalar conservation law with locally Lipschitz-continuous flux and L^1 data, and show the equivalence of a renormalized dissipative solution and a renormalized entropy solution in the sense of Bénilan et al. We apply our result to contractive relaxation systems in merely an L^1-setting and construct a renormalized dissipative solution as a relaxation limit. Contents of this chapter is based on the paper [KT2005] with Professor Kazuo Kobayasi. This research was supported by Waseda University Grant for Special Research Projects ♯2003A–856.

Chapter 5 deals with the extension of the notion of renormalized dissipative solutions to second order degenerate parabolic equations with locally Lipschitz-continuous flux and L^1 data. We shall show the equivalence of such solutions and renormalized entropy solutions in the sense of Bendahmane and Karlsen [BK2004]. In this case, there is another difficulty due to the appearance of derivative of the Dirac mass. In order to overcome this, we try to multiply a test function and convolute them. We apply our result to certain relaxation systems in general L^1-setting and construct a renormalized dissipative solution. Contents of this chapter is based on the paper [T2005]. This research was supported by Waseda University Grant for Special Research Projects $\sharp$2004A–108.

A list of our original papers shall be drawn up at the tail of this dissertation.

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Chapter 2

Local center unstable manifolds

We start with construction of local center unstable manifolds for nonlinear parabolic evolution equations. In the case of unbounded domain, the main difficulty arises from the appearance of continuous spectrum of the linearized equation. This means that there is no well-defined spectral gap, and therefore the existing center unstable manifold theorem can not be used. In order to apply the partial differential equations on unbounded domains, we shall present a useful theorem to formulate a local center unstable manifolds for the evolution equations in Banach spaces. Furthermore, our result is used in Kobayasi [Ko2002] to construct a local invariant manifold for a nonlinear parabolic equation on the whole space $\mathbf{R}^N$. Contents of this chapter is based on the paper [KT2003] which is a joint work with Professor Kazuo Kobayasi.

2.1 Introduction

We consider the existence of local C^k center unstable manifolds for time dependent evolution equations of parabolic type in Banach spaces. The center unstable manifold theorem is a standard and useful idea in studying the long-time behavior of solutions to a class of partial differential equations in the neighborhood of a stationary point. In its formulation, up to now almost all theorems can apply to only a partial differential equation on a bounded domain. These frameworks are, however, too restrictive for many interesting applications, especially in the application of the equations on unbounded domain. In the case of unbounded domain, the main difficulty arises from the appearance of continuous spectrum of the linearized equation; there is no well-defined spectral gap. Nevertheless, a nonlinear heat equation of the form $u_t = \triangle u + F(u)$ on $\mathbf{R}^N$ does possess finite-dimensional local center unstable manifold, see Wayne [W1997].

It is thus useful to formulate a local center unstable manifold theorem in order to apply the partial differential equations on unbounded domains. In this chapter, we present such an abstract theorem for the evolution equations in Banach spaces as to treat a class of partial differential equations on unbounded domains. Indeed, our result is used in Kobayasi [K2002] to construct a local invariant manifold for a nonlinear parabolic equation on the whole space $\mathbf{R}^N$.

Our approach is based on the classical method of Lyapunov-Perron and follows closely Chow and Lu [CL1988]. Related results can be found in Miklavčič [Mik1991], Galley [G1993], Mielke [Mie1991], Carr [C1983], Kobayasi [K1999], and so on.

Let X, Y and Z be Banach spaces. The norms of X and Y will be denoted by $\|\cdot\|$ and $|\cdot|$, respectively. Suppose that both X and Y are continuously embedded in Z. Note that X is not necessarily embedded in Y. Let $\{S(t)\,;\, t \geq 0\}$ be a C_0-semigroup in Z and $f : \mathbf{R} \times X \to Y$ a nonlinear map of class C^k for some $k \geq 1$. Instead of evolution equations, we would rather consider the integral equation

$$u(t) \;=\; S(t)\,x_0 + \int_0^t S(t-s)\,f(s,u(s))\,ds, \qquad t \geq 0. \tag{2.1}$$

We are interested in the asymptotic behavior of the solution of (2.1).

We assume the following conditions on the C_0-semigroup:

(H1) $Z = Z_1 \oplus Z_2$, $\dim Z_1 < \infty$ and $S(t)\,P_i = P_i\,S(t)$, $i = 1,2$, where P_i is a continuous projection from Z onto Z_i.

(H2) $Z_1 \subset X \times Y$ and the restriction of $S(t)$ to X also forms a C_0-semigroup on X.

(H3) There exist constants $\alpha, \beta, \gamma, \eta, M, M^*$ such that $\alpha > 0$, $\beta + (k-1)\,\eta > 0$, $\eta < 0$, $0 \le \gamma < 1$, $M \ge 1$, $M^* \ge 0$,

$$\begin{aligned} \left\| e^{-\eta t}\, S(t)\, P_1\, y \right\| &\le M\, e^{\alpha t} \left| y \right| && \text{for} \quad t \le 0,\ y \in Y, \\ \left\| e^{-\eta t}\, S(t)\, P_2\, x \right\| &\le M\, e^{-\beta t} \left\| x \right\| && \text{for} \quad t \ge 0,\ x \in X, \\ \left\| e^{-\eta t}\, S(t)\, P_2\, y \right\| &\le (M\, t^{-\gamma} + M^*)\, e^{-\beta t} \left| y \right| && \text{for} \quad t > 0,\ y \in Y. \end{aligned}$$

Remark 2.1. (a) *Condition* (H2) *implies* $X_1 = Y_1 = Z_1$, *where* $X_1 = P_1 X$ *and* $Y_1 = P_1 Y$, *for* $X_1 \subset Z_1 = P_1 Z \subset X_1$. *Therefore, there is a constant* $M_1 \ge 1$ *such that* $M_1^{-1} \left| y \right| \le \left\| y \right\| \le M \left| y \right|$ *for* $y \in X_1 = Y_1$.

(b) *The restriction of* P_2 *to* X *becomes a continuous projection from* X *onto* $X_2 = P_2 X$, *for* $\left\| P_2\, x \right\| \le \left\| x \right\| + \left\| P_1\, x \right\| \le \left\| x \right\| + C \left| P_1\, x \right|_Z \le \left\| x \right\| + C \left| x \right|_Z \le C \left\| x \right\|$ *for* $x \in X$.

(c) *Under the conditions* (H1)-(H3) *there exists* $M_0 \ge 1$ *such that*

$$\left\| S(t)\, y \right\| \le M_0\, t^{-\gamma} \left| y \right| \qquad \text{for} \quad t \in (0,1],\ y \in Y. \tag{2.2}$$

We have our primary conclusion.

Theorem 2.2. *Assume that the hypotheses* (H1)-(H3) *above are satisfied. Let the map* $f : \mathbf{R} \times X \to Y$ *satisfy the following conditions:*

(a) *For each* $t \in \mathbf{R}$, $f(t,\cdot)$ *is of class* C^k. *For each* $x \in X$, $f(\cdot, x)$ *is continuous.*

(b) $f(t,0) = 0$ *and* $Df(t,0) = 0$ *for* $t \in \mathbf{R}$, *where* $Df(t,x)$ *is the derivative of* $f(t,x)$ *with respect to* x *evaluated at* (t,x).

(c) $f(t,x)$ *and* $Df(t,x)$ *converge, as* $\left\| x \right\| \to 0$ *uniformly in* t, *to* 0 *in* Y *and* $\mathcal{B}(X,Y)$, *respectively.*

Then there exist neighborhoods $U_1 \subset X_1$, $U_2 \subset X_2$ of zero and a continuous function $h : \mathbf{R} \times U_1 \to U_2$ with the following properties:

(M1) *The set $\mathcal{M} = \bigcup_{t\in\mathbf{R}} \mathcal{M}(t)$, $\mathcal{M}(t) := \{(t, \xi + h(t,\xi))\ ;\ \xi \in U_1\}$, is a local invariant manifold of* (2.1)*, that is, for each $x_0 \in \mathcal{M}(0)$ there exist $T_0,\ T_1 \in (0,\infty]$ such that a solution u of* (2.1) *uniquely exists on $(-T_0, T_1)$ and $u(t) \in \mathcal{M}(t)$ for all $t \in (-T_0, T_1)$.*

(M2) *For each $t \in \mathbf{R}$, $h(t,\cdot)$ is of class C^k, $h(t,0) = 0$ and $Dh(t,0) = 0$.*

(M3) *For each $x_0 \in U_1 \times U_2$,* (2.1) *has a unique solution on some interval $[0,T)$. If in addition $T = \infty$, then there exists a unique solution $\widetilde{u}$ of* (2.1) *on $\mathcal{M}$ such that*

$$\sup_{t>0} e^{-\eta t} \left\|u(t) - \widetilde{u}(t)\right\| \ <\ \infty.$$

The proof of Theorem 2.2 is obtained from the global center unstable manifold theorem by using an appropriate cut off function. We thus consider the global theorem in Section 2 and complete the proof of Theorem 2.2 in Section 3. In Section 4, we shall introduce an application of our theory to nonlinear parabolic equations on the whole space which was studied by Kobayasi [K2002].

2.2 Global center unstable manifold theorem

In this section, as a nonlinear map let us take the continuous map $F : \mathbf{R}\times X \to Y$ satisfying the following conditions:

(H4) For each $t \in \mathbf{R}$, $F(t,\cdot)$ is of class C^k, $F(t,0) = 0$ and $DF(t,0) = 0$.

(H5) There exist constants $L,\ r > 0$ such that $\left|DF(t,x)\right|_{\mathcal{B}(X,Y)} \le L$ and $F(t,\widetilde{x}) = 0$ for all $t \in \mathbf{R}$, $x \in B_2(r)$ and $\widetilde{x} \in X$ with $\left\|P_1 \widetilde{x}\right\| > r$, where

$$B_2(r)\ :=\ \{x \in X\ ;\ \left\|P_2 x\right\| \le r\}.$$

Let $J \subset \mathbf{R}$ be an interval. For any $\mu \in \mathbf{R}$ we denote by $C_\mu(J, X)$ the Banach space

$$C_\mu(J,X) \ := \ \Big\{ v \in C(J,X) \ ; \ \sup_{t \in J} e^{-\mu t} \, \|v(t)\| < \infty \Big\}$$

with the norm

$$\|v\|_{C_\mu(J,X)} \ := \ \sup_{t\in J} e^{-\mu t} \, \|v(t)\|.$$

Let

$$C(J, B_2(r)) \ := \ \big\{ u \in C(J,X) \ ; \ u(t) \in B_2(r) \qquad \text{for all} \ \ t \in J \big\}$$

and

$$C_\eta(r) \ := \ C_\eta(\mathbf{R}^-, X) \cap C(\mathbf{R}^-, B_2(r)).$$

Clearly, $C_\eta(r)$ is a closed subset of $C_\eta(\mathbf{R}^-, X)$. Set

$$\begin{aligned} \mathcal{J}_\tau(\varphi,\xi)(t) \ \ := \ \ & S(t)\,\xi + \int_t^0 S(t-s)\, P_1\, F(s+\tau, \varphi(s))\, ds \\ & + \int_{-\infty}^t S(t-s)\, P_2\, F(s+\tau,\varphi(s))\, ds, \qquad t \in \mathbf{R}^- \end{aligned}$$

for $\xi \in X_1$, $\varphi \in C_\eta(r)$ and $\tau \in \mathbf{R}$. Note that by virtue of (H3)-(H5), these integrals exist.

Lemma 2.3. *If*

$$K(\alpha, \beta + (k-1)\eta, \gamma)\, L \ < \ \frac{1}{2(M+1)},$$

then there exists $\varepsilon_0 \in (0, \alpha)$ *such that for each* $\varepsilon \in [0, \varepsilon_0]$, $\tau \in \mathbf{R}$ *and* $\xi \in X_1$, *the equation*

$$\varphi(t) \ = \ \mathcal{J}_\tau(\varphi, \xi)(t), \qquad t \in \mathbf{R}^- \tag{2.3}$$

has a unique solution $\varphi(\tau,\xi)(\cdot) \in C_{\eta+\varepsilon}(r/(M+1))$ *independent of* ε*, where*

$$K(\alpha,\beta,\gamma) \; := \; M(\alpha^{-1} + \Gamma(1-\gamma)\beta^{\gamma-1}) + M^*\beta^{-1}$$

and Γ *is the gamma function. Moreover, the map* $\varphi(\tau,\cdot) : X_1 \to C_{k\eta}(r/(M+1))$ *is of class* C^k*.*

Proof. By the continuity of $K(\alpha,\beta,\gamma)$ in α and β, there exists $\varepsilon_0 > 0$ such that

$$K(\alpha-\varepsilon,\beta+\varepsilon,\gamma)\, L \; < \; \frac{1}{2(M+1)}$$

for every $\varepsilon \in [0,\varepsilon_0]$. We show that $\mathcal{J}_\tau(\cdot,\xi) : C_{\eta+\varepsilon}(r/(M+1)) \to C_{\eta+\varepsilon}(r/(M+1))$ is a uniform contraction with respect to ξ and τ. We first prove that $\mathcal{J}_\tau(\cdot,\xi)$ maps $C_{\eta+\varepsilon}(r/(M+1))$ into itself. Let $\varphi \in C_{\eta+\varepsilon}(r/(M+1))$. By (H3)-(H5), we have $\mathcal{J}_\tau(\varphi,\xi) \in C_{\eta+\varepsilon}(\mathbf{R}^-,X)$ and

$$\begin{aligned}\big\|P_2\,\mathcal{J}_\tau(\varphi,\xi)(t)\big\| \; &= \; \left\|\int_{-\infty}^t S(t-s)\,P_2\,F(s+\tau,\varphi(s))\,ds\right\| \\ &\le \; \int_{-\infty}^t (M(t-s)^{-\gamma} + M^*)\,e^{-(\beta-\eta)(t-s)}\,\big|F(s+\tau,\varphi(s))\big|\,ds.\end{aligned}$$

Since $F(s+\tau,\varphi(s)) = 0$ if $\big\|P_1\varphi(s)\big\| > r$ by (H5), we have

$$\begin{aligned}\big|F(s+\tau,\varphi(s))\big| \; &= \; \big|F(s+\tau,\varphi(s)) - F(s+\tau,0)\big| \\ &\le \; L\left(r + \big\|P_2\,\varphi(s)\big\|\right) \\ &= \; \frac{M+2}{M+1}\,L\,r.\end{aligned}$$

Therefore,

$$\big\|P_2\,\mathcal{J}_\tau(\varphi,\xi)(t)\big\| \; \le \; K(\alpha,\beta-\eta,\gamma)\,\frac{M+2}{M+1}\,L\,r \; \le \; \frac{r}{M+1},$$

and so $\mathcal{J}_\tau(\varphi,\xi) \in C_{\eta+\varepsilon}(r/(M+1))$.

Next we prove that $\mathcal{J}_\tau(\cdot,\xi)$ is a contraction uniformly with respect to ξ and τ. For $\varphi_1,\varphi_2 \in C_{\eta+\varepsilon}(r/(M+1))$, $\xi \in X_1$ and $\tau \in \mathbf{R}$, from (H3) and (H5) we have (see Chow and Lu [CL1988])

$$\begin{aligned}\big\|\mathcal{J}_\tau(\varphi_1,\xi)-\mathcal{J}_\tau(\varphi_2,\xi)\big\|_{C_{\eta+\varepsilon}(\mathbf{R}^-,X)} &\le K(\alpha-\varepsilon,\beta+\varepsilon,\gamma)\,L\,\big\|\varphi_1-\varphi_2\big\|_{C_{\eta+\varepsilon}(\mathbf{R}^-,X)}\\ &\le \frac{1}{2(M+1)}\big\|\varphi_1-\varphi_2\big\|_{C_{\eta+\varepsilon}(\mathbf{R}^-,X)}.\end{aligned}$$

The strict contraction theorem assures that there exists a unique $\varphi_\varepsilon(\tau,\xi)\in C_{\eta+\varepsilon}(r/(M+1))$ such that $\mathcal{J}_\tau(\varphi_\varepsilon(\tau,\xi),\xi)=\varphi_\varepsilon(\tau,\xi)$. Since $C_{\eta+\varepsilon}(r/(M+1))\subset C_\eta(r/(M+1))$, by the uniqueness we have $\varphi_\varepsilon(\tau,\xi)=\varphi_0(\tau,\xi)$ for every $\varepsilon\in[0,\varepsilon_0]$.

Finally, according to [CL1988, Lemma 3.4], $\varphi(\tau,\cdot)$ is C^k as a mapping from X_1 into $C_{k\eta}(r/(M+1))$. We notice that the proof of [CL1988] works well only by replacing $C_{\eta+\delta}(\mathbf{R}^-,X)$ with $C_{\eta+\delta}(r/(M+1))$. □

Now consider the equation

$$u(t) \;=\; S(t-t_0)\,u(t_0)+\int_{t_0}^{t}S(t-s)\,F(s,u(s))\,ds,\qquad t\ge t_0. \tag{2.4}$$

We shall say that $u\in C(J,X)$ is a solution of (2.4) on J if it satisfies (2.4) for all $t,\ t_0\in J$ with $t_0\le t$. Proceeding as in the proof of [CL1988, Lemma 4.2] we can also obtain that a function $u\in C((-\infty,\tau],B_2(r))$ is a solution of (2.4) on $(-\infty,\tau]$ if and only if the function $\varphi(t)$ defined by $\varphi(t)=u(t+\tau)$ is a solution of (2.3) with $\xi=P_1\,u(\tau)$.

Lemma 2.4. *Let $1<\rho<1+1/M$ and*

$$K(\alpha,\beta,\gamma)\,L \;<\; \frac{1-(\rho-1)M}{2(M+1)}.$$

Then, for each $x_0\in B_2(\rho\,r/(M+1))$ and $t_0\in\mathbf{R}$, the equation (2.4) has a unique solution $u\in C([t_0,\infty),B_2(r))$ such that $u(t_0)=x_0$.

Proof. We may assume that $t_0=0$. Let $x_0\in B_2(\rho\,r/(M+1))$. For $w\in C([0,T],B_2(r))$ set

$$(Gw)(t) \;:=\; S(t)\,x_0+\int_0^t S(t-s)\,F(s,w(s))\,ds,\qquad t\in[0,T].$$

We first show that G maps $C([0,T],B_2(r))$ into itself. Indeed, by our hypotheses

$$\begin{aligned}
&\|P_2(Gw)(t)\| \\
&\le \frac{M\,\rho\, r}{M+1} + \int_0^t (M\,(t-s)^{-\gamma} + M^*)\, e^{-\beta(t-s)}\, L\left(r + \left\|P_2\, w(s)\right\|\right) ds \\
&\le \frac{M\,\rho\, r}{M+1} + 2K(\alpha,\beta,\gamma)\, L\, r \;\le\; r.
\end{aligned}$$

It follows from (H5) and (2.2) that for $w, \widetilde{w} \in C([0,T],B_2(r))$

$$\left\|(Gw)(t) - (G\widetilde{w})(t)\right\| \;\le\; \frac{L\,M_0}{1-\gamma}\, t^{1-\gamma}\, \left\|w - \widetilde{w}\right\|_{C([0,T],X)}.$$

By induction on n it follows easily that

$$\left\|(G^n w)(t) - (G^n \widetilde{w})(t)\right\| \;\le\; \frac{(L\,M_0\,\Gamma(1-\gamma))^n}{\Gamma(n+1-n\,\gamma)}\, t^{n-n\gamma}\, \left\|w - \widetilde{w}\right\|_{C([0,T],X)}$$

Since

$$\lim_{n\to\infty} \frac{(LM_0\Gamma(1-\gamma)T)^n}{\Gamma(n+1-n\,\gamma)} \;=\; 0,$$

by the fixed point theorem G has a unique fixed point u_T in $C([0,T],B_2(r))$. We then define $u \in C(\mathbf{R}^+, B_2(r))$ by $u(t) = u_T(t)$ for $t \in [0,T]$, which is well-defined by the uniqueness of fixed points. Clearly, u becomes a solution of (2.4).

The uniqueness of u is a consequence of the following argument.Let $\widetilde{u}$ be another solution. Let $t_1 \ge 0$ and $t \in [t_1, t_1+1]$. By (H5) and (2.2) we have

$$\left\|u(t) - \widetilde{u}(t)\right\| \;\le\; K\left\|u(t_1) - \widetilde{u}(t_1)\right\| + M_0\, L \int_{t_1}^t (t-s)^{n-n\gamma-1}\left\|u(s) - \widetilde{u}(s)\right\| ds,$$

where $K = \max_{0\le t\le 1}\left\|S(t)\right\|_{\mathcal{B}(X)}$. Therefore, it follows from [P1983, Lemma 6.7] that

$$\begin{aligned}
\left\|u(t) - \widetilde{u}(t)\right\| \;&\le\; K\left\|u(t_1) - \widetilde{u}(t_1)\right\| + \sum_{j=0}^{n-1}\left(\frac{M_0\, L\, (t_1+1)^{1-\gamma}}{1-\gamma}\right)^j \\
&\quad + \frac{(M_0\, L\,\Gamma(1-\gamma))^n}{\Gamma(n-n\,\gamma)} \int_{t_1}^t (t-s)^{n-n\gamma-1}\left\|u(s) - \widetilde{u}(s)\right\| ds.
\end{aligned}$$

We now fix n sufficiently large such that $n(1-\gamma) > 1$. Then, we find that there exist positive constants $C_1(t_1)$ and $C_2(t_1)$ such that

$$\big\|u(t) - \widetilde{u}(t)\big\| \;\leq\; C_1(t_1)\,\big\|u(t_1) - \widetilde{u}(t_1)\big\| + C_2(t_1)\int_{t_1}^{t}\big\|u(s) - \widetilde{u}(s)\big\|\,ds.$$

Using Gronwall's inequality, we obtain

$$\big\|u(t) - \widetilde{u}(t)\big\| \;\leq\; C_1(t_1)\,e^{C_2(t_1)}\,\big\|u(t_1) - \widetilde{u}(t_1)\big\|.$$

Since $t_1 \geq 0$ is arbitrary, this inequality immediately yields the uniqueness of u. □

Proposition 2.5. *Suppose that* (H1)-(H5) *are satisfied. Let*

$$1 \,<\, \rho \,<\, 1 + \frac{1}{M} \quad \text{and} \quad K(\alpha,\beta,\gamma)\,L \,<\, \frac{1-(\rho-1)M}{2(M+1)}.$$

For $\tau \in \mathbf{R}$, *define*

$$\mathcal{M}(\tau) = \big\{u(\tau)\ ;\ u \in C((-\infty,\tau], B_2(r/(M+1))) \text{ is a solution of (2.4) on } (-\infty,\tau]\big\}.$$

Then we have that

(a) *There exists a function* $h \in C(\mathbf{R} \times X_1, B_2(r/(M+1)))$ *such that* $h(t,\xi)$ *is* C^k *in* ξ *and*

$$\mathcal{M}(\tau) \;=\; \{\xi + h(\tau,\xi)\ ;\ \xi \in X_1\}.$$

(b) *For a solution* u *of* (2.4) *on* $[\tau,\infty)$, *we have that* $u(\tau) \in \mathcal{M}(\tau)$ *implies* $u(t) \in \mathcal{M}(t)$ *for* $t \geq \tau$.

Proof. By Lemma 2.3 we see that $\mathcal{M}(\tau) \neq \emptyset$. Let $x_0 \in \mathcal{M}(\tau)$ and

$$u \in C((-\infty,\tau], B_2(r/(M+1)))$$

a solution of (2.4) with $u(\tau) = x_0$. As noted above, $\varphi(\tau,\xi)(\cdot) \equiv u(\cdot + \tau)$ is the unique solution of (2.3) with $\xi_1 = P_1 x_0$. Then we set

$$h(\tau,\xi) \;=\; \int_{-\infty}^{0} S(-s)\, P_2\, F(s+\tau, \varphi^{\tau}(s))\, ds.$$

It is easy to see that $x_0 = \xi + h(\tau,\xi)$ and $h(\tau,\xi) = \varphi(\tau,\xi)(0) - \xi$. Hence, by Lemma 2.3, $h(\tau,\xi)$ is a C^k mapping from X_1 into X_2 with respect to ξ. To see the continuity of $h(t,\xi)$ in t, we write

$$\begin{aligned} & h(t, P_1\, u(t)) - h(\sigma, P_1\, u(t)) \\ & \quad = \; P_2\,(u(t) - u(\sigma)) + h(\sigma, P_1\, u(\sigma)) - h(\sigma, P_1\, u(t)). \end{aligned}$$

Hence,

$$\big\| h(t, P_1\, u(t)) - h(\sigma, P_1\, u(t)) \big\| \;\le\; C\, \big\| u(\sigma) - u(t) \big\|$$

for some constant C. Thus (a) is proved.

Next, let $x_0 \in \mathcal{M}(\tau)$. Since $x_0 \in B_2(r/(M+1))$, by Lemma 2.4 we can extend u to the solution of (2.4) on $\mathbf{R}$ satisfying $u(t) \in B_2(r)$ for all $t \in \mathbf{R}$. In particular, we have for $\widetilde{\tau} > \tau$

$$u(t+\widetilde{\tau}) \;=\; \mathcal{J}_{\widetilde{\tau}}(u(\cdot + \widetilde{\tau}), P_1\, u(\widetilde{\tau}))(t), \quad t \in \mathbf{R}^-.$$

Hence, for $t \le \widetilde{\tau}$

$$\begin{aligned} \big\| P_2\, u(t) \big\| \;&=\; \Big\| \int_{-\infty}^{t-\widetilde{\tau}} S(t - \widetilde{\tau} - s)\, P_2\, F(s+\widetilde{\tau}, u(s+\widetilde{\tau}))\, ds \Big\| \\ &\le\; \int_{-\infty}^{t} (M\,(t-s)^{-\gamma} + M^*)\, e^{-\beta(t-s)}\, L\, \big(r + \big\| P_2\, u(s) \big\|\big)\, ds \\ &\le\; 2K(\alpha,\beta,\gamma)\, L\, r \;\le\; \frac{r}{M+1}. \end{aligned}$$

Therefore, by definition we have $u(\widetilde{\tau}) \in \mathcal{M}(\widetilde{\tau})$. This proves (b). □

Proposition 2.6. *Suppose that* (H1)-(H5) *are satisfied. Let* $1 < \rho < 1 + 1/M$,

$$K(\alpha,\beta,\gamma)\, L \;<\; \frac{1 - (\rho - 1)\, M}{2(M+1)} \quad \text{and} \quad \frac{M K(\alpha,\beta,\gamma)\, L}{1 - K(\alpha,\beta,\gamma)\, L} \;<\; 1.$$

Then, for each $x_0 \in B_2(\rho\, r/(M+1))$*, there exists a unique* $x_0^* \in \mathcal{M}(0)$ *such that*

$$\sup_{t\geq 0} e^{-\eta t}\left\|u(t,x_0)-u(t,x_0^*)\right\| \;<\; \infty,$$

where $u(t,x_0)$ *is the solution of* (2.4) *on* $\mathbf{R}^+$ *with* $u(0,x_0)=x_0$ *and* M_1 *is the constant given in* Remark 2.1 (a)*.*

Proof. Fix the solution $u(t)=u(t,x_0)\in C(\mathbf{R}^+,B_2(r))$ of (2.4) and put

$$\dot{E}_r \;=\; \left\{w\in C_\eta(\mathbf{R}^+,X)\;;\;w(t)+u(t)\in B_2(r) \qquad \text{for all } \; t\geq 0\right\}.$$

Let $\omega_2\in B_2(\rho\, r/(M+1))$. For $w\in \dot{E}_r$ define

$$\begin{aligned}\mathcal{L}(w)(t) \;&=\; S(t)\,(\omega_2-x_0)+\int_0^t S(t-s)\,P_2\,(F(s,w+u)-F(s,u))\,ds\\ &\quad -\int_t^\infty S(t-s)\,P_1\,(F(s,w+u)-F(s,u))\,ds \qquad \text{for } \; t\geq 0.\end{aligned}$$

Then, we have for $w,\ \widetilde{w}\in \dot{E}_r$ and $t\geq 0$

$$\left\|P_2(\mathcal{L}(w)(t)+u(t)\right\| \;\leq\; \frac{M\,\rho\, r}{M+1}+2K(\alpha,\beta,\gamma)\,L\,r \;\leq\; r$$

and

$$\left\|e^{-\eta t}\,(\mathcal{L}(w)(t)-\mathcal{L}(\widetilde{w})(t))\right\| \;\leq\; K(\alpha,\beta,\gamma)\,L\left\|w-\widetilde{w}\right\|_{C_\eta(\mathbf{R}^+,X)}.$$

Thus, $\mathcal{L}$ is a strict contraction from $\dot{E}_r$ into itself and hence there exists a unique $\hat{w}(\omega_2)(\cdot)\in \dot{E}_r$ such that $\mathcal{L}(\hat{w}(\omega_2))=\hat{w}(\omega_2)$. Define

$$g(\omega_2) \;=\; P_1\hat{w}(\omega_2)(0) \qquad \text{for } \; \omega_2\in B_2(\rho\, r/(M+1)).$$

Since we have

$$\left\|g(\omega_2)-g(\widetilde{\omega}_2)\right\| \;\leq\; \frac{M\,K(\alpha,\beta,\gamma)\,L}{1-K(\alpha,\beta,\gamma)\,L}\left\|\omega_2-\widetilde{\omega}_2\right\|, \quad \omega_2,\ \widetilde{\omega}_2\in B_2(\rho\, r/(M+1))$$

and

$$\left\|h(0,\xi)-h(0,\widetilde{\xi})\right\| \leq \frac{M\,M_1\,K(\alpha,\beta,\gamma)\,L}{1-K(\alpha,\beta,\gamma)\,L}\left|\xi-\widetilde{\xi}\right|, \quad \xi,\ \widetilde{\xi}\in X_1,$$

there exists a unique $\omega_1^* \in X_1$ such that $\omega_1^* = g(h(0, \omega_1^* + P_1 x_0))$. Then we set $\omega_2^* = h(0, \omega_1^* + P_1 x_0)$ and $\hat{w}^*(t) = \hat{w}(\omega_2^*)(t)$. Obviously, we get $\omega_1^* = g(\omega_2^*) = P_1 \hat{w}^*(0)$ and $\hat{w}^*$ satisfies the equation

$$\hat{w}^*(t) + u(t) \;=\; S(t)(\hat{w}^*(0) + x_0) + \int_0^t S(t-s)\, F(s, \hat{w}^* + u)\, ds.$$

Now set $x_0^* = \hat{w}^*(0) x_0$. Since $x_0^* = \omega_1^* + P_1 x_0 + h(0, \omega_1^* + P_1 x_0) \in \mathcal{M}(0)$, by Lemma 2.4 we must have $\hat{w}^*(t) + u(t) = u(t, x_0^*)$ the unique solution of (2.4) on $\mathbf{R}^+$ with $u(0, x_0^*) = x_0^*$. Hence, $u(t, x_0^*) - u(t) = \hat{w}^*(t) \in C_\eta(\mathbf{R}^+, X)$. □

2.3 Proof of Theorem 2.2

We are now in the position to prove Theorem 2.2.

Proof of Theorem 2.2. Let $\rho : X_1 \to \mathbf{R}$ be a smooth function such that

$$0 \;\le\; \rho(\xi) \;\le\; 1 \qquad \text{for} \;\; \xi \in X_1$$

and

$$\rho(\xi) \;:=\; \begin{cases} 1 & \text{if} \;\; \|\xi\| \le \dfrac{1}{2} \\[2ex] 0 & \text{if} \;\; \|\xi\| \ge 1. \end{cases}$$

Since X_1 is finite dimensional, the existence of such a function is obvious. For $r > 0$ set

$$F_r(t, x) \;=\; f(t, x)\, \rho\left((P_1\, x)/r\right), \quad x \in X,$$

and denote by $L(t, r)$ the maximum of the Lipschitz constant of F_r with respect to x over $\{x \in X \;;\; \|P_2\, x\| \le r\}$. Then, by assumption (c) of Theorem 2.2, we have

$$\lim_{r \downarrow 0} L(t, r) \;=\; 0 \qquad \text{uniformly in} \;\; t.$$

Hence, applying Propositions 2.5 and 2.6 to the nonlinear map $F_r(t,u)$ with sufficiently small $r > 0$, we obtain the conclusion of Theorem 2.2 with

$$U_1 = \{x_1 \in X_1 \; ; \; \|x_1\| \le r\}$$

and

$$U_2 = \{x_2 \in X_2 \; ; \; \|x_2\| \le r\}.$$

□

2.4 Application

We now apply our theory to nonlinear parabolic equations on the whole space. This application was studied by Kobayasi [K2002]. For this reason, we shall mention only an outline.

We consider the problem about existence of finite dimensional invariant manifolds for nonlinear heat equations of the form

$$u_t = \triangle u + F(u, \nabla u) \quad \text{in} \;\; [1,\infty) \times \mathbf{R}^N,$$

where $u = u(t,x)$, $t \ge 1$ and $x \in \mathbf{R}^N$. The linearized equation is the heat equation on the whole space $\mathbf{R}^N$ which has continuous spectrum extending from negative infinity to zero, so that there is no gap in the spectrum. The applications of invariant manifold theorems in nonlinear partial differential equations, however, require that the linearized equation has an appropriate spectral gap in order to split the spectrum into the parts associated with center or stable manifold. Nevertheless, we shall prove that there are still finite dimensional invariant manifolds for these partial differential equations which control the long-time behavior of solution near the origin.

We here consider the asymptotic behavior of equations of the form

$$u_t = \triangle u - |u|^{\gamma-1} u + F(u, \nabla u) \quad \text{in} \;\; [1,\infty) \times \mathbf{R}^N, \tag{2.5}$$

where $\gamma > 1$. We assume the following condition on F:

(F) F is C^1, $F(0,0)=0$ and there exist constants $L>0$, $q_1, q_2 \geq 1$ such that

$$\left|r\, F_r(r,z)\right| + \left|z\cdot\nabla_z F(r,z)\right| \;\leq\; L\left|r\right|^{q_1}\left|z\right|^{q_2}$$

for all $r\in\mathbf{R}$ and $z\in\mathbf{R}^N$, and

$$q_1+\frac{\gamma+1}{2}\,q_2 \;\geq\; \gamma.$$

Let $p>1$ and $m=0,1,\cdots$. For any positive continuous function K, we define a weighted Sobolev space as follows:

$$\begin{aligned} L^p(K) &:= \left\{u\,;\int_{\mathbf{R}^N}\left|u(x)\right|^p K(x)\,dx \,<\, \infty\right\}, \\ W^{m,p}(K) &:= \left\{u\,;\,D^\alpha u\in L^p(K),\ \left|\alpha\right|\leq m\right\}, \end{aligned}$$

the Banach spaces with the usual norms.

The main result is the following:

Theorem 2.7 (Kobayasi [K2002]). *Suppose that $\gamma>1$ and the condition* (F) *holds. Let*

$$p \,>\, \max\left\{q_2+1, q_2\,N\right\} \quad \text{and} \quad n \,>\, \frac{2}{\gamma-1}-N-1.$$

Then, we can choose a weighted Sobolev space $W^{1,p}(K_r)$ with $K_r(x)=(1+|x|^2)^{r/2}$ for $r>0$, and a neighborhood U of $[1,\infty)\times\{0\}$ in $[1,\infty)\times W^{1,p}(K_r)$ with the following properties:

(i) *There exists a $\sum_{j=0}^{n}\binom{j+N-1}{N-1}$ dimensional local invariant manifold $\mathcal{M}$ for* (2.5) *in U. More precisely, for each $(1,x_0)\in\mathcal{M}$, there exist $T_0\in[1,\infty)$ and a unique mild solution u of* (2.5) *on $[1,T_0)$ such that $u(1)=x_0$ and $(t,u(t))\in\mathcal{M}$ for all $t\in[1,T_0)$.*

(ii) *For each* $(1, x_1) \in U$*, there exist* $T_1 > 0$ *and a unique mild solution* u_1 *of* (2.5) *on* $[1, T_1)$ *with* $u_1(1) = x_1$*. If, moreover,* $(t, u(t)) \in U$ *for all* $t \geq 1$*, then* $T_1 = \infty$ *and for every* $\varepsilon > 0$*, there exist a unique mild solution* $\overline{u}_1$ *on the invariant manifold* $\mathcal{M}$ *and a constant* $C > 0$ *such that*

$$\left(1 + \frac{|x|^2}{t}\right)^{r/2p} \left|u_1(t,x) - \overline{u}_1(t,x)\right| \leq C\, t^{-N/2-(n+1)/2+\varepsilon}$$

for all $t \geq 1$ *and* $x \in \mathbf{R}^N$*.*

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Chapter 3

Renormalized solutions

In this chapter, we consider renormalized solutions for degenerate quasilinear elliptic equations. The existence result of renormalized solutions for this problem was obtained by Kobayasi [K1998]. In terms of this, we focus on uniqueness of renormalized solutions, and apply our theory to the stationary problem of p-Laplace equations. Contents of this chapter is based on the paper [KTU2000] which is a joint work with Professor Kazuo Kobayasi and Mr. Takeshi Uehara.

3.1 Introduction

Let Ω denote an arbitrary open set in $\mathbf{R}^N$, $N \geq 2$. We shall study the nonlinear elliptic equation

$$\text{(P)} \qquad \begin{cases} \beta(u) - \operatorname{div} a(\cdot, u, \nabla u) \ni f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in L^1(\Omega)$, $\nabla u = (u_{x_1}, \cdots, u_{x_N})$ denotes the gradient of u, β is a maximal monotone graph in $\mathbf{R}^2$ with $\beta(0) \ni 0$ and $a : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function, that is, measurable in $x \in \Omega$ for any $r \in \mathbf{R}$, any $\xi \in \mathbf{R}^N$ and continuous in $(r, \xi) \in \mathbf{R} \times \mathbf{R}^N$ for almost every $x \in \Omega$.

Many authors considered the problem of type (P) as well as the evolution problem associated with (P) under various hypotheses on the vector field a, cf.

e.g. Atik and Rakotoson [AR1996], Bénilan et al. [BBGGPV1995], Bénilan and Gariepy [BeG1995], Otto [O1996], Rakotoson [R1991] and Xu [X1994]. Boccardo et al. [BGDM1993] dealt with the problem of existence and regularity of renormalized solutions for some elliptic equations with convection term. Rakotoson [R1994] treated the problem about existence and uniqueness of renormalized solutions of (P) in the case that a may depend on u under the assumption that Ω is bounded, β is an increasing continuous function and $a(x, u, \nabla u)$ has power growth in u as well as $|\nabla u|$ of an appropriate order. The notion of renormalized solutions was introduced by DiPerna and Lions [DPL1989] dealing with the existence of a solution of the Boltzmann equation and various existence and uniqueness results have been obtained, cf. e.g. Carrillo and Wittbold [CW1999], Kobayasi [K1998] and the references therein.

In this chapter, we shall adopt the notion of renormalized solutions for the nonlinear elliptic equation (P). Our goal is to establish the existence and uniqueness of renormalized solutions of (P) under the hypothesis that Ω is an arbitrary domain, not necessarily bounded set in $\mathbf{R}^N$, and $a(x, u, \nabla u)$ may depend on u so as to contain the convection term with no growth condition. In particular our theory applies to the stationary problem

$$\beta(u) - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + h(u)\right) \ni f \quad \text{in } \Omega,$$

where $1 < p < N$, β is maximal monotone in $\mathbf{R}^2$ and $h \in C(\mathbf{R})^N$, which is associated with the nonlinear diffusion-convection problem. See Gagneux and Madaune-Tort [GMT1994], for example.

Let us state our precise assumptions. Let $1 < p < N$ and $p \leq q < \infty$.

(H1) β is a maximal monotone graph in $\mathbf{R}^2$ such that $0 \in \beta(0)$, $D(\beta) = \mathbf{R}$ and

$$\limsup_{r \to 0} \frac{|r|^q}{|\beta^0(r)|} < \infty$$

where β^0 denotes the minimal section of β (see Brezis [Br1973]).

We will denote by H^q_β the set of continuous functions $h : \mathbf{R} \to \mathbf{R}^N$ such that

$$\limsup_{r\to 0} \frac{|h(r)|^{q'}}{|\beta^0(r)|} \;<\; \infty \quad \text{with} \;\; q' = \frac{q}{q-1}.$$

(H2) $a : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function and there exist $\lambda > 0$ and $h \in H^q_\beta$ such that

$$\langle a(x,r,\xi), \xi\rangle \;\geq\; \lambda\,|\xi|^p + \langle h(r), \xi\rangle$$

holds for almost every $x \in \Omega$, any $r \in \mathbf{R}$ and $\xi \in \mathbf{R}^N$, where $\langle\cdot,\cdot\rangle$ denotes scalar product in $\mathbf{R}^N$.

(H3) There exist nonnegative and nondecreasing functions d and ω defined on $\mathbf{R}^+$ and $b_0 \in L^1(\Omega)$ such that $\int_0^1 \omega(s)^{-1} ds = \infty$ and

$$\begin{aligned}&\langle a(x,r,\xi) - a(x,s,\eta), \xi - \eta\rangle \\ &\qquad > \;\; -d(|r| + |s|)\,\omega(|r-s|)\,\big(b_0(x) + |\xi|^p + |\eta|^p + |\beta^0(s)| + |\beta^0(r)|\big)\end{aligned}$$

holds for almost every $x \in \Omega$, any $r, s \in \mathbf{R}$ and any $\xi, \eta \in \mathbf{R}^N$ with $\xi \neq \eta$.

(H4) There exist $\Lambda \geq 0$, $a_0 \in L^{p'}(\Omega)$ and $h \in H^q_\beta$ such that

$$\langle a(x,r,\xi), \eta\rangle \;\leq\; \Lambda\,\big(a_0(x) + \rho(r)^{1/p'} + |\xi|^{p-1}\big)\,|\eta| + \langle h(r), \eta\rangle$$

holds for almost every $x \in \Omega$, any $r \in \mathbf{R}$ and any $\xi, \eta \in \mathbf{R}^N$, where

$$\rho(r) \;=\; \begin{cases} |\beta^0(r)| & \text{if } |r| \leq 1, \\ \max\big\{|\beta^0(r)|, |r|^{p_1}\big\} & \text{if } |r| > 1,\end{cases}$$

with

$$p_1 \;:=\; \frac{N(p-1)}{N-p} \quad \text{and} \quad p' \;:=\; \frac{p}{p-1}.$$

This chapter is organized as follows. In Section 2 we state the definition of the functional space $\mathcal{T}^{1,p}_0(\Omega)$ and renormalized solutions, and mention the existence of such solutions which was obtained by Kobayasi [K1998]. We state the main uniqueness result of renormalized solutions and devote to its proof in Section 3. We show an example of our theory in Section 4.

3.2 Renormalized solutions

We denote the usual Lebesgue and Sobolev spaces by $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively, and $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$, the space of compactly supported C^∞-functions on Ω, in $W^{1,p}(\Omega)$. $\|\cdot\|_p$ denotes the L^p-norm in Ω. We also use the local spaces $L^p_{loc}(\Omega)$ and $W^{1,p}_{loc}(\Omega)$.

For $k > 0$ and a measurable function u on Ω we define the truncated function $T_k u$ by $(T_k u)(x) := T_k(u(x))$ for almost every $x \in \Omega$, where

$$T_k(r) \;:=\; \frac{|k+r| - |k-r|}{2}.$$

We also define $T_k^+(r)$ and $T_k^-(r)$, respectively, by

$$T_k^+(r) \;:=\; T_k(r^+) \qquad \text{and} \qquad T_k^-(r) \;:=\; T_k((-r)^+),$$

where $r^+ := \max\{r, 0\}$. Obviously, $T_k(r) = T_k^+(r) - T_k^-(r)$.

Let us state the definition of $\mathcal{T}_0^{1,p}(\Omega)$. Firstly, $\mathcal{T}^{1,p}(\Omega)$ is the space of measurable functions $u : \Omega \to \mathbf{R}$ such that for every $k > 0$ the truncated function $T_k(u)$ belongs to $W^{1,1}_{loc}(\Omega)$ and $\nabla T_k(u) \in L^p(\Omega)^N$. Secondary, $\mathcal{T}_0^{1,p}(\Omega)$ is the subset of $\mathcal{T}^{1,p}(\Omega)$ defined as follows: A function $u \in \mathcal{T}^{1,p}(\Omega)$ belongs to $\mathcal{T}_0^{1,p}(\Omega)$ if for every $k > 0$ there exists a sequence $\psi_n \in C_0^\infty(\Omega)$ such that

$$\begin{aligned} \nabla\psi_n \;&\to\; \nabla T_k(u) \qquad \text{in } \; L^p(\Omega)^N, \\ \psi_n \;&\to\; T_k(u) \qquad \text{in } \; L^1_{loc}(\Omega). \end{aligned}$$

We list a few fundamental properties of those spaces. See Bénilan et al. [BBGGPV1995] for more details.

(P1) $W_0^{1,p}(\Omega) \subset \mathcal{T}_0^{1,p}(\Omega)$.

(P2) For every $u \in \mathcal{T}^{1,p}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbf{R}$ such that

$$\nabla T_k(u) \;=\; v\,\chi_{\{|u|<k\}} \qquad \text{for } \; k > 0,$$

where χ_A denotes the indicator function of a subset A of Ω. In what follows we denote this function v by ∇u.

(P3) If $u \in \mathcal{T}_0^{1,p}(\Omega)$ and $1 < p < N$, then $T_k(u) \in L^{p^*}(\Omega)$ for $k > 0$, where p^* denotes the Sobolev conjugate of p, that is,

$$p^* := \frac{pN}{N-p}.$$

(P4) If $u \in \mathcal{T}_0^{1,p}(\Omega)$, then $\nabla\theta(u) = \theta'(u)\,\nabla u$ for any $\theta \in \Theta$. Moreover, if $u \in \mathcal{T}_0^{1,p}(\Omega)$, then $\theta(u) \in \mathcal{T}_0^{1,p}(\Omega)$ for any $\theta \in \Theta$ with $\theta(0) = 0$, where

$$\Theta := \{\theta \in \mathrm{Lip}(\mathbf{R})\,;\,\mathrm{spt}\,\theta' \text{ is bounded}\}.$$

We here present related lemmas. These proofs can be found in [K1998].

Lemma 3.1. *Let $H(r) = \int_0^r h(\sigma)\,d\sigma$ and $a \geq 0$. If $v \in \mathcal{T}_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\varphi \in W^{1,1}_{loc}(\Omega) \cap L^\infty(\Omega)$ satisfy*

$$\beta^0(v) \in L^1(\Omega) \quad \text{and} \quad \nabla\varphi \in L^\infty(\{a \leq v\}),$$

then we have

$$\int_{\{a\leq v\}} \mathrm{div}\,H(v)\,\varphi\,dx = -\int_{\{a\leq v\}} \langle H(v) - H(a), \nabla\varphi\rangle\,dx.$$

Lemma 3.2. (i) *If $u \in \mathcal{T}^{1,p}(\Omega)$ and $v \in \mathcal{T}_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $J_L(u)\,v \in \mathcal{T}_0^{1,p}(\Omega)$ for every $L > 0$, where*

$$J_L(r) := \frac{1 + \big|L + 1 - |r|\big| - \big|L - |r|\big|}{2} \quad \text{for} \quad r \in \mathbf{R}.$$

(ii) *If $u, v \in \mathcal{T}_0^{1,p}(\Omega)$, then $u - T_L(v) \in \mathcal{T}_0^{1,p}(\Omega)$ for every $L > 0$.*

We now state the definition of renormalized solutions.

Definition 3.3. *Let $f \in L^1(\Omega)$ and β a maximal monotone graph in $\mathbf{R}^2$. We say that a function u is a* renormalized solution *of* (P ; β, f)*, or* (P) *for short, if the following properties hold:*

(R1) $u \in \mathcal{T}_0^{1,p}(\Omega)$, *and there exists* $w \in L^1(\Omega)$ *such that* $w(x) \in \beta(u(x))$ *almost every* $x \in \Omega$.

(R2) *For any* $\theta \in \Theta_c$ *and* $\varphi \in L^\infty(\Omega)$ *with* $\theta(u)\,\varphi \in \mathcal{T}_0^{1,p}(\Omega)$,

$$\int_\Omega w\,\theta(u)\,\varphi\,dx + \int_\Omega \langle a(x,u,\nabla u), \nabla(\theta(u)\varphi)\rangle\,dx \;=\; \int_\Omega f\,\theta(u)\,\varphi\,dx,$$

where $\Theta_c = \{\theta \in \mathrm{Lip}(\mathbf{R})\,;\, \mathrm{spt}\,\theta \text{ is compact and } \mathrm{spt}\,\theta' \text{ is bounded}\}$.

(R3) $\lim_{M\to\infty} I_u(M) = 0$, *where* $I_u(M) = \left(\int_{\{M<|u|<M+1\}} |\nabla u|^p\,dx\right)^{1/p}$.

(R4) $\int_{\{|u|<M\}} |\nabla u|^p\,dx \le CM$ *for any* $M > 0$.

Let us make a few remarks about the definition. We first note that $\beta^0(u) \in L^1(\Omega)$. Fix $\theta \in \Theta_c$ and $\varphi \in L^\infty(\Omega)$ as in (R2). Then $\nabla(\theta(u)\,\varphi) \in L^p(\Omega)$ by $\theta(u)\,\varphi \in \mathcal{T}_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. By virtue of (2.3) and (2.4) in [K1998], for each $k > 0$ we have that whenever $|u| < k$

$$\rho(u) \;\le\; |\beta^0(u)| + |u|^{p_1}\chi_{\{|u|>1\}} \;\le\; |\beta^0(u)| + k^{p_1}|u|^q \;\le\; C_k\,|\beta^0(u)|$$

and

$$\begin{aligned} |h(u)| \;&\le\; C\,|\beta^0(u)|^{1/q'} + |h(u)|\,\chi_{\{|u|>\alpha\}} \\ &\le\; C_k\left(|\beta^0(u)|^{1/p'} + (|u|/\alpha)^{q/p'}\right) \\ &\le\; C_{k,\alpha}\,|\beta^0(u)|^{1/p'}. \end{aligned}$$

Hence by (H4) we obtain for $|u| < k$

$$|a(x,u,\nabla u)|^{p'} \;\le\; C_{k,\alpha}\left(a_0^{p'} + |\beta^0(u)| + |\nabla u|^p\right). \tag{3.1}$$

Consequently, we get

$$\begin{aligned} &\int_\Omega \left|\langle a(x,u,\nabla u), \nabla(\theta(u)\,\varphi)\rangle\right| dx \\ &\quad\le\; C_{k,\alpha}\left(\|a_0\|_{p'} + \|\beta^0(u)\|_1^{1/p'} + \|\nabla T_k(u)\|_p^{p-1}\right)\|\nabla(\theta(u)\,\varphi)\|_p, \end{aligned} \tag{3.2}$$

which is finite by (R4), and so the integrals in (R2) of the definition make sense under assumptions (H1) and (H4).

We here mention the existence of renormalized solutions of (P). This result was obtained in [K1998], therefore we only state the existence theorem and related lemmas.

Lemma 3.4. *Assume that* $u \in \mathcal{T}_0^{1,p}(\Omega)$ *satisfies* (R4). *Let* $\ell_1 \in (0, p_1)$, $\ell_2 \in (0, p_2)$ *and* $K \subset \Omega$ *a measurable set with finite measure, where*

$$p_1 := \frac{N(p-1)}{N-p} \quad \text{and} \quad p_2 := \frac{N(p-1)}{N-1}.$$

Then, we have

$$\int_K |u|^{\ell_1} \, dx \le C_1 \quad \text{and} \quad \int_K |\nabla u|^{\ell_2} \, dx \le C_2,$$

where C_1 *and* C_2 *are constants depending on* C_0, $\operatorname{meas} K$, ℓ_1 *and* ℓ_2.

Lemma 3.5. *For* $i = 1, 2$, *let* $f_i \in L^1(\Omega)$, β_i *a maximal monotone graph in* $\mathbf{R}^2$, u_i *a renormalized solution of* $(\mathrm{P}\,; \beta_i, f_i)$ *and* w_i *the section of* $\beta_i(u_i)$. *Suppose that* $|\beta_1^0(r)| \le |\beta_2^0(r)|$ *for* $r \in \mathbf{R}$ *and* (H1)-(H4) *hold with* $\beta = \beta_i$ *and* $h = h_i$ *for some* $h_i \in H^q_{\beta_i}$. *If either* u_1 *or* u_2 *belongs to* $L^\infty(\Omega)$, *then*

$$\int_\Omega (w_1 - w_2)^+ \, dx \le \int_\Omega (f_1 - f_2)^+ \, dx.$$

Theorem 3.6 (Kobayasi [K1998]). *Suppose that* $1 < p < N$ *and* (H1)-(H4) *hold. Then, for each* $f \in L^1(\Omega)$, *there exists at least one renormalized solution of* (P).

3.3 Uniqueness

We consider uniqueness of solutions under the following additional assumptions.

(H5) For each $k > 0$ there exist constants $\mu \geq 0$ and $\widetilde{p} \geq 0$ such that $\widetilde{p} < p_2$ and

$$\begin{aligned}&\langle a(x,r,\xi) - a(x,s,\eta), \xi - \eta\rangle \\ &\quad \geq \quad -\mu\left((1+|\xi|+|\eta|)^{\widetilde{p}} + |\beta^0(r)| + |\beta^0(s)|\right)\end{aligned}$$

for almost every $x \in \Omega$, any $\xi, \eta \in \mathbf{R}^N$ and any $r, s \in \mathbf{R}$ satisfying $|r-s| \leq k$.

(H6) $a(x,r,\xi) = a(x,s,\xi)$ for almost every x and any ξ, whenever $\beta(r) \cap \beta(s) \neq \emptyset$.

The first uniqueness result is the following:

Theorem 3.7. *Suppose that* $1 < p < N$ *and* (H1)-(H6) *hold. Let* u *be the particular solution obtained in* Theorem 3.6 *by the approximation process and* $\widehat{u}$ *an arbitrary renormalized solution of* (P)*. Then we have* $u = \widehat{u}$*.*

To prove this theorem we begin with a direct consequence of Lemma 3.5.

Lemma 3.8. *Suppose that* (H1)-(H4) *hold. Let* $u, \widehat{u}$ *be the same solutions as in* Theorem 3.7 *and* $w, \widehat{w}$ *the corresponding sections of* $\beta(u), \beta(\widehat{u})$ *in the definition renormalized solutions. Then* $w = \widehat{w}$*.*

Proof. We denote by u_n the approximation of u and by w_n the section of $\beta(u_n)$ given by $(\mathrm{P}\,; \beta + \gamma_n, g_n)$. By Lemma 3.5 we have for $v = w$ or $\widehat{w}$

$$\int_\Omega |v - (w_n + \gamma_n(u_n))|\, dx \;\leq\; \int_\Omega |f - g_n|\, dx.$$

Therefore, letting $n \to \infty$ we obtain $\int_\Omega |w - \widehat{w}|\, dx = 0$, and so $w = \widehat{w}$. □

Proof of Theorem 3.7. Let $k > 0$. In the same manner as in the proof of (4.3) in [K1998] we can prove that

$$\begin{aligned}&\int_\Omega \langle a(x,\widehat{u},\nabla\widehat{u}) - a(x,u_n,\nabla u_n), \nabla T_k(\widehat{u} - u_n)\rangle\, dx \\ &\quad = \int_\Omega (f - g_n - \widehat{w} + w_n + \gamma_n(u_n))\, T_k(\widehat{u} - u_n)\, dx. \qquad (3.3)\end{aligned}$$

Fix $\eta \in (0,1)$ and write

$$E_n^\eta = \big(\{|u| \geq \eta\} \cup \{|u_n| \geq \eta\}\big) \cap \{|u_n - \widehat{u}| < k\}$$

and

$$F_n^\eta = \{|u| < \eta\} \cap \{|u_n| < \eta\} \cap \{|u_n - \widehat{u}| < k\},$$

and so $E_n^\eta \cup F_n^\eta = \{|u_n - \widehat{u}| < k\}$. In view of (H5) we have

$$\begin{aligned} &\chi_{E_n^\eta} \langle a(\cdot, \widehat{u}, \nabla\widehat{u}) - a(\cdot, u_n, \nabla u_n), \nabla\widehat{u} - \nabla u_n\rangle \\ &\qquad \geq \; -\mu\, \chi_{E_n^\eta} \Big((1 + \big|\nabla\widehat{u}\big| + \big|\nabla u_n\big|)^{\widetilde{p}} + \big|\beta^0(\widehat{u})\big| + \big|\beta^0(u_n)\big|\Big). \end{aligned}$$

Since $\widetilde{p} < p_2$ by assumption, it follows from (4.4) in [K1998] and Lemma 3.4 that the function $\chi_{E_n^\eta}\big(1+\big|\nabla\widehat{u}\big|+\big|\nabla u_n\big|\big)$ is bounded in $L^r(\Omega)$ uniformly in n, provided that $\widetilde{p} < r < p_2$. Hence, by Vitali's convergence theorem we may as well assume that as $n \to \infty$ it converges in $L^{\widetilde{p}}(\Omega)$ to the function

$$\chi_{\{|u|\geq\eta, |u-\widehat{u}|<k\}}\big(1 + \big|\nabla\widehat{u}\big| + \big|\nabla u\big|\big).$$

Furthermore, since $\big|\beta^0(u_n)\big| \leq \big|\gamma_n(u_n) + w_n\big|$ and $\gamma_n(u_n) + w_n \to w$ in $L^1(\Omega)$, we can consequently employ Fatou's lemma to get

$$\begin{aligned} &\liminf_{n\to\infty} \int_{E_n^\eta} \langle a(x, \widehat{u}, \nabla\widehat{u}) - a(x, u_n, \nabla u_n), \nabla\widehat{u} - \nabla u_n\rangle \; dx \\ &\qquad \geq \int_{\{|u|\geq\eta, |u-\widehat{u}|<k\}} \langle a(x, \widehat{u}, \nabla\widehat{u}) - a(x, u, \nabla u), \nabla\widehat{u} - \nabla u\rangle \; dx. \end{aligned} \tag{3.4}$$

On the other hand, we have from (3.1) that the function

$$\chi_{F_n^\eta}\big(a(\cdot, \widehat{u}, \nabla\widehat{u}) - a(\cdot, u_n, \nabla u_n)\big)$$

is bounded in $L^{p'}(\Omega)$ uniformly in n as well as η. This implies that as $n \to \infty$ it converges weakly in $L^{p'}(\Omega)$ to the function

$$\chi_{\{|u|<\eta, |u-\widehat{u}|<k\}}\big(a(\cdot, \widehat{u}, \nabla\widehat{u}) - a(\cdot, u, \nabla u)\big),$$

and hence

$$\lim_{n\to\infty}\int_{F_n^\eta}\langle a(x,\widehat{u},\nabla\widehat{u})-a(x,u_n,\nabla u_n),\nabla\widehat{u}-\nabla u_n\rangle\;dx$$
$$=\int_{\{|u|<\eta,|u-\widehat{u}|<k\}}\langle a(x,\widehat{u},\nabla\widehat{u})-a(x,u,\nabla u),\nabla\widehat{u}-\nabla u\rangle\;dx. \quad (3.5)$$

Owing to (4.4) in [K1998] the integral of $\left|\nabla u-\nabla u_n\right|^p$ over F_n^η is bounded by $2^{p-1}\,C_1\,\eta$, and so

$$\int_{F_n^\eta}\left|\langle a(x,\widehat{u},\nabla\widehat{u})-a(x,u_n,\nabla u_n),\nabla\widehat{u}-\nabla u_n\rangle\right|dx\;\le\;C_k\,\eta \quad (3.6)$$

for some constant C_k probably depending on k but not on n and η.

We now intend to pass to limits in (3.3). Thanks to (3.4)-(3.6) and the fact that $w=\widehat{w}$ by Lemma 3.8, we obtain

$$\int_{\{|u-\widehat{u}|<k\}}\langle a(x,\widehat{u},\nabla\widehat{u})-a(x,u,\nabla u),\nabla\widehat{u}-\nabla u\rangle\;dx\;\le\;C_k\,\eta.$$

Since $\beta(u)\cap\beta(\widehat{u})\neq\emptyset$ almost everywhere by Lemma 3.8 again, (H6) implies that $a(\cdot,u,\nabla u)=a(\cdot,u,\nabla u)$ almost everywhere. Therefore, by letting $\eta\downarrow 0$ we conclude that

$$\int_{\{|u-\widehat{u}|<k\}}\langle a(x,\widehat{u},\nabla\widehat{u})-a(x,\widehat{u},\nabla u),\nabla\widehat{u}-\nabla u\rangle\;dx\;\le\;0. \quad (3.7)$$

Since this is true for any $k>0$, we conclude by (H3) that $\nabla u=\nabla\widehat{u}$ almost everywhere. Taking into account that $u,\widehat{u}\in\mathcal{T}_0^{1,p}(\Omega)$ we conclude that $u=\widehat{u}$. $\square$

Remark 3.9. *The main interest of our treatment lies in the uniqueness of unbounded renormalized solutions. If we restrict ourselves to the bounded solutions, then the uniqueness is a simpler matter. We have the following: Suppose* $1<p<N$ *and that* (H1)-(H4) *and* (H6) *hold. If* u_1 *and* u_2 *are arbitrary renormalized solutions of* (P) *satisfying* $u_2\in L^\infty(\Omega)$, *then* $u_1=u_2$. *Indeed, since* (4.3) *in* [K1998] *is still valid with* F_ε *replaced by* T_k *and* $w_1=w_2$ *by* Lemma 3.5, *it follows from* (H6) *that* (3.7) *again holds for every* $k>0$. *Thus we have the conclusion.*

We next consider uniqueness of solutions under the following additional assumptions.

(H5)′ There exist a constant $\lambda > 0$ and an increasing function $\omega_1 \in C(\mathbf{R}^+, \mathbf{R}^+)$ such that $\int_0^1 \omega_1(s)^{-1/\widehat{p}}\, ds = \infty$ and

$$\begin{aligned} &\langle a(x,r,\xi) - a(x,s,\eta), \xi - \eta\rangle \\ &\geq \frac{\lambda|\xi-\eta|^{\widehat{p}}}{(1+|\xi|+|\eta|)^{\widetilde{p}}} - \omega_1(|r-s|)\left((1+|\xi|+|\eta|)^{\widetilde{p}} + |\beta^0(r)| + |\beta^0(s)|\right)\end{aligned}$$

for almost every $x \in \Omega$, any $\xi, \eta \in \mathbf{R}^N$ and any $r, s \in \mathbf{R}$, where $\widehat{p} = \max\{p, 2\}$ and $\widetilde{p} = \max\{2-p, 0\}$.

(H6)′ There exists a constant $\alpha > 0$ satisfying that if $r, s \in (-\alpha, \alpha)$ and $\beta(r) \cap \beta(s) \neq \emptyset$, then $r = s$.

Theorem 3.10. *Assume* $2 - N^{-1} < p < N$ *and that* (H1)-(H4), (H5)′ *and* (H6)′ *hold. Let* u *and* $\widehat{u}$ *be the same solutions as in* Theorem 3.7. *Then we have* $u = \widehat{u}$.

Proof. Fix $\varepsilon > 0$ and define the function G_ε by

$$G_\varepsilon(r) = \int_\varepsilon^{r_\varepsilon} \omega_1(\sigma)^{-1}\, d\sigma \qquad \text{for} \quad r \in \mathbf{R},$$

where $r_\varepsilon = \max\{\varepsilon, \min\{r, 1\}\}$ as before. In the same fashion as in the proof of (4.3) in [K1998] we can again prove that

$$\begin{aligned} &\int_\Omega \langle a(x,\widehat{u},\nabla\widehat{u}) - a(x,u_n,\nabla u_n), \nabla G_\varepsilon(\widehat{u}-u_n)\rangle\, dx \\ &= \int_\Omega \left(f - g_n - \widehat{w} + w_n + \gamma_n(u_n)\right) G_\varepsilon(\widehat{u}-u_n)\, dx. \end{aligned} \tag{3.8}$$

Of course u_n and w_n denote the approximation of u and the section of $\beta(u_n)$. For simplicity, we write

$$\begin{aligned} h_n &= 1 + |\nabla\widehat{u}| + |\nabla u_n|, \\ E_n &= \left(\{|\widehat{u}| \geq \alpha\} \cup \{|u_n| \geq \alpha\}\right) \cap \{\varepsilon < \widehat{u} - u_n < 1\}, \\ F_n &= \{|\widehat{u}| < \alpha\} \cap \{|u_n| < \alpha\} \cap \{\varepsilon < \widehat{u} - u_n < 1\}. \end{aligned}$$

As before, measE_n is finite and uniformly bounded in n. Since $\widetilde{p} < p_2$ by assumption, it follows from Lemma 3.4 that $\int_{E_n} h_n^{\widetilde{p}}\,dx$ is uniformly bounded in n. Hence, by (H5)$'$

$$\begin{aligned}\int_{E_n} &\langle a(x,\widehat{u},\nabla\widehat{u}) - a(x,u_n,\nabla u_n), \nabla G_\varepsilon(\widehat{u}-u_n)\rangle \,dx \\ &\geq \int_{E_n}\left(\frac{\lambda\left|\nabla\widehat{u}-\nabla u_n\right|^{\widehat{p}}}{\omega_1(\widehat{u}-u_n)h_n^{\widetilde{p}}} - \mu\left(h_n^{\widetilde{p}} + \left|\beta^0(\widehat{u})\right| + \left|\beta^0(u_n)\right|\right)\right)dx \\ &\geq \lambda\int_{E_n}\frac{\left|\nabla\widehat{u}-\nabla u_n\right|^{\widehat{p}}}{\omega_1(\widehat{u}-u_n)h_n^{\widetilde{p}}}\,dx - C \qquad (3.9)\end{aligned}$$

with some constant C independent of n.

Take $t \in [1, p_2)$. This choice of t is possible by the assumption that $2 - N^{-1} < p$. By Hölder's inequality

$$\int_{E_n}\frac{\left|\nabla\widehat{u}-\nabla u_n\right|^{t}}{\omega_1(\widehat{u}-u_n)^{t/\widehat{p}}}\,dx \;\leq\; C\left(\int_{E_n}\frac{\left|\nabla\widehat{u}-\nabla u_n\right|^{\widehat{p}}}{\omega_1(\widehat{u}-u_n)\,h_n^{\widetilde{p}}}\,dx\right)^{t/\widehat{p}}. \qquad (3.10)$$

On the other hand, to calculate the integrals over F_n we notice that

$$F_n \subset \{|\widehat{u}| \geq \varepsilon/2\} \cup \{|u_n| \geq \varepsilon/2\}.$$

Then, by Hölder's inequality, for $1 < s < p/p_2$

$$\begin{aligned}\int_{F_n}&\left(\frac{\left|\nabla\widehat{u}-\nabla u_n\right|^{t}}{\omega_1(\widehat{u}-u_n)^{t/\widehat{p}}}\right)^{s} dx \\ &\leq\; \omega_1(\varepsilon)^{-ts/\widehat{p}}\,(\mathrm{meas}F_n)^{1-(ts/p)}\left(\int_{F_n}\left|\nabla\widehat{u}-\nabla u_n\right|^{p}dx\right)^{ts/p},\end{aligned}$$

which is bounded in n by (4.4) in [K1998]. In view of Lemma 3.8 and (H6)$'$, $\widehat{u}$ must coincide with u on the set $\{|\widehat{u}| < \alpha\} \cap \{|u| < \alpha\}$ and hence the set $\{|\widehat{u}| < \alpha,\ |u| < \alpha,\ \varepsilon < \widehat{u} - u < 1\}$ is empty for almost everywhere. Thus

$$\lim_{n\to\infty}\int_{F_n}\frac{\left|\nabla\widehat{u}-\nabla u_n\right|^{t}}{\omega_1(\widehat{u}-u_n)^{t/\widehat{p}}}\,dx \;=\; 0. \qquad (3.11)$$

Thanks to (H5)$'$, we have

$$\begin{aligned}\chi_{F_n} &\langle a(\cdot,\widehat{u},\nabla\widehat{u}) - a(\cdot,u_n,\nabla u_n), \nabla G_\varepsilon(\widehat{u}-u_n)\rangle \\ &\geq \; -C\chi_{F_n}(h_n^{\widetilde{p}} + |\beta^0(\widehat{u})| + |\beta^0(u_n)|).\end{aligned}$$

From (4.4), (4.5) in [K1998] and Lemma 3.4 we find that the right-hand side of this inequality is bounded in $L^s(\Omega)$, uniformly with respect to n, whenever $1 < s < p_2/\widetilde{p}$. Therefore we can apply Fatou's lemma to obtain

$$\liminf_{n\to\infty} \int_{F_n} \langle a(x,\widehat{u},\nabla\widehat{u}) - a(x,u_n,\nabla u_n), \nabla G_\varepsilon(\widehat{u}-u_n)\rangle \, dx \;\geq\; 0. \tag{3.12}$$

Next, on the analogy of G_ε define the function θ_ε by

$$\theta_\varepsilon(r) \;=\; \int_\varepsilon^{r_\varepsilon} \omega_1(\sigma)^{-1/\widehat{p}}\, d\sigma \qquad \text{for } \; r \in \mathbf{R}.$$

Since $\theta_\varepsilon(\widehat{u}-u_n) \in \mathcal{T}_0^{1,p}(\Omega)$ by Lemma 3.2, Sobolev's embedding implies

$$\begin{aligned}\left\|\theta_\varepsilon(\widehat{u}-u_n)\right\|_{t^*}^t &\leq\; C\left\|\nabla\theta_\varepsilon(\widehat{u}-u_n)\right\|_t^t \\ &=\; C\int_{E_n\cup F_n} \frac{|\nabla\widehat{u}-\nabla u_n|^t}{\omega_1(\widehat{u}-u_n)^{t/\widehat{p}}}\, dx.\end{aligned} \tag{3.13}$$

Combining (3.8)-(3.13), we immediately calculate that

$$\left\|\theta_\varepsilon(\widehat{u}-u_n)\right\|_{t^*} \;\leq\; \liminf_{n\to\infty}\left\|\theta_\varepsilon(\widehat{u}-u_n)\right\|_{t^*} \;\leq\; C$$

with a finite constant C.

To prove $\widehat{u} \leq u$, we assume to the contrary that meas$\{\widehat{u}-u > \rho\}$ for some $\rho > 0$; it follows that for $0 < \varepsilon < \rho$

$$\theta_\varepsilon(\rho)\,(\text{meas}\{\widehat{u}-u > \rho\})^{1/r^*} \;\leq\; \left\|\theta_\varepsilon(\widehat{u}-u)\right\|_{r^*} \;\leq\; C.$$

But, $\theta_\varepsilon(\rho) \to \int_0^\rho \omega_1(\sigma)^{-1/\widehat{p}}\, d\sigma = \infty$ as $\varepsilon \downarrow 0$. This contradicts that C is finite. Thus we obtain $\widehat{u} \leq u$. Likewise we can prove the converse inequality and hence $\widehat{u} = u$. □

3.4 Application

Let us consider as an example of our theory the following problem

$$\begin{cases} v - \triangle_p \phi(v) + \operatorname{div} F(v) = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

where Ω is an open subset in $\mathbf{R}^N$ with $N \geq 2$, $\triangle_p$ is the so-called p-Laplacian, that is, $\triangle_p v = \operatorname{div}(|\nabla v|^{p-2} \nabla v)$, $\phi \in C(\mathbf{R})$ and $F \in C(\mathbf{R})^N$. This is the stationary problem corresponding to the associated degenerate parabolic equation with convection. See Gagneux and Madaune-Tort [GMT1994] for more details.

We make the assumptions that ϕ is nondecreasing, $\phi(\mathbf{R}) = \mathbf{R}$, $\phi(0) = 0$, $F(0) = 0$, and that there exists a constant C such that

$$|F(r) - F(s)| \leq C \, |\phi(r) - \phi(s)|^{1/\widehat{p}'} \, |r - s|^{1/q'}, \qquad r, \ s \in \mathbf{R}. \tag{3.15}$$

Here recall that $\widehat{p} = \max\{p, 2\}$ and $1 < p \leq q$. We see from (3.15) that for each $r \in \mathbf{R}$ the function F takes a constant value on the set $\phi^{-1}(r)$. Put

$$\beta = \phi^{-1} \qquad \text{and} \qquad h = F \circ \phi^{-1}.$$

Then β is maximal monotone in $\mathbf{R}^2$, $\beta(0) \ni 0$, $D(\beta) = \mathbf{R}$, $h \in C(\mathbf{R}, \mathbf{R}^N)$, and (3.14) may be written as

$$\begin{cases} \beta(u) - \triangle_p u + \operatorname{div} h(u) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

To apply our results above we furthermore assume that

$$\limsup_{r \to 0} \frac{|r|^q}{|\beta^0(r)|} < \infty \tag{3.17}$$

and

$$\sup_{|r-s|<k} |h(r) - h(s)| < \infty \qquad \text{for each } \ k > 0. \tag{3.18}$$

Then from (3.15) and (3.17) it is easily seen that

$$\lim_{r\to 0} \frac{\left|h(r)\right|^{q'}}{\left|\beta^0(r)\right|} = 0,$$

and hence condition (H1) holds. To check the rests of conditions postulated above put $a(x,r,\xi) = \left|\xi\right|^{p-2}\xi + h(r)$ and $\widetilde{p} = \max\{2-p, 0\}$ as before. An elementary calculation shows that

$$\left\langle \left|\xi\right|^{p-2}\xi - \left|\eta\right|^{p-2}\eta, \xi-\eta \right\rangle \;\geq\; \lambda_p \left(\left|\xi\right| + \left|\eta\right|\right)^{-\widetilde{p}} \left|\xi-\eta\right|^{\widehat{p}}, \quad \xi,\ \eta \in \mathbf{R}^N, \qquad (3.19)$$

the constant λ_p depending on p. By (3.15), (3.19) and Hölder's inequality, we have for $\xi \neq \eta$

$$\begin{aligned} &\langle a(x,r,\xi) - a(x,s,\eta), \xi-\eta \rangle \\ &\quad\geq\; \frac{\lambda_p \left|\xi-\eta\right|^{\widehat{p}}}{2(|\xi|+|\eta|)^{\widetilde{p}}} - C\left(|\xi|+|\eta|\right)^{\widetilde{p}} \left|h(r)-h(s)\right|^{\widehat{p}'} \\ &\quad\geq\; -C\left(|\xi|+|\eta|\right)^{\widetilde{p}} \left|\beta^0(r)-\beta^0(s)\right|^{\widehat{p}'/q'} \left|r-s\right| \\ &\quad\geq\; -C\, d(|r|+|s|) \left(\left|\xi\right|^p + \left|\eta\right|^p + \left|\beta^0(r)\right| + \left|\beta^0(s)\right|\right) \left|r-s\right| \end{aligned}$$

with some nondecreasing function $d : \mathbf{R}^+ \to \mathbf{R}^+$. This implies that condition (H3) holds. Furthermore, if (3.18) is satisfied then condition (H5) follows from the first inequality stated just above provided $\widetilde{p} < p_2$ or, equivalently,

$$\frac{3N-2}{2N-1} < p < N.$$

Since conditions (H2), (H4) and (H6) are immediately satisfied, we can conclude from Theorem 3.6 and Theorem 3.7 the following result.

Theorem 3.11. *Suppose* $1 < p < N$, $f \in L^1(\Omega)$ *and that* $\phi \in C(\mathbf{R})$ *and* $F \in C(\mathbf{R})^N$ *satisfy* $\phi(0) = 0$, $F(0) = 0$ *and* $\phi(\mathbf{R}) = \mathbf{R}$. *If* (3.15) *and* (3.17) *hold, then* (3.14) *admits at least one solution* v *in* $L^1(\Omega)$ *in the sense that* $u = \phi(v)$ *is a renormalized solution of* (3.16). *In addition, if*

$$\frac{3N-2}{2N-1} < p < N$$

and (3.18) *is satisfied, then the solution is unique.*

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Chapter 4

Renormalized dissipative solutions

In this chapter, we consider the Cauchy problem of a scalar conservation law (CP): $u_t + \operatorname{div} \mathbf{F}(u) = f,\ u(0,\cdot) = u_0$ with locally Lipschitz continuous $\mathbf{F}$. In the case that the flux $\mathbf{F}$ is globally Lipschitz continuous, Portilheiro introduced a notion of dissipative solutions for (CP) and proved the equivalence of such solutions and entropy solutions. The dissipative solutions are more suitable to obtain relaxation limits for some hyperbolic systems than entropy solutions. Indeed, Portilheiro used this notion to obtain certain relaxation limits for hyperbolic systems describing discrete velocity models and chemical reaction models. Our purpose of this chapter is to extend some results obtained by Portilheiro [P2003a, P2003b] to the case of locally Lipschitz-continuous flux. We introduce a new notion of renormalized dissipative solutions which is a generalization of dissipative solutions in the sense of Portilheiro for a scalar conservation law (CP) with locally Lipschitz $\mathbf{F}$ and L^1 data, and show the equivalence of such solutions and renormalized entropy solutions in the sense of Benilan et al. As an example, we apply our result to contractive relaxation systems in merely an L^1-setting and construct a renormalized dissipative solution via relaxation. Contents of this chapter is based on the paper [KoT2005] which is a joint work with Professor Kazuo Kobayasi. This research was supported by Waseda University Grant for

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4.1 Introduction

We consider the following Cauchy problem

$$\text{(CP)}\qquad \begin{cases} u_t + \operatorname{div}\mathbf{F}(u) &= f \quad \text{in} \quad Q := (0,T)\times\mathbf{R}^N, \\ u(0,\cdot) &= u_0 \quad \text{in} \quad \mathbf{R}^N, \end{cases}$$

where $T > 0$ and $N \geq 1$. Here $f \in L^1(Q)$ and $u_0 \in L^1(\mathbf{R}^N)$ are given functions and the flux $\mathbf{F} : \mathbf{R} \to \mathbf{R}^N$ is a locally Lipschitz continuous function.

Kružkov [Kr1970] proved that if $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, then (CP) has a unique weak solution $u \in C([0,T); L^1(\mathbf{R}^N)) \cap L^\infty(Q)$ satisfying the entropy inequalities, which is the so-called entropy solution. In the case that the flux $\mathbf{F}$ is globally Lipschitz, Portilheiro introduced a notion of dissipative solutions for (CP) and proved the equivalence of such solutions and entropy solutions. The relationship between the notions of various solutions for degenerate parabolic equations is also investigated in Kobayasi [Ko2003], and Perthame and Souganidis [PS2003]. The dissipative solutions are more suitable to obtain relaxation limits for some hyperbolic systems than entropy solutions. Indeed, Portilheiro [P2003b] used this notion to obtain certain relaxation limits for hyperbolic systems describing discrete velocity models and chemical reaction models. His idea is also based on the perturbed test function method introduced by Evans [E1989] for conservation laws. These systems have already been studied by Katsoulakis and Tzavaras [KaT1997, KaT1999], who obtained several important results including comparison results. On the other hand, it is known that if $f \in L^1(Q)$ and $u_0 \in L^1(\mathbf{R}^N)$, then the mild solution u of (CP) constructed by nonlinear semigroup theory is a unique entropy solution which is unbounded in general. In the case that $\mathbf{F}$ is only locally Lipschitz, the flux function $\mathbf{F}(u)$ may fail to be locally integrable since no growth condition is assumed on the flux $\mathbf{F}$, and hence (CP) does not possess a solution even in the sense of distributions. To overcome this the notion of renormalized entropy solutions has been introduced by Bénilan et al. [BCW2000], where the existence and uniqueness of a renormalized entropy solution of (CP) has been established and the semigroup solutions of (CP) in L^1 spaces are characterized. Renormalized solutions have been introduced first by DiPerna and Lions [DPL1989] for the Boltzmann equation and utilized for degenerate elliptic and parabolic problems in the L^1-setting in the last decade. However, the argument in Portilheiro [P2003a] does not work well in the case that $\mathbf{F}$ is only locally Lipschitz and the solution u is unbounded.

Our purpose of this chapter is to extend some results in [P2003a] to the case of locally Lipschitz continuous flux. In Section 2, we introduce a new notion of renormalized dissipative solutions which is a generalization of dissipative solutions in [P2003a], and we prove that the equivalence of renormalized dissipative solutions and renormalized entropy solutions in Section 3. In Section 4, as an application, we apply our result to contractive relaxation systems in merely an L^1-setting and construct a renormalized dissipative solution via relaxation.

4.2 Equivalence

We begin with some notations and definitions. For $r, s \in \mathbf{R}$, we set $r \wedge s := \min(r, s)$, $r \vee s := \max(r, s)$, $r^+ := r \vee 0$ and $r^- := (-r) \vee 0$. For $r \in \mathbf{R}$ and $j = 0, 1$, we define a sign function S_j by $S_j(r) = 1$ if $r > 0$, $S_j(r) = -1$ if $r < 0$, $S_j(0) = j$. Then we denote $S_j^+(r) := S_j(r) \vee 0$ and $S_j^-(r) := S_j(r) \wedge 0$.

Let $u \in L^1(Q)$. For $(t, x) \in Q$ and $r > 0$, we set

$$B_r(t,x) \, := \, \big\{(s,y) \in Q\,;\, (s-t)^2 + \big|y-x\big|^2 \leq r^2\big\},$$

and define the upper and lower semicontinuous envelopes of u as

$$u^*(t,x) \, := \, \lim_{r \downarrow 0} \sup \big\{u(s,y)\,;\, (s,y) \in B_r(t,x)\big\}$$

and

$$u_*(t,x) \, := \, \lim_{r \downarrow 0} \inf \big\{u(s,y)\,;\, (s,y) \in B_r(t,x)\big\},$$

respectively. Then we see that $u_* \leq u \leq u^*$, u^* is upper semicontinuous and u_* is lower semicontinuous.

We now recall from Bénilan et al. [BCW2000] the definition of renormalized entropy solutions.

Definition 4.1. (i) *We say* $u \in L^1(Q)$ *is a* renormalized entropy subsolution *of* (CP) *if for any* $k, \ell \in \mathbf{R}$,

$$\begin{aligned} \mu_{k,\ell} \;&:=\; (u \wedge \ell - k)^+_{\;t} + \operatorname{div}\left\{S_0^+(u \wedge \ell - k)\,(\mathbf{F}(u \wedge \ell) - \mathbf{F}(k))\right\} \\ &\quad - S_0^+(u \wedge \ell - k)\, f \end{aligned} \tag{4.1}$$

is a Radon measure on Q *such that for each* $k \in \mathbf{R}$,

$$\lim_{\ell \to \infty} \mu^+_{k,\ell}(Q) \;=\; 0,$$

and for each $\ell \in \mathbf{R}$,

$$(u(t,\cdot) \wedge \ell - u_0 \wedge \ell)^+ \;\to\; 0 \;\text{ in }\; L^1_{loc}(\mathbf{R}^N) \;\text{ as }\; t \to 0 \;\text{ essentially.}$$

(ii) *We say* $u \in L^1(Q)$ *is a* renormalized entropy supersolution *of* (CP) *if for any* $k, \ell \in \mathbf{R}$,

$$\begin{aligned} \nu_{k,\ell} \;&:=\; (u \vee \ell - k)^-_{\;t} + \operatorname{div}\left\{S_0^-(u \vee \ell - k)\,(\mathbf{F}(u \vee \ell) - \mathbf{F}(k))\right\} \\ &\quad - S_0^-(u \vee \ell - k)\, f \end{aligned} \tag{4.2}$$

is a Radon measure on Q *such that for each* $k \in \mathbf{R}$,

$$\lim_{\ell \to -\infty} \nu^+_{k,\ell}(Q) \;=\; 0,$$

and for each $\ell \in \mathbf{R}$,

$$(u(t,\cdot) \vee \ell - u_0 \vee \ell)^- \;\to\; 0 \;\text{ in }\; L^1_{loc}(\mathbf{R}^N) \;\text{ as }\; t \to 0 \;\text{ essentially.}$$

(iii) *We say* $u \in L^1(Q)$ *is a* renormalized entropy solution *of* (CP) *if* u *is a renormalized entropy subsolution of* (CP) *and also a renormalized entropy supersolution of* (CP).

Next, we introduce a new notion of renormalized dissipative solutions of (CP).

Definition 4.2. (i) *We say* $u \in L^1(Q)$ *is a* renormalized dissipative subsolution *of* (CP) *if there is a sequence* $\{\mu_\ell\} \subset \mathcal{M}_b(Q)^+$ *with* $\mu_\ell(Q) \to 0$ *as* $\ell \to \infty$ *such that for each* $\ell \geq 1$ *and* $\phi \in \mathcal{T}_\ell$,

$$\iint_Q S_0^+(u \wedge \ell - \phi)\,(f - \phi_t - \operatorname{div}\mathbf{F}(\phi))\,dxdt + \iint_Q S_0^+(u^* \wedge \ell - \phi)\,d\mu_\ell \;\geq\; 0 \tag{4.3}$$

and

$$(u(t,\cdot) \wedge \ell - u_0 \wedge \ell)^+ \;\to\; 0 \;\text{ in }\; L^1_{loc}(\mathbf{R}^N) \;\text{ as }\; t \to 0 \;\text{ essentially,}$$

where $\mathcal{T}_\ell := C_0^1(Q) \cap \{\phi\,;\,\phi(t,x) \equiv k \text{ for } (t,x) \in Q \text{ if } |x| > R \text{ for some } k \in (-\ell,\ell) \text{ and } R > 0\}$ *and* $\mathcal{M}_b(Q)^+$ *denotes the space of all nonnegative bounded measures on* Q.

(ii) *We say* $u \in L^1(Q)$ *is a* renormalized dissipative supersolution *of* (CP) *if there is a sequence* $\{\nu_\ell\} \subset \mathcal{M}_b(Q)^+$ *with* $\nu_\ell(Q) \to 0$ *as* $\ell \to \infty$ *such that for each* $\ell \geq 1$ *and* $\phi \in \mathcal{T}_\ell$,

$$\iint_Q S_0^-(u \vee (-\ell) - \phi)\,(f - \phi_t - \operatorname{div}\mathbf{F}(\phi))\,dxdt + \iint_Q S_0^-(u_* \vee (-\ell) - \phi)\,d\nu_\ell \;\geq\; 0 \tag{4.4}$$

and

$$(u(t,\cdot) \vee \ell - u_0 \vee \ell)^- \;\to\; 0 \;\text{ in }\; L^1_{loc}(\mathbf{R}^N) \;\text{ as }\; t \to 0 \;\text{ essentially.}$$

(iii) *We say* $u \in L^1(Q)$ *is a* renormalized dissipative solution *of* (CP) *if* u *is a renormalized dissipative subsolution of* (CP) *and also a renormalized dissipative supersolution of* (CP).

Then we obtain the following main result.

Theorem 4.3. *Suppose that* $u \in L^1(Q)$ *and* $u^*(t,x) < \infty$ *and* $u_*(t,x) > -\infty$ *for almost every* $(t,x) \in Q$. *Then* u *is a renormalized entropy subsolution (respectively supersolution) of* (CP) *if and only if* u *is a renormalized dissipative subsolution (respectively supersolution) of* (CP).

4.3 Proof of Theorem 4.3

Claim 1: *If* $u \in L^1(Q)$ *and* $u^*(t,x) < \infty$ *(respectively* $u_*(t,x) > -\infty$*) for almost every* $(t,x) \in Q$*, then a renormalized entropy subsolution (respectively supersolution)* u *of* (CP) *implies a renormalized dissipative subsolution (respectively supersolution).*

The proof of Claim 1 will be divided by several parts.

Step 1: It follows from [BCW2000, Proposition 2.7] that there exists a sequence $\{\mu_\ell\} \subset \mathcal{M}_b(Q)^+$ such that $\mu_\ell(Q) \to 0$ as $\ell \to \infty$ and $\mu_{k,\ell} = \mu_\ell - \mu_k - \chi_{\{u>\ell\}} f$ for $k < \ell$, where χ_A denotes the indicator function of A. Then we have

$$\iint_{u^* \wedge \ell < k} \theta \, d\mu_\ell = \iint_{u^* \wedge \ell < k} \theta \, d\mu_k$$

for each $\theta \in C_0^\infty(Q)$. Indeed, since $u^* \wedge \ell$ is upper semicontinuous, $\{u^* \wedge \ell < k\}$ is open and hence for any $\varphi \in C_0^\infty(\{u^* \wedge \ell < k\})$,

$$\begin{aligned} &\iint_Q \varphi \, d\mu_{k,\ell} \\ &\quad = -\iint_Q S_0^+(u \wedge \ell - k) \Big\{ (u \wedge \ell - k)\, \varphi_t + (\mathbf{F}(u \wedge \ell) - \mathbf{F}(k)) \cdot \nabla \varphi + f\, \varphi \Big\} \, dx dt. \end{aligned}$$

On the other hand, since $\{u > \ell\} = \emptyset$ whenever $u^* \wedge \ell < k$ and $k < \ell$, we have

$$\begin{aligned} \iint_Q \varphi \, d\mu_{k,\ell} &= \iint_Q \varphi \, d\mu_\ell - \iint_Q \varphi \, d\mu_k - \iint_{u>\ell} f\, \varphi \, dx dt \\ &= \iint_Q \varphi \, d\mu_\ell - \iint_Q \varphi \, d\mu_k. \end{aligned}$$

Therefore, we obtain that

$$\iint_Q \varphi\,d\mu_k - \iint_Q \varphi\,d\mu_\ell$$
$$= \iint_Q S_0^+(u\wedge\ell - k)\Big\{(u\wedge\ell-k)\,\varphi_t + (\mathbf{F}(u\wedge\ell)-\mathbf{F}(k))\cdot\nabla\varphi + f\varphi\Big\}dxdt. \quad (4.5)$$

We now use the partition of unity (see Yosida [Y1965]). Since $\{u^*\wedge\ell < k\}$ is open, there exists a system of functions $\{\sigma_j\}\subset C_0^\infty(Q)$ such that

$$\operatorname{spt}\sigma_j \subset \{u^*\wedge\ell < k\}$$

for each j, $0\le\sigma_j(t,x)\le 1$ for every j and $\sum_j \sigma_j(t,x) = 1$ for $(t,x)\in\{u^*\wedge\ell<k\}$. Let $\theta\in C_0^\infty(Q)$ and put $\theta\,\sigma_j$ into φ in (4.5). Then, since

$$S_0^+(u\wedge\ell-k) = 0 \qquad \text{for any} \quad (t,x)\in\{u^*\wedge\ell<k\}$$

and $\operatorname{spt}(\theta\,\sigma_j)\subset\{u^*\wedge\ell<k\}$, we see that

$$\iint_{u^*\wedge\ell<k}\theta\,\sigma_j\,d\mu_k - \iint_{u^*\wedge\ell<k}\theta\,\sigma_j\,d\mu_\ell \;=\; 0.$$

Therefore, summing up with respect to j, we get

$$\iint_{u^*\wedge\ell<k}\theta\,d\mu_\ell \;=\; \iint_{u^*\wedge\ell<k}\theta\,d\mu_k \quad (4.6)$$

for each $\theta\in C_0^\infty(Q)$. Then, from (4.6) we have for any $\theta\in C_0^\infty(Q)^+$ and $k<\ell$,

$$\begin{aligned}\iint_Q\theta\,d\mu_{k,\ell} &= \iint_Q\theta\,d\mu_\ell - \iint_Q\theta\,d\mu_k - \iint_Q\chi_{\{u>\ell\}}\,f\,\theta\,dxdt\\ &= \iint_{u^*\wedge\ell\ge k}\theta\,d\mu_\ell - \iint_{u^*\wedge\ell\ge k}\theta\,d\mu_k - \iint_{u^*\wedge\ell>k}\chi_{\{u>\ell\}}\,f\,\theta\,dxdt\\ &\le \iint_{u^*\wedge\ell\ge k}\theta\,d\mu_\ell - \iint_{u^*\wedge\ell>k}\chi_{\{u>\ell\}}\,f\,\theta\,dxdt.\end{aligned}$$

We now check that

$$\iint_{u^*\wedge\ell=k}\theta\,d\mu_\ell \;=\; 0. \quad (4.7)$$

To this end, let C be a countable subset of $(-\ell,\ell)$. Then, we have

$$\begin{aligned}\sum_{k\in C}|k|\,\mu_\ell(\{u^*\wedge\ell=k\}) &= \sum_{k\in C}|k|\iint_{u^*\wedge\ell=k}d\mu_\ell\\ &\le \iint_Q|u^*\wedge\ell|\,d\mu_\ell\\ &\le \iint_Q|u^*|\,d\mu_\ell\;<\;\infty.\end{aligned}$$

From this, we see that the cardinality of the set

$$\big\{k\in(-\ell,\ell)\,;\,\mu_\ell(\{u^*\wedge\ell=k\})>0\big\}$$

is at most countable, and therefore the set

$$\big\{k\in(-\ell,\ell)\,;\,\mu_\ell(\{u^*\wedge\ell=k\})=0\big\}$$

is dense in $(-\ell,\ell)$. This means that (4.7) holds.

Combining these estimates, we obtain that

$$\begin{aligned}&\iint_Q S_0^+(u\wedge\ell-k)\Big\{(u\wedge\ell-k)\,\theta_t+(\mathbf{F}(u\wedge\ell)-\mathbf{F}(k))\cdot\nabla\theta+f\,\theta\Big\}\,dxdt\\ &\quad= -\iint_Q\theta\,d\mu_{k,\ell}\\ &\quad\ge -\iint_{u^*\wedge\ell>k}\theta\,d\mu_\ell+\iint_{u^*\wedge\ell>k}\chi_{\{u>\ell\}}\,f\,\theta\,dxdt,\end{aligned}$$

which implies

$$\begin{aligned}&\iint_Q S_0^+(u\wedge\ell-k)\Big\{(u\wedge\ell-k)\,\theta_t+(\mathbf{F}(u\wedge\ell)-\mathbf{F}(k))\cdot\nabla\theta+f\,\theta\Big\}\,dxdt\\ &\qquad+\iint_Q S_0^+(u^*\wedge\ell-k)\Big\{\theta\,d\mu_\ell-\chi_{\{u>\ell\}}\,f\,\theta\,dxdt\Big\}\;\ge\;0 \qquad (4.8)\end{aligned}$$

for any $\theta\in C_0^\infty(Q)^+$ and $k,\ell\in\mathbf{R}$.

Step 2: Let η_ε and ρ_λ be standard mollifiers on $\mathbf{R}^N$ and $\mathbf{R}$, respectively, and let ζ_n be a nonnegative smooth function satisfying

$$\zeta_n(t,x)\;:=\;\begin{cases}1 & \text{if }\;|x|\le n,\\ 0 & \text{if }\;|x|\ge 2n,\end{cases}$$

and $|\nabla\zeta_n| \le C/n$ with positive constant C. Take $\phi \in \mathcal{T}_\ell$. Then we put $\theta = \eta_\varepsilon(x-y)\,\rho_\lambda(t-s)\,\zeta_n(t,x)$ and $k = \phi(s,y)$ in (4.8), and integrate in s and y over Q to obtain

$$\begin{aligned}
0 \;\le\;& \iiiint_{Q^2} S_0^+(u\wedge\ell-\phi)\Big\{(u\wedge\ell-\phi)\,(\eta_\varepsilon\,\rho_\lambda\,\zeta_n)_t + f\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n \\
&\qquad\qquad + (\mathbf{F}(u\wedge\ell)-\mathbf{F}(\phi))\cdot\nabla_x(\eta_\varepsilon\,\rho_\lambda\,\zeta_n)\Big\}\,dydsdxdt \\
&+ \iiiint_{Q^2} S_0^+(u^*\wedge\ell-\phi)\Big\{\eta_\varepsilon\,\rho_\lambda\,\zeta_n\,d\mu_\ell - \chi_{\{u>\ell\}}\,f\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n\,dxdt\Big\}\,dyds \\
=\;& \iiiint_{Q^2} S_0^+(u\wedge\ell-\phi)\Big[(u\wedge\ell-\phi)\,\eta_\varepsilon\,\rho_\lambda\,(\zeta_n)_t - \eta_\varepsilon\,\rho_\lambda\,\zeta_n\,\phi_s \\
&\qquad - \Big((u\wedge\ell-\phi)\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n\Big)_s + f\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n \\
&\qquad - \mathrm{div}_y\mathbf{F}(\phi)\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n + \eta_\varepsilon\,\rho_\lambda\,(\mathbf{F}(u\wedge\ell)-\mathbf{F}(\phi))\cdot\nabla_x\zeta_n \\
&\qquad - \mathrm{div}_y\big\{(\mathbf{F}(u\wedge\ell)-\mathbf{F}(\phi))\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n\big\}\Big]\,dydsdxdt \\
&+ \iiiint_{Q^2} S_0^+(u^*\wedge\ell-\phi)\Big\{\eta_\varepsilon\,\rho_\lambda\,\zeta_n\,d\mu_\ell - \chi_{\{u>\ell\}}\,f\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n\,dxdt\Big\}\,dyds \\
=:\;& \sum_{j=1}^{9} I_j^{\varepsilon,\lambda,n}.
\end{aligned} \tag{4.9}$$

We begin with $I_4^{\varepsilon,\lambda,n}$. For $p,q>0$ we set

$$\Phi(p,q) := \sup\Big\{\big|\phi(t,x)-\phi(s,y)\big|\,;\,(t,x),(s,y)\in Q,\ \big|t-s\big|\le p,\ \big|x-y\big|\le q\Big\}.$$

Then we have

$$\begin{aligned}
I_4^{\varepsilon,\lambda,n} \;=\;& \iiiint_{f\ge 0} S_0^+(u\wedge\ell-\phi(s,y))\,f\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n\,dydsdxdt \\
&+ \iiiint_{f<0} S_0^+(u\wedge\ell-\phi(s,y))\,f\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n\,dydsdxdt \\
\le\;& \iiiint_{f\ge 0} S_0^+(u\wedge\ell-\phi(t,x)+\Phi(\lambda,\varepsilon))\,f\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n\,dydsdxdt \\
&+ \iiiint_{f<0} S_0^+(u\wedge\ell-\phi(t,x)-\Phi(\lambda,\varepsilon))\,f\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n\,dydsdxdt,
\end{aligned}$$

which implies

$$\begin{aligned}\limsup_{\varepsilon,\,\lambda\downarrow 0} I_4^{\varepsilon,\lambda,n} &\le \iint_{f\ge 0} S_1^+(u\wedge\ell-\phi)\,f\,\zeta_n\,dxdt + \iint_{f<0} S_0^+(u\wedge\ell-\phi)\,f\,\zeta_n\,dxdt\\ &= \iint_Q S_0^+(u\wedge\ell-\phi)\,f\,\zeta_n\,dxdt + \iint_{u\wedge\ell=\phi} f^+\,\zeta_n\,dxdt.\end{aligned}$$

In a similar way we also have that

$$\begin{aligned}\limsup_{\varepsilon,\,\lambda\downarrow 0} I_1^{\varepsilon,\lambda,n} &\le \iint_Q S_0^+(u\wedge\ell-\phi)\,(u\wedge\ell-\phi)\,(\zeta_n)_t\,dxdt,\\ \limsup_{\varepsilon,\,\lambda\downarrow 0} I_2^{\varepsilon,\lambda,n} &\le -\iint_Q S_0^+(u\wedge\ell-\phi)\,\phi_t\,\zeta_n\,dxdt\\ &\quad + \iint_{u\wedge\ell=\phi} \big(\phi_t\big)^-\,\zeta_n\,dxdt,\\ \limsup_{\varepsilon,\,\lambda\downarrow 0} I_5^{\varepsilon,\lambda,n} &\le -\iint_Q S_0^+(u\wedge\ell-\phi)\,\mathrm{div}\,\mathbf{F}(\phi)\,\zeta_n\,dxdt\\ &\quad + \iint_{u\wedge\ell=\phi} \big(\mathrm{div}\,\mathbf{F}(\phi)\big)^-\,\zeta_n\,dxdt,\\ \limsup_{\varepsilon,\,\lambda\downarrow 0} I_6^{\varepsilon,\lambda,n} &\le \iint_Q S_0^+(u\wedge\ell-\phi)\,(\mathbf{F}(u\wedge\ell)-\mathbf{F}(\phi))\cdot\nabla\zeta_n\,dxdt\\ &\quad + \iint_{u\wedge\ell=\phi} \big((\mathbf{F}(u\wedge\ell)-\mathbf{F}(\phi))\cdot\nabla\zeta_n\big)^+\,dxdt\\ &= \iint_Q S_0^+(u\wedge\ell-\phi)\,(\mathbf{F}(u\wedge\ell)-\mathbf{F}(\phi))\cdot\nabla\zeta_n\,dxdt,\\ \limsup_{\varepsilon,\,\lambda\downarrow 0} I_8^{\varepsilon,\lambda,n} &\le \iint_Q S_0^+(u^*\wedge\ell-\phi)\,\zeta_n\,d\mu_\ell + \iint_{u^*\wedge\ell=\phi} \zeta_n\,d\mu_\ell,\end{aligned}$$

and

$$\begin{aligned}&\limsup_{\varepsilon,\,\lambda\downarrow 0} I_9^{\varepsilon,\lambda,n}\\ &\quad\le \iint_Q S_0^+(u^*\wedge\ell-\phi)\,\chi_{\{u>\ell\}}\,f\,\zeta_n\,dxdt + \iint_{u^*\wedge\ell=\phi} \big(\chi_{\{u>\ell\}}\,f\big)^+\,\zeta_n\,dxdt.\end{aligned}$$

As to $I_7^{\varepsilon,\lambda,n}$, we introduce a sequence $\{\alpha_m\}\subset C^1(\mathbf{R})$ with

$$0 \le \alpha_m'(r) \le C\,m\,\chi_{\{|r|\le 1/m\}}$$

which approximates $S_0^+(r)$. Then we have

$$\begin{aligned} I_7^{\varepsilon,\lambda,n} &= -\iiiint_{Q^2} S_0^+(u\wedge\ell-\phi)\,\mathrm{div}_y\big\{(\mathbf{F}(u\wedge\ell)-\mathbf{F}(\phi))\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n\big\}\,dydsdxdt \\ &= -\lim_{m\to\infty}\iiiint_{Q^2} \alpha_m(u\wedge\ell-\phi)\,\mathrm{div}_y\big\{(\mathbf{F}(u\wedge\ell)-\mathbf{F}(\phi))\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n\big\}\,dydsdxdt \\ &= -\lim_{m\to\infty}\iiiint_{Q^2} \alpha_m'(u\wedge\ell-\phi)\,\eta_\varepsilon\,\rho_\lambda\,\zeta_n\,(\mathbf{F}(u\wedge\ell)-\mathbf{F}(\phi))\cdot\nabla_y\phi\,dydsdxdt. \end{aligned}$$

Let us denote by L_ℓ the Lipschitz constant of $\mathbf{F}$ on $[-\ell,\ell]$. Then, for any $(t,x)\in Q$ and m large enough, we get

$$\begin{aligned} &\big|\alpha_m(u\wedge\ell-\phi)\,(\mathbf{F}(u\wedge\ell)-\mathbf{F}(\phi))\big|\,\big|\nabla_y\phi\big| \\ &\qquad\le\; C\,m\,\chi_{\{|u\wedge\ell-\phi|\le 1/m\}}\,L_\ell\,\big|u\wedge\ell-\phi\big|\,\big|\nabla_y\phi\big| \\ &\qquad\le\; C\,L_\ell\,\chi_{\{|u\wedge\ell-\phi|\le 1/m\}}\,\big|\nabla_y(u\wedge\ell-\phi)\big| \\ &\qquad\to\; C\,L_\ell\,\chi_{\{|u\wedge\ell-\phi|=0\}}\,\big|\nabla_y(u\wedge\ell-\phi)\big| \;=\; 0 \qquad \text{as} \quad m\to\infty \end{aligned}$$

and hence $I_7^{\varepsilon,\lambda,n}=0$. Similarly, we also get $I_3^{\varepsilon,\lambda,n}=0$. Passing to the limit in (4.9) as ε, $\lambda\downarrow 0$ gives

$$\begin{aligned} 0 \;\le\; & \iint_Q S_0^+(u\wedge\ell-\phi)\,(u\wedge\ell-\phi)\,(\zeta_n)_t\,dxdt \\ & -\iint_Q S_0^+(u\wedge\ell-\phi)\,\phi_t\,\zeta_n\,dxdt + \iint_{u\wedge\ell=\phi}\big(\phi_t\big)^-\,\zeta_n\,dxdt \\ & +\iint_Q S_0^+(u\wedge\ell-\phi)\,f\,\zeta_n\,dxdt + \iint_{u\wedge\ell=\phi} f^+\,\zeta_n\,dxdt \\ & -\iint_Q S_0^+(u\wedge\ell-\phi)\,\mathrm{div}\,\mathbf{F}(\phi)\,\zeta_n\,dxdt + \iint_{u\wedge\ell=\phi}\big(\mathrm{div}\,\mathbf{F}(\phi)\big)^-\,\zeta_n\,dxdt \\ & +\iint_Q S_0^+(u\wedge\ell-\phi)\,(\mathbf{F}(u\wedge\ell)-\mathbf{F}(\phi))\cdot\nabla\zeta_n\,dxdt \\ & +\iint_Q S_0^+(u^*\wedge\ell-\phi)\,\zeta_n\,d\mu_\ell + \iint_{u^*\wedge\ell=\phi}\zeta_n\,d\mu_\ell \\ & +\iint_Q S_0^+(u^*\wedge\ell-\phi)\,\chi_{\{u>\ell\}}\,f\,\zeta_n\,dxdt + \iint_{u^*\wedge\ell=\phi}\big(\chi_{\{u>\ell\}}\,f\big)^+\,\zeta_n\,dxdt. \end{aligned}$$

Passing to the limit as $n \to \infty$, we have

$$\begin{aligned}
0 \;\leq\; & \iint_Q S_0^+(u \wedge \ell - \phi)\,(f - \phi_t - \operatorname{div}\mathbf{F}(\phi))\,dxdt \\
& + \iint_Q S_0^+(u^* \wedge \ell - \phi)\,\Big\{d\mu_\ell + \chi_{\{u>\ell\}}\,f\,dxdt\Big\} \\
& + \iint_{u\wedge\ell=\phi} \Big\{f^+ + \big(\phi_t\big)^- + \big(\operatorname{div}\mathbf{F}(\phi)\big)^-\Big\}\,dxdt \\
& + \iint_{u^*\wedge\ell=\phi} \Big\{d\mu_\ell + \big(\chi_{\{u>\ell\}}\,f\big)^+\,dxdt\Big\} \\
& + \limsup_{n\to\infty} \iint_Q S_0^+(u \wedge \ell - \phi)\,(u \wedge \ell - \phi)\,(\zeta_n)_t\,dxdt \\
& + \limsup_{n\to\infty} \iint_Q S_0^+(u \wedge \ell - \phi)\,(\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi))\cdot\nabla\zeta_n\,dxdt. \quad (4.10)
\end{aligned}$$

Step 3: We calculate the last term on the right hand in (4.10). Since $S_0^+(u \wedge \ell - \phi)\,\mathbf{F}(u \wedge \ell) \in L^1(Q)^N$, we see that

$$\lim_{n\to\infty} \iint_Q S_0^+(u \wedge \ell - \phi)\,\mathbf{F}(u \wedge \ell)\cdot\nabla\zeta_n\,dxdt \;=\; 0.$$

Suppose that $\phi(t,x) \equiv k$ for large $|x|$. If $k = 0$, then $\phi \in L^1(Q)$, and hence

$$\lim_{n\to\infty} \iint_Q S_0^+(u \wedge \ell - \phi)\,\mathbf{F}(\phi)\cdot\nabla\zeta_n\,dxdt \;=\; 0.$$

We have

$$\begin{aligned}
& \iint_Q S_0^+(u \wedge \ell - \phi)\,\mathbf{F}(\phi)\cdot\nabla\zeta_n\,dxdt \\
& \quad = \iint_Q S_0^+(u \wedge \ell - \phi)\,(\mathbf{F}(\phi) - \mathbf{F}(k))\cdot\nabla\zeta_n\,dxdt \\
& \qquad + \iint_Q S_0^+(u \wedge \ell - \phi)\,\mathbf{F}(k)\cdot\nabla\zeta_n\,dxdt \\
& \quad = \iint_Q S_0^+(u \wedge \ell - \phi)\,(\mathbf{F}(\phi) - \mathbf{F}(k))\cdot\nabla\zeta_n\,dxdt + \iint_{u\wedge\ell>\phi} \mathbf{F}(k)\cdot\nabla\zeta_n\,dxdt.
\end{aligned}$$

The first integral converges to 0 as $n \to \infty$ since $\mathbf{F}(\phi) - \mathbf{F}(k) \in L^1(Q)^N$. As to the second integral, we first note that for fixed $k \in (-\ell, \ell)$ and $\xi \in C_0^\infty(Q)$ we may write $\phi = \xi + k$. Assume $0 < k < \ell$. Then by Chebyshev's inequality we get

$$\begin{aligned}\mathcal{L}^{N+1}(\{u \wedge \ell > \phi\}) &= \mathcal{L}^{N+1}(\{u \wedge \ell - \xi > k\}) \\ &\leq \frac{1}{k} \iint_Q \big|u \wedge \ell - \xi\big|\, dxdt \;<\; \infty,\end{aligned}$$

where $\mathcal{L}^{N+1}$ denotes the $(N+1)$-dimensional Lebesgue measure on Q. Hence this integral converges to 0 as $n \to \infty$.

Next assume that $-\ell < k < 0$. Since the second integral equals

$$-\iint_{u\wedge\ell\leq\phi} \mathbf{F}(k)\cdot\nabla\zeta_n \, dxdt,$$

we have

$$\begin{aligned}\mathcal{L}^{N+1}(\{u \wedge \ell \leq \phi\}) &= \mathcal{L}^{N+1}(\{u \wedge \ell - \xi \leq k\}) \\ &\leq \frac{1}{|k|} \iint_Q \big|u \wedge \ell - \xi\big|\, dxdt \;<\; \infty.\end{aligned}$$

Therefore, the second integral also converges to 0 as $n \to \infty$. Thus we obtain

$$\lim_{n\to\infty} \iint_Q S_0^+(u \wedge \ell - \phi)\, \mathbf{F}(\phi)\cdot\nabla\zeta_n \, dxdt \;=\; 0$$

for $\phi \in \mathcal{T}_\ell$. In a similar way, we also see that

$$\lim_{n\to\infty} \iint_Q S_0^+(u \wedge \ell - \phi)\, (u \wedge \ell - \phi)\, (\zeta_n)_t \, dxdt \;=\; 0$$

for $\phi \in \mathcal{T}_\ell$.

Step 4: Recall that we may write $\phi = \xi + k$ for $k \in (-\ell, \ell)$ and $\xi \in C_0^\infty(Q)$. Then we see that the set $\big\{k \in (-\ell,\ell)\,;\, \mu(\{u\wedge\ell = \xi + k\}) = 0\big\}$ is dense in $(-\ell,\ell)$ because $\sum_{k\in C} |k|\, \mu(\{u \wedge \ell = \xi + k\})$ is finite for any countable set $C \subset (-\ell, \ell)$, where μ denotes the $(N+1)$-dimensional Lebesgue measure $\mathcal{L}^{N+1}$ or μ_ℓ. Hence the cardinality of the set $\big\{k \in (-\ell,\ell)\,;\, \mu(\{u \wedge \ell = \xi + k\}) > 0\big\}$ is at most countable.

We now fix any $k \in (-\ell, \ell)$ and choose a sequence $\{k_n^+\}$ such that $k_n^+ \downarrow k$ as $n \to \infty$ and $\mu(\{u \wedge \ell = \xi + k_n^+\}) = 0$ for any $n \geq 1$. It follows from (4.10) with

$\phi = \xi + k_n^+$ that

$$\begin{aligned} 0 \;\le\; & \iint_Q S_0^+(u\wedge\ell-\phi)\,(f-\phi_t-\operatorname{div}\mathbf{F}(\phi))\,dxdt \\ & + \iint_Q S_0^+(u^*\wedge\ell-\phi)\,\Big\{d\mu_\ell+\chi_{\{u>\ell\}}\,f\,dxdt\Big\} \end{aligned}$$

which means that (4.3) holds with $d\mu_\ell$ replaced by $d\mu_\ell+\chi_{\{u>\ell\}}f\,dxdt$.

Claim 2: *If $u\in L^1(Q)$ and $u^*(t,x)<\infty$ for a.e. $(t,x)\in Q$, then a renormalized dissipative subsolution u of* (CP) *implies a renormalized entropy subsolution.*

The proof of Claim 2 will be also divided by several parts.

Step 5: Let $k\in(-\ell,\ell)$ and $\theta\in C_0^\infty(Q)^+$. For each $\delta,\varepsilon>0$, choose a function $\psi_{\delta,\varepsilon}\in C_0^\infty(Q)$ such that

$$\psi_{\delta,\,\varepsilon}(t,x) \;:=\; \begin{cases} 0 & \text{if } \;(t,x)\in B_\delta(0,0), \\ 1/\varepsilon & \text{if } \;(t,x)\in Q\setminus B_{\delta+\varepsilon}(0,0), \end{cases}$$

where $B_\delta(0,0)=\big\{(t,x)\in Q\,;\,t^2+|x|^2\le\delta^2\big\}$. Putting $\phi_{\delta,\varepsilon}:=k+\psi_{\delta,\varepsilon}(t-s,x-y)$ in (4.3) for each $(s,y)\in Q$, multiplying (4.3) by an arbitrary function $\theta(s,y)\in C_0^\infty(Q)^+$ and integrating in s and y over Q, we obtain

$$\begin{aligned} & \iiiint_{Q^2} S_0^+(u\wedge\ell-\phi_{\delta,\varepsilon})\,\Big\{f-(\phi_{\delta,\varepsilon})_t-\operatorname{div}_x\mathbf{F}(\phi_{\delta,\varepsilon})\Big\}\,\theta\,dydsdxdt \\ & \qquad + \iiiint_{Q^2} S_0^+(u^*\wedge\ell-\phi_{\delta,\varepsilon})\,\theta\,dydsd\mu_\ell \;\ge\; 0. \qquad (4.11) \end{aligned}$$

Note that $S_0^+(u\wedge\ell-\phi_{\delta,\varepsilon})\to S_0^+(u\wedge\ell-k)\,\chi_{B_\delta(t,x)}(s,y)$ and $S_0^+(u^*\wedge\ell-\phi_{\delta,\varepsilon})\to S_0^+(u^*\wedge\ell-k)\,\chi_{B_\delta(t,x)}(s,y)$ as $\varepsilon\downarrow 0$, and also note that

$$\begin{aligned} & -\iiiint_{Q^2} S_0^+(u\wedge\ell-\phi_{\delta,\varepsilon})\,(\phi_{\delta,\varepsilon})_t\,\theta\,dydsdxdt \\ & \quad = \iiiint_{Q^2} S_0^+(u\wedge\ell-\phi_{\delta,\varepsilon})\,(u\wedge\ell-\phi_{\delta,\varepsilon})\,\theta_s\,dydsdxdt \\ & \qquad - \iiiint_{Q^2} S_0^+(u\wedge\ell-\phi_{\delta,\varepsilon})\,\big((u\wedge\ell-\phi_{\delta,\varepsilon})\,\theta\big)_s\,dydsdxdt \end{aligned}$$

and

$$
\begin{aligned}
-\iiiint_{Q^2} & S_0^+(u\wedge \ell - \phi_{\delta,\varepsilon})\,\mathrm{div}_x\mathbf{F}(\phi_{\delta,\varepsilon})\,\theta\,dydsdxdt \\
&= \iiiint_{Q^2} S_0^+(u\wedge \ell - \phi_{\delta,\varepsilon})\,(\mathbf{F}(u\wedge\ell) - \mathbf{F}(\phi_{\delta,\varepsilon}))\cdot\nabla_y\theta\,dydsdxdt \\
&\quad - \iiiint_{Q^2} S_0^+(u\wedge \ell - \phi_{\delta,\varepsilon})\,\mathrm{div}_y\big\{(\mathbf{F}(u\wedge\ell) - \mathbf{F}(\phi_{\delta,\varepsilon}))\,\theta\big\}\,dydsdxdt. \quad (4.12)
\end{aligned}
$$

Step 6: We first compute the second integral on the right hand in (4.12). As in the same argument as above, by using again the approximating functions $\{\alpha_m\}\subset C^1(\mathbf{R})$ we see that this integral vanishes. As to the first integral on the right hand in (4.12), note that for $k\in(-\ell,\ell)$ and $\varepsilon>0$ sufficiently small,

$$
\big|S_0^+(u\wedge \ell - \phi_{\delta,\varepsilon})\,(\mathbf{F}(u\wedge\ell) - \mathbf{F}(\phi_{\delta,\varepsilon}))\big| \;\le\; L_\ell\,(u\wedge\ell - \phi_{\delta,\varepsilon})^+,
$$

which implies that $S_0^+(u\wedge \ell - \phi_{\delta,\varepsilon})\,(\mathbf{F}(u\wedge\ell) - \mathbf{F}(\phi_{\delta,\varepsilon}))\cdot\nabla_y\theta\in L^1(Q^2)$. Therefore, passing to the limit in (4.11) as $\varepsilon\downarrow 0$ yields

$$
\begin{aligned}
\iiiint_{Q\times B_\delta(t,x)} & S_0^+(u\wedge\ell - k) \\
&\times\Big\{f\theta + (u\wedge\ell - k)\,\theta_s + (\mathbf{F}(u\wedge\ell) - \mathbf{F}(k))\cdot\nabla_y\theta\Big\}\,dydsdxdt \\
&+ \iiiint_{Q\times B_\delta(t,x)} S_0^+(u^*\wedge\ell - k)\,\theta\,dydsd\mu_\ell \;\ge\; 0.
\end{aligned}
$$

Dividing by volume of the ball $B_\delta(t,x)$ and passing to the limit as $\delta\downarrow 0$, we obtain

$$
\begin{aligned}
0 \;\le\; & \iint_Q S_0^+(u\wedge\ell - k)\,\Big\{f\theta + (u\wedge\ell - k)\,\theta_t + (\mathbf{F}(u\wedge\ell) - \mathbf{F}(k))\cdot\nabla\theta\Big\}\,dxdt \\
& + \iint_Q S_0^+(u^*\wedge\ell - k)\,\theta\,d\mu_\ell \\
= \; & \iint_Q \theta\,d\Big(S_0^+(u\wedge\ell - k)\,f - (u\wedge\ell - k)_t^+ + S_0^+(u^*\wedge\ell - k)\,\mu_\ell \\
& \qquad - \mathrm{div}\,\big\{S_0^+(u\wedge\ell - k)\,(\mathbf{F}(u\wedge\ell) - \mathbf{F}(k))\big\}\Big) \quad (4.13)
\end{aligned}
$$

for each $\theta \in C_0^\infty(Q)^+$. This means that

$$(u \wedge \ell - k)^+_t + \operatorname{div}\left\{S_0^+(u \wedge \ell - k)\,(\mathbf{F}(u \wedge \ell) - \mathbf{F}(k))\right\}$$
$$-S_0^+(u \wedge \ell - k)\,f - S_0^+(u^* \wedge \ell - k)\,\mu_\ell$$

is a Radon measure on Q and hence

$$\begin{aligned}\mu_{k,\ell} \;&=\; (u \wedge \ell - k)^+_t + \operatorname{div}\left\{S_0^+(u \wedge \ell - k)\,(\mathbf{F}(u \wedge \ell) - \mathbf{F}(k))\right\} \\ &\quad - S_0^+(u \wedge \ell - k)\,f\end{aligned}$$

is also a Radon measure on Q. Moreover, we see from (4.13) that for any $\theta \in C_0^\infty(Q)^+$,

$$\begin{aligned}&\iint_Q \theta\, d\mu_{k,\ell} \\ &= -\iint_Q S_0^+(u \wedge \ell - k)\left\{f\theta + (u \wedge \ell - k)\,\theta_t + (\mathbf{F}(u \wedge \ell) - \mathbf{F}(k))\cdot\nabla\theta\right\} dxdt \\ &\le \iint_Q S_0^+(u^* \wedge \ell - k)\,\theta\, d\mu_\ell.\end{aligned}$$

Taking a sequence $\{\theta_n\} \subset C_0^\infty(Q)^+$ which tends to $\chi_{\{\mu_{k,\ell}>0\}}$ as $n \to \infty$, and passing to the limit as $n \to \infty$, we obtain that

$$\mu^+_{k,\ell}(Q) \;\le\; \iint_Q S_0^+(u^* \wedge \ell - k)\, d\mu_\ell \;\le\; \mu_\ell(Q) \;\to\; 0 \qquad \text{as} \quad \ell \to \infty.$$

We also see that for each $\ell \in \mathbf{R}$, $(u(t,\cdot) \wedge \ell - u_0 \wedge \ell)^+ \to 0$ in $L^1_{loc}(\mathbf{R}^N)$ as $t \to 0$ essentially, and thus we complete the proof of the theorem.

4.4 Application

We prove the existence of renormalized dissipative solutions of (CP) via relaxation methods.

Let $\omega_i > 0$ and suppose that for $k = 1, 2, \cdots$ and $i = 1, 2, \cdots, N$, $V_{k,i}$ satisfy the conditions

$$1 + \sum_{i=1}^N V_{k,i}^{-1} \inf_{|u|\le k} F_i'(u) \;>\; 0,$$

$$\frac{1+\Omega}{1+\sum_{j=1}^{N} V_{k,j}^{-1}\inf_{|u|\le k} F_j'(u)}\, V_{k,i}^{-1} \sup_{|u|\le k} F_i'(u) \;<\; \omega_i,$$

where $\Omega = \sum_{i=1}^{N}\omega_i$. It is proved in [KaT1997, Lemma 4.1] that there are a strictly increasing function $r_k : [-k,k] \to \mathbf{R}$ defined by

$$w \;=\; r_k(u) \;:=\; \frac{1}{1+\Omega}\left(u+\sum_{i=1}^{N} V_{k,i}^{-1} F_i(u)\right)$$

and functions $h_{k,i} : [r_k(-k), r_k(k)] \to \mathbf{R}$, satisfying the conditions $dh_{k,i}/dw < 0$, $h_{k,i}(0) = 0$ such that

$$w - \sum_{i=1}^{N} h_{k,i}(w) \;=\; u,$$

$$\omega_i\, V_{k,i}\, w + V_{k,i}\, h_{k,i}(w) \;=\; F_i(u), \quad u \in [-k,k].$$

We consider the following family of relaxation systems for w^ε and $\mathbf{z}^\varepsilon = (z_1^\varepsilon, \cdots, z_N^\varepsilon)$:

$$\text{(CRS)} \quad \left\{ \begin{aligned} \frac{\partial w^\varepsilon}{\partial t} + \sum_{i=1}^{N} \omega_i\, V_{k,i}\, \frac{\partial w^\varepsilon}{\partial x_i} \;&=\; \frac{1}{\varepsilon}\sum_{i=1}^{N}(h_{k,i}(w^\varepsilon) - z_i^\varepsilon), \\ \frac{\partial z_i^\varepsilon}{\partial t} - V_{k,i}\, \frac{\partial z_i^\varepsilon}{\partial x_i} \;&=\; \frac{1}{\varepsilon}(h_{k,i}(w^\varepsilon) - z_i^\varepsilon), \quad i = 1, \cdots, N,\ \varepsilon > 0 \end{aligned} \right.$$

with the initial conditions

$$w^\varepsilon(0,x) = w_0(x), \quad \mathbf{z}^\varepsilon(0,x) = \mathbf{z}_0(x), \quad x \in \mathbf{R}^N, \tag{4.14}$$

$$a \le w_0 \le b, \quad h_{k,i}(b) \le z_{0i} \le h_{k,i}(a), \tag{4.15}$$

where $a < 0$ and $b > 0$ are constants such that

$$-k \;\le\; a + \sum_{i=1}^{N} h_{k,i}(b) \;\le\; b + \sum_{i=1}^{N} h_{k,i}(a) \;\le\; k.$$

The following result is obtained in [KaT1997, Theorem 4.2].

Proposition 4.4. *Let $k \geq 1$, $u^\varepsilon = w^\varepsilon - \sum_{i=1}^N z_i^\varepsilon$ and let $u_0 = w_0 - \sum_{i=1}^N z_{0i} \in L^1(\mathbf{R}^N)$. Then $\overline{u}_k = \lim_{\varepsilon \downarrow 0} u^\varepsilon$ exists in $L^1(Q)$ and $\overline{u}_k$ is an entropy solution of* (CP) *with $f = 0$ satisfying $-k \leq \overline{u}_k \leq k$.*

Now, let $u_0 \in L^1(\mathbf{R}^N)$ and choose sequences of functions $\{w_{0,k}\}_{k\geq 1}$ and $\{\mathbf{z}_{0,k}\}_{k\geq 1}$ which satisfy condition (4.15). Moreover, we assume that $u_{0,k} = w_{0,k} - \sum_{i=1}^N z_{0i,k}$ converges as $k \to \infty$ to u_0 in $L^1(\mathbf{R}^N)$. Since the function $\overline{u}_k$ is a bounded entropy solution of (CP) with $f = 0$ the comparison property of entropy solutions leads to

$$\left\|\overline{u}_k(t) - \overline{u}_{k'}(t)\right\|_{L^1(\mathbf{R}^N)} \leq \left\|u_{0,k} - u_{0,k'}\right\|_{L^1(\mathbf{R}^N)}$$

for $t \in [0, T]$ and $k, k' \geq 1$. Therefore, $\{\overline{u}_k\}$ converges as $k \to \infty$ to some function $\overline{u}$ in $L^1(Q)$. In fact, we can prove:

Theorem 4.5. *The limit function $\overline{u}$ above is a unique renormalized dissipative solution of* (CP) *with $f = 0$.*

Proof. We shall show that inequalities (4.3) and (4.4) with $\mu_\ell = \nu_\ell = 0$. To this end we fix $\ell \geq 1$. Define t_0 by $t_0 = 0$ if $u(t) \leq \ell$ for all $t \geq 0$ and by $t_0 = \inf\{t > 0\,;\, u(t) = \ell\}$ otherwise. We take any test function $\phi \in \mathcal{T}_\ell$ and let $\zeta = r_k(\phi)$ and $\psi_i = h_{k,i}(\zeta)$. Choose $\beta > 0$ such that $\beta - \sum_{i=1}^N h_{k,i}(\beta) = \ell$. This choice is possible if k is taken sufficiently large. Since the constant functions $w \equiv \beta$ and $z_i = h_{k,i}(\beta)$ satisfy the contractive relaxation system (CRS), we have (see (2.11) in [KaT1997])

$$\begin{aligned} 0 \leq \iint_{(0,t_0)\times\mathbf{R}^N} \Big\{ & S_0^+(\beta - \zeta)\Big[-\zeta_t - \sum_{i=1}^N \omega_i V_{k,i}\zeta_{x_i} + \frac{1}{\varepsilon}\sum_{i=1}^N (h_{k,i}(\zeta) - \psi_i)\Big] \\ & + \sum_{i=1}^N S_0^-(h_{k,i}(\beta) - h_{k,i}(\zeta))\Big[-(\psi_i)_t + V_{k,i}(\psi_i)_{x_i} + \frac{1}{\varepsilon}(h_{k,i}(\zeta) - \psi_i)\Big]\Big\}\, dxdt. \end{aligned}$$

We notice that $-S_0^-(h_{k,i}(\beta) - h_{k,i}(\zeta)) = S_0^+(\beta - \zeta) = S_0^+(r_k(\ell) - r_k(\phi)) = S_0^+(\ell - \phi)$, $\beta = r_k(\ell)$, $\zeta - \sum_{i=1}^N \psi_i = \phi$ and $\omega_i V_{k,i}\zeta + V_{k,i}\psi_i = F_i(\phi)$. Thus, the inequality becomes

$$0 \leq \iint_{(0,t_0)\times\mathbf{R}^N} S_0^+(\ell-\phi)\,(-\phi_t - \operatorname{div}\mathbf{F}(\phi))\,dxdt.$$

On the other hand, the comparison property for (CRS) yields that $u(t) \leq \ell$ for $t \in [t_0, T]$. A similar argument as in [P2003b, Theorem 2.1] shows that

$$\iint_{(t_0,T)\times\mathbf{R}^N} S_0(\overline{u}_k-\phi)\,(-\phi_t - \operatorname{div}\mathbf{F}(\phi))\,dxdt \geq 0.$$

Since $2S_0^+(r) = S_0(r) + 1 - \chi_{\{r=0\}}$, we see that

$$\begin{aligned}
&\iint_{(t_0,T)\times\mathbf{R}^N} S_0^+(\overline{u}_k-\phi)\,(-\phi_t - \operatorname{div}\mathbf{F}(\phi))\,dxdt \\
&= \frac{1}{2}\iint_{(t_0,T)\times\mathbf{R}^N} S_0(\overline{u}_k-\phi)\,(-\phi_t - \operatorname{div}\mathbf{F}(\phi))\,dxdt \\
&\quad + \frac{1}{2}\iint_{(t_0,T)\times\mathbf{R}^N}(-\phi_t - \operatorname{div}\mathbf{F}(\phi))\,dxdt - \frac{1}{2}\iint_{\overline{u}_k=\phi}(-\phi_t - \operatorname{div}\mathbf{F}(\phi))\,dxdt \\
&\geq \frac{1}{2}\iint_{(t_0,T)\times\mathbf{R}^N}(-\phi_t - \operatorname{div}\mathbf{F}(\phi))\,dxdt.
\end{aligned}$$

The above inequality comes from the facts that $\overline{u}_k$ is a dissipative solution and satisfies (CP) with $f = 0$. Moreover, we see that the last integral also vanishes due to the divergence theorem. Consequently, we deduce that

$$\iint_{(t_0,T)\times\mathbf{R}^N} S_0^+(\overline{u}_k-\phi)\,(-\phi_t - \operatorname{div}\mathbf{F}(\phi))\,dxdt \geq 0.$$

Notice, however, that

$$\begin{aligned}
&\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\Big(\big\|(f+\lambda\,g)^+\big\|_{L^1((t_0,T)\times\mathbf{R}^N)} - \big\|f^+\big\|_{L^1((t_0,T)\times\mathbf{R}^N)}\Big) \\
&= \inf_{\lambda>0}\frac{1}{\lambda}\Big(\big\|(f+\lambda\,g)^+\big\|_{L^1((t_0,T)\times\mathbf{R}^N)} - \big\|f^+\big\|_{L^1((t_0,T)\times\mathbf{R}^N)}\Big) \\
&= \iint_{f>0} S_0^+(f)\,g\,dxdt + \iint_{f=0} g^+\,dxdt
\end{aligned}$$

for $f, g \in L^1((t_0,T)\times \mathbf{R}^N)$. Indeed, we can check as follows:

$$\begin{aligned}
&\frac{1}{\lambda}\Big(\big\|(f+\lambda\, g)^+\big\|_{L^1((t_0,T)\times\mathbf{R}^N)} - \big\|f^+\big\|_{L^1((t_0,T)\times\mathbf{R}^N)}\Big)\\
&\quad= \frac{1}{\lambda}\left(\iint_{(t_0,T)\times\mathbf{R}^N}\big((f+\lambda\, g)^+ - f^+\big)\,dxdt\right)\\
&\quad= \frac{1}{\lambda}\left(\iint_{(t_0,T)\times\mathbf{R}^N}\big(S_0^+(f+\lambda\, g)\,(f+\lambda\, g) - S_0^+(f)\,f\big)\,dxdt\right)\\
&\quad= \iint_{(t_0,T)\times\mathbf{R}^N} S_0^+(f+\lambda\, g)\,g\,dxdt\\
&\qquad+\frac{1}{\lambda}\iint_{(t_0,T)\times\mathbf{R}^N}\big(S_0^+(f+\lambda\, g)-S_0^+(f)\big)\,f\,dxdt. \qquad (4.16)
\end{aligned}$$

Dividing the first term into two integrals by the sign of g, we have

$$\begin{aligned}
&\iint_{(t_0,T)\times\mathbf{R}^N} S_0^+(f+\lambda\, g)\,g\,dxdt\\
&\quad= \iint_{g>0} S_0^+(f+\lambda\, g)\,g\,dxdt + \iint_{g<0} S_0^+(f+\lambda\, g)\,g\,dxdt\\
&\quad\to \iint_{g>0} S_1^+(f)\,g\,dxdt + \iint_{g<0} S_0^+(f)\,g\,dxdt \qquad (\lambda\downarrow 0)\\
&\quad= \iint_{f>0} S_0^+(f)\,g\,dxdt + \iint_{f=0} g^+\,dxdt.
\end{aligned}$$

As to the last term in (4.16), we first see that

$$\begin{aligned}
&\frac{1}{\lambda}\iint_{(t_0,T)\times\mathbf{R}^N}\big(S_0^+(f+\lambda\, g)-S_0^+(f)\big)\,f\,dxdt\\
&\quad= \frac{1}{\lambda}\left(\iint_{f<0,\,f+\lambda g>0} f\,dxdt - \iint_{f>0,\,f+\lambda g\le 0} f\,dxdt\right) \;\le\; 0.
\end{aligned}$$

On the other hand, we also calculus as

$$\begin{aligned}
&\frac{1}{\lambda}\iint_{(t_0,T)\times\mathbf{R}^N}\big(S_0^+(f+\lambda\, g)-S_0^+(f)\big)\,f\,dxdt\\
&\quad= \frac{1}{\lambda}\left(\iint_{f<0,\,f+\lambda g>0} f\,dxdt - \iint_{f>0,\,f+\lambda g\le 0} f\,dxdt\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \iint_{f>0,\, f+\lambda g\leq 0} g\,dxdt - \iint_{f<0,\, f+\lambda g>0} g\,dxdt \\
&\to \iint_{f>0,\, f\leq 0} g\,dxdt - \iint_{f<0,\, f>0} g\,dxdt \qquad (\lambda\downarrow 0) \\
&= 0.
\end{aligned}
$$

Hence, these estimates leads to

$$
\begin{aligned}
&\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\Big(\|(f+\lambda\,g)^+\|_{L^1((t_0,T)\times\mathbf{R}^N)} - \|f^+\|_{L^1((t_0,T)\times\mathbf{R}^N)}\Big) \\
&\quad = \iint_{f>0} S_0^+(f)\,g\,dxdt + \iint_{f=0} g^+\,dxdt.
\end{aligned}
$$

We thus have that for any $\lambda>0$,

$$
0 \leq \frac{1}{\lambda}\iint_{(t_0,T)\times\mathbf{R}^N}\Big((\overline{u}_k-\phi-\lambda\,\phi_t-\lambda\,\mathrm{div}\,\mathbf{F}(\phi))^+ - (\overline{u}_k-\phi)^+\Big)\,dxdt.
$$

Passing to the limit as $k\to\infty$ first and then as $\lambda\downarrow 0$ yields

$$
\begin{aligned}
0 &\leq \iint_{\overline{u}-\phi>0} S_0^+(\overline{u}-\phi)\,(-\phi_t-\mathrm{div}\,\mathbf{F}(\phi))\,dxdt \\
&\quad + \iint_{\overline{u}-\phi=0}\big(-\phi_t-\mathrm{div}\,\mathbf{F}(\phi)\big)^+\,dxdt \\
&= \iint_{(t_0,T)\times\mathbf{R}^N} S_0^+(\overline{u}-\phi)\,(-\phi_t-\mathrm{div}\,\mathbf{F}(\phi))\,dxdt.
\end{aligned}
$$

Consequently, we conclude that

$$
\iint_Q S_0^+(\overline{u}\wedge\ell-\phi)\,(-\phi_t-\mathrm{div}\,\mathbf{F}(\phi))\,dxdt \geq 0.
$$

The inequality (4.4) can be proved similarly. Therefore, $\overline{u}$ is a renormalized dissipative solution of (CP) and hence by Theorem 4.3 it is a renormalized entropy solution of (CP). By virtue of the uniqueness theorem in [BCW2000], $\overline{u}$ is a unique solution. □

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Chapter 5

Renormalized dissipative solutions for second order equations

In this chapter, we introduce a new notion of renormalized dissipative solutions for the Cauchy problem of a quasilinear anisotropic degenerate parabolic equation $u_t + \operatorname{div} \mathbf{F}(u) = \operatorname{div}(A(u)\nabla u) + f$ with locally Lipschitz-continuous flux $\mathbf{F}$ and L^1 data, and prove the equivalence of such solutions and renormalized entropy solutions in the sense of Bendahmane and Karlsen. The structure of renormalized dissipative solutions is flexible and suitable to deal with relaxation systems than the renormalized entropy scheme. The proof of our main theorem is based on the method of doubling variables established by Kružkov. As applications, we apply our result to certain relaxation systems in general L^1-setting and construct a renormalized dissipative solution. Contents of this chapter is based on the paper [T2005]. This research was supported by Waseda University Grant for Special Research Projects ♯2004A–108.

5.1 Introduction

We consider the following Cauchy problem:

$$\text{(CP)} \qquad \begin{cases} u_t + \operatorname{div} \mathbf{F}(u) = \operatorname{div}(A(u)\nabla u) + f & \text{in} \quad Q := (0,T) \times \mathbf{R}^N, \\ u(0,\cdot) = u_0 & \text{in} \quad \mathbf{R}^N, \end{cases}$$

where $T > 0$ and $N \geq 1$. Here $f \in L^1(Q)$ and $u_0 \in L^1(\mathbf{R}^N)$ are given functions, the diffusion function $A(u)$ is a nonnegative symmetric $N \times N$ matrix and the flux $\mathbf{F} : \mathbf{R} \to \mathbf{R}^N$ is a locally Lipschitz-continuous function.

In the case of $A(u) \equiv O$, the diffusion term degenerates, and therefore the equation becomes a hyperbolic equation $u_t + \operatorname{div} \mathbf{F}(u) = f$. It is known that (CP) has many solutions in the sense of distributions called weak solutions. Finding a suitable criterion which would ensure the uniqueness of a weak solution is one of the most interesting problems, and therefore many researchers have studied hyperbolic equations and degenerate parabolic equations including conservation laws. In consequence, various important results have been clarified for the last few decades.

In 1970, Kružkov [Kr1970] proved that if $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, then the equation has a unique weak solution $u \in C([0,T); L^1(\mathbf{R}^N)) \cap L^\infty(Q)$ satisfying the entropy inequality, which is the so-called entropy solution. He also introduced the method of doubling variables which is a practical tool and on the basis of the proof of uniqueness. Around three decades later, Chen and Perthame [CP2003] extended the notion of entropy solutions to general degenerate parabolic equations with anisotropic nonlinearity, and obtained uniqueness of an entropy solution by utilizing a kinetic formulation and regularization by convolution. At the same time, Portilheiro [P2003a] defined a dissipative solution of scalar conservation laws with globally Lipschitz-continuous flux $\mathbf{F}$, which was established first by Evans, and showed the equivalence of such solutions and entropy solutions by accretive operator theory. Furthermore, the notion of dissipative solutions was extended by Perthame and Souganidis [PS2003] to the second order degenerate parabolic balance laws and the equivalence result was obtained. The definition of dissipative solutions is more simple and flexible, and also suitable to study asymptotic problems handling relaxation systems than entropy solutions. Direct proofs of existence and uniqueness of dissipative solutions, however, have not been obtained yet.

On the other hand, it is known that if $u_0 \in L^1(\mathbf{R}^N)$ and $f \in L^1(Q)$, then the mild solution u of (CP) constructed by nonlinear semigroup theory is a unique entropy solution, which is unbounded in general. In the case where $\mathbf{F}$ is only locally Lipschitz-continuous, the flux function $\mathbf{F}(u)$ may fail to be locally integrable since no growth condition is assumed on the flux $\mathbf{F}$, and hence (CP) does not possess a solution even in the sense of distributions. To overcome this, the notion of renormalized entropy solutions has been introduced by Bénilan et al. [BCW2000] for scalar conservation laws and by Bendahmane and Karlsen [BK2004] for second order degenerate parabolic equations. Furthermore, the existence and uniqueness of a renormalized entropy solution of these equations have been established and the semigroup solutions of (CP) in L^1 spaces are characterized. The arguments in [PS2003] and [P2003a], however, do not work well in the case where $\mathbf{F}$ is only locally Lipschitz-continuous and the solution u is unbounded. The notion of renormalized solutions has been introduced by DiPerna and Lions [DPL1989] for dealing with the existence of a solution of the Boltzmann equation and utilized for degenerate elliptic and degenerate parabolic problems in the L^1-setting in the last decade.

A new concept of renormalized dissipative solutions for a hyperbolic equation with L^1 data has been established in [KoT2005] and the equivalence of such solutions and renormalized entropy solutions in the sense of [BCW2000] was proved. Existence of renormalized dissipative solutions for a contractive relaxation system describing discrete velocity models and chemical reaction models has been also shown in general L^1-settings in [KoT2005] and solutions of the system were characterized. The purpose of this paper is to extend this notion to quasilinear anisotropic degenerate parabolic equations including hyperbolic conservation laws. In Section 2, we recall some important definitions and extend the notion of renormalized dissipative solutions which is a generalization of dissipative solutions in [PS2003]. We next show the equivalence of renormalized dissipative solutions and renormalized entropy solutions in the sense of [BK2004] in Section 3. As applications, we shall apply the notion of renormalized dissipative solutions to contractive relaxation systems and construct a renormalized dissipative solution for the Cauchy problem of a scalar conservation law and the generalized Stefan problem in Section 4.

5.2 Equivalence

We begin with some notations and definitions. Let $s \in \mathbf{R}$ and $j \in [-1, 1]$. We set $s^+ := \max\{s, 0\}$ and $s^- := -\min\{s, 0\}$. Note that $s^- \geq 0$ and $s = s^+ - s^-$. Define a sign function S_j by $S_j(s) = 1$ if $s > 0$, $S_j(s) = -1$ if $s < 0$ or $S_j(0) = j$, and set $S_j^+(s) := \max\{S_j(s), 0\}$ and $S_j^-(s) := \min\{S_j(s), 0\}$.

For $s \in \mathbf{R}$, the diffusion function $A(s) = (a_{ij}(s))$ is a nonnegative symmetric $N \times N$ matrix of the form

$$a_{ij}(s) = \sum_{m=1}^{M} \sigma_{im}(s)\, \sigma_{jm}(s), \qquad \sigma_{im} \in L^\infty_{loc}(\mathbf{R}) \tag{5.1}$$

for $i, j = 1, \cdots, N$ and $m = 1, \cdots, M$, where $M \leq N$ can be thought to be the maximal rank of the matrix. Let $T_\ell : \mathbf{R} \to [-\ell, \ell]$ denote the truncation function with height $\ell > 0$, that is, $T_\ell(s) := \min\big\{\max\{s, -\ell\}, \ell\big\}$ for any $s \in \mathbf{R}$. For $1 \leq m \leq M$, $1 \leq i \leq N$ and $s \in \mathbf{R}$, we set

$$\beta_{im}(s) := \int_0^s \sigma_{im}(r)\, dr, \qquad \boldsymbol{\beta}_m(s) = (\beta_{1m}(s), \cdots, \beta_{Nm}(s)),$$

and for any $\psi \in C(\mathbf{R})$

$$\beta^\psi_{im}(s) := \int_0^s \psi(r)\, \sigma_{im}(r)\, dr, \qquad \boldsymbol{\beta}^\psi_m(s) = (\beta^\psi_{1m}(s), \cdots, \beta^\psi_{Nm}(s)).$$

Following [BK2004] we define an entropy-entropy flux triple and a renormalized entropy solution of (CP).

Definition 5.1. *For any convex C^2 entropy function $\eta : \mathbf{R} \to \mathbf{R}$, the corresponding entropy fluxes*

$$\mathbf{q} = (q_1, \cdots, q_N) : \mathbf{R} \to \mathbf{R}^N \quad \text{and} \quad R = (r_{ij}) : \mathbf{R} \to \mathbf{R}^{N\times N}$$

are defined by $q_i'(s) = \eta'(s)\, F_i'(s)$ and $r_{ij}'(s) = \eta'(s)\, a_{ij}(s)$ for $i, j = 1, \cdots, N$ and $s \in \mathbf{R}$. Then, we define $(\eta, \mathbf{q}, R)$ as an entropy-entropy flux triple.

Definition 5.2. *We say $u \in L^\infty(0, T; L^1(\mathbf{R}^N))$ is a* renormalized entropy solution *of* (CP) *if a measurable function $u : Q \to \mathbf{R}^N$ satisfies the following conditions:*

(E1) *For any $m = 1, \cdots, M$,*

$$\boldsymbol{\beta}_m(T_\ell(u)) \in L^2(Q)^N \quad \text{and} \quad \operatorname{div} \boldsymbol{\beta}_m(T_\ell(u)) \in L^2(Q) \quad \text{for all} \quad \ell > 0.$$

(E2) *For any $m = 1, \cdots, M$ and $\psi \in C(\mathbf{R})$,*

$$\operatorname{div} \boldsymbol{\beta}_m^\psi(T_\ell(u)) = \psi(T_\ell(u)) \operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))$$

a.e. in Q and in $L^2(Q)$ for all $\ell > 0$.

(E3) *For any $\ell > 0$ and any entropy-entropy flux triple $(\eta, \mathbf{q}, R)$ with $|\eta'| \le K$ for some given $K > 0$, there exists for any $\ell > 0$ a nonnegative bounded Radon measure μ_ℓ^K on Q with $\mu_\ell^K(Q) \to 0$ as $\ell \to \infty$ such that*

$$\begin{aligned} &\eta(T_\ell(u))_t + \operatorname{div} \mathbf{q}(T_\ell(u)) - \sum_{i,j=1}^N r_{ij}(T_\ell(u))_{x_i x_j} - \eta'(T_\ell(u))\, f \\ &\le \; -\eta''(T_\ell(u)) \sum_{m=1}^M \big(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))\big)^2 + \mu_\ell^K \qquad \text{in} \;\; \mathcal{D}'(Q). \end{aligned} \tag{5.2}$$

(E4) *$u(t, \cdot) \to u_0$ in $L^1(\mathbf{R}^N)$ as $t \downarrow 0$ essentially.*

Note that all terms in (5.2) are well-defined since $T_\ell(u) \in L^\infty(Q)$, and also note that (E3) implies there exists a nonnegative bounded Radon measure μ_ℓ on Q with $\mu_\ell(Q) \to 0$ as $\ell \to \infty$ such that $\mu_\ell^K = K_0 \mu_\ell$ with $K_0 := \sup_{s\in[-\ell,\ell]} \left|\eta'(s)\right|$. Indeed, for each i, j, putting $\widetilde{\eta} := K_0^{-1}\eta$, $\widetilde{q}_i := K_0^{-1} q_i$ and $\widetilde{r}_{ij} := K_0^{-1} r_{ij}$, the triple $(\widetilde{\eta}, \widetilde{\mathbf{q}}, \widetilde{R})$ should be an entropy-entropy flux triple with $\left|\widetilde{\eta}'\right| \le 1$.

Next, we introduce a new notion of renormalized dissipative solutions which is a generalization of dissipative solutions in the sense of [PS2003].

Definition 5.3. *We say $u \in L^\infty(0,T; L^1(\mathbf{R}^N))$ is a* renormalized dissipative solution *of* (CP) *if a measurable function $u : Q \to \mathbf{R}^N$ satisfies the following conditions:*

(D1) *For any $m = 1, \cdots, M$,*

$$\boldsymbol{\beta}_m(T_\ell(u)) \in L^2(Q)^N \quad \text{and} \quad \operatorname{div} \boldsymbol{\beta}_m(T_\ell(u)) \in L^2(Q) \quad \text{for all} \quad \ell > 0.$$

(D2) *For any $m = 1, \cdots, M$ and $\psi \in C(\mathbf{R})$,*

$$\operatorname{div} \boldsymbol{\beta}_m^\psi(T_\ell(u)) = \psi(T_\ell(u)) \operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))$$

a.e. in Q and in $L^2(Q)$ for all $\ell > 0$.

(D3) *For any $\ell > 0$, $\xi \in C_0^2(\mathbf{R}^N)$ and $\theta \in C_0^2(\mathbf{R})^+$ with $\operatorname{spt}\theta \subset (-\ell, \ell)$, there exists a nonnegative bounded Radon measure ν_ℓ on Q with $\nu_\ell(Q) \to 0$ as $\ell \to \infty$ such that*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{R}^N}\int_{\mathbf{R}} \theta(k) \left(T_\ell(u) - k - \xi\right)^+ dk dx \\
&\quad\le \int_{\mathbf{R}^N}\int_{\mathbf{R}} \theta(k)\, S_0^+(T_\ell(u) - k - \xi) \\
&\qquad\qquad \times \left(f - \operatorname{div} \mathbf{F}(k+\xi) + \sum_{i,j=1}^N A_{ij}(k+\xi)_{x_i x_j} \right) dk dx \\
&\qquad - \int_{\mathbf{R}^N} \theta(T_\ell(u) - \xi) \sum_{m=1}^M \left(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u)) - \boldsymbol{\sigma}_m(T_\ell(u)) \cdot \nabla \xi \right)^2 dx \\
&\qquad + \int_{\mathbf{R}^N}\int_{\mathbf{R}} \theta(k)\, S_0^+(\ell - k - \xi)\, dk d\nu_\ell \qquad \text{in} \quad \mathcal{D}'(0,T),
\end{aligned} \tag{5.3}$$

where $A'_{ij}(\cdot) := a_{ij}(\cdot)$, $\boldsymbol{\sigma}_m(\cdot) = (\sigma_{1m}(\cdot), \cdots, \sigma_{Nm}(\cdot))$ *and* $C_0^2(\mathbf{R})^+$ *denotes the space of all nonnegative functions in* $C_0^2(\mathbf{R})$ *as usual.*

(D4) $u(t,\cdot) \to u_0$ *in* $L^1(\mathbf{R}^N)$ *as* $t \downarrow 0$ *essentially.*

Then we obtain the following main result.

Theorem 5.4. *Suppose that* $u \in L^\infty(0,T;L^1(\mathbf{R}^N))$. *Then,* u *is a renormalized entropy solution of* (CP) *if and only if* u *is a renormalized dissipative solution of* (CP).

Note that if a renormalized entropy (respectively renormalized dissipative) solution u belongs to $L^\infty(Q)$, then it is also an entropy (respectively a dissipative) solution in the sense of [BK2004, Definition 2.2] (respectively [PS2003, Definition 1.3]). As we mentioned in Section 1, uniqueness of an entropy solution in the sense of [BK2004, Definition 2.2] was proved in [CP2003] utilizing a kinetic formulation, and the equivalence result of such solutions and dissipative solutions was obtained in [PS2003].

If $A(u)$ is a diagonal matrix, for example the isotropic case $u_t + \operatorname{div}\mathbf{F}(u) = \triangle b(u) + f$, the assumptions (E2) and (D2) are automatically fulfilled. In this case, the notion of renormalized dissipative solutions was introduced and the equivalence result of renormalized entropy solutions and renormalized dissipative solutions was obtained in [T2004]. If $A(u) \equiv O$, then the equation becomes a hyperbolic equation $u_t + \operatorname{div}\mathbf{F}(u) = f$. In this case, the equivalence of renormalized entropy solutions and renormalized dissipative solutions was proved in Chapter 4. Due to appearance of the Dirac mass, however, the definition of renormalized dissipative solutions for hyperbolic equations differs from Definition 5.3. Then we shall reconsider afterward the contractive relaxation system studied in Chapter 4 as an application for the hyperbolic case.

5.3 Proof of Theorem 5.4

Claim 1: *If $u \in L^\infty(0,T;L^1(\mathbf{R}^N))$ is a renormalized entropy solution of* (CP), *then u is a renormalized dissipative solution.*

Proof. We see from the definition of renormalized entropy solutions that for any $\ell > 0$ and any entropy-entropy flux triple $(\eta, \mathbf{q}, R)$ with $|\eta'| \le K$ for some given $K > 0$, there exists a nonnegative bounded Radon measure μ_ℓ on Q with $\mu_\ell(Q) \to 0$ as $\ell \to \infty$ such that

$$\begin{aligned} 0 \ \le\ & \iint_Q \eta(T_\ell(u))\,\zeta_t\,dxdt + \iint_Q \sum_{i=1}^N q_i(T_\ell(u))\,\zeta_{x_i}\,dxdt \\ & + \iint_Q \sum_{i,j=1}^N r_{ij}(T_\ell(u))\,\zeta_{x_ix_j}\,dxdt + \iint_Q \eta'(T_\ell(u))\,f\,\zeta\,dxdt \\ & - \iint_Q \eta''(T_\ell(u)) \sum_{m=1}^M \big(\operatorname{div}\boldsymbol{\beta}_m(T_\ell(u))\big)^2 \zeta\,dxdt + \iint_Q K_0\zeta\,d\mu_\ell \qquad (5.4) \end{aligned}$$

for any $\zeta \in C_0^2(Q)^+$, where $K_0 := \sup_{s\in[-\ell,\ell]} \big|\eta'(s)\big|$.

On the other hand, for given $\xi \in C_0^2(\mathbf{R}^N)$ and $\theta \in C_0^2(\mathbf{R})^+$ with $\operatorname{spt}\theta \subset (-\ell,\ell)$, we observe that

$$\eta(T_\ell(u)) \ =\ \int_{\mathbf{R}} \big(T_\ell(u) - k - \xi(y)\big)^+\,\theta(k)\,dk$$

is a smooth entropy. Moreover, we see that

$$\eta'(T_\ell(u)) = \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi(y))\,\theta(k)\,dk$$

and

$$K_0 = \int_{\mathbf{R}} S_0^+(\ell - k - \xi(y))\,\theta(k)\,dk.$$

Let ϕ and ρ be standard mollifiers on $(0,T)$ and $\mathbf{R}^N$, respectively. Define ρ_ε by

$$\rho_\varepsilon(x-y) \ :=\ \varepsilon^{-N}\,\rho((x-y)/\varepsilon),$$

and let ψ_n be a nonnegative smooth function satisfying

$$\psi_n(x) := \begin{cases} 1 & \text{if } |x| \le n \\ 0 & \text{if } |x| \ge 2n, \end{cases}$$

and $|\nabla \psi_n| \le C/n$ for some $C > 0$. We now recall the definition of an entropy-entropy flux triple and properties of the Dirac mass. Putting

$$\zeta = \rho_\varepsilon(x-y)\,\phi(t)\,\psi_n(t,x)$$

in (5.4), integrating with respect to y over $\mathbf{R}^N$ and using $(\rho_\varepsilon)_{y_i} = -(\rho_\varepsilon)_{x_i}$ yield

$$\begin{aligned}
0 \le\ & \int_{\mathbf{R}^N}\iint_Q \eta(T_\ell(u))\,\big(\rho_\varepsilon(x-y)\,\phi(t)\,\psi_n(t,x)\big)_t\,dxdtdy \\
& + \int_{\mathbf{R}^N}\iint_Q \sum_{i=1}^N q_i(T_\ell(u))\,\big(\rho_\varepsilon(x-y)\,\phi(t)\,\psi_n(t,x)\big)_{x_i}\,dxdtdy \\
& + \int_{\mathbf{R}^N}\iint_Q \sum_{i,j=1}^N r_{ij}(T_\ell(u))\,\big(\rho_\varepsilon(x-y)\,\phi(t)\,\psi_n(t,x)\big)_{x_ix_j} dxdtdy \\
& + \int_{\mathbf{R}^N}\iint_Q \eta'(T_\ell(u))\,f\,\rho_\varepsilon(x-y)\,\phi(t)\,\psi_n(t,x)\,dxdtdy \\
& - \int_{\mathbf{R}^N}\iint_Q \eta''(T_\ell(u))\,\sum_{m=1}^M \big(\mathrm{div}_x\boldsymbol{\beta}_m(T_\ell(u))\big)^2\rho_\varepsilon(x-y)\,\phi(t)\,\psi_n(t,x)\,dxdtdy \\
& + \int_{\mathbf{R}^N}\iint_Q K_0\rho_\varepsilon(x-y)\,\phi(t)\,\psi_n(t,x)\,d\mu_\ell dy \\
=\ & \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi(y))\,\theta(k)\,\rho_\varepsilon\Big((T_\ell(u)-k-\xi(y))\,\phi\,\psi_n\Big)_t\,dkdxdtdy \\
& - \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi(y))\,T_\ell(u)_t\,\theta(k)\,\rho_\varepsilon\,\phi\,\psi_n\,dkdxdtdy \\
& - \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi(y))\,\theta(k)\,\mathrm{div}_y\mathbf{F}(k+\xi(y))\,\rho_\varepsilon\,\phi\,\psi_n\,dkdxdtdy \\
& + \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi(y))\,\theta(k) \\
& \qquad\qquad \times \big(\mathbf{F}(T_\ell(u)) - \mathbf{F}(k+\xi(y))\big)\cdot\nabla_x\psi_n\,\rho_\varepsilon\,\phi\,dkdxdtdy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}}\sum_{i,j=1}^N S_0^+(T_\ell(u)-k-\xi(y))\,\theta(k)\,A_{ij}(T_\ell(u))_{x_ix_j}\,\rho_\varepsilon\,\phi\,\psi_n\,dkdxdtdy\\
&+\int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi(y))\,\theta(k)\,f\,\rho_\varepsilon\,\phi\,\psi_n\,dkdxdtdy\\
&+\int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(\ell-k-\xi(y))\,\theta(k)\,\rho_\varepsilon\,\phi\,\psi_n\,dkd\mu_\ell dy\\
=:&\sum_{h=1}^7 I_h^{\varepsilon,n}. \qquad (5.5)
\end{aligned}
$$

We begin with $I_5^{\varepsilon,n}$. For $p>0$, we set

$$\omega(p) := \sup\left\{\left|\xi(x)-\xi(y)\right|;\, x,y\in\mathbf{R}^N,\ \left|x-y\right|\le p\right\}.$$

Note that $\omega(p)\ge 0$ for any $p>0$ and $\omega(p)\to 0$ as $p\downarrow 0$. Then we see that

$$
\begin{aligned}
I_5^{\varepsilon,n} \le & \iiiint_{\sum_{i,j}A_{ij}(T_\ell(u))_{x_ix_j}\ge 0} S_0^+(T_\ell(u)-k-\xi(x)+\omega(\varepsilon))\\
&\qquad\times\sum_{i,j=1}^N A_{ij}(T_\ell(u))_{x_ix_j}\,\theta\,\rho_\varepsilon\phi\,\psi_n\,dkdxdtdy\\
&+\iiiint_{\sum_{i,j}A_{ij}(T_\ell(u))_{x_ix_j}< 0} S_0^+(T_\ell(u)-k-\xi(x)-\omega(\varepsilon))\\
&\qquad\times\sum_{i,j=1}^N A_{ij}(T_\ell(u))_{x_ix_j}\,\theta\,\rho_\varepsilon\phi\,\psi_n\,dkdxdtdy
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} I_5^{\varepsilon,n}\\
&\le \iiint_{\sum_{i,j}A_{ij}(T_\ell(u))_{x_ix_j}\ge 0} S_1^+(T_\ell(u)-k-\xi)\,\sum_{i,j=1}^N A_{ij}(T_\ell(u))_{x_ix_j}\,\theta\,\phi\,\psi_n\,dkdxdt\\
&\quad+\iiint_{\sum_{i,j}A_{ij}(T_\ell(u))_{x_ix_j}< 0} S_0^+(T_\ell(u)-k-\xi)\,\sum_{i,j=1}^N A_{ij}(T_\ell(u))_{x_ix_j}\,\theta\,\phi\,\psi_n\,dkdxdt\\
&= \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,\sum_{i,j=1}^N A_{ij}(T_\ell(u))_{x_ix_j}\,\theta\,\phi\,\psi_n\,dkdxdt\\
&\quad+\iiint_{T_\ell(u)=k+\xi}\left(\sum_{i,j=1}^N A_{ij}(T_\ell(u))_{x_ix_j}\right)^+\theta\,\phi\,\psi_n\,dkdxdt.
\end{aligned}
$$

As to other integrals, we see from the same arguments as above that

$$\begin{aligned}
I_1^{\varepsilon,n} &= 0,\\
\limsup_{\varepsilon\downarrow 0} I_2^{\varepsilon,n} &\le -\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,T_\ell(u)_t\,\theta\,\phi\,\psi_n\,dkdxdt,\\
\limsup_{\varepsilon\downarrow 0} I_3^{\varepsilon,n} &\le -\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,\mathrm{div}\,\mathbf{F}(k+\xi)\,\theta\,\phi\,\psi_n\,dkdxdt\\
&\quad+\iiint_{T_\ell(u)=k+\xi}\big(\mathrm{div}\,\mathbf{F}(k+\xi)\big)^-\,\theta\,\phi\,\psi_n\,dkdxdt,\\
\limsup_{\varepsilon\downarrow 0} I_4^{\varepsilon,n} &\le \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,\big(\mathbf{F}(T_\ell(u))-\mathbf{F}(k+\xi)\big)\cdot\nabla\psi_n\,\theta\,\phi\,dkdxdt,\\
\limsup_{\varepsilon\downarrow 0} I_6^{\varepsilon,n} &\le \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,f\,\theta\,\phi\,\psi_n\,dkdxdt\\
&\quad+\iiint_{T_\ell(u)=k+\xi} f^+\,\theta\,\phi\,\psi_n\,dkdxdt,\\
\limsup_{\varepsilon\downarrow 0} I_7^{\varepsilon,n} &\le \iint_Q\int_{\mathbf{R}} S_0^+(\ell-k-\xi)\,\theta\,\phi\,\psi_n\,dkd\mu_\ell.
\end{aligned}$$

Hence, passing to the limit in (5.5) as $\varepsilon\downarrow 0$ first and then $n\to\infty$ gives

$$\begin{aligned}
0 \le &-\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,T_\ell(u)_t\,\theta\,\phi\,dkdxdt\\
&-\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,\mathrm{div}\,\mathbf{F}(k+\xi)\,\theta\,\phi\,dkdxdt\\
&+\iiint_{T_\ell(u)=k+\xi}\big(\mathrm{div}\,\mathbf{F}(k+\xi)\big)^-\,\theta\,\phi\,dkdxdt\\
&+\limsup_{n\to\infty}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,\big(\mathbf{F}(T_\ell(u))-\mathbf{F}(k+\xi)\big)\cdot\nabla\psi_n\theta\phi\,dkdxdt\\
&+\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,\sum_{i,j=1}^N A_{ij}(T_\ell(u))_{x_ix_j}\,\theta\,\phi\,dkdxdt\\
&+\iiint_{T_\ell(u)=k+\xi}\left(\sum_{i,j=1}^N A_{ij}(T_\ell(u))_{x_ix_j}\right)^+\theta\,\phi\,dkdxdt\\
&+\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,f\,\theta\,\phi\,dkdxdt+\iiint_{T_\ell(u)=k+\xi} f^+\,\theta\,\phi\,dkdxdt\\
&+\iint_Q\int_{\mathbf{R}} S_0^+(\ell-k-\xi)\,\theta\,\phi\,dkd\mu_\ell. \qquad (5.6)
\end{aligned}$$

Note that the set $\{k \in (-\ell,\ell)\,;\, \mathcal{L}^{N+1}(\{T_\ell(u) = k+\xi\}) = 0\}$ is dense in $(-\ell,\ell)$ because $\sum_{k\in C} |k|\, \mathcal{L}^{N+1}(\{T_\ell(u) = k+\xi\})$ is finite for any countable set $C \subset (-\ell,\ell)$, where $\mathcal{L}^{N+1}$ denotes the $(N+1)$-dimensional Lebesgue measure. Hence the cardinality of the set $\{k \in (-\ell,\ell)\,;\, \mathcal{L}^{N+1}(\{T_\ell(u) = k+\xi\}) > 0\}$ is at most countable.

We now fix any $k \in (-\ell,\ell)$ and choose a sequence $\{k_n^+\}$ such that $k_n^+ \downarrow k$ as $n \to \infty$ and $\mathcal{L}^{N+1}(\{T_\ell(u) = k_n^+ + \xi\}) = 0$ for any $n \geq 1$. It follows from (5.6) with $k = k_n^+$ that

$$\begin{aligned}
0 \;\leq\; & -\iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\, T_\ell(u)_t\, \theta\, \phi\, dkdxdt \\
& -\iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\, \mathrm{div}\, \mathbf{F}(k+\xi)\, \theta\, \phi\, dkdxdt \\
& + \limsup_{n\to\infty} \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\, \big(\mathbf{F}(T_\ell(u)) - \mathbf{F}(k+\xi)\big)\cdot \nabla\psi_n \theta\phi\, dkdxdt \\
& + \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi) \sum_{i,j=1}^N A_{ij}(T_\ell(u))_{x_i x_j}\, \theta\, \phi\, dkdxdt \\
& + \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\, f\theta\phi\, dkdxdt + \iint_Q \int_{\mathbf{R}} S_0^+(\ell-k-\xi)\, \theta\phi\, dkd\mu_\ell \\
=: & \sum_{h=1}^6 I_h.
\end{aligned} \tag{5.7}$$

To this end, using the properties of the Dirac mass, we have

$$I_1 \;=\; \iint_Q \int_{\mathbf{R}} \big(T_\ell(u)-k-\xi\big)^+\, \theta\, \phi'\, dkdxdt.$$

As to I_3, we first note that $\mathbf{F}(T_\ell(u)) - \mathbf{F}(k+\xi) + \mathbf{F}(k) \in L^1(Q)^N$. From this, we see that

$$\lim_{n\to\infty} \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\big(\mathbf{F}(T_\ell(u)) - \mathbf{F}(k+\xi) + \mathbf{F}(k)\big)\cdot \nabla\psi_n\, \theta\, \phi\, dkdxdt \;=\; 0.$$

On the other hand, thanks to Chebyshev's inequality, we have for $k > 0$ that

$$\mathcal{L}^{N+1}\big(\{T_\ell(u) - \xi > k\}\big) \;\leq\; \frac{1}{k} \iint_Q \big|T_\ell(u) - \xi\big|\, dxdt \;<\; \infty,$$

and therefore we see that

$$\lim_{n\to\infty} \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi)\, \mathbf{F}(k)\cdot\nabla\psi_n\, \theta\, \phi\, dkdxdt \;=\; 0.$$

For $k < 0$, the same result can be also obtained. From these observations, we conclude that $I_3 = 0$. We now calculate I_4 as

$$\begin{aligned}
I_4 \;=\; & \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi) \sum_{i,j=1}^N \big(A_{ij}(T_\ell(u)) - A_{ij}(k+\xi)\big)_{x_i x_j}\, \theta\, \phi\, dkdxdt \\
& + \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi) \sum_{i,j=1}^N A_{ij}(k+\xi)_{x_i x_j}\, \theta\, \phi\, dkdxdt \\
=\; & -\iint_Q \theta(T_\ell(u) - \xi) \sum_{m=1}^M \Big(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u)) - \boldsymbol{\sigma}_m(T_\ell(u))\cdot\nabla\xi\Big)^2 \phi\, dxdt \\
& + \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi) \sum_{i,j=1}^N A_{ij}(k+\xi)_{x_i x_j}\, \theta\, \phi\, dkdxdt.
\end{aligned}$$

Combining these estimates, we obtain that

$$\begin{aligned}
0 \;\le\; & \iint_Q \int_{\mathbf{R}} (T_\ell(u) - k - \xi)^+\, \theta\, \phi'\, dkdxdt \\
& - \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi)\, \operatorname{div}\mathbf{F}(k+\xi)\, \theta\, \phi\, dkdxdt \\
& + \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi) \sum_{i,j=1}^N A_{ij}(k+\xi)_{x_i x_j}\, \theta\, \phi\, dkdxdt \\
& - \iint_Q \theta(T_\ell(u) - \xi) \sum_{m=1}^M \Big(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u)) - \boldsymbol{\sigma}_m(T_\ell(u))\cdot\nabla\xi\Big)^2 \phi\, dxdt \\
& + \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi)\, f\, \theta\, \phi\, dkdxdt \\
& + \iint_Q \int_{\mathbf{R}} S_0^+(\ell - k - \xi)\, \theta\, \phi\, dkd\mu_\ell.
\end{aligned}$$

This is exactly (D3). □

Claim 2: *If $u \in L^\infty(0,T;L^1(\mathbf{R}^N))$ is a renormalized dissipative solution of* (CP)*, then u is a renormalized entropy solution.*

Proof. Let $u \in L^\infty(0,T;L^1(\mathbf{R}^N))$ be a renormalized dissipative solution of (CP). We consider a function $\alpha \in C_0^2(\mathbf{R}^N)^+$ and for each ε, $\lambda > 0$ a nondecreasing smooth function $\xi_{\varepsilon,\lambda}$ defined by

$$\xi_{\varepsilon,\lambda}(s) := \begin{cases} 0 & \text{for } |s| \le \lambda \\ \text{strictly increasing} & \text{for } \lambda \le |s| \le \lambda+\varepsilon \\ 1/\varepsilon & \text{for } |s| \ge \lambda+\varepsilon. \end{cases}$$

Let $V(N)$ denote the volume of the unit ball in $\mathbf{R}^N$. Using the test function $\xi_{\varepsilon,\lambda}(x-y)$ in (5.3), multiplying by

$$\alpha_\lambda(y) := \frac{1}{V(N)\,\lambda^N}\,\alpha(y)$$

and integrating with respect to y yield for any $\phi \in C_0^1(0,T)^+$,

$$\begin{aligned}
0 \le\ & \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} \theta(k)\,(T_\ell(u)-k-\xi_{\varepsilon,\lambda}(x-y))^+\,\phi'(t)\,\alpha_\lambda(y)\,dkdxdtdy \\
& + \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} \theta(k)\,S_0^+(T_\ell(u)-k-\xi_{\varepsilon,\lambda}(x-y))\,f\,\phi(t)\,\alpha_\lambda(y)\,dkdxdtdy \\
& - \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} \theta(k)\,S_0^+(T_\ell(u)-k-\xi_{\varepsilon,\lambda}(x-y)) \\
& \qquad\qquad \times \mathrm{div}_x\mathbf{F}(k+\xi_{\varepsilon,\lambda}(x-y))\,\phi(t)\,\alpha_\lambda(y)\,dkdxdtdy \\
& + \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} \theta(k)\,S_0^+(T_\ell(u)-k-\xi_{\varepsilon,\lambda}(x-y)) \\
& \qquad\qquad \times \sum_{i,j=1}^{N} A_{ij}(k+\xi_{\varepsilon,\lambda}(x-y))_{x_ix_j}\,\phi(t)\,\alpha_\lambda(y)\,dkdxdtdy \\
& - \int_{\mathbf{R}^N}\iint_Q \theta(T_\ell(u)-\xi_{\varepsilon,\lambda}(x-y)) \\
& \quad \times \sum_{m=1}^{M}\Big(\mathrm{div}_x\boldsymbol{\beta}_m(T_\ell(u)) - \boldsymbol{\sigma}_m(T_\ell(u))\cdot\nabla_x\xi_{\varepsilon,\lambda}(x-y)\Big)^2\,\phi(t)\,\alpha_\lambda(y)\,dxdtdy \\
& + \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} \theta(k)\,S_0^+(\ell-k-\xi_{\varepsilon,\lambda}(x-y))\,\phi(t)\,\alpha_\lambda(y)\,dkd\nu_\ell dy \\
=:\ & \sum_{h=1}^{6} J_h^{\varepsilon,\lambda}.
\end{aligned} \tag{5.8}$$

We begin with $J_1^{\varepsilon,\lambda}$. Thanks to the Lebesgue differentiation theorem, we have

$$\begin{aligned}
J_1^{\varepsilon,\lambda} &= \iiiint_{\xi_{\varepsilon,\lambda}=0} (T_\ell(u)-k)^+\,\theta(k)\,\phi'\,\alpha_\lambda(y)\,dkdxdtdy \\
&\quad + \iiiint_{\xi_{\varepsilon,\lambda}>0} (T_\ell(u)-k-\xi_{\varepsilon,\lambda})^+\,\theta(k)\,\phi'\,\alpha_\lambda(y)\,dkdxdtdy \\
&\to \iiiint_{|x-y|\le\lambda} (T_\ell(u)-k)^+\,\theta(k)\,\phi'\,\alpha_\lambda(y)\,dkdxdtdy \qquad (\varepsilon\downarrow 0) \\
&= \iint_Q\int_{\mathbf{R}} \frac{1}{V(N)\,\lambda^N}\int_{|x-y|\le\lambda} \alpha(y)\,dy\,(T_\ell(u)-k)^+\,\theta(k)\,\phi'\,dkdxdt \\
&\to \iint_Q\int_{\mathbf{R}} (T_\ell(u)-k)^+\,\theta(k)\,\phi'\,\alpha(x)\,dkdxdt \qquad (\lambda\downarrow 0).
\end{aligned}$$

Let $\Theta'(\cdot) := \theta(\cdot)$ with $\Theta(-\infty)=0$. Calculating other integrals similarly, we obtain that

$$\begin{aligned}
J_2^{\varepsilon,\lambda} &\to \iiiint_{|x-y|\le\lambda} S_0^+(T_\ell(u)-k)\,f\,\theta(k)\,\phi\,\alpha_\lambda(y)\,dkdxdydy \qquad (\varepsilon\downarrow 0) \\
&\to \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k)\,f\,\theta(k)\,\phi\,\alpha(x)\,dkdxdt \qquad (\lambda\downarrow 0),
\end{aligned}$$

$$\begin{aligned}
J_3^{\varepsilon,\lambda} &\to -\iiiint_{|x-y|\le\lambda} S_0^+(T_\ell(u)-k)\,\mathbf{F}(k)\cdot\nabla_y\alpha_\lambda(y)\,\theta(k)\,\phi\,dkdxdtdy \\
&\quad + \iiint_{|x-y|\le\lambda} \Theta(T_\ell(u))\,\mathbf{F}(T_\ell(u))\cdot\nabla_y\alpha_\lambda(y)\,\phi\,dxdtdy \qquad (\varepsilon\downarrow 0) \\
&\to -\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k)\,\mathbf{F}(k)\cdot\nabla_x\alpha_\lambda(x)\,\theta(k)\,\phi\,dkdxdt \\
&\quad + \iint_Q \Theta(T_\ell(u))\,\mathbf{F}(T_\ell(u))\cdot\nabla_x\alpha_\lambda(x)\,\phi\,dxdt \qquad (\lambda\downarrow 0) \\
&= \iint_Q \Theta(T_\ell(u))\,\mathbf{F}(T_\ell(u))\cdot\nabla\alpha(x)\,\phi\,dxdt,
\end{aligned}$$

$$
\begin{aligned}
&J_4^{\varepsilon,\lambda} + J_5^{\varepsilon,\lambda} \\
&\quad = \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi_{\varepsilon,\lambda}) \\
&\qquad\qquad\qquad \times \sum_{i,j=1}^N \big(A_{ij}(k+\xi_{\varepsilon,\lambda}) - A_{ij}(T_\ell(u))\big)\, \alpha_\lambda(y)_{y_iy_j}\, \theta(k)\, \phi\, dkdxdtdy \\
&\qquad - \iint_Q\int_{\mathbf{R}} \theta(T_\ell(u) - \xi_{\varepsilon,\lambda}) \sum_{m=1}^M \Big(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))\Big)^2 \phi\, \alpha_\lambda(y)\, dxdtdy \\
&\qquad + 2\iint_Q\int_{\mathbf{R}} \theta(T_\ell(u) - \xi_{\varepsilon,\lambda}) \\
&\qquad\qquad\qquad \times \sum_{m=1}^M \Big(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))\Big)\Big(\boldsymbol{\sigma}_m(T_\ell(u))\cdot\nabla_x \xi_{\varepsilon,\lambda}\Big)\phi\, \alpha_\lambda(y)\, dxdtdy \\
&\quad \to \iiiint_{|x-y|\le\lambda} S_0^+(T_\ell(u) - k) \\
&\qquad\qquad\qquad \times \sum_{i,j=1}^N \big(A_{ij}(k) - A_{ij}(T_\ell(u))\big)\, \alpha_\lambda(y)_{y_iy_j}\, \theta(k)\, \phi\, dkdxdtdy \\
&\qquad - \iiint_{|x-y|\le\lambda} \theta(T_\ell(u)) \sum_{m=1}^M \Big(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))\Big)^2 \phi\, \alpha_\lambda(y)\, dxdtdy \\
&\qquad - 2\iiint_{|x-y|\le\lambda} \Theta(T_\ell(u)) \\
&\qquad\qquad \times \sum_{m=1}^M \Big(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))\Big)\Big(\boldsymbol{\sigma}_m(T_\ell(u))\cdot\nabla_y \alpha_\lambda(y)\Big)\phi\, dxdtdy \qquad (\varepsilon\downarrow 0) \\
&\quad \to \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u) - k) \sum_{i,j=1}^N \big(A_{ij}(k) - A_{ij}(T_\ell(u))\big)\, \alpha(x)_{x_ix_j}\, \theta(k)\, \phi\, dkdxdt \\
&\qquad - \iint_Q \theta(T_\ell(u)) \sum_{m=1}^M \Big(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))\Big)^2 \phi\, \alpha(x)\, dxdt \\
&\qquad - 2\iint_Q \Theta(T_\ell(u)) \sum_{m=1}^M \Big(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))\Big)\Big(\boldsymbol{\sigma}_m(T_\ell(u))\cdot\nabla \alpha(x)\Big)\phi\, dxdt \qquad (\lambda\downarrow 0)
\end{aligned}
$$

and

$$
\begin{aligned}
J_6^{\varepsilon,\lambda} &\to \iiiint_{|x-y|\le\lambda} S_0^+(\ell - k)\, \theta(k)\, \phi\, \alpha_\lambda(y)\, dkd\nu_\ell dy \qquad (\varepsilon\downarrow 0) \\
&\to \iint_Q\int_{\mathbf{R}} S_0^+(\ell - k)\, \theta(k)\, \phi\, \alpha(x)\, dkd\nu_\ell \qquad (\lambda\downarrow 0).
\end{aligned}
$$

Combining these estimates, (5.8) can be written as

$$\begin{aligned}
0 \ \le\ & \iint_Q \int_{\mathbf{R}} (T_\ell(u)-k)^+ \,\theta(k)\, \phi' \,\alpha(x)\, dk dx dt \\
&+ \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u)-k)\, f\, \theta(k)\, \phi\, \alpha(x)\, dk dx dt \\
&+ \iint_Q \Theta(T_\ell(u))\, \mathbf{F}(T_\ell(u)) \cdot \nabla\alpha(x)\, \phi\, dx dt \\
&+ \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u)-k) \sum_{i,j=1}^{N} \Big(A_{ij}(k) - A_{ij}(T_\ell(u))\Big)\, \alpha(x)_{x_i x_j}\, \theta(k)\, \phi\, dk dx dt \\
&- \iint_Q \theta(T_\ell(u)) \sum_{m=1}^{M} \Big(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))\Big)^2 \phi\, \alpha(x)\, dx dt \\
&- 2 \iint_Q \Theta(T_\ell(u)) \sum_{m=1}^{M} \Big(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))\Big) \Big(\boldsymbol{\sigma}_m(T_\ell(u)) \cdot \nabla\alpha(x)\Big) \phi\, dx dt \\
&+ \iint_Q \int_{\mathbf{R}} S_0^+(\ell-k)\, \theta(k)\, \phi\, \alpha(x)\, dk d\nu_\ell. \qquad (5.9)
\end{aligned}$$

Following the definition of an entropy-entropy flux triple, we see that

$$\begin{aligned}
\eta(T_\ell(u)) &= \int_{\mathbf{R}} \eta''(k)\, (T_\ell(u)-k)^+\, dk, \\
\eta'(T_\ell(u)) &= \int_{\mathbf{R}} \eta''(k)\, S_0^+(T_\ell(u)-k)\, dk, \\
q_i(T_\ell(u))_{x_i} &= \eta'(T_\ell(u))\, F_i(T_\ell(u))_{x_i}, \\
r_{ij}(T_\ell(u))_{x_i x_j} &= \eta'(T_\ell(u))_{x_j}\, A_{ij}(T_\ell(u))_{x_i} + \eta'(T_\ell(u))\, A_{ij}(T_\ell(u))_{x_i x_j}.
\end{aligned}$$

Putting $\theta = \eta''$ and $\Theta = \eta'$ in (5.9), we obtain that

$$\begin{aligned}
0 \ \le\ & \iint_Q \eta(T_\ell(u))\, \phi'\, \alpha\, dx dt + \iint_Q \eta'(T_\ell(u))\, f\, \phi\, \alpha\, dx dt \\
&+ \iint_Q \mathbf{q}(T_\ell(u)) \cdot \nabla\alpha\, \phi\, dx dt + \iint_Q \sum_{i,j=1}^{N} r_{ij}(T_\ell(u))\, \alpha_{x_i x_j}\, \phi\, dx dt \\
&- \iint_Q \eta''(T_\ell(u)) \sum_{m=1}^{M} \Big(\operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))\Big)^2 \phi\, \alpha\, dx dt + \iint_Q \eta'(\ell)\, \phi\, \alpha\, d\nu_\ell, \quad (5.10)
\end{aligned}$$

which is exactly (E3). Thus we complete the proof of the theorem. □

5.4 Applications

We now present two examples of renormalized dissipative solutions for relaxation systems.

Example 1: The definition of renormalized dissipative solutions for (CP) differs from the definition for hyperbolic equations mentioned in Chapter 4. For this reason, we reconsider the same relaxation system in Chapter 4.

Let a parabolic-hyperbolic equation (E): $u_t + \mathrm{div}\,\mathbf{F}(u) = \mathrm{div}\,(A(u)\nabla u) + f$ be given. We assume that the initial data $u_0(x)$ takes values in some interval and $\mathbf{F}(0) = \mathbf{0}$. Let $\omega_i > 0$ and suppose that $V_{n,i}$ satisfy the conditions

$$\sum_{i=1}^{N} V_{n,i}^{-1} \inf_{|u|\le n} F_i'(u) \;>\; -1$$

and

$$\left(1 + \sum_{j=1}^{N} \omega_j\right) V_{n,i}^{-1} \sup_{|u|\le n} F_i'(u) \;<\; \omega_i \left(1 + \sum_{j=1}^{N} V_{n,j}^{-1} \inf_{|u|\le n} F_j'(u)\right)$$

for $n = 1, 2, \cdots$ and $i = 1, 2, \cdots, N$. Following [KaT1997, Lemma 4.1], we see that there exist a strictly increasing function $r_n : [-n, n] \to \mathbf{R}$ defined by

$$w \;=\; r_n(u) \;:=\; \left(1 + \sum_{i=1}^{N} \omega_i\right)^{-1} \left(u + \sum_{i=1}^{N} V_{n,i}^{-1} F_i(u)\right)$$

and strictly decreasing functions $h_{n,i} : [r_n(-n), r_n(n)] \to \mathbf{R}$ with $h_{n,i}(0) = 0$ such that

$$w - \sum_{i=1}^{N} h_{n,i}(w) \;=\; u \quad \text{and} \quad \omega_i\, V_{n,i}\, w + V_{n,i}\, h_{n,i}(w) \;=\; F_i(u), \quad u \in [-n, n].$$

Now we consider the following relaxation system for w^ε and $\mathbf{z}^\varepsilon = (z_1^\varepsilon, \cdots, z_N^\varepsilon)$

with relaxation parameter $\varepsilon > 0$:

$$\text{(RS1)}\quad \begin{cases} w^\varepsilon_t + \sum_{i=1}^N \omega_i V_{n,i} w^\varepsilon_{x_i} = \frac{1}{\varepsilon}\sum_{i=1}^N (h_{n,i}(w^\varepsilon) - z^\varepsilon_i) & \text{in } Q, \\ (z^\varepsilon_i)_t - V_{n,i}\,(z^\varepsilon_i)_{x_i} = \frac{1}{\varepsilon}(h_{n,i}(w^\varepsilon) - z^\varepsilon_i) & \text{in } Q, \quad i = 1, \cdots, N, \\ w^\varepsilon(0,\cdot) = w_0 & \text{in } \mathbf{R}^N, \\ z^\varepsilon_i(0,\cdot) = z_{i0} & \text{in } \mathbf{R}^N,\ i = 1, \cdots, N, \end{cases}$$

with

$$a \le w_0 \le b \quad \text{and} \quad h_{n,i}(b) \le z_{i0} \le h_{n,i}(a). \tag{5.11}$$

Here $a < 0$ and $b > 0$ are constants satisfying

$$-n \le a + \sum_{i=1}^N h_{n,i}(b) \le b + \sum_{i=1}^N h_{n,i}(a) \le n.$$

We next set $u^\varepsilon = w^\varepsilon - \sum_{i=1}^N z^\varepsilon_i$ and $u_0 = w_0 - \sum_{i=1}^N z_{i0} \in L^1(\mathbf{R}^N)$. Then, from the result of Katsoulakis and Tzavaras [KaT1997], we see that $\overline{u}_n = \lim_{\varepsilon\downarrow 0} u^\varepsilon$ exists in $L^1(Q)$ and $\overline{u}_n$ is an entropy solution of (CP) with $A \equiv O$ and $f = 0$ satisfying $-n \le \overline{u}_n \le n$. Let $u_0 \in L^1(\mathbf{R}^N)$ and choose sequences of functions $\{w_{0,n}\}_{n\ge 1}$ and $\{z_{i0,n}\}_{n\ge 1}$ satisfying (5.11) for $i = 1, \cdots, N$. Moreover, we assume that $u_{0,n} = w_{0,n} - \sum_{i=1}^N z_{i0,n}$ converges to u_0 in $L^1(\mathbf{R}^N)$ as $n \to \infty$. Then we obtain an L^1 contraction property. Indeed, since the function $\overline{u}_n$ is a bounded entropy solution of (CP) with $A \equiv O$ and $f = 0$, we can apply the comparison property of entropy solutions. From these observations, we obtain that

Theorem 5.5. *The limit function $\overline{u} = \lim_{n\to\infty} \overline{u}_n$ in $L^1(Q)$ is a unique renormalized dissipative solution of* (CP) *with $A \equiv O$ and $f = 0$.*

Proof. We check that $\overline{u}$ satisfies (D3). Fix $\ell \ge 1$ and assume first $u_0 \ge -\ell$. Note that $u(t) \ge -\ell$ for all $t \ge 0$ whenever $u_0 \ge -\ell$ due to the invariant region property. Define t_0 by

$$t_0 = \begin{cases} 0 & \text{if } \ u(t) \in [-\ell, \ell] \ \text{ for all } \ t \geq 0, \\ \inf \{ t > 0 \,;\, u(t) = \ell \} & \text{otherwise}, \end{cases}$$

and set $Q_1 := (0, t_0] \times \mathbf{R}^N$ and $Q_2 := (t_0, T) \times \mathbf{R}^N$. We take any test functions $\xi \in C_0^2(\mathbf{R}^N)$ and $\theta \in C_0^2(\mathbf{R})^+$ with $\operatorname{spt} \theta \subset (-\ell, \ell)$, and let $\zeta = r_n(k + \xi)$ and $\psi_i = h_{n,i}(\zeta)$. Taking n large, we can choose $\gamma > 0$ such that $\gamma - \sum_{i=1}^N h_{n,i}(\gamma) = \ell$. Notice that constant functions $w \equiv \gamma$ and $z_i = h_{n,i}(\gamma)$ satisfy the contractive relaxation system (RS1), and therefore we see from [KoT2005] that

$$\begin{aligned} 0 \ \leq \ & \iint_{Q_1} \int_{\mathbf{R}} \theta(k) \Big\{ (\gamma - \zeta)^+ + \sum_{i=1}^N \big(h_{n,i}(\gamma) - \psi_i \big)^- \Big\} \phi' \, dk dx dt \\ & + \iint_{Q_1} \int_{\mathbf{R}} \theta(k) \Big\{ S_0^+(\gamma - \zeta) \Big[\frac{1}{\varepsilon} \sum_{i=1}^N (h_{n,i}(\zeta) - \psi_i) - \sum_{i=1}^N \omega_i V_{n,i} \zeta_{x_i} \Big] \\ & \qquad + \sum_{i=1}^N S_0^-(h_{n,i}(\gamma) - \psi_i) \Big[\frac{1}{\varepsilon} (h_{n,i}(\zeta) - \psi_i) + V_{n,i}(\psi_i)_{x_i} \Big] \Big\} \phi \, dk dx dt. \end{aligned}$$

The first term on the right hand side is 0 since the integrand without ϕ' is independent on t. We also note that

$$-S_0^-(h_{n,i}(\gamma) - h_{n,i}(\zeta)) \ = \ S_0^+(\gamma - \zeta) \ = \ S_0^+(r_n(\ell) - r_n(k + \xi)) \ = \ S_0^+(\ell - k - \xi),$$

$\gamma = r_n(\ell)$, $\zeta - \sum_{i=1}^N \psi_i = k + \xi$ and $\omega_i V_{n,i} \zeta + V_{n,i} \psi_i = F_i(k + \xi)$. Thus, the inequality becomes

$$0 \ \leq \ \iint_{Q_1} \int_{\mathbf{R}} \theta(k) \, S_0^+(\ell - k - \xi) \, \big(- \operatorname{div} \mathbf{F}(k + \xi) \big) \, \phi \, dk dx dt.$$

On the other hand, thanks to the comparison property for (RS1), we see that $u(t) \in [-\ell, \ell]$ for $t \in [t_0, T]$. A similar argument as in [P2003b, Theorem 2.1] leads to

$$
\begin{aligned}
0 \;\le\; & \iint_{Q_2}\int_{\mathbf{R}} \theta(k)\,(\overline{u}_n - k - \xi)^+\,\phi'\,dkdxdt \\
& + \iint_{Q_2}\int_{\mathbf{R}} \theta(k)\,S_0^+(\overline{u}_n - k - \xi)\,\big(-\operatorname{div}\mathbf{F}(k+\xi)\big)\,\phi\,dkdxdt \\
= \; & \iint_{Q_2}\int_{\mathbf{R}} \theta(k)\,S_0^+(\overline{u}_n - k - \xi)\Big\{\big(\overline{u}_n - k - \xi\big)\,\phi' - \operatorname{div}\mathbf{F}(k+\xi)\,\phi\Big\}\,dkdxdt,
\end{aligned}
$$

which implies that for any $\lambda > 0$,

$$
\begin{aligned}
0 \;\le\; & \frac{1}{\lambda}\iint_{Q_2}\int_{\mathbf{R}} \Bigg\{\bigg(\overline{u}_n - k - \xi + \lambda\Big\{\big(\overline{u}_n - k - \xi\big)\,\theta\,\phi' - \operatorname{div}\mathbf{F}(k+\xi)\,\theta\,\phi\Big\}\bigg)^+ \\
& \qquad\qquad - (\overline{u}_n - k - \xi)^+\Bigg\}\,dkdxdt.
\end{aligned}
$$

Passing to the limit as $n \to \infty$ first and then as $\lambda \downarrow 0$ yields

$$
\begin{aligned}
0 \;\le\; & \iiint_{\overline{u}-k-\xi>0} S_0^+(\overline{u} - k - \xi)\,\Big\{\big(\overline{u} - k - \xi\big)\,\theta\,\phi' - \operatorname{div}\mathbf{F}(k+\xi)\,\theta\,\phi\Big\}\,dkdxdt \\
& + \iiint_{\overline{u}-k-\xi=0} \Big(\big(\overline{u} - k - \xi\big)\,\theta\,\phi' - \operatorname{div}\mathbf{F}(k+\xi)\,\theta\,\phi\Big)^+\,dkdxdt \\
= \; & \iint_{Q_2}\int_{\mathbf{R}} (\overline{u} - k - \xi)^+\,\theta\,\phi'\,dkdxdt \\
& + \iint_{Q_2}\int_{\mathbf{R}} S_0^+(\overline{u} - k - \xi)\,\big(-\operatorname{div}\mathbf{F}(k+\xi)\big)\,\theta\,\phi\,dkdxdt.
\end{aligned}
$$

The same result can be obtained if $u_0 < -\ell$. Consequently, we conclude that

$$
\begin{aligned}
0 \;\le\; & \iint_{Q}\int_{\mathbf{R}} \theta(k)\,(T_\ell(\overline{u}) - k - \xi)^+\,\phi'\,dkdxdt \\
& + \iint_{Q}\int_{\mathbf{R}} \theta(k)\,S_0^+(T_\ell(\overline{u}) - k - \xi)\,\big(-\operatorname{div}\mathbf{F}(k+\xi)\big)\,\phi\,dkdxdt.
\end{aligned}
$$

This means $\overline{u}$ is a renormalized dissipative solution of (CP). By Theorem 5.4 and the uniqueness theorem in [BCW2000], we conclude that $\overline{u}$ is a unique solution of (CP). □

Example 2: We next consider the following system for w^ε and z^ε with relaxation parameter $\varepsilon > 0$:

$$\text{(RS2)} \qquad \begin{cases} w^\varepsilon_t + \operatorname{div} \mathbf{G}(w^\varepsilon) - \displaystyle\sum_{i,j=1}^{N} B_{ij}(w^\varepsilon)_{x_i x_j} = -\frac{1}{\varepsilon} w^\varepsilon z^\varepsilon & \text{in } Q, \\ z^\varepsilon_t = -\dfrac{1}{\varepsilon} w^\varepsilon z^\varepsilon & \text{in } Q, \\ w^\varepsilon(0,\cdot) = w_0 & \text{in } \mathbf{R}^N, \\ z^\varepsilon(0,\cdot) = z_0 & \text{in } \mathbf{R}^N, \end{cases}$$

with

$$0 \le z_0 \le a \quad \text{and} \quad 0 \le w_0 \le g_n(a) \qquad \text{a.e. in } \mathbf{R}^N,$$

where $g_n : [0,n] \to \mathbf{R}^+$ is a strictly increasing function and a is a nonnegative constant such that $-n \le -a \le g_n(a) \le n$ for $n = 1, 2, \cdots$. In addition, we assume on the data as follows:

(H1) $B_{ij} = B_{ji} \in C^2(\mathbf{R})$ and $B = (b_{ij}) \ge 0$ with $b_{ij}(\cdot) := B'_{ij}(\cdot)$ and $b_{ij}(0) = 0$ for $i, j = 1, \cdots, N$.

(H2) $\mathbf{G} : \mathbf{R} \to \mathbf{R}^N$ is a locally Lipschitz-continuous flux with $\mathbf{G}(0) = \mathbf{0}$.

(H3) $w_0, z_0 \in \big(L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)\big)^+$ with $\int_{\mathbf{R}^N} |x|^2\, w_0\, dx < \infty$.

(H4) For $i, j, m = 1, \cdots, N$,

$$\sum_{m=1}^{N} \tau_{im}(s)\, \tau_{jm}(s) = b_{ij}(s) \quad \text{and} \quad \gamma'_{im}(s) = \tau_{im}(s) \quad \text{for } s \in \mathbf{R},$$

and $\boldsymbol{\gamma}_m(w^\varepsilon) \in L^2(Q)^N$ with $\boldsymbol{\gamma}_m(s) := (\gamma_{1m}(s), \cdots, \gamma_{Nm}(s))$ for $s \in \mathbf{R}$.

This system describes the evolution of a chemical or a biological species which is called a tracer in a porous medium. This tracer is supposed to be stuck on the surface of the solid frame. Belhadj et al. [BGP2003] studied this system and obtained the existence of entropy solutions with continuously differentiable flux $\mathbf{G}$. In case of locally Lipschitz-continuous $\mathbf{G}$, as in the analogous argument we obtain the following results:

Proposition 5.6. *Suppose that* (H1)-(H4)*. Then, the problem* (RS2) *has a unique entropy solution* $(w^\varepsilon, z^\varepsilon) \in C((0,T); L^1(\mathbf{R}^N))^2$ *satisfying the following properties:*

(P1) $0 \le w^\varepsilon(t,x) \le \|w_0\|_{L^\infty(\mathbf{R}^N)}$ *and* $0 \le z^\varepsilon(t,x) \le \|z_0\|_{L^\infty(\mathbf{R}^N)}$ *almost every* $(t,x) \in Q$.

(P2) *If* $(w^\varepsilon, z^\varepsilon)$ *and* $(\overline{w}^\varepsilon, \overline{z}^\varepsilon)$ *are two solutions corresponding to the initial data* (w_0, z_0) *and* $(\overline{w}_0, \overline{z}_0)$*, respectively, then we have*

$$\begin{aligned} &\|w^\varepsilon(t) - \overline{w}^\varepsilon(t)\|_{L^1(\mathbf{R}^N)} + \|z^\varepsilon(t) - \overline{z}^\varepsilon(t)\|_{L^1(\mathbf{R}^N)} \\ &\quad \le \ \|w_0 - \overline{w}_0\|_{L^1(\mathbf{R}^N)} + \|z_0 - \overline{z}_0\|_{L^1(\mathbf{R}^N)} \qquad \text{for all} \ \ t \ge 0. \end{aligned}$$

(P3) *Let* $(w^\varepsilon, z^\varepsilon)$ *and* $(\overline{w}^\varepsilon, \overline{z}^\varepsilon)$ *be two solutions corresponding to the initial data* (w_0, z_0) *and* $(\overline{w}_0, \overline{z}_0)$*, respectively. If* $w_0 \le \overline{w}_0$ *and* $z_0 \le \overline{z}_0$*, then we have*

$$w^\varepsilon(t) \le \overline{w}^\varepsilon(t) \quad \text{and} \quad z^\varepsilon(t) \le \overline{z}^\varepsilon(t) \quad \text{a.e. in} \ \ \mathbf{R}^N.$$

(P4) $\operatorname{div} \boldsymbol{\gamma}_m(w^\varepsilon) \in L^2(Q)$ *for* $m = 1, \cdots, N$.

Proposition 5.7. *Suppose that* (H1)-(H4)*. Let* $n \ge 1$*,* $u^\varepsilon = w^\varepsilon - z^\varepsilon$ *and* $u_0 = w_0 - z_0 \in L^1(\mathbf{R}^N)$*. Then,* $\overline{u}_n = \lim_{\varepsilon \downarrow 0} u^\varepsilon$ *exists in* $L^1(Q)$ *and* $\overline{u}_n \in [-n, n]$ *is a unique entropy solution of the following generalized Stefan problem:*

$$\text{(GSP)} \qquad \begin{cases} u_t + \operatorname{div} \mathbf{G}(u^+) - \displaystyle\sum_{i,j=1}^N B_{ij}(u^+)_{x_i x_j} \ = \ 0 & \text{in} \ \ Q, \\ \qquad\qquad\qquad\qquad\qquad u(0,\cdot) \ = \ u_0 & \text{in} \ \ \mathbf{R}^N. \end{cases}$$

From these propositions, we finally obtain that

Theorem 5.8. *Suppose that* (H1)-(H4)*. Then, the limit function* $\overline{u} = \lim_{n\to\infty} \overline{u}_n$ *in* $L^1(Q)$ *is a unique renormalized dissipative solution of the generalized Stefan problem* (GSP)*.*

Proof. Recall the definition of renormalized dissipative solutions and show that for any $\ell > 0$, $\xi \in C_0^2(\mathbf{R}^N)$ and $\theta \in C_0^2(\mathbf{R})^+$ with $\operatorname{spt}\theta \subset (-\ell, \ell)$, there exists a sequence $\{\nu_\ell\} \subset \mathcal{R}_b(Q)^+$ with $\nu_\ell(Q) \to 0$ as $\ell \to \infty$ such that

$$
\begin{aligned}
0 \le & \iint_Q \int_{\mathbf{R}} \theta(k) \left(T_\ell(\overline{u}) - k - \xi\right)^+ \phi' \, dk dx dt \\
& + \iint_Q \int_{\mathbf{R}} \theta(k) \, S_0^+(T_\ell(\overline{u}) - k - \xi) \\
& \qquad \times \left(-\operatorname{div} \mathbf{G}((k+\xi)^+) + \sum_{i,j=1}^N B_{ij}((k+\xi)^+)_{x_i x_j} \right) \phi \, dk dx dt \\
& - \iint_Q \theta(T_\ell(\overline{u}) - \xi) \sum_{m=1}^N \left(\operatorname{div} \boldsymbol{\gamma}_m(T_\ell(\overline{u})^+) - \boldsymbol{\tau}_m(T_\ell(\overline{u})^+) \cdot \nabla \xi \right)^2 \phi \, dx dt \\
& + \iint_Q \int_{\mathbf{R}} \theta(k) \, S_0^+(\ell - k - \xi) \, \phi \, dk d\nu_\ell \qquad \text{for any} \quad \phi \in C_0^1(0,T)^+,
\end{aligned}
\tag{5.12}
$$

where $\boldsymbol{\tau}_m(\cdot) := (\tau_{1m}(\cdot), \cdots, \tau_{Nm}(\cdot))$.

To this end, we fix $\ell \ge 1$. In a similar argument as Example 1, we first assume that $u_0 \ge -\ell$, and define t_0 by

$$
t_0 = \begin{cases} 0 & \text{if} \quad u(t) \in [-\ell, \ell] \quad \text{for all} \quad t \ge 0, \\ \inf \left\{ t > 0 \, ; \, u(t) = \ell \right\} & \text{otherwise.} \end{cases}
$$

We now set $Q_1 := (0, t_0] \times \mathbf{R}^N$ and $Q_2 := (t_0, T) \times \mathbf{R}^N$, and take any test functions $\xi \in C_0^2(\mathbf{R}^N)$ and $\theta \in C_0^2(\mathbf{R})^+$ with $\operatorname{spt}\theta \subset (-\ell, \ell)$. We can consider constant functions $w \equiv \ell$ and $z \equiv 0$ by taking n large. Since this pair satisfies the contractive relaxation system (RS2) with appropriate test functions, we see that

$$
\begin{aligned}
0 \ \le\ & \iint_{Q_1}\int_{\mathbf{R}} \theta(k)\,(\ell-k-\xi)^+\phi'\,dkdxdt \\
&+\iint_{Q_1}\int_{\mathbf{R}} \theta(k)\,S_0^+(\ell-k-\xi) \\
&\qquad\qquad \times\left(-\mathrm{div}\,\mathbf{G}((k+\xi)^+)+\sum_{i,j=1}^N B_{ij}((k+\xi)^+)_{x_ix_j}\right)\phi\,dkdxdt \\
&-\iint_{Q_1} \theta(\ell-\xi)\sum_{m=1}^N \big(-\boldsymbol{\tau}_m(\ell)\cdot\nabla\xi\big)^2\phi\,dkdxdt.
\end{aligned}
$$

On the other hand, if $t\in[t_0,T]$, then by the comparison property for (RS2) we see that $u(t)\in[-\ell,\ell]$. From Proposition 5.7 and the equivalence result [PS2003, Theorem 1.1], we obtain that

$$
\begin{aligned}
0 \ \le\ & \iint_{Q_2}\int_{\mathbf{R}} \theta(k)\,(\overline{u}_n-k-\xi)^+\,\phi'\,dkdxdt \\
&+\iint_{Q_2}\int_{\mathbf{R}} \theta(k)\,S_0^+(\overline{u}_n-k-\xi) \\
&\qquad\qquad \times\left(-\mathrm{div}\,\mathbf{G}((k+\xi)^+)+\sum_{i,j=1}^N B_{ij}((k+\xi)^+)_{x_ix_j}\right)\phi\,dkdxdt \\
&-\iint_{Q_2} \theta(\overline{u}_n-\xi)\sum_{m=1}^N \big(\mathrm{div}\,\boldsymbol{\gamma}_m(\overline{u}_n^+)-\boldsymbol{\tau}_m(\overline{u}_n^+)\cdot\nabla\xi\big)^2\phi\,dkdxdt \\
=\ & \iint_{Q_2}\int_{\mathbf{R}} S_0^+(\overline{u}_n-k-\xi)\,h(\overline{u}_n,k)\,dkdxdt,
\end{aligned}
$$

where

$$
h(\overline{u}_n,k)\ :=\ (\overline{u}_n-k-\xi)\,\theta\,\phi'+\left(-\,\mathrm{div}\,\mathbf{G}((k+\xi)^+)+\sum_{i,j=1}^N B_{ij}(\overline{u}_n^+)_{x_ix_j}\right)\theta\,\phi.
$$

From this, we obtain that for any $\lambda>0$,

$$
0\le \frac{1}{\lambda}\iint_{Q_2}\int_{\mathbf{R}}\Big\{\big(\overline{u}_n-k-\xi+\lambda\,h(\overline{u}_n,k)\big)^+-(\overline{u}_n-k-\xi)^+\Big\}\,dkdxdt.
$$

Passing to the limit as $n \to \infty$ first and then as $\lambda \downarrow 0$ yields

$$\begin{aligned}
0 \;\le\; & \iiint_{\overline{u}-k-\xi>0} S_0^+(\overline{u}-k-\xi)\, h(\overline{u},k)\, dkdxdt + \iiint_{\overline{u}-k-\xi=0} h(\overline{u},k)^+\, dkdxdt \\
= \; & \iint_{Q_2}\int_{\mathbf{R}} (\overline{u}-k-\xi)^+\, \theta\, \phi'\, dkdxdt \\
& + \iint_{Q_2}\int_{\mathbf{R}} S_0^+(\overline{u}-k-\xi) \\
& \qquad \times \left(-\operatorname{div} \mathbf{G}((k+\xi)^+) + \sum_{i,j=1}^{N} B_{ij}((k+\xi)^+)_{x_i x_j} \right) \theta\, \phi\, dkdxdt \\
& - \iint_{Q} \theta(\overline{u}-\xi) \sum_{m=1}^{N} \big(\operatorname{div} \boldsymbol{\gamma}_m(\overline{u}^+) - \boldsymbol{\tau}_m(\overline{u}^+)\cdot\nabla\xi \big)^2 \phi\, dxdt.
\end{aligned}$$

The same result can be obtained if $u_0 < -\ell$. Consequently, we prove that

$$\begin{aligned}
0 \;\le\; & \iint_{Q}\int_{\mathbf{R}} \theta(k)\, \big(T_\ell(\overline{u})-k-\xi\big)^+\, \phi'\, dkdxdt \\
& + \iint_{Q}\int_{\mathbf{R}} \theta(k)\, S_0^+(T_\ell(\overline{u})-k-\xi) \\
& \qquad \times \left(-\operatorname{div} \mathbf{G}((k+\xi)^+) + \sum_{i,j=1}^{N} B_{ij}((k+\xi)^+)_{x_i x_j} \right) \phi\, dkdxdt \\
& - \iint_{Q} \theta(T_\ell(\overline{u})-\xi) \sum_{m=1}^{N} \big(\operatorname{div} \boldsymbol{\gamma}_m(T_\ell(\overline{u})^+) - \boldsymbol{\tau}_m(T_\ell(\overline{u})^+)\cdot\nabla\xi \big)^2\, \phi\, dxdt
\end{aligned}$$

for any $\phi \in C_0^1(0,T)^+$. This means $\overline{u}$ is a renormalized dissipative solution of (GSP). Moreover, by the uniqueness theorem in [BCW2000], we conclude that $\overline{u}$ is a unique solution. □

Remark 5.9. *We now check that* (5.12) *is meaningful. In other words, we shall* *prove that for* $u \in L^1(Q) \cap L^\infty(Q)$ *and any* $\phi \in C_0^1(0,T)^+$

$$\begin{aligned}
0 \;\le\; & \iint_Q \int_{\mathbf{R}} \theta(k)\,(u-k-\xi)^+\,\phi'\,dkdxdt \\
& + \iint_Q \int_{\mathbf{R}} \theta(k)\,S_0^+(u-k-\xi) \\
& \qquad \times \left(-\mathrm{div}\,\mathbf{G}((k+\xi)^+) + \sum_{i,j=1}^N B_{ij}((k+\xi)^+)_{x_i x_j} \right) \phi\,dkdxdt \\
& - \iint_Q \theta(u-\xi) \sum_{m=1}^N \Big(\mathrm{div}\,\boldsymbol{\gamma}_m(u^+) - \boldsymbol{\tau}_m(u^+)\cdot\nabla\xi \Big)^2 \phi\,dxdt.
\end{aligned}$$

Let $u \in L^1(Q) \cap L^\infty(Q)$*. For any test function* $\zeta \in C_0^2(Q)$ *we have the following estimates:*

$$\begin{aligned}
& \iint_Q \int_{\mathbf{R}} \theta(k) \big(S_0^+(u-k-\xi)\,(u-k-\xi) \big)_t\,\zeta\,dkdxdt \\
& \quad = \iint_Q \int_{\mathbf{R}} \theta(k)\,\delta(u-k-\xi)\,(u-k-\xi)\,u_t\,\zeta\,dkdxdt \\
& \qquad + \iint_Q \int_{\mathbf{R}} \theta(k)\,S_0^+(u-k-\xi)\,u_t\,\zeta\,dkdxdt \\
& \quad = \iint_Q \int_{\mathbf{R}} \theta(k)\,S_0^+(u-k-\xi)\,u_t\,\zeta\,dkdxdt,
\end{aligned}$$

$$\begin{aligned}
& \iint_Q \int_{\mathbf{R}} \theta(k)\,\mathrm{div}\Big\{ S_0^+(u-k-\xi)\,\big(\mathbf{G}(u^+) - \mathbf{G}((k+\xi)^+)\big) \Big\}\,\zeta\,dkdxdt \\
& \quad = \iint_Q \int_{\mathbf{R}} \theta(k)\,\delta(u-k-\xi)\,\big(\mathbf{G}(u^+) - \mathbf{G}((k+\xi)^+)\big)\cdot\nabla(u-\xi)\,\zeta\,dkdxdt \\
& \qquad + \iint_Q \int_{\mathbf{R}} \theta(k)\,S_0^+(u-k-\xi)\,\mathrm{div}\big(\mathbf{G}(u^+) - \mathbf{G}((k+\xi)^+)\big)\,\zeta\,dkdxdt \\
& \quad = \iint_Q \int_{\mathbf{R}} \theta(k)\,S_0^+(u-k-\xi)\,\mathrm{div}\big(\mathbf{G}(u^+) - \mathbf{G}((k+\xi)^+)\big)\,\zeta\,dkdxdt
\end{aligned}$$

and

$$
\begin{aligned}
&-\iint_Q\int_{\mathbf{R}}\theta(k)\sum_{i,j=1}^{N}\Big\{S_0^+(u-k-\xi)\left(B_{ij}(u^+)-B_{ij}((k+\xi)^+)\right)\Big\}_{x_ix_j}\zeta\,dkdxdt \\
&= -\iint_Q\int_{\mathbf{R}}\theta(k)\,\zeta\sum_{i,j=1}^{N}\Big\{\delta(u-k-\xi)\left(B_{ij}(u^+)-B_{ij}((k+\xi)^+)\right)(u-\xi)_{x_i}\Big\}_{x_j}dkdxdt \\
&\quad -\iint_Q\int_{\mathbf{R}}\theta(k)\sum_{i,j=1}^{N}\Big\{S_0^+(u-k-\xi)\left(B_{ij}(u^+)-B_{ij}((k+\xi)^+)\right)_{x_i}\Big\}_{x_j}\zeta\,dkdxdt \\
&= \iint_Q\int_{\mathbf{R}}\theta(k)\sum_{i,j=1}^{N}\delta(u-k-\xi)\left(B_{ij}(u^+)-B_{ij}((k+\xi)^+)\right)(u-\xi)_{x_i}\,\zeta_{x_j}\,dkdxdt \\
&\quad -\iint_Q\int_{\mathbf{R}}\theta(k)\,\zeta\sum_{i,j=1}^{N}\delta(u-k-\xi)\left(B_{ij}(u^+)-B_{ij}((k+\xi)^+)\right)_{x_i}(u-\xi)_{x_j}dkdxdt \\
&\quad -\iint_Q\int_{\mathbf{R}}\theta(k)\sum_{i,j=1}^{N}S_0^+(u-k-\xi)\left(B_{ij}(u^+)-B_{ij}((k+\xi)^+)\right)_{x_ix_j}\zeta\,dkdxdt.
\end{aligned}
$$

As to the second term of the last estimate, we see that

$$
\begin{aligned}
&-\iint_Q\int_{\mathbf{R}}\theta(k)\sum_{i,j=1}^{N}\delta(u-k-\xi)\left(B_{ij}(u^+)-B_{ij}((k+\xi)^+)\right)_{x_i}(u-\xi)_{x_j}\zeta\,dkdxdt \\
&= -\iint_Q\int_{\mathbf{R}}\theta(k)\,\delta(u-k-\xi)\,\zeta\sum_{i,j=1}^{N}\left(b_{ij}(u^+)\,u^+_{x_i}-b_{ij}((k+\xi)^+)\,(k+\xi)^+_{x_i}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\,(u-\xi)_{x_j}\,dkdxdt \\
&= -\iint_Q\int_{\mathbf{R}}\theta(k)\,\delta(u-k-\xi)\,\zeta\sum_{i,j=1}^{N}b_{ij}(u^+)\,u^+_{x_i}\,(u-\xi)_{x_j}\,dkdxdt \\
&\quad +\iint_Q\int_{\mathbf{R}}\theta(k)\,\delta(u-k-\xi)\,\zeta\sum_{i,j=1}^{N}b_{ij}((k+\xi)^+)\,(k+\xi)^+_{x_i}\,(u-\xi)_{x_j}\,dkdxdt
\end{aligned}
$$

$$
\begin{aligned}
= & -\iint_Q \int_{\mathbf{R}} \theta(k)\, \delta(u-k-\xi)\, \zeta \sum_{i,j=1}^{N} b_{ij}(u^+)\, u^+_{x_i}\, (u-\xi)_{x_j}\, dkdxdt \\
& + \iint_Q \int_{\mathbf{R}} \theta(k)\, \delta(u-k-\xi)\, \zeta \sum_{i,j=1}^{N} \delta(k+\xi)\, b_{ij}((k+\xi)^+)\, (k+\xi)\, \xi_{x_i} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times (u-\xi)_{x_j}\, dkdxdt \\
& + \iint_Q \int_{\mathbf{R}} \theta(k)\, \delta(u-k-\xi)\, \zeta \sum_{i,j=1}^{N} S_0^+(k+\xi)\, b_{ij}((k+\xi)^+)\, \xi_{x_i}\, (u-\xi)_{x_j}\, dkdxdt \\
= & -\iint_Q \theta(u-\xi)\, \zeta \sum_{i,j=1}^{N} b_{ij}(u^+)\, u^+_{x_i}\, (u^+-\xi)_{x_j}\, dxdt \\
& + \iint_Q \theta(u-\xi)\, \zeta \sum_{i,j=1}^{N} S_0^+(u)\, b_{ij}(u^+)\, \xi_{x_i}\, (u^+-\xi)_{x_j}\, dxdt \\
= & -\iint_Q \theta(u-\xi)\, \zeta \sum_{i,j=1}^{N} b_{ij}(u^+)\, (u^+-\xi)_{x_i}\, (u^+-\xi)_{x_j}\, dxdt \\
= & -\iint_Q \theta(u-\xi)\, \zeta \sum_{i,j=1}^{N} \sum_{m=1}^{N} \tau_{im}(u^+)\, \tau_{jm}(u^+) \\
& \qquad\qquad\qquad\qquad \times \left(u^+_{x_i}\, u^+_{x_j} - u^+_{x_i}\, \xi_{x_j} - u^+_{x_j}\, \xi_{x_i} + \xi_{x_i}\, \xi_{x_j} \right) dxdt \\
= & -\iint_Q \theta(u-\xi)\, \zeta \sum_{i,j=1}^{N} \sum_{m=1}^{N} \big(\gamma_{im}(u^+)_{x_i}\, \gamma_{jm}(u^+)_{x_j} - \gamma_{im}(u^+)_{x_i}\, \tau_{jm}(u^+)\, \xi_{x_j} \\
& \qquad\qquad - \gamma_{jm}(u^+)_{x_j}\, \tau_{im}(u^+)\, \xi_{x_i} + \tau_{im}(u^+)\, \tau_{jm}(u^+)\, \xi_{x_i}\, \xi_{x_j} \big)\, dxdt \\
= & -\iint_Q \theta(u-\xi)\, \zeta \sum_{i,j=1}^{N} \sum_{m=1}^{N} \left(\gamma_{im}(u^+)_{x_i} - \tau_{im}(u^+)\, \xi_{x_i} \right) \\
& \qquad\qquad\qquad\qquad\qquad \times \left(\gamma_{jm}(u^+)_{x_j} - \tau_{jm}(u^+)\, \xi_{x_j} \right) dxdt \\
= & -\iint_Q \theta(u-\xi)\, \zeta \sum_{m=1}^{N} \left(\sum_{i=1}^{N} \left(\gamma_{im}(u^+)_{x_i} - \tau_{im}(u^+)\, \xi_{x_i} \right) \right)^2 dxdt \\
= & -\iint_Q \theta(u-\xi)\, \zeta \sum_{m=1}^{N} \Big(\operatorname{div} \boldsymbol{\gamma}_m(u^+) - \boldsymbol{\tau}_m(u^+) \cdot \nabla \xi \Big)^2 dxdt.
\end{aligned}
$$

Combining these estimates and putting $\phi \in C_0^1(0,T)^+$ into ζ, we obtain that

$$\begin{aligned}
&\iint_Q \int_{\mathbf{R}} \theta(k)\,\phi\,\Big[\big(S_0^+(u-k-\xi)\,(u-k-\xi)\big)_t \\
&\qquad + \operatorname{div}\Big\{S_0^+(u-k-\xi)\,\big(\mathbf{G}(u^+)-\mathbf{G}((k+\xi)^+)\big)\Big\} \\
&\qquad - \sum_{i,j=1}^N \Big\{S_0^+(u-k-\xi)\,\big(B_{ij}(u^+)-B_{ij}((k+\xi)^+)\big)\Big\}_{x_i x_j}\Big]\,dkdxdt \\
&\le \iint_Q \int_{\mathbf{R}} \theta(k)\,S_0^+(u-k-\xi)\,u_t\,\phi\,dkdxdt \\
&\qquad + \iint_Q \int_{\mathbf{R}} \theta(k)\,S_0^+(u-k-\xi)\operatorname{div}\big(\mathbf{G}(u^+)-\mathbf{G}((k+\xi)^+)\big)\,\phi\,dkdxdt \\
&\qquad - \iint_Q \int_{\mathbf{R}} \theta(k) \sum_{i,j=1}^N S_0^+(u-k-\xi)\,\big(B_{ij}(u^+)-B_{ij}((k+\xi)^+)\big)_{x_i x_j}\phi\,dkdxdt \\
&\qquad - \iint_Q \theta(u-\xi) \sum_{m=1}^N \Big(\operatorname{div}\boldsymbol{\gamma}_m(u^+) - \boldsymbol{\tau}_m(u^+)\cdot\nabla\xi\Big)^2 \phi\,dxdt \\
&= \iint_Q \int_{\mathbf{R}} \theta(k)\,S_0^+(u-k-\xi) \\
&\qquad\quad \times \left(-\operatorname{div}\mathbf{G}((k+\xi)^+) + \sum_{i,j=1}^N B_{ij}((k+\xi)^+)_{x_i x_j}\right)\phi\,dkdxdt \\
&\qquad - \iint_Q \theta(u-\xi) \sum_{m=1}^N \Big(\operatorname{div}\boldsymbol{\gamma}_m(u^+) - \boldsymbol{\tau}_m(u^+)\cdot\nabla\xi\Big)^2 \phi\,dxdt.
\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}
&\iint_Q \int_{\mathbf{R}} \theta(k)\,\phi\,\Big[\big(S_0^+(u-k-\xi)\,(u-k-\xi)\big)_t \\
&\qquad + \operatorname{div}\Big\{S_0^+(u-k-\xi)\,\big(\mathbf{G}(u^+)-\mathbf{G}((k+\xi)^+)\big)\Big\} \\
&\qquad - \sum_{i,j=1}^N \Big\{S_0^+(u-k-\xi)\,\big(B_{ij}(u^+)-B_{ij}((k+\xi)^+)\big)\Big\}_{x_i x_j}\Big]\,dkdxdt \\
&= -\iint_Q \int_{\mathbf{R}} \theta(k)\,S_0^+(u-k-\xi)\,(u-k-\xi)\,\phi'\,dkdxdt \\
&= -\iint_Q \int_{\mathbf{R}} \theta(k)\,(u-k-\xi)^+\,\phi'\,dkdxdt.
\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}
0 \;\le\; & \iint_Q \int_{\mathbf{R}} \theta(k)\,(u-k-\xi)^+\,\phi'\,dk dx dt \\
& + \iint_Q \int_{\mathbf{R}} \theta(k)\,S_0^+(u-k-\xi) \\
& \qquad \times \left(-\operatorname{div} \mathbf{G}((k+\xi)^+) + \sum_{i,j=1}^{N} B_{ij}((k+\xi)^+)_{x_i x_j} \right) \phi\,dk dx dt \\
& - \iint_Q \theta(u-\xi) \sum_{m=1}^{N} \Big(\operatorname{div} \boldsymbol{\gamma}_m(u^+) - \boldsymbol{\tau}_m(u^+)\cdot\nabla\xi \Big)^2 \phi\,dx dt.
\end{aligned}$$

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