ON VASSILIEV INVARIANTS AND C_n-MOVES FOR KNOTS

（結び目のヴァシリエフ不変量とC_n-movesについて）

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ON VASSILIEV INVARIANTS AND C_n-MOVES FOR KNOTS

BY
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Introduction

In 1990, V. A. Vassiliev [39] defined a series of $\mathbb{Z}$-valued knot invariants to study on the cohomology of the space of all knots. Afterwards, J. S. Birman and X.-S. Lin [4] proposed a combinatorial way to calculate them.

Is Vassiliev invariant complete invariant? We have not got the answer to it, but when we fix a natural number n, order of Vassiliev invariant, Lin, Y. Ohyama, T. Stanford proved Proposition A [19, 27, 35].

Proposition A. *Let n be a natural number and K an oriented knot. Then there are infinitely many knots J_m $(m = 1, 2, \cdots)$ such that $v(J_m) = v(K)$ for any Vassiliev invariant v of order less than or equal to n.*

M. N. Goussarov [7] and K. Habiro [9] gave one answer to the question: when two knots have the same values for Vassiliev invariants of order less than or equal to n, what kind of topological properties do they have. They introduced a local move which is defined as C_n-*move* and proved Proposition B independently.

Proposition B. *Two oriented knots have the same Vassiliev invariant of order less than or equal to n if and only if they are transformed into each other by C_{n+1}-moves.*

If two knots K and K' can be transformed into each other by C_n-moves, we denote the minimal number of C_n-moves needed to transform K into K' by $d_{C_n}(K, K')$ and call it the *C_n-distance between K and K'*.

There is much still unknown part what property Vassiliev invariants have, and many approaches have been done. From Proposition B we set problems below.

Problem 1. *For given natural number n, do Vassiliev invariant of order less than or equal to n have any information about C_k-distance between two knots?*

Problem 2. *For given natural numbers m and n ($m \geq n$), if $d_{C_n}(K, K') = 1$ what is the value of $v_m(K) - v_m(K')$, where $v_m(K)$ is a Vassiliev invariant of order m of the knot K?*

Problem 2 relates to the distance of knots on the C_n-moves.

On Problem 1, in the case of $k = 1$, Ohyama, K. Taniyama and S. Yamada [30], and Ohyama [28] showed that Vassiliev invariants have no information on C_1-distance. On Problem 2, for the case $m = n$, M. Okada [32] ($n = 2$), T. Tsukamoto [38] ($n = 3$), and B. Matsuzaka [20] ($n = 4$) determined concrete value and for any n Ohyama and Tsukamoto [31] showed the relation between C_n-move and Vassiliev invariant of order n.

In Chapter 1, we give the results related to Problem 1 for $k = 2, 3$ by restricting the property of J_m in Proposition A.

In Chapter 2, on Problem 2 we consider m-th coefficient of Conway polynomial a_m for most elementary Vassiliev invariant of order m and study the relation between C_n-move and a_m. In this paper, we treat the case of $n = 2, 3, 4$. We are getting on with general case.

In Chapter 3, we apply the result giving in Chapter 2 to the argument of Dehn surgery.

It is known that any closed orientable 3-manifold is a surgery manifold of some framed link in $\mathbb{S}^3$, and if two framed links determine the same surgery manifold, then they are related by a finite sequence of Kirby moves [14]. We do not have similar relation when we restrict it to knot. But if we specify the framing number, there are some results. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be framed knots. In the case the framings are 0, it is known that $\nabla_{K_1}(z) = \nabla_{K_1}(z)$ if $\chi(\mathbb{S}^3; \mathcal{K}_1) = \chi(\mathbb{S}^3; \mathcal{K}_2)$, where $\nabla_K(z)$ is a Conway polynomial of K. So we give Problem 3.

Problem 3. *When $\chi(\mathbb{S}^3; \mathcal{K}_1) = \chi(\mathbb{S}^3; \mathcal{K}_2)$, how do the Conway polynomial $\nabla_{K_1}(z)$ and $\nabla_{K_2}(z)$ relate to each other?*

On Problem 3, when the framings are ± 1, we have $|a_2(K_1)| = |a_2(K_2)|$ from Casson surgery formula [34]. Moreover, W. B. R. Lickorish showed that their Conway polynomials can differ [18]. In Chapter 3, we will show that there is no restriction on the coefficient of higher order of Conway polynomial by applying Theorem 2.1.3.

Chapter 1

Delta and clasp-pass distances and Vassiliev invariants of knots

1.1 Introduction and results

In 1990, V. A. Vassiliev defined a sequence of knot invariants which is now called Vassiliev invariants [39]. After that, for any knot K and any integer n, some examples of knots have been constructed whose Vassiliev invariants of order less than or equal to n coincide with those of K [7, 19, 27, 35]. Recently Y. Ohyama, K. Taniyama and S. Yamada [30], and Ohyama [28] gave such examples of knots whose unknotting numbers are equal to one.

Theorem 1.1.1 ([30, 28])**.** *Let n be a natural number and K an oriented knot in $\mathbb{S}^3$. Then there are infinitely many unknotting number one knots J_m $(m = 1, 2, \cdots)$ such that $v(J_m) = v(K)$ for any Vassiliev invariant v of order less than or equal to n.*

Therefore, for fixed n, all the Vassiliev invariants of order less than or equal to n do not detect the knots whose unknotting number is greater than one.

In this chapter, we consider similar problem for C_k-distance, and give the result on delta and clasp-pass distances. Namely we get the result for C_2 and C_3-distances.

A *delta move* is a local move as illustrated in Fig. 1.1.1. Delta move is defined in [21, 24] and it is shown that any oriented knots K and K' are transformed into each other by delta moves. We denote the minimal number of delta moves that is needed to transform K into K' by $d_{\triangle}(K, K')$ and call it the *delta distance* of K and K'. Let T be a trivial knot. Then we denote $d_{\triangle}(K, T)$ by $u_{\triangle}(K)$ and call it the *delta unknotting number* of K. We denote the second coefficient of the Conway polynomial of K by $a_2(K)$. It is well-known that $a_2(K)$ is a Vassiliev invariant of order 2, and that any Vassiliev invariant of order 2 is determined by it. M. Okada showed Theorem 1.1.2.

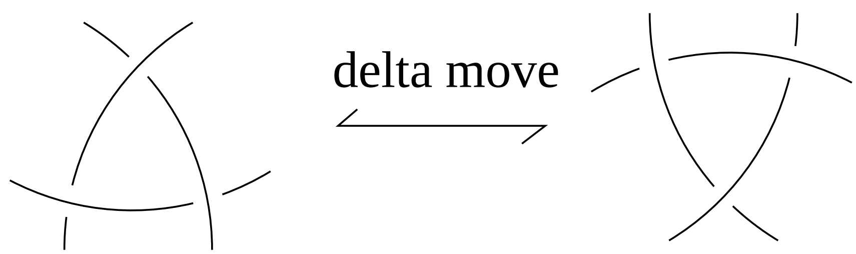


Fig. 1.1.1

Theorem 1.1.2 ([32]). *Let K and K' be oriented knots. If K' is obtained from K by a delta move, then $a_2(K) = a_2(K') \pm 1$.*

By Theorem 1.1.2, we can estimate the delta unknotting number of a knot K.

Corollary 1.1.3. *Let K be a nontrivial knot and $a_2(K) = p$. Then if $p \neq 0$, $u_\Delta(K) \geq |p|$ and if $p = 0$, $u_\Delta(K) \geq 2$.*

See [25] for an application of Corollary 1.1.3.

When we think about Problem 1 only from the view of delta distance, we have Theorem 1.1.4.

Theorem 1.1.4. *Let n be a natural number and K and M oriented knots in $\mathbb{S}^3$.*

(1) *Suppose that $a_2(K) \neq a_2(M)$. Then there are infinitely many knots J_m $(m = 1, 2, \cdots)$ with $d_\triangle(J_m, M) = |a_2(K) - a_2(M)|$ such that $v(J_m) = v(K)$.*
(2) *Suppose that $a_2(K) = a_2(M)$. Then there are infinitely many knots J_m $(m = 1, 2, \cdots)$ with $d_\triangle(J_m, M) = 2$ such that $v(J_m) = v(K)$.*

Where v is any Vassiliev invariant of order less than or equal to n.

Corollary 1.1.5. *Let n be a natural number and K an oriented knot in $\mathbb{S}^3$.*

(1) *Suppose that $a_2(K) \neq 0$. Then there are infinitely many knots J_m $(m = 1, 2, \cdots)$ with $u_\triangle(J) = |a_2(K)|$ such that $v_n(J_m) = v_n(K)$.*
(2) *Suppose that $a_2(K) = 0$. Then there are infinitely many knots J_m $(m = 1, 2, \cdots)$ with $u_\triangle(J) = 2$ such that $v_n(J_m) = v_n(K)$.*

Where v is any Vassiliev invariant of order less than or equal to n.

Remark. Note that if $n \geq 2$ and $v(J_m) = v(K)$ for any Vassiliev invariants of order less than or equal to n, then $a_2(J_m) = a_2(K)$. Therefore, by Corollary 1.1.3 $d_\triangle(J_m, M) \geq |a_2(K) - a_2(M)|$. Theorem 1.1.4 says that a_2 is the only

Vassiliev invariant of order less than or equal to n that detects delta distance of knots.

Proof of Theorem 1.1.4 can be deduced from that of Theorem 1.1.9 in Section 3 of Chapter 1. If $a_2(K) \neq a_2(M)$ ($a_2(K) = a_2(M)$, resp.), we set $a_2(K) - a_2(M) = p$ in Theorem 1.1.9, and start the proof with a composite knot $M\#K_p$, where K_p is the knot illustrated in Fig. 1.3.1 (Fig. 1.3.2, resp.), and we give similar procedure.

M. N. Goussarov [8] and K. Habiro [9, 10] showed independently that two oriented knots have the same Vassiliev invariant of order less than or equal to n if and only if they are transformed into each other by C_{n+1}-moves, where C_n-move is a local move illustrated in Fig. 1.1.2.

Now we define ${C_n}'$-move as illustrated in Fig. 1.1.3 for $n \geq 4$, and when $n \leq 3$, we regard ${C_n}'$-move as C_n-move. It is easy to see that ${C_n}'$-move is equivalent to C_n-move. Therefore Goussarov and Habiro's theorem can be rephrased as :

Theorem 1.1.6. *Two oriented knots have the same Vassiliev invariant of order less than or equal to n if and only if they are transformed into each other by C'_{n+1}-moves.*

A C_2-move is equivalent to a delta move and a C_3-move is equivalent to a local move in Fig. 1.1.4. A C_3-move as in Fig. 1.1.4 is also called *a clasp-pass move.*

By the result of Goussarov and Habiro, two knots K and K' with the same order two Vassiliev invariant can be transformed into each other by C_3-moves. In [38], T. Tsukamoto described the difference of the order three Vassiliev

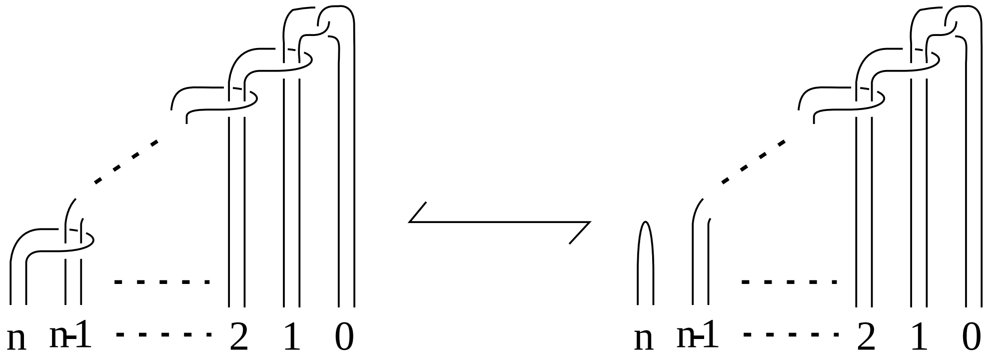


Fig. 1.1.2

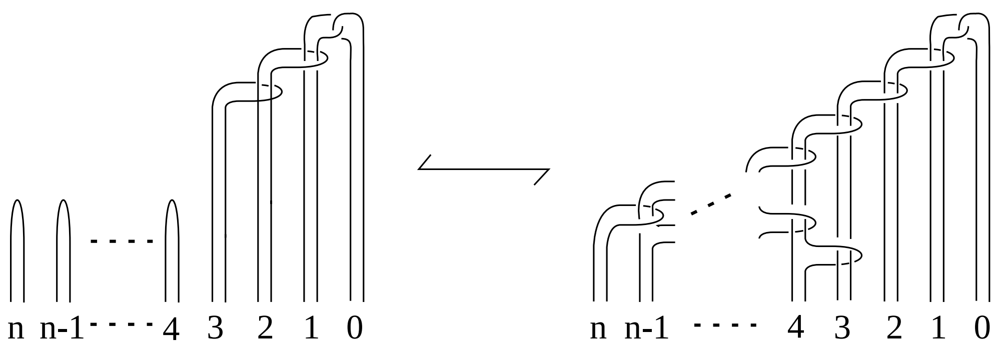


Fig. 1.1.3

invariant between K and K' by the chord diagram. Let $V_K^{(3)}(t)$ be the third derivative of the Jones polynomial $V_K(t)$ [12] of K then $V_K^{(3)}(1)$ is a Vassiliev invariant of order three.

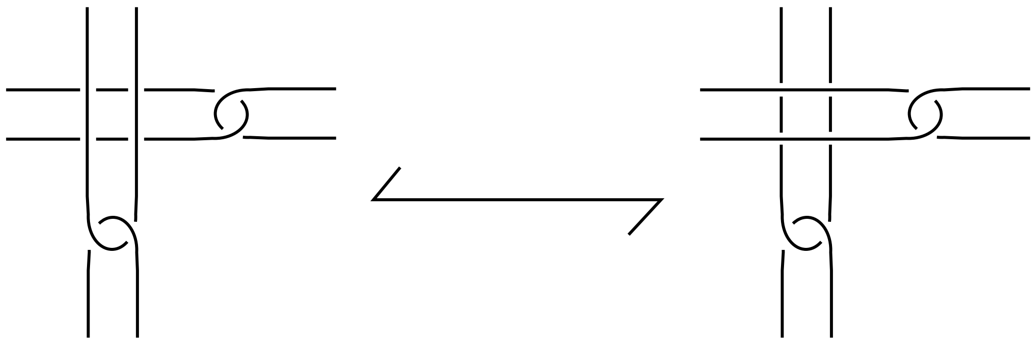


Fig. 1.1.4

Theorem 1.1.7 ([38])**.** *If a knot K is transformed into K' by a clasp-pass move, then*

$$V_K^{(3)}(1) - V_{K'}^{(3)}(1) = 0, \, or \pm 36.$$

If two knots K and K' have the same order two Vassiliev invariant, by $d_{cp}(K, K')$, we denote the minimal number of clasp-pass moves needed to transform K into K'. By Theorem 1.1.7, we have Corollary 1.1.8.

Corollary 1.1.8. *If a knot K is transformed into K' by clasp-pass moves, then*

$$d_{cp}(K, K') \geq \frac{1}{36} \left| V_K^{(3)}(1) - V_{K'}^{(3)}(1) \right|.$$

In this chapter, by modifying the way to prove Theorem 1.1.1 in [28], we will construct examples of knots that satisfy more conditions than those of Theorem 1.1.1 and Corollary 1.1.5. Namely we will prove Theorem 1.1.9.

Theorem 1.1.9. *Let n be a natural number and K a knot with $a_2(K) = p$. And let T_p be the twist knot with $a_2(T_p) = p$ and suppose $V_K^{(3)}(1) - V_{T_p}^{(3)}(1) = 36q$. Then there exist infinitely many unknotting number one knots J_m ($m = 1, 2, \cdots$) such that $v(J_m) = v(K)$ for any Vassiliev invariant v of order less than or equal to n and each J_m satisfies the followings:*

(1) *If $p \neq 0$, $u_\Delta(J_m) = |p|$ and if $p = 0$, $u_\Delta(J_m) = 2$.*

(2) *If $|q| \geq 2$, $d_{cp}(J_m, T_p) = |q|$, if $|q| = 1$, $d_{cp}(J_m, T_p) \leq 3$ and if $q = 0$, $d_{cp}(J_m, T_p) \leq 2$.*

1.2 Vassiliev invariants and one-branch tree diagrams

In the next section, we prove Theorem 1.1.9 by the argument about the relation between Jacobi diagram and C_n-move.

Whenever we have a knot invariant v which takes values in some abelian group, we can extend it to an invariant of singular knots by the Vassiliev skein relation:

$$v(K_D) = v(K_+) - v(K_-).$$

Here a *singular knot* is an immersion of a circle into $\mathbb{R}^3$ whose only singularities are transversal double points and K_D, K_+ and K_- denote the diagrams of singular knots which are identical except near one point as is shown in Fig. 1.2.1.

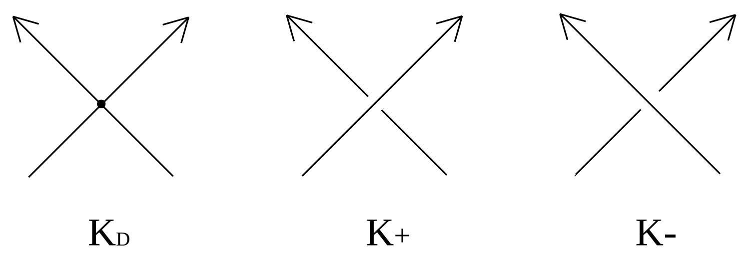


Fig. 1.2.1

An invariant v is called a *Vassiliev invariant of order* n, if n is the smallest integer such that v vanishes on all singular knots with more than n double points and we denote it by v_n [39].

To compute Vassiliev invariants, a notion of chord diagram is introduced in [4] and it is generalized to Jacobi diagram in [2]. In this paper we consider a special kind of Jacobi diagrams called a one-branch tree diagram which is defined by K. Y. Ng and T. Stanford in [26]. A *one-branch tree diagram* T *of order* n is a trivalent graph with $2n$ vertices. It is a union of a circle and a graph G which is isomorphic to a standard n-tree in Fig. 1.2.2. Only the circle is oriented and each vertex has a cyclic ordering of the edges incident to it.

Jacobi diagrams satisfy the STU-relation in Fig. 1.2.3 and, as a consequence of the STU-relation, the IHX-relation in Fig. 1.2.4 and the antisymmetry relation in Fig. 1.2.5. Since a one-branch tree diagram T is a kind of Jacobi diagrams, it satisfies the IHX-relation and the antisymmetry relation.

Label the branches of the standard n-tree as in Fig. 1.2.2. Under the isomorphism between the standard n-tree and the graph G of T, the branches of G are also labelled. And number the vertices on the circle of T by 0, 1, 2, $\cdots$,

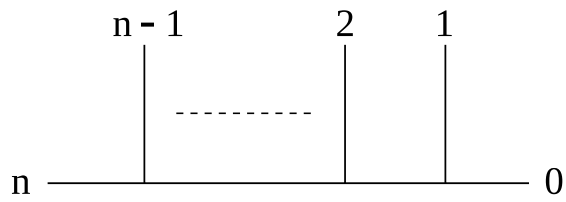


Fig. 1.2.2

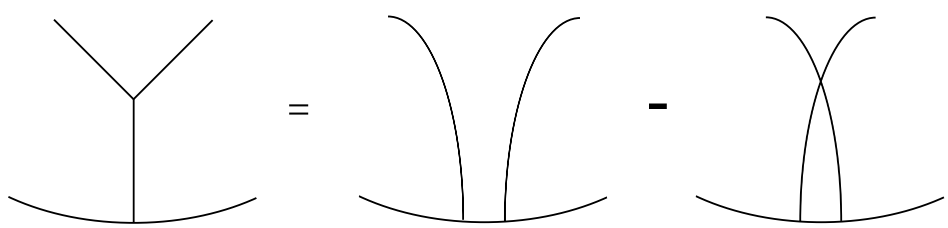


Fig. 1.2.3

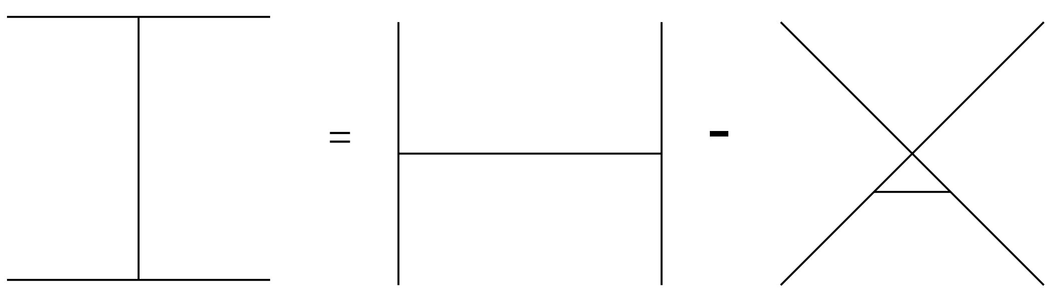


Fig. 1.2.4

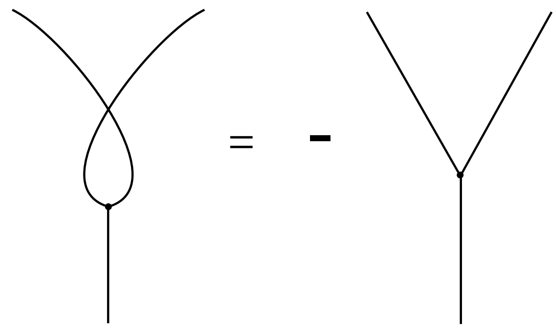


Fig. 1.2.5

n in the counterclockwise direction such that the end of branch 0 of G is numbered by 0. Then the correspondence between the label of branches of G and the number of their end points on the circle determines a permutation $\sigma \in S_n$. Conversely, if a permutation $\sigma \in S_n$ is given, we can construct a unique one-branch tree diagram T, denoted by T_σ. For one-branch tree diagrams and Vassiliev invariants, we have Lemma 1.2.1.

Lemma 1.2.1 ([26, 28])**.** *If K and K' are two knots with $w(K) = w(K')$ for any Vassiliev invariants w of order less than n, then there are integers a_σ and one-branch tree diagrams T_σ $(\sigma \in S_n)$ of order n such that $v(K) - v(K') = \sum_{\sigma \in S_n} a_\sigma v(T_\sigma)$ for any Vassiliev invariant v of order n.*

The value of a Vassiliev invariant of order n for a singular knot with n double points only depends on the chord diagram corresponding to it [3]. A Vassiliev invariant of order n for a chord diagram with n chords is that for a singular knot representing the chord diagram. By STU-relation, a one-branch tree diagram is the signed sum of chord diagrams. Then $v(T_\sigma)$ in Lemma 1.2.1 means the signed sum of the values for chord diagrams.

Remark. Since one-branch tree diagrams satisfy the antisymmetry relation and the IHX-relation, we have Fig. 1.2.6. By Fig. 1.2.6, it is enough to consider the one-branch tree diagrams T_σ whose permutation $\sigma \in S_n$ satisfies that $\sigma(1) < \sigma(2)$ and $\sigma(1) < \sigma(3)$ in Lemma 1.2.1.

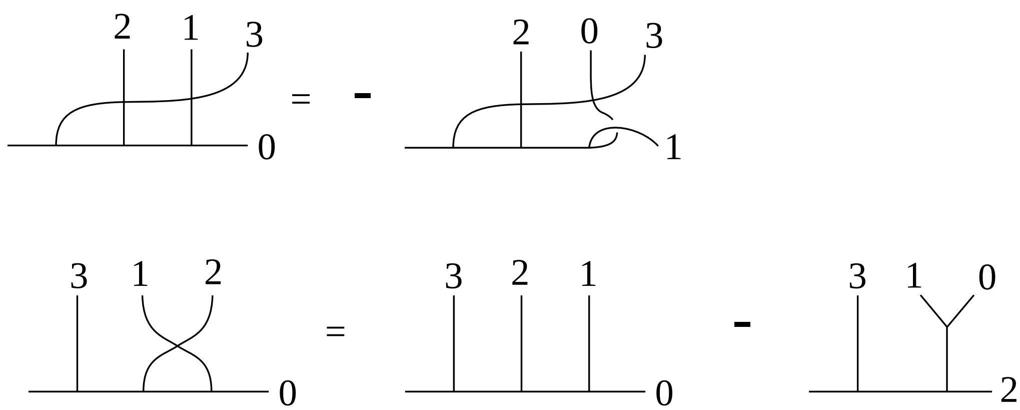


Fig. 1.2.6

A one-branch tree diagram is closely related to a C_n-move in Fig. 1.1.2. Y. Ohyama and T. Tsukamoto showed the following.

Theorem 1.2.2 ([31]). *Let v_n be a Vassiliev invariant of order n. If a knot K' is obtained from a knot K by a C_n-move, then*

$$v_n(K) - v_n(K') = \pm v_n(T_\sigma),$$

where T_σ is a one-branch tree diagram of order n.

A one-branch tree diagram in Theorem 1.2.2 is determined by the position of bands in the C_n-move on a knot K and a sign in Theorem 1.2.2 depends only on the signs of crossings in the C_n-move. And we note that for any permutation σ and any sign $\varepsilon \in \{-1, 1\}$, we can choose a C_n-move that changes the Vassiliev invariant by $\varepsilon v_n(T_\sigma)$.

1.3 Proof of Theorem 1.1.9

In this section, we will prove Theorem 1.1.9 by using Lemma 1.2.1 and Theorem 1.2.2. For $p \neq 0$, let K_p be a diagram of the twist knot T_p with $a_2(T_p) = p$ as is shown in Fig. 1.3.1. For $p = 0$, let K_p be a trivial knot in Fig. 1.3.2.

In the case $|q| \geq 2$, we perform the C_3'-move on the band A by $|q|$ times as in Fig. 1.3.3 and we have the knot $K_{p,q}$. Since C_n'-moves cannot change the Vassiliev invariants of order less than n, $a_2(K_{p,q}) = p$. By Theorem 1.1.7, Lemma 1.2.1 and Theorem 1.2.2, $V^{(3)}_{K_{p,q}}(1) - V^{(3)}_{K_p}(1) = 36q$. If we perform C_2'-moves on the center band in $K_{p,q}$ by p times, we have a trivial knot. Then we have $u_\Delta(K_{p,q}) = |p|$ if $p \neq 0$ and $u_\Delta(K_{p,q}) = 2$ if $p = 0$. If we perform C_3'-moves on the band A by q times, we have T_p. Then it follows that $d_{cp}(K_{p,q}, T_p) = |q|$.

Since $K_{p,q}$ and K have the same Vassiliev invariants of order less than 4, there are integers a_σ such that

$$v_4(K) - v_4(K_{p,q}) = \sum_{\sigma \in S_4} a_\sigma v_4(T_\sigma),$$

for any Vassiliev invariants v_4 of order 4. Here, we may suppose that $a_\sigma = 0$ unless $\sigma(1) < \sigma(2)$ and $\sigma(1) < \sigma(3)$ by Remark in Section 2 of Chapter 1.

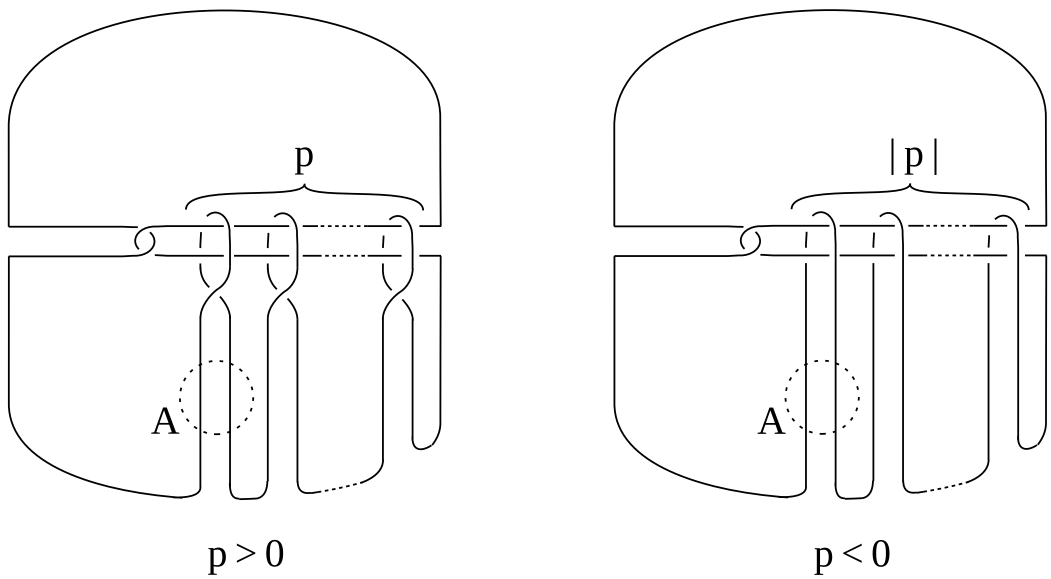


Fig. 1.3.1

Then we consider two cases $\sigma(1) < \sigma(2) < \sigma(3)$ and $\sigma(1) < \sigma(3) < \sigma(2)$. In the case σ of T_σ satisfies $\sigma(1) < \sigma(2) < \sigma(3)$, if $a_\sigma > 0$ we perform C'_4-moves that change the Vassiliev invariant by $v_4(T_\sigma)$ on the band B by a_σ times and if $a_\sigma < 0$ we perform C'_4-moves that change the Vassiliev invariant by $-v_4(T_\sigma)$ on the band B by $|a_\sigma|$ times. In the case σ of T_σ satisfies $\sigma(1) < \sigma(3) < \sigma(2)$, we perform C'_4-moves on the band C in the same way as the case $\sigma(1) < \sigma(2) < \sigma(3)$. Let $K^4_{p,q}$ be the knot obtained from $K_{p,q}$ by C'_4-moves as above. We continue this process, that is, if we have the knot $K^i_{p,q}$ such that $v_k(K^i_{p,q}) = v_k(K)$ $(k = 1, 2, \cdots, i)$, we construct the $K^{i+1}_{p,q}$ by C'_{i+1}-moves in the same way for the construction for $K^4_{p,q}$. Then we have the knot

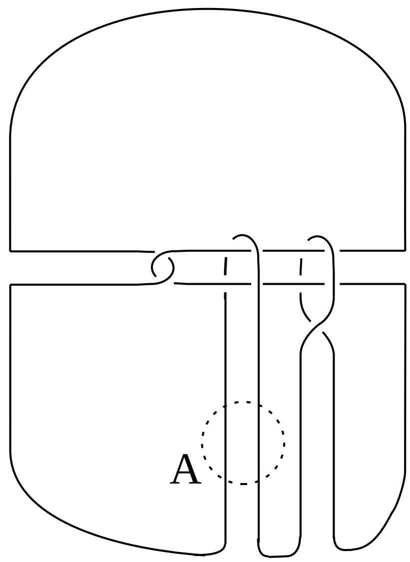


Fig. 1.3.2

$K^n_{p,q}$. By Lemma 1.2.1 and Theorem 1.2.2, it follows that $v_k(K^n_{p,q}) = v_k(K)$ $(k = 1, 2, \cdots, n)$. And as the case for $K_{p,q}$, we have $u_\Delta(K^n_{p,q}) = |p|$ if $p \neq 0$ and $u_\Delta(K^n_{p,q}) = 2$ if $p = 0$. Moreover the unknotting number of $K^n_{p,q}$ is equal to one and $d_{cp}(K^n_{p,q}, T_p) = |q|$. Here, we choose a C'_{n+1}-move which corresponds to T_σ of order $n+1$ such that $v_{n+1}(T_\sigma)$ is not zero. By performing the C'_{n+1}-moves on $K^n_{p,q}$ repeatedly, we have an infinite sequence of knots $K^n_{p,q} = J_1, J_2, J_3, \cdots$, no two of whose Vassiliev invariants of order $n+1$ coincide, and we have the case $|q| \geq 2$.

In the case $|q| = 1$, let $K_{p,q}$ be the knot in Fig. 1.3.4 and in the case $q = 0$ let $K_{p,q}$ be the knot in Fig. 1.3.5. By a similar way of the case $|q| \geq 2$, we can obtain the case $q = 0, \pm 1$. □

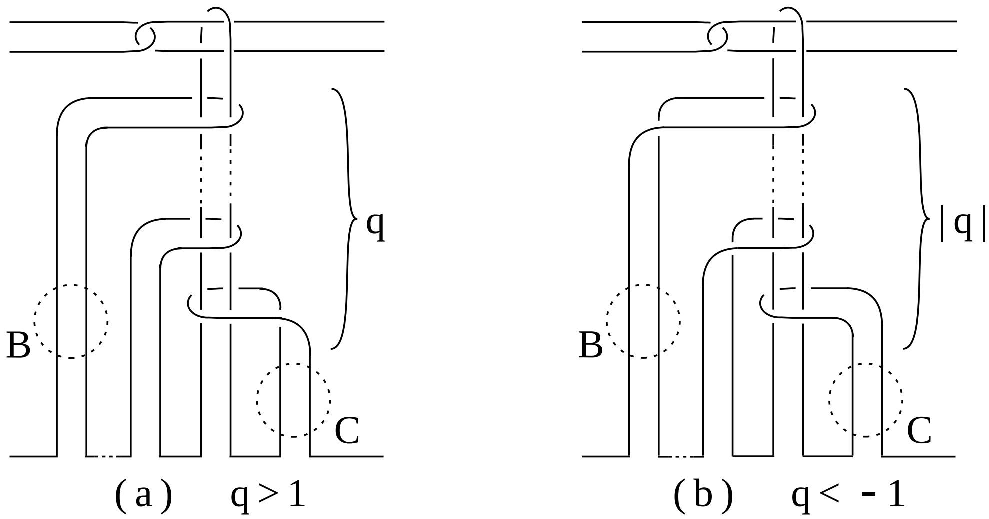


Fig. 1.3.3

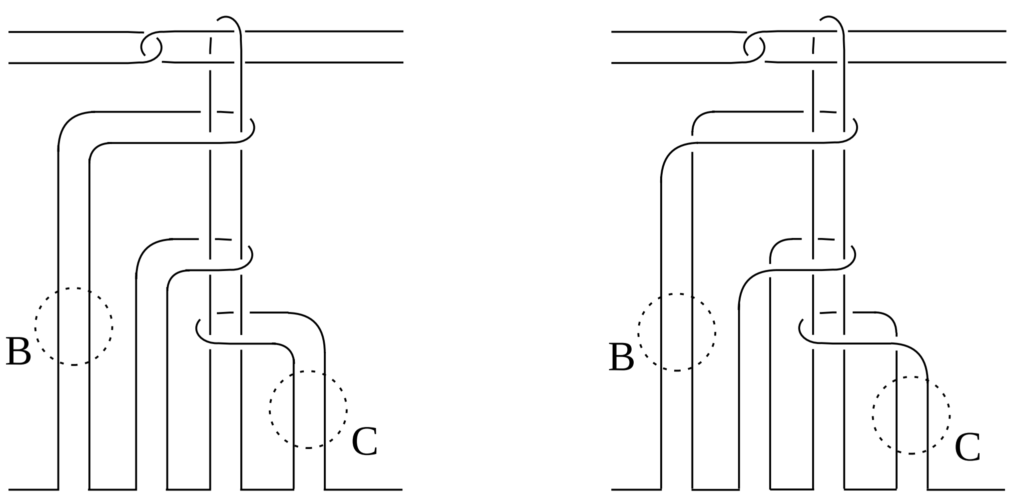


Fig. 1.3.4

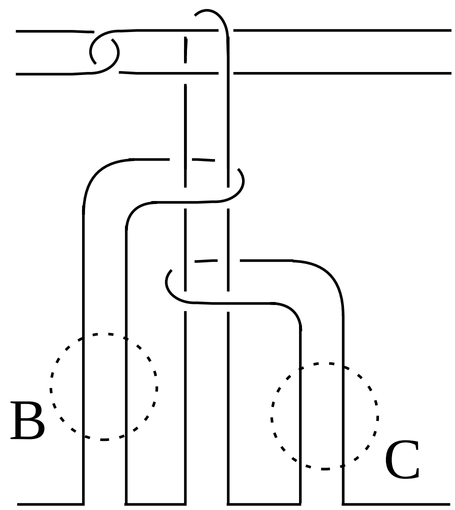


Fig. 1.3.5

Remark. In the case $|q| = 1$ in Theorem 1.1.9, there exists the case with $d_{cp}(J_m, T_p) = 1$ for a knot K. In the case $q = 0$, it is not clear for the author whether there exists the case with $d_{cp}(J_m, T_p) < 2$ or not.

Chapter 2

C_2, C_3 and C_4-moves and the coefficient of the Conway polynomial for knots

2.1 Introduction and results

If two knots K and K' can be transformed into each other by C_n-moves, we denote the minimal number of C_n-moves needed to transform K into K' by $d_{C_n}(K, K')$ and call it the *C_n-distance between K and K'*.

Based on M. N. Goussarov and K. Habiro's work that we mention in Chapter 1, some researches about the Vassiliev invariant of order n and C_n-move has been done [20, 31, 38]. In such a situation, it is natural that we have a problem as below.

Problem 2.1.1. *For given natural numbers m and n, if $d_{C_n}(K, K') = 1$ what is the value of $v_m(K) - v_m(K')$, where $v_m(K)$ is a Vassiliev invariant of order m of the knot K.*

This is a problem related for the distance of knots on the C_n-moves.

In this chapter, we investigate the variance of the value of a_m, the m-th coefficient of the Conway polynomial of knots as a concrete Vassiliev invariant of order m, by a C_n-move.

Let K and K' be knots. When they are transformed into each other by C_n-moves, the following equation is easily deduced from the result by Goussarov and Habiro:

$$v_m(K) - v_m(K') = 0 \qquad\qquad (0 \leq m < n).$$

Then we only consider the case $m \geq n$.

Problem 2.1.2. *For given natural numbers m and n with $m \geq n$, if $d_{C_n}(K, K') = 1$ what is the value of $a_m(K) - a_m(K')$?*

Remark. It is known that the Conway polynomial $\nabla_K(z)$ of a knot K can be expressed as $\nabla_K(z) = 1 + \sum_{i \in N} a_{2i}(K) z^{2i}$. Therefore we have only to consider the case that m is even.

On Problem 2.1.2, $a_m(K) - a_m(K') \equiv 0\ (2)$ for $m = n > 2$ [22, 29]. Moreover it is shown that $a_2(K) - a_2(K') = \pm 1$ for $m = n = 2$ [32] and $a_4(K) - a_4(K') = 0$ or ± 2 for $m = n = 4$ in [20]. In the case $n = 1$, for given any integer sequence $(n_1, n_2, \cdots, n_l)$, there are knots K and K' satisfying that $d_{C_1}(K, K') = 1$, $a_{2k}(K) - a_{2k}(K') = n_k$ $(1 \leq k \leq l)$ and $a_{2p}(K) - a_{2p}(K') = 0$ $(l < p)$. This is induced immediately by the fact that "there exist unknotting number one knots whose Conway polynomial coincides with any given polynomial with constant term being 1 in $Z[z^2]$ " [15, 33].

In the case $m \geq 2n$, we have Theorem 2.1.3 from Proposition 2.2.1 in Section 2 of Chapter 1.

Theorem 2.1.3. *For any natural number n and integer sequence $(p_n, p_{n+1}, \cdots, p_l)$, there are knots K and K' satisfying that*

$$d_{C_n}(K, K') = 1,$$
$$a_{2k}(K) - a_{2k}(K') = p_k \quad (n \leq k \leq l) \quad \textit{and}$$
$$a_{2q}(K) - a_{2q}(K') = 0 \quad (l < q).$$

By the above result in [32] and the case $n = 2$ in Theorem 2.1.3, we have the answer for C_2-moves on Problem 2.1.2.

Theorem 2.1.3 concerns $m \geq 6$ for C_3-moves and $m \geq 8$ for C_4-moves. For the rest case, we have Theorems 2.1.4 and 2.1.5 for $n < m < 2n$ on $n = 3$ and $n = 4$ from Propositions 2.2.2, 2.2.3 and 2.2.4.

Theorem 2.1.4. *For any natural number k, there are knots K and K' satisfying that*

$$d_{C_3}(K, K') = 1 \quad \textit{and}$$
$$a_4(K) - a_4(K') = k.$$

Theorem 2.1.5. *For any natural number k, there are knots K and K' satisfying that*

$$d_{C_4}(K, K') = 1 \quad \textit{and}$$
$$a_6(K) - a_6(K') \geq k.$$

2.2 Proofs of Theorems

Let $K(\alpha_1, \alpha_2, \cdots, \alpha_k)$ ($\alpha_i \in \mathbb{Z}$) be a knot depicted in Fig. 2.2.1. Let $K_{b3}(\alpha)$, $K_{b4}(\alpha)$ and $K_{b5}(\alpha)$ ($\alpha \in \{0\} \cup \mathbb{N}$) be knots depicted in Figs. 2.2.2, 2.2.3 and 2.2.4.

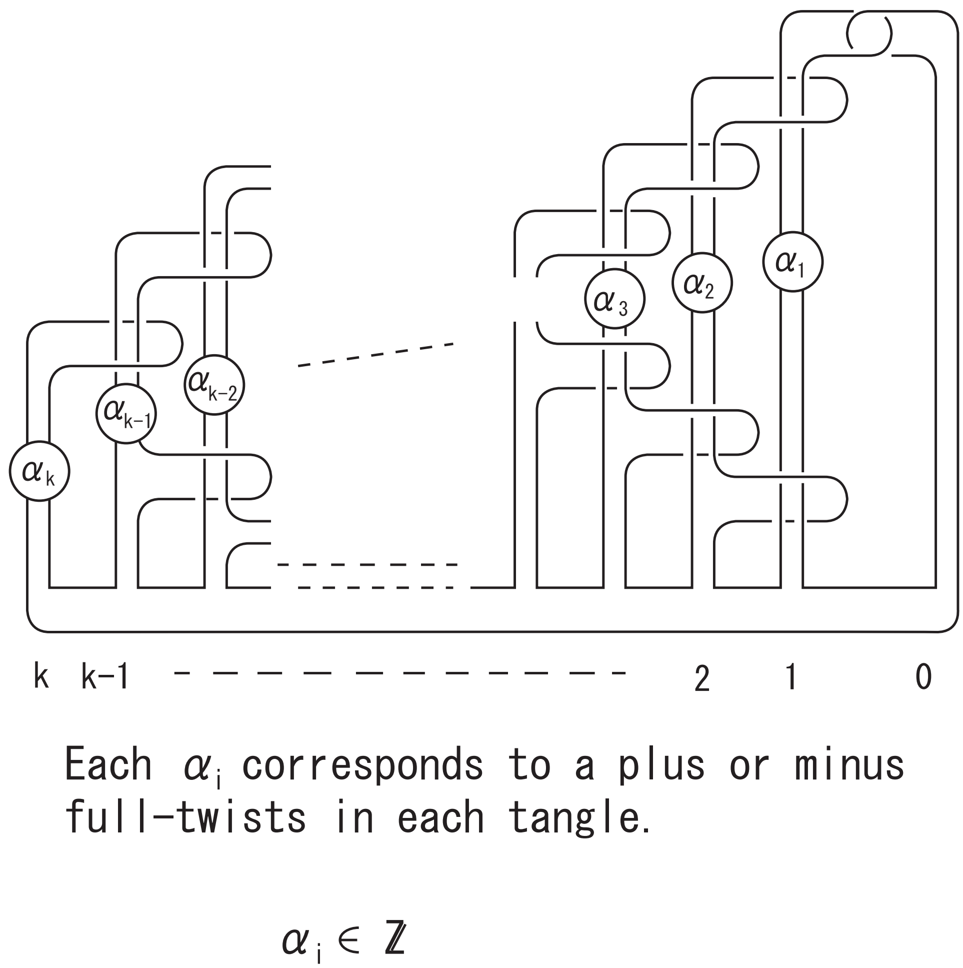


Fig. 2.2.1

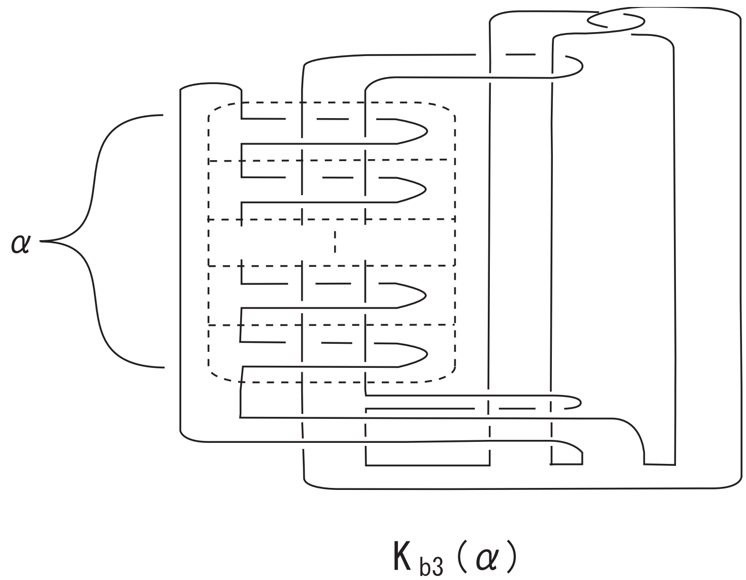


Fig. 2.2.2

For $K(\alpha_1, \alpha_2, \cdots, \alpha_k)$, we can know the Conway polynomial of it immediately by Proposition 3.2.1.

Proposition 2.1.1. $\nabla_{K(\alpha_1,\alpha_2,\cdots,\alpha_k)}(z) = 1 + (-1)^{k-1} z^{2(k-1)} + \sum_{i=1}^{k} (-1)^{i-1} \alpha_i z^{2i}$.

We also get the coefficient of minimum degrees except constant of $K_{b3}(\alpha)$, $K_{b4}(\alpha)$ and $K_{b5}(\alpha)$.

Proposition 2.1.2. $\nabla_{K_{b3}(\alpha)}(z) = 1 + (-\alpha^2 - \alpha) z^4 + \cdots\cdots$.

Proposition 2.1.3. $\nabla_{K_{b4}(\alpha)}(z) = 1 + (2\alpha + 1) z^4 + \cdots\cdots$.

Proposition 2.1.4. $\nabla_{K_{b5}(\alpha)}(z) = 1 + (-\alpha^2 - 4\alpha - 1) z^6 + \cdots\cdots$.

We prepare some definitions and Lemmas to show Proposition 2.1.1. In this paper, all coefficients of homology groups are assumed to be the integers $\mathbb{Z}$. It is known that any oriented knot or link L bounds a *Seifert surface* S, that is, a compact connected oriented 2-manifold S embedded in $\mathbb{S}^3$ with oriented

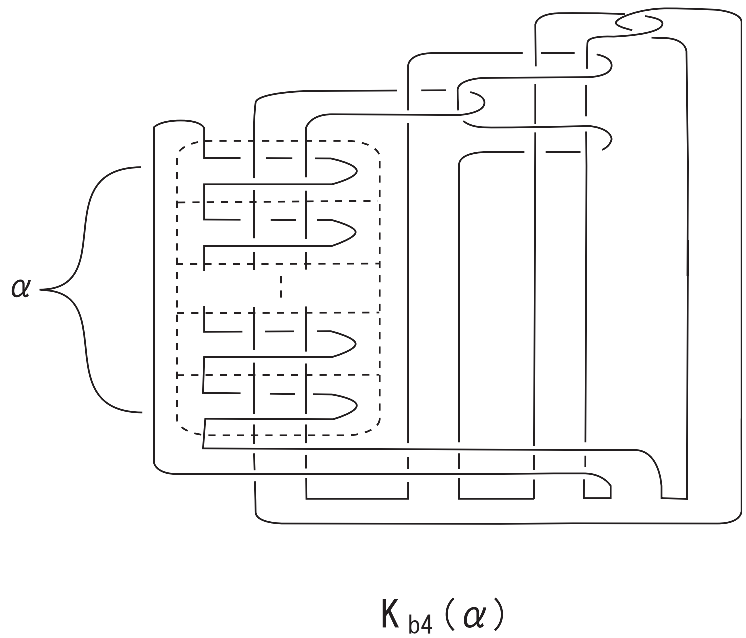
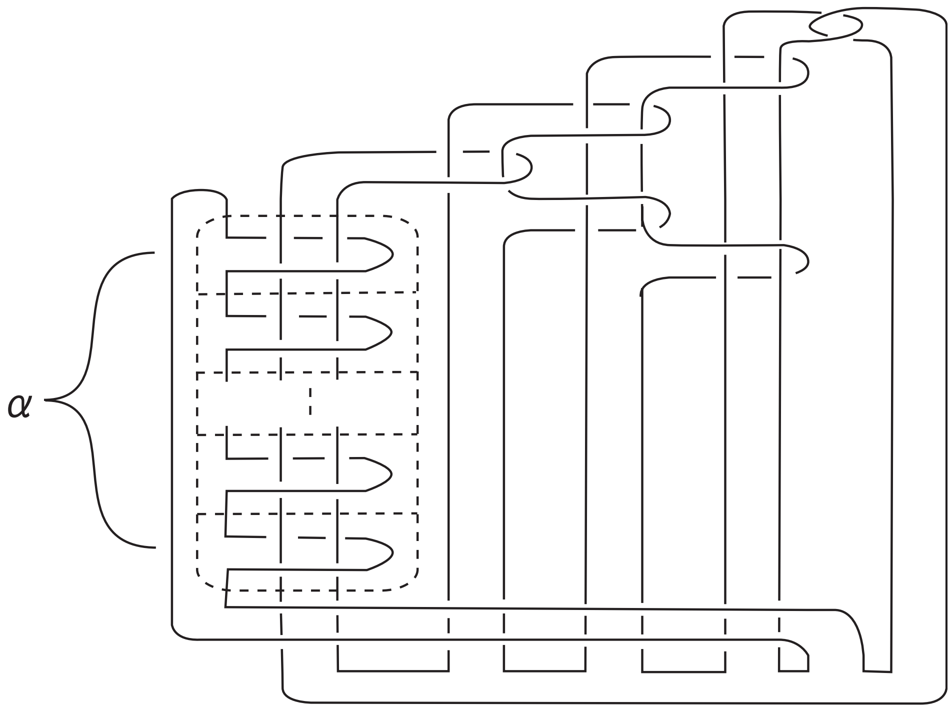


Fig. 2.2.3

boundary $\partial S = L = S \cap L$. A family $\vec{v} = (J_1, \cdots, J_n)$ of oriented simple closed curves J_i's in S is called a *basis of* S (*or* $H_1(S)$) if the homology classes $[J_1]$, ..., $[J_n]$ generates $H_1(S)$ and $n = \text{rank}(H_1(S))$. For a simple closed curve J in S, we let J^+ denote a simple closed curve in $\mathbb{S}^3$ which is obtained from J by pushing forward to the positive side of S.

Let L be an oriented link, and S a Seifert surface for L. Let $\vec{v} = (v_1, \cdots, v_n)$ be a basis of $H_1(S)$. We denote the matrix $(\text{lk}(v_i, v_j^+))$ by $V_{S,\vec{v}}$, or simply by V_S and we call it the associated *Seifert matrix* of S. The polynomial $\det\left(t^{\frac{1}{2}}V_S - t^{-\frac{1}{2}}V_S^T\right)$ is called the *Alexander polynomial of* L *associated with* S. It is known that the associated Alexander polynomial is independent of the



$K_{b5}(\alpha)$

Fig. 2.2.4

choice of S and $\vec{v}$, and the polynomial is called the *Alexander polynomial of* L and it is denoted by $\Delta_L(t)$. (See [34, Lecture 7], [17, Appendix] for details.)

The *Conway polynomial* $\nabla_L(z)$ and the Alexander polynomial $\Delta_L(t)$ are related to each other via $z = t^{-\frac{1}{2}} - t^{\frac{1}{2}}$.

For an n-tuple $(\alpha_1, \cdots, \alpha_n)$ of integers, we set $A_{(\alpha_1,\cdots,\alpha_n)}$ the following $(2n \times 2n)$-matrix:

$$A_{(\alpha_1,\cdots,\alpha_n)} = \begin{pmatrix} 1 & -1 & & & & & & & & \\ 0 & \alpha_1 & 1 & & & & & & & \\ & 1 & 0 & -1 & & & & & & \\ & & 0 & \alpha_2 & 1 & & & & & \\ & & & \ddots & \ddots & \ddots & & & & \\ & & & & 1 & 0 & -1 & & & \\ & & & & & 0 & \alpha_{n-1} & 1 & & \\ & & & & & & 1 & 0 & -1 \\ & & & & & & & 0 & \alpha_n \end{pmatrix}.$$

It is noticed that $A_{(\alpha_1,\cdots,\alpha_n)}$ is realized as a Seifert matrix of the knot $K(\alpha_1, \cdots, \alpha_n)$. The corresponding Seifert surface of genus n and the basis $\{x_1, y_1, \cdots, x_n, y_n\}$ are indicated in Fig. 2.2.5.

Lemma 2.2.5.

$$\det\left(t^{\frac{1}{2}} A_{(\alpha_1,\cdots,\alpha_n)} - t^{-\frac{1}{2}} A^T_{(\alpha_1,\cdots,\alpha_n)}\right) = 1 - \sum_{i=1}^{n} (-1)^i \alpha_i (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{2i}.$$

Proof. The proof will be done by induction on n. When $n = 1$, $A_{(\alpha_1)} = \begin{pmatrix} 1 & -1 \\ 0 & \alpha_1 \end{pmatrix}$ and $\det\left(t^{\frac{1}{2}} A_{(\alpha_1)} - t^{-\frac{1}{2}} A^T_{(\alpha_1)}\right) = 1 + \alpha_1 (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2$. The conclusion follows.

Assume that $n > 1$. First we observe the following:

Lemma 2.2.6. *Let $U_{(\alpha_1,\cdots,\alpha_n)}$ be the $((2n-1)\times(2n-1))$-submatrix of $A_{(\alpha_1,\cdots,\alpha_n)}$ obtained by removing the $2n$-th row and column. Then,*

$$\det\left(t^{\frac{1}{2}} U_{(\alpha_1,\cdots,\alpha_n)} - t^{-\frac{1}{2}} U^T_{(\alpha_1,\cdots,\alpha_n)}\right) = (-1)^{n-1} (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{2n-1}.$$

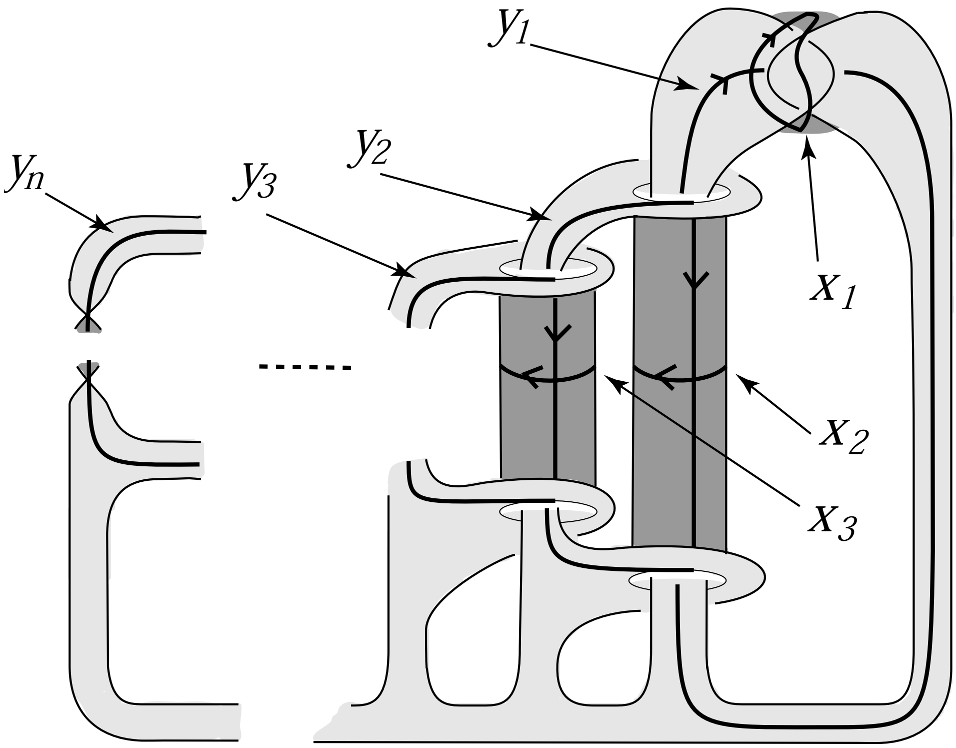


Fig. 2.2.5

Proof. This follows by inductively since

$$
\begin{aligned}
&\det\left(t^{\frac{1}{2}}U_{(\alpha_1,\cdots,\alpha_n)} - t^{-\frac{1}{2}}U^T_{(\alpha_1,\cdots,\alpha_n)}\right) \\
= \ &\det\left(\begin{array}{c|cc}
t^{\frac{1}{2}}U_{(\alpha_1,\cdots,\alpha_{n-1})} - t^{-\frac{1}{2}}U^T_{(\alpha_1,\cdots,\alpha_{n-1})} & -t^{\frac{1}{2}} & \\
\hline
t^{-\frac{1}{2}} & -z\alpha_{n-1} & -z \\
 & -z & 0
\end{array}\right)
\end{aligned}
$$

where $z = t^{-\frac{1}{2}} - t^{\frac{1}{2}}$. □

By using Lemma 2.2.6 and the hypothesis on induction, we have:

$$
\begin{aligned}
& \det\left(t^{\frac{1}{2}}A_{(\alpha_1,\cdots,\alpha_{n-1},\alpha_n)} - t^{-\frac{1}{2}}A^T_{(\alpha_1,\cdots,\alpha_{n-1},\alpha_n)}\right) \\
= \; & \det\left(\begin{array}{cccccccc|c|ccc}
-z & -t^{\frac{1}{2}} & & & & & & & & & & \\
t^{-\frac{1}{2}} & -z\alpha_1 & -z & & & & & & & & & \\
 & -z & 0 & -t^{\frac{1}{2}} & & & & & & & & \\
 & & t^{-\frac{1}{2}} & -z\alpha_2 & & & & & & & & \\
 & & & & \ddots & -z & & & & & & \\
 & & & & -z & 0 & & & -t^{\frac{1}{2}} & & & \\
\hline
 & & & & & t^{-\frac{1}{2}} & & & -z\alpha_{n-1} & -z & & \\
\hline
 & & & & & & & & -z & 0 & -t^{\frac{1}{2}} & \\
 & & & & & & & & & t^{-\frac{1}{2}} & -z\alpha_n &
\end{array}\right) \\
= \; & \det\left(t^{\frac{1}{2}}A_{(\alpha_1,\cdots,\alpha_{n-1})} - t^{-\frac{1}{2}}A^T_{(\alpha_1,\cdots,\alpha_{n-1})}\right) \times -(-t^{\frac{1}{2}} \cdot t^{-\frac{1}{2}}) \\
& -\det\left(t^{\frac{1}{2}}U_{e,(\alpha_1,\cdots,\alpha_{n-1})} - t^{-\frac{1}{2}}U^T_{(\alpha_1,\cdots,\alpha_{n-1})}\right) \times (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2 \cdot (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\alpha_n \\
= \; & 1 - \sum_{i=1}^{n-1} \alpha_i(-1)^i(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{2i} - (-1)^{n-2}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{2n-3}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^3\alpha_n \\
= \; & 1 - \sum_{i=1}^{n-1} \alpha_i(-1)^i(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{2i} - \cdot(-1)^n\alpha_n(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{2n} \\
= \; & 1 - \sum_{i=1}^{n} \alpha_i(-1)^i(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{2i}.
\end{aligned}
$$

This completes the proof. □

From Lemma 2.2.5 and the relation between Conway polynomial and Alexander polynomial, we have the Proposition 2.2.1 immediately.

Propositions 2.2.2, 2.2.3 and 2.2.4 can also be proven by inductions on α respectively.

Remark. In Proposition 2.2.1, we embedded knot to $\mathbb{S}^3$. More generary, when we embed knot in homology three sphere, it also holds and can be proved by the same way. We will use this fact in Chapter 3.

Proof of Theorem 2.1.3. Suppose $n \leq k$, we can choose and perform a C_n-move on $K(\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots, \alpha_k)$ to produce $K(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, 0)$ then we have

$$d_{C_n}(K(\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots, \alpha_k), K(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, 0)) = 1,$$

and from Proposition 2.2.1, comparing the value of Conway polynomial of $K(\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots, \alpha_k)$ to $K(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, 0)$, we have Theorem 2.1.3 immediately. $\square$

Examples of Theorem 2.1.3. Here we suppose each α_i is an integer.

(1) For given vector $(\alpha_2, \alpha_3, \cdots, \alpha_6)$, we take a knot K in Fig. 2.2.6 to get a pair of knots satisfying the condition of Theorem 2.1.3.

Let K' be a trivial knot, then we can find a C_2-move from K to K'. Now we know their values of Conway polynomial from Proposition 2.2.1.

$$\nabla_K(z) = 1 + (-1)^5 z^{10} - z^2 + \alpha_2 z^4$$
$$+ \alpha_3 z^6 + \alpha_4 z^8 + (\alpha_5 + 1) z^{10} + \alpha_6 z^{12}$$
$$= 1 - z^2 + \alpha_2 z^4 + \alpha_3 z^6 + \alpha_4 z^8 + \alpha_5 z^{10} + \alpha_6 z^{12}$$
$$\nabla_{K'}(z) = 1,$$

so we have

$$\begin{cases} d_{C_2}(K, K') = 1 \\ a_{2k}(K) - a_{2k}(K') = \alpha_k & (2 \leq k \leq 6) \\ a_{2p}(K) - a_{2p}(K') = 0 & (6 < p). \end{cases}$$

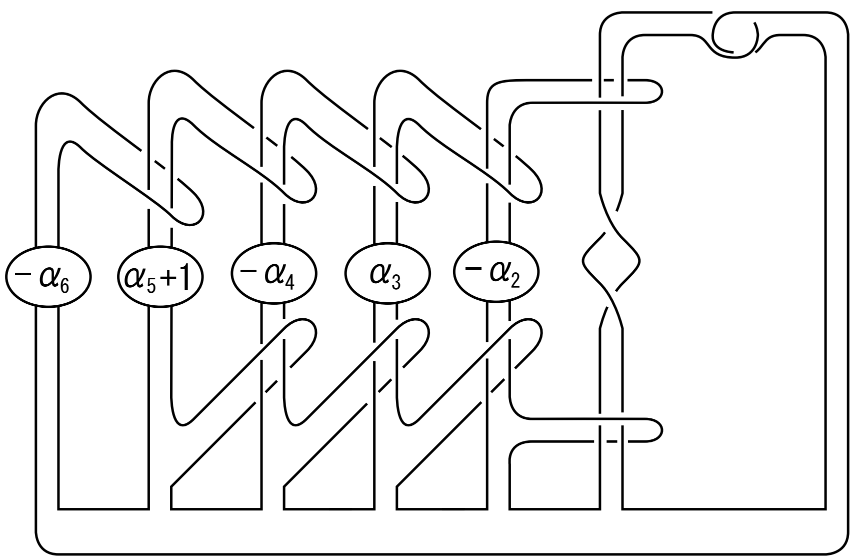


Fig. 2.2.6

(2) For given vector $(\alpha_3, \alpha_4, \alpha_5)$, we take a knot K in Fig. 2.2.7.

Let K' be trivial knot in Fig. 2.2.8, so same as above examples we have

$$\nabla_K(z) = 1 + z^4 + \alpha_3 z^6 + \alpha_4 z^8 + \alpha_5 z^{10}$$

$$\nabla_{K'}(z) = 1,$$

so we have

$$\begin{cases} d_{C_3}(K, K') = 1 & \\ a_{2k}(K) - a_{2k}(K') = \alpha_k & (3 \le k \le 6) \\ a_{2p}(K) - a_{2p}(K') = 0 & (5 < p). \end{cases}$$

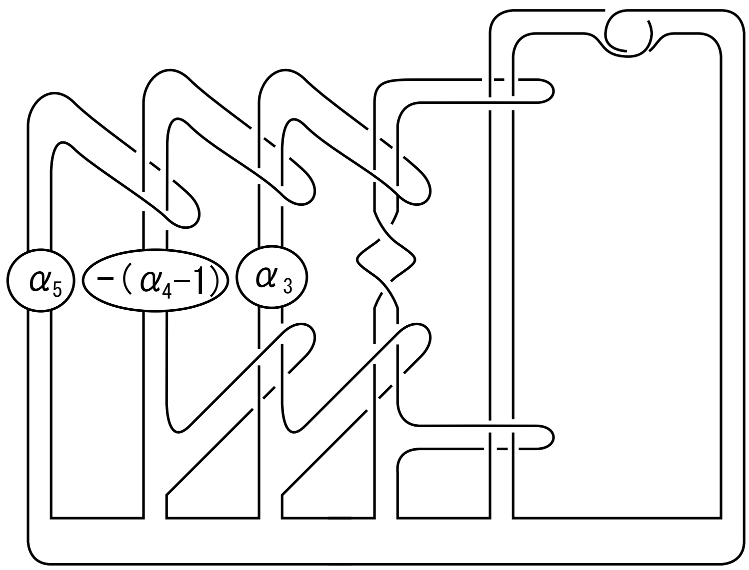


Fig. 2.2.7

Proof of Theorem 2.1.4. We consider two cases that k is even and is odd.

Case 1: If k is even, we use the knots of Proposition 2.2.2. Then, we have

$$d_{C_3}(K_{b3}(\alpha), K_{b3}(\alpha+1)) = 1 \quad \text{and}$$

$$a_4(K_{b3}(\alpha)) - a_4(K_{b3}(\alpha+1)) = 2(\alpha+1),$$

for any positive integer $\alpha \in \mathbb{N} \cup \{0\}$.

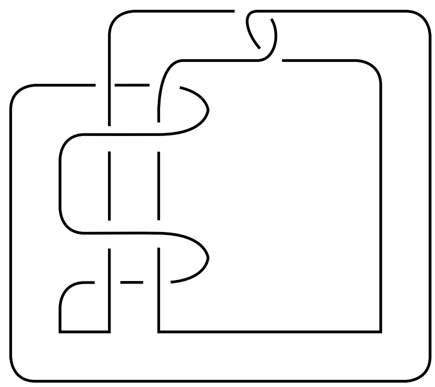


Fig. 2.2.8

Therefore we get the pair of knots $K_{b3}(\alpha)$ and $K_{b3}(\alpha+1)$ satisfying the condition of Theorem 2.1.4, by setting $\alpha = \dfrac{k}{2} - 1$.

Case 2: If k is odd, we use the knot of Proposition 2.2.3 and the trivial knot. Then, we have

$$d_{C_3}(K_{b4}(\alpha), T) = 1 \quad \text{and}$$

$$a_4(K_{b4}(\alpha)) - a_4(T) = 2\alpha + 1,$$

for any positive integer $\alpha \in \mathbb{N} \cup \{0\}$. Where by T, we denote the trivial knot.

Therefore we get the pair of knots $K_{b4}(\alpha)$ and trivial knot satisfying the condition of Theorem 2.1.4, by setting $\alpha = \dfrac{k-1}{2}$. □

Proof of Theorem 2.1.5. The following equations hold for the knot of Proposition 2.2.4 and the trivial knot.

$$d_{C_4}(K_{b5}(\alpha), T) = 1 \quad \text{and}$$

$$a_6(T) - a_6(K_{b5}(\alpha)) = \alpha^2 + 4\alpha + 1 > \alpha,$$

for any positive integer $\alpha \in \mathbb{N} \cup \{0\}$.

Therefore we get the pair of knots $K_{b5}(\alpha)$ and trivial knot satisfying the condition of Theorem 2.1.5, by setting $\alpha = n$. $\square$

Chapter 3

Variation of the Alexander-Conway polynomial under Dehn surgery

3.1 Introduction and results

Let H be an integral homology 3-sphere. A *framed knot* (*colored knot*, resp.) in H is a pair $\mathcal{K} = (K, \gamma)$ such that K is a knot in H and γ is an integer (a rational number $\gamma = q/p$ or ∞, resp.) which is called the *framing for* K. (*coloring for* K, resp.) A *framed link* (*colored link*, resp.) is a link $L = K_1 \cup \cdots \cup K_n$ with an n-tuple $\mathcal{L} = (\mathcal{K}_1, \cdots, \mathcal{K}_n)$ where $\mathcal{K}_i = (K_i, \gamma_i)$ a framed knot (a colored knot, resp.) in H. We let $E(L)$ denote the exterior $H - \mathring{N}(L)$ of a link L in H. For a framed (colored, resp.) link $\mathcal{L}$ in H, a simple closed curve l_i in each component of $\partial E(L)$ corresponding to $\partial N(K_i)$ is determined uniquely up to isotopy by γ_i for K_i in such a way that $[l_i]$ represents an element $(p_i, q_i) \in H_1(\partial N(K_i))$ such that $\gamma_i = q_i/p_i$ where $(1, 0)$ represents the homology class of the preferred longitude and $(0, 1)$ the meridian of K_i. By attaching a solid torus V_i to each component of $\partial E(L)$ so that the boundary of a meridian disk of V_i is glued to l_i, we obtain a closed 3-manifold $\chi(H;\mathcal{L}) = E(L) \cup \bigcup_{i=1}^{n} V_i$, so called a *surgery manifold*, and the construction $H \to \chi(H;\mathcal{L})$ is called *surgery along* $\mathcal{L}$. It is known that any closed orientable 3-manifold is a surgery manifold of some framed link in S^3, and if two framed links determine the same surgery manifold, then they are related by a finite sequence of Kirby moves [14].

Let $\mathcal{K}_1 = (K_1, \gamma_1)$ and $\mathcal{K}_2 = (K_2, \gamma_2)$ be framed knots yielding the same surgery manifold. We study the following problem. How do the Conway polynomials $\nabla_{K_1}(z)$ and $\nabla_{K_2}(z)$ relate to each other? Here we shall specify each framing to ± 1 and 0 to simplify arguments. The Alexander-Conway polynomial is a typical example of classical polynomial invariants for knots and links in homology spheres.

When $\gamma_1 = \gamma_2 = 0$, the surgery manifold M is a *homology handle*, that is, a 3-manifold with the infinite cyclic homology group $H_1(M) = \mathbb{Z}$, and it is well-known that the Conway polynomials of K_1 and K_2 coincide and the polynomial is called the *associated Conway polynomial of* M. Several examples of non-equivalent knots which yields the same homology handle via 0-framed surgery have been constructed. In [37], M. Teragaito gave finite sequences of pairwise distinct such satellite knots of arbitrarily large numbers, and in [13], A. Kawauchi constructed mutative hyperbolic knots such that they yield the same hyperbolic homology handle and non-isometric but mutative 1-surgery hyperbolic homology spheres.

In the case where $\gamma_1 = \varepsilon_1 \in \{-1, +1\}$ and $\gamma_2 = \varepsilon_2 \in \{-1, +1\}$, the surgery manifold is an integral homology sphere. In this case, the Alexander polynomials can differ [18]. In 1985, A. Casson introduced an integer valued invariant for oriented integral homology spheres, that counts the $SU(2)$-representations of their fundamental groups in some sense. See [1, 34] for reviews and see [17, 40] for more general surgery formula and extension of Casson invariant for general 3-manifolds. This *Casson invariant* is denoted by $\lambda(\cdot)$. It satisfies the following Casson surgery formula for any knot K in a homology sphere H, and for any $\varepsilon \in \{-1, +1\}$:

$$\lambda(\chi(H; (K, \varepsilon))) - \lambda(H) = \varepsilon a_2(K)$$

where the coefficient of z^n in the Conway polynomial $\nabla_K(z)$ is denoted by $a_n(K)$. In particular, when $\chi(H; (K_1, \varepsilon_1)) = \chi(H; (K_2, \varepsilon_2))$, then $\varepsilon_1 a_2(K_1) = \varepsilon_2 a_2(K_2)$. In this paper, we show that there is no other restriction for the Alexander polynomials of K_1 and K_2 by proving the following theorem.

Theorem 3.1.1. *Let H be a homology sphere. Let $f_1(z) = \sum_{i=2}^{n} c_i z^{2i}$ and $f_2(z) = \sum_{i=2}^{m} d_i z^{2i}$ be two polynomials in z^2. For any $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ and for any integer $a \in \mathbb{Z}$, there exist framed knots $\mathcal{K}_1 = (K_1, \varepsilon_1)$ and $\mathcal{K}_2 = (K_2, \varepsilon_2)$ in H such that $\nabla_{K_1}(z) = 1 + \varepsilon_2 a z^2 + f_1(z)$, $\nabla_{K_2}(z) = 1 + \varepsilon_1 a z^2 + f_2(z)$, and $\chi(H; \mathcal{K}_1) = \chi(H; \mathcal{K}_2)$.*

The construction of the knots K_1 and K_2 will be explicit. As soon as $f_1(z)$ is different from $f_2(z)$, 0-surgeries along K_1 and K_2 produce distinct manifolds.

3.2 Proof of Theorem

For a link L in H and a colored knot $\mathcal{K}$ in H which is disjoint from L, let $\chi(L;\mathcal{K})$ denote the link in $\chi(H;\mathcal{K})$ which is obtained from L by surgery along $\mathcal{K}$. Note that if $\mathcal{K} = (K, 1/n)$ and if K is a trivial knot, then $\chi(H;\mathcal{K})$ is homeomorphic to H and $L' = \chi(L;\mathcal{K})$ is obtained from L by the $(-n)$-full twists along K.

Note the following lemma.

Lemma 3.2.1. *Let K_1 and K_2 be two disjoint knots in H. Let (J,ε) be a $1/n$-colored knot in H disjoint from the link $K_1 \cup K_2$. Then in the surgery manifold $H' = \chi(H;(J,1/n))$,*

$$\begin{aligned} & \mathrm{lk}_{H'}(\chi(K_1;(J,1/n)), \chi(K_2;(J,1/n))) \\ = \ & \mathrm{lk}_H(K_1,K_2) - n \cdot \mathrm{lk}_H(K_1,J) \cdot \mathrm{lk}_H(K_2,J). \end{aligned}$$

Proof. This follows by a homological argument. (cf. Fig. 3.2.1. Crossings encircled contribute $-\mathrm{lk}(K_1,J) \cdot \mathrm{lk}(K_2,J)$.) □

Let $L^{(c_1,\cdots,c_n)}_{(d_1,\cdots,d_m)} = C_1 \cup C_2$ be the two-component link locally viewed as Fig. 3.2.2. It is clear that each component C_i is unknotted and $\mathrm{lk}(C_1,C_2) = 0$. Put $K_1 = \chi(C_1;(C_2,1/n_2))$ and $K_2 = \chi(C_2;(C_1,1/n_1))$. Since C_2 is unknotted, K_1 is obtained from C_1 by performing $-n_2$ full twists along C_2. Similarly, K_2 is obtained by twisting C_2 along C_1.

Then, we show the following.

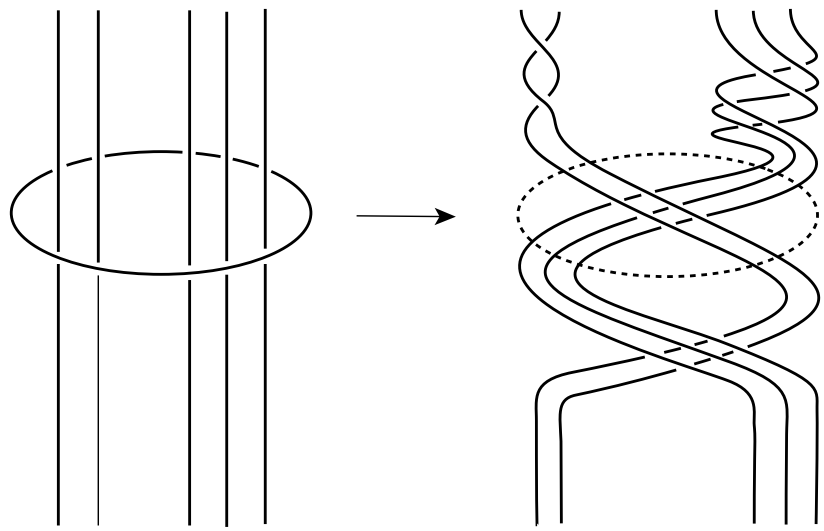


Fig. 3.2.1

Lemma 3.2.2.

$$\nabla_{K_1}(z) = 1 - n_2(c_1 + d_1)z^2 + n_2 \sum_{i=2}^{n} c_i(-z^2)^i, \quad \textit{and}$$

$$\nabla_{K_2}(z) = 1 - n_1(c_1 + d_1)z^2 + n_1 \sum_{i=2}^{m} d_i(-z^2)^i.$$

Proof. Span a Seifert surface S_1 of genus n to C_1 disjoint from C_2 as in the figure, by performing a peripheral tubing on the side indicated in Fig. 3.2.2. Take a basis $\vec{v}_1 = (x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$ of $H_1(S_1)$ so that:

- x_1 represents a meridian of the tube,
- y_1 goes through the tube once satisfying $\mathrm{lk}(y_1, C_2) = 0$,
- $x_2, y_2, \ldots, x_n, y_n$ are the same as in Fig. 3.3.3, and

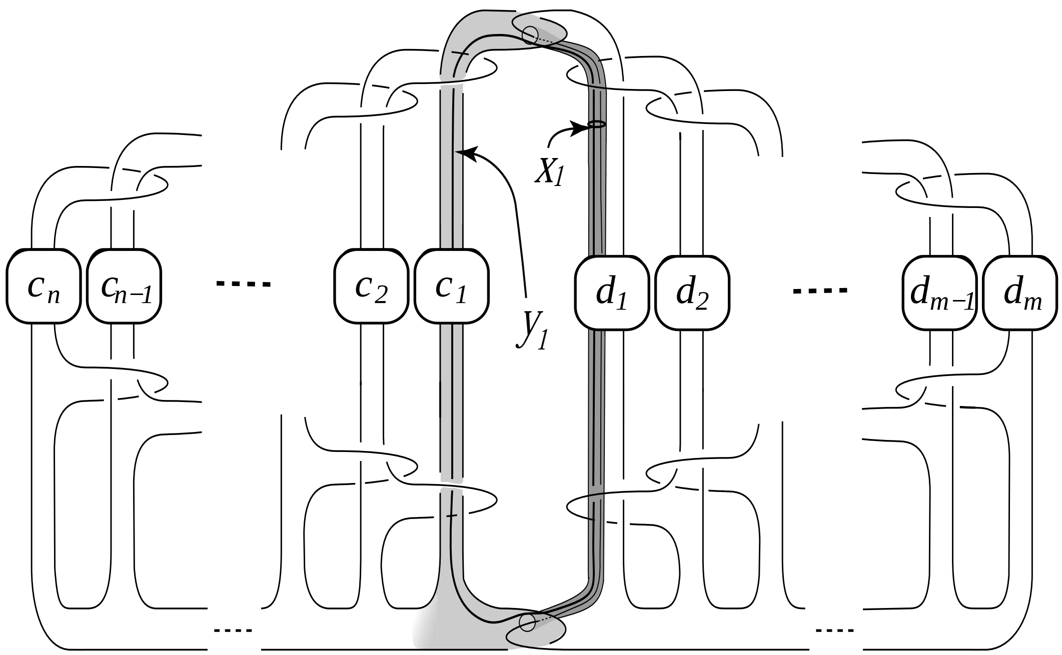


Fig. 3.2.2

- $V_{S_1,\vec{v}_1} = A_{0,(c_1,c_2,\cdots,c_n)}$.

Since K_1 is obtained from C_1 by performing the $(1/n_2)$-surgery on C_2, we see that the Seifert form matches $A_{-n_2,(c_1+d_1,c_2,\cdots,c_n)}$ by Lemma 3.2.1. Now it follows from Lemma 2.2.5 that $\nabla_{K_1}(z) = 1 - n_2(c_1+d_1)z^2 + n_2\sum_{i=2}^{n} c_i(-z^2)^i$.

By the same argument, we get the K_2 from C_2 by surgery along $(C_1, 1/n_1)$ such that $\nabla_{K_2}(z) = 1 - n_1(c_1+d_1)z^2 + n_1\sum_{i=2}^{m} d_i(-z^2)^i$. □

Now we are ready to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Put $c_1' = -a$, $d_1' = 0$, $c_i' = (-1)^i\varepsilon_2 c_i$, and $d_i' = (-1)^i\varepsilon_1 d_i$. Let $L^{(c_1',c_2',\cdots,c_n')}_{(d_1',d_2',\cdots,d_m')} = C_1 \cup C_2$ be the two-component link in H locally viewed as Fig. 3.2.2. Put $K_1 = \chi(C_1;(C_2,\varepsilon_2))$ and $K_2 = \chi(C_2;(C_1,\varepsilon_1))$.

Since C_i is unknotted, the ε_i-surgery on C_i does not change the ambient manifold H. Thus each K_i is a knot in H. Note that $H' = \chi(H; (K_1, \varepsilon_1)) = \chi(H; (K_2, \varepsilon_2)) = \chi(H; (C_1, \varepsilon_1), (C_2, \varepsilon_2))$ and $\varepsilon_i^2 = 1$.

It follows form Lemma 3.2.2 that $\nabla_{K_1}(z)$ and $\nabla_{K_2}(z)$ have the desired forms. This completes the proof. □

In the rest of this section, we state related problems. In order to construct two knots K_1 and K_2 yielding the homeomorphic homology spheres, one may begin with a two-component Brunnian link $C_1 \cup C_2$ with linking number 0, twist n_1 times along C_1 (n_2 times along C_2, resp.) and obtain K_2 from C_2 as the result $\chi(C_2; (K_1, 1/n_1))$. (K_1, C_1 and $\chi(C_1; (K_2, 1/n_2))$ resp.) Note the following proposition.

Proposition 3.2.3. *Let K be a knot in a homology sphere H. Let C be a knot in H disjoint from K such that* $\mathrm{lk}(K, C) = 0$. *Put $H' = \chi(H; (C, -1/n))$ and $K' = \chi(K; (C, -1/n))$. Then, $\nabla_{K'}(z) - \nabla_K(z) = nz^2 f(z)$ for some polynomial $f(z)$ in z^2.*

Proof. Use Lemma 3.2.1 and consider a Seifert surface of K disjoint from C to compute $\Delta_K(t)$ and $\Delta_{K'}(t)$. □

Our construction of colored knots $(K_1, 1/n_1)$, $(K_2, 1/n_2)$ defining the same homology sphere always produces ones with the property that $n_1 a_{2i}(K_1) - n_2 a_{2i}(K_2) \equiv 0 \mod n_1 n_2$ for each $i > 0$. In general, this does not hold. For example, let K_1 be a fibered knot in S^3, and K_2 the $(2, 1)$-cable about K_1. Then it follows from [23, Proposition 1.1] that $\chi(S^3; (K_1, 1/4)) = \chi(S^3; (K_2, 1))$ and K_2 is also fibered of twice genus of K_1. Thus, both $\nabla_{K_1}(z)$ and $\nabla_{K_2}(z)$ are monic.

Now it is natural to ask the following.

Question. Let $n_1, \ldots, n_k$ be k integers. Let $f_1, \cdots, f_i(z) = \sum_{j=2}^{m_i} c_{i,j} z^{2j}, \ldots, f_k$ be k polynomials in z^2. For some $a \in \mathbb{Z}$ such that n_i divides into a for each i, do there exist k knots $K_1, \ldots, K_k$ in a homology sphere H such that $\nabla_{K_i}(z) = 1 + \dfrac{a}{n_i} z^2 + f_i(z)$ and they have surgeries defining the same surgery homology sphere $H' = \chi(H; (K_i, 1/n_i))$?

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