SPATIAL GRAPHS BOUNDING INTERIOR DISJOINT SURFACES

BY
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Contents

Introduction

It is well known that any graph can be embedded in the three dimensional Euclidean space R^3 (or the 3-sphere S^3). A graph can be embedded in R^3 or S^3 in various way. For a planar graph, the simplest sort of embedding would be one that lies in a plane R^2 (resp. a 2-sphere S^2) in R^3 (resp. S^3) and the embedding is unique up to ambient isotopy [12]. So we can regard an embedding of planar graph into $R^2 \subset R^3$ (resp. $S^2 \subset S^3$) as a standard embedding. However for a non-planar graph, it is difficult to decide which embedding is the simplest. Hence we are interested in a natural way to define a standard embedding or embeddings of any graph. As a part of the study of a standard embedding, Kobayashi introduced a locally unknotted spatial embedding of a graph [10]. We say that a spatial embedding f of G is locally unknotted if there is a set of cycles of G which forms a basis of $H_1(G;\mathbb{Z})$ and the knots in $f(G)$ corresponding to the set of cycles bound a set of disks with disjoint interiors. Endo and Otsuki proved Proposition A concerning a locally unknotted spatial embedding [6].

Proposition A. *Any graph has a locally unknotted spatial embedding.*

In general the rank of $H_1(G;\mathbb{Z})$ is not an upper bound of the number of such disks with disjoint interiors. Hence, we can naturally think about the following questions.

Question A. *What is the least upper bound of the number of such disks?*

Question B. *What property does a spatial embedding which realize the least upper bound have?*

In Chapter 1, we consider about Question A. First in order to give an upper bound of the number of such disks, we discuss about compact connected orientable surfaces with disjoint interiors bounded by knots in a spatial graph. Then we will show that this upper bound is the least upper bound by realizing the upper bound with disks.

In Chapter 2, we define the boundary spatial embedding of a graph which is a generalization of the boundary link as an answer to Question B. By the definition of the boundary spatial embedding, we have that every graph does not have a boundary spatial embedding. In Section 2.2, we give a complete characterization of a graph which has a boundary spatial embedding. In Section 2.3, we classify boundary spatial embeddings of graphs completely up to self pass-equivalence. From the classification we have that any boundary spatial embedding of a graph is trivial up to edge-homotopy. This result is a generalization of the fact that any boundary link is trivial up to link-homotopy [4, 5].

In Chapter 3, we give spatial-graph-homology classification of spatial graphs by linking numbers and Simon invariants. Spatial-graph-homology [28] is an equivalence relation of spatial graphs defined by Taniyama. We show that two spatial embeddings f and g are spatial-graph-homologous if and only if for each subgraph H of G that is homeomorphic to a disjoint union of two circles, the restriction maps $f|_H$ and $g|_H$ have the same linking number, and for each subgraph H of G that is homeomorphic to a complete graph on five vertices K_5 or a complete bipartite graph on three-three vertices $K_{3,3}$, the restriction maps $f|_H$ and $g|_H$ have the same Simon invariant. In [14], it is shown that two spatial embeddings are spatial-graph-homologous if and only if they are transformed into each other by delta-moves and ambient isotopies. It is known that a delta-move does not change any finite type invariant of order 1 of spatial graph in the sense of [27]. Therefore we have that linking number and Simon invariant determine all of finite type invariants of order 1 of spatial graph.

The matter of Chapter 2 is a joint work with Ryo Nikkuni and the matter of Chapter 3 is a joint work with Kouki Taniyama.

Chapter 0

Definitions and notations

Throughout this dissertation, we work in the piecewise linear category. In this chapter, we introduce some definitions and notations which we use throughout this dissertation. First we prepare terminology of graph theory. Every graph which we deal with a finite graph. Let G be a graph. We consider G as a topological space in the usual way. We denote the set of all vertices and edges of G by $V(G)$ and $E(G)$ respectively. The *degree of v in G*, denoted by $\deg(v, G)$, is the number of the edges of G incident to v where a loop is counted twice. Let W be a subset of $V(G)$. By $G - W$ we denote the maximal subgraph of G with $V(G - W) = V(G) - W$. Let F be a subset of $E(G)$. By $G - F$ we denote the subgraph of G with $V(G - F) = V(G)$ and $E(G - F) = E(G) - F$. By $|X|$ we denote the number of the elements of a finite set X. A graph G is said to be *n-connected* if $|V(G)| \geq n + 1$ and for any subset W of $V(G)$ with $|W| \leq n - 1$ the graph $G - W$ is connected. We say that a graph is *topologically n-connected* if the graph is homeomorphic to an n-connected graph. A *simple graph* is a graph without loops and multiple edges. A graph is said to be *topologically simple* if it is not homeomorphic to any non-simple graph. A *path* is a graph that is homeomorphic to a closed interval. A *path of G* is a subgraph of G that is a path. Therefore we consider an edge as a path. For a graph $H = H_1 \cup H_2$, we say that H is obtained from H_1 by a *path addition* if H_2 is a path of H and $H_1 \cap H_2$ is the end points of H_2. A *cycle* is a graph that is homeomorphic to a circle. A *cycle of* G is a subgraph of G that is a cycle. We denote the set of all cycles of G by $\Gamma(G)$. Let $E' \neq \emptyset$ be a subset of $E(G)$. The *induced subgraph* $G[E']$ is the subgraph of G such that $E(G[E']) = E'$ and $V(G[E'])$ is the set of the all end points of the edges which belong to E'. A vertex $v \in V(G)$ is called a *cut vertex* if there are subsets E_1 and E_2 of $E(G)$ such that $E(G) = E_1 \cup E_2$, $E_i \neq \emptyset$ $(i = 1, 2)$ and $G[E_1] \cap G[E_2] = v$. A *block* is a graph which is connected and does not contain a cut vertex. A subgraph $H \subset G$ is called a *block of* G if H is a block and there does not exist a block $H' \subset G$ such that $H \subset H'$ and $H \neq H'$. For any graph G, there are blocks $H_1, H_2, \cdots, H_k$ of G such that $G = \bigcup_{i=1}^{k} H_i$. This decomposition is unique. We call this the *block decomposition* of G. For an edge $e \in E(G)$ that is not a loop, the *edge-contraction* G/e is the graph obtained from $G - \mathrm{int} e$ by identifying the end points of e. A graph H is called a *minor* of G, denoted by $H < G$, if there exists a subgraph G' of G and $e_1, e_2, \ldots, e_m \in E(G')$ such that $H = (\cdots((G'/e_1)/e_2)/\cdots)/e_m$. For other standard terminology of graph theory, see [1] and [2] for example.

Next we prepare terminology of spatial graph theory. We call an embedding $f : G \to S^3$ (resp. $f : G \to R^3$) of G into S^3 (resp. R^3) a *spatial embedding* of a graph G or simply a *spatial graph*. A graph G is said to be *planar* if there exists an embedding of G into S^2. A spatial embedding of a planar graph is said to be *trivial* if it is ambient isotopic to an embedding into $S^2 \subset S^3$ (resp. $R^2 \subset R^3$).We use notations of graph theory for spatial graphs by extending them. For example, for a graph G and a spatial embedding f of G we denote the set of all embedded vertices and edges of $f(G)$ by $V(f(G))$ and $E(f(G))$ respectively.

Other definitions and notations which we need will be properly introduced in each chapter.

Chapter 1

Bounding disks to a spatial graph

1.1 Introduction and results

Throughout this chapter, we assume that the graph does not have vertices of degree 0 or 1. Let G be a graph and $f : G \to S^3$ a spatial embedding. For a cycle $\gamma \in \Gamma(G)$, we can regard $f(\gamma)$ as a knot in the spatial graph. We consider about connected, compact and orientable surfaces with disjoint interiors bounded by such knots. Let $\Gamma_m(G) = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ be a subset of $\Gamma(G)$. Let $S_1, S_2, \ldots, S_m$ be connected, compact and orientable surfaces in S^3. Then we say that $\mathfrak{S} = \{S_1, S_2, \ldots, S_m\}$ is a *collection of m-spanning surfaces of* $f(G)$ (*with respect to* $\Gamma_m(G)$) if $\mathfrak{S}$ satisfies the following conditions;

(S1) $S_i \cap f(G) = \partial S_i = f(\gamma_i)$ $(i = 1, 2, \ldots, m)$,

(S2) $\mathrm{int}S_i \cap \mathrm{int}S_j = \emptyset$ $(i \neq j)$.

If each S_i is homeomorphic to a disk, then $\mathfrak{S}$ is called a *collection of m-spanning disks* of $f(G)$. There have been some studies on spatial graphs using such surfaces. See for example [3], [6], [17] and [24]. As regards a collection of spanning disks, some results have been known. In [10], Kobayashi introduced a locally unknotted spatial embedding of a graph. We denote the first Betti number of G by $\beta(G)$. If $m = \beta(G)$, $\mathfrak{S}$ is a collection of m-spanning disks of $f(G)$ and $\Gamma_m(G)$ represents a basis of $H_1(G;\mathbb{Z})$, then f is said to be *locally unknotted* (*with respect to* $\Gamma_m(G)$). In [6] Endo and Otsuki showed that for any graph G there exist $\Gamma_{\beta(G)}(G) \subset \Gamma(G)$ that represents a basis of $H_1(G;\mathbb{Z})$ and a locally unknotted spatial embedding $f : G \to S^3$ with respect to $\Gamma_{\beta(G)}(G)$. We note that in general $\beta(G)$ is not an upper bound of the number of the spanning surfaces. For example, we can see that there exists a collection of 3-spanning disks of $f(\Theta)$ where Θ and f are the graph with $\beta(\Theta) = 2$ and the embedding as illustrated in Fig. 1.1.1 respectively. Hence, we are interested in the least upper bound of the number of such surfaces of a spatial embedding of G. To simplify our arguments, we introduce the following property. Let G be a block with $\beta(G) = g$, $f : G \to S^3$ a spatial embedding of G, $\mathfrak{D} = \{D_1, D_2, \ldots, D_{3g-3}\}$ a collection of spanning disks of $f(G)$ and $\mathfrak{C} = \{C_1, C_2, \ldots, C_n\}$ the set of the closures of the connected components of $S^3 - \bigcup_{i=1}^{3g-3} D_i$. We say that $\mathfrak{D}$ *satisfies* $(\star)$ if $\mathfrak{D}$ satisfies the following conditions;

($\star$i) Each C_i is a 3-ball,

($\star$ii) $C_i \cap f(G) = \partial C_i \cap f(G)$ is homeomorphic to the graph Θ as illustrated in Fig. 1.1.1 ($i = 1, 2, \ldots, n$).

Now we ready to state our results of Chapter 1.

Theorem 1.1.1. *Let G be a graph with $\beta(G) = g$. Let $G = \left(\bigcup_{i=1}^{k} B_i\right) \cup \left(\bigcup_{i=1}^{l} K_i\right) \cup \left(\bigcup_{i=1}^{n} P_i\right)$ be the block decomposition of G such that $\beta(B_i) \geq 2$,*

$\beta(K_i) = 1$ and $\beta(P_i) = 0$ for each i. We regard $\bigcup_{i=1}^{0} B_i$, $\bigcup_{i=1}^{0} K_i$ and $\bigcup_{i=1}^{0} P_i$ as $\emptyset$. If for a spatial embedding $f : G \to S^3$ there is a collection of m-spanning surfaces $\mathfrak{S}$ of $f(G)$, then $m \leq 3g - 3k - 2l$.

We will show that any graph has a spatial embedding which has a collection of spanning disks realizing the upper bound of m given in Theorem 1.1.1 in order to give the least upper bound of m. First we consider the case that G is a block. We actually prove a slightly stronger result.

Theorem 1.1.2. *Let G be a block with $\beta(G) = g \geq 2$ and $\gamma \in \Gamma(G)$. Then there is a spatial embedding $f : G \to S^3$ of G such that there exists a collection of $(3g-3)$-spanning disks $\mathfrak{D} = \{D_1, D_2, \ldots, D_{3g-3}\}$ of $f(G)$ satisfying $(\star)$ and $\partial D_1 = f(\gamma)$.*

Remark 1.1.3. In Theorem 1.1.2 if $\mathfrak{D}$ satisfies $(\star)$ and $\partial D_1 = f(\gamma)$, the following can be seen. Let G^* be the dual graph of $f(G)$ in S^3 such that $V(G^*)$ and $E(G^*)$ correspond to $\mathfrak{C}$ and $\mathfrak{D}$ respectively. The graph G^* is a block and 3-regular, i.e. all vertices of G^* have the same degree 3. We can take a maximal tree T^* of G^* which does not contain the edge corresponding to D_1. Let $\{D'_1 = D_1, D'_2, \ldots, D'_g\} \subset \mathfrak{D}$ be the set of the disks which correspond to $E(G^*) - E(T^*)$. Since $N(f(G) \cup \bigcup_{i=1}^{g} D'_i) = cl(S^3 - N(T^*)) = cl(S^3 - B^3) = B^3$, f is locally unknotted with respect to $\{f^{-1}(D'_1), f^{-1}(D'_2), \ldots, f^{-1}(D'_g)\}$, where $N(\cdot)$ denotes a regular neighborhood and B^3 is a 3-ball. Though we did not make mention of the relation between $|\mathfrak{C}|$ and g, we have the equality

$|\mathfrak{C}| = 2g-2$ by the equality $|E(G^*)| = |\mathfrak{D}| = 3g-3$, $|V(G^*)| = |\mathfrak{C}|$ and $3|V(G^*)| = 2|E(G^*)|$.

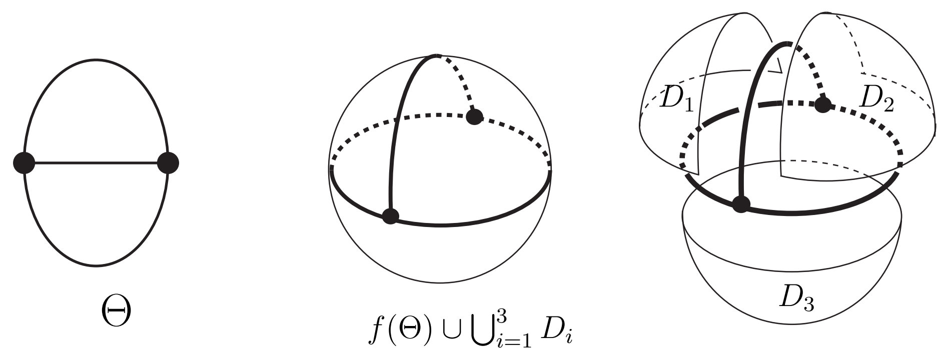

Fig. 1.1.1

We have the following as a corollary of Theorem 1.1.2.

Corollary 1.1.4. *Let G be a graph with $\beta(G) = g$ and $G = \left(\bigcup_{i=1}^{k} B_i\right) \cup \left(\bigcup_{i=1}^{l} K_i\right) \cup \left(\bigcup_{i=1}^{n} P_i\right)$ the block decomposition of G such that $\beta(B_i) \geq 2$, $\beta(K_i) = 1$ and $\beta(P_i) = 0$ for each i. Let γ_i be a cycle of B_i $(1 \leq i \leq k)$. Then there is a spatial embedding $f : G \to S^3$ of G such that there exists a collection of $(3g-3k-2l)$-spanning disks $\mathfrak{D} = \{D_1, D_2, \ldots, D_{3g-3k-2l}\}$ of $f(G)$ such that the set of cycles $\{f^{-1}(\partial D_1), f^{-1}(\partial D_2), \ldots, f^{-1}(\partial D_{3g-3k-2l})\}$ of G contains a set of cycles including each γ_i and K_j that represents a basis of $H_1(G;\mathbb{Z})$.*

This corollary generalizes Endo-Otsuki's result. Here we give a proof of Corollary 1.1.4 by using Theorem 1.1.2.

Proof of Corollary 1.1.4. For each B_i, there is a spatial embedding f_i of B_i such that there exists a collection of spanning disks of $f_i(B_i)$ satisfying the conditions of Theorem 1.1.2. For each K_i, there is a spatial embedding f_i' of K_i such that there exists a spanning disk of $f_i'(K_i)$. By Remark 1.1.3 and considering one-point sum and split sum of these embeddings, we can construct a spatial embedding f of G satisfying the conditions of this corollary. $\square$

We see that the estimation of Corollary 1.1.4 is best possible by Theorem 1.1.1.

1.2 Proof of Theorems

First we prove Theorem 1.1.1 which give an upper bound of the number of spanning surfaces.

Proof of Theorem 1.1.1. First we consider the case that G is a block. If $\beta(G) = 1$, G is homeomorphic to a circle. Since G has only one cycle, the theorem holds. Suppose that $\beta(G) \geq 2$. We take a regular neighborhood $N(f(G))$ of $f(G)$ in S^3 so that $N(f(G)) \cap S_i$ is an annulus for each i. Note that $\partial N(f(G))$ is a closed connected orientable surface of genus g. For any distinct surfaces S_i and S_j, there is an edge $e \in E(G)$ such that $f(e) \subset \partial S_i$ and $f(e) \not\subset \partial S_j$. Let e^* be the meridian curve corresponding to e, then $|(S_i \cap \partial N(f(G))) \cap e^*| = 1$ and $|(S_j \cap \partial N(f(G))) \cap e^*| = 0$. That means $[S_i \cap \partial N(f(G))] \neq [S_j \cap \partial N(f(G))] \in H_1(\partial N(f(G); \mathbb{Z}))$. Hence we see that $\bigcup_{i=1}^m S_i \cap \partial N(f(G))$ is a disjoint union of m non-parallel essential simple closed curves on $\partial N(f(G))$. It is well known that there are at most $3g - 3$ mutually disjoint non-parallel essential simple closed curves on a closed connected orientable surface of genus g. Therefore $m \leq 3g - 3$. Now we consider the general case. By considering the block decomposition of G, we have the inequality $m \leq \sum_{i=1}^k (3\beta(G_i) - 3) + l$. Since $g = \sum_{i=1}^k \beta(B_i) + l$, we have the desired inequality $m \leq 3g - 3k - 2l$. $\square$

In order to prove Theorem 1.1.2, we introduce the *amalgamation* of two graphs and prepare some lemmas. Let G_1 and G_2 be graphs and $h : H_1 \to H_2$ a homeomorphism from a subgraph H_1 of G_1 to a subgraph H_2 of G_2. The *amalgamation* $G_1 *_h G_2$ is the graph obtained from the union of G_1 and G_2 by identifying H_1 with H_2 by h. We regard a subgraph H of G_i as a subgraph of $G_1 *_h G_2$ naturally ($i = 1, 2$).

Lemma 1.2.1. *Let G_i be a block with $\beta(G_i) = g_i \geq 2$, γ_i a cycle of G_i and $f_i : G_i \to S^3$ a spatial embedding of G_i ($i = 1, 2$). Suppose that $\mathfrak{D}_i = \{D_{i,1}, D_{i,2}, \ldots, D_{i,3g_i-3}\}$ is a collection of $(3g_i - 3)$-spanning disks of $f_i(G_i)$ satisfying ($\star$) and $\partial D_{i,1} = f_i(\gamma_i)$. Let $h : H_1 \to H_2$ be a homeomorphism where $H_i = \gamma_i$ and G the amalgamation $G_1 *_h G_2$. We set $g = \beta(G)$. Then there exists a spatial embedding $f : G \to S^3$ such that there is a collection of $(3g - 3)$-spanning disks $\mathfrak{D} = \{D_1, D_2, \cdots, D_{3g-3}\}$ of $f(G)$ satisfying ($\star$), $\partial D_1 = f(\gamma_i)$ and $\mathfrak{D}_i \subset \mathfrak{D}$.*

Proof. Note that $g = g_1 + g_2 - 1$. By the assumption, we may assume that there exist 3-balls $B_1, B_2 \subset S^3$ such that $f_i(G_i) \cup \bigcup_{k=1}^{3g_i-3} D_{i,k} \subset B_i$, ∂B_i consists of $D_{i,1}$ and two elements of $\mathfrak{D}_i$, say $\Delta_{i,1}$ and $\Delta_{i,2}$, and $f_1(G_1) \cup f_2(G_2)$ is equal to G embedded in S^3 and $B_1 \cap B_2 = \partial B_1 \cap \partial B_2 = D_{1,1} = D_{2,1}$. Let f be the corresponding spatial embedding of G with $f(G) = f_1(G_1) \cup f_2(G_2)$ and $B = B_1 \cup B_2$ the 3-ball. We note that there exists a collection of $(3g-4)$ -spanning disks of $f(G)$ corresponding to $\mathfrak{D}_1 \cup \mathfrak{D}_2$. Let C' be the closure of $S^3 - B$. The closures of the connected components of $S^3 - (\bigcup_{k=1}^{3g_1-3} D_{1,k} \cup \bigcup_{k=2}^{3g_2-3} D_{2,k})$ except for C' satisfy ($\star$i) and ($\star$ii). At least one of $\Delta_{2,1}$ and $\Delta_{2,2}$, say $\Delta_{2,1}$, intersects $\Delta_{1,1}$ by an arc. Let Δ' be a properly embedded disk in C' such that $\partial \Delta' = \partial(\Delta_{1,1} \cup \Delta_{2,1})$. The disk Δ' divides C' into two 3-balls C_1 and C_2 such that $\partial C_1 = \Delta_{1,1} \cup \Delta_{2,1} \cup \Delta'$ and $\partial C_2 = \Delta_{1,2} \cup \Delta_{2,2} \cup \Delta'$. We note that C_1 and

C_2 satisfy ($\star$i) and ($\star$ii). After renumbering the set of disks $\mathfrak{D}_1 \cup \mathfrak{D}_2 \cup \{\Delta'\}$ is a collection of $(3g-3)$-spanning disks of $f(G)$ satisfying $\partial D_1 = f(\gamma_i)$. $\square$

Lemma 1.2.2. *Let G be a block and G_i a subgraph of G such that $G = G_1 \cup G_2$ $(i = 1, 2)$. If there is a cycle γ of G such that $\gamma \in \Gamma(G_1)$ and $G_1 \cap G_2 \subset \gamma$. Then G_1 is a block.*

Proof. Suppose that G_1 is disconnected. Then there are subgraphs H_1 and H_2 of G_1 such that $G_1 = H_1 \cup H_2$, $H_1 \cap H_2 = \emptyset$, $\gamma \subset H_1$ and $H_2 \neq \emptyset$. Since $G = H_2 \cup (H_1 \cup G_2)$ and $H_2 \cap (H_1 \cup G_2) = \emptyset$, G is disconnected. This is a contradiction. Thus G_1 is connected. Suppose that G_1 has a cut vertex v. Then there are subsets E_1 and E_2 of $E(G_1)$ such that $E_i \neq \emptyset$, $E(G) = E_1 \cup E_2$, $G[E_1] \cap G[E_2] = v$ and $\gamma \subset G[E_1]$ $(i = 1, 2)$. Since $G = G[E(G_2) \cup E_1] \cup G[E_2]$ and $G[E(G_2) \cup E_1] \cap G[E_2] = v$, v is a cut vertex of G. This is a contradiction. Thus G_1 has no cut vertices. Hence we have that G_1 is a block. $\square$

Now we are ready to prove Theorem 1.1.2.

Proof of Theorem 1.1.2. We may assume that the minimum degree of vertices of G is greater than or equal to 3. The proof proceeds by induction on $\beta(G)$. If $\beta(G) = 2$, then G is homeomorphic to Θ and the result follows from Fig. 1.1.1. Now assume that Theorem 1.1.2 holds for any block B with $\beta(B) < \beta(G)$. Since G is a block, γ is not a loop. First, we consider the case that $G - \gamma$ is connected. For any $e \in E(\gamma)$ there is a path P of $G[E(G) - E(\gamma)]$ such that $e \cup P$ is a cycle and $G_1 = G[E(\gamma) \cup E(P)]$ is homeomorpic to Θ. If we put $G_2 = G[E(G) - e]$ and $\gamma_2 = (\gamma - e) \cup P \in \Gamma(G_2)$, we can regard G as the amalgamation of G_1 and G_2 at γ_2. By Lemma 1.2.2, G_2 is a block. Since $\beta(G_2) = \beta(G) - 1$, the conclusion for the pair (G_2, γ_2) follows by induction hypothesis. Now the conclusion for the pair (G, γ) follows from Lemma 1.2.1. Next we consider the case that $G - \gamma$ is disconnected. Let $G'_1, G'_2, \ldots, G'_n$ be the connected components of $G - \gamma$. If we put $G_i = G'_i \cup \gamma$, we can regard G as the amalgamation of G_1 and $\bigcup_{i=2}^{n} G_i$ at γ. By Lemma 1.2.2, G_i is a block. It is easy to see that $\beta(G_i) < \beta(G)$. Then the conclusion for the pair (G_i, γ) follows by induction hypothesis. Now the conclusion follows from Lemma 1.2.1 inductively. $\square$

Chapter 2

Boundary spatial embeddings of a graph

2.1 Introduction and results

Let G be a graph. We set $\Gamma(G) = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$. We call a spatial embedding $f : G \to S^3$ of G a *boundary spatial embedding* if there exists a collection of m-spanning surfaces of $f(G)$. We note that if G is homeomorphic to the disjoint union of circles, then a boundary spatial embedding of G is a *boundary link* [25].

Example 2.1.1. Let f be a spatial embedding of Θ as illustrated in Fig. 2.1.1. Then it is easy to see that there exist connected, compact and orientable surfaces S_2 with $\partial S_1 = f(e_1) \cup f(e_3)$ and S_2 with $\partial S_1 = f(e_2) \cup f(e_3)$ such that $S_1 \cap S_2 = f(e_3)$. We note that $S_1 \cup S_2$ is also a connected, compact and orientable surface. We define $S_3 = (S_1 \cup S_2)^+$, where S^+ (resp. S^-) denotes a parallel copy of a connected, compact and oriented surface S with boundary in S^3 obtained by pushing S slightly in the positive (resp. negative) normal direction of S relative to ∂S, namely $S \cap S^+ = \partial S = \partial S^+$ (resp. $S \cap S^- = \partial S = \partial S^-$) and $\mathrm{int}S \cap \mathrm{int}S^+ = \emptyset$ (resp. $\mathrm{int}S \cap \mathrm{int}S^- = \emptyset$). Then we have that S_3 is also a connected, compact and orientable surface whose boundary is $f(e_1) \cup f(e_2)$ and the interiors of S_1, S_2 and S_3 are mutually disjoint. Therefore we have that f is a boundary spatial embedding.

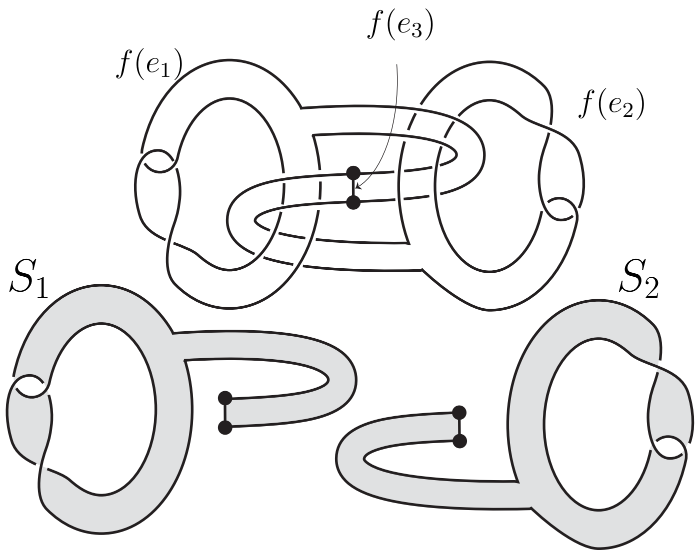

Fig. 2.1.1

Every graph does not always have a boundary spatial embedding. We give the following characterization of graphs which have boundary spatial embeddings.

Theorem 2.1.2. *Let $G = B_1 \cup B_2 \cup \cdots \cup B_n$ be a graph and its block decomposition. Then the following are equivalent.*

(1) *There exists a boundary spatial embedding of G.*

(2) *Each B_i is an edge or a graph which is homeomorphic to one of the graphs $G_1, G_2, \dots, G_5$ as illustrated in Fig. 2.1.2.*

(3) *Each B_i $(i = 1, 2, \dots, n)$ does not have a minor which is homeomorphic to one of the graphs G'_1, G'_2 and G'_3 as illustrated in Fig. 2.1.3.*

We prove Theorem 2.1.2 in Section 2.2.

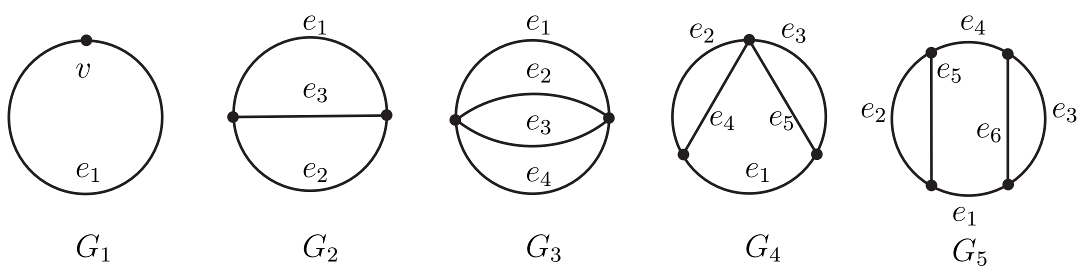

Fig. 2.1.2

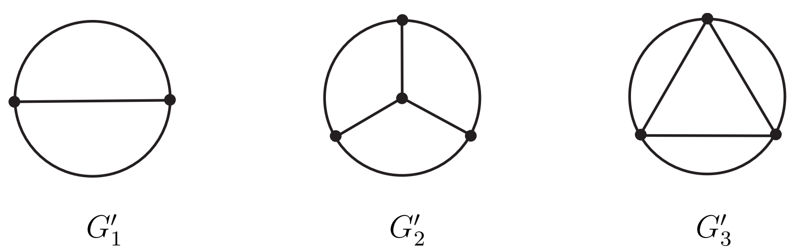

Fig. 2.1.3

It is well known that a graph is non-planar if and only if it contains a subgraph which is homeomorphic to K_5 or $K_{3,3}$ as illustrated in Fig. 2.1.4 [11]. Since each of K_5 and $K_{3,3}$ has a subgraph which is homeomorphic to G'_2, we have the following.

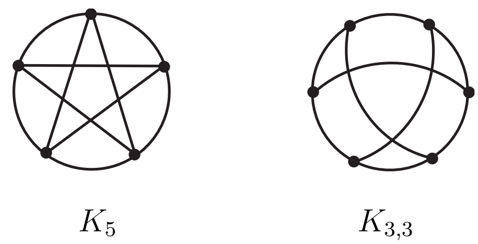

Fig. 2.1.4

Corollary 2.1.3. *Any non-planar graph does not have a boundary spatial embedding.*

Let G be an oriented graph, namely an orientation is given to each edge of G. For a spatial embedding f of G, we give the orientation to each spatial edge induced by G. A *pass-move* [8] and a *sharp-move* [15] on a spatial graph are local moves which are illustrated in Fig. 2.1.5 and Fig. 2.1.6 respectively. We also refer the reader to [18] for a related work.

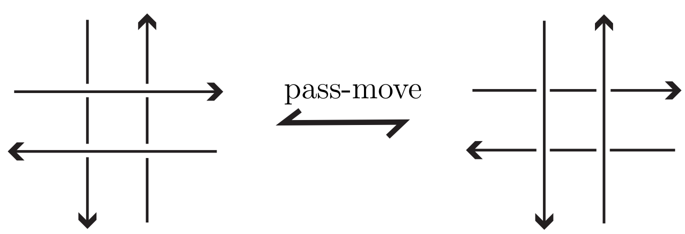

Fig. 2.1.5

In Section 2.3 we consider a specific pass-move (resp. sharp-move) on a spatial graph. We call a pass-move (resp. sharp-move) on a spatial graph is a *self pass-move* (resp. *self sharp-move*) [21] if all four strings in the move belong to the same spatial edge. We say that two spatial embeddings f and g of G are

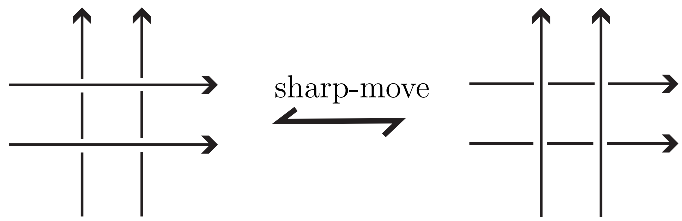

Fig. 2.1.6

self pass-equivalent (resp. *self sharp-equivalent*) if they are transformed into each other by self pass-moves (resp. self sharp-moves) and ambient isotopies. It is easy to see that these equivalences do not depend on the choice of orientations of edges of G. In particular for oriented links, the following results are known.

Theorem 2.1.4. (1) (Murakami [15]) *Any two oriented knots are self sharp-equivalent.*
(2) (Kauffman [8]) *Two oriented knots J and K are self pass-equivalent if and only if* $\mathrm{Arf}(J) = \mathrm{Arf}(K)$, *where* $\mathrm{Arf}(\cdot)$ *denotes the Arf invariant* [19].
(3) (Shibuya [20]) *Any two boundary links are self sharp-equivalent.*
(4) (Cervantes and Fenn [4]) *Two boundary links $L = J_1 \cup J_2 \cup \cdots \cup J_n$ and $M = K_1 \cup K_2 \cup \cdots \cup K_n$ are self pass-equivalent if and only if* $\mathrm{Arf}(J_i) = \mathrm{Arf}(K_i)$ $(i = 1, 2, \ldots, n)$.

We note that $\mathrm{Arf}(K)$ coincides with the modulo two reduction of the second coefficient of the *Conway polynomial* of a knot K [8]. We extend Theorem 2.1.4 to boundary spatial embeddings of a graph G as follows.

Theorem 2.1.5. (1) *Any two boundary spatial embeddings of a graph are self sharp-equivalent.*

(2) *Two boundary spatial embeddings f and g of G are self pass-equivalent if and only if* $\mathrm{Arf}(f(\gamma)) = \mathrm{Arf}(g(\gamma))$ *for any* $\gamma \in \Gamma(G)$.

Two spatial embeddings of a graph G are said to be *edge-homotopic* [28] if they are transformed into each other by *self crossing changes* and ambient isotopies, where a self crossing change is a crossing change on the same spatial edge. This is a generalization of *link-homotopy* on oriented links in the sense of Milnor [13]. Since a self sharp-move is realized by self crossing changes, we have the following by Theorem 2.1.5 (1) and Corollary 2.1.3 as a generalization of the fact that any boundary link is trivial up to link-homotopy [4, 5].

Corollary 2.1.6. *Any boundary spatial embedding of a graph is trivial up to edge-homotopy.*

We prove Theorem 2.1.5 in Section 2.3. We remark here that all oriented links were classified up to self pass-equivalence by Shibuya and Yasuhara in terms of the Arf invariant of proper sublinks and link-homotopy [22].

2.2 Complete characterization of a graph which has a boundary spatial embedding

In this section we prove Theorem 2.1.2 which gives a complete characterization of a graph which has a boundary spatial embedding. By Theorem 1.1.1, we have the following.

Lemma 2.2.1. *Let G be a block with $\beta(G) \geq 2$ and $|\Gamma(G)| = n$. If $n > 3\beta(G) - 3$, then G does not have a boundary spatial embedding.*

For graphs $G_1, G_2, \ldots, G_5$ as illustrated in Fig. 2.1.2, we set

$$\begin{aligned}
\Gamma(G_1) &= \{\gamma_1\}, \\
& \gamma_1 = e_1, \\
\Gamma(G_2) &= \{\gamma_1, \gamma_2, \gamma_3\}, \\
& \gamma_1 = e_1 \cup e_3,\ \gamma_2 = e_2 \cup e_3,\ \gamma_3 = e_1 \cup e_2, \\
\Gamma(G_3) &= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}, \\
& \gamma_1 = e_1 \cup e_4,\ \gamma_2 = e_2 \cup e_4,\ \gamma_3 = e_3 \cup e_4, \\
& \gamma_4 = e_1 \cup e_2,\ \gamma_5 = e_1 \cup e_3,\ \gamma_6 = e_2 \cup e_3, \\
\Gamma(G_4) &= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}, \\
& \gamma_1 = e_1 \cup e_4 \cup e_5,\ \gamma_2 = e_2 \cup e_4,\ \gamma_3 = e_3 \cup e_5, \\
& \gamma_4 = e_1 \cup e_2 \cup e_5,\ \gamma_5 = e_1 \cup e_4 \cup e_3,\ \gamma_6 = e_1 \cup e_2 \cup e_3, \\
\Gamma(G_5) &= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}, \\
& \gamma_1 = e_1 \cup e_4 \cup e_5 \cup e_6,\ \gamma_2 = e_2 \cup e_5,\ \gamma_3 = e_3 \cup e_6,
\end{aligned}$$

$$\gamma_4 = e_1 \cup e_2 \cup e_4 \cup e_6,\ \gamma_5 = e_1 \cup e_3 \cup e_4 \cup e_5,$$

$$\gamma_6 = e_1 \cup e_2 \cup e_3 \cup e_4.$$

Lemma 2.2.2. *Let G be a block. Then the following are equivalent.*

(1) *There exists a boundary spatial embedding of G.*

(2) *G is homeomorphic to one of the graphs $G_1, G_2, \ldots, G_5$ as illustrated in Fig. 2.1.2.*

(3) *G does not have a minor which is homeomorphic to one of the graphs G'_1, G'_2 and G'_3 as illustrated in Fig. 2.1.3.*

Proof. We first show (3) $\Rightarrow$ (2). It is well known that any block is homeomorphic to a graph which can be obtained from G_1 by path additions. Then it can be easily seen that G_3, G_4, G_5 and G'_2 are all blocks which can be obtained from G_1 by two path additions, see Fig. 2.2.1. Then we can check that any of the graphs which can be obtained from G_3, G_4 and G_5 by a path addition has a minor which is homeomorphic to G'_1, G'_2 or G'_3. Thus we have the result.

Next we show (2) $\Rightarrow$ (1). It is sufficient to show that each of G_1, G_2, $\ldots$, G_5 has a boundary spatial embedding. Let B_1 and B_2 be 3-balls such that $S^3 = B_1 \cup B_2$ and $\partial B_1 = \partial B_2 = S^2$. We regard each G_i as illustrated in Fig. 2.1.2 as a trivial spatial embedding $h_i : G_i \to S^2 = \partial B_1 = \partial B_2 \subset S^3$ of G_i ($i = 1, 2, \ldots, 5$). It is clear that h_1 is a boundary spatial embedding, namely there exists a disk D_1 in S_2 such that $\partial D_1 = h_1(\gamma_1)$. Next we consider h_2. There exist disks D_1 and D_2 in S^2 such that $\partial D_i = h_2(\gamma_i)$ ($i = 1, 2$). Besides we can obtain a disk D_3 which is properly embedded in B_1 such that $\partial D_3 = h_2(\gamma_3)$. Since D_1, D_2 and D_3 have mutually disjoint interiors, we have

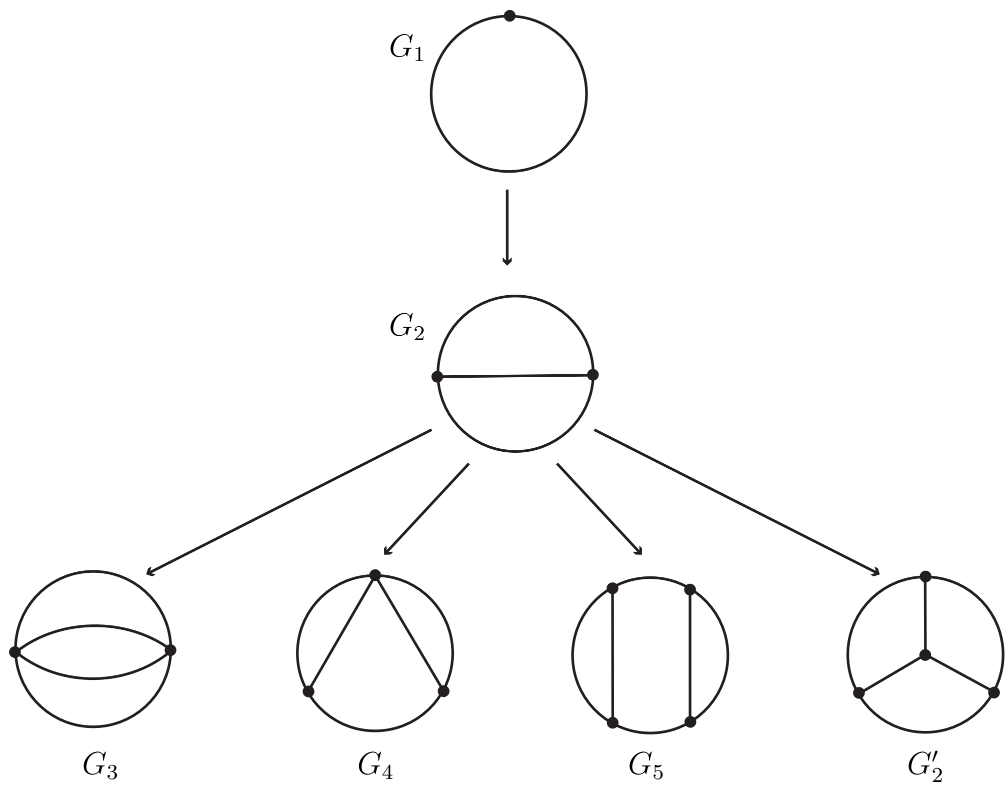

Fig. 2.2.1

that h_2 is a boundary spatial embedding. Next we consider h_3. There exist disks D_3, D_4 and D_6 in S^2 such that $\partial D_i = h_3(\gamma_i)$ $(i = 3, 4, 6)$. Besides we can obtain a disk D_5 which is properly embedded in B_1 such that $\partial D_5 = h_3(\gamma_5)$ and a disk D_2 which is properly embedded in B_2 such that $\partial D_2 = h_3(\gamma_2)$. Then $S^2 \cup D_5 - \mathrm{int}(D_4 \cup D_6)$ is a 2-sphere in B_1 which bounds a 3-ball B_3 and we can obtain a disk D_1 which is properly embedded in B_3 such that $\partial D_1 = h_3(\gamma_1)$. Since $D_1, D_2, \ldots, D_6$ have mutually disjoint interiors, we have that h_3 is a boundary spatial embedding. We have that h_4 and h_5 are boundary spatial embeddings in the same way as the case of h_3. Thus we have the result.

Finally we show (1) $\Rightarrow$ (3). Assume that G has a boundary spatial embedding f. For any subgraph H of G, it is easy to see that $f|_H$ is a boundary spatial embedding of H. Let e be an edge of G that is not a loop. Then the contraction of e induces a bijection from $\Gamma(G)$ to $\Gamma(G/e)$ and we can see that $f(G)/f(e)$ represents a boundary spatial embedding of G/e naturally. Therefore we have that each minor of G has a boundary spatial embedding. But we can see that each of G'_1, G'_2 and G'_3 is a block and does not have a boundary spatial embedding by Lemma 2.2.1. Thus G cannot have a minor which is homeomorphic to G'_1, G'_2 or G'_3. This completes the proof. $\square$

Now we are ready to Theorem 2.1.2.

Proof of Theorem 2.1.2. By considering the block decomposition of any graph, we have the result immediately by Lemma 2.2.2. $\square$

2.3 Classification of boundary spatial embeddings of a graph up to self pass-equivalence

In this section we prove Theorem 2.1.5. First we prepare some lemmas. It is known that a pass-move is realized by sharp-moves and ambient isotopies as illustrated in Fig. 2.3.1 [16]. Thus we have the following.

Lemma 2.3.1. *Self pass-equivalence implies self sharp-equivalence.*

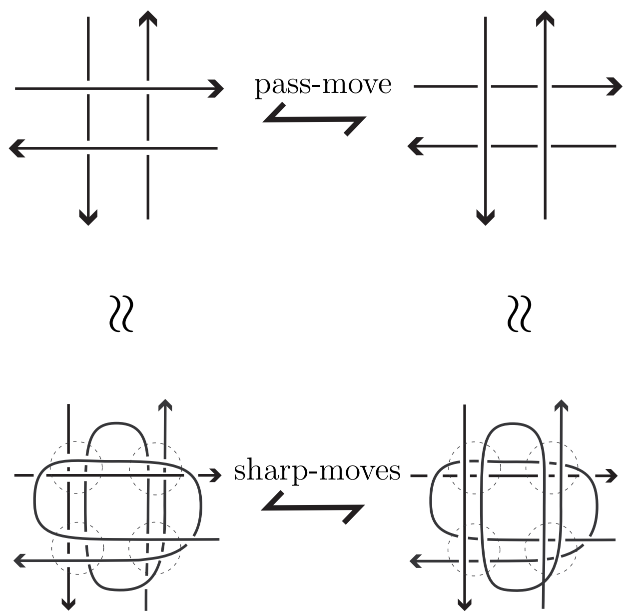

Fig. 2.3.1

A Γ-*move* [8] is a local move on a spatial graph as illustrated in Fig. 2.3.2. We call a Γ-move a *self* Γ-*move* if all three strings in the move belong to the same spatial edge. It is known that a Γ-move is realized by a pass-move [8] and ambient isotopies, see Fig. 2.3.3. Thus we have the following.

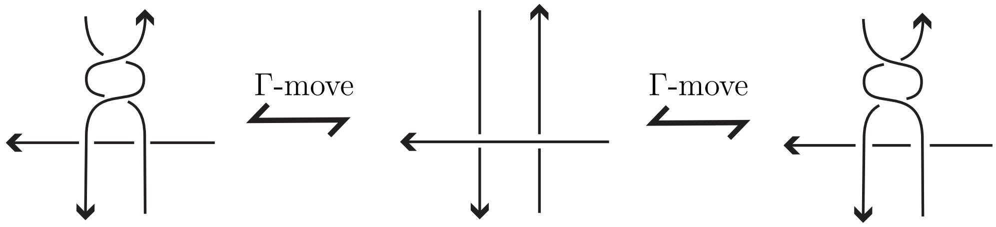

Fig. 2.3.2

Lemma 2.3.2. *A self* Γ*-move is realized up to self pass-equivalence.*

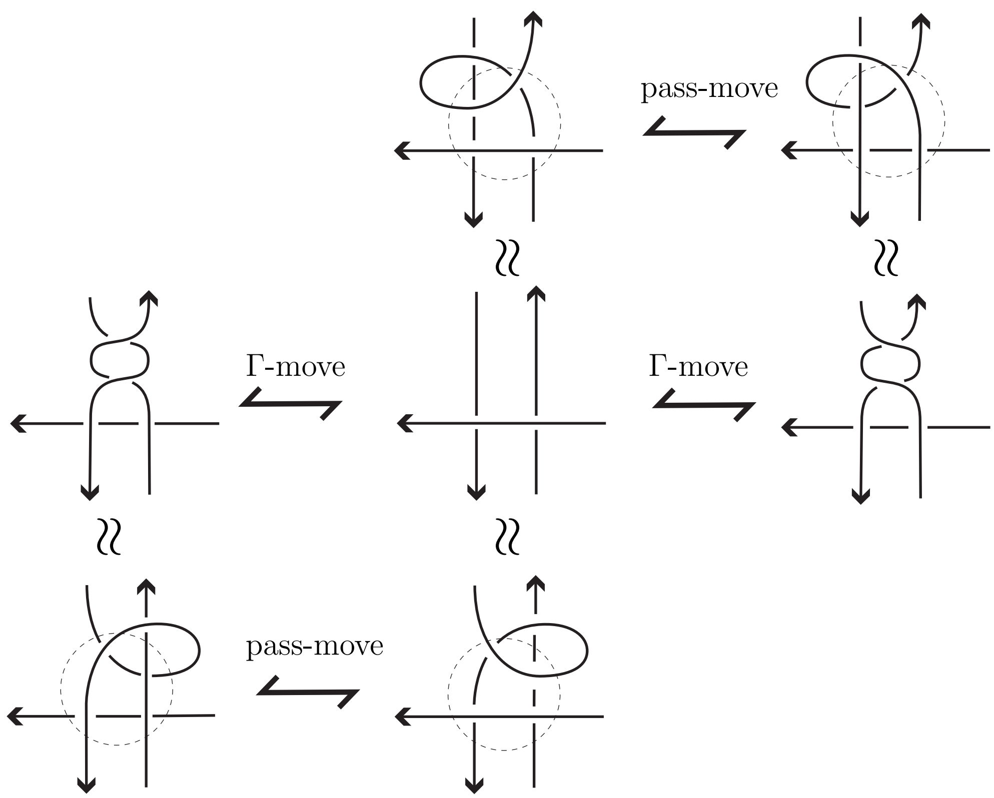

Fig. 2.3.3

Lemma 2.3.3. *If two spatial embeddings f and g of G are self pass-equivalent, then* $\mathrm{Arf}(f(\gamma)) = \mathrm{Arf}(g(\gamma))$ *for any* $\gamma \in \Gamma(G)$.

Proof. If two spatial embeddings f and g of G are self pass-equivalent, it is clear that $f(\gamma)$ and $g(\gamma)$ are self pass-equivalent for any $\gamma \in \Gamma(G)$. Thus by Theorem 2.1.4 (2) we have that $\mathrm{Arf}(f(\gamma)) = \mathrm{Arf}(g(\gamma))$. $\square$

Now we are ready to prove Theorem 2.1.5.

Proof of Theorem 2.1.5. We first prove (2). By Lemma 2.3.3, we have the 'only if' part. So we show the 'if' part. Let f and g be boundary spatial embeddings of G such that $\mathrm{Arf}(f(\gamma)) = \mathrm{Arf}(g(\gamma))$ for any $\gamma \in \Gamma(G)$. In the following we show that f can be transformed into a canonical spatial embedding ψ_f up to self pass-equivalence.

Let

$$G = \bigcup_{l=0}^{5} \bigcup_{i_l=1}^{n_l} B_{i_l}^{(l)}$$

be the block decomposition of G such that $B_{i_0}^{(0)}$ is an edge ($i_0 = 1, 2, \ldots, n_0$) and $B_{i_l}^{(l)}$ is homeomorphic to G_l ($i_l = 1, 2, \ldots, n_l$ and $l = 1, 2, \ldots, 5$). We fix a homeomorphism $\varphi_{i_l}^{(l)} : G_l \to B_{i_l}^{(l)}$ and put

$$\Gamma(B_{i_l}^{(l)}) = \left\{ \gamma_{i_l,j}^{(l)} = \varphi_{i_l}^{(l)}(\gamma_j) \ \mid \ \gamma_j \in \Gamma(G_l) \right\}$$

($i_l = 1, 2, \ldots, n_l$ and $l = 1, 2, \ldots, 5$). Let $T_1 = v$, $T_2 = e_3$, $T_3 = e_4$, $T_4 = e_4 \cup e_5$ and $T_5 = e_4 \cup e_5 \cup e_6$ be spanning trees of $G_1, G_2, \ldots, G_5$, respectively. Namely $T_{i_l}^{(l)} = \varphi_{i_l}^{(l)}(T_l)$ is a spanning tree of $B_{i_l}^{(l)}$.

Since f is a boundary spatial embedding of G, there exist connected, compact and orientable surfaces $S^{(l)}_{i_l,j}$ ($l = 1,2,\dots,5$, $i_l = 1,2,\dots,n_l$, $j = 1,2,\dots$) such that the interiors of them are mutually disjoint and

$$f(G)\cap S^{(l)}_{i_l,j} = f(G)\cap \partial S^{(l)}_{i_l,j} = f(\gamma^{(l)}_{i_l,j}).$$

Let us consider

$$\begin{aligned} P = f(G) \quad &\cup \quad \bigcup_{i_1=1}^{n_1} S^{(1)}_{i_1,1} \cup \bigcup_{i_2=1}^{n_2}\left(\bigcup_{j=1}^{2} S^{(2)}_{i_2,j}\right) \cup \bigcup_{i_3=1}^{n_3}\left(\bigcup_{j=1}^{3} S^{(3)}_{i_3,j}\right) \\ &\cup \quad \bigcup_{i_4=1}^{n_4}\left(\bigcup_{j=1}^{3} S^{(4)}_{i_4,j}\right) \cup \bigcup_{i_5=1}^{n_5}\left(\bigcup_{j=1}^{3} S^{(5)}_{i_5,j}\right). \end{aligned}$$

Let $N^{(1)}_{i_1}, N^{(2)}_{i_2}, N^{(3)}_{i_3}, N^{(4)}_{i_4}$ and $N^{(5)}_{i_5}$ be regular neighbourhoods of $f(T^{(1)}_{i_1})$, $f(T^{(2)}_{i_2})$, $f(T^{(3)}_{i_3})$, $f(T^{(4)}_{i_4})$ and $f(T^{(5)}_{i_5})$ in $S^{(1)}_{i_1,1}$, $S^{(2)}_{i_2,1}\cup S^{(2)}_{i_2,2}$, $S^{(3)}_{i_3,1}\cup S^{(3)}_{i_3,2}\cup S^{(3)}_{i_3,3}$, $S^{(4)}_{i_4,1}\cup S^{(4)}_{i_4,2}\cup S^{(4)}_{i_4,3}$ and $S^{(5)}_{i_5,1}\cup S^{(5)}_{i_5,2}\cup S^{(5)}_{i_5,3}$, respectively such that $N^{(l)}_{i_l}$ contains all cut vertices between $f(B^{(l)}_{i_l})$ and the other blocks of $f(G)$ ($i_l = 1,2,\dots,n_l$, $l = 1,2,\dots,5$) as illustrated in Fig. 2.3.4. Then we can regard

$$\bigcup_{l=1}^{5}\bigcup_{i_l=1}^{n_l}\left(f(T^{(l)}_{i_l})\cup \partial N^{(l)}_{i_l}\right)$$

as a trivial spatial embedding h of G and

$$F = \mathrm{cl}\left(P - \bigcup_{l=1}^{5}\bigcup_{i_l=1}^{n_l} N^{(l)}_{i_l}\right)$$

is the disjoint union of spanning surfaces of a boundary link $L = \partial F$. Therefore we may assume that there exist mutually disjoint $n_1 + 2n_2 + 3n_3 + 3n_4 + 3n_5$ disks $b^{(l)}_{i_l,j}$ embedded in S^3 such that $b^{(l)}_{i_l,j}\cap F = \partial b^{(l)}_{i_l,j}\cap \partial F$ is an arc, $b^{(l)}_{i_l,j}\cap h(G) = \partial b^{(l)}_{i_l,j}\cap \mathrm{int} h(\varphi^{(l)}_{i_l}(e_j))$ is also an arc ($i_l = 1,2,\dots,n_l$, $l = 1,2,\dots,5$ and

$j = 1, 2, \dots)$ and

$$f(G) \;=\; h(G) \cup \bigcup \partial b^{(l)}_{i_l,j} \cup L - \bigcup \mathrm{int}\left(h(\varphi^{(l)}_{i_l}(e_j)) \cap b^{(l)}_{i_l,j}\right) - \mathrm{int}\left(\partial F \cap b^{(l)}_{i_l,j}\right),$$

see Fig. 2.3.5. We call this a *band sum of a boundary link L and $h(G)$*.

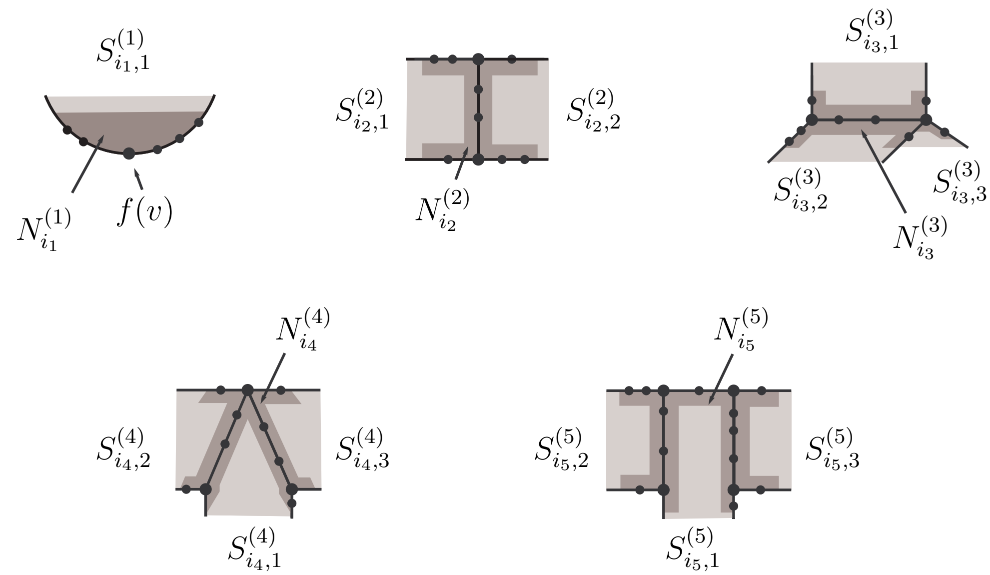

Fig. 2.3.4

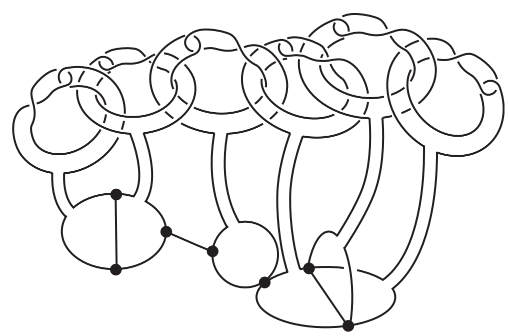

Fig. 2.3.5

By Theorem 2.1.4 (4), L can be transformed into a completely split link L' up to self pass-equivalence such that each of the components of L is a trivial knot or a trefoil knot. Thus we have that f can be transformed into a band sum of L' and $h(G)$ up to self pass-equivalence, see Fig. 2.3.6. Then by using self Γ-moves, namely up to self pass-equivalence by Lemma 2.3.2, we can shrink each band with the component of L' one by one, see Fig. 2.3.6. By shrinking all bands in such a way, we obtain a spatial embedding ψ_f which is a trivial spatial embedding with some local trefoil knots.

We note that a local trefoil knot attached to $\psi_f(\varphi_{i_l}^{(l)}(e_j))$ is unique up to ambient isotopy. We have that g also can be transformed into a canonical spatial embedding ψ_g up to self pass-equivalence in the same way. Since a trivial spatial embedding of a planar graph is unique up to ambient isotopy [12], by the assumption we have that $\psi_f = \psi_g$. Therefore we have that f and g are self pass-equivalent.

Next we prove (1). By Lemma 2.3.1, we have that any boundary spatial embedding f of a graph can be transformed into ψ_f by self sharp-equivalence in the same way as the proof of (2). We note that the self sharp-move is an unknotting operation (Theorem 2.1.4 (1)). Thus we can undo each of the local knots by self sharp-moves. So we have that f is trivial up to self sharp-equivalence. □

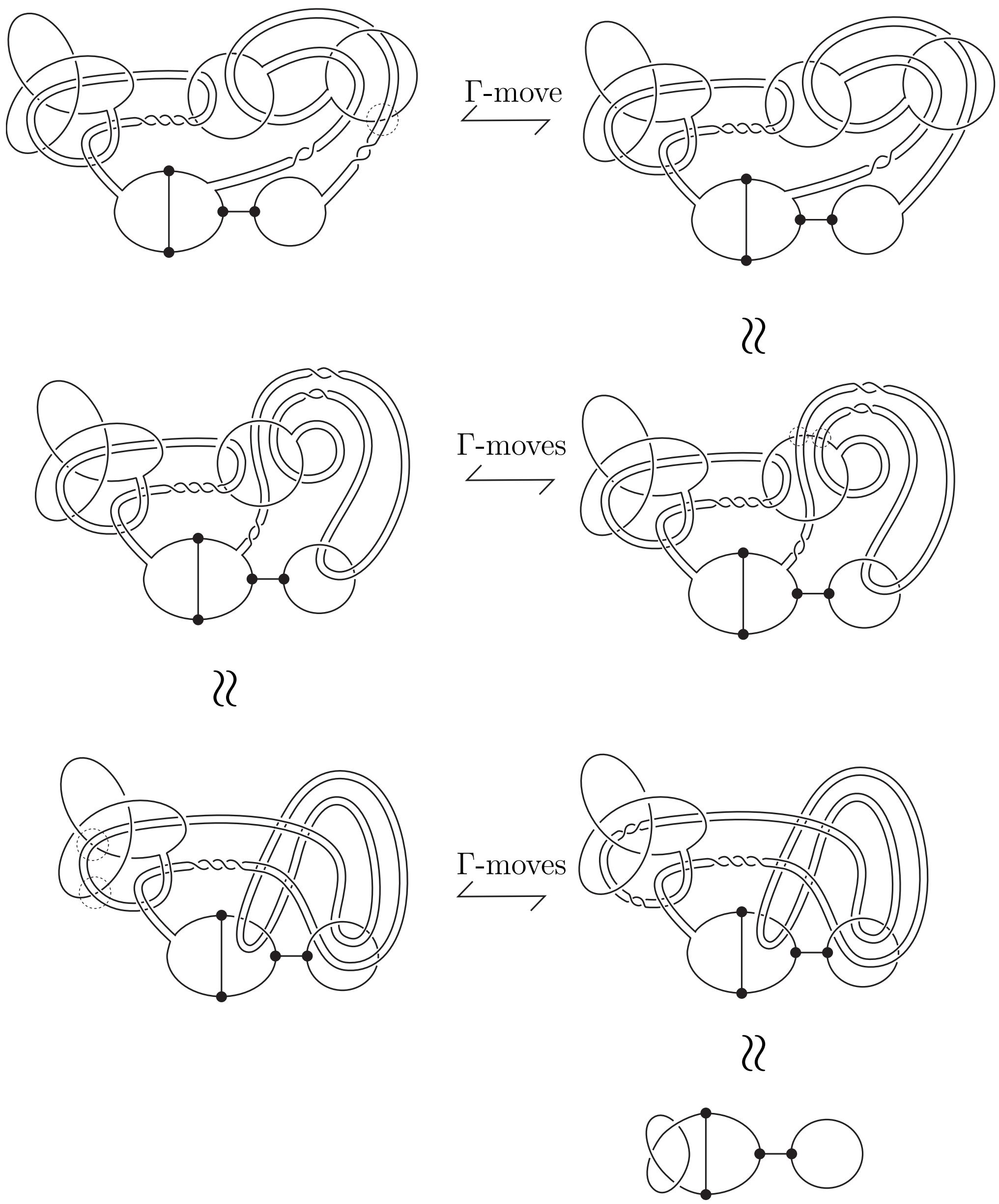

Fig. 2.3.6

Chapter 3

Homology classification of spatial graphs by linking numbers and Simon invariants

3.1 Introduction and results

Let G be a graph. In this chapter, we mean an embedding $f : G \to R^3$ of G into R^3 by a spatial embedding of G. In [29] Taniyama showed that two spatial embeddings are spatial-graph-homologous if and only if they have the same Wu invariant. Spatial-graph-homology is an equivalence relation of spatial graphs introduced in [28]. We note that in [28] and [29] spatial-graph-homology is simply called homology. See [28] or [29] for the definition of spatial-graph-homology. It is known that Wu invariant coincides with linking number if G is homeomorphic to a disjoint union of two circles, and it coincides with Simon invariant if G is homeomorphic to a complete graphs on five vertices K_5 or a complete bipartite graph on three-three vertices $K_{3,3}$. Note that both linking number and Simon invariant are integral invariants that are easily calculated from a regular diagram of a spatial graph. The purpose of this chapter is to show that f and g are spatial-graph-homologous if and only if all of their linking numbers and Simon invariants coincide. Namely f and g are spatial-graph-homologous if and only if for each subgraph H of G that is homeomorphic to a disjoint union of two circles, the restriction maps $f|_H$ and $g|_H$ have the same linking number, and for each subgraph H of G that is homeomorphic to K_5 or $K_{3,3}$, the restriction maps $f|_H$ and $g|_H$ have the same Simon invariant. In [14] it is shown that two spatial embeddings are spatial-graph-homologous if and only if they are transformed into each other by delta-moves and ambient isotopies. It is known that a delta-move does not change any finite type invariant of order 1 of spatial graphs in the sense of [27]. Therefore we have that linking number and Simon invariant determine all of finite type invariants of order 1 of spatial graphs.

Now we state the definition of Wu invariant. See [29] for more detail. For a topological space X let $C_2(X)$ be the configuration space of ordered two points on X. Let σ be an involution on $C_2(X)$ that is the exchange of the order of two points, i.e. $\sigma(x, y) = (y, x)$. Let $f : G \to R^3$ be a spatial embedding. Let $f^2 : C_2(G) \to C_2(R^3)$ be a map defined by $f^2(x, y) = (f(x), f(y))$. Then f^2 induces a homomorphism

$$(f^2)^{\#} : H^2(C_2(R^3), \sigma) \to H^2(C_2(G), \sigma)$$

where $H^2(C_2(X), \sigma)$ denotes the skew-symmetric second cohomology of the pair $(C_2(X), \sigma)$. It is known that $H^2(C_2(R^3), \sigma)$ is an infinite cyclic group. Let τ be a fixed generator of $H^2(C_2(R^3), \sigma)$. Then Wu defined an invariant of f by $(f^2)^{\#}(\tau)$ [31].

We denote this element of $H^2(C_2(G), \sigma)$ by $\mathcal{L}(f)$ and call it the Wu invariant of f.

Theorem 3.1.1. (Taniyama [29]) *Two spatial embeddings $f, g : G \to R^3$ are spatial-graph-homologous if and only if $\mathcal{L}(f) = \mathcal{L}(g)$.*

Thus Wu invariant classifies spatial graphs completely up to spatial-graph-homology. See [32] for another spatial-graph-homology classification using disk/band surfaces of spatial graphs.

In the summer of 1990, J. Simon gave a lecture at Tokyo. In the lecture he defined an invariant for spatial embeddings of K_5 and $K_{3,3}$ as follows.

We give an orientation of the edges of K_5 and $K_{3,3}$ as illustrated in Fig. 3.1.1.

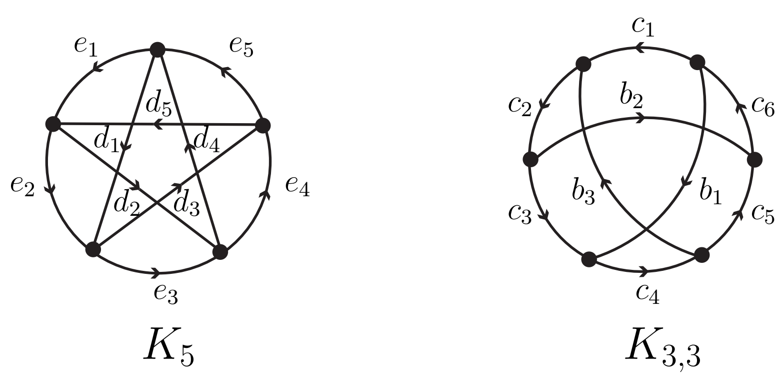

Fig. 3.1.1

Let $G = K_5$ or $K_{3,3}$. For two disjoint edges x, y, we define the sign $\epsilon(x, y) = \epsilon(y, x)$ as follows;

$\epsilon(e_i, e_j) = 1$, $\epsilon(d_i, d_j) = -1$ and $\epsilon(e_i, d_j) = -1$ for $i, j \in \{1, 2, 3, 4, 5\}$.

$\epsilon(c_i, c_j) = 1$, $\epsilon(b_k, b_l) = 1$ and

$$\epsilon(c_i, b_k) = \begin{cases} 1 & \text{if } c_i \text{ and } b_k \text{ are parallel in Fig. 3.1.1} \\ -1 & \text{if } c_i \text{ and } b_k \text{ are anti-parallel in Fig. 3.1.1} \end{cases}$$

for $i, j \in \{1, 2, 3, 4, 5, 6\}$, $k, l \in \{1, 2, 3\}$.

Let $f : G \to R^3$ be a spatial embedding and $\pi : R^3 \to R^2$ a natural projection. Suppose that $\pi \circ f$ is a regular projection. For disjoint oriented edges x and y of G, let $\ell(f(x), f(y))$ be the sum of the signs of the mutual crossings $\pi \circ f(x) \cap \pi \circ f(y)$ where the sign of a crossing is defined by Fig. 3.1.2.

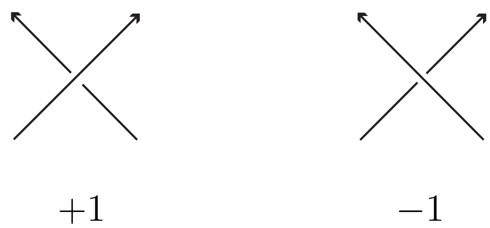

Fig. 3.1.2

Now we define an interger $L(f)$ by

$$L(f) = \sum_{x \cap y = \emptyset} \epsilon(x, y)\ell(f(x), f(y))$$

where the summation is taken over all unordered pairs of disjoint edges of G.

It is known that two regular projections represent ambient isotopic embeddings if and only if they are connected by a sequence of generalized Reidemeister moves [9]. Then it is easy to check that $L(f)$ is invariant under these moves. Therefore $L(f)$ is a well-defined ambient isotopy invariant. We call $L(f)$ the *Simon invariant* of f.

The following are known in [29]. If G is homeomorphic to a disjoint union of two circles, K_5 or $K_{3,3}$, the group $H^2(C_2(G), \sigma)$ is an infinite cyclic group.

Then we may suppose that $\mathcal{L}(f)$ is an integer. If G is homeomorphic to a disjoint union of two circles, $\mathcal{L}(f)$ is equal to twice the linking number of $f(G)$ up to sign. If G is homeomorphic to K_5 or $K_{3,3}$, $\mathcal{L}(f)$ is equal to $L(f)$ up to sign. In [28, Theorem C] it is shown that if a graph G does not contain any subgraph that is homeomorphic to a disjoint union of two circles, K_5 or $K_{3,3}$ then any two spatial embeddings of G are spatial-graph-homologous. Corresponding to this result it is shown in [29] that the group $H^2(C_2(G), \sigma)$ is trivial for such G. Namely if G is a planar graph that does not contain disjoint circles then $H^2(C_2(G), \sigma) = 0$.

In [26] the following is shown.

Theorem 3.1.2. (Soma, Sugai and Yasuhara [26]) *Let G be a connected planar graph and $f, g : G \to R^3$ spatial embeddings of G. Then f and g are spatial-graph-homologous if and only if for any subgraph H of G that is homeomorphic to a disjoint union of two circles the restriction maps $f|_H$ and $g|_H$ have the same linking number.*

In this chapter we generalize Theorem 3.1.2 to an arbitrary finite graph. Our main theorem in this chapter is the following theorem.

Theorem 3.1.3. *Let G be a graph and $f, g : G \to R^3$ spatial embeddings of G. Then f and g are spatial-graph-homologous if and only if for each subgraph H of G that is homeomorphic to a disjoint union of two circles the restriction maps $f|_H$ and $g|_H$ have the same linking number, and for each subgraph H of G that is homeomorphic to K_5 or $K_{3,3}$ the restriction maps $f|_H$ and $g|_H$*

have the same Simon invariant. In other words $\mathcal{L}(f) = \mathcal{L}(g)$ *if and only if* $\mathcal{L}(f|_H) = \mathcal{L}(g|_H)$ *for each subgraph* H *of* G *that is homeomorphic to a disjoint union of two circles,* K_5 *or* $K_{3,3}$.

In [29, §2] a method of calculation of Wu invariant from a regular diagram of a spatial graph is explained. By using this calculation it is easily seen that Wu invariant is a finite type invariant of order 1 in the sense of [27]. It is shown in [30] that two spatial embeddings have the same finite type invariants of order 1 if they are transformed into each other by delta-moves and ambient isotopies. Since spatial-graph-homologous embeddings are transformed into each other by delta-moves and ambient isotopies [14] we have that every finite type invariant of order 1 is determined by linking numbers and Simon invariants. Namely we have the following theorem.

Theorem 3.1.4. *Let* G *be a finite graph and* $f, g : G \to R^3$ *spatial embeddings of* G*. Then the following conditions are equivalent.*

(1) f *and* g *are spatial-graph-homologous,*

(2) $\mathcal{L}(f) = \mathcal{L}(g)$,

(3) $v(f) = v(g)$ *for any finite type invariant* v *of order* 1 ,

(4) *for each subgraph* H *of* G *that is homeomorphic to a disjoint union of two circles the restriction maps* $f|_H$ *and* $g|_H$ *have the same linking number, and for each subgraph* H *of* G *that is homeomorphic to* K_5 *or* $K_{3,3}$ *the restriction maps* $f|_H$ *and* $g|_H$ *have the same Simon invariant.*

Remark 3.1.5. In [29] it is shown that $H^2(C_2(G), \sigma)$ is torsion free. This fact is essentially used in the proof of the 'if' part of Theorem 3.1.1. We give a new proof of this fact as a corollary of Theorem 3.2.1 in Section 2.

We say that two spatial embeddings $f, g : G \to R^3$ are *minimally different* if f and g are not ambient isotopic and for each proper subgraph H of G, the restriction maps $f|_H$ and $g|_H$ are ambient isotopic. Let G be a planar graph and $u : G \to R^3$ a trivial spatial embedding. Then a spatial embedding $f : G \to R^3$ is called *minimally knotted* if f and u are minimally different. A graph G is called a *generalized bouquet* if there is a vertex v of G such that $G - \{v\}$ contains no cycle. It is shown in [7] that a minimally knotted embedding is not isotopic to u unless G is a generalized bouquet. Note that isotopy is an equivalence relation of spatial graphs that is weaker than ambient isotopy, but stronger than spatial-graph-homology [28]. As an application of Theorem 3.1.3 we have the following result that is a contrast to the result stated above.

Theorem 3.1.6. *Let G be a graph which is homeomorphic to none of a disjoint union of two circles, K_5 and $K_{3,3}$. Then any two minimally different embeddings of G are spatial-graph-homologous.*

Proof. Let $f, g : G \to R^3$ be minimally different embeddings. Let H be a subgraph of G that is homeomorphic to J, K_5 or $K_{3,3}$. By the assumption we have that H is a proper subgraph of G. Then we have $f|_H$ and $g|_H$ are ambient isotopic. Therefore $\mathcal{L}(f|_H) = \mathcal{L}(g|_H)$. Then by Theorem 3.1.3 we have that f and g are spatial-graph-homologous. $\square$

Note that each of a disjoint union of two circles, K_5 and $K_{3,3}$ has minimally different embeddings that are not spatial-graph-homologous. Examples are illustrated in Fig. 3.1.3. Since they have different linking numbers or different Simon invariants they are not spatial-graph-homologous. Then it is easily checked that they are minimally different.

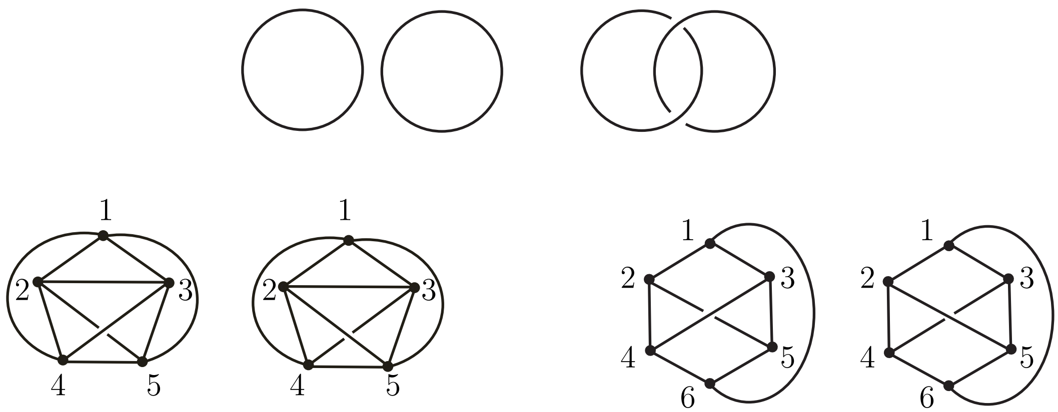

Fig. 3.1.3

3.2 Proof of Main Theorem

For the simplicity we denote the group $H^2(C_2(G), \sigma)$ by $L(G)$. Let H be a subgraph of G then the inclusion $C_2(H) \subset C_2(G)$ induces a homomorphism $\varphi_H : L(G) \to L(H)$. By $J = C_1 \cup C_2$ we denote a disjoint union of two circles C_1 and C_2.

Theorem 3.2.1. *Let G be a finite graph. Let x, y be elements of $L(G)$. Suppose that $\varphi_H(x) = \varphi_H(y)$ for any subgraph H of G that is homeomorphic to J, K_5 or $K_{3,3}$. Then $x = y$.*

Corollary 3.2.2. (Taniyama [29]) *Let G be a finite graph. Then $L(G)$ is torsion free.*

Proof. Let x be an element of $L(G)$ and n an integer greater than one such that $nx = 0$. Suppose that a subgraph H of G is homeomorphic to J, K_5 or $K_{3,3}$. Then $L(H)$ is an infinite cyclic group. Therefore we have $0 = \varphi_H(0) = \varphi_H(nx) = n\varphi_H(x)$. Thus we have $\varphi_H(x) = 0$. Thus we have $\varphi_H(x) = \varphi_H(0)$ if H is homeomorphic to J, K_5 or $K_{3,3}$. Then by Theorem 3.2.1 we have $x = 0$. This completes the proof. □

Proof of Theorem 3.1.3. Suppose that $\mathcal{L}(f|_H) = \mathcal{L}(g|_H)$ for each subgraph H of G that is homeomorphic to J, K_5 or $K_{3,3}$. Since $\mathcal{L}(f|_H) = \varphi_H(\mathcal{L}(f))$ and $\mathcal{L}(g|_H) = \varphi_H(\mathcal{L}(g))$ we have $\varphi_H(\mathcal{L}(f)) = \varphi_H(\mathcal{L}(g))$. Then by Theorem 3.2.1 we have $\mathcal{L}(f) = \mathcal{L}(g)$. □

We prove Theorem 3.2.1 step by step. First we prove Theorem 3.2.1 when G is a simple 3-connected graph. This case is the core of Theorem 3.2.1.

Proposition 3.2.3. *Theorem 3.2.1 is true if G is a simple 3-connected graph.*

For the proof of Proposition 3.2.3 we prepare some lemmas. Let P be a path. Let u and v be the degree one vertices of P. We call u and v the end points of P. Then we say that P joins u and v and by ∂P we denote the set $\{u, v\}$. Let P and Q be paths of G. If $Q \subset P$ then we say that Q is a *subpath* of P. A subpath of P joining u and v is denoted by $(u, v; P)$. Let H be a subgraph of G. Let u and v be vertices of H. Let X and Y be subsets of $V(H)$. Suppose that there uniquely exists a path P of H joining u and v such that $V(P) \supset X$ and $V(P) \cap Y = \emptyset$. Then we denote P by $(u, v, H, \mathrm{in}X, \mathrm{ex}Y)$. We denote $(u, v, H, \mathrm{in}\emptyset, \mathrm{ex}Y)$ by $(u, v, H, \mathrm{ex}Y)$, $(u, v, H, \mathrm{in}X, \mathrm{ex}\emptyset)$ by $(u, v, H, \mathrm{in}X)$ and $(u, v, H, \mathrm{in}\emptyset, \mathrm{ex}\emptyset)$ by (u, v, H) for simplicity. The following Lemma 3.2.4 is well-known in graph theory. See [2] for example.

Lemma 3.2.4. *Let G be a 2-connected graph and e_1 and e_2 disjoint edges of G. Then there is a cycle of G containing both of e_1 and e_2.*

Lemma 3.2.5. *Let G be a finite graph and e_1 and e_2 disjoint edges of G. Suppose that $G - \{e_1\}$, $G - \{e_2\}$ and $G - \{e_1, e_2\}$ are topologically simple and topologically 3-connected. Note that then G is topologically 3-connected. However G is not necessarily topologically simple. Then there is a subgraph H of G satisfying one of the following conditions.*

(1) *There is a homeomorphism* $h : J \to H$ *such that* $h(C_i)$ *contains* e_i $(i = 1, 2)$.

(2) *There is a homeomorphism* $h : K_5 \to H$ *such that* $h(d_i)$ *contains* e_i $(i = 1, 2)$ *where* d_1 *and* d_2 *are disjoint edges of* K_5.

(3) *There is a homeomorphism* $h : K_{3,3} \to H$ *such that* $h(d_i)$ *contains* e_i $(i = 1, 2)$ *where* d_1 *and* d_2 *are disjoint edges of* $K_{3,3}$.

Proof.

Claim 1. There is a cycle Ω of G containing e_1 and e_2.

Since G is 2-connected this is an immediate consequence of Lemma 3.2.4.

Claim 2. There is a subgraph H of G, disjoint edges f_1 and f_2 of a complete graph on four vertices K_4, and a homeomorphism $h : K_4 \to H$ such that $e_1 \subset h(f_1)$ and $e_2 \subset h(f_2)$.

Let e_3 and e_4 be edges on Ω such that e_1, e_3, e_2, e_4 are lying on Ω in this cyclic order. Since $G - \{e_1, e_2\}$ is 2-connected there is a cycle Λ of $G - \{e_1, e_2\}$ containing e_3 and e_4. Then it is not hard to see that either the condition (1) holds or there is a subgraph H in $\Omega \cup \Lambda$ that satisfies the desired conditions.

Claim 3. Suppose that the condition (1) does not hold. Then there is a subgraph H of G, disjoint edges f_1 and f_2 of a complete graph on four vertices K_4 and a homeomorphism $h : K_4 \to H$ such that $e_1 = h(f_1)$ and $e_2 = h(f_2)$.

Suppose that e_i is a proper subset of $h(f_i)$ for some $i \in \{1, 2\}$, say $i = 1$. Let u and v be the end points of $h(f_1)$. Since $(G - \{e_2\}) - \{u, v\}$ is connected there is a path of $(G - \{e_2\}) - \{u, v\}$ joining a vertex of $h(f_1)$ and a vertex of $H - h(f_1)$. Then we either have the condition (1) or find H' and a homeomorphism $h' : K_4 \to H'$ with $e_1 \subset h'(f_1)$ and $e_2 \subset h'(f_2)$ such that $h'(f_1 \cup f_2)$ is a proper subset of $h(f_1 \cup f_2)$. By repeating this replacement we finally have the desired situation.

Let u_i and v_i be the vertices incident to e_i for $i = 1, 2$. Let Γ be the cycle $H - \{e_1, e_2\}$. Since $(G - \{e_1, e_2\}) - \{u_2, v_2\}$ is connected there is a path P of $G - \{e_1, e_2\}$ joining a vertex, say w_1, in $(v_2, u_2, \Gamma, \mathrm{in}\{u_1\}) - \{u_2, v_2\}$ and a vertex, say w_2, in $(u_2, v_2, \Gamma, \mathrm{in}\{v_1\}) - \{u_2, v_2\}$. We may suppose that $P \cap H = \partial P$. Up to the symmetry of H it is sufficient to consider the following two cases.

Case 1. $w_1 \in (u_1, u_2, \Gamma, \mathrm{ex}\{v_1\}) - \{u_1\}$ and $w_2 \in (v_1, v_2, \Gamma, \mathrm{ex}\{u_1\}) - \{v_1\}$.

In this case we have that $H \cup P$ is homeomorphic to $K_{3,3,}$ and we have the condition (3).

Case 2. $w_1 \in (u_1, u_2, \Gamma, \mathrm{ex}\{v_1\})$ and $w_2 \in (u_2, v_1, \Gamma, \mathrm{ex}\{v_2\})$.

We choose P so that w_1 is closest to u_1 among all paths with $w_1 \in (u_1, u_2, \Gamma, \mathrm{ex}\{v_1\})$ and $w_2 \in (u_2, v_1, \Gamma, \mathrm{ex}\{v_2\})$. Note that we consider the case $w_1 = u_1$ as the closest case. Suppose that $w_1 \neq u_1$ and there is a path Q with $Q \cap (H \cup P) = \partial Q$ joining a vertex, say s, of $(u_1, w_1, \Gamma, \mathrm{ex}\{u_2\}) - \{w_1\}$ and a vertex, say t, of $(w_1, u_2, \Gamma, \mathrm{ex}\{v_1\}) - \{w_1\}$. Then we replace $(s, t, \Gamma, \mathrm{in}\{w_1\})$ by Q and have a new subgraph, still denoted by H. Then we choose for this new H new P with new $w_1 \in (u_1, u_2, \text{new } \Gamma, \mathrm{ex}\{v_1\})$ and new $w_2 \in (u_2, v_1, \text{new } \Gamma, \mathrm{ex}\{v_2\})$ so that w_1 is closest to u_1. Note that such new P exists because old $P \cup (s, \text{old } w_1, \text{old } \Gamma, \mathrm{ex}\{u_2\})$ satisfies the condition for new P. If new w_1 is still not equal to u_1 and there still exists a path Q as above then we perform the same replacement. We continue these replacements so that there are no such paths. Then among all paths with the same w_1 we choose P so that w_2 is closest to v_1. Suppose that $w_2 \neq v_1$ and there is a path Q with $Q \cap (H \cup P) = \partial Q$ joining a vertex of $(u_2, w_2, \Gamma, \mathrm{ex}\{v_1\}) - \{w_2\}$ and a vertex of $(w_2, v_1, \Gamma, \mathrm{ex}\{v_2\}) - \{w_2\}$. Then we perform a similar replacement. Then we rechoose P so that w_2 is closest to v_1. We continue these replacements until there are no such paths. Note that these operations do not change w_1.

Case 2.1. $w_2 \neq v_1$ and there is another path Q with $Q \cap (H \cup P) = \partial Q$ joining a vertex, say x_1, of $(w_1, u_2, \Gamma, \mathrm{ex}\{v_1\}) - \{w_1, u_2\}$ and a vertex, say x_2, of $(w_2, v_1, \Gamma, \mathrm{ex}\{v_2\}) - \{w_2\}$.

We choose Q such that x_2 is closest to v_1. If $x_2 \neq v_1$ and there is a path R with $R \cap (H \cup P \cup Q) = \partial R$ joining a vertex of $(w_2, x_2, \Gamma, \mathrm{ex}\{v_1\}) - \{x_2\}$ and a vertex of $(x_2, v_1, \Gamma, \mathrm{ex}\{v_2\}) - \{x_2\}$ then we perform a similar replacement. By repeating the operations we have that there are no such paths and x_2 is closest to v_1.

Now we consider the graph $(G - \{e_1, e_2\}) - \{w_1, x_2\}$. Since this graph is connected we find a path W joining the components of $(\Gamma \cup P \cup Q) - \{w_1, x_2\}$ and find the condition (1) in $H \cup P \cup Q \cup W$.

Case 2.2. There are no such path Q, and $w_1 \neq u_1$ or $w_2 \neq v_1$.

In this case we consider the graph $(G - \{e_1, e_2\}) - \{w_1, w_2\}$. Since this graph is connected we find a path W and find either condition (1) or (3) in $H \cup P \cup W$.

Case 2.3. $w_1 = u_1$ and $w_2 = v_1$.

In this case we consider the graph $(G - \{e_1, e_2\}) - \{u_1, v_1\}$. Since this graph is connected we find the condition (1), or find paths W_1 and W_2 joining the components of $(\Gamma \cup P) - \{u_1, v_1\}$ and find either condition (2) or (3) in $H \cup P \cup W_1 \cup W_2$. This completes the proof of Lemma 3.2.5. $\square$

Lemma 3.2.6. *Let G be a simple 3-connected graph and e an edge of G such that $G - \{e\}$ is topologically simple and topologically 3-connected. Then there is a subgraph G_0 of $G - \{e\}$ with $\partial e \subset G_0$ that is homeomorphic to K_4, and there is an increasing sequence $G_0 \subset G_1 \subset \cdots \subset G_n = G - \{e\}$ with the following properties;*

(1) *each G_i is topologically simple and topologically 3-connected,*

(2) *each G_i is obtained from G_{i-1} by a path addition,*

(3) *for each i the following* (a) *or* (b) *holds;*

(a) *$G_i \cup e$ is topologically simple and topologically 3-connected,*

(b) *$G_{i+1} \cup e$ is topologically simple and topologically 3-connected.*

Proof. Let u and v be the vertices incident to e. First we show that there is a cycle Γ of $G - \{e\}$ containing u and v. Let e_1 and e_2 be edges of $G - \{e\}$ incident to u and v respectively. Note that $G - \{e\}$ is 2-connected. If e_1 and e_2 are disjoint then by Lemma 3.2.4 we have a cycle containing e_1 and e_2. Suppose that $e_1 \cap e_2$ is a vertex, say w. Since $(G - \{e\}) - \{w\}$ is connected there is a path, say Q, of $(G - \{e\}) - \{w\}$ joining u and v. Then $e_1 \cup e_2 \cup Q$ is a desired cycle. Note that $e_1 \neq e_2$ since G has no multiple edges. Since $G - \{e\}$ is 2-connected there is a path, say P, with $P \cap \Gamma = \partial P$ joining some vertices of Γ. Then $\Gamma \cup P$ is a graph homeomorphic to Θ as illustrated in Fig. 1.1.1. Since

$G - \{e\}$ is topologically 3-connected there is a path Q with $Q \cap (\Gamma \cup P) = \partial Q$ such that $\Gamma \cup P \cup Q$ is homeomorphic to K_4. Then we set $G_0 = \Gamma \cup P \cup Q$.

Now suppose inductively that there is an increasing sequence $G_0 \subset G_1 \subset \cdots \subset G_k$ of subgraphs of G satisfying the conditions (1), (2) and (3). Suppose that $G_k \neq G - \{e\}$.

Case 1. $G_k \cup e$ is topologically simple.

Case 1.1. There is a vertex v of G_k that has degree two in $G_k \cup e$.

Let P be the longest path of G_k that contains v so that each vertex of $P - \partial P$ has degree two in $G_k \cup e$. Let $\partial P = \{s, t\}$. Since $G - \{s, t\}$ is connected there is a path Q of $(G - \{e\}) - \{s, t\}$ with $Q \cap G_k = \partial Q$ joining a vertex of P and a vertex of $G_k - V(P)$. Set $G_{k+1} = G_k \cup Q$. Then it is easy to check that G_{k+1} is topologically simple and topologically 3-connected.

Case 1.2. No vertex of G_k has degree two in $G_k \cup e$.

There is a path P of $G - \{e\}$ with $P \cap G_k = \partial P$. Let $\partial P = \{s, t\}$. If s and t are not adjacent in G_k then we set $G_{k+1} = G_k \cup P$. Suppose that s and t are incident to an edge d of G_k. Since G has no multiple edges P is not an edge. Then we replace d by P. Note that this replacement changes the increasing sequence $G_0 \subset G_1 \subset \cdots \subset G_k$. However it is clear that the new increasing sequence still satisfies the required conditions. Thus this case is reduced to Case 1.1.

Case 2. $G_k \cup e$ is not topologically simple.

Suppose that $G_k = G_{k-1} \cup P$ where P is a path of $G - \{e\}$ with $P \cap G_{k-1} = \partial P$. By the assumption we have that $G_{k-1} \cup e$ is topologically simple. Therefore we have that P joins u and v. Since G has no multiple edges we have that

P is not an edge. Since G is 3-connected there is a path Q of $G - \{e\}$ with $Q \cap G_k = \partial Q$ joining a vertex of $P - \partial P$ and a vertex of $G_{k-1} - \{u, v\}$. Set $G_{k+1} = G_k \cup Q$. Then it is easy to check that G_{k+1} is topologically simple and topologically 3-connected. $\square$

Lemma 3.2.7. *Let G be a simple 3-connected graph. Suppose that G is not isomorphic to K_4. Then there is an edge e of G such that $G - \{e\}$ is topologically simple and topologically 3-connected.*

Proof. Let e_1 and e_2 be distinct edges of G. We subdivide e_1 and e_2 by taking vertices v_1 and v_2 on them respectively. We add an edge d joining v_1 and v_2 to G. It is easy to see that the resultant graph G' is simple and 3-connected. Note that $G' - \{d\}$ is homeomorphic to G hence topologically simple and topologically 3-connected. Then we apply Lemma 3.2.6 to G' and d. Then we have that $G' - \{d\}$ is obtained from a topologically simple and topologically 3-connected graph by adding a path P. Let e be an edge of G corresponding the path P. Then we have that $G - \{e\}$ is topologically simple and topologically 3-connected. $\square$

Proof of Proposition 3.2.3. We give a proof by an induction on $|E(G)|$. The first step is the case that G is isomorphic to K_4. As we explained in the previous section, $L(K_4) = 0$. Thus Theorem 3.2.1 is true in this case. Suppose that Theorem 3.2.1 is true for all simple 3-connected graphs with less than n edges. Let G be a simple 3-connected graph with n edges. Now we review an explicit presentation of $L(G) = H^2(C_2(G), \sigma)$. See [29] for more details. We set $E(G) = \{e_1, e_2, \cdots, e_n\}$ and $V(G) = \{v_1, v_2, \cdots, v_m\}$. We choose a fixed orientation on each edge of G. For a pair of integers (i, j) with $1 \leq i < j \leq n$ and $e_i \cap e_j = \emptyset$, we denote the pair (e_i, e_j) by E^{ij}. For a pair of integers (i, s) with $1 \leq i \leq n$, $1 \leq s \leq m$ and v_s is not incident to e_i, we denote the pair (e_i, v_s) by V^{is}. We set

$$\delta^1(V^{is}) = \sum_{I(j)=s} E^{\rho(ij)} - \sum_{T(k)=s} E^{\rho(ik)}$$

where $I(j) = s$ means that the initial vertex of e_j is v_s and $T(k) = s$ means that the terminal vertex of e_k is v_s, and

$$\rho(ij) = \begin{cases} ij & \text{if } i < j \\ ji & \text{if } i > j. \end{cases}$$

Here the sum is taken over all j with $I(j) = s$ and $e_i \cap e_j = \emptyset$ and all k with $T(k) = s$ and $e_i \cap e_k = \emptyset$. Then $L(G)$ has an abelian group presentation

$$\begin{aligned} &\langle E^{ij}\ (1 \leq i < j \leq n,\ e_i \cap e_j = \emptyset) \mid \\ &\qquad\qquad \delta^1(V^{is})\ (1 \leq i \leq n,\ 1 \leq s \leq m,\ v_s \text{ is not incident to } e_i)\rangle. \end{aligned}$$

By Lemma 3.2.7 there is an edge e of G such that $G - \{e\}$ is topologically simple and topologically 3-connected. We may suppose without loss of generality that $e = e_n$ and e_n is incident to v_1 and v_2. Let x, y be elements of $L(G)$ such that $\varphi_H(x) = \varphi_H(y)$ for any subgraph H of G that is homeomorphic to J, K_5 or $K_{3,3}$. We will show that $x - y = 0$. Let $(G - \{e_n\})'$ be the 3-connected graph that is homeomorphic to $G - \{e_n\}$. Then $(G - \{e_n\})'$ is simple. Since $G - \{e_n\} = (G - \{e_n\})'$ or $G - \{e_n\}$ is a subdivision of $(G - \{e_n\})'$ we have that $(G - \{e_n\})'$ has at most $n - 1$ edges. Therefore we may apply the hypothesis of induction and have that $\varphi_{G-\{e_n\}}(x - y) = 0$. This implies that $x - y$ can be represented by an element as

$$x - y = \left[\sum a_{in} E^{in} \right]$$

where i varies over all $i \in \{1 \leq i < n\}$ with $e_i \cap e_n = \emptyset$ and a_{in} is an integer. We will change the representative element of $x - y$ step by step so that the range of i becomes smaller and smaller as follows. Let $G_0 \subset G_1 \subset \cdots \subset G_k = G - \{e_n\}$ be an increasing sequence satisfying the conditions of Lemma 3.2.6. Let P_i be a path of G such that $G_i = G_{i-1} \cup P_i$. We may suppose without loss of generality that there are integers $1 < r_0 < r_1 < r_2 < \ldots < r_{k-1} < r_k = n - 1$ such that $E(G_i) = \{e_1, e_2, \ldots, e_{r_i}\}$ for each i. Similarly we may suppose that there are integers $1 < s_0 \leq s_1 \leq s_2 \leq \cdots \leq s_{k-1} \leq s_k = m$ such that $V(G_i) = \{v_1, v_2, \ldots, v_{s_i}\}$ for each i. Up to the symmetry of K_4 there are six cases of the topological type of $G_0 \cup e_n$ as illustrated in Fig. 3.2.1.

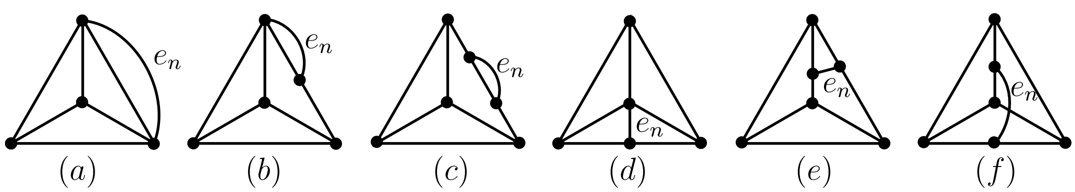

Fig. 3.2.1

In any case it is easy to see that there is an element $\sum b_{ns} \delta^1(V^{ns})$ where s varies over the set $\{1, 2, \ldots, s_1\}$ and b_{ns} is an integer such that

$$\sum a_{in} E^{in} + \sum b_{ns} \delta^1(V^{ns}) = \sum c_{jn} E^{jn}$$

where j varies over the set $\{r_0 + 1, r_0 + 2, \ldots, n - 2, n - 1\}$ and c_{jn} is an integer. Note that in (b), (c) and (e) we use the fact that $\varphi_H(x - y) = 0$ where H is homeomorphic to J. In (f) we use the fact that $\varphi_H(x-y)=0$ where H is homeomorphic to $K_{3,3}$.

Now suppose inductively that $x-y$ is represented as

$$x-y=\left[\sum a_{in}E^{in}\right]$$

where i varies over the set $\{r_j+1, r_j+2, \ldots, n-2, n-1\}$. We consider the following three cases.

Case 1. $G_j \cup e_n$ is not topologically simple.

In this case $\partial P_j = \partial e_n = \{v_1, v_2\}$ and ∂P_{j+1} contains a vertex on P_j. By adding some $\sum b_{ns}\delta^1(V^{ns})$ where s varies over the set $\{s_{j-1}+1, s_{j-1}+2, \ldots, s_{j+1}\}$ we have the result. Namely we have

$$\sum a_{in}E^{in}+\sum b_{ns}\delta^1(V^{ns})=\sum c_{jn}E^{jn}$$

where j varies over the set $\{r_{j+1}+1, r_{j+1}+2, \ldots, n-2, n-1\}$ and c_{jn} is an integer.

Case 2. $G_j \cup e_n$ is topologically simple and $P_{j+1}\cap e_n \neq \emptyset$.

By adding some $\sum b_{ns}\delta^1(V^{ns})$ where s varies over the set $\{s_j+1, s_j+2, \ldots, s_{j+1}\}$ we have the result.

Case 3. $G_j \cup e_n$ is topologically simple and $P_{j+1}\cap e_n = \emptyset$.

In this case we regard e_n and P_{j+1} as disjoint edges and apply Lemma 3.2.5. Namely by adding some $\sum b_{ns}\delta^1(V^{ns})$ where s varies over the set $\{s_j+1, s_j+2, \ldots, s_{j+1}\}$ we have the result. This complete the proof. □

Next we prove Theorem 3.2.1 for simple 2-connected graphs.

Proposition 3.2.8. *Theorem 3.2.1 is true if G is a simple 2-connected graph.*

Lemma 3.2.9. *Let G be a simple 2-connected graph and u, v vertices of G. Suppose that the graph $G - \{u, v\}$ is not connected. Let $Q_1, Q_2, \ldots, Q_p$ be the connected components of the topological space $G - \{u, v\}$. Let H_i be the closure of Q_i in G. Let G_i be a graph obtained from H_i by adding a new edge joining u and v. Suppose that Theorem 3.2.1 is true for each G_i. Then Theorem 3.2.1 is true for G.*

Proof. Let x, y be elements of $L(G)$ such that $\varphi_H(x) = \varphi_H(y)$ for any subgraph H of G that is homeomorphic to J, K_5 or $K_{3,3}$. We will show that $x-y = 0$. We set $E(G) = \{e_1, e_2, \ldots, e_n\}$ and $V(G) = \{v_1 = u, v_2 = v, v_3, \ldots, v_m\}$. Suppose that $x - y$ is represented by an element as

$$x - y = \left[\sum a_{ij} E^{ij} \right]$$

where a_{ij} is an integer and the summation is taken for all pair (i, j) with $1 \leq i < j \leq n$ and $e_i \cap e_j = \emptyset$. We will change the representative $\sum a_{ij} E^{ij}$ step by step as follows. Let T_i be a spanning tree of H_i such that the degree of u in T_i is one and the degree of v in T_i is one. Let $T = T_1 \cup T_2 \cup \cdots \cup T_p$. Note that the maximal subgraph of T without vertices of degree one is homeomorphic to a graph on two vertices and p edges joining them. Therefore it is easy to see that the representative $\sum a_{ij} E^{ij}$ of $x - y$ can be chosen such that the following condition (1) holds.

(1) $a_{ij} = 0$ if $e_i \in E(T_k)$ and $e_j \in E(T_l)$ for some $k \neq l$.

Next we show that in addition to the condition (1) the following condition (2) also holds.

(2) $a_{ij} = 0$ if one of e_i, e_j is in $E(T_k)$ for some k and the other is not in $E(H_k)$.

Suppose that $e_i \in E(H_l) - E(T_l)$. If e_i is incident to u or v then it is easy to erase all a_{ij} and a_{ji} with $e_j \in E(T_k)$ for some $k \neq l$. Suppose that e_i is not incident to u nor v. Let γ be the unique cycle of $T_l \cup e_i$. Then by using the condition on the disjoint cycles of $T \cup e_i$ containing γ as a component we can erase all a_{ij} and a_{ji} with $e_j \in E(T_k)$ for some $k \neq l$ without breaking the previous conditions. Next we show that the following condition (3) holds.

(3) $a_{ij} = 0$ unless $\{e_i, e_j\} \subset E(H_k)$ for some k.

Note that the condition (3) implies both (1) and (2). Let $e_i \in E(H_k) - E(T_k)$ and $e_j \in E(H_l) - E(T_l)$ with $k \neq l$. Let γ_1 be the unique cycle of $T_k \cup e_i$ and γ_2 the unique cycle of $T_l \cup e_j$. Then by the condition on these disjoint cycles we have that $a_{ij} = 0$. Thus we have a representative $\sum a_{ij} E^{ij}$ of $x - y$ that satisfies the condition (3).

Finally we will erase the term a_{ij} with $e_i, e_j \in E(H_k)$ for some k. We will do this step by step. First we will erase all the terms $a_{ij} E^{ij}$ with $e_i, e_j \in E(H_1)$ as follows. Let P be a path in T_2 joining u and v. Then $H_1 \cup P$ is homeomorphic to G_1. Let e_0 be the edge of G_1 joining $u = v_1$ and $v = v_2$. Then by the assumption on G_1 we have that

$$\sum a'_{ij} E^{ij} = \sum b_{is} \tilde{\delta}^1(V^{is})$$

where $a'_{ij} = a_{ij}$ if $e_i, e_j \in E(H_1)$ and $a'_{ij} = 0$ otherwise, and the summation of the second term is taken over some pair i, s with $e_i \in E(G_1)$ and $v_s \in V(G_1)$, and each of $\tilde{\delta}^1(V^{i1})$ and $\tilde{\delta}^1(V^{i2})$ expresses the signed sum of some E^{jk} with

$e_j, e_k \in E(G_1)$, one of them is e_i the other incident to v_1 or v_2, not the signed sum of some E^{jk} with $e_j, e_k \in E(G)$, and otherwise $\tilde{\delta}^1(V^{is}) = \delta^1(V^{is})$. We will modify the second summation as follows. First we replace each term $b_{i1}\tilde{\delta}^1(V^{i1})$ by $b_{i1}\delta^1(V^{i1})$. Next we replace each term $b_{i2}\tilde{\delta}^1(V^{i2})$ by $b_{i2}\sum\delta^1(V^{is})$ where the summation is taken over all s with $v_s \in (V(G) - V(H_1)) \cup \{v_2\}$. Finally we replace each term $b_{0s}\tilde{\delta}^1(V^{0s})$ by $b_{0s}\sum\delta^1(V^{is})$ where the summation is taken over all i with e_i incident to $u = v_1$ and $e_i \in E(G) - E(H_1)$. Let $\sum c_{is}\delta^1(V^{is})$ be the summation obtained from $\sum b_{is}\tilde{\delta}^1(V^{is})$ by the replacement stated above. Then we have that the new representative $\sum a_{ij}E^{ij} - \sum c_{is}\delta^1(V^{is}) = \sum d_{ij}E^{ij}$ of $x - y$ satisfies $d_{ij} = 0$ if $e_i, e_j \in E(H_1)$, and still satisfies the condition (3). Repeating this replacement p times we have 0 as a representative of $x - y$. This completes the proof. $\square$

Proof of Proposition 3.2.8. We will give a proof by an induction on the number of the edges of a simple 2-connected graph. The minimal number of the edges of a simple 2-connected graph is three and then the graph is K_3. Since $L(K_3)$ is trivial Theorem 3.2.1 is true for K_3. Suppose that Theorem 3.2.1 is true for each simple 2-connected graph that has k or less edges. Let G be a simple 2-connected graph that has $k+1$ edges. If G is 3-connected then by Proposition 3.2.3 we have the result. Suppose that G is not 3-connected. Then there are vertices u and v of G such that the graph $G - \{u, v\}$ is not connected. Let $Q_1, Q_2, \ldots, Q_p$ be the connected components of the topological space $G - \{u, v\}$. Let H_i be the closure of Q_i in G. Let G_i be a graph obtained from H_i by adding a new edge joining u and v. Suppose that u and

v are not adjacent in G. Then we have that each G_i is a simple 2-connected graph. Suppose that u and v are adjacent in G. Then we have $p \geq 3$, one of $G_1, \ldots, G_p$ is a cycle, and other graphs are simple 2-connected graphs. Note that Theorem 3.2.1 is true for a cycle since $L(G)$ is trivial if G is a cycle. Since each G_i has k or less edges we have the result by the induction hypothesis and by Lemma 3.2.9. $\square$

Next we prove Theorem 3.2.1 for simple connected graphs.

Proposition 3.2.10. *Theorem 3.2.1 is true if G is a simple connected graph.*

Lemma 3.2.11. *Let G be a simple connected graph and v a vertex of G. Suppose that the graph $G - \{v\}$ is not connected. Let $Q_1, Q_2, \ldots, Q_p$ be the connected components of the topological space $G - \{v\}$. Let G_i be the closure of Q_i in G. Suppose that Theorem 3.2.1 is true for each G_i. Then Theorem 3.2.1 is true for G.*

Proof. Let x, y be elements of $L(G)$ such that $\varphi_H(x) = \varphi_H(y)$ for any subgraph H of G that is homeomorphic to J, K_5 or $K_{3,3}$. We will show that $x - y = 0$. We set $E(G) = \{e_1, e_2, \ldots, e_n\}$ and $V(G) = \{v_1 = v, v_2, \ldots, v_m\}$. Suppose that $x - y$ is represented by an element as

$$x - y = \left[\sum a_{ij} E^{ij} \right]$$

where a_{ij} is an integer and the summation is taken for all pair (i, j) with $1 \leq i < j \leq n$ and $e_i \cap e_j = \emptyset$. We will change the representative $\sum a_{ij} E^{ij}$ step by step as follows. Let T_i be a spanning tree of G_i such that the degree of v in T_i is one. First we change the representative element of $x-y$ by using the assumption on each G_i such that

(1) $a_{jk} = 0$ if $e_j, e_k \in E(G_i)$ for some i.

To do this we first consider the case $i = 1$. By the assumption on G_1 we have

$$\sum a'_{ij} E^{ij} = \sum b_{is} \tilde{\delta}^1(V^{is})$$

where the meanings of a'_{ij} and $\tilde{\delta}^1$ are similar to those in the proof of Lemma 3.2.9. Then we replace each $b_{i1}\tilde{\delta}^1(V^{i1})$ by $b_{i1}\sum \delta^1(V^{is})$ where the summation is taken over all s with $v_s \in (V(G) - V(G_1)) \cup \{v_1\}$. Let $\sum c_{is}\delta^1(V^{is})$ be the summation obtained from $\sum b_{is}\tilde{\delta}^1(V^{is})$ by this replacement. Then we have that the new representative $\sum a_{ij}E^{ij} - \sum c_{is}\delta^1(V^{is}) = \sum d_{ij}E^{ij}$ of $x-y$ satisfies $d_{ij} = 0$ if $e_i, e_j \in E(G_1)$. Repeating this replacement p times we have the condition (1). Next we change the representative element such that in addition to the condition (1),

(2) $a_{jk} = 0$ if one of e_j and e_k is in $E(T_i)$ for some i.

This is easily done by using the fact that each T_i is a tree. Then by considering appropriate disjoint cycles we have that $a_{jk} = 0$ for any j and k. This completes the proof. □

Proof of Proposition 3.2.10. We will give a proof by an induction on the number of the vertices of a simple connected graph. It is clear that Theorem 3.2.1 is true for all graphs of one or two vertices. Suppose that Theorem 3.2.1 is true for each simple connected graph that has k or less vertices. Let G be a simple connected graph that has $k+1$ vertices. If G is 2-connected then by Proposition 3.2.8 we have the result. Suppose that G is not 2-connected. Then there is a vertex v of G such that the graph $G - \{v\}$ is not connected. Let $Q_1, Q_2, \dots, Q_p$ be the connected components of the topological space $G - \{v\}$. Let G_i be the closure of Q_i in G. Then we have that each G_i is a simple connected graph. Since each G_i has k or less vertices we have the result by the induction hypothesis and by Lemma 3.2.11. $\square$

Proposition 3.2.12. *Theorem 3.2.1 is true if G is a simple graph.*

Proof. Let G be a simple graph and $G_1, \dots, G_p$ the connected components of G. Then each G_i is a simple connected graph. We choose a spanning tree T_i for each G_i. Then the proof is similar to that of Lemma 3.2.11 and we omit it. $\square$

Proof of Theorem 3.2.1. By Proposition 3.2.12 it is sufficient to consider the case that G is not simple. Let G' be a simple graph that is a subdivision of G. Then by Proposition 3.2.12 we have that Theorem 3.2.1 is true for G'. Since $L(G)$ is isomorphic to $L(G')$ we have that Theorem 3.2.1 is true for G. $\square$

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List of papers by Reiko Shinjo

[1] *Homology classification of spatial graphs by linking numbers and Simon invariants*, Topology Appl. **134** (2003), no. 1, 53-67 (with Kouki Taniyama).

[2] *On boundary spatial embeddings of a graph,* Q. J. Math. **56** (2005), no. 2, 239-249 (with Ryo Nikkuni).

[3] *Bounding disks to a spatial graph*, J. Knot Theory Ramifications, to appear.