2. T-norm and its Family

T-norm is a product operation of fuzzy set and fuzzy logic. As the fuzzy set operation, T-norm means 'fuzzy intersection operation', as the fuzzy logic operation, this means 'fuzzy AND operation'.

In this chapter, we would explain T-norm, T-norm family and its characteristics

2.1 T-norm and its Family

T-norm (Triangular Norm) is a binary operation that is defined by the following:

Definition.2.1 T-norm is a binary operation

$$p, q \in [0,1] \rightarrow T(p,q) \in [0,1]$$

satisfying the following properties:

(1) Commutativity : $T(p,q) = T(q,p)$

(2) Associativity : $T(p,T(q,r)) = T(T(p,q),r)$

(3) Monotonicity : $p \leq q, \quad r \leq s \Rightarrow T(p,r) \leq T(q,s)$

(4) Boundary conditions : $T(p,0) = 0, \quad T(p,1) = p$

The typical T-norms are the followings:

(1) Logical product : $T_L(p,q) = p \wedge q$

(2) Algebraic product : $T_A(p,q) = p \times q$

(3) Multi-valued product (Lukaszewicz product) : $T_M(p,q) = (p+q-1) \vee 0$

(4) Drastic product : $T_D(p,q) = \begin{cases} 0 & , p \vee q < 1 \\ p \wedge q & , p \vee q = 1 \end{cases}$

T-norms of 3-dimension graphs are illustrated in Figure.2.1.1-2.1.4.

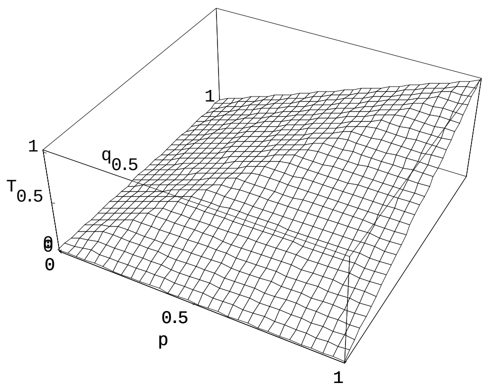


Figure.2.1.1 Logical Product

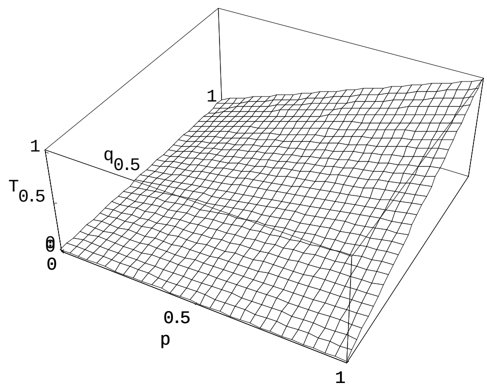


Figure.2.1.2 Algebraic Product

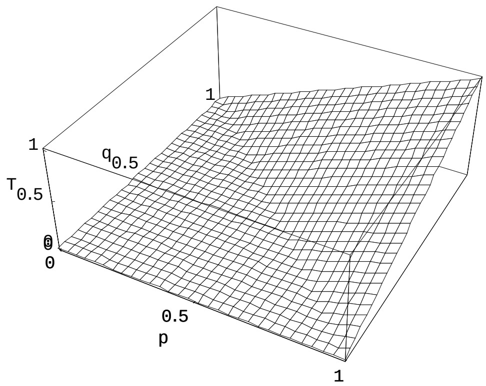


Figure.2.1.3 Multi-Valued Product

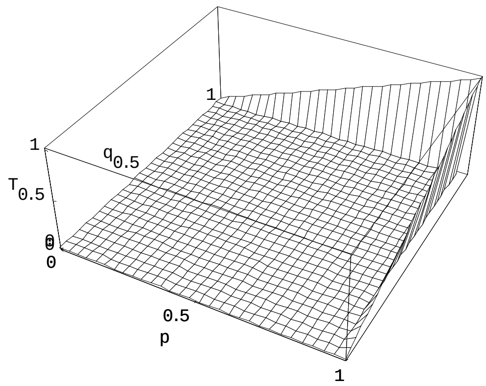


Figure.2.1.4 Drastic Product

Definition.2.2 Order of T-norms

For two T-norms T_α and T_β, if a relation

$$T_\alpha (p,q) \leq T_\beta (p,q),\ (p,q) \in [0,1]^2$$

holds, we denote it by $T_\alpha \leq T_\beta$.

Theorem.2.1

For any T-norm T, an inequality $T_D \leq T \leq T_L$ holds.

Proof

If T-norm $T \geq T_L$ exists, then

$$T_L (p,q) \leq T\ (p,q)$$

holds.

Let $p>q$,

$$q \leq T(p,q)$$

From monotonicity and boundary conditions,

$$q \leq T(p,q) \leq T(1,q) = q$$

$$\therefore T(p,q) = q$$

Let $p \leqq q$,

$$p \leq T(p,q)$$

From monotonicity and boundary conditions,

$$p \leq T(p,q) \leq T(p,1) = p$$

$$\therefore T(p,q) = p$$

$$\therefore T(p,q) = p \wedge q = T_L(p,q) \quad \blacksquare$$

If T-norm $T \leq T_D$ exists, then

$$T\ (p,q) \leq T_D (p,q)$$

holds.

Let $p \vee q \neq 1$,

$$T(p,q) \leq 0$$

From monotonicity and boundary conditions,

$$T(p,0) \leq T(p,q) \leq 0$$
$$0 \leq T(p,q) \leq 0$$
$$\therefore T(p,q) = 0$$

Let $p \vee q = 1$,

from boundary conditions,

$$T(p,q) = p \wedge q .$$

$$\therefore T(p,q) = \begin{cases} 0 & , p \vee q \neq 1 \\ p \wedge q , & p \vee q = 1 \end{cases}$$
$$= T_D(p,q) \quad \blacksquare$$

Definition.2.3 T-norm Family

For any $\lambda \in [a,b]$, if T_λ is T-norm, then we say that $\{T_\lambda\}$ is T-norm family that connects T_a with T_b.

The typical T-norm families are the followings:

(1) Dombi product : $T_\lambda(p,q) = \dfrac{1}{1+\sqrt[\lambda]{((1-p)/p)^\lambda + ((1-q)/q)^\lambda}}, \lambda > 0$

(2) Weber product : $T_\lambda(p,q) = \max(0, (1+\lambda)(p+q-1) - \lambda pq), \lambda \geq -1$

(3) Yager product : $T_\lambda(p,q) = 1 - \min(1, \sqrt[\lambda]{(1-p)^\lambda + (1-q)^\lambda}), \lambda > 0$

(4) Hamacher product : $T_\lambda(p,q) = \dfrac{pq}{\lambda + (1-\lambda)(p+q-pq)}, \lambda \geq 0$

(5) Schweizer product (i) : $T_\lambda(p,q) = \sqrt[\lambda]{\max(0, p^\lambda + q^\lambda - 1)}, \lambda > 0$

(6) Schweizer product (ii) : $T_\lambda(p,q) = \dfrac{1}{\sqrt[\lambda]{1/p^\lambda + 1/q^\lambda - 1}}, \lambda > 0$

(7) Schweizer product (iii) : $T_\lambda(p,q) = 1 - \sqrt[\lambda]{(1-p)^\lambda + (1-q)^\lambda - (1-p)^\lambda(1-q)^\lambda}, \lambda > 0$

(8) Dubois product : $T_\lambda(p,q) = \dfrac{pq}{\max(p,q,\lambda)}, \lambda \in [0,1]$

(9) Frank product : $T_\lambda(p,q) = \log_\lambda\left[1 + \dfrac{(\lambda^p - 1)(\lambda^q - 1)}{\lambda - 1}\right], \lambda > 0, \lambda \neq 1$

Theorem 2.2

(1) Dombi product : $T_\lambda(p,q)=\dfrac{1}{1+\sqrt[\lambda]{((1-p)/p)^\lambda+((1-q)/q)^\lambda}}, \lambda>0$

$$\lambda\to\infty \Rightarrow T_\lambda(p,q)=T_L(p,q)$$
$$\lambda\to 0 \Rightarrow T_\lambda(p,q)=T_D(p,q)$$

Proof

Let denote $f(\lambda)=\sqrt[\lambda]{((1-p)/p)^\lambda+((1-q)/q)^\lambda}$,

$$\lim_{\lambda\to\infty}\log f(\lambda)=\lim_{\lambda\to\infty}\log\sqrt[\lambda]{((1-p)/p)^\lambda+((1-q)/q)^\lambda}$$
$$=\lim_{\lambda\to\infty}\frac{\log\left(\left(\frac{1-p}{p}\right)^\lambda+\left(\frac{1-q}{q}\right)^\lambda\right)}{\lambda}$$
$$=\lim_{\lambda\to\infty}\frac{\log\left(\left(\frac{1}{p}-1\right)^\lambda+\left(\frac{1}{q}-1\right)^\lambda\right)}{\lambda}$$

From l'Hospital's theorem,

$$\lim_{\lambda\to\infty}\log f(\lambda)=\lim_{\lambda\to\infty}\frac{\left(\frac{1}{p}-1\right)^\lambda\log\left(\frac{1}{p}-1\right)+\left(\frac{1}{q}-1\right)^\lambda\log\left(\frac{1}{q}-1\right)}{\left(\left(\frac{1}{p}-1\right)^\lambda+\left(\frac{1}{q}-1\right)^\lambda\right)}$$

Let $p>q$, $\frac{1}{p}-1<\frac{1}{q}-1$.

$$\lim_{\lambda\to\infty}\log f(\lambda)=\lim_{\lambda\to\infty}\frac{\frac{\left(\frac{1}{p}-1\right)^\lambda}{\left(\frac{1}{q}-1\right)^\lambda}\log\left(\frac{1}{p}-1\right)+\log\left(\frac{1}{q}-1\right)}{\left(\frac{\left(\frac{1}{p}-1\right)^\lambda}{\left(\frac{1}{q}-1\right)^\lambda}+1\right)}$$
$$=\log\left(\frac{1}{q}-1\right)$$

Let $p\le q$, as in the previous proof,

$$\therefore \lim_{\lambda\to\infty} f(\lambda) = \left(\frac{1}{p}-1\right) \vee \left(\frac{1}{q}-1\right).$$

$$\lim_{\lambda\to\infty} T_\lambda(p,q) = \frac{1}{1+\left(\frac{1}{p}-1\right)\vee\left(\frac{1}{q}-1\right)}$$
$$= p \wedge q$$
$$= T_L(p,q) \quad \blacksquare$$

Let denote $f(\lambda) = \sqrt[\lambda]{((1-p)/p)^\lambda + ((1-q)/q)^\lambda}$, $p \neq 0,1,\ q \neq 0,1,$

$$\lim_{\lambda\to 0} f(\lambda) = \lim_{\lambda\to 0} \sqrt[\lambda]{((1-p)/p)^\lambda + ((1-q)/q)^\lambda}$$
$$= \infty$$

$$\therefore \lim_{\lambda\to 0} T_\lambda(p,q) = 0 \quad (p \neq 0,1,\ q \neq 0,1)$$

$$\therefore \lim_{\lambda\to 0} T_\lambda(p,q) = T_D(p,q) \quad \blacksquare$$

(2) Weber product : $T_\lambda(p,q) = \max(0,(1+\lambda)(p+q-1)-\lambda pq), \lambda \geq -1$

$$\lambda \to \infty \Rightarrow T_\lambda(p,q) = T_D(p,q)$$

Proof

$$T_\lambda(p,q) = \max(0,(1+\lambda)(p+q-1)-\lambda pq)$$
$$= \max(0,(p+q-1)-\lambda(1-p)(1-q))$$

Let $p \vee q \neq 1$, $\lim_{\lambda\to\infty} T_\lambda(p,q) = 0$.

Let $p \vee q = 1$, $\lim_{\lambda\to\infty} T_\lambda(p,q) = p \wedge q$.

$$\therefore \lim_{\lambda\to\infty} T_\lambda(p,q) = T_D(p,q) \quad \blacksquare$$

(3) Yager product : $T_\lambda(p,q) = 1-\min(1, \sqrt[\lambda]{(1-p)^\lambda + (1-q)^\lambda}), \lambda > 0$

$$\lambda \to 0 \Rightarrow T_\lambda(p,q) = T_D(p,q)$$
$$\lambda \to \infty \Rightarrow T_\lambda(p,q) = T_L(p,q)$$

Proof

Let denote $f(\lambda)=\sqrt[\lambda]{(1-p)^{\lambda}+(1-q)^{\lambda}},\ p\neq 0,1,\ q\neq 0,1,$

$$\lim_{\lambda\to 0} f(\lambda)=\lim_{\lambda\to 0}\sqrt[\lambda]{(1-p)^{\lambda}+(1-q)^{\lambda}}$$
$$=\infty$$

$$\lim_{\lambda\to 0} T_{\lambda}(p,q)=0$$

$$\therefore \lim_{\lambda\to 0} T_{\lambda}(p,q)=T_{D}(p,q) \quad \blacksquare$$

Let denote $f(\lambda)=\sqrt[\lambda]{(1-p)^{\lambda}+(1-q)^{\lambda}},\ p\neq 0,1,\ q\neq 0,1,$

$$\lim_{\lambda\to\infty}\log f(\lambda)=\lim_{\lambda\to\infty}\log\sqrt[\lambda]{(1-p)^{\lambda}+(1-q)^{\lambda}}$$
$$=\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\left((1-p)^{\lambda}+(1-q)^{\lambda}\right)$$

From l'Hospital's theorem,

$$=\lim_{\lambda\to\infty}\frac{(1-p)^{\lambda}\log(1-p)+(1-q)^{\lambda}\log(1-q)}{\left((1-p)^{\lambda}+(1-q)^{\lambda}\right)}$$

Let $p>q$, then $1-p<1-q$.

$$=\lim_{\lambda\to\infty}\frac{\frac{(1-p)^{\lambda}}{(1-q)^{\lambda}}\log(1-p)+\log(1-q)}{\left(\frac{(1-p)^{\lambda}}{(1-q)^{\lambda}}+1\right)}$$

$$=\log(1-q)$$

$$\therefore \lim_{\lambda\to\infty} f(\lambda)=1-q$$

$$\therefore \lim_{\lambda\to\infty} T_{\lambda}(p,q)=q$$

Let $p\leq q$, as in the previous proof,

$$\therefore \lim_{\lambda\to\infty} T_{\lambda}(p,q)=T_{L}(p,q) \quad \blacksquare$$

(4) Hamacher product : $T_{\lambda}(p,q)=\dfrac{pq}{\lambda+(1-\lambda)(p+q-pq)},\lambda\geq 0$

$$\lambda\to\infty\Rightarrow T_{\lambda}(p,q)=T_{D}(p,q)$$

Proof

$$T_\lambda(p,q) = \frac{pq}{\lambda + (1-\lambda)(p+q-pq)}$$
$$= \frac{pq}{(p+q-pq) + \lambda(1-p)(1-q)}$$

Let $p \neq 0,1,\ q \neq 0,1$,

$$\lim_{\lambda\to\infty} T_\lambda(p,q) = \lim_{\lambda\to\infty} \frac{pq}{(p+q-pq) + \lambda(1-p)(1-q)}$$
$$= 0$$

$$\therefore \lim_{\lambda\to\infty} T_\lambda(p,q) = T_D(p,q) \quad \blacksquare$$

(5) Schweizer product (i) : $T_\lambda(p,q) = \sqrt[\lambda]{\max(0,\ p^\lambda + q^\lambda - 1)}, \lambda > 0$

$$\lambda \to 0 \Rightarrow T_\lambda(p,q) = T_A(p,q)$$
$$\lambda \to \infty \Rightarrow T_\lambda(p,q) = T_D(p,q)$$

Proof

Let denote $f(\lambda) = \sqrt[\lambda]{p^\lambda + q^\lambda - 1},\ p \neq 0,1,\ q \neq 0,1$

$$\lim_{\lambda\to 0} \log f(\lambda) = \lim_{\lambda\to 0} \log \sqrt[\lambda]{p^\lambda + q^\lambda - 1}$$
$$= \lim_{\lambda\to 0} \frac{1}{\lambda} \log\left(p^\lambda + q^\lambda - 1\right)$$

From l'Hospital's theorem,

$$= \lim_{\lambda\to 0} \frac{p^\lambda \log p + q^\lambda \log q}{p^\lambda + q^\lambda - 1}$$
$$= \log p + \log q$$
$$= \log pq$$

$$\therefore \lim_{\lambda\to\infty} T_\lambda(p,q) = T_A(p,q) \quad \blacksquare$$

Let denote $p \neq 0,1,\ q \neq 0,1$

$$\lim_{\lambda\to\infty} T_\lambda(p,q) = \lim_{\lambda\to\infty} \sqrt[\lambda]{\max(0,\ p^\lambda + q^\lambda - 1)}$$
$$= 0$$

$$\therefore \lim_{\lambda\to\infty} T_\lambda(p,q) = T_D(p,q) \quad \blacksquare$$

(6) Schweizer product (ii) : $T_\lambda(p,q) = \dfrac{1}{\sqrt[\lambda]{1/p^\lambda + 1/q^\lambda - 1}}, \lambda > 0$

$$\lambda \to 0 \Rightarrow T_\lambda(p,q) = T_A(p,q)$$
$$\lambda \to \infty \Rightarrow T_\lambda(p,q) = T_D(p,q)$$

Proof

Let denote $f(\lambda) = \sqrt[\lambda]{p^{-\lambda} + q^{-\lambda} - 1},\ p \neq 0,1,\ q \neq 0,1$

$$\lim_{\lambda\to 0} \log f(\lambda) = \lim_{\lambda\to 0} \log \sqrt[\lambda]{p^{-\lambda} + q^{-\lambda} - 1}$$
$$= \lim_{\lambda\to 0} \frac{1}{\lambda} \log\left(p^{-\lambda} + q^{-\lambda} - 1\right)$$

From l'Hospital's theorem,

$$= \lim_{\lambda\to 0} - \frac{p^{-\lambda}\log p + q^{-\lambda}\log q}{p^\lambda + q^\lambda - 1}$$
$$= -(\log p + \log q)$$
$$= -\log pq$$
$$= \log\frac{1}{pq}$$

$$\therefore \lim_{\lambda\to\infty} T_\lambda(p,q) = T_A(p,q) \quad \blacksquare$$

Let denote $f(\lambda) = \sqrt[\lambda]{p^{-\lambda} + q^{-\lambda} - 1},\ p \neq 0,1,\ q \neq 0,1$

$$\lim_{\lambda\to\infty} \log f(\lambda) = \lim_{\lambda\to\infty} \log \sqrt[\lambda]{p^{-\lambda} + q^{-\lambda} - 1}$$
$$= \lim_{\lambda\to\infty} \frac{1}{\lambda} \log\left(p^{-\lambda} + q^{-\lambda} - 1\right)$$

From l'Hospital's theorem,

$$= \lim_{\lambda\to\infty} - \frac{p^{-\lambda}\log p + q^{-\lambda}\log q}{p^\lambda + q^\lambda - 1}$$
$$= \infty$$

$$\therefore \lim_{\lambda\to\infty} T_\lambda(p,q) = T_D(p,q) \quad \blacksquare$$

(7) Schweizer product (iii) : $T_\lambda(p,q) = 1 - \sqrt[\lambda]{(1-p)^\lambda + (1-q)^\lambda - (1-p)^\lambda(1-q)^\lambda}, \lambda > 0$

$$\lambda \to 0 \Rightarrow T_\lambda(p,q) = T_D(p,q)$$
$$\lambda \to \infty \Rightarrow T_\lambda(p,q) = T_L(p,q)$$

Proof

Let denote $f(\lambda) = \sqrt[\lambda]{(1-p)^\lambda + (1-q)^\lambda - (1-p)^\lambda(1-q)^\lambda}$, $p \neq 0,1,\ q \neq 0,1$

$$\lim_{\lambda\to 0} \log f(\lambda) = \lim_{\lambda\to 0} \log \sqrt[\lambda]{(1-p)^\lambda + (1-q)^\lambda - (1-p)^\lambda(1-q)^\lambda}$$
$$= \lim_{\lambda\to 0} \frac{1}{\lambda} \log\left((1-p)^\lambda + (1-q)^\lambda - (1-p)^\lambda(1-q)^\lambda\right)$$

From l'Hospital's theorem,

$$= \lim_{\lambda\to 0} \frac{(1-p)^\lambda \log(1-p) + (1-q)^\lambda \log(1-q) - (1-p)^\lambda(1-q)^\lambda \log(1-p) - (1-p)^\lambda(1-q)^\lambda \log(1-q)}{(1-p)^\lambda + (1-q)^\lambda - (1-p)^\lambda(1-q)^\lambda}$$
$$= 0$$

$$\therefore \lim_{\lambda\to 0} T_\lambda(p,q) = T_D(p,q) \quad \blacksquare$$

Let denote $f(\lambda) = \sqrt[\lambda]{(1-p)^\lambda + (1-q)^\lambda - (1-p)^\lambda(1-q)^\lambda}$, $p \neq 0,1,\ q \neq 0,1$

$$\lim_{\lambda\to \infty} \log f(\lambda) = \lim_{\lambda\to \infty} \log \sqrt[\lambda]{(1-p)^\lambda + (1-q)^\lambda - (1-p)^\lambda(1-q)^\lambda}$$
$$= \lim_{\lambda\to \infty} \frac{1}{\lambda} \log\left((1-p)^\lambda + (1-q)^\lambda - (1-p)^\lambda(1-q)^\lambda\right)$$

From l'Hospital's theorem,

$$= \lim_{\lambda\to \infty} \frac{(1-p)^\lambda \log(1-p) + (1-q)^\lambda \log(1-q) - (1-p)^\lambda(1-q)^\lambda \log(1-p) - (1-p)^\lambda(1-q)^\lambda \log(1-q)}{(1-p)^\lambda + (1-q)^\lambda - (1-p)^\lambda(1-q)^\lambda}$$

Let $p>q$, then $(1-p)<(1-q)$,

$$= \lim_{\lambda\to \infty} \frac{\frac{(1-p)^\lambda}{(1-q)^\lambda} \log(1-p) + \log(1-q) - (1-p)^\lambda \log(1-p) - (1-p)^\lambda \log(1-q)}{\frac{(1-p)^\lambda}{(1-q)^\lambda} + 1 - (1-p)^\lambda}$$
$$= \log(1-q)$$

$$\lim_{\lambda\to \infty} T_\lambda(p,q) = 1-(1-q)$$
$$= q$$

Let $p \leqq q$, as in the previous proof,

$$\lim_{\lambda\to \infty} T_\lambda(p,q) = p$$

$$\therefore \lim_{\lambda\to\infty} T_\lambda(p,q) = T_L(p,q) \quad \blacksquare$$

(9) Frank product : $T_\lambda(p,q) = \log_\lambda\left[1+\frac{(\lambda^p-1)(\lambda^q-1)}{\lambda-1}\right], \lambda>0, \lambda\neq 1$

$$\lambda\to\infty \Rightarrow T_\lambda(p,q) = T_M(p,q)$$

Proof

Let $p\neq 0{,}1,\ q\neq 0{,}1,$

$$\lim_{\lambda\to\infty} T_\lambda(p,q) = \lim_{\lambda\to\infty}\frac{\log\left[1+\frac{(\lambda^p-1)(\lambda^q-1)}{\lambda-1}\right]}{\log\lambda}$$

$$= \lim_{\lambda\to\infty}\frac{\log\left[\frac{\lambda-1+(\lambda^p-1)(\lambda^q-1)}{\lambda-1}\right]}{\log\lambda}$$

$$= \lim_{\lambda\to\infty}\frac{\log\left(\lambda-1+(\lambda^p-1)(\lambda^q-1)\right)-\log(\lambda-1)}{\log\lambda}$$

From l’Hospital’s theorem,

$$= \lim_{\lambda\to\infty}\lambda\left[\frac{1+(p+q)\lambda^{p+q-1}-p\lambda^{p-1}-q\lambda^{q-1}}{\lambda-1+(\lambda^p-1)(\lambda^q-1)}-\frac{1}{\lambda-1}\right]$$

$$= \lim_{\lambda\to\infty}\left[\frac{\lambda+(p+q)\lambda^{p+q}-p\lambda^p-q\lambda^q}{\lambda-1+(\lambda^p-1)(\lambda^q-1)}\right]-1$$

Let $p+q>1$,

$$= \lim_{\lambda\to\infty}\left[\frac{\lambda^{-(p+q-1)}+(p+q)-p\lambda^{-p}-q\lambda^{-q}}{\lambda^{-(p+q-1)}-\lambda^{-(p+q)}+(1-\lambda^{-p})(1-\lambda^{-q})}\right]-1$$

$$= p+q-1$$

Let $p+q\leqq 1$,

$$= \lim_{\lambda\to\infty}\left[\frac{1+(p+q)\lambda^{p+q-1}-p\lambda^{p-1}-q\lambda^{q-1}}{1+\lambda^{p+q-1}-\lambda^{p-1}-\lambda^{q-1}}\right]-1$$

$$= 0$$

$$\therefore \lim_{\lambda\to 0} T_\lambda(p,q) = T_M(p,q) \quad \blacksquare$$

T-norm family connects one typical T-norm to another. The relation between λ and T_λ is appeared in Table.2.1.

T-Norm Families \ T-Norms	Logical	Algebraic	Multi-valued	Drastic
(1) Dombi	Infinity			$\to 0$
(2) Weber		-1	0	Infinity
(3) Yager	Infinity		1	$\to 0$
(4) Hamacher		1		Infinity
(5) Schweizer (i)	$\to 0$		1	Infinity
(6) Schweizer (ii)	Infinity	$\to 0$		
(7) Schweizer (iii)	Infinity	1		$\to 0$
(8) Dubois	0	1		
(9) Frank			Infinity	

Table.2.1 Relation between λ and T_λ

The continuity and the monotonicity of T-norm family are defined as the followings.

Definition.2.5 Continuity of T-norm

T-norm family $\{T_\lambda\}$ is continuous, if for any $\varepsilon > 0$, $\lambda \in [0,1]$, $(p, q) \in [0,1]^2$, a positive number δ exists, a proposition

$$\forall \mu \in [0,1] : |\lambda - \mu| < \delta \Rightarrow |T_\lambda(p,q) - T_\mu(p,q)| < \varepsilon$$

holds.

Definition.2.6 Monotonicity of T-norm

T-norm family $\{T_\lambda\}$ is monotonous, if for any $\lambda \in [0,1]$, $(p, q) \in [0,1]^2$, $T_\lambda(p, q)$ is monotonous.

2.2 The New T-norm Family "Quasi-Logical Product"

Definition.2.7 Quasi-Logical Product (Uesu Product)

Quasi-logical product (Uesu product) is defined as the following:

$$T_\lambda(p,q)=\begin{cases}0 & ,p\vee q<1-\lambda\\ p\wedge q & ,p\vee q\geq 1-\lambda\end{cases},\lambda\in[0,1]$$

Theorem 2.3

Quasi-logical product T_λ is T-norm family.

Proof

(1) Commutativity : $T(p,q)=T(q,p)$

$$T_\lambda(p,q)=\begin{cases}0 & ,p\vee q<1-\lambda\\ p\wedge q & ,p\vee q\geq 1-\lambda\end{cases}$$

$$=\begin{cases}0 & ,q\vee p<1-\lambda\\ q\wedge p & ,q\vee p\geq 1-\lambda\end{cases}$$

$$=T_\lambda(q,p)$$

(2) Associativity : $T(p,T(q,r))=T(T(p,q),r)$

$$T_\lambda(q,r)=\begin{cases}0 & ,q\vee r<1-\lambda\\ q\wedge r & ,q\vee r\geq 1-\lambda\end{cases}$$

If $q\vee r\geq 1-\lambda$, then

$$T_\lambda(p,T_\lambda(q,r))=p\wedge q\wedge r,$$

if $q\vee r<1-\lambda$, then

$$T_\lambda(p,T_\lambda(q,r))=\begin{cases}0 & ,p<1-\lambda\\ p\wedge q\wedge r & ,p\geq 1-\lambda\end{cases}$$

Then,

$$T_\lambda(p,T_\lambda(q,r))=\begin{cases}0 & ,p\vee q\vee r<1-\lambda\\ p\wedge q\wedge r & ,p\vee q\vee r\geq 1-\lambda\end{cases}$$

$$T_\lambda(p,q)=\begin{cases}0 & ,p\vee q<1-\lambda\\ p\wedge q & ,p\vee q\geq 1-\lambda\end{cases}$$

If $p\vee q\geq 1-\lambda$, then

$$T_\lambda(T_\lambda(p,q),r)=p\wedge q\wedge r,$$

if $p\vee q<1-\lambda$, then

$$T_\lambda(T_\lambda(p,q),r) = \begin{cases} 0 & , r < 1-\lambda \\ p \wedge q \wedge r & , r \geq 1-\lambda \end{cases}$$

Then,

$$T_\lambda(T_\lambda(p,q),r) = \begin{cases} 0 & , p \vee q \vee r < 1-\lambda \\ p \wedge q \wedge r & , p \vee q \vee r \geq 1-\lambda \end{cases}$$

(3) Monotonicity : $p \leq q, \quad r \leq s \Rightarrow T(p,r) \leq T(q,s)$

From $T_\lambda(p,r) = \begin{cases} 0 & , p \vee q < 1-\lambda \\ p \wedge q, & p \vee q \geq 1-\lambda \end{cases}$ and $T_\lambda(q,s) = \begin{cases} 0 & , q \vee s < 1-\lambda \\ q \wedge s, & q \vee s \geq 1-\lambda \end{cases}$, it is clearly.

(4) Boundary conditions : $T(p,0) = 0, \quad T(p,1) = p$

$$T_\lambda(p,0) = \begin{cases} 0 & , p \vee 0 < 1-\lambda \\ p \wedge 0, & p \vee 0 \geq 1-\lambda \end{cases}$$
$$= 0$$

$$T_\lambda(p,1) = \begin{cases} 0 & , p \vee 1 < 1-\lambda \\ p \wedge 1, & p \vee 1 \geq 1-\lambda \end{cases}$$
$$= p$$

From (1) ,(2), (3) and (4), quasi-logical product T_λ is T-norm family.■

The relation of typical T-norm families and quasi-logical product could be illustrated in Figure.2.2.

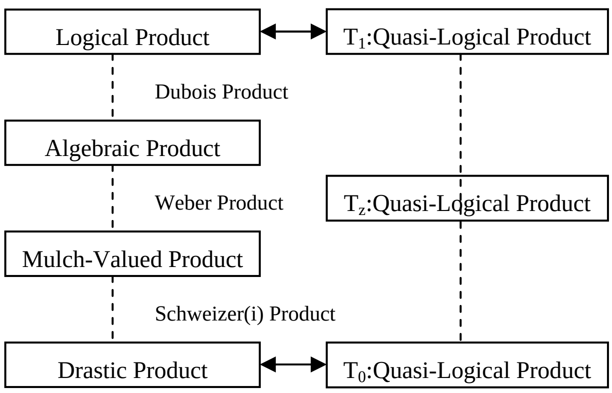
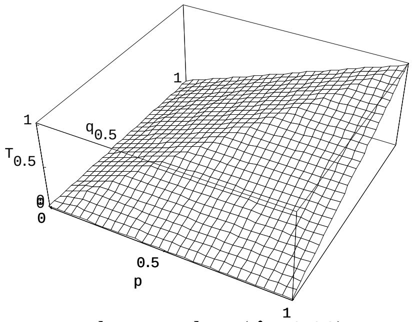
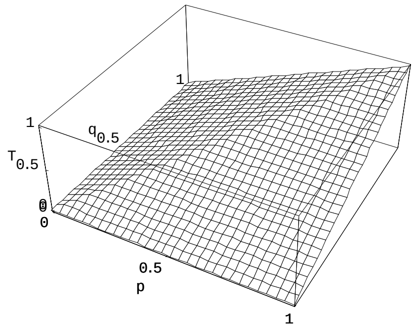


Figure.2.2 Relation of T-norm Families

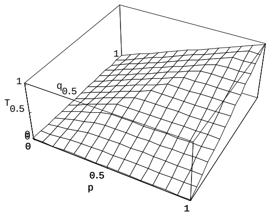
This relation of 3-dimension graphs is illustrated in Figure.2.3.



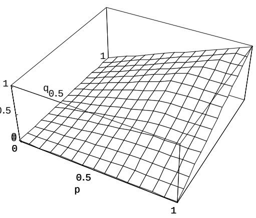
Dubois Product (λ =0.00)



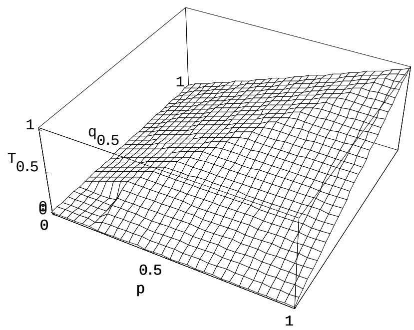
Quasi-Logical Product (λ =1.00)



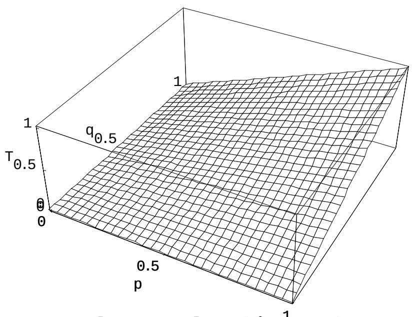
Dubois Product (λ =0.33)



Dubois Product (λ =0.67)

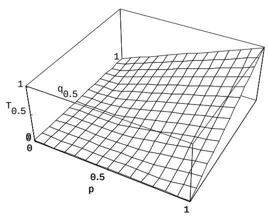


Quasi-Logical Product (λ =0.80)

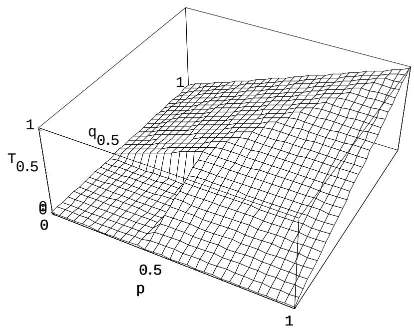


Dubois Product (λ =1.00)

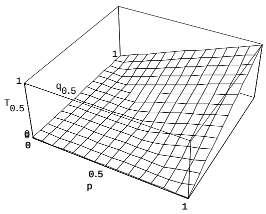
Weber Product (λ =-1.00)



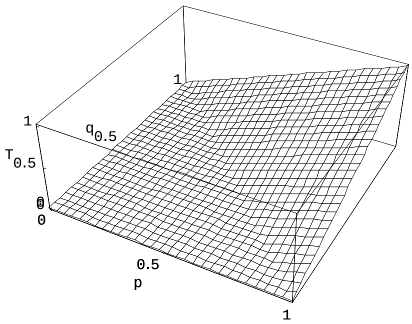
Weber Product (λ =-0.67)



Quasi-Logical Product (λ =0.60)

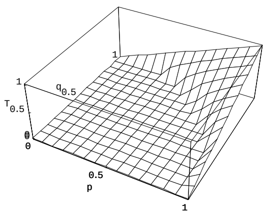


Weber Product (λ =-0.33)

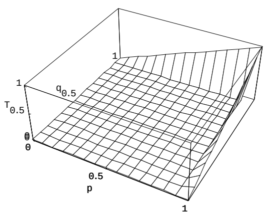


Weber Product (λ =0.00)

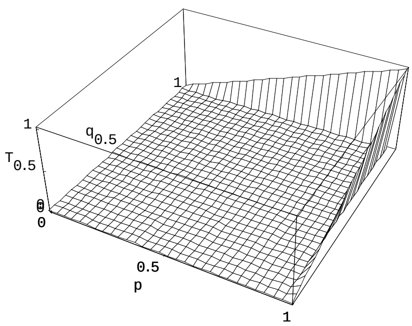
Shweizer Product(i) (λ =1.00)



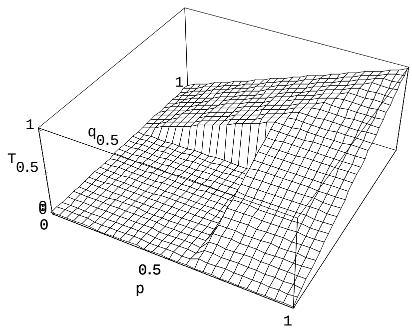
Shweizer Product(i) (λ =2.00)



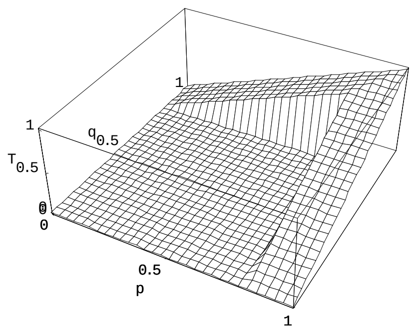
Shweizer Product(i) (λ =10)



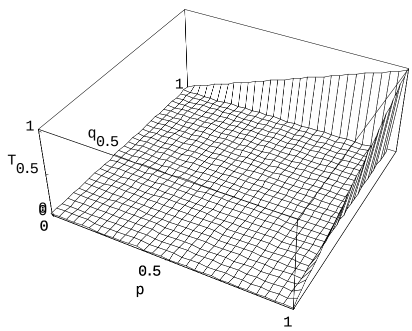
Shweizer Product(i) (λ →∞)



Quasi-Logical Product (λ =0.40)



Quasi-Logical Product (λ =0.20)



Quasi-Logical Product (λ =0.00)

Figure.2.3 The Relation of T-norm Family and Quasi-Logical Product