3. Fuzzy Node Fuzzy Graph and its Analysis

The fuzzy node fuzzy graph is usually too complicated to analyze its structural feature. Therefore, we would propose a transformation method from the fuzzy node fuzzy graph to the crisp node fuzzy graph by applying T-norms. Secondly, we would propose a cluster analysis method of the fuzzy node fuzzy graph applying the fuzzy decision. Thirdly, we would propose the global structure analysis of the fuzzy node fuzzy graph and discuss the optimal approximate fuzzy graph.

3.1 Transformation from the Fuzzy Node Fuzzy Graph to the Crisp Node Fuzzy Graph

Definition 3.1.1 Transformation from the Fuzzy Node Fuzzy Graph to the Crisp Node Fuzzy Graph

The fuzzy node fuzzy graph G=(V, Y) can be transformed to the crisp node fuzzy graph G=(V,F) by the following method:

Let the fuzzy node fuzzy graph

$$G=(V,Y) : V=\{v_i(u_i)\},\ Y=(y_{ij}),$$

the crisp node fuzzy graph

$$G=(V,F) : V=\{v_i\},\ F=(f_{ij})$$

could be defined by applying T-norms.

$$f_{ij}=T(u_i,\ y_{ij}).$$

For example, let the fuzzy node fuzzy graph G=(V,Y);

$$G=(V,Y) : V=\{v_1(0.42),\ v_2(0.65),\ v_3(0.54),\ v_4(0.87)\},\quad Y=\begin{pmatrix} 1 & 0.84 & 0.12 & 0.84 \\ 0.39 & 1 & 0.65 & 0.71 \\ 0 & 0.43 & 1 & 0.32 \\ 1.00 & 0.93 & 0.56 & 1 \end{pmatrix}$$

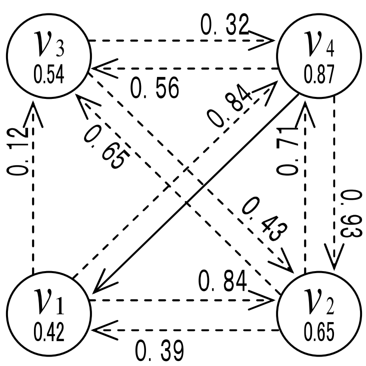

Figure 3.1.1 Example of the Fuzzy Node Fuzzy Graph G=(V,Y)

The fuzzy node fuzzy graph G=(V,Y) could be transformed to the crisp node fuzzy graph G=(V,F) by using the logical product $T_L(p,q) = p \wedge q$ in Figure.3.2.

$$G=(V,F) : V=\{v_1, v_2, v_3, v_4\}, \quad F = \begin{pmatrix} 0.42 & 0.42 & 0.12 & 0.42 \\ 0.39 & 0.65 & 0.65 & 0.65 \\ 0 & 0.43 & 0.54 & 0.32 \\ 0.87 & 0.87 & 0.56 & 0.87 \end{pmatrix}$$

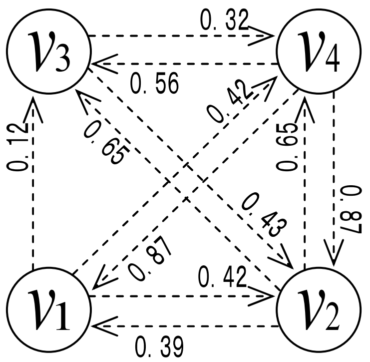

Figure 3.1.2　Example of the Crisp Node Fuzzy Graph G=(V,F)

Here, we would obtain many crisp node fuzzy graphs $G_\lambda = (V, F_\lambda)$ from the fuzzy node fuzzy graph G by using T-norm family T_λ. The series $\{G_\lambda\}$ is called the fuzzy graph series $\{G_\lambda\}$.

Theorem 3.1.1

The fuzzy graph series $\{G_\lambda\}$ is the fuzzy partial graph series by using the quasi-logical product.

Proof.

Let the fuzzy node fuzzy graph;

$$G=(V,Y) : V=\{v_i(u_i)\}, Y=(y_{ij})$$

If $\lambda_1 < \lambda_2$, we obtained two crisp node fuzzy graphs;

$$G_{\lambda_1} = (V, F) : V=\{v_i\}, F=(f_{ij}), \quad f_{ij} = \begin{cases} 0 & , u_i \vee y_{ij} < 1-\lambda_1 \\ u_i \wedge y_{ij} & , u_i \vee y_{ij} \geq 1-\lambda_1 \end{cases}$$

$$G_{\lambda_2} = (V, F') : V=\{v_i\}, F'=(f_{ij}'), \quad f_{ij}' = \begin{cases} 0 & , u_i \vee y_{ij} < 1-\lambda_2 \\ u_i \wedge y_{ij} & , u_i \vee y_{ij} \geq 1-\lambda_2 \end{cases}$$

Since the quasi-logical product is monotonous, then $f_{ij} \leq f_{ij}'$.

$\therefore G_{\lambda_1} \pi G_{\lambda_2}$ ■

For example, the fuzzy node fuzzy graph G and the fuzzy graph series $\{G_\lambda\}$ are illustrated in Figure.3.1.3-3.1.10.

$$G=(V,Y) : V=\{v_1(0.65),\ v_2(0.24),\ v_3(0.89),\ v_4(0.96)\},\quad Y=\begin{pmatrix} 1 & 0.65 & 0.50 & 0.05 \\ 0.85 & 1 & 0.44 & 0.50 \\ 0.96 & 0.65 & 1 & 1.00 \\ 0 & 0.05 & 0.89 & 1 \end{pmatrix}$$

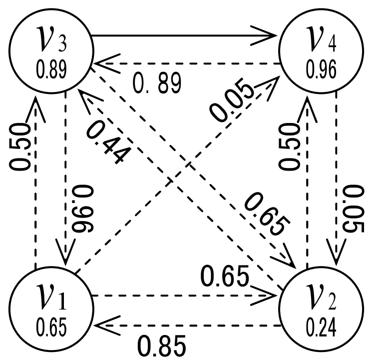

Figure 3.1.3 Fuzzy Node Fuzzy Graph G=(V,Y)

$$G_\lambda=(V,F_\lambda) : V=\{v_1,\ v_2,\ v_3,\ v_4\},\quad F_\lambda=\begin{pmatrix} 0.65 & 0.65 & 0.50 & 0.05 \\ 0.24 & 0.24 & 0.24 & 0.24 \\ 0.89 & 0.65 & 0.89 & 0.89 \\ 0 & 0.05 & 0.89 & 0.96 \end{pmatrix}$$

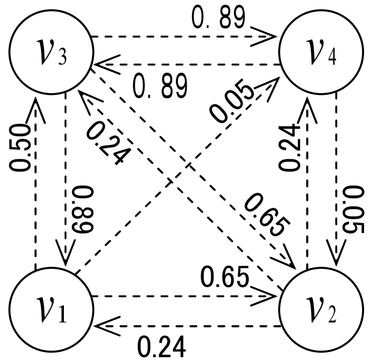

Figure 3.1.4 Crisp Node Fuzzy Graph G_λ=(V,F_λ), $\lambda \in$[0.56, 1.00]

$G_\lambda=(V,F_\lambda)$: $V=\{v_1, v_2, v_3, v_4\}$, $F_\lambda = \begin{pmatrix} 0.65 & 0.65 & 0.50 & 0.05 \\ 0.24 & 0.24 & 0 & 0.24 \\ 0.89 & 0.65 & 0.89 & 0.89 \\ 0 & 0.05 & 0.89 & 0.96 \end{pmatrix}$

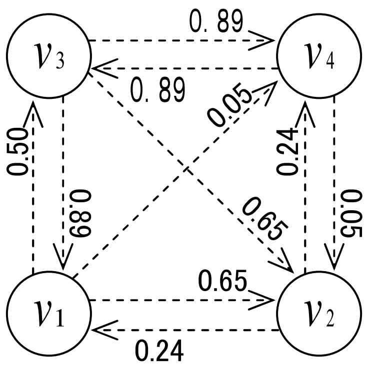

Figure 3.1.5 Crisp Node Fuzzy Graph $G_\lambda=(V,F_\lambda)$, $\lambda \in [0.50, 0.56)$

$G_\lambda=(V,F_\lambda)$: $V=\{v_1, v_2, v_3, v_4\}$, $F_\lambda = \begin{pmatrix} 0.65 & 0.65 & 0.50 & 0.05 \\ 0.24 & 0.24 & 0 & 0 \\ 0.89 & 0.65 & 0.89 & 0.89 \\ 0 & 0.05 & 0.89 & 0.96 \end{pmatrix}$

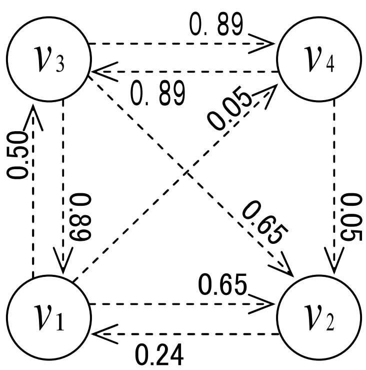

Figure 3.1.6 Crisp Node Fuzzy Graph $G_\lambda=(V,F_\lambda)$, $\lambda \in [0.35, 0.50)$

$$G_\lambda=(V,F_\lambda): V=\{v_1, v_2, v_3, v_4\},\quad F_\lambda=\begin{pmatrix}0.65 & 0 & 0 & 0\\ 0.24 & 0.24 & 0 & 0\\ 0.89 & 0.65 & 0.89 & 0.89\\ 0 & 0.05 & 0.89 & 0.96\end{pmatrix}$$

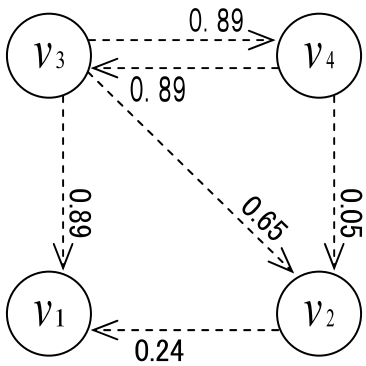

Figure 3.1.7 Crisp Node Fuzzy Graph $G_\lambda=(V,F_\lambda)$, $\lambda\in[0.15, 0.35)$

$$G_\lambda=(V,F_\lambda): V=\{v_1, v_2, v_3, v_4\},\quad F_\lambda=\begin{pmatrix}0.65 & 0 & 0 & 0\\ 0 & 0.24 & 0 & 0\\ 0.89 & 0.65 & 0.89 & 0.89\\ 0 & 0.05 & 0.89 & 0.96\end{pmatrix}$$

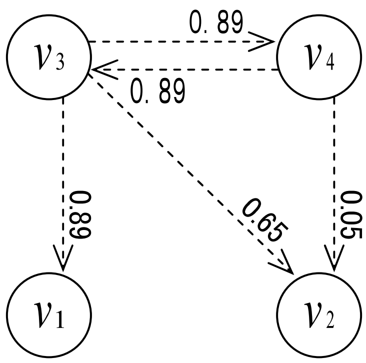

Figure 3.1.8 Crisp Node Fuzzy Graph $G_\lambda=(V,F_\lambda)$, $\lambda\in[0.11, 0.15)$

$$G_\lambda=(V,F_\lambda): V=\{v_1, v_2, v_3, v_4\}, \quad F_\lambda = \begin{pmatrix} 0.65 & 0 & 0 & 0 \\ 0 & 0.24 & 0 & 0 \\ 0.89 & 0 & 0.89 & 0.89 \\ 0 & 0.05 & 0.89 & 0.96 \end{pmatrix}$$

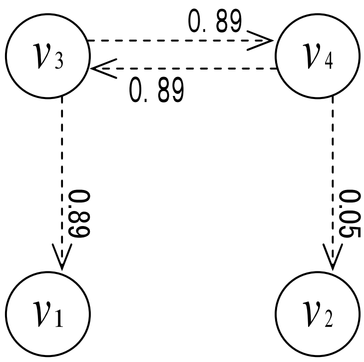

Figure 3.1.9 Crisp Node Fuzzy Graph $G_\lambda=(V,F_\lambda)$, $\lambda \in [0.04, 0.11)$

$$G_\lambda=(V,F_\lambda): V=\{v_1, v_2, v_3, v_4\}, \quad F_\lambda = \begin{pmatrix} 0.65 & 0 & 0 & 0 \\ 0 & 0.24 & 0 & 0 \\ 0 & 0 & 0.89 & 0.89 \\ 0 & 0 & 0 & 0.96 \end{pmatrix}$$

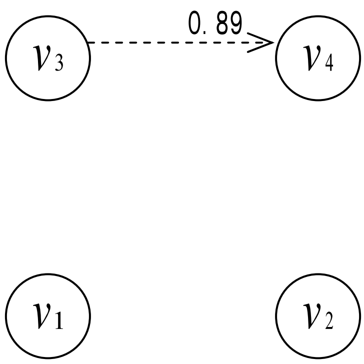

Figure 3.1.10 Crisp Node Fuzzy Graph $G_\lambda=(V,F_\lambda)$, $\lambda \in [0.00, 0.04)$

This fuzzy node fuzzy graph analysis would be applied to the sociometry analysis, the instruction structure analysis and so on.

3.2 Clustering Structure Analysis of Fuzzy Node Fuzzy Graph

In order to analyze the similarity structure of nodes for a fuzzy graph $G=(V, F)$, we use the symmetric relation matrix $S=(s_{ij})$.

A symmetric relation matrix S is defined by

$$S=(s_{ij}),\quad \frac{2}{s_{ij}} = \frac{1}{f_{ij}} + \frac{1}{f_{ji}}$$

where $s_{ij}=0$ if $f_{ij} \cdot f_{ji}=0$.

In order to analyze the clustering structure among nodes, we have its transitive closure $\hat{S} = (\hat{s}_{ij})$ which is computed by $\hat{S} = S^n,\ n = \#(V)$.

After that, we define the c-cut matrix S_c of $\hat{S} = (\hat{s}_{ij})$ as follows:

$$S_c = (s_{ij}^c),\quad s_{ij}^c = \begin{cases} 1 & (\hat{s}_{ij} \geq c) \\ 0 & (\hat{s}_{ij} < c) \end{cases},\ 0 \leq c \leq 1$$

From the matrix S_c, we define the cluster $CL_c(i)$.

$$CL_c(i) = \{x_j \mid s_{ij}^c = 1\}$$

The cluster $CL_c(i)$ gives an equivalence relation among nodes.

From the level c of the cluster $CL_c(i)$, we define the clustering situation R_c.

$$R_c = \{CL_c(i) \mid 1 \leq i \leq n\}$$

Thence, we can construct the partition tree by changing the level c of the c-cut matrix which represents the clustering situation of nodes in a fuzzy graph.

For example, if a fuzzy matrix F is in Fig.4, then we have obtain the symmetric matrix S in Fig.5 and the transitive closure $\hat{S} = (\hat{s}_{ij})$ is in Fig.6, by changing the value of c, we have obtained the c-cut matrix S_c in Fig.6.1-6.5.

From the c-cut matrix S_c, we have constructed the partition tree in Fig.7. For example, the partition tree shows that cluster 2 and 4 were merged at the value of c=0.83, we say that 2 and 4 are merged at cluster level $R_{0.83}$.

	1	2	3	4	5	6	7	8
1	1		0.67	0.33	0.50	0.17	0.83	1.00
2		1	1.00	0.83	0.67		0.33	0.50
3	1.00	0.33	1	0.17	0.50		0.67	0.83
4		0.83	1.00	1	0.33	0.17	0.67	0.50
5	0.83	0.50	0.67		1	1.00	0.33	0.17
6					1.00	1		
7	0.83	0.33	0.67	0.50	0.17		1	1.00
8	0.67	0.17	1.00	0.50	0.33		0.83	1

Figure 3.2.1　Fuzzy Matrix *F*

	1	2	3	4	5	6	7	8
1	1		0.80		0.62		0.83	0.80
2		1	0.50	0.83	0.57		0.33	0.25
3	0.80	0.50	1	0.29	0.57		0.67	0.91
4		0.83	0.29	1			0.57	0.50
5	0.62	0.57	0.57		1	1.00	0.22	0.22
6					1.00	1		
7	0.83	0.33	0.67	0.57	0.22		1	0.91
8	0.80	0.25	0.91	0.50	0.22		0.91	1

Figure 3.2.2　Synmetric Matrix *S*

	1	2	3	4	5	6	7	8
1	1	0.57	0.83	0.57	0.62	0.62	0.83	0.83
2	0.57	1	0.57	0.83	0.57	0.57	0.57	0.57
3	0.83	0.57	1	0.57	0.62	0.62	0.91	0.91
4	0.57	0.83	0.57	1	0.57	0.57	0.57	0.57
5	0.62	0.57	0.62	0.57	1	1.00	0.62	0.62
6	0.62	0.57	0.62	0.57	1.00	1	0.62	0.62
7	0.83	0.57	0.91	0.57	0.62	0.62	1	0.91
8	0.83	0.57	0.91	0.57	0.62	0.62	0.91	1

Figure 3.2.3　Transitive Closure $\hat{S} = (\hat{s}_{ij})$

	1	2	3	4	5	6	7	8
1	1							
2		1						
3			1					
4				1				
5					1	1		
6					1	1		
7							1	
8								1

Figure 3.2.4 *c*-cut Matrix S_c ($c \in (0.91, 1.00]$)

	1	2	3	4	5	6	7	8
1	1							
2		1						
3			1				1	1
4				1				
5					1	1		
6					1	1		
7			1				1	1
8			1				1	1

Figure 3.2.5 *c*-cut Matrix S_c ($c \in (0.83, 0.91]$)

	1	2	3	4	5	6	7	8
1	1		1				1	1
2		1		1				
3	1		1				1	1
4		1		1				
5					1	1		
6					1	1		
7	1		1				1	1
8	1		1				1	1

Figure 3.2.6 *c*-cut Matrix S_c ($c \in (0.62, 0.83]$)

	1	2	3	4	5	6	7	8
1	1		1		1	1	1	1
2		1		1				
3	1		1		1	1	1	1
4		1		1				
5	1		1		1	1	1	1
6	1		1		1	1	1	1
7	1		1		1	1	1	1
8	1		1		1	1	1	1

Figure 3.2.7 c-cut Matrix S_c ($c \in (0.57, 0.62]$)

	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1
6	1	1	1	1	1	1	1	1
7	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1

Figure 3.2.8 c-cut Matrix S_c ($c \in [0.00, 0.57]$)

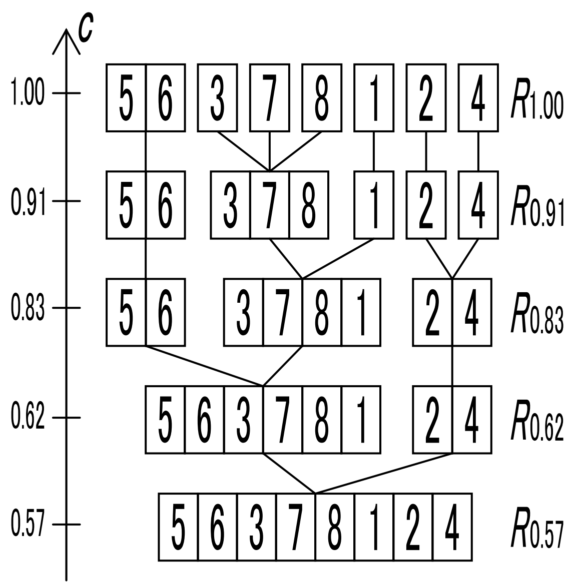

Figure 3.2.9 Example of Partition Tree

Concerning the cluster analysis of a fuzzy graph, it is important to decide the optimal level of fuzzy clustering as to the partition tree. Here, we would propose a new analysis method to decide the optimal cut level c_0.

Since a partition tree P shows cluster situation Rc, we could measure the branching number $x(c)$ and maximum cluster size $y(c)$ of each Rc.

$$x(c) = \#(R_c)$$
$$y(c) = \max_{1 \le i \le n} \{\#(CL_c(i))\}$$

Based on $x(c)$ and $y(c)$, we could define the cluster branch function $p(c)$ and the cluster size function $q(c)$ as follows:

Definition 3.3.1 Cluster Branch Function and Cluster Size Function

$$p(c) = \frac{x(c)-1}{x(1)-1}$$
$$q(c) = \frac{y(c)-1}{y(0)-1}$$

According to the maximal decision of the fuzzy decision, we could reasonably find the optimal cluster level Rc with $c=c_0$ concerning the partition tree P.

Definition 3.3.2 Decision of optimal cut level c_0

$$r(c) = p(c) \wedge q(c)$$
$$c_0 = \max\{c : r(c) = \max_{0 \le x \le 1} r(x)\}$$

Then, we could decide an optimal cut level c_0 by using this method.

In case of the partition tree in Fig.7, the cluster branch function $p(c)$ and the cluster size function $q(c)$ are followings:

$$p(c) = \frac{x(c)-1}{7-1}$$
$$q(c) = \frac{y(c)-1}{8-1}$$

By change the parameter c, we have obtained Table 2 and Fig.8.

From the Table, we could decide the optimal cut level c_0=0.83 in Fig.9.

c	$p(c)$	$q(c)$	$r(c)$
[0.00, 0.57]	0.00	1.00	0.00
(0.57, 0.62]	0.17	0.71	0.17
(0.62, 0.83]	0.33	0.43	0.33
(0.83, 0.91]	0.67	0.29	0.29
(0.91, 1.00]	1.00	0.14	0.14

Table 2 Value of $p(c)$, $q(c)$ and $r(c)$

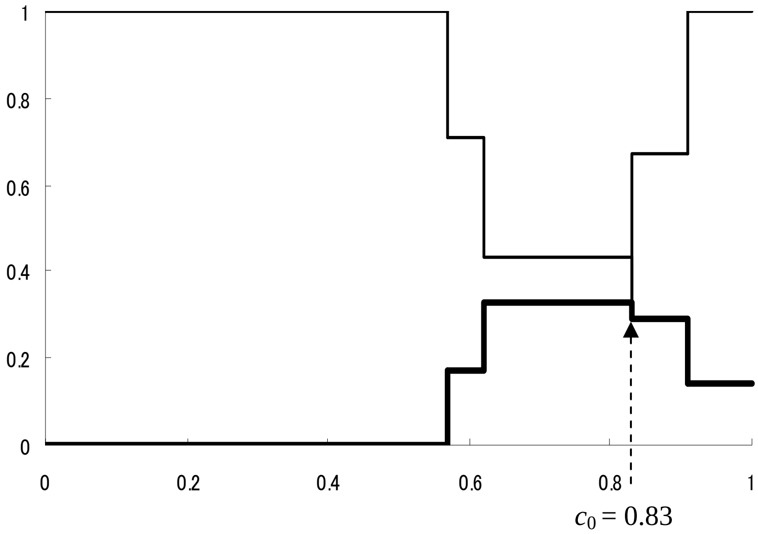

Figure 3.2.10 Fuzzy Decision $r(c) = p(c) \wedge q(c)$

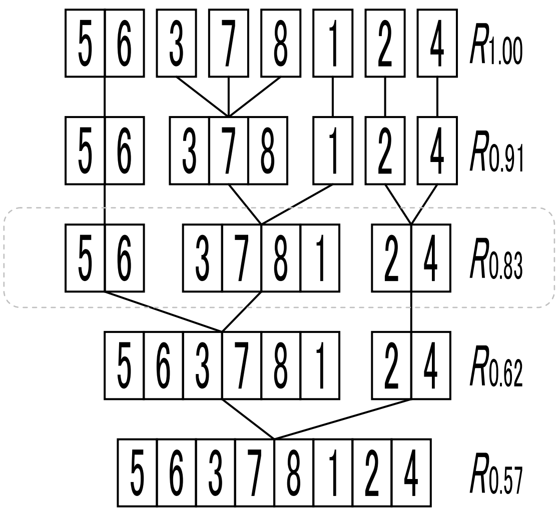

Figure 3.2.11 Optimal Cut Level c_0=0.83

3.3 Global Structure Analysis of Fuzzy Node Fuzzy Graph

Concerning the transformation from a fuzzy node fuzzy graph $G=(V,Y)$ to a fuzzy graph $G_\lambda=(V,F_\lambda)$, we could analyze a fuzzy node fuzzy graph gradually by using quasi-logical product T_λ. By changing the parameter λ, a sequence $\{G_\lambda\}$ of fuzzy graphs is composed.

Here, we would choose the optimal fuzzy graph G_{λ_0} from the fuzzy graph sequence.

Then, we would present the decision method of an optimal parameter λ_0.

In order to decide the optimal fuzzy graph G_{λ_0}, we would define two functions $d(\lambda)$ and $e(\lambda)$ as follows:

Definition 3.3.1 Distance Function and Connectivity Function

$$d(\lambda)=d(F_\lambda,S_{c_0})=\frac{1}{n^2-n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|f_{ij}-s_{ij}^{c_0}\right|$$

$$e(\lambda)=e(F_\lambda)=\frac{\gamma(F_\lambda)}{n^2-n}$$

where, $\gamma(F_\lambda)=\#(\Gamma), \Gamma=\{f_{ij}\in F_\lambda \mid f_{ij}>0\}$.

Here, $d(\lambda)$ evaluates the feature of the optimal clustering cut level c_0. If the value of $d(\lambda)$ is large enough, then G_λ reasonably shows the feature of the clustering level c_0.

On the other hand, $e(\lambda)$ evaluates the connectivity information of G_λ. If the value of $e(\lambda)$ is large enough, then G_λ reasonably shows the feature of the connectivity information.

Here, we would normalize the values of $d(\lambda)$ and $e(\lambda)$ respectively, and define $f_d(\lambda)$ and $f_e(\lambda)$ as follows:

Definition 3.3.2 Fuzzy Distance Function and Fuzzy Connectivity Function

$$f_d(\lambda)=\frac{d_M-d(\lambda)}{d_M-d_m}$$

$$f_e(\lambda)=\frac{e(\lambda)-e_m}{e_M-e_m}$$

where,

$d_M=Max\{d(\lambda)\}, d_m=Min\{d(\lambda)\}$,
$e_M=Max\{e(\lambda)\}$ and $e_m=Min\{e(\lambda)\}$.

By applying the maximal decision of the fuzzy decision, we could reasonably find the optimal value λ_0 concerning the sequence $\{G_\lambda\}$.

Definition 3.3.3 Decision of Optimal Value λ_0

$$f_m(\lambda) = f_d(\lambda) \wedge f_e(\lambda)$$
$$\lambda_0 = \min\{\lambda : f_m(\lambda) = \max_{0 \le x \le 1} f_m(x)\}$$

According to this decision method, we could obtained the optimal fuzzy graph G_{λ_0} .

For example, if a fuzzy node fuzzy graph is in Figure.3.3.1, then we have obtained a fuzzy graph sequence $\{G_\lambda\}$ in Figure.3.3.2-9.8.

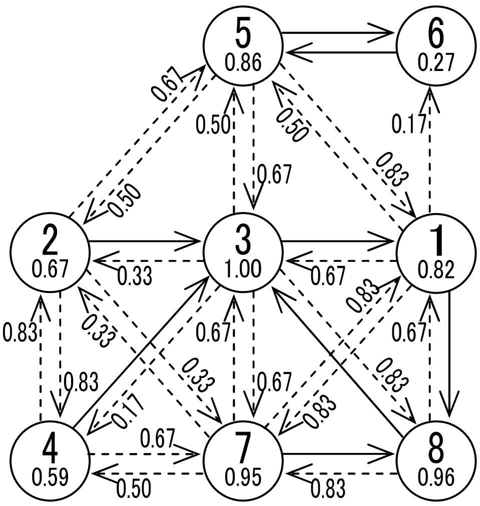

Figure 3.3.1 Fuzzy Node Fuzzy Graph *G*=(*V*, *F*)

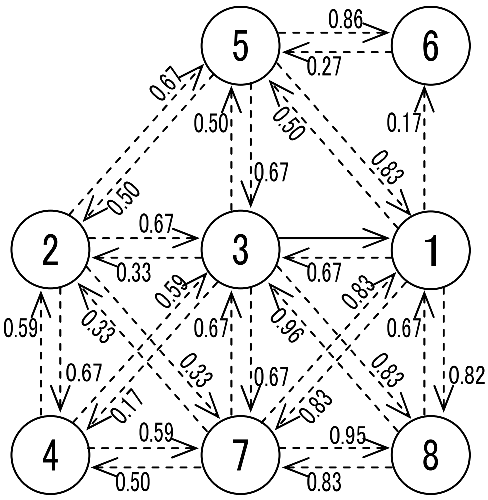

Figure 3.3.2 Fuzzy Graph $G_\lambda = (V, F_\lambda)$, $\lambda \in [0.73, 1.00]$

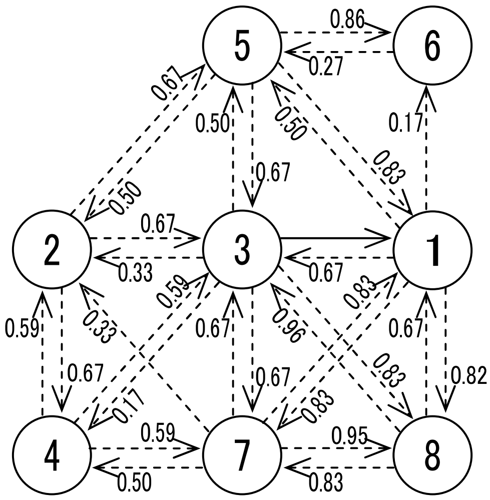

Figure 3.3.3 Fuzzy Graph $G_\lambda = (V, F_\lambda)$, $\lambda \in [0.33, 0.71)$

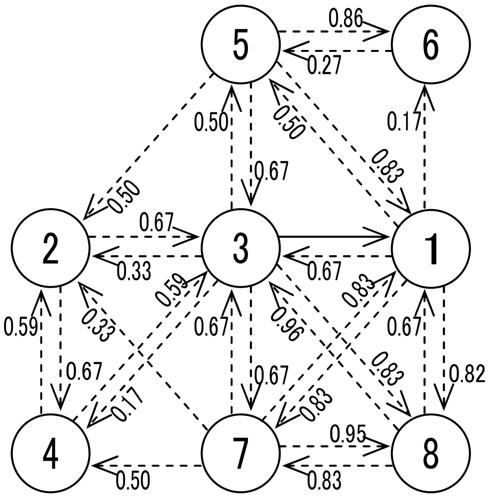

Figure 3.3.3 Fuzzy Graph $G_{\lambda} = (V, F_{\lambda})$

$, \lambda \in [0.18, 0.33)$

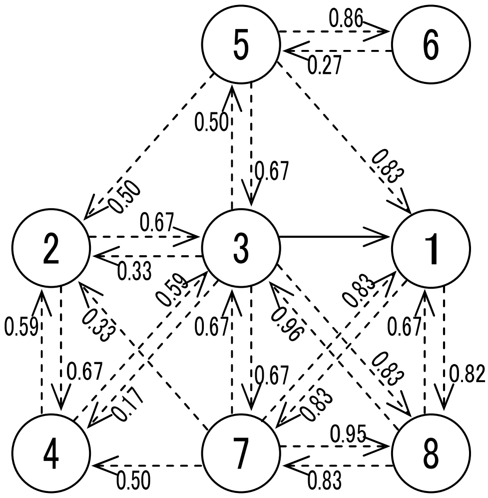

Figure 3.3.4 Fuzzy Graph $G_{\lambda} = (V, F_{\lambda})$

$, \lambda \in [0.17, 0.18)$

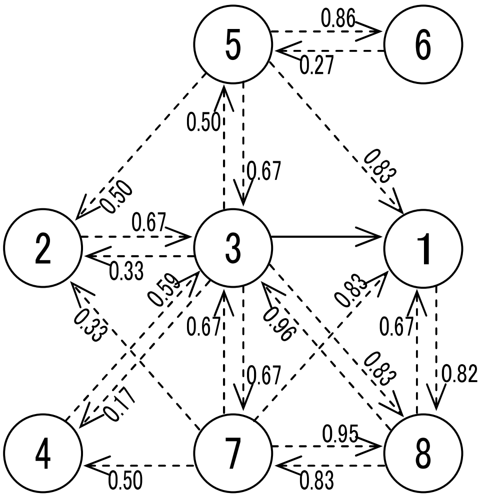

Figure 3.3.5 Fuzzy Graph $G_{\lambda} = (V, F_{\lambda})$

$, \lambda \in [0.14, 0.17)$

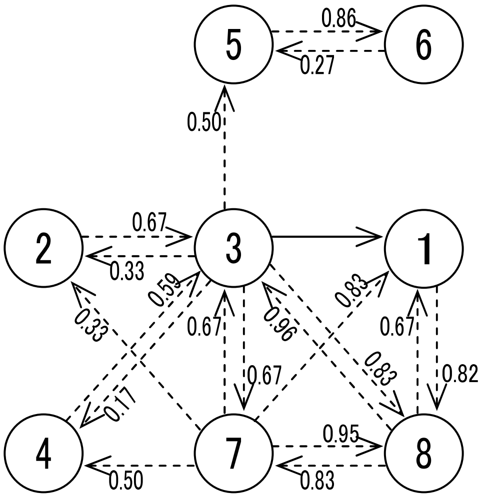

Figure 3.3.6 Fuzzy Graph $G_{\lambda} = (V, F_{\lambda})$

$, \lambda \in [0.05, 0.14)$

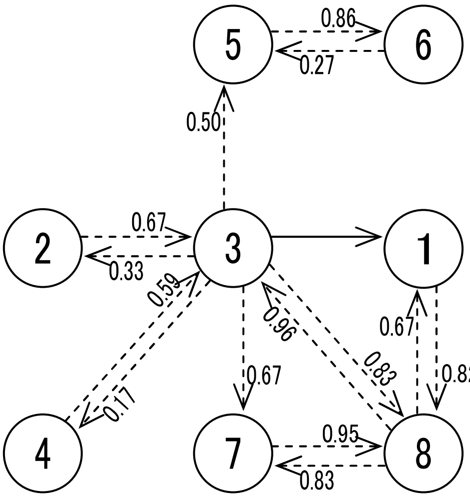

Figure 3.3.7 Fuzzy Graph $G_{\lambda} = (V, F_{\lambda})$, $\lambda \in [0.04, 0.05)$

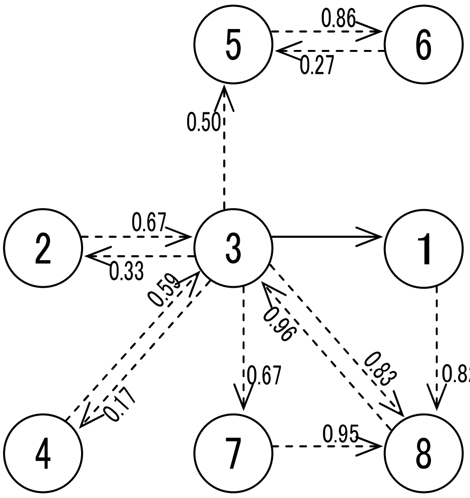

Figure 3.3.8 Fuzzy Graph $G_{\lambda} = (V, F_{\lambda})$, $\lambda \in [0.00, 0.04)$

Here, we could calculate the values of the distance function $d(\lambda)$ and the connectivity function $e(\lambda)$ respectively, and obtain Table 3.

λ	$d(\lambda)$	$e(\lambda)$
[0.000, 0.038)	0.212	0.232
[0.038, 0.053)	0.204	0.321
[0.053, 0.139)	0.195	0.411
[0.139, 0.167)	0.239	0.500
[0.167, 0.180)	0.202	0.554
[0.180, 0.333)	0.208	0.625
[0.333, 0.334)	0.231	0.661
[0.334, 0.413)	0.245	0.696
[0.413, 0.732)	0.263	0.750
[0.732, 1.000]	0.263	0.750

Table 3 Values of $d(\lambda)$ and $e(\lambda)$

From the values of $d(\lambda)$ and $e(\lambda)$, we could obtain the values of the fuzzy distance function $f_d(\lambda)$ and the connectivity function $f_e(\lambda)$ respectively as shown in Table 4.

λ	$f_d(\lambda)$	$f_e(\lambda)$
[0.000, 0.038)	0.740	0.00
[0.038, 0.053)	0.870	0.172
[0.053, 0.139)	1.000	0.345
[0.139, 0.167)	0.350	0.517
[0.167, 0.180)	0.889	0.621
[0.180, 0.333)	0.802	0.759
[0.333, 0.334)	0.477	0.828
[0.334, 0.413)	0.260	0.897
[0.413, 0.732)	0.000	1.000
[0.732, 1.000]	0.000	1.000

Table 4 Values of $f_d(\lambda)$ and $f_e(\lambda)$

By applying the fuzzy decision, we could had the optimal value $\lambda_0 = 0.180$ as shown in Figure3.3.9.

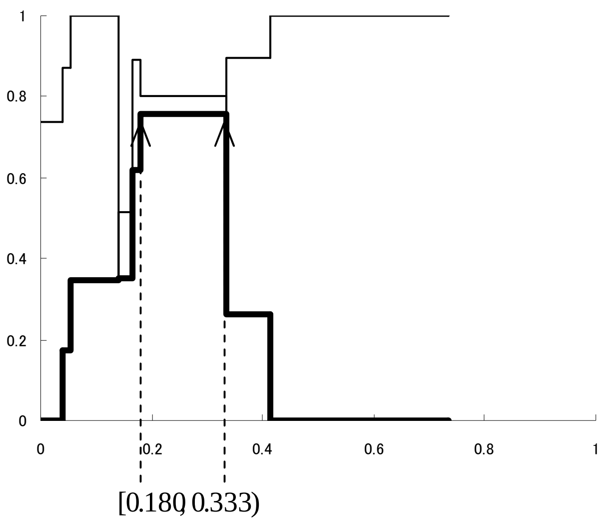

Figure 3.3.9 Fuzzy Decision $f_m(\lambda) = f_d(\lambda) \wedge f_e(\lambda)$

From this result, concerning the fuzzy graph sequence $\{G_\lambda\}$, we could obtain the optimal fuzzy graph G_{λ_0}, $\lambda_0 = 0.180$ which is the fuzzy graph in Figure 3.3.3.