

## 4. Application of Fuzzy Node Fuzzy Graph

*In Chapter 4, we would present the practical applications of the fuzzy node fuzzy graph which are the sociometry analysis in social psychology and the cognition analysis in mathematics education, and illustrate the positive effectiveness of the fuzzy node fuzzy graph.*

### 4.1 Application to Sociometry Analysis

As a practical application of fuzzy node fuzzy graph, we would describe a fuzzy sociometry analysis in psychology.

If we execute the questionnaires in Figure 4.1.1 to the  $M$  students, we have a response table  $K = (k_{ij}^{(q)})$ . It shows the members  $v_j$ 's with whom the member  $v_i$  has selected for activities  $B_q$ ,  $1 \leq q \leq N$ .

From the response table  $K$ , we have an evaluation matrix where

$$r_{ij} = \sum_{q=1}^N (n - k_{ij}^{(q)} + 1)$$

$$n = \left[ \frac{\sum_{i=1}^M n_i}{MN} + 0.5 \right]$$

and  $r_{ij}$  shows the evaluation that the member  $v_i$  preferred the member  $v_j$ .

By analyzing the matrix  $R=(r_{ij})$ , we define the fuzzy amicable degrees and preferring degrees among the members as follows:

We have the fuzzy preferring degree  $y_{ij}$  from  $v_i$  to  $v_j$  and obtain a fuzzy matrix  $Y=(y_{ij})$

$$y_{ij} = \frac{r_{ij}}{nN}$$

where  $y_{ij}=1$  if  $i=j$ .

Next, we have the fuzzy amicable degree between  $v_i$  and  $v_j$ , and obtain a fuzzy matrix  $S=(s_{ij})$

$$s_{ij} = \frac{2}{r_{ij} + r_{ji}} + \frac{nN}{r_{ij}}$$

where  $s_{ii}=1$  if  $i=j$ .

Then, we could define the fuzzy graphs  $G=(V, Y)$  and  $G'=(V, S)$ .

If  $s_{ij}$  is equal to 1,  $v_i$  and  $v_j$  are strongly amicable, and if  $s_{ij}$  is 0, they never are amicable. By analyzing the fuzzy graph  $G'$ , we have a partition tree  $P$ . It shows the clustering situation among the members.

We have the fuzziness  $u_i$  of member  $v_i$  by applying the Shapley-value as follows.

$$u_i = \left[ \frac{\phi(i)}{\text{Max}\{\phi(k)\}_{1 \leq k \leq M}} \right]^r, \quad 0 < r \leq 1$$

$$\phi(i) = \frac{\sum_{j=1}^M s'_{ij}}{\sum_{k=1}^M \sum_{m=1}^M s'_{km}} \quad (\text{Shapley-value})$$

$$S' = (s'_{ij}) = S - E$$

Then, we have obtain the fuzzy node fuzzy graph  $G=(V, Y) : V=\{v_i(u_i)\}, Y=(y_{ij}), 0 \leq y_{ij} \leq 1$ .

Generally, the fuzzy node fuzzy graph  $G=(V, Y)$  is complicated, we usually construct its  $N$ -nary approximate graph  $G_N^*$  with  $p^*$ .

The fuzzy node fuzzy graph is complicated too, as mentioned before, we transform the fuzzy node fuzzy graph  $G=(V, Y)$  to the fuzzy graph  $G=(V, F)$  by using T-norm. And, from these fuzzy graphs, we have made the  $N$ -nary approximate graphs  $G_N^*$  with  $p^*$  which was obtained from the fuzzy node fuzzy graph  $G=(V, Y)$ .

By summarizing the fuzzy graph  $G_N^*$  and the partition tree  $P$  with the optimal cut level  $z=c_0$ , we could obtain the approximate sociogram  $U^z$ , where  $z$  shows the amicable level among the members.

We would present a practical case study of the fuzzy sociogram analysis that has been experimented in high school.

If we executed the questionnaires in Figure 4.1.1 to the students, we could obtain the response matrix  $K$  in Figure 4.1.2.

From the matrix  $K$ , we have obtained the evaluation matrix  $R$  in Figure 4.1.3.

By analyzing the matrix  $R$ , we have the amicability matrix  $Y=(y_{ij})$  in Figure 4.1.4 and the connectivity matrix  $S=(s_{ij})$  in Figure 4.1.5. From the amicability matrix  $Y$  and the connectivity matrix  $S$ , we have obtained the fuzziness  $u_i$  of member  $s_i$  by applying the Shapley-value and illustrated the fuzzy node fuzzy graph  $G=(V, Y)$  in Figure 4.1.6.

Write your friends in order with whom you want to do each group activity.

Q<sub>1</sub> : Study in Group  
Q<sub>2</sub> : Take Lunch Together  
Q<sub>3</sub> : Do Sport Together

	1	2	3	4
Q <sub>1</sub>				
Q <sub>2</sub>				
Q <sub>3</sub>				

Figure 4.1.1 Questionnaires

		1	2	3	4
S01	Q1	7	6	2	4
	Q2	7	6	2	4
	Q3	7	4	2	6
S02	Q1	4	3	1	9
	Q2	7	4	9	8
	Q3	4	7	9	3
S03	Q1	2	4	7	10
	Q2	2	10	4	7
	Q3	7	2	4	0
S04	Q1	1	9	2	0
	Q2	1	9	7	2
	Q3	1	9	7	2
S05	Q1	2	8	7	3
	Q2	8	4	3	9
	Q3	9	8	3	4
S06	Q1	2	7	4	8
	Q2	7	2	4	1
	Q3	0	0	2	7
S07	Q1	2	9	6	1
	Q2	9	2	1	0
	Q3	1	0	0	0
S08	Q1	5	7	9	6
	Q2	5	2	10	3
	Q3	5	9	7	2
S09	Q1	1	4	2	7
	Q2	4	1	7	2
	Q3	0	4	2	7
S10	Q1	2	2	1	0
	Q2	7	2	9	0
	Q3	7	7	9	0

Figure 4.1.2 Response Matrix *K*

	1	2	3	4	5	6	7	8	9	10
1		6		5		7	12			
2	2		4	11			7	1	5	
3		11		7			7			4
4	12	4					4		9	
5		4	5	4			2	10	5	
6	1	9		4			8	1		
7	7	7				2				7
8		4	1		12	1	5		5	2
9	7	5		10			4			
10	2	10					11		4	

Figure 4.1.3 Evaluation Matrix *R*

	1	2	3	4	5	6	7	8	9	10
1	1	0.50		0.42		0.58	1.00			
2	0.17	1	0.33	0.92			0.58	0.08	0.42	
3		0.92	1	0.58			0.58			0.33
4	1.00	0.33		1			0.33		0.75	
5		0.33	0.42	0.33	1		0.17	0.83	0.42	
6	0.08	0.75		0.33		1	0.67	0.08		
7	0.58	0.58				0.17	1		0.58	
8		0.33	0.08		1.00	0.08	0.42	1	0.42	0.17
9	0.58	0.42		0.83			0.33		1	
10	0.17	0.83					0.92		0.33	1

Figure 4.1.4 Connectivity Matrix  $Y=(y_{ij})$

	1	2	3	4	5	6	7	8	9	10
1	1	0.25		0.59		0.15	0.74			
2	0.25	1	0.49	0.49			0.58	0.13	0.42	
3		0.49	1							
4	0.59	0.49		1					0.79	
5					1			0.91		
6	0.15					1	0.27	0.08		
7	0.74	0.58				0.27	1		0.42	
8		0.13			0.91	0.08		1		
9		0.42		0.79			0.42		1	
10										1

Figure 4.1.5 Amicability Matrix  $S=(s_{ij})$

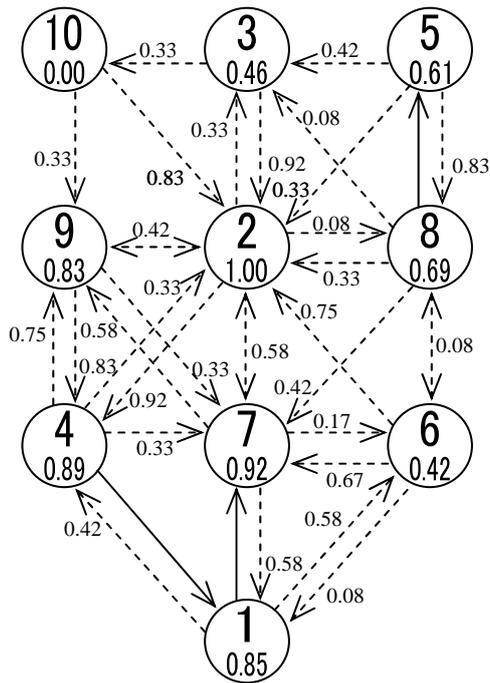


Figure 4.1.6 Fuzzy Node Fuzzy Graph  $G=(V, Y)$

We could transform the fuzzy node fuzzy graph  $G=(V, Y)$  to the fuzzy graph  $G=(V, F)$  by applying Quasi-Logical Product . By changing parameter  $\lambda$  , we have obtained fuzzy graphs in Figure 4.1.7 – 4.1.12.

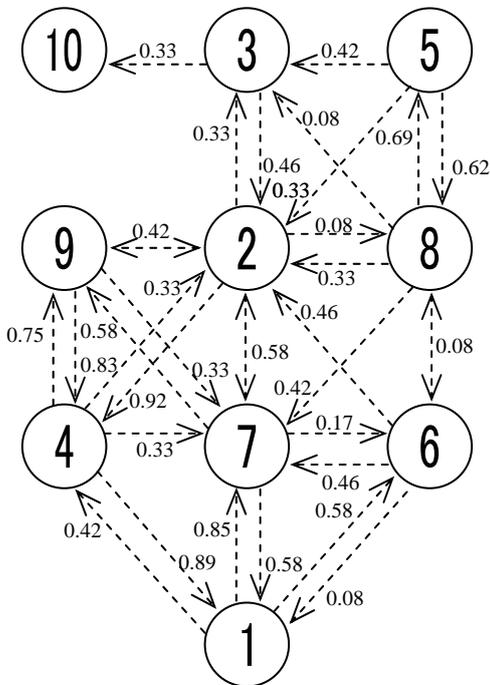


Figure 4.1.7 Fuzzy Graph  $G_\lambda=(V, F_\lambda)$ ,  
 $\lambda \in [0.545, 1.000]$

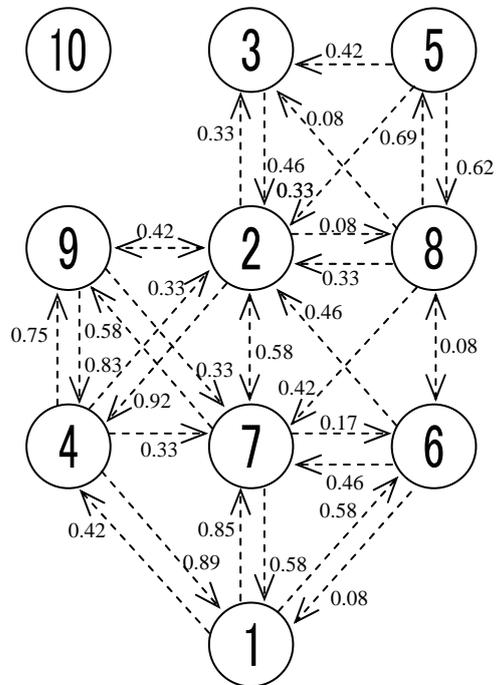


Figure 4.1.8 Fuzzy Graph  $G_\lambda=(V, F_\lambda)$ ,  
 $\lambda \in [0.542, 0.545]$

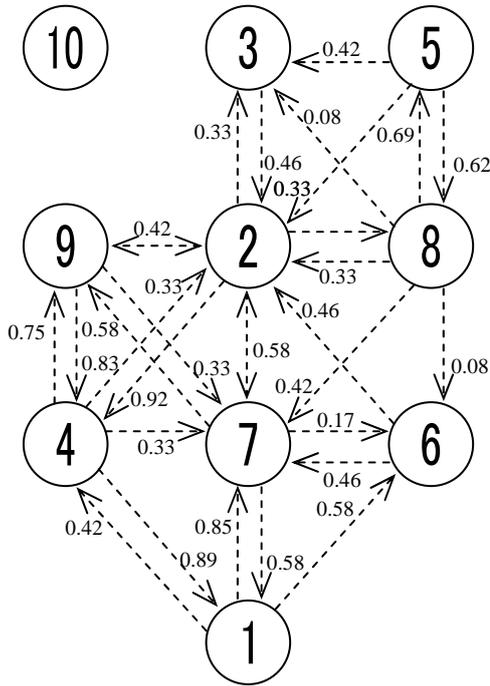


Figure 4.1.9 Fuzzy Graph  $G_\lambda=(V,F_\lambda)$ ,  
 $\lambda \in [0.417, 0.542)$

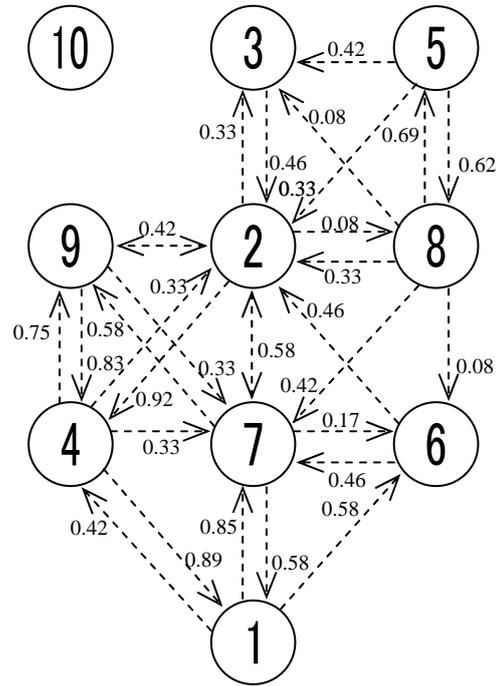


Figure 4.1.10 Fuzzy Graph  $G_\lambda=(V,F_\lambda)$ ,  
 $\lambda \in [0.379, 0.417)$

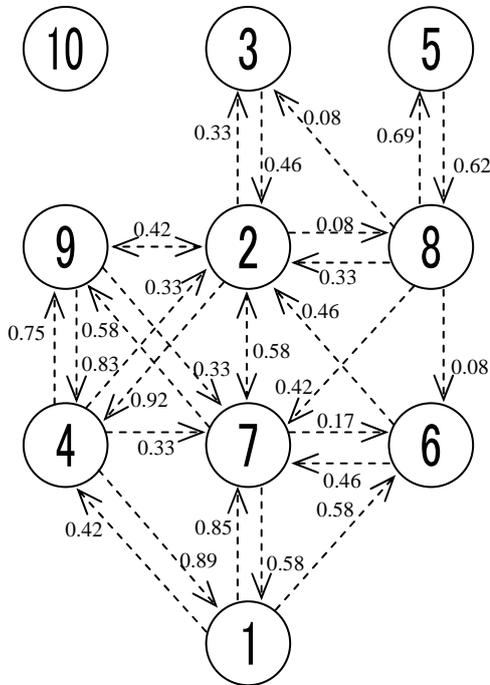


Figure 4.1.11 Fuzzy Graph  $G_\lambda=(V,F_\lambda)$ ,  
 $\lambda \in [0.333, 0.379)$

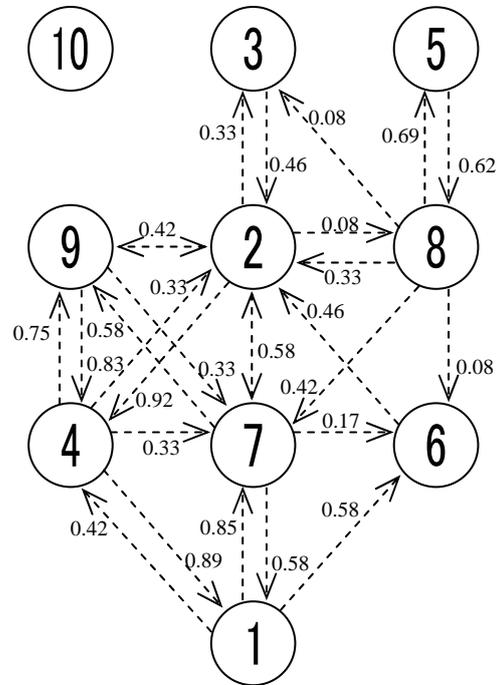


Figure 4.1.12 Fuzzy Graph  $G_\lambda=(V,F_\lambda)$ ,  
 $\lambda \in [0.309, 0.333)$

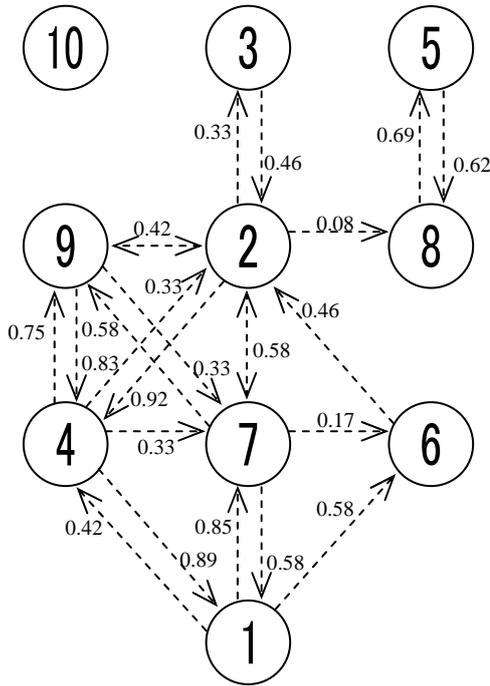


Figure 4.1.13 Fuzzy Graph  $G_\lambda=(V,F_\lambda)$ ,  
 $\lambda \in [0.250, 0.309)$

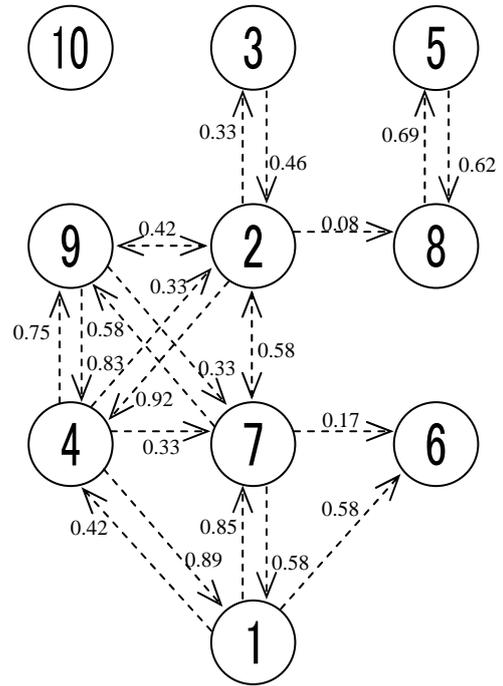


Figure 4.1.14 Fuzzy Graph  $G_\lambda=(V,F_\lambda)$ ,  
 $\lambda \in [0.169, 0.250)$

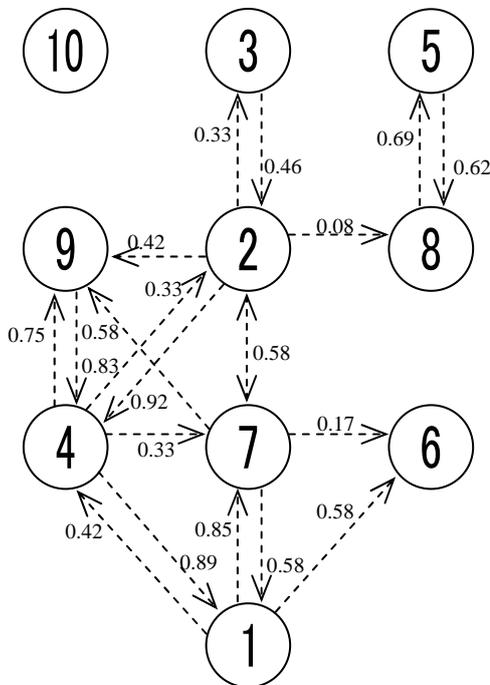


Figure 4.1.15 Fuzzy Graph  $G_\lambda=(V,F_\lambda)$ ,  
 $\lambda \in [0.167, 0.169)$

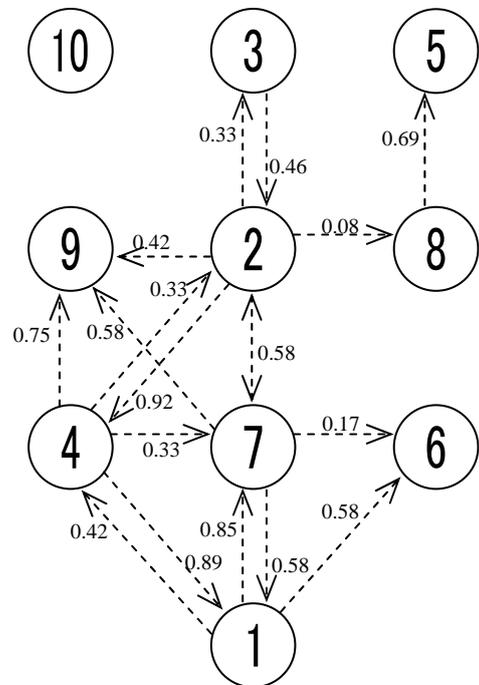


Figure 4.1.16 Fuzzy Graph  $G_\lambda=(V,F_\lambda)$ ,  
 $\lambda \in [0.146, 0.167)$

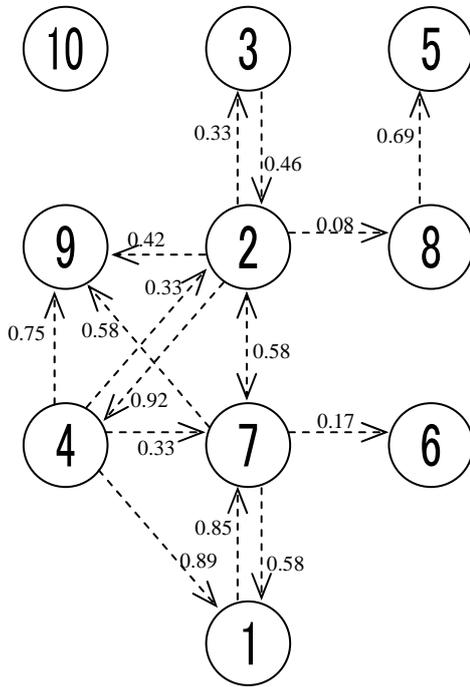


Figure 4.1.17 Fuzzy Graph  $G_\lambda=(V,F_\lambda)$ ,  
 $\lambda \in [0.111, 0.146]$

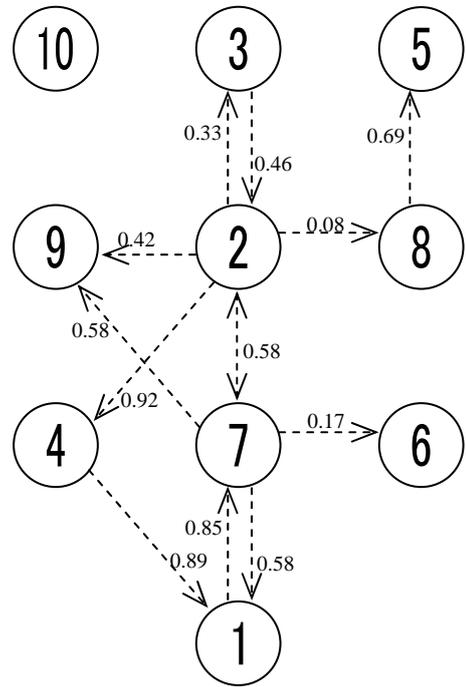


Figure 4.1.18 Fuzzy Graph  $G_\lambda=(V,F_\lambda)$ ,  
 $\lambda \in [0.083, 0.111]$

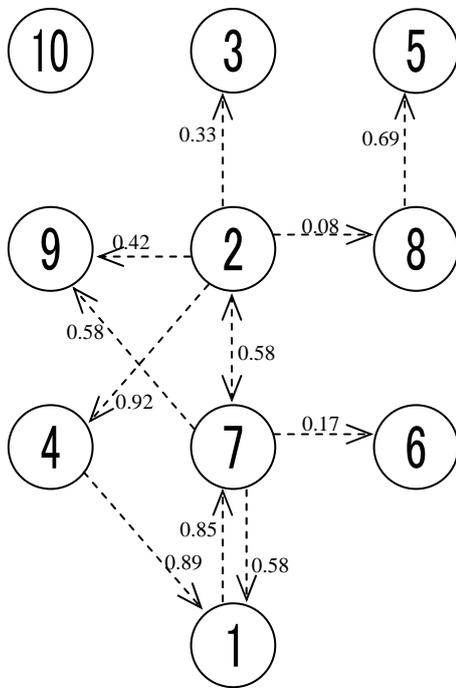


Figure 4.1.19 Fuzzy Graph  $G_\lambda=(V,F_\lambda)$ ,  
 $\lambda \in [0.077, 0.083]$

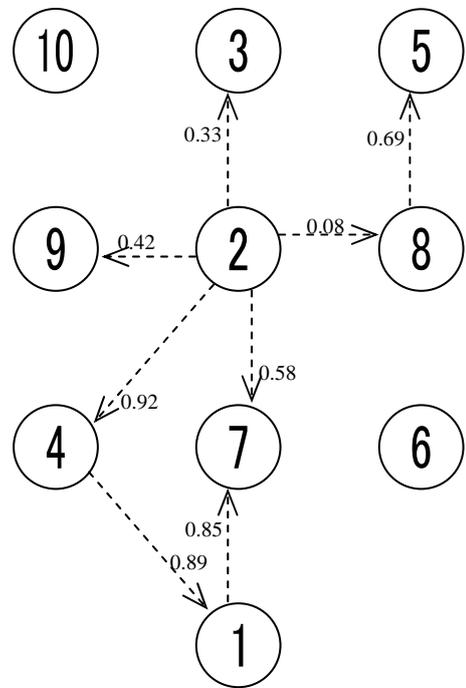


Figure 4.1.20 Fuzzy Graph  $G_\lambda=(V,F_\lambda)$ ,  
 $\lambda \in [0.000, 0.077]$

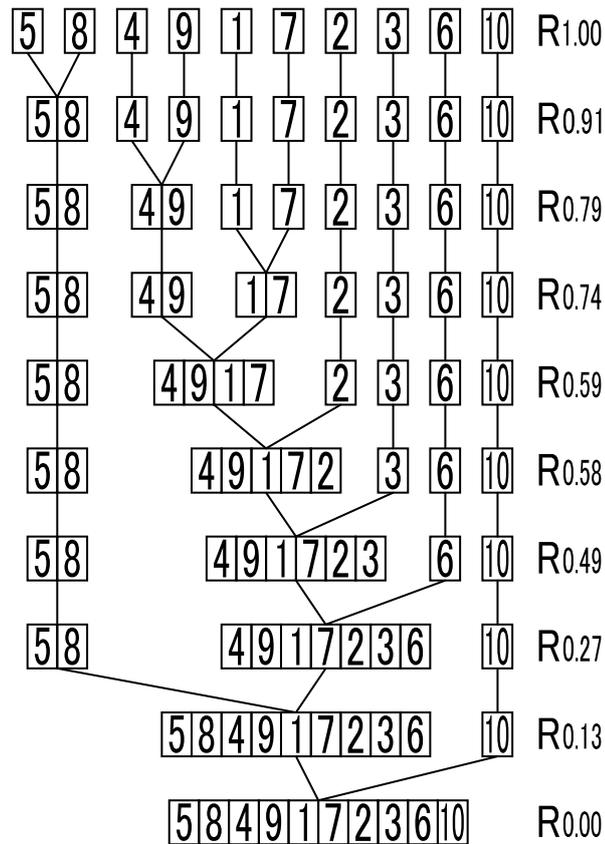


Figure 4.1.21 Partition Tree  $P$

On the other hand, we could obtain the partition tree  $P$  in Figure 4.1.21, if we execute the cluster analysis concerning the amicability matrix  $S$ .

As we mentioned before, the function  $p(c)$  and  $q(c)$  are

$$p(c) = \frac{x(c) - 1}{10 - 1}$$

$$q(c) = \frac{y(c) - 1}{10 - 1}$$

Here, we could decide the optimal cut level  $c_0 = 0.58$  as shown in Figure 4.1.22.

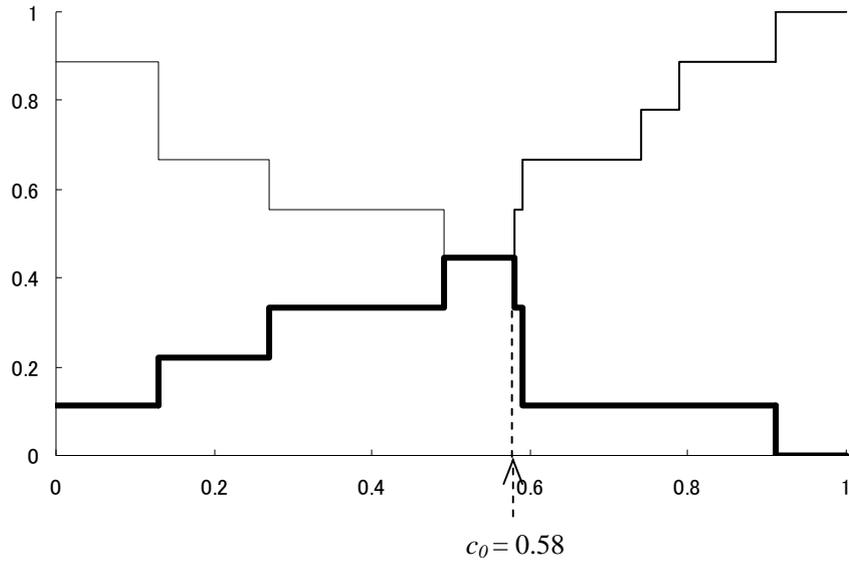


Figure 4.1.22 Fuzzy Decision  $r(c)$

By using  $c_0$ -cut matrix  $S_{c_0}$  and the fuzzy graph sequence  $\{G_\lambda\}$ , we could decide the optimal fuzzy graph  $G_{\lambda_0}$ . Here, by calculating the distance function  $d(\lambda)$  and the connectivity function  $e(\lambda)$ , we have obtained Table 5.

$$d(\lambda) = \frac{1}{90} \sum_{i=1}^{10} \sum_{j=1}^{10} |f_{ij} - s_{ij}^{c_0}|$$

$$e(\lambda) = \frac{\gamma(F_\lambda)}{90}$$

where,  $\gamma(F_\lambda) = \#\{\Gamma\}$ ,  $\Gamma = \{f_{ij} \in F_\lambda \mid f_{ij} > 0\}$ .

	$d(\lambda)$	$e(\lambda)$
[0.545, 1.000]	0.203	0.489
[0.542, 0.545]	0.199	0.478
[0.417, 0.542]	0.193	0.444
[0.379, 0.417]	0.183	0.422
[0.333, 0.379]	0.165	0.367
[0.309, 0.333]	0.160	0.356
[0.250, 0.309]	0.143	0.289
[0.169, 0.250]	0.138	0.278
[0.167, 0.169]	0.153	0.244
[0.146, 0.167]	0.169	0.222
[0.111, 0.146]	0.172	0.189
[0.083, 0.111]	0.186	0.156
[0.077, 0.083]	0.181	0.144
[0.000, 0.077]	0.199	0.100

Table 5 Values of  $d(\lambda)$  and  $e(\lambda)$

Then, we have normalized the value of  $d(\lambda)$  and  $e(\lambda)$ , so we have calculated  $f_d(\lambda)$  and  $f_e(\lambda)$  as follows:

$$f_d(\lambda) = \frac{d_M - d(\lambda)}{d_M - d_m}$$

$$f_e(\lambda) = \frac{e(\lambda) - e_m}{e_M - e_m}$$

where,  $d_M = 0.203$ ,  $d_m = 0.138$ ,  $e_M = 0.489$  and  $e_m = 0.100$ .

From  $f_d(\lambda)$  and  $f_e(\lambda)$ , we have obtained Table 6.

	$f_d(\lambda)$	$f_e(\lambda)$
[0.545, 1.000]	0.000	1.000
[0.542, 0.545)	0.057	0.971
[0.417, 0.542)	0.143	0.886
[0.379, 0.417)	0.299	0.829
[0.333, 0.379)	0.585	0.686
[0.309, 0.333)	0.664	0.657
[0.250, 0.309)	0.921	0.486
[0.169, 0.250)	1.000	0.457
[0.167, 0.169)	0.771	0.371
[0.146, 0.167)	0.522	0.314
[0.111, 0.146)	0.465	0.229
[0.083, 0.111)	0.250	0.143
[0.077, 0.083)	0.328	0.114
[0.000, 0.077)	0.057	0.000

Table 6 Values of  $f_d(F)$  and  $f_e(F)$

By applying the fuzzy decision, we could obtain the optimal value  $\lambda_0 = 0.309$  as shown in Figure 4.1.23.

We would illustrate the fuzzy sociogram  $U^z$ ,  $z=0.58$  in Figure 4.1.24 which is based on the fuzzy graph  $G_{\lambda_0} = (V, F_{\lambda_0})$ ,  $\lambda_0 = 0.309$  in Figure 4.1.12.

This fuzzy sociogram  $U^z$  effectively represents the following results:

- (1) The opinion leaders are 2 and 7 because these nodes have been connected with many other nodes.
- (2) The relatively isolated member is 10 because this node has not been connected with the others.
- (3) The subgroups are {5, 8} and {1,2,4,7, 9}.

In this way, we could effectively analyze and clarify the interpersonal relation among members by applying the fuzzy node fuzzy graph and the quasi-logical product.

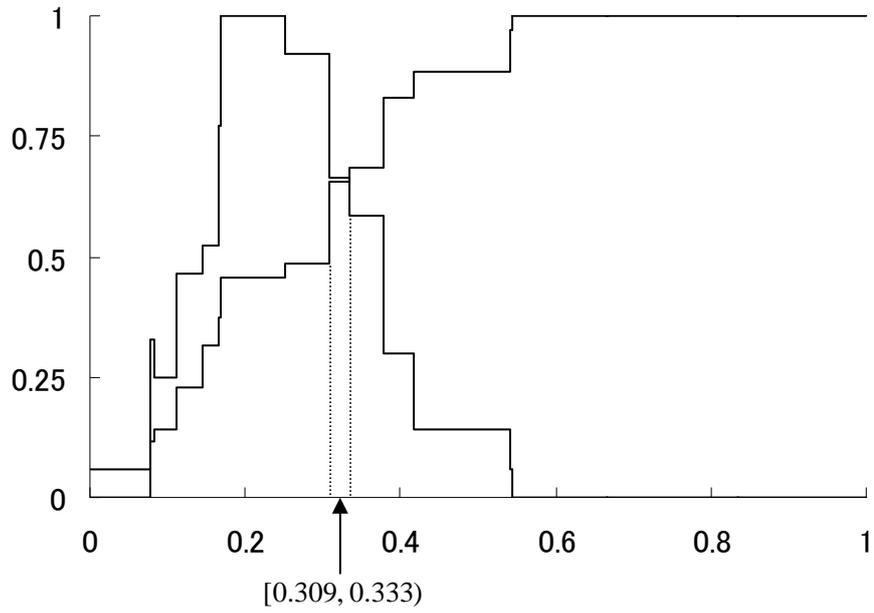


Figure 4.1.23 Fuzzy Decision  $f_d(F) \wedge f_c(F)$

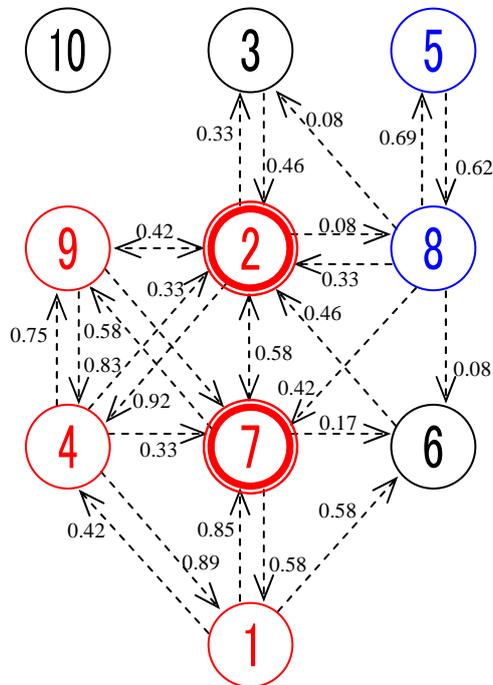


Figure 4.1.24 Fuzzy Sociogram  $U^z$ ,  $\lambda_0 = 0.309$ ,  $z = 0.58$