Remarks on BV estimates for vanishing viscosity approximations to hyperbolic systems

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Abstract

We consider the Cauchy problems for a parabolic $n\times n$ system in one-space dimension: $u_t+A(u)u_x=u_{xx}$, assuming that $A(u)$ has n real, distinct eigenvalues and that the initial data have small total variation. We discuss local existence and decay estimates of the solutions in detail.

1. Introduction

The Cauchy problem for a system of conservation laws in one space dimension takes the form

$$u_t + f(u)_x = 0 \ , \tag{1.1}$$

$$u(0,x) = \bar{u}(x) \ . \tag{1.2}$$

Here $u = (u_1, ..., u_n)$ is the vector of *conserved quantities*, while the components of $f = (f_1, ..., f_n)$ are the *fluxes*. We assume the flux function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is smooth and that the system is *strictly hyperbolic*; i.e., at each point u the Jacobian matrix $A(u) = Df(u)$ has n real, distinct eigenvalues

$$\lambda_1 < \cdots < \lambda_n \ . \tag{1.3}$$

One can then select bases of right and left eigenvectors $r_i(u)$, $l_i(u)$, normalized so that

$$| r_i |= 1, \qquad l_i \cdot r_j = \begin{cases} 1 & \text{if} \quad i=j \ , \\ 0 & \text{if} \quad i \neq j \ . \end{cases} \tag{1.4}$$

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Since the hyperbolic Cauchy problem is known to be well posed within a space of functions with small total variation, it seems natural to develop a theory of vanishing viscosity approximations within the same space BV. This was indeed accomplished in [1], in the more general framework of nonlinear hyperbolic systems not necessarily in conservation form. The only assumptions needed here are the strict hyperbolicity of system and the small total variation of the initial data.

Theorem 1.1. Consider the Cauchy problem for the hyperbolic system with viscosity

$$u_t^\epsilon + A(u^\epsilon)u_x^\epsilon = \epsilon u_{xx}^\epsilon \qquad u^\epsilon(0,x) = \bar{u}(x) \ . \tag{1.5}$$

Assume that $A(u)$ are strictly hyperbolic, smoothly depending on u in a neighborhood of the origin. Then there exist constants C, L, L' and $\delta > 0$ such that the following holds. If

$$\text{Tot.Var.}\{\bar{u}\} < \delta \ , \quad \|\bar{u}\|_{\boldsymbol{L}^\infty} < \delta \ , \tag{1.6}$$

then for each $\epsilon > 0$ the Cauchy problem $(2.5)_\epsilon$ has a unique solution u^ϵ, defined for all $t \geq 0$. Adopting a semigroup notation, this will be written as $t \mapsto u^\epsilon(t,\cdot) \doteq S_t^\epsilon \bar{u}$. In addition, one has

$$\textbf{BV bounds :} \qquad \text{Tot.Var.}\{S_t^\epsilon \bar{u}\} \leq C \ \text{Tot.Var.}\{\bar{u}\} \ . \tag{1.7}$$

$$L^1 \textbf{ stability :} \qquad \left\| S_t^\epsilon \bar{u} - S_t^\epsilon \bar{v} \right\|_{\boldsymbol{L}^1} \leq L \left\| \bar{u} - \bar{v} \right\|_{\boldsymbol{L}^1} \ , \tag{1.8}$$

$$\left\| S_t^\epsilon \bar{u} - S_s^\epsilon \bar{u} \right\|_{\boldsymbol{L}^1} \leq L' \Big(|t-s| + |\sqrt{\epsilon t} - \sqrt{\epsilon s}| \Big) \ . \tag{1.9}$$

Convergence: As $\epsilon \to 0+$, the solutions u^ϵ converge to the trajectories of a semigroup S such that

$$\left\| S_t \bar{u} - S_s \bar{v} \right\|_{\boldsymbol{L}^1} \leq L \|\bar{u} - \bar{v}\|_{\boldsymbol{L}^1} + L' |t-s| \ . \tag{1.10}$$

These vanishing viscosity limits can be regarded as the unique **vanishing viscosity solutions** of the hyperbolic Cauchy problem

$$u_t + A(u)u_x = 0 \ , \quad u(0,x) = \bar{u}(x) \ . \tag{1.11}$$

In the conservation case $A(u) = Df(u)$, every vanishing viscosity solution is a weak solution of

$$u_t + f(u)_x = 0 \ , \quad u(0,x) = \bar{u}(x) \ , \tag{1.12}$$

satisfying the Liu admissibility conditions (cf. [2]).

We observe that u^ϵ is a solution of (1.5) if and only if the rescaled function $u(t,x) \doteq u^\epsilon(\epsilon t, \epsilon x)$ is a solution of the parabolic system with unit viscosity

$$u_t + A(u)u_x = u_{xx} \ , \tag{1.13}$$

with initial data $u(0,x) = \bar{u}(\epsilon x)$. Clealy, the stretching of the space variable has no effect on the total variation. Notice however that the value of u^ϵ on a fixed time interval $[0,T]$ correspond to the values of u on the much longer time interval $[0, \frac{T}{\epsilon}]$. To obtain the desired BV bounds for the viscous solutions u^ϵ, we can confine all our analysis to solutions of (1.13), but we need estimates uniformly valid for all times $t \geq 0$, depending only on the total variation of the initial data $\bar{u}$.

In the proof of Theorem 1.1 in [1], it is proved that (1.13) with initial data

$$u(0,x) = \bar{u}(x) \tag{1.14}$$

having small total variation, say

$$\text{Tot.Var.}\{\bar{u}\} \leq \delta_0 \ ,$$

and

$$z_t + [DA(u)\cdot z]u_x + A(u)z_x = z_{xx} \tag{1.15}$$

with initial data

$$z(0,x) = \bar{z}(x) \in \boldsymbol{L}^1 \tag{1.16}$$

have solutions $u = u(t,x)$ and $z = z(t,x)$, respectively, defined on an initial time interval $[0,\hat{t}]$ with $\hat{t} \approx \delta_0^{-2}$. Moreover, it is proved that all higher derivatives of the solutions, decay quickly.

In this paper, we discuss this situation in detail.

2. Parabolic estimates

Following a standard approach, one can write the parabolic system (1.13) in the form

$$u_t - u_{xx} = -A(u)u_x \tag{2.1}$$

regarding the hyperbolic term $A(u)u_x$ as a first order perturbation of the heat equation. The general solution of (2.1) can then be written as

$$u(t) = G(t) * u(0) - \int_0^t G(t-s) * [A(u(s))u_x(s)]ds$$

in terms of convolutions with the Gauss kernel

$$G(t,x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \ . \tag{2.2}$$

From the above integral formura, one can derive local existence, uniqueness and regularity estimates for solutions of (2.1). Since we shall be dealing with a solution $u = u(t,x)$ having small total variation, a more effective representation is following. Consider the state

$$u^* \doteq \lim_{x \to -\infty} u(t,x) \ , \tag{2.3}$$

which is independent of time. We then define the matrix $A^* \doteq A(u^*)$ and let $\lambda_i^*, r_i^*, l_i^*$ be the corresponding eigenvalues and right and left eigenvectors, normalized as in (1.4). It will be convenient to use " $\bullet$ " to denote a directional derivative, so that $z \bullet A(u) \doteq DA(u) \cdot z$ indicates the derivative of the matrix valued functon $u \mapsto A(u)$ in the direction of the vector z. The systems (1.13) and (1.15) can now be written respectively as

$$u_t + A^* u_x - u_{xx} = (A^* - A(u))u_x \ , \tag{2.4}$$

$$z_t + A^* z_x - z_{xx} = (A^* - A(u))z_x - (z \bullet A(u))u_x \ . \tag{2.5}$$

Observing that, if u is a solution of (2.4), then $z = u_x$ is a paticular solution of the variational equation (2.5). Therefore, as soon as one proves an a priori bound on z_x or z_{xx}, the same estimate will be valid also for the corresponding derivatives u_{xx}, u_{xxx}.

In both of the equations (2.4), (2.5), we regard the right hand side as a perturbation of the linear parabolic system with constant coefficients

$$w_t + A^* w_x - w_{xx} = 0 \ . \tag{2.6}$$

We denote by G^* the Green kernel for (2.6), so that any solution admits the integral representation

$$w(t,x) = \int G^*(t, x-y) w(0,y) dy \ .$$

The matrix valued function G^* can be explicitly computed. If w solves (2.6), then the i-th component $w_i \doteq l_i^* \cdot w$ satisfies the scalar equation

$$w_{i,t} + \lambda_i^* w_{i,x} - w_{i,xx} = 0 \ .$$

Therefore $w_i(t) = G_i^*(t) * w_i(0)$, where

$$G_i^*(t,x) = \frac{1}{2\sqrt{\pi t}} \exp\Big\{ -\frac{(x-\lambda_i^* t)^2}{4t} \Big\} \ .$$

It is now clear that this Green kernel $G^* = G^*(t,x)$ satisfies the bounds

$$\|G^*(t)\|_{\boldsymbol{L}^1} \le \kappa \ , \quad \|G_x^*(t)\|_{\boldsymbol{L}^1} \le \frac{\kappa}{\sqrt{t}} \ , \quad \|G_{xx}^*(t)\|_{\boldsymbol{L}^1} \le \frac{\kappa}{t} \ , \tag{2.7}$$

for some constant κ and all $t > 0$.

In the following, we consider an initial data $u(0,\cdot)$ having small total variation, but possibly discontinuous. We shall prove the local existence of solutions and some estimates on the decay of higher order derivatives. To get a feeling on this rate of decay, let us first take a look at this most elementary case.

Example 2.1. The solution to the Cauchy problem for the heat equation

$$u_t - u_{xx} = 0 \ , \qquad u(0,x) = \begin{cases} 0 & \text{if} \quad x < 0 \ , \\ \delta_0 & \text{if} \quad x > 0 \ , \end{cases}$$

is computed explicitly as

$$u(t,x) = \delta_0 \int_0^\infty G(t, x-y)dy \ .$$

The Gauss kernel (2.2) satisfies

$$\Big\| \frac{\partial^{k-1}}{\partial x^{k-1}} G(t) \Big\|_{\boldsymbol{L}^\infty} \le \Big\| \frac{\partial^k}{\partial x^k} G(t) \Big\|_{\boldsymbol{L}^1} = O(1) t^{-\frac{k}{2}} \quad \text{for } 1 \le k \le 3 \ .$$

In the present example we have $u_x(t,x) = \delta_0 G(t,x)$. Therefore

$$\|u_x(t)\|_{\boldsymbol{L}^\infty} \le \|u_{xx}(t)\|_{\boldsymbol{L}^1} = O(1)\frac{\delta_0}{\sqrt{t}} \ , \tag{2.8}$$

$$\|u_{xx}(t)\|_{\boldsymbol{L}^\infty} \le \|u_{xxx}(t)\|_{\boldsymbol{L}^1} = O(1)\frac{\delta_0}{t} \ , \tag{2.9}$$

$$\|u_{xxx}(t)\|_{\boldsymbol{L}^\infty} = O(1)\frac{\delta_0}{t\sqrt{t}} \ . \tag{2.10}$$

In this section, our analysis will show that the same decay rates hold for solutions of the perturbed equation (2.4), restricted to some initial interval $[0,\hat{t}]$. More precisely, let δ_0 measure the order of magnitude of the total variation data, so that

$$\text{Tot.Var.}\{u(0,\cdot)\} = O(1)\delta_0 \ .$$

Then:

• There exists an initial interval $[0, \hat{t}]$, with $\hat{t} = O(1)\delta_0^{-2}$ on which the solution of (2.4) is well defined. Its derivatives decay according to the estimates (2.8)-(2.10).

• As long as the total variation remains small, say

$$\|u_x(t)\|_{\boldsymbol{L}^1} \leq \delta_0 \ , \tag{2.12}$$

the solution can be prolonged in time. In this case, for $t > \hat{t}$ the higher derivatives satisfy the bounds

$$\|u_x(t)\|_{\boldsymbol{L}^\infty} \ , \ \|u_{xx}(t)\|_{\boldsymbol{L}^1} = O(1)\delta_0^2 \ , \tag{2.13}$$

$$\|u_{xx}(t)\|_{\boldsymbol{L}^\infty} \ , \ \|u_{xxx}(t)\|_{\boldsymbol{L}^1} = O(1)\delta_0^3 \ , \tag{2.14}$$

$$\|u_{xxx}(t)\|_{\boldsymbol{L}^\infty} = O(1)\delta_0^4 \ . \tag{2.15}$$

Proposition 2.2 (Local existence). For $\delta_0 > 0$ sufficiently small, consider intial data

$$u(0, x) = \bar{u}(x) \ , \quad z(0, x) = \bar{z}(x) \tag{2.16}$$

such that

$$\text{Tot.Var.}\{\bar{u}\} \leq \frac{\delta_0}{2\kappa} \ , \quad \bar{z} \in \boldsymbol{L}^1 \ . \tag{2.17}$$

Then the equations (2.4), (2.5) have solutions $u = u(t, x)$, $z = z(t, x)$ defined on the time interval $[0, \hat{t}]$, where

$$\hat{t} \doteq \left(\frac{1}{220\kappa\kappa_A\delta_0}\right)^2 , \quad \kappa_A \doteq \sup_u\{\|DA\| \ , \ \|D^2A\|\}$$

and κ is the constant in (2.7). Moreover one has

$$\|u_x(t)\|_{\boldsymbol{L}^1} \leq 2\kappa\text{Tot.Var.}\{\bar{u}\} \ , \ \|z(t)\|_{\boldsymbol{L}^1} \leq 2\kappa\|\bar{z}\|_{\boldsymbol{L}^1} \quad \text{for all } t \in (0, \hat{t}] \ . \tag{2.18}$$

Proof. The couple (u, u_x), consisting of the solution of (2.2) together with its derivative, will be obtained as the unique fixed point of a contractive transformation. For simplicity, we assume here

$$u^* \doteq \lim_{x \to -\infty} \bar{u}(x) = 0 \ .$$

Of course this is not restrictive, since it can always be achieved by $\bar{u}(x) - \bar{u}(-\infty)$. Cosider the Banach space

$$E = \left\{(u, v); u \in C\big([0, \hat{t}] : \boldsymbol{L}^\infty(\mathbb{R})\big), v \in \boldsymbol{L}^\infty\big((0, \hat{t}] : \boldsymbol{L}^1(\mathbb{R})\big)\right\}$$

with norm

$$\big\|(u,v)\big\|_E \doteq \sup_t \max\{\|u(t)\|_{\boldsymbol{L}^\infty}\ ,\ \|v(t)\|_{\boldsymbol{L}^1}\}\ .$$

On E we define the transformation $\mathcal{T}(u,v) = (\hat{u},\hat{v})$ for $(u,v)\in E$ by setting

$$\hat{u}(t) \doteq G^*(t)*\bar{u} + \int_0^t G^*(t-s)*[A^*-A(u(s))]v(s)ds\ ,$$
$$\hat{v}(t) \doteq G^*_x(t)*\bar{u} + \int_0^t G^*_x(t-s)*[A^*-A(u(s))]v(s)ds\ .$$

Of course, the above definition implies $\hat{v} = \hat{u}_x$. Observing that

$$\lim_{x\to-\infty} G^*(t)*\bar{u} = \lim_{x\to-\infty}\bar{u} = u^* = 0\ ,$$

we can compute

$$\|G^*(t)*\bar{u}\|_{\boldsymbol{L}^\infty} \le \|G^*_x(t)*\bar{u}\|_{\boldsymbol{L}^1} \le \|G^*(t)\|_{\boldsymbol{L}^1}\int_{\mathbb{R}} d|\bar{u}(y)| \le \frac{\delta_0}{2}\ . \tag{2.19}$$

Moreover, if

$$\|u(s)\|_{\boldsymbol{L}^\infty} \le \delta_0, \quad \|v(s)\|_{\boldsymbol{L}^1} \le \delta_0 \quad \text{for all } s\in(0,\hat{t}]\ ,$$

then

$$\begin{aligned}
&\Big\|\int_0^t G^*(t-s)*[A^*-A(u(s))]v(s)ds\Big\|_{\boldsymbol{L}^\infty}\\
&\quad\le \Big\|\int_0^t G^*_x(t-s)*[A^*-A(u(s))]v(s)ds\Big\|_{\boldsymbol{L}^1}\\
&\quad\le \int_0^t \frac{\kappa}{\sqrt{t-s}}\Big\|[A^*-A(u(s))]v(s)\Big\|_{\boldsymbol{L}^1}ds\\
&\quad\le \int_0^t \frac{\kappa}{\sqrt{t-s}}\kappa_A\delta_0 \sup_{s\in(0,\hat{t}]}\|v(s)\|_{\boldsymbol{L}^1}ds\\
&\quad= \kappa\kappa_A\delta_0 \sup_{s\in(0,\hat{t}]}\|v(s)\|_{\boldsymbol{L}^1}\\
&\quad\le 2\sqrt{t}\kappa\kappa_A\delta_0^2\ .
\end{aligned}$$

Therefore, it follows that

$$\|(\hat{u},\hat{v})\|_E \le \frac{\delta_0}{2} + 2\sqrt{\hat{t}}\kappa\kappa_A\delta_0^2 < \delta_0\ .$$

Hence we see that the transformation $\mathcal{T}$ maps the domain

$$D \doteq \Big\{(u,v)\in E : \|u(t)\|_{\boldsymbol{L}^\infty}, \|v(t)\|_{\boldsymbol{L}^1} \le \delta_0 \quad \text{for all } t\in(0,\hat{t}]\Big\}$$

into itself. To prove that $\mathcal{T}$ is a strict contraction, we compute the difference $\mathcal{T}(u,v) - \mathcal{T}(u^{'},v^{'})$. The norm is estimated as

$$
\begin{aligned}
\|\hat{u}-\hat{u}'\|_{\boldsymbol{L}^\infty} &\le \|\hat{v}-\hat{v}'\|_{\boldsymbol{L}^1}\\
&= \Big\| \int_0^t G_x^*(t-s) * \big\{[A^* - A(u(s))](v(s)-v'(s))\\
&\qquad\qquad + [A(u'(s)) - A(u(s))]v'(s)\big\} ds \Big\|_{\boldsymbol{L}^1}\\
&\le \int_0^t \|G_x^*(t-s)\|_{\boldsymbol{L}^1} \|[A^* - A(u(s))](v(s)-v'(s))\\
&\qquad\qquad + [A(u'(s)) - A(u(s))]v'(s)\|_{\boldsymbol{L}^1} ds\\
&\le \int_0^t \frac{\kappa}{\sqrt{t-s}} \{\|DA\|\|u(s)\|_{\boldsymbol{L}^\infty}\|v(s)-v'(s)\|_{\boldsymbol{L}^1}\\
&\qquad\qquad + \|DA\|\|u(s)-u'(s)\|_{\boldsymbol{L}^\infty}\|v'(s)\|_{\boldsymbol{L}^1}\} ds\\
&\le 2\kappa\kappa_A\delta_0 \|(u-u', v-v')\|_E \int_0^t \frac{1}{\sqrt{t-s}} ds\\
&\le 4\sqrt{t}\kappa\kappa_A\delta_0\|(u-u', v-v')\|_E .
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|\mathcal{T}(u,v) - \mathcal{T}(u', v')\|_E &\le 4\sqrt{\hat{t}}\kappa\kappa_A\delta_0\|(u-u', v-v')\|_E\\
&= \frac{1}{55}\|(u-u', v-v')\|_E .
\end{aligned}
$$

Therefore, the map $\mathcal{T}$ is a strict contraction. By the contraction mapping theorem, a unique fixed point exists in D. Clearly, this provides the solution of (2.4) with the prescribed initial data.

Having constructed a solution u of (2.4), we now prove the existence of a solution z of the linearized variational system (2.5), with initial data $\bar{z} \in \boldsymbol{L}^1$. Cosider the Banach space

$$
E' = \big\{z; z \in C\big([0,\hat{t}] : \boldsymbol{L}^1(\mathbb{R})\big), z_x \in \boldsymbol{L}^\infty\big((0,\hat{t}] : \boldsymbol{L}^1(\mathbb{R})\big)\big\}
$$

with norm

$$
\big\|z\big\|_{E'} \doteq \sup_t \max\{\|z(t)\|_{\boldsymbol{L}^1} \ , \ \sqrt{t}\|z_x(t)\|_{\boldsymbol{L}^1}\} .
$$

On E' we define the transformation $\mathcal{T}(z) = \hat{z}$ for $z \in E'$ by setting

$$
\begin{aligned}
\hat{z}(t) \doteq G^*(t) * \bar{z} + \int_0^t G^*(t-s) * \Big\{&[A^* - A(u(s))]z_x(s)\\
&- [z(s) \bullet A(u(s))]u_x(s)\Big\} ds .
\end{aligned}
$$

Thus u is a solution of (2.4). The bounds (2.7) now yield

$$
\|G^*(t) * \bar{z}\|_{\boldsymbol{L}^1} \le \kappa\|\bar{z}\|_{\boldsymbol{L}^1}, \quad \|G_x^*(t) * \bar{z}\|_{\boldsymbol{L}^1} \le \frac{\kappa}{\sqrt{t}}\|\bar{z}\|_{\boldsymbol{L}^1} . \tag{2.20}
$$

Moreover, using the identity

$$\int_0^t \frac{1}{\sqrt{s(t-s)}} ds = \int_0^1 \frac{1}{\sqrt{\sigma(1-\sigma)}} = \pi < 4 \tag{2.21}$$

one obtains the bound

$$\begin{aligned}
&\Big\| \int_0^t G^*(t-s) * \big\{ [A^* - A(u(s))] z_x(s) - [z(s) \bullet A(u(s))] u_x(s) \big\} ds \Big\|_{\boldsymbol{L}^1} \\
&\le \int_0^t \frac{\kappa}{\sqrt{s}} \big\{ \|DA\| \|u(s)\|_{\boldsymbol{L}^\infty} \sqrt{s} \|z_x(s)\|_{\boldsymbol{L}^1} + \|DA\| \sqrt{s} \|z_x(s)\|_{\boldsymbol{L}^1} \|u_x(s)\|_{\boldsymbol{L}^1} \big\} ds \\
&\le 4\sqrt{t}\kappa\kappa_A\delta_0 \sup_{s\in(0,\hat{t}]} \sqrt{s}\|z_x(s)\|_{\boldsymbol{L}^1} \\
&\le 4\sqrt{t}\kappa\kappa_A\delta_0 \|z\|_{E'}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\Big\| \int_0^t G_x^*(t-s) * \big\{ [A^* - A(u(s))] z_x(s) - [z(s) \bullet A(u(s))] u_x(s) \big\} ds \Big\|_{\boldsymbol{L}^1} \\
&\le \int_0^t \frac{\kappa}{\sqrt{s(t-s)}} \big\{ \|DA\| \|u(s)\|_{\boldsymbol{L}^\infty} \sqrt{s} \|z_x(s)\|_{\boldsymbol{L}^1} \\
&\qquad\qquad + \|DA\| \sqrt{s} \|z_x(s)\|_{\boldsymbol{L}^1} \|u_x(s)\|_{\boldsymbol{L}^1} \big\} ds \\
&\le 8\kappa\kappa_A\delta_0 \sup_{s\in(0,\hat{t}]} \sqrt{s}\|z_x(s)\|_{\boldsymbol{L}^1} \\
&\le 8\kappa\kappa_A\delta_0 \|z\|_{E'} \ .
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\|\hat{z}\|_{E'} &\le \|\hat{z}(t)\|_{\boldsymbol{L}^1} + \sqrt{t}\|\hat{z}_x(t)\|_{\boldsymbol{L}^1} \\
&\le \|\hat{z}(t)\|_{\boldsymbol{L}^1} + \sup_{t\in(0,\hat{t}]} \sqrt{t}\|\hat{z}_x(t)\|_{\boldsymbol{L}^1} \\
&\le 2\kappa\|\bar{z}\|_{\boldsymbol{L}^1} + 12\sqrt{\hat{t}}\kappa\kappa_A\delta_0 \sup_{t\in(0,\hat{t}]} \sqrt{t}\|z_x(t)\|_{\boldsymbol{L}^1} \\
&\le 2\kappa\|\bar{z}\|_{\boldsymbol{L}^1} + \frac{3}{55}\|z\|_{E'} \\
&< \infty \ .
\end{aligned}$$

Hence we see that the transformation $\mathcal{T}$ maps the domain E' into itself. Observing that

$$\begin{aligned}
&\|\hat{z} - \hat{z}'\|_{\boldsymbol{L}^1} \le 4\sqrt{t}\kappa\kappa_A\delta_0\|z - z'\|_{E'} \\
&\|\hat{z}_x - \hat{z}_x'\|_{\boldsymbol{L}^1} \le 8\kappa\kappa_A\delta_0\|z - z'\|_{E'} \ ,
\end{aligned}$$

it follows that

$$\begin{aligned}\|\hat{z}(t)-\hat{z}^{'}(t)\|_{E'} &\le \|\hat{z}(t)-\hat{z}^{'}(t)\|_{\boldsymbol{L}^1}+\sqrt{t}\|\hat{z}_x(t)-\hat{z}_x^{'}(t)\|_{\boldsymbol{L}^1}\\ &\le 12\sqrt{\hat{t}}\kappa\kappa_A\delta_0\|z-z^{'}\|_{E'}\\ &= \frac{3}{55}\|z-z^{'}\|_{E'}\ .\end{aligned}$$

Therefore, the map $\mathcal{T}$ is a strict contraction. By the contraction mapping theorem, a unique fixed point exists in $E^{'}$. Clearly, this provides the solution of (2.5) with the prescribed initial data.

Finally, we prove (2.18). Because of $v=\hat{v}=u_x$, it follows that

$$\begin{aligned}\|u_x(t)\|_{\boldsymbol{L}^1} &\le \|G_x^*(t)*\bar{u}\|_{\boldsymbol{L}^1}+\|\int_0^t G_x^*(t-s)*[A^*-A(u(s))]v(s)ds\|_{\boldsymbol{L}^1}\\ &\le \kappa\text{Tot.Var.}\{\bar{u}\}+2\sqrt{\hat{t}}\kappa\kappa_A\delta_0\sup_{t\in(0,\hat{t}]}\|u_x(t)\|_{\boldsymbol{L}^1}\ .\end{aligned}$$

Hence

$$\left(1-2\sqrt{\hat{t}}\kappa\kappa_A\delta_0\right)\sup_{t\in(0,\hat{t}]}\|u_x(t)\|_{\boldsymbol{L}^1}\le\kappa\text{Tot.Var.}\{\bar{u}\}\ .$$

Since

$$1-2\sqrt{\hat{t}}\kappa\kappa_A\delta_0=\frac{109}{110}\ ,$$

we obtain

$$\sup_{t\in(0,\hat{t}]}\|u_x(t)\|_{\boldsymbol{L}^1}\le 2\kappa\text{Tot.Var.}\{\bar{u}\}\ .$$

Hence

$$\|u_x(t)\|_{\boldsymbol{L}^1}\le 2\kappa\text{Tot.Var.}\{\bar{u}\}\ .$$

Moreover, because of $z=\hat{z}$, it follows that

$$\|z(t)\|_{\boldsymbol{L}^1}+\sup_{t\in(0,\hat{t}]}\sqrt{t}\|z_x(t)\|_{\boldsymbol{L}^1}\le 2\kappa\|\bar{z}\|_{\boldsymbol{L}^1}+12\sqrt{\hat{t}}\kappa\kappa_A\delta_0\sup_{t\in(0,\hat{t}]}\sqrt{t}\|z_x(t)\|_{\boldsymbol{L}^1}\ .$$

Hence

$$\|z(t)\|_{\boldsymbol{L}^1}+\left(1-12\sqrt{\hat{t}}\kappa\kappa_A\delta_0\right)\sup_{t\in(0,\hat{t}]}\|z_x(t)\|_{\boldsymbol{L}^1}\le 2\kappa\|\bar{z}\|_{\boldsymbol{L}^1}\ .$$

Since

$$1-12\sqrt{\hat{t}}\kappa\kappa_A\delta_0=\frac{52}{55}\ ,$$

we obtain

$$\|z(t)\|_{\boldsymbol{L}^1} \le 2\kappa \|\bar{z}\|_{\boldsymbol{L}^1} \ . \qquad \blacksquare$$

Having established the local existence of a solution, we now prove the decay of its higher order derivatives.

Proposition 2.3. Let u and z be solutions of the systems (2.4) and (2.5), respectively, constructed in Proposition 2.2, satisfying the bounds

$$\|u_x(t)\|_{\boldsymbol{L}^1} \le \delta_0, \quad \|z(t)\|_{\boldsymbol{L}^1} \le \delta_0 \ , \tag{2.22}$$

for $\delta_0 > 0$ sufficiently small and all $t \in (0, \hat{t}]$, where

$$\hat{t} = \Big(\frac{1}{220\kappa\kappa_A\delta_0}\Big)^2, \quad \kappa_A = \sup_u\{\|DA\|,\ \|D^2A\|\} \tag{2.23}$$

and κ is the constant in (2.7). Then for $t \in (0, \hat{t}]$ the following estimates hold.

$$\|u_{xx}(t)\|_{\boldsymbol{L}^1},\ \|z_x(t)\|_{\boldsymbol{L}^1} \le \frac{2\kappa\delta_0}{\sqrt{t}} \ , \tag{2.24}$$

$$\|u_{xxx}(t)\|_{\boldsymbol{L}^1},\ \|z_{xx}(t)\|_{\boldsymbol{L}^1} \le \frac{5\kappa^2\delta_0}{t} \ , \tag{2.25}$$

$$\|u_{xxx}(t)\|_{\boldsymbol{L}^\infty},\ \|z_{xx}(t)\|_{\boldsymbol{L}^\infty} \le \frac{16\kappa^3\delta_0}{t\sqrt{t}} \ . \tag{2.26}$$

Proof. It will suffice to establish the desired estimates under the additional assumption that $z(0)$ is smooth, because the general case will then follow by completion.

We begin with (2.24). The function z_x can be represented in terms of convolutions with the Green kernel G^*, as

$$\begin{aligned} z_x = G^*_x\Big(\frac{t}{2}\Big) * z\Big(\frac{t}{2}\Big) - \int_{\frac{t}{2}}^{t} G^*_x(t-s) * \Big\{&(z \bullet A(u))u_x(s) \\ &+ (A(u) - A^*)z_x\Big\}ds \ . \end{aligned} \tag{2.27}$$

Recalling the identity (2.21) we compute

$$\begin{aligned} \|z_x(t)\|_{\boldsymbol{L}^1} \ &\le \ \Big\|G^*_x\Big(\frac{t}{2}\Big)\Big\|_{\boldsymbol{L}^1}\Big\|z\Big(\frac{t}{2}\Big)\Big\|_{\boldsymbol{L}^1} \\ &\quad + \int_{\frac{t}{2}}^{t} \|G^*_x(t-s)\|_{\boldsymbol{L}^1}\Big\{\|z(s)\|_{\boldsymbol{L}^\infty}\|DA\|\|u_x(s)\|_{\boldsymbol{L}^1} \\ &\qquad\qquad + \|u(s)\|_{\boldsymbol{L}^\infty}\|DA\|\|z_x(s)\|_{\boldsymbol{L}^1}\Big\}ds \\ &\le \ \frac{\sqrt{2}\kappa}{\sqrt{t}}\Big\|z\Big(\frac{t}{2}\Big)\Big\|_{\boldsymbol{L}^1} + 2\kappa\kappa_A\delta_0\int_{\frac{t}{2}}^{t}\frac{1}{\sqrt{t-s}}\|z_x(s)\|_{\boldsymbol{L}^1}ds \end{aligned}$$

$$\leq \frac{\sqrt{2}\kappa}{\sqrt{t}}\Big\|z\Big(\frac{t}{2}\Big)\Big\|_{\boldsymbol{L}^1} + 8\kappa\kappa_A\delta_0 \sup_{s\in(0,\hat{t}]}\sqrt{s}\|z_x(s)\|_{\boldsymbol{L}^1} \ .$$

Hence

$$\begin{aligned}\sqrt{t}\|z_x(t)\|_{\boldsymbol{L}^1} &\leq \sqrt{2}\kappa\Big\|z\Big(\frac{t}{2}\Big)\Big\|_{\boldsymbol{L}^1} + 8\sqrt{\hat{t}}\kappa\kappa_A\delta_0 \sup_{t\in(0,\hat{t}]}\sqrt{t}\|z_x(t)\|_{\boldsymbol{L}^1} \\ &= \sqrt{2}\kappa\Big\|z\Big(\frac{t}{2}\Big)\Big\|_{\boldsymbol{L}^1} + \frac{2}{55}\sup_{t\in(0,\hat{t}]}\sqrt{t}\|z_x(t)\|_{\boldsymbol{L}^1} \ .\end{aligned}$$

Observing that

$$\begin{aligned}\frac{53}{55}\sup_{t\in(0,\hat{t}]}\sqrt{t}\|z_x(t)\|_{\boldsymbol{L}^1} &\leq \sqrt{2}\kappa\Big\|z\Big(\frac{t}{2}\Big)\Big\|_{\boldsymbol{L}^1} \\ &\leq \sqrt{2}\kappa\delta_0 \ ,\end{aligned}$$

we obtain

$$\sqrt{t}\|z_x(t)\|_{\boldsymbol{L}^1} \leq \frac{55\sqrt{2}}{53}\kappa\delta_0 < 2\kappa\delta_0 \ ,$$

which proves (2.24).

A similar technique is used to establish (2.25). Indeed, we can write

$$\begin{aligned}z_{xx} = G_x^*\Big(\frac{t}{2}\Big) * z_x\Big(\frac{t}{2}\Big) - \int_{\frac{t}{2}}^{t} G_x^*(t-s) * \Big\{&(z \bullet A(u))u_x(s) \\ &+ (A(u)-A^*)z_x\Big\}_x ds \ .\end{aligned} \tag{2.28}$$

Since

$$\int_{\frac{t}{2}}^{t}\frac{1}{s\sqrt{t-s}}ds < \frac{1}{\sqrt{\frac{t}{2}}} < \frac{2\sqrt{\hat{t}}}{t} \ ,$$

we compute

$$\begin{aligned}\|z_{xx}\|_{\boldsymbol{L}^1} &\leq \frac{\kappa}{\sqrt{\frac{t}{2}}}\frac{2\kappa\delta_0}{\sqrt{\frac{t}{2}}} + \int_{\frac{t}{2}}^{t}\frac{\kappa}{\sqrt{t-s}}\Big\|(DA\cdot z)_x u_x \\ &\qquad +(DA\cdot z)u_{xx} + (DA\cdot u_x)z_x + (A(u)-A^*)z_{xx}\Big\|_{\boldsymbol{L}^1} ds \\ &\leq \frac{4\kappa^2\delta_0}{t} + \int_{\frac{t}{2}}^{t}\frac{\kappa}{\sqrt{t-s}}\Big\{\|D^2A\|\|u_{xx}\|_{\boldsymbol{L}^1}^2\|z\|_{\boldsymbol{L}^1} \\ &\qquad +\|DA\|\|z_{xx}\|_{\boldsymbol{L}^1}\|u_x\|_{\boldsymbol{L}^1} + \|DA\|\|z\|_{\boldsymbol{L}^1}\|u_{xxx}\|_{\boldsymbol{L}^1} \\ &\qquad +\|DA\|\|u_x\|_{\boldsymbol{L}^1}\|z_{xx}\|_{\boldsymbol{L}^1} + \|DA\|\|u_x\|_{\boldsymbol{L}^1}\|z_{xx}\|_{\boldsymbol{L}^1}\Big\}ds \\ &\leq \frac{4\kappa^2\delta_0}{t} + 4\kappa\kappa_A\delta_0\Big\{\kappa^2\delta_0^2 + \sup_{s\in(0,\hat{t}]}s\|z_{xx}(s)\|_{\boldsymbol{L}^1}\Big\}\int_{\frac{t}{2}}^{t}\frac{1}{s\sqrt{t-s}}ds\end{aligned}$$

$$
\begin{aligned}
&< \frac{4\kappa^2\delta_0}{t} + \frac{8\sqrt{\hat{t}}}{t}\kappa\kappa_A\delta_0\Big\{\kappa^2\delta_0^2 + \sup_{s\in(0,\hat{t}]} s\|z_{xx}(s)\|_{\boldsymbol{L}^1}\Big\} \\
&= \frac{4\kappa^2\delta_0}{t} + \frac{2}{55t}\Big\{\kappa^2\delta_0^2 + \sup_{s\in(0,\hat{t}]} s\|z_{xx}(s)\|_{\boldsymbol{L}^1}\Big\} .
\end{aligned}
$$

Hence

$$
\begin{aligned}
t\|z_{xx}(t)\|_{\boldsymbol{L}^1} &< 4\kappa^2\delta_0 + \frac{2}{55}\kappa^2\delta_0 + \frac{2}{55}\sup_{t\in(0,\hat{t}]} t\|z_{xx}(t)\|_{\boldsymbol{L}^1} \\
&= \frac{222\kappa^2\delta_0}{55} + \frac{2}{55}\sup_{t\in(0,\hat{t}} t\|z_{xx}(t)\|_{\boldsymbol{L}^1} .
\end{aligned}
$$

Observing that

$$
\Big(1-\frac{2}{55}\Big)\sup_{t\in(0,\hat{t}]} t\|z_{xx}(t)\|_{\boldsymbol{L}^1} \leq \frac{222\kappa^2\delta_0}{55} ,
$$

we obtain

$$
t\|z_{xx}(t)\|_{\boldsymbol{L}^1} \leq \frac{222\kappa^2\delta_0}{53} < 5\kappa^2\delta_0 ,
$$

which proves (2.25).

Finally, since

$$
\int_{\frac{t}{2}}^{t} \frac{1}{s^{\frac{3}{2}}\sqrt{t-s}}ds < \frac{4}{t} < \frac{4\sqrt{\hat{t}}}{t\sqrt{t}} ,
$$

using (2.28) we compute

$$
\begin{aligned}
\|z_{xx}\|_{\boldsymbol{L}^\infty} &\leq \frac{\kappa}{\sqrt{\frac{t}{2}}}\frac{5\kappa^2\delta_0}{\frac{t}{2}} + \int_{\frac{t}{2}}^{t}\frac{\kappa}{\sqrt{t-s}}\Big\|(DA\cdot z)_x u_x \\
&\qquad +(DA\cdot z)u_{xx} + (DA\cdot u_x)z_x + (A(u)-A^*)z_{xx}\Big\|_{\boldsymbol{L}^\infty} ds \\
&\leq \frac{10\sqrt{2}\kappa^3\delta_0}{t\sqrt{t}} + \int_{\frac{t}{2}}^{t}\frac{\kappa}{\sqrt{t-s}}\Big\{\|D^2A\|\|u_x\|_{\boldsymbol{L}^\infty}^2\|z\|_{\boldsymbol{L}^\infty} \\
&\qquad +\|DA\|\|z_x\|_{\boldsymbol{L}^\infty}\|u_x\|_{\boldsymbol{L}^\infty} + \|DA\|\|z\|_{\boldsymbol{L}^\infty}\|u_{xx}\|_{\boldsymbol{L}^\infty} \\
&\qquad +\|DA\|\|u_x\|_{\boldsymbol{L}^\infty}\|z_x\|_{\boldsymbol{L}^\infty} + \|DA\|\|u\|_{\boldsymbol{L}^\infty}\|z_{xx}\|_{\boldsymbol{L}^\infty}\Big\}ds \\
&\leq \frac{10\sqrt{2}\kappa^3\delta_0}{t\sqrt{t}} + \int_{\frac{t}{2}}^{t}\frac{\kappa}{s^{\frac{3}{2}}\sqrt{t-s}}\Big\{8\kappa^3\kappa_A\delta_0^3 \\
&\qquad +30\kappa^3\kappa_A\delta_0^2 + \kappa_A\delta_0\sup_{s\in(0,\hat{t}]} s^{\frac{3}{2}}\|z_{xx}(s)\|_{\boldsymbol{L}^\infty}\Big\}ds \\
&< \frac{15\kappa^3\delta_0}{t\sqrt{t}} + \frac{4\sqrt{\hat{t}}\kappa\kappa_A\delta_0}{t\sqrt{t}}\Big\{8\kappa^3\delta_0^2 + 30\kappa^3\delta_0 + \sup_{s\in(0,\hat{t}]} s^{\frac{3}{2}}\|z_{xx}(s)\|_{\boldsymbol{L}^\infty}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{15\kappa^3\delta_0}{t\sqrt{t}} + \frac{1}{55t\sqrt{t}}\Big\{8\kappa^3\delta_0^2 + 30\kappa^3\delta_0 + \sup_{s\in(0,\hat{t}]} s^{\frac{3}{2}}\|z_{xx}(s)\|_{\boldsymbol{L}^\infty}\Big\} \\
&< \frac{863\kappa^3\delta_0}{55t\sqrt{t}} + \frac{1}{55t\sqrt{t}} \sup_{s\in(0,\hat{t}]} s^{\frac{3}{2}}\|z_{xx}(s)\|_{\boldsymbol{L}^\infty} \ .
\end{aligned}
$$

Hence

$$
t^{\frac{3}{2}}\|z_{xx}(t)\|_{\boldsymbol{L}^\infty} \le \frac{863}{55}\kappa^3\delta_0 + \frac{1}{55}\sup_{t\in(0,\hat{t}]} t^{\frac{3}{2}}\|z_{xx}(t)\|_{\boldsymbol{L}^\infty} \ .
$$

Observing that

$$
\Big(1-\frac{1}{55}\Big)\sup_{t\in 0,\hat{t}]} t^{\frac{3}{2}}\|z_{xx}(t)\|_{\boldsymbol{L}^\infty} \le \frac{863}{55}\kappa^3\delta_0 \ ,
$$

we obtain

$$
t^{\frac{3}{2}}\|z_{xx}(t)\|_{\boldsymbol{L}^\infty} \le \frac{863}{54}\kappa^3\delta_0 < 16\kappa^3\delta_0 \ ,
$$

which proves (2.26). ■

The estimates in the following corollary show that as long as Tot.Var.$\{u(t)\}$ and $\|z(t)\|_{\boldsymbol{L}^1}$ satisfy the desired bounds, all higher order derivatives of u_x and z are small, with expotentially decaying $\boldsymbol{L}^1$ norms.

Corollary 2.4. In the same setting as in Proposition 2.3, assume that the boubds (2.22) hold on a larger interval $[0,T]$. Then for all $t\in[\hat{t},T]$,

$$
\|u_{xx}(t)\|_{\boldsymbol{L}^1}\ , \quad \|u_x(t)\|_{\boldsymbol{L}^\infty}\ , \quad \|z_x(t)\|_{\boldsymbol{L}^1} = O(1)\delta_0^2\ , \tag{2.29}
$$

$$
\|u_{xxx}(t)\|_{\boldsymbol{L}^1}\ , \quad \|u_{xx}(t)\|_{\boldsymbol{L}^\infty}\ , \quad \|z_{xx}(t)\|_{\boldsymbol{L}^1} = O(1)\delta_0^3\ , \tag{2.30}
$$

$$
\|u_{xxx}(t)\|_{\boldsymbol{L}^\infty}\ , \quad \|z_{xx}(t)\|_{\boldsymbol{L}^\infty} = O(1)\delta_0^4\ . \tag{2.31}
$$

Proof. These follow by applying Proposition 2.3 on the interval $[t-\hat{t},t]$. ■

References

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