境界をもつ３次元多様体のハンドル体分解

（課題番号　10640094）

平成 10 ～ 11 年度科学研究費補助金（基盤研究(C)(2)）

研究成果報告書

平成 12 年 3 月

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はしがき

本研究は，境界をもつコンパクトで向き付け可能な３次元多様体に対しても，境界をもたない場合（＝閉多様体）の Heegaard 分解に対応するような，２つの同相なハンドル体に分解可能であるという事実を基に，それを実用に耐えるようにやや特殊化し，その性質を詳しく論じたものである。

研究組織

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研究経費

平成 10 年度	４００千円
平成 11 年度	５００千円
計	９００千円

研究発表

口頭発表：Handlebody splittings of compact 3-manifolds with boundary, Workshop "Knot Theory"（Dedicated to Professor Kunio Murasugi for his 70th birthday）, 1999 年 7 月 13 日

論　　文：Handlebody splittings of compact 3-manifolds with boundary, 上記 Workshop の Proceedings, pp.323-330, March 2000.

HANDLEBODY SPLITTINGS OF COMPACT 3-MANIFOLDS WITH BOUNDARY

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1. Introduction

Throughout this paper we work in the piecewise-linear category, consisting of simplicial complexes and piecewise-linear maps.

We call a compact, connected, orientable 3-manifold M with nonempty boundary ∂M a *bordered* 3-manifold. A bordered 3-manifold H is said to be a *handlebody of genus* g iff H is the disk-sum (= the boundary connected-sum) of g copies of the solid-torus $D^2 \times S^1$ (cf. Gross [3], Swarup [15], etc.). A handlebody of genus g is characterazed by a regular neighborhood $N(P; R^3)$ of a connected 1-polyhedron P with the Euler chracteristic $\chi(P) = 1 - g$ in the 3-dimensional Euclidean space R^3, and by an irreducible bordered 3-manifold M with a connected boundary whose fundamental group $\pi_1(M)$ is a free group of rank g (see Ochiai [11]).

It is well-known that a closed(=compact, without boundary), connected, orientable 3-manifold M is decomposed into two homeomorphic handlebodies; that is,

Theorem 1.1. (Heegaard Splittings; cf. Seifert-Threlfoll [14], etc.)

(A) *For every closed, connected, orientable 3-manifold M, there exist handlebodies H_1 and H_2 in M such that*

(0) $H_1 \cong H_2$ *; say, genus*$(H_1) =$ *genus*$(H_2) = g$,

(1) $M = H_1 \cup H_2$, *and*

(2) $H_1 \cap H_2 = \partial H_1 \cap \partial H_2 = \partial H_1 = \partial H_2 = F$.

(B) *For every bordered 3-manifold M, there exist a handlebody H_1 and a set of disjoint 2-handles(=3-balls) $H_2 = h_1 \cup \cdots \cup h_s$ such that*

1991 *Mathematics Subject Classification.* 57M99.

Research supported by Grant-in-Aid for Scientific Research (C) 10640094, The Ministry of Education, Science, Sports and Culture, Japan.

(0) *genus*$(H_1) = g$,

(1) $M = H_1 \cup H_2$, *and*

(2) *each* h_i *attaches for* H_1 *at* $\partial H_1 = F$. □

We call such a $(M; H_1, H_2; F)$ a *Heegaard splitting* (or *H-splitting*) for M of genus g, and call the minimum genus of such splittings for M the *Heegaard genus* (or *H-genus*) of M and denote it by $Hg(M)$.

For an H-splitting for a closed orientable 3-manifold, Haken [4] proved the following fundamental theorem. (See Hempel [6], Jaco [7] and also Ochiai [12].):

Theorem 1.2. (Haken [4]) *If a closed orientable 3-manifold* M *with a given Heegaard splitting* $(M; H_1, H_2; F)$ *contains an essential 2-sphere, then* M *contains a 2-sphere which meets* F *in a single circle.* □

Since H_2 of a H-splitting for a bordered 3-manifold M is a set of disjoint 3-balls and so $\partial H_2 \neq F$, a Haken type theorem could not formulate for a H-splitting for M. Casson-Gordon [1] has introdued a concept of compression bodies as a generalization of handlebodies, and for a bordered 3-manifold defined a new Heegaard splitting using compression bodies, and formulated and proved a generalization of the Haken theorem.

On the other hand, in 1970 Downig [2] proved that every bordered 3-manifold can be decomposed into two homeomorphic handlebodies, and Roeling [13] discussed on these decompositions for bordered 3-manifolds with connected boundary. The purpose of the paper is to report the Downing's results [2] and Roeling's results [13] in slightly modified and generalized forms, and formulate a Haken type theorem for these decompositions in the way of Casson-Gordon [1].

2. Handlebody-splittings for bordered 3-manifolds

For a bordered 3-manifold M, let $\partial M = B_1 \cup B_2 \cup \cdots \cup B_m$, here B_i is a connected component for $i = 1, 2, \cdots, m$, and let $g_i = genus(B_i)$.

Theorem 2.1. (Downing [2]) *For every bordered 3-manifold* M*, there exist handlebodies* H_1 *and* H_2 *in* M *which satisfy the followings:*

(0) $H_1 \cong H_2$; *say,* $genus(H_1) = genus(H_2) = g$,

(1) $M = H_1 \cup H_2$,

(2) $H_1 \cap H_2 = \partial H_1 \cap \partial H_2 = F_0$ *is a connected surface,*

(3) $H_j \cap Bi = \partial H_j \cap B_i = F_{ji}$ *is a disk with* g_i *holes, and* $F_{1i} \cong F_{2i}$ $(j = 1, 2; i = 1, 2, \cdots, m)$,

(4) *the homomorphism induced from the inclusion*

$$\iota : \pi_1(F_{ji}; xi) \longrightarrow \pi_1(H_j; x_i),\ x_i \in \partial F_{ji}\ (j = 1, 2; i = 1, 2, \cdots, m)$$

is injective. □

We call such a $(M; H_1, H_2; F_0)$ a *Downing splitting* (or *D-splitting*) *for* M of genus g , and call the minimum genus of such splittings for M the *Downing genus* (or *D-genus*) of M and denote it by $Dg(M)$. By the way, Roeling [13] has pointed out that $\pi_1(F_{ji}; x_i)$ in Theorem 2.1(4) injects not only into $\pi_1(H_j; x_i)$ but also onto a free factor of $\pi_1(H_j; x_i)$, when the boundary ∂M is connected. In fact, it holds the following :

Theorem 2.2. *For every bordered 3-manifold* M*, there exists a D-splitting* $(M; H_1, H_2; F_0)$ *which satisfies the followings :*

(4) *the homomorphism induced from the inclusion*

$$\iota : \pi_1(F_{ji}; x_i) \longrightarrow \pi_1(H_j; x_i),\ x_i \in \partial F_{1i} = \partial F_{2i}\ (j = 1, 2; i = 1, 2, \cdots, m)$$

is injective, and every image $\iota\pi_1(F_{ji}; x_i)$ *is a free factor of the free group* $\pi_1(H_j; x_i)$ *of rank* g ,

(5) *there exists a tree* T *in* F_0 *connecting* $x_1, x_2, \cdots, x_m$ *such that the homomorphism induced from inclusion*

$$\iota : \pi_1(F_{j1} \cup \cdots \cup F_{jm} \cup T; x) \longrightarrow \pi_1(H_j; x), x \in T\ (j = 1, 2)$$

is injective, and the image is a free factor of $\pi_1(H_j; x)$. □

By Kaneto [8] or Zieschang [16], the conditions (4) and (5) are equivalent to the following geometric condition :

(5*) there exists a complete system of meridian-disks $\mathcal{D}_j = \{D_{j1}, \cdots, D_{jg}\}$ of H_j satisfying the following :

 (i) $D_{jk} \cap (F_{j1} \cup \cdots \cup F_{jm}) = \partial D_{jk} \cap (F_{j1} \cup \cdots \cup F_{jm})$ consists of at most one simple arc $(j = 1, 2; k = 1, 2, \cdots, m)$, and

(ii) $Cl(F_{ji} - N(D_{j1} \cup \cdots \cup D_{jg}))$ is a disk ($j = 1, 2$).

We call a D-splitting for M satisfying the conditions (4) and (5) in Theorem 2.2 or the condition (5*) a *Special Downing splitting* (or *SD-splitting*) for M of genus g, and call the minimum genus of such splittings for M the *Special Downing genus* (or *SD-genus*) of M and denote it by $SDg(M)$.

It will be noticed that for a closed, connected, orientable 3-manifold, the three splittings, an H-splitting, a D-splitting and an SD-splitting, are considered as the same one.

In order to prove Theorems 2.1 and 2.2, we need a lemma which is a generalization of Lemma 1 of Downing [2]. In proving a lemma, the notation and definitions of Downing [2] will be helpful. If g is a nonnegative integer, let Y(g) be the set of all points (x, y) in the plane R^2 which satisfy

$$x \in \{0, 1, \cdots, g\} \text{ and } -1 \leq y \leq 1; \text{ or}$$
$$0 \leq x \leq g \text{ and } |y| = 1.$$

We put

$$\begin{aligned} X(g) &= \{(x, y) \in Y(g) | y \geq 0\}, \\ \partial X(g) &= \{(x, 0) \in X(g)\}, \\ Z(g) &= \{(x, y) \in R^2 | 0 \leq x \leq g, 0 \leq y \leq 1\}. \end{aligned}$$

Let H be a handlebody with X a copy of $X(g)$ embedded as a PL subspace of H. X is said to be *proper* in H if $X \cap \partial H = \partial X$, and X is said to be *unknotted* if X is proper in H and the embedding of $X(g)$ can be extended to an embedding of $Z(g)$. Let $X_1 \cup \cdots \cup X_m$ be a copy of $X(g_1) \cup \cdots \cup X(g_m)$ properly embedded as a PL subspace of H. We say that $X_1 \cup \cdots \cup X_m$ is *unknotted* if the embedding of $X(g_1) \cup \cdots \cup X(g_m)$ can be extended to an embedding of $Z(g_1) \cup \cdots \cup Z(g_m)$.

Lemma 2.3. (Downing [2]) *Let M' ba a closed, connected orientable 3-manifold, and $(M'; W_1, W_2; F)$ be an H-splitting for M'. Let S be a 1-dimensional spine of W_1. We suppose that $Y_1 \cup \cdots \cup Y_m$ is a copy of $Y(g_1) \cup \cdots \cup Y(g_m)$ embedded in S. Then there exists an ambient isotopy $\{\eta_t\}$ of M' satisfying the following :*

(*) $\eta_1(Y_1 \cup \cdots \cup Y_m) \cap W_j = X_{j1} \cup \cdots \cup X_{jm}$ is a copy of $X(g_1) \cup \cdots \cup X(g_m)$ which is proper and unknotted in W_j for $j = 1, 2$.

Proof. The case $m = 1$ is Lemma 1 of Downing [2], and the proof of the case $m \geq 2$, which is omitted here, is the same as that of the case $m = 1$. $\square$

Proof of Theorems 2.1 and 2.2. The proof of Theorems 2.1 and 2.2 is the same as that of Theorem 1 of Downing [2], but for the future reference, we record it here.

Let V_i be a handlebody of genus g_i $(i = 1, 2, \cdots, m)$. We sew V_i into the boundary component B_i of M to form a closed, connected, orientable 3-manifold $M' = M \cup V_1 \cup \cdots \cup V_m$. Let Y_i be a copy of $Y(g_i)$ which is embedded as a 1-dimensional spine of V_i and we triangulate M' so that $Y_1 \cup \cdots \cup Y_m$ is contained in the 1-skeleton S.

Let $W_1 = N(S; M')$, a regular neighborhood of S in M', and let $W_2 = Cl(M' - W_1; M')$. Then these form an H-splitting $(M'; W_1, W_2; F)$ for M', where $F = \partial W_1 = \partial W_2$. By Lemma 2.3, there exists an ambient isotopy $\{\eta_t\}$ of M' so that

(*) $\eta_1(Y_1 \cup \cdots \cup Y_m) \cap W_j = X_{j1} \cup \cdots \cup X_{jm}$ is a copy of $X(g_1) \cup \cdots \cup X(g_m)$ which is proper and unknotted in W_j $(j = 1, 2)$.

We put

$$N = N(\eta_1(Y_1 \cup \cdots \cup Y_m); M'),$$
$$N_1 = N(X_{11} \cup \cdots \cup X_{1m}; W_1),\ N_2 = N(X_{21} \cup \cdots \cup X_{2m}; W_2).$$

Then, $N = N_1 \cup N_2$, and $Cl(M' - N)$ is homeomorphic to M because $\{\eta_t\}$ is an ambient isotopy. From the unknotted condition (*),

$$H_1 = Cl(W_1 - N_1),\ H_2 = Cl(W_2 - N_2)$$

are homeomorphic handlebodies decomposing $Cl(M' - N) = M$, and it is easily checked that this splitting satisfies the conditions (1)$\sim$(5) in Theorem, completing the proof. $\square$

3. Remarks on genera of bordered 3-manifolds

From the definitions and the proof of Theorems 2.1 and 2.2, we know :

Proposition 3.1. *For every bordered 3-manifold M, it holds the following :*

(1) $SDg(M) \geq Dg(M)$.

(2) $SDg(M) \geq g_1 + \cdots + g_m =$ *the total genus of* ∂M.

The following theorem has proved by Roeling [13] when $m = 1$, and the proof of general case is the same as that of $m = 1$ under the condition (5*).

Theorem 3.2. (Roeling [13, Theorem 1]) *If a bordered 3-manifold M has an SD-splitting $(M; H_1, H_2; F_0)$ of genus g , then M has an H-splitting of genus g. Thus, for every bordered 3-manifold M, it holds that*

$$Hg(M) \leq SDg(M) \ \square$$

Theorem 3.3. (Roeling [13, Theorem 2]) *If a bordered 3-manifold M with connected boundary has an H-splitting $(M; H_1, H_2; F)$ of genus g , then M has a D-splitting of genus g . Thus, for every bordered 3-manifold M with connected boundary, it holds that :*

$$Dg(M) \leq Hg(M) \leq SDg(M). \ \square$$

Closed 3-manifolds of H-genus 0 are chracteralized as the 3-dimensional sphere S^3. Corresponding to this fact, it holds the following :

Proposition 3.4. *Let M be a bordered 3-manifold with m boundary components.*

$$\begin{aligned} SDg(M) = 0 \quad &\Longleftrightarrow \quad Hg(M) = 0 \\ &\Longleftrightarrow \quad M = S^3 \text{ with } M \text{ holes} \\ &\Longleftrightarrow \quad M \text{ is the connected sum of } m \text{ copies of the 3-ball } D^3. \end{aligned}$$

4. Haken Type Theorem (1)

A 2-sphere in a 3-manifold M is *essential* if it does not bound a 3-ball in M. A 3-manifold M is *irreducible* if it contains no essential 2-sphere.

The following corresponds to the Haken Theorem 1.2.

Theorem 4.1. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for a bordered 3-manifold M. If there exists an essential 2-sphere in M, then there exists an essential 2-sphere Σ in M such that $\Sigma \cap F_0$ consists of a single loop.*

Proof. We will give a mild generalization of this theorem in Theorem 4.3 below, and so we will not include a proof of Theorem 4.1, but simply refer the reader to Jaco's account of Haken's proof [4, Chapter II] or the proof of Theorem 4.3 below. $\square$

Corollary 4.2. *Suppose that a bordered 3-manifold M has a decomposition*

$$M = M_1 \# \cdots \# M_u$$

(as a connected-sum; see Hempel [6]). Then it holds that :

$$SDg(M) = SDg(M_1) + \cdots + SDg(M_u). \ \square$$

Let F_0 be a compact orientable surface, and let $\mathcal{J}_1$ and $\mathcal{J}_2$ be proper 1-dimensional submanifolds in F_0. We shall say that $\mathcal{J}_1$ and $\mathcal{J}_2$ are in *reduced position*, if $\mathcal{J}_1 \cap \mathcal{J}_2$ consists of a finite number of points crossing one another, and there is no disk on F_0 whose boundary consists of an arc in $\mathcal{J}_1$ and an arc in $\mathcal{J}_2$.

Let M be a bordered 3-manifold and let $(M; H_1, H_2; F_0)$ be an SD-splitting for M. We call the complete systems of meridian-disks $\mathcal{D}_1$ of H_1 and $\mathcal{D}_2$ of H_2 which satisfy the condition (5*) a special complete systems of meridian-disks. These special complete systems of meridian-disks $\mathcal{D}_1$ of H_1 and $\mathcal{D}_2$ of H_2 are said to be *irreducible* if $\mathcal{J}_1 = \mathcal{D}_1 \cap F_0$ and $\mathcal{J}_2 = \mathcal{D}_2 \cap F_0$ are in reduced position in F_0.

Theorem 4.3. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for a bordered 3-manifold M, and let $\mathcal{D}_j = \{D_{j1}, \cdots, D_{jg}\}$ be a special complete system of meridian-disks of H_j $(j = 1, 2)$, and we suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are irreducible. Let Σ be a disjoint union of essential 2-spheres in M. Then there exist a disjoint union of essential 2-spheres Σ^* and a complete system of meridian-disks $\mathcal{D}_2^*$ of H_2 such that*

(1) *Σ^* is obtained from Σ by ambient 1-surgery and isotopy,*
(2) *each component of Σ^* meets F_0 in a single loop,*
(3) *$\mathcal{D}_1 \cap \Sigma^* = \emptyset$, $\mathcal{D}_2^* \cap \Sigma^* = \emptyset$, and $\mathcal{D}_2^* \cap (F_{j1} \cup \cdots \cup F_{jm}) = \mathcal{D}_2 \cap (F_{j1} \cup \cdots \cup F_{jm})$, where F_{ji} is the planar surface $\partial H_j \cap B_i$, B_i a connected component of ∂M.*

Proof. We choose a 1-dimensional spine S_{2i} of the planar surface F_{ji} so that S_{2i} consists of simple loops based at the point x_i and each loop intersectd with D_2 at a single point $(i = 1, 2, \cdots, m)$. Then we can choose a 1-dimensional spine S_2 of H_2 so that $S_2 \cap D_{2i}$ consists of a single point $(i = 1, 2, \cdots, m)$ and $S_2 \cap \partial H_2 = S_{21} \cup \cdots \cup S_{2m}$. We may suppose that S_2 intersects transversally with Σ at a finite number of points. Since H_2 is a regular neighborhood of S_2 , we may assume that Σ intersects with H_2 at a finite number of disks, say $\sigma_1, \cdots, \sigma_n$.

Let $\Sigma_0 = Cl(\Sigma - (\sigma_1 \cup \cdots \cup \sigma_n); \Sigma)$. Then $\Sigma_0 \cap (D_{11} \cup \cdots \cup D_{1g})$ consists of a finite number of simple loops and proper arcs. Since H_1 is irreducible, we can remove all simple loops by cut-and-paste, and so we may assume that $\Sigma_0 \cap (D_{11} \cup \cdots \cup D_{1g})$ consists of a finite number of proper arcs, say $\alpha_1, \cdots, \alpha_k$. Since $\Sigma_0 \cap F_{1j} = \emptyset$ for $i = 1, 2, \cdots, m$, we can choose an innermost arc , say α_1, on one of $D_{11}, \cdots, D_{1g}$, say D_{11}, if $\Sigma_0 \cap (D_{11} \cup \cdots \cup D_{1g}) \neq \emptyset$. Let $\Delta \subset D_{11}$ be the disk cut off by α_1 so that

$$\Delta \cap \Sigma_0 = \partial\Delta \cap \Sigma_0 = \alpha_1, \ \Delta \cap (F_{11} \cup \cdots \cup F_{1m}) = \emptyset.$$

Now, we may apply the same argument as that of Jaco [7, 7~ 9]; that is, we can deform Σ along Δ (by isotopy of type A) so that new Σ^* does not meet at α_1. By the repetition of the procedure, we can get rid of all intersections $\alpha_1, \cdots, \alpha_k$ of $\Sigma^* \cap \mathcal{D}_1$. Now, it is easy to see that the new Σ^* satisfies the conditions (1), (2) and (3) $\mathcal{D}_1 \cap \Sigma^* = \emptyset$.

Since $H_2 \cap \Sigma^*$ consists of a finite number of disks and $\Sigma^* \cap (F_{21} \cup \cdots \cup F_{2m}) = \emptyset$, we can choose, if necessary, a complete system of meridian-disks $\mathcal{D}_2^*$ of H_2 so that $\mathcal{D}_2^*$ satisfies the other conditions in (3), and completing the proof. □

5. Haken type theorem (2)

A proper disk in a bordered 3-manifold M is said to be *essential* if it does not cut off a 3-ball from M. Using essential disks, Gross [3] and Swarup [15] have formulated another prime decomposition theorem under the boundary connected sum (= disk sum) for a bordered 3-manifold.

Now the following question immediately come to mind :

Question and Example 5.1. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for a bordered 3-manifold M. If there exists an essential proper disk in M, then does there exist an essential proper disk Δ in M such that $\Delta \cap F_0$ consists of a single arc ?*

The answer is NO in general. The following counter example is due to Dr Kanji Morimoto. Let K be a simple loop on the boundary $S^1 \times S^1$ of the solid torus $D^2 \times S^1$ such that $K \cap D^2 = K \cap \partial D^2$ consists of two crossing points, where D^2 is a standard meridian-disk of $D^2 \times S^1$. Let $J \subset D^2$ be a simple proper arc joining the two points. Let $H_1 = N(K \cup J; D^2 \times S^1)$, and $H_2 = Cl(D^2 \times S^1 - H_1; D^2 \times S^1)$. Then we have an SD-splitting $(D^2 \times S^1; H_1, H_2; F_0)$ for $D^2 \times S^1$ of genus 2, where F_0 is the surface $Cl(\partial H_1 \cap Int(D^2 \times S^1); D^2 \times S^1)$. The meridian-disk D^2 is an essential proper disk in $D^2 \times S^1$ which is unique up to ambient isotopy of $D^2 \times S^1$, and $D^2 \cap F_0$ consists of two arcs. It will be noticed that $D^2 \times S^1$ has an SD-splitting of genus 1, and the above splitting is of genus 2. $\square$

Proposition 5.2. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for a bordered 3-manifold M. If there exists an essential 2-sphere in M which is not boundary parallel, then there exists an essential proper disk Δ in M such that $\Delta \cap F_0$ consists of a single arc.*

Proof. By Theorem 4.1 (or 4.3), we have an essential 2-sphere Σ in M such that $\Sigma \cap F_0$ consists of a single loop. Using this Σ, we can easily obtain a required essential disk Δ. $\square$

The following lemma corresponds to Theorem 4.3.

Lemma 5.3. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for an irreducible bordered 3-manifold M. If there exists an essential proper disk in M, then there exist an essential proper disk Δ in M and a special complete system of meridian-disks $\mathcal{D}_j = \{D_{j1}, \cdots, D_{jg}\}$ of H_j $(j = 1, 2)$ satisfying the followings :*

(1) $\Delta \cap F_0$ *consists of a finite number of proper arcs,*

(2) $\Delta \cap H_j$ *consists of a finite number of proper disks, and each component is essential in* H_j $(j = 1, 2)$, *and*

(3) $\Delta \cap D_2 = \emptyset$.

Proof. We choose a 1-dimensional spine S_2 of H_2 as the same way as that of the proof of Theorem 4.3. Then, we may consider H_2 as a regular neighborhood of S_2.

Let $\square$ be an essential proper disk in M. We may assume that $\square$ intersects with S_2 transversally in a finite number of points, and so $\square \cap H_2$ consists of a finite number of proper disks, which are regular neighborhoods of $\square \cap S_2$ in $\square$. Now $\square \cap F_0$ consists of a finite number of proper arcs and loops. We can remove the loops by the same way as that of the proof of Theorem 4.3 (cf. Jaco [7]), and let Δ be the new disk. It is easy to see that Δ satisfies the conditions (1) and (2). If we cut H_2 along Δ, then we have some handlebodies, and so we can choose a complete system of meridian-disks D_2 of H_2 with the condition (3), and completing the proof. $\square$

Using this Lemma, we can prove the following :

Proposition 5.4. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for an irreducible bordered 3-manifold M with connected boundary B of genus g. If there exists an essential proper disk in M and $SDg(M) = g$, then there exists an essential proper disk Δ in M such that $\Delta \cap F_0$ consists of a single arc.*

Proof. Let $\Delta \subset M$ be an essential proper disk, and $\mathcal{D}_j$ be a special complete system of meridian-disks of H_j $(j = 1, 2)$ such that these satisfy the conditions of Lemma 5.3. We cut H_j along $\mathcal{D}_j$; we have a 3-ball D^3_j. On the boundary ∂D^3_j, F_{j1} apears as a disk from the condition (5*)-(ii). Using Δ we construct a required disk by the condition (3). The proof is not so hard but fairly complicated, and we omit here. □

As a corollary to this Proposition, we have the following characterization of handlebodies by SD-splittings.

Corollary 5.5. *Let M be an irreducible bordered 3-manifold with connected boundary B of genus g, and we suppose that M contains an essential proper disk. Then it holds that :*

$$SDg(M) = g \iff M \text{ is a handlebody of genus } g.$$

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