非線形動学の方法による経済変動の研究

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１． 研究の目的

本研究の目的は、非線形動学の手法を用いて経済変動を分析することにある。国民所得や株価などの様々な経済データが時間を通じて複雑な挙動を示すことは経験的にもよく知られている。従来、このような経済現象を定式化・分析するにあたっては、以下に述べる二つの方法のいずれかが用いられてきた。

第一の方法は、現象の主要部分を取り扱いの容易な線形モデルとして定式化し、それだけでは説明できない複雑変動部分は単なるノイズとして取り扱うというものである。第一次近似としてはこの方法で十分なことも多い。しかし実際にこの方法を用いて分析を行うと、説明できない残差であるはずのノイズ部分が、説明されるべき経済変動の大部分を占めてしまう状況がしばしば生じ得る。言い換えれば、線形モデルの説明力は場合によっては非常に低くなるのである。この欠点を補うために現れたのが、非線形動学の方法である。この第二の分析手法は、複雑な経済変動を、単なるノイズではなく経済システムの構造が産み出すより本質的な現象として捉えるというものである。すなわち、ノイズがまったく存在しない状況であっても、経済システムの持つ非線形性によって複雑なダイナミクスが産み出されると考えるのである。分析のための数学的手法としては、主として決定論的カオス理論が用いられる。

本研究では、主に後者の方法を採用する。経済システムの様々な局面（在庫循環、調整過程、人口変動 など）に内包される非線形性をモデル化し、そこから極めて複雑な変動が生じうることを示す。また、本研究独自の試みとして、非線形動学にノイズを付加した経済モデルの分析を行う。これは前述の二つの方法のいずれにも当て嵌まらない第三の方法というべきものであり、これによって経済変動におけるノイズの新たな役割を明らかにすることが可能である。

２． 研究内容

本研究の内容は、具体的には以下のとおりである。

(i) 決定論的な（ノイズを含まない）一次元力学系理論を用いて、在庫循環、クモの巣型調整過程、人口と資源賦存の変動、経済利得の外部性といった様々な要因が引き起こす経済変動を、個々に定式化し解析する。一次元カオスについては既にかなり多くのことが知られており、そのためそれらを応用した経済モデルについても詳細な分析が可能である。

(ii) 二次元以上の高次元動学を応用した経済モデルを分析する。これらの高次元力学系については、二次元については研究が比較的進展しつつあるものの、三次元以上では未だ良く分かっていない部分が多い。その一方で、例えば二国からなる国際マクロモデルの分析では五次元という高い次元が必要となる。そのため、三次元を越える高次元モデルの分析には、主として計算機による数値シミ

ュレーションを用いることになる。さらに発展的研究として、非線形構造に確率ノイズを付加したモデルを分析する。経済系は孤立したシステムではなく、社会の他のサブシステムや他国との不断の相互干渉にさらされている。このような他のシステムなどからの影響を、確率的ノイズをはじめとした外力(強制振動モデル)や他の形態を導入することによって、動学的性質が如何に変化するかを調べる。

(iii) 一般的に、非線形確率微分方程式で記述される系を陽に解くことは困難であり、したがってそれを分析するためには何らかの数値的・近似的手法が必要となる。新たな数値的。近似的手法の開発を試みる。

3. 研究成果

前項で述べた研究内容に対して、以下のような結果が得られた。

(i) 一次元非線形差分方程式モデル

生産量調整の遅れのような経済行動のラグが、複雑な変動を生み出す可能性があることを示した(添付資料①, ②)。これは「くもの巣調整過程モデル」と呼ばれる良く知られたモデルを発展させたものである。従来の線形モデルでは比較的簡単な周期変動しか取り扱えなかったのに対し、モデルに非線形性を積極的に取り込むことによりさらに複雑な挙動を説明できるようになった。また、複雑な変動が経済厚生に与える影響を分析した(添付資料③)。

(ii) 高次元非線形方程式モデルおよびノイズを付加した高次元モデル

まず、数学的に厳密な取り扱いが比較的容易な二次元モデルについて数理的な考察を行った。動学モデルは、次元を問わずに大別して、連続時間上で定義されるものと離散時間上のものに分けられる。連続時間上の二次元モデルでは、ホップ分岐定理とポアンカレ・ベンディクソン定理の両者を援用することで、カルドアモデルやベナシーモデルといった既存モデルを、新たな視点から再解釈する方法を与えた(添付資料④)。また離散時間上の二次元モデルでは、政府介入の効果を世代重複モデルによって分析し、馬蹄写像を用いてカオスの存在を証明した(添付資料⑤)。

つぎに、三次元カルドア型の小国開放モデルを用いて景気循環の分析を行い、それにより得られた知見をもとにして、ノイズを付加することによってモデルの性質がどのように変わるかを調べた(添付資料⑥、⑦、⑧、⑨)。この結果、ノイズを付加することは必ずしもモデルの動学的性質を不明瞭にするわけではなく、逆に背後に隠された構造を顕在化させ得ることを示した。このことは以下のことを意味する。ノイズがない場合、好況・不況を繰り返す状態にあったとしても、海外との資金移動の容易さの程度を表すパラメータがノイズによって揺らぐとき、好況の状態が相当期間持続し得る。また、変動相場制の下で、ノイズがない場合為替レートが激しく変動していても、ノイズが付加されることで為替レートの変動幅が縮小される。

さらに今までは外性的に扱っていた外国の存在を内性化し、モデルを五次元の２国モデルに拡張、分析した（添付資料⑨）。その結果、２国間の景気循環の同期・非同期の関係は、一方の国の調整速度パラメータに鋭敏に依存することがわかった。

(iii)　一般的に、非線形確率微分方程式で記述される系を陽に解くことは困難であり、したがってそれを分析するためには何らかの数値的・近似的手法が必要となる。従来の確率系の近似法にはなかった、系の持つ動学的特性を保つ新しい数値近似法を提唱した（添付資料⑩、⑪）。特にこの手法は、コンピュテーショナル・ファイナンスで扱われる累乗型ボラティリティを持つ非線形確率系の数値近似に有効であることを検証した。

(iv)　上記 (i)、(ii)、(iii)とは異なるモデルに関しても検討した（添付資料⑫、⑬、⑭、⑮）。企業を情報処理システムとしてとらえ、構造変化などの外部環境の変化によって組織の効率性が変化し、そのことが組織変更を引き起こすことを実証的に分析した(⑫)。株価収益率の分布が、正規分布に比べて厚い裾野を持つことは広く知られている。視野の限定された多数のエージェントが局所的に相互作用するモデルを考え、それによって株価収益率分布の厚い裾野が再現されることを示した。モデル化に際しては、統計力学の手法であるパーコレーション理論が用いられた(⑮)。

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(ii) 高次元非線形方程式モデルおよびノイズを付加した高次元モデル

CHAPTER 12

Ergodic chaos in a piecewise linear cobweb model

Akio Matsumoto

The emergence of complex dynamics in a cobweb model augmented with upper and lower bounds for output variations is demonstrated. The purpose is to consider the implications of the output constraints on the dynamic behavior of an agricultural economy.

The traditional cobweb model, which has monotonic specifications of demand and supply and naive or adaptive expectations formation, can produce only three types of dynamics: convergence to an equilibrium, convergence to period-2 cycles, or divergence. None of these types, however, is satisfactory to explain the irregular and asymmetric fluctuations of agricultural goods markets.

To overcome those limitations, the literature on nonlinear cobweb dynamics has been expanding with the help of new developments in nonlinear dynamics.[1] Several stability results have been established that show the existence of chaotic fluctuations as well as the convergence to stable periodic cycles. The literature fall into two groups. In the first, we have endogenous nonlinear cobweb models in which the supply-and/or-demand curves are nonlinear (see Jensen and Urban 1984, Chiarella 1988, Finkenstädt and Kuhbier 1992, and Hommes 1994). By nonlinear behavioral assumptions, transition maps in the first group are more or less similar to the logistic map that is able to give rise to complex dynamics involving chaos. In the second, we find a linear cobweb model with a upper bound for variations of output. Owing to the upper-quantity constraint, the transition map in the second ground is also nonlinear (or, more precisely, piecewise linear) in spite of the linear behavioral specifications. In particular, it is similar to the tent map that is able to generate complex dynamics. The upper bound not only prevents the price (or quantity) dynamics from explosive oscillations but also works to generate persistent irregular fluctuations (see Cugno and Montrucchio 1980, Nusse and Hommes 1990, and Huang 1995). However,

[1] The traditional cobweb model has been extended in other directions in which the effects of several production time lags or the effects of exogenous stochastic factors are analyzed.

the piecewise linear model with the upper bound has a possibility of almost all trajectories escaping from the domain of the transition map for some parameter constellations. In such a case, trajectories are negatively unbounded and thus the model is unable to track the price–quantity evolution in an economically meaningful region. Not much has yet been revealed with respect to a device that makes output trajectories remain nonnegative. The purpose of this chapter is to demonstrate the possibility of economically meaningful complex dynamics for the case in which a linear cobweb model augments the lower bound for variations of output in addition to the upper bound. It is intuitively clear that unstable trajectories bounce back to a bounded region and keep fluctuating within it. However, it has not yet been investigated whether complex dynamics may appear in the case in which output variations are bounded from above and below. This chapter demonstrates, through numerical examples, that the linear cobweb model augmented with the upper and the lower bounds can generate a wide spectrum of dynamic behavior ranging from stable periodic cycles to ergodic chaos.

This chapter is organized as follows. In Section 1 the linear cobweb model is set up with upper and lower bounds for output variations. In Section 2 the model is simulated to explore the relations between the output constraints and the output dynamics. Concluding remarks are made in Section 4.

1 Piecewise linear cobweb model

The simplest version of the traditional cobweb model is made up of the following four equations in discrete time:

$$\begin{aligned} q_t^d &= D(p_t), \text{ demand,} \\ q_t^s &= S(p_t^e), \text{ supply,} \\ q_t &= q_t^d = q_t^s, \text{ temporary equilibrium,} \\ p_t^e &= p_{t-1}, \text{ naive expectation.} \end{aligned} \tag{12.1}$$

This model can be reduced to a one-dimensional difference equation of output or price:

$$q_{t+1} = S[D^{-1}(q_t)] \quad \text{or} \quad p_{t+1} = D^{-1}[S(p_t)]. \tag{12.2}$$

In the simple version, the demand function as well as the supply function is assumed to be monotonic and thus the composite map, $S[D^{-1}(q_t)]$ or $D^{-1}[S(p_t)]$, is also monotonic. Its slope evaluated at the equilibrium point characterizes dynamics, as it equals an eigenvalue of the dynamic equation. As long as the slope lies between 0° and −45°, the equilibrium point is stable. The stable trajectories of price or quantity converge to a stationary state, which does not go with persistent fluctuations observed in the real world. As the slope steepens beyond -45°, the equilibrium point is unstable. The unstable trajectories explosively oscillate, which also contradicts the actual dynamic behavior. When the slope is equal to -45°, period-2 cycles can appear. However, the nature of regular cycles is unlike the irregular nature of the actual cycles. Thus such a simple cobweb model has difficulties in explaining cyclical and erratic movements observed in statistical data of the agricultural good (e.g., see the data provided by Finkenstädt and Kuhbier 1992).

By using linear supply-and-demand functions, Cugno and Montrucchio (1984) modified the traditional cobweb model by introducing an upper bound a imposed on the positive growth rate of q_t:

$$
\begin{aligned}
&S(p_{t-1}) = -a + bp_{t-1}, \quad a > 0, b > 0, \\
&D^{-1}(q_t) = c - dq_t, \quad c > 0, d > 0, \\
&q_{t+1} \leq q^U_{t+1} = (1+\alpha)q_t, \quad \alpha > 0,
\end{aligned}
\tag{12.3}
$$

where the last device is found in Day[2] (1980, 1994). Substituting the second equation of Eqs. (12.3) into the first and taking account of the third equation yield the following piecewise linear transition map:

$$
q_{t+1} = \max[(1+a)q_t, bc - a - bdq_t]. \tag{12.4}
$$

It is a tent map in which a maximizer is

$$
Q_M = \frac{bc - a}{1 + a + bd}. \tag{12.5}
$$

Constructing this piecewise linear model, Cugno and Montrucchio (1984) clarify the following conditions under which the transition map can give rise to chaotic dynamics:

$$
\frac{1+a}{a} \geq bd \geq \frac{2+a}{1+a}. \tag{12.6}
$$

In condition (12.6), the first condition guarantees trajectories to be nonnegative and the second condition with equality guarantees Q_M to be a period-3 point. By the Li–Yorke chaos theorem, the period-3 point implies that transition map (12.4) satisfying condition (12.6) generates the persistent and irregular (i.e., chaotic) motions of output. Furthermore, recognizing the fact that transition map (12.4) is expansive (i.e., the slope in the absolute value is greater than 1) and unimodal in an unstable economy where $bd > 1$, Nusse and Hommes (1990) replace condition (12.6) with a weaker condition,

$$
\frac{1+a}{a} \geq bd > 1, \tag{12.7}
$$

[2] Day (1980, p. 197) considers the symmetric upper and lower constraints.

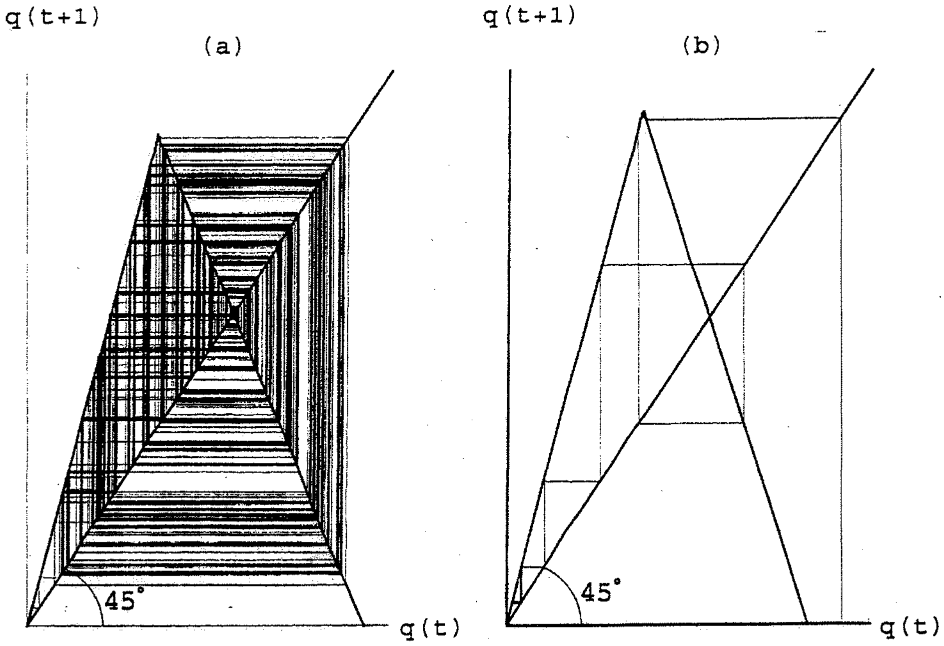

Figure 12.1. Examples of chaotic trajectory and escaping trajectory.

and demonstrate that transition map (12.4) generates the Li–Yorke chaos as well as the "sensitive dependence on initial values" chaos.

There is a possibility that a set of parameters may violate the first inequality condition of condition (12.6) or (12.7). In such a case, almost all trajectories (with respect to the Lebesgue measure) eventually escape from a economically meaningful interval between 0 and $[(bc - a)/(bd)]$ and fall to zero if a nonnegative constraint is implicitly assumed or go to negative infinity if not. Figures 12.1(a) and 12.1(b) are graphical representations of output trajectories generated by transition map (12.4). In those simulations, we start with the same initial value of q_t and set it to be 0.05. We take a and $bc - a$ to be 1.5 and 5, respectively. The simulations are performed under two different values of bd: bd is taken to be $20/13 \simeq 1.54$ in Fig. 12.1(a) and to be 2 in Fig. 12.1(b). Because $[(1 + a)/a] = 5/3 \simeq 1.67$, it can be checked that the former value of bd satisfies condition (7) but the latter violates it. In consequence, as those figures suggested, the trajectory in Fig. 12.1(a) is chaotically oscillating within the bounded interval and that in Fig. 12.1(b) escapes from the domain of the transition map.

To get rid of such economically meaningless dynamics, we modify the Cugno–Montrucchio version of the cobweb model by introducing the lower bound for variations of output. If we observe the real economic world, it is not surprising that a competitive firm prevents output tomorrow from changing drastically from output today, taking into account various constraints such as capacity constraints, financial constraints, and cautious response to demand uncertainty.[3] Having recognized this fact, we impose a lower bound β on the negative growth rate of output,

$$q_{t+1} \geq q_t^L = (1-\beta)q_{t-1}, \quad 1 > \beta > 0, \tag{12.8}$$

to the Cugno–Montrucchio model given by Eqs. (12.3). The lower-bound constraint, analogous to the upper-bound constraint, has the effect of preventing output in period t from decreasing by more than $100\beta\%$ from the output of period $t-1$.

The lower bound and the upper bound work to compress trajectories into a cone spanned by $q_{t+1}^U = (1+\alpha)q_t$ and $q_t^L = (1-\beta)q_{t-1}$ in the phase space. In consequence, the transition map becomes a map with three line segments and two kinked points,

$$f(q_t) = \begin{cases} (1+\alpha)q_t & \text{for } q_t \leq Q_M \\ bc - a - bdq_t & \text{for } Q_M \leq q_t \leq Q_m, \\ (1-\beta)q_t & \text{for } q_t \geq Q_m \end{cases} \tag{12.9}$$

where a minimizer Q_m is calculated as

$$Q_m = \frac{bc-a}{1-\beta+bd}. \tag{12.10}$$

In a compact form, transition map (12.9) is

$$f(q_t) = \min\{(1-\beta)q_t, \max[(1+\alpha)q_t, bc-a-bdq_t]\}.$$

Under the assumptions of positive parameters, $bd > 0, bc-a > 0, \alpha > 0$, and $\beta > 0$, transition map (12.9) has the tilted-z profile and its nonlinearity becomes more pronounced when bd gets larger. Let q^* be a stationary state satisfying $q^* = f(q^*)$ {i.e., $q^* = [(bc-a)/(1+bd)]$}. If $bd > 1$, it is oscillatory unstable but fluctuations are bounded by the upper and the lower constraints of output. Three cases can be distinguished, depending on the relative magnitudes between α and β: (1) $(1+\alpha)(1-\beta) = 1$; (2) $(1+\alpha)(1-\beta) < 1$; and (3) $(1+\alpha)(1-\beta) > 1$. The typical profiles of the transition map under conditions (1), (2), and (3) are depicted in Figs. 12.2(a), 12.2(b), and 12.2(c), respectively, where the horizontal axis is q_t and the vertical axis is q_{t+1}. We call condition (1) the symmetric condition as the upper-bound locus and the lower-bound locus deviated from the 45° line by the same degree – the former positively and the latter negatively. We call condition (2) the upper asymmetric condition as the upper-bound locus is deviated from the 45° less than the lower-bound locus, and we call condition (3) the lower asymmetric condition as the lower-bound locus is deviated less than the upper-bound locus. By simulating the model

[3] Huang (1995) presumes that those constraints limit the positive growth rate of output. We assume that those constraints also can work in the opposite direction to limit the negative growth rate.

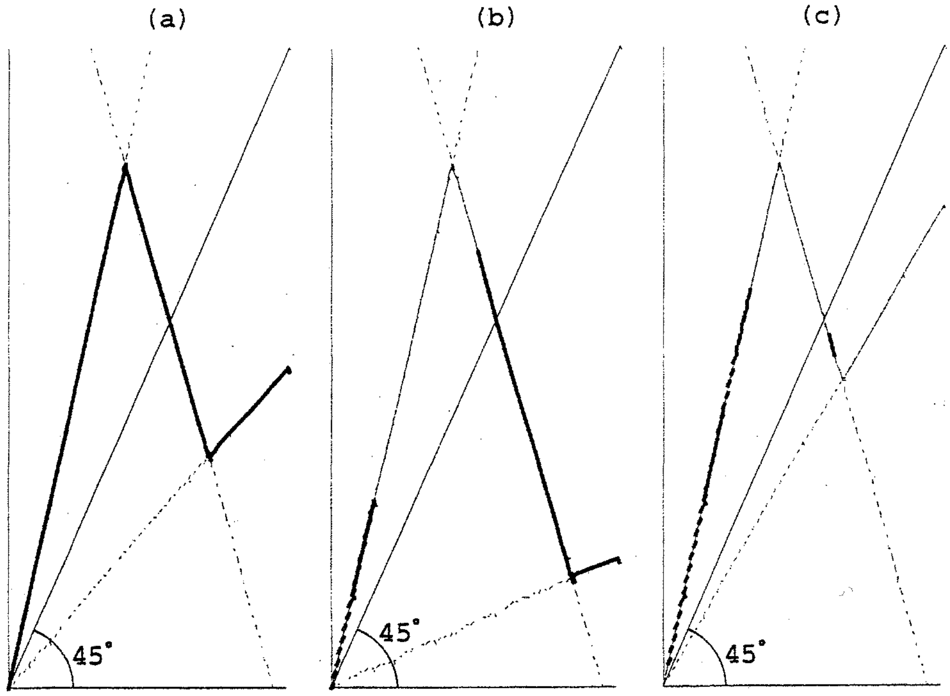

Figure 12.2. Three profiles of the transition map $f(q_t) = \min\{(1-\beta)q_t, \max[(1+a)q_t, bc - a - bdq_t]\}$.

under the different values of $bd(>1)$, we demonstrate that piecewise linear cobweb model (12.9) can generate various dynamics under each of these three parameter constellations for α and β.

2 Simulation of the model

To explore the dynamic behavior of output q, we simulate the model under the different values of bd, in which b and $1/d$ reflect the slopes of supply and demand curves, respectively. In bifurcation diagrams below, bd is taken to be a bifurcation parameter and the calculations are done under different values of α and β.

Symmetric case

The first simulation is shown in Fig. 12.3. It is performed with $\alpha = 1$ and $\beta = 0.5$, which, as verified, satisfy the symmetric-constraint condition, $(1+\alpha)(1-\beta) \doteq 1$. The inverse of the bifurcation parameter, $1/bd$, is varied in decrements of 0.02 from 1 to 0. For each value of $1/bd, f(q_t)$ is iterated 400 times. Although the last 300 iterates are plotted on the vertical axis, only two points are shown in the bifurcation diagram. Thus Fig. 12.3 suggests that the symmetric-constrained cobweb model can give rise to period-2 cycles. These results are verified as follows. In Fig. 12.4, the dashed line is a graph of $f(q)$,

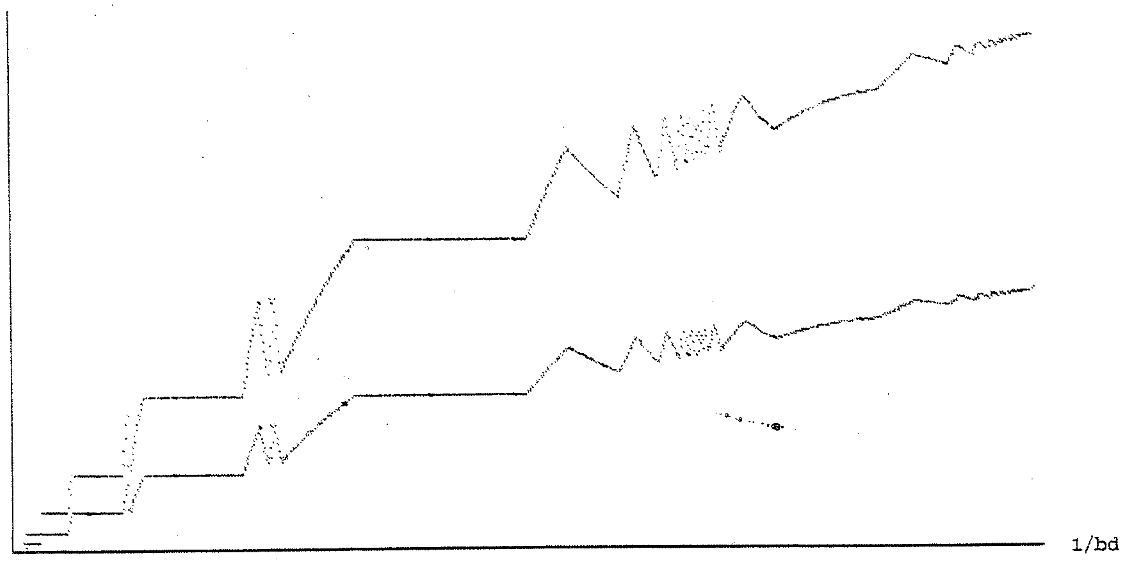

Figure 12.3. Bifurcation diagram in the symmetric case $(1+\alpha)(1-\beta)=1$.

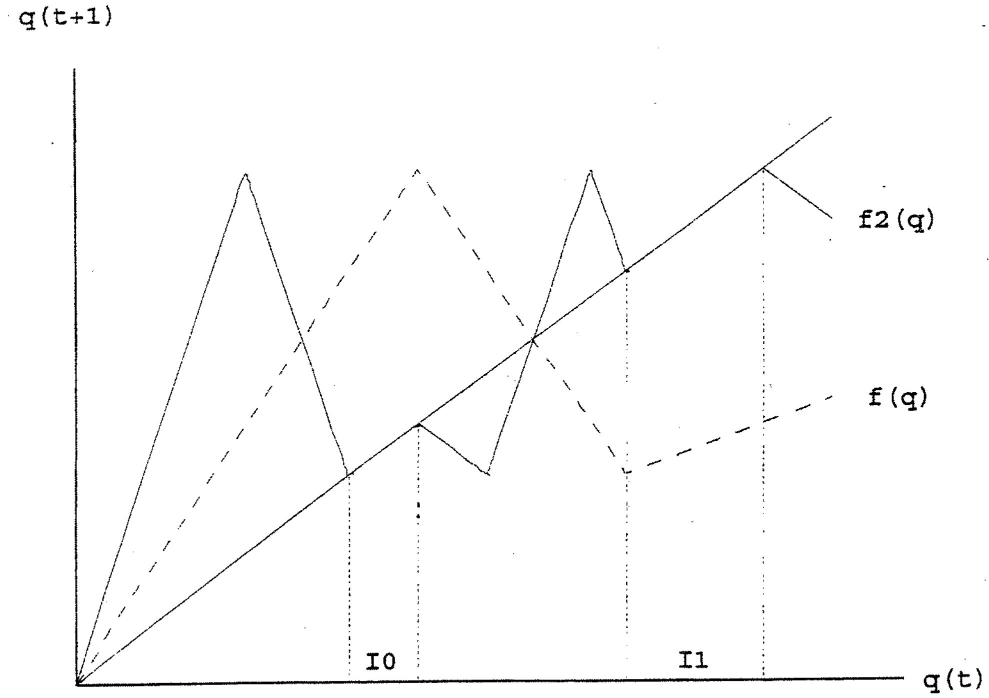

Figure 12.4. Graphs of $f(q)$ and $f^2(q) := f[f(q)]$.

and the solid straight line is a graph of $f^2(q) := f[f(q)]$, the second iteration of $f(q)$. We denote a local minimum by $Q_{\min} := f(Q_m)$ and a local maximum by $Q_{\max} := f(Q_M)$. Because $Q_{\min} < Q_M < Q_m < Q_{\max}$ holds in the symmetric case, an interval $I := [Q_{\min}, Q_{\max}]$ is the trapping interval, that is, any output trajectories starting outside the interval enter it within finite iterations and one starting inside remains there. As can be seen, the graph of $f^2(q)$ has two parts that are identical to the 45° line. This means that the corresponding two subintervals of I, $I_0 := [Q_{\min}, Q_M]$ and $I_1 := [Q_m, Q_{\max}]$ are sets of fixed points of $f^2(q)$ or sets of periodic points with period 2 of $f(q)$ [i.e., $f(I_0) = I_1$, $f(I_1) = I_0$.] Every trajectory emanating from an interval $\overline{I_0 \cup I_1 \cup \{q^*\}}$ enters into $I_0 \cup I_1$ after finite iterations. Thus we have stable period-2 cycles in the symmetric case.

Upper asymmetric case

Because the transition map, $f(q)$, has the upper and the lower bounds, it induces any trajectories, which are repelled by the unstable equilibrium, to bounce back to a vicinity of the equilibrium point. Thus we can define a trapping interval by an interval that eventually traps all trajectories. A restriction of $f(q)$ to the trapping interval governs the asymptotic behavior of q. Two distinct trapping intervals can be identified, which depends on the relation between the maximum $Q_{\max}$ and the minimizer Q_m: one interval is defined when $Q_{\max} \leq Q_m$, and the other is defined when $Q_{\max} > Q_m$. Rewriting the upper asymmetric condition as $[\alpha/(1+\alpha)] < \beta$, we can define $(bd)^U$ by $(bd)^U := \beta[(1+\alpha)/\alpha] > 1$. It can be checked that $Q_{\max} \leq Q_m$ holds for $1 < bd \leq (bd)^U$ and $Q_{\max} > Q_m$ for $bd > (bd)^U$.

Let $V_1 := [f(Q_{\max}), Q_{\max}]$; It is the trapping interval for $1 < bd \leq (bd)^U$. Consequently the restriction of $f(q_t)$ to V_1,

$$f|_{V_1}(q_t) = \min[(1+\alpha)q_t, bc - a - bdq_t], \tag{12.11}$$

generates trajectories that are eventually confined in V_1. It is an asymmetric tent map that is essentially the same as (12.2). Because it is expansive and unimodal, the restricted map $f|_{V_1}(q_t)$ can generate ergodic chaos.[4] Furthermore, we can obtain a complete characterization of transition map (12.9) by applying the results of Ito et al. (1979) and Day and Shafer (1987). A trajectory generated by $f|_{V_1}(q_t)$ is depicted in Fig. 12.5(a) in which bd is taken to be 1.2 and the bold line on the $q(t)$ axis is the trapping interval V_1. It can be seen that a trajectory keeps aperiodically oscillating around the equilibrium point within the trapping interval. Let $V_2 := [Q_{\min}, Q_{\max}]$, which is the trapping interval for $bd > (bd)^U$. Because V_2 contains two kinked points, Q_M and Q_m, a restriction of $f(q_t)$ to V_2,

$$f|_{V_2}(q_t) = \max\{(1-\beta)q_t, \min[(1+\alpha)q_t, bc - a - bdq_t]\} \tag{12.12}$$

takes on a tilted-z shape. Day and Schafer (1987) also make some characterizations for such a map with three line segments and two turning points.

[4] See Eq. (3) in property 5 of Nusse and Hommes (1990, p. 13). Also see theorem 3 of Day and Schafer (1987, pp. 352–53).

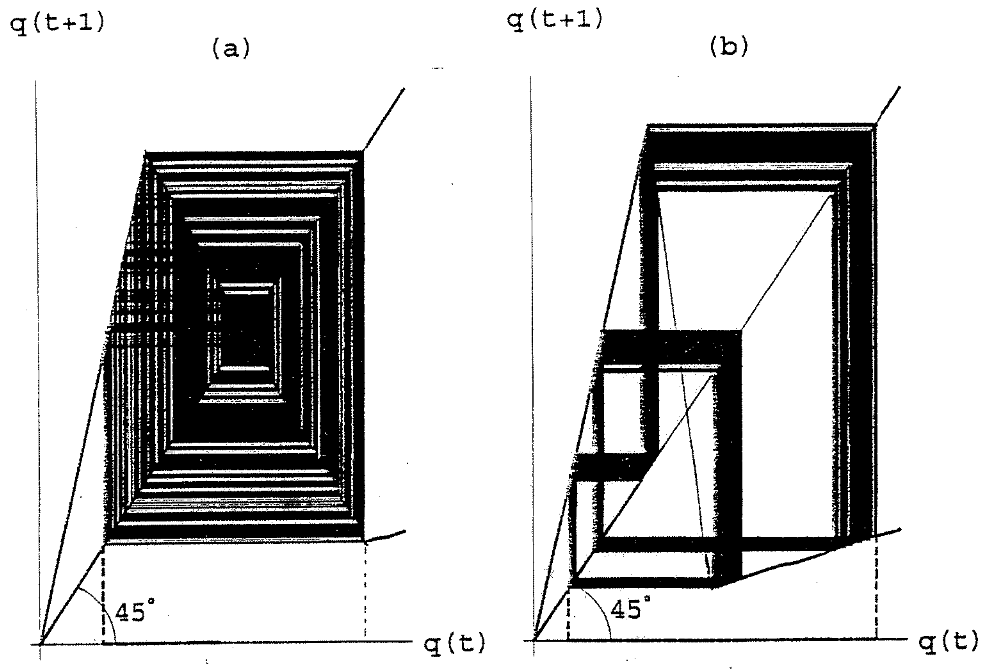

Figure 12.5. Two time paths of chaotic trajectory generated by $f|_{V_1}(q)$ and of noisy chaos with period 5 by $f|_{V_2}(q)$.

Figure 12.5(b) depicts trajectories generated by $f|_{V_2}(q_t)$ in which bd is taken to be 5 and the bold line on the $q(t)$ axis is the trapping interval V_2. It can be seen that trajectories periodically visit from one interval to another but aperiodically oscillate within the interval, it is called noisy chaos with period 5.

A bifurcation diagram in the upper asymmetric case is shown in Fig. 12.6, which has been calculated by fixing $\alpha = 2$ and $\beta = 0.8$ and decreasing $1/bd$ from 1 to 0. The switching of the transition map from $f|_{V_1}(q_t)$ to $f|_{V_2}(q_t)$ takes place at $1/(bd)^U$. Chaotic dynamics in Fig. 12.5(a) emerges along a vertical dashed line aa, passing through $1/1.2 \simeq 0.83$ whereas noisy chaos with period 5 in Fig. 12.5(b) emerges along a vertical dashed line bb, passing through a point $1/5 = 0.2$.

Lower asymmetric case

Analysis in the lower asymmetric case is similar to that in the upper asymmetric case. In Fig. 12.7 the bifurcation diagram in the lower asymmetric case is shown. The simulation is done with $\alpha = 3.2$ and $\beta = 4/9$, which satisfy the lower asymmetric condition $(1 + \alpha)(1 - \beta) > 1$. Once the bifurcation parameter $1/bd$ is less than unity, the stable-quantity equilibrium becomes unstable and bifurcates to asymptotically stable period-2 cycles, noisy chaos with period 2^2, noisy chaos with period 2^1, and noisy chaos with period 2^0 (i.e., chaos). As $1/bd$ is decreased further, the asymptotically stable period-3 cycle emerges

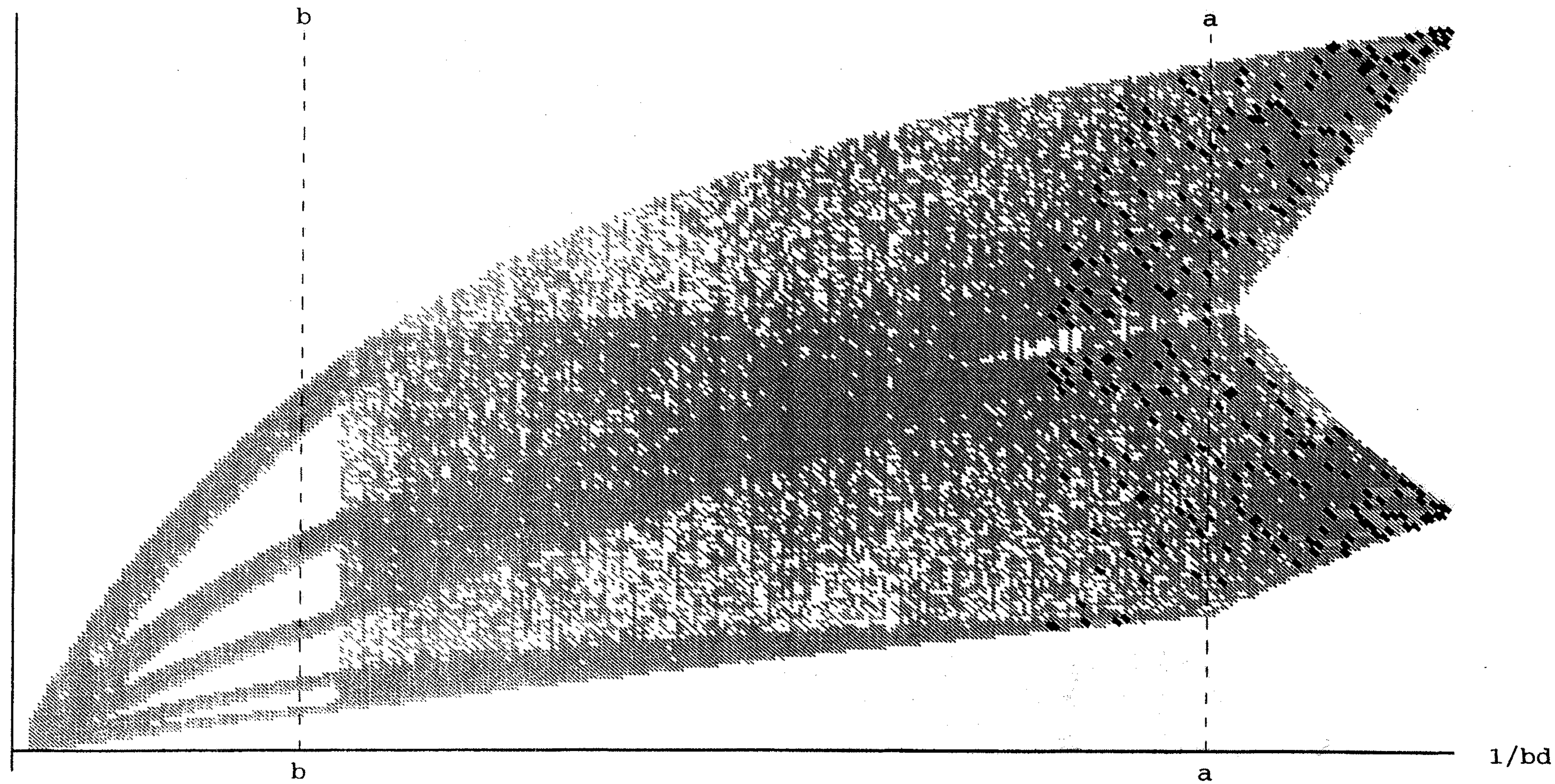

Figure 12.6. Bifurcation diagram in the upper asymmetric case $(1+a)(1-\beta) < 1$.

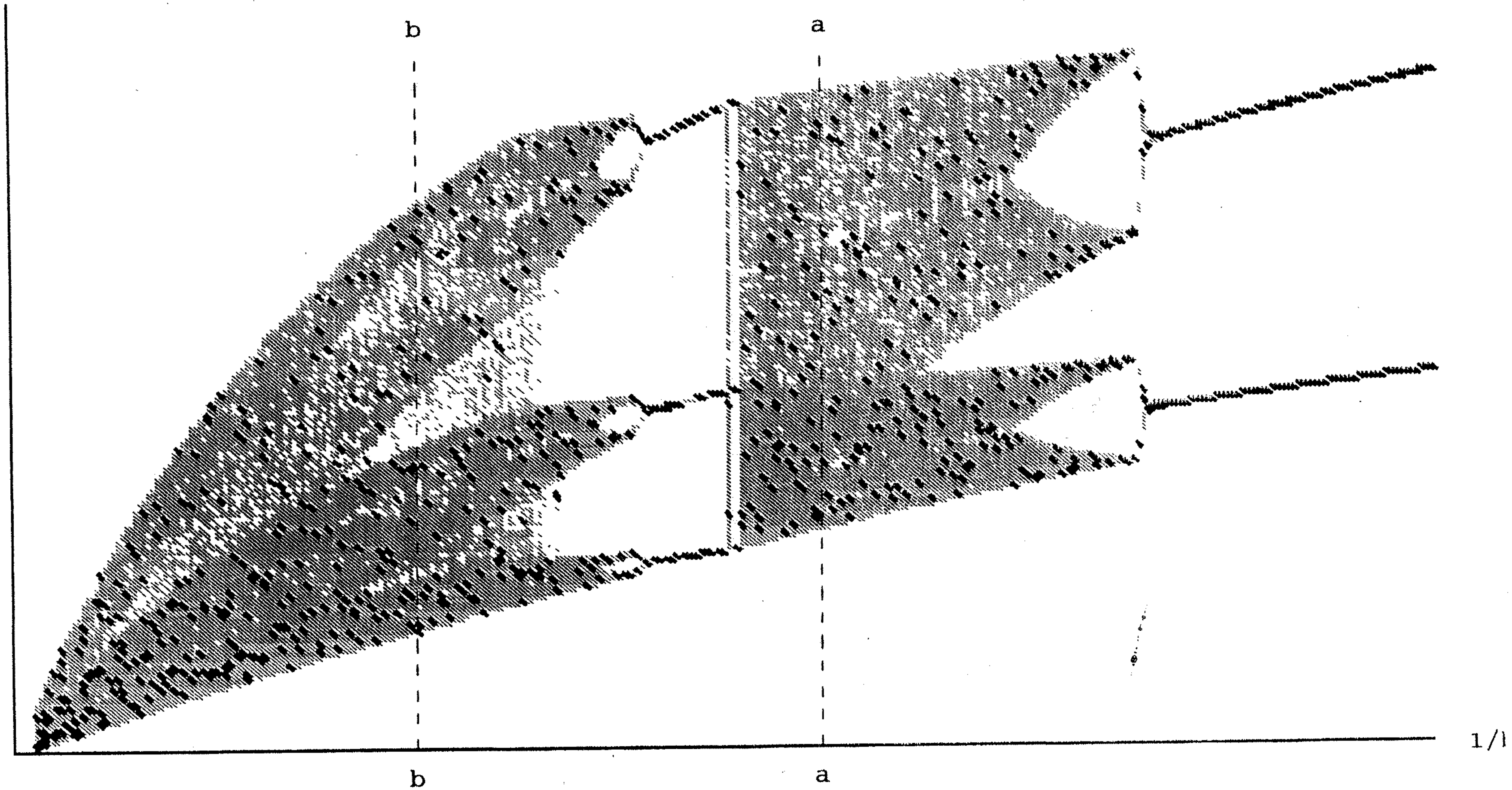

Figure 12.7. Bifurcation diagram in the lower asymmetric case $(1 + a)(1 - \beta) > 1$.

(i.e., window phenomenon occurs). When $1/bd$ is decreased further, the period-3 cycle bifurcates to the noisy chaos with period 3×2^1, one with period 3×2^0, and then to chaos. As in the upper asymmetric case, the switching of a transition map takes place. In order to see this, let $(bd)^L := \{a[(1-\beta)/\beta]\} > 1$, where the last inequality is due to the lower asymmetric condition. For $1 < bd \leq (bd)^L$, we have $Q_M < Q_{\min}$ so that $U_1 := [Q_{\min}, f(Q_{\min})]$ can be a trapping interval. A restriction of $f(q)$ to U_1,

$$f|_{U_1}(q) = \max[b\alpha - a - bdq_t, (1-\beta)q_t], \tag{12.13}$$

has an asymmetric tent-shaped profile. $f|_{U_1}(q)$ is, however, not expansive because it has a slope less than unity (i.e., $\{[\partial f|_{U_1}(q)]/\partial q\} = 1-\beta < 1$) on an interval $[Q_{\min}, Q_M)$ and one greater than unity (i.e., $|\{[\partial f|_{U_1}(q)]/\partial q\}| = bd > 1$) on $[Q_M, f(Q_{\min})]$. Although the results of Day and Schafer (1987) cannot be applied to a nonexpansive map, some results of Ito et al. (1979) are useful to characterize it. For $bd > (bd)^L$, we have $Q_{\min} < Q_M$ so that $U_2 := [Q_{\min}, Q_{\max}]$ can be a trapping interval that has two kinked points, Q_M and Q_m.

The restriction of the transition map to U_2, denoted by $f|_{U_2}(q)$, is linearly conjugate to Eq. (12) and possesses a tilted-z-shaped profile. Figure 12.8 shows two numerical examples of chaotic trajectories; Fig. 12.8(a) is a trajectory generated by $f|_{U_1}(q)$ that emerges along the vertical dashed line aa in Fig. 12.7, and Fig. 12.8(b) is a trajectory generated by $f|_{U_2}(q)$ that emerges along the vertical dashed line bb.

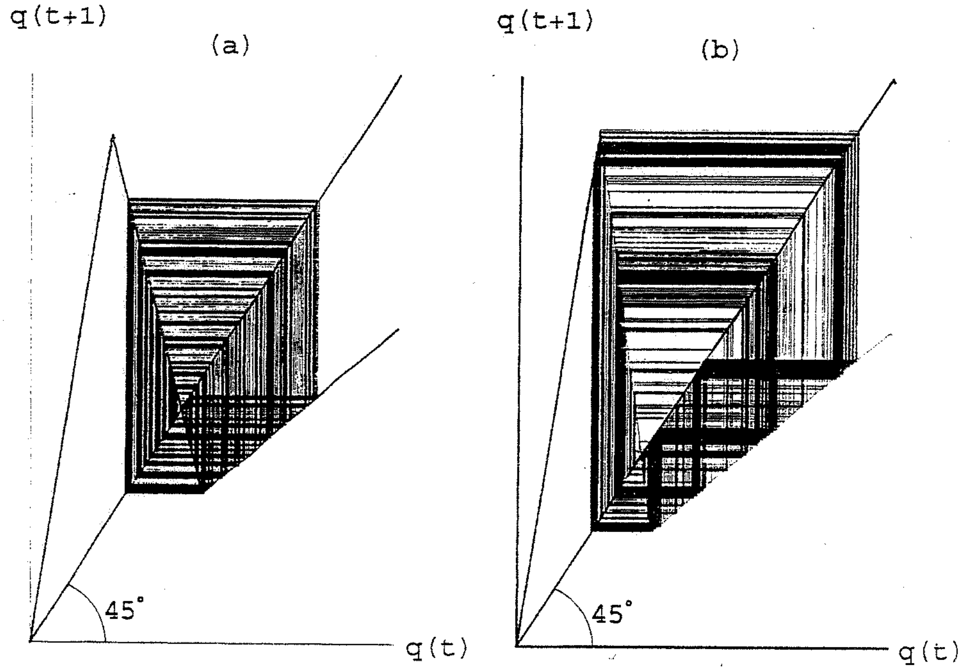

Figure 12.8. Two chaotic time paths generated by $f|_{U_1}(q)$ and $f|_{U_2}(q)$.

3 Concluding remarks

This chapter investigates the dynamic structure of the linear cobweb model with upper and lower bounds for variations on output. By the exogenous bounds, the transition map becomes nonlinear (i.e., piecewise linear) even though the behavioral specifications of supply and demand are linear. Simulating the model under different values of bd, it was demonstrated that the piecewise linear cobweb model may generate chaotic behavior if the output constraints are asymmetric and that it can generate stable period-2 cycles whose amplitudes depend on the prevailing parameter constellations (i.e., choice of initial point, values of bd, etc.) if the output constraints are symmetric. Although the bifurcation diagrams in asymmetric cases imply the emergence of chaotic behavior, the bifurcation example of chaos is different according to whether the output constraints are upper or lower asymmetric. In the upper symmetric case, the transition map generates persistent and aperiodic cycles for any values of bd greater than unity. The control system (i.e., the restriction of the transition map to the trapping interval) switches from the expansive map to the tilted-z-shaped map as bd is increased. In consequence, noisy chaos with period 2 and then true chaos appear as bd exceeds unity. Noisy chaos with period 5 appears if bd is further increased. On the other hand, in the lower asymmetric case, the control system switches from a nonexpansive tent map to the tilted-z-shaped map. As bd is increased from unity, the stable-output equilibrium is unstable and bifurcates to stable period-2 cycles, to noisy chaos with period 2^2, to chaos with period 2, and then to true chaos (i.e., noisy chaos with period 1), the last of which shrinks to zero as bd goes to infinity.

The fact that this model can justify the persistent and aperiodic fluctuations in a disequilibrium agricultural economy is itself worth note. What implications do the simulations have for a linear cobweb model augmented with the upper and lower bounds? The emergence of complex dynamics involving ergodic chaos depends on a combination of parameters, bd, α, and β. $bd > 1$ is necessary for generations of chaos in the asymmetric cases. Here b measures the steepness of the supply curve, α is the upper bound imposed on the positive growth rate of output, and β is the lower bound on the negative growth rate. These are decision variables of a producer. d measures the steepness of the inverse demand curve and is a decision variable of a consumer. Thus it can be stated that one source of such complex dynamics is an interaction between agents in the disequilibrium market. Although those parameters' values are exogenously given within our framework, it is of interest to investigate how each agent determines values of his or her decision variables.

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Complex dynamics in a cobweb model with adaptive production adjustment ☆

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Abstract

Chaos occurs in a nonlinear cobweb model with normal demand and supply, naive expectations and adaptive production adjustment. The model differs from existing ones in that it includes adaptive production adjustment instead of adaptive expectations. The model exhibits observable chaos (strange attractors) as well as topological chaos (horseshoes) associated with homoclinic points. As numerical simulations show, the faster suppliers adjust their production and the more inelastic demand is, the more likely the market behaves chaotically. ©2000 Elsevier Science B.V. All rights reserved.

JEL classification: D21; E32

Keywords: Nonlinear cobweb model; Adaptive production adjustment; Topological chaos; Observable chaos; Homoclinic point

1. Introduction

A farmer decides how much to produce in a certain period before price is determined and sales revenues are received. Some economists represent anticipated sales prices by model-consistent expectations. According to their view, farmers have the true model of the

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product market. Many empirical researchers, for example Ito (1990) and Hey (1994), test the hypothesis that expectations are model-consistent and reject it.

Nevertheless, the farmer has to forecast the price. Experimental evidence suggests that subjects use past market prices to forecast and follow rules of thumb. Williams (1987), e.g., shows that the adaptive expectation hypothesis (including naive forecasts, which are a special case of adaptive expectations) describes expectations in an experimental double-auction market better than the extrapolative one. This result is consistent with the common behavior of the farmer who uses the most recently received price as his prediction for the next period.

Such naive expectations are investigated in the formal cobweb literature starting with Kaldor (1934), Leontief (1934) and Ezekiel (1938). In models with normal supply and demand only three types of simple dynamics are possible: convergence to an equilibrium, two-period cycles or exploding oscillations. If expectations are not naive but adaptive, price behavior in the model with linear supply and demand is also simple (Nerlove, 1958). Unfortunately, the behavior of prices in agricultural markets is not so simple. [1]

Artstein (1983), Jensen and Urban (1984), Lichtenberg and Ujihara (1989) and Day and Hanson (1991) show that complex price behavior is possible if at least either demand or supply is non-monotonic. Hommes (1991, 1994), Finkenstädt and Kuhbier (1992) and Finkenstädt (1995) find complex behavior in normal markets with adaptive expectations when supply or demand is nonlinear. [2] This literature assumes that farmers make a best (optimizing) response given current expectations.

In this paper we investigate an alternative rule which specifies that farmers adjust partially in the direction of the best current response. Such adjustment is a behavioral response to uncertainty and adjustment costs. In order to show that our model exhibits topological and observable chaos, we exploit some mathematical results concerning homoclinic points. The occurrence of topological chaos is proved by applying the classical Homoclinic Point Theorem which asserts that a transverse homoclinic orbit implies a horseshoe. Under differentiability, this result is a little sharper than that by the Li–Yorke Theorem in regard to continuous maps on interval. Furthermore, the occurrence of observable chaos (i.e., strange attractors) for a large set of parameter values is shown, under some minor assumption, with the aid of a recent result concerning the homoclinic bifurcation by Mora and Viana (1993). In this way, we detect chaotic behavior in a theoretically large and empirically relevant region of price elasticities of demand and adjustment speeds. The faster suppliers adjust their production and the more inelastic the demand is, the more likely the market behaves chaotically.

2. The model

At period t, a supplier decides his production x_{t+1} for period $t+1$. As he knows well, even a production plan that maximizes profits may turn out to be a disaster in reality. He calculates the profit maximum $\tilde{x}_{t+1}$ and uses it as a target of adjustment. The calculation

[1] Whereas many agricultural markets are regulated to stabilize prices, see for example, Finkenstädt (1995) for chaotic price movements of egg, potato and pig in Northern Germany.

[2] See Lorenz (1993) for a general introduction to complex economic behavior due to various kind of nonlinearities.

is done subject to the quadratic cost function $(b/2)x^2$, $b > 0$ and naive price expectation, which means that his price expectation for the next period is equal to the current price p_t. The resulting amount is

$$\tilde{x}_{t+1} = \frac{p_t}{b}. \tag{1}$$

He has to adjust cautiously since every theoretically advantageous change may or may not enlarge real profits. Therefore, he is assumed not to produce $\tilde{x}_{t+1}$ immediately but to adjust adaptively his last period's production in the direction of $\tilde{x}_{t+1}$. This is a simple hedging rule in the uncertain real world and is expressed as the equation:

$$x_{t+1} = x_t + \alpha\left(\tilde{x}_{t+1} - x_t\right), \tag{2}$$

where $\alpha \in (0, 1)$ is the speed of adjustment. This equation, which can be rationalized by adjustment costs, is one of the earliest ways of incorporating adaptive processes explicitly into economic models (Nerlove, 1958) and is often used in econometric studies of macroeconomic behaviors.[3]

In order to bridge the gap between a single supplier and the market as a whole, we suppose that all n suppliers are homogeneous and behave identically. Therefore, the aggregate supply X is given by

$$X_t = nx_t. \tag{3}$$

We assume the following monotonic inverse demand function with constant price elasticity of $1/\beta(\beta > 0)$:

$$p_t = \frac{c}{Y_t^{\beta}}, \tag{4}$$

where Y_t is demand at period t and c is a positive shift parameter, which can be regarded as the extent of the market. Finally, price is set so that the market clears at each period:

$$Y_t = X_t. \tag{5}$$

Summarizing the model, we substitute Eqs. (1) and (3)–(5) into Eq. (2) and obtain the one-dimensional, discrete-time dynamical equation:

$$X_{t+1} = (1-\alpha)X_t + \frac{\alpha c n}{bX_t^{\beta}}. \tag{6}$$

To make the analysis below easier, let us consider the variable transformation:

$$X_t := \left(\frac{b}{cn}\right)^{-1/(1+\beta)} z_t.$$

Substituting into Eq. (6), we get

$$z_{t+1} = (1-\alpha)z_t + \frac{\alpha}{z_t^{\beta}}. \tag{7}$$

[3] For a survey of adaptive behavior, see Day (1998).

As the transformation is linear, X_t behaves periodically if z_t does so, and X_t behaves aperiodically if z_t does so, regardless of the number of suppliers n, the slope of marginal cost b and the extent of the market c. These parameters change the value of z_t into that of X_t through the scalar $(b/cn)^{-1/(1+\beta)}$. Essential parameters for the qualitative behavior of our model are the adjustment speed of production α and price elasticity of demand $1/\beta$. In what follows, we concentrate only on the dynamics of Eq. (7).

3. Analysis of the model

Our model (7) can be reformulated by the two-parameter family of maps $f_{\alpha,\beta}:\mathbb{R}_{++}\to\mathbb{R}_{++}$ as

$$f_{\alpha,\beta}(z)=(1-\alpha)z+\frac{\alpha}{z^{\beta}},\qquad (\alpha,\beta)\in(0,1)\times(0,\infty), \tag{8}$$

which are also expressed as f_β or simply f. Note that for any pair $(\alpha,\beta)\in(0,1)\times(0,\infty)$, $z^*=1$ is the unique fixed point for f, i.e.,

$$f(z^*)=z^*\Leftrightarrow z^*=1.$$

The first and second derivatives are calculated as

$$f'(z)=1-\alpha-\frac{\alpha\beta}{z^{1+\beta}},\qquad f''(z)=\frac{\alpha\beta(1+\beta)}{z^{2+\beta}}>0,\qquad z\in\mathbb{R}_{++},$$

which imply that f is a strictly convex and unimodal function on $\mathbb{R}_{++}$ (see Fig. 1) with its minimum at the critical point

$$z=\left[\frac{\alpha\beta}{1-\alpha}\right]^{1/(1+\beta)}=:\theta(\alpha,\beta)\,(=\theta(\beta)=\theta).$$

The fixed point is a repeller if

$$f'(1)=1-\alpha-\alpha\beta<-1,$$

which is equivalently rewritten as

$$\beta>\frac{2-\alpha}{\alpha}. \tag{9}$$

In the following subsections we present two propositions concerning the complex behavior of our model. The proofs and the related fundamental notions of symbolic dynamics are given in the Appendix.

3.1. Existence of horseshoes

First we present a proposition which states that for every sufficiently large β, the map f_β exhibits a *horseshoe*. By a horseshoe we mean here a compact invariant set on which some iterate of f_β is topologically conjugate to the one-sided full-shift on two symbols. The

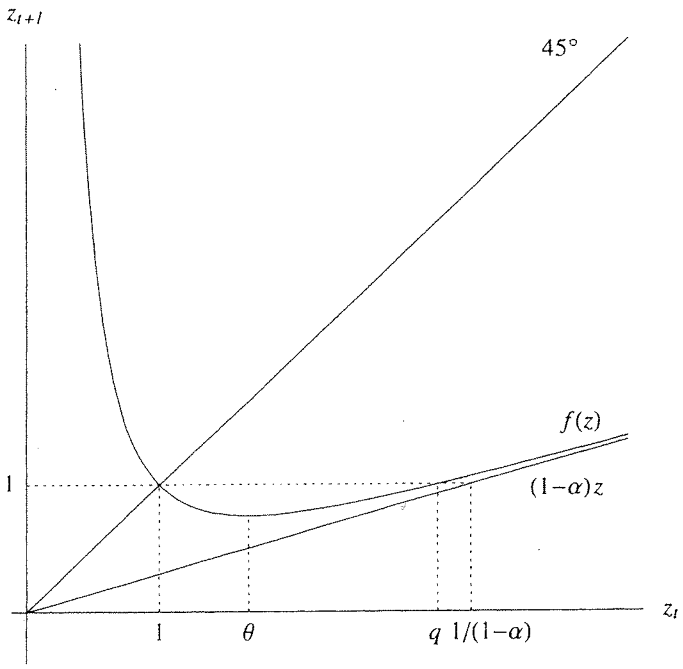

Fig. 1. Graph of the map (8).

existence of a horseshoe is assured by that of a transverse homoclinic point. We say that a map exhibits *topological chaos* either if it has a horseshoe or, alternatively, if the topological entropy of the map is positive.[4] Although a map restricted on horseshoes behaves in a complicated way, the existence of horseshoes itself does not assure complex dynamics in the long run; the economic system may eventually settle down to a periodic motion even if horseshoes are present. Nevertheless, horseshoes may often generate long-lasting complicated transient dynamics, and even small external shocks are likely to give rise to erratic motions of a system which are otherwise periodic in the long run. Finding horseshoes in our model is, therefore, not insignificant even from an empirical viewpoint.

Our result is summarized as follows:

Proposition 1. *Fix an $\alpha \in (0, 1)$ arbitrarily. Then there exists a number $\bar{\beta} = \bar{\beta}(\alpha) > (2-\alpha)/\alpha$ such that f_β in Eq.* (8) *has a horseshoe for each $\beta \geq \bar{\beta}$.*

One of the important features of a horseshoe (more precisely, a hyperbolic set) is the stability of the associated map against C^r-perturbations ($r \geq 1$) (see e.g. de Melo and van Strien (1993), p. 225, Theorem 2.3). Roughly speaking, once the economic system described by f_β possesses horseshoes, they will be preserved despite small changes in the underlying economic structure.

[4] See, e.g. Block and Coppel (1992) for details about topological entropy.

3.2. Existence of strange attractors

Next we present a proposition which states that, under some generic condition, our model frequently exhibits *observable chaos* in the sense of *strange attractors*. While horseshoes do not assure complex dynamics in the long run, strange attractors do assure that we can observe erratic behavior for some large set of initial conditions. Hence, frequent occurrence of observable chaos seems to be useful in explaining irregular behavior of economic time series.

To state our proposition, we first introduce some basic notions.

Definition. A compact invariant set $\mathcal{A} \subset \mathbb{R}$ for f is called an *attractor* if its stable set $W^s(\mathcal{A}) = \{z \in \mathbb{R} | \lim_{n\to\infty} d(f^n(z), \mathcal{A}) = 0\}$ contains a nonempty interior and f has a dense orbit in $\mathcal{A}$. An attractor $\mathcal{A}$ here is said to be *strange* if it contains a dense orbit with positive Lyapunov exponent, i.e., there is a point $z \in \mathcal{A}$ for which $\overline{\{f^n(z)\}_{n\geq 0}} = \mathcal{A}$ and $\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \log|f'(f^{i-1}(z))| > 0$.

Strange attractors will appear via *homoclinic bifurcation*, that is, when a non-degenerate homoclinic tangency unfolds generically. In other words, they will appear when the critical point $\theta(\beta)$ with $f''(\theta) \neq 0$ which is contained in a homoclinic orbit to the repeller $z^* = 1$ for some $\beta = \beta^*$ passes through z^* at non-zero speed for some iterate of f_β as β varies. Formally, $(\mathrm{d}/\mathrm{d}\beta) f_\beta^n(\theta(\beta)) \neq 0$ at $\beta = \beta^*$ for some n with $f_{\beta^*}^n(\theta(\beta^*)) = 1$. See Mora and Viana (1993) for a full explanation.

The following result shows that our model may exhibit observable chaos for measure-theoretically large sets of parameter values.

Proposition 2. *For any $\alpha \in (0, 1)$, there is generically a positive measure set of parameter values of β, $E \subset \mathbb{R}_+$, such that for every $\beta \in E$ the map f_β exhibits a strange attractor.*

4. Numerical simulations of the model

In this section we perform some numerical simulations and show that those are supported by the theoretical results in the previous section.

First we depict the standard, one-parameter bifurcation plot of Eq. (8) with respect to β in Fig. 2 and the corresponding topological entropy (*TE*) and Lyapunov exponent (*LE*) in Figs. 3 and 4.[5] Intuitively, topological entropy measures the exponential growth rate of the number of foldings of the graph of the nth iterate of a map. By definition $TE \geq 0$, and if $TE > 0$ then the map exhibits topological chaos. From Fig. 3 we can observe that topological chaos occurs in our model for values of β larger than $\beta_{TE>0} \approx 3.0008$. Proposition 1 states that our model may exhibit topological chaos when β satisfies the condition (9). We can confirm this as follows: the value of α used in the calculation of *TE* is 0.7 and substituting into $\beta = (2-\alpha)/\alpha$ gives $\hat{\beta} \approx 1.857 < \beta_{TE>0}$ which is found in Fig. 2.[6]

[5] To calculate topological entropy, we utilized the algorithm presented by Block et al. (1989).

[6] $\hat{\beta}$ is exactly the period-doubling bifurcation value at which a single stable fixed point splits into a stable period-2 cycle.

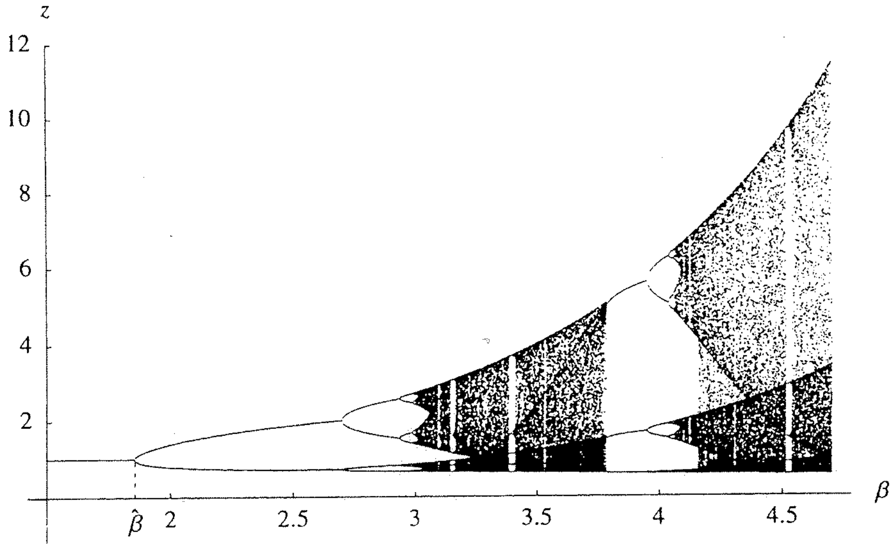

Fig. 2. Bifurcation diagram with respect to $\beta (1.5 \leq \beta \leq 4.7)$ with $\alpha = 0.7$.

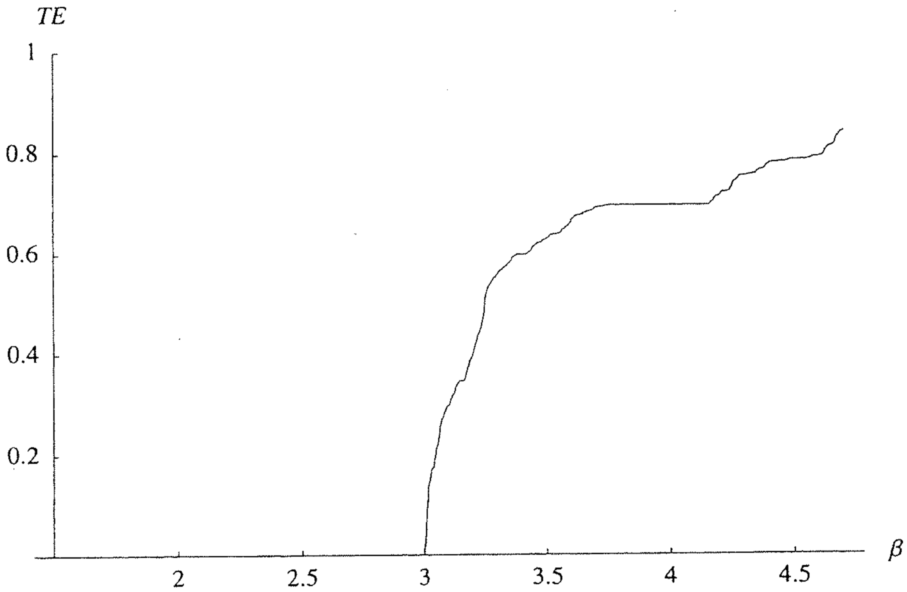

Fig. 3. Graph of topological entropy corresponding to Fig. 1 with $\alpha = 0.7$. Topological entropy is measured using base 2 logarithms so that the vertical axis is [0, 1]. A positive value for topological entropy indicates topological chaos.

The Lyapunov exponent expresses the exponential rate of divergence between two arbitrarily close orbits as time elapses. If $LE > 0$ then the map exhibits observable chaos in the sense that it has strange attractors, and no stable-periodic orbit has a positive Lyapunov exponent. Values of β such that Lyapunov exponents have a positive sign in Fig. 4 correspond to those of shaded area in Fig. 2, and observable chaos occurs for such values of β.

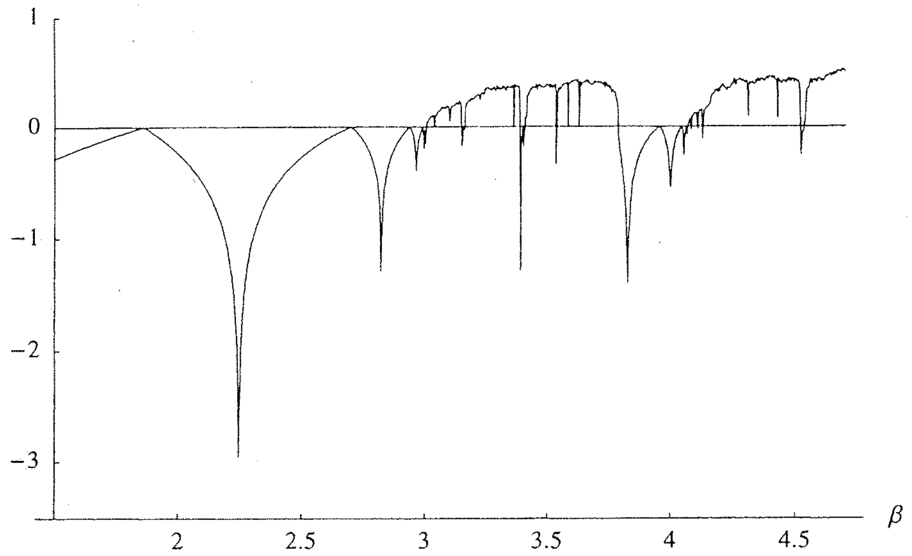

Fig. 4. Graph of Lyapunov exponent corresponding to Fig. 1 with $\alpha = 0.7$. A positive value of the Lyapunov exponent indicates observable chaos.

As stated in the previous section, the observable motions may be indeed periodic even if topological chaos is present. Comparing Figs. 2 and 3, it is realized that in the region of $TE > 0$ there exist windows of periodic behavior. On the other hand, if there appear periodic motions then $LE \leq 0$. Therefore, we can classify chaos to be present in our model as follows:

$$TE > 0: \quad \text{topological chaos} \quad \begin{cases} LE > 0: & \text{observable chaos,} \\ LE \leq 0: & \text{windows (latent chaos).} \end{cases}$$

In addition, the Schwarzian derivative of f at z is

$$Sf(z) = -\frac{\alpha\beta(1+\beta)\left[\alpha\beta(\beta-1) + 2(1-\alpha)(2+\beta)z^{1+\beta}\right]}{2\left[(\alpha-1)z^{2+\beta} + \alpha\beta z\right]^2},$$

and it is negative if Eq. (9) is satisfied. Therefore, f has at most one periodic attractor.[7]

As mentioned above, the essential parameters in our model are α and β. Thus a question arises here: what happens to the above bifurcation plot if α also varies? To answer this, we draw two kinds of 2-parameter diagrams after the manner of Gallas and Nusse (1996): one is an *iso-period plot* and the other is an *observable chaos plot*. The former is the union of all iso-period-p plots for $p \in [1, \bar{p}] \subset \mathbb{N}$ (in this paper $\bar{p} = 64$). And each iso-period-p plot is made of the set in the parameter space such that for each element in this set the trajectory through some fixed initial point x_0 converges to a stable period-p cycle. The latter consists of the set in the parameter space such that for each element in this set the orbit through

[7] See, e.g. Devaney (1989), pp. 69f.

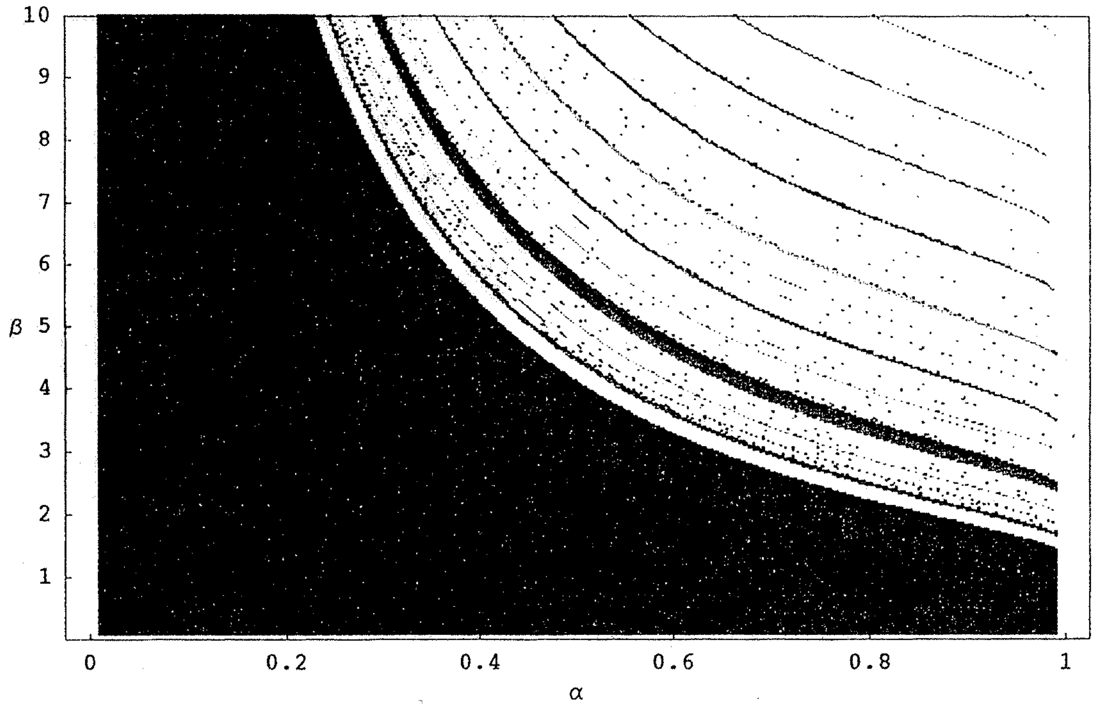

Fig. 5. Iso-period plot for the map (8). The green area corresponds to period-1 cycles (a stable fixed point), the dark blue area to period-2 cycles, the yellow area to period-4 cycles, and the red area to period-8 cycles. The blue and magenta areas correspond to period-3 and period-6 cycles.

some fixed initial point x_0 is observably chaotic in the sense that it has a positive Lyapunov exponent.

We consider the region $K = \{(\alpha, \beta)|0 < \alpha < 1, 0 < \beta \leq 10\}$. The resulting iso-period plot is shown in Fig. 5. In this figure, the green area exhibits pairs of parameter values for which every trajectory converges to a unique stable fixed point. The dark blue area consists of pairs of parameter values for which every trajectory converges to a period-2 cycle. The blue area corresponds to a period-3 cycle, the yellow area corresponds to a period-4 cycle, the magenta area corresponds to a period-6 cycle, and the red area corresponds to a period-8 cycle, etc. We emphasize that the border between the green area and the blue area is expressed by the equation $\beta = (2 - \alpha)/\alpha$, the upper region of which satisfies Eq. (9); therefore, the fixed point of the model is unstable there.

The resulting chaos plot is presented in Fig. 6. The black area in this figure is the set of parameters for which our model exhibits observable chaos in the sense of a positive Lyapunov exponent. This figure implies that observable chaos occurs when α and β are large. In other words, the faster suppliers adjust their production and the more inelastic demand is, the more likely the market behaves chaotically.

Finally, it should be stressed that these figures suggest that periodic behavior and observably chaotic behavior are complementary in the sense that the union of the colored area in Fig. 5 and the black region in Fig. 6 is equal to the whole region of K. But unfortu-

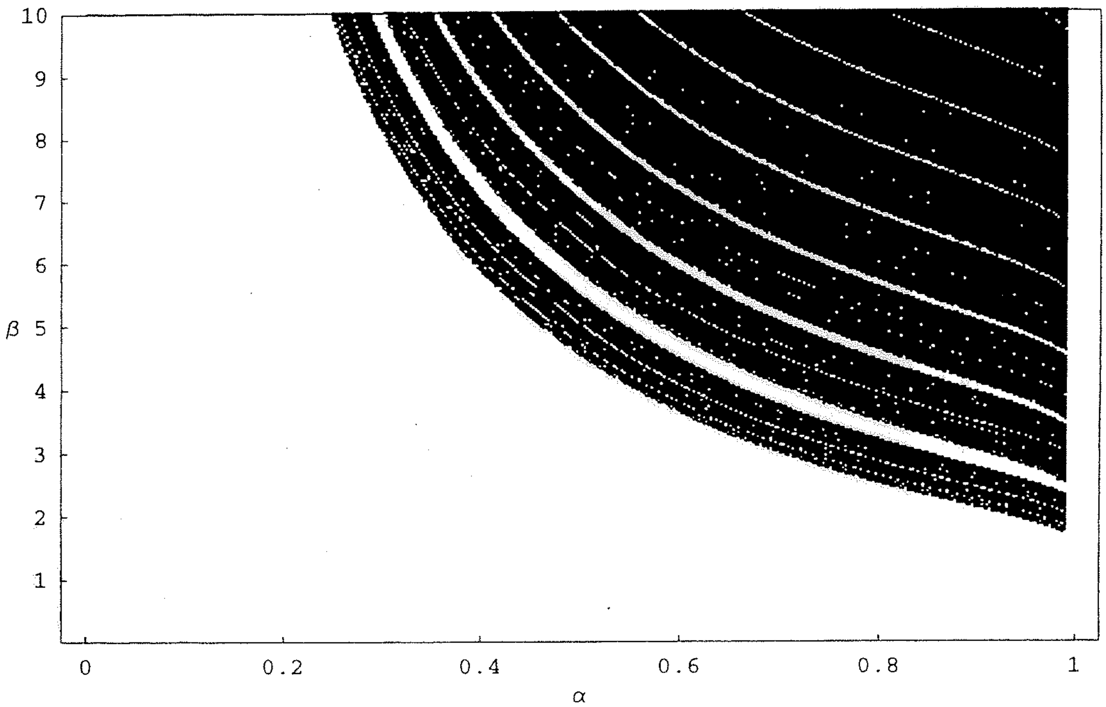

Fig. 6. Observable chaos plot for the map (8). The black area is the set of (α, β) for which the model has strange attractors.

nately, as Gallas and Nusse (1996) point out, there exists no theoretical result which assures this fact.

5. Concluding remarks

We have investigated the dynamics of a nonlinear cobweb model where suppliers adjust cautiously to hedge against the uncertain world. If suppliers adapt slowly, they may stabilize the market. Adaptive adjustment could be a reasonable strategy to prevent large price fluctuations. Whether the cautious behavior stabilizes the market effectively depends on how much consumers change their demand as price changes.

It is well-known that price elasticities of demand for essential goods like food are relatively low. In fact, according to estimates by Pagoulatos and Sorensen (1986), the majority of US food and tobacco industries has price elasticities of less than 1/5. In such markets, chaos occurs even if adjustment is rather cautious.

The main difference between our model and existing ones consists of the adaptive adjustment hypothesis. Future research is required to better understand the adaptive behavior of economic agents. Our model is so simple that we consider it a mere stepping-stone; nevertheless, it shows that the adaptive adjustment approach is a promising research agenda for explaining complex economic phenomena.

Appendix

Some fundamental notions of symbolic dynamics

Let Σ_2 denote the set of all infinite sequences $s = (s_1, s_2, s_3, \dots)$, where $s_i = 0$ or 1 for $i \in \mathbb{N}$. We define a metric on Σ_2 by the function

$$d(s,t) = \sum_{i=1}^{\infty} \frac{|s_i - t_i|}{2^i}, \qquad s, t \in \Sigma_2.$$

The metric space (Σ_2, d) is then compact, totally disconnected, and perfect, i.e., it is a Cantor set. The *shift* map $\sigma : \Sigma_2 \to \Sigma_2$ is defined by $\sigma((s_1, s_2, \dots)) = (s_2, s_3, \dots)$, which is referred to as the *one-sided shift on two symbols*.

The shift map $\sigma : \Sigma_2 \to \Sigma_2$ has the following properties:

(i) Σ_2 contains a countably infinite set of periodic orbits;

(ii) Σ_2 contains an uncountably infinite set of aperiodic orbits;

(iii) the set of periodic points is dense in Σ_2;

(iv) $\sigma : \Sigma_2 \to \Sigma_2$ is *topologically mixing*, i.e., for every pair of nonempty open sets $U, V \subset \Sigma_2$, there is $m \geq 1$ such that $\sigma^n(U) \cap V \neq \phi$ for all $n \geq m$;

(v) $\sigma : \Sigma_2 \to \Sigma_2$ is *expansive*, i.e., there is $\delta > 0$ such that for any $s, t \in \Sigma_2 (s \neq t)$, there is $m \geq 1$ with $d(\sigma^m(s), \sigma^m(t)) \geq \delta$.

By definition, topological mixing property implies *topological transitivity*, and expansiveness implies *sensitive dependence on initial conditions*.

Let X and Y be metric spaces, and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. The map f is said to be *topologically conjugate* (or *equivalent*) to g if there exists a homeomorphism $h : X \to Y$ (one-to-one, onto, continuous map with continuous inverse) such that

$$h \circ f(x) = g \circ h(x) \quad \text{for every } x \in X,$$

i.e., the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[g]{} & Y \end{array}$$

commutes. The map f is said to be *topologically semi-conjugate* to g if there is an endomorphism h of X *onto* Y (i.e., continuous map of X onto Y) such that $h \circ f = g \circ h$.

Let us next consider a C^1-map f of (an interval of) $\mathbb{R}$ into itself. Let $p \in \mathbb{R}$ be a repelling hyperbolic fixed point (or repeller) of f, i.e., $f(p) = p$ and $|f'(p)| > 1$. Let $q \in \mathbb{R} (q \neq p)$ be a point such that $f^n(p) = q$ for some $n \in \mathbb{N}$ and that there exists a sequence $\{q_{-i}\}_{i=0}^{\infty}$ with the property that $q = q_0$, $f(q_{-i-1}) = q_{-i} (i \geq 0)$ and $q_{-i} \to p (i \to \infty)$. The sequence $\mathcal{HO}_f(q, p) := \{q, f(q), f^2(q), \dots, f^n(q) = p\} \cup \{q_{-i}\}_{i=0}^{\infty}$ is said to be a *homoclinic orbit* of q to p. An element of the homoclinic orbit $x \in \mathcal{HO}_f(q, p)$ is a homoclinic point to p. A homoclinic orbit $\mathcal{HO}_f(q, p)$ is said to be *transverse* if $f'(z) \neq 0$ for every $z \in \mathcal{HO}_f(q, p)$.

Finally, we introduce an important theorem[8] which is utilized to prove Proposition 1.

Theorem (Homoclinic Point Theorem for C^1-Map on Interval). *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-map and $p \in \mathbb{R}$ be a repelling hyperbolic fixed point. If f has a transverse homoclinic orbit to p, then there exist a number $n \in \mathbb{N}$ and a compact set $\Lambda \subset \mathbb{R}$ such that*

(i) $f^n(\Lambda) = \Lambda$;

(ii) $p \in \Lambda$;

(iii) $f^n|\Lambda : \Lambda \to \Lambda$ *is topologically conjugate to the one-sided full-shift on two symbols* $\sigma : \Sigma_2 \to \Sigma_2$, *i.e., there is a homeomorphism* $h : \Lambda \to \Sigma_2$ *with* $h \circ f^n|\Lambda = \sigma \circ h$.

The set Λ here is called a *horseshoe*, and if the map f has such a set, we say that the map f has a horseshoe. By topological conjugacy, $f^n|\Lambda$ inherits complicated dynamical properties of the shift map $\sigma|\Sigma_2$ described above. Furthermore, if f is a continuous map on interval and has a periodic point of period three, then for some n and some invariant set Λ for f^n, $f^n|\Lambda$ is topologically *semi*-conjugate to $\sigma|\Sigma_2$. See Block and Coppel (1992) for more detail.

5.1. *Proof of Proposition 1*

By the Homoclinic Point Theorem in the previous subsection, it suffices to show that the map f has a transverse homoclinic orbit. To show this we have to make some preparations.

Lemma 1. *The following statements hold:*

(i) $0 < f(\theta) < 1 < \theta$;

(ii) *there is a unique point* $q = q(\beta) > \theta$ *such that* $f(q) = 1$.

Proof. (i) From Eq. (9), we get the last inequality. Since f has its global minimum at θ, we get the second inequality (see Fig. 1). (ii) Since $f(\theta) < 1$ and $f(x) > 1$ for large x, there is $q > \theta$ with $f(q) = 1$. Since $f(x)$ is strictly increasing on the interval (θ, ∞), the uniqueness follows (see Fig. 1). □

Let us define a piecewise linear map $L_\beta : \mathbb{R} \to \mathbb{R}$ by

$$L_\beta(z) := \begin{cases} l_1(z) = f'_\beta(1)(z-1) + 1, & z \leq 1, \\ l_2(z) = -(z-1) + 1, & z > 1. \end{cases}$$

Clearly, $L(1) = 1$ and $L^{-n}(z) \to 1 (n \to \infty)$ for each $z \in \mathbb{R}$ (see Fig. 7).

Lemma 2. *There is a number* $\beta_1 > (2-\alpha)/\alpha$ *such that for every* $\beta \geq \beta_1$ *and for every* $z \in I := [f_\beta(\theta), \theta]$ *there is a unique sequence* $\{z_{-i}\}_{i=0}^{\infty} \subset I$ *such that* $z_0 = z$, $f(z_{-i-1}) = z_{-i}$ *for* $i \geq 0$, *and* $z_{-i} \to 1$ *as* $i \to \infty$.

Proof. Since f_β is one-to-one on the interval $[f_\beta(\theta), \theta]$, the conclusion holds if given sufficiently large β, $f_\beta(z) > l_1(z)$ for $z \in [f_\beta(\theta), 1)$ and $f_\beta(z) < l_2(z)$ for $z \in (1, \theta]$. By

[8] For the proof of this theorem see e.g. Devaney (1989).

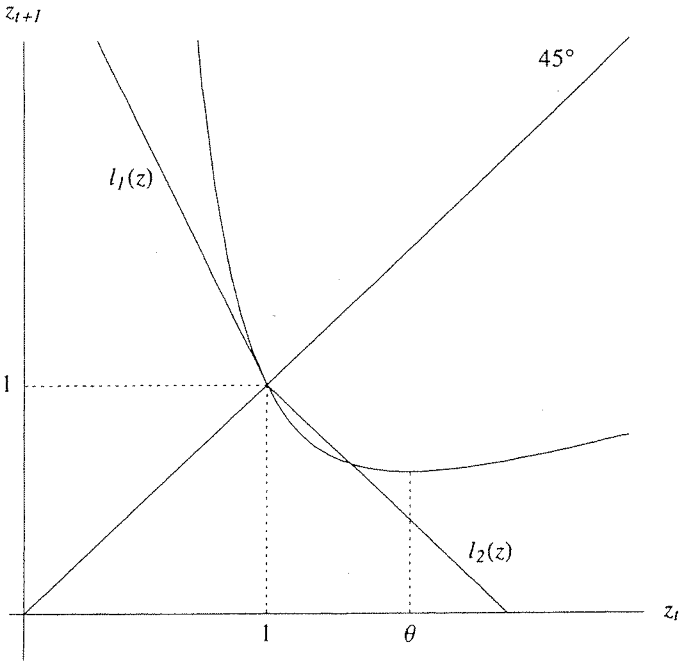

Fig. 7. Graph of the map (8) and the piecewise linear map $L_\beta(z)$.

the strict convexity of f and by the construction of L, it is sufficient to show that for β large enough, the inequality $f(\theta) < l_2(\theta)$, i.e.,

$$f_\beta(\theta(\beta)) + \theta(\beta) < 2$$

holds. This is verified by the fact

$$\lim_{\beta\to\infty}\theta(\beta) = \lim_{\beta\to\infty}\left[\frac{\alpha\beta}{1-\alpha}\right]^{1/(\beta+1)} = 1,$$
$$\lim_{\beta\to\infty} f_\beta(\theta(\beta)) = \lim_{\beta\to\infty}\left[(1-\alpha)\theta(\beta) + \alpha\theta(\beta)^{-\beta}\right] = 1-\alpha$$

which completes the proof. □

Lemma 3. *There is a number $\beta_2 > (2-\alpha)/\alpha$ such that for every $\beta \geq \beta_2$ the following inequality holds:*

$$0 < f_\beta(\theta(\beta)) < 1 < \theta(\beta) < q(\beta) < f_\beta^2(\theta(\beta)).$$

Proof. By Lemma 1, it suffices to show that for any arbitrarily large β, the following inequality holds:

$$q < f^2(\theta). \tag{A.1}$$

Let us define a function l by

$$l(z) := f'(1)(z-1) + 1 = (1-\alpha-\alpha\beta)(z-1) + 1.$$

Again by the strict convexity of f, we obtain

$$l \circ f_\beta(\theta(\beta)) < f_\beta^2(\theta(\beta)).$$

Note that $f_\beta(z) > (1-\alpha)z$ holds for every $z \in \mathbb{R}_{++}$ (see Fig. 1), so we have

$$q(\beta) < \frac{1}{1-\alpha}.$$

To obtain the inequality (A.1), it is therefore sufficient to show

$$\frac{1}{1-\alpha} \leq l \circ f_\beta(\theta(\beta))$$

for any sufficiently large β. Noting that

$$l \circ f_\beta(\theta(\beta)) = (1-\alpha-\alpha\beta)\left[(1-\alpha)\theta(\beta) + \alpha\theta(\beta)^{-\beta} - 1\right] + 1,$$

we get $\lim_{\beta\to\infty} l \circ f_\beta(\theta(\beta)) = \infty$. So the lemma follows. □

Proof of Proposition 1. Let $\overline{\beta} = \max\{\beta_1, \beta_2\}$ and pick $\beta \geq \overline{\beta}$ arbitrarily. By Lemma 3, there is a point $q' \in (f(\theta), 1)$ such that $f(q') = q$. By Lemma 2, there is a sequence $\{q'_{-i}\}_{i=0}^{\infty} \subset I$ such that $q'_0 = q'$, $f(q'_{-i-1}) = q'_{-i} (i \geq 0)$, and $q'_{-i} \to 1 (i \to \infty)$. Hence, together with $f^2(q') = 1$, q' is a homoclinic point to the repeller 1. Clearly, for any homoclinic point $z \in \mathcal{HO}_f(q', 1) = \{q', f(q'), f^2(q') = 1\} \cup \{q'_{-i}\}_{i=0}^{\infty}$, we have $z \neq \theta$ and so $f'(z) \neq 0$, which implies that the homoclinic orbit of q' to 1, $\mathcal{HO}_f(q', 1)$, is transverse. By the Homoclinic Point Theorem, the statement is proved. □

5.2. *Proof of Proposition 2*

To find the abundance of strange attractors for the family of maps $\{f_\beta\}_\beta$, we exploit the theorem by Mora and Viana (1993), Theorem C.

Proof of Proposition 2. We show that the (non-degenerate) critical point $\theta(\beta)$ is contained in the homoclinic orbit of the repelling fixed point $z^* = 1$ for some sequence of β-values. Note first that the critical point θ is always non-degenerate, i.e., $f''(\theta) \neq 0$, since $f''(x) > 0$ for all $x \in \mathbb{R}_{++}$.

We can observe that there is a sequence of eventually fixed points depending smoothly on β,

$$Q(\beta) = \{q_i(\beta) | q_i(\beta) = f_\beta(q_{i+1}(\beta)) \quad \text{for} \quad i \in \mathbb{N}, \quad q = q_1 < q_2 < \cdots < q_n < \cdots\},$$

where $q_i(\beta) \to (1-\alpha)^{-i} < \infty$ as $\beta \to \infty$ for every $i \in \mathbb{N}$.

Let us fix $\alpha \in (0, 1)$ arbitrarily and take $\beta = \beta_1$ as in Lemma 2. Then, from the observation above, there is $q_i(\beta_1) \in Q(\beta_1)$ such that $f_{\beta_1}^2(\theta(\beta_1)) < q_i(\beta_1)$. Since $f_\beta^2(\theta(\beta)) \to \infty$ as $\beta \to \infty$ by the proof of Lemma 3, and $q_i(\beta) \to (1-\alpha)^{-i}$ as $\beta \to \infty$, there is $\beta^* > \beta_1$ such that $f_{\beta^*}^2(\theta(\beta^*)) = q_i(\beta^*)$, which implies that the backward orbit of $\theta(\beta^*)$ converges to the repelling fixed point $z^* = 1$ and there is an integer n such that $f_{\beta^*}^n(\theta(\beta^*)) = 1$ and

$f^m_{\beta^*}(\theta(\beta^*)) \neq 1$ for $m < n$. Hence, the non-degenerate critical point $\theta(\beta^*)$ lies in a homoclinic orbit to the fixed point $z^* = 1$ (homoclinic tangency). Since $(\partial/\partial\beta)f(x, \beta) = 0$ if and only if $x = 1$, we may generically assume that $(d/d\beta)f^n_\beta(\theta(\beta)) \neq 0$ at $\beta = \beta^*$, which implies that, in our case, this homoclinic tangency unfolds generically. By Theorem C in Mora and Viana (1993), the statement of Proposition 2 follows. □

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Can inventory chaos be welfare improving☆

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Abstract

This study demonstrates the possibility that chaotic fluctuations may be preferable to a steady state. For this purpose, it uses a simple macro disequilibrium model in which inventory can be chaotically fluctuated. In such a model, the profit of a firm as well as the utility of a consumer fluctuates. This raises the question of their average profitability and preferability in the long run. This study, with the aid of numerical examples, demonstrates that the long-run average profit can be greater than the profit obtained at the steady state while the long-run average utility is the same as the utility obtained at the steady state. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the recent literature, periodic or aperiodic motions of aggregate inventory have been demonstrated in à la Metzlerian inventory cycle models with the help of the nonlinear dynamic theory. Honkapohja and Ito [1] (H-I henceforth), Eckalbar [2], Hommes [3], Franke and Lux [4], and Matsumoto [5] are only a few. Among them, of particular relevance to this study are [1,3]. In particular, H-I show a possibility of the long-run market instability, that is, the market diverges from a long-run steady-state equilibrium while remaining bounded. Hommes extends some results of H-I: he shows that a deterministic nonlinear endogenous relationship generates chaotic motions, that is, seemingly random behavior. In those studies, the focus is mainly placed on the existence of persistent inventory fluctuations. The continuance of inventory fluctuations means the continuing presence of disequilibrium between supply and demand. However, such persistent disequilibrium of the market is regarded as an unfavorable phenomenon in the traditional economics. It is thus imperative to investigate the nature of long-run market dynamics in order to understand the economic implications of chaotic fluctuations.

The present study constructs a simple, macroeconomic, disequilibrium model with inventory and attempts to characterize the market instability for the case in which inventory trajectories are bounded but never reach a steady state. In such an unstable economy, the profits of firms as well as the

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utilities of consumers also fluctuate. This raises the question of their average profits and utilities in the long run. This study demonstrates, specifying the dynamic system, that the long-run average profit of the firm can be larger than the steady-state profit while the long-run average utility of the consumer is the same as the steady-state utility. These results indicate that the inventory fluctuations may be preferable to a steady state and provide new insights into the chaotic fluctuations in perpetual disequilibrium.

This study is organized as follows. Section 2 outlines the basic macroeconomic model. Section 3 derives a dynamical system of the model. Section 4 provides possible microeconomic underpinnings of the macroeconomic model and develops a method of characterizing the chaotic inventory fluctuations. Section 5 performs numerical simulations to show the main result of this study, namely, that chaotic fluctuations can be preferable to a steady state in the long term. Section 6 is the conclusion.

2. Model

In this section, we recapitulate Hommes's version of the H-I disequilibrium model with buffer stock inventories.[1] The model has two markets and two representative agents: one market for labor and the other for storable consumption goods; one agent is the consumer and the other the firm. The markets are assumed to operate sequentially; labor transactions are carried out first, and then production and the trade of goods take place. This is a disequilibrium model where demand is not necessarily compatible with supply, and prices are held constant. Without price adjustment, actual transactions in disequilibrium markets are determined by the minimum of supply and demand (i.e., the *min-rule* or the *short-side rule*),

$$L_t = \mathrm{Min}[L_t^{\mathrm{D}}, L_t^{\mathrm{S}}] \quad \text{and} \quad Y_t = \mathrm{Min}[Y_t^{\mathrm{D}}, Y_t^{\mathrm{S}}], \qquad (1)$$

where L_t^{D}, respectively L_t^{S}, denotes the demand for, respectively, the supply of, labor, Y_t^{D}, respectively Y_t^{S}, indicates the demand for, respectively, the supply of consumption goods, and L_t, respectively Y_t, denotes the actual transaction of labor, respectively the consumption goods. t is an integer and denotes the given time period. This is an inventory model where the difference between supply and demand determines the initial inventory of the following period and serves as the main source of dynamics.

The consumer demands consumption goods and supplies labor:

$$Y_t^{\mathrm{D}} = a + bL_t, \quad a > 0, \ b > 0, \qquad (2)$$

$$L_t^{\mathrm{S}} = d > 0. \qquad (3)$$

Eq. (2) is an aggregate linear demand function for consumption goods in which a is the minimum demand and b is a proxy of the marginal propensity to consume. As the labor market operates first, it depends on employment. Eq. (3) describes the supply of labor which, for the sake of simplicity, is assumed to be inelastic.

The firm supplies consumption goods, demands labor and carries inventory under the following four assumptions in which I_t^{d} denotes the desired stock of inventory and S_t^{e} the expected sales.[2]

Assumption 1. The firm tries to maintain a fixed ratio β of expected sales to desired stock, $I_t^{\mathrm{d}} = \beta S_t^{\mathrm{e}}$, $\beta > 0$.

Assumption 2. The firm determines its production so as to aim at having the quantity, $I_t^{\mathrm{d}} + S_t^{\mathrm{e}}$.

Assumption 3. The firm produces output from employment with constant returns to scale, δL_t, $\delta > 0$.

Assumption 4. The firm has perfect foresight, $S_t^{\mathrm{e}} = Y_t^{\mathrm{D}}$.

Assumption 1 means that the desired stock is an increasing function of the expected sales. It captures the spirit of the micro-behavior of inventory holding firms and is compatible with the (almost)

[1] See Honkapohja and Ito [1] and Hommes [3] for detail.

[2] H-I make the first three assumptions and Hommes replaces rational expectations with perfect foresight. This replacement does not affect the existence results significantly.

constant ratio of actual stock to sales observed in the time series data for the past years.[3] By Assumption 2, ex-ante production at period t, denoted as y_t, is equal to production for expected sales S_t^e plus production for desired inventory investment $I_t^d - I_t$. By Assumption 1, y_t is written as

$$y_t = (1+\beta)S_t^e - I_t. \tag{4}$$

By Assumption 3, ex-ante demand for labor is determined by the inverse of the production function:

$$L_t^D = \frac{(1+\beta)S_t^e - I_t}{\delta}. \tag{5}$$

By Assumption 4, the firm can correctly forecast the incoming demand. Thus substituting (2) for S_t^e in (5) and solving the resultant equation for L_t^D on condition that the firm can realize its demand (i.e., $L_t = L_t^D$) yield the demand for labor as a function of initial stock of inventory:

$$L^D(I_t) = \frac{(1+\beta)a - I_t}{\delta - (1+\beta)b}. \tag{6}$$

The microeconomic study of inventory holding firms confirms that optimal behavior can be characterized by production smoothing, that is, an increase in initial inventory leads to a reduction in production (and hence a reduction in demand for labor). To assert production smoothing, the denominator of (6) is assumed to be positive in Assumption 5 below. Here δ is the marginal productivity of labor and $(1+\beta)b$ is obtained by substituting (2) into (4) and differentiating it with respect to L_t. Thus the positive denominator means that the marginal productivity of labor is greater than the marginal output the firm expects to sell by each unit of labor employed.[4]

Assumption 5. $\delta > (1+\beta)b$.

The macroeconomic model functions as follows. At the beginning of period t, holding the initial level of inventory, I_t, the firm demands labor, $L_t^D = L^D(I_t)$, and the consumer supplies labor, $L_t^S = d$. The min-rule determines the actual employment that is less than or equal to the full-employment and must be nonnegative. Thus the actual employment is a piecewise linear function of the initial level of inventory:

$$L(I_t) = \mathrm{Max}[0, \mathrm{Min}[d, L^D(I_t)]]. \tag{7}$$

Substituting $L(I_t)$ into (2) determines the aggregate demand for goods while the sum of initial inventory and current production determines the total supply of goods, which will be denoted as $Y^S(I_t)$:

$$Y^D(I_t) = a + bL(I_t) \quad \text{and} \quad Y^S(I_t) = I_t + \delta L(I_t). \tag{8}$$

Again the min-rule determines the actual transactions of goods. As it must be nonnegative, the actual transaction is also piecewise linear in I_t:

$$Y(I_t) = \mathrm{Max}[0, \mathrm{Min}[Y^D(I_t), Y^S(I_t)]]. \tag{9}$$

The difference between the total supply and the aggregate demand determines the inventory of the following period at the end of period t, and then the process repeats itself with a new value of initial inventory. The inventory dynamics is, therefore, generated by a first-order difference equation,

$$I_{t+1} = \Theta(I_t) \quad \text{where } \Theta(I_t) \equiv Y^S(I_t) - Y^D(I_t). \tag{10}$$

As the result of the piecewise linearities of $Y^D(I_t)$ and $Y^S(I_t)$, $\Theta(I_t)$ is also piecewise linear.

3. Dynamic equation

In this section, we derive a dynamic system of the model, using the behavioral functions defined in the previous section. In their dynamic analysis, H-I distinguish two cases, depending on the sign of $\delta d - (a + bd)$. They demonstrate the existence of a fixed point of $\Theta(I_t)$ in the case where $\delta d - (a+bd) > 0$ and call it a *Keynesian steady state*.[5] Hommes confines his dynamic analysis to the Keynesian steady state. Since we attempt to extend some of Hommes's results in this study, we make the same assumption as he does; production

[3] See Fig. 1 of Eckalbar [2].

[4] A referee suggests this interpretation of Assumption 5.

[5] See Theorem 1 of Honkaphja and Ito [1].

is greater than demand when labor supply determines employment,

Assumption 6. $\delta d > (a + bd)$.

Taking account of the Eqs. (5) to (10), we can obtain the following piecewise linear function of inventory:

$$\begin{aligned} \Theta(I_t) &= \Theta_1(I_t) \quad \text{if } I_t \leqslant I_M, \\ \Theta(I_t) &= \Theta_2(I_t) \quad \text{if } I_M \leqslant I_t \leqslant I_m, \\ \Theta(I_t) &= \Theta_3(I_t) \quad \text{if } I_m \leqslant I_t, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \Theta_1(I_t) &\equiv I_t + d\delta - (a + bd), \\ \Theta_2(I_t) &\equiv -\frac{b\beta}{\delta - b(1+\beta)} I_t + \frac{a\beta\delta}{\delta - b(1+\beta)}, \\ \Theta_3(I_t) &\equiv I_t - a. \end{aligned} \tag{12}$$

In (11), two critical levels of inventory, denoted as I_M and I_m, are defined by $L^D(I_M) = d$ and $L^D(I_m) = 0$, respectively and are spelled out as

$$\begin{aligned} I_M &= (1+\beta)a - d[\delta - b(1+\beta)], \\ I_m &= (1+\beta)a. \end{aligned} \tag{13}$$

Under Assumptions 5 and 6, (12) indicates that $\Theta_1(I_t)$ shifts upward from the $I_{t+1} = I_t$ locus and $\Theta_3(I_t)$ shifts downward and that $\Theta_1(I_t)$ and $\Theta_3(I_t)$ have positive slopes while $\Theta_2(I_t)$ has a negative slope. Thus $\Theta(I_t)$, as a whole, takes on a tilted Z-shaped profile and kinks at two critical levels of inventory, I_M and I_m. As is evident from the profile of $\Theta(I_t)$, $\Theta_2(I_t)$ has a unique fixed point that is a stationary level of inventory. The corresponding labor is determined through (6). We denote those stationary values by

$$I^* = \frac{a\beta\delta}{\delta - b} \quad \text{and} \quad L^* = \frac{a}{\delta - b}. \tag{14}$$

Both are positive as Assumption 5 implies $\delta > b$. Moreover, it can be observed that the stationary level of employment is less than the full-employment level, $L^* < d$, and the stock-out is impossible if the labor demand determines employment (i.e., $L_t = L^D(t)$). At the stationary state, consumers can achieve their intended transactions in the goods market but not in the labor market while firms can achieve their intended transactions in the labor market but not in the goods market. This is the reason why H-I call this state the Keynesian steady state.

The local stability of I^* depends on the eigenvalue of the dynamic equation $\Theta(I_t)$ evaluated at the fixed point, the eigenvalue of which is equal to a slope of $\Theta_2(I_t)$. Accordingly, the stationary state is locally stable if the slope is less than unity in absolute value. The piecewise linearity of $\Theta(I_t)$ further implies that the stationary state is globally stable as well. There are no persistent oscillations in the stable case. Since the main purpose of this study is to investigate the nature of the long-run unstable market dynamics, we focus on the unstable case in which the slope of $\Theta_2(I_t)$ is assumed to be greater than unity in absolute value.

Assumption 7. $b\beta/\delta - b(1 + \beta) > 1$.

Assumptions 5 and 7 limit the range of the marginal productivity to $b(1 + \beta) < \delta < b(1 + 2\beta)$. These inequalities imply that the following analysis is concentrated on a case where the marginal productivity δ is neither much smaller nor much larger. If δ is larger than $b(1 + 2\beta)$, the market is stabilized as stated above. If δ is smaller than $b(1 + \beta)$, the firm behaves paradoxically; demand for labor increases with the stock of inventory and zero employment prevails for lower levels of inventory.

Under Assumption 7, the fixed point losses its stability and forces trajectories of inventory to move away from its neighborhood. However, the upper and/or lower bounds (i.e., $\Theta_1(I_t)$ and/or $\Theta_3(I_t)$) induces trajectories to bounce back to a neighborhood of the fixed point, and, then, the trajectories move away again. Thus, trajectories stay in a bounded region and perpetually fluctuate within it. Hommes characterizes such complex dynamics fully and demonstrates that $\Theta(I_t)$ can generate various dynamics ranging from periodic cycles to chaos, depending on the parametric values.[6]

[6] See Theorems 2B.1, 2 and 3 of Hommes [3].

Here we go one step further and consider the economic implications of chaotic inventory fluctuations. We know, by the theorem of Lasota and Yorke [6], that $\Theta(I_t)$ is *ergodically chaotic*.[7] This means that trajectories are unstable, highly erratic and unpredictable. Such behavior suggests that trajectories are represented by densities (or distributions) in the long-run. Indeed, the Mean Ergodic Theorem of Birkhoff-von Newman[8] implies that if $\Theta(I_t)$ is ergodic, its time average converges to its space average,

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} f(\Theta^t(I)) = \int_{\mathbb{I}} f(I)\Phi(I)\,\mathrm{d}I$$

$$\text{for almost all } I\in\mathbb{I}, \tag{15}$$

where $f(\cdot)$ is an integrable function, $\mathbb{I}$ is its domain and $\Phi(\cdot)$ is an invariant density function. Eq. (15) implies that it is possible to characterize the long-run average behavior of chaotic trajectories once we obtain the functions $f(\cdot)$ and $\Phi(\cdot)$. Thus our next job is to find appropriate integrable functions to evaluate a chaotic inventory trajectory and to construct an explicit form of its density function. To this end, our approach consists of three steps: (1) we will find a profit function and a utility function that are consistent with the behavioral specifications of the macroeconomic model. Those functions are clues to calculate the long-run average profit and utility; (2) we will develop a method of constructing density functions of chaotic trajectories; (3) applying the long-run property described by (15), we will compare the long-run behavior with the stationary behavior. In particular, we will investigate the following problem: *Can chaotic fluctuations be preferable to a steady state from the long-run perspective*?

[7] The theorem can be stated as follows: Let $\theta: D\to D$ be piecewise C^2 where D is an interval. If $|\theta'(I_t)|\geqslant\sigma>1$ for almost everywhere in D, then there exists an absolutely continuous invariant measure. Although $\Theta(I_t)$ does not satisfy the condition of the theorem, its iterated equation, $\Theta^n(I_t)$ for $n\geqslant 2$ does. Consequently the theorem asserts the existence of a unique absolutely continuous measure that is invariant for $\Theta^n(I_t)$, namely, $\Theta(I_t)$ is ergodically chaotic.

[8] See Day [7] and its reference to this theorem.

4. Microeconomic underpinning

This section is divided into three subsections. We will formulate the maximization problems of the firm and the consumer in order to construct a consistent microeconomic underpinning of the macroeconomic model in the first two subsections and develop a method of constructing densities of chaotic trajectories in the third.

4.1. Profit maximization

A firm determines its production so as to maximize an expected profit that is the expected sales minus costs associated with production and holding inventories:

$$\Pi_t = pS_t^e - C(y_t) - H(I_{t+1} - I_t^d) + V(I_{t+1}), \tag{16}$$

subject to the inventory identity and the perfect foresight,

$$I_{t+1} = I_t + y_t - S_t^e, \quad \text{and} \quad S_t^e = a + bL_t. \tag{17}$$

p is the price of the goods. The firm incurs two costs in Π_t: $C(y_t)$ is the cost of producing y_t and $H(I_{t+1} - I_t^d)$ is the cost of holding inventory, which depends on the deviation of an actual stock of inventory from the desired stock. $V(I_{t+1})$ is the discounted maximum of the expected profit that the firm can achieve by employing the best policy from the next period and onwards. It takes into account the discount factor. Labor is the only input for production and the production function is linearly homogenous by Assumption 3 so that the production cost is equal to the labor cost:

$$C(y) = \frac{w}{\delta}y, \tag{18}$$

where w is the nominal wage rate. To permit explicit solutions in this maximization problem, we specify each of the remaining functions involved in Π_t as follows:

$$H(I_{t+1} - I_t^d) = \frac{\alpha}{2}(I_{t+1} - I_t^d)^2 \quad \text{and} \quad V(I_{t+1}) = \gamma I_{t+1}. \tag{19}$$

The quadratic cost function of holding inventory implies that as an actual stock of inventories deviates from the desired stock, the cost of holding inventory increases due to the increasing possibility of losing goodwill for negative deviation ($I_{t+1} < I_t^d$) and due to the increment of the storage cost for positive deviation ($I_{t+1} > I_t^d$). The linearity of $V(I_{t+1})$ shows that the imputed real value increases as inventories increase.[9] Differentiating (16) with respect to L_t yields the following condition for maximizing expected profit:

$$p\frac{\mathrm{d}S_t^e}{\mathrm{d}L_t} - C'(y_t)\frac{\mathrm{d}y_t}{\mathrm{d}L_t} - H'(I_{t+1} - I_t^d)\frac{\mathrm{d}(I_{t+1} - I_t^d)}{\mathrm{d}L_t} + V'(I_{t+1})\frac{\mathrm{d}I_{t+1}}{\mathrm{d}L_t} = 0, \tag{20}$$

which roughly implies that the marginal sales revenue minus the cost of producing one more unit today minus the cost of holding it until tomorrow is equal to the expected revenue for selling one more unit out of inventory tomorrow. Our main aim in this subsection is to construct a possible microeconomic underpinning of Eq. (4), the supply side of the macroeconomic model. We may find, by observing the optimal condition (20), that the following condition is sufficient for our purpose:

$$p\frac{\mathrm{d}S_t^e}{\mathrm{d}L_t} + V'(I_{t+1})\frac{\mathrm{d}I_{t+1}}{\mathrm{d}L_t} = C'(y_t)\frac{\mathrm{d}y_t}{\mathrm{d}L_t}. \tag{21}$$

The left-hand side is the sum of the current expected sales revenue and the discounted future profit induced by a unit change in employment while the right-hand side implies increases in production cost induced by the unity change. Thus if, as indicated by (21), the marginal revenue is equal to the marginal cost of production, the optimal demand of labor is determined so as to minimize the marginal cost of holding inventory, $H'(I_{t+1} - I_t^d) = 0$ which is achieved for $I_{t+1} = I_t^d$. Rearranging it to take account of the inventory identity and Assumption 1, we have the following form of the optimal production:

$$y_t = (1 + \beta)S_t^e - I_t, \tag{22}$$

which is identical to Eq. (4). Thus the above profit maximization can provide a possible microeconomic underpinning of the supply side of the macroeconomic model. Inserting the functions specified above into (21) gives the following relationship among parameters, which we assume to hold in the sequel.

Assumption 8. $pb + \gamma(\delta - b) = w$.

According to the piecewise linearity of the actual employment, the maximized expected profit (16) is also piecewise linear:

$$\Pi(I_t) \equiv \mathrm{Min}\{\Pi_3(I_t), \mathrm{Min}\{\Pi_1(I_t), \Pi_2(I_t)\}\}, \tag{23}$$

where

$$\begin{aligned} \Pi_1(I_t) &\equiv (p - \gamma)a + \gamma I_t - \frac{\alpha}{2}(I_t - I_M)^2 \quad \text{for } I_t \leqslant I_M, \\ \Pi_2(I_t) &\equiv (p - \gamma)a + \gamma I_t \quad \text{for } I_M < I_t < I_m, \\ \Pi_3(I_t) &\equiv (p - \gamma)a + \gamma I_t - \frac{\alpha}{2}(I_t - I_m)^2 \quad \text{for } I_m \leqslant I_t. \end{aligned} \tag{24}$$

4.2. Utility maximization

Supposing that a consumer demands money,[10] we give one possible microeconomic underpinning of the demand side of the macroeconomic model. Possessing the actual quantity traded in the labor

[9] In this study which is a starting point for a more complete study, we adopt the linear assumption for three reasons. First, it is basically an approximation to inventory holding behavior in which the present value function is supposed to be $V'(I_{t+1}) > 0$ and $V''(I_{t+1}) \leqslant 0$. Second it makes the formidable mathematical problem simpler and manageable. Third, only such a simplification enables us to derive rigorous results and economically interesting insights.

[10] Although it is not explicitly stated, we can presume that Honkapohja and Ito [1] and Hommes [3] deal with a monetary economy in which exchange takes place through money in markets. Thus consumers as well as firms may hold money balance. For the sake of analytical simplicity, we assume (1) the stock of money in the economy is exogenously given; (2) the firm always pays dividends equal to its profit to consumers at the beginning of the period.

market, L, the consumer makes a choice of consumption demand and demand for money by maximizing the utility function subject to the sum of the wage income wL and the initial amount of money holding $\bar{M}$. If we assume the Cobb–Douglas utility function,

$$U \equiv \sigma S^x \left(\frac{M}{p}\right)^{1-x}, \quad \sigma > 0, \quad 1 > x > 0, \tag{25}$$

where S is the demand for consumption goods and M/p is the desired end-of-period real balance, routine calculations yield the demand functions of consumption goods and money:

$$S = \frac{x}{p}(wL + \bar{M}) \quad \text{and} \quad \frac{M}{p} = \frac{(1-x)}{p}(wL + \bar{M}). \tag{26}$$

If we set $a = x\bar{M}/p$ and $b = xw/p$, the optimal demand is written as

$$S = a + bL, \tag{27}$$

which is identical with the aggregate demand function defined in (2). Thus the above specification for utility maximization can be a possible microeconomic underpinning of the demand side of the macroeconomic model. Again, due to the piecewise linearity of actual employment, the indirect utility function is also piecewise linear:

$$U(I_t) \equiv \mathrm{Min}\{U_3(I_t), \mathrm{Max}\{U_1(I_t), U_2(I_t)\}\}, \tag{28}$$

where

$$U_1(I_t) \equiv \sigma x^x (1-x)^{1-x}\left(\frac{\bar{M}}{p} + \frac{w}{p}d\right) \quad \text{for } I_t \leqslant I_M,$$

$$U_2(I_t) \equiv \sigma x^x (1-x)^{1-x}\left(\frac{\bar{M}}{p} + \frac{w}{p}L^D(I_t)\right)$$

$$\text{for } I_M < I_t < I_m, \tag{29}$$

$$U_3(I_t) \equiv \sigma x^x (1-x)^{1-x}\frac{\bar{M}}{p} \quad \text{for } I_m \leqslant I_t.$$

4.3. Density function

Given $\Pi(I_t)$ and $U(I_t)$, the long-run properties of ergodically chaotic trajectories can be investigated when the densities of trajectories are known. It has been demonstrated that an invariant density is a fixed point of the Frobenius–Perron operator.[11] Boyarsky and Goyarsky [8] show that it is possible to construct closed-form expressions for densities of chaotic trajectories if a dynamic equation is piecewise linear, continuous and *Markov*.[12] $\Theta(I_t)$ is clearly continuous and piecewise linear. Furthermore, in our analysis, it can be Markov if, for some natural number N, the kinked point is either eventually fixed or a periodic point with period-N. According to their theorem, the following procedure is appropriate for constructing the density function which $\Theta(I_t)$ permits.

(1) Find a condition under which the kinked point is eventually fixed or periodic.
(2) Define a trapping interval that eventually traps all trajectories and restrict $\Theta(I_t)$ to it.
(3) Divide the trapping interval into subintervals by the points of the periodic orbit.
(4) Construct a matrix version of the Frobenius–Perron operator, $\boldsymbol{M}_N = (m_{ij})$ in which its entries are defined by

$$m_{ij} = |\Theta'_j|^{-1}\delta_{ij},$$

Θ'_j being the slope of Θ on $\mathbb{I}_j$ where $\mathbb{I}_j$ is a subinterval of the trapping interval $(j = 1, 2, \ldots, N)$ and $\delta_{ij} = 1$ if $\mathbb{I}_i \subset \Theta(\mathbb{I}_j)$ and 0 otherwise.
(5) Solve the matrix equation, $\boldsymbol{x}\boldsymbol{M}_N = \boldsymbol{x}$ where $\boldsymbol{x} = (x_i) \in R^N$. Then elements of solution $\boldsymbol{x}$ satisfying the requirement $\sum_{i=1}^{N-1} x_i(\mathbb{I}_i - \mathbb{I}_{i+1}) = 1$ are constant steps of a unique absolutely continuous invariant density function under $\Theta(I_t)$.

When I_0 is an initial point and not a periodic point of period-N cycle, we can define the *long-run*

[11] See Section 8.6 of Day [7] for a full account of the Frobenius–Perron operator.

[12] See their Theorem 3. A continuous map is called *Markov* if the domain of the map is divided into disjointed subintervals and the restriction of the map to one subinterval is a homeomorphism onto some subinterval(s).

average profit and the *long-run average utility* by

$$\pi'_N \equiv \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \Pi(\Theta^t(I_0)) = \sum_{i=1}^{N} \int_{\mathbb{I}_i} \Pi_i(I)\Phi_i(I)\,dI,$$
$$u'_N \equiv \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} U(\Theta^t(I_0)) = \sum_{i=1}^{N} \int_{\mathbb{I}_i} U_i(I)\Phi_i(I)\,dI, \tag{30}$$

where $\Phi_i(I) = x_i$ is a density function defined over the partition, $\mathbb{I}_i$. We call the profit that a firm obtains at the stationary state the *stationary state profit* and the utility that a consumer obtains at the stationary state the *stationary state utility*, and denote those by

$$\pi^*_N \equiv \pi(I^*_N) \quad \text{and} \quad u^*_N \equiv u(I^*_N), \tag{31}$$

where I^*_N is the fixed point of $\Theta(I)$.

5. Simulation results

In this section, we present one numerical example concerning the long-run average behavior of consumers and firms. We distinguish two cases depending on the relative magnitude between the upward shift of $\Theta_1(I_t)$, $\delta d - (a + bd)$, and the downward shift of $\Theta_3(I_t)$, a. We call the case in which $\delta d - (a + bd) < a$ holds *upper-constrained* and the case in which $\delta d - (a + bd) > a$ *lower-constrained*. Following the procedure given above, we construct density functions of chaotic trajectories and calculate the long-run average profit and utility in each case. $\Theta(I_t)$ is nonlinear (i.e., piecewise linear) and its nonlinearity becomes more pronounced as the slope of $\Theta_2(I)$ gets larger in absolute value. Among the salient features of generations of complex dynamics is the steepness of the slope. For notational simplicity, we denote it by $B \equiv b\beta/\delta - b(1 + \beta)$.

5.1. Condition for eventual convergence

This example deals with a *period-1 cycle* (i.e., $N = 1$) where the kinked point of $\Theta(I)$ eventually converges at the stationary state. We consider the upper-constrained case first. As illustrated in Fig. 1A where the upward shift of $\Theta_1(I_t)$ is less than the downward shift of $\Theta_3(I_t)$, $\delta d - (a + bd) < a$, a trajectory starting from I_M passes through $I_{\text{MAX}} \equiv \Theta(I_M)$, the local maximum, then visits a point, $I_{M2} \equiv \Theta_2(I_{\text{MAX}})$, and finally converges at the stationary point, I^*. Since the local maximizer is mapped to the local maximum by definition, the kinked point, I_M, converges to the steady state if $\Theta_1(I_{M2}) = I^*$ holds. Alternatively put, $\Theta_1(\Theta_2(\Theta_1(I_M))) = I^*$ or more generally, $\Theta^3(I_M) = I^*$ is the condition under which $\Theta(I_t)$ generates a period-1 cycle in the upper-constrained case. Returning to (11)–(13), we have, after rearranging terms,

$$\Theta^3(I_M) - I^* = \frac{(\delta d - a - bd)(B^2 - B - 1)}{1 + B}. \tag{32}$$

Solving $B^2 - B - 1 = 0$ gives the slope of $\Theta_2(I_t)$ being equal to $B_1 = (1 + \sqrt{5})/2$ where the subscript indicates the number of periods.[13] Therefore, for $B = B_1, I_M$ is mapped to I^* with the third iterations (i.e., $\Theta^3(I_M) = I^*$). We call the set $\{I_M, I_{\text{MAX}}, I_{M2}, I^*\}$ a stationary trajectory of I_M under Θ. Since it can be shown that $I_{M2} \leqslant \Theta^n(I_0) \leqslant I_{\text{MAX}}$ holds for any initial point I_0 if n is large enough, an interval $\mathbb{I}_M \equiv [I_{M2}, I_{\text{MAX}}]$ can be the trapping interval which all trajectories eventually enter and cannot leave. As far as the asymptotic or long-run behavior is concerned, it is enough to consider a restriction of the dynamic equation to $\mathbb{I}_M$ that will be denoted by $\Theta(I_t)_{|\mathbb{I}_M} : \mathbb{I}_M \to \mathbb{I}_M$.

By the same token, we can clarify the condition under which $\Theta(I_t)$ generates the period-1 cycle in the lower-constrained case. As seen from Fig. 1B where $\delta d - (a + bd) > a$, I_m converges to the steady-state passing though the local minimum, $I_{\min} \equiv \Theta_3(I_m)$, and a point $I_{m2} \equiv \Theta_2(I_{\min})$. Thus $\Theta_3(\Theta_2(\Theta_3(I_m))) = I^*$ or $\Theta^3(I_m) = I^*$ is the condition for the generation of period-1 cycle. Solving $\Theta^3(I_m) = I^*$ yields the same steepness of the slope $B_1 = 1 + \sqrt{5}/2$ in the lower-constrained case as one in the upper constrained case. An interval $\mathbb{I}_m \equiv [I_{\min}, I_{m2}]$ can be the trapping interval and

[13] Although $B^2 - B - 1 = 0$ has a positve root and a negative root, B must be positive by Assumption 5.

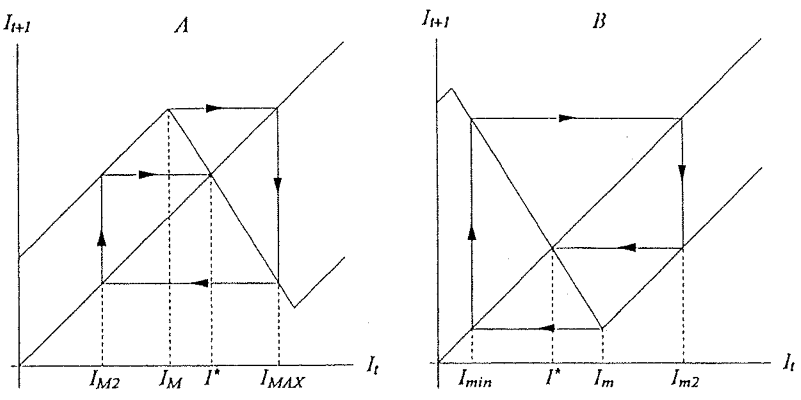

Fig. 1. Period-1 cycles.

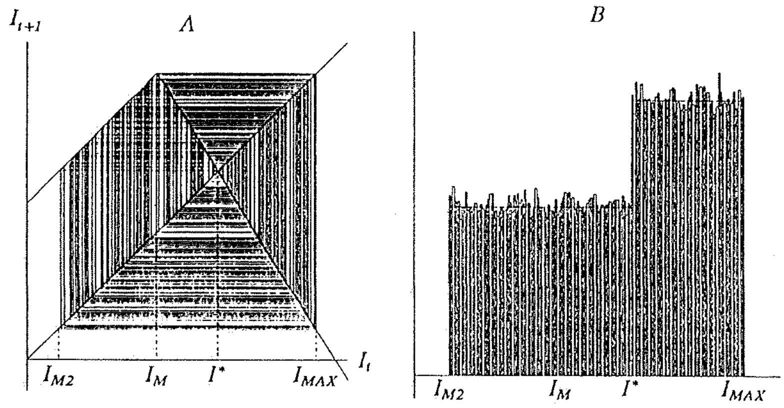

Fig. 2. Nonperiodic fluctuations and numerical histogram for $\Theta(I_t)$.

a restriction of the dynamic equation to $\mathbb{I}_m$ is defined, $\Theta(I_t)_{|\mathbb{I}_m} : \mathbb{I}_m \to \mathbb{I}_m$. It is possible to transform $\Theta(I_t)_{|\mathbb{I}_m}$ to $\Theta(I_t)_{|\mathbb{I}_M}$ through a coordinate change. That means that $\Theta(I_t)_{|\mathbb{I}_M}$ generates essentially the *same* dynamics as $\Theta(I_t)_{|\mathbb{I}_m}$. To avoid unnecessary repetition, we choose $\Theta(I_t)_{|\mathbb{I}_M}$ as the canonical map for a while. However, as we will see, the same dynamics has different economic implications under different economic circumstances.

Fig. 2A shows a return map in which we fix the parameters $a = 0.2$, $b = 0.75$, $d = \delta = 1$, and $\beta = (\sqrt{5} - 1)/6$.[14] There, $\Theta(I_t)_{|\mathbb{I}_M}$ generates a cobweb type trajectory that apparently alternates nonperiodic ups and downs within the trapping interval. Fig. 2B shows the histogram obtained from 100,000 iterates of $\Theta(I)$ under the same parameters values on the intervals of width $\frac{1}{200}$. It can be seen that inventories are distributed through the full range of the trapping interval. Two horizontal lines in Fig. 2B indicate the heights of the density function that is analytically derived below.

5.2. Construction of density function

5.2.1. The upper-constrained case

We proceed to determine an explicit form of a density function in the upper-constrained case. Points, I_{M2}, I_M, I^* and $I_{\rm MAX}$, can be the partition points of the trapping interval so that $\mathbb{I}_M$ is divided

[14] The choice of the parameters a, b, d and δ is the same as the one Hommes [3] set out in his numerical investigations. Rearranging the definition of B yields $\beta = B_1(\delta - b)/(1 + B_1)b$ where $B_1 = (1 + \sqrt{5}/2)$ in this example.

into three subintervals, $\mathbb{I}_1 = [I_{M2,}, I_M]$, $\mathbb{I}_2 = [I_M, I^*]$ and $\mathbb{I}_3 = [I^*, I_{\mathrm{MAX}}]$. Since $\mathbb{I}_3 \subset \Theta(\mathbb{I}_1)|_{\mathbb{I}_M}$, $\mathbb{I}_3 \subset \Theta(\mathbb{I}_2)|_{\mathbb{I}_M}$ and $\mathbb{I}_1, \mathbb{I}_2 \subset \Theta(\mathbb{I}_3)|_{\mathbb{I}_M}$, a matrix version of the Frobenius–Perron operator is

$$M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{1}{B_1} \\ \frac{1}{B_1} & \frac{1}{B_1} & 0 \end{pmatrix}. \tag{33}$$

The equations $xM_1 = x$, $x = (x_i) \in R^3$ has a solution, $x = (X, X, B_1 X)$. X is then determined by solving $\sum_{i=1}^{3} \int_{\mathbb{I}_i} \Phi_i(I)\,\mathrm{d}I = 1$ where $\Phi_1(I) \equiv X$ on $\mathbb{I}_1$, $\Phi_2(I) \equiv X$ on $\mathbb{I}_2$, and $\Phi_3(I) \equiv B_1 X$ on $\mathbb{I}_3$. We find that the density function is a step function with constant step on the intervals, $\mathbb{I}_i$, $i = 1, 2, 3$,

$$\begin{aligned}
\Phi_1(I) &= \frac{1}{2(\delta d - (a + bd))} \quad \text{for } I \in \mathbb{I}_1 = [I_{M2,}, I_M], \\
\Phi_2(I) &= \frac{1}{2(\delta d - (a + bd))} \quad \text{for } I \in \mathbb{I}_2 = [I_M, I^*], \\
\Phi_3(I) &= \frac{B_1}{2(\delta d - (a + bd))} \quad \text{for } I \in \mathbb{I}_3 = [I^*, I_{\mathrm{MAX}}].
\end{aligned} \tag{34}$$

We are now ready to investigate the long-run average behavior. The long-run average profit and utility can be computed explicitly by substituting the explicit form of the density function $\Phi_i(I)$ in (34) and the piecewise linear profit function, $\Pi_i(I)$ in (24), respectively, the piecewise linear utility function $U_i(I)$ in (29), into the most right expressions of (30):

$$\begin{aligned}
\pi_1' &= \int_{\mathbb{I}_1} \Pi_1(I)\Phi_1(I)\,\mathrm{d}I + \int_{\mathbb{I}_2} \Pi_2(I)\Phi_2(I)\,\mathrm{d}I \\
&\quad + \int_{\mathbb{I}_3} \Pi_2(I)\Phi_3(I)\,\mathrm{d}I, \\
u_1' &= \int_{\mathbb{I}_1} U_1(I)\Phi_1(I)\,\mathrm{d}I + \int_{\mathbb{I}_2} U_2(I)\Phi_2(I)\,\mathrm{d}I \\
&\quad + \int_{\mathbb{I}_3} U_2(I)\Phi_3(I)\,\mathrm{d}I.
\end{aligned} \tag{35}$$

The difference between the long-run average profit and the stationary state profit, the long-run average utility and the stationary utility are computed as follows:

$$\begin{aligned}
\pi_1' - \pi_1^e &= \frac{(a + bd - d\delta)\{3B_1\gamma - \alpha(a + bd - d\delta)\}}{12B_1(1 + B_1)} \\
&< 0, \\
u_1' = u_1^e &= \frac{\sigma x^x(1 - x)^{1-x}\bar{M}\delta}{p(\delta - b)}.
\end{aligned} \tag{36}$$

The direction of inequality in the first equation of (36) is due to Assumption 6, and indicates that the stationary state profit is greater than the long-run average profit. The second results imply that the long-run average utility is the same as the stationary utility.

5.2.2. The lower-constrained case

In the lower-constrained case, using the same procedure as in the upper-constrained case, we have the following density function:

$$\begin{aligned}
\Phi_1(I) &= \frac{B_1}{2a} \quad \text{for } I \in [I_{\min,}, I^*], \\
\Phi_2(I) &= \frac{1}{2a} \quad \text{for } I \in [I^*, I_m], \\
\Phi_3(I) &= \frac{1}{2a} \quad \text{for } I \in [I_m, I_{m2}],
\end{aligned} \tag{37}$$

and the following computational results:

$$\begin{aligned}
\pi_1' - \pi_1^e &= \frac{a\{3B_1\gamma - a\alpha\}}{12B_1(1 + B_1)} \lesseqgtr 0 \\
&\text{according to } 3B_1\gamma - a\alpha \lesseqgtr 0, \\
u_1' = u_1^w &= \frac{\sigma x^x(1 - x)^{1-x}\bar{M}\delta}{p(\delta - b)}.
\end{aligned} \tag{38}$$

The equality in the second line of (38) implies that the consumer, again, has the same utility. The inequalities in the first line imply that the firm can yield a larger long-run average profit than a stationary state profit. To clarify the direction of inequality, we note the following relationships:

$$a = \frac{x\bar{M}}{p} \quad \text{where}$$

$$x = \frac{bp}{w} \quad \text{and} \quad \gamma = \frac{w - pb}{\delta - b}, \tag{39}$$

where a and x are set to induce the macro demand function from the utility maximization and the last is due to Assumption 8. Substituting those relationships into $3B_1 - \gamma a\alpha \gtreqless 0$ and then rearranging the expression yield

$$\Delta \equiv 3B_1 \frac{w - pb}{\delta - b} - \alpha \frac{\bar{M}b}{w} \gtreqless 0. \tag{40}$$

$\delta - b > 0$ by Assumption 5 and the positive marginal future revenue $\mathrm{d}V/\mathrm{d}I_{t+1} = \gamma > 0$ implies $w - pb > 0$. Further it is reasonable to assume that the marginal productivity is greater than the real wage rate so as to induce a positive level of production, $\delta > w/p$. In order to see the effects caused by the changes in parameters' values on the direction of inequalities in (40), we differentiate Δ with respect to w, δ and b:

$$\begin{cases} \dfrac{\partial \Delta}{\partial w} = \dfrac{bM\alpha}{w^2} + \dfrac{3B_1}{\delta - b} > 0 & \text{if } \delta - b > 0, \\ \dfrac{\partial \Delta}{\partial \delta} = \dfrac{3B_1(pb - w)}{(\delta - b)^2} < 0 & \text{if } w - pb > 0, \\ \dfrac{\partial \Delta}{\partial b} = \dfrac{-M\alpha(\delta - b)^2 - 3B_1 w(p\delta - w)}{(\delta - b)} < 0 & \text{if } \delta > \dfrac{w}{p}. \end{cases} \tag{41}$$

Given α and $\bar{M}$, (40) and (41) imply that a higher wage rate, a lower marginal productivity and a lower propensity to consume lead to $\pi_1' > \pi_1^e$, that is, the long-run average profit is larger than the steady-state profit.

Those results are summarized as follows:

Theorem. *If the slope of $\Theta_2(I_t)$ in absolute value is equal to $B_1 = 1 + \sqrt{5}/2$, the kinked point converges at the stationary state, and the dynamic equation $\Theta(I_t)$ generates chaotic fluctuations if the initial level of inventory is not on the stationary trajectories. The long-run average profit is smaller than the stationary state profit in the upper-constrained case; $\pi_1' < \pi_1^e$ if $\delta d - (a + bd) < a$, and can be larger in the lower-constrained case; $\pi_1' \lesseqgtr \pi_1^e$ if $\delta d - (a + bd) > a$. The long-run average utility is the same as the stationary state utility.*

5.3. *Interpretation: Disequilibrium dominates equilibrium*

We attempt to explain intuitively why the long-run average profit can be larger than the steady-state profit in the lower-constrained case but not in the upper-constrained case. To this end, we juxtapose the graphs of the profit function defined in (34) and the density functions defined in (34) and (37). Fixing the parameters $b = 0.75$, $d = \delta = 1$, we set $a = 0.2$ in Fig. 3A that corresponds to the upper-constrained case (i.e., $\delta d - (a + bd) = 0.05 < 0.2 = a$) and $a = 0.08$ in Fig. 3B that corresponds to the lower-constrained case (i.e., $\delta d - (a + bd) = 0.17 > 0.08 = a$). In both figures, the heights of the density (step) functions are appropriately adjusted.We observe that the profit function is increasing for I_t. We also observe from Fig. 2A that the inventory dynamics is characterized by regularly alternating behavior with respect to the stationary level of inventory, I^*, that is, if the level of inventory is smaller than I^*, it is mapped to a point larger than I^*; if the level of inventory is larger than I^*, then it is mapped to a point less than I^*. Consequently the corresponding profit also alternate with respect to the stationary-state profit; the profit in one period is smaller than the steady state profit while the profit in the next is larger than the stationary state profit. If we denote π_2 as the average profit for a two-period iteration $\{I_t, \Theta(I_t)\}$ (i.e., $\pi_2 \equiv \Pi(I_t) + \Pi(\Theta(I_t))/2)$ and π^e the stationary state profit for I^* (i.e., $\pi^e \equiv \Pi(I^*)$). Then the difference between the average profit and the stationary profit is

$$\pi_2 - \pi^e = \tfrac{1}{2}\{(\Pi(I_t) - \pi^e) + (\Pi(\Theta(I_t)) - \pi^e)\}. \tag{42}$$

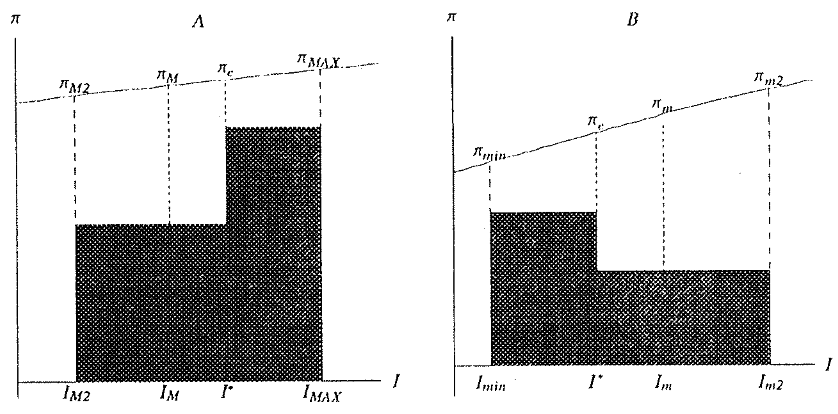

Fig. 3. Profit Functions and density functions.

Suppose $I_t \in \mathbb{I}_1 \cup \mathbb{I}_2$ for a moment.[15] Then $(\Pi(I_t) - \pi^e) < 0$ and $(\Pi(\Theta(I_t)) - \pi^e) > 0$. The sum of those terms can be positive if I_t is close to I^* (i.e., $(\Pi(I_t) - \pi^e) < 0$ is small) and $\Theta(I_t)$ is close to I_{MAX} ($(\Pi(\Theta(I_t)) - \pi^e) > 0$ is large). It may be negative if I_t is close to I_{M2} (i.e., $(\Pi(I_t) - \pi^e) < 0$ is large) and $\Theta(I_t)$ is close to I^* ($(\Pi(\Theta(I_t)) - \pi^e) > 0$ is small). The long-run average profit is the average of π_2 over the entire period and inventory trajectory visits whole the interval $[I_{M2}, I_{MAX}]$ chaotically. Thus it depends on the shape of the density function whether the long-run average profit is larger or smaller than the stationary state profit. In Fig. 3A, the length of the interval $\mathbb{I}_3$ is shorter than the length of the interval, $\mathbb{I}_1 \cup \mathbb{I}_2$. Since $(\Pi(\Theta(I_t)) - \pi^e) > 0$ on $\mathbb{I}_3$ and $(\Pi(I_t) - \pi^e) < 0$ on $\mathbb{I}_1 \cup \mathbb{I}_2$. This means $\pi_2 - \pi^e < 0$ on average, which the first line of (36) indicates. On the other hand, in Fig. 3B, $(\Pi(I_t) - \pi^e) < 0$ for $I_t \in [I_{min}, I^*]$ and $(\Pi(\Theta(I_t)) - \pi^e) > 0$ for $I_t \in [I^*, I_{m2}]$ where $|I_{m2} - I^*| > |I^* - I_{min}|$. $\pi_2 - \pi^e$ could be positive on average, which the first line of (38) indicates.

6. Concluding remarks

This study presents a disequilibrium macro dynamical model constructed on the basis of a firm's profit maximizing behavior and a consumer's utility maximizing behavior. Transactions in markets are determined by the minimum of supply and demand. The differences are stored as inventory which serves as the main source of dynamics. In this study, we focus on a case where inventory fluctuations are chaotic and analyze the qualitative properties of such fluctuations. In particular, we construct densities of chaotic inventory trajectories and calculate the long-run average profit and utility. We demonstrate, with the aid of numerical examples, that the firm could earn a higher long-run average profit than the steady-state profit in the lower-constrained case under some parameter constellations, while the consumer's long-run average utility is the same as the stationary state utility. These results suggest that even in a disequilibrium market, there is a possibility that perpetual disequilibrium can be preferable to the steady state.

The fact that our model can justify the persistent and chaotic fluctuations in a disequilibrium economy is itself worthy of note. Although we present only one example in this study, it can be verified that qualitatively similar results hold in all other cases in which the kinked points are eventually fixed or periodic.

[15] The similar results hold if we choose $I_t \in \mathbb{I}_3$.

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A New Perspective on the Generating Mechanism of the Business Cycle

Kazuyuki SASAKURA*

Abstract

This paper presents a new perspective on the generating mechanism of the business cycle. In the new perspective the economic system has a stable equilibrium and gives rise to undulatory processes with the help of small disturbances. But a certain once-and-for-all shock can trigger a persistent periodic oscillation. This corresponds to the case of a hard generation oscillation in the dynamical systems theory. It is shown that a subcritical Hopf bifurcation is useful to explain such a phenomenon. Existing business cycle models can be reinterpreted in the new perspective. As examples of application Kaldor's model and Benassy's model are considered.

1. Introduction

For a long time there have been two conflicting perspectives on why economy fluctuates periodically. Needless to say, one is formally called the exogenous business cycle theory, while the other the endogenous business cycle theory. In the first perspective the economic system has a mechanism which is being subjected to damped vibrations of a periodic character and to causal disturbances accumulating energy just sufficient to counterbalance the damping (Slutzky (1937, p. 131)). In fact, most macroeconomists now share this perspective (Blanchard and Fisher (1989, p. 277)).

Frisch (1933), though advocate of the first perspective, also suggested the analytical usefulness of the second perspective, i.e., the idea of an auto-maintained oscillation in mathematical language. In the second perspective the economic system itself can generate a periodic fluctuation without disturbances from outside. Recent revival of the perspective is due to the deepened knowledge of the chaotic

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dynamics.[1] The most conspicuous distinction between the first and second perspectives in model building appears as the stability of the economic system in question. Of course, the equilibrium of the economic system ought to be stable in the first perspecitve, while unstable in the second.

This paper points out a possibility of the third perspective, so to speak, a mixture of the first and the second. In this perspective the economic system has a stable equilibrium and gives rise to undulatory processes with the help of small disturbances. But a certain once-and-for-all shock can trigger a persistent periodic oscillation. From the dynamical system's point of view, this third perspective corresponds to the case of a hard generation oscillation, while the second perspective to that of a soft generation oscillation. In economic literature sufficient conditions for the occurrence of a hard generation oscillation are not discussed in detail. Thus, in Section 2 such conditions are considered in rather general circumstances. In Section 3 existing business cycle models are reinterpreted in the third perspective. Section 4 reviews economic literature on a hard generation oscillation and Section 5 concludes this paper.

2. Subcritical Hopf Bifurcation and Stable Limit Cycle

To obtain a stable limit cycle from an economic system with a stable equilibrium point, we look at the Hopf bifurcation theorem and the Poincaré-Bendixson theorem *at once*. Although each theorem is not new in the business cycle theory, they have been used only separately. Furthermore their use has been limited to the case of a soft generation oscillation where a stable limit cycle appears from an economic system with an unstable equilibrium. It is their joint use that is proposed here.[2]

Consider the following planar dynamical system with a bifurcation parameter μ

$$\begin{aligned} \dot{x} &= f(x, y, \mu), \\ \dot{y} &= g(x, y, \mu). \end{aligned} \tag{A}$$

Assume system (A) to be nonlinear and the equilibrium point of (A) to be the origin for all μ.[3] Expanding (A) around the equilibrium point, we have

[1] For a survey of chaotic economic dynamics, see Grandmont and Malgrange (1986), Boldrin and Woodford (1990) and Scheinkman (1990).

[2] For example, only the Poincaré-Bendixson theorem is used in Varian (1979), Schinasi (1982) and Benassy (1984), whereas only the Hopf bifurcation theorem in Torre (1977), Semmler (1987) and Lux (1992).

[3] If the equilibrium point (x^*, y^*) is not the origin $(0, 0)$, it can be transferred to the origin by the transformation $X = x - x^*$, $Y = y - y^*$.

$$\dot{x} = ax + by + p(x, y),$$
$$\dot{y} = cx + dy + q(x, y).$$

where $p(x, y)$ and $q(x, y)$ are analytic functions

$$p(x, y) = \sum_{i+j\geq 2} a_{ij}x^i y^j$$
$$= a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2 y + a_{12}xy^2 + a_{03}y^3 + \cdots,$$

$$q(x, y) = \sum_{i+j\geq 2} b_{ij}x^i y^j$$
$$= b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2 y + b_{12}xy^2 + b_{03}y^3 + \cdots.$$

Coefficients a, b, c, d, a_{ij} and b_{ij} may be functions of μ. Suppose a and/or d to be so and write $D_A(\mu) = ad - bc$ and $T_A(\mu) = a + d$. Moreover make a formula $\sigma_A(\mu)$ as follows:

$$\begin{aligned}\sigma_A(\mu) = -3\pi\{2bD_A(\mu)\}^{-3/2}[&\{ac(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) + ab(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02})\\ &+c^2(a_{11}a_{02} + a_{02}b_{02}) - 2ac(b_{02}^2 - a_{20}a_{02}) - 2ab(a_{20}^2 - b_{20}b_{02}) - b^2(2a_{20}b_{20}\\ &+b_{11}b_{20}) + (bc - 2a^2)(b_{11}b_{02} - a_{11}a_{20})\} - (a^2 + bc)\{3(cb_{03} - ba_{30})\\ &+2a(a_{21} + b_{12}) + (ca_{12} - bb_{21})\}].\end{aligned}$$

Now the Hopf bifurcation theorem can be stated correctly (see Perko (1991)).

> *Hopf bifurcation theorem*: Assume that the following three conditions hold.
>
> H1. There exists μ_0 such that $D_A(\mu_0) > 0$ and $T_A(\mu_0) = 0$.
>
> H2. $\left.\dfrac{dT_A(\mu)}{d\mu}\right|_{\mu = \mu_0} \neq 0.$
>
> H3. $\sigma_A(\mu_0) \neq 0.$
>
> Then at $\mu = \mu_0$ the Hopf bifurcation occurs in (A). If $\sigma_A(\mu_0) < 0$, it is a supercritical bifurcation. If $\sigma_A(\mu_0) > 0$, it is a subcritical bifurcation.

As can easily be seen, the equilibrium point is a focus when μ is close to μ_0. Condition H3 implies that the equilibrium point is also a focus at $\mu = \mu_0$.[4] If $\sigma_A(\mu_0) < 0$, it is stable. On the other hand, if $\sigma_A(\mu_0) > 0$, it is unstable. Figures 1(a) and (b)

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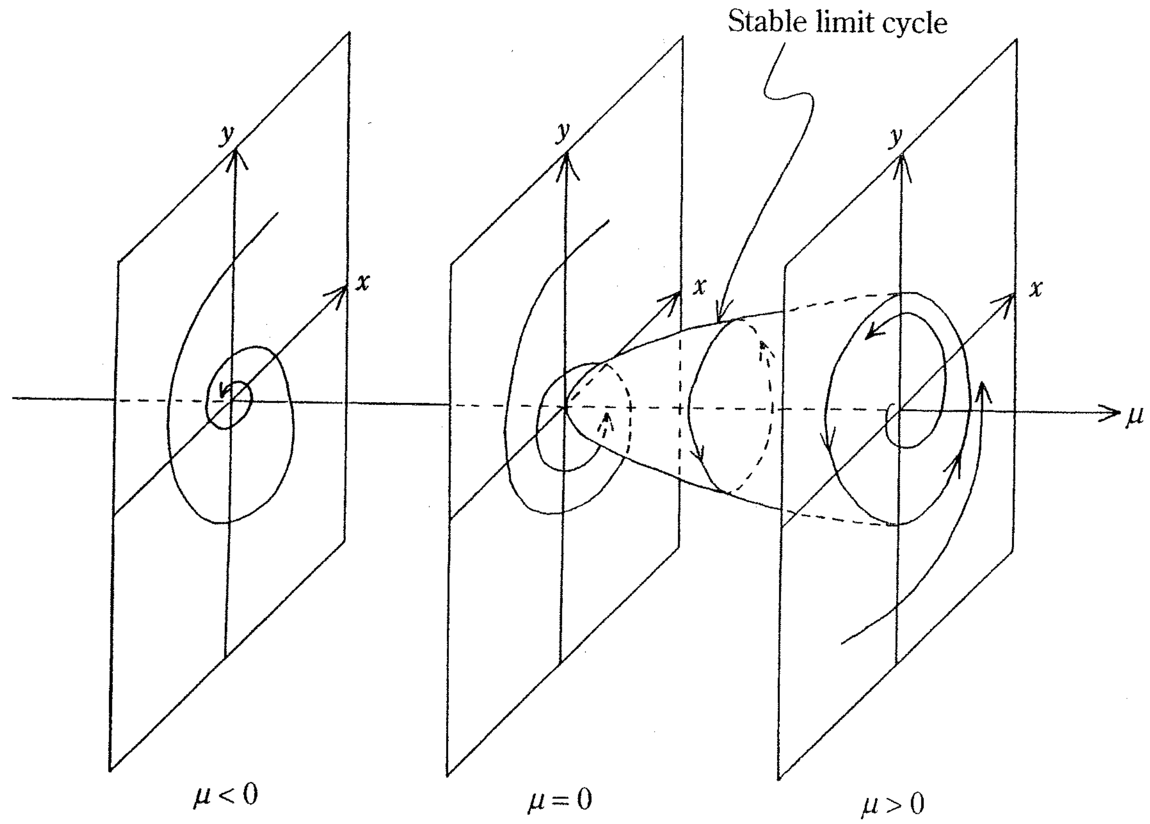
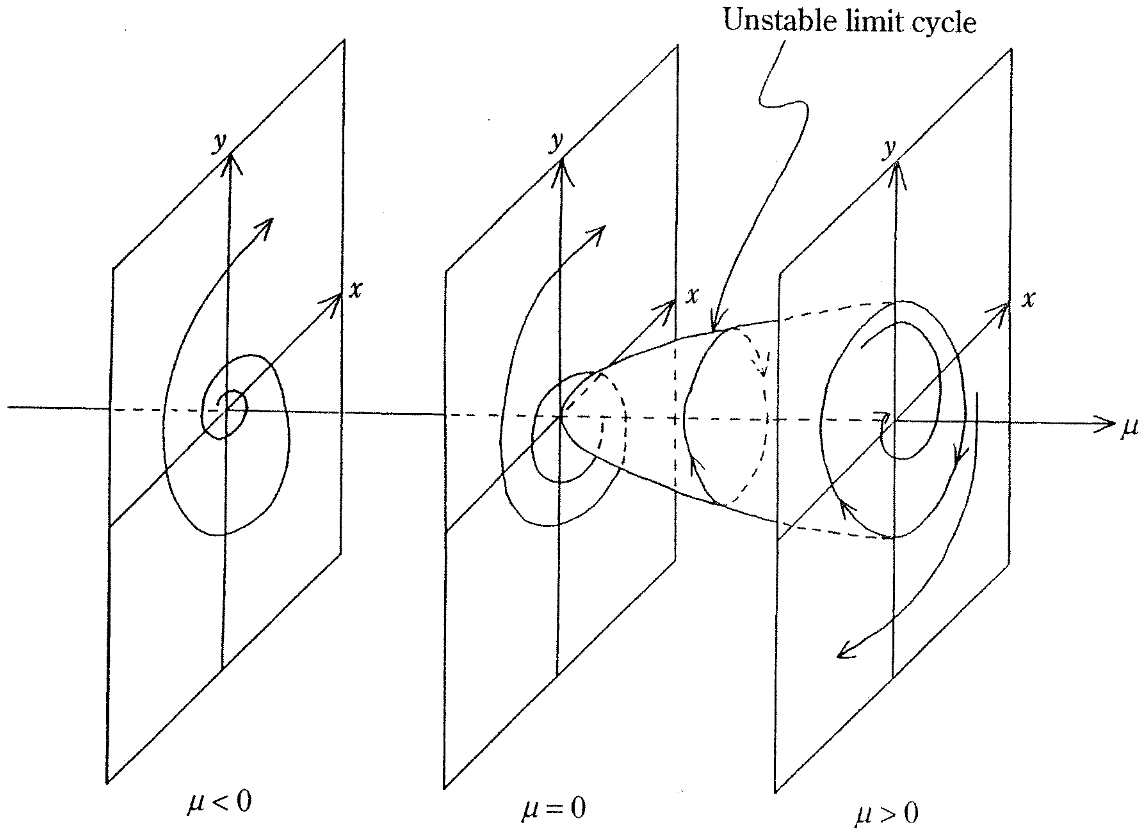
[4] Such an equilibrium point is called a weak focus of multiplicity one. In a nonlinear dynamical system condition H3 generally holds. Non-zero $\sigma_A(\mu_0)$ is called the Liapunov number.



(a) Supercritical case.



(b) Subcritical case.

Fig. 1. Hopf bifurcations.

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show typical examples of bifurcation processes. A supercritical case is characterized by a combination of an unstable equilibrium point and a stable limit cycle, while a subcritical case by that of a stable equilibrium point and an unstable limit cycle.

In the second perspective or endogenous business cycle theory only a supercritical case is regarded as useful to explain business cycles since a stable limit cycle matters.[5] In fact the formula $\sigma_A(\mu)$ is very complicated and we can hardly interpret it from the economic point of view. Attempts have been made to establish the occurrence of a supercritical Hopf bifurcation, e.g., by a mere assumption of $\sigma_A(\mu_0) < 0$ as in Torre (1977), by computer simulation as in Semmler (1987), and by relative simplification of functions as in Lux (1992).

It should be emphasized, however, that the Hopf bifurcation is a local bifurcation, i.e., it takes place near the equilibrium point. Thus the Hopf bifurcation theorem does not provide global information by itself unlike the Poincaré-Bendixson theorem. To see this, I would like to introduce an interesting example reported by Kohda *et al.* (1984):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \gamma y - px^2 - axy - qx^3 - bx^2 y, \end{aligned} \tag{B}$$

where it is assumed that $a > 0$, $b > 0$, $p > 0$ and $p^2 < 4q$. The last assumption guarantees the origin to be a unique equilibrium point. With γ as a bifurcation parameter we can confirm by the Hopf bifurcation theorem the occurrence of a supercritical (subcritical) bifurcation at $\gamma = 0$ for $ap - b <$ (>) 0.[6]

Using the Hopf bifurcation theorem, a stable limit cycle can be found only for $ap - b < 0$ (supercritical case). But Kohda *et al.* (1984) also shows by the Poincaré-Bendixson theorem alone the existence of a stable limit cycle for $ap - b > 0$ (subcritical case).[7] Figure 2 illustrates the relation between the stable and unsta-

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5 See, for example, Lorenz (1993, p. 105).

6 The following conditions obtain:

H1. $D_B(0) = 1, T_B(0) = 0,$

H2. $\left.\dfrac{dT_B(\gamma)}{d\gamma}\right|_{\gamma=0} = -1,$

H3. $\sigma_B(0) = 3\pi\,\{ap - b\} \lesseqgtr 0$ for $ap - b \lesseqgtr 0$.

7 For $ap - b > 0$ an unstable limit cycle found by the Hopf bifurcation theorem can also be proved by the Poincaré-Bendixson theorem, but the stable limit cycle found by the Poincaré-Bendixson theorem is not covered by the Hopf bifurcation theorem. See Kohda *et al.* (1984).

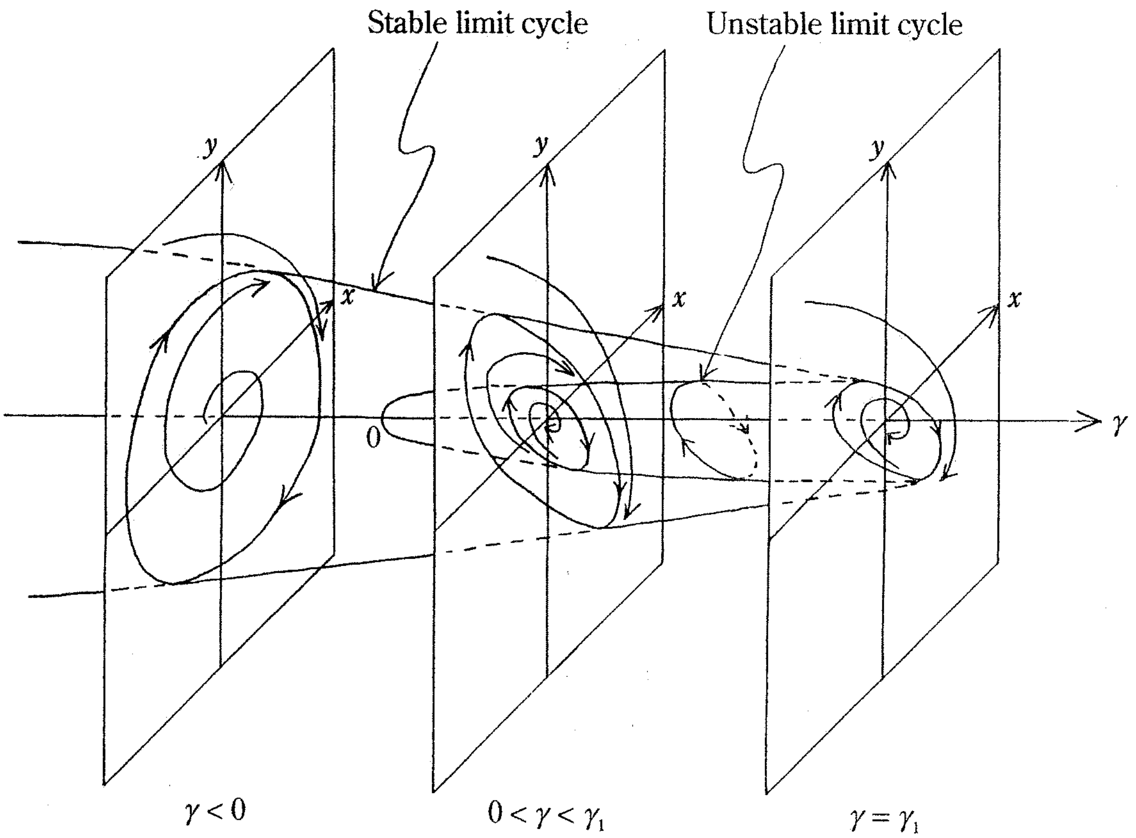

Fig. 2. Subcritical Hopf bifurcation and hard generation oscillation.

ble limit cycles for $ap - b > 0$. For $\gamma < 0$ a soft generation oscillation is observed. At $\gamma = 0$ a subritical bifurcation occurs and for $0 < \gamma < \gamma_1$ a hard generation oscillation is observed, where the stable and unstable limit cycles coexist with the former outside the latter. At $\gamma = \gamma_1$ the two merge into a semistable one.

From the above arguments a subcritical bifurcation should not be disregarded because it generates no stable limit cycle. It is closely related to a hard generation oscillation. In my view it would be easier to prove the occurrence of a hard generation oscillation using the Hopf bifurcation theorem and the Poincaré-Bendixson theorem jointly than separately. In the next section I show this.

3. Reinterpretation in the Third Perspective

Existing business cycle models are based on the first perspective or the second. The formal distinction appears as the stability of an equilibrium point. Therefore, as far as existing business cycle models are concerned, it may safely be said that the models with a stable (an unstable) equilibrium point are based on the first (second) perspective. In the third perspective, however, such a distinction does not hold true any longer. In what follows I reinterpret well-known endogenous models in the third perspective. They are the Chang and Smyth's (1971) version of Kaldor's (1940) model and Benassy's (1984, 1986) non-Walrasian model.[8] Note that the construction of the former and the latter is respectively before and aftre Torre (1977) introduced the Hopf bifurcation theorem to economics for the first time, but both use only the Poincaré-Bendixson theorem.

3. 1. Kaldor's model

Chang and Smyth (1971) analyzed Kaldor's (1940) model in the following form:

$$\begin{aligned}\dot{Y} &= \alpha\{I(Y,K)-S(Y,K)\},\\ \dot{K} &= I(Y,K),\end{aligned} \tag{C}$$

where Y, K, I, S and α are national income, capital stock, *ex-ante* net investment, *ex-ante* savings and the speed of adjustment, respectively. As is well known, the essence of Kaldor's model lies in a sigmoid shape of investment function $I(Y, K)$. The assumptions made are as follows:

C1. $I_Y > 0, S_Y > 0$ and $I_K < S_K < 0$.

C2. There exists the equilibrium point (Y^*, K^*) in the positive orthant such that $\alpha\{I_Y^* - S_Y^*\} + I_K^* > 0$ and $I_K^* S_Y^* < S_K^* I_Y^*$.

C3. There exists a finite $\tilde{K} > 0$ such that $I(0, \tilde{K}) = 0$, and also a finite $Y_1 > 0$ such that $I(Y_1, 0) = S(Y_1, 0)$. Furthermore, whenever $I(Y, K) = S(Y, K)$, $K \to \infty$ as $Y \to 0$.

On these assumptions a soft generation oscillation occurs in (C).

As is suggested in Section 2, however, there is also a possibility of a hard generation oscillation in (C). To confirm this let us regard α as a bifurcation parameter and put $\alpha_0 = -I_K^*/(I_Y^* - S_Y^*)$. Assume

C4. $\sigma_C(\alpha_0) > 0$,

and, instead of C2,

C2′. there exists the equilibrium point (Y^*, K^*) in the positive orthant such that $I_Y^* - S_Y^* > 0$ and $I_K^* S_Y^* < S_K^* I_Y^*$.

Then it is easy to check that under Cl, C2′ and C4 a subcritical Hopf bifurcation occurs at $\alpha = \alpha_0$, i.e., an unstable limit cycle emerges for $\alpha < \alpha_0$.

Next consider a compact region $P = \{(Y, K) \mid 0 \le Y \le Y_1, 0 \le K \le K_1\}$ such that

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[8] For the books dealing with both models, see Gabisch and Lorenz (1987), Sordi (1990) and Dore (1993).

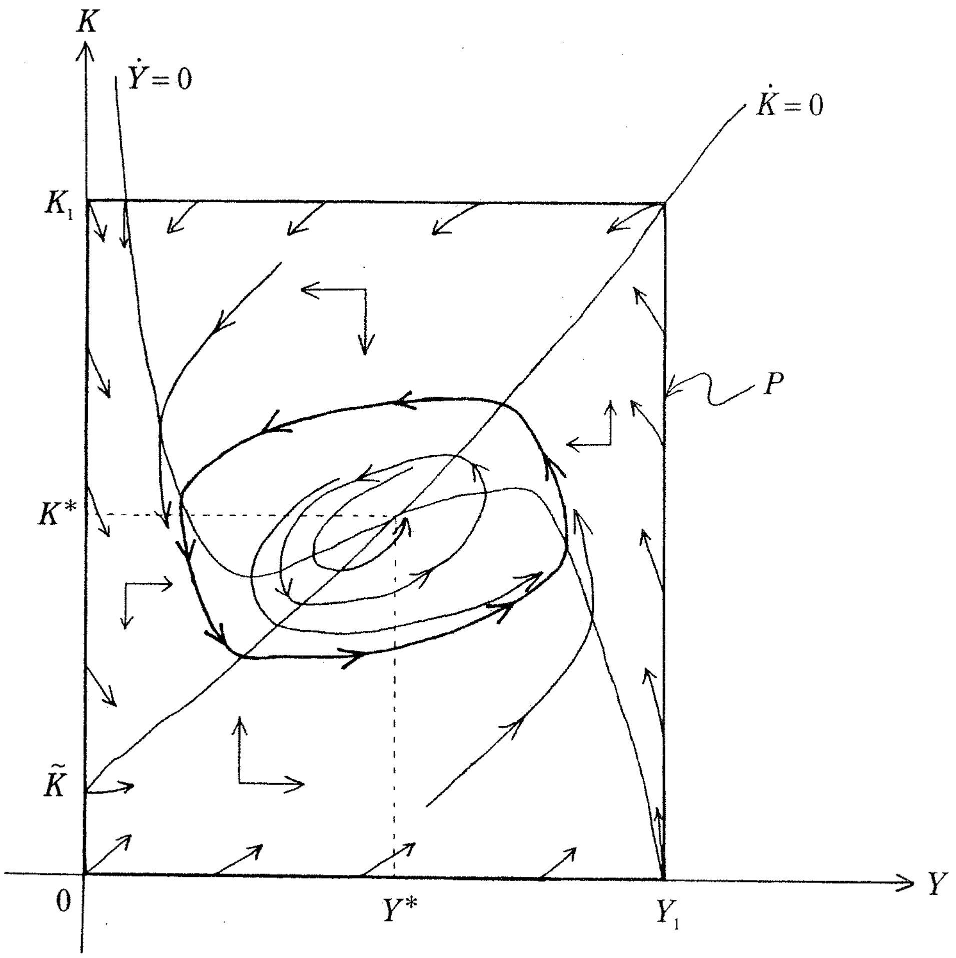

Fig. 3. Kaldor's model in the third perspective.

$I(Y_1, K_1) = 0$. Then it can be proved from C1, C2′ and C3 that all positive semi-orbits starting from the boundary of P enter the interior of P and do not leave P, so that the positive limit set is contained in P. However, in a subcritical case, neither the stable equilibrium point nor the unstable limit cycle can be the limit set. According to the Poincaré-Bendixson theorem, therefore, the limit set is a stable limit cycle. Figure 3 shows the stable limit cycle outside an unstable limit cycle. If ecomony starts from the neighborhood of the equilibrium or inside the unstable limit cycle, the first perspective is useful for understanding the generating mechanism of the business cycle. On the other hand, once economy is thrown out of the absorbing region, the second perspective wins. As a whole the third perspective enters the business cycle theory.

3. 2. Benassy's model

The business cycle in Benassy's model is described as a succession of short-run non-Walrasian equilibria, which are represented by the solution (y, r, p) of the following simultaneous equations.

$$
\begin{aligned}
&y = C(y, p) + I(x, r), \\
&L(y, r, p) = \overline{M}, \\
&y = F[F'^{-1}(w / p)],
\end{aligned}
\tag{D}
$$

where y, r, p, x and w are respectively output, interest rate, price, expected demand and money wage. x and w are assumed to be constant in the short run. Functions C, I, L and F represent consumption, investment, demand for money and production, respectively. The quantity of money in the economy is always fixed. The first two equations represent the equilibrium of demand and supply in the goods and money markets. On the other hand, the last equation implys the excess supply in the labor market, i.e., it is assumed that $F'^{-1}(w/p) < l_0$, where l_0 is the total supply of labor.

By solving (D), y can be written as the function of x and w, i.e., $y = Z(x, w)$. In the short run, since x and w are constant, so is $Z(x, w)$. In the long run, however, they vary over time. Expected demand x adaptively adjusts towards the current demand $y = Z(x, w)$. Money wage w evolves according to a traditional Phillips curve $\dot{w} = H(u)$, where u represents the level of unemployment. Therefore the business cycle is described by the long-run dynamic relationship between x and w

$$
\begin{aligned}
&\dot{x} = \mu\{Z(x, w) - x\}, \\
&\dot{w} = G[Z(x, w)],
\end{aligned}
\tag{E}
$$

where μ is the adjustment speed and $G(y) \equiv H[l_0 - F^{-1}(y)]$.

The following are assumed in (D) and (E):

E1. $0 < C_y < 1, C_p < 0, I_x > 0, I_r < 0, L_y > 0, L_r < 0, L_p > 0, F' > 0$ and $F'' < 0$.

E2. $H' < 0$, $H(u) \to \infty$ as $u \to 0$, and there exists $\bar{u} > 0$ such that $H(\bar{u}) = 0$.

E3. $\lim_{p \to 0} D(p, 0) > y_0$ and $\lim_{p \to +\infty} D(p, y_0) < \bar{y} = F(l_0 - \bar{u})$, where $y_0 = F(l_0)$ and $D(p, x)$ is the traditional aggregate demand function derived from the first two equations of (D).

E4. At the equilibrium point (x^*, w^*) of (E), $\mu(Z_x^* - 1) + G'^* Z_w^* > 0$.

E5. Whenever $Z(x, w) = x$, $w \to \infty$ as $x \to 0$.

It can be proved on these assumptions that the long-urn model (E) has a stable limit cycle based on the Poincaré-Bendixson theorem.

As shown in the previous subsection, we can show that (E) can also have a hard generation oscillation. Take μ as a bifurcation parameter and put μ_0 =

$-G^{*}Z_w^{*}/(Z_w^{*}-1)$. Assume

E6. $\sigma_E(\mu_0) > 0$,

and, instead of E4,

E4′. at the equilibrium point (x^*, w^*) of (E), $Z_x^* - 1 > 0$.

Then, under E1-3, E4′ and E6 the Hopf bifurcation theorem warrants the occurrence of a subcritical bifurcation at $\mu = \mu_0$, i.e., an unstable limit cycle appears for $\mu < \mu_0$.

Moreover consider a compact region $Q = \{(x, w) \mid 0 \le x \le y_0,\ w \le \hat{w},\ Z(x, w) \le \hat{y}\}$, where $Z(y_0, \hat{w}) < \hat{y}$ and $G(\hat{y}) > -\mu y_0\, Z_x/Z_w$. It has been ingeniously checked by Benassy (1984) that under E1-3 and E5 every positive semi-orbit starting from the boundary of Q enters the interior of Q and does not leave Q. Following the same line as in the preceding subsection, we can say that (E) has also a stable limit cycle for $\mu < \mu_0$. Figure 4 illustrates both limit cycles in a subcritical case.[9] If economy starts from outside the unstable limit cycle, a periodic fluctuation persists without disturbances and the second perspective applies for the time being. How-

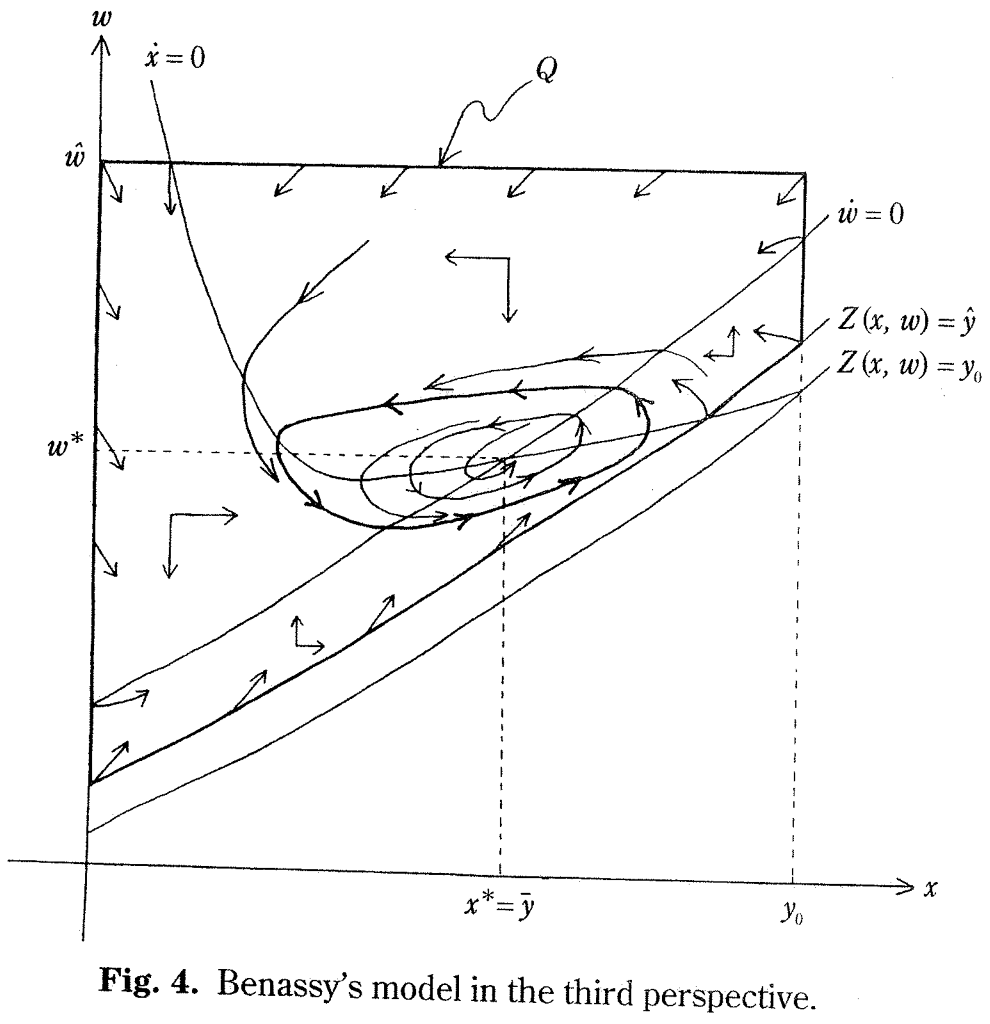

Fig. 4. Benassy's model in the third perspective.

ever, a certain once-and-for-all shock can shift economy to the domain of attraction bounded by the unstable limit cycle and then the first perspective seems to be more suitable. Nevertheless this phase will not last forever, either. It is the third perspective that is capable of capturing both the phases at the same time.

4. Hard Generation Oscillation in Economic Dynamics

The case of a hard generation oscillation has not extensively treated to understand economic phenomena. There are only a limited number of works concerned. As far as I know, a possibility of a hard generation oscillation in economic dynamics was mentioned for the first time by Medio (1980), where only the Poincaré-Bendixson theorem is used. Using the Hopf bifurcation theorem alone, Benhabib and Miyao (1981) related a subcitical bifurcation to the "corridor" concept proposed by Leijonhufvud (1973). According to the concept, the economic system is likely to behave differently for large than for moderate displacements from the equilibrium path. Whithin some range from the path (called the corridor), the system's homeostatic mechanisms work well. The system tends to home in on the ideal path and, in the absence of disturbances, to stay on it. Outside the corridor, multiplier-repercussions are strong enough for effects of shocks to the prevailing state to be endogenously amplified. Up to a point, multiplier-coefficients are expected to increase with distance from the ideal path. Applying the concept to the results in the last section, the inner unstable limit cycle corresponds to the wall of the corridor, while the outer stable limit cycle can be interpreted as the set of "a point" outside the corridor.

The only example suggesting the joint use of the Hopf bifurcation theorem and the Poincaré-Bendixson theorem is Mas-Colell (1986) who studied the price and quantity tâtonnement dynamics with one input and one output.[10] Since there is not trade out of equilibrium in Walras' tâtonnement theory, however, a limit cycle lacks any real singnificance. Puu (1986) provided a simulation result exhibiting a hard generation oscillation by the singular perturbation method. Grasman and Wentzel (1994) also presented a numerical example of a hard gen-

[9] In Figure 4 it is assumed that $Z_x-1>0$ for $x \geq x^*$ as in Benassy (1984). This leads to the upward slope of the $\dot{x}=0$ locus for $x \geq x^*$. The one-bend shape of the locus, a kind of the Goodwin characteristic, is a remarkable feature of Benassy's model. However, the assumption is not necessary to obtain the compact region Q. Even if there exists $x_1 > x^*$ such that $Z_x-1<0$ for $x>x_1$, we have Q. In such a case investment function $I(x, r)$ may have a sigmoid shape as $I(Y, K)$ in Kaldor's model.

[10] A computer simulation of the price and quantity dynamics can be found in Flaschel *et al.* (1997, pp. 41-43).

eration oscillation in Kaldor's model with a sufficiently large.[11]

5. Conclusion

This paper proposed a new perspective on the generating mechanism of the business cycle based on the standard dynamical systems theory. This perspective is, so to speak, a mixture of the traditional two perspectives and also resembles the above-mentioned corridor concept. It has been shown that a subcritical phenomenon during a bifurcation process is useful to explain the perspective.

To conclude, I point out two characteristics of the business cycle in the new perspective. First, the amplitude of the business cycle depends on from which side of an unstable limit cycle or the corridor economy starts. Of course, the amplitude of the business cycle represented by a stable limit cycle is larger than that of the business cycle generated by disturbances hitting damped vibrations inside an unstable limit cycle. Thus we can see the present phase of the economy by examining the amplitude of the cycle, not the stability of the economic system. Second, the business cycle with a larger amplitude is robust to structural change. Consider, e.g., a parameter α in model (C). Everyone agrees that α is not always constant. So imagine the situation where α varies across the bifurcation value α_0, i.e., the occurrence of structural change. This paper concentrated on a subcritical case for $\alpha < \alpha_0$. However, a stable limit cycle also exists for $\alpha > \alpha_0$, which is the case of a soft generation oscillation (see also Figure 2). Thus structural change does not necessarily leads to the extinction of the present endogenous business cycle. On the contrary, the amplitude of the cycle may be increased.

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[11] Note that this papper deals with the case of relatively small speed of adjustment (α in Kaldor's model and μ in Benassy's model). The result of Grasman and Wentzel (1994) is related to a rather peculiar oscillation, i.e., a so-called relaxation (or discontinuous) oscillation. The existence of two junction points as well as sufficiently large α is crucial for the occurrence of such an oscillation. Thus, their result cannot be applied to Benassy's model which has only one junction point. For junction points, see Sasakura (1995).

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Chaotic dynamics in a two-dimensional overlapping generations model

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Abstract

This paper investigates the global dynamics of a two-dimensional Diamond-type overlapping generations model extended to allow for government intervention. Using a singular perturbation method, we identify conditions under which transverse homoclinic points to the golden rule steady state are generated. For a parametric example with a CES production function, the occurrence of complicated dynamics (e.g. strange attractors) associated with homoclinic bifurcations is demonstrated. © 2000 Elsevier Science B.V. All rights reserved.

JEL classification: E32

Keywords: 2-D OLG model; Homoclinic point; Horseshoe; Strange attractor; Coexisting attractors

1. Introduction

In the recent literature on economic dynamics it has been widely recognized that a variety of fluctuating patterns in economic variables can emerge even in deterministic systems. In particular, beginning in the 1980s, many economic

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models exhibiting periodic as well as more complicated motion such as so-called *chaos* have been studied.[1] Overlapping generations (OLG) models have played an important role in the development of chaotic nonlinear business cycle theory compatible with the competitive framework. See Benhabib and Day (1982) and Grandmont (1985) for one-dimensional (1-D) OLG models, which are early examples of the use of chaos in economics.

In this paper we study the dynamics of a standard discrete-time two-dimensional (2-D) OLG model with a Cobb–Douglas utility function, inelastic labour supply, productive capital, and the government following a balanced budget policy. Our model is based on that of Farmer (1986), which is, in turn, a version of the seminal model of Diamond (1965) extended to two dimensions. Farmer aimed to derive, using local bifurcation theory, a necessary condition for his 2-D OLG model to generate persistent cycles on an invariant closed curve around the golden rule steady state. He showed that such cycles appear, in his setting, only if the net worth of the government is positive at the golden rule steady state.[2] Our main interest, however, is in the global and complicated behaviour of the model, rather than the local and regular behaviour.

Several numerical results suggesting the occurrence of chaos in 2-D OLG models have appeared in the economic literature; Medio (1992), Böhm (1993), and Medio and Negroni (1996) have provided interesting simulation results for 2-D OLG models in various settings. Such numerical experiments have given rise to the question whether these models displaying seemingly complicated behaviour would *really* be chaotic in a certain strict sense. While dynamical systems theory explains that so-called *homoclinic points*, and in particular their creation and destruction are responsible for chaotic dynamics,[3] it is, in general, very hard to detect them in higher-dimensional concrete systems.

De Vilder (1996; also 1995) has offered, however, a promising approach for studying higher-dimensional nonlinear systems; he has presented a 'computer-assisted proof' that an explicit 2-D OLG model with elastic labour supply and Leontief technology, essentially based on that of Reichlin (1986), can really exhibit complicated dynamics generated by *homoclinic bifurcation*[4] associated with the stable and unstable manifolds of the autarkic steady state. Although his

[1] See e.g. Boldrin and Woodford (1990), Hommes (1991), Lorenz (1993) for general surveys and examples of chaos in endogenous economic dynamics.

[2] For other related production 2-D OLG models exhibiting periodic or quasi-periodic fluctuations, see e.g. Reichlin (1986) and also Jullien (1988).

[3] See e.g. Palis and Takens (1993) for recent dynamical systems theory with an emphasis on homoclinic bifurcations.

[4] For an excellent analysis concerning homoclinic and heteroclinic bifurcations in a 2-D cobweb model with heterogeneous beliefs, see Brock and Hommes (1997), who proved, using a geometric configuration of the stable and unstable manifolds in a 'limiting case' where all agents choose the optimal predictor, that their model undergoes homoclinic bifurcations.

method is based on numerical computations, the proof itself is rigorous because of his accurate estimation of computational errors.

As a complementary approach to that of de Vilder, we use a *singular perturbation method* suggested by Marotto (1979); see also van Strien (1981). Without requiring a computer and numerical specification of parameter values, this approach allows us to rigorously establish the occurrence of *horseshoes* (topological chaos). These horseshoes are assured by the presence of a transverse homoclinic point to the golden rule steady state for our 2-D model with a small constant rate of savings. This is done by finding a transverse homoclinic point (or a 'snap-back repeller') for the reduced singular (i.e., 1-D) system and then by perturbing the latter again into the corresponding nonsingular 2-D (but nearly 1-D) system without destroying the transverse homoclinic point.[5] Moreover, in analyzing a parametric example with a CES production function, the perturbation technique is used to detect not only horseshoes but also *strange attractors* (observable chaos) as well as *infinitely many coexisting periodic attractors* which are created by homoclinic bifurcation.

This paper is organized as follows: Section 2 introduces the basic model. In Section 3 conditions and implications of the existence of transverse homoclinic points are discussed. In Section 4 we consider the complicated dynamics of a parametric example with a CES production function. In Section 5 some concluding remarks are given. Proofs of lemmas and propositions are assembled in the Appendix.

2. Basic model

We introduce a 2-D version of a Diamond-type OLG model, which is essentially based on that of Farmer (1986). See also Azariadis (1993) for intensive studies of models of this type and Jullien (1988) for a similar OLG model with productive capital and money.

We are concerned with a discrete-time OLG economy with productive capital and government intervention. The population is constant over time. The representative consumer lives for two periods and he supplies his labour inelastically

[5] Various strategies to establish the occurrence of chaos in discrete-time 2-D models are presented by several authors: Jullien (1988) restricts his 2-D OLG model with real money balances onto a 1-D invariant curve on which chaotic behaviour is possible. In Hommes (1991), a return map technique reduces a 2-D piecewise linear inventory cycle model to circle maps which exhibit (quasi-)periodic as well as chaotic attractors; in de Vilder (1995), a similar method is applied to a 2-D OLG model with an investment constraint. Dohtani et al. (1996) use a 1-D reduction and perturbation method, similar to ours, to prove the occurrence of topological chaos in a 2-D discrete version of Kaldor-type business cycle model. For a 2-D 'addiction' model, Feichtinger et al. (1997) use a 1-D reduction method to show the existence of horseshoes, and then use a perturbation method to show that the horseshoes are preserved for nearby 2-D systems.

only in youth. In order to emphasize the role of the production side, the consumer at period t is assumed to have a simple linearly homogeneous Cobb–Douglas utility function with constant weight $s \in (0,1)$ on his consumption in old age:

$$u(c_{1,t}, c_{2,t+1}) := a c_{1,t}^{1-s} c_{2,t+1}^{s}, \quad a > 0, \tag{1}$$

where $c_{i,\ j}$, $i = 1, 2$, denotes the quantity of consumption by the young and old at period j, respectively. Given the wage rate at period t, w_t, and the gross real interest rate at the next period, r_{t+1}, utility maximizing behaviour yields the savings function represented by

$$\begin{aligned} S(w_t) &:= \left\{ z_t \in [0, w_t] \mid \max_{0 \le z_t \le w_t} u(w_t - z_t, r_{t+1} z_t) \right\} \\ &= s w_t. \end{aligned} \tag{2}$$

By our choice of the utility function, this savings function is independent of the interest rate, and the parameter s can be referred to as the *savings rate*, i.e., the propensity to save out of wage income in youth.

The representative firm is characterized by a well-behaved production function $f(k_t)$ defined on $\mathbb{R}_+$, where $k_t \in \mathbb{R}_+$ stands for the capital–labour ratio at period t. Using capital and labour, the firm produces a single perishable commodity (e.g. rice) which depreciates totally in one period. We assume that the production function f satisfies the following:

Condition (A):

(A.1) f is C^2 on $\mathbb{R}_+$,
(A.2) $f(0) = 0$, $f(x) > 0$, $f'(x) > 0$, and $f''(x) < 0$ for all $x > 0$,
(A.3) $f'(0) > 1$, and $\lim_{x \to +\infty} f'(x) \to 0$,
(A.4) $f'(x) = x \Leftrightarrow x = 1$, (normalization)
(A.5) $f''(1) < -2$,
(A.6) the elasticity of marginal production function $\eta(x) := -xf''(x)/f'(x)$ is strictly increasing with respect to the capital–labour ratio.

Conditions (A.1)–(A.3) are standard in economics. Condition (A.4) is justified by Condition (A.3). Conditions (A.5) and (A.6) look a bit unusual. These last two conditions will be discussed later.

Competition implies that the marginal product of each factor is equal to its factor price (w_t or r_t), that is,

$$r_t = f'(k_t), \tag{3}$$

$$w_t = f(k_t) - k_t f'(k_t). \tag{4}$$

According to Farmer (1986), we will assume that the government follows a policy maintaining a zero budget deficit at all times. This implies, from the government budget constraint, that

$$b_{t+1} = r_t b_t, \tag{5}$$

where $b_t \in \mathbb{R}$ denotes the debt–labour ratio (i.e., the government debt per worker) at time t. Requiring that the asset market be cleared, we have

$$k_{t+1} + b_{t+1} = S(w_t). \tag{6}$$

Combining Eqs. (3)–(6), we obtain a second-order difference equation which characterizes the system

$$B(k_t, k_{t+1}) = f'(k_t)B(k_{t-1}, k_t), \tag{7}$$

where

$$B(k_t, k_{t+1}) := s(f(k_t) - k_t f'(k_t)) - k_{t+1}. \tag{8}$$

This expression represents the net indebtedness of the government to the private sector.

Note that b_t is negative if $B(k_{t-1}, k_t) < 0$, i.e., if the economywide capital stock exceeds net private ownership. In this case the government is a net creditor to the private sector.

Let

$$g(x) := xf'(x), \quad w(x) := f(x) - xf'(x) \quad \text{and} \quad h(x,y) := w(y) - f'(y)w(x), \tag{9}$$

then (7) is transformed into

$$k_{t+1} - g(k_t) - sh(k_{t-1}, k_t) = 0. \tag{10}$$

We see that $f(k_t) \equiv g(k_t) + w(k_t)$. Given the capital–labour ratio k_t, $w(k_t)$ represents the competitive wage rate or the share of labour in output per worker $f(k_t)$, while $g(k_t)$ can be viewed as the share of capital. We will call this function g the *capital–share function*.

Setting $k_t = x_t$ and $k_{t+1} = y_t$, we obtain a second-order difference equation with one parameter equivalent to (7):

$$(x_{t+1}, y_{t+1}) = F_s(x_t, y_t), \tag{11}$$

where

$$F_s(x, y) = (y, g(y) + sh(x, y)).$$

Suppose now that not only the initial capital–labour ratio, $k_0 \in \mathbb{R}_+$, but also the initial debt–labour ratio, $b_0 \in \mathbb{R}$, are given historically. This pair of initial states (k_0, b_0) determines $(k_0, k_1) = (x_0, y_0)$ by $k_1 = sw(k_0) - f'(k_0)b_0$. Because of the nonnegativity of the capital–labour ratio, we have to restrict ourselves to the set of initial states whose iterates by F_s will never leave the nonnegative quadrant, i.e.,

$$X_s := \{(x, y) \in \mathbb{R}^2_+ \mid F^n_s(x, y) \in \mathbb{R}^2_+ \text{ for all } n \geq 0\}. \tag{12}$$

Hence, for each parameter value $s \in (0, 1)$, the difference equation (11) induces a map from X_s to itself:

$$F_s\colon X_s \subset \mathbb{R}^2_+ \to X_s. \tag{13}$$

The set of initial states X_s might be very small. Nevertheless, as Lemma 2 below indicates, this is not the case at least when the parameter s is small.

Before ending this section, we will give some remarks on steady states (i.e., fixed points) of (11). The set of steady states of (11), denoted by $\mathrm{Fix}(F_s)$, is defined as

$$\mathrm{Fix}(F_s) := \{(k^*, k^*) \in \mathbb{R}^2_+ \mid B(k^*, k^*)(f'(k^*) - 1) = 0\}. \tag{14}$$

Since $f'(k^*) = 1$ if and only if $k^* = 1$ by Condition (A.4), the *golden rule steady state* $p = (1, 1) \in \mathbb{R}^2_+$ is well-defined and independent of the parameter s. Since $B(1, 1) = s(f(1) - 1) - 1$, the government is a net creditor at the golden rule steady state whenever the propensity to save is sufficiently small. In this paper we do not take account of balanced steady states[6] satisfying $B(k^*, k^*) = sw(k^*) - k^* = 0$.

3. Characterization of global dynamics

3.1. Preliminaries

In this subsection, we briefly discuss some notions and implications of homoclinic points and homoclinic bifurcations, which will be used in what follows. Guckenheimer and Holmes (1983) and Palis and Takens (1991) offer mathematical treatments of the subject. For a discussion of these topics in an economic context, the reader is referred to de Vilder (1995, 1996) and Brock and Hommes (1997).

For simplicity, we mostly treat here differentiable invertible maps (i.e. diffeomorphisms), but some similar results are provided even for differentiable noninvertible maps. Let $F\colon\mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable invertible map and $p \in \mathbb{R}^2$ be a hyperbolic fixed point for F (i.e., where the Jacobian matrix evaluated at p,

[6] The autarkic steady state $(0, 0) \in \mathrm{Fix}(F_s)$ may well be very influential for the complicated global dynamics in our model. See de Vilder (1996) for the creation of homoclinic points to the autarkic steady state.

D_pF, has no eigenvalues with norm 1). If p is a periodic point of period k, then we may replace F by $G = F^k$. For a small neighbourhood U of p, the *local stable and unstable manifolds* of p are defined as

$$W^s_{loc}(p) = \left\{ x \in U \,\middle|\, \lim_{k \to \infty} F^k(x) \to p \right\},$$

$$W^u_{loc}(p) = \left\{ x \in U \,\middle|\, \lim_{k \to \infty} F^{-k}(x) \to p \right\},$$

respectively. Even if F is not invertible, such invariant manifolds do exist. The *global stable and unstable manifolds* of p are then defined as

$$W^s(p) = \bigcup_{n=0}^{\infty} F^{-n}(W^s_{loc}(p)),$$

$$W^u(p) = \bigcup_{n=0}^{\infty} F^{n}(W^u_{loc}(p)),$$

respectively. Note that if F is not invertible, then $W^s(p)$ and $W^u(p)$ may no longer be manifolds in the global sense (see e.g. Palis and Takens, 1993 for more information). A point $q \in W^s(p) \cap W^u(p) \backslash \{p\}$ is said to be a *homoclinic point* to p. If $W^s(p)$ and $W^u(p)$ intersect transversely at this homoclinic point q, then we say that q is a *transverse homoclinic point* to p, and the orbit of q, $O(q) := \{F^i(q)\}_{i \in \mathbb{Z}}$, is called a transverse homoclinic orbit. If $W^s(p)$ and $W^u(p)$ intersect tangentially at this homoclinic point q, then we say that q is a *homoclinic tangency*. The Homoclinic Point Theorem[7] assures that a transverse homoclinic orbit to a hyperbolic fixed point p implies the existence of a *horseshoe* near the homoclinic orbit. This is defined as a Cantor set which is invariant under (some iterate of) the map $G = F^n$ and on which G is topologically equivalent to a shift map with a countable infinity of periodic orbits, an uncountable infinity of aperiodic orbits, topological transitivity, and sensitive dependence on initial conditions. In addition, horseshoes as well as transverse homoclinic orbits to a hyperbolic fixed point (more generally, hyperbolic sets) have a kind of semi-local structural stability, that is, they are persistent against small perturbations of the map.

The existence of horseshoes does not imply that a typical trajectory exhibits complicated long-run dynamical behaviour, since a horseshoe will not attract nearby points in the phase space. Topological chaos in the sense of horseshoes is, therefore, not observable in general.[8] To describe the asymptotic behaviour

[7] See e.g. Smale (1967), Guckenheimer and Holmes (1983), Palis and Takens (1993) for various versions of this theorem; for noninvertible maps, see e.g. Marotto (1978).

[8] This does not mean that topological chaos would be empirically insignificant at least in the short run. In fact, it is very likely to generate long-lasting complicated *transient* motion, which will be often sustained under the influence of very small noise. See e.g. Dohtani et al. (1996) for the effect of noise on a chaotic Kaldor-type business cycle model.

which can be observed in the long run, we need some notions concerning 'attractors'. A compact invariant set A of the map F is called an *attractor* if it contains a dense orbit (i.e., $F|A$ is topologically transitive) and its basin of attraction, i.e., a set of points x such that $\mathrm{dist}(F^n(x), A) \to 0$ as $n \to \infty$, has nonempty interior. An attractor A of the map F is said to be *strange* if $F|A$ has sensitive dependence on initial conditions. Contrary to the case of the horseshoe, chaotic dynamics will be observed in the long run for a large set of initial states if strange attractors are present in the system.

Now consider a one-parameter family of maps $\{F_\mu: \mu \in I \subset \mathbb{R}\}$ with a hyperbolic saddle $p = p(\mu)$. We say that the family of maps $\{F_\mu\}$ exhibits a *homoclinic bifurcation*, associated with p, at $\mu = 0$ if

(i) for $\mu < 0$, $W^s(p)$ and $W^u(p)$ have no intersection;
(ii) for $\mu = 0$, $W^s(p)$ and $W^u(p)$ have a tangency at $q \neq p$;
(iii) for $\mu > 0$, $W^s(p)$ and $W^u(p)$ have a transverse intersection.

Furthermore, if we can choose a μ-dependent local coordinate (x, y) near q so that $W^s(p)$ is given by $y = 0$ and $W^u(p)$ by

$$y = ax^2 + b\mu, \quad a \neq 0 \text{ and } b \neq 0,$$

then we say that the tangency q is *quadratic* ($a \neq 0$) and *unfolds generically* ($b \neq 0$). Some results of dynamical systems theory guarantee that for the families of maps $\{F_\mu\}$ exhibiting a generically unfolding quadratic homoclinic tangency at $\mu = 0$, several interesting dynamical complexities arise for μ-values near $\mu = 0$ such as

- creation (or destruction) of horseshoes;
- coexistence of infinitely many periodic attractors or repellers (Newhouse, 1979);
- existence of strange attractors or repellers for a positive Lebesgue measure set of μ-values (Mora and Viana, 1993).

3.2. Chaotic dynamics

In the present section we try to identify a sufficient condition under which the dynamics of F_s in (11) are *topologically chaotic* in the sense that the system has a horseshoe. To this end, we will use a perturbation method (see e.g. Marotto, 1979; van Strien, 1981; Palis and Takens, 1993) to detect a transverse homoclinic orbit to the golden rule steady state when the system is 2-D but nearly 1-D.

As a first step, it is convenient to consider the extreme case in which $s = 0$ in (11). This corresponds to a world in which generations do not virtually overlap because the representative consumer consumes all his income in youth and nothing in old age. Consequently, all the capital needed for production is owned by the government, i.e., $k_t = -b_t$. In this limiting case, the two-dimensional map (11) collapses, formally, to a *singular map* from $\mathbb{R}^2_+$ onto the graph of g, i.e., $C = \{(x, y) \in \mathbb{R}^2_+ \mid y = g(x), x \geq 0\}$, as follows:

$$F_0 : \mathbb{R}^2_+ \to C \subset \mathbb{R}^2_+; \quad F_0(x, y) = (y, g(y)), \tag{15}$$

whose dynamics are clearly governed by an equivalent one-dimensional map

$$g : \mathbb{R}_+ \to \mathbb{R}_+. \tag{16}$$

Hence, the system F_s in (11) with small $s > 0$ can be regarded as a perturbation of F_0 in (15).

By Condition (A), the following properties of the map g can be easily checked:

Lemma 1. Under Condition (A), *the following statements hold*:

(L1.1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ *is* C^1.
(L1.2) $g(0) = 0$ *and* $g'(0) > 1$.
(L1.3) $g(x) \gtrless x$ *as* $x \lessgtr 1$ ($x = 1$ *is a fixed point*).
(L1.4) $g'(1) < -1$ (*the fixed point* $x = 1$ *is a repeller*).
(L1.5) *there exist unique points* q *and* $\theta \in \mathbb{R}_{++}$ *such that*
  (L1.5.1) $g'(x) \gtrless 0$ *as* $x \lessgtr \theta$ (*unimodality*),
  (L1.5.2) $g(q) = 1$,
  (L1.5.3) $0 < q < \theta < 1 < g(\theta)$ *and* $0 < g^2(\theta) < 1$.

Assertion (1.4) in Lemma 1 follows from Condition (A.5), and (1.5) essentially follows from (A.6). From Lemma 1, we see that the graph of the marginal production function $y = f'(x)$ satisfying Condition (A) must have two and only two intersections with the hyperbola of $y = 1/x$, at $x = 1$ and at $x = q$. On the interval $[q,1]$ the graph of f' is located above the hyperbola, and below otherwise. A typical situation with a 'reversed sigmoidal' marginal production function[9] is depicted in Fig. 1.

It is worth noting that the capital-share function g is not monotone[10] with respect to the capital-labour ratio, whereas both the wage function w ($w'(x) = -xf''(x) > 0$ for $x > 0$) and the production function f are strictly increasing (see Fig. 2). Nonmonotonicity of the function g requires that the elasticity of marginal production function η straddles unity because for $x > 0$, $\eta(x) = 1$ if and only if $g'(x) = f'(x)(1 - \eta(x)) = 0$. In case that Condition (A.6) is not satisfied, the function g may have more than one hump. Moreover, even if neither Condition (A.5) nor (A.6) is satisfied, g cannot be a monotone function as long as $\sup_{x \in \mathbb{R}_+} \eta(x) > 1$.

[9] Figs. 1 and 2 are drawn based on a CES production function satisfying Condition (A), which will be discussed in Section 4.

[10] For a similar 2-D OLG model, Jullien (1988) assumed that the function g is nondecreasing.

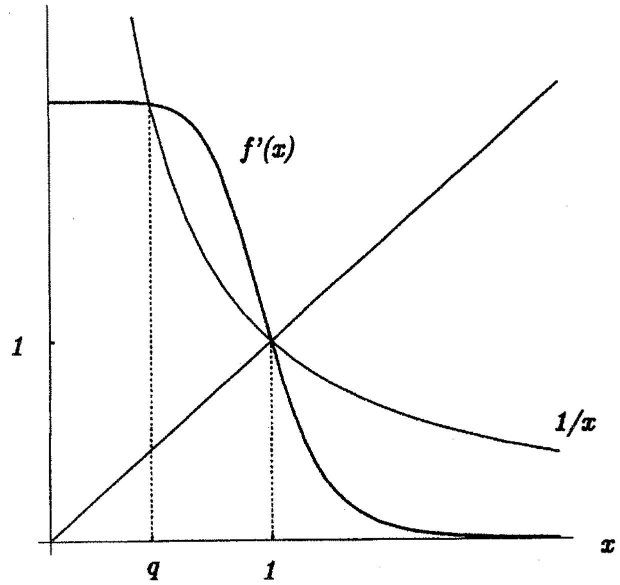

Fig. 1. Nonlinearity in the marginal production function f'.

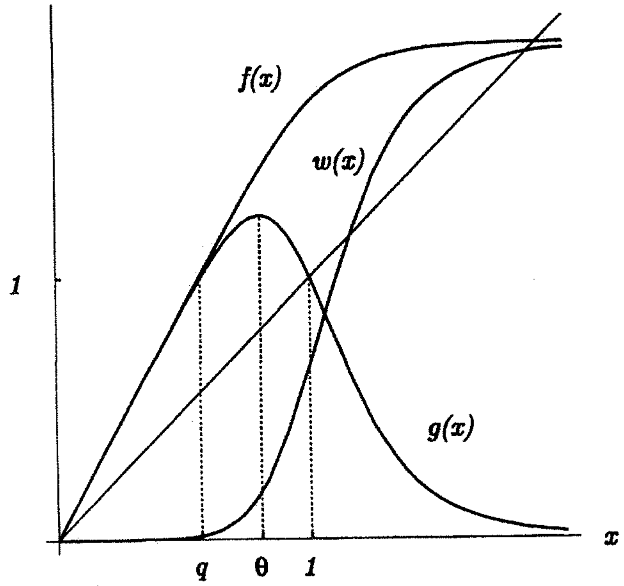

Fig. 2. Graphs of the functions f, w, and g.

In order to guarantee the existence of bounded and positive-valued equilibrium paths in terms of the capital–labour ratio $\{k_t\}_{t \geq 0}$ for a *large* set of initial states, it is meaningful to show that, on the strictly positive phase plane, we can find a compact region M such that the forward orbit of every initial state in M cannot escape from there. We will then show that F_s has a *trapping region* in $\mathbb{R}^2_{++}$.

Lemma 2. Suppose Condition (A) *holds. Then there exist a compact region* $M \subset \mathbb{R}^2_{++}$ $(p \in \operatorname{int} M)$ *and a number* $\varepsilon > 0$ *such that for every* $s \in (0, \varepsilon)$ *the following assertions hold*:

(L2.1) M *is a trapping region for* F_s, *i.e.*, $F_s(M) \subset \operatorname{int} M$,
(L2.2) $F_s|M : M \to M$ *is a* C^1-*diffeomorphism onto its image*,
(L2.3) p *is a hyberbolic saddle, i.e., the Jacobian matrix* D_pF_s *evaluated at* p *has two real eigenvalues* $\lambda_1(s)$ *and* $\lambda_2(s)$ *with* $|\lambda_1(s)| > 1 > |\lambda_2(s)| > 0$,
(L2.4) p *is dissipative, i.e.*, $|\det D_pF_s| = |\lambda_1(s)\lambda_2(s)| < 1$.

Of course, for every $s \in (0, \varepsilon)$, the trapping region M is contained in X_s.

We first attempt to identify conditions under which F_0 has a transverse homoclinic point to p. To do this, it suffices, using the argument presented by Marotto (1979), to identify conditions under which the 1-D map g has a so-called *snap-back repeller*, introduced by Marotto (1978), for $x = 1$:

Lemma 3. (Marotto, 1979, Lemma 2.2). *If* g *has a snap-back repeller, then* $F_0(x, y) = (y, g(y))$ *has a transverse homoclinic point.*

Note that the fixed point $x = 1$ of g is a hyperbolic repeller, i.e., $|g'(1)| > 1$, from (1.4) in Lemma 1. In order to show that the fixed point $x = 1$ of g is a snap-back repeller, it is then sufficient to find a point $z \in \mathbb{R}_+$ $(z \neq 1)$ which has an orbit $O(z) = O^+(z) \cup O^-(z)$ satisfying the following:

(S1) $O^+(z) = \{x_i \in \mathbb{R}_+ \mid x_0 = z,\ g^m(x_0) = 1$ for some $m \geq 1$, and $x_{i+1} = g(x_i)$ f or $i \geq 0\}$,
(S2) $O^-(z) = \{x_{-i} \in \mathbb{R}_+ \mid x_0 = z,\ x_{-i} \to 1$ as $i \to \infty$, and $x_{-i} = g(x_{-i-1})$ for $i \geq 0\}$,
(S3) $g'(x) \neq 0$ for each $x \in O(z)$.

In addition to Condition (A), we impose further conditions on g:

Condition (B):

(B.1) $g^2(\theta) < q$,
(B.2) $g^2(x) \neq x$ for any $x \in (\theta, 1)$,

where θ and q are unique points with $g'(\theta) = 0$ and $g(q) = 1$ $(0 < q < \theta < 1)$ as given in Lemma 1.

Note that the statement of (B.1) can be replaced by $g^3(\theta) < 1$. Condition (B.2) requires that the map g has no periodic point of period two on the interval $(\theta, 1)$.

We can see that, under Conditions (A) and (B), the map g has a snap-back repeller, which implies that the singular map F_0 has a transverse homoclinic point.

Lemma 4. Under Conditions (A) *and* (B), *the map* $F_0(x, y) = (y, g(y))$ *in* (15) *has a transverse homoclinic point to the golden rule steady state* p.

This situation is depicted in Fig. 3.The stable 'manifold' for F_0, $W^s(p, F_0)$, consists of horizontal line segments passing through points which are eventually mapped onto the golden rule steady state p. In particular, $W^s(p, F_0)$ contains a horizontal line segment $\gamma^s = \{(x, y) \in \mathbb{R}^2_+ \mid x \in [g^2(\theta), g(\theta)], y = 1\}$ passing through p. Furthermore, the unstable manifold $W^u(p, F_0)$ contains an arc on the graph C of g, $\gamma^u = \{(x, y) \in C \subset \mathbb{R}^2_+ \mid x \in [g^2(\theta), g(\theta)], y = g(x)\}$, because each point on this arc γ^u has a backward orbit converging to p (see the proof of Lemma 4 in the Appendix).

By the perturbation argument of invariant manifolds (see e.g. Palis and Takens, 1993, Appendices 1 and 4), we can perturb the singular map F_0 with a transverse homoclinic point, by making the parameter s slightly bigger than zero, so that every nearby nonsingular map F_s retains a transverse homoclinic point. Hence, by applying the Homoclinic Point Theorem, we obtain the following result:

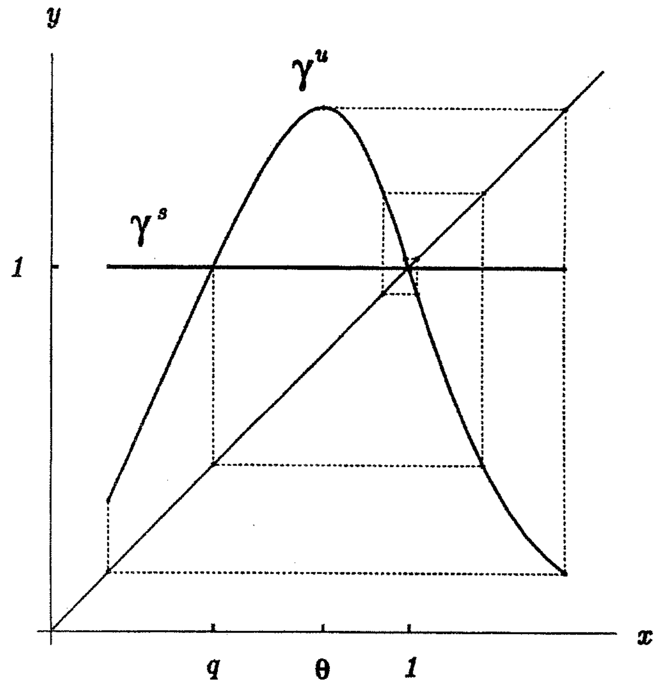

Fig. 3. Transverse homoclinic orbit for the singular map F_0.

Proposition 1. Under Conditions (A) *and* (B), *there exists* $\varepsilon > 0$ *such that for every* $s \in (0, \varepsilon)$, F_s *has a horseshoe* $\Lambda_s \subset M$ *with* $p \in \Lambda_s$, *where* $M \subset \mathbb{R}^2_{++}$ *is a trapping region for* F_s.

3.3. Asymptotic behaviour near homoclinic orbits

Before proceeding to the next section, we will briefly discuss the asymptotic behaviour for a large set of initial states when homoclinic orbits exist *and* the dissipativity of the system is strong. Since $M \subset \mathbb{R}^2_{++}$ in Proposition 1 is a trapping region containing a horseshoe Λ_s, every positive orbit of a point starting in M is indeed bounded and does not leave M. But this fact does not imply that the asymptotic behaviour of every such orbit would be approximated by $F_s|\Lambda_s$ because Λ_s itself is not an attractor even though it may be a part of such a set. Some points in M might settle down to periodic attractors. We can, however, show at least that there is an open set surrounded by the segments of the stable and unstable manifolds of p such that every point in the set is drawn near the unstable manifold by the iteration process provided the dissipativity is strong enough,[11] i.e., provided $|\det D_x F_s| < 1$ for every $x \in M$. More precisely, there is an open set $U_s \subset M$ (depending on s) such that the ω-limit set of each point $x \in U_s$, $\omega(x) := \{y \in M \mid \exists n_i \to +\infty;\ F_s^{n_i}(x) \to y\}$, is contained in the closure of the unstable manifold $\overline{W^u(p)}$ of the golden rule p. This implies that certain attractors are contained in $\overline{W^u(p)}$:

Proposition 2. Let Conditions (A) *and* (B) *hold, and let* $M \subset \mathbb{R}^2_{++}$ *be a compact region as in Proposition* 1. *Then there is* $\varepsilon' > 0$ *such that for every* $s \in (0, \varepsilon')$, M *contains an open set* U_s *with* $\omega(x) \subset \overline{W^u(p)}$ *for every* $x \in U_s$.

4. Example with CES production function

4.1. Horseshoes and homoclinic tangles

In this section we give an example with a constant elasticity of substitution (CES) production function $f_\beta(x)$ with two parameters: one is the distribution factor, $\alpha \in (0, 1)$, and the other is the substitution factor, $\beta \in (-1, 0) \cup (0, \infty)$, which is related to the elasticity of factor substitution. The CES production

[11] A similar result for a different 2-D OLG model with strong dissipativity has been shown by de Vilder (1995).

function is assumed to be of the following form:

$$f_\beta(x) := \frac{1}{\alpha}[1 - \alpha + \alpha x^{-\beta}]^{-1/\beta}$$

$$= \frac{x}{\alpha[(1-\alpha)x^\beta + \alpha]^{1/\beta}}, \tag{17}$$

where the elasticity of substitution is given by $(1+\beta)^{-1}$. The first and second derivative of f_β and the elasticity of f_β' are calculated as follows:

$$f_\beta'(x) = [\alpha + (1-\alpha)x^\beta]^{-(\beta+1)/\beta},$$

$$f_\beta''(x) = -\frac{(1-\alpha)(1+\beta)x^{\beta-1}}{[\alpha + (1-\alpha)x^\beta]^{(2\beta+1)/\beta}},$$

$$\eta_\beta(x) = -\frac{x f_\beta''(x)}{f_\beta'(x)} = \frac{(1-\alpha)(1+\beta)x^\beta}{\alpha + (1-\alpha)x^\beta}.$$

One can easily see that if $\beta > (\alpha+1)/(1-\alpha) > 1$ then the production function f_β satisfies Condition (A). The functions $g_\beta(x) := x f_\beta'(x)$ and $h_\beta(x,y) := w_\beta(y) - f_\beta'(y) w_\beta(x)$ are then represented by

$$g_\beta(x) = x[\alpha + (1-\alpha)x^\beta]^{-(1+\beta)/\beta} \tag{18}$$

and

$$h_\beta(x,y) = \frac{(1-\alpha)[y^{\beta+1}[\alpha + (1-\alpha)x^\beta]^{(\beta+1)/\beta} - x^{\beta+1}]}{\alpha[\alpha + (1-\alpha)x^\beta]^{(\beta+1)/\beta}[\alpha + (1-\alpha)y^\beta]^{(\beta+1)/\beta}}. \tag{19}$$

The point θ at which g_β attains its maximum can be calculated by solving $\eta_\beta(\theta) = 1$:

$$\theta = \theta(\beta) = \left(\frac{\alpha}{\beta(1-\alpha)}\right)^{1/\beta}.$$

Note that $\theta(\beta) \in (0,1)$ whenever $\beta > \alpha/(1-\alpha)$ and that $\theta(\beta) \to 1$ as $\beta \to \infty$.

The dynamics of this economic system can then be characterized by

$$F_{s,\beta}(x,y) = (y, g_\beta(y) + s h_\beta(x,y)). \tag{20}$$

It can be shown that g_β satisfies Condition (B) for every sufficiently large β (see Appendix). According to Proposition 1, we can therefore state the following proposition:

Proposition 3. Fix $\alpha \in (0,1)$. *Then there exists* $\beta^* > (\alpha+1)/(1-\alpha)$ *such that, given* $\beta \geq \beta^*$, $F_{s,\beta}$ *in* (20) *has a horseshoe for every* $s \in (0, \varepsilon)$ *for some* $\varepsilon = \varepsilon(\beta) > 0$.

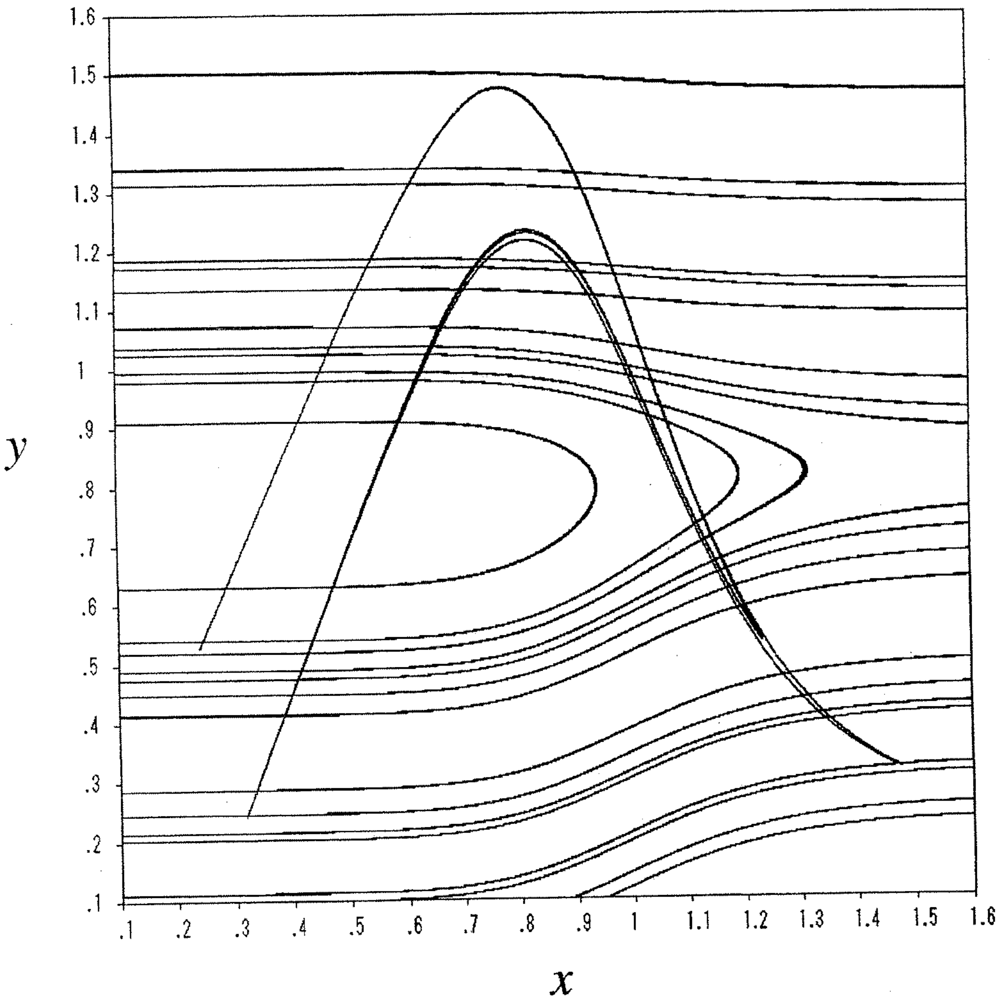

Fig. 4. Homoclinic tangles: the wildly winding stable und unstable manifolds of the golden rule steady state have infinitely many intersections for $(\alpha, \beta, s) = (0.5, 7.5, 0.1)$.

Proposition 3 says that if the elasticitiy of substitution between capital and labour is sufficiently small and the representative consumer consumes "too much" out of his wage income in youth, then the economic system given by (20) may give rise to topological chaos.

For example, given $\alpha = 0.5$ fixed, one can numerically derive that $\beta^* = 4.85$ is sufficient. So, given $\beta \geq 4.85$, topological chaos occurs for every sufficiently small s. The parameter s may have to be very small indeed for chaos to occur. However, the simulation results illustrate that the system $F_{s,\beta}$ can have a transverse homoclinic point to the golden rule even for relatively large values of s: Fig. 4 illustrates the case of so-called *homoclinic tangles* for the parameter set $(\alpha, \beta, s) = (0.5, 7.5, 0.1)$. These appear as a result of the infinitely many intersections of the stable and unstable manifolds of the saddle-type golden rule[12] and the resulting wild oscillations of these manifolds.

4.2. Homoclinic bifurcations and complicated dynamics

The singular perturbation argument developed in Section 3 to detect horseshoes can be extended to establish the occurrence of homoclinic bifurcations and the resulting complex dynamics for (20) (see e.g. van Strien (1981) for a similar argument). This allows us to find β-values for which $F_{s,\beta}$ has a quadratic homoclinic tangency which unfolds generically, associated with the hyperbolic and dissipative saddle golden rule steady state p for small s. Applying the theorems of Newhouse (1979, Theorem 3) and Mora and Viana (1993, Theorem A) (see also Palis and Takens, 1993) yields the following proposition:

Proposition 4. Fix $\alpha \in (0,1)$ *arbitrarily. Then there exists* $\varepsilon > 0$ *such that for each* $s \in (0, \varepsilon)$ *and for some* $\hat{\beta} = \hat{\beta}(s)$, $F_{s,\hat{\beta}}$ *has a quadratic homoclinic tangency, unfolding generically, associated with the golden rule steady state p. Thus the following assertions hold for all* $\delta > 0$:

(i) *Coexistence of Infinitely Many Periodic Attractors: There is a nontrivial subinterval* $I \subset [\hat{\beta} - \delta, \hat{\beta} + \delta]$ *and a dense subset* $J \subset I$ *such that for each* $\beta \in J$, $F_{s,\beta}$ *has infinitely many coexisting periodic attractors of arbitrarily large period* (*The Newhouse Phenomenon*).
(ii) *Abundance of Strange Attractors: There is a positive Lebesgue measure set of* β*-values* $E \subset [\hat{\beta} - \delta, \hat{\beta} + \delta]$ *such that* $F_{s,\beta}$ *exhibits a strange attractor for each* $\beta \in E$.

While the occurrence of horseshoes in Proposition 3 does not assure the observability of chaotic behaviour in $F_{s,\beta}$ in the long run, the second assertion in Proposition 4 *does*, even for a measure-theoretically large set of parameter values. In this sense, the occurrence of observable chaos is one of the typical dynamical phenomena for system (20).

However, the coexistence of infinitely many periodic attractors demonstrated in the first assertion of Proposition 4 might be a *rare* phenomenon; the set of parameter values for which $F_{s,\beta}$ has infinitely many coexisting periodic attractors is conjectured to be of measure zero (see Tedeschini-Lalli and Yorke, 1986). Nevertheless, as is known (see e.g. van Strien, 1981; Guckenheimer and Holmes,

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[12] For $\alpha = \frac{1}{2}$ and $\beta \in (3, 9)$, one may easily check that the golden rule steady state $p = (1, 1)$ of $F_{s,\beta}$ in (20) is a hyperbolic saddle if $0 < s < s_{PD} = (\beta - 3)/3(1 + \beta)$, and it is a hyperbolic attractor if $s_{PD} < s < s_{NS} = 2/(1 + \beta)$, where s_{PD} denotes the period-doubling bifurcation point and s_{NS} denotes the Neimark–Sacker (or Hopf) bifurcation point ($\beta \neq 5, 7$).

1983, Chapter 6), Newhouse Phenomena cannot occur for the singular (i.e., 1-D) map $F_{0,\beta}$. This fact implies that, however small the savings rate $s > 0$ may be, there will be a big qualitative difference in the global dynamics between the singular ($s = 0$) and nonsingular ($s > 0$) maps.

4.3. *Multiple attractors for a large savings rate: Numerical observation*

Even though the infinitely many attractors associated with homoclinic bifurcations may hardly occur, the coexistence of a finite number of attractors is certainly a common feature of nonlinear systems (see e.g. Hommes, 1991). In this subsection, we observe using computer simulations that the system $F_{s,\beta}$ in (20)

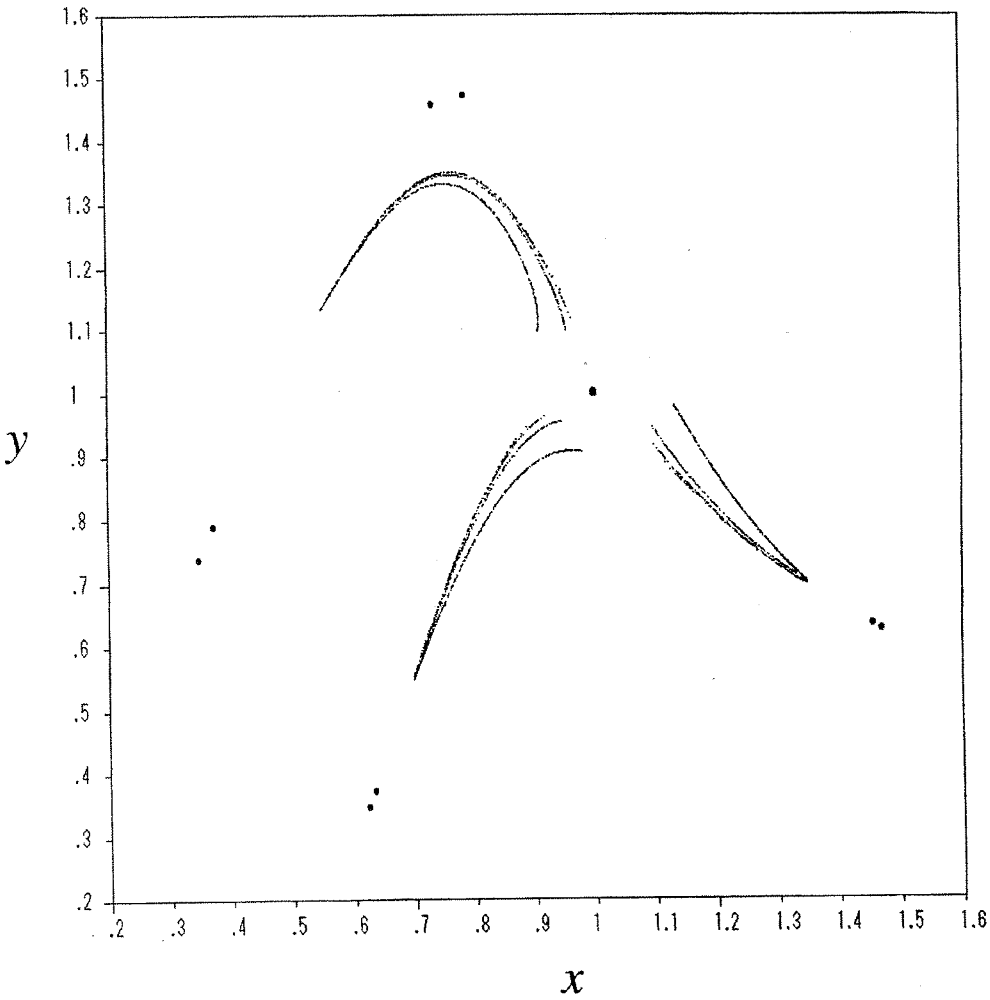

Fig. 5. Three coexisting attractors: attracting golden rule, periodic attractor of period eight, and three-piece strange attractor coexist simultaneously for $(\alpha, \beta, s) = (0.5, 6.5, 0.23)$.

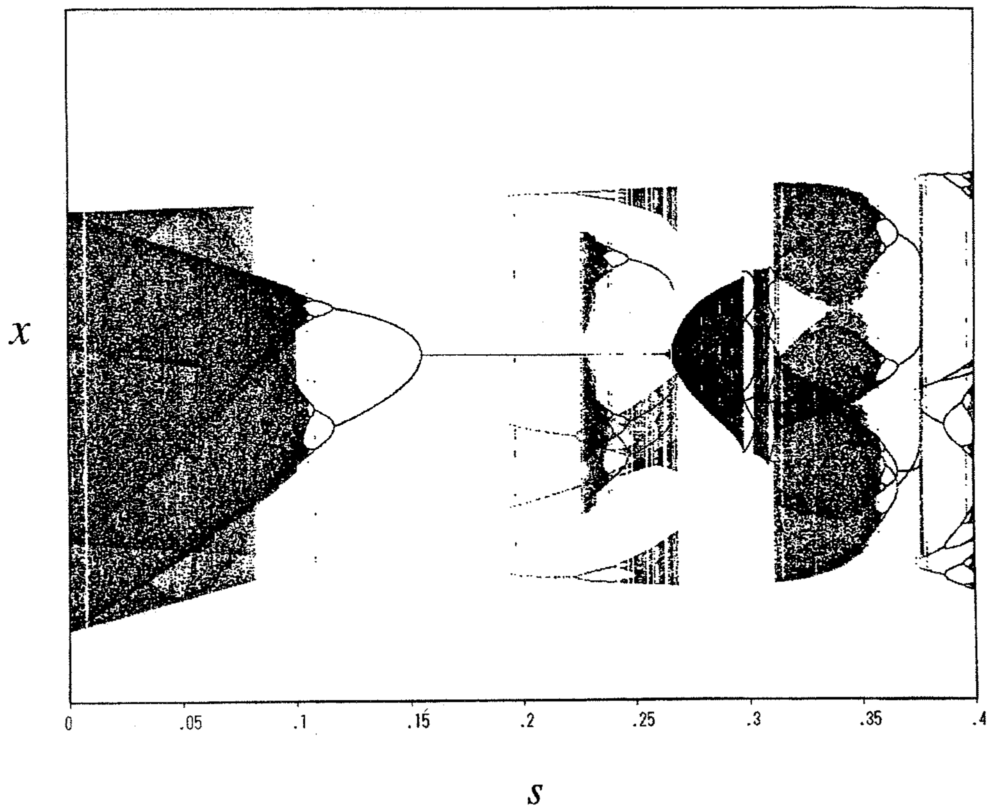

Fig. 6. Bifurcation diagram: multiple attractors are observed.

can simultaneously exhibit stationary, periodic, and chaotic attractors for certain parameter values.

Let us fix the parameter values as follows:

$$\alpha = 0.5, \quad \beta = 6.5 \quad \text{and} \quad s = 0.23.$$

Then at least three coexisting attractors can be observed numerically (see Fig. 5). Plotting a bifurcation diagram[13] by computer helps us find these attractors (Fig. 6). In fact, from Fig. 6 we can see multiple attractors appearing in an overlapping way around $s = 0.23$. The first attractor is the attracting golden rule steady state $p = (1, 1)$. Recall that, as mentioned previously in the footnote, if the savings rate s lies between the period-doubling and Neimark–Sacker bifurcation points, i.e., if $s \in (s_{PD}, s_{NS}) = ((\beta - 3)/3(1 + \beta), 2/(1 + \beta)) \approx (0.156, 0.267)$, then the golden rule p is an attractor. The second is a periodic attractor of period eight. The third is (probably) a strange attractor, which may be called a "three-piece" strange attractor because it seems to consist of three isolated pieces.

[13] The initial points are randomized for every increment of the parameter value s.

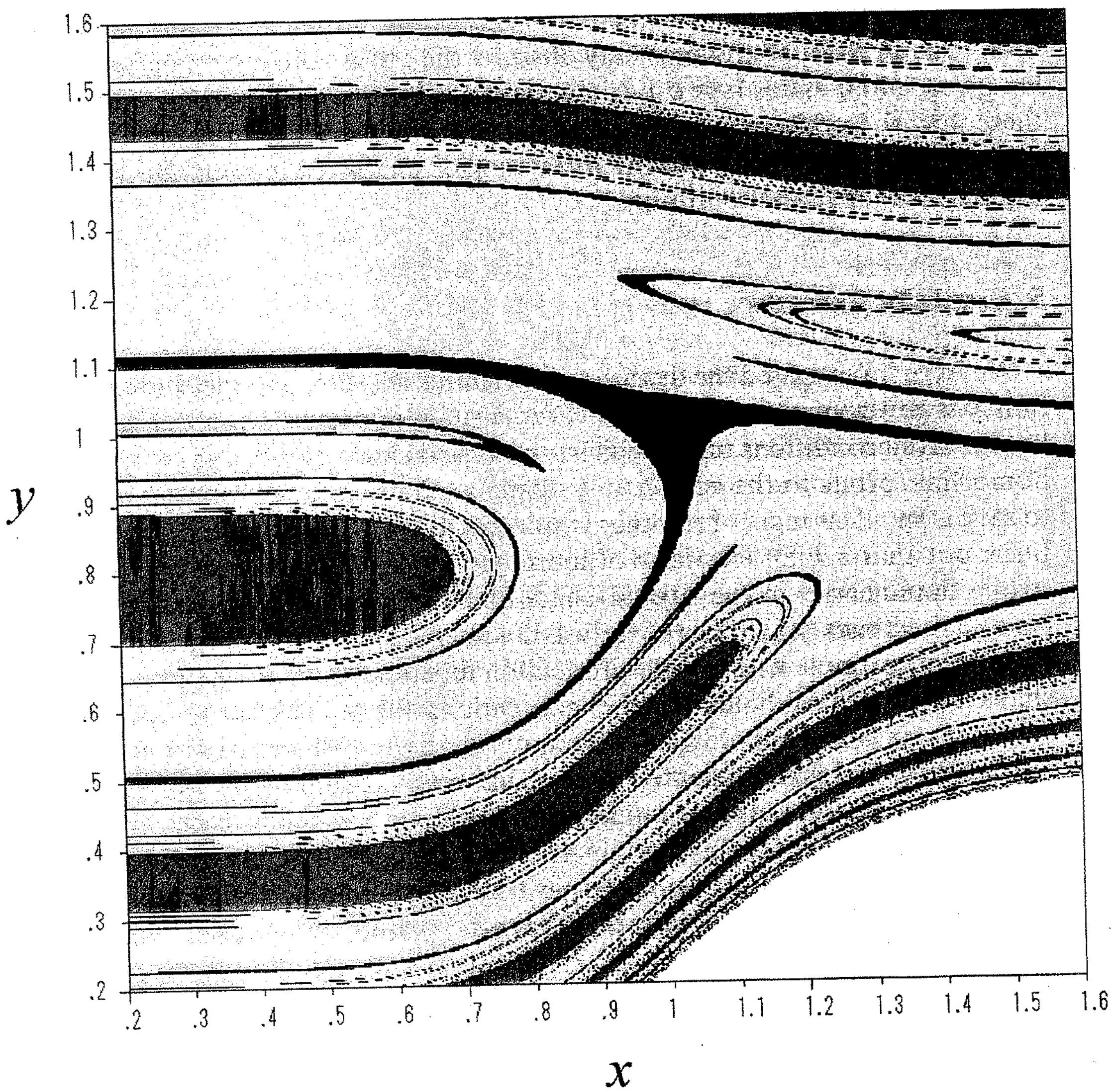

Fig. 7. Basins of attraction for three attractors (blue: attracting golden rule; red: period eight attractor; yellow: strange attractor) and fractal basin boundaries.

Since an attractor will attract all its nearby points, it is interesting to know by which attractor the initial points randomly given on the phase plane are attracted; Fig. 7 depicts how the basins of attraction of these three attractors share the phase plane. The boundary of the closure of a basin is called a *basin boundary*. One can see that the structure of some basin boundaries looks very complicated. Such so-called *fractal basin boundaries* may arise due to homoclinic or heteroclinic bifurcations to some periodic points; Brock and Hommes (1997) have presented a computer assisted proof that in a cobweb model with heterogeneous beliefs, fractal basin boundaries are created by heteroclinic bifurcation between the stable and unstable manifolds of two different saddles of period four. Fractal basin boundaries may obstruct the precise prediction of *final states* for given initial states (see e.g. McDonald et al. (1985) for more details about these topics). In this sense, the complexity of basin boundaries provides another type of unpredicability different from that of chaos, defined as the sensitive dependence on initial conditions.

5. Concluding remarks

We have investigated the dynamics of a simple 2-D OLG model with production and government intervention. Using a singular perturbation technique, we have derived conditions under which topological chaos occurs due to transverse homoclinic orbits to the golden rule steady state when the constant propensity to save is small enough, or in other words, when the 2-D system is nearly 1-D. It turns out that a high elasticity of marginal production function may lead to strong nonlinearity in the capital–share function, which is responsible for the chaotic dynamics of (at least) nearly 1-D systems. We have also given a useful parametric example with a CES production function which exhibits observable chaos associated with homoclinic bifurcations for a large set of parameter values. From a methodological viewpoint, there are several advantages in the perturbation method presented here. This method allows us to prove the existence of transverse and/or tangential homoclinic points in 2-D or even much higher-dimensional systems without the use of a computer, provided the systems can be transformed into tractable 1-D. Furthermore, it may require less specification of function forms or parameter values than other computer-assisted methods. Of course, we should also point out that our method, so far, does not take account of the global dynamics far from nearly 1-D. For instance, geometric structures and generation mechanisms for fractal basin boundaries as observed by computer simulations in our OLG model are not well analyzed yet. This will be an important topic for future research.

Appendix

Proof of Lemma 1. (L1.1): Obvious from (A.1). (L1.2): From (A.3). (L1.3): From (L1.2) and (A.4). (L1.4): Since $g'(x) = f'(x) + xf''(x)$, it follows from (A.4) and (A.5) that $g'(1) = 1 + f''(1) < -1$. (L1.5): Note first that $g'(x) = f'(x)(1 - \eta(x))$. Since $\eta(1) = 2$ and $\eta(0) = 0$, and η is strictly increasing by (A.6), there is a unique point $\theta \in (0, 1)$ such that $\eta(\theta) = 1$, implying (L1.5.1). Since g is unimodal with its global maximum at $\theta \in (0, 1)$ and $g(x) > 0$ for all $x > 0$, it follows that $g(0) = 0 < \theta < 1 < g(\theta)$ and $0 < g^2(\theta) < 1$. Thus there is a unique point q with $g(q) = 1$, which proves (L1.5.2) and (L1.5.3). □

Proof of Lemma 2. (L2.1): First, consider the case when $s = 0$. Putting $a = g^2(\theta)$ and $b = g(\theta)$, we have $0 < a < 1 < b$. We prove the case when $a < \theta$ (the argument for the case when $\theta \leq a < 1$ is similar). Note that $[a, b]$ is invariant under g, i.e., $g([a, b]) = [a, b]$. Given $\varepsilon_0 \in (0, a)$, we can choose $\varepsilon_1 > 0$ such that $g(b + \varepsilon_1) > a - \varepsilon_0$. Let $I_y := [a - \varepsilon_0, b + \varepsilon_1] \subset \mathbb{R}_{++}$, then $g(I_y) \subset \operatorname{int} I_y$. Similarly, given $\varepsilon_0' \in (\varepsilon_0, a)$, we can choose $\varepsilon_1' > \varepsilon_1$ such that $g(b + \varepsilon_1') > a - \varepsilon_0'$. Then $I_x := [a - \varepsilon_0', b + \varepsilon_1'] \subset \mathbb{R}_{++}$ satisfies $I_y \subset \operatorname{int} I_x$ and $g(I_x) \subset \operatorname{int} I_x$. Let $M \subset \mathbb{R}^2_{++}$ be a compact rectangle defined by $M := I_x \times I_y$, then $F_0(M) = I_y \times g(I_y) \subset \operatorname{int} I_x \times \operatorname{int} I_y = \operatorname{int} M$. By definition of M, $p = (1, 1) \in \operatorname{int} M$. Since $h(x, y)$ is continuous on the compact set M, we see that, for any sufficiently small $s > 0$, $F_s(M) \subset \operatorname{int} M$.

(L2.2): Since $h(x, y)$ is obviously C^1 on $\mathbb{R}^2_{++}$, $F_s(x, y) = (y, g(y) + sh(x, y))$ is C^1 on $\mathbb{R}^2_{++}$. Therefore it is sufficient to show that F_s is injective (one-to-one) on $\mathbb{R}^2_{++}$ and that the Jacobian matrix of F_s is nonsingular, i.e., $\det D_x F_s \neq 0$ for $x \in \mathbb{R}^2_{++}$.

Suppose that F_s is not injective. Then there exist two distinct points $a = (a_1, a_2) \in \mathbb{R}^2_{++}$ and $b = (b_1, b_2) \in \mathbb{R}^2_{++}$ $(a \neq b)$ such that $F_s(a_1, a_2) = F_s(b_1, b_2)$. This implies that $a_2 = b_2$ and $h(a_1, a_2) = h(b_1, b_2)$. Thus $h(a_1, b_2) = h(b_1, b_2)$, which implies $w(a_1) = w(b_1)$. But the function $w(x)$ is strictly increasing, since $w'(x) = -xf''(x) > 0$ for $x > 0$. Hence $a_1 = b_1$, which contradicts the hypothesis.

On the other hand, the Jacobian matrix of F_s at every point $z = (x, y) \in \mathbb{R}^2_{++}$ is given by

$$D_z F_s = \begin{bmatrix} 0 & 1 \\ sh_1(x, y) & g'(y) + sh_2(x, y) \end{bmatrix}. \tag{21}$$

Hence we have $\det D_z F_s = -sh_1(x, y) = -sxf'(y)f''(x) > 0$.

(L2.3) *and* (L2.4): Let $\lambda_1(s)$ and $\lambda_2(s)$ with $|\lambda_1(s)| \geq |\lambda_2(s)|$ be the two eigenvalues of (21) at $p = (1, 1)$, then $\lim_{s \to 0} \lambda_1(s) = g'(1) < -1$ and $\lim_{s \to 0} \lambda_2(s) = 0$. By continuity of $\lambda_i(s)$ $(i = 1, 2)$ with respect to s and by $|\det D_p F_s| = |\lambda_1(s) \lambda_2(s)| > 0$ $(s \neq 0)$, the claims follow. □

Proof of Lemma 4. According to Lemma 3, it is sufficient to prove that the 1-D map g has a snap-back repeller for $x = 1$. By Condition (B.1) and (1.5.3) in Lemma 1, the ordering $0 < g^2(\theta) < q < \theta < 1 < g(\theta)$ holds. Let $I := [g^2(\theta), g(\theta)]$, $I_1 := [g^2(\theta), \theta]$, $I_2 := [\theta, 1]$, and $I_3 := [1, g(\theta)]$ be intervals with $I = \bigcup_{i=1}^3 I_i$, then g is stricly increasing on I_1 and g is strictly decreasing on $I_2 \cup I_3$. We claim that

(C1) every $x \in I$ has a backward orbit which is contained in I and converges to 1, and

(C2) every $x \in I_2 \cup I_3 \setminus \{\theta, g(\theta)\}$ has a backward orbit which is contained in $I_2 \cup I_3 \setminus \{\theta, g(\theta)\}$ and converges to 1.

(C1): since $g(I_2) = I_3$ and $g(I_3) = g^2(I_2) = I_1 \cup I_2$, it follows that for every $x \in I$ there is a point $y \in I_2$ with $g^m(x) = y$ for some $2 \geq m \geq 0$. Hence it suffices to verify that every $x \in I_2$ has a backward orbit for g^2 which is contained in I_2 and converges to 1. Condition (B.2), together with $(g^2)'(1) = (g'(1))^2 > 1$ and $g^2(\theta) < \theta$, implies that $g^2(x) < x$ holds for every $x \in I_2 \backslash \{1\}$. Since $g^2(x_0) < x_0$ and $g^2(1) = 1 > x_0$ for every $x_0 \in I_2 \backslash \{1\}$, it follows that there is $x_{-1} \in (x_0, 1)$ with $x_0 = g^2(x_{-1})$, and, inductively, that there is a strictly increasing sequence of points $\{x_0, x_{-1}, x_{-2}, \ldots\} \subset I_2$ such that $g^2(x_{-i-1}) = x_i$ for $i \geq 0$ and $x_{-i} \to 1$ as $i \to \infty$. This proves (C1). (C2): recall that g is strictly monotone on $I_2 \cup I_3$ and maps I_2 onto I_3 homeomorphically; then (C2) is obvious from the proof of (C1).

On the other hand, it follows by Condition (B.1) and (1.5.3) in Lemma 1, that $g^2(\theta) < q$ and $g^2(1) = 1 > q$. So there is a point $q' \in I_2 \backslash \{\theta, 1\}$ such that $g^2(q') = q$. By (C2), q' has a backward orbit $O^-(q') \subset I_2 \cup I_3 \backslash \{\theta, g(\theta)\}$ satisfying (S2). Since the forward orbit of q', $O^+(q') = \{q', g(q') \in I_3, g^2(q') = q, g^3(q') = 1\}$, satisfies (S1) and does not contain θ, the so obtained orbit of q', $O(q') = O^+(q') \cup O^-(q')$ satisfies (S1)–(S3). □

Proof of Proposition 1. By Lemma 4, some compact parts $\gamma^{s/u}$ of the stable und unstable manifolds $W^{s/u}(p, F_0)$ have a transverse intersection. In particular, $W^u(p, F_0)$ contains a parabolic arc $\gamma^u = \{(x, y) \in \mathbb{R}^2_+ \mid x \in [g^2(\theta), g(\theta)], y = g(x)\}$ and $W^s(p, F_0)$ contains a compact horizontal line segment $\gamma^s = \{(x, y) \in \mathbb{R}^2_+ \mid x \in [g^2(\theta), g(\theta)], y = 1\}$. Clearly, γ^u and γ^u have a transverse intersection at $(q,1) \in \mathbb{R}^2_+$. By the perturbation argument of invariant manifolds for noninvertible maps (see Palis and Takens, 1993, Appendices 1 and 4), we see that the compact arc of $W^u(p, F_0)$ and the compact arc consisting of regular points[14] of $W^s(p, F_0)$ vary continuously on the map in the C^1 sense. This means that, for every sufficiently small $s > 0$, some compact arcs $\hat{\gamma}^{s/u}$ sufficiently C^1-close to $\gamma^{s/u}$ are contained in $W^{s/u}(p, F_s)$, respectively. Since transverse intersections are stable in the C^1 sense, $\hat{\gamma}^s$ and $\hat{\gamma}^u$ above do have a transverse intersection. Since for all sufficiently small $s > 0$, the fixed point p is hyperbolic and the homoclinic orbit to p, obtained above, is contained in int M by Lemma 2, the map F_s has, by the Homoclinic Point Theorem, a horseshoe $\Lambda_s \subset M$ for each $s \in (0, \varepsilon)$, for some $\varepsilon > 0$. □

Proof of Proposition 2. Note that $\det D_x F_s = -sxf'(y)f''(x)$. We can pick a small number $\delta > 0$ so that $1 > \delta \cdot \max_{(x,y) \in M} |xf'(y)f''(x)| > 0$. Let $\varepsilon > 0$ be as in Proposition 1 and let $\varepsilon' = \min\{\varepsilon, \delta\}$; then for every $s \in (0, \varepsilon')$, M is a trapping

[14] See Appendix 4 in Palis and Takens (1993). Let $K \subset W^s(p)$ be any compact set. Then, for some n, $F^n(K) \subset W^s_{loc}(p)$. We say that the points of K are *regular* points of $W^s(p)$ if for each $x \in K$, $\mathrm{Im}(D_x F^n) + T_{F^n(x)}(W^s_{loc}(p)) = \mathbb{R}^2$. So the arc γ^s contains only regular points of $W^s(p)$.

region for F_s and $F_s|M$ satisfies the following:

- $W^s(p) \cap W^u(p) \setminus \{p\} \neq \phi$,
- $W^u(p) \subset M$, and
- $|\det D_x F_s| < 1$ for every $x \in M$.

We can take a bounded region $U_s \subset M$ whose boundary consists of segments of $W^s(p)$ and $W^u(p)$, and now apply Proposition 1 in Appendix 3 in Palis and Takens (1993). □

Proof of Proposition 3. We know that if $\beta > (1+\alpha)/(1-\alpha)$ then f_β satisfies Condition (A). By Proposition 1 in Section 3, all we need is to verify that the function g_β satisfies Condition (B) for every sufficiently large β. In what follows, we assume $\beta > (1+\alpha)/(1-\alpha)$.

(B.1): We first note that $f_\beta(0) = g_\beta(0) = 0$ and $f_\beta'(0) = \alpha^{-(\beta+1)/\beta}$. Since $w(x) = f_\beta(x) - g_\beta(x)$ is strictly increasing and f_β is strictly concave, it follows that $g_\beta(x) < f_\beta(x) < x/\alpha^{1+1/\beta}$ for $x > 0$ and $0 < \alpha^{1+1/\beta} < q(\beta)$, where $g_\beta(q(\beta)) = 1$ and $q(\beta) \in (0, \theta(\beta))$. Since $\lim_{\beta\to\infty} \alpha^{1+1/\beta} = \alpha$, it is sufficient to show that $\lim_{\beta\to\infty} g_\beta^2(\theta(\beta)) \to 0$. Considering that

$$g_\beta(\theta(\beta)) = \left[\alpha(1-\alpha)^{1/\beta}\beta^{1/\beta}\left(1+\frac{1}{\beta}\right)^{1+1/\beta}\right]^{-1} \to \frac{1}{\alpha}(>1) \quad \text{as } \beta \to \infty,$$

and that $\lim_{\beta\to\infty} g_\beta(x) \to 0$ holds for each $x > 1$, we obtain the last claim.

(B.2): Since

$$g_\beta''(x) = \frac{(1-\alpha)(1+\beta)x^{\beta+1}[\beta(1-\alpha)x^\beta - \alpha(1+\beta)]}{[\alpha + (1-\alpha)x^\beta]^{(3\beta+1)/\beta}},$$

it follows that g_β has a unique inflection point

$$\tilde{\theta}(\beta) = \left(\frac{\alpha(1+\beta)}{\beta(1-\alpha)}\right)^{1/\beta} > \theta(\beta),$$

so that $g_\beta''(x) \gtrless 0$ as $x \gtrless \tilde{\theta}(\beta)$. Note for future reference that the critical point $\theta(\beta)$ is quadratic, i.e., $g_\beta''(\theta(\beta)) \neq 0$.

Let us define a continuous piecewise linear map $\psi_{a,b}: \mathbb{R} \to \mathbb{R}$ with two parameters $a > 0$ and $b > 0$ by

$$\psi_{a,b}(x) := \begin{cases} -a(x-1)+1 =: l_1(x) & \text{for } x \leq 1, \\ -b(x-1)+1 =: l_2(x) & \text{for } x > 1. \end{cases}$$

By virtue of the uniqueness of the inflection point, one can check that if we let

$$a(\beta) := \min\left\{|g'_\beta(1)|,\ \left|\frac{g_\beta(\theta(\beta)) - 1}{\theta(\beta) - 1}\right|\right\} \quad \text{and}$$

$$b(\beta) := \min\left\{|g'_\beta(1)|,\ \left|\frac{g^2_\beta(\theta(\beta)) - 1}{g_\beta(\theta(\beta)) - 1}\right|\right\},$$

then $\psi_{a(\beta),b(\beta)}(x) = l_1(x) < g_\beta(x)$ for $x \in (\theta(\beta), 1)$ and $\psi_{a(\beta),b(\beta)}(x) = l_2(x) > g_\beta(x)$ for $x \in (1, g_\beta(\theta(\beta)))$. We claim that if $a(\beta)b(\beta) > 1$ then $g^2_\beta(x) < x$ for all $x \in (\theta(\beta), 1)$. To see this, note first that for $x \in (\theta(\beta), 1)$, we have $\psi^2_{a(\beta),b(\beta)}(x) = l_2 \circ l_1(x) > l_2 \circ g_\beta(x) > g^2_\beta(x)$. And note that if $a(\beta)b(\beta) > 1$ and $x < 1$, then $\psi^2_{a(\beta),b(\beta)}(x) - x = l_2 \circ l_1(x) - x = (1 - a(\beta)b(\beta))(1 - x) < 0$. Combining these inequalities, we get the claim.

To complete the proof, it is then sufficient to show that $a(\beta)b(\beta) > 1$ for β large enough. But this follows from the fact that

$$\lim_{\beta\to\infty} |g'_\beta(1)| = \lim_{\beta\to\infty} |\alpha - \beta(1-\alpha)| \to \infty, \quad \lim_{\beta\to\infty} |\theta(\beta) - 1| = 0,$$

$$\lim_{\beta\to\infty} |g_\beta(\theta(\beta)) - 1| = \frac{1-\alpha}{\alpha} \quad \text{and} \quad \lim_{\beta\to\infty} |g^2_\beta(\theta(\beta)) - 1| = 1. \qquad \square$$

Proof of Proposition 4. Given $\alpha \in (0,1)$, let $\beta \geq \beta^*$, where β^* is as given in Proposition 3. Note first that the unstable manifold $W^u(p, F_{0,\beta})$ contains a compact parabolic arc $\gamma^u(\beta) = \{(x,y) \in \mathbb{R}^2_+ \mid x \in [q(\beta), 1], y = g_\beta(x)\}$. Next, we can observe that there is a sequence of points, depending upon β, which are eventually mapped to the fixed point $x = 1$:

$$Q(\beta) = \{q_i(\beta) \in [0, \theta(\beta)] \mid q_i(\beta) = g_\beta(q_{i+1}(\beta)) \text{ for } i \in \mathbb{N},$$
$$q = q_1 > q_2 > \cdots, q_i \to 0 \text{ as } i \to \infty\}.$$

Since for every $x \in (0,1)$, $g_\beta(x) \to x/\alpha$ as $\beta \to \infty$ and so the increasing part of the graph of g_β converges to the line segment $y = x/\alpha$ as $\beta \to \infty$, we have $q_i(\beta) \in Q(\beta) \to \alpha^i$ as $\beta \to \infty$. On the other hand, by the proof of Proposition 3, we have $g^2_\beta(\theta(\beta)) \to 0$ as $\beta \to 0$. Hence, given $\beta_1 \geq \beta^*$, there is $k \in \mathbb{N}$ with $g^2_{\beta_1}(\theta(\beta_1)) > q_k(\beta_1)$, and there is $\beta_2 > \beta_1$ with $g^2_{\beta_2}(\theta(\beta_2)) < q_k(\beta_2)$. So $g^2_{\beta_3}(\theta(\beta_3)) = q_k(\beta_3)$ for some $\beta_3 \in (\beta_1, \beta_2)$. Consequently, $W^s(p, F_{0,\beta})$ contains a horizontal compact line segment $\gamma^s(\beta)$ such that (i) $\gamma^u(\beta_1)$ and $\gamma^s(\beta_1)$ have no intersection, (ii) $\gamma^u(\beta_2)$ and $\gamma^s(\beta_2)$ have two transverse intersections, and (iii) $\gamma^u(\beta_3)$ and $\gamma^s(\beta_3)$ have a quadratic tangency at $(\theta(\beta_3), g_{\beta_3}(\theta(\beta_3))) \in \mathbb{R}^2_+$.

By the same perturbation argument used in the proof of Proposition 1, for every sufficiently small $s > 0$, the stable and unstable manifolds $W^{s/u}(p, F_{s,\beta})$ contain arcs $\hat{\gamma}^{s/u}(\beta)$, sufficiently C^r-close ($r \geq 1$) to $\gamma^{s/u}(\beta)$. These satisfy the 'inevitable tangency' condition (see Takens, 1992 for weakened generic conditions for real-analytic diffeomorphisms):

(i) $\hat{\gamma}^s(\beta_1) \subset W^s(p, F_{s,\beta_1})$ and $\hat{\gamma}^u(\beta_1) \subset W^u(p, F_{s,\beta_1})$ have no intersection;
(ii) $\hat{\gamma}^s(\beta_2) \subset W^s(p, F_{s,\beta_2})$ and $\hat{\gamma}^u(\beta_2) \subset W^u(p, F_{s,\beta_2})$ have two transverse intersections.

Hence, for $s > 0$ small enough, we get a homoclinic bifurcation value $\beta = \hat{\beta}(s) \in (\beta_1, \beta_2)$ at which $\hat{\gamma}^s(\hat{\beta}) \subset W^s(p, F_{s,\hat{\beta}})$ and $\hat{\gamma}^u(\hat{\beta}) \subset W^u(p, F_{s,\hat{\beta}})$ have a homoclinic tangency.

Since Takens' generic conditions (inevitable tangency, analyticity of the map, and non-constantness of $-\log(\lambda_1(\beta))/\log(\lambda_2(\beta))$ with respect to β, where $\lambda_1(\beta)$ and $\lambda_2(\beta)$ are eigenvalues of $D_pF_{s,\beta}$) are satisfied, the homoclinic tangency obtained above is quadratic and unfolds generically. Recall that the fixed point p is hyperbolic and dissipative for every sufficiently small $s > 0$ by Lemma 2. Now apply Theorem 3 from Newhouse (1979) for the first assertion of Proposition 4, and Theorem A from Mora and Viana (1993) for the second assertion. □

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非線形経済理論研究会

変動相場制のマクロ動学分析：カオス的変動とノイズ効果

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1．はじめに

一般に，経済は孤立したシステムではなく，社会における他のサブシステムとの相互干渉にさらされている．この事実は，複数の国または地域の間の経済的な相互作用を分析する国際経済学や地域経済学の理論的・実証的な研究にとって特に重要である．Asada（1995）は，カルドア型の景気循環理論に国際貿易と国際資本移動を導入した小国開放経済のマクロ動学モデルを提出した[1)]．Asada（1995）は，確率的な攪乱が存在しない連続時間モデル（微分方程式モデル）のフレームワークのもとで，固定相場制と変動相場制の双方のケースを分析している．浅田・稲葉・三澤（1998）は，確率的な攪乱を含む離散時間モデル（確率差分方程式モデル）のフレームワークのもとで固定相場制のケースを数値シミュレーションによって分析し，このシステムがカオスを含む極めて複雑な挙動をもたらすことを示した．

本稿でも，確率的攪乱（ノイズ）を含む離散時間モデルを用いて小国開放経済のカルドア型景気循環モデルを考察するが，本稿では，変動相場制のシステムに考察の対象が絞られる．我々は，期待為替レート E^e が適応期待仮説に従って変動し，期待の調整速度 γ が確率的な攪乱にさらされていることを仮定する．我々のシステムは，3変数の非線形確率差分方程式にまとめられる．浅田・稲葉・三澤（1998）は，固定相場制システムにおいて，「ノイズの存在は経済に埋め込まれている基本的な構造を不明瞭にする効果を持っているだけではなく，場合によっては逆に背後に隠されている構造を浮かび上がらせることもある」（p. 182）

ということ，たとえば，「窓」の近傍の背後に隠れたカオス・アトラクターを浮かび上がらせることがあり得る，ということを示した．本稿では，変動相場制システムにおける数値シミュレーションによって以下のことを示す．(1)分岐パラメーター（財市場における不均衡の調整速度）α が増加するにつれて突然為替レートEの変動域が急拡大する「バースト」(burst，爆発）という現象が発生するが，ノイズによってむしろ為替レートの「バースト」が抑制されることがあり得る．(2)周期的変動の「窓」(windows）がノイズによって崩され，カオス的変動が発生し得る．(3)背後に隠された構造がノイズによって浮かび上がることがあり得る．

2．モデルの定式化

本稿のモデルにおける基本的な方程式体系は，以下のように定式化される．

$$Y_{t+1}-Y_t=\alpha\left[C_t+I_t+G+J_t-Y_t\right];\ \alpha>0 \tag{1}$$

$$K_{t+1}-K_t=I_t \tag{2}$$

$$C_t=c(Y_t-T_t)+C_0;\ 0<c<1,\ C_0\geqq 0 \tag{3}$$

$$I_t=I(Y_t,K_t,r_t);\ I_Y\equiv\frac{\partial I_t}{\partial Y_t}>0,\ I_K\equiv\frac{\partial I_t}{\partial K_t}<0,\ I_r\equiv\frac{\partial I_t}{\partial r_t}<0 \tag{4}$$

$$T_t=\tau Y_t-T_0;\ 0<\tau<1,\ T_0\geqq 0 \tag{5}$$

$$\frac{M_t}{P}=L(Y_t,r_t);\ L_Y\equiv\frac{\partial L_t}{\partial Y_t}>0,\ L_r\equiv\frac{\partial L_t}{\partial r_t}<0 \tag{6}$$

$$J_t=J(Y_t,E_t);\ J_Y\equiv\frac{\partial J_t}{\partial Y_t}<0,\ J_E\equiv\frac{\partial J_t}{\partial E_t}>0 \tag{7}$$

$$Q_t=\beta\left\{r_t-r_f-\frac{E^e_t-E_t}{E_t}\right\};\ \beta>0 \tag{8}$$

$$A_t=J_t+Q_t \tag{9}$$

$$A_t=0 \tag{10}$$

$$E^e_{t+1}-E^e_t=(\gamma+\sigma\varepsilon_t)(E_t-E^e_t);\ \gamma>0,\ \sigma\geqq 0 \tag{11}$$

$$M_t=\overline{M}>0 \tag{12}$$

ここで，記号の意味は，以下のとおりである．Y＝実質国民純生産，C＝実質消費支出，I＝実質民間純投資支出，G＝実質政府支出（定数），K＝実質（物的）資本ストック，T＝実質所得税，M＝名目貨幣供給，P＝物価水準（定数），E＝内貨建て為替レート（外貨1単位＝内貨E単位），E^e＝期待為替レート（近い将来に成立すると人々に期待されるEの期待値），J＝実質経常収支（純輸出），Q＝実質資本収支，A＝実質総合収支，α＝財市場における調整速度を表わすパラメーター，β＝国際資本移動の流動性の程度を表わすパラメーター，

γ＝期待為替レートの調整速度を表わすパラメーター，ε＝正規分布に従う擬乱数 $N(0, 1)$，σ＝ノイズの大きさを表わす標準偏差パラメーター．サブスクリプト t は，時点を表わす．

(1)式は，財市場における数量調整プロセスを表現している．(2)式は，「資本蓄積方程式」であり，純投資が資本ストックの変化に等しいことを示している．(3)〜(5)の各式はそれぞれ，消費関数，投資関数，租税関数である．(6)式は，貨幣市場における均衡条件である．(7)式は，標準的なタイプの経常収支関数である．(8)式は，国内債券と外国債券の収益率の差が正か負かに応じて資本収支が正か負かが決まることを示している．βが大きければ大きいほど，国際資本移動は活発になる．(9)式は，総合収支の定義式である．(10)式は，変動相場制システムにおける為替レート決定メカニズムを表わしている．この式は，総合収支を均衡させるように為替レート E が内生的に決まることを示している．(11)式は，期待為替レートに関する「適応期待仮説」を定式化している．期待の調整係数γがノイズによって確率的に変動することが仮定されている．(12)式は，変動相場制システムにおいては通貨当局が貨幣供給をコントロールすることができるので，M を通貨当局によって決定される外生変数とみなすことができることを示している[2]．

方程式体系(1)〜(12)を，以下のようなよりコンパクトなシステムにまとめることができる[3]．

$$\left.\begin{aligned}
&\text{(i)}\quad Y_{t+1}-Y_t=\alpha\,[c(1-\tau)Y_t+cT_0+C_0+G+I(Y_t,K_t,r(Y_t,\overline{M}))\\
&\qquad +J(Y_t,E_t)-Y_t]\ ;\ \alpha>0\\
&\text{(ii)}\quad K_{t+1}-K_t=I(Y_t,K_t,r(Y_t,\overline{M}))\\
&\text{(iii)}\quad A_t=J(Y_t,E_t)+\beta\left\{r(Y_t,\overline{M})-r_f-\frac{E_t^e}{E_t}-1\right\}=0\\
&\text{(iv)}\quad E_{t+1}^e-E_t^e=(\gamma+\sigma\,\varepsilon_t)(E_t-E_t^e)
\end{aligned}\right\}\qquad(13)$$

(13) (iii)式を E_t に関して解くことにより，以下の表現を得る．

$$E_t=E(Y_t,E_t^e\,;\,\beta)\ ;\ E_Y\equiv\frac{\partial E_t}{\partial Y_t}=\frac{-J_Y-\beta r_Y}{J_E+\beta\,(E_t^e/E_t^2)}=\frac{m-\beta r_Y}{J_E+\beta\,(E_t^e/E_t^2)}\gtreqless 0$$

$$\Leftrightarrow\beta\lesseqgtr\frac{m}{r_Y},\quad E_{E^e}\equiv\frac{\partial E_t}{\partial E_t^e}=\frac{\beta}{J_EE_t+\beta\,(E_t^e/E_t^2)}>0\qquad(14)$$

ここで，$m\equiv-J_Y\equiv-\partial J_t/\partial Y_t>0$，$r_Y\equiv\partial r_t/\partial Y_t>0$，$J_E\equiv\partial J_t/\partial E_t>0$ である[4]．(14)式は，資本移動の流動性βが十分に小さければ E_t は Y_t の増加関数になり，βが十分に大きければ E_t は Y_t の減少関数になることを示している[5]．

(14)式を(13)式に代入すれば，次のような 3 変数の非線形確率差分方程式体系を得る．

$$\left.\begin{aligned}
&\text{(i)}\quad Y_{t+1}=Y_t+\alpha\,[c(1-\tau)Y_t+cT_0+C_0+G+I(Y_t,K_t,r(Y_t,\overline{M}))\\
&\qquad +J(Y_t,E(Y_t,E_t^e\,;\,\beta))-Y_t]\equiv F_1(Y_t,K_t,E_t^e\,;\,\alpha,\beta)
\end{aligned}\right\}\qquad(S_1)$$

$$
\left.\begin{array}{ll}
\text{(ii)} & K_{t+1}=K_t+I(Y_t,K_t,r(Y_t,\overline{M}))\equiv F_2(Y_t,K_t) \\
\text{(iii)} & E^e_{t+1}=E^e_t+(\gamma+\sigma\varepsilon_t)\{E(Y_t,E^e_t;\beta)-E^e_t\}\equiv F_3(Y_t,E^e_t;\beta,\gamma,\sigma)
\end{array}\right\}
$$

他方，Asada (1995) によって定式化されたノイズが存在しない連続時間モデル（微分方程式モデル）は，以下のように書かれる．

$$
\left.\begin{array}{ll}
\text{(i)} & \dot{Y}=\alpha[c(1-\tau)Y+cT_0+C_0+G+I(Y,K,r(Y,\overline{M})) \\
 & \quad +J(Y,E(Y,E^e;\beta))-Y]\equiv f_1(Y,K,E^e;\alpha,\beta) \\
\text{(ii)} & \dot{K}=I(Y,K,r(Y,\overline{M}))\equiv f_2(Y,K) \\
\text{(iii)} & \dot{E}^e=\gamma\{E(Y,E^e;\beta)-E^e\}\equiv f_3(Y,E^e;\beta,\gamma)
\end{array}\right\} \qquad (S_2)
$$

Asada (1995) は，システム (S_2) が一意的な均衡点 $(Y^*, K^*, E^{e*})>(0, 0, 0)$ を持つことを一定の仮定のもとで証明したが，システム (S_1) の均衡点はシステム (S_2) の均衡点と同一であることが容易にわかる．本稿の以下の部分では，システム (S_1) が経済的に有意味な均衡点を持つことを仮定して分析を進める．

3．若干の解析的アプローチ

Asada (1995) は，次の諸命題を証明した．

(1) もしβが十分に大きければ，システム (S_2) の均衡点は小域的に安定である．

(2) 均衡点において $I_Y+I_r r_Y>1-c(1-\tau)$ であると仮定しよう．このとき，βが十分に小さくてαが十分に大きければ，システム (S_2) の均衡点は小域的にサドルポイントになる．

これらの諸命題は，我々のモデルの連続時間ヴァージョンでは，国（または地域）の間の資本移動の高い流動性はシステムの安定化要因であることを意味している．この結論は，果たして離散時間モデル（差分方程式モデル）にもあてはまるのであろうか．実は，離散時間モデルにおいては，「オーバーシュート現象」の存在を無視できず，あまりにもβが大きい場合にはシステム (S_1) の均衡点が不安定になるので，上述の命題(1)は，システム (S_1) には適用できないのである．次に，$\sigma=0$（確率的攪乱が存在しない）と仮定して，以上の主張をフォーマルに証明することにしよう．

システム (S_1) の均衡点で評価されたヤコービ行列 J_1 は，以下のように書ける．

$$
J_1\equiv\begin{bmatrix} F_{11}(\alpha,\beta) & F_{12}(\alpha) & F_{13}(\alpha,\beta) \\ F_{21} & F_{22} & 0 \\ F_{31}(\beta,\gamma) & 0 & F_{33}(\beta,\gamma) \end{bmatrix} \qquad (15)
$$

ここで，$F_{11}(\alpha,\beta)=1+\alpha[\underset{(+)}{I_Y}+\underset{(-)}{I_r}\underset{(+)}{r_Y}-\{1-c(1-\tau)+\underset{(+)}{m}\}+\underset{(+)}{J_E}\underset{(?)}{E_Y}(\beta)]$，$F_{12}(\alpha)=\alpha I_K<0$，$F_{13}(\alpha,\beta)=\alpha\underset{(+)}{J_E}\underset{(+)}{E_{E^e}}(\beta)>0$，$F_{21}=\underset{(+)}{I_Y}+\underset{(-)}{I_r}\underset{(+)}{r_Y}$，$F_{22}=1+\underset{(-)}{I_K}$，$F_{31}(\beta,\gamma)=\gamma\underset{(?)}{E_Y}(\beta)$，$F_{33}(\beta,\gamma)=1+\gamma\{E_{E^e}(\beta)-1\}=1-(\underset{(+)}{J_E}E\gamma)/(\underset{(+)}{J_E}E+\beta)$

である．このシステムの特性方程式は，

$$\Delta(\lambda) \equiv |\lambda I - J_1| = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{16}$$

となる．ここで，a_1〜a_3は，以下のように与えられる．

$$\left.\begin{aligned}
&\text{(i)}\quad a_1 = -\operatorname{trace} J_1 \\
&\qquad\; = -F_{11}(\alpha,\beta) - F_{22} - F_{33}(\beta,\gamma) \\
&\text{(ii)}\quad a_2 = \begin{vmatrix} F_{22} & 0 \\ 0 & F_{33} \end{vmatrix} + \begin{vmatrix} F_{11} & F_{13} \\ F_{31} & F_{33} \end{vmatrix} + \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} \\
&\qquad\; = F_{22}F_{33}(\beta,\gamma) + F_{11}(\alpha,\beta)F_{33}(\beta,\gamma) - F_{13}(\alpha,\beta)F_{31}(\beta,\gamma) \\
&\qquad\quad + F_{11}(\alpha,\beta)\,F_{22} - F_{12}(\alpha,\beta)\,F_{21} \\
&\text{(iii)}\quad a_3 = -\det J_1 \\
&\qquad\; = -F_{11}(\alpha,\beta)F_{22}\,F_{33}(\beta,\gamma) + F_{13}(\alpha,\beta)F_{22}\,F_{31}(\beta,\gamma) \\
&\qquad\quad + F_{12}(\alpha)F_{21}\,F_{33}(\beta,\gamma)
\end{aligned}\right\} \tag{17}$$

差分方程式に関するコーン＝シューアの安定条件によれば，システム（S_1）が小域的に安定になるための必要十分条件は，以下のように与えられる[6)]．

$$\left.\begin{aligned}
&\text{(i)}\quad 1 + a_2 - |a_1 + a_3| > 0 \\
&\text{(ii)}\quad 1 - a_2 + a_1a_3 - a_3^2 > 0 \\
&\text{(iii)}\quad a_2 < 3
\end{aligned}\right\} \tag{18}$$

従って，もし$a_2>3$ならばシステム（S_1）は小域的に不安定になる．以下に提出する命題は，この事実の単純な系論である．

〔命題〕

$0 < \underset{(+)}{I_Y} + \underset{(-)}{I_r}\,\underset{(+)}{r_Y} < \overset{(+)}{\underbrace{1 - c\,(1-\tau)}} + \underset{(+)}{m}$ および $\underset{(-)}{I_K} < -1$ がシステム（S_1）の均衡点において成立していると仮定しよう．このとき，次の（A）または（B）のいずれかの条件が満たされていれば，システム（S_1）の均衡点は小域的に不安定になる．

（A）　パラメーターβおよびαが十分に大きい．

（B）　パラメーターβおよびγが十分に大きい．

〔証明〕

(14)式，$F_{31}(\beta,\gamma)$および$F_{33}(\beta,\gamma)$の定義および均衡において$E^e=E$となることを用いれば，次の関係を得る．

$$\lim_{\beta\to\infty} J_E E_Y = \lim_{\beta\to\infty} \frac{m - \beta r_Y}{1 + (\beta/J_E E)} = -r_Y J_E E < 0 \tag{19}$$

$$\lim_{\beta\to\infty} F_{31}(\beta,\gamma) = \lim_{\beta\to\infty} \frac{\gamma\,(m - \beta r_Y)}{J_E + \beta} = -\gamma r_Y < 0 \tag{20}$$

$$\lim_{\beta\to\infty} F_{33}(\beta,\gamma) = \lim_{\beta\to\infty}\left\{1-\gamma\left(\frac{\beta}{J_E E+\beta}-1\right)\right\}=1 \tag{21}$$

従って，次式が成立する．

$$A \equiv \lim_{\beta\to\infty} a_2 = \underset{(-)}{F_{22}} + \underset{(-)}{F_{11}^*}(\alpha)\underbrace{(1+F_{22})}_{(-)} - \underset{(+)}{F_{13}^*}(\alpha)\underset{(-)}{F_{31}^*}(\gamma) - \underset{(-)}{F_{12}}(\alpha)\underset{(+)}{F_{21}} \tag{22}$$

ここで，$F_{ij}^* = \lim_{\beta\to\infty} F_{ij}$, $\partial F_{11}^*(\alpha)/\partial\alpha<0$, $\partial F_{12}(\alpha)/\partial\alpha<0$, $\partial F_{13}^*(\alpha)/\partial\alpha>0$, $\partial F_{31}^*(\gamma)/\partial\gamma<0$ である．(22)式より（i）β と α が十分に大きい場合か，あるいは（ii）β と γ が十分に大きい場合には，$A>3$ となり，システム（S_1）は小域的に不安定になることがわかる．

（証明了）

この命題は，財市場における速い調整速度（α）または期待為替レートの速い調整速度（γ）と結び付いた国際資本移動の高い流動性（β）は，オーバーシュート現象のためにシステム（S_1）の均衡点を不安定化させることを，若干の追加的仮定のもとで示している．

4．数値シミュレーション分析

本節では，前節で提出されたモデルの若干の数値シミュレーションの結果を紹介することにしよう．我々は，以下のように関数形およびパラメーター値を特定化してシミュレーションを行なった．

$$I(Y_t, K_t, r_t) = f(Y_t) - 0.3K_t - r_t + 100 \tag{23}$$

$$f(Y_t) = \frac{80}{3.1415}\mathrm{Arctan}\left\{\left(\frac{2.25\times 3.1415}{20}\right)(Y_t-112)\right\}+35 \tag{24}$$

$$r_t = r(Y_t, M) = 10\sqrt{Y_t} - M \tag{25}$$

$$J(Y_t, E_t) = -0.3Y_t + 100 - \frac{100}{E_t} \tag{26}$$

$$E_{t+1}^e - E_t^e = (\gamma + \sigma\varepsilon_t)(E_t - E_t^e) \tag{27}$$

$$c=0.8,\ \tau=0.2,\ r_f=6,\ cT_0+C_0+G=238,\ M=100,\ \beta=15,\ \gamma=1.2 \tag{28}$$

(24)式は，カルドア型のＳ字型投資関数をアークタンジェント関数で表現している．(25)式と(26)式を国際収支の均衡条件（(13)(iii)式）に代入すれば，

$$-0.3Y_t + 100 - \frac{100}{E_t} + 15\left(10\sqrt{Y_t} - \frac{E_t^e}{E_t} - 105\right) = 0 \tag{29}$$

という式を得る．(29)式を E_t に関して解けば，以下のような為替レート E_t の表現が得られる．

$$E_t = \frac{100+15E_t^e}{-0.3Y_t + 150\sqrt{Y_t} - 1475} \tag{30}$$

これらのデータを用いて，我々は数値シミュレーションを行なった．分岐パラメーターとしては，財市場における調整速度 α を採用した．確率的攪乱が存在しない（$\sigma=0$）場合には，均衡点は $(Y^*, K^*, E^{e*})=(Y^*, K^*, E^*)\simeq(112, 450, 1.56)$ となるが，これらの値は，パラメーター α からは独立に決まる．

ノイズ効果を分析するために，期待為替レートの調整係数（γ）がノイズによって変動することが仮定された．一般的に言えば，2種類のノイズ，すなわち，「加法的ノイズ」(additive noise) と「パラメトリック・ノイズ」(parametric noise) が存在する．加法的ノイズの場合には，ある種の方程式にノイズが足し算の形で加えられる．パラメトリック・ノイズの場合には，方程式におけるあるパラメーターがノイズによって変動する．Crutchfield et al. (1982) は，ロジスティック方程式におけるノイズ効果を分析し，パラメトリック・ノイズと加法的ノイズが同値になることを示した．しかし，この結論は，高次元（多変数）のシステムについては必ずしもあてはまらない．本稿の研究においては，我々は，外国為替レートがノイズによって変動する効果を分析するために，パラメトリック・ノイズのケースをとりあげることにする．閉鎖経済におけるパラメトリック・ノイズを伴うカルドア型景気循環モデルの数値シミュレーションの詳細については，Dohtani et al. (1996) を参照されたい．

我々の数値シミュレーションの主な結果は，以下のように要約される．

(1) ノイズが存在しないシステム（$\sigma=0$）において財市場の調整速度を表わすパラメーター α が増加するにつれて，為替レートEの変動域が突然急拡大する「バースト」という現象が発生する．

(2) ノイズによって「バースト」が抑制され得る．換言すれば，ノイズは隠された構造を浮かび上がらせることがある．

(3) 周期解を発生させる「窓」がノイズによって崩され，背後に隠されたカオス的な構造が現われることがある．

4-1 「バースト」の出現と抑制

図1は，ノイズが存在しない場合の外国為替レートEの分岐図である．分岐パラメーターは，財市場における調整速度 α である．図2は，このシステムにおける最大リャプノフ指数を示している[7]．これらの図より，$\alpha\simeq0.33$ において「バースト」が出現し，$\alpha\geqq0.33$ の領域で頻繁にカオス的な変動が発生することがわかる[8]．図3は，$\alpha=0.33$ におけるカオス・アトラクターをE-Y平面上に描いたものである．

ところで，ノイズによって為替レートの「バースト」が抑制されることがあり得る．図4は，$\alpha=0.33$ のときわずかのノイズ（$\sigma=0.01$）によって浮かび上がらされたE-Y平面上の

アトラクターである．このアトラクターは，ノイズが存在しない場合の $\alpha=0.32$ におけるアトラクターと似ているが，このアトラクターは周期的であり，「バースト」も出現していない．経済学的な観点からこのことを解釈すれば，以下のようになるであろう．確率的な攪乱（ノイズ）がない場合に外国為替レートが激しく変動する場合でも，その激しい変動がわずかのノイズによって抑制され得るのである．図5と図6は，$\alpha=0.33$ における為替レートの軌道をノイズがない場合（$\sigma=0$）とわずかのノイズが存在する場合（$\sigma=0.01$）に分けて比較している．

以上の例は，ノイズがシステムを安定化させ得るということを示している．しかし，このことは，ノイズがシステムを常に安定化させるということを意味するわけではない．図7と図8は，幾分大きなノイズ（$\sigma=0.05$）が存在する場合の数値実験の2つの例である．この例は，ノイズがシステムを不安定化させることもあり得るということを示している．図9は，$\alpha=0.33$ および $\sigma=0.05$ のときに出現したE-Y平面上のアトラクターである．このアトラクターはカオス的であり，ノイズが存在しない場合の $\alpha=0.34$ におけるアトラクターに似ている．以上の結果を要約すれば，ノイズは隠された構造を浮かび上がらせることがあり得る，ということになるであろう．

図1　ノイズが存在しない場合のEの分岐図

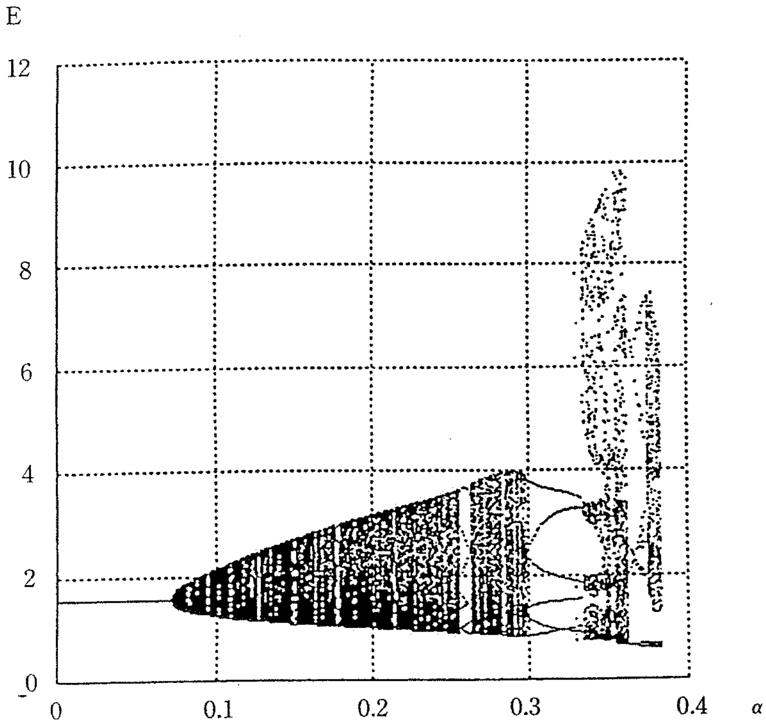


図 2　ノイズが存在しない場合の最大リャプノフ指数

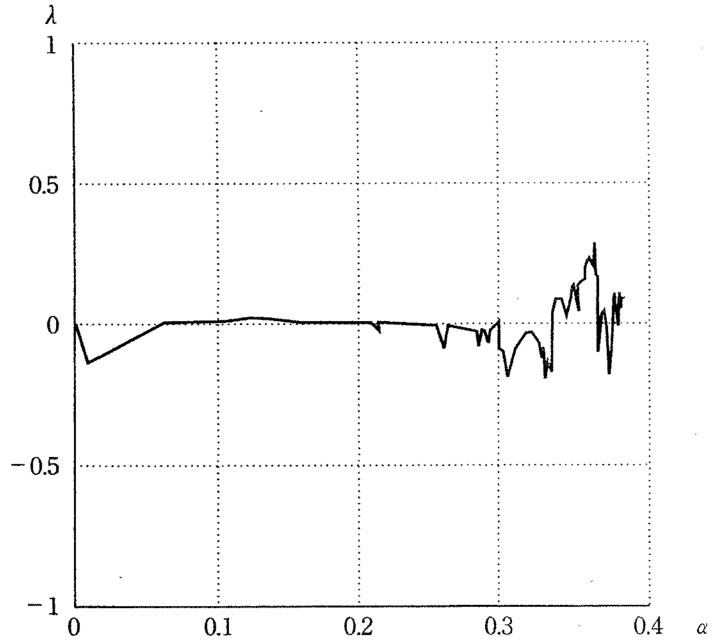


図 3　ノイズが存在しない場合の E-Y 平面上のアトラクター（α＝0.33）

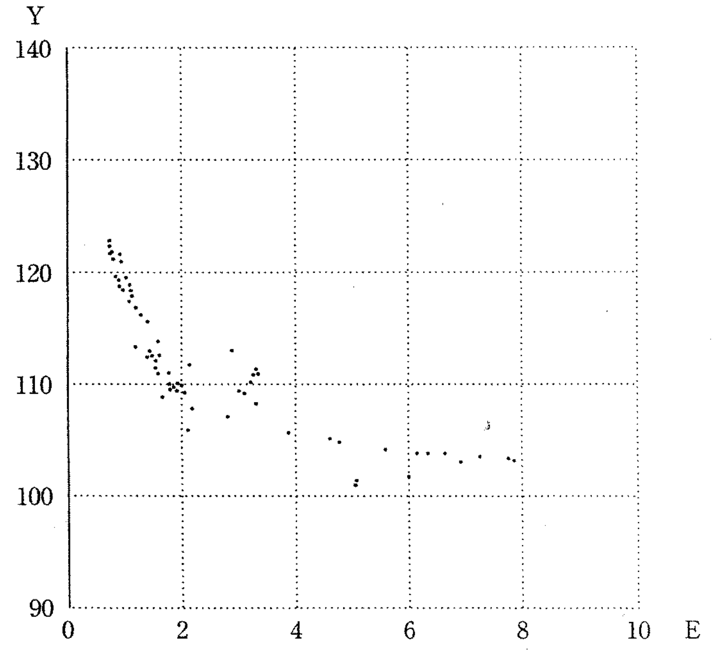


図4　ノイズが存在する場合のE-Y平面上のアトラクター（$\alpha=0.33$,　$\sigma=0.01$）

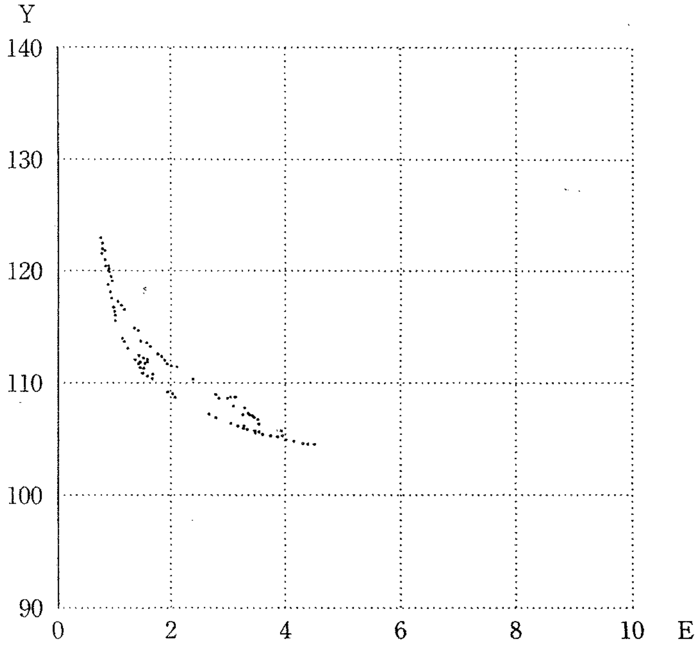


図5　ノイズが存在しない場合のEの軌道（$\alpha=0.33$,　$\sigma=0.01$）

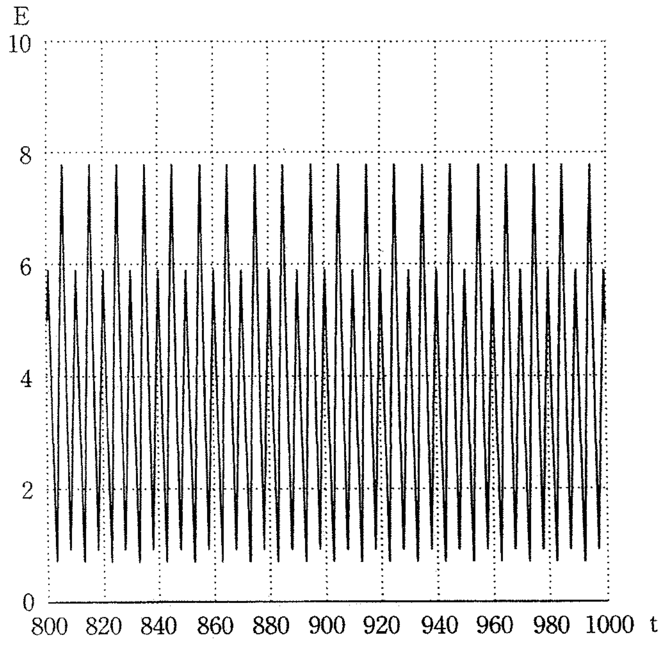


図 6　ノイズが存在する場合の E の軌道（$\alpha=0.33$，$\sigma=0.01$）

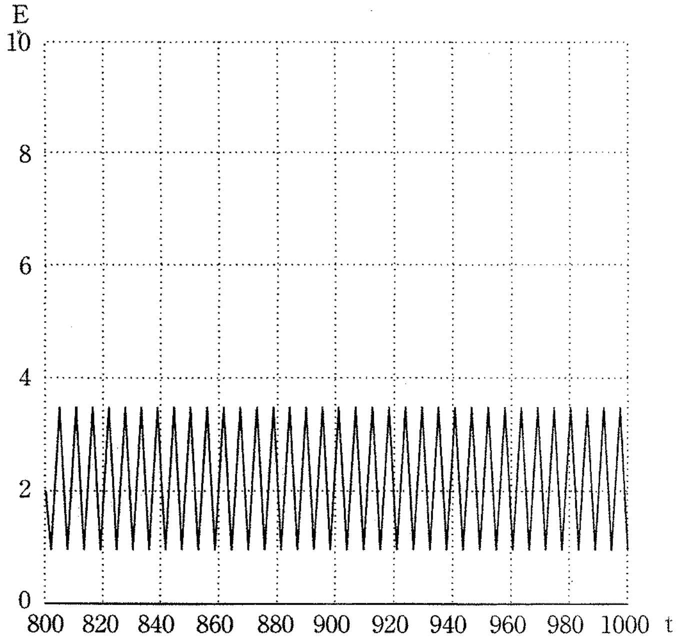


図 7　ノイズが存在する場合の E の軌道（$\alpha=0.33$，$\sigma=0.05$）

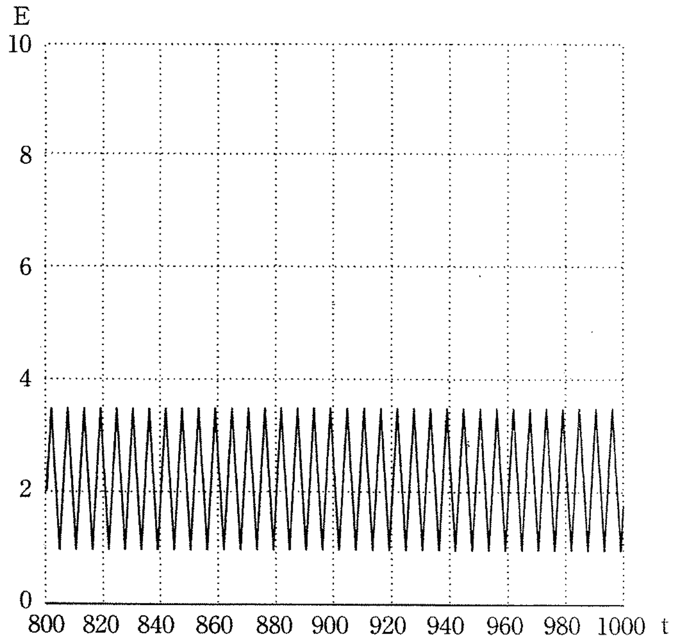


図8　ノイズが存在する場合のＥの軌道（$\alpha=0.33$,　$\sigma=0.05$）

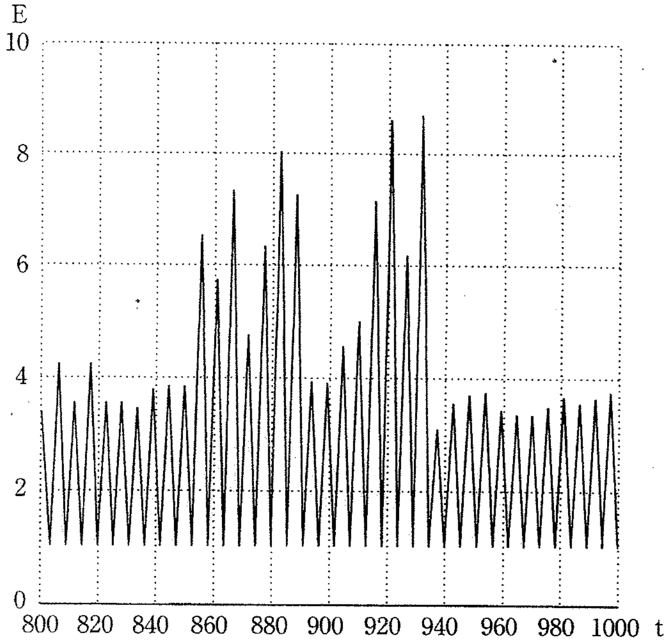
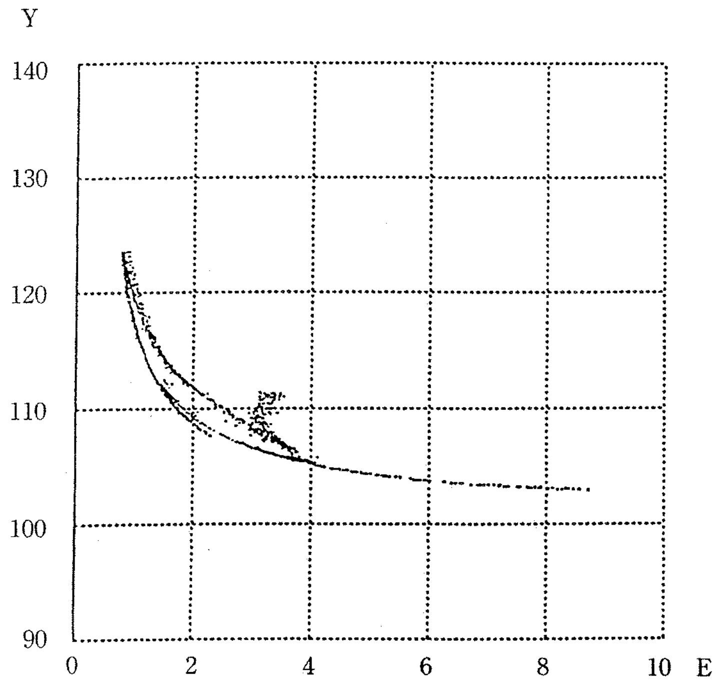


図9　ノイズが存在する場合のE-Y平面上のアトラクター（$\alpha=0.33$,　$\sigma=$ ~~0.01~~ 0.05）



4-2 「窓」の崩壊

ノイズが存在する場合の分岐図と最大リャプノフ指数（図10および図11）をノイズが存在

図10　ノイズが存在する場合のＥの分岐図

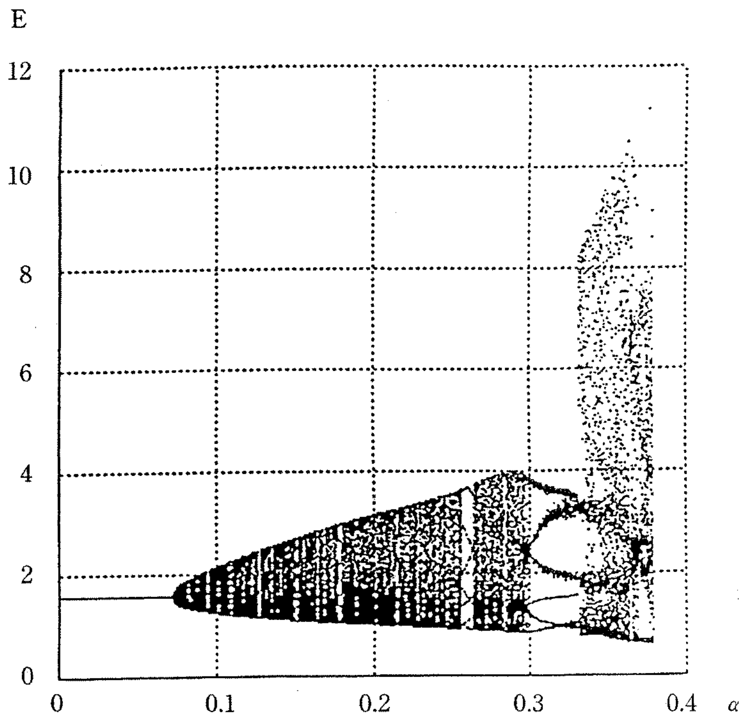
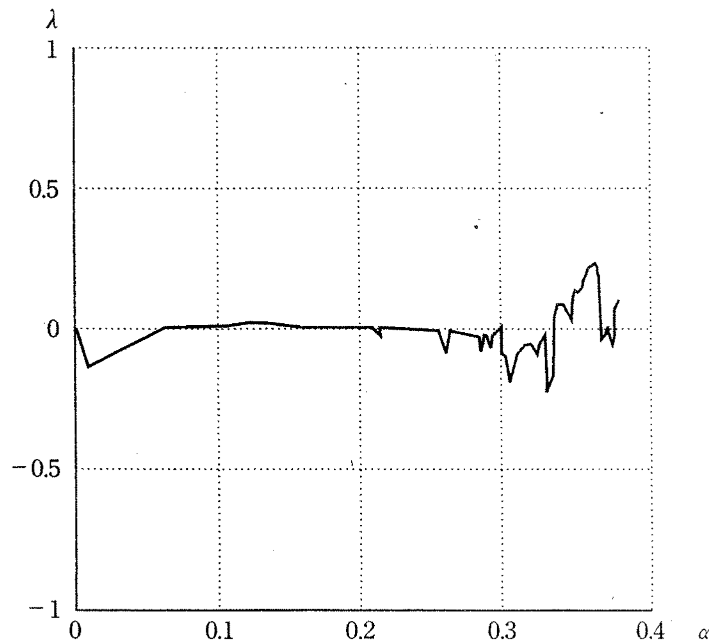


図11　ノイズが存在する場合の最大リャプノフ指数



しない場合のそれら（図１および図２）と比較することにより，ノイズは，為替レートＥに関する周期解の「窓」を崩壊させ得ることがわかる．このことは，国民所得Ｙに関してもあてはまる．この例は，ノイズが存在しない場合のシステムの挙動が周期的であったとしても，背景に隠れたカオス的な構造がノイズによって浮かび上がることがある，ということを示している．

5．結　　論

本稿においては，変動相場制下の小国開放経済におけるカルドア型景気循環モデルの離散時間ヴァージョンを用いることにより，ノイズ効果の影響を分析した．このモデルは，国または地域の間の経済的な相互依存関係を分析するための出発点としては，十分に意味のあるものであろう．しかし，小国開放経済モデルにおいては，外国の国民所得や利子率のような重要な変数が外生変数として扱われているという意味で，その分析の射程距離は限られている．地域間の動学的な相互依存関係を分析するためには，多数国モデル（あるいは多数地域モデル）が最適であろう．多数国モデルの最も単純なヴァージョンは，２国モデルである．動学的な２国モデルの本格的な分析は，我々に残された将来の課題である．

〔付記〕 本稿は，平成11年度学術振興会科学研究費補助金に基づく共同研究「非線形動学の方法による経済変動の研究」の成果の一部である．なお，本稿は，1999年７月15日韓国のソウルで開催されたPRSCO16（The 16th Pacific Regional Science Conference）で報告された論文“Chaotic Dynamics in a Flexible Exchange Rate System : A Study of Noise Effects”に基づいている．

注

1) この論文の拡張されたヴァージョンは，浅田（1997）第３章として再録されている．
2) 固定相場制のもとでは，貨幣供給は，国際収支（総合収支）が正か負かに応じて変動する内生変数になる．Asada（1995），浅田（1997）第３章，浅田・稲葉・三澤（1998）を参照されたい．
3) ⒀式において，$r(Y_t, \bar{M})$ は，⑹式を r_t に関して解くことによって得られる「LM 方程式」である．
4) ⑹式を r_t に関して解けば，$r_t = r(Y_t, \bar{M})$；$r_t \equiv \partial r_t / \partial Y_t = -L_Y/L_r > 0$を得る．ただし，$L_Y \equiv \partial L_t / \partial Y_t > 0$，$L_r \equiv \partial L_t / \partial r_t < 0$ である．
5) この結果の経済学的な意味は，極めて明快である．Y_tが増加すれば，輸入の増加を通じて経常収支 J_t は減少するが，国内利子率の上昇を通じて資本収支 Q_t は増加する．もしβが小さければ，「経常収支効果」が「資本収支効果」を圧倒して，総合収支 A_t は減少する．この場合，為替レートの変化によって総合収支の均衡（$A_t = 0$）を回復するためには，内貨建て為替レート E_t は増加しなければならない．他方，βが大きい場合には，「資本収支効果」が優勢になるので A_t は増加する．この場合に $A_t = 0$ を回復するためには，E_t は減少しなければならない．
6) この条件については，たとえば Gandolfo (1996) chap. 7 を参照されたい．実際には，不等式⒅(iii)は，他の２つの不等式から自動的に導出されるので，余分なものとみなすことができるが，現在の我々の目的にとっては便利な表現である．
7) 最大リャプノフ指数については，浅田・稲葉・三澤（1997），Lorenz (1993) chap. 6 を参照されたい．
8) 最大リャプノフ指数λがを０を上回るとき，システムはカオス的であるとみなし得る．Lorenz

(1993) chap. 6 参照.

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A Nonlinear Macrodynamic Model with Fixed Exchange Rates: Its Dynamics and Noise Effects

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In this paper, we formulate a discrete time version of the Kaldorian macrodynamic model in a small open economy with fixed exchange rates. The model is described by a system of the three-dimensional nonlinear difference equations with and without stochastic disturbances (noise effects). We study the local stability/instability properties analytically by using the linear approximation method, and chaotic dynamics with and without noise effects are investigated by means of numerical simulations. In general, it is believed that the effect of the noise is to obscure the basic structure of the system. But, this is not necessarily the case. We show by means of numerical analysis that the noise can reveal the hidden structure of the model contrary to the usual intuition in some situations.

Keywords: Fixed exchange rates, Noise effects, Nonlinear macrodynamics, Small open economy

1. INTRODUCTION

The purpose of this paper is to formulate a macrodynamic model which is described by a system of the three-dimensional nonlinear difference equations with and without stochastic disturbances (noise effects), and to investigate its behavior by means of analytical method and numerical simulations. The model presented in this paper is a discrete time version of the Kaldorian business cycle model in an open economy which was formulated by Asada (1995) as a continuous time model.[†] Contrary to Asada (1995)'s original model, we introduce the noise effects.

Generally speaking, the economy is not isolated system, but it is subject to the interactions with other subsystems of the society. One of the effective methods to model such influences is to introduce the 'noise' (stochastic disturbance). In our model, it is supposed that another subsystem named 'foreign country' exists outside the system, and the dynamics of the economy are affected by the

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[†] The original version of the Kaldorian business cycle model in a closed economy was presented by Kaldor (1940)'s classical paper, and it was later refined by several authors. See, for example, Chang and Smyth (1971), Gabisch and Lorenz (1989), and Lorenz (1993).

transactions with 'foreign country'. We assume that the parameter β which reflects the 'degree of capital mobility' is subject to the stochastic disturbances, and study the effects of the noise on the dynamics of the system by means of numerical simulations.

A seminal paper which introduced noise into the Kaldorian business cycle model in a closed economy is Kosobad and O'Nell (1972)'s model. Dohtani *et al.* (1996)'s work is a more recent contribution. In particular, Dohtani *et al.* (1996) introduced the noise effects into the Kaldorian business cycle model which is described by the two-dimensional nonlinear difference equations, and showed by means of numerical experimentation that the noise can reveal rather than obscure the hidden structure of the system in some situations contrary to the usual intuition. In this paper, we show that such a conclusion also applies in an extended version of the three-dimensional Kaldorian system in an open economy with fixed exchange rates. In particular, it is shown by means of numerical approach that the noise can reveal the hidden chaotic attractors at the vicinity of the 'window', and two separate chaotic attractors can be combined under the influence of the noise.

2. THE MODEL

Asada (1995) tried to extend the Kaldorian type of the nonlinear business cycle model to the small open economy by using the deterministic continuous time model. In Asada (1995), both the system of fixed exchange rates and that of flexible exchange rates were formulated and investigated by analytical method and numerical simulations. In particular, the effect of the change of the parameter β which represents the 'degree of capital mobility' were analyzed, and it was shown by means of the Hopf bifurcation theorem that the cyclical fluctuation can occur at some parameter values in the system of fixed exchange rates.

In this paper, we shall consider a stochastic version of Asada (1995)'s model of fixed exchange rates.[‡] We can describe the basic system of equations as follows:

$$Y_{t+1} - Y_t = \alpha[C_t + I_t + G + J_t - Y_t],\ \alpha > 0; \tag{1}$$

$$K_{t+1} - K_t = I_t; \tag{2}$$

$$C_t = c(Y_t - T_t) + C_0;\quad 0 < c < 1,\ C_0 > 0; \tag{3}$$

$$\begin{aligned} I_t = I(Y_t, K_t, r_t);\quad & I_Y \equiv \partial I_t/\partial Y_t > 0,\\ I_K \equiv \partial I_t/\partial K_t < 0,\ & I_r \equiv \partial I_t/\partial r_t < 0; \end{aligned} \tag{4}$$

$$T_t = \tau Y_t - T_0;\quad 0 < \tau < 1,\ T_0 > 0; \tag{5}$$

$$\begin{aligned} M_t/p = L(Y_t, r_t);\quad & L_Y \equiv \partial L_t/\partial Y_t > 0,\\ L_r \equiv \partial L_t/\partial r_t < 0; & \end{aligned} \tag{6}$$

$$\begin{aligned} J_t = J(Y_t, E_t);\quad & J_Y \equiv \partial J_t/\partial Y_t < 0,\\ J_E \equiv \partial J_t/\partial E_t > 0; & \end{aligned} \tag{7}$$

$$\begin{aligned} & Q_t = (\beta + \sigma\gamma_t)\{r_t - r_f - (E_t^e - E_t)/E_t\};\\ & \beta > 0,\ \sigma \geq 0; \end{aligned} \tag{8}$$

$$A_t = J_t + Q_t; \tag{9}$$

$$E_t = \bar{E}; \tag{10}$$

$$E_t^e = \bar{E}; \tag{11}$$

$$M_{t+1} - M_t = pA_t; \tag{12}$$

where the meanings of the symbols are as follows: Y = net real national income, C = real consumption expenditure, I = net real private investment expenditure, G = real government expenditure (fixed), K = real physical capital stock, T = real income tax, M = nominal money supply, p = price level

[‡] The model of flexible exchange rates will be considered separately in another paper.

(fixed), r = nominal domestic rate of interest, r_f = nominal foreign rate of interest (fixed), E = value of a unit of foreign currency in terms of domestic currency (exchange rate), E^e = expected exchange rate of near future, J = balance of current account (net export) in real terms, Q = balance of capital account in real terms, A = total balance of payments in real terms, α = adjustment speed in the goods market, β = parameter which represents the "degree of capital mobility" ($\beta > 0$), γ = normal pseudo-random number $N(0, 1)$, σ = standard deviation parameter ($\sigma \geq 0$). The subscript t denotes time period.

Equation (1) formulates the quantity adjustment process in the goods market, i.e., Y_t fluctuates according as the excess demand in the goods market is positive or negative. Equation (2) is the capital accumulation equation. Equations (3)–(5) are consumption function, investment function, and income tax function respectively. Equation (6) is the equilibrium condition in the money market. Equation (7) is the current account function. Equation (8) says that the balance of capital account depends on the difference between the rates of return of domestic and foreign bonds. It is assumed that the parameter β (degree of capital mobility) is fluctuated by noise. Equation (9) is the definition of the total balance of payments. Equations (10) and (11) express the institutional arrangement of the system of fixed exchange rates. Equation (12) says that money supply endogenously fluctuates according as the total balance of payments is positive or negative under the system of fixed exchange rates.

These equations can be reduced to the following set of three-dimensional nonlinear difference equations:¶

$$\begin{aligned}
\text{(i)}\quad Y_{t+1} &= Y_t + \alpha[c(1-\tau)Y_t + cT_0 + C_0 \\
&\quad + G + I(Y_t, K_t, r(Y_t, M_t)) \\
&\quad + J(Y_t, \bar{E}) - Y_t] \\
&\equiv F_1(Y_t, K_t, M_t; \alpha);
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad K_{t+1} &= K_t + I(Y_t, K_t, r(Y_t, M_t)) \\
&\equiv F_2(Y_t, K_t, M_t); \qquad\qquad (S_1) \\
\text{(iii)}\quad M_{t+1} &= M_t + pJ(Y_t, \bar{E}) \\
&\quad + (\beta + \sigma\gamma_t)p\{r(Y_t, M_t) - r_f\} \\
&\equiv F_3(Y_t, M_t; \beta, \sigma).
\end{aligned}$$

We shall call the system (S_1) 'model 1'. By the way, Chang and Smyth (1971)'s version of the Kaldorian business cycle model adopts the following type of the saving function:

$$S_t = S(Y_t, K_t); \quad 1 > S_Y \equiv \partial S_t/\partial Y_t > 0, \quad S_K \equiv \partial S_t/\partial K_t < 0. \tag{13}$$

Since the saving S_t is the difference between the disposable income and the consumption C_t, Eq. (13) implies the following type of the consumption function:

$$C_t = C(Y_t, K_t); \quad 1 > C_Y \equiv \partial C_t/\partial Y_t > 0, \quad C_K \equiv \partial C_t/\partial K_t > 0. \tag{14}$$

This consumption function represents a sort of the 'wealth effect', i.e., the increase of the real capital stock stimulates the consumption expenditure. If we adopt this type of consumption function, we must replace Eq. (S_1)-(i) with the following equation:

$$Y_{t+1} = Y_t + \alpha[C(Y_t, K_t) + I(Y_t, K_t, r(Y_t, M_t)) + G + J(Y_t, \bar{E}) - Y_t]. \tag{15}$$

In this case, we have

$$\frac{\partial Y_{t+1}}{\partial K_t} = \alpha(\underset{(+)}{C_K} + \underset{(-)}{I_K}) > \alpha \underset{(-)}{I_K}. \tag{16}$$

In particular, in the special case of $C_K = |I_K|$, we obtain

$$\frac{\partial Y_{t+1}}{\partial K_t} = 0. \tag{17}$$

¶The expression $r(Y_t, M_t)$ is the 'LM equation' which is derived from Eq. (6). It is easy to see that $r_Y \equiv \partial r_t/\partial Y_t > 0$ and $r_M \equiv \partial r_t/\partial M_t < 0$.

In other words, the negative effect of the change of K_t on Y_{t+1} through the negative effect on the investment expenditure tends to be canceled out by the positive effect on the consumption expenditure when the 'wealth effect' exists. If Eq. (17) is satisfied, the system (S_1) must be modified as follows:

$$\begin{aligned}
&\text{(i)} \quad Y_{t+1} = F_1(Y_t, M_t; \alpha);\\
&\text{(ii)} \quad K_{t+1} = F_2(Y_t, K_t, M_t); \qquad (S_2)\\
&\text{(iii)} \quad M_{t+1} = F_3(Y_t, M_t; \beta, \sigma).
\end{aligned}$$

In this system, Y_{t+1} is independent of K_t so that the system becomes 'decomposable'. In other words, the path of K_t depends on the paths of Y_t and M_t, but the movements of Y_t and M_t are independent of the path of K_t. We shall refer to the system (S_2) as 'model 2'.

3. LOCAL STABILITY–INSTABILITY ANALYSIS OF 'MODEL 1'

First, let us consider the local stability–instability analysis of 'model 1' by assuming $\sigma = 0$ (no stochastic disturbance). Asada (1995) proved that the system (S_1) has the unique equilibrium point $(Y^*, K^*, M^*) > (0,\ 0,\ 0)$ under some reasonable conditions. In this paper, we shall *assume* that such an equilibrium point in fact exists. The Jacobian matrix of this system which is evaluated *at the equilibrium point* can be written as follows:

$$J_1 = \begin{bmatrix} F_{11}(\alpha) & F_{12}(\alpha) & F_{13}(\alpha) \\ F_{21} & F_{22} & F_{23} \\ F_{31}(\beta) & 0 & F_{33}(\beta) \end{bmatrix}, \tag{18}$$

where

$$F_{11}(\alpha) = 1 + \alpha[\underset{(+)}{I_Y} + \underset{(-)(+)}{I_r r_Y} - \{1 - c(1-\tau) + \underset{(+)}{m}\}],$$

$$F_{12}(\alpha) = \alpha \underset{(-)}{I_K} < 0, \quad F_{13}(\alpha) = \alpha \underset{(-)(-)}{I_r r_M} > 0,$$

$$F_{21} = \underset{(+)(-)(+)}{I_Y + I_r r_Y}, \quad F_{22} = 1 + \underset{(-)}{I_K}, \quad F_{23} = \underset{(-)(-)}{I_r r_M} > 0,$$

$$F_{31}(\beta) = p(\underset{(+)}{-m} + \beta \underset{(+)}{r_Y}),$$

$$F_{33}(\beta) = 1 + \beta p \underset{(-)}{r_M}.$$

In these expressions, $m = -J_Y > 0$ is the 'marginal propensity to import'. The characteristic equation of this system is expressed as

$$\begin{aligned} \Delta_1(\lambda) &= |\lambda I - J_1| \\ &= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \end{aligned} \tag{19}$$

where

$$a_1 = -\operatorname{trace} J_1 = -F_{11}(\alpha) - F_{22} - F_{33}(\beta), \tag{20}$$

$$\begin{aligned} a_2 &= \begin{vmatrix} F_{22} & F_{23} \\ 0 & F_{33}(\beta) \end{vmatrix} + \begin{vmatrix} F_{11}(\alpha) & F_{13}(\alpha) \\ F_{31}(\beta) & F_{33}(\beta) \end{vmatrix} \\ &\quad + \begin{vmatrix} F_{11}(\alpha) & F_{12}(\alpha) \\ F_{21} & F_{22} \end{vmatrix} \\ &= F_{22}F_{33}(\beta) + F_{11}(\alpha)F_{33}(\beta) - F_{13}(\alpha)F_{31}(\beta) \\ &\quad + F_{11}(\alpha)F_{22} - F_{12}(\alpha)F_{21}, \end{aligned} \tag{21}$$

$$\begin{aligned} a_3 &= -\det J_1 \\ &= -F_{11}(\alpha)F_{22}F_{33}(\beta) - F_{12}F_{23}F_{31}(\beta) \\ &\quad + F_{13}(\alpha)F_{22}F_{31}(\beta) + F_{12}(\alpha)F_{21}F_{33}(\beta). \end{aligned} \tag{22}$$

The Cohn–Schur conditions for local stability can be expressed as follows:[§]

$$1 + a_2 - |a_1 + a_3| > 0, \tag{23}$$

$$1 - a_2 + a_1 a_3 - a_3^2 > 0, \tag{24}$$

$$a_2 < 3. \tag{25}$$

[§] See Gandolfo (1996), p. 90. In fact, the condition (25) is redundant because this condition can be derived from other two conditions. However, for our purpose, this expression is convenient.

From these local stability conditions we can derive a very simple *sufficient condition for local instability*, i.e., $a_2 > 3$. By using this local instability condition, we can derive the following proposition.

PROPOSITION 1 *Suppose that $I_Y < 1 - c(1-\tau) + m$. Then, the equilibrium point of the system* (S_1) *is locally unstable if $\alpha > 0$ and $\beta > 0$ are sufficiently large.*

Proof Differentiating Eq. (21), we have

$$\begin{aligned}
&\partial a_2/\partial\alpha \\
&\quad = \{F_{22} + F_{33}(\beta)\}\{\partial F_{11}(\alpha)/\partial\alpha\} \\
&\qquad - F_{31}(\beta)\{\partial F_{13}(\alpha)/\partial\alpha\} - F_{21}\{\partial F_{12}(\alpha)/\partial\alpha\} \\
&\quad = (2\beta + I_K + \beta p r_M)[I_Y + I_r r_Y \\
&\qquad - \{1 - c(1-\tau) + m\}] \\
&\qquad - p(-m + \beta r_Y) I_r r_M - (I_Y + I_r r_Y) I_K \\
&\quad = \underset{(-)}{\beta r_M p}\underbrace{[I_Y - \{1 - c(1-\tau) + m\}]}_{(-)} \\
&\qquad + pm - (I_Y + I_r r_Y) I_K + (2 + I_K) \\
&\qquad \times [I_Y - \{1 - c(1-\tau) + m\} + I_r r_Y]. \qquad (26)
\end{aligned}$$

From Eq. (26) we have $\lim_{\beta\to+\infty} \partial a_2/\partial\alpha = +\infty$ so that $\partial a_2/\partial\alpha$ becomes positive for sufficiently large $\beta > 0$. In this case, $a_2 > 3$ for sufficiently large α and β.

Proposition 1 implies that under certain conditions, the increase of the adjustment speed in the goods market (α) and the degree of capital mobility (β) tends to *destabilize* the system under the system of fixed exchange rates. This conclusion is in line with the result which was derived by Asada (1995)'s continuous time version of the model of fixed exchange rates.

4. LOCAL STABILITY–INSTABILITY ANALYSIS OF 'MODEL 2'

Next, we shall consider the local stability–instability analysis of 'model 2'. We also assume in this section that $\sigma = 0$ (absence of noise effect). The Jacobian matrix of the system (S_2) becomes

$$J_2 = \begin{bmatrix} F_{11}(\alpha) & 0 & F_{13}(\alpha) \\ F_{21} & F_{22} & F_{23} \\ F_{31}(\beta) & 0 & F_{33}(\beta) \end{bmatrix}, \qquad (27)$$

where the meanings of the symbols are the same as those of the previous chapter. The characteristic equation of this system is

$$\Delta_2(\lambda) = |\lambda I - J_2| = (\lambda - F_{22})(\lambda^2 + b_1\lambda + b_2) = 0. \qquad (28)$$

where

$$b_1 = -F_{11}(\alpha) - F_{33}(\beta), \qquad (29)$$

$$b_2 = F_{11}(\alpha)\,F_{33}(\beta) - F_{13}(\alpha)\,F_{31}(\beta). \qquad (30)$$

We can express the Cohn–Schur conditions for local stability as follows:[‖]

$$|F_{22}| < 1, \qquad (31)$$

$$1 + b_2 > |b_1|, \qquad (32)$$

$$b_2 < 1. \qquad (33)$$

Equation (31) is equivalent to the condition $|I_K| < 1$. We assume that in fact this condition is satisfied. By the way, we can easily see that $b_2 > 1$ is a *sufficient* condition for *local instability*.

Differentiating Eq. (30), we have

$$\begin{aligned}
\partial b_2/\partial\alpha &= F_{33}(\beta)\{\partial F_{11}(\alpha)/\partial\alpha\} \\
&\quad - F_{31}(\beta)\{\partial F_{13}(\alpha)/\partial\alpha\} \\
&= \underset{(-)}{\beta r_M p}\underbrace{[I_Y - \{1 - c(1-\tau)\} + m]}_{(-)} \\
&\quad + pm + [I_Y - \{1 - c(1-\tau) + m\} + I_r r_Y]. \qquad (34)
\end{aligned}$$

[‖] See Okuguchi (1977), p. 238.

From Eq. (34), we have $\lim_{\beta\to+\infty} \partial h_2/\partial\alpha = +\infty$ so that we have $h_2 > 1$ for sufficiently large $\alpha > 0$ and $\beta > 0$. This proves that Proposition 1 also applies to the system (S_2).

5. NUMERICAL EXPERIMENTATION OF 'MODEL 1'

Analytical approach by means of linear approximation of the system without stochastic disturbance which was developed in the previous sections gives us relatively little information on the behavior of the original nonlinear system with stochastic disturbance. Numerical approach will provide us some useful insight, which cannot be obtained if we stick to the analytical approach. In this section, we shall summarize the results of our numerical experimentation of 'model 1'.

We specify the functional forms of the relevant functions and the parameter values as follows:

$$I(Y_t, K_t, r_t) = f(Y_t) - 0.3K_t - r_t + 147; \quad (35)$$

$$f(Y_t) = (80/\pi)\,\mathrm{Arc\,tan}\{(2.25\pi/20) \times (Y_t - 165/0.66)\} + 35; \quad (36)$$

$$r_t = r(Y_t, M_t) = 10\sqrt{Y_t} - M_t; \quad (37)$$

$$J(Y_t, \bar{E}) = -0.3Y_t + 50; \quad (38)$$

$$c = 0.8, \quad \tau = 0.2, \quad p = 1, \quad r_f = 6; \quad (39)$$

$$cT_0 + C_0 + G = 115. \quad (40)$$

The function $f(Y_t)$ in Eq. (36) represents the Kaldorian S-shaped investment function (see Fig. 1). Equation (37) is the LM equation which describes the equilibrium condition in the money market, and Eq. (38) is the current account function.# Substituting Eqs. (35)–(40) into the

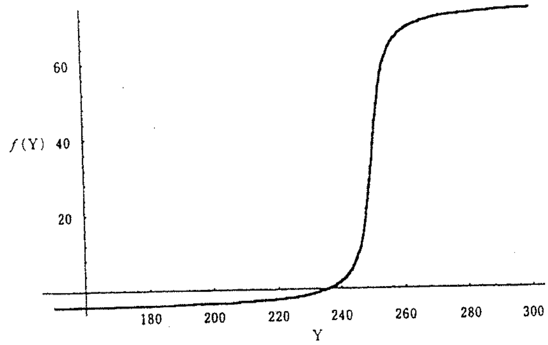

FIGURE 1

system (S_1) in Section 2, we obtain the following expression:

$$\text{(i)} \quad Y_{t+1} - Y_t = \alpha\{-0.66Y_t + f(Y_t) - 0.3K_t + 147 - 10\sqrt{Y_t} + M_t + 165,$$

$$\text{(ii)} \quad K_{t+1} - K_t = f(Y_t) - 0.3K_t - 10\sqrt{Y_t} + M_t + 147,$$

$$\text{(iii)} \quad M_{t+1} - M_t = -0.3Y_t + 50 + (\beta + \sigma\gamma_t) \times (10\sqrt{Y_t} - M_t - 6). \quad (41)$$

In this system, it is assumed that the 'degree of capital mobility' (β) is fluctuated by noise, and we select the parameters α and β as the bifurcation parameters.**

If we assume that $\beta = 1$ and $\sigma = 0$, the equilibrium solution of the system (41) becomes $(Y^*, K^*, M^*) \simeq (250, 503, 127)$. The equilibrium national income Y^* is independent of the values of α and β. On the other hand, K^* and M^* depend on the values of α and β. Our numerical simulation shows that the behavior of this model can be very complex even if the noise does not exist ($\sigma = 0$), and the hidden structure of the system may be sometimes revealed rather than obscured when the system is fluctuated by noise.

Note that under the system of fixed exchange rates E is fixed, so that we need not explicitly introduce E as a variable into the model.
** In other words, in this model the noise effect is modeled by means of the 'parametric noise' rather than usual additive noise.

5.1. Dynamics of National Income

In this subsection, we shall consider the dynamics of national income (Y). First, let us consider the case without noise ($\sigma = 0$). Figures 2 and 3 are the bifurcation diagrams of national income with respect to the parameters α and β respectively.¶¶ Figure 2 shows that the period of income fluctuation increases rapidly as the adjustment speed in the goods market (α) increases, and eventually the chaotic behavior emerges. However, as the parameter α increases further more, the 'window' which represents the periodical behavior emerges, and then the chaotic region reappears. We can confirm this statement by observing the largest Lyapunov exponent (see Fig. 4). Figures 5 and 6 are the bifurcation diagram and the largest Lyapunov exponent with small stochastic disturbance ($\sigma = 0.01$). We can see from these figures that the window of periodical solution disappears and

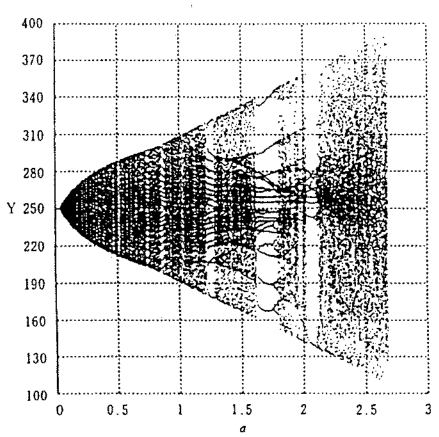

FIGURE 2 Bifurcation diagram of Y without noise (parameter: α).

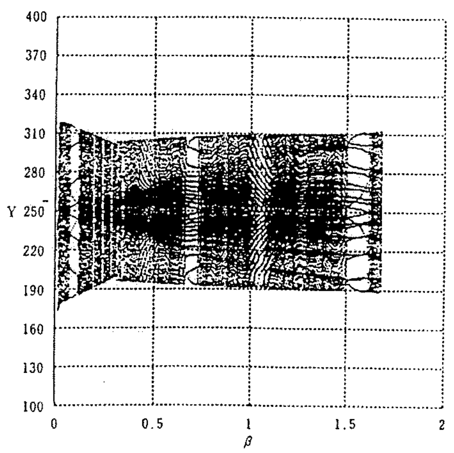

FIGURE 3 Bifurcation diagram of Y without noise (parameter: β).

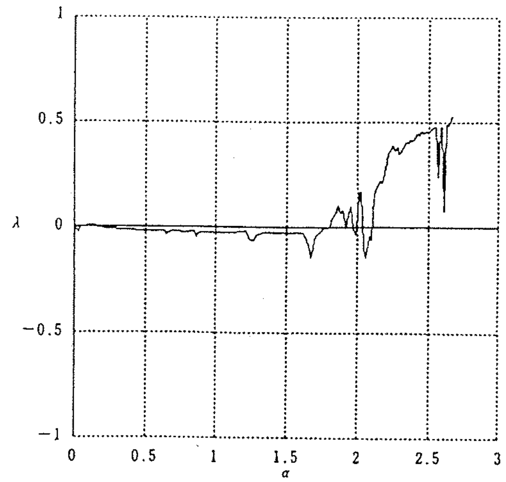

FIGURE 4 The largest Lyapunov exponent (λ) without noise.

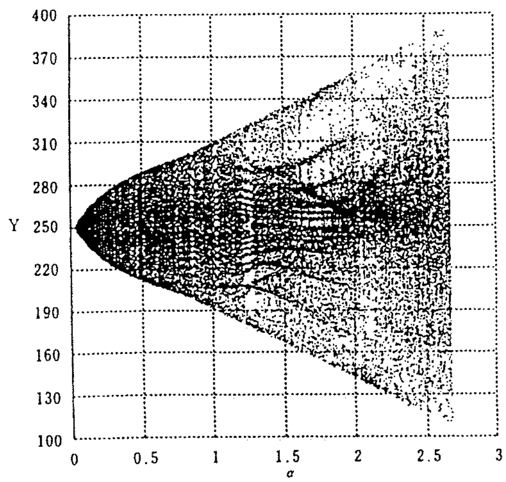

FIGURE 5 Bifurcation diagram of Y with noise ($\sigma = 0.01$).

¶¶ It is assumed that $\beta = 1$ when α is selected as a bifurcation parameter, and $\alpha = 1$ is assumed when β is selected as a bifurcation parameter.

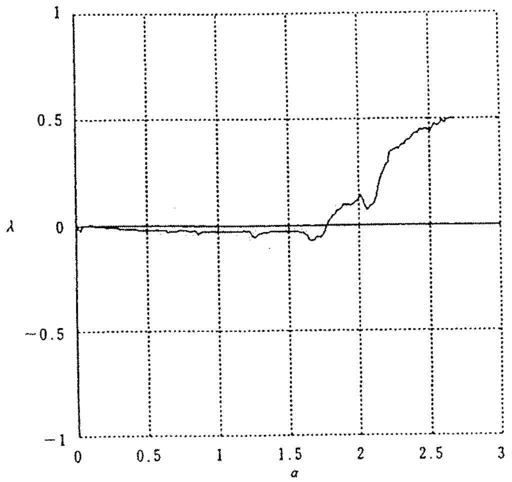

FIGURE 6 The largest Lyapunov exponent with noise ($\sigma = 0.01$).

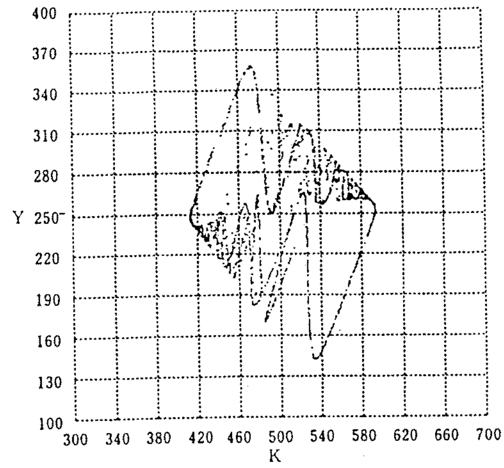

FIGURE 7 Attractor in Y–K plane without noise when $\alpha = 2.0$.

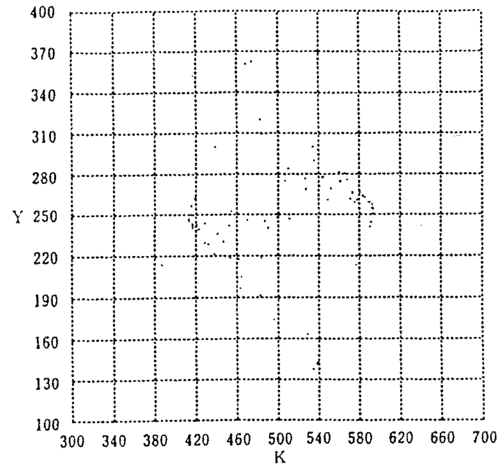

FIGURE 8 Attractor in Y–K plane without noise when $\alpha = 2.1$.

the behavior of the system becomes more chaotic because of the noise effects.

Now, let us compare the system with noise effects and that without noise effects. Figures 2 and 4 show that the behavior of the system without noise is chaotic when $\alpha = 2.0$, while the 'window' of the periodical solution appears when $\alpha = 2.1$. However, the behavior of this system becomes chaotic again when $\alpha = 2.2$. Figures 7–9 give the attractors of the system in K–Y plane in these three cases. Figure 10 is the attractor of the system with small noise effects ($\sigma = 0.01$) when $\alpha = 2.1$. The shape of the attractor in Fig. 10 is similar to that in Fig. 7 or 9. If there is no stochastic disturbance, the periodical trajectory is stable and chaotic trajectory is unstable at the 'window'. This implies that the chaotic structure is invisible and hidden at the 'window' if there is no stochastic disturbance. However, our numerical experimentation shows that the stochastic noise can make visible this hidden chaotic structure in some situations. Hence, it is *not* correct to say that the noise only obscures the basic structure of the system.

Figure 11 is the bifurcation diagram of Y with respect to the parameter β when β is subject to the small stochastic disturbance ($\sigma = 0.01$). Also in this case, some 'windows' of the periodical solution disappear because of the noise effects (compare Figs. 3 and 11).

5.2. Dynamics of Capital Stock

Dynamics of capital stock are given by Figs. 12–15. Figures 12 and 13 compare the bifurcation diagram of K with respect to α without noise and that with small noise. Figures 14 and 15 are bifurcation diagrams of K with respect to β.

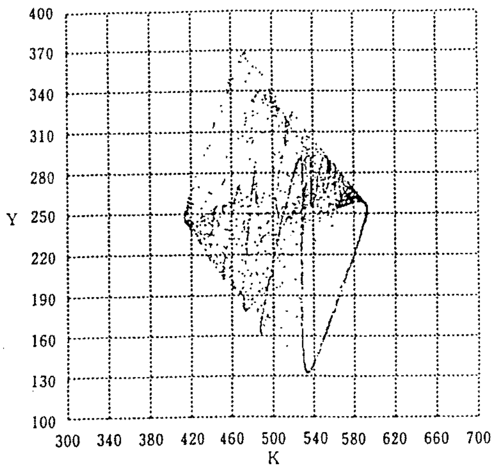

FIGURE 9 Attractor in Y–K plane without noise when $\alpha = 2.2$.

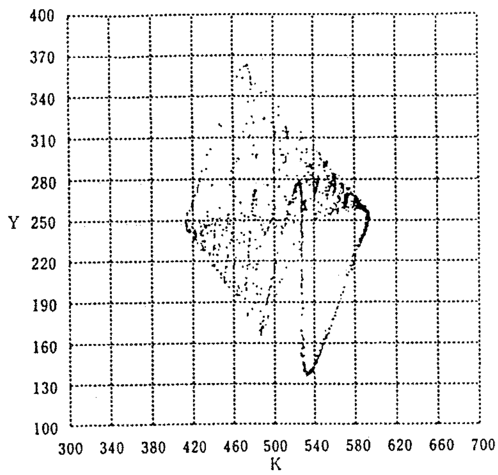

FIGURE 10 Attractor in Y–K plane with noise when $\alpha = 2.1$.

6. NUMERICAL EXPERIMENTATION OF 'MODEL 2'

We can construct a numerical example of 'model 2' by slightly modifying Eq. (41). In fact, we can obtain such a model by replacing $0.3\,K_t$ in Eq. (41)(i) with zero and keeping other two equations of (41)(ii) and (iii) intact. However, this slight modification changes the behavior of the system considerably.

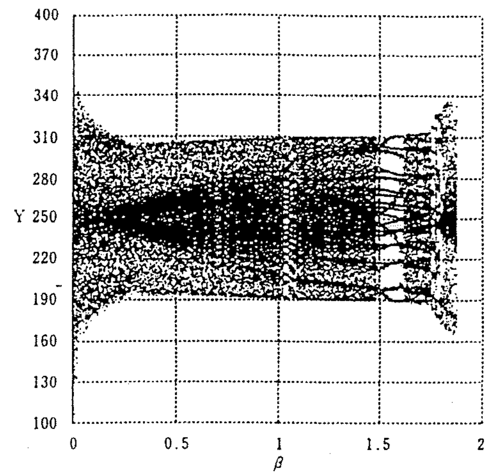

FIGURE 11 Bifurcation diagram of Y with noise ($\sigma = 0.01$).

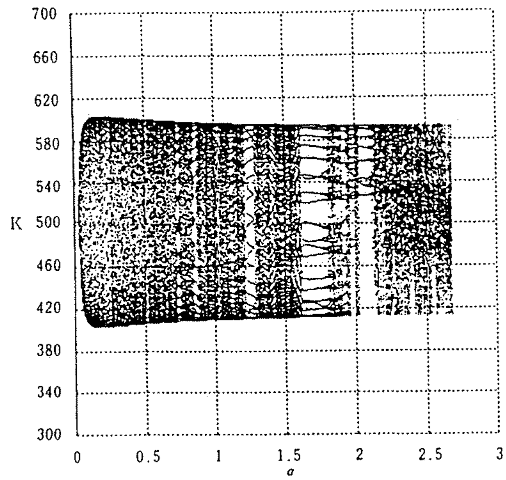

FIGURE 12 Bifurcation diagram of K without noise (parameter: α).

Figure 16 is the bifurcation diagram of Y with respect to the changes of the parameter α without noise effects, and Fig. 17 shows the largest Lyapunov exponent in this case. The equilibrium point is stable when α is small, but it becomes unstable and two period cycle becomes stable when α exceeds 2.25. Then, the period-doubling bifurcations occur rapidly, and the behavior of the system becomes chaotic.

It is worth to note that this system has two equilibrium points. In fact, we obtained Fig. 16 by

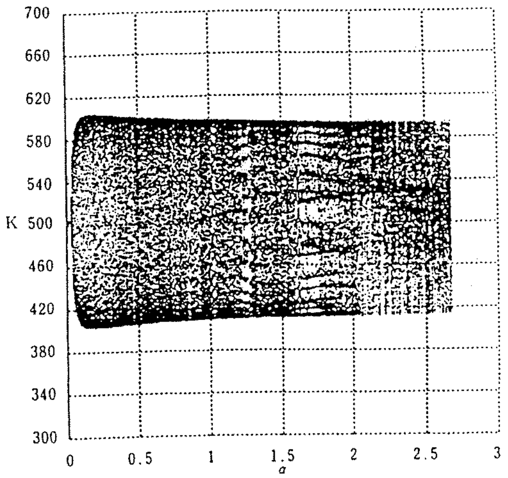

FIGURE 13 Bifurcation diagram of K with noise (parameter: α) ($\sigma = 0.01$).

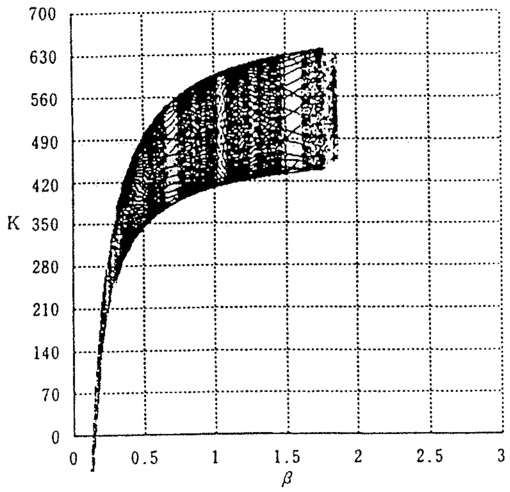

FIGURE 14 Bifurcation diagram of K without noise (parameter: β).

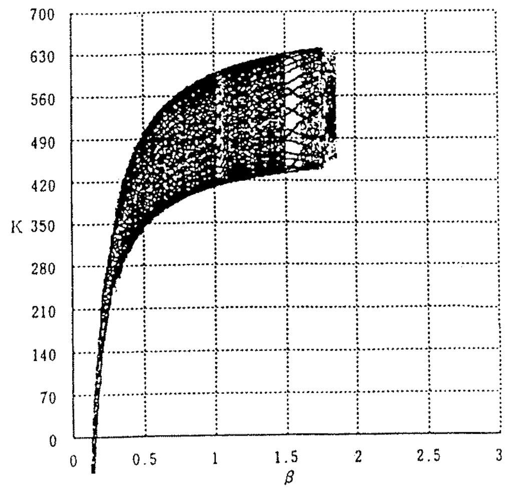

FIGURE 15 Bifurcation diagram of K with noise (parameter: β) ($\sigma = 0.01$).

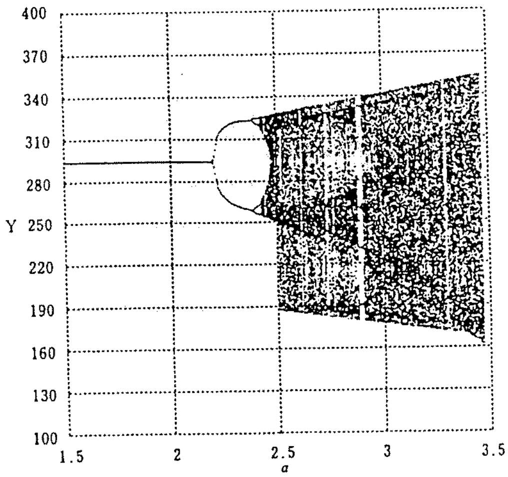

FIGURE 16 Bifurcation diagram of Y without noise (parameter: α).

adopting the initial condition (Y_0, K_0, M_0), which is near from the equilibrium point of 'model 1', i.e. $(Y^*, K^*, M^*) \simeq (250, 503, 127)$. If we adopt another initial condition, we can obtain another attractor and another bifurcation diagram. However, Fig. 16 shows that the fusion of two attractors occurs so that the fluctuating area of Y expands suddenly when α exceeds 2.5. Figure 17 shows that there are several 'windows' of periodic solutions in the area of $\alpha > 2.5$.

Figure 18 is the bifurcation diagram which is fluctuated by noise ($\sigma = 0.08$). This figure shows that the 'windows' of the periodic solutions disappear because of the noise effect. Furthermore, in this figure the fusion of the attractors occur even if $\alpha < 2.5$. We can interpret this phenomenon that the hidden structure of the system is revealed because of the noise effects. For convenience, let us say that the economy is in 'boom' when $Y_t > 250$ and it is in 'slump' when $Y_t < 250$. Comparing Figs. 16 and 18,

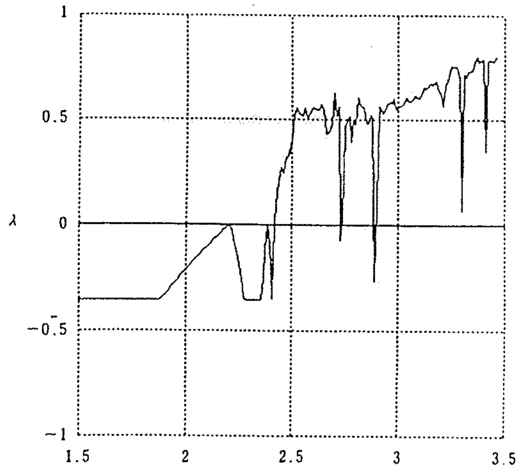

FIGURE 17 The largest Lyapunov exponent (λ) without noise.

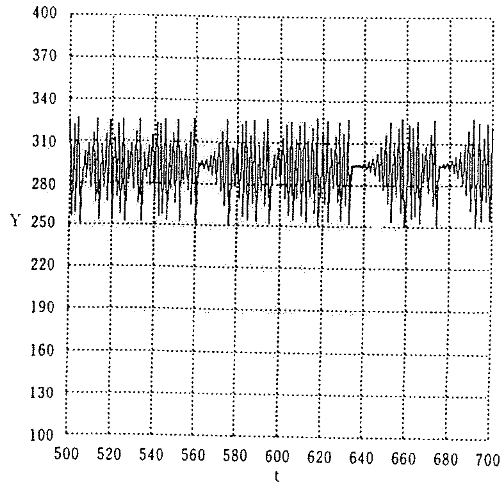

FIGURE 19 Trajectory of Y without noise when $\alpha = 2.5$.

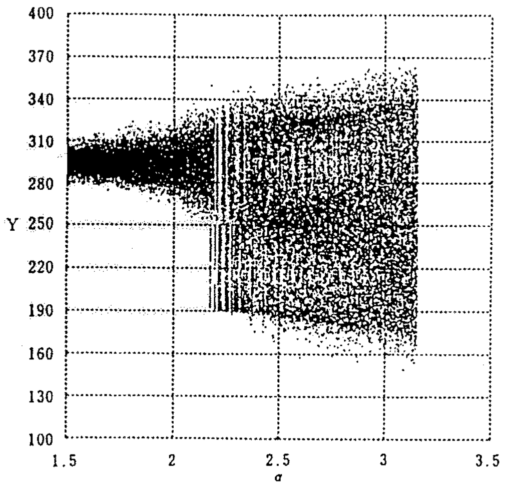

FIGURE 18 Bifurcation diagram of Y with noise ($\sigma = 0.08$).

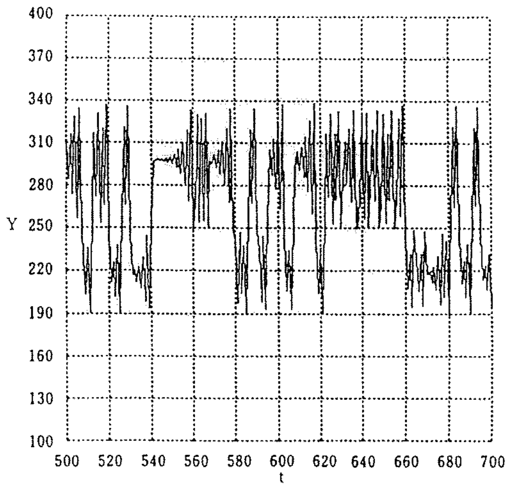

FIGURE 20 Trajectory of Y with noise when $\alpha = 2.5$ ($\sigma = 0.08$).

we can conclude that even if the economy is in boom, some stochastic disturbance can bring abut slump if the depressing structure is hidden in the system.

Figures 19 and 20 compare the trajectories of Y_t without noise and that with noise in the case of $\alpha = 2.5$. Figures 21 and 22 show the result of the similar experimentation in the case of $\alpha = 2.55$. These examples show that the noise may transform the lasting booms into the violent alternations of booms and slumps, but the opposite case is also possible. This is one of the important lessons of our numerical simulations.

7. COMPARISON WITH ASADA (1995)'s VERSION

Before closing this paper, let us make a comparison between the model in this paper and Asada (1995)'s original version with continuous time without noise. Asada (1995)'s system of equations, which

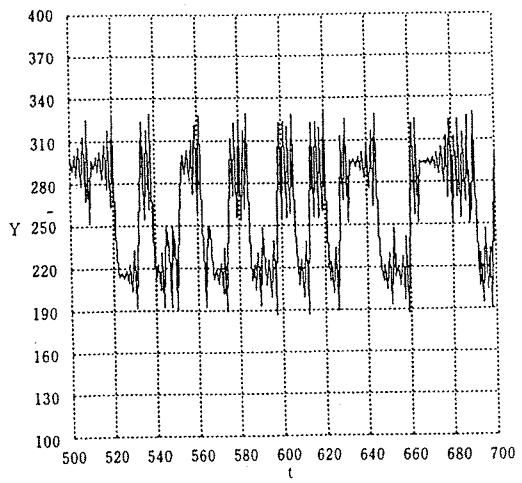

FIGURE 21 Trajectory of Y without noise when $\alpha = 2.55$.

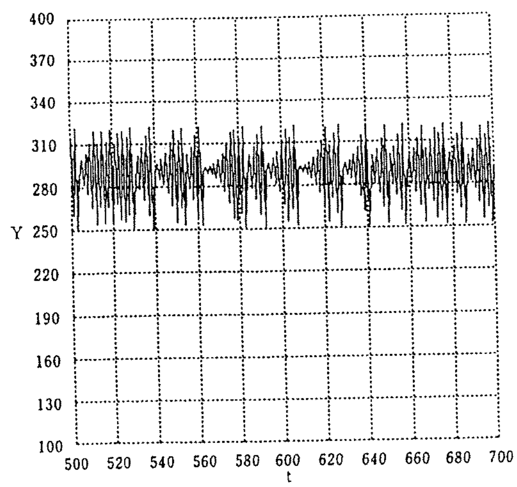

FIGURE 22 Trajectory of Y with noise when $\alpha = 2.55$ ($\sigma = 0.08$).

corresponds to Eq. (S_1) in this paper, is given as follows:

$$\text{(i)} \quad \mathrm{d}Y/\mathrm{d}t = \alpha[c(1-\tau)Y + cT_0 + C_0 + G + I(Y, K, r(Y, M)) + J(Y, \bar{E}) - Y] \equiv f_1(Y, K, M),$$

$$\text{(ii)} \quad \mathrm{d}K/\mathrm{d}t = I(Y, K, r(Y, M)) \equiv f_2(Y, K, M),$$

$$\text{(iii)} \quad \mathrm{d}M/\mathrm{d}t = pJ(Y, \bar{E}) + \beta p\{r(Y, M) - r_f\} \equiv f_3(Y, M; \beta). \qquad (S_1')$$

Asada (1995) derived the following results analytically under some assumptions:

(1) The equilibrium point of the system S_1' is locally stable if $\beta > 0$ is sufficiently small, and it becomes a saddle point if β is sufficiently large.
(2) There exists the parameter value $\beta_0 > 0$ at which the Hopf bifurcation occurs. In other words, there exist some nonconstant periodic solutions at some values of β which is sufficiently close to β_0.

Asada (1995) also presented some numerical simulations which support the above analytical results. However, Asada (1995)'s original version could not produce chaotic motion, but it produced rather 'regular' movement. Compared to Asada (1995)'s version, the discrete time version with and without noise which is presented in this paper can produce much complex and richer behavior, and it provides us a foundation to further research.

8. CONCLUDING REMARKS

In this paper, we investigated the discrete time version of the Kaldorian business cycle model in an open economy with and without noise effects by means of analytical method and numerical simulations. As a result, we could find some interesting behaviors of the system including chaotic movement. However, the model which was presented in this paper is restricted to the system of fixed exchange rates. While in the system of fixed exchange rates the money supply becomes an endogenous variable, in the system of flexible exchange rates we can consider the money supply as the exogenous variable which is controlled by the central bank.[##] Obviously, the next step must be the analysis of the system of flexible exchange rates. This is the theme which we shall study in another paper.

[##] See Asada (1995).

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Chaotic Dynamics in a Flexible Exchange Rate System: A Study of Noise Effects

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In this paper, we investigate by means of analytical method and numerical simulations the properties of three-dimensional business cycle model, in which foreign exchange rate is flexible and a parameter is fluctuated by noise. The model is a discrete time version of Asada (*Journal of Economics*, 62, 239–269, 1995)'s continuous time open economy model without noise. We show (1) noise may suppress the burst of flexible foreign exchange rate when its behavior begins to burst as a bifurcation parameter (adjustment speed of the goods market) is increased, (2) the windows of cycles can be broken by noise, and (3) noise may reveal the hidden structures.

Keywords: Chaotic dynamics, Flexible exchange rates, Noise effects, Small open economy

1. INTRODUCTION

In general, an economy is not an isolated system but it is subject to the disturbances from other subsystems of the society. This observation is particularly important for theoretical and empirical investigations in international economics or regional sciences, which study the economic interactions between several regions. Asada (1995) presented a dynamic model of small open economy by introducing international trade and international capital movement into the Kaldorian business cycle theory. Asada (1995) investigated both of the fixed exchange rate system and the flexible exchange rate system in a framework of the continuous time model without stochastic disturbance (noise). Asada *et al.* (1998) studied a discrete time version of the fixed exchange rate system with noise effects by means of numerical simulations, and showed that such a system can produce very complex behavior including chaos.

This paper also considers a discrete time version of the Kaldorian business cycle model in a small open economy with noise effects, but in this study

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we concentrate on the system of flexible exchange rates. We investigate by means of analytical method and numerical simulations the properties of three-dimensional system, in which a parameter which represents the adjustment speed of the adaptive expectation of exchange rate is fluctuated by noise. Asada *et al.* (1998) showed in a framework of the fixed exchange rate system that the noise may not only obscure the underlying structures, but also reveal the hidden structures, for example, chaotic attractors near the window. In this paper, we show (1) noise may suppress the burst of flexible exchange rate when its behavior begins to burst as a bifurcation parameter (adjustment speed of the goods market) is increased, (2) the windows of cycle can be broken by noise, and (3) noise may reveal the hidden structures.

2. FORMULATION OF THE MODEL

The basic system of equations is given as follows*:

$$Y_{t+1} - Y_t = \alpha[C_t + I_t + G + J_t - Y_t]; \quad \alpha > 0, \tag{1}$$

$$K_{t+1} - K_t = I_t, \tag{2}$$

$$C_t = c(Y_t - T_t) + C_0; \quad 0 < c < 1, \ C_0 > 0, \tag{3}$$

$$I_t = I(Y_t, K_t, r_t); \quad I_Y \equiv \partial I_t/\partial Y_t > 0, \quad I_K \equiv \partial I_t/\partial K_t < 0, \quad I_r \equiv \partial I_t/\partial r_t < 0, \tag{4}$$

$$T_t = \tau Y_t - T_0; \quad 0 < \tau < 1, \ T_0 > 0, \tag{5}$$

$$M_t/p = L(Y_t, r_t); \quad L_Y \equiv \partial L_t/\partial Y_t > 0, \quad L_r \equiv \partial L_t/\partial r_t < 0, \tag{6}$$

$$J_t = J(Y_t, E_t); \quad J_Y \equiv \partial J_t/\partial Y_t < 0, \quad J_E \equiv \partial J_t/\partial E_t > 0, \tag{7}$$

$$Q_t = \beta\{r_t - r_f - (E_t^e - E_t)/E_t\}; \quad \beta > 0, \tag{8}$$

$$A_t = J_t + Q_t, \tag{9}$$

$$A_t = 0, \tag{10}$$

$$E_{t+1}^e - E_t^e = (\gamma + \sigma\varepsilon_t)(E_t - E_t^e); \quad \gamma > 0, \ \sigma \geqq 0, \tag{11}$$

$$M_t = \bar{M}, \tag{12}$$

where the meanings of the symbols are as follows. Y = net real national income, C = real consumption expenditure, I = net real private investment expenditure, G = real government expenditure (fixed), K = real physical capital stock, T = real income tax, M = nominal money supply, p = price level (fixed), r = nominal domestic rate of interest, r_f = nominal foreign rate of interest (fixed), E = value of a unit of foreign currency in terms of domestic currency (exchange rate), E^e = expected exchange rate of near future, J = balance of current account (net export) in real terms, Q = balance of capital account in real terms, A = total balance of payments in real terms, α = adjustment speed in the goods market, β = parameter which represents the 'degree of capital mobility', γ = parameter which represents the "speed of adaptation" of the expected exchange rate, ε = normal pseudo random number $N(0, 1)$, σ = standard deviation parameter. The subscript t denotes time period.

Equation (1) represents the quantity adjustment process in the goods market. Equation (2) says that the physical capital stock increases or decreases according as the net investment is positive or negative. Equations (3), (4), and (5) are consumption function, investment function, and income tax function respectively. Equation (6) is the equilibrium condition for the money market. Equation (7) says that the current account is determined by Y_t and E_t, which is a standard type of the current account function. Equation (8) formalizes the idea that the capital account becomes positive or negative according as the difference between the rates of return of domestic and foreign bonds is positive or negative. We can consider β as the index of the degree of the capital mobility. Equation (9) is

* Equations (1)–(7) in this paper are identical to those in a fixed exchange rate system which was presented in Asada *et al.* (1998).

the definition of the total balance of payments. Equation (10) is a characterization of the flexible exchange rate system, i.e., it is assumed that the exchange rate is adjusted instantaneously to keep the equilibrium of the total balance of payments ($A_t = 0$). Equation (11) is a formalization of the 'adaptive expectation hypothesis' concerning the expected exchange rate. It is assumed that the speed of adaptation is fluctuated by noise. Equation (12) says that under flexible exchange rate system the domestic monetary authority can control money supply contrary to the case of fixed exchange rate system, so that we can consider the money supply (M) as an exogenous variable.[†]

We can reduce the above system (1)–(12) to the following system of equations[‡]:

$$\begin{aligned}
&\text{(i)}\ Y_{t+1} - Y_t = \alpha[(1-\tau)Y_t + cT_0 + C_0 \\
&\qquad + G + I(Y_t, K_t, r(Y_t, \bar{M})) \\
&\qquad + J(Y_t, E_t) - Y_t]; \quad \alpha > 0, \\
&\text{(ii)}\ K_{t+1} - K_t = I(Y_t, K_t, r(Y_t, \bar{M})), \\
&\text{(iii)}\ A_t = J(Y_t, E_t) + \beta\{r(Y_t, \bar{M}) \\
&\qquad - r_f - E_t^e/E_t + 1\} = 0, \\
&\text{(iv)}\ E_{t+1}^e - E_t^e = (\gamma + \sigma\varepsilon_t)(E_t - E_t^e).
\end{aligned} \tag{13}$$

Solving Eq. (13)(iii) with respect to E_t, we obtain

$$\begin{aligned}
&E_t = E(Y_t, E_t^e; \beta), \\
&E_Y \equiv \partial E_t/\partial Y_t = (-J_Y - \beta r_Y)/(J_E + \beta E_t^e/E_t^2) \\
&\quad = (m - \beta r_Y)/(J_E + \beta E_t^e/E_t^2) \\
&\quad \gtreqless 0 \iff \beta \lesseqgtr m/r_Y, \\
&E_E^e = \partial E_t/\partial E_t^e = \beta/(J_E E_t + \beta E_t^e/E_t) > 0,
\end{aligned} \tag{14}$$

where $m \equiv -J_Y \equiv -\partial J_t/\partial Y_t > 0$, $r_Y \equiv \partial r_t/\partial Y_t > 0$, and $J_E \equiv \partial J_t/\partial E_t > 0$.[¶] Equation (14) implies that E_t is an increasing function of Y_t when the degree of capital mobility (β) is sufficiently small, but it becomes a decreasing function of Y_t when β is sufficiently large.[§]

Substituting Eq. (14) into (13), we obtain the following system of 'fundamental dynamical equations':

$$\begin{aligned}
&\text{(i)}\ Y_{t+1} = Y_t + \alpha[(1-\tau)Y_t + cT_0 \\
&\qquad + C_0 + G + I(Y_t, K_t, r(Y_t, \bar{M})) \\
&\qquad + J(Y_t, E(Y_t, E_t^e; \beta)) - Y_t] \\
&\qquad \equiv F_1(Y_t, K_t, E_t^e; \alpha, \beta), \\
&\text{(ii)}\ K_{t+1} = K_t + I(Y_t, K_t, r(Y_t, \bar{M})) \equiv F_2(Y_t, K_t), \\
&\text{(iii)}\ E_{t+1}^e = E_t^e + (\gamma + \sigma\varepsilon_t)\{E(Y_t, E_t^e; \beta) - E_t^e\} \\
&\qquad \equiv F_3(Y_t, E_t^e; \beta, \gamma, \sigma),
\end{aligned} \tag{S$_1$}$$

On the other hand, the continuous time version without noise effect which was formulated in Asada (1995) is read as

$$\begin{aligned}
&\text{(i)}\ \mathrm{d}Y/\mathrm{d}t = \alpha[(1-\tau)Y + cT_0 + C_0 \\
&\qquad + G + I(Y, K, r(Y, \bar{M})) \\
&\qquad + J(Y, E(Y, E^e; \beta)) - Y] \\
&\qquad \equiv f_1(Y, K, E^e; \alpha, \beta), \\
&\text{(ii)}\ \mathrm{d}K/\mathrm{d}t = I(Y, K, r(Y, \bar{M})) \equiv f_2(Y, K), \\
&\text{(iii)}\ \mathrm{d}E^e/\mathrm{d}t = \gamma\{E(Y, E^e; \beta) - E^e\} \\
&\qquad \equiv f_3(Y, E^e; \beta, \gamma).
\end{aligned} \tag{S$_2$}$$

It is easily shown that the equilibrium point (Y^*, K^*, E^{e*}) of the system (S$_1$) is identical to that of the system (S$_2$), and Asada (1995) showed that there exists an equilibrium point $(Y^*, K^*, E^{e*}) > (0, 0, 0)$ in the system (S$_2$) under some reasonable conditions. From now on, we *assume* that there exists an economically meaningful equilibrium point in the system (S$_1$).

[†] Under the fixed exchange rate system, money supply endogenously fluctuates according as the total balance of payments is positive or negative. See Asada (1995) and Asada *et al.* (1998).

[‡] In Eq. (13), $r(Y_t, \bar{M})$ is the 'LM equation' which is the solution of Eq. (6) with respect to r_t.

[¶] Solving Eq. (6) with respect to r_t, we have $r_t = r(Y_t, \bar{M})$; $r_Y \equiv \partial r_t/\partial Y_t = -L_Y/L_r > 0$, where $L_Y \equiv \partial L_t/\partial Y_t > 0$ and $L_t \equiv \partial L_t/\partial r_r < 0$.

[§] The economic implication of this result is very clear. When Y_t increases, the current account (J_t) decreases through the increase of the import, while the capital account (Q_t) increases through the increase of the domestic rate of interest. If β is small, the 'current account effect' dominates the 'capital account effect' so that the total balance of payments (A_t) *decreases*. In this case, the exchange rate (E_t) must *increase* to keep the equilibrium of the balance of payments ($A_t = 0$). On the other hand, if β is large, the 'capital account effect' dominates so that A_t *increases*. In this case, E_t must *decrease* to keep $A_t = 0$.

3. LOCAL STABILITY – INSTABILITY ANALYSIS

Asada (1995) proved the following propositions.

(1) The equilibrium point of the system (S_2) is locally stable if β is sufficiently large.
(2) Suppose that $I_Y + I_r r_Y > 1 - c(1-\tau)$ *at the equilibrium point*. Then, the equilibrium point of the system (S_2) becomes locally unstable when β is sufficiently small and α is sufficiently large.

This proposition implies that in a continuous time version of our model, large capital mobility between countries (or regions) tends to *stabilize* the system. Does this conclusion also apply to the discrete time version? In fact, proposition (1) does not apply to the system (S_1), because in the discrete time version the 'overshooting phenomena' are not negligible so that the system becomes unstable when the degree of capital mobility is too large. Now, let us prove this assertion formally by assuming $\sigma = 0$ (no stochastic disturbance).

We can write the Jacobian matrix of the system (S_1) which is evaluated *at the equilibrium point* as follows:

$$J_1 = \begin{bmatrix} F_{11}(\alpha,\beta) & F_{12}(\alpha) & F_{13}(\alpha,\beta) \\ F_{21} & F_{22} & 0 \\ F_{31}(\beta,\gamma) & 0 & F_{33}(\beta,\gamma) \end{bmatrix}, \qquad (15)$$

where

$$F_{11}(\alpha,\beta) = 1 + \alpha[\underset{(+)}{I_Y} + \underset{(-)}{I_r}\,\underset{(+)}{r_Y} - \{1 - c(1-\tau) + \underset{(+)}{m}\} + \underset{(+)}{J_E}\,\underset{(?)}{E_Y(\beta)}],$$

$$F_{12}(\alpha) = \alpha I_K < 0, \quad F_{13}(\alpha,\beta) = \alpha\,\underset{(+)}{J_E}\,\underset{(+)}{E_E(\beta)} > 0,$$

$$F_{21} = \underset{(+)}{I_Y} + \underset{(-)}{I_r}\,\underset{(+)}{r_Y}, \quad F_{22} = 1 + \underset{(-)}{I_K},$$

$$F_{31}(\beta,\gamma) = \gamma\,\underset{(?)}{E_Y(\beta)},$$

$$F_{33}(\beta,\gamma) = 1 + \gamma\{E_E(\beta) - 1\} = 1 - (\underset{(+)}{J_E}\,E\gamma)/(\underset{(+)}{J_E}\,E + \beta).$$

We can write the characteristic equation of this system as

$$\psi(\lambda) = |\lambda I - J_1| = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \qquad (16)$$

where

$$\begin{aligned}
\text{(i)}\ a_1 &= -\text{trace}\, J_1 \\
&= -F_{11}(\alpha,\beta) - F_{22} - F_{33}(\beta,\gamma), \\
\text{(ii)}\ a_2 &= \begin{vmatrix} F_{22} & 0 \\ 0 & F_{33} \end{vmatrix} + \begin{vmatrix} F_{11} & F_{13} \\ F_{31} & F_{33} \end{vmatrix} + \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} \\
&= F_{22}F_{33}(\beta,\gamma) + F_{11}(\alpha,\beta)F_{33}(\beta,\gamma) \\
&\quad - F_{13}(\alpha,\beta)F_{31}(\beta,\gamma) + F_{11}(\alpha,\beta)F_{22} \\
&\quad - F_{12}(\alpha,\beta)F_{21}, \\
\text{(iii)}\ a_3 &= -\det J_1 \\
&= -F_{11}(\alpha,\beta)F_{22}F_{33}(\beta,\gamma) \\
&\quad + F_{13}(\alpha,\beta)F_{22}F_{31}(\beta,\gamma) \\
&\quad + F_{12}(\alpha)F_{21}F_{33}(\beta,\gamma).
\end{aligned} \qquad (17)$$

It follows from the Cohn–Schur conditions for local stability that the system (S_1) is locally stable *if and only if* the following conditions are satisfied[‖]

$$\begin{aligned}
&\text{(i)}\ 1 + a_2 - |a_1 + a_3| > 0, \\
&\text{(ii)}\ 1 - a_2 + a_1a_3 - a_3^2 > 0, \\
&\text{(iii)}\ a_2 < 3.
\end{aligned} \qquad (18)$$

Therefore, the equilibrium point of the system (S_1) becomes locally unstable if the inequality $a_2 > 3$ is satisfied. The following proposition is a simple corollary of this fact.

PROPOSITION *Suppose that*

$$0 < \underset{(+)}{I_Y} + \underset{(-)}{I_r}\,\underset{(+)}{r_Y} < \underbrace{1 - c(1-\tau)}_{(+)} + \underset{(+)}{m}$$

‖ See, for example, Gandolfo (1996) Chap. 7. In fact, the inequality (18)(iii) is redundant because this inequality can be derived from other two inequalities. Nevertheless, the inequality (18)(iii) as a *necessary* condition for local stability is useful for our purpose.

and

$$I_K < -1.$$

Then, the equilibrium point of the system (S_1) *is locally unstable when either of the following conditions* (A) *or* (B) *is satisfied.*

(A) *The parameters* β *and* α *are sufficiently large.*
(B) *The parameters* β *and* γ *are sufficiently large.*

Proof From Eq. (14), the definitions of $F_{31}(\beta,\gamma)$ and $F_{33}(\beta,\gamma)$, the fact that $E^e = E$ at the equilibrium point we have the following relationships:

$$\lim_{\beta\to\infty} J_E E_Y = \lim_{\beta\to\infty} (m - \beta r_Y)/(1 + \beta/J_E E) = -r_Y J_E E < 0, \tag{19}$$

$$\lim_{\beta\to\infty} F_{31}(\beta,\gamma) = \lim_{\beta\to\infty} \gamma(m - \beta r_Y)/(J_E + \beta) = -\gamma r_Y < 0, \tag{20}$$

$$\lim_{\beta\to\infty} F_{33}(\beta,\gamma) = \lim_{\beta\to\infty} [1 + \gamma\{\beta/(J_E E + \beta) - 1\}] = 1. \tag{21}$$

Therefore, we obtain the following expression

$$A \equiv \lim_{\beta\to\infty} a_2 = \underset{(-)}{F_{22}} + \underset{(-)}{F^*_{11}(\alpha)}\underbrace{(1 + F_{22})}_{(-)} - \underset{(+)}{F^*_{13}(\alpha)}\,\underset{(-)}{F^*_{31}(\gamma)} - F^*_{12}(-)(\alpha)\,\underset{(+)}{F_{21}}, \tag{22}$$

where $F^*_{ij} = \lim_{\beta\to\infty} F_{ij}$, $\partial F^*_{11}(\alpha)/\partial\alpha < 0$, $\partial F^*_{12}(\alpha)/\partial\alpha < 0$, $\partial F^*_{13}(\alpha)/\partial\alpha > 0$, and $\partial F^*_{31}(\gamma)/\partial\gamma < 0$. It follows from Eq. (22) that we have $A > 3$ for sufficiently large β and α, or alternatively, for sufficiently large β and γ. This proves the proposition.

This proposition shows that under some additional conditions the high degree of capital mobility (β) combined with high adjustment speed in the goods market (α) *or* high speed of adaptation of the expected exchange rate (γ) tends to *destabilize* the system (S_1) because of the overshooting phenomena contrary to the continuous time model.

4. NUMERICAL SIMULATIONS

In this section, we shall present the results of some numerical simulations of the model which was formulated in the previous section. We adopt the following specifications of the functions and parameter values:

$$I(Y_t, K_t, r_t) = f(Y_t) - 0.3K_t - r_t + 100, \tag{23}$$

$$f(Y_t) = (80/3.1415)\mathrm{Arc\,tan}\{(2.25 \times 3.1415/20) \times (Y_t - 112)\} + 35, \tag{24}$$

$$r_t = r(Y_t, M) = 10\sqrt{Y_t} - M, \tag{25}$$

$$J(Y_t, E_t) = -0.3Y_t + 100 - 100/E_t, \tag{26}$$

$$E^e_{t+1} - E^e_t = (\gamma + \sigma\varepsilon_t)(E_t - E^e_t), \tag{27}$$

$$c = 0.8,\quad \tau = 0.2,\quad r_f = 6,\quad cT_0 + C_0 + G = 238,\quad M = 100,\quad \beta = 15,\quad \gamma = 1.2. \tag{28}$$

Equation (24) is a formalization of the Kaldorian S-shaped investment function. Substituting Eqs. (25) and (26) into the equilibrium condition of the balance of payments (Eq. (13)(iii)), we have

$$-0.3Y_t + 100 - 100/E_t + 15(10\sqrt{Y_t} - E^e_t/E_t - 105) = 0. \tag{29}$$

Solving Eq. (29) with respect to E_t, we obtain the following expression for the exchange rate

$$E_t = (100 + 15E^e_t)/(-0.3Y_t + 150\sqrt{Y_t} - 1475). \tag{30}$$

By using these data, we studied the numerical simulation. We selected the parameter α as a bifurcation parameter. When there is no stochastic disturbance ($\sigma = 0$), the equilibrium point is $(Y^*, K^*, E^{e*}) = (Y^*, K^*, E^*) \simeq (112, 450, 1.56)$, which is independent of the value of the parameter α.

In order to investigate the noise effect, the speed of adjustment in adaptive expectation hypothesis of exchange rate (γ) is fluctuated by noise. In general, there are two types of noise, i.e., additive noise and parametric noise. In the case of additive noise, noise is added to a certain deterministic equation, and in the case of parametric noise, a certain parameter of a deterministic equation is fluctuated by noise. Crutchfield *et al.* (1982) studied the effect of noise for logistic equation and showed that the effect of parametric noise is equivalent to that of additive noise. However, it is not always true for high dimensional system. In this study, we treat with the case of parametric noise because it is likely for the foreign exchange rate to be fluctuated by noise. For the details of numerical simulation of the Kaldorian business cycle model with parametric noise in a closed economy, see Dohtani *et al.* (1996). The following statement summarizes the results of our numerical simulation:

(1) The behavior of flexible foreign exchange rate (E) begins to burst as the adjustment speed in the goods market (α) is increased in the system without noise.
(2) Noise may suppress the burst of exchange rate, in other words, noise may reveal the hidden structures.
(3) The windows of periodic solution can be broken by noise and the hidden chaotic structure may appear.

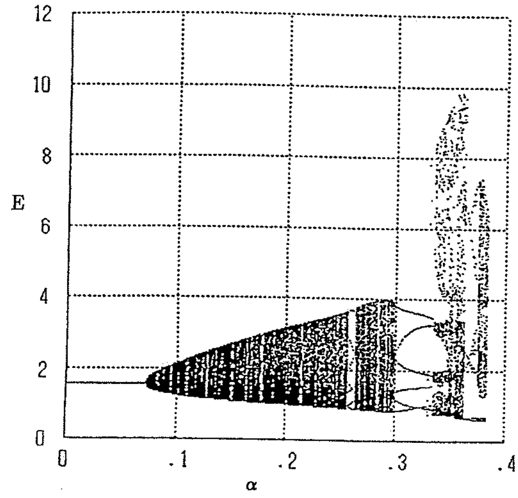

FIGURE 1 The bifurcation diagram of E without noise.

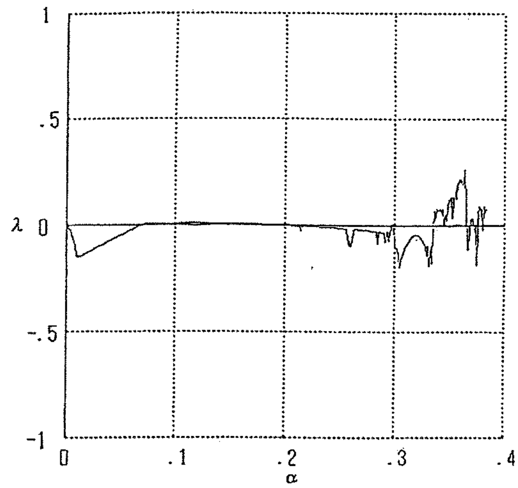

FIGURE 2 The largest Lyapunov exponent without noise.

4.1. The Appearing and the Suppressing of Burst

Figure 1 is the bifurcation diagram of the foreign exchange rate (E) without noise. The bifurcation parameter is the adjustment speed in the goods market (α). Figure 2 shows the largest Lyapunov exponent in this system. These figures indicate that the behavior of the exchange rate begins to burst and chaotic behaviors appear frequently when $\alpha \geq 0.33$. Figure 3 is the chaotic attractor in the E–Y plane without noise when $\alpha = 0.33$.

However, noise can suppress the burst of the exchange rate. Figure 4 is the attractor which is revealed by the small noise ($\sigma = 0.01$) when $\alpha = 0.33$. This attractor is similar to that in the case of $\alpha = 0.32$, which is cyclical and not burst. From the economic point of view, this means that even if the foreign exchange rate fluctuates heavily when there is no stochastic disturbance, this heavy

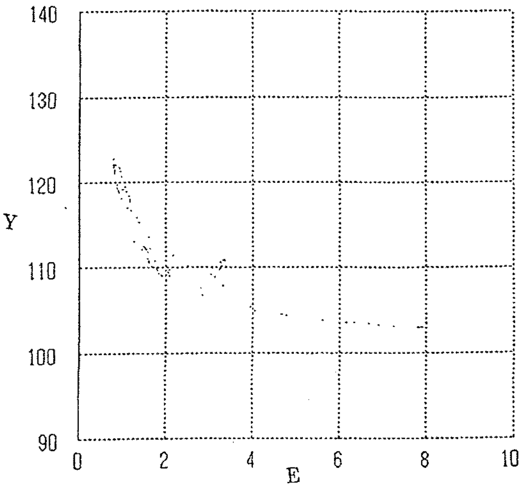

FIGURE 3 The attractor in $E-Y$ plane without noise ($\alpha=0.33$).

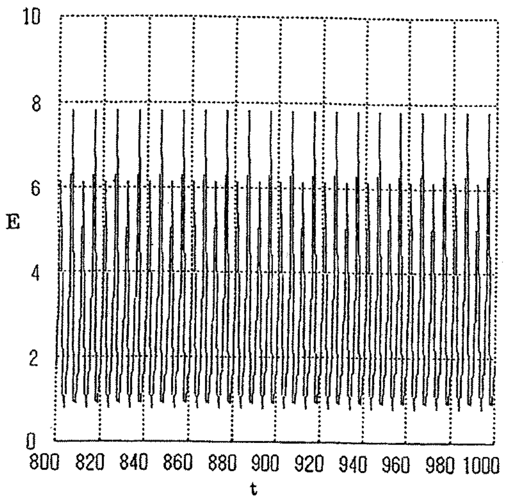

FIGURE 5 A trajectory of E without noise ($\alpha=0.33$).

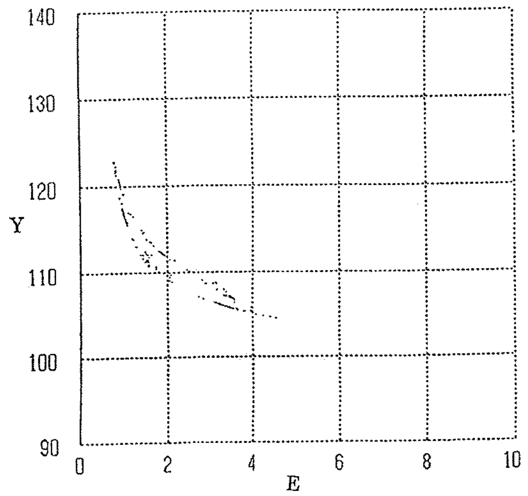

FIGURE 4 The attractor in $E-Y$ plane with noise ($\alpha=0.33$, $\sigma=0.01$).

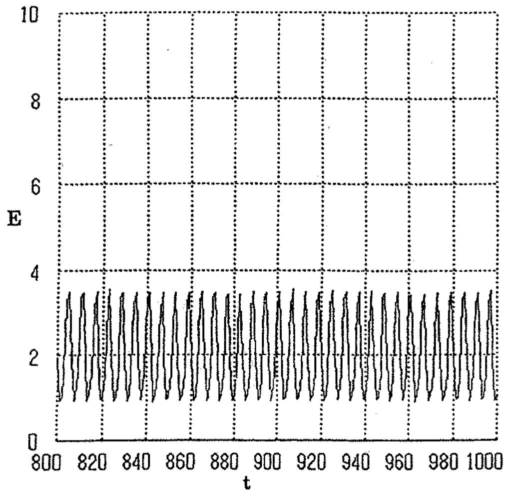

FIGURE 6 A trajectory of E with noise ($\alpha=0.33$, $\sigma=0.01$).

fluctuation may be suppressed by a small noise. Figures 5 and 6 compare the trajectory of the exchange rate when $\alpha=0.33$ without noise to that with small noise ($\sigma=0.01$).

The above example shows that the noise *may* stabilize the system. But, this does *not* mean that the noise *always* stabilize the system. Figures 7 and 8 are two examples of the experiments with somewhat larger noise ($\sigma=0.05$). These examples show that the noise can destabilize rather than stabilize the system in some situations. Figure 9 is a revealed attractor in the $E-Y$ plane when $\alpha=0.33$ and $\sigma=0.05$, which is chaotic and similar to that for $\alpha=0.34$ without noise. To sum up, noise can reveal the hidden structures.

4.2. The Broken Windows

By comparing the largest Lyapunov exponent and bifurcation diagram without noise (Figs. 1 and 2)

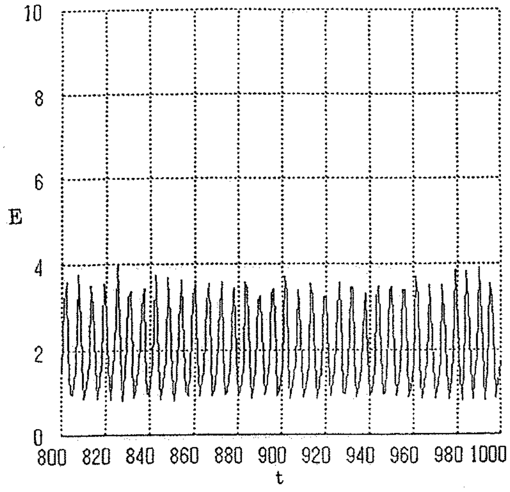

FIGURE 7 A trajectory of E with noise ($\alpha=0.33$, $\sigma=0.05$).

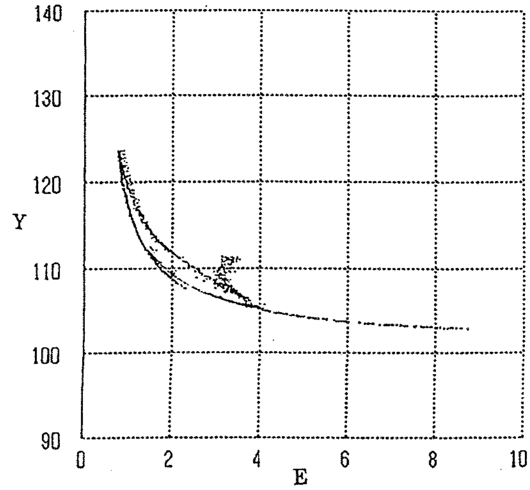

FIGURE 9 The attractor in E–Y plane with noise ($\alpha=0.33$, $\sigma=0.05$).

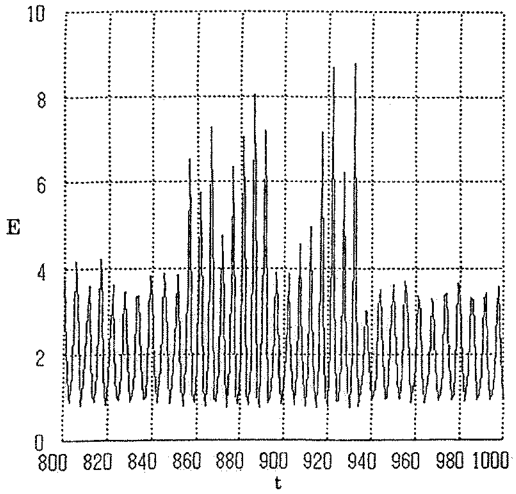

FIGURE 8 A trajectory of E with noise ($\alpha=0.33$, $\sigma=0.05$).

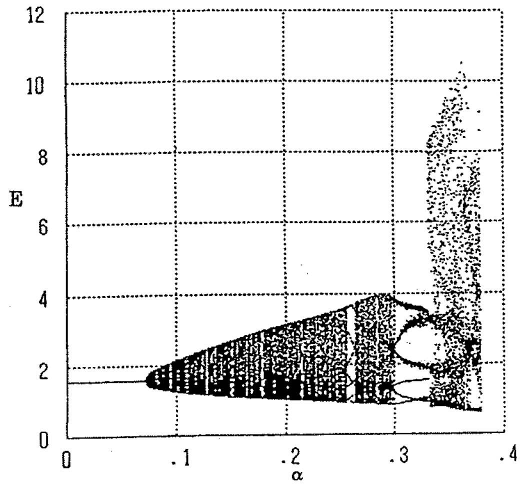

FIGURE 10 The bifurcation diagram of E with noise.

with those with noise (Figs. 10 and 11), we can see that noise may obscure the windows of cycle for the exchange rate (E). This is also true for the behavior of national income (Y). This example shows that noise can reveal the hidden chaotic structure even if the behavior of the system without noise is periodic.

5. CONCLUDING REMARKS

In this paper, we studied the economic implications of the noise effects by using an analytical framework of the discrete time version of Kaldorian business cycle model in a small open economy with flexible exchange rates. This is a good starting point to the study of the economic interactions between countries or regions. The effective range of the model of small open economy is, however, rather restricted, because many variables such as national income or rate of interest of foreign country are supposed to be given outside of the system. Multi-country or multi-regional model may be more appropriate for the study of the

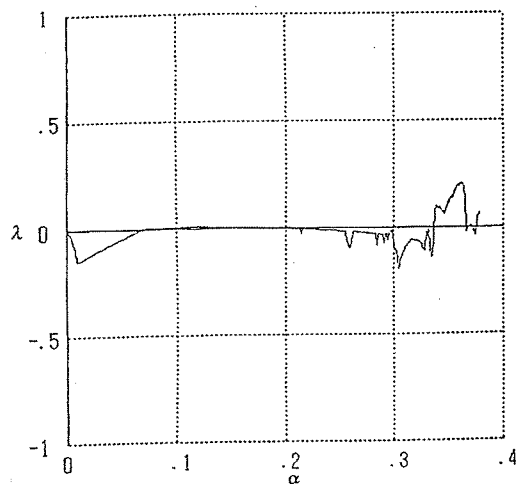

FIGURE 11 The largest Lyapunov exponent with noise.

dynamic interactions between regions. The simplest version of such a model is two country model. The analysis of such a complicated system is beyond the scope of the present paper and it is left for our research in future.

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Nonlinear Macroeconomic Dynamics : A Study of Noise Effects

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あらまし 小国開放経済モデルと二国経済モデルについて、モデルのパラメータが摂動を受けた場合、ノイズは経済に埋め込まれている基本的構造を不明瞭にする効果を持っているだけではなく、場合によっては逆に背後の隠されている構造を浮かび上がらせることもあるということを、数値シミュレーションによって示す。

キーワード ノイズ、景気循環、為替レート、景気変動の逆転現象

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Abstract In this paper we show by means of numerical experimentation that noise may not only obscure the underlying structures, but also reveal the hidden structures for a small open economy model and two-country model.

Key words noise, business cycle, foreign exchange rates, reversion of economic fluctuations

1.　はじめに

本稿では、3 次元および 5 次元の非線形差分方程式体系で記述されるマクロ動学モデルにノイズ摂動を与えた場合系の挙動がどのように変化するのか、そしてそれらの経済的な意味は何であるかを調べる。本稿で使用されるモデルはAsada[1]で定式化された開放経済におけるカルドア型景気循環モデルに基づいている。カルドア型景気循環モデルでは財市場での総供給と総需要とが毎期毎期(毎年毎年)均衡する必要がなく、不均衡タイプのモデルという特徴がある。他方、貨幣市場は毎期毎期需給がバランスするものと想定する。オリジナルモデルが確定的な連続時間モデルであるのに対し、本稿のモデルは、離散時間モデルであるとともに確率的な摂動が導入されている。

一般的に言えば、経済系は孤立したシステムではなく、社会における他のサブシステムとの不断の相互干渉にさらされている。このような他のシステムからの影響をモデル化する 1 つの方法は、「ノイズ」を導入することである。2 通りのノイズのタイプが考えられる。一つは、決定論的な方程式に加法的な形でノイズを加える方式で、ここでは「加法的」ノイズと呼ぶことにする。他の一つは、方程式のパラメータに摂動を与える形のノイズで、「乗法的」ノイズと呼ぶことにする。本稿では「乗法的」ノイズを主として扱う。

3 次元モデルは、小国開放経済モデルといえる。外国との輸出、輸入、資金の移動があり、開放経済であるものの、輸出、輸入が海外の経済に影響を与えないという意味で、小国経済となっている。これに対して、5 次元モデルは貿易と資金(資本)移動を通じて2国は相互依存の関係にある。2 国間の景気循環の関連が新たな関心事として追加される。

ノイズが系の動学的性質に与える影響については、動学的振る舞いを乱す[2]、逆に秩序を作り出す[3]など多数の先行研究がある。しかし、経済学においてはノイズ効果についての研究は極めて少ない。

本研究の構成は以下の通りである。2 章では3次元の開放経済モデルを提示する。外国為替相場が変動しない、固定相場制の場合と変動相場制の場合のモデルを説明する。次の 3 章では、固定相場制の場合について5次元の2国モデルを提示する。4 章では、数値シミュレーションによる計算結果を、2、3章のモデルに関して示す。5 章は要約と今後の課題を述べる。

2.　小国開放経済(3次元)モデル

生産されたモノ(財・サービス)が取引される財市場における需給バランスの関係は所得調整方程式と資本蓄積方程式とで表現される。総供給と総需要とが毎期毎期必ずしも均衡する必要がなく、需給は不均衡であっても構わない。むしろ調整には時間がかかり、その時々の環境変化に迅速に対応できない状態が普通であるとの認識に基づく。これに対して、カネ(貨幣)が取引される市場における需給バランスは極めて短時間で達成されると想定する。

2.1　固定相場制

固定相場制の下では、モデルは Y、K、M の 3 変数からなる以下の基本的方程式体系で表される[2]。

$$Y_{t+1} - Y_t = \alpha[c(1-\tau)Y_t + cT_0 + G + I(Y_t, K_t, r(Y_t, M_t)) + J(Y_t, E) - Y_t] \qquad (1)$$

$$K_{t+1} - K_t = I(Y_t, K_t, r(Y_t, M_t)) \qquad (2)$$

$$M_{t+1} - M_t = pJ(Y_t, E) + (\beta + \sigma\varepsilon_t)p\{r(Y_t, M_t) - r_f\} \qquad (3)$$

ここで、記号の意味は以下の通りである。Y=実質国民純生産、K=実質資本ストック量、M=名目貨幣ストック量、I=実質民間純投資支出、J=実質経常収支)、r=国内名目利子率、r_f =外国の名目利子率、E=内貨建て為替レート(一定、固定相場)、P=物価水準(一定)、c =可処分所得からの限界消費性向、τ =限界税率、G=実質政府支出(一定)、T_0=基礎的な補助金、C_0=基礎的消費、α=財市場における調整速度を示すパラメータ(α>0)、β=国際資本移動の流動性の程度を表すパラメータ(β>0)、ε=正規分布に従う擬似乱数N(0、1)、σ=ノイズの大きさの程度を表す標準偏差パラメータ($\sigma \geqq 0$)。

(1)式は、財市場における数量調整メカニズムを定式化している。すなわち、財市場における超過需要に応じてYが変動することを表している。(2)式は純

投資が資本ストックの変化に等しいことを示す資本蓄積方程式である。(3)式は、固定相場制の下では総合収支(経常収支＋資本収支)が正か負に応じてマネーサプライが内生的に変化するメカニズムを定式化している。固定相場制の下では、Eは固定されている。

所得調整方程式(1)においては、消費は所得にのみ依存すると想定されている。このようなモデルを「モデル1」と呼ぶことにする。これに対して、消費が所得だけではなく資産にも依存すると想定した場合を「モデル2」と言うことにする。その場合、所得方程式はは資本ストック変数Kに依存しない。

2．2　変動相場制

外国為替レートが変動する下では、Y、K、E^e(期待為替レート)の3変数からなる基本的方程式体系は以下のように表される[3]。

$$Y_{t+1} - Y_t = \alpha[c(1-\tau)Y_t + cT_0 + G + I(Y_t, K_t, r(Y_t, \overline{M})) + J(Y_t, E_t) - Y_t] \quad (4)$$

$$K_{t+1} - K_t = I(Y_t, K_t, r(Y_t, \overline{M})) \quad (5)$$

$$E^e{}_{t+1} - E^e{}_t = (\underset{\gamma}{\cancel{\chi}} + \sigma\varepsilon_t)\{E(Y_t, E^e{}_t) - E^e{}_t\} \quad (6)$$

(4)、(5)式は、固定相場制の場合とほぼ同じであるが、簡単化のため、貨幣ストック$\overline{M}$は定数として扱われる。(6)式は、変動相場制の下での為替決定メカニズムを表している。ここでは、総合収支を均衡させるように為替レートEが内生的に決まるものと想定している。また、為替レートの見通しに関しては、標準的な「適応的期待」が採用される。

3．二国モデル（固定相場制）

二国モデルは5次元の差分方程式体系で表される。そこでは貿易と資金(資本)移動を通じて2国は相互依存の関係にある。小国開放経済モデルとは相違し、2国間の景気循環の関連が新たな関心事として追加される。

固定相場制の下での5次元モデルは以下のように表される。方程式(7)、(8)は第1国の所得調整方程式と資本蓄積方程式を表す。これに対して方程式(9)、(10)式は第2国のそれらを表す。最後の(11)式は第1国と第2国の幣供給量が両国の貿易収支と資本収支とによって決定されるのを表している。

$$\begin{aligned} Y_1(t+1) &= Y_1(t) + \alpha_1[c_1(1-\tau_1)Y_1(t) + c_1T_{01} + C_{01} \\ &\quad + \delta H_1(Y_1(t), Y_2(t), \overline{E}) - Y_1(t)] \\ &\equiv F_1(Y_1(t), K_1(t), Y_2(t), M_1(t); \delta, \alpha_1) \end{aligned} \quad (7)$$

$$\begin{aligned} K_1(t+1) &= K_1(t) + I_1(Y_1(t), K_1(t), r_1(Y_1(t), M_1(t))) \\ &\equiv F_2(Y_1(t), K_1(t), M_1(t)) \end{aligned} \quad (8)$$

$$\begin{aligned} Y_2(t+1) &= Y_2(t) + \alpha_2[c_2(1-\tau_2)Y_2(t) + c_2T_{02} + C_{02} \\ &\quad + G_2 + I_2(Y_2(t), K_2(t), r_2(Y_2(t), \{\overline{M} - M_1(t)\}/\overline{E})) \\ &\quad - (1/\overline{E})\delta H_1(Y_1(t), Y_2(t), \overline{E}) - Y_2(t)] \\ &\equiv F_3(Y_1(t), Y_2(t), K_2(t), M_1(t); \delta, \alpha_2) \end{aligned} \quad (9)$$

$$\begin{aligned} K_2(t+1) &= K_2(t) + I_2(Y_2(t), K(t), r_2(Y_2(t), \\ &\quad (1/\overline{E})\{\overline{M} - M_1(t)\}))_2 \\ &\equiv F_4(Y_2(t), K_2(t), M_1(t)) \end{aligned} \quad (10)$$

$$\begin{aligned} M_1(t+1) &= M_1(t) + \delta H_1(Y_1(t), Y_2(t), \overline{E}) \\ &\quad + \beta\{r_1(Y_1(t), M_1(t)) - r_2(Y_2(t), \{\overline{M} - M_1(t)\}/\overline{E})\} \\ &\equiv F_5(Y_1(t), Y_2(t), M_1(t); \delta, \beta) \end{aligned} \quad (11)$$

ここで記号は以下の通りである。

国(地域)　　$(i = 1,2)$

J_i ＝ i国の経常収支$(\overline{E}J_2 = -J_1)$、Q_i ＝i国の資本収支$(\overline{E}Q_2 = -Q_1)$、$A_i = J_i + Q_i$ ＝i国の資本収支$(\overline{E}A_2 = -A_1)$、$\overline{E}$＝ 外国為替レート(国1の通貨で評価した国2の価値、固定相場)、Y_i ＝ i国の実質国民所得、C_i ＝ i国の実質消費支出、I_i ＝i国の実質純民間設備投資、G_i ＝i国の実質政府支出(一定)、

K_i＝i国の実質物的資本ストック、T_i ＝ i国の実質所得税、M_i ＝ i 国の名目貨幣ストック量（$M_1(t)+\bar{E}M_2(t)=\overline{M}$）、$p_i$＝ i国の物価水準(固定)、r_i ＝ i国の名目利子率、t ＝ 期間 、δ ＝ 国際取引の容易さの程度、β ＝ 国際資本移動の流動性の程度。

4.数値計算結果

4．1　小国開放経済(固定相場制)

2章で論じた開放経済系のモデルについて 、数値実験を行うことによってその性質を調べてみる。われわれが対象としている経済系に対して、他のサブシステムからの影響（確率的ノイズ）がない場合とある場合の両者について検討する。貨幣量の調整を表す方程式において国際資本移動の程度を示すパラメータを主として想定する。なお、数値実験の詳細については [4] を参照されたい。

2章の小国開放経済モデルの固定相場制の場合については、以下のような特定化をした[5]。

$$Y_{t+1}-Y_t=\alpha\{-0.66Y_t+f(Y_t)-0.3K_t+147-\sqrt{Y_t}+M_t+165 \quad (12)$$

$$K_{t+1}-K_t=f(Y_t)-0.3K_t-10\sqrt{Y_t}+M_t+147$$

(13)

$$M_{t+1}-M_t=-0.3Y_t+50+(\beta+\sigma\chi_t)\{10\sqrt{Y_t}-M_t-6\}$$

(14)

ここで

$$f(Y_t)=\frac{80}{\pi}Arc\tan\{\frac{2.25\pi}{20}(Y_t-\frac{165}{0.66})\}+35$$

2章において、「モデル1」と資本ストックの財市場への影響を遮断した「モデル2」を述べた。モデル1の均衡点は（Ye,Ke,Me）≒（250,503,127）となる。所得の均衡値は、所得調整方程式の調整係数と貨幣量の調整を表す方程式において国際資本移動の流動性の程度を示すパラメータの値に依存しない。他方、資本ストックと貨幣量の均衡値は国際資本移動の流動性程度を示すパラメータの値に依存する。これらモデル1と2のそれぞれについて、ノイズの影響を調べる。分岐パラメータとして、所得調整方程式の調整係数（α）を考える。モデル1、2ともにノイズが存在しない場合においても複雑な挙動を示す。ノイズによって国際資本移動の流動性の程度を示すパラメータが摂動されるとき、系の背後にある構造が顕在化することがある。

モデル1

所得調整方程式の調整係数を分岐パラメータとした場合の所得についての分岐図は図1で与えられる。貨幣量調整方程式の国際資本移動の流動性の程度を示すパラメータについては、$\beta=1$と仮定する。所得調整方程式の調整係数が増加するにつれ、所得変動の周期が急激に増大し、高周期となり、やがてカオス的になることが図1より分かる。しかしさらに増加させると、時折周期的な挙動を示す「窓」が現れる。このことは、初期値の僅かな差が指数関数的に乖離するかどうかを示すリャプノフ指数からも分かる（図2）。パラメータ値が小さい場合リャプノフ指数は僅かであるが負の値をとり、周期的であることを示している。パラメータが2以前のところでリャプノフ指数は正の値となりカオス的になる。しかし2を超えたところで再び負の値となり、周期の窓が現れたことを示している。

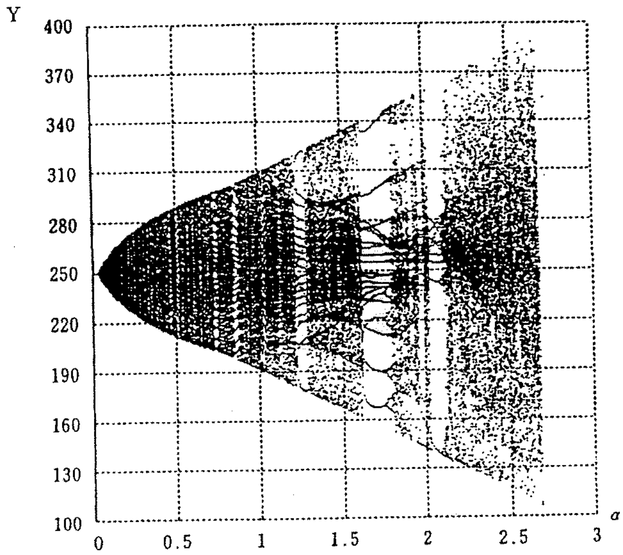

図1　分岐図

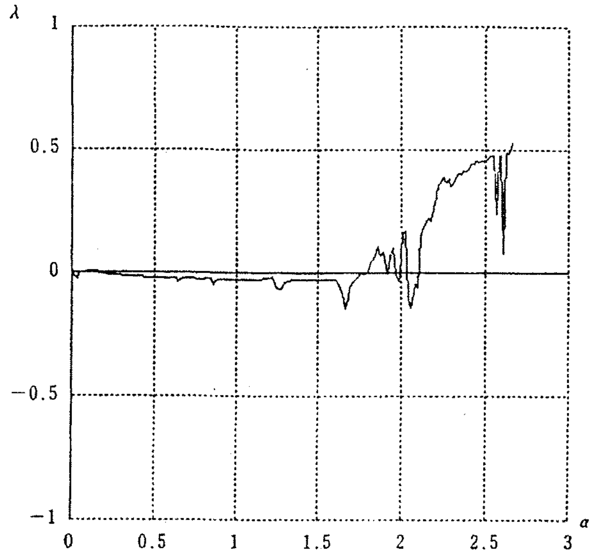

図２　リャプノフ指数

ノイズがない場合とある場合とを比較してみよう。ノイズがなければ、先の図１と２から、所得調整方程式の調整係数の値がα＝２．０のときカオス的であるが、α＝２．１のとき「窓」が現れ周期的となる。さらにα＝２．２のとき再びカオス的となる。これら３つの場合のアトラクターは図３、４、５で与えられる。α＝２．１の「窓」においてノイズによって貨幣量調整方程式の国際資本移動の流動性の程度を示すパラメータを摂動した場合のアトラクターは図６で与えられる。図４のノイズがない場合は周期的であった挙動に対して、ノイズによってカオス

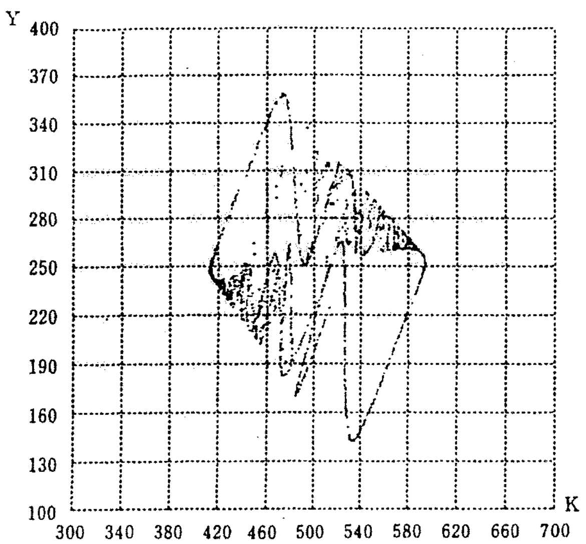

図３　α=2.0 のときのＹとＫのアトラクター

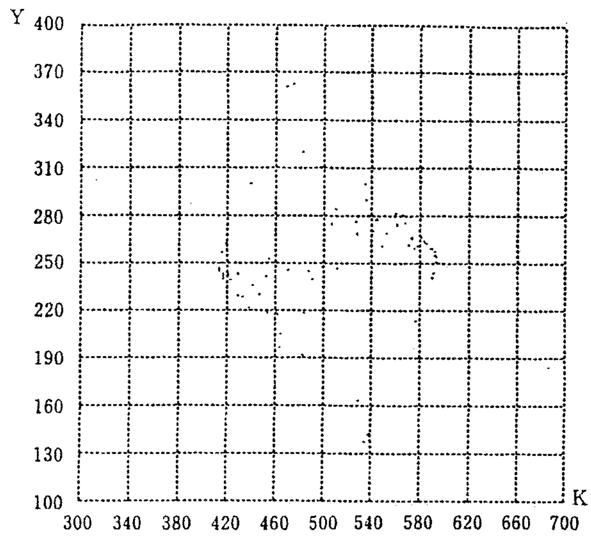

図４　α=2.1(窓)のときのＹとＫのアトラクター

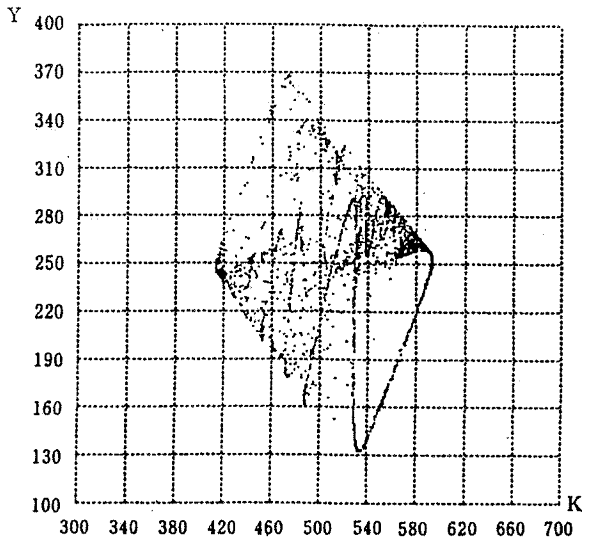

図５　α=2.2 のときのＹとＫのアトラクター

的振る舞いになる。このカオス的振る舞いは窓の前後のそれと類似していることが窺える。窓においては、周期的軌道が安定的であるが、不安定なカオス的軌道は現れず、潜在的である。しかし、ノイズを付加することによって背後の構造が顕在化し得ることが分かる。本来は周期的な変動を示す経済であったとしても、国際資本移動の流動性の程度を示すパラメータの僅かな摂動によっても変動がカオス的になることを意味している。

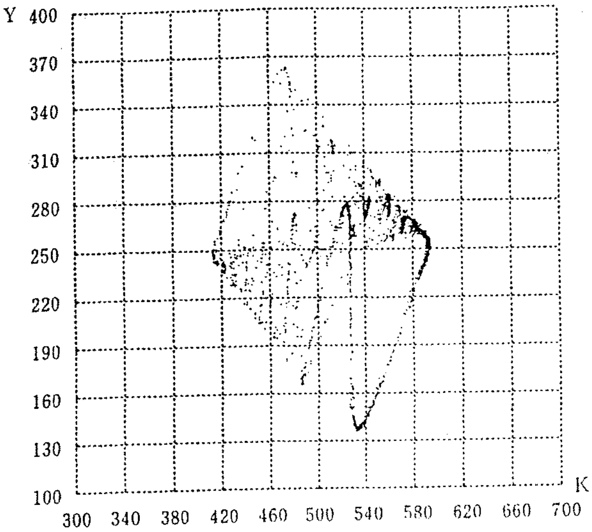

図6　βがノイズによって摂動された場合の
α=2.1(窓)のときのYとKのアトラクター

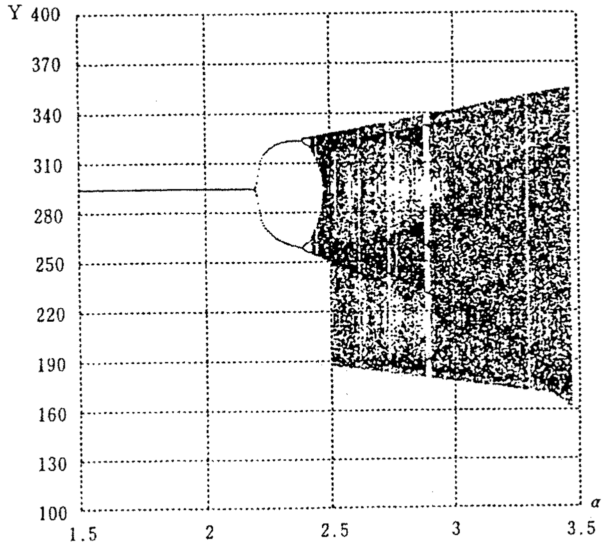

図7　ノイズがない場合の分岐図

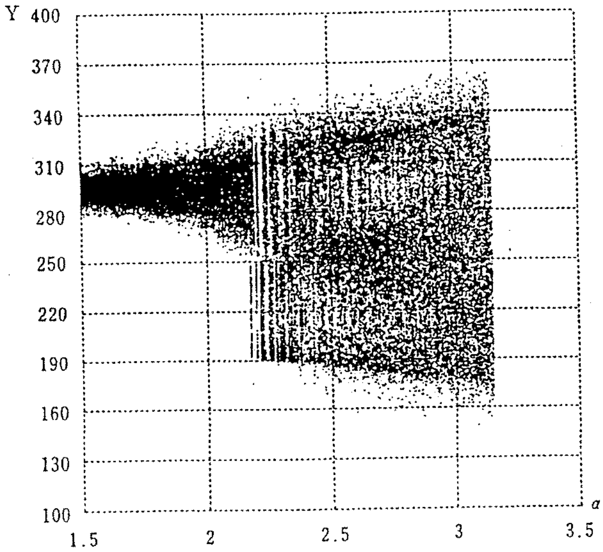

図8　βがノイズによって摂動された場合の分岐図

モデル　2

富効果(資産効果)を考慮したモデル2の性質を数値実験によって調べてみよう。この場合の分岐図(図7)はモデル1のそれとはかなり異なる。分岐パラメータ値が小さいとき均衡は安定的であるが、パラメータ値がおおよそ2.25を超えると不安定となり、2周期のサイクルが現れる。しかし、周期倍分岐が極めて速やかに起こりカオス的となる。さらにパラメータが2.5を超えるとYの変動域は「モデル1」の均衡値Y=250の下側にも拡大する。実は、初期値を「モデル1」の定常解（Ye,Ke,Me）≒（250,503,127）の近くにとった場合、図7の分岐図が得られている。別の初期値をとれば、アトラクターが少なくとももう1つ存在することから、Y=250を中心に反転した分岐図が得られる。したがって、$\alpha>2.5$におけるY=250の下側にも拡大したYの変動域は、下側にある別のアトラクターとの融合が行われたことを反映している。また周期の窓が複数存在することも観察できる。

図8は、βをある程度の大きさのノイズによって摂動することによりYの軌道が複雑になり周期の窓も崩れていくことを示している。興味深い現象として、ノイズによってY=250の下側にある軌道が$\alpha<2.5$の場合に既に現れてくるのが観察される。これは、この系の背後にある隠れた「構造」がノイズによって浮かび上がらされたと解釈できる。Yが250より上方にあれば好況、下方にあれば不況と呼ぶことにすれば、表面上は好況であるとしても、不況に変化しうる構造が隠れていたら、何らかの外乱でそれが顕在化することもあり得る。

次にパラメータαをある値に固定し、Yの時系列を調べてみる。アトラクターが融合する直前のα=2.5と融合直後のα=2.55について、ノイズの影響を調べる。図9と10は融合直前の場合である。図9は、ノイズがなければ好況局面にとどまりつづけることを表している。他方、図10はノイズがそれほど大きくなければ、好況から不況へと切り替わりまた好況へ切り替わるという景気の交替が起きることを示している。しかしノイズが大きくなると好・不

況の間を激しく振動する。やはりこの場合も背後にある隠された「構造」がノイズによって顕在化したと解釈される。

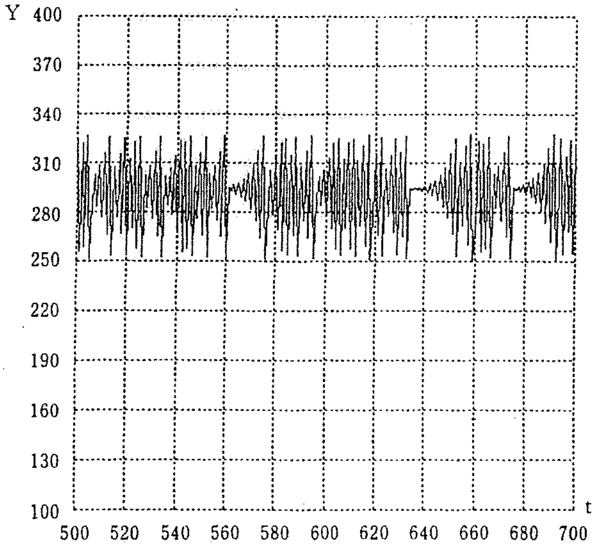

図９　ノイズがない場合のＹの軌道
(融合直前 α=2.5)

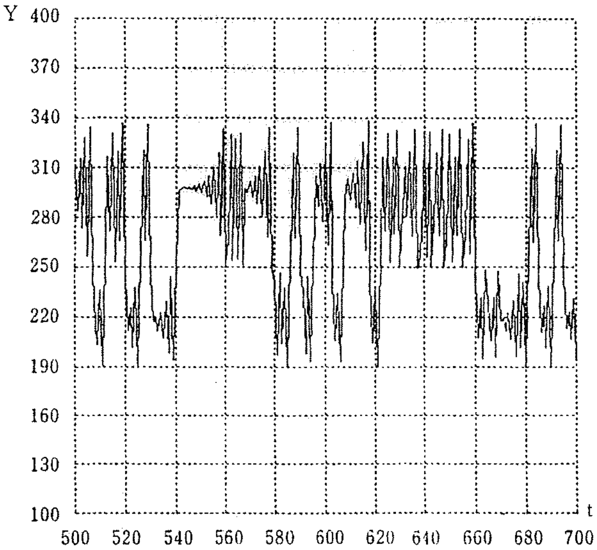

図 10　βがノイズによって摂動された場合の Y の軌道(融合直前 α=2.5、ノイズ偏差=0.08)

融合直後については、逆に「時として」好・不況の一方だけに留まる軌道が出ないかどうかを試してみた。その結果、ある程度大きなパラメトリックノイズをかけたとき、このようなことが起こり得ることがわかった。図 11 はノイズがない場合の軌道を表している。好況・不況の切り替わりがみられる。これに対して、ノイズがかかると例えば図 12 にみられるように、好況の局面に留まる。この現象はパラメトリックノイズについて顕著に見られた。勿論、この現象は常に起こるのではなく、しかも未来永劫、一方に留まるわけでもない（乱数の出方によると考えられる)。したがって、一方に留まっていても、また大きな外乱が来れば、再び正負を横断する軌道も起こり得る。この現象は、好・不況を頻繁に繰り返している場合でも、うまく（まずく）外乱が入ってくれば、なかなか終わらない好況（不況）が発生しうる」あるいは「なかなか終わらない好況（不況気）も構造的にそれが発生しているとは限らず、幸運（不運）な外乱の影響で見かけ上そうなっているだけかもしれないと解釈できるであろう。

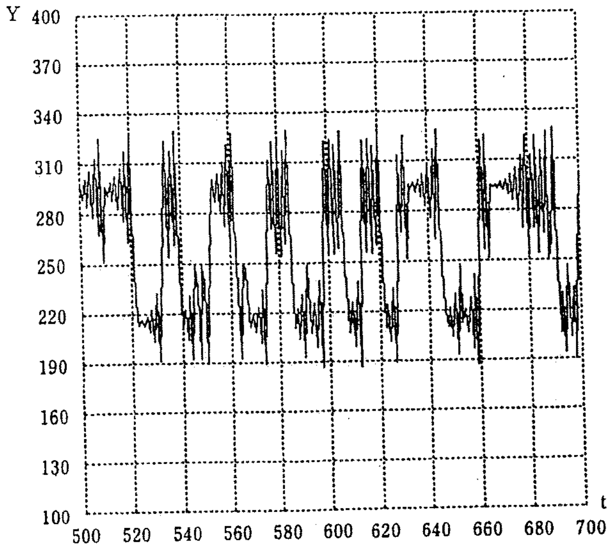

図 11　ノイズがない場合のＹの軌道
(融合直後 α=2.5)

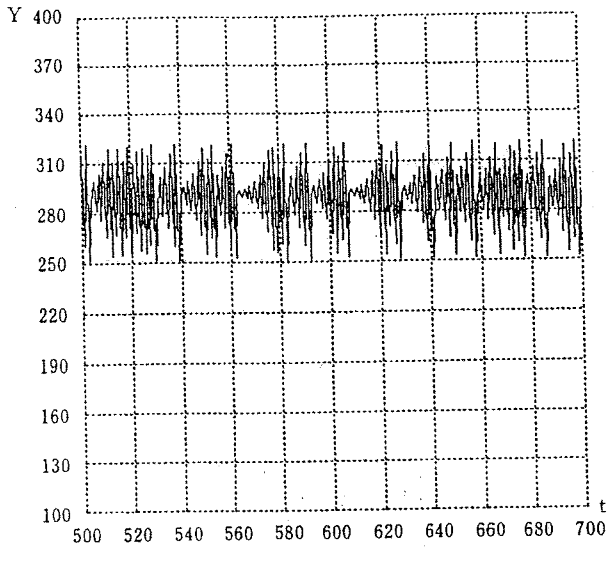

図 12　βがノイズによって摂動された場合の Y の軌道(融合直後 α=2.5、ノイズ偏差=0.08)

4．2　小国開放経済(変動相場制)

数値計算を行う際、(4)式の所得調整方程式と(5)式の資本蓄積方程式は、固定相場制の場合と同じく、(11)式と(12)式のような特定化をした[6]。ただし、国際資本移動の流動性の程度を示すパラメータβは固定された値とし、ノイズによる摂動を受けない。それに代わって、(6)式の為替調整方程式の期待為替レートの調整係数γがノイズによって微細な変動を与えられるものとの想定した。数値シミュレーションの結果は、以下のように要約される。

(1)ノイズが存在しないシステムにおいて財市場の調整速度を表すパラメータが増加するにつれて、為替レートの変動域が突然急拡大する「バースト」と言う現象が発生する。

(2)ノイズによって「バースト」が抑制され得る。換言すれば、ノイズは隠された構造を浮かび上がらせることになる。

(3)周期解を発生させる「窓」がノイズによって崩され、背後に隠されたカオス的な構造が現れることがある。

図 13 はノイズが存在しない場合の外国為替レート E の分岐図である。分岐パラメータは財市場における調整速度αである。図 14 はこのシステムにおける最大リャプノフ指数を示している。これらの図より、α=0.33 において「バースト」が出現し、$\alpha \geqq 0.33$ の領域で頻繁にカオス的変動が発生することがわかる。図 15 はα=0.33 におけるカオス・アトラクターをE-Y平面上に描いたものである。

ところで、ノイズによって為替レートの「バースト」が抑制されることがあり得る。図 16 はα=0.33 のとき小さなノイズ(σ=0.01)によって浮かび上がらされたE-Y平面上のアトラクターである。このアトラクターはノイズが存在しない場合のα=0.32 におけるアトラクターと似ているが、このアトラクターは周期的であり、「バースト」は出現していない。すなわち、確率的な撹乱(ノイズ)がない場合に外国為替レートが激しく変動することがあっても、その激しい変動がわずかのノイズによって抑制され得る。図 17,18 はα=0.33 における為替レートの軌道をノイズがない場合と小さなノイズが存在す場合に分けて比較している。以上の例はノイズがシステムを安定化させ得ることを示している。

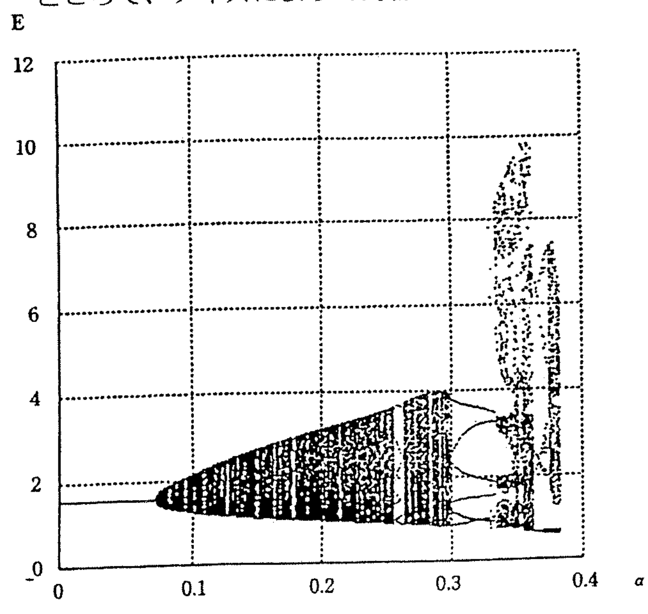

図 13　E の分岐図

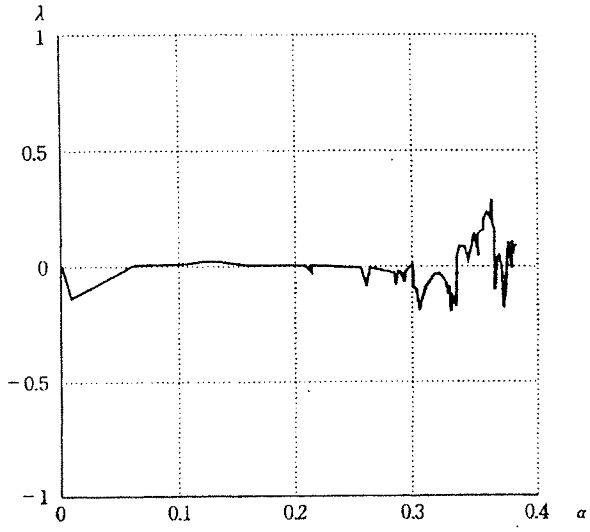

図 14　リャプノフ指数

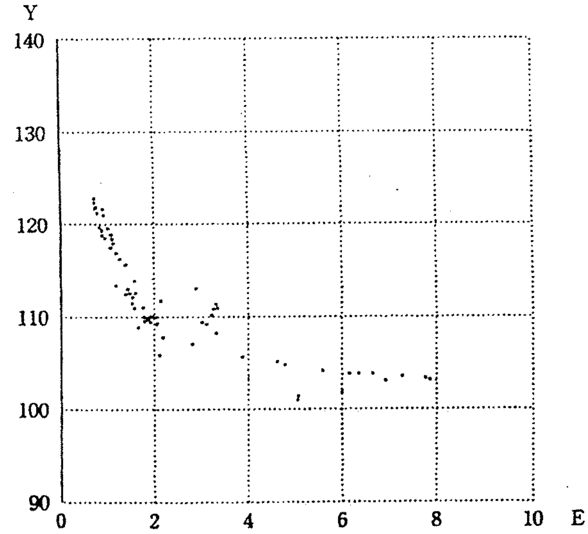

図 15　α=0.33 のときの Y と E のアトラクター

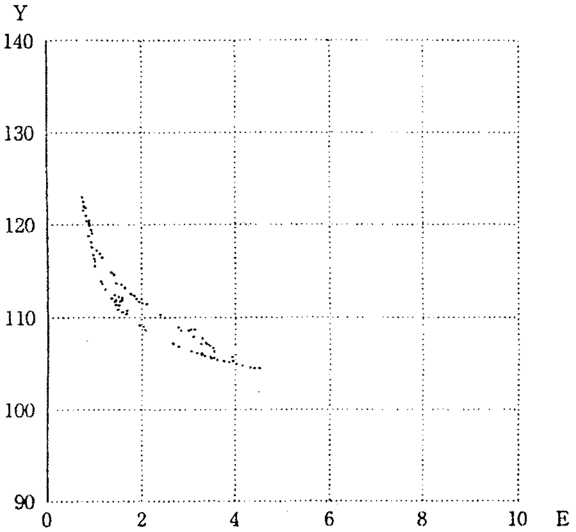

図 16　γがノイズによって摂動された場合の
Y と E のアトラクター（α=0.33、ノイズ偏差=0.01）

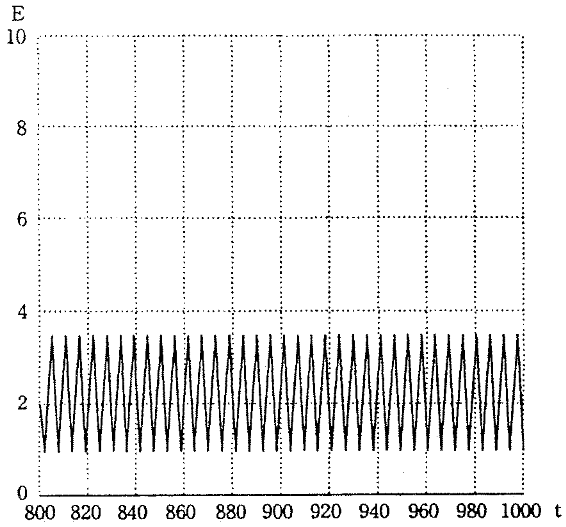

図 18　γがノイズによって摂動された場合の
E の軌道（α=0.33,ノイズ偏差=0.01）

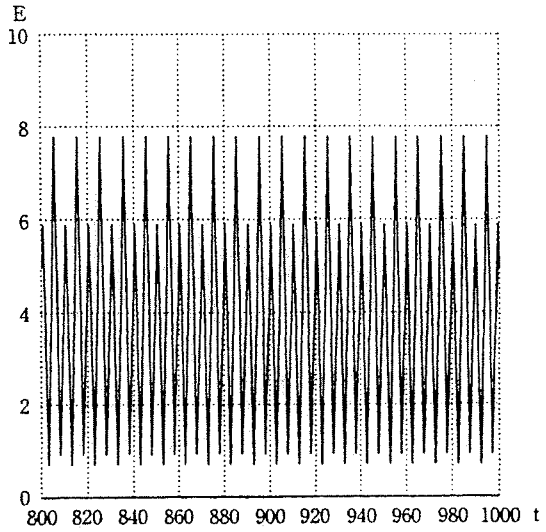

図 17　ノイズが存在しない場合の E の軌道
（α=0.33）

4．3　二国モデル(固定相場制)

二国は経済規模および行動様式においてほぼ同一であるものとして、二国の所得調整方程式と資本蓄積方程式は小国開放経済モデルの場合とほぼ同じ特定化をする。国際資本移動の流動性の程度を示すパラメータβがノイズによって摂動を受けるものと想定する。国際取引の容易さの程度を示すパラメータδ(=1)は固定された値を取り、摂動を受けない。固定された為替レートは簡単化のためE=1と仮定する。

分岐パラメータとして、第 1 国の所得調整方程式の調整係数（α_1）を考える。第 2 国の所得調整方程式の調整係数についてはα_2=0.1 と仮定する。このとき、第 1 国の所得についての分岐図は図 19 で与えられる。α_1=0.1 の場合、2 国は同一の調整速度となる。

国際資本移動の流動性程度を示すパラメータβがノイズによって摂動を受けないとき、両国の景気の変動は同じものとなる(図 20)。第 1 国が好況のときは第 2 国も好況で、第 1 国が不況のときは第 2 国もそうなり、景気変動は同期化する。

しかし、国際資本移動の流動性程度を示すパラメータβがノイズによって摂動を受けるとき、二国の景気は逆の動きを示す(図 21)。

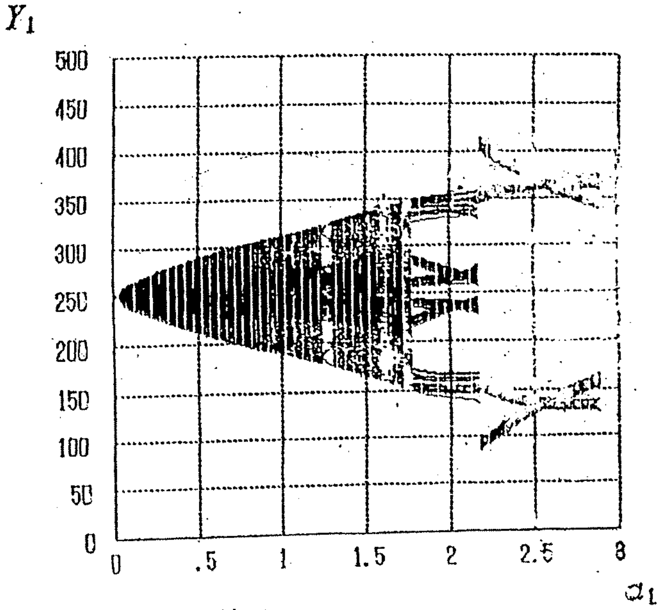

図 19　Y_1 の分岐図

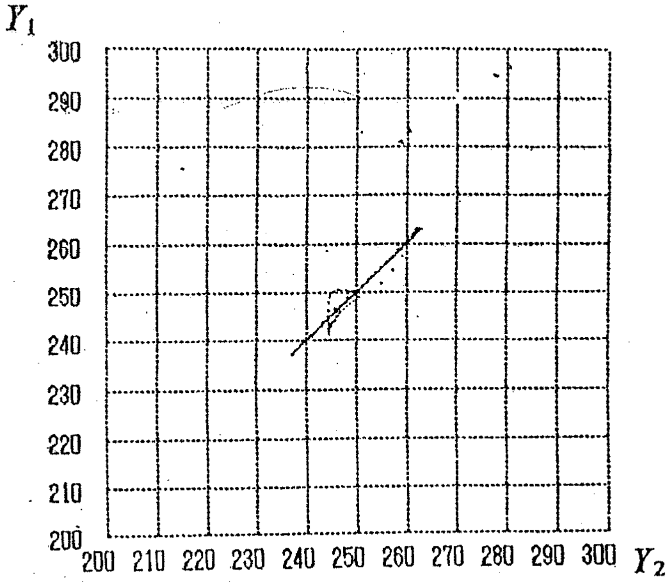

図 20　$\alpha_1 = \alpha_2 = 0.1$ のときの Y_1 と Y_2 の
アトラクター

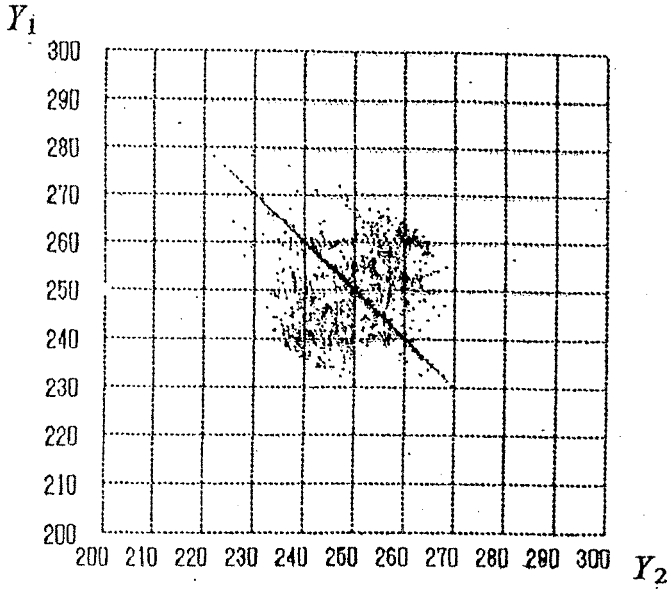

図 21　β がノイズによって摂動された場合の
Y_1 と Y_2 のアトラクター

5．結語

小国開放経済モデルと二国経済モデルについて、モデルのパラメータが摂動を受けた場合、ノイズは経済に埋め込まれている基本的構造を不明瞭にする効果を持っているだけではなく、場合によっては逆に背後の隠されている構造を浮かび上がらせることもあるということを、数値シミュレーションによって示した。今後の課題として、二国モデルにおける変動相場制の場合の分析、ノイズ効果についての可能な限りの解析的分析などがある。

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Numerical integration of stochastic differential equations by composition methods

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Abstract

In this paper, "composition methods (or operator splitting methods) "for autonomous stochastic differential equations (SDEs) are formulated to make numerical approximation schemes for the equations. In the methods, the exponential map, which is given by solution of a stochastic differential equation, is approximated by composition of the stochastic flows derived from simpler and exact integrable vector field operators having stochastic coefficients. The local errors of the numerical schemes derived from the stochastic composition methods are investigated in detail. The new schemes are advantageous to preserve the special character of SDEs numerically and are useful for approximations of the solutions to stochastic non-linear equations. To examine the superiority, several numerical simulations on the basis of the schemes are carried out for stochastic differential equations which are treated in the mathematical finance and stochastic Hamilton dynamical systems.

Key words. stochastic differential equations, Lie algebra, composition methods (operator splitting method), mathematical finance, stochastic Hamilton dynamical systems, stochastic non-linear system

AMS subject classifications. 65C30, 17B66, 91B28, 70-08, 70H33

1 Introduction

The theory of stochastic differential equations is understood as a fundamental tool for the description of random-phenomena treated in physics, engineering, economics and mathe- matical finance. However, it is often difficult to obtain the solutions of stochastic differential equations explicitly. Hence, there has been increasing interest in numerical analysis of stochastic differential equations, and many numerical schemes for stochastic differential equations has been formulated (*e.g.* Gard (1988), Kloeden and Platen (1992), Saito and Mitsui (1993)).

The purpose of the present paper is to propose some new numerical schemes for autonomous stochastic differential equations on the basis of "composition methods ". The reasons why we address such a topic are as follows:

In numerical analysis for deterministic ordinary differential equations, whether or not some special character or structure of the equations is preserved precisely is an important point in performing reliable numerical calculations. From this point of view, various numerical methods to realize the characters of differential equations have been investigated. Indeed, we can find out such examples as energy conservative methods (Greenspan (1984), Ishimori (1994)), symplectic integrators for Hamilton dynamical systems (Suzuki (1990), Forest and Ruth (1990), Yoshida (1990)), and composition methods (McLachlan (1995), Moreau and Vandewalle (1996)). Particularly, the composition methods are useful to make numerical schemes which leave some structure or character of general differential equations numerically invariant. Hence, it seems to be quite natural that we investigate the methods for stochastic differen- tial equations to make numerical schemes having the conservation properties; this is the first reason for setting up the purpose mentioned above.

Moreover, in the theory of differential equations, composition methods are also known as operator-splitting methods, and they are often utilized for approximations of non-linear equation of which solutions are not obtained explicitly (Yanenko (1959), Iserles (1984)). In consideration of this, we may expect that the methods bring us a convenient and powerful way of approximations for "stochastic non-linear"differential equations, and this is another reason for our purpose.

Here, we will outline the original composition methods for ordinary differential equations. Let X denote vector fields on some space with coordinates x, with flows $\exp(tX)$, that is, the solutions of differential equations of the form $\dot{x}(t) = X(x)$ are given by the form $x(t) = \exp(tX)(x(0))$. Then, the vector field X is to be integrated numerically with fixed time step t. In the framework, we can apply composition methods to the differential equation, if one can write $X = A + B$ in such a way that $\exp(tA)$ and $\exp(tB)$ can both be calculated *explicitly* (more generally, this can be relaxed by approximations of the exponential maps). In the most elementary case, the method gives the approximation for $x(t)$ through

$$\tilde{x}(t) = \exp(tA)\exp(tB)(x) = x(t) + O(t^2);$$

the last equality is shown by using Baker-Campbell-Hausdorff (BCH) formula in Lie algebraic theory.

Thus, in composition methods, we use the exponential representation of solutions to differential equations as an important tool. To formulate the methods for stochastic differential equations, therefore, one needs the same notion for the equations. In Kunita (1980), such a topic has been investigated in detail. Hence, Section 2 is devoted to review his work and to set up some notations of stochastic differential equations and vector fields. In addition, to estimate approximation errors of our new numerical schemes, we will prove some propositions with respect to time asymptotics of multiple stochastic integrals.

On the basis of the results in Section 2, composition methods are formulated for autonomous stochastic differential equations in Section 3. Through the methods, we will obtain some numerical schemes for the stochastic equations. Then, the approximation-error of numerical solutions derived from the schemes must be estimated. In this paper, as the first step, we apply the local error estimation in mean-square sense, which has established by Saito and Mitsui (1993), to the obtainable numerical solutions. BCH formula will be useful for calculating the errors as in case of composition methods for deterministic differential equations.

In Section 4, to examine the superiority of the new schemes, we investigate some examples of the numerical simulations on the basis of the schemes. In the first example, the following non-linear scalar stochastic differential equation is treated, which is often adopted as a model of an asset price process in mathematical finance (Geman and Yor (1993)):

$$dS(t) = S(t)dt + 2\sqrt{S(t)} \circ dW(t), \quad S(0) = s(> 0). \tag{1.1}$$

It is known that a solution $S(t)$ to this equation takes always a "non-negative "value for any $t \in [0, T]$. Through the standard stochastic numerical schemes, however, such a character of this equation is not always preserved numerically. In contrast with this, we will see that our new schemes by composition methods leave the structure invariant numerically, and thereby we can examine the first advantage of composition methods. In the second and third examples, we investigate the second advantage of composition methods as a tool of approximation for stochastic non-linear systems. Particularly, in case of deterministic differential equations, such a superiority is often found out in Hamilton dynamical systems as dimensional splitting methods. Therefore, in the third example, we treat the composition methods for "stochastic "Hamilton systems (Misawa (1999)). In the end of the section, we further touch upon a way to make numerical schemes which realize the numerical preservation of conserved quantities for stochastic systems (Misawa (2000)) .

Finally, some concluding remarks and future problems are given in Section 5.

2 Representation of solutions of SDEs

Now, we start with a review of representation of solutions of stochastic differential equations (SDEs) in the framework of Kunita's work (Kunita (1980)). Let us consider an autonomous SDE of Stratonovich type (*e.g.* Ikeda and Watanabe (1989), Arnold (1973)) under the probability space $(\Omega, \mathcal{F}, \mathcal{P})$

$$dS(t) = b(S(t))dt + \sum_{j=1}^{r} g_j(S(t)) \circ dW^j(t) \tag{2.1}$$

defined on a connected C^∞-manifold M of dimension d, where $b = (b^i)_{i=1}^d$ and $g_j = (g_j^i)_{i=1}^d$ $(j = 1, \cdots, r)$ are d-dimensional C^∞ functions on M, respectively, and $W(t) = (W^1(t), \cdots W^r(t))$ is a standard Wiener Process. Here $S(t)$ is assumed to be adapted with a non-decreasing family of sigma-algebra $(\mathcal{F}_t)_{t\geq 0} \subset \mathcal{F}$. Note that Eq.(2.1) is rewritten in the form of SDE of Itô type as follows:

$$dS_t = \{b(S(t)) + \frac{1}{2}\sum_{k=1}^{r}\sum_{i=1}^{d} g_k^i \partial_i g_k(S(t))\}dt + \sum_{j=1}^{r} g_j(S(t))dW^j(t), \tag{2.2}$$

where $\partial_i = \partial/\partial S^i$. Using the coefficient-functions in (2.1), we define C^∞-vector fields $X_0, X_1, \cdots, X_r$ as follows:

$$X_0 = \sum_{i=1}^{d} b^i \partial_i, \quad X_j = \sum_{i=1}^{d} g_j^i \partial_i \quad (j = 1, \cdots, r). \tag{2.3}$$

The proof of Kunita's lemma (lemma 2.1 in Kunita (1980)) suggests that the solution of SDE (2.1) with an initial value $S(0) = s$ is *formally* represented as

$$S(t) = (\exp Y_t)(s), \tag{2.4}$$

where $Y_t(\omega)$ is the vector field for each t and a.s. $\omega \in \Omega$ given by

$$\begin{aligned} Y_t &= \sum_{i=0}^{r} W^i(t)X_i + \frac{1}{2}\sum_{i<j}^{r}[W^i, W^j](t)[X_i, X_j] + \frac{1}{36}\sum_{i=0}^{r}\sum_{j=1}^{r} tW^i(t)[[X_i, X_j], X_j] \\ &+ \frac{1}{18}\sum_{i<j,k}^{r}[[W^i, W^j], W^k](t)[[X_i, X_j], X_k] \\ &+ \sum_{J;4\leq|J|}\{\sum_{\Delta J}{}^{*} c_{\Delta J} W^{\Delta J}(t)\}X^J, \ (i = 0, 1, \cdots, r; j, k = 1, \cdots, r). \end{aligned} \tag{2.5}$$

In (2.5), $[X_i, X_j]$ is the Lie bracket defined by $X_iX_j - X_jX_i$, and $X^J = [\cdots[X_{j_1}, X_{j_2}]\cdots]X_{j_m}]$ $(J = (j_1, \cdots, j_m)$. Moreover, we set $W^0(t) = t$ and define $[W^i, W^j](t)$ and $[[W^i, W^j], W^k](t)$ as multiple Wiener-Stratonovich integrals of degrees equal to 2 and 3 given by

$$[W^i, W^j](t) = \int_0^t W^i(\tau) \circ dW^j(\tau) - \int_0^t W^j(\tau) \circ dW^i(\tau),$$

and

$$[[W^i, W^j], W^k](t) = \int_0^t [W^i, W^j](\tau) \circ dW^k(\tau) - \int_0^t W^k(\tau) \circ d[W^i, W^j](\tau),$$

respectively. Moreover, $|J|$ denotes the length of a multi-index J; that is, if $J = (j_1, \cdots, j_m)$, then $|J| = m$. $W^{\Delta J}(t)\}$ and $c_{\Delta J}$ are the multiple Wiener-Stratonovich integrals with respect to $(W^0(t), W^1(t), \cdots, W^r(t))$ and the constant coefficients which are determined from a single or double divided index ΔJ of J, respectively, and $\sum_{\Delta J}{}^*$ denotes the sum for all single and double divided indices of J; on the details of the definitions of ΔJ, $W^{\Delta J}(t)$ and $c_{\Delta J}$, see pp.285-289 in Kunita's paper (1980).

The equation (2.4) with (2.5) mean that the solution $S(t,\omega)$ equals $\phi(1,s,\omega)$ a.s., where $\phi(\tau,s,\omega)$ is the solution of the *ordinary* differential equation

$$\frac{d\phi(\tau)}{d\tau} = Y_t(\omega)(\phi(\tau)), \quad \phi(0) = s \tag{2.6}$$

regarding t and ω as parameters.

Remark 2.1: If the Lie algebra generated by $X_0, X_1, \cdots, X_r$ is of finite dimension, Ben Arous has proved that the stochastic infinite series in Eq.(2.5) actually converges before a stopping time. Therefore, in the case, the representation of solution (2.4) with (2.5) is well-defined (Theorem 20 in Ben Arous (1989)). We will see such examples in 4.1 and 4.2 of Section 4.

Now, in what follows, we suppose that one may obtain the explicit representation of solution (2.4) with (2.5). Moreover, we restrict ourselves to the SDEs (2.1) with a one-dimensional Wiener process; that is, we consider

$$dS(t) = b(S(t))dt + g(S(t)) \circ dW(t), \tag{2.7}$$

where b and g are d-dimensional vector-valued C^∞ functions. Then, we may rewrite (2.5) in the more simpler form in terms of Kloeden-Platen's representation for multiple Wiener-Stratonovich integrals and multiple Wiener-Itô ones (Kloeden and Platen (1989)); they are defined by

$$J_{(\alpha)}(t,s) = \int_s^{s+t}\int_s^{\tau_k}\cdots\int_s^{\tau_2} \circ dY^{(j_1)}(\tau_1)\cdots \circ dY^{(j_{k-1})}(\tau_{k-1}) \circ dY^{(j_k)}(\tau_k) \tag{2.8}$$

$$I_{(\alpha)}(t,s) = \int_s^{s+t}\int_s^{\tau_k}\cdots\int_s^{\tau_2} dY^{(j_1)}(\tau_1)\cdots dY^{(j_{k-1})}(\tau_{k-1})dY^{(j_k)}(\tau_k), \tag{2.9}$$

respectively, where $\alpha = (j_1, \cdots, j_k)$, $(j_i = 0,1; i = 1, \cdots, k)$ and

$$dY^{(j)}(u) = \begin{cases} du & \text{for } j = 0 \\ dW(u) & \text{for } j = 1 \end{cases}$$

In what follows, we denote $J_{(\alpha)}(t,s)$ and $I_\alpha(t,s)$ by $J_{(\alpha)}(t)$ and $I_\alpha(t)$, respectively, if $s = 0$

Remark 2.2: Note that $J_{(\alpha)}(t,s)$ can be rewritten by $I_{(\alpha)}(t,s)$ (see pp.174-175 in Kloeden and Platen's book (1992)). For example, we have

$$J_{(j_1)} = I_{(j_1)} \quad (j_1 = 0,1) \tag{2.10}$$

$$J_{(j_1,j_2)} = I_{(j_1,j_2)} + \frac{1}{2}1_{\{j_1=j_2=1\}}I_{(0)} \quad (j_1,j_2 = 0,1) \tag{2.11}$$

$$J_{(j_1,j_2,j_3)} = I_{(j_1,j_2,j_3)} + \frac{1}{2}(1_{\{j_1=j_2=1\}}I_{(0,j_3)} + 1_{\{j_2=j_3=1\}}I_{(j_1,0)}) \quad (j_1, j_2, j_3 = 0, 1), \tag{2.12}$$

where $1_{\{\cdot\}}$ denotes the defining function. In general, any multiple Stratonovich integral $J_{(\alpha)}$ can be written as a multiple Itô integral $I_{(\alpha)}$ or a finite sum of $I_{(\alpha)}$ and multiple Itô integrals $I_{(\beta)}$ satisfying

$$\ell(\alpha) + n(\alpha) \leq \ell(\beta) + n(\beta), \tag{2.13}$$

where $\ell(\alpha)$ ={the number of elements of α} and $n(\alpha)$ ={the number of 0 in the elements of α} (Remark 5.2.8 in Kloeden and Platen (1992)).

In terms of (2.8) and stochastic Stratonovich integration by parts formula for $W^i(t)$ $(i = 0, 1)$ and $[W^0, W^1](t)$, we can rewrite the equation (2.5) into the following form:

$$\begin{aligned} Y_t &= J_{(0)}(t)X_0 + J_{(1)}(t)X_1 + \frac{1}{2}(J_{(0,1)}(t) - J_{(1,0)}(t))[X_0, X_1] \\ &+ \frac{1}{18}\{2J_{(0,1,0)}(t) - 2J_{(1,0,0)}(t) + J_{(0)}(t)J_{(1,0)}(t) - J_{(0)}(t)J_{(0,1)}(t)\}[[X_0, X_1], X_0] \\ &+ \frac{1}{18}\{2J_{(0,1,1)}(t) - 2J_{(1,0,1)}(t) + J_{(1)}(t)J_{(1,0)}(t) - J_{(1)}(t)J_{(0,1)}(t)\}[[X_0, X_1], X_1] \\ &+ \frac{1}{36}\{J_{(0)}(t)\}^2[[X_0, X_1], X_1] + \sum_{J;4\leq|J|} K^J(t)X^J. \end{aligned} \tag{2.14}$$

Here in the last term of the right-hand side of the above equation, $K^J(t) = \{\sum_{\triangle J}{}^* c_{\triangle J} W^{\triangle J}(t)\}$, and $J = (j_1, \cdots, j_\ell)$ $(j_i = 0, 1; i = 1, \cdots, \ell; \ell \geq 4)$. In terms of Remark 2.2, this can be described by multiple Wiener-Itô integrals as follows:

$$\begin{aligned} Y_t &= I_{(0)}(t)X_0 + I_{(1)}(t)X_1 + \frac{1}{2}(I_{(0,1)}(t) - I_{(1,0)}(t))[X_0, X_1] \\ &+ \frac{1}{18}\{(2I_{(0,1,0)}(t) - 2I_{(1,0,0)}(t) + I_{(0)}(t)I_{(1,0)}(t) - I_{(0)}(t)I_{(0,1)}(t))[[X_0, X_1], X_0] \\ &+ \frac{1}{18}\{(2I_{(0,1,1)}(t) - 2I_{(1,0,1)}(t) + I_{(1)}(t)I_{(1,0)}(t) - I_{(1)}(t)I_{(0,1)}(t))[[X_0, X_1], X_1] \\ &+ \frac{1}{36}\{I_{(0,0)}(t) + \{I_{(0)}(t)\}^2\}[[X_0, X_1], X_1] + \sum_{J;4\leq|J|} H^J(t)X^J, \end{aligned} \tag{2.15}$$

where $H^J(t)$ is another version of $K^J(t)$ under each multi-index J, which is derived by transforming multiple Stratonovich integrals in $K^J(t)$ into Itô ones through Remark 2.2. Therefore, for each multi-index J, $H^J(t)$ is described as a polynomial function of multiple-Itô integrals. In the next section, we will formulate the numerical schemes of SDE (2.7) on the basis of (2.4) with (2.15).

Now, in the remainder of this section, as a preparation for the error estimation to the new schemes in the next section, we will prove a proposition concerning with the coefficients $H^J(t)$ of X^J for multiple indeces J in (2.15). For this purpose, we first give a lemma for multiple Itô integrals (2.9) proved by Kloeden and Platen (lemma 5.7.5 in Kloeden and Platen (1992); Gard

(1988)). Let $E[\cdot|\mathcal{F}_s]$ be the conditional expectation with respect to a non-decreasing family of σ-subalgebra $\mathcal{F}_s$.

Lemma 2.1 : For any $\alpha = (j_1, \cdots, j_k), (j_i = 0, 1; i = 1, \cdots, k)$ and $q = 1, 2, \cdots,$

$$E[|I_{(\alpha)}(\Delta t, s)|^{2q}|\mathcal{F}_s] = O((\Delta t)^{q(\ell(\alpha)+n(\alpha))}) \quad (\Delta t \downarrow 0), \tag{2.16}$$

where $\ell(\alpha)$ and $n(\alpha)$ are the indices defined in Remark 2.2; that is, $\ell(\alpha)$ ={the number of elements of α} and $n(\alpha)$ ={the number of 0 in the elements of α}

Let $F(t, s)$ be a function of multiple stochastic integrals $I_{(\alpha)}(t, s)$. Suppose that

$$E[\{F(\Delta t, s)\}^2|\mathcal{F}_s] = O((\Delta t)^m) \quad (\Delta t \downarrow 0) \tag{2.17}$$

holds. Then, in what follows, we call the real number m "*mean-square order (MSO)*" of $F(t, s)$.

From Lemma 2.1 we see that MSO of a multiple Itô integral $I_{(\alpha)}(t, s)$ is equal to $\ell(\alpha)+n(\alpha)$. Moreover, the following lemma shows MSO of $I_{(\alpha)}(t, s)I_{(\beta)}(t, s)$ is given by $\ell(\alpha) + n(\alpha) + \ell(\beta) + n(\beta)$; that is, MSO of a product of multiple Itô integrals equals to the sum of MSOs of the each stochastic integral.

Lemma 2.2 : For any multi-indeces α and β,

$$E[|I_{(\alpha)}(\Delta t, s)I_{(\beta)}(\Delta t, s)|^2|\mathcal{F}_s] = O((\Delta t)^{(\ell(\alpha)+n(\alpha)+\ell(\beta)+n(\beta))}) \quad (\Delta t \downarrow 0). \tag{2.18}$$

holds.

Proof: Through Schwartz inequality, we obtain

$$E[|I_{(\alpha)}(\Delta t, s)I_{(\beta)}(\Delta t, s)|^2|\mathcal{F}_s] \leq \sqrt{E[\{I_{(\alpha)}(\Delta t, s)\}^4|\mathcal{F}_s]E[\{I_{(\beta)}(\Delta t, s)\}^4|\mathcal{F}s]}.$$

The lemma is straightforwardly proved by this inequality and Lemma 2.1.

Remark 2.3: By Lemma 2.1 and Lemma 2.2 together with Remark 2.2, we may verify that multiple Stratonovich integrals $J_{(\alpha)}(t, s)$ also satisfy the results in Lemma 2.1 and Lemma 2.2. Hence, for a given α, we find that MSO of a multiple Stratonovich integral $J_{(\alpha)}(t, s)$ is also equal to $\ell(\alpha) + n(\alpha)$; that is, MSO of $J_{(\alpha)}(t, s)$ agrees with that of $I_{(\alpha)}(t, s)$ for the same multi-index α.

We are now to proceed to our purpose. Using Lemma 2.1 and Lemma 2.2, we can estimate MSO of the each coefficient for X^J in (2.15) which is given by a polynomial function of multiple Itô integrals on $[0, t]$. For example, in case of $|J| = 1$, MSOs of the coefficients for X_0 and X_1, that is, MSOs of $I_0(t)$ and $I_1(t)$ are equal to 2 and 1, respectively. In case of $|J| = 2$, MSO of $I_{(0,1)}(t)$ or $I_{(1,0)}(t)$ in the coefficient for $[X_0, X_1]$ is given by 3, and thereby we can easily prove that MSO of the coefficient $(I_{(0,1)}(t) - I_{(1,0)}(t))/2$ itself is also equal to 3. In the same manner, we find MSOs of the coefficients for X^J when $|J| = 3$; that is, MSOs of the coefficients for $[X_0, X_1], X_0]$ and $[X_0, X_1], X_1]$ are given by 5 and 4, respectively. Hence, in this case, *the least value* of MSOs of the coefficients for X^J equals 4. These facts suggest we may verify the following proposition, and it is just one we want.

Proposition 2.1: Suppose that k is a given integer more than or equal to 2, and that the multi-indices J satisfy $|J| = k$; that is, $J = (j_1, \cdots, j_k)$ $(j_i = 0 \text{ or } 1; i = 1, \cdots, k)$. Then the least value of MSOs of the coefficients $H^J(t)$ for X^J in (2.15) is equal to $k + 1$.

Proof: We may prove this by induction. From the above examples, this proposition is obvious in case of $|J| = 2, 3$. We assume that the assertion of this proposition holds for the case of $|J| = \ell (\geq 3)$. That is, the least value of MSOs of the coefficients $H^J(t)$ for X^J under $J = (j_1, \cdots, j_\ell)$ in (2.15) equals $\ell + 1$. Then, note that under for the same J, the least value of MSOs of $K^J(t)$ in (2.14) agrees with that of $H^J(t)$, since $H^J(t)$ is only another version of $K^J(t)$ in terms of multiple Itô integrals.

Now, let us consider the coefficients $K^{\tilde{J}}(t)$ for $X^{\tilde{J}}$ under $|\tilde{J}| = \ell + 1$. According to Kunita (1980) pp.285-289, one can obtain them by adding Stratonovich integral with respect to dw or dt to the multiple Stratonovich integrals in the coefficients $K^J(t)$ for $|J| = \ell$. From the assumption, the least value of MSOs of $K^J(t)$ for $|J| = \ell$ is equal to $\ell + 1$. On the other hand, the following equations show that the MSOs for the integrals by increments dw and dt correspond to 1 and 2, respectively:

$$E[|\int_s^{s+\Delta t} dw(\tau)|^2 | \mathcal{F}_s] = O(\Delta t), \quad E[|\int_s^{s+\Delta t} dt|^2 | \mathcal{F}_s] = O((\Delta t)^2) \ (\Delta t \downarrow 0).$$

Hence, in consideration these facts, one see that the least value of MSOs of $K^{\tilde{J}}(t)$ is given by $\ell + 1 + 1 = \ell + 2$, and thereby, the least value of MSOs of $H^{\tilde{J}}(t)$ is also so. Thus, the assertion in our proposition is proved.

3 Composition methods for numerical integration of SDEs

Now, we proceed to the new stochastic numerical schemes of SDEs on the basis of composition methods. We start with a numerical integration of the stochastic equation (2.7) on the discretized time series in the framework of the previous results on representation of solutions to SDEs. It adopts an equidistant discretization of the time interval $[0, T]$ with stepsize

$$\Delta t = \frac{T}{N}$$

for fixed natural number N. Let $t_n = n\Delta t (n = 0, 1, 2, \cdots, N)$ be the n-th step-point. Then, for all $n \in \{0, \cdots, N\}$, we abbreviate $S_n = S(t_n)$. Moreover, we use ΔW_n for $n = 0, 1, \cdots, N$ to denote the increments $W(t_{n+1}) - W(t_n)$; they are independent Gaussian random variables with mean 0 and variance Δt, that is, $N(0, \Delta t)$-distributed random variables.

On account of (2.4), we may find the numerical solutions $S_n (n = 0, 1, \cdots, N)$ to SDE (2.7) by

$$S_{n+1} = \exp(Y_{n\Delta t})(S_n) \ (n = 0, 1, 2, \cdots N-1), \tag{3.1}$$

formally, where $Y_{n\Delta t}$ is a vector field derived by replacing all the multiple Wiener integrals $I_{(\alpha)}(t) = I_{(\alpha)}(t, 0)$ in (2.15) by $I_{(\alpha)}(\Delta t, n\Delta t)$. In the followings, $I_{(\alpha)}(\Delta t, n\Delta t)$ is denoted by $I_{(\alpha),n}(\Delta t)$. Moreover, we set $S_0 = S(0) = s_0$. According to the theory of ordinary differential equations, $\exp(Y_{n\Delta t})(\cdot)$ is often called the time-Δt map or exponential map. However, it is usually difficult to find out the explicit form of the exponential map, and hence, we need to build an approximation for (3.1).

To carry out this, we formulate a new stochastic numerical scheme as the following two procedures, which are composed of the truncation of the vector field (2.15) and a composition method (or operator splitting method) to the exponential map derived from the truncated vector field:

Procedure 1: For the vector field Y_t described by (2.15), we define a "truncated"vector field $\hat{Y}_t$ which is given by a truncation of the higher-order terms with respect to MSO of the coefficients for X^J in (2.15). Then, we define a numerical sequence $(\hat{S}_n)_{n=0}^N$ through

$$\hat{S}_{n+1} = \exp(\hat{Y}_{n\Delta t})(\hat{S}_n) \ (n = 0, 1, \cdots .N-1), \tag{3.2}$$

where $\hat{S}_0 = S(0) = s_0$.

Procedure 2: For $\hat{S}_{n+1} = \exp(\hat{Y}_{n\Delta t})(\hat{S}_n)$, we apply a "composition method "in a way analogous to that in the theory of ordinary differential equations. Suppose that the vector field $\hat{Y}_{n\Delta t}$ is of the form

$$\hat{Y}_{n\Delta t} = A_{n\Delta t} + B_{n\Delta t}, \tag{3.3}$$

where $\exp(A_{n\Delta t})$ and $\exp(B_{n\Delta t})$ can both be explicitly calculated through (2.6). Then an approximation to the exponential map $\hat{Y}_{n\Delta t}$ is given by $\exp(A_{n\Delta t})\exp(B_{n\Delta t})$. Hence, the sequence of $(\hat{S}_n)_{n=0}^N$ in Procedure 1 is approximated by

$$\tilde{S}_{n+1} = \exp(A_{n\Delta t})\exp(B_{n\Delta t})(\tilde{S}_n), \ (n = 0, 1, \cdots N-1), \tag{3.4}$$

where $\tilde{S}_0 = S(0) = s_0$.

After all, we regard $(\tilde{S}_n)_{n=0}^N$ as a numerical approximation to the exact discretized solutions $(S_n)_{n=0}^N$.

Next, we will turn to estimate local errors of mean-square sense for the numerical approximation scheme mentioned above. For this purpose, in this paper, we use the notion of "local error order "defined by Saito and Mitsui (1993).

Definition 3.1: Suppose that $S(t)$ and $\bar{S}_n$ are an exact solution and the numerical approximation to SDE (2.7), respectively. Moreover, let $E_{\tau,\xi}$ be the expectation conditioned on starting at ξ at time τ. Then *the local error order* α is defined by

$$E_{t_n,s}[|\bar{S}_{n+1} - S(t_{n+1})|^2] = O((\Delta t)^{\alpha+1}),\ (\Delta t \downarrow 0), \tag{3.5}$$

where $t_k = k\Delta t$ $(k = n, n+1)$, $|\cdot|$ denotes the Euclidean norm on the space R^d, and we set $S(t_n) = \bar{S}_n = s$.

We note that the accuracy of a numerical scheme improves with increasing the local order.

Remark 3.1: In the framework of another definition of local error order by Kloeden and Platen (1992), the local order of $\bar{S}_n$ satisfying (3.5) is given by $(\alpha+1)/2$, since the difference of $\bar{S}_{n+1}$ and $S(t_{n+1})$ is squared. Hence, if the above-mentioned local order for a certain numerical scheme is equal to α, one can calculate Kloeden and Platen's local error order β through $\beta = (\alpha+1)/2$.

Now, we will to apply thus local error estimation to our approximation procedures. In what follows, we set $S_n = s$, where s is a given value.

Local error estimation for the truncation error in Procedure 1:

First, we investigate a truncation error in Procedure 1. Let ${H_n}^J(\Delta t)$ be the coefficient for X^J in $Y_{n\Delta t}$ which is represented by a polynomial function of multiple-Itô integrals for a given multi-index J as in (2.15).

Proposition 3.1: Suppose that a truncation vector field $\hat{Y}_{n\Delta t}$ is given in the following form:

$$\hat{Y}_{n\Delta t} = \sum_{J;1\le|J|\le\gamma} {H_n}^J(\Delta t)X^J. \tag{3.6}$$

That is, we assume the terms in $Y_{n\Delta t}$ satisfying $|J| \ge \gamma+1$ are neglected. Then,

$$E_{t_n,s}[|\hat{S}_{n+1} - S_{n+1}|^2] = O((\Delta t)^{\gamma+2}),\ (\Delta t \downarrow 0). \tag{3.7}$$

Proof: In terms of Proposition 2.1, we can easily show that the least value of MSOs of ${H_n}^J(\Delta t)$ in the neglected terms equals $\gamma+2$, since $|J| \ge \gamma+1$. This fact together with the definition of exponential map (2.15) straightforwardly indicate (3.7).

Thus, we obtain the local order $\gamma+1$ for numerical approximation solutions $(\hat{S}_n)_{n=0}^N$ in the sense of Definition 3.1.

Remark 3.2: If the Lie algebra generated by X_0 and X_1 is of finite dimension, our error estimation mentioned above agree with that on truncation of stochastic exponential maps by

Ben Arous (1989), since the convergence of Eq.(2.14) (or (2.15)) is actually guaranteed (*cf.* Remark 2.1). That is, under the assumption that such a convergence holds, the local error estimation mentioned above exactly holds. In general, however, our result may give only a formal error estimation. Indeed, according to Castell (1993), when the stochastic series does not always converge, the asymptotic expansion of stochastic exponential maps is estimated only in a "probability "sense. We will investigate this problem in future works.

Local error estimation for the composition scheme in Procedure 2:

Next, we will proceed to the local error estimation for Procedure 2. We can carry out it by the Baker-Campbell-Hausdorff (BCH) formula (Bourbaki (1989)) together with Lemma 2.2; the formula is given by the following form:

$$\begin{aligned}\exp\left(\epsilon(\Delta t)X\right)\exp\left(\delta(\Delta t)Y\right) &= \exp(\epsilon(\Delta t)X+\delta(\Delta t)Y+\frac{1}{2}\epsilon(\Delta t)\delta(\Delta t)[X,Y] \\ &+\frac{1}{12}(\epsilon(\Delta t)^2\delta(\Delta t)[X,[X,Y]]+\epsilon(\Delta t)\delta(\Delta t)^2[Y,[Y,X]])+\cdots),\end{aligned} \tag{3.8}$$

where X and Y are C^∞ vector fields, and $\epsilon(\Delta t)$ and $\delta(\Delta t)$ are any functions with respect to Δt; in our case, they are corresponding to polynomial functions of multiple Itô stochastic integrals $I_{(\alpha),n}(\Delta t)$.

Proposition 3.2: Let $\hat{Y}_{n\Delta t}$ be a truncated vector field given by (3.6). Suppose that the vector fields $A_{n\Delta t}$ and $B_{n\Delta t}$ in a decomposition (3.3) for $\hat{Y}_{n\Delta t}$ are described by

$$A_{n\Delta t}=\sum_{J;1\le|J|\le\gamma}{F_n}^J(\Delta t)X^J,\quad B_{n\Delta t}=\sum_{J;1\le|J|\le\gamma}{G_n}^J(\Delta t)X^J, \tag{3.9}$$

respectively, and that the least values of MSOs of ${F_n}^J(\Delta t)$ and ${G_n}^J(\Delta t)$ in (3.9) are given by α and β, respectively. If $X_A^{J_\alpha}$ and $X_B^{J_\beta}$, which are vector fields corresponding to the coefficients with α and β as MSO, respectively, satisfy $[X_A^{J_\alpha},X_B^{J_\beta}]\neq 0$, then

$$E_{t_n,s}[|\tilde{S}_{n+1}-\hat{S}_{n+1}|^2]=O((\Delta t)^{\alpha+\beta}),\ (\Delta t\downarrow 0). \tag{3.10}$$

Proof: Let ${F_n}^{J_\alpha}(\Delta t)$ and ${G_n}^{J_\beta}(\Delta t)$ be the coefficients in (3.9) of which MSOs are equal to α and β, respectively. Then, in an analogous way to that in Lemma 2.2, we can prove that

$$E_{t_n,s}[|{F_n}^{J_\alpha}(\Delta t){G_n}^{J_\beta}(\Delta t)|^2]=O((\Delta t)^{\alpha+\beta})\ (\Delta t\downarrow 0). \tag{3.11}$$

Therefore, on account of BCH formula (3.8), (3.11) and the assumption of α and β, one may find that

$$\begin{aligned}E_{t_n,s}[|\tilde{S}_{n+1} &- \hat{S}_{n+1}|^2]=E_{t_n,s}[|\exp\left(A_{n\Delta t}+B_{n\Delta t}\right)(s)-\exp\left(A_{n\Delta t}\right)\exp\left(B_{n\Delta t}\right)(s)|^2] \\ &= E_{t_n,s}[|\exp\left(A_{n\Delta t}+B_{n\Delta t}\right)(s)-\exp\left(A_{n\Delta t}+B_{n\Delta t}+\frac{1}{2}[A_{n\Delta t},B_{n\Delta t}]+\cdots\right)(s)|^2] \\ &= O((\Delta t)^{\alpha+\beta}).\end{aligned} \tag{3.12}$$

Thus, the local order between $(\tilde{S}_n)_{n=0}^N$ and $(\hat{S}_n)_{n=0}^N$ is given by $\alpha+\beta-1$.

Remark 3.3: By further manipulating the BCH formula to eliminate higher order terms, we can obtain the schemes which give higher-order approximations to the exponential map. For example, the scheme corresponding to "leapfrog ", which is well-known in deterministic numerical analysis, is given by

$$\exp((\Delta t)(X+Y)) = \exp(\frac{\Delta tY}{2})\exp(\Delta tX)\exp(\frac{\Delta tY}{2}) + O((\Delta t)^3). \tag{3.13}$$

In a way analogous to that in (3.4), we define a stochastic leapfrog scheme as follows:

$$\tilde{S}_{n+1} = \exp(\frac{B_{n\Delta t}}{2})\exp(A_{n\Delta t})\exp(\frac{B_{n\Delta t}}{2})(\tilde{S}_n), \ (n=0,1,\cdots,N-1). \tag{3.14}$$

Then, using BCH formula (3.8) and (3.11) repeatedly, we can estimate the local error for this scheme as follows:

$$E_{t_n,s}[|\tilde{S}_{n+1}-\hat{S}_{n+1}|^2] = O((\Delta t)^{\alpha+2\beta}). \tag{3.15}$$

Thus, we can make another numerical scheme having the better local order than that of (3.4). Moreover, using this scheme as a basis element for further leapfrog schemes, we may also produce an approximation to exponential map up to any order in a similar way to that in ordinary numerical analysis. This will be investigated in the future work.

Total local error estimation for the numerical scheme by Procedure 1 and 2: Finally, we estimate the local error order between the exact discretized solutions $(S_n)_{n=0}^N$ and the numerical approximation solutions $(\tilde{S}_n)_{n=0}^N$. This is easily carried out by using Proposition 3.1 and Proposition 3.2 (or Remark 3.3) for the local orders in the above two procedures together with

$$E_{t_n,s}[|S_{n+1}-\tilde{S}_{n+1}|^2] \le E_{t_n,s}[|S_{n+1}-\hat{S}_{n+1}|^2] + E_{t_n,s}[|\hat{S}_{n+1}-\tilde{S}_{n+1}|^2], \tag{3.16}$$

and thereby, we obtain the following theorem:

Theorem 3.1: Under the conditions of Proposition 3.1 and Proposition 3.2,

$$E_{t_n,s}[|S_{n+1}-\tilde{S}_{n+1}|^2] \le O((\Delta t)^{\delta}) \ (\Delta t \downarrow 0) \tag{3.17}$$

holds, where $\delta = \min(\alpha+\beta, \gamma+2)$ in case of (3.4) and $\delta = \min(\alpha, \beta+2\gamma)$ in case of (3.14).

We call a value of $\delta - 1$ the local order "*of weak sense* "for the scheme giving the numerical approximation solutions $(\tilde{S}_n)_{n=0}^N$, since the estimation of error order is indirectly derived from the inequality (3.16).

In the followings, we will investigate some examples of new numerical schemes for (2.7) which are derived from the procedures mentioned above, and estimate the local error on the basis of Theorem 3.1.

Example 3.1: Suppose that a truncated vector field $\hat{Y}_{n\Delta t}$ in Procedure 1 is given by

$$\begin{aligned}\hat{Y}_{n\Delta t} &= I_{(0),n}(\Delta t)X_0 + I_{(1),n}(\Delta t)X_1, \\ &= \Delta t X_0 + \Delta W_n X_1. \end{aligned} \tag{3.18}$$

On account of (2.15), we see that γ in Proposition 3.1 for this truncated vector field equals 1. We further set $A_{n\Delta t} = \Delta t X_0$ and $B_{n\Delta t} = \Delta W_n X_1$ in the decomposition (3.3), and assume that the explicit forms of both exponential maps for them are obtained through (2.6). In this case, α and β in Proposition 3.2 become 2 and 1, respectively, because of Lemma 2.1. Then, the scheme (3.4) is put into the following form:

Scheme 3.1:

$$\tilde{S}_{n+1} = \exp(\Delta t X_0)\exp(\Delta W_n X_1)(\tilde{S}_n). \tag{3.19}$$

Assume that $[X_0, X_1] \neq 0$. Then, Theorem 3.1 indicates

$$E_{t_n,s}[|S_{n+1} - \tilde{S}_{n+1}|^2] \leq O((\Delta t)^3). \tag{3.20}$$

Thus, we find that the local order of weak sense for Scheme 3.1 equals 2, and this result shows that the accuracy of this scheme corresponds to that of the stochastic Taylor schemes of local order 2 (Saito and Mitsui (1993)).

Example 3.2: For $\hat{Y}_n(\Delta t)$ in Example 3.1, we set $A_{n\Delta t} = \Delta t {X^A}_0 + \Delta W_n {X^A}_1$ and $B_{n\Delta t} = \Delta t {X^B}_0 + \Delta W_n {X^B}_1$ in (3.3), where $X_0 = {X^A}_0 + {X^B}_0$ and $X_1 = {X^A}_1 + {X^B}_1$. We assume that $[{X^A}_1, {X^B}_1] \neq 0$ and that the explicit forms of both exponential maps for them are obtained. In this case, α and β in Proposition 3.2 become 1 and 1, respectively, and hence the local error order of (3.4) becomes 1; the accuracy for the scheme corresponds to that of Euler-Maruyama scheme (Saito and Mitsui 1993). In order to make a scheme having better accuracy than that of it, we use (3.14) instead of (3.4):

Scheme 3.2:

$$\tilde{S}_{n+1} = \exp\left(\frac{B_{n\Delta t}}{2}\right)\exp(A_{n\Delta t})\exp\left(\frac{B_{n\Delta t}}{2}\right), \tag{3.21}$$

where $A_{n\Delta t} = \Delta t {X^A}_0 + \Delta W_n {X^A}_1$ and $B_{n\Delta t} = \Delta t {X^B}_0 + \Delta W_n {X^B}_1$ under $X_0 = {X^A}_0 + {X^B}_0$ and $X_1 = {X^A}_1 + {X^B}_1$.

Then, Theorem 3.1 indicates (3.20) also holds in this case; that is, the local error order of this scheme is equal to 2. This means that the accuracy of Scheme 3.2 corresponds to that of Taylor schemes of local order 2.

Example 3.3: We will make a scheme with more better accuracy than that of the schemes mentioned above. For this purpose, we choose the following vector field as $\hat{Y}_{n\Delta t}$ in (3.6):

$$\hat{Y}_{n\Delta t} = \Delta t X_0 + \Delta W_n X_1 + \frac{1}{2}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t))[X_0, X_1]. \tag{3.22}$$

Then, from (2.15), we see that γ in Proposition 3.1 for this truncated vector field becomes 2. Moreover, we set

$$\exp(A_{n\Delta t}) = \exp(\Delta t X_0) \tag{3.23}$$

and

$$\exp(B_{n\Delta t}) = \exp(\Delta W_n X_1 + \frac{1}{2}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t))[X_0, X_1]). \tag{3.24}$$

Assume that the explicit forms of both exponential maps for them are obtained through (2.6), respectively. In this case, α and β in Proposition 3.2 become 2 and 1, respectively, because of Lemma 2.1. Moreover, we adopt the scheme (3.14) for these vector fields:

Scheme 3.3:

$$\tilde{S}_{n+1} = \exp(\frac{B_{n\Delta t}}{2}) \exp(A_{n\Delta t}) \exp(\frac{B_{n\Delta t}}{2})(\tilde{S}_n), \tag{3.25}$$

where $\exp(A_{n\Delta t})$ is given by (3.23) and $\exp(B_{n\Delta t}/2)$ is derived from replacing $B_{n\Delta t}$ by $B_{n\Delta t}/2$ in (3.24).

Then, because of Theorem 3.1, we find that

$$E_{t_n,s}[|S_{n+1} - \tilde{S}_{n+1}|^2] \leq O((\Delta t)^4), \tag{3.26}$$

and hence that the local order of weak sense for Scheme 3.3 equals 3.

4 Examples

In this section, we will give several examples of applying our new stochastic numerical schemes to stochastic differential equations concretely.

4.1 Numerical simulation to a non-linear asset price process in mathematical finance

As was mentioned in Section 1, we first work with the following non-linear scalar SDE which is often treated as a model of an asset price process of Bessel type in mathematical finance (Geman and Yor (1993); *cf.* Remark 4.1):

$$dS(t) = S(t)dt + 2\sqrt{S(t)} \circ dW(t), \quad S(0) = s(> 0). \tag{4.1}$$

This system has a structure that the value of solution becomes to be "non-negative"for any $t \in [0, T]$. In standard stochastic numerical schemes, however, this is not always preserved numerical- ly; especially, if an initial value s is close to zero, the numerical solutions often go into the domain of negative values in the midst of numerical simulations. Such a trouble will be observed in the numerical results mentioned later. In contrast with this, through the results in previous section, we may obtain the scheme which leaves the structure of the stochastic system (4.1) invariant numerically. We will examine it.

First, from the equations (2.3) and (4.1), we see that the vector fields X_0 and X_1 become

$$X_0 = S\frac{d}{dS}, \; X_1 = 2\sqrt{S}\frac{d}{dS}, \tag{4.2}$$

respectively. Here, we note that $[X_0, X_1] = -X_1/2$, and the Lie algebra generated by X_0 and X_1 is of finite dimension. Hence, Remark 2.1 indicates that Eq.(2.14) and (2.15) actually converge in this case.

We proceed to investigate Scheme 3.1 in Example 3.1 to SDE (4.1). On account of (3.19), we suppose that $A_{n\Delta t}$ and $B_{n\Delta t}$ in (3.3) are given by

$$A_{n\Delta t} = \Delta t X_0 = \Delta t S\frac{d}{dS}, \quad B_{n\Delta t} = \Delta W_n X_1 = \Delta W_n 2\sqrt{S}\frac{d}{dS}. \tag{4.3}$$

Then, in consideration of (2.6), we obtain the exponential maps for $A_{n\Delta t}$ and $B_{n\Delta t}$ explicitly as follows:

$$\exp(A_{n\Delta t})(s) = s\exp(\Delta t), \quad \exp(B_{n\Delta t})(s) = \{\Delta W_n + \sqrt{s}\}^2. \tag{4.4}$$

Inserting them into (3.19), we find that Scheme 3.1 for SDE (4.1) is given by

$$\tilde{S}_{n+1} = \{\Delta W_n + \sqrt{\tilde{S}_n}\}^2 \exp(\Delta t), \tag{4.5}$$

where $\tilde{S}_0 = S(0) = s$. Evidently, the numerical solutions derived from our scheme "never "take negative values, and this is just a result we want.

Next we will obtain Scheme 3.3 for (4.1). On account of (3.23), in this case, we also obtain $s\exp(\Delta t)$ as $\exp(A_{n\Delta t})(s)$. In contrast with this, the equation (3.24) is put into

$$\exp(B_{n\Delta t}) = \exp\{(\Delta W_n - \frac{1}{4}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t)))X_1\}, \tag{4.6}$$

since $[X_0, X_1] = -X_1/2$. In similar to way in that of Scheme 3.1, this is also calculated explicitly as follows:

$$\exp(B_{n\Delta t})(s) = \{\Delta W_n - \frac{1}{4}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t)) + \sqrt{s}\}^2, \tag{4.7}$$

and thereby we obtain Scheme 3.3 for the SDE (4.1) as

$$\tilde{S}_{n+1} = \{\frac{\Delta W_n}{2} - \frac{1}{8}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t)) + \sqrt{\hat{s}}\}^2 \tag{4.8}$$

together with

$$\hat{s} = \{\frac{\Delta W_n}{2} - \frac{1}{8}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t)) + \sqrt{\tilde{S}_n}\}^2 \exp(\Delta t), \tag{4.9}$$

where $\tilde{S}_0 = S(0) = s$. This also indicates that the numerical solutions derived from this scheme take non-negative values.

Now, we proceed to the numerical simulations of (4.1) on the basis of our schemes. As mentioned in Section 3, Theorem 3.1 indicates the schemes (4.5) and (4.8) with (4.9) have the local order of weak sense 2 and 3, respectively. To examine these results, we will compare the numerical accuracy of our schemes with that of the standard numerical schemes. For this purpose, we adopt here Euler-Maruyama scheme (Taylor scheme of local order 1) and Kloeden's Taylor scheme of local order 3 (Kloeden and Platen (1992), Saito and Mitsui (1993)); they are given in the following forms for the SDE (4.1):

Euler-Maruyama scheme:

$$S_{n+1} = S_n + (S_n + 1)\Delta t + 2\sqrt{S_n}\Delta W_n. \tag{4.10}$$

Kloeden's Taylor scheme of local order 3:

$$\begin{aligned} S_{n+1} &= S_n + (S_n+1)\Delta t + 2\sqrt{S_n}\Delta W_n \\ &+ \{(\Delta W_n)^2 - \Delta t\} \\ &+ 2\sqrt{S_n} I_{(1,0),n}(\Delta t) + \sqrt{S_n} I_{(0,1),n}(\Delta t) \\ &+ \frac{1}{2}(S_n+1)(\Delta t)^2. \end{aligned} \tag{4.11}$$

In the schemes (4.5), (4.8) with (4.9), (4.10) and (4.11), ΔW_n, $I_{(1,0),n}(\Delta t)$ and $I_{(0,1),n}(\Delta t)$ are numerically realized by the independent $N(0,1)$ random numbers γ_n and $\hat{\gamma}_n$ ($n = 0, 1, \cdots$) as follows (Kloeden and Platen (1992)):

$$\begin{aligned} \Delta W_n &= \gamma_n \sqrt{\Delta t} \\ I_{(1,0),n}(\Delta t) &= \frac{1}{2}(\gamma_n + \frac{1}{\sqrt{3}}\hat{\gamma}_n)(\Delta t)^{3/2} \\ I_{(0,1,n)}(\Delta t) &= \frac{1}{2}(\gamma_n - \frac{1}{\sqrt{3}}\hat{\gamma}_n)(\Delta t)^{3/2}. \end{aligned} \tag{4.12}$$

Moreover, we here choose $T = 1$ and $N = 1000$, and hence the stepsize $\Delta t = 10^{-3}$.

Table 4.1 and Table 4.2 indicate the examples of the numerical solutions from these schemes mentioned above (in case of Table 4.2, those schemes except (4.10)) with the initial value $s = 0.01$ and $s = 0.001$, respectively. Here we have used the same sequences of random numbers for each scheme together with (4.12). As was mentioned in the introductory part of this section, from these results we observe that the values of numerical solutions derived from the standard schemes become to be negative in midst of their simulations, if their initial values are close to zero; in contrast with these results, our each scheme is free from such a trouble. Thus, our scheme (4.5) and (4.8) with (4.9) have a superiority with respect to numerical realization of the character of (4.1), that is, of non-negativity of solutions than the standard schemes.

Table 4.3 indicates the results with the initial value $s = 1$. By the estimation of local order with respect to these schemes, we see that the accuracy for the scheme (4.5) corresponds to that for Taylor scheme of local order 2, and the accuracy for the scheme (4.8) with (4.9) corresponds to Taylor scheme of local order 3. We may consider that Table 4.3 supports such a result (note that Euler scheme is just a Taylor scheme of local order 1).

Remark 4.1: Let us consider the following SDE:

$$dS(t) = S(t)dt + \frac{1}{1-\gamma}\{S(t)\}^{\gamma} \circ dW(t), \quad S(0) = s(>0),$$

where $0 < \gamma < 1$. This is also often treated as a model of an asset price process in mathematical finance, which is a generalization of (4.1). For this process, we can also construct the numerical schemes as mentioned above in a similar way. Indeed, Scheme 3.1 for this SDE, of which local order equals 2, is given by

$$\tilde{S}_{n+1} = [\{\Delta W_n + \tilde{S}_n^{1-\gamma}\}^2]^{1/\{2(1-\gamma)\}} \exp(\Delta t),$$

where $\tilde{S}_0 = S(0) = s$. Note that the numerical solutions derived from this scheme also satisfy "non-negativity ".

4.2 Example of Scheme 3.2 for a non-linear SDE

We now turn into the example of Scheme 3.2 given by (3.21). Let us consider the following non-linear scalar SDE:

$$dS(t) = S(t)dt + \{S(t) + 2\sqrt{S(t)}\} \circ dW(t), \quad S(0) = s(>0). \tag{4.13}$$

In this case, the vector fields X_0 and X_1 are set by

$$X_0 = S\frac{d}{dS}, \ X_1 = (S + 2\sqrt{S})\frac{d}{dS} \tag{4.14}$$

respectively. Then, we remark that $[X_0, X_1] = -\sqrt{S}(d/dS)$, $[X_0, [X_0, X_1]] = -[X_0, X_1]/2$ and $[X_1, [X_0, X_1]] = -[X_0, X_1]/2$ hold, respectively; hence the Lie algebra generated by X_0 and X_1 is of finite dimension. Hence, as in 4.1, Eq.(2.14) and (2.15) also actually converge in this case.

We may regard the SDE (4.13) as a linear SDE with the random perturbation $2\sqrt{S(t)}\circ dW(t)$. On account of this, as $A_{n\Delta t}$ and $B_{n\Delta t}$ in Example 3.2, we adopt

$$A_{n\Delta t} = \Delta t S\frac{d}{dS} + \Delta W_n S\frac{d}{dS}, \quad B_{n\Delta t} = \Delta W_n 2\sqrt{S}\frac{d}{dS}; \tag{4.15}$$

that is, we set $X_0^A = S(d/dS)$, $X_1^A = S(d/dS)$, $X_0^B = 0$ and $X_1^B = 2\sqrt{S}(d/dS)$. Then, in consideration of (2.6), we obtain the exponential maps for them explicitly as follows:

$$\exp(A_{n\Delta t})(s) = s\exp(\Delta t + \Delta W_n), \quad \exp(B_{n\Delta t})(s) = \{\Delta W_n + \sqrt{s}\}^2. \tag{4.16}$$

Inserting these equations into (3.21), we find Scheme 3.2 for the SDE (4.13); it is given by

$$\tilde{S}_{n+1} = \{\Delta W_n/2 + \sqrt{\{\Delta W_n/2 + \sqrt{\tilde{S}_n}\}^2 \exp(\Delta t + \Delta W_n)}\}^2, \tag{4.17}$$

where $\tilde{S}_0 = S(0) = s$.

Now, in a similar way to that in Subsection 4.1, we will compare the numerical accuracy of this scheme with that of the standard numerical schemes for (4.13). In this case, Euler-Maruyama scheme and Kloeden's Taylor scheme of local order 3 are written in the following forms:

Euler-Maruyama scheme:

$$S_{n+1} = S_n + \{\frac{3}{2}(S_n + \sqrt{S_n}) + 1\}\Delta t + (S_n + 2\sqrt{S_n})\Delta W_n. \tag{4.18}$$

Taylor scheme of local order 3:

$$\begin{aligned} S_{n+1} &= S_n + \{\frac{3}{2}(S_n + \sqrt{S_n}) + 1\}\Delta t + (S_n + 2\sqrt{S_n})\Delta W_n \\ &+ \frac{1}{2}(S_n + 3\sqrt{S_n} + 2)\{(\Delta W_n)^2 - \Delta t\} \\ &+ \frac{3}{2}(S_n + \frac{5}{2}\sqrt{S_n} + 1)I_{(1,0),n}(\Delta t) + \frac{3}{2}(S_n + \frac{11}{6}\sqrt{S_n} + 1)I_{(0,1),n}(\Delta t) \\ &+ \frac{1}{6}(S_n + \frac{7}{2}\sqrt{S_n} + 3)\{(\Delta W_n)^3 - 3\Delta t\Delta W_n\} \\ &+ \frac{9}{8}(S_n + \frac{17}{12}\sqrt{S_n} + \frac{5}{6})(\Delta t)^2. \end{aligned} \tag{4.19}$$

Therefore, inserting (4.12) into the schemes (4.17)-(4.19), we obtain numerical solutions through the schemes.

Table 4.4 shows the results with the initial value $s = 1$, $T = 1$, $N = 1000$ and $\Delta t = 10^{-3}$. According to Theorem 3.1 and Example 3.2, we may expect the accuracy for our scheme (4.17) is corresponding to that for Taylor scheme of local order 2. Table 4.4 indicates that such an expectation is practically valid.

4.3 Composition method to stochastic Hamilton dynamical systems

As mentioned in Section 1, composition methods (or operator splitting methods) are not only a superior integrating method for differential equations in preserving the special character or structure of the equations, but also often useful for approximations of non-linear equations of which solutions are not obtained explicitly. The examples of 4.1 and 4.2 mentioned above show that these facts are also true in case of stochastic systems. As also mentioned, such a advantage is remarkable in case of dynamical systems with multiple space dimensions or Hamilton dynamical systems as standing for dimensional splitting methods (*e.g.* Yanenko (1959); Iserles (1984)). In consideration of this, we will investigate numerical schemes by composition methods for stochastic dynamical systems with "Hamiltonian structure "(Misawa (1999),(2000)).

First we review stochastic Hamilton dynamical systems (Misawa (1999)). Let us consider the following 2ℓ-dimensional stochastic dynamical systems:

$$d\begin{pmatrix} x^i(t) \\ x^{\ell+i}(t) \end{pmatrix} = \begin{pmatrix} \partial_{\ell+i}H_0(x(t)) \\ -\partial_i H_0(x(t)) \end{pmatrix} dt + \begin{pmatrix} \partial_{\ell+i}H_1(x(t)) \\ -\partial_i H_1(x(t)) \end{pmatrix} \circ dW(t), \quad (i = 1, \cdots \ell) \tag{4.20}$$

where $x = (x^k)_{k=1}^{2\ell}$ and $\partial_j = \partial/\partial x^j$ $(j = 1, 2, \cdots 2\ell)$, respectively. In (4.20), $H_\alpha(x)$ $(\alpha = 0, 1)$ are smooth scalar functions on $R^{2\ell}$. Formally, one may regard this as a Hamilton dynamical system

$$\frac{d}{dt}\begin{pmatrix} x^i \\ x^{\ell+i} \end{pmatrix} = \begin{pmatrix} \partial_{\ell+i}\hat{H}(x) \\ -\partial_i\hat{H}(x), \end{pmatrix} \quad (i = 1, \cdots, \ell)$$

with a "randomized "Hamiltonian $\hat{H}$ given by

$$\hat{H} = H_0 + H_1\gamma_t,$$

where γ_t is a one-dimensional Gaussian white noise. On account of this fact, we call (4.20) and H_α $(\alpha = 0, 1)$ an (ℓ-dimensional) *stochastic Hamilton dynamical system* and the *Hamiltonian.*

Now, we proceed to an example of our new scheme for stochastic Hamilton systems. For simplicity, we set $\ell = 1$ and denote $x^1(t)$ and $x^2(t)$ by $q(t)$ and $p(t)$, respectively. Let us consider the class of Hamilton systems with the typical Hamiltonian $H_0 = p^2/2 + V_0(q)$ and $H_1 = p^2/2 + V_1(q)$, where $V_0(q)$ and $V_1(q)$ are any potential functions. Then the equation (4.20) turns to be

$$d\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} p(t) \\ -V_0'(q(t)) \end{pmatrix} dt + \begin{pmatrix} p(t) \\ -V_1'(q(t)) \end{pmatrix} \circ dW(t). \tag{4.21}$$

In general, this is a stochastic "non-linear"system. For this system, the vector fields X_0 and X_1 become

$$X_0 = p\partial_q - V_0'(q)\partial_p, \quad X_1 = p\partial_q - V_1'(q)\partial_p, \tag{4.22}$$

respectively.

We are to apply our scheme to this system. As an important example of the decomposition of (3.3) for the above system, we choose the following splitting of Scheme 3.2 type:

$$A_{n\Delta t} = p(\Delta t + \Delta W_n)\partial_q, \quad B_{n\Delta t} = -(V_0'(q)\Delta t + V_1'(q)\Delta W_n)\partial_p. \tag{4.23}$$

This corresponds to the decomposition mentioned in Example 3.2; that is, X_0^A, X_1^A, X_0^B and X_1^B in Example 3.2 are given by $p\partial_q$, $p\partial_q$, $-V_0'(q)\partial_p$ and $-V_1'(q)\partial_p$, respectively. Then we note that $\exp(A_{n\Delta t})$ and $\exp(B_{n\Delta t})$ are exponential maps which correspond to the flows of solutions to the following SDEs, respectively:

$$d\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} p(t) \\ 0 \end{pmatrix} dt + \begin{pmatrix} p(t) \\ 0 \end{pmatrix} \circ dW(t).$$

$$d\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -V_0'(q(t)) \end{pmatrix} dt + \begin{pmatrix} 0 \\ -V_1'(q(t)) \end{pmatrix} \circ dW(t).$$

Therefore, we can obtain the explicit forms of them; this may be regarded as an example of dimensional splitting. The results are given by

$$\exp(A_{n\Delta t})\begin{pmatrix} q_n \\ p_n \end{pmatrix} = \begin{pmatrix} p_n(\Delta t + \Delta W_n) + q_n \\ p_n \end{pmatrix}$$

$$\exp(B_{n\Delta t})\begin{pmatrix} q_n \\ p_n \end{pmatrix} = \begin{pmatrix} q_n \\ -(\Delta t V_0'(q_n) + \Delta W_n V_1'(q_n)) + p_n \end{pmatrix}.$$

Inserting these equations into (3.21), we finally find the numerical scheme 3.2 for this system as follows:

$$\begin{pmatrix} \tilde{q}_{n+1} \\ \tilde{p}_{n+1} \end{pmatrix} = \begin{pmatrix} \hat{q}_n \\ -\frac{1}{2}(\Delta t V_0'(\hat{q}_n) + \Delta W_n V_1'(\hat{q}_n)) + \hat{p}_n \end{pmatrix} \tag{4.24}$$

where

$$\begin{pmatrix} \hat{q}_n \\ \hat{p}_n \end{pmatrix} = \begin{pmatrix} \hat{p}_n(\Delta t + \Delta W_n) + \tilde{q}_n \\ -\frac{1}{2}(\Delta t V_0'(\tilde{q}_n) + \Delta W_n V_1'(\tilde{q}_n)) + \tilde{p}_n \end{pmatrix}. \tag{4.25}$$

As in the example in Subsection 4.2, this scheme has the local order of weak sense 2. Thus, for the class of stochastic Hamilton dynamical systems with typical Hamiltonians mentioned above, we can numerically approximate them through our scheme (4.24) with (4.25) having the accuracy corresponding to Taylor scheme of local order 2.

4.4 Remark on composition methods and conserved quantities in stochastic dynamical systems

Finally, we remark on numerical schemes for stochastic dynamical systems which preserve "conserved quantities "of the systems. It is well-known that conserved quantities play an essential role to determine the structure of dynamical systems; hence, it is important to find a numerical scheme which has the conservation properties on the quantities for stochastic systems. On the other hands, composition methods often give such schemes in deterministic systems. Therefore, we may expect that one may obtain the schemes through our results, which have the advantageous to preserve them for stochastic systems, and in the remainder of this section, we will briefly examine it.

Let us consider d-dimensional stochastic dynamical systems (2.7). Suppose that a smooth function $I = I(S)$ satisfies

$$X_0 I = 0, \quad X_1 I = 0, \tag{4.26}$$

where X_0 and X_1 are the vector fields given by (2.3). According to Misawa (1999), $I(S)$ becomes a constant quantity; that is, $I(S(t)) = constant$ holds on the diffusion process $S(t)$ governed by (2.7).

Under some conditions, we may straightforwardly make a stochastic scheme satisfying numerical preservation of conserved quantities. Assume that the exponential maps of $A_{n\Delta t} = \Delta t X_0$ and $B_{n\Delta t} = \Delta W_n X_1$ are explicitly calculated. Then, it is obvious that Scheme 3.1 preserves the conserved quantity I numerically, because of the definition of exponential map and (4.26).

Now, we investigate a trivial example of a stochastic dynamical system with a conserved quantity and the numerical scheme through composition methods. Let us consider

$$d\begin{pmatrix} S^1(t) \\ S^2(t) \end{pmatrix} = \begin{pmatrix} S^2(t) \\ -S^1(t) \end{pmatrix} dt + \begin{pmatrix} S^2(t) \\ -S^1(t) \end{pmatrix} \circ dW(t); \tag{4.27}$$

this is a stochastic system with the conserved quantity $I(S) = \frac{1}{2}((S^1)^2 + (S^2)^2)$, since (4.26) holds. However, as mentioned in Misawa (2000), the ordinary schemes do not conserve $I(S)$ numerically. On the other hand, for this system, we adopt Scheme 3.1 with $A_{n\Delta t} = \Delta t X_0 = \Delta t(S^2\partial_1 - S^1\partial_2)$ and $B_{n\Delta t} = \Delta W_n X_1 = \Delta W_n(S^2\partial_1 - S^1\partial_2)$; then through (2.6), the numerical scheme is explicitly given by

$$\begin{pmatrix} \tilde{S}^1_{n+1} \\ \tilde{S}^2_{n+1} \end{pmatrix} = \begin{pmatrix} \cos(\Delta t) & \sin(\Delta t) \\ -\sin(\Delta t) & \cos(\Delta t) \end{pmatrix} \begin{pmatrix} \cos(\Delta W_n) & \sin(\Delta W_n) \\ -\sin(\Delta W_n) & \cos(\Delta W_n) \end{pmatrix} \begin{pmatrix} \tilde{S}^1_n \\ \tilde{S}^2_n \end{pmatrix}. \tag{4.28}$$

Therefore, for any n, the numerical solutions (4.28) satisfy $I(\tilde{S}^1_n, \tilde{S}^2_n) = constant$. Thus, our scheme numerically preserve a conserved quantity I of the stochastic system (4.27), and this fact also shows the superiority of the scheme derived through composition methods.

5 Concluding remarks

In this paper, we have formulated composition methods for stochastic differential equations (SDEs), and thereby we have made some stochastic numerical schemes. Then, the several examples have indicated that the new schemes have a superiority in conservation properties on character of SDEs and they are useful for approximations of the solutions to SDEs. Moreover, we have estimated local error orders for our schemes within the framework of Saito and Mitsui's definition. Finally, we give some remarks and future problems concerning with this work.

(i) As mentioned in Remark 3.2, we should carry out a more analytical error estimation for our schemes through the result on time asymptotics of exponential maps for SDEs by Castell (1993), since the stochastic series (2.14) is only a formal representation.

(ii) In our error estimation, we have addressed "local error order "for our schemes. In general, it is more important to estimate "global error order "for numerical schemes. In Burrage and Burrage (1999), the relationship between local order and global one has been discussed in some detail. Using the results, we may carry out global error estimation for our new schemes.

(iii) In this paper, we have treated the SDEs with one-dimensional Wiener process. On the other hand, it often happens that the error order of a numerical method collapses, if there is more than 1 Wiener process. Hence, we should investigate the error orders of our schemes in the case of SDEs with multi-dimensional Wiener process.

(iv) Moreover, Li and Liu (1997) and Kunita (2000) have studied on stochastic exponential maps for a more general class of stochastic processes, *e.g.* Lévy processes. Hence, through the works, we may formulate stochastic composition methods for such general stochastic processes.

The author will report these subjects in future works.

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n	Euler-Maruyama	Taylor of order 3	Scheme (4.5)	Scheme (4.8)
179	.0273725	.0105996	.0105927	.0105912
180	.0391646	.0183733	.0183659	.0183624
181	.0576639	.0323048	.0323008	.0322903
182	.0506609	.0265831	.0265762	.0265694
183	.0512129	.0262494	.0262425	.0262353
184	.0596316	.0318198	.0318131	.031804
185	.0561177	.0285982	.0285888	.0285826
186	.0478919	.0223767	.02237	.022363
187	.0507206	.0236332	.0236273	.0236186
188	.0462954	.0200634	.0200576	.0200496
189	.0373054	.014018	.0140095	.0140059
190	.0459159	.0190609	.0190539	.0190461
191	.0428106	.0164982	.01649	.0164839
192	.0524778	.0223075	.0222992	.0222907
193	.0365604	.0126323	.0126217	.0126192
194	.0190258	4.07906E-03	.0040724	4.07239E-03
195	.0120115	1.20801E-03	1.20498E-03	1.20484E-03
196	7.86052E-04	4.42925E-04	4.46361E-04	4.45822E-04
197	2.07421E-03	6.85707E-04	6.89868E-04	6.8875E-04
198	-9.47943E-05	7.45417E-05	7.31484E-05	7.32705E-05

Table 4.1: An example of numerical solutions from Euler-Maruyama scheme (4.10), Taylor scheme of local order 3 (4.11), Scheme (4.5) and Scheme (4.8) with (4.9) for (4.1) with an initial value s=0.01.

n	Taylor of order 3	Scheme (4.5)	Scheme (4.8)
348	.012517	.0126881	.0124984
349	.0128065	.012981	.0127872
350	8.45901E-03	8.60197E-03	8.44348E-03
351	4.89666E-03	5.00384E-03	4.88441E-03
352	.006811	6.93784E-03	6.79605E-03
353	.0119832	.0121529	.0119632
354	.0100548	.0102105	.010036
355	7.71649E-03	7.85315E-03	7.69978E-03
356	5.17094E-03	5.28222E-03	5.15692E-03
357	5.46758E-03	5.58249E-03	5.45263E-03
358	4.28647E-03	4.38672E-03	4.27267E-03
359	7.06959E-03	7.19804E-03	7.05171E-03
360	2.92672E-04	3.21066E-04	.0002912
361	5.76257E-04	6.15458E-04	5.7371E-04
362	2.57058E-03	2.65334E-03	2.56494E-03
363	3.09359E-03	3.18363E-03	3.08695E-03
364	8.20704E-04	8.66758E-04	8.17091E-04
365	2.12819E-04	1.90873E-04	2.14876E-04
366	5.3766E-04	5.01956E-04	5.40526E-04
367	-2.97776E-07	7.52643E-07	7.81725E-10

Table 4.2: An example of numerical solutions from Taylor scheme of local order 3 (4.11), Scheme (4.5) and Scheme (4.8) with (4.9) for (4.1) with an initial value s=0.001.

n	Euler-Maruyama	Taylor of order 3	Scheme (4.5)	Scheme (4.8)
980	3.72261	3.68752	3.68641	3.68752
981	3.7145	3.67843	3.67732	3.67843
982	3.73492	3.69778	3.69667	3.69778
983	3.92852	3.89192	3.89084	3.89192
984	4.10458	4.06816	4.06707	4.06816
985	4.21954	4.18242	4.18134	4.18242
986	4.19385	4.15585	4.15479	4.15586
987	4.30738	4.26862	4.26757	4.26862
988	4.4013	4.36165	4.3606	4.36165
989	4.46073	4.42002	4.41896	4.42002
990	4.46056	4.41882	4.41777	4.41882
991	4.33234	4.29109	4.29002	4.29109
992	4.25444	4.21288	4.2118	4.21288
993	3.97593	3.9392	3.93813	3.9392
994	3.72468	3.69204	3.69095	3.69204
995	3.48053	3.4519	3.45081	3.4519
996	3.6467	3.61833	3.61729	3.61834
997	3.79678	3.76838	3.76736	3.76838
998	3.74771	3.71863	3.7176	3.71863
999	3.72407	3.69411	3.69307	3.69412
1000	3.62125	3.5914	3.59035	3.59141

Table 4.3: An example of numerical solutions from Euler-Maruyama scheme (4.10), Taylor scheme of local order 3 (4.11), Scheme (4.5) and Scheme (4.8) with (4.9) for (4.1) with an initial value s=1.

n	Euler-Maruyama	Taylor of order 3	Scheme (4.17)
980	8.8134	9.51844	9.52098
981	9.09664	9.813	9.81558
982	9.33128	10.0542	10.0568
983	10.5972	11.4613	11.4644
984	11.3247	12.2459	12.2492
985	11.4263	12.3423	12.3456
986	10.7926	11.6704	11.6735
987	10.4671	11.3161	11.3191
988	10.1213	10.9412	10.9441
989	9.46724	10.2515	10.2542
990	9.70748	10.4991	10.502
991	9.07412	9.83114	9.83382
992	9.77413	10.5897	10.5926
993	9.35542	10.1398	10.1426
994	9.79685	10.6085	10.6114
995	9.86074	10.6656	10.6686
996	9.91146	10.7086	10.7115
997	9.57355	10.3428	10.3456
998	9.27501	10.0183	10.021
999	9.64218	10.4042	10.407
1000	9.47959	10.2216	10.2244

Table 4.4: An example of numerical solutions from Euler-Maruyama scheme (4.18), Taylor scheme of local order 3 (4.19), Scheme (4.17) for (4.13) with an initial value s=1.

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A LIE ALGEBRAIC APPROACH TO NUMERICAL INTEGRATION OF STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract. In this article, "composition methods (or operator splitting methods)" for autonomous stochastic differential equations (SDEs) are formulated to produce numerical approximation schemes for the equations. In the proposed methods, the exponential map, which is given by the solution of an SDE, is approximated by composition of the stochastic flows derived from simpler and exactly integrable vector field operators having stochastic coefficients. The local and global errors of the numerical schemes derived from the stochastic composition methods are investigated. The new schemes are advantageous to preserve the special character of SDEs numerically and are useful for approximations of the solutions to stochastic nonlinear equations. To examine their superiority, several numerical simulations on the basis of the proposed schemes are carried out for SDEs which arise in mathematical finance and stochastic Hamiltonian dynamical systems.

Key words. stochastic differential equations, Lie algebra, composition methods (operator splitting method), mathematical finance, stochastic Hamiltonian dynamical systems, stochastic nonlinear system

AMS subject classifications. 65C30, 17B66, 91B28, 70-08, 70H33

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1. Introduction.

The theory of stochastic differential equations (SDEs) is understood as a fundamental tool for the description of random phenomena treated in physics, engineering, economics, and so on. Particularly, as the famous Black–Scholes option pricing model, the stochastic equations are used to describe the price process of underlying asset in mathematical finance. However, it is often difficult to obtain the solutions of SDEs explicitly, and hence there has been increasing interest in numerical analysis of SDEs. Indeed, many numerical schemes for SDEs have been proposed (e.g., [8], [15], [23]).

The purpose of the present article is to propose some new numerical schemes for autonomous SDEs on the basis of "composition methods." The reasons why we address this topic are as follows.

In numerical analysis for deterministic ordinary differential equations, whether or not some special character or structure of the equations is preserved precisely is an important point in performing reliable numerical calculations. From this point of view, various numerical methods to realize the characters of differential equations have been proposed. Indeed, we can find such examples as energy conservative methods [10], [13], symplectic integrators for Hamiltonian dynamical systems [24], [7], [26], and composition methods [19], [22]. In particular, the composition methods are useful to produce numerical schemes which leave some structure or character of general differential equations numerically invariant. Hence, it seems to be quite natural that we investigate the methods for SDEs to produce numerical schemes having the conservation properties; this is the first reason for setting up the numerical schemes proposed here.

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Moreover, in the theory of differential equations, composition methods are known as operator splitting methods, and they are often utilized for approximation of nonlinear equations to which solutions are not obtained explicitly [25], [12]. From this viewpoint, we may expect that the methods bring us a convenient and powerful way of approximations for "stochastic nonlinear" differential equations, and this is the second reason for our investigation.

We will first outline the original composition methods for ordinary differential equations. Let X denote vector fields on some space with coordinates x with flows $\exp(tX)$; that is, the solutions of differential equations of the form $\dot{x}(t) = X(x)$ are given in the form $x(t) = \exp(tX)(x(0))$. Then, the vector field X is to be integrated numerically with fixed time step t. In this framework, we can apply composition methods to the differential equation if we can write $X = A + B$ in such a way that $\exp(tA)$ and $\exp(tB)$ can both be calculated *explicitly*. More generally, this can be relaxed by approximations of the exponential maps. In the most elementary case, the method gives the approximation for $x(t)$ through

$$\tilde{x}(t) = \exp(tA)\exp(tB)(x) = x(t) + O(t^2);$$

the last equality is obtained by using Baker–Campbell–Hausdorff (BCH) formula, which is well known in Lie algebraic theory.

Thus, in composition methods, we use the exponential representation of solutions to differential equations as an important tool. To formulate the methods for SDEs, therefore, one needs a stochastic version of the notion of exponential representation of solutions. In [16], this topic has been investigated in detail. Hence, in section 2, we first review Kunita's work on an explicit expression of solutions of SDEs as a functional of multiple Wiener integrals. We next give a formal extension of Kunita's explicit expression of solutions of SDEs, and on the basis of the formal result, we will propose new schemes for SDEs. Moreover, to estimate the error for the new schemes, we touch upon some lemmas and a proposition concerning mean-square order of multiple Wiener integrals.

In section 3, on the basis of the results in section 2, composition methods are formulated for autonomous SDEs with a one-dimensional Wiener process. Through these methods, we will obtain some numerical schemes for the stochastic equations. Then, the approximation error of numerical solutions derived from the schemes must be estimated. In this paper, we first apply the local error estimation in mean-square sense to the obtainable numerical solutions. The BCH formula will be useful for calculating the approximation error in a way analogous to that in the deterministic case. Using the result on local error estimation, we next address global error estimation for our new schemes. At the present stage, we obtain only local and global error orders through indirect estimation, and hence such orders will be said to be "in a weak sense."

In section 4, in order to examine the superiority of the new schemes, we investigate some illustrative examples of the numerical simulations using the proposed schemes. In the first example, the following nonlinear scalar SDE is treated, which is often adopted as a model of an asset price process in mathematical finance [9]:

$$dS(t) = S(t)dt + 2\sqrt{S(t)} \circ dW(t), \quad S(0) = s(> 0). \tag{1.1}$$

It is known that the value of the solution to this equation is "nonnegative" for any $t \in [0, T]$. Through the standard stochastic numerical schemes, however, such a character of this equation is not always preserved numerically. In contrast with this, we will see that our new numerical schemes by composition methods in section 3 leave the structure invariant numerically, and thereby the first advantage of composition methods is revealed. In the second and third examples, we examine the second advantage of composition methods as a tool of approximation for stochastic nonlinear systems. Particularly, such a superiority is often found in Hamiltonian dynamical systems through the dimensional splitting methods. Therefore, in the third example, we treat the composition methods for "stochastic" Hamiltonian systems. At the end of that section, to show the advantage of composition methods, we further touch upon a way to produce numerical schemes which realize the numerical preservation of conserved quantities for stochastic systems [20], [21].

Some concluding remarks and a discussion of future problems are given in section 5.

Finally, we note the recent remarkable work by K. Burrage and P.M. Burrage [4] as another numerical approach to SDEs through Lie algebra.

2. Representation of solutions of SDEs. In this section, we first review Kunita's work mentioned in section 1 [16]. Next we will give a formal extension of Kunita's explicit expression of solutions of SDEs. On the basis of this formal expression, our new schemes for SDEs are formulated in the next section. At the end of this section, as a preparation for the error estimation of the new schemes, we will discuss mean-square order of multiple Wiener integrals dealt with in the representation of solutions of SDEs.

2.1. Review of Kunita's explicit expression of solutions of SDEs. In order to explain Kunita's main result, we first address Campbell–Hausdorff formula for n vector fields. Let $Y_1, \ldots, Y_n$ be n C^∞ vector fields on a connected C^∞-manifold M of dimension d. Suppose that $Y^{i_1\cdots i_m} = [[\ldots[Y_{i_1}, Y_{i_2}]\ldots]Y_{i_m}]$, $m = 1, 2, \ldots$, and their sums are all complete vector fields, where $[X, Y]$ is the Lie bracket defined by $XY - YX$. Moreover, we assume that the power series

$$Z = \sum_{m=1}^{\infty} \sum_{(i_1\cdots i_m)} c_{i_1\cdots i_m} Y^{i_1\cdots i_m} \tag{2.1}$$

is absolutely convergent and defines a complete vector field. Here, for a given multi-index $I = (i_1, \ldots, i_m)$, each coefficient $c_{i_1\cdots i_m}$ is determined through (2.3) given below. Then (2.2) holds

$$\exp Y_n \cdots \exp Y_1 = \exp Z. \tag{2.2}$$

This is the Campbell–Hausdorff formula for n vector fields [16].

We will touch upon the explicit form of the coefficients $c_{i_1\cdots i_m}$. Let us divide the multi-index $I = (i_1, \ldots, i_m)$ into a sequence of shorter ones $I_j, j = 1, \ldots, \ell$, and denote it by $\hat{I}$;

$$\hat{I} = (I_1, \ldots, I_{k_1})(I_{k_1+1}, \ldots, I_{k_2}) \cdots (I_{k_{\ell-1}+1}, \ldots, I_{k_\ell}),$$

where each I_k consists of the same number $\hat{i}_k$ and the numbers $\hat{i}_k, k = 1, \ldots, k_\ell$, satisfy

$$\hat{i}_1 > \hat{i}_2 > \cdots > \hat{i}_{k_1} < \hat{i}_{k_1+1} > \cdots > \hat{i}_{k_2} \cdots < \hat{i}_{k_{\ell-1}+1} > \cdots > \hat{i}_{k_\ell}.$$

We call $\hat{I}$ a natural division. Moreover, we denote the number of elements in I_k by n_k, where $\sum_{k=1}^{k_\ell} n_k = m$. Divide again each index I_k into j_k indices, each of which consists of $n_k^{(i)}(i = 1, \dots, j_k)$ elements (hence $\sum_{i=1}^{j_k} n_k^{(i)} = n_k$). Then according to [16], the coefficients $c_{i_1 \cdots i_m}$ are given by (2.3):

$$
\begin{aligned}
c_{i_1,\cdots i_m} &= \frac{1}{m}\sum_{s=0}^{\ell-1}\sum_{*} {}_{\ell-1}C_s(-1)^{j_1+\cdots+j_{k_\ell}-s-1}(j_1+\cdots+j_{k_\ell}-s)^{-1} \\
&\quad\times \frac{1}{n_1^{(1)}!\cdots n_1^{(j_1)}!\cdots n_{k_\ell}^{(1)}!\cdots n_{k_\ell}^{(j_\ell)}!},
\end{aligned}
\tag{2.3}
$$

where the sum $\sum_*$ is taken for all subdivisions of $I_k (k = 1, \dots, k_\ell)$, that is, for positive integers $n_k^{(i)}(i = 1, \dots, j_k; k = 1, \dots, k_\ell)$ under $\sum_{i=1}^{j_k} n_k^{(i)} = n_k$.

Let I' be another multi-index of length m such that the natural division is given by $\hat{I}' = (I'_1, \dots, I'_{k'_1}) \cdots (I'_{k'_{\ell-1}+1}, \dots, I'_{k'_\ell})$. We say I and I' are equivalent if for each k, I_k and I'_k contain the same number of elements and $k'_j = k_j (j = 1, \dots, \ell)$ hold. Then we note that $c_I = c_{I'}$ holds.

Now, we proceed to our stochastic systems governed by SDEs. Let us consider an autonomous SDE of Stratonovich type (e.g., [11], [1]) under the probability space $(\Omega, \mathcal{F}, \mathcal{P})$:

$$
dS(t) = b(S(t))dt + \sum_{j=1}^{r} g_j(S(t)) \circ dW^j(t) \tag{2.4}
$$

defined on a connected C^∞-manifold M of dimension d, where $b = (b^i)_{i=1}^d$ and $g_j = (g_j^i)_{i=1}^d \quad (j = 1, \dots, r)$ are d-dimensional C^∞ functions on M, respectively, and $W(t) = (W^1(t), \dots, W^r(t))$ is a standard Wiener process. Here $S(t)$ is assumed to be adapted with a nondecreasing family of sigma-algebra $(\mathcal{F}_t)_{t \geq 0} \subset \mathcal{F}$. Note that (2.4) is rewritten in the form of SDE of Itô type as follows:

$$
dS_t = \{b(S(t)) + \frac{1}{2}\sum_{k=1}^{r}\sum_{i=1}^{d} g_k^i \partial_i g_k(S(t))\}dt + \sum_{j=1}^{r} g_j(S(t))dW^j(t), \tag{2.5}
$$

where $\partial_i = \partial/\partial S^i$. Using the coefficient-functions in (2.4), we define C^∞ vector fields $X_0, X_1, \dots, X_r$ as follows:

$$
X_0 = \sum_{i=1}^{d} b^i \partial_i, \quad X_j = \sum_{i=1}^{d} g_j^i \partial_i \quad (j = 1, \dots, r). \tag{2.6}
$$

We will now review Kunita's result (Lemma 2.1 in [16]). For this purpose, we begin with notations on multi-index. Let us divide a multi-index $I = (i_1, \dots, i_m)$ $(i_1, \dots, i_m \in 0, 1, \dots, r)$ into shorter ones $I_1 \cdots I_q (q \leq m)$, where each I_k consists of the same element $\hat{i}_j$. For given positive integers $k_1 < k_2 < \cdots < k_\ell = q$, we define a division of I as

$$
\Delta I = (I_1, \dots, I_{k_1})(I_{k_1+1}, \dots, I_{k_2}) \cdots (I_{k_{\ell-1}+1}, \dots, I_{k_\ell}). \tag{2.7}
$$

We call ΔI single or double when each I_k contains a single element or at most two. Here we remark that ΔI may not be equal to a natural division of I; but if there is an index $\tilde{I}$ such that its natural division is equivalent to ΔI, we set $c_{\Delta I} = c_{\tilde{I}}$ in (2.3) for convention.

Now suppose that we are given an index I and a divided index ΔI. Moreover, we set $W^0(t) = t$. For a single divided index ΔI, we define the multiple Wiener–Stratonovich integral $W^{\Delta I}(t)$ as

$$W^{\Delta I}(t) = \int \cdots \int_A \circ dW^{i_1}(t_1) \cdots \circ dW^{i_m}(t_m), \tag{2.8}$$

where $A = \{t_{k_1} < \cdots < t_1 < t, \ldots, t_{k_\ell} < \cdots < t_{k_{\ell-1}+1} < t, t_{k_i} < t_{k_i+1}\ (i = 1, \ldots, \ell)\}$. On the other hand, if ΔI is a double index, we define

$$W^{\Delta I}(t) = \int \cdots \int_A \circ dW^{I_1}(t_1) \cdots \circ dW^{I_{k_\ell}}(t_\ell), \tag{2.9}$$

where

$$\begin{aligned} W^{I_k}(t) &= W^{i_k}(t) \quad (I_k : \text{single}; \quad I_k = \{i_k\}) \\ &= t \quad (I_k : \text{double}; \quad I_k = \{i_k, i_k\} \quad \text{and} \quad i_k \neq 0) \\ &= 0 \quad (I_k = \{0, 0\}). \end{aligned} \tag{2.10}$$

Under the notations defined above, Kunita proved the following lemma.

Kunita's lemma. Suppose that the Lie algebra $L = L(X_0, \ldots, X_r)$ generated by $X_0, \ldots, X_r$ is nilpotent of step p. Then the solution $S(t)$ of (2.4) with $S(0) = s$ is represented as

$$S(t) = \exp(Y_t)(s), \tag{2.11}$$

where $Y_t(\omega)$ is the vector field for each t and almost surely (a.s.) $\omega \in \Omega$ given by

$$Y_t = \sum_{i=0}^{r} W^i(t) X_i + \sum_{J;2 \leq |J| \leq p} \left\{ \sum_{\Delta J}{}^{*} c_{\Delta J} W^{\Delta J}(t) \right\} X^J, \tag{2.12}$$

and $X^J = [[\cdots[X_{j_1}, X_{j_2}]\cdots]X_{j_m}]$ $(J = (j_1, \ldots, j_m))$. Here, $\sum^{*}_{\Delta J}$ is the sum for all single and double divided indices of J, and $c_{\Delta J}$ are the coefficients determined through (2.3) for the divided indices of J. Moreover, $|J|$ denotes the length of a multi-index J.

We remark that (2.11) with (2.12) means that the solution $S(t, \omega)$ equals $\phi(1, s, \omega)$ a.s., where $\phi(\tau, s, \omega)$ is the solution of the *ordinary* differential equation

$$\frac{d\phi(\tau)}{d\tau} = Y_t(\omega)(\phi(\tau)), \quad \phi(0) = s, \tag{2.13}$$

regarding t and ω as parameters.

The proof of Kunita's lemma is outlined as follows. For a fixed positive integer n and positive time t, we define $\delta W^i_k = W^i(\frac{k}{n}t) - W^i(\frac{(k-1)}{n}t)$ $(i = 1, \ldots, r)$ and define $Z_j = \frac{t}{n} X_0 + \sum_{i=1}^{r} \delta W^i_j(t) X_i$ $(j = 1, \ldots, n)$. We further set

$$S^n(t) = (\exp Z_n \cdots \exp Z_1)(s). \tag{2.14}$$

Then, there is a subsequence of $S^n(t)$ converging to $S(t)$ a.s.. On the other hand, applying the Campbell–Hausdorff formula to the right-hand side of (2.14), we obtain

$$\sum_J \left\{ \sum_{\Delta J} c_{\Delta J} \sum_{I:\hat{I} \sim \Delta J} \delta W^{j_1}_{i_1} \cdots \delta W^{j_m}_{i_m} \right\} X^J,$$

where $\sum_{I:\hat{I}\sim\Delta J}\delta W_{i_1}^{j_1}\cdots\delta W_{i_m}^{j_m}$ means the sum for all indices I such that the natural division $\hat{I}$ is equivalent to ΔJ. Then the sum converges to $W^{\Delta J}(t)$ if ΔJ is a single or double index, and it converges to 0 if ΔJ is more than double. Note that the last assertion corresponds to the following result from formal calculus with respect to stochastic differentials dt and $dW^i \quad (i = 1,\ldots,r)$ [11]; $dW^i dW^j = \delta_{ij}dt$, $dtdW^i = 0$, $dtdt = 0$, and $dW^i dW^j dW^k = 0$. Finally, we can also prove that the sum converges to the right-hand side of (2.11) with (2.12) a.s..

Remark 2.1. According to Lemma 2.2 in Kunita's paper [16], we can calculate the coefficients of X^J in the case of $|J| = 2,3$. On account of these results, the vector field (2.12) is rewritten as

$$\begin{aligned}(2.15)\quad Y_t = &\sum_{i=0}^{r} W^i(t)X_i + \frac{1}{2}\sum_{i<j}^{r}[W^i,W^j](t)[X_i,X_j]\\ &+\frac{1}{36}\sum_{i=0}^{r}\sum_{j=1}^{r} tW^i(t)[[X_i,X_j],X_j]\\ &+\frac{1}{18}\sum_{i<j,k}^{r}[[W^i,W^j],W^k](t)[[X_i,X_j],X_k]\\ &+\sum_{J;4\le|J|\le p}\left\{\sum_{\Delta J}{}^{*} c_{\Delta J}W^{\Delta J}(t)\right\}X^J \quad (i=0,1,\ldots,r;j,k=1,\ldots,r).\end{aligned}$$

In (2.15), we define $[W^i,W^j](t)$ and $[[W^i,W^j],W^k](t)$ as multiple Wiener–Stratonovich integrals of degrees equal to 2 and 3 given by

$$[W^i,W^j](t) = \int_0^t W^i(\tau)\circ dW^j(\tau) - \int_0^t W^j(\tau)\circ dW^i(\tau)$$

and

$$[[W^i,W^j],W^k](t) = \int_0^t [W^i,W^j](\tau)\circ dW^k(\tau) - \int_0^t W^k(\tau)\circ d[W^i,W^j](\tau),$$

respectively.

Note that the condition that Lie algebra L is nilpotent of step p is required only in order to make the sum in (2.12) finite. This suggests that if L is general, we may *formally* represent the solution $S(t)$ of (2.4) as

$$S(t) = (\exp Y_t)(s), \tag{2.16}$$

where

$$Y_t = \sum_{i=0}^{r} W^i(t)X_i + \sum_{J;2\le|J|}\left\{\sum_{\Delta J}{}^{*} c_{\Delta J}W^{\Delta J}(t)\right\}X^J, \tag{2.17}$$

and this is just the formal expression we want. In what follows, we suppose that one may obtain the explicit representation of solution (2.16) with (2.17).

Remark 2.2. If the Lie algebra generated by $X_0, X_1, \ldots, X_r$ is of finite dimension, Ben Arous has proved that the stochastic infinite series in (2.17) actually converges before a stopping time. Therefore, in such a case, the representation of solution (2.16) with (2.17) is well-defined (Theorem 20 in [2]). We will see such examples in subsections 4.1 and 4.2 of section 4.

Moreover, we restrict ourselves to the SDEs (2.4) with a one-dimensional Wiener process; that is, we consider

$$dS(t) = b(S(t))dt + g(S(t)) \circ dW(t), \tag{2.18}$$

where b and g are d-dimensional vector-valued C^∞ functions. Then, we may rewrite (2.17) in a simpler form in terms of Kloeden–Platen's representation for multiple Wiener–Stratonovich integrals and multiple Wiener–Itô ones [14]; they are defined by

$$J_{(\alpha)}(t,s) = \int_s^{s+t}\int_s^{\tau_k}\cdots\int_s^{\tau_2} \circ dY^{(j_1)}(\tau_1)\cdots \circ dY^{(j_{k-1})}(\tau_{k-1}) \circ dY^{(j_k)}(\tau_k), \tag{2.19}$$

$$I_{(\alpha)}(t,s) = \int_s^{s+t}\int_s^{\tau_k}\cdots\int_s^{\tau_2} dY^{(j_1)}(\tau_1)\cdots dY^{(j_{k-1})}(\tau_{k-1}) dY^{(j_k)}(\tau_k), \tag{2.20}$$

respectively, where $\alpha = (j_1, \dots, j_k)$ $(j_i = 0,1; i = 1,\dots,k)$ and

$$dY^{(j)}(u) = \begin{cases} du & \text{for } j = 0, \\ dW(u) & \text{for } j = 1. \end{cases}$$

In what follows, we denote $J_{(\alpha)}(t,s)$ and $I_{(\alpha)}(t,s)$ by $J_{(\alpha)}(t)$ and $I_{(\alpha)}(t)$, respectively, if $s = 0$.

Remark 2.3. Note that $J_{(\alpha)}(t,s)$ can be rewritten in terms of $I_{(\alpha)}(t,s)$ (see pp. 174–175 in [15]). For example, we have

$$J_{(j_1)} = I_{(j_1)} \quad (j_1 = 0,1), \tag{2.21}$$

$$J_{(j_1,j_2)} = I_{(j_1,j_2)} + \frac{1}{2}1_{\{j_1=j_2=1\}}I_{(0)} \quad (j_1,j_2 = 0,1), \tag{2.22}$$

$$J_{(j_1,j_2,j_3)} = I_{(j_1,j_2,j_3)} + \frac{1}{2}(1_{\{j_1=j_2=1\}}I_{(0,j_3)} + 1_{\{j_2=j_3=1\}}I_{(j_1,0)}) \quad (j_1,j_2,j_3 = 0,1), \tag{2.23}$$

where $1_{\{\cdot\}}$ denotes the defining function. In general, any multiple Stratonovich integral $J_{(\alpha)}$ can be written as a multiple Itô integral $I_{(\alpha)}$ or a finite sum of $I_{(\alpha)}$ and multiple Itô integrals $I_{(\beta)}$ satisfying

$$\ell(\alpha) + n(\alpha) \le \ell(\beta) + n(\beta), \tag{2.24}$$

where $\ell(\alpha) = \{$the number of elements of $\alpha\}$ and $n(\alpha) = \{$the number of 0's in the elements of $\alpha\}$ (Remark 5.2.8 in [15]).

In terms of (2.19) and stochastic Stratonovich integration by parts formula for $W^i(t)$ $(i = 0,1)$ and $[W^0, W^1](t)$, we can rewrite (2.17) in the following form:

$$\begin{aligned} Y_t = {} & J_{(0)}(t)X_0 + J_{(1)}(t)X_1 + \frac{1}{2}(J_{(0,1)}(t) - J_{(1,0)}(t))[X_0,X_1] \\ & + \frac{1}{18}\{2J_{(0,1,0)}(t) - 2J_{(1,0,0)}(t) + J_{(0)}(t)J_{(1,0)}(t) - J_{(0)}(t)J_{(0,1)}(t)\}[[X_0,X_1],X_0] \\ & + \frac{1}{18}\{2J_{(0,1,1)}(t) - 2J_{(1,0,1)}(t) + J_{(1)}(t)J_{(1,0)}(t) - J_{(1)}(t)J_{(0,1)}(t)\}[[X_0,X_1],X_1] \\ & + \frac{1}{36}\{J_{(0)}(t)\}^2[[X_0,X_1],X_1] + \sum_{J;4\le|J|} K^J(t)X^J. \end{aligned} \tag{2.25}$$

Here, in the last term on the right-hand side of the above equation, $K^J(t) = \{\sum_{\Delta J}{}^* c_{\Delta J} W^{\Delta J}(t)\}$, and $J = (j_1, \ldots, j_\ell)$ $(j_i = 0, 1; i = 1, \ldots, \ell; \ell \geq 4)$. In terms of Remark 2.3, this can be described by multiple Wiener–Itô integrals as follows:

$$
\begin{aligned}
(2.26) \qquad Y_t &= I_{(0)}(t)X_0 + I_{(1)}(t)X_1 + \frac{1}{2}(I_{(0,1)}(t) - I_{(1,0)}(t))[X_0, X_1] \\
&+ \frac{1}{18}\{2I_{(0,1,0)}(t) - 2I_{(1,0,0)}(t) + I_{(0)}(t)I_{(1,0)}(t) - I_{(0)}(t)I_{(0,1)}(t)\}[[X_0, X_1], X_0] \\
&+ \frac{1}{18}\{2I_{(0,1,1)}(t) - 2I_{(1,0,1)}(t) + I_{(1)}(t)I_{(1,0)}(t) - I_{(1)}(t)I_{(0,1)}(t)\}[[X_0, X_1], X_1] \\
&+ \frac{1}{36}\{I_{(0,0)}(t) + \{I_{(0)}(t)\}^2\}[[X_0, X_1], X_1] + \sum_{J;4\leq|J|} H^J(t)X^J,
\end{aligned}
$$

where $H^J(t)$ is another version of $K^J(t)$ under each multi-index J, which is derived by transforming multiple Stratonovich integrals in $K^J(t)$ into Itô ones through Remark 2.3. Therefore, for each multi-index J, $H^J(t)$ is described as a polynomial function of multiple Itô integrals. In the next section, we will formulate the numerical schemes for the solution of SDE (2.18) on the basis of (2.16) with (2.26).

2.2. Mean-square order of multiple Wiener integrals. Now, in the remainder of this section, as a preparation for the error estimation for the new schemes in the next section, we will prove a proposition concerning the coefficients $H^J(t)$ of X^J for multiple indices J in (2.26). For this purpose, we first touch upon a lemma for multiple Itô integrals (2.20) proved by Kloeden and Platen (Lemma 5.7.5 in [15]). Let $E[\cdot|\mathcal{F}_s]$ be the conditional expectation with respect to a nondecreasing family of σ-subalgebra $\mathcal{F}_s$.

LEMMA 2.1. *For any $\alpha = (j_1, \ldots, j_k)$, $(j_i = 0, 1;\ i = 1, \ldots, k)$, and $q = 1, 2, \ldots$,*

$$
(2.27) \qquad E[|I_{(\alpha)}(\Delta t, s)|^{2q}|\mathcal{F}_s] = O((\Delta t)^{q(\ell(\alpha)+n(\alpha))}) \quad (\Delta t \downarrow 0),
$$

where $\ell(\alpha)$ and $n(\alpha)$ are the indices defined in Remark 2.3; that is, $\ell(\alpha) =$ {the number of elements of α} and $n(\alpha) =$ {the number of 0's in the elements of α}.

Let $F(t, s)$ be a function of multiple stochastic integrals $I_{(\alpha)}(t, s)$. Suppose that

$$
(2.28) \qquad E[\{F(\Delta t, s)\}^2|\mathcal{F}_s] = O((\Delta t)^m) \quad (\Delta t \downarrow 0)
$$

holds. Then, in what follows, we call the real number m "*mean-square order (MSO)*" of $F(t, s)$.

From Lemma 2.1 we see that MSO of a multiple Itô integral $I_{(\alpha)}(t, s)$ is equal to $\ell(\alpha) + n(\alpha)$. Moreover, the following lemma shows that MSO of $I_{(\alpha)}(t, s)I_{(\beta)}(t, s)$ is given by $\ell(\alpha) + n(\alpha) + \ell(\beta) + n(\beta)$; that is, MSO of a product of multiple Itô integrals equals the sum of MSO of each of the stochastic integrals.

LEMMA 2.2. *For any multi-indices α and β,*

$$
(2.29) \quad E[|I_{(\alpha)}(\Delta t, s)I_{(\beta)}(\Delta t, s)|^2|\mathcal{F}_s] = O((\Delta t)^{(\ell(\alpha)+n(\alpha)+\ell(\beta)+n(\beta))}) \quad (\Delta t \downarrow 0)
$$

holds.

Proof. Through the Schwartz inequality, we obtain

$$
E[|I_{(\alpha)}(\Delta t, s)I_{(\beta)}(\Delta t, s)|^2|\mathcal{F}_s] \leq \sqrt{E[\{I_{(\alpha)}(\Delta t, s)\}^4|\mathcal{F}_s]E[\{I_{(\beta)}(\Delta t, s)\}^4|\mathcal{F}_s]}.
$$

The lemma is straightforwardly proved using this inequality and Lemma 2.1.

Remark 2.4. By Lemmas 2.1 and 2.2 together with Remark 2.3, we may verify that multiple Stratonovich integrals $J_{(\alpha)}(t,s)$ also satisfy the results in Lemmas 2.1 and 2.2. Hence, for a given α, we find that MSO of a multiple Stratonovich integral $J_{(\alpha)}(t,s)$ is also equal to $\ell(\alpha)+n(\alpha)$; that is, MSO of $J_{(\alpha)}(t,s)$ agrees with that of $I_{(\alpha)}(t,s)$ for the same multi-index α.

We now proceed to our goal. Using Lemmas 2.1 and 2.2, we can estimate MSO of each coefficient of X^J in (2.26) which is given by a polynomial function of multiple Itô integrals on $[0,t]$. For example, in the case of $|J|=1$, MSOs of the coefficients of X_0 and X_1, that is, MSOs of $I_0(t)$ and $I_1(t)$ are equal to 2 and 1, respectively. In the case of $|J|=2$, MSO of $I_{(0,1)}(t)$ or $I_{(1,0)}(t)$ in the coefficient of $[X_0,X_1]$ is given by 3, and thereby we can easily prove that MSO of the coefficient $(I_{(0,1)}(t)-I_{(1,0)}(t))/2$ itself is also equal to 3. In the same manner, we find MSOs of the coefficients of X^J when $|J|=3$; that is, MSOs of the coefficients of $[X_0,X_1],X_0]$ and $[X_0,X_1],X_1]$ are given by 5 and 4, respectively. Hence, in this case, *the least value* of MSOs of the coefficients of X^J equals 4. These facts suggest that we may verify the following proposition which we wish to establish.

PROPOSITION 2.1. *Suppose that k is a given integer greater than or equal to 2, and that the multi-indices J satisfy $|J|=k$; that is, $J=(j_1,\ldots,j_k)$ ($j_i=0$ or 1; $i=1,\ldots,k$). Then the least value of MSOs of the coefficients $H^J(t)$ of X^J in* (2.26) *is equal to $k+1$.*

Proof. We may prove this by induction. From the above examples, this proposition is obvious in the case of $|J|=2,3$. We assume that the assertion of this proposition holds for the case of $|J|=\ell(\geq 3)$. That is, the least value of MSOs of the coefficients $H^J(t)$ of X^J under $J=(j_1,\ldots,j_\ell)$ in (2.26) equals $\ell+1$. Then, note that under the same J, the least value of MSOs of $K^J(t)$ in (2.25) agrees with that of $H^J(t)$, since $H^J(t)$ is only another version of $K^J(t)$ in terms of multiple Itô integrals. Now, let us consider the coefficients $K^{\tilde{J}}(t)$ of $X^{\tilde{J}}$ under $|\tilde{J}|=\ell+1$. On account of the definition of multiple integrals $W^{\Delta J}(t)$ (2.8) or (2.9) with (2.10) in the coefficients, one may obtain $K^{\tilde{J}}(t)$ by adding a stochastic integral with respect to dW or dt to each multiple integral in $K^J(t)$. Moreover, the following equations show that the MSOs for the integrals with increments dW and dt correspond to 1 and 2, respectively:

$$E\left[\left|\int_s^{s+\Delta t} dW(\tau)\right|^2\Big|\mathcal{F}_s\right]=O(\Delta t),\quad E\left[\left|\int_s^{s+\Delta t} dt\right|^2\Big|\mathcal{F}_s\right]=O((\Delta t)^2)\ (\Delta t\downarrow 0).$$

Hence, the least value of MSOs of $K^{\tilde{J}}(t)$ is equal to $\ell+2$, and thereby, the least value of MSOs of $H^{\tilde{J}}(t)$ is also so. Thus, the assertion in our proposition is proved. $\square$

3. Composition methods for numerical integration of SDEs. In this section, we will formulate some new stochastic numerical schemes for SDEs on the basis of composition methods and estimate the approximation errors for the schemes in the local and global senses.

3.1. Procedures of composition methods for SDEs. We start with a numerical integration of the stochastic equation (2.18) on the discretized time series in the framework of the previous results on representation of solutions to SDEs. It adopts a uniform discretization of the time interval $[0,T]$ with stepsize

$$\Delta t=\frac{T}{N}$$

for fixed natural number N. Let $t_n = n\Delta t (n = 0, 1, 2, \dots, N)$ be the nth step-point. Then, for all $n \in \{0, \dots, N\}$, we abbreviate $S_n = S(t_n)$. Moreover, we use ΔW_n for $n = 0, 1, \dots, N$ to denote the increments $W(t_{n+1}) - W(t_n)$; they are independent Gaussian random variables with mean 0 and variance Δt, that is, $N(0, \Delta t)$-distributed random variables.

On account of (2.16), we may find the numerical solutions $S_n (n = 0, 1, \dots, N)$ to SDE (2.18) by using

$$S_{n+1} = \exp(Y_{n\Delta t})(S_n) \ (n = 0, 1, 2, \dots, N-1), \tag{3.1}$$

formally, where $Y_{n\Delta t}$ is a vector field derived by replacing all the multiple Wiener integrals $I_{(\alpha)}(t) = I_{(\alpha)}(t, 0)$ in (2.26) by $I_{(\alpha)}(\Delta t, n\Delta t)$. In the following, $I_{(\alpha)}(\Delta t, n\Delta t)$ is denoted by $I_{(\alpha),n}(\Delta t)$. Moreover, we set $S_0 = S(0) = s_0$. According to the theory of ordinary differential equations, $\exp(Y_{n\Delta t})(\cdot)$ is often called the time-Δt map or exponential map. However, it is usually difficult to find the explicit form of the exponential map, and hence we need to build an approximation for (3.1).

To do this, we formulate a new stochastic numerical scheme as the following two procedures, which are composed of the truncation of the vector field (2.26) and a composition method (or operator splitting method) applied to the exponential map derived from the truncated vector field.

Procedure 1. For the vector field Y_t described by (2.26), we define a "truncated" vector field $\hat{Y}_t$ which is given by a truncation of the higher-order terms with respect to MSO of the coefficients of X^J in (2.26). Then, we define a numerical sequence $(\hat{S}_n)_{n=0}^N$ through

$$\hat{S}_{n+1} = \exp(\hat{Y}_{n\Delta t})(\hat{S}_n) \ (n = 0, 1, \dots, N-1), \tag{3.2}$$

where $\hat{S}_0 = S(0) = s_0$.

Procedure 2. For $\hat{S}_{n+1} = \exp(\hat{Y}_{n\Delta t})(\hat{S}_n)$, we apply a composition method in a way analogous to that in the theory of ordinary differential equations. Suppose that the vector field $\hat{Y}_{n\Delta t}$ is of the form

$$\hat{Y}_{n\Delta t} = A_{n\Delta t} + B_{n\Delta t}, \tag{3.3}$$

where $\exp(A_{n\Delta t})$ and $\exp(B_{n\Delta t})$ can both be explicitly calculated through (2.13). Then an approximation to the exponential map of $\hat{Y}_{n\Delta t}$ is given by $\exp(A_{n\Delta t}) \exp(B_{n\Delta t})$. Hence, the sequence of $(\hat{S}_n)_{n=0}^N$ in Procedure 1 is approximated by

$$\tilde{S}_{n+1} = \exp(A_{n\Delta t}) \exp(B_{n\Delta t})(\tilde{S}_n) \ (n = 0, 1, \dots, N-1), \tag{3.4}$$

where $\tilde{S}_0 = S(0) = s_0$.

We regard $(\tilde{S}_n)_{n=0}^N$ as a numerical approximation to the exact discretized solutions $(S_n)_{n=0}^N$.

3.2. Local error estimation for the new numerical scheme. We will estimate the approximation error of the numerical scheme described above. For this purpose, we first examine "local errors in the mean-square sense" for the scheme. Using this result, we will finally address the "global" error estimation in the next subsection.

We start with the definition of "local error order" in a manner analogous to that in [15].

DEFINITION 3.1. *Suppose that $S(t)$ and $(\bar{S}_n)_{n=0}^N$ are an exact solution and the numerical approximation solutions to SDE* (2.18), *respectively. Let $E_{\tau,\xi}$ be the expectation conditioned on starting at ξ at time τ. Then the local error order α is defined by*

$$E_{t_n,s}[|\bar{S}_{n+1} - S_{n+1}|^2] = O((\Delta t)^{2\alpha}) \ (\Delta t \downarrow 0), \tag{3.5}$$

where $t_n = n\Delta t$ $(n = 0, \ldots, N-1)$, $S_n = S(n\Delta t)$, and $|\cdot|$ denotes the Euclidean norm on the space R^d. Note that the condition in the expectation (3.5) *means $S_n = \bar{S}_n = s$.*

We note that the accuracy of a numerical scheme improves with increasing local order.

Remark 3.1. In the framework of local error order defined by [23], the local order of $\bar{S}_n$ satisfying (3.5) is given by $2\alpha - 1$.

On the basis of this definition of local error estimation, we will investigate the error estimation of our approximation procedures. In what follows, we set $S_n = s$, where s is a given value.

Local error estimation for the truncation error in Procedure 1. First, we investigate the truncation error in Procedure 1. Let ${H_n}^J(\Delta t)$ be the coefficient of X^J in $Y_{n\Delta t}$ which is represented by a polynomial function of multiple Itô integrals for a given multi-index J as in (2.26).

PROPOSITION 3.1. *Suppose that a truncation vector field $\hat{Y}_{n\Delta t}$ is given in the following form:*

$$\hat{Y}_{n\Delta t} = \sum_{J;1\le|J|\le\gamma} {H_n}^J(\Delta t)X^J. \tag{3.6}$$

That is, we assume the terms in $Y_{n\Delta t}$ satisfying $|J| \ge \gamma + 1$ are neglected. Then,

$$E_{t_n,s}[|\hat{S}_{n+1} - S_{n+1}|^2] = O((\Delta t)^{\gamma+2}) \ (\Delta t \downarrow 0). \tag{3.7}$$

Proof. In terms of Proposition 2.1, we can easily show that the least value of MSOs of ${H_n}^J(\Delta t)$ in the neglected terms equals $\gamma + 2$, since $|J| \ge \gamma + 1$. This fact, together with the definition of exponential map (see 2.11 with 2.13 [16], [3]), straightforwardly shows (3.7). Thus, we obtain the local order $(\gamma/2) + 1$ for numerical approximation solutions $(\hat{S}_n)_{n=0}^N$ in the sense of Definition 3.1.

Remark 3.2. If the Lie algebra generated by X_0 and X_1 is of finite dimension, our error estimation derived above agrees with that of truncation of stochastic exponential maps by [2], since the convergence of (2.17), and hence (2.26), are actually guaranteed (cf. Remark 2.2). That is, under the assumption that such a convergence holds, the local error estimation derived above holds exactly. In general, however, our result may give only a formal error estimation. Indeed, according to [6], when the convergence is not guaranteed, the asymptotic expansion of stochastic exponential maps is estimated only in a "probability" sense.

Local error estimation for the composition scheme in Procedure 2. Next, we will proceed to the local error estimation for Procedure 2. We can carry this out using the BCH formula [3] together with Lemma 2.2; the formula is given by the following form:

$$\begin{aligned}\exp(\epsilon(\Delta t)X)\exp(\delta(\Delta t)Y) &= \exp(\epsilon(\Delta t)X + \delta(\Delta t)Y + \frac{1}{2}\epsilon(\Delta t)\delta(\Delta t)[X,Y] \\ &+\frac{1}{12}(\epsilon(\Delta t)^2\delta(\Delta t)[X,[X,Y]] + \epsilon(\Delta t)\delta(\Delta t)^2[Y,[Y,X]]) + \cdots),\end{aligned} \tag{3.8}$$

where X and Y are C^∞ vector fields, and $\epsilon(\Delta t)$ and $\delta(\Delta t)$ are any functions of Δt; in our case, they correspond to polynomial functions of multiple Itô stochastic integrals $I_{(\alpha),n}(\Delta t)$.

PROPOSITION 3.2. *Let $\hat{Y}_{n\Delta t}$ be a truncated vector field given by (3.6). Suppose that the vector fields $A_{n\Delta t}$ and $B_{n\Delta t}$ in a decomposition (3.3) for $\hat{Y}_{n\Delta t}$ are described by*

$$A_{n\Delta t} = \sum_{J;1\le|J|\le\gamma} F_n{}^J(\Delta t)X^J, \quad B_{n\Delta t} = \sum_{J;1\le|J|\le\gamma} G_n{}^J(\Delta t)X^J, \tag{3.9}$$

respectively, and that the least values of MSOs of $F_n{}^J(\Delta t)$ and $G_n{}^J(\Delta t)$ in (3.9) are given by α and β, respectively. If $X_A^{J_\alpha}$ and $X_B^{J_\beta}$, which are vector fields corresponding to the coefficients with α and β as MSO, respectively, satisfy $[X_A^{J_\alpha}, X_B^{J_\beta}] \neq 0$, then

$$E_{t_n,s}[|\tilde{S}_{n+1} - \hat{S}_{n+1}|^2] = O((\Delta t)^{\alpha+\beta}) \ (\Delta t \downarrow 0). \tag{3.10}$$

Proof. Let $F_n{}^{J_\alpha}(\Delta t)$ and $G_n{}^{J_\beta}(\Delta t)$ be the coefficients in (3.9) whose MSOs are equal to α and β, respectively. Then, in an analogous way to that in Lemma 2.2, we can prove that

$$E_{t_n,s}[|F_n{}^{J_\alpha}(\Delta t)G_n{}^{J_\beta}(\Delta t)|^2] = O((\Delta t)^{\alpha+\beta}) \ (\Delta t \downarrow 0). \tag{3.11}$$

Therefore, on account of the BCH formula (3.8), (3.11), and the definition of α and β given above, one may find that

(3.12)
$$\begin{aligned}
E_{t_n,s}[|\tilde{S}_{n+1} - \hat{S}_{n+1}|^2] &= E_{t_n,s}[|\exp(A_{n\Delta t} + B_{n\Delta t})(s) - \exp(A_{n\Delta t})\exp(B_{n\Delta t})(s)|^2] \\
&= E_{t_n,s}\Big[\Big| \exp(A_{n\Delta t} + B_{n\Delta t})(s) \\
&\qquad - \exp\Big(A_{n\Delta t} + B_{n\Delta t} + \frac{1}{2}[A_{n\Delta t}, B_{n\Delta t}] + \cdots\Big)(s)\Big|^2\Big] \\
&= O((\Delta t)^{\alpha+\beta}).
\end{aligned}$$

Thus, the local order between $(\tilde{S}_n)_{n=0}^N$ and $(\hat{S}_n)_{n=0}^N$ is given by $(\alpha+\beta)/2$.

Remark 3.3. By further manipulating the BCH formula to eliminate higher-order terms, we can obtain various schemes which give higher-order approximations to the exponential map. For example, the scheme corresponding to "leapfrog," which is well known in deterministic numerical analysis, is given by

$$\exp((\Delta t)(X+Y)) = \exp\left(\frac{\Delta t Y}{2}\right)\exp(\Delta t X)\exp\left(\frac{\Delta t Y}{2}\right) + O((\Delta t)^3). \tag{3.13}$$

In a way analogous to that in (3.4), we define a stochastic leapfrog scheme as follows:

$$\tilde{S}_{n+1} = \exp\left(\frac{B_{n\Delta t}}{2}\right)\exp(A_{n\Delta t})\exp\left(\frac{B_{n\Delta t}}{2}\right)(\tilde{S}_n) \ (n = 0, 1, \ldots, N-1). \tag{3.14}$$

Then, using the BCH formula (3.8) and (3.11) repeatedly, we can find that the local error for this scheme is given by $(\alpha+2\beta)/2$ as follows:

$$E_{t_n,s}[|\tilde{S}_{n+1} - \hat{S}_{n+1}|^2] = O((\Delta t)^{\alpha+2\beta}). \tag{3.15}$$

Thus, we can produce another numerical scheme having a better local order than that of the scheme (3.4). Moreover, using this scheme as a basis element for further leapfrog schemes, we may also produce an approximation to exponential map up to any order in a similar way to that in ordinary numerical analysis.

Total local error estimation for the numerical scheme of Procedures 1 and 2. Finally, we estimate the local error order between the exact discretized solutions $(S_n)_{n=0}^N$ and the numerical approximation solutions $(\tilde{S}_n)_{n=0}^N$. This is easily carried out by using Propositions 3.1 and 3.2 (or Remark 3.3) for the local orders in the above two procedures together with

$$E_{t_n,s}[|S_{n+1} - \tilde{S}_{n+1}|^2] \le E_{t_n,s}[|S_{n+1} - \hat{S}_{n+1}|^2] + E_{t_n,s}[|\hat{S}_{n+1} - \tilde{S}_{n+1}|^2], \tag{3.16}$$

and thereby we obtain the following theorem.

THEOREM 3.1. *Under the conditions of Propositions* 3.1 *and* 3.2,

$$E_{t_n,s}[|S_{n+1} - \tilde{S}_{n+1}|^2] \le O((\Delta t)^\delta)\ (\Delta t \downarrow 0) \tag{3.17}$$

holds, where $\delta = \min(\alpha + \beta, \gamma + 2)$ *in the case of* (3.4) *and* $\delta = \min(\alpha + 2\beta, \gamma + 2)$ *in the case of* (3.14).

On account of Definition 3.1, if (3.17) holds, we may regard $(\delta/2)$ as a local order for the scheme giving the numerical approximation solutions $(\tilde{S}_n)_{n=0}^N$. In this article, however, we call the value a local order in a "weak sense" for the scheme, since the estimation of error order is "indirectly" derived from the inequality (3.16).

3.3. Global error estimation and some examples for the new schemes. Now, we proceed to global error estimation for our schemes. We start with the definition of "global error order" in a manner analogous to that in [15] mentioned in Definition 3.1.

DEFINITION 3.2. *Suppose that* $S(t)$ *and* $(\tilde{S}_n)_{n=0}^N$ *are an exact solution and the numerical approximation solutions to SDE* (2.18), *respectively. Moreover, let* $E_{0,\xi}$ *be the expectation conditioned on starting at* ξ *at "initial time"* τ. *Then the global error order* λ *is defined by*

$$E_{0,s}[|\tilde{S}_N - S_N|^2] = O((\Delta t)^{2\lambda})\ (\Delta t \downarrow 0), \tag{3.18}$$

where $S_N = S(N\Delta t)$, $N\Delta t = T$ *and* $|\cdot|$ *denotes the Euclidean norm on the space* R^d.

Note that the condition in the expectation (3.18) means $S_0 = \tilde{S}_0 = s$. It is obvious that the accuracy of a numerical scheme improves with increasing global order.

Remark 3.4. In the framework of global error order defined by [23], the global order for the numerical solutions $\tilde{S}_n$ satisfying (3.18) is given by 2λ.

Using Theorem 3.1, we can prove the following lemma and thereby find the global order of our schemes in a weak sense.

LEMMA 3.1. *Suppose that the numerical approximation solutions* $(\tilde{S}_n)_{n=0}^N$ *to SDE* (2.18) *given by* (3.4) *or* (3.14) *satisfy* (3.17); *that is, the local order in a weak sense for the solutions is equal to* $(\delta/2)$. *Then*

$$E_{0,s}[|S_{n+1} - \tilde{S}_{n+1}|^2] \le (n+1) \times O((\Delta t)^\delta)\ (\Delta t \downarrow 0). \tag{3.19}$$

Proof. Here, we treat only the case of $(\tilde{S}_n)_{n=0}^N$ given by (3.4) because we can prove (3.19) for the case of numerical solutions by (3.14) in a way analogous to that shown below.

We prove this lemma by induction. First, if $n = 0$, the inequality (3.19) is obvious, since it reduces to (3.17). We assume that (3.19) holds in the case of $n = k-1$. Then, for $n = k$, we rewrite the left-hand side of (3.19) as follows:

$$
\begin{aligned}
(3.20)\qquad & E_{0,s}[|S_{k+1} - \tilde{S}_{k+1}|^2] \\
&= E_{0,s}[|S_{k+1} - \exp(A_{(k+1)\Delta t})\exp(B_{(k+1)\Delta t})(S_k) \\
&\qquad\qquad + \exp(A_{(k+1)\Delta t})\exp(B_{(k+1)\Delta t})(S_k) - \tilde{S}_{k+1}|^2] \\
&\le E_{0,s}[|S_{k+1} - \exp(A_{(k+1)\Delta t})\exp(B_{(k+1)\Delta t})(S_k)|^2] \\
&+ E_{0,s}[|\exp(A_{(k+1)\Delta t})\exp(B_{(k+1)\Delta t})(S_k) - \tilde{S}_{k+1}|^2],
\end{aligned}
$$

where $\exp(A_{(k+1)\Delta t})\exp(B_{(k+1)\Delta t})$ is the composition exponential map given by (3.4). Using (3.4) for $n = k$, that is, $\tilde{S}_{k+1} = \exp(A_{(k+1)\Delta t})\exp(B_{(k+1)\Delta t})(\tilde{S}_k)$, we may rewrite the first term on the right-hand side in (3.20) in the following form:

$$E_{0,s}[|S_{k+1} - \exp(A_{(k+1)\Delta t})\exp(B_{(k+1)\Delta t})(S_k)|^2] = E[|S_{k+1} - \tilde{S}_{k+1}|^2 | S_k = \tilde{S}_k].$$

This fact, together with the assumption about the local order for $(\tilde{S}_n)_{n=0}^N$, gives

$$E_{0,s}[|S_{k+1} - \exp(A_{(k+1)\Delta t})\exp(B_{(k+1)\Delta t})(S_k)|^2] \le O((\Delta t)^\delta).$$

In a similar manner, using (3.4), we can put the second term on the right-hand side in (3.20) into the following form:

$$
\begin{aligned}
& E_{0,s}[|\exp(A_{(k+1)\Delta t})\exp(B_{(k+1)\Delta t})(S_k) - \tilde{S}_{k+1}|^2] \\
&= E_{0,s}[|\exp(A_{(k+1)\Delta t})\exp(B_{(k+1)\Delta t})(S_k - \tilde{S}_k)|^2].
\end{aligned}
$$

Then, the definition of exponential map [16], [3], together with (3.8) and the assumption of the induction, proves

$$E_{0,s}[|\exp(A_{(k+1)\Delta t})\exp(B_{(k+1)\Delta t})(S_k - \tilde{S}_k)|^2] \le k \times O((\Delta t)^\delta).$$

Therefore, we find

$$E_{0,s}[|S_{k+1} - \tilde{S}_{k+1}|^2] \le (k+1) \times O((\Delta t)^\delta) \quad (\Delta t \downarrow 0), \tag{3.21}$$

and hence (3.19) holds for $n = k$, thus completing the proof. □

We note that $N\Delta t = T = constant$. Therefore, Lemma 3.1, together with this fact, straightforwardly proves the following theorem.

THEOREM 3.2. *Suppose that the local order in a weak sense for numerical approximation solutions $(\tilde{S}_n)_{n=0}^N$ given by (3.4) or (3.14) is equal to $(\delta/2)$. Then*

$$E_{0,s}[|S_N - \tilde{S}_N|^2] \le O((\Delta t)^{\delta-1}) \ (\Delta t \downarrow 0). \tag{3.22}$$

Thus, we can estimate the global error order for our schemes, although the estimation is indirectly derived through the inequality (3.22). Hence, on account of Definition 3.2, if (3.22) holds, we call the value $(\delta-1)/2$ a global order in a weak sense for the schemes of (3.4) or (3.14).

Now, in the following, we will investigate some examples of new numerical schemes for (2.18), which are derived from the procedures given above, and estimate the local and global errors on the basis of Theorems 3.1 and 3.2.

Example 3.1. Suppose that a truncated vector field $\hat{Y}_{n\Delta t}$ in Procedure 1 is given by

$$\begin{aligned} \hat{Y}_{n\Delta t} &= I_{(0),n}(\Delta t)X_0 + I_{(1),n}(\Delta t)X_1, \\ &= \Delta t X_0 + \Delta W_n X_1. \end{aligned} \tag{3.23}$$

On account of (2.26), we see that γ in Proposition 3.1 for this truncated vector field equals 1. We further set $A_{n\Delta t} = \Delta t X_0$ and $B_{n\Delta t} = \Delta W_n X_1$ in the decomposition (3.3) and assume that the explicit forms of both exponential maps for them are obtained through (2.13). In this case, α and β in Proposition 3.2 become 2 and 1, respectively, because of Lemma 2.1. Then, the scheme (3.4) can be put into the following form.

Scheme 3.1.

$$\tilde{S}_{n+1} = \exp(\Delta t X_0) \exp(\Delta W_n X_1)(\tilde{S}_n). \tag{3.24}$$

Assume that $[X_0, X_1] \neq 0$. Then, through Theorems 3.1 and 3.2, we obtain

$$E_{t_n,s}[|S_{n+1} - \tilde{S}_{n+1}|^2] \le O((\Delta t)^3) \tag{3.25}$$

and

$$E_{0,s}[|S_N - \tilde{S}_N|^2] \le O((\Delta t)^2). \tag{3.26}$$

Thus, we find that the local order and the global order in a weak sense for Scheme 3.1 equals 1.5 and 1, respectively.

Example 3.2. For $\hat{Y}_n(\Delta t)$ in Example 3.1, we set $A_{n\Delta t} = \Delta t X^A{}_0 + \Delta W_n X^A{}_1$ and $B_{n\Delta t} = \Delta t X^B{}_0 + \Delta W_n X^B{}_1$ in (3.3), where $X_0 = X^A{}_0 + X^B{}_0$ and $X_1 = X^A{}_1 + X^B{}_1$. We assume that $[X^A{}_1, X^B{}_1] \neq 0$ and that the explicit forms of both exponential maps for them are obtained. In this case, α and β in Proposition 3.2 are both equal to 1; hence the local order of the scheme (3.4) for the above $A_{n\Delta t}$ and $B_{n\Delta t}$ becomes 1. In order to produce a scheme having better accuracy than that of this scheme, we use (3.14) instead of (3.4).

Scheme 3.2.

$$\tilde{S}_{n+1} = \exp\left(\frac{B_{n\Delta t}}{2}\right) \exp(A_{n\Delta t}) \exp\left(\frac{B_{n\Delta t}}{2}\right), \tag{3.27}$$

where $A_{n\Delta t} = \Delta t X^A{}_0 + \Delta W_n X^A{}_1$ and $B_{n\Delta t} = \Delta t X^B{}_0 + \Delta W_n X^B{}_1$ under $X_0 = X^A{}_0 + X^B{}_0$ and $X_1 = X^A{}_1 + X^B{}_1$.

Then, Theorems 3.1 and 3.2 indicate that (3.25) and (3.26) also hold in this case. That is, the local error order and the global order of this scheme are equal to 1.5 and 1, respectively.

Example 3.3. We will formulate a scheme with better accuracy than that of the schemes described above. For this purpose, we choose the following vector field as $\hat{Y}_{n\Delta t}$ in (3.6):

$$\hat{Y}_{n\Delta t} = \Delta t X_0 + \Delta W_n X_1 + \frac{1}{2}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t))[X_0, X_1]. \tag{3.28}$$

Then, from (2.26), we see that γ in Proposition 3.1 for this truncated vector field becomes 2. Moreover, we set

$$\exp(A_{n\Delta t}) = \exp(\Delta t X_0) \tag{3.29}$$

and

$$\exp(B_{n\Delta t}) = \exp\left(\Delta W_n X_1 + \frac{1}{2}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t))[X_0, X_1]\right). \tag{3.30}$$

Assume that the explicit forms of both exponential maps for these are obtained through (2.13). In this case, Lemmas 2.1 and 2.2 show that α and β in Proposition 3.2 are equal to 2 and 1, respectively. Moreover, we adopt (3.14) for these vector fields which leads to the following scheme.

Scheme 3.3.

$$\tilde{S}_{n+1} = \exp\left(\frac{B_{n\Delta t}}{2}\right)\exp(A_{n\Delta t})\exp\left(\frac{B_{n\Delta t}}{2}\right)(\tilde{S}_n), \tag{3.31}$$

where $\exp(A_{n\Delta t})$ is given by (3.29) and $\exp(B_{n\Delta t}/2)$ is derived by replacing $B_{n\Delta t}$ by $B_{n\Delta t}/2$ in (3.30).

Then, because of Theorems 3.1 and 3.2, we find that

$$E_{t_n,s}[|S_{n+1} - \tilde{S}_{n+1}|^2] \leq O((\Delta t)^4) \tag{3.32}$$

and

$$E_{0,s}[|S_N - \tilde{S}_N|^2] \leq O((\Delta t)^3), \tag{3.33}$$

and hence conclude that the local order and the global order in a weak sense for Scheme 3.3 equals 2 and 1.5, respectively.

4. Examples. In this section, we will give several illustrative examples of applying our new stochastic numerical schemes to SDEs.

4.1. Numerical simulation of a nonlinear asset price process in mathematical finance. As was mentioned in section 1, we first work with the following nonlinear scalar SDE which is often treated as a model of an asset price process of Bessel type in mathematical finance [9] (cf. Remark 4.1):

$$dS(t) = S(t)dt + 2\sqrt{S(t)} \circ dW(t), \quad S(0) = s(> 0). \tag{4.1}$$

This system has a structure such that the value of solution remains nonnegative for any $t \in [0, T]$. In standard stochastic numerical schemes, however, this property is not always preserved numerically; in particular, if an initial value s is close to zero, the numerical solutions often go into the domain of negative values in the midst of numerical simulations. Such troublesome behavior will be observed in the numerical results given later. In contrast with this, through the results in previous section, we may obtain a scheme which leaves the structure of the stochastic system (4.1) invariant numerically. We will examine it now.

First, from (2.6) and (4.1), we see that the vector fields X_0 and X_1 become

$$X_0 = S\frac{d}{dS}, \; X_1 = 2\sqrt{S}\frac{d}{dS}, \tag{4.2}$$

respectively. Here, we note that $[X_0, X_1] = -X_1/2$, and the Lie algebra generated by X_0 and X_1 is of finite dimension. Hence, Remark 2.2 indicates that (2.26) actually converges in this case.

We proceed to investigate the application of Scheme 3.1 to SDE (4.1). On account of (3.24), we suppose that $A_{n\Delta t}$ and $B_{n\Delta t}$ in (3.3) are given by

$$A_{n\Delta t} = \Delta t X_0 = \Delta t S\frac{d}{dS}, \quad B_{n\Delta t} = \Delta W_n X_1 = \Delta W_n 2\sqrt{S}\frac{d}{dS}. \tag{4.3}$$

Then, in view of (2.13), we obtain the exponential maps for $A_{n\Delta t}$ and $B_{n\Delta t}$ explicitly as follows:

$$\exp(A_{n\Delta t})(s) = s\exp(\Delta t), \quad \exp(B_{n\Delta t})(s) = \{\Delta W_n + \sqrt{s}\}^2. \tag{4.4}$$

Inserting these into (3.24), we find that Scheme 3.1 applied to SDE (4.1) leads to

$$\tilde{S}_{n+1} = \left\{\Delta W_n + \sqrt{\tilde{S}_n}\right\}^2 \exp(\Delta t), \tag{4.5}$$

where $\tilde{S}_0 = S(0) = s$. Clearly, the numerical solutions derived from our scheme *never* take negative values, and this is just the result we want. Moreover, as mentioned in section 3.3, the local and global orders in a weak sense for this scheme are given by 1.5 and 1, respectively.

Next we will apply Scheme 3.3 to SDE (4.1). On account of (3.29), in this case, we also obtain $s\exp(\Delta t)$ as $\exp(A_{n\Delta t})(s)$. In contrast with this, (3.30) takes the form

$$\exp(B_{n\Delta t}) = \exp\left\{\left(\Delta W_n - \frac{1}{4}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t))\right) X_1\right\}, \tag{4.6}$$

since $[X_0, X_1] = -X_1/2$. In a way similar to that of Scheme 3.1, this is also calculated explicitly as follows:

$$\exp(B_{n\Delta t})(s) = \left\{\Delta W_n - \frac{1}{4}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t)) + \sqrt{s}\right\}^2, \tag{4.7}$$

and thereby we obtain the result of Scheme 3.3 applied to SDE (4.1) as

$$\tilde{S}_{n+1} = \left\{\frac{\Delta W_n}{2} - \frac{1}{8}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t)) + \sqrt{\hat{s}}\right\}^2, \tag{4.8}$$

together with

$$\hat{s} = \left\{\frac{\Delta W_n}{2} - \frac{1}{8}(I_{(0,1),n}(\Delta t) - I_{(1,0),n}(\Delta t)) + \sqrt{\tilde{S}_n}\right\}^2 \exp(\Delta t), \tag{4.9}$$

where $\tilde{S}_0 = S(0) = s$. This also indicates that the numerical solutions derived from this scheme take only nonnegative values. Then, as mentioned in section 3.3, the local and global orders for this scheme are equal to 2 and 1.5, respectively.

Here, we will examine the numerical solutions of SDE (4.1) given by these schemes. Then, we will compare these numerical results with those of several standard numerical schemes. For this purpose, we adopt the Euler–Maruyama scheme (Taylor scheme of

TABLE 1

An example of numerical solutions to SDE (4.1) from the Euler–Maruyama scheme (4.10), Kloeden's Taylor scheme (4.11), Scheme (4.5) and Scheme (4.8) with (4.9) for an initial value $s = 0.01$.

n	Euler–Maruyama	Kloeden's Taylor	Scheme (4.5)	Scheme (4.8)
189	0.0373054	0.014018	0.0140095	0.0140059
190	0.0459159	0.0190609	0.0190539	0.0190461
191	0.0428106	0.0164982	0.01649	0.0164839
192	0.0524778	0.0223075	0.0222992	0.0222907
193	0.0365604	0.0126323	0.0126217	0.0126192
194	0.0190258	4.07906E-03	0.0040724	4.07239E-03
195	0.0120115	1.20801E-03	1.20498E-03	1.20484E-03
196	7.86052E-04	4.42925E-04	4.46361E-04	4.45822E-04
197	2.07421E-03	6.85707E-04	6.89868E-04	6.8875E-04
198	-9.47943E-05	7.45417E-05	7.31484E-05	7.32705E-05

global order 0.5) and Kloeden's Taylor scheme of global order 1.5 [15], [23]; they are given in the following forms for the SDE (4.1):

Euler–Maruyama scheme.

$$S_{n+1} = S_n + (S_n + 1)\Delta t + 2\sqrt{S_n}\Delta W_n. \tag{4.10}$$

Kloeden's Taylor scheme of global order 1.5.

$$\begin{aligned} S_{n+1} = S_n &+ (S_n + 1)\Delta t + 2\sqrt{S_n}\Delta W_n \\ &+ \{(\Delta W_n)^2 - \Delta t\} \\ &+ 2\sqrt{S_n} I_{(1,0),n}(\Delta t) + \sqrt{S_n} I_{(0,1),n}(\Delta t) \\ &+ \frac{1}{2}(S_n + 1)(\Delta t)^2. \end{aligned} \tag{4.11}$$

In the schemes (4.5), (4.8) with (4.9), (4.10), and (4.11), ΔW_n, $I_{(1,0),n}(\Delta t)$, and $I_{(0,1),n}(\Delta t)$ are numerically realized by the independent $N(0,1)$ random numbers γ_n and $\hat{\gamma}_n$ ($n = 0, 1, \ldots$) as follows [15]:

$$\begin{aligned} \Delta W_n &= \gamma_n \sqrt{\Delta t}, \\ I_{(1,0),n}(\Delta t) &= \frac{1}{2}\left(\gamma_n + \frac{1}{\sqrt{3}}\hat{\gamma}_n\right)(\Delta t)^{3/2}, \\ I_{(0,1),n}(\Delta t) &= \frac{1}{2}\left(\gamma_n - \frac{1}{\sqrt{3}}\hat{\gamma}_n\right)(\Delta t)^{3/2}. \end{aligned} \tag{4.12}$$

Moreover, we choose $T = 1$ and $N = 1000$ here, and hence the stepsize $\Delta t = 10^{-3}$.

Tables 1 and 2 display the results of the numerical solutions from these schemes listed above for the initial value $s = 0.01$ and $s = 0.001$, respectively. (Note: in the case of Table 2, the scheme (4.10) is excluded.) Here we have used the same sequences of random numbers for each scheme together with (4.12). As was mentioned in the introductory part of this section, from these results we observe that the values of numerical solutions derived from the standard schemes become negative during their computation if their initial values are close to zero. In contrast with these results, all of our schemes are free from such problems. Thus, our scheme (4.5) and (4.8) with (4.9) are superior with respect to numerical realization of the character of SDE (4.1); that is, nonnegativity of solutions is preserved in contrast to the results of the standard schemes.

TABLE 2

An example of numerical solutions to SDE (4.1) *from Kloeden's Taylor scheme* (4.11), *Scheme* (4.5) *and Scheme* (4.8) *with* (4.9) *for an initial value* $s = 0.001$.

n	Kloeden's Taylor	Scheme (4.5)	Scheme (4.8)
358	4.28647E-03	4.38672E-03	4.27267E-03
359	7.06959E-03	7.19804E-03	7.05171E-03
360	2.92672E-04	3.21066E-04	0.0002912
361	5.76257E-04	6.15458E-04	5.7371E-04
362	2.57058E-03	2.65334E-03	2.56494E-03
363	3.09359E-03	3.18363E-03	3.08695E-03
364	8.20704E-04	8.66758E-04	8.17091E-04
365	2.12819E-04	1.90873E-04	2.14876E-04
366	5.3766E-04	5.01956E-04	5.40526E-04
367	-2.97776E-07	7.52643E-07	7.81725E-10

TABLE 3

An example of numerical solutions to SDE (4.1) *from the Euler–Maruyama scheme* (4.10), *Kloeden's Taylor scheme* (4.11), *Scheme* (4.5) *and Scheme* (4.8) *with* (4.9) *for an initial value* $s = 1$.

n	Euler–Maruyama	Kloeden's Taylor	Scheme (4.5)	Scheme (4.8)
991	4.33234	4.29109	4.29002	4.29109
992	4.25444	4.21288	4.2118	4.21288
993	3.97593	3.9392	3.93813	3.9392
994	3.72468	3.69204	3.69095	3.69204
995	3.48053	3.4519	3.45081	3.4519
996	3.6467	3.61833	3.61729	3.61834
997	3.79678	3.76838	3.76736	3.76838
998	3.74771	3.71863	3.7176	3.71863
999	3.72407	3.69411	3.69307	3.69412
1000	3.62125	3.5914	3.59035	3.59141

Table 3 displays the results for the initial value $s = 1$. In Table 3, it is observed that numerical results of the schemes (4.5) and (4.8) are closer to those of Kloeden's scheme (4.11) than to the results of Euler's scheme (4.10). In particular, the numerical solutions from (4.8) are very similar to those of (4.11); from theoretical consideration of local and global orders in section 3.3, these observations are as expected.

Remark 4.1. Let us consider the following SDE:

$$dS(t) = S(t)dt + \frac{1}{1-\gamma}\{S(t)\}^{\gamma} \circ dW(t), \quad S(0) = s(> 0),$$

where $0 < \gamma < 1$. This is also often treated as a model of an asset price process in mathematical finance, and it is a generalization of (4.1). For this process, we can also construct numerical schemes like those described above in a similar way. Indeed, Scheme 3.1 for this SDE, of which local and global orders equal 1.5 and 1, respectively, is given by

$$\tilde{S}_{n+1} = [\{\Delta W_n + \tilde{S}_n^{1-\gamma}\}^2]^{1/\{2(1-\gamma)\}} \exp(\Delta t),$$

where $\tilde{S}_0 = S(0) = s$. Note that the numerical solutions derived from this scheme also satisfy nonnegativity.

4.2. Example of Scheme 3.2 for a nonlinear SDE. We study an example for Scheme 3.2 given by (3.27). Let us consider the following nonlinear scalar SDE:

$$dS(t) = S(t)dt + \{S(t) + 2\sqrt{S(t)}\} \circ dW(t), \quad S(0) = s(> 0). \tag{4.13}$$

In this case, the vector fields X_0 and X_1 are set as

$$X_0 = S\frac{d}{dS}, \quad X_1 = (S + 2\sqrt{S})\frac{d}{dS}, \tag{4.14}$$

respectively. Then, we remark that $[X_0, X_1] = -\sqrt{S}(d/dS)$, $[X_0, [X_0, X_1]] = -[X_0, X_1]/2$, and $[X_1, [X_0, X_1]] = -[X_0, X_1]/2$ hold; hence the Lie algebra generated by X_0 and X_1 is of finite dimension. Therefore, as in section 4.1, (2.26) also actually converges in this case.

We may regard the SDE (4.13) as a linear SDE with the additional random perturbation $2\sqrt{S(t)} \circ dW(t)$. On account of this, as $A_{n\Delta t}$ and $B_{n\Delta t}$ in (3.27), we adopt

$$A_{n\Delta t} = \Delta t S\frac{d}{dS} + \Delta W_n S\frac{d}{dS}, \quad B_{n\Delta t} = \Delta W_n 2\sqrt{S}\frac{d}{dS}; \tag{4.15}$$

that is, we set $X_0^A = S(d/dS)$, $X_1^A = S(d/dS)$, $X_0^B = 0$, and $X_1^B = 2\sqrt{S}(d/dS)$. Then, in view of (2.13), we obtain the exponential maps for them explicitly as follows:

$$\exp(A_{n\Delta t})(s) = s\exp(\Delta t + \Delta W_n), \quad \exp(B_{n\Delta t})(s) = \{\Delta W_n + \sqrt{s}\}^2. \tag{4.16}$$

Inserting these equations into (3.27), we find Scheme 3.2 for the SDE (4.13); it is given by

$$\tilde{S}_{n+1} = \left\{\Delta W_n/2 + \sqrt{\{\Delta W_n/2 + \sqrt{\tilde{S}_n}\}^2 \exp(\Delta t + \Delta W_n)}\right\}^2, \tag{4.17}$$

where $\tilde{S}_0 = S(0) = s$. Then, the theoretical consideration of error stimation in section 3.3 proves that the local and global orders for this scheme equal 1.5 and 1, respectively.

Here, in a way similar to that in section 4.1, we will observe the results of numerical solutions through this scheme and compare them with the results of the following Euler–Maruyama scheme and Kloeden's Taylor scheme for (4.13):

Euler–Maruyama scheme.

$$S_{n+1} = S_n + \left\{\frac{3}{2}(S_n + \sqrt{S_n}) + 1\right\}\Delta t + (S_n + 2\sqrt{S_n})\Delta W_n. \tag{4.18}$$

Kloeden's Taylor scheme of global order 1.5.

$$\begin{aligned} S_{n+1} = S_n &+ \left\{\frac{3}{2}(S_n + \sqrt{S_n}) + 1\right\}\Delta t + (S_n + 2\sqrt{S_n})\Delta W_n \\ &+ \frac{1}{2}\left(S_n + 3\sqrt{S_n} + 2\right)\{(\Delta W_n)^2 - \Delta t\} \\ &+ \frac{3}{2}\left(S_n + \frac{5}{2}\sqrt{S_n} + 1\right) I_{(1,0),n}(\Delta t) + \frac{3}{2}\left(S_n + \frac{11}{6}\sqrt{S_n} + 1\right) I_{(0,1),n}(\Delta t) \\ &+ \frac{1}{6}\left(S_n + \frac{7}{2}\sqrt{S_n} + 3\right)\{(\Delta W_n)^3 - 3\Delta t\Delta W_n\} \\ &+ \frac{9}{8}\left(S_n + \frac{17}{12}\sqrt{S_n} + \frac{5}{6}\right)(\Delta t)^2. \end{aligned} \tag{4.19}$$

Finally, inserting (4.12) into the schemes (4.17)–(4.19), we obtain numerical solutions which are shown in Table 4 for the initial value $s = 1$, $T = 1$, $N = 1000$, and $\Delta t = 10^{-3}$.

TABLE 4
An example of numerical solutions to SDE (4.13) from the Euler–Maruyama scheme (4.18), Kloeden's Taylor scheme (4.19) and Scheme (4.17) for an initial value $s = 1$.

n	Euler–Maruyama	Kloeden's Taylor	Scheme (4.17)
991	9.07412	9.83114	9.83382
992	9.77413	10.5897	10.5926
993	9.35542	10.1398	10.1426
994	9.79685	10.6085	10.6114
995	9.86074	10.6656	10.6686
996	9.91146	10.7086	10.7115
997	9.57355	10.3428	10.3456
998	9.27501	10.0183	10.021
999	9.64218	10.4042	10.407
1000	9.47959	10.2216	10.2244

From Table 4 it is also observed that the numerical results of the scheme (4.17) are closer to those of Kloeden's scheme than to the results of Euler's scheme. On account of local and global orders for each scheme, this is also to be expected.

4.3. Composition method applied to stochastic Hamiltonian dynamical systems. As mentioned in section 1, composition methods (or operator splitting methods) are not only a superior integration method for differential equations in preserving the special character or structure of the equations but also often useful for approximations of nonlinear equations whose solutions are not obtained explicitly. The results in sections 4.1 and 4.2 given above show that this is also true in the case of stochastic systems. As also mentioned, such an advantage is remarkable in the case of dynamical systems with multiple space dimensions or Hamiltonian dynamical systems using dimensional splitting methods (e.g., [25], [12]). To illustrate this, we will investigate numerical schemes by composition methods for stochastic dynamical systems with "Hamiltonian structure" [20], [21].

First we review stochastic Hamiltonian dynamical systems [20]. Let us consider the following 2ℓ-dimensional stochastic dynamical system:

$$d\begin{pmatrix} x^i(t) \\ x^{\ell+i}(t) \end{pmatrix} = \begin{pmatrix} \partial_{\ell+i}H_0(x(t)) \\ -\partial_i H_0(x(t)) \end{pmatrix} dt + \begin{pmatrix} \partial_{\ell+i}H_1(x(t)) \\ -\partial_i H_1(x(t)) \end{pmatrix} \circ dW(t) \quad (i = 1, \cdots \ell), \tag{4.20}$$

where $x = (x^k)_{k=1}^{2\ell}$ and $\partial_j = \partial/\partial x^j$ $(j = 1, 2, \ldots, 2\ell)$, respectively. In (4.20), $H_\alpha(x)$ $(\alpha = 0, 1)$ are smooth scalar functions on $R^{2\ell}$. Formally, one may regard this as a Hamiltonian dynamical system

$$\frac{d}{dt}\begin{pmatrix} x^i \\ x^{\ell+i} \end{pmatrix} = \begin{pmatrix} \partial_{\ell+i}\hat{H}(x) \\ -\partial_i\hat{H}(x), \end{pmatrix} \quad (i = 1, \ldots, \ell)$$

with a "randomized" Hamiltonian $\hat{H}$ given by

$$\hat{H} = H_0 + H_1\gamma_t,$$

where γ_t is a one-dimensional Gaussian white noise. With these definitions, we call (4.20) and H_α $(\alpha = 0, 1)$ an (ℓ-dimensional) *stochastic Hamiltonian dynamical system* and the *Hamiltonian*, respectively.

Now, we proceed to an illustration of our new scheme for stochastic Hamiltonian systems. For simplicity, we set $\ell = 1$ and denote $x^1(t)$ and $x^2(t)$ by $q(t)$ and $p(t)$, respectively. Let us consider the class of Hamiltonian systems with the typical Hamiltonian $H_0 = p^2/2 + V_0(q)$ and $H_1 = p^2/2 + V_1(q)$, where $V_0(q)$ and $V_1(q)$ are any potential functions. With this notation, (4.20) becomes

$$d\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} p(t) \\ -V_0'(q(t)) \end{pmatrix} dt + \begin{pmatrix} p(t) \\ -V_1'(q(t)) \end{pmatrix} \circ dW(t). \tag{4.21}$$

In general, this is a stochastic nonlinear system. For this system, the vector fields X_0 and X_1 become

$$X_0 = p\partial_q - V_0'(q)\partial_p, \quad X_1 = p\partial_q - V_1'(q)\partial_p, \tag{4.22}$$

respectively.

We will apply our scheme to this system. As an important example of the decomposition of (3.3) for the above system, we choose the following splitting:

$$A_{n\Delta t} = p(\Delta t + \Delta W_n)\partial_q, \quad B_{n\Delta t} = -(V_0'(q)\Delta t + V_1'(q)\Delta W_n)\partial_p. \tag{4.23}$$

This corresponds to the decomposition mentioned in Scheme 3.2; that is, X_0^A, X_1^A, X_0^B, and X_1^B in Scheme 3.2 are given by $p\partial_q$, $p\partial_q$, $-V_0'(q)\partial_p$, and $-V_1'(q)\partial_p$, respectively. Then we note that $\exp(A_{n\Delta t})$ and $\exp(B_{n\Delta t})$ are exponential maps which correspond to the flows of solutions to the following SDEs, respectively:

$$d\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} p(t) \\ 0 \end{pmatrix} dt + \begin{pmatrix} p(t) \\ 0 \end{pmatrix} \circ dW(t),$$

$$d\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -V_0'(q(t)) \end{pmatrix} dt + \begin{pmatrix} 0 \\ -V_1'(q(t)) \end{pmatrix} \circ dW(t).$$

Therefore, we can obtain the explicit forms of them; this may be regarded as an example of dimensional splitting. The results are given by

$$\exp(A_{n\Delta t})\begin{pmatrix} q_n \\ p_n \end{pmatrix} = \begin{pmatrix} p_n(\Delta t + \Delta W_n) + q_n \\ p_n \end{pmatrix},$$

$$\exp(B_{n\Delta t})\begin{pmatrix} q_n \\ p_n \end{pmatrix} = \begin{pmatrix} q_n \\ -(\Delta t V_0'(q_n) + \Delta W_n V_1'(q_n)) + p_n \end{pmatrix}.$$

Inserting these equations into (3.27), we finally find Scheme 3.2 for this system as follows:

$$\begin{pmatrix} \tilde{q}_{n+1} \\ \tilde{p}_{n+1} \end{pmatrix} = \begin{pmatrix} \hat{q}_n \\ -\frac{1}{2}(\Delta t V_0'(\hat{q}_n) + \Delta W_n V_1'(\hat{q}_n)) + \hat{p}_n \end{pmatrix}, \tag{4.24}$$

where

$$\begin{pmatrix} \hat{q}_n \\ \hat{p}_n \end{pmatrix} = \begin{pmatrix} \hat{p}_n(\Delta t + \Delta W_n) + \tilde{q}_n \\ -\frac{1}{2}(\Delta t V_0'(\tilde{q}_n) + \Delta W_n V_1'(\tilde{q}_n)) + \tilde{p}_n \end{pmatrix}. \tag{4.25}$$

As in the example in section 4.2, the local and global orders in a weak sense for this scheme are equal to 1.5 and 1, respectively. Thus, for the class of stochastic Hamiltonian dynamical systems with typical Hamiltonians mentioned above, we can numerically approximate them through our scheme (4.24) with (4.25) and achieve an accuracy corresponding to Taylor scheme of global order 1.

4.4. Remark on composition methods and conserved quantities in stochastic dynamical systems. Finally, we remark on numerical schemes for stochastic dynamical systems which preserve "conserved quantities" of the systems. It is well known that conserved quantities play an essential role to determine the structure of dynamical systems; hence, it is important to find a numerical scheme which has the conservation properties for the quantities related to stochastic systems. On the other hand, composition methods often give such schemes for deterministic systems. Therefore, we may expect that one may obtain such schemes through our results, which have the advantage of numerically preserving the conserved quantities for stochastic systems; and in the remainder of this section, we will briefly examine this feature of our schemes.

Let us consider d-dimensional stochastic dynamical systems (2.18). Suppose that a smooth function $I = I(S)$ satisfies

$$X_0 I = 0, \quad X_1 I = 0, \tag{4.26}$$

where X_0 and X_1 are the vector fields given by (2.6). According to [20], $I(S)$ becomes a constant quantity; that is, $I(S(t)) = \textit{constant}$ holds for the diffusion process $S(t)$ governed by SDE (2.18).

Under some conditions, we may straightforwardly formulate a stochastic scheme satisfying numerical preservation of conserved quantities. Assume that the exponential maps of $A_{n\Delta t} = \Delta t X_0$ and $B_{n\Delta t} = \Delta W_n X_1$ are explicitly calculated. Then, it is obvious that Scheme 3.1 preserves the conserved quantity I numerically because of the definition of exponential map and (4.26).

Now, we investigate a trivial example of a stochastic dynamical system with a conserved quantity and the numerical scheme through composition methods. Let us consider

$$d\begin{pmatrix} S^1(t) \\ S^2(t) \end{pmatrix} = \begin{pmatrix} S^2(t) \\ -S^1(t) \end{pmatrix} dt + \begin{pmatrix} S^2(t) \\ -S^1(t) \end{pmatrix} \circ dW(t); \tag{4.27}$$

this is a stochastic system with the conserved quantity $I(S) = \frac{1}{2}((S^1)^2 + (S^2)^2)$, since (4.26) holds. However, as mentioned in [21], ordinary schemes do not conserve $I(S)$ numerically. On the other hand, for this system, we adopt Scheme 3.1 with $A_{n\Delta t} = \Delta t X_0 = \Delta t(S^2\partial_1 - S^1\partial_2)$ and $B_{n\Delta t} = \Delta W_n X_1 = \Delta W_n(S^2\partial_1 - S^1\partial_2)$; then through (2.13), the numerical scheme is explicitly given by

$$\begin{pmatrix} \tilde{S}^1_{n+1} \\ \tilde{S}^2_{n+1} \end{pmatrix} = \begin{pmatrix} \cos(\Delta t) & \sin(\Delta t) \\ -\sin(\Delta t) & \cos(\Delta t) \end{pmatrix} \begin{pmatrix} \cos(\Delta W_n) & \sin(\Delta W_n) \\ -\sin(\Delta W_n) & \cos(\Delta W_n) \end{pmatrix} \begin{pmatrix} \tilde{S}^1_n \\ \tilde{S}^2_n \end{pmatrix}. \tag{4.28}$$

Therefore, for any n, the numerical solutions (4.28) for (4.27) satisfy $I(\tilde{S}^1_n, \tilde{S}^2_n) = \textit{constant}$. Thus, our scheme numerically preserves a conserved quantity I of the stochastic system (4.27), and this fact also shows the superiority of the scheme derived through composition methods.

5. Concluding remarks. In this article, we have formulated composition methods for SDEs, and through these we have proposed some stochastic numerical schemes. Several illustrative examples show that the new schemes are superior in their conservation properties related to the character of SDEs, and they are useful for approximating the solutions to SDEs. Moreover, we have investigated local and global error orders for our schemes. Finally, we offer some remarks and note some future research problems related to this work.

(i) As mentioned in Remark 3.2, we plan to carry out a more detailed analytical error estimation for our schemes using the result on time asymptotics of exponential maps for SDEs by [6], since the stochastic series (2.26) is only a formal representation.

(ii) In our error estimation, we have addressed local and global error orders "in a weak sense" for our new schemes, since we have *indirectly* estimated the error orders for numerical solutions using these schemes. To obtain a precise local or global error order, it would be necessary to carry out a direct error estimation for such numerical solutions.

(iii) In this article, we have treated the SDEs with a one-dimensional Wiener process. However, it often happens that the error order of a numerical method collapses if there is more than one Wiener process. Hence, it would be important to investigate the error orders of our schemes in the case of SDEs with a multidimensional Wiener process. Moreover, in any approach to this problem, account should be taken of the remarkable work by K. Burrage and P.M. Burrage [4], [5]. In their papers, they have developed stochastic Runge–Kutta schemes of higher order for such SDEs through the Magnus formula related to Lie algebra. Therefore, based on their work, we should be able to improve our composition methods and offer new schemes with high order for SDEs with a multidimensional Wiener process.

(iv) It is to be noted that Li and Liu [18] and Kunita [17] have studied stochastic exponential maps for a more general class of stochastic processes (e.g., Lévy processes). Hence, using their results, it would be interesting to formulate stochastic composition methods for such general stochastic processes.

The research on these topics will be reported in future papers.

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■統一論題

「情報」からみた企業の競争力と経営戦略
ー企業行動の実証分析に基づいてー

Competitive Position of Firms and Management Strategy from the Perspective of "information" -on the Basis of Empirical Analysis of Firms Behavior-

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1．はじめに

外部環境の急激な変化の中で、企業の競争力が問われている。「情報」あるいは「情報化」、「グローバル化」が、外部環境の変化として取り上げられる。競争の実態は、グローバルな広がりを持った情報化経済システムにおける企業間競争として理解される。経済システムを情報システムとして理解する試みは、それほど新しいことではなく（北川、1966）、システムや組織一般、あるいは主体の行動一般が、情報の伝達と処理の観点から論じられてきた。

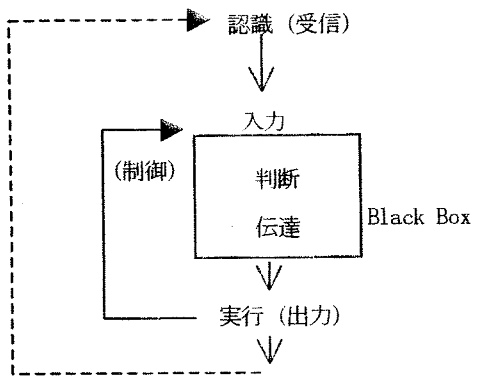
本稿では、企業の活動を「情報」の観点からとらえ直すことにより、情報化経済システムにおける企業の競争力について分析、検討する。その際、企業の競争力についての時系列分析と成果を検討材料として用いる。

以下では、情報処理システムとして企業をとらえ（2-1）、経営戦略を位置付け、生産効率を通して経営戦略と競争力の関係について検討する（2-2）。次に、経営戦略の実行主体としての組織と組織を動かす人的資源について情報処理プロセスの視点から検討し、企業競争力との関係を論じる（2-3, 2-4）。最後に、企業活動をマクロシステムとしての情報化経済システムの中に位置付け、マクロシステムの競争力について論じ、結びとする。

2－1．情報処理システムとしての企業活動

企業活動を「情報」の観点からとらえたとき、その基本的構造は図1によって与えられる。矢印は「情報の伝達」を意味する。

図1



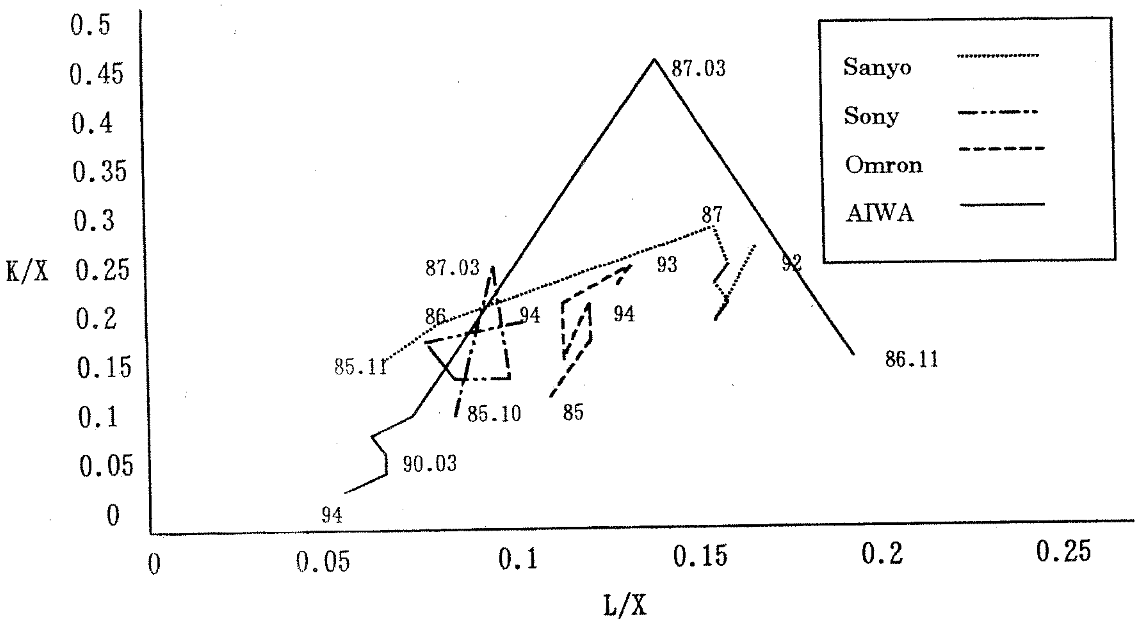
企業は、状況を認識し、認識した情報をデータとして入力し、出力するシステムとしてとらえられる。出力は、情報のフィードバックを通して制御される。入力から出力に至る情報処理プロセスは、ブラックボックスとして扱われることもあるが、その内容は、判断・意思決定プロセスとデータの伝達プロセスからなる。認識、判断をはじめとしてシステム全体のあるべき姿

は、何らかの価値基準によって規定されることになる。その基準と方向性を与えるのが企業の目標システムであり、経営戦略である。マクロ的構造変化に見られるような外部環境の変化に対する適応は、通常、経営戦略の変更を伴う。点線のフィードバックにより、システム全体のあり方に変更がもたらされる。入力システム、情報処理システム、出力システムは、組織とそこで組み合わされる生産資源・人的資源の問題としてとらえられる。

2－2．経営戦略と企業競争力

経営戦略は、企業の目標システムとともに、図1で示されるシステムのあり方を規定する。図2は、フロンティア生産関数を用いた分析をもとに（Albach,1988）、日本の電気産業の生産効率性経路を描いたものである（Waragai & Watanabe,1998）．

図2



効率性経路の原点への接近は、効率性の向上を意味する。図2は4つの代表的パターンを示しているが、一般に、1985年～1987年の円高の時期に効率性は悪化し、1987年～1990年の「バブル期」に改善、それ以後のバブルの崩壊とともに効率性は悪化した。1985年当時と、1997年現在を比較したとき、検討対象企業54社中、43社の生産効率性は大きく悪化した。日本企業・日本経済のグローバル市場における競争力の弱体化を、生産サイドから説明するものである。ここでは、ソニーとアイワに注目する。両社はともにバブル期に積極的な投資活動を展開した。ソニーは、国内生産にウエートを置いた戦略を採用し、商品開発に力を注いだ。アイワは生産拠点をアジアへ移し、積極的な海外戦略を展開した。効率性経路の動きは、「バブル期」におけるそれら経営戦略の展開に裏打ちされており、以後の国際市場における両社の競争力の高さを示している。しかし、両社の戦略における国外生産の位置付けの相違は、円高の影響に違いをもたらした。ソニーにおいては一層の合理化とリストラが強く論じられることとなった。

2－3．組織の最適化と企業競争力

経営戦略は、入力～出力までの情報処理システム、すなわち組織を通して実現が図られる。ここに、組織の最適化と企業競争力が問題となる。組織の最適化の問題を、理論的・実証的に扱ったのが Albach(1989) である。そこでは階層的組織の効率性と最適化が、取引費用最少化を通して論じられている。すなわち、情報を費用の観点から考察することにより、組織の効率性についての経営経済学的、ミクロ経済学的根拠付けと理論的枠組の構築が図られている。

組織の効率性について、組織変更の観点から実証的に検討したのが Waragai(1989) である。赤池情報量(AIC)に基づく構造変化検証モデル(Waragai,1990; Waragai & Akiba,1989)を用いて、組織変更が企業指標にいかなる影響を与えるかについて論じている(注１)。旧西ドイツにおける化学企業３社(Bayer, BASF, Hoechst)が、1980年代前半の不況期に、石油危機以降の厳しい資源制約とマルク高、そしてグローバル化を内容とする外部環境の変化に対して、いかなる戦略的組織変更をおこなったか、それについて明かにするとともに、その戦略的組織変更が、いかに企業の競争力に影響を与えたかについて、企業指標の変化から読み取ることを行った。その結果、検討対象である組織変更が、利潤関連指標に改善をもたらしたことが検証された。すなわち、Bayer社は、1984年に国際関連部門の再編成と部門間責任範囲の見直しを内容とする大規模な組織変更を行ったが、1984年第四四半期～85年の第一四半期の時点を境に、利益改善指標が改善した（表１）。BASF

表１

インデックス	推計期間	変化時点
利益	1975Ⅲ-86Ⅳ	1984Ⅳ-85Ⅰ
利益/雇用者数	1975Ⅲ-86Ⅳ	1984Ⅳ-85Ⅰ

社においても同様な変化が見られた。他方、この時期に組織変更を行わなかったHoechst社の利益指標は、マクロ経済的動向に対応して停滞した動きを示した。

２－４．人的資源と企業の競争力

Waragai(1990)は、人的資源が与える投資活動への影響について、賃金の観点から論じた。投資活動を取り上げ、石油危機時に発揮した日本経済・企業の適応力の大きさについて、日独比較を通して、説明が与えられた。表２は、化学産業のパラメータ値を示している（注２）。推計された「投資関数」の賃金パラメータ aw, aw-1 に注目する。第II期の負のパラメータ値は、我が国経済の高度成長を支えてきた外部環境が1960年代後半に崩れ、適応・合理化のプロセスが始まる中、賃金上昇がが投資の阻害要因であったことを示している。第III期のパラメータ値の回復は、石油危機後の「労使協調」が賃金上昇を押さえ、賃金が投資の阻害要因になっていないことを意味している。ドイツの場合、石油危機以後の賃金上昇が投資活動の大きな阻害要因となっており、他の欧米諸国と同様に、当該時期のパラメータ値は、大きな負の値を示した。日本企業の適応力を支え、日本企業が発揮した競争力に、一つの説明を与えるものである。

表２

期間	定数	aub	aub-1	aw	aw-1	acf	al/K-1	az	az-1	aB	aB-1
Ⅰ	-.0038	-	-	.1192	.8698	.8583	-	-.8693	-.0083	-	.0312
Ⅱ	-.1076	-	-	-.2.5284	-	2.9954	-	-	-	-	-
Ⅲ	.1048	-	.5593	.2349	-	.229*	-	-	-	-	.052
Ⅳ	-.6095	-	-	-.8529	-	3.5536	-	-	-	-	-

期間：Ⅰ（1966-69年），Ⅱ（1970-73年），Ⅲ（1974-81年），Ⅳ（1982-85年）。　a　はパラメータを示し、添字は　I:投資、K：資本、UB：業種売上、W:賃金、CF:キャッシュフロー、Z:市場利子率、B:株式指数　を意味する。－１は1期のタイムラグを示す.

３．結び

情報システム全体のあり方を決定する経営戦略の変更は、企業の競争的立場を左右する。また、戦略的組織変更と利潤指標の改善の事例は、情報処理システムの効率化が競争に持つ意味を示している。情報処理システムの効率化は情報

費用の最少化を内容とする最適化問題として把握される。そして、システムのパフォーマンスは、情報処理の担い手の行動様式と結びついていることが時系列的に明かにされた。競争力の強化に情報システムの最適化を図る計画が求められよう。それでは計画に基準を与え、システム全体のあり方を決定するフィードバック部分（参照図1；点線部分）、即ち経営戦略はいかに機能するのであろうか。

Futagami&Waragai(1998)は、燕・三条を例にとり、ノウハウ・技術の蓄積とそれに結びつく地域の産業構造が、地域とそこに位置する企業の適応力と競争力に決定的意味を有することを示した。1980年代後半以降の海外への生産拠点の移動により、外部環境の変化への適応力の前提である情報蓄積が脅かされているのであり、そこに「産業の空洞化」の深刻さがある。図1から地域経済を眺めたとき、フィードバックが機能し、社会あるいは地域において新たな「経営戦略」が展開するためのエネルギーは、社会が有する情報蓄積の大きさに依存しよう。企業の競争力強化に不可欠な同様部分、即ち新しい経営戦略の展開に情報蓄積の持つ意味の大きさを示している。

（注1） 構造変化検証モデルは赤池情報量

$$AIC=N\log\sigma^2+2(p+2)$$

但し N：データ数

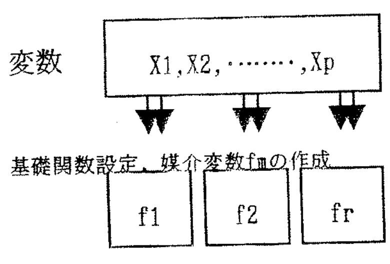
$$\sigma^2=-\Sigma a_k\gamma_k,\quad a_0=-1$$

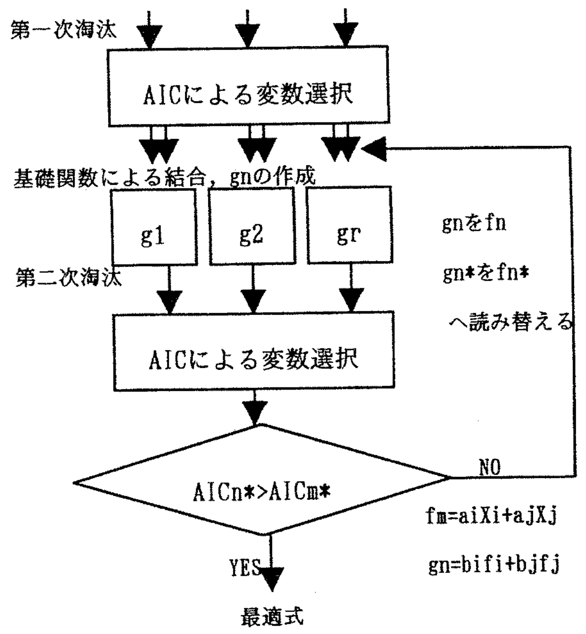
$$\gamma_j^2=\Sigma a_k\gamma_{k-j},\quad j=1,\ldots,p$$

$$\gamma_k=\Sigma z(i)z(i+k)/N$$

を用いて非定常プロセスの最適化を図り、構造変化の有無と変化の時点について検討するモデルである。

（注2） 推計で用いた自己組織化モデル（(Group Method of Data Handling)のアルゴリズムは以下のとおりである。





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第11章

企業・経営モデル

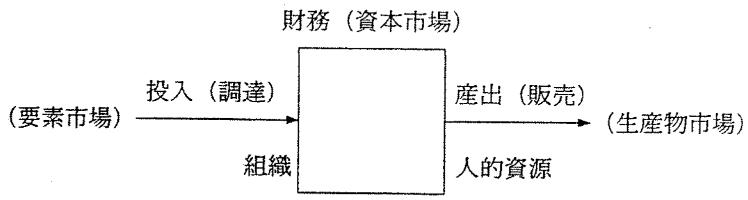
1　企業と市場目標と戦略の検討を中心に

1　はじめに

企業を取り巻く経済システムは、市場経済システムと計画経済システムに分類される。

計画経済システムは、集権的に、一定の価値基準に基づき策定された経済計画に従って運営される。生産担当部門は、計画に基づき配分された生産資源を用いて生産活動を行い、生産された財・サービスは、計画に従って供給先に向けられる。他方、市場経済システムにおいては、経済のメカニズムは市場における企業と家計の取引に従う。企業と家計の行動は、それぞれの自律的意思決定に基づくことから、その分権的性格が指摘される。

それでは、市場経済システムの中の企業は、これまでどのように把握されてきたであろうか。企業や家計を自利心に従う経済主体として位置づけ、経済主体の合理的行動とその調整のメカニズムについて、はじめて検討を手がけたのはスミス（A. Smith）であった。企業間の競争と企業・家計間の利害の調整、その両者を含む市場調整のプロセスは、ワルラス（L. Warlas）により模索過程として示され、一般均衡分析の枠組みがつくりあげられた。その枠組みの下、市場経済の調整メカニズムが十分に機能することによって、パレート最適と称される資源の最適配分に到達することが明らかにされた。ミクロ経済学の成果である。ここでは、市場と資源配分が分析対象であり、企業の活動は分析目的に従う形で理念的市場行動として把握され、利潤最大化



図II-1　オープン・システムとしての企業

をはかる経済主体としてその基本的性格が与えられた。

本節では、市場における企業の実際がいかに把握されるかについて、モ
ルと仮説の検討を通して明らかにする。具体的には、まず、企業とはいか
るものか、市場といかなる関係をもってとらえられるかについて、システ
論的に検討する。次に、システム論的に把握された企業が、市場経済シス
ムの中でいかに自律的に意思を決定するかについて検討する。自律性を規
する企業目標と市場構造との関連について論じる。企業活動に対する市場
造の制約性が検討される。最後に、その制約条件の変化が企業にいかなる
響を与えるかについて検討する。そこでは、市場を含む外部環境の変化の
での企業の活動について、経営戦略の観点から論じることになる。

2　企業をどうとらえるか、そしてそれは市場とどう関連しているか

（1）　生産主体としての企業：企業をオープン・システムとしてとらえ

企業は、市場経済システムの中で生産主体として位置づけられる。生産
動の核心部分は、財・サービスの生産を通した価値の創造にあり、企業活
は、投入・産出関係をもって把握される（図II-1）。システム論的にブラ
クボックスとして表現される生産プロセスにおいては、投入された生産要
が結合され、その結果として生産物が産出される。生産要素の投入にあた
て、企業は生産要素を調達しなくてはならず、産出した生産物は売却され
くてはならない。ここに企業は、オープン・システムとして理解されるこ
になる。

（2）　ブラックボックスとしての生産関係をいかに把握するか

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生産主体としての企業を理解するために、ブラックボックスの中身、すなわち生産要素の結合の仕方が問われてきた。

伝統的ミクロ理論では、まず、市場の完全競争性が想定され、生産要素市場において決定された要素価格の下、最適組合せとして生産要素の結合が決定される。その関係は、新古典派的“well-behaved”な生産関数によって示される。要素価格の変化に応じて生産要素の結合のあり方は変化し、その都度、最適な組合せが求められることになる。生産要素は完全に代替的であると考えられている。

これに対して、グーテンベルク (E. Gutenberug) は、生産要素の結合は、働く者と用いられる機械に依存しており、要素価格の変化に応じて要素結合が自在に変化することはなく、生産要素の結合のあり方は自然科学的・工学的特性に基づいていることを明らかにした。生産要素間の代替性の厳しい制約は、いわゆるB型生産関数を用いて示され、その後の生産理論の展開と生産関係を中心に据えた理論の体系化に基礎を与えた (アルバッハ他, 1987を参照)。

（3） オープン・システムとしての企業の活動と機能

投入・産出関係として理解される企業活動の実際の現れは、組織活動として把握され、組織を動かす人的資源についての管理が求められる。これらの活動を支えるのが財務活動であり、一連の活動の統一体として企業が把握される。統一体としてのオープン・システムは、入口 (調達) と出口 (販売) で生産要素市場と生産物市場と結びつき、また支援活動である財務活動を通して資本市場と結びつくオープン・システムである[1)]。

それでは統一的オープン・システムとして理解される企業は、何をめざすのであろうか。ここに企業の目標が問題となり、次項の検討課題である。企業の目標設定は、市場のあり方・構造と無関係ではない。

3 市場構造と企業目標

前項の（2）では、企業活動を生産関係からとらえたときの、技術関係の

制約性についてとりあげた。企業活動をオープン・システムとしてとらえたとき、システムの入口と出口でつながる市場の構造は企業活動に対するもう1つの制約条件となる。

（1） 市場構造についての伝統的分類と企業目標：利潤最大化仮説 (profit maximization hypothesis)

従来、市場は、市場参加者、情報、市場参入の観点から分類されてきた。伝統的企業理論においては、市場参加者が多数存在し、価格についての情報が完全であり、新規企業の参入が自由である完全競争 (perfect competition) 市場における企業が検討された。企業は、プライス・テーカーとして把握され、価格与件の下で利潤最大化をめざすものとされた。さらに、伝統的枠組みの下で完全競争条件が緩められ、より現実的市場構造の下での企業活動が検討された。

市場にただ1つの生産者が存在する独占 (monopoly) 市場の分析においては、企業の需要曲線は市場需要曲線そのものであり、プライス・メーカーとしての独占企業の分析が進められた。企業が少数存在する寡占 (oligopoly) 市場では、競争者の行動に反応する企業の行動が分析された。反応をいかに特定化するかによって、クールノー理論 (Cournot theory)、屈折需要曲線の理論 (theory of kinky demand curve)、シュタッケルベルクの理論 (Stackelberg theory) など、さまざまな理論が展開されるとともに、ゲーム理論は、新しい展望を切り開いた。また、完全競争条件は満たすものの、各企業の生産物が質的に少しずつ異なる場合、その市場構造は独占的競争 (monopolistic competition) 状況にあるとされ、その下での企業行動が論じられた。

伝統的企業理論においては、いずれも目的関数として利潤最大化をめざす企業が想定されてきた。利潤最大化の論理的帰結としての限界収入＝限界費用の均衡条件がそれぞれの市場構造に従っていかなる内容を持つか、市場構造の相違は決定される価格と生産量にいかなる相違をもたらすかが、論じられた。

（2） 「所有と経営の分離」と伝統的企業論批判：売上高最大化仮説

(sales-revenue maximization hypothesis) と経営者効用最大化仮説 (managerial utility maximization hypothesis)

バーリとミーンズ (A. A. Berle and G. C. Means) が指摘した「所有と経営の分離」という現状認識から、伝統的企業論に対する批判が展開された。

企業の所有者と経営者が一致している場合、所有者の利益最大化目的は企業経営に反映するであろう。しかし、株式市場の発達に伴い、株式は多数の所有者に分散所有されることになる。所有者の実態は多数の株主であり、企業経営に実質的影響力を持たない。したがって所有と経営が分離する状況においては、所有者の利益最大化は、経営目標とはなりえないとされた。

ボーモル (W. Baumol) は、経営者の経営目標を売上収入の最大化にあるとした (売上高最大化仮説)。経営者の報酬やその地位は、利潤でなく売上収入に依存するとするこの仮説において、利潤は一定水準の確保が求められる制約条件として位置づけられているにすぎない。ボーモル・モデルにおいて、企業は売上収入を最大にするように生産量と広告支出を決定することになる。

ボーモルのモデルが静学的枠組みにとどまったのに対し、企業動態、すなわち企業成長を明示的にとりあげながら、経営者効用の最大化を論じたのがマリス (R. Marris) である。マリスのモデルにおいては、研究開発支出や販売促進支出などからなる成長促進支出 (growth-creating expenditure) が制約条件として明示的に組み入れられた。経営者効用は、報酬、地位、権力と相関を持つと考えられる企業成長と、株主の利益を反映するネット・キャッシュフロー (net cash flow) の両者に依存するとされ、企業は経営者効用の最大化をめざすとされた[2)]。

ボーモル・モデルについては、売上収入の最大化は長期的には利潤最大化にほかならないとの反批判がなされた。また、需要曲線が屈折する場合、利潤最大化仮説と売上収入最大化仮説の間で、価格・産出量水準の決定に大方差がないことが指摘された。他方、マリスの仮説に対しては、長期においては、企業成長率とネット・キャッシュフローの最大化は、ほぼ同一方向に向かうであろうし、ネット・キャッシュフローの最大化は株主の資産価値の最

大化であり、企業所有者の利益の最大化と対立しないことが指摘された。資本市場が完全である場合には、株価最大化を意味することも明らかにされた。

両仮説は、利潤目標が犠牲にされ、売上収入の最大化、あるいは経営者効用の最大化がめざされると主張したのであるが、反批判は、犠牲にされる利潤が短期利潤であり、長期的にはいずれの仮説も、利潤極大化仮説と変わりないことを指摘したのである。

（3） 市場における情報の不完全性と組織としての企業：最適化原理の批判と修正

企業が何をめざすかについてのこれまでの議論は、目的関数についての相違はあるものの、制約条件つきの最適問題として定式化されたものである。これに対し、市場構造として情報の不完全性を認め、企業活動を組織活動としてとらえる観点から、企業の目標を最適化原理に従って解く枠組みそのものへの批判と修正が展開された。

行動科学的企業理論（behavioral theory of firm）は、不完全な情報の下、企業はすべての代替案の中から最適案を選択するのではなく、制限された合理性（bounded rationality）に従うであろうこと、行動は逐次的でありより満足をもたらす案がみつかれば、それ以上に代替案を探すことは止めるであろうことを指摘した。すなわち，企業は最適化原理に従うのではなく、満足化原理（satisficing principle）に従うものとされた。さらに、企業目的は組織目的として論じられることになる。組織に参加する構成員、利害関係者の個人的目標は互いに一致するものではなく、それらが組織に持ち込まれることにより、個人目標は組織への制約の意味を持ち、それら制約の集合として組織目標を理解することが主張された。

他方、企業の組織的側面が強調される中で、最適化原理の修正がはかられた。企業組織の問題を、権威の発生と資源配分の問題としてとらえたのが青木昌彦（1978）である。経営者は、雇用者と株主間の利害の対立・交渉を調整する一方、調整のプロセスの中で企業組織の目的関数が定まるとした。経営者は、企業内の資源配分を委託され、その権限の行使に伴い発生する権威

に基づき資源配分を決定することになる。委託の問題を、依頼主（principal）と代理人（agent）の問題として論じるのが、エージェンシー（agency）理論であり、階層的組織における意思決定が論じられることとなった。

（4） 市場か企業か：内部組織の経済学（economics of internal organization）

情報の不完全性と企業組織の重視は、もう一方で内部組織の経済学をもたらした。情報の不完全性・非対称性の下、市場における企業の取引活動は、情報収集・探索のためのコストと契約のためのコスト、すなわち取引費用（transaction costs）を発生させる。企業は、取引費用をできるだけ小さくするために、取引の内部化をはかるとされる。たとえば、労働市場において人材を雇用するには、探索し契約するための取引費用が発生するが、雇用契約を長期化させ労働市場を企業内に内部化することにより、自社内で人材をみつけることができ、（外部）労働市場で発生する取引費用を回避することができる。労働市場に依存する場合に発生する取引費用と、内部市場に依存する場合に発生する費用を比較することにより、市場か企業かが決定される。ここに市場と企業は代替的性質を持つことになる。コース（R. Coase）が示した基本的論点に対し、アロー（K. Arrow）などにより取引費用の概念が導入され、内部組織の理論としての体系化がはかられてきた。

4 市場の変化と企業：戦略の展開

（1） 外部環境の急激な変化と緊急課題としての市場競争力の強化

企業がオープン・システムとしてとらえられ、市場構造が企業活動の制約として位置づけられるとき、市場構造の変化は、オープン・システムのあり方を変えることになる。ここでは市場における変化を、企業を取り巻く外部環境一般にまで拡張し、その変化の問題としてとらえる。市場の変化、あるいは企業の外部環境の変化がゆっくりであり、変化の方向が予測できる場合には、それらの変化を与件として扱うことが可能である。しかし、現状は、外部環境が急激に変化し市場競争が激化する中、企業は競争的地位の維持・強化という緊急の課題に直面している。

（2） 外部環境とは何か

外部環境は、一般に、市場・経済環境、政治環境、社会・文化環境、そして物的・自然的環境からなるとされる。アダム・スミスが考えた企業は、自利心に従い利益の極大化をはかる企業であり、そこでは市場・経済環境が主たる環境であった。経済規模の拡大とシステムの複雑化に伴い規制や政策が重視され、法制度の整備がはかられ、政治的諸要因の影響力は増大した。企業が属する社会の影響要因は、企業の社会貢献への期待と社会的責任の増大に伴い、その影響力を増大させた。社会はそれぞれ固有の文化を有することから、それら影響要因には、しばしば文化的特徴が反映する。環境の浪費と破壊の深刻さは、経済社会の発展と企業活動の維持に、自然との「共生」が不可欠であるとの認識を広めることとなった。

現在、外部環境は大きく急激に変化しつつあることが指摘され、その変化は、国際化、情報化、規制緩和、あるいは持続的発展などの「イシュー」により、特徴づけられる[3]。これらの特徴は、しばしば相互に影響しあうことにより、その特徴を一層際立たせることになる。たとえば、企業活動の国際化とそれに伴う市場の国際化、競争の激化に対し、企業は迅速な対応を求められている。情報化は企業の迅速な対応を可能とするとともに、企業間ネットワークの根本的あり方に変化をもたらしつつある。

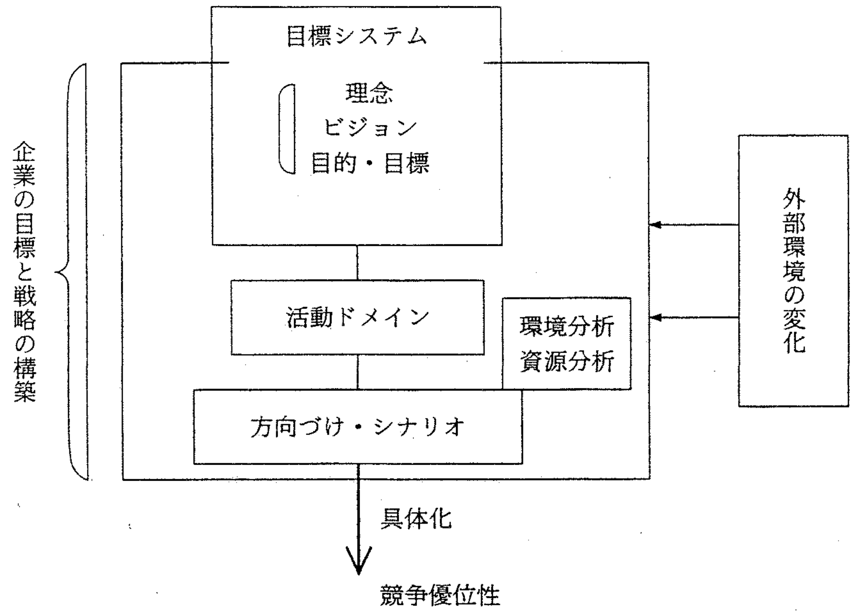
（3） 企業目標システムの明確化

企業の外部環境が変化する中、競争力の維持と強化のために向かうべき中長期的な基本方向を明確にすること、これが企業目標の設定である（図II-2を参照)。

基本的方向づけには、企業の存在そのものを規定する企業目標を明らかにしなくてはならない。コーポレート・ストラテジーを策定する場合、経営哲学や経営ビジョンを示す経営理念と、抽象的経営理念に対し到達点を示す経営政策的目標からなる目標システムの明確化が求められる。

（4） コーポレート・ストラテジーの策定１：活動ドメインの決定

明確化された目標システムの下、いかなる事業活動を通して目標達成をは



図II-2　経営戦略アプローチの概念図

かるか、事業活動の領域である活動ドメインが設定されなくてはならない。外部環境の変化に伴い、活動ドメインを新しく設定することの必要性が検討される。活動ドメインが設定されることにより、企業の目標システムを企業活動に結びつけることが可能となる。具体的にいかなる財・サービスを生産するか、その競争力をいかにして強化するかが問題となる。

（5）　コーポレート・ストラテジーの策定２：SWOT 分析

外部環境の変化への適応の中で競争力の維持・強化をはかるには、外部環境とその変化の分析（環境分析）とそれに立ち向かう自身の分析（経営資源分析）が求められる。環境の内、事業展開に直接影響を与える環境要因はタスク環境と呼ばれる。タスク環境の分析は、一般に市場分析と競合分析に大別され、機会（opportunity）と脅威（threat）が検討される。経営資源の分析は自身の強み（strength）と弱み（weakness）についての検討であり、PPM（Product Portfolio Management）はその代表的分析手法である。強み、弱み、機会、脅威についてのこれら分析は、その頭文字から、SWOT 分析と呼ば

れる（たとえば、二神，1996，5章4節を参照)。

（6） 戦略の策定3：競争戦略の決定

SWOT分析の結果をもとに、競争力強化のための戦略、競争戦略が具体的に策定されることになる。ポーター（M. Porter）は、競争戦略を構成する基本戦略として、コスト・リーダーシップ、差別化、集中を掲げた。また、実践的競争戦略の策定のために、量的経営資源と質的経営資源の観点から企業の市場における競争地位を明らかにし、それに応じた競争戦略の定石が論じられる（たとえば、二神，1996，5章5節を参照)。

こうして策定された戦略は、組織を通して実現することから、戦略論的組織論が論じられることになる。また組織を動かす人間について、戦略的人的資源管理が問題となる。

5 企業活動の能動的役割について

本節では、企業をオープン・システムとしてとらえ、市場とのかかわりの中で企業を把握し、その自律的存在を規定する目標と意思決定のあり方について、市場構造との関連の下、論じた。さらには、企業活動に対する制約としての市場構造の変化を外部環境の変化として広くとらえることにより、競争的優位性について戦略の観点から論じた。外部環境の変化に適応し、市場競争力の強化をはかる企業の姿が描かれた。

外部環境の変化への適応は、オープン・システムとしての企業を制約する条件の変化に対する受身的対応である。他方、企業の活動、あるいは適応的活動は、市場あるいは広く外部環境へのはたらきかけでもある。自身の競争優位性を確保するために外部環境にはたらきかける、より能動的企業の姿が現れてくる。

企業活動の能動的役割を社会の構造変化の動因にまで高め、その理論的体系化をはかったのがシュンペーター（J. Schumpeter）である。シュンペーターは、イノベーションが発展現象の動因であるとし、真の企業家の果たす役割について論じた。イノベーションは新結合の遂行であり、技術革新に基づ

く生産プロセスの変更を意味する。オープン・システムとしての生産主体である企業が、市場経済のダイナミズムに果たす根源的役割が明らかにされた。

理を実践している経営者がいないわけではない。

注

1）対照的に、企業の内部関連性を重視し、経営・管理的、人間組織的側面の検討を中心に据えたのが、アメリカの伝統的経営学である。そこでは、計画活動、組織活動、動機づけ、統制といった一連の活動が、目標達成という統一意思の下でいかに合理的、整合的に結びつくかについて、企業内部の問題として分析・検討が加えられた。その場合、企業の目標設定と内部関連的最適性が問われることになる（二神，1996，第1章参照）。

2）ウィリアムソン（O. Williamson）は、経営者効用関数に経営者の裁量をとり入れ、裁量的行動理論（theory of discretionary behavior）を展開した。

3）外部環境の変化の詳細は、第V章で現代イシューとして論じられる。

4）本項のソニーの事例はNayak and Ketteringham（1986）の第3章から議論に必要な部分を要約したものである。

5）本項のナイキ社の事例はNayak and Ketteringham（1986）の第9章から議論に必要な部分を要約したものである。

6）コーエン、マーチらの研究における概念で、規定しがたい目標、不明瞭な因果律、流動的な参加という一般特性を持つ状態をさす。

7）本項でとりあげた以外にもRoberts（1989）には数多くのセレンディピティー、擬セレンディピティーの事例が紹介されている。なお、セレンディピティーは幸運な発見でそれまで考えてもいなかった目的をみつけ出す能力を意味し、擬セレンディピティーは追い求めた目的達成の道を偶然の中からみつけ出す能力を意味している。

8）Utterback（1994）はここで紹介する事例以外にも数多くの事例を紹介しながら、イノベーションのダイナミズムを解説している。

9）Nayak and Ketteringham（1986）の第1章。

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Iterated Elimination and No Trade Theorem

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Abstract

"No Trade Theorem," presented by Milgrom and Stokey (1982), implies that purely speculative trades are impossible. The basic idea is that, if the participating agents are rational enough, each of them can reason that "the trade is acceptable for all of them only if the trade is acceptable for all of them only if $\cdots$," and they will conclude that the trade is possible if and only if all of agents are indifferent between accepting and rejecting the trade. The rationality of the agents is usually represented by the assumption that the acceptability of the trade is "common knowledge" between the agents.

However, the concept of common knowledge does not exactly correspond to the iterated reasoning above. Common knowledge is knowledge without uncertainty while the agent who reasons iteratedly "the trade is acceptable only if $\cdots$" need not know whether the trade is actually acceptable or not. In this paper, we will introduce notions of the iterated reasoning and the acceptability of a trade instead of common knowledge, and show that our results contain usual No Trade Theorem.

1 Introduction

Milgrom and Stokey (1982) show the impossibility of the purely speculative trade. The main assumption used to derive their result is that it is common knowledge between the participating agents that all of them expect some gain from trading. In this paper, we will formulate the problem in another way, and generalize "No Trade Theorem" for the case in which it may not be common knowledge that all the agents prefer a given speculative trade. The basic idea is that while it is not common knowledge that the trade is accepted by all, the iterated reasoning of the type "if each agent still wants to trade even when he knows that I want to trade when I know that he wants to trade when $\cdots$" may make it common knowledge. This is illustrated by the following example, which is essentially the same as one in Milgrom and Stokey (1982).

Suppose two agents to be risk-neutral. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ be a set of the states of the world. Each agent gets private information, which has the following structure:

$$\phi_1 = \{(\omega_1,\, \omega_2),\, (\omega_3)\},$$
$$\phi_2 = \{(\omega_1),\, (\omega_2,\, \omega_3)\}.$$

For example, if ω_2 is realized, then agent 1 gets $\phi_1(\omega_2) = (\omega_1,\, \omega_2)$ and agent 2 $\phi_2(\omega_2) = (\omega_2,\, \omega_3)$ before the realized state is revealed. Assume that the probability measure is uniform and that the following bet is proposed: if $\omega = \omega_1$ agent 2 pays one dollar to agent 1, if $\omega = \omega_3$ agent 1 pays one dollar to agent 2, and if $\omega = \omega_2$ the bet is drawn. The following table illustrates this bet.

	ω_1	ω_2	ω_3
Agent1	1	0	-1
Agent2	-1	0	1

The event "the trade is profitable for agent 1" is $\{\omega_1,\, \omega_2\}$, and the event "the trade is profitable for agent 2" is $\{\omega_2,\, \omega_3\}$. Therefore, the event "the trade is profitable to both of them" is $\{\omega_2\} = \{\omega_1,\, \omega_2\} \cap \{\omega_2,\, \omega_3\}$. It is, however, not common knowledge because each agent cannot know $\{\omega_2\}$ from his private information: agent 1, who gets private information $\phi_1(\omega_2) = \{\omega_1,\, \omega_2\}$, does not know which state has realized. Since it is not common knowledge that the trade is profitable to both of them, we cannot apply "No Trade Theorem" to this bet.

As long as both of the agents are rational, however, they can reason whether the bet is profitable or not. For example, agent 1 will reason as follows: "If $\omega = \omega_1$, agent 2 will refuse the bet. Therefore if my private information is (ω_1, ω_2) and agent 2 accepts the bet, then the state is ω_2, at which I am indifferent between accepting and rejecting the bet." Agent 2 uses a similar reasoning, therefore the bet is accepted by both only at ω_2.

Thus, the rational reasoning reveals that a trade is not strictly profitable if the trade is accepted. Even for a more complicated trade, the iterated reasoning can eliminate states at which the bet will be rejected. For example, suppose the following bet.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
Agent1	1	−1	1	−1	2	3	−5
Agent2	−1	1	−1	1	−2	−3	5

Then, the sequence of eliminated states is

$$\omega_7 \quad \Rightarrow \quad \omega_5 \text{ and } \omega_6 \quad \Rightarrow \quad \omega_4 \quad \Rightarrow \quad \omega_3,$$

and ω_1 and ω_2 survive. As a result, the expected profits of the trade on (ω_1, ω_2) are 0 for both of the agents.

In the case of the following bet, all of the states are eliminated, and the trade will be rejected.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
Agent1	1	−1.1	1	−1	2	3	−5
Agent2	−1	1.1	−1	1	−2	−3	5

Therefore, we can derive the "no-trade" result from the iterated reasoning even if the assumption of common knowledge is not satisfied. In this paper, we will introduce a notion of the acceptability of a trade by using the iterated reasoning. We also show that our result is more general than the usual No Trade Theorem. Volij (2000) also formulates the iterated reasoning, but it assumes unlimited communication among agents, while our definition does not need any communication.

The paper is organized as follows. Section 2 gives some basic notions, such as common knowledge, an economy with asymmetric information, and the original description of No Trade Theorem. Section 3 gives our main result, and the acceptability of trades is formulated.

2 Common Knowledge and No Trade Theorem – Preliminary Results

In this section, we give some basic concepts, such as information functions, knowledge operators, and common knowledge. All propositions and lemmas are described without proofs. For more detail, see Fudenberg and Tirole (1991), Geanakoplos (1994), or Osborne and Rubinstein (1996).

Definition 2.1 (Information Functions) *: Let Ω be a finite set and X a random variable on $(\Omega, 2^{\Omega})$. Ω represents the set of states of the world, and X a private signal.* ***An information function*** *$\phi : \Omega \to 2^{\Omega}$ is a map defined by*

$$\phi(\omega) := X^{-1} \circ X(\omega) = \{\rho \in \Omega \,|\, X(\rho) = X(\omega)\} \quad (\omega \in \Omega). \tag{1}$$

An information function ϕ gives a partition of Ω: both

$$\textit{either} \quad \phi(\omega) \cap \phi(\omega') = \emptyset \quad \textit{or} \quad \phi(\omega) = \phi(\omega') \quad ({}^{\forall}\omega, \omega' \in \Omega)$$

and

$$\bigcup_{\omega \in \Omega} \phi(\omega) = \Omega$$

are always satisfied.

Definition 2.2 (Knowledge Operators) *: A map* $\mathrm{K} : 2^{\Omega} \to 2^{\Omega}$ *is a* ***knowledge operator*** *defined by*

$$K(E) := \{\omega \in \Omega \,|\, \phi(\omega) \subset E\} \quad (E \in 2^{\Omega}). \tag{2}$$

An agent knows event $E \in 2^{\Omega}$ at $\omega \in \Omega$ if he knows that the true state surely lies in E, that is, if $\phi(\omega) \subset E$. Since any information function must satisfy $\omega \in \phi(\omega)$, E is always true when an agent knows E.

Proposition 2.3 (Basic Properties of Knowledge Operators) *: For each $A, B \in 2^{\Omega}$,*

1. $K(\Omega) = \Omega$
2. $K(A) \cap K(B) = K(A \cap B)$
3. $A \subset B \;\Rightarrow\; K(A) \subset K(B)$
4. $K(A) \subset A$

5. $K \circ K(A) = K(A)$

6. $K(A)^c = K\left(K(A)^c\right)$.

Definition 2.4 (Self-Evident) : *$E \in 2^\Omega$ is **self-evident** if $K(E) = E$.*

It is easy to see that E is self-evident if and only if

$$\omega \in E \Rightarrow \phi(\omega) \subset E \tag{3}$$

is satisfied. Moreover,

Proposition 2.5 : *An event $E \in 2^\Omega$ is self-evident if and only if E satisfies*

$$E = \bigcup_{\omega \in E} \phi(\omega). \tag{4}$$

Definition 2.6 : *Let $\mathcal{I} = \{1, \cdots, I\}$ be a set of agents. The event **"everyone knows E"** is a set*

$$K_{\mathcal{I}}(E) := \bigcap_{i \in \mathcal{I}} K_i(E) \tag{5}$$

where K_i is the knowledge operator of agent i, whose information function is ϕ_i. An event $E \in 2^\Omega$ is self-evident for every agent $i \in \mathcal{I}$ if $K_{\mathcal{I}}(E) = E$.

Proposition 2.7 : *For all $i \in \mathcal{I}$,*

$$K_{\mathcal{I}}(E) = E \quad \Longleftrightarrow \quad K_i(E) = E \Longleftrightarrow \quad E = \bigcup_{\omega \in E} \phi_i(\omega).$$

Definition 2.8 (Common Knowledge) : *Let $\{K_{\mathcal{I}}^n(E)\}_{n \in N}$ be a decreasing sequence of events defined inductively by*

$$K_{\mathcal{I}}^1(E) := K_{\mathcal{I}}(E), \quad K_{\mathcal{I}}^n(E) := K_{\mathcal{I}} \circ K_{\mathcal{I}}^{n-1}(E) \qquad (n = 1, 2, 3, \cdots),$$

*and $K_{\mathcal{I}}^\infty(E) := \bigcap_{n=1}^\infty K_{\mathcal{I}}^n(E)$. An event $E \in 2^\Omega$ is **common knowledge at** $\omega \in \Omega$ if*

$$\omega \in K_{\mathcal{I}}^\infty(E). \tag{6}$$

Note that $K_{\mathcal{I}}^2(E) = K_{\mathcal{I}}(K_{\mathcal{I}}(E))$ means that "everyone knows that everyone knows E," and $K_{\mathcal{I}}^{\infty}(E)$ is the event that "everyone knows that everyone knows that everyone knows $\cdots$ E."

It is important that, if $E \in 2^{\Omega}$ is common knowledge at $\omega \in \Omega$, then $\omega \in E$ must hold. This means that any events which are not true cannot be common knowledge.

Lemma 2.9 : *$E \in 2^{\Omega}$ is common knowledge at $\omega \in \Omega$ if and only if there exits an event $F \in 2^{\Omega}$ which satisfies both $K_{\mathcal{I}}(F) = F$ and $\omega \in F \subset E$.*

Definition 2.10 (An Economy with Asymmetric Information) : *Consider a pure exchange economy with I agents in an uncertain environment $(\Omega, 2^{\Omega}, \boldsymbol{P})$. There are l commodities in each state of the world. Let L_+^l (or L^l) be the set of the $\boldsymbol{R}_+^l$-valued (or $\boldsymbol{R}^l$-valued) random variables on $(\Omega, 2^{\Omega}, \boldsymbol{P})$ and we assume that the consumption set is L_+^l.*

A pure exchange economy with asymmetric information *$\mathcal{E}^{\phi}$ is described by*

$$\mathcal{E}^{\phi} = \{\ (E[U_i], e_i, \phi_i)_{i \in \mathcal{I}}\ ,\ \mathcal{Y}\ \}\ , \tag{7}$$

where

- *$\mathcal{Y} \subset (L^l)^I$ is the convex feasible set,*
- *$\mathcal{I} = \{1,\ 2,\ \cdots,\ I\}$ is the set of agents,*
- *$U_i\ :\ \boldsymbol{R}_+^l \to \boldsymbol{R}$ is the utility function of agent $i \in \mathcal{I}$,*
- *$e_i \in L_+^l$ is the initial endowment of agent i, and*
- *$\phi_i\ :\ \Omega \to 2^{\Omega}$ is the information function of agent i.*

A net trade $Y = (y_i)_{i \in \mathcal{I}} \in (L^l)^I$ is feasible if $Y \in \mathcal{Y}$. Usually, the feasible set $\mathcal{Y}$ is specified by

$$\mathcal{Y} = \left\{ Y = (y_1, \cdots, y_I) \in (L^l)^I \;\middle|\; \sum_{i \in \mathcal{I}} y_i \leq 0 \right\} .$$

The expected utility function of agent i $E[U_i(\cdot + e_i)] : L^l \to \boldsymbol{R}$ is assumed to be increasing for all i, and simply denoted by $E[U_i](\cdot)$. As a usual assumption, all the agents have a common prior. Then, the value of $E[U_i](\cdot)$ is calculated by

$$E[U_i](y_i) = \sum_{\omega \in \Omega} U_i\left(y_i(\omega) + e_i(\omega)\right) \boldsymbol{P}(\omega) \quad (y_i \in \mathcal{Y}).$$

Theorem 2.11 (No Trade Theorem, Milgrom and Stokey (1982)) *: At a pure exchange economy with asymmetric information $\mathcal{E}^\phi = \{(E[U_i], e_i, \phi_i)_{i\in\mathcal{I}}, \mathcal{Y}\}$, suppose that all the agents are weakly risk-averse (all the agents' utility functions are concave) and that a net trade $\hat{Y} = (\hat{y}_i)_{i\in\mathcal{I}}$ is Pareto-optimal ex ante.*

If it is common knowledge at ω^* that each agent weakly prefers a feasible net trade $Y = (y_i)_{i\in\mathcal{I}} \in \mathcal{Y}$ to $\hat{Y}$, then every agent is indifferent between Y and $\hat{Y}$.

Proof: *See Milgrom and Stokey (1982) or Fudenberg and Tirole (1991).*

[*Q.E.D.*]

The event "each agent weakly prefers a feasible net trade $Y = (y_i)_{i\in\mathcal{I}} \in \mathcal{Y}$ to $\hat{Y}$" is given as follows:

$$\Pi = \bigcap_{i\in\mathcal{I}} \left\{ \omega \in \Omega \;\middle|\; E[U_i \mid \phi_i(\omega)](y_i) \geq E[U_i \mid \phi_i(\omega)](\hat{y}_i) \right\}.$$

The assumption that Π is common knowledge at ω^* implies $\omega^* \in \Pi$, namely, all of the agents actually prefer Y to $\hat{Y}$ at ω^*.

3 Iterated Elimination – Main Results

Theorem 2.11 assumes that the weak profitability of the trade is common knowledge. This assumption implies that all of the agents know the trade is actually profitable for all. We show this assumption can be relaxed since, as mentioned in section 1, the rational agent can reason whether the trade is acceptable or not for him without knowing the trade is profitable for the others. To this end, we give an formal definition of the iterated reasoning process. First, we introduce the desirability, which is used to judge whether the trade is profitable or not.

Second, information functions and knowledge operators are extended on a family of sets. Finally, the rational-reasoning process is formulated as a sequence of iteratedly-refined knowledge operators.

Definition 3.1 (Desirability) *: Suppose that agent i is faced with two trades, $Y = (y_i)_{i \in \mathcal{I}}$ and $\hat{Y} = (\hat{y}_i)_{i \in \mathcal{I}}$.*

*At an event $A \in 2^{\Omega}$, Y is **desirable** for agent i relative to $\hat{Y}$ if*

$$E\,[U_i\,,\,A]\,(y_i) \geq E\,[U_i\,,\,A]\,(\hat{y}_i).$$

A set of all events which make Y desirable for agent i is denoted by $\mathcal{A}_i(y_i, \hat{y}_i)$, that is,

$$\mathcal{A}_i(y_i, \hat{y}_i) := \left\{ A \in 2^{\Omega} \;\middle|\; E\,[U_i\,,\,A]\,(y_i) \geq E\,[U_i\,,\,A]\,(\hat{y}_i) \right\}. \tag{8}$$

Definition 3.2 (Knowledge Operators on a Family of Sets) *: A map $\mathcal{K}_i$ is Agent i's **knowledge operator on a family of sets** defined by*

$$\mathcal{K}_i[\mathcal{A}] := \{\omega \in \Omega \mid \phi_i(\omega) \in \mathcal{A}\} \quad ({}^{\forall}\mathcal{A} \subset 2^{\Omega}),$$

where ϕ_i is the information function of agent i.

Proposition 3.3 *: At $\omega^* \in \Omega$, agent i weakly prefers Y to $\hat{Y}$ if and only if $\omega^* \in \mathcal{K}_i[\mathcal{A}_i(y_i\,,\,\hat{y}_i)]$.*

Proof:

$$\begin{aligned}
\omega^* \in \mathcal{K}_i[\mathcal{A}_i(y_i\,,\,\hat{y}_i)] \quad &\Longleftrightarrow \quad \phi_i(\omega^*) \in \mathcal{A}_i(y_i\,,\,\hat{y}_i) \\
&\Longleftrightarrow \quad E\,[U_i\,,\,\phi_i(\omega^*)]\,(y_i) \geq E\,[U_i\,,\,\phi_i(\omega^*)]\,(\hat{y}_i).
\end{aligned}$$

[*Q.E.D.*]

Suppose that an event $\Sigma \in 2^{\Omega}$ is revealed to all the agents. Then, each agent can refine his private information. The refined information of agent i is defined by the cap product of his private information $\phi_i(\,\cdot\,)$ and the revealed information Σ, that is, $\phi_i(\,\cdot\,) \cap \Sigma$.

Definition 3.4 (Refinement of Knowledge Operators) *: A **knowledge operator refined by an event** $\Sigma \in 2^\Omega$ is the restriction of $\mathcal{K}_i$ on Σ, and denoted by $\mathcal{K}_i|_\Sigma$, that is,*

$$\mathcal{K}_i\Big|_\Sigma[\mathcal{A}] := \{\omega \in \Sigma \mid \phi_i(\omega) \cap \Sigma \in \mathcal{A}\} \qquad ({}^\forall \mathcal{A} \subset 2^\Omega).$$

Note that $\mathcal{K}_i|_\emptyset[\mathcal{A}]$ is empty for any $\mathcal{A} \subset 2^\Omega$ by the definition.

Definition 3.5 (Iterated Elimination) *: A sequence of sets $\{\Omega^n\}_{n\in N}$ is the **iterated-eliminating sequence** if Ω^n is recursively defined as follows:*

$$\begin{cases} \Omega^0 := \Omega \\ \Omega^{n+1} := \bigcap_{i\in\mathcal{I}} \mathcal{K}_i|_{\Omega^n}[\mathcal{A}_i(y_i, \hat{y}_i)] \qquad (n = 0, 1, 2, \cdots). \end{cases} \tag{9}$$

By Proposition 3.3, the event Ω^1 equals to the event "Y is desirable for all the agents relative to $\hat{Y}$." The event Ω^2 means "Y is desirable for all the agents even after their refining of their information functions by Ω^1," and so on. The states eliminated at each step of reasoning are $\Omega\backslash\Omega^1$, $\Omega^1\backslash\Omega^2$, $\Omega^2\backslash\Omega^3, \cdots$.

It is important that Definition 3.5 needs no communication between the participating agents. Each agent can refine hypothetically his information function, and decide whether he accepts the trade or not.

Definition 3.6 (Acceptability) *: At a state $\omega \in \Omega$, a trade $Y = (y_i)_{i\in\mathcal{I}}$ is **acceptable** for agent i relative to $\hat{Y} = (\hat{y}_i)_{i\in\mathcal{I}}$ if*

$$\omega \in \mathcal{K}_i\Big|_{\Omega^n}[\mathcal{A}_i(y_i, \hat{y}_i)] \qquad (n = 0, 1, 2, \cdots). \tag{10}$$

If (10) holds for all $i \in \mathcal{I}$, that is,

$$\omega \in \bigcap_{i\in\mathcal{I}} \mathcal{K}_i\Big|_{\Omega^n}[\mathcal{A}_i(y_i, \hat{y}_i)] \ \left(= \Omega^{n+1}\right) \qquad (n = 0, 1, 2, \cdots), \tag{11}$$

then Y is acceptable for all the agents relative to $\hat{Y}$ at ω.

The acceptability means that "all the agents don't refuse Y even though each knows that all the agents don't refuse Y even though each knows that all the agents don't refuse $Y \cdots$." By the definition, it is clear that the sequence $\{\Omega^n\}_{n\in \boldsymbol{N}}$ is decreasing. Therefore (11) equals to $\omega \in \Omega^\infty (= \bigcap_{n=0}^\infty \Omega^n)$.

Note that agent i needs only his private information $\phi_i(\omega^*)$ in order to see that the trade Y is acceptable at ω^*. In other words, he cannot predict that the trade Y will be really accepted even when he knows the trade is acceptable.

Theorem 3.7 (No Trade Theorem under Acceptability) : *At a pure exchange economy with asymmetric information* $\mathcal{E}^\phi = \{(E[U_i], e_i, \phi_i)_{i\in\mathcal{I}}\ \ \mathcal{Y}\}$, *suppose that all the agents are weakly risk-averse and that a net trade* $\hat{Y} = (\hat{y}_i)_{i\in\mathcal{I}}$ *is Pareto-optimal ex ante.*

If Y *is acceptable for all the agents relative to* $\hat{Y}$ *at* ω^*, *then every agent is indifferent between* Y *and* $\hat{Y}$.

Proof: *Since* Y *is acceptable,* $\omega^* \in \Omega^n$ *holds for all* $n \in \boldsymbol{N}$. *It is clear that* $\Omega = \Omega^0 \supset \Omega^1 \supset \Omega^2 \supset \cdots\cdots \ni \omega^*$. *Since* Ω *is finite, there is* $\bar{n} \in \boldsymbol{N}$ *for which*

$$\Omega^{\bar{n}} = \Omega^{\bar{n}+1} = \Omega^{\bar{n}+2} = \cdots \ni \omega^*. \tag{12}$$

Let $\Omega^* := \Omega^{\bar{n}}$ *and* $\phi_i^* := \phi_i \cap \Omega^*$. *Then* Ω^* *satisfies*

$$\Omega^* = \bigcap_{i\in\mathcal{I}} \mathcal{K}_i\big|_{\Omega^*}[\mathcal{A}_i(y_i, \hat{y}_i)] = \bigcap_{i\in\mathcal{I}} \left\{\omega \in \Omega^* \;\middle|\; \phi_i^*(\omega) \in \mathcal{A}_i(y_i, \hat{y}_i)\right\}. \tag{13}$$

By Proposition 3.3,

$$E[U_i(y_i),\ \phi_i^*(\omega)] \geq E[U_i(\hat{y}_i),\ \phi_i^*(\omega)] \tag{14}$$

for all $\omega \in \Omega^*$ *and* $i \in \mathcal{I}$.

Suppose that the inequality in (14) is strict for agent j *at* $\omega^* \in \Omega^*$. *Since* Ω^* *satisfies*

$$\Omega^* = \bigcup_{\omega\in\Omega^*} \phi_i^*(\omega) \quad (i \in \mathcal{I})$$

by Proposition 2.5,

$$\begin{aligned} E\left[U_i(y_i), \Omega^*\right] &= E\left[U_i(y_i),\ \bigcup_{\omega\in\Omega^*} \phi_i^*(\omega)\right] \\ &= \sum_{\phi_i^*(\omega)} E\left[U_i(y_i), \phi_i^*(\omega)\right] \end{aligned}$$

$$\geq \sum_{\phi_i^*(\omega)} E\left[U_i(\hat{y}_i), \phi_i^*(\omega)\right]$$
$$= E\left[U_i(\hat{y}_i), \Omega^*\right] \tag{15}$$

holds for any i, and the inequality is strict for j.

Consider a new trade $Y^* = (y_i^*)_{i\in\mathcal{I}}$ defined by

$$y_i^* = y_i \mathbf{1}_{\Omega^*} + \hat{y}_i \mathbf{1}_{(\Omega^*)^c} \qquad (i \in \mathcal{I}), \tag{16}$$

where $\mathbf{1}_{\Omega^*}$ is the indicator function, that is,

$$\mathbf{1}_{\Omega^*}(\omega) = \begin{cases} 1 & \text{if } \omega \in \Omega^* \\ 0 & \text{if } \omega \notin \Omega^*. \end{cases}$$

Then, for any i,

$$\begin{aligned} E\left[U_i(y_i^*)\right] &= E\left[U_i(y_i), \Omega^*\right] + E\left[U_i(\hat{y}_i), (\Omega^*)^c\right] \\ &\geq E\left[U_i(\hat{y}_i), \Omega^*\right] + E\left[U_i(\hat{y}_i), (\Omega^*)^c\right] = E\left[U_i(\hat{y}_i)\right] \end{aligned} \tag{17}$$

follows from (15), and the inequality is strict for j. This contradicts to our hypothesis about the ex ante Pareto optimality of $\hat{Y}$.

[*Q.E.D.*]

In Theorem 3.8, we will show that acceptability is a weaker condition than common knowledge. Therefore, our result Theorem 3.7 is a generalization of the usual No Trade Theorem.

Theorem 3.8 : *If it is common knowledge at ω^* that each agent weakly prefers Y to $\hat{Y}$, then Y is acceptable for all the agents relative to $\hat{Y}$.*

Proof: *Observe first that the event " each agent weakly prefers Y to $\hat{Y}$" is equivalent to*

$$\Omega^1 = \bigcap_{i\in\mathcal{I}} \mathcal{K}_i\Big|_{\Omega^0}[\mathcal{A}_i(y_i\,,\,\hat{y}_i)]. \tag{18}$$

By Proposition 3.3, $\phi_i(\rho) \in \mathcal{A}_i(y_i\,,\,\hat{y}_i)$ holds for any $\rho \in \Omega^1$ and $i \in \mathcal{I}$.

By Lemma 2.9, there is a set F which satisfies $\omega^* \in F \subset \Omega^1$ and $K_{\mathcal{I}}(F) = F$ since (18) is common knowledge at ω^*. Since F is self-evident, $\rho \in F \Rightarrow \phi_i(\rho) \subset F \subset \Omega^1$ for any $i \in \mathcal{I}$. Thus,

$$\phi_i(\rho) \cap \Omega^1 = \phi_i(\rho) \in \mathcal{A}_i(y_i\,,\,\hat{y}_i)$$

holds for any $i \in \mathcal{I}$ and $\rho \in F$. Moreover,

$$\rho \in \mathcal{K}_i\Big|_{\Omega^1}[\mathcal{A}_i(y_i\,,\,\hat{y}_i)] = \left\{\omega \in \Omega^1 \;\Big|\; \phi_i(\omega) \cap \Omega^1 \in \mathcal{A}\right\}. \tag{19}$$

By Definition 3.5,

$$\omega^* \in F \subset \Omega^2 = \bigcap_{i \in \mathcal{I}} \mathcal{K}_i\Big|_{\Omega^1}[\mathcal{A}_i(y_i\,,\,\hat{y}_i)], \tag{20}$$

which means that Ω^2 is also common knowledge at ω^*.

Thus, $\omega^* \in F \subset \Omega^n$ holds for any $n \in \boldsymbol{N}$. By definition, Y is acceptable for all the agents.

[*Q.E.D*]

4 Conclusion

In this paper, we formulate the iterated-eliminating reasoning and generalize No Trade Theorem. We have shown that the agent can judge whether the newly proposed trade is acceptable or not even when the assumption of common knowledge is not satisfied, and that the trade is accepted by all agents if and only if they are indifferent between trading and not trading.

In order to represent the rationality of the agents, the concept of common knowledge seems to be misleading and confusing. Common knowledge means exact mutual understanding. In betting or the other games with asymmetric information, however, every agent does not have any incentives to communicate and reveal his private information. In such cases, only from his private information, he will reason on his expected gain. The definition of the iterated-eliminating reasoning given in this paper is a little more complicated, but seems to be more straightforward and appropriate than common knowledge.

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株価変動のパーコレーションモデル

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概 要

株価収益率の分布が，正規分布に較べて厚い裾野(fat tails)を持つことは広く知られている．この論文では，視野の限定された多数のエージェントが局所的に相互作用するモデルを考え，それによって株価収益率分布のfat tail性を再現することを試みる．モデル化にあたっては，統計力学の手法であるパーコレーション(浸透)理論が用いられる．パーコレーションには，パラメータが臨界値を超えると相転移を起こすという性質がある．この論文では，この性質を用いて株価の天井あるいは床を定義し，さらに計算機シュミレーションによって日経平均の天井と床を計算する．

1 Introduction

Black-Sholes(1973)をはじめとする標準的なファイナンス理論では，一般に株価過程S_tの収益率$\frac{dS_t}{S_t}$は正規分布に従うものと仮定されている．しかしながら，Mandelbrot(1963)あるいはFama(1965)以降の殆ど全ての実証研究は，株価の収益率分布が正規分布よりも厚い裾野(fat tails)を持つことを示している．図１と図２はこの11年間(1991年１月-2001年9月)の日経平均の終値と，その収益率の対数経験分布である．太線は日経平均の対数分布，細線は同一の平均・分散を持つ対数正規分布である．収益率の大きさが４％を越える領域では，経験分布が正規分布を大きく上回っていることがわかる．この事実は，変動のガウス性に大きく依存している金融派生商品の価格付け理論にとって無視し得ない意味を持つばかりでなく，株価の不安定性を直接的に意味しているものでもある．例えば，この11年間で日経平均は７％以上の変動を６回経験しているが，仮に変化率が正規分布に従うとするならば，同様の変動が６回起こるためにはおよそ2,000年が必要である．

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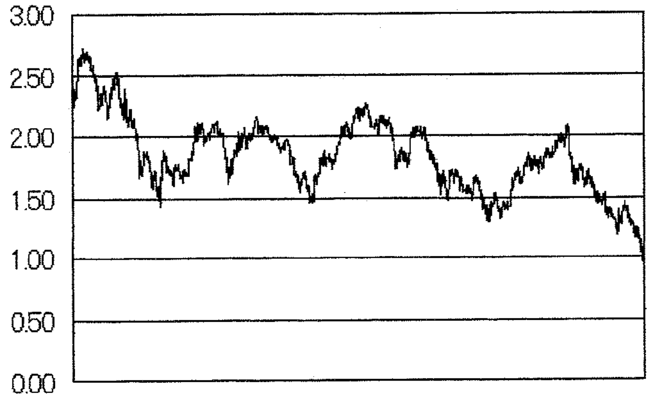


図１：日経平均終値 (1991.1-2001.9)

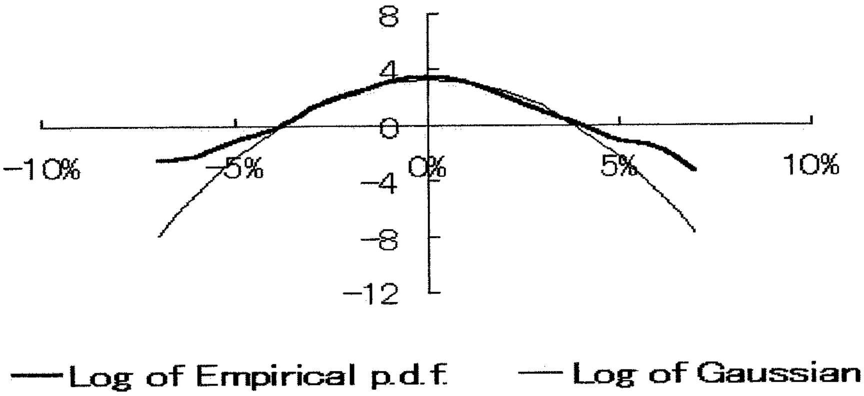


図２：正規分布との比較

この論文では，限定的な視野を持つ多数のエージェントたちから構成される単純な株式市場モデルを考え，エージェントたちの局所的相互作用を通じて株価収益率分布のfat tail性が再現されることを示す．モデル化にあたっては，統計物理の重要な手法のひとつであるパーコレーション(浸透)理論が援用される．パーコレーション理論とは，その名が示すように，ある粒子の物理状態が周囲の粒子にどのように浸透・伝染していくかを分析する理論であり，統計物理のみならず，伝染病や山火事，生物の縄張りの形成等のモデル分析にも応用されている．さらにパーコレーション系には，伝染確率がある臨界値を超えると正の確率

2

で伝染の連鎖が無限に続くようになるという相転移を思わせる性質があり，この現象に対する興味から離散確率論の対象として数学的にも盛んに研究されている(Grimmett,1999). パーコレーションに関するより詳しい説明は，第２章で与えられる.

パーコレーション理論を株式市場に応用する試みは，Stauffer-Penna(1998)によって既になされている．彼らのモデルはfat tailsの再現に成功しているが，しかしながらそのモデルの仮定は経済学的な解釈にやや難を残し，またパーコレーションの重要な特性である相転移現象を十分に応用していない点でも不満が残る．株式市場をパーコレーションとしてモデル化する場合，無限連鎖の発生を株価の大幅高騰あるいは下落として解釈することはごく自然な発想であると思われるが，このモデルの仮定のもとではこの解釈を行うことが難しいのである．詳細は次章に譲る.

この論文では，Stauffer-Penna(1998)とは異なる仮定に基づく第二のモデルを提示する. 第２章においてパーコレーション理論の概説を与えた後に，第３章で本論文のモデルを定義し，計算機シュミレーションによってfat tailsを再現する．ついで第４章で日経平均の天井と床が試算される．最後の第５章において，結論と今後の課題が述べられる.

2 パーコレション理論とStauffer-Pennaモデル

『平面上の正方格子Z^2を考え，その各点に果樹を植える．原点$(0,0)$に植えられた木から山火事が発生したとき，この果樹園はどの程度の被害を被るだろうか？ただし，一本の木からこの周囲の木々に延焼する確率をpとする.』

これは，統計物理学においてパーコレーション理論と呼ばれる分野の典型的な問題のひとつである．一本の木から，周囲の４本の木々にそれぞれ確率pで被害が広がり，さらに新たに燃え始めた木々から，その周りの木々へと連鎖的に火災が伝染していく．このようにして，最終的に焼き尽くされた木からなる集団(クラスター)が形成されることになる.

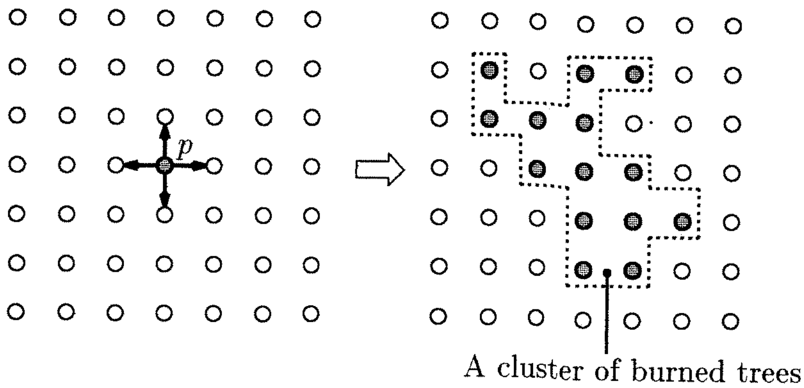


図３：山火事のパーコレーション

このクラスターに含まれる点からなる集合をC，そのサイズを$|C|$と表すとき，ある木から近傍の木へと延焼する確率p(一般には感染確率という)が1ならば，$|C|=\infty$となることは明らかである．逆に$p=0$ならば，$|C|=0$となる．実は$p<1$であっても，ある確率でクラスターのサイズが無限大になり得ることが数学的に証明されている．すなわち：

Definition 1 パーコレーション確率$\theta(p)$とは，感染確率pのもとで原点が無限クラスターに属する確率であり

$$\theta(p)=\boldsymbol{P}_p(\,|C|=\infty\,)=1-\sum_{n=1}^{\infty}\boldsymbol{P}_p(\,|C|=n\,), \tag{1}$$

で定義される．ここで$\boldsymbol{P}_p(\,|C|=n\,)$とは，$p$が与えられたときにサイズ$n$のクラスターが形成される確率である．

Theorem 1 ある感染確率$p^c\in(0,1)$が一意に存在して，以下を満たす．

$$\theta(p)\begin{cases}=0 & (p<p^c)\\ \\ >0 & (p>p^c)\quad .\end{cases} \tag{2}$$

このp^cを臨界確率という．

(Proof) *Grimmett (1999), pp.15-19.*

山火事モデルのような$\boldsymbol{Z}^2$上のパーコレーションの場合には

$$p^c=0.592745\cdots$$

であることが計算機シミュレーションによって予想されている(詳細はGrimmett(1999)を参照)．この定理から，パーコレーションモデルが一種の相転移の性質を備えていることが分かる．この性質から，パーコレーション理論は統計力学の分野で現在もっとも盛んに研究されている数学モデルのひとつになっている．

Stauffer-Penna(1998)は，パーコレーションを株式市場に応用し，彼らのモデルがfat tail性を持つことを数値実験によって示した．(彼らの研究は，既に一部で発表されていたCont-Bouchaudモデルに若干手を加えたうえで，大規模な計算機シミュレーションをおこなったものである．筆者は，もととなったCont-Bouchaudの論文を入手していない．) このStauffer-Pennaモデルの設定は以下のとおりである．平面上に密度$p\in(0,1)$でエージェントが分布しているとする．彼らは周囲のエージェントたちと浸透確率pでクラスターを形成し，各クラスターは株価の先行きに関する期待を共有する(例えば，各クラスターは同一のアドバイザー

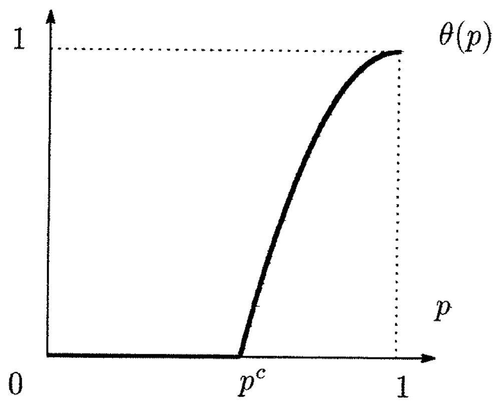
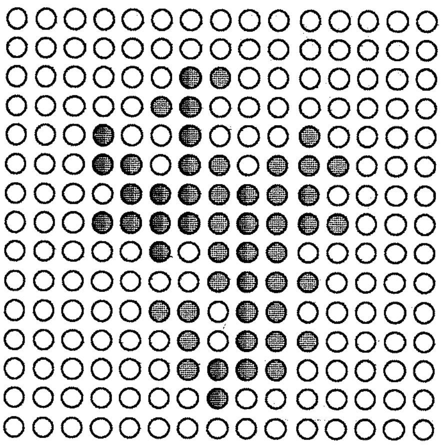
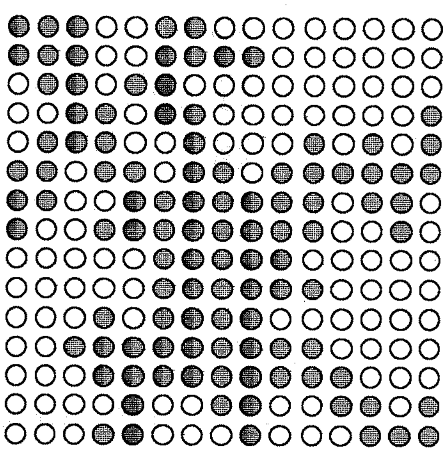


図4：パーコレーション確率



(i) A Finite Cluster: $p < p^c$



(ii) An Infinite Cluster: $p > p^c$

図5：パーコレーションにおける相転移現象

から投資アドバイスを得ていると考える). 具体的には, 各クラスターは確率aで「買い」, aで「売り」, $1-2a$で「静観」のポジションを取る.「買い」クラスターを$C_+^1, C_+^2, C_+^3, \cdots$,「売り」クラスターを$C_-^1, C_-^2, C_-^3, \cdots$とするとき, 価格の変化率は超過需要$\sum_j |C_+^j| - \sum_j |C_-^j|$の定数倍として与えられるとする. すなわち

$$\frac{\Delta S_t}{S_t} \quad \propto \quad \sum_j |C_+^j| - \sum_j |C_-^j| \tag{3}$$

である.

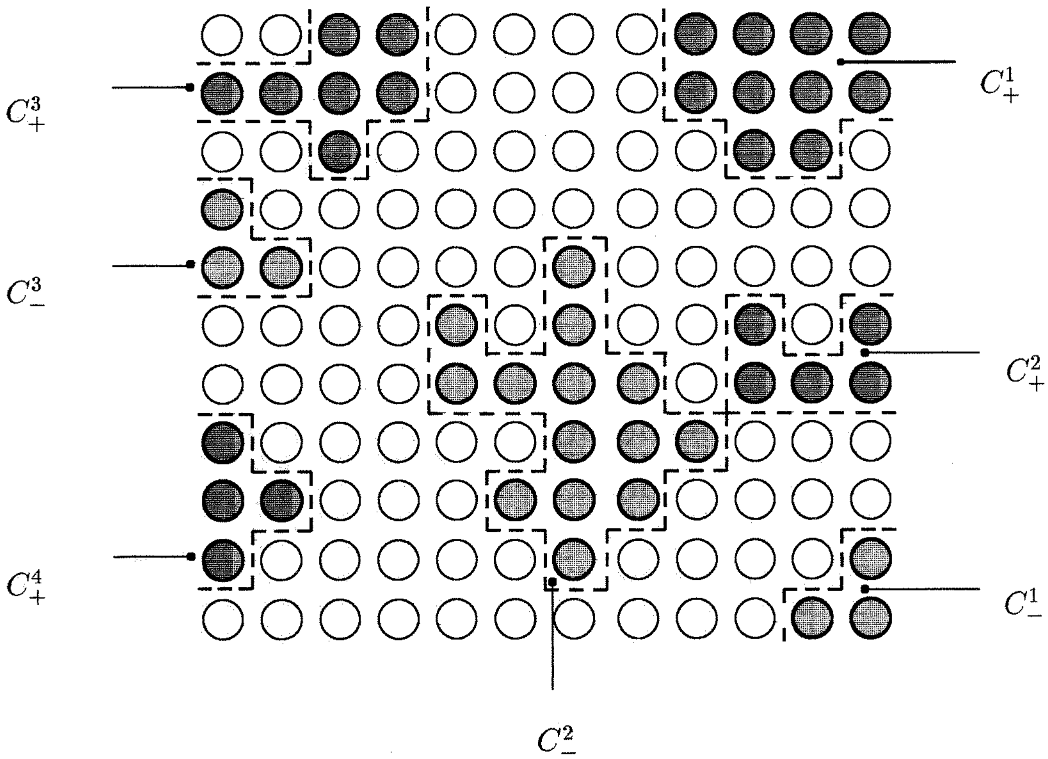


図6 : Cont-Bouchaudモデル

彼らのモデルはfat tail性を再現したという点で成功しているが, エージェントたちがクラスターを形成する理由が些か苦しいのが難点である. またこの設定下では浸透確率pを定数とせざるを得ず, パーコレーションの最大の魅力である相転移現象をモデルに取り込むことができない.

3 モデル

この論文では，シュミレーションの容易さを考慮して以下のような仮定をおこなう．

- N^2人のエージェントと多数の株式銘柄を考え，各エージェントを(i,j) で表す$(i,j=1,2,\cdots,N)$.
- 各エージェントは，それぞれ4銘柄の株式を保有し，また各銘柄は，それぞれある2人のエージェントだけによって保有される．例えばエージェント(i,j)は，銘柄$B_{i,j}^{i+1,j}$，$B_{i,j}^{i,j+1}$，$B_{i-1,j}^{i,j}$，および$B_{i,j-1}^{i,j}$を保有する．これらの添え字の意味は，下図によって示されている．すなわち，各エージェントは保有する株式を通じて4人の近傍エージェントと繋がっている．ただし，このネットワークの境界に位置するエージェントは，反対境界上のエージェントと株式を持ち合うものとする．例えば，エージェント$(1,j)$

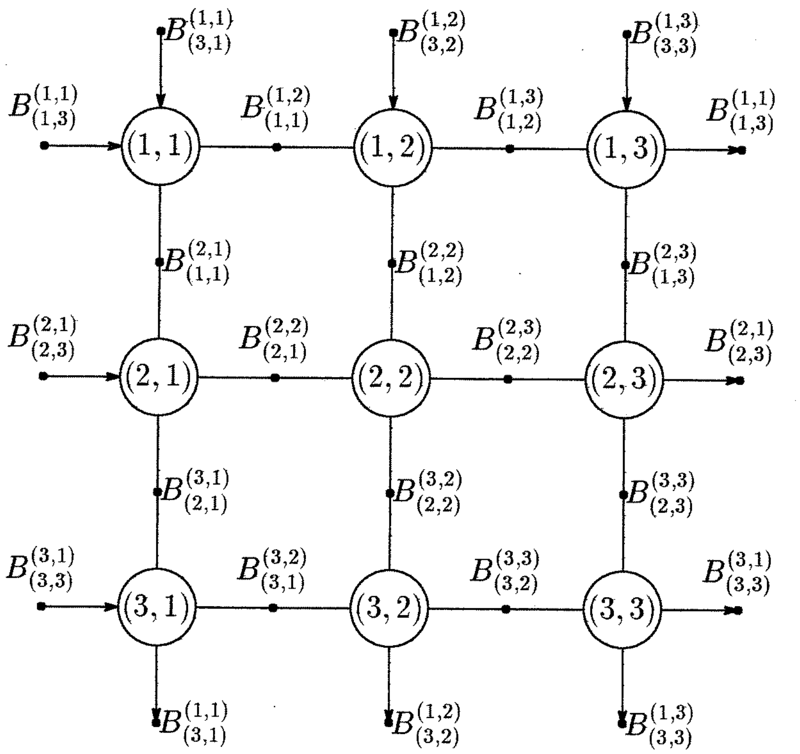


図７：エージェントのネットワーク構造（N = 3）

は株式$B_{N,j}^{1,j}$を通してエージェント(N,j)と繋がっている．言い換えれば，エージェントたちは２次元トーラス上の正方格子ネットワークを形成している．

- 各エージェントは常に，自分の保有する4銘柄を一斉に買い，あるいは売る．単一銘柄だけを売り買いすることはない．また，保有する銘柄を変更することもない（すなわち，ネットワークの組換えは想定しない）．
- 各エージェントは，自分の保有する銘柄の価格と，適当な株価指数(例えば日経平均株価あるいはS＆P500)のt時点における値だけに注目して意思決定を行う．したがって，あるエージェントの売り買いは，その近傍の4人のエージェントだけに知られる．

以上の仮定のもと，t期における株価指数の値S_tが以下のように逐次的に定まるとする（時間は離散的とする）．

1. t期末において，任意に選ばれたエージェントが，ニュース$I_t(=\pm 1)$を知る．$I_t = +1$のとき，このエージェントは株価の先行きを楽観して保有する銘柄全てに「買い」の注文を出す．逆に$I_t = -1$のときには，悲観的に予想して「売り」の注文を行う．
2. このエージェントの行動は，前述の仮定により，同じ株式を持ち合っている近傍の4人のエージェントたちにだけ伝わる．各エージェントたちは，最初のエージェントの買い注文に対しては確率p^uで楽観に転じ，また確率$1-p^u$で静観のままであるとする．逆に最初のエージェントが売り注文を出した場合には，確率p^dで悲観的な予想を抱き，確率$1-p^d$でやはり静観するものと仮定する．言い換えれば，楽観と悲観の感染確率はそれぞれp^u, p^dである．
   感染確率p^u, p^dを，それぞれ株価指数Sの減少関数および増加関数であると仮定する．ここでは
   $$\begin{cases} p^u(S) = e^{-\alpha S} \\ \\ p^d(S) = 1 - e^{-\delta S} \end{cases} \tag{4}$$
   のように特定化しておく．

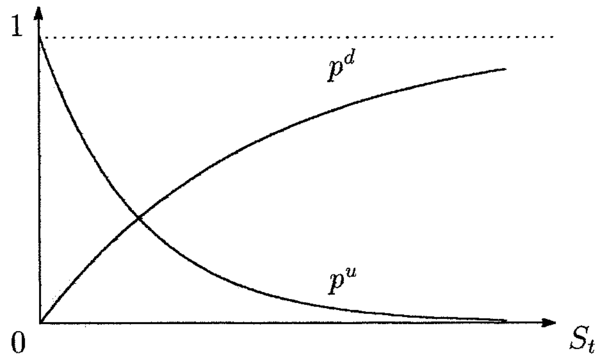


図８：感染確率

3. さらにその近傍のエージェントへと，強気・弱気の感染が同様に繰り返される．その結果，最初のエージェントが受け取ったニュースI_tの値に応じて，楽観的あるいは悲観的な予想のいずれかを共有するクラスターC_tが形成される．
4. 株価指数の収益率$\frac{S_{t+1}-S_t}{S_t}$が，クラスターサイズに比例して定まる．すなわち

$$\frac{\Delta S_t}{S_t} = \rho \times I_t \times \frac{|C_t|}{N^2} \tag{5}$$

によって，$t+1$期における株価指数の値S_{t+1}が定まる．ここで定数ρは，株価指数過程のヴォラティリティあるいは価格弾力性の逆数であり，超過需要が価格変動に与える影響度を定める．

各パラメータを$\alpha = 1.4, \delta = 0.44, \rho = 3.5, N = 21$としてシミュレーションをおこない，その結果得られた株価指数$\{S_t\}$のサンプルパスと，その変化率の対数頻度グラフを下に示す．変動規模が大きくなるにつれて，正規分布からの乖離が大きくなっていく様子がはっきりと示されている．

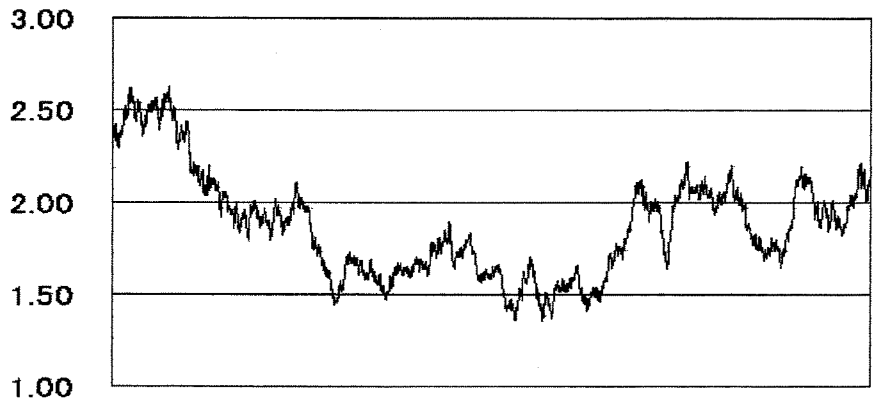


図9：シミュレーション$(\alpha = 1.4, \delta = 0.44, \rho = 3.5, \mathrm{N} = 21)$

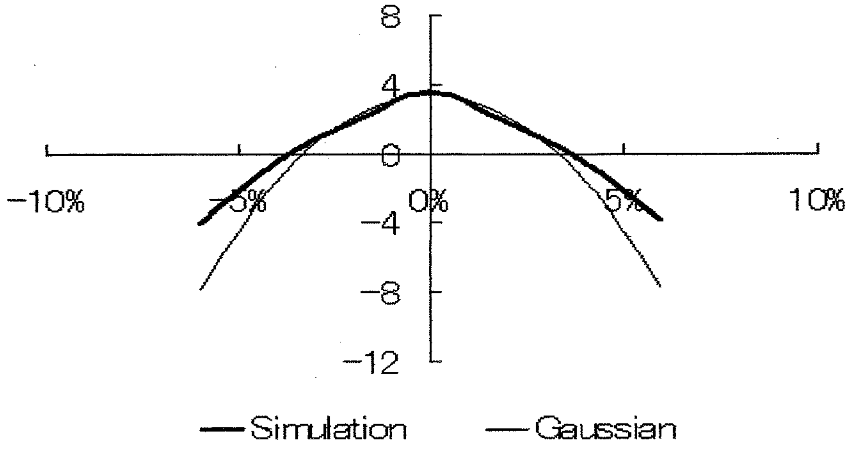


図10：シミュレーション($\alpha = 1.4, \delta = 0.44, \rho = 3.5, \mathrm{N} = 21$)

シミュレーション言語は`Mathematica`を用い，プログラミングにあたっては大部分をGaylord-Wellin(1995)に依拠した．プログラムの入手を希望される方は，筆者宛にメールを送られたい．

4　経験データを用いたパラメータの決定

パーコレーションに対する解析的研究の成果により，モデルのサイズNが十分に大きいならば，その挙動は一定の分布に収束することが知られている．したがって，第３章で定義したモデルにとって本質的なパラメータは，α, δ, ρ の３つである．この章では，1991年1月から2001年10月までの日経平均終値を用いてこれらの値を求めることを試みる．

しかしながら，パーコレーションは非常に複雑度の高いモデルであり，その性質を解析的に調べることは難しい．したがって，前章で導入したモデルを数学的に解くことは現時点では不可能に近く，計算機によるシミュレーションに頼らざるを得ない．パラメータの値を決定するにあたっても，既存の統計的な手法を応用することが難しため，この論文では，原データに近い特性を持つサンプルパスを生成するパラメータが見つかるまでシミュレーションを繰り返す，という些か原始的な方法によってパラメータを探すこととする．

３つのパラメータを決定するためには，経験データから最低でも３つの特性値を取り出す必要がある．ここでは，1991年1月から2001年10月までの日経平均終値の平均値$\hat{S} = 1.8265$(万円)，標準偏差$\hat{\sigma} = 0.3205$(万円)のほかに，以下で定義される$\hat{\gamma}$を用いることにする．まず，株価収益率の分布は左右対称であるとし，収益率の絶対値は指数分布に従うとする．すな

わち，収益率の密度関数fを

$$f(x) := P\left(\left|\frac{\Delta S}{S}\right| = x\right) = C_0 e^{-\gamma x} \tag{6}$$

と特定化する．したがって$\log f(x) = c_0 - \gamma x$であるが，経験データはこの特定化によくフィットしている．通常の回帰分析により，$\hat{c}_0 = 4.5769$, $\hat{\gamma} = 90.9691$ と推定される．

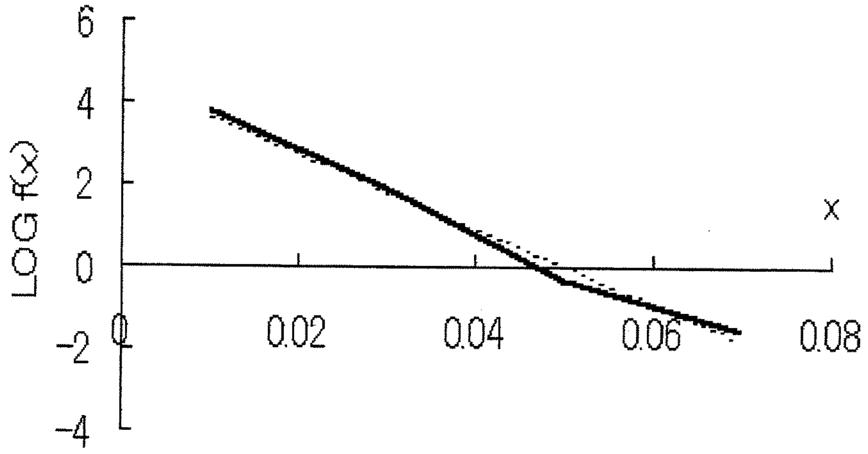


図11：絶対値の対数経験頻度(1991.1-2001.10)

以上の値，$\hat{S} = 1.8265$，$\hat{\sigma} = 0.3205$，および$\hat{\gamma} = 90.9691$を用いて，モデルのパラメータα，δ，およびρを求める．しかしながら，上述のとおりパラメータの決定はいわゆる“試行錯誤”によっておこなうため，決定すべきパラメータ数を事前にひとつでも減らしておきたい．そこで，シミュレーションの観察から得られる以下の予想を用いて，決定すべきパラメータ数を3から2に減らすことにする．

《予想》　楽観と悲観の浸透確率p^u, p^dが等しくなる価格を$\bar{S}$とする．すなわち

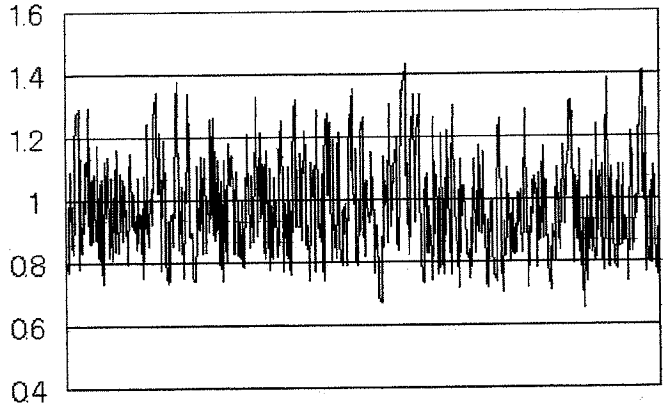
$$p^u(\bar{S}) = e^{-\alpha\bar{S}} = 1 - e^{-\delta\bar{S}} = p^d(\bar{S}). \tag{7}$$

このときシミュレーションから生成される価格過程$\{S_t\}$について

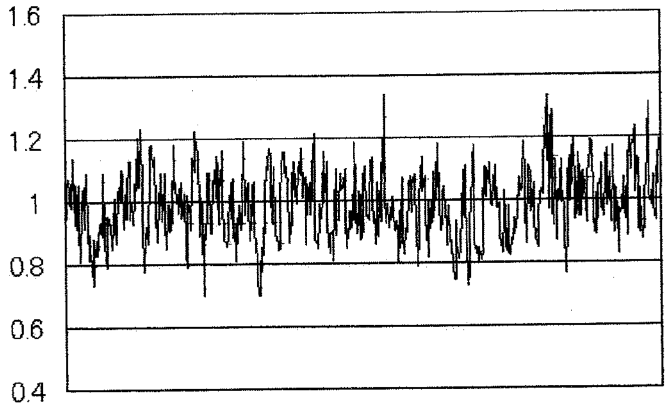
$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} S_t = \bar{S} \tag{8}$$

が成り立つ．

$\bar{S}=1$を満たす二つのパラメータの組について，それぞれのサンプルパスを以下に示す．いずれも$\bar{S}=1$を中心に変動しているように見える．



(i) $\alpha=0.6, \delta=0.7959, \rho=0.5, N=441 \;\Rightarrow\; \hat{S}=0.9773$



(ii) $\alpha=0.7, \delta=0.6863, \rho=0.5, N=441 \;\Rightarrow\; \hat{S}=0.9925$

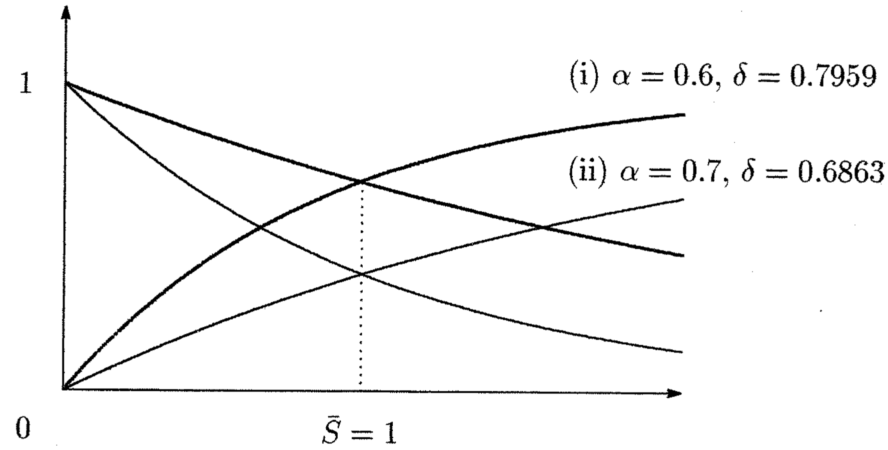
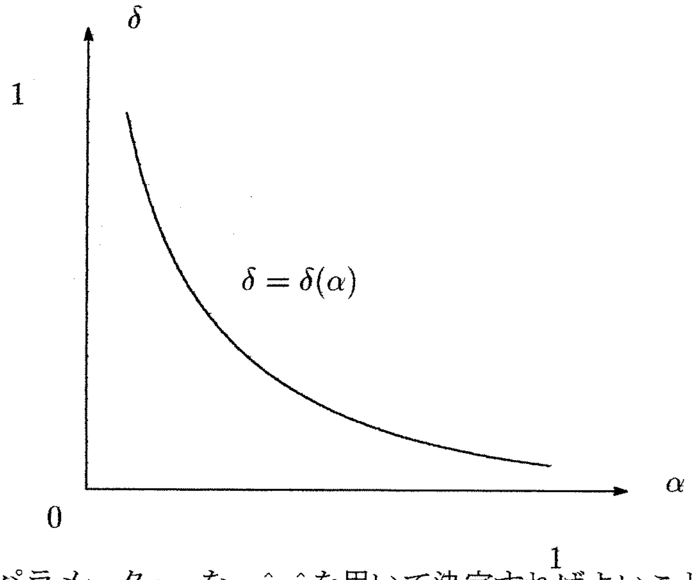


図12：ケース(i)および(ii)における感染確率グラフ

この予想が正しいものとすれば，経験データの平均値$\hat{S} = 1.8265$を用いて$\hat{S} = \bar{S}$ とすることにより，パラメータをひとつ減らすことができる．すなわち

$$e^{-1.8265\alpha} = 1 - e^{-1.8265\delta} \tag{9}$$

としてδに関して解き，$\delta = \delta(\alpha)$とする．



これにより，二つのパラメータα, ρを，$\hat{\sigma}, \hat{\gamma}$を用いて決定すればよいことになる．以上の準備の下で，$N = 21$とし，またαを0.1刻み，ρを0.5刻みで10,000期分を計算させた．この結果から，およそ

$$\alpha = 1.6, \quad \delta = 0.030, \quad \rho = 6.5$$

であると推測される．図13と14に，このパラメータのもとでのサンプルパスと，対数頻度グラフを示す．このサンプルパスの平均は1.8333，標準偏差は0.3032であり，実際の日経平均の値$\hat{S} = 1.8265$，$\hat{\sigma} = 0.3205$に近い．また，シミュレーションによって得られた対数頻度グラフが，経験データによくフィットしていることも見て取れる．

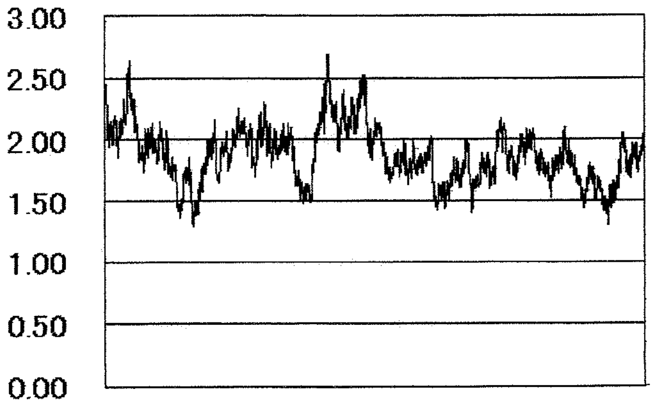


図13：シミュレーション（$\alpha = 1.6$, $\delta = 0.030$, $\rho = 6.5$）

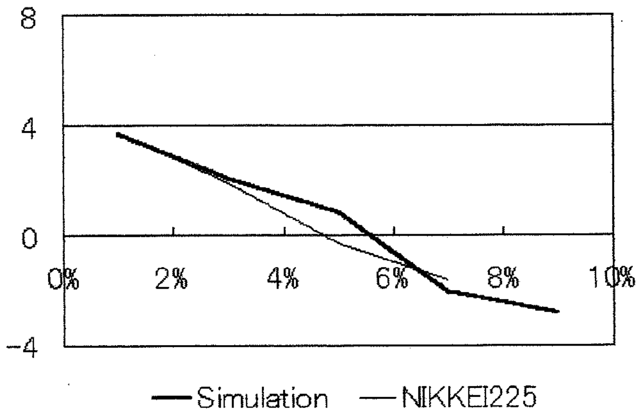


図14：対数頻度分布による比較

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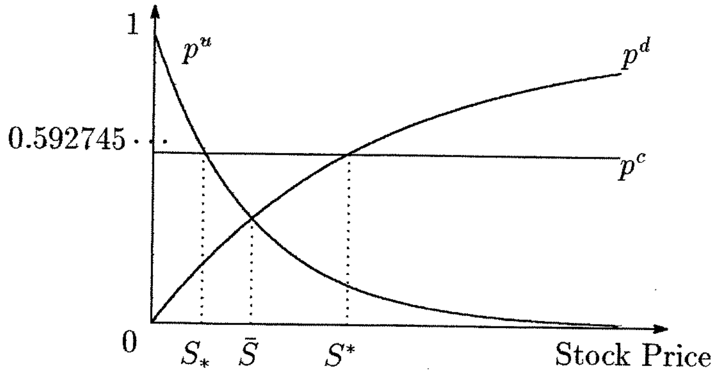
5　天井と床

株価指数過程$\{S_t\}$の天井S^*と床S_*を，臨界確率p^cを用いて

$$p^d(S^*) = p^c, \quad p^u(S_*) = p^c \tag{10}$$

によってそれぞれ定義する．



すなわち，$S_t > S^*$となった場合には$p^d > p^c$となるため,「売り」の無限クラスターが発生する確率が正になる．逆に$S_t < S^*$となった場合には，$p^u > p^c$となるため,「買い」の無限クラスターが正の確率で発生することになる．それぞれの場合が株式市場の大暴落あるいは大高騰に対応していると考えれば，上述の定義は自然であると思われる．

しかしながら，前章で求めたパラメータのもとで天井と床を計算すると$S^* = 29.6672$(万円)，$S_* = 0.3269$(万円)となって現実味が薄い．これは，データの当てはめをおこなった期間が長すぎることにも一因があるものと思われる．したがって，今度は2001年下半期６ヶ月分のデータを用いて再計算すると

$$\alpha = 0.7, \quad \delta = 0.58, \quad \rho = 0.4 \tag{11}$$

となる．このときの天井と床は，それぞれ

$$S^* = 1.5489, \quad S_* = 0.7471 \tag{12}$$

となり，より現実的な値となっている．より確かな推定をおこなうためには，取引毎データのようなより細かいデータを用いる必要がある．

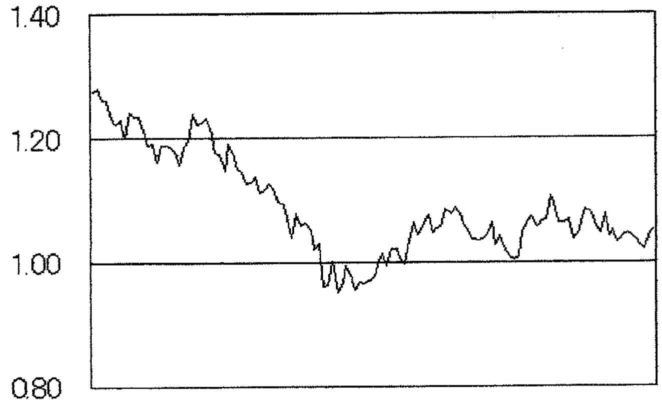


図15：日経平均2001年下半期

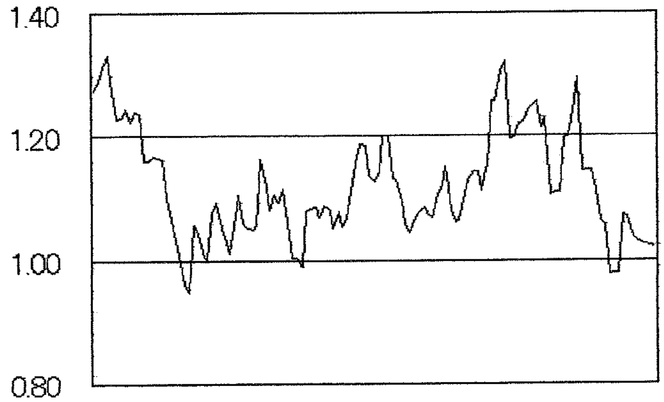


図16：シミュレーション（$\alpha = 0.7, \delta = 0.58, \rho = 0.4$）

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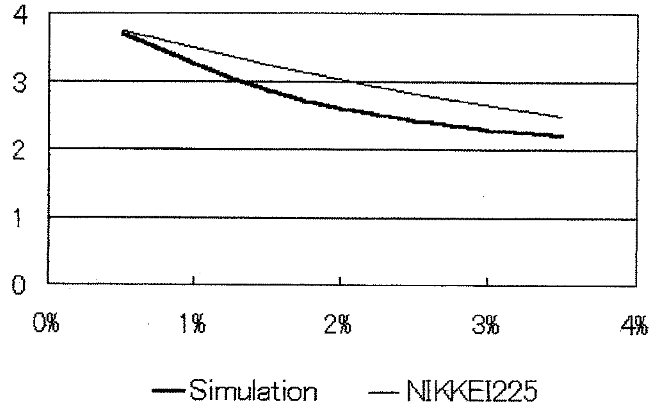

図17：対数頻度分布による比較

6　結論と今後の課題

本論文では，パーコレーションを用いて株式市場をモデル化し，限定視野を持つエージェントの局所的な相互依存関係が株価の大規模な変動の一因になっている可能性があることを示した．また，臨界確率の概念を応用して株価の天井と床を定義し，さらにその値を日経平均終値の経験データを用いて算出した．すなわち，2001年下半期のデータによれば，床はおよそ7,500円，天井はおよそ15,500円となる．

ここで示されたモデルは，最低限の特徴だけを与えたもっとも簡単なものである．したがって，このモデルをもとにした様々な拡張が容易に考えられる．例えば，本稿ではシミュレーションの容易さを考えて２次元のモデルだけを分析したが，より高次元のモデルを考えることもできる．あるいは，時間・空間的に至るところ均一な構造をもつ正方格子ネットワークではなく，毎期にその構造を組み変えていくランダムネットワークモデルのほうが，現実に近いであろう．

モデルそのものの拡張だけでなく，経験データからモデルのパラメータを推定する効率的な手法を開発する必要もある．楽観・悲観の浸透確率p^u, p^dをデータから直接求めることができれば，分析は著しく容易になる．また，エージェントの振舞いや株価の決定メカニズムに経済学的な基礎を与えることも重要であろう．

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