概周期関数の拡張と零点の分布について

課題番号　10640180

平成10年度—平成13年度科学研究費補助金（基盤研究（C）(2)）研究成果報告書

平成14年4月

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この報告書は平成10年度から同13年度までの科学研究費補助金 ，基盤研究 C (2), 10640180, による研究内容をまとめたものである.

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交付決定額（配分額）

（金額単位：千円）

	直接経費	間接経費	合計
平成 10 年度	500	0	500
平成 11 年度	500	0	500
平成 12 年度	500	0	500
平成 13 年度	500	0	500
総計	2,000	0	2,000

研究発表
（1）学会誌など.

J. Tanaka, The Riemann Zeta-function and ergodic Hardy spaces,
to appear.

J. Tanaka, Extension of almost periodic functions and analyticity on flows,
Amer. Math. Soc. Transl. (2), 204, 2001, 63-80.

H. Helson and J. Tanaka, Singular cocycles and the generator problem
Proceeding of Operator Theory Conference, Theta, 2000,
173-186.

田中純一，概周期関数の拡張と流れの上の解析性，岩波「数学」，論説，
51 (2), 1999, 113-128.

THE RIEMANN ZETA-FUNCTION AND ERGODIC HARDY SPACES

BY

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Dedicated to Professor Kōzo Yabuta on his 60th birthday

ABSTRACT

We consider the Riemann zeta-function $\zeta(s)$ as an outer function in Hardy spaces defined by an ergodic flow on $\mathbf{T}^\omega$, the infinite dimensional torus. This enables us to investigate collectively some properties of Dirichlet series of the form $\mathfrak{z}(x,s) = \prod_p \left(1 - a(p)p^{-s}\right)^{-1}$, $\{a(p)\} \in \mathbf{T}^\omega$, where p ranges over primes. We then discuss sequences of rectangles in the strip $1/2 < \sigma < 1$ free from zeros of $\zeta(\sigma + it)$, with the aid of nomal families argument. By using them it is shown that a mean-value theorem for all powers of $\zeta(s)$ holds in a weak sense.

1. Introduction.

A *Dirichlet series* has the form

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad s = \sigma + it, \tag{1.1}$$

where the coefficients $a(n)$ are given complex numbers, and such a series converges in a half-plane $\sigma > \sigma_c$. Then $f(s)$ is said to have an *Euler product* if there is a representation

$$f(s) = \prod_p \left(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \cdots\right),$$

where p runs through all primes. The most important Dirichlet series is the *Riemann zeta-function* given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1.$$

Partially supported by Grant 10640180 from the Japanese Ministry of Education.
April, 2002

It can be analytically continued over complex plane $\mathbf{C}$ except at $s = 1$, where it has a simple pole with residue 1, and satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{1}{2} s\pi \, \Gamma(1-s)\zeta(1-s). \tag{1.2}$$

Thus the *Riemann hypothesis* amounts to the assertion that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1/2$.

In what follows, we are concerned with a certain class of Dirichlet series with Euler products associated with $\zeta(s)$. Let $\mathbf{T}^\omega$ be the infinite-dimensonal torus, the complete direct sum of countably many copies $\mathbf{T}_p$ of the unit circle $\mathbf{T}$, where $\mathbf{T}_p$ are indexed by primes p. The dual group of $\mathbf{T}^\omega$ is the direct sum $\mathbf{Z}^\infty$ of countably many copies $\mathbf{Z}_p$ of the group $\mathbf{Z}$ of integers (see [18, 2.2.5]). We denote by $\boldsymbol{\sigma}_P$ the normalized Haar measure on $\mathbf{T}^\omega$. On the other hand, the fundamental theorem of arithmetic asserts that each integer $n > 1$ can be expressed as a product of prime factors,

$$n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_\ell^{k_\ell},$$

in only one way apart from the order of the factors. This implies that each $x = \{a(p)\}$ in $\mathbf{T}^\omega$ induces a strongly multiplicative function $a(n)$ defined by

$$a(n) = a(p_1)^{k_1} \cdot a(p_2)^{k_2} \cdots a(p_\ell)^{k_\ell}. \tag{1.3}$$

We thus obtain a Dirichlet series with Euler product

$$\mathfrak{z}(x,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \left(1 - \frac{a(p)}{p^s}\right)^{-1}, \quad \sigma > 1, \tag{1.4}$$

which is intimately akin to $\zeta(s)$. Indeed, in a suitable sense, such a Dirichlet series may be regarded as a limit of translations of $\zeta(s)$.

Our objective in this note is to investigate some properties of $\zeta(s)$ as an element of the class of all $\mathfrak{z}(x,s)$. By extending $\zeta(s)$ to an analytic function on a compact abelian group with ordered dual, it is shown that $\mathfrak{z}(x,s)$ is analytically continued to and has no zeros on the half-plane $\sigma > 1/2$, outside $\boldsymbol{\sigma}_P$−null set in $\mathbf{T}^\omega$. Although $\zeta(s)$ is exceptional in some ways, it inherits a certain measure of properties common to the class. This fact provides a method of investigating the asymptotic behavior of $\zeta(s)$ in the critical strip. With the aid of the ergodic theorem, we also give another approach to the mean-value theorems of $\zeta(s)$. Our direction stems from the theory of invariant subspaces based on uniform algebras, and some of the ideas are especially motivated by [8] and [15]. Throughout the paper we restrict our attention to the case of $\zeta(s)$ for simplicity, although most of our techniques are applicable to more general class of Dirichlet series, containing Dirichlet L-functions. In [1] and [2], the Riemann zeta-function is also discussed in connection with functional analysis.

In the next section, we establish notation and present some known lemmas about analyticity on almost periodic flows, which are modified to suit our purposes. In Section 3, $\zeta(s)$ is extended to an outer function in Hardy spaces on a compact abelian group. We develop our techniques in Section 4 to investigate the outer function obtained by $\zeta(s)$. In Section 5, after preparing some lemmas, we prove our mean-value theorem in a weak sense. We close with complementary remarks in Section 6.

2. Analyticity on compact groups.

Let Γ be a dense subgroup of the real line $\mathbf{R}$, endowed with the discrete topology, and let K be the dual group of Γ. For each t in $\mathbf{R}$, e_t denotes the element of K defined by $e_t(\lambda) = e^{i\lambda t}$ for any λ in Γ. Then the map of t to e_t embeds $\mathbf{R}$ continuously onto a dense subgroup of K. Define a one-parameter group $\{T_t\}_{t\in\mathbf{R}}$ of homeomorphisms of K onto itself by

$$T_t x = x + e_t, \quad x \in K. \tag{2.1}$$

Then the pair $(K, \{T_t\}_{t\in\mathbf{R}})$ is a uniquely ergodic flow, of which the unique invariant probability measure is the normalized Haar measure $\boldsymbol{\sigma}$ on K. Let $C(K)$ be the space of all continuous complex-valued functions on K. A *trigonometric polynomial* p is a function of the form

$$p(x) = \sum_{n=1}^{N} a_n \chi_{\lambda_n}(x), \quad x \in K,$$

where χ_λ is the character on K defined by $\chi_\lambda(x) = x(\lambda)$. Since

$$p(e_t) = \sum_{n=1}^{N} a_n e^{i\lambda_n t},$$

the Stone-Weierstrass theorem assures that $C(K)$ may be identified with the space of all uniformly almost periodic functions with exponents in Γ. Then the continuous flow $(K, \{T_t\}_{t\in\mathbf{R}})$ is called an *almost periodic flow.*

A function ϕ in $L^1(\boldsymbol{\sigma})$ is *analytic* if its Fourier coefficients

$$a_\lambda(\phi) = \int_K \overline{\chi_\lambda}\, \phi \, d\boldsymbol{\sigma}$$

vanish for all negative λ in Γ. The *Hardy space* $H^q(\boldsymbol{\sigma})$, $1 \le q \le \infty$, is defined to be the space of all analytic functions in $L^q(\boldsymbol{\sigma})$. We also denote by $A(K)$ the uniform algebra of all analytic functions in $C(K)$. Then it is easy to see that $A(K)$ is a Dirichlet algebra on K and $\boldsymbol{\sigma}$ is a representing measure for $A(K)$. If $1 \le q < \infty$, $H^q(\boldsymbol{\sigma})$ is the closure of $A(K)$ in $L^q(\boldsymbol{\sigma})$, while $H^\infty(\boldsymbol{\sigma})$ is the weak-* closure of $A(K)$ in $L^\infty(\boldsymbol{\sigma})$. In analogy with the classical theory, an analytic function of modulus one is said to be *inner*, and a function ϕ in $H^q(\boldsymbol{\sigma})$ is called *outer* if ϕ satisfies

$$\log |a_0(\phi)| = \int_K \log |\phi| \, d\boldsymbol{\sigma} > -\infty.$$

It follows from Szegö's theorem that a function ϕ in $H^q(\boldsymbol{\sigma})$, $1 \le q < \infty$, is outer if and only if the invariant subspace generated by ϕ equals $H^q(\boldsymbol{\sigma})$, that is, the closure of $A(K)\cdot\phi$ in $L^q(\boldsymbol{\sigma})$ equals $H^q(\boldsymbol{\sigma})$. When $q = \infty$, the same result holds with respect to the weak-* topology in $L^\infty(\boldsymbol{\sigma})$.

We denote by $H^\infty(dt/\pi(1+t^2))$ the space of all the boundary-value functions of bounded analytic functions in the upper half-plane $\mathbf{R}^2_+$. The closure of $H^\infty(dt/\pi(1+t^2))$ in $L^q(dt/\pi(1+t^2))$, $1 \le q < \infty$, is denoted by $H^q(dt/\pi(1+t^2))$. The *Poisson kernel* $P_z(t)$ for $\mathbf{R}^2_+$ is defined by $P_z(t) = v/\pi((u-t)^2 + v^2)$, where $z = u + iv$ with $v > 0$. We usually identify each $f(t)$ in $H^q(dt/\pi(1+t^2))$ with its analytic extension $f(z)$ to $\mathbf{R}^2_+$ obtained by the convolution

$$f(z) \;=\; f * P_{iv}(u) \;=\; \int_{-\infty}^{\infty} f(t)\, P_z(t)\, dt.$$

Recall that a function f in $H^q(dt/\pi(1+t^2))$ is outer in the ordinary sense if and only if

$$\log|f(z)| \;=\; \int_{-\infty}^{\infty} \log|f(t)|\, P_z(t)\, dt \;>\; -\infty,$$

and any outer function does not vanish on $\mathbf{R}^2_+$.

It is known that a function ϕ in $L^q(\boldsymbol{\sigma})$ is analytic if and only if $t \to \phi(x+e_t)$ lies in $H^q(dt/\pi(1+t^2))$, for $\boldsymbol{\sigma} - a.e.\, x$ in K (see [10, 3.2]). Using this pointwise criterion, we characterize the outer functions in $H^q(\boldsymbol{\sigma})$ as follows:

Lemma **2.1.** *Let ϕ be a function in $H^q(\boldsymbol{\sigma}), 1 \le q \le \infty$. Then ϕ is outer in $H^q(\boldsymbol{\sigma})$ if and only if $a_0(\phi) \ne 0$ and $t \to \phi(x+e_t)$ is outer in $H^q(dt/\pi(1+t^2))$ for $\boldsymbol{\sigma} - a.e.\, x$ in K.*

Proof. Since $H^1(\boldsymbol{\sigma}) \cap L^q(\boldsymbol{\sigma}) \;=\; H^q(\boldsymbol{\sigma})$ by [10, 1.6 Lemma], it suffices to show the case where $q = 1$. Suppose that $A(K)\cdot\phi$ is not dense in $H^1(\boldsymbol{\sigma})$. Then there is a nonconstant function ψ in $H^\infty(\boldsymbol{\sigma})$ which is orthogonal to $A(K)\cdot\phi$, that is,

$$\int_K g\,\phi\,\overline{\psi}\, d\boldsymbol{\sigma} \;=\; 0\,, \quad g \in A(K)\,.$$

Since $\phi\overline{\psi}$ is in $H^1(\boldsymbol{\sigma})$, we see that almost every $t \to \phi(x+e_t)\overline{\psi(x+e_t)}$ lies in $H^1(dt/\pi(1+t^2))$. Since $t \to \psi(x+e_t)$ is not constant, $t \to \phi(x+e_t)$ cannot be an outer function in $H^1(dt/\pi(1+t^2))$ for $\boldsymbol{\sigma} - a.e.\, x$ in K.

For the converse, suppose that the set E of all x in K such that

$$\log\,|\phi * P_i(x)| \;<\; \int_{-\infty}^{\infty} \log|\phi(x+e_t)|\, P_i(t)\, dt$$

has positive measure. We notice that Jensen's inequality holds for $P_i(t)\,dt \;=\; dt/\pi(1+t^2)$. On the other hand, since $\phi * P_i$ lies in $H^1(\boldsymbol{\sigma})$, we also obtain

$$\log\,\Big|\int_K \phi * P_i\, d\boldsymbol{\sigma}\,\Big| \;\le\; \int_K \log\,|\phi * P_i|\, d\boldsymbol{\sigma}$$

by Jensen's inequality. Observe that

$$a_0(\phi) \;=\; \int_K \phi\, d\boldsymbol{\sigma} \;=\; \int_K \phi * P_i\, d\boldsymbol{\sigma}\,.$$

It follows from Fubini's theorem that

$$\log|a_0(\phi)| \;<\; \int_K \log\,|\phi|\, d\boldsymbol{\sigma}\,,$$

thus ϕ cannot be an outer function in $H^1(\boldsymbol{\sigma})$. □

There is a local product decomposition which is useful for understanding the structure of K. Fix an $\ell > 0$ and suppose $2\pi/\ell$ lies in Γ. Let $K_{2\pi/\ell}$ be the subgroup of all x in K such that $\chi_{2\pi/\ell}(x) \;=\; 1$. Then K may be identified measure theoretically, and almost topologically, with $K_{2\pi/\ell} \times [0,\ell)$ via the map of $y+e_u$ to (y,u). Let T be the homeomorphism of $K_{2\pi/\ell}$ onto itself by

$$Ty \;=\; y + e_\ell\,, \quad y \in K_{2\pi/\ell}. \tag{2.2}$$

Then the dynamical system $(K_{2\pi/\ell}, T)$ is uniquely ergodic, where the normalized Haar measure $\boldsymbol{\tau}$ on $K_{2\pi/\ell}$ is the unique invariant probability measure. Let $m_{\mathbf{R}}$ be Lebesgue measure on $\mathbf{R}$, and let m_I be the restriction of $(1/\ell)dm_{\mathbf{R}}$ to $[0,\ell)$. Then $\boldsymbol{\sigma}$ is carried by $\boldsymbol{\tau} \times m_I$. The flow $(K, \{T_t\}_{t\in\mathbf{R}})$ is also represented as

$$T_t x \;=\; T_t(y,u) \;=\; (T^{[(t+u)/\ell]}y\,, t+u-[(t+u)/\ell]\ell)\,, \quad x=(y,u)\in K\,, \tag{2.3}$$

where $[t]$ denotes the largest integer not exceeding t.

A Borel function f on $K_{2\pi/\ell} \times \mathbf{R}$ is *automorphic* if

$$f(y, t+\ell) \;=\; f(Ty,t)\,, \quad \boldsymbol{\tau} \times m_{\mathbf{R}} - a.e.\ (y,t) \in K_{2\pi/\ell} \times \mathbf{R}. \tag{2.4}$$

Then each function ϕ on K has the automorphic extension $\phi^\sharp$ defined by

$$\phi^\sharp(y,t) \;=\; \phi(T^{[t/\ell]}y\,, t-[t/\ell]\ell)\,, \quad (y,t) \in K_{2\pi/\ell} \times \mathbf{R}. \tag{2.5}$$

We next introduce several notions of density (refer to [4, Chapter 3]). The cardinality of finite set S is denoted by $|S|$. Let J be a subset of the set $\mathbf{Z}^+$ of all nonnegative integers. Then the *upper Banach density* $\mathcal{BD}^*(J)$ of J is defined by

$$\mathcal{BD}^*(J) \;=\; \limsup_{|I|\to\infty} \frac{|J\cap I|}{|I|}\,,$$

where I ranges over intervals of $\mathbf{Z}^+$. The *upper density* $\mathcal{D}^*(J)$ of J is defined by

$$\mathcal{D}^*(J) \;=\; \limsup_{N\to\infty} \frac{|J\cap[0,N-1]|}{N}\,,$$

and the *lower density* $\mathcal{D}_*(J)$ is defined similarly. When $\mathcal{D}^*(J) \;=\; \mathcal{D}_*(J)$, the value $\mathcal{D}(J)$ is called the *density* of J. It is easy to give examples of J with $\mathcal{D}^*(J) \;<\; \mathcal{BD}^*(J)$.

For a given e in $K_{2\pi/\ell}$, let $E(J)$ be the closure of $\{T^n e\,;\, n\in J\}$ in $K_{2\pi/\ell}$. Then we have

$$\mathcal{D}^*(J) \;\le\; \mathcal{BD}^*(J) \;\le\; \boldsymbol{\tau}(E(J)). \tag{2.6}$$

Indeed, for each $\epsilon > 0$, there is a function p in $C(K_{2\pi/\ell})$ with $0 \le p \le 1$ such that $p \equiv 1$ on $E(J)$ and

$$\boldsymbol{\tau}(E(J)) \;\le\; \int_{K_{2\pi/\ell}} p\, d\boldsymbol{\tau} \;<\; \boldsymbol{\tau}(E(J)) + \epsilon.$$

Since $(K_{2\pi/\ell}, T)$ is uniquely ergodic, we see

$$\frac{1}{N}\sum_{n=0}^{N-1} p\circ T^n \;\longrightarrow\; \int_{K_{2\pi/\ell}} p\, d\boldsymbol{\tau}\,, \quad \text{as} \quad N\to\infty\,,$$

uniformly on $K_{2\pi/\ell}$. This implies that

$$\limsup_{N\to\infty}\left\{\sup_{k\ge 0}\frac{1}{N}\sum_{n=0}^{N-1} I_{E(J)}(T^{k+n}e)\right\} \;<\; \boldsymbol{\tau}(E(J)) + \epsilon\,,$$

where $I_{E(J)}$ denotes the characteristic function of $E(J)$. Thus the desired inequality (2.6) follows immediately.

The following lemma plays an important role in the proof of our mean-value theorem for $\zeta(s)$.

LEMMA 2.2. *Let J, e, and $E(J)$ be as above. Suppose that, for each $\epsilon > 0$, there is a subset J_1 of $\mathbf{Z}^+ \setminus J$ such that $\boldsymbol{\tau}(E(J) \cap E(J_1)) = 0$ and $\mathcal{D}^*(\mathbf{Z}^+ \setminus (J \cup J_1)) < \epsilon$. If p_0 lies in $C(E(J))$, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} p_0(T^n e) = \int_{E(J)} p_0 \, d\boldsymbol{\tau}, \tag{2.7}$$

where p_0 is extended to $K_{2\pi/\ell}$ by setting $p_0 = 0$ outside $E(J)$. We also have $\mathcal{D}(J) = \boldsymbol{\tau}(E(J))$.

Proof. We may assume that p_0 is nonnegative and $E(J) \cap E(J_1) = \emptyset$. Indeed, when $E(J) \cap E(J_1) \neq \emptyset$, we choose a neighborhood V of $E(J) \cap E(J_1)$ such that $\boldsymbol{\tau}(\overline{V})$ is sufficiently small, where $\overline{V}$ denotes the closure of V. Then we replace J_1 with $J_1 \setminus \{n \in J_1; ne \in V\}$.

Let $\epsilon > 0$ be given, and choose a subset J_1 of $\mathbf{Z}^+ \setminus J$ such that $E(J) \cap E(J_1) = \emptyset$ and $\mathcal{D}^*(\mathbf{Z}^+ \setminus (J \cup J_1)) < \epsilon / \|p_0\|_{E(J)}$, where $\|p_0\|_{E(J)}$ denotes the uniform norm of p_0 on $E(J)$. It follows from Tietze's extension theorem that p_0 extends to a nonnegative function p in $C(K_{2\pi/\ell})$ with the same norm such that $p \equiv 0$ on $E(J_1)$. Since

$$1 \le \mathcal{D}^*(\mathbf{Z}^+ \setminus (J \cup J_1)) + \mathcal{D}^*(J \cup J_1),$$

we obtain by (2.6) that

$$1 - \frac{\epsilon}{\|p_0\|_{E(J)}} < \mathcal{D}^*(J) + \mathcal{D}^*(J_1) \le \boldsymbol{\tau}(E(J)) + \boldsymbol{\tau}(E(J_1)).$$

This shows that

$$\boldsymbol{\tau}((E(J) \cup E(J_1))^c) < \frac{\epsilon}{\|p_0\|_{E(J)}}.$$

Since $(K_{2\pi/\ell}, T)$ is uniquely ergodic, we observe

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} p_0(T^n e) &\le \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} p(T^n e) \\
&= \int_{K_{2\pi/\ell}} p \, d\boldsymbol{\tau} \\
&< \int_{E(J)} p_0 \, d\boldsymbol{\tau} + \epsilon.
\end{aligned}$$

On the other hand, it is easy to see that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} I_{(E(J)\cup E(J_1))^c}(T^n e) \le \mathcal{D}^*(\mathbf{Z}^+ \setminus (J \cup J_1)).$$

since $p\, I_{E(J)\cup E(J_1)}$ is the extension of p_0 and

$$p \le p\, I_{E(J)\cup E(J_1)} + \|p_0\|_{E(J)}\, I_{(E(J)\cup E(J_1))^c},$$

we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} p(T^n e) - \epsilon \le \liminf_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} p_0(T^n e),$$

thus the equation (2.7) holds.

The last statement follows from (2.6) and the inequality

$$\begin{aligned} \mathcal{D}_*(J) &= 1 - \mathcal{D}^*(\mathbf{Z}^+ \setminus J) \\ &\ge 1 - \boldsymbol{\tau}(E(J_1)) - \epsilon \\ &\ge \boldsymbol{\tau}(E(J)) - \epsilon. \end{aligned}$$

□

We refer the reader to [10] and [5, Chapter VII] for further details of analyticity on compact abelian groups, and to [4] and [19]for required results in ergodic theory. Besides, [3] and [6] are useful references for the classical theory of Hardy spaces.

3. **Extension of the Riemann zeta-function.**

From now on we assume that Γ is the discrete group of all $\log r$ where r runs through positive rationals. If $\mathcal{G}$ is the subset of Γ of all $\log p$ where p is prime, then $\mathcal{G}$ is an independent set generating Γ. Let $\mathbf{T}^\omega$, $\mathbf{Z}^\infty$ and $\boldsymbol{\sigma}_{\boldsymbol{P}}$ be as in Section 1, and let τ be the group isomorphism of $\mathbf{Z}^\infty$ onto Γ by

$$\tau(\{n_p\}) = \sum_p n_p \log p, \quad \{n_p\} \in \mathbf{Z}^\infty.$$

Then the dual group K of Γ is isomorphic to $\mathbf{T}^\omega$ via the adjoint map τ^* of τ defined by

$$\langle \tau^*(x), \{n_p\}\rangle = \langle x, \tau(\{n_p\})\rangle, \quad x \in K,$$

so we may identify $\mathbf{T}^\omega$ with K. Since $e_t(\log p) = e^{it\log p}$, the one-parameter group $\{T_t\}_{t\in\mathbf{R}}$ by (2.1) is concretely represented as

$$T_t(\{e^{i\theta_p}\}) = \{e^{i(\theta_p + t\log p)}\}, \quad \{e^{i\theta_p}\} \in \mathbf{T}^\omega.$$

We note also that

$$d\boldsymbol{\sigma}_{\boldsymbol{P}} = \prod_p \frac{1}{2\pi} d\theta_p.$$

Using the flow $(\mathbf{T}^\omega, \{T_t\}_{t\in\mathbf{R}})$, we can extend Dirichlet series to analytic functions on $\mathbf{T}^\omega$ under suitable conditions. Let $f(s)$ be a Dirichlet series by (1.1). Suppose that $a(n) = O(n^\epsilon)$ for any $\epsilon > 0$. Fix $u > 1/2$, and write formally

$$f(u+it) = \sum_{n=1}^{\infty} \frac{a(n)}{n^u} e^{-it\log n}.$$

Notice that if $u > 1$, then $t \to f(u-it)$ is an analytic almost periodic function on $\mathbf{R}$. We define the analytic function F_u on $\mathbf{T}^\omega$ by

$$F_u(x) = \sum_{n=1}^{\infty} \frac{a(n)}{n^u} \chi_{\log n}(x), \quad x \in \mathbf{T}^\omega .$$

Since $a(n) = O(n^\epsilon)$, we observe that

$$\|F_u\|_2^2 = \sum_{n=1}^{\infty} \left\{ \frac{|a(n)|}{n^u} \right\}^2 < \infty,$$

Then F_u lies in the Hardy space $H^2(\boldsymbol{\sigma}_P)$ defined in Section 2. We need to strengthen this fact as follows:

LEMMA **3.1.** *Let F_u be as above, and let $1 \le q < \infty$. Then F_u belongs to $H^q(\boldsymbol{\sigma}_P)$.*

Proof. Let

$$G_u = \sum_{n=1}^{\infty} \frac{b(n)}{n^u} \chi_{\log n} ,$$

with $b(n) = O(n^\epsilon)$ for any $\epsilon > 0$, and put

$$c(n) = \sum_{d|n} a(d) b(\frac{n}{d}) ,$$

where $d|n$ means that d is a divisor of n. It is known that the number $d_2(n)$ of divisors of n satisfies that $d_2(n) = O(n^\epsilon)$ (see [7, Theorem 315] for a proof). This shows that $c(n) = O(n^\epsilon)$, so the product

$$F_u G_u = \sum_{n=1}^{\infty} \frac{c(n)}{n^u} \chi_{\log n}$$

lies also in $H^2(\boldsymbol{\sigma}_P)$. Since by induction F_u^m is in $H^2(\boldsymbol{\sigma}_P)$ for each integer $m \ge 1$, F_u belongs to $H^q(\boldsymbol{\sigma}_P)$. □

We note that the class of all such F_u makes a subalgebra of the algebra $\bigcap_{1 \le q < \infty} H^q(\boldsymbol{\sigma}_P)$.

Let

$$Z_u = \sum_{n=1}^{\infty} \frac{1}{n^u} \chi_{\log n} , \tag{3.1}$$

and

$$Z_u^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^u} \chi_{\log n} , \tag{3.2}$$

where $\mu(n)$ is the Möbius function defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n \text{ is the product of } k \text{ diffrent primes}, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from Lemma 3.1 that both Z_u and Z_u^{-1} lie in $H^q(\boldsymbol{\sigma}_P)$, $1 \le q < \infty$. Restrictiong Z_u to the orbit $\mathcal{O}(0) = \{e_t\,;\, t \in \mathbf{R}\}$ of the unit element $0 = \{1\}$ of $\mathbf{T}^\omega$, we can represent $\zeta(s)$ on $\sigma > u$. More precisely, let us denote by $\chi_{\log n}(e_z)$ the analytic extension of $t \to \chi_{\log n}(e_t)$ to $\mathbf{R}^2_+$. Then we have formally

$$\begin{aligned} Z_u(e_z) &= \sum_{n=1}^{\infty} \frac{1}{n^u} \chi_{\log n}(e_z) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^u} e^{i(\log n)z} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^u} \frac{1}{n^{-iz}} . \end{aligned}$$

If we put $z = is = -t + i\sigma$, that is, the $\pi/2$ rotation of the right half-plane $\sigma > 0$, then $Z_u(e_z)$ represents $\zeta(u+s) = \zeta((u+\sigma)+it)$. Thus $Z_u(x)$ is regarded as an extension of $\zeta(s)$ to $\mathbf{T}^\omega$, and each $z \to Z_u(x+e_z)$, $Im\, z > 0$, is analogous to $\zeta(s)$ by its almost periodicity. In particular, since $Z_u(x+e_{i/2})$ lies in $A(\mathbf{T}^\omega)$, $\zeta((u+1/2)+s)$ is completely determined by the restriction of $Z_u(x+e_{i/2})$ to the orbit $\mathcal{O}(0)$. Similarly Z_u^{-1} may be regarded as an analytic extension of $\zeta^{-1}(s)$ to $\mathbf{T}^\omega$.

Lemma **3.2.** *Let $u > 1/2$, and let $1 \le q < \infty$. If Z_u and Z_u^{-1} are the functions by* (3.1) *and* (3.2), *respectively, then both Z_u and Z_u^{-1} are outer functions in $H^q(\boldsymbol{\sigma}_P)$.*

Proof. Since Z_u and Z_u^{-1} lie in $H^q(\boldsymbol{\sigma}_P)$ by Lemma 3.1, we see that $t \to Z_u(x+e_t)$ and $t \to Z_u^{-1}(x+e_t)$ lie in $H^q(dt/\pi(1+t^2))$, outside $\boldsymbol{\sigma}_P$−null set in $\mathbf{T}^\omega$. Recall that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \sigma > 1.$$

This shows that $Z_u(x+e_z)Z_u^{-1}(x+e_z) \equiv 1$ on $Im\, z > 1/2$ for each x in $\mathbf{T}^\omega$. From this fact the equation $Z_u(x)Z_u(x)^{-1} \equiv 1$ follows. Therefore, since $A(\mathbf{T}^\omega)$ is dense in $H^q(\boldsymbol{\sigma}_P)$, it is easy to see that both $A(\mathbf{T}^\omega) \cdot Z_u$ and $A(\mathbf{T}^\omega) \cdot Z_u^{-1}$ are dense in $H^q(\boldsymbol{\sigma}_P)$. □

It follows from Lemma 2.1 that almost every $t \to Z_u(x+e_t)$ is outer in $H^q(dt/\pi(1+t^2))$, so it has no zeros in $\mathbf{R}^2_+$. However, since $\zeta(s)$ has a pole at $s = 1$, $t \to Z_u(e_t)$ may not lie in $H^q(dt/\pi(1+t^2))$. Then the orbit $\mathcal{O}(0)$ must be contained wholly in the exceptional null set in Lemma 2.1, although we always consider $t \to Z_u(e_t)$ as the boundary function of the meromorphic function $z \to \zeta(u - iz)$ on $\mathbf{R}^2_+$. For $\boldsymbol{\sigma}_P - a.e.\ x$ in $\mathbf{T}^\omega$, we see that

$$Z_u(x+e_{i\sigma}) = Z_u * P_{i\sigma}(x) = \int_{-\infty}^{\infty} Z_u(x+e_t)P_{i\sigma}(t)\,dt, \quad \sigma > 0,$$

and if $\sigma > 1-u$, then $Z_u(x+e_{i\sigma})$ represents a function in $A(\mathbf{T}^\omega)$. We also notice that when $1/2 < u \le 1$, Z_u cannnot be in $H^\infty(\boldsymbol{\sigma}_P)$.

Let us look into the relation between Z_u and the class of $\mathfrak{z}(x,s)$ by (1.4). Let $a(n) = \chi_{\log n}(x)$. Since $a(n)$ is strongly multiplicative, $a(n)$ is of the form (1.3). Then x equals $\{a(p)\}$ in $\mathbf{T}^\omega$. Conversely, each $x = \{a(p)\}$ in $\mathbf{T}^\omega$ determines $\chi_{\log n}(x)$ by putting

$\chi_{\log p}(x) = a(p)$. Therefore each $\mathfrak{z}(x,s)$ is obtained by restrictiong Z_u to the orbit $\mathcal{O}(x) = \{x + e_t; t \in \mathbf{R}\}$ of x. Precisely, we have

$$Z_u(x + e_{is}) = \mathfrak{z}(x, u+s), \quad s = \sigma + it.$$

Thus Lemmas 2.1 and 3.2 provide the following:

THEOREM **3.3.** *Let u and $\mathfrak{z}(x,s)$ be as above. Then there is a $\boldsymbol{\sigma}_P$-null set $N = N(u,q)$ in $\mathbf{T}^\omega$ such that, for each x in $\mathbf{T}^\omega \setminus N$, both $\mathfrak{z}(x,s)$ and $\mathfrak{z}(x,s)^{-1}$ are analytic and free from zeros on $\sigma > u$. Moreover, $t \to \mathfrak{z}(x, u-it)$ and $t \to \mathfrak{z}(x,u-it)^{-1}$ belong to $H^q(dt/\pi(1+t^2)), 1 \le q < \infty$.*

In practice, it would be difficult to decide whether a given $\{a(p)\}$ in $\mathbf{T}^\omega$ lies in the null set N or not.

We know analogous methods, dealing with Dirichlet series, have been developed in [12], [16] and [13]. Nonetheless, our approach is essentially due to Helson (see [8], [9], and [10]), which enables us to utilize advantageously the theory of Hardy spaces based on uniform algebras. We also note that a part of Theorem 3.3 has been obtained in [13, Chapter 5] by a probabilistic method.

Although almost every $\mathfrak{z}(x,s)$ extends analytically to the half-plane $\sigma > 1/2$ and has no zeros, there are many sorts of exceptions in the class of $\mathfrak{z}(x,s)$. Indeed, we know that $\mathfrak{z}(\{1\},s) = \zeta(s)$ has a pole at $s=1$. Since $\mathfrak{z}(\{-1\},s)$ behaves like $\zeta(s)^{-1}$, it has a zero at $s=1$. Let S be the subset of all $x = \{a(p)\}$ in $\mathbf{T}^\omega$ of which all but finite number of terms $a(p)$ are 1. Then S is a dense subgroup of $\mathbf{T}^\omega$. If $a(p) = 1$ for $p \ge m$, then

$$\mathfrak{z}(x,s) = \zeta(s) \cdot \prod_{p \le m} \left(1 - \frac{1}{p^s}\right)\left(1 - \frac{a(p)}{p^s}\right)^{-1}.$$

Therefore $\mathfrak{z}(x,s)$ is similar to $\zeta(s)$ for each x in S, and oppositely $\mathfrak{z}(x,s)$ has the only zero at $s=1$ for x in $\{-1\}+S$. Furthermore, by using a property of alternating series, it is not difficult to find $\mathfrak{z}(x,s)$ having zeros, or poles, or both in the strip $1/2 < \sigma < 1$.

PROPOSITION **3.4.** *Let $1/2 < \alpha < 1$, and let m be a positive integer. Then there is a dense subset E of $\mathbf{T}^\omega$ on which each $\mathfrak{z}(x,s)$ extends analytically to $\sigma > 1/2$ and has a zero of order m at $s = \alpha$.*

Proof. For a given $x = \{a(p)\}$ in $\mathbf{T}^\omega$, it is easy to verify the inequality

$$\left|\log\left(1 - \frac{a(p)}{p^s}\right)^{-1} - \frac{a(p)}{p^s}\right| \le \frac{3}{p^{2\sigma}}, \quad \sigma > 1/2,$$

for each prime p. This implies that $\mathfrak{z}(x,s)$ is represented as

$$\mathfrak{z}(x,s) = \exp\left\{\sum_p \frac{a(p)}{p^s}\right\} \cdot h(x,s), \tag{3.3}$$

where $h(x,s)$ is analytic and free from zeros on $\sigma > 1/2$.

Let $0 < \beta < 1/2$. We then obseve that

$$\lim_{p\to\infty} \frac{m}{p^{\beta+(1-\alpha)}} p^\beta = 0 \quad \text{and} \quad \sum_p \frac{m}{p^{\beta+(1-\alpha)}} = \infty.$$

If we choose a suitable sequence $\{p(j)\}$ of primes and put

$$s(p(k)) \;=\; \sum_{p(k-1)<p\le p(k)} \frac{m}{p^{\beta+(1-\alpha)}}\,, \qquad k=2,3,\dots,$$

then the sequence $\{s(p(k))\}$ has the property

$$0 \;<\; \sum_{k=2}^{j} s(p(k)) \;-\; \sum_{k=1}^{j-1} \frac{1}{p(k)^{\beta}} \;<\; \frac{1}{p(j)^{\beta}}\,. \tag{3.4}$$

Then the inequality (3.4) shows that the Dirichlet series

$$\sum_{p} \frac{m}{p^{(1-\alpha)}}\frac{1}{p^{s}} \;-\; \sum_{k=1}^{\infty} \frac{1}{p(k)^{s}}$$

converges on $\sigma > \beta$. Let $\{q(j)\}$ be the remaining primes outside $\{p(j)\}$. A property of alternating series asserts that the Dirichilet series

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{q(j)^s}\,, \qquad \sigma > 0\,,$$

also converges. We observe by (3.3) that the behavior of

$$\exp\left\{\sum_{p} \frac{-m}{p^{(1-\alpha)+s}}\right\}, \qquad \sigma > 1/2\,,$$

is similar to $\zeta((1-\alpha)+s)^{-m}$. Define

$$\epsilon(p) \;=\; \begin{cases} -1\,, & \text{if } p=p(j),\\ (-1)^j\,, & \text{if } p=q(j). \end{cases}$$

Then $\mathfrak{z}(\{\epsilon(p)\},s)$ has a zero of order m at $s=\alpha$. Let S the dense subgroup of $\mathbf{T}^{\omega}$ above. Then the coset $E \;=\; \{\epsilon(p)\}+S$ satisfies the desired property. $\square$

Let us make some remarks on Proposition 3.4. Using the same notation, we observe that $\mathfrak{z}(x,s)$ has a pole of order m on the dense subset $\{-1\}+E$. Next we divide m as m_1+m_2. Then we may choose a subsequence $\{p_1(j)\}$ of $\{p(j)\}$ such that

$$\sum_{p} \frac{m_1}{p^{(1-\alpha)}}\frac{1}{p^{s}} \;-\; \sum_{k=1}^{\infty} \frac{1}{p_1(k)^{s}}$$

converges. Define

$$a(p) \;=\; \begin{cases} -e^{it\log p}, & \text{if } p=p_1(j),\\ 1\,, & \text{if } p \text{ is in } \{p(j)\}\setminus\{p_1(j)\},\\ (-1)^j, & \text{if } p=q(j). \end{cases}$$

Then $\mathfrak{z}(\{a(p)\},s)$ has a zero of order m_1 at $s=\alpha+it$ as well as a pole of order m_2 at $s=\alpha$. By extending this method, it is not difficult to construct a $\mathfrak{z}(x,s)$ with the property that

$$\lim_{\sigma\to\alpha+0} \mathfrak{z}(x,\sigma+ir) \;=\; 0\,, \quad r\in\mathbf{Q}\,,$$

where $\mathbf{Q}$ denotes the set of all rationals. This asserts that the line $s = \alpha + it$ is the natural boundary of $\mathfrak{z}(x, s)$ (compare with Proposition 6.4 in Section 6). However, replacing m with $1/2$ in Proposition 3.4, we find a $\mathfrak{z}(x, s)$ which cannot be extend meromorphically across the line $s = \alpha + it$. Indeed, since

$$\lim_{\sigma \to \alpha + 0} (\sigma - \alpha)^{1/2} \mathfrak{z}(x, \sigma)$$

exists, such $\mathfrak{z}(x, s)$ cannnot be analytic at $s = \alpha$.

We now point out other properties of Z_u. As we have seen in the above proof, $\mathfrak{z}(x + \{-1\}, s)$ behaves like $\mathfrak{z}(x, s)^{-1}$. Since $\chi_{\log p}(\{-1\}) = -1$ for each prime p, we see that

$$\frac{\zeta(s)}{\zeta(2s)} = \prod_p \left(1 - \frac{\chi_{\log p}(\{-1\})}{p^s}\right), \quad s = \sigma + it$$

and

$$Z_{2u}^{-1}(2x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2u}} \chi_{\log n}(x)^2 = \prod_p \left(1 - \left(\frac{\chi_{\log n}(x)}{p^u}\right)^2\right),$$

It follows from them that

$$Z_u(x)\, Z_{2u}^{-1}(2x) = Z_u^{-1}(x + \{-1\}).$$

Since $Z_{2u}^{-1}(2x)$ has an absolutely convergent Fourier series, $Z_{2u}^{-1}(2x)$ is an outer function in $A(\mathbf{T}^\omega)$. Therefore $Z_u(x)$ plays as if it were self-reciprocal by the translation $x \to x + \{-1\}$. The set of zeros of $Z_u(x + e_z)$ corresponds to the one of poles of $Z_u(x + \{-1\} + e_z)$, and *vice versa*, although both are generally empty. The exceptional null set N in Theorem 3.3 is closed under the translation by $\{-1\}$. Incidentally, since $\zeta(s) = \overline{\zeta(\bar{s})}$ by the reflection principle, the equation $Z_u(x) = \overline{Z_u(-x)}$ follows. This shows that $Z_u(x + e_{is}) = \overline{Z_u(-x + e_{i\bar{s}})}$, thus N is also closed under the inverse operation on $\mathbf{T}^\omega$.

4. Normal families and value-distribution.

We now turn to the distribution of values of $\zeta(s)$ in the strip $1/2 < \sigma < 1$. In the previous section, the extension Z_u of $\zeta(s)$ is defined by the Fourier series (3.1). Let us reconstruct Z_u by using certain normal families. This enables us to investigate the asymptotic behavior of $\zeta(s)$ from another point of view. For instance, one of our techniques runs as follows: Let $1/2 < u < 1$. Since $\zeta(s)$ extends to an outer function Z_u in $H^q(\boldsymbol{\sigma}_P), 1 \le q < \infty$, the function $z \to Z_u(x + e_z)$ is usually analytic and free from zeros on $\mathbf{R}_+^2$. Then it is shown that almost all such functions are represented as limits of translations of $\zeta(u - iz + it)$ in a sense. Thus if $\zeta(s)$ should have zeros in $u < \sigma < 1$, the distribution would be very rare by Hurwitiz's theorem.

One can find in [17, Chapter 7] for information about normal families and related topics.

Let $\ell > 0$. For each n in $\mathbf{Z}^+$, define the rectangles $Q(u, n)$ and $\mathcal{D}(u, n)$ by

$$Q(u, n) = \{\sigma + it\,;\; u < \sigma < 2\,,\; n\ell \le t < (n+1)\ell\} \tag{4.1}$$

and

$$(4.2)\qquad \mathcal{D}(u,n) = \{\sigma + it\,;\, u-\delta < \sigma < 2,\, n\ell - \delta < t < (n+1)\ell + \delta\},$$

where $0 < \delta < \min(\ell/3, u-1/2)$.

We need a relation between the mean-value theorems for $\zeta(s)$ and certain normal families. It follows from [22, Theorem 7.2(A)] that

$$\int_1^T |\zeta(\sigma+it)|^2\, dt \;<\; A\,T \min\left(\log T, \frac{1}{\sigma - \frac{1}{2}}\right)$$

uniformly for $1/2 \le \sigma \le 2$, where A is a constant. Let $1/2 < \alpha < u - \delta$. Then the mean-value theorem [22, Theorem 7.2] implies that

$$(4.3)\qquad \lim_{T\to\infty} \frac{1}{T}\int_1^T dt \int_\alpha^2 |\zeta(\sigma+it)|^2\, d\sigma = \int_\alpha^2 \zeta(2\sigma)\, d\sigma,$$

with the aid of the bounded convergence theorem. Let

$$B(n) = \left\{\iint_{Q(\alpha,n)} |\zeta(\sigma+it)|^2\, d\sigma \times dt\right\}^{\frac{1}{2}},$$

where $Q(\alpha,n)$ is the rectangle by (4.1) with u replaced by α. Then $B(n)$ is the *Bergman L^2–norm* of $\zeta(s)$ on $Q(\alpha,n)$. We represent (4.3) as

$$(4.4)\qquad \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{\ell} B(n)^2 = \int_\alpha^2 \zeta(2\sigma)\, d\sigma.$$

For a given $M > 0$, we denote by $J(M)$ the set of all n in $\mathbf{Z}^+$ such that

$$\|\zeta\|_{\mathcal{D}(u,n)} = \sup\{|\zeta(s)|\,;\, s\in\mathcal{D}(u,n)\} \le M$$

LEMMA **4.1.** *If $\epsilon > 0$, then there is an $M_0 > 0$ for which*

$$\mathcal{D}_*(J(M)) > 1-\epsilon, \quad M \ge M_0,$$

where $\mathcal{D}_*(J(M))$ is the lower density of $J(M)$.

Proof. By (4.4) we choose an $L > 0$ so large that $\mathcal{D}_*(\{n\in\mathbf{Z}^+; B(n)\le L\}) > 1-\epsilon/3$. This implies that

$$\mathcal{D}_*\left(\left\{n\in\mathbf{Z}^+; B(n-1),\, B(n),\, \text{and } B(n+1) \le L\right\}\right) > 1-\epsilon.$$

On the other hand, since

$$\mathcal{D}(u,n) \subset Q(\alpha,n-1)\cup Q(\alpha,n)\cup Q(\alpha,n+1),$$

Bergman's inequality [17, p 155] yields

$$|\zeta(s)| \le \frac{1}{\sqrt{\pi}(u-\delta-\alpha)}\{B(n-1)+B(n)+B(n+1)\}, \quad s\in\mathcal{D}(u,n).$$

Thus $M_0 = \{\sqrt{\pi}(u-\delta-\alpha)\}^{-1} 3L$ satisfies the desired property. $\square$

Define the rectangle $\mathcal{D}$ by
$$\mathcal{D} = \{t+i\sigma\,; -\delta < t < \ell+\delta,\ 0 < \sigma < 2-(u-\delta)\},$$
and, let
$$F_n(z) = \zeta(u-\delta-iz-in\ell).$$
Then $F_n(z)$ is analytic on $\mathcal{D}$ for each $n \geq 1$. If n is in $J(M)$, then
$$\|F_n\|_{\mathcal{D}} = \|\zeta\|_{\mathcal{D}(u,n)} \leq M,$$
because $\zeta(\sigma - it) = \overline{\zeta(\sigma+it)}$. Therfore $\mathcal{F} = \{F_n\,; 0 < n \in J(M)\}$ makes a normal family on $\mathcal{D}$. We eliminate F_0 from $\mathcal{F}$ since F_0 has a pole at $z = i(1-u+\delta)$.

THEOREM **4.2.** *Let $1/2 < u < 1$, and let $Q(u,n)$ and $J(M)$ be as above. Define J_0 be the set of all n in $\mathbf{Z}^+$ for which $\zeta(s)$ has zeros in $Q(u,n)$. Then we have*
$$\mathcal{BD}^*(J_0 \cap J(M)) = 0, \tag{4.5}$$
for each $M > 0$. Consequently, J_0 has density zero, that is, $\mathcal{D}^*(J_0) = 0$.

Proof. We first assume that $2\pi/\ell$ lies in Γ. Then the dual group $\mathbf{T}^\omega$ of Γ is identified with $K_{2\pi/\ell} \times [0,\ell)$ as we have seen in Section 2. Since $1/2 < u-\delta$, $\zeta(s)$ extends to an outer function $Z_{u-\delta}$ in $H^q(\boldsymbol{\sigma}_P)$, $1 \leq q < \infty$, by Lemma 3.2. Let $Z^\sharp_{u-\delta}(y,z)$ be the automorphic extension of $Z_{u-\delta}$ to $K_{2\pi/\ell} \times \mathbf{R}^2_+$. Restricting $Z^\sharp_{u-\delta}(y,z)$ to $K_{2\pi/\ell} \times \mathcal{D}$, we observe that
$$Z^\sharp_{u-\delta}(ne_\ell, z) = \zeta(u-\delta-iz-in\ell) = F_n(z), \quad z \in \mathcal{D}.$$
For each $\epsilon > 0$, we choose an $M > 0$ by Lemma 4.1 such that $\mathcal{D}_*(J(M)) > 1-\epsilon$. If $E(J(M))$ denotes the closure of $\{ne_\ell\,; n \in J(M)\}$ in $K_{2\pi/\ell}$, then $\boldsymbol{\tau}(E(J(M))) > 1-\epsilon$ by (2.6).

We note that
$$Z_{u-\delta}(x + e_{i/2}) = \sum_{n=1}^{\infty} \frac{1}{n^{u-\delta+1/2}} \chi_{\log n}(x)$$
converges uniformly on $\mathbf{T}^\omega$. If $n_j e_\ell$ tends to y in $K_{2\pi/\ell}$, then $F_{n_j}(z)$ converges uniformly to $Z^\sharp_{u-\delta}(y,z)$ on the subregion
$$\{t+i\sigma\,;\ -\delta < t < \ell+\delta,\ 1/2 < \sigma < 2-(u-\delta)\}$$
of $\mathcal{D}$. Furthermore, when each n_j lies in $J(M)$, that is, y is in $E(J(M))$, $F_{n_j}(z)$ converges uniformly to $Z^\sharp_{u-\delta}(y,z)$ on every compact set in $\mathcal{D}$, because the family $\mathcal{F}$ above is normal. Then we see that $Z^\sharp_{u-\delta}(y,z)$ is represented as a limit of subsequence of $\mathcal{F}$ on $E(J(M)) \times \mathcal{D}$.

Define the subregion $\mathcal{D}_0$ of $\mathcal{D}$ by
$$\mathcal{D}_0 = \left\{t+i\sigma\,;\ -\frac{\delta}{2} < t < \ell+\frac{\delta}{2},\ \frac{\delta}{2} < \sigma < 2-(u-\delta)\right\}.$$
Notice that, for each n in J_0, $F_n(z)$ has zeros in $\mathcal{D}_0$. It then follows from Hurwitiz's theorem that if y is in the closure $E(J_0 \cap J(M))$ of $J_0 \cap J(M)$ in $K_{2\pi/\ell}$, then $z \to Z^\sharp_{u-\delta}(y,z)$ must have zeros in $\mathcal{D}$. Since $Z_{u-\delta}$ is outer in $H^q(\boldsymbol{\sigma}_P)$, we thus have
$$\boldsymbol{\tau}(E(J_0 \cap J(M))) = 0.$$
This yields (4.5) by applying (2.6) again.

Let us show that J_0 has density zero. Let $\epsilon > 0$ be given. It follows from Lemma 4.1 that $\mathcal{D}^*(\mathbf{Z}^+ \setminus J(M)) < \epsilon$ for some $M > 0$. Since

$$\mathcal{D}^*(J_0) \leq \mathcal{D}^*(J_0 \cap J(M)) + \mathcal{D}^*(J_0 \cap (\mathbf{Z}^+ \setminus J(M))) < \epsilon,$$

we see that $\mathcal{D}^*(J_0) = 0$.

When $2\pi/\ell$ does not lie in Γ, we let Γ^* be the discrete group generated by $2\pi/\ell$ and Γ. Then Z_u can be interpreted as an outer function in the Hardy space $H^q(\boldsymbol{\sigma}), 1 \leq q < \infty$, on the dual group K^* of Γ^*. Since $\{ne_\ell; n \in \mathbf{Z}^+\}$ is also dense in $K^*_{2\pi/\ell}$, we obtain the same conclusion by the same way as in above. $\square$

It is customary to denote by $N(T)$ the number of zeros of $\zeta(s)$ in the region $\{\sigma + it\,;\, 0 \leq \sigma \leq 1,\ 0 < t \leq T\}$. It follows immediately from Jensen's theorem that

$$N(T+\ell) - N(T) = O(\log T) \quad \text{as} \quad T \to \infty,$$

and it is also well-known that $N(T) = O(T \log T)$ (see [22, 9.2 and 9.4] for details). Similarly, $N(\sigma, T)$ denotes the number of zeros $\beta + i\gamma$ of $\zeta(s)$ such that $\beta > \sigma$ and $0 < \gamma \leq T$. Thus our result assures that if $1/2 < \sigma < 1$, then

$$N(\sigma, T) = o(T \log T) \quad \text{as} \quad T \to \infty. \tag{4.6}$$

However, from a point of view, Theorem 4.2 seems to carry new information on the distribution of zeros of $\zeta(s)$. Indeed, rapidly as $g(i)$ increases, there is a sequence $\{n_i\}$ with $n_i = o(g(i))$ such that $\boldsymbol{\tau}(E(\{n_i\}) \cap E(J(M))) > 0$. Therefore $J_0 \cap J(M)$ never contains such sequences. Furthermore (4.5) shows that the numbers in J_0 may not be in long succession. These observations would restrict the behavior of $N(\sigma, T)$ when T tends to infinity.

With the aid of functional equation (1.2), we may strengthen (4.6) as follows (refer to [22, 9.24]).

COROLLARY **4.3.** *There is a positive decreasing function $f(t)$ tending to zero such that all but an infinitesimal proportion of zeros of $\zeta(s)$ in $\mathbf{R}^2_+$ lie in the region*

$$\left\{ \sigma + it\,;\, |\sigma - \frac{1}{2}| < f(t) \right\}.$$

By the argument preceding Theorem 3.3, we see that almost all $\mathfrak{z}(x, s)$ extends analytically to $\sigma > 1/2$. The following theorem should be compared with the 'universal' property for $\zeta(s)$ (see [22, 11.11]).

THEOREM **4.4.** *For $\boldsymbol{\sigma}_P - a.e.\, x$ in $\mathbf{T}^\omega$, there is a sequence $\{T_n\}$ such that*

$$\zeta(s + iT_n) \longrightarrow \mathfrak{z}(x, s), \quad as \quad n \to \infty, \tag{4.7}$$

uniformly on each compact set in the half-plane $\sigma > 1/2$.

Proof. Fix an $\ell > 0$ such that $2\pi/\ell$ is in Γ. Since $\mathbf{T}^\omega$ is identified with $K_{2\pi/\ell} \times [0, \ell)$, we obtain $\mathfrak{z}(x, s) = \mathfrak{z}(y, s - iu)$ for some (y, u) in $K_{2\pi/\ell} \times [0, \ell)$. So it suffices to show that (4.7) holds for $\boldsymbol{\tau} - a.e.\, y$ in $K_{2\pi/\ell}$. Let $1/2 < u < 1$, and let $\epsilon > 0$. We denote by $Z_u^\sharp(y, z)$ the automorphic extension of Z_u to $K_{2\pi/\ell} \times \mathbf{R}^2_+$ as usual. Notice that $Z_u^\sharp(ne_\ell, z) = \zeta(u - iz - in\ell)$. For any positive integer k, we choose M large enough in Lemma 4.1 and put$E(k,\epsilon) = E(J(M))$. By the same argument as used in the proof of Theorem 4.2, we make $E(k,\epsilon)$ satisfy that $\boldsymbol{\tau}(E(k,\epsilon)) > 1-\epsilon$ and

$$Z_u^\sharp(n_j e_\ell, z) \longrightarrow Z_u^\sharp(y,z), \quad j \to \infty, \tag{4.8}$$

uniformly on

$$\mathfrak{D}(k) = \{t + i\sigma\,;\, -k\ell \le t \le k\ell,\, \sigma \ge 0\},$$

where $\{n_j e_\ell\}$ is a sequence in $E(k,\epsilon)$ that tends to y. This shows that there is a $\boldsymbol{\tau}-$null set $N(u)$ in $K_{2\pi/\ell}$ on which (4.8) holds uniformly on $\mathfrak{D}(k)$ for all k. Let $\{u_i\}$ be a decreasing sequence that tends to $1/2$. Let us define the $\boldsymbol{\tau}-$null set N by $N = \cup N(u_i)$. Then if y is not in N, then

$$\zeta(\frac{1}{2} + \sigma - it - n_j\ell) \longrightarrow \mathfrak{z}(y, \frac{1}{2} + \sigma - it)$$

uniformly on each compact set in $\sigma > 0$. Taking the complex conjugate and replacing $-y$ with y, we observe easily that the property (4.7) holds. □

From Jensen's inequality, it is derived that the number of zeros of $\zeta(s)$ in $Q(u,s)$ is uniformly bounded on $J_0 \cap J(M)$. However, this fact also follows from the following property of Gleason parts of $A(\mathbf{T}^\omega)$: Let ϕ be a function in $A(\mathbf{T}^\omega)$. Then $\phi \equiv 0$ on a part $\{x + e_z; z \in \mathbf{R}^2_+\}$ if and only if $\phi \equiv 0$ on $\mathbf{T}^\omega$ (refer to [5, Chapter VI] for Gleason parts).

Proposition 4.5. *Let D be a domain in $\sigma > 1/2$ such that $D \cap \{\sigma > 1\} \neq \emptyset$, and let E be a compact subset of D. Suppose that there is a constant $M > 0$ for which*

$$\|\zeta\|_{D+iT_n} = \sup\{|\zeta(s+iT_n)|\,;\, s \in D\} \le M$$

for a sequence $\{T_n\}$ in $\mathbf{R}$. Then the the number $N_E(n)$ of zeros of $\zeta(s)$ in $E + iT_n$ is bounded.

Proof. Suppose on the contrary that $N_E(n)$ is unbounded. On passing to a subsequence, we assume $N_E(n)$ diverges and $-e_{T_n}$ converges to a point x in $\mathbf{T}^\omega$. This implies that $\zeta(s+iT_n)$ converges to $Z_u(x + e_{i(s-u)})$ uniformly in compact subsets of D. It follows from Hurwitz's theorem that $Z_u(x + e_{i(s-u)}) \equiv 0$ on D. Then $z \to Z_{u+1/2}(x + e_z)$ is identically zero on $\mathbf{R}^2_+$. Since $Z_{u+1/2}(x) = Z_u(x + e_{i/2})$ lies in $A(\mathbf{T}^\omega)$, the above remark shows that $Z_u(x) \equiv 0$, which is a contradiction. □

We observe that Proposition 3.4 assures the existence of a sequence $\{T_n\}$ for which $\{\zeta(s+iT_n)\}$ is not a normal family on D. It is well-known that if a family $\mathcal{F}$ of analytic functions omits two fixed values in $\mathbf{C}$, then $\mathcal{F}$ is normal. This yields directly a version of [22, Theorem 11.1].

Proposition 4.6. *Let D be a domain as in Proposition 4.5. Then there is a sequence $\{T_n\}$ with the property that $\zeta(s)$ takes every value, with one possible exception, an infinity of times in*

$$\bigcup_{n=1}^{\infty} D + iT_n.$$

5. **Weak mean-value theorem.**

It is known that under the Riemann hypothesis

$$\int_1^T \frac{dt}{|\zeta(\sigma+it)|^2} \sim \frac{\zeta(2\sigma)}{\zeta(4\sigma)}T, \quad \text{as} \quad T \to \infty, \tag{5.1}$$

for a fixed $\sigma > 1/2$ (as discussed in [22, 14.2]). In this section, by discarding a small part of $[1,\infty)$, we show these kinds of limits always hold. Let $-\infty < k < \infty$, and fix $u > 1/2$. Since both $|Z_u|$ and $|Z_u^{-1}|$ lie in $L^2(\boldsymbol{\sigma}_P)$, $\log |Z_u|$ lies in $L^2(\boldsymbol{\sigma}_P)$. This shows that the conjugate function V_u of $\log |Z_u|$ is also in $L^2(\boldsymbol{\sigma}_P)$ (see [5, Chapter IV, §1] for conjugate functions). Recall that both Z_u and Z_u^{-1} are outer functions in $H^q(\boldsymbol{\sigma}_P)$, $1 \le q < \infty$, by Lemma 3.2. Hence Z_u^k is defined by

$$Z_u^k = \exp\{ k(\log |Z_u| + iV_u)\}, \tag{5.2}$$

where

$$a_0(V_u) = \int_{\mathbf{T}^\omega} V_u \, d\boldsymbol{\sigma}_P = 0.$$

Since Z_u^k lies in $H^2(\boldsymbol{\sigma}_P)$, it follows from the individual ergodic theorem that

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T |Z_u(x-e_t)|^{2k}\, dt = \int_{\mathbf{T}^\omega} |Z_u|^{2k}\, d\boldsymbol{\sigma}_P, \tag{5.3}$$

for $\boldsymbol{\sigma}_P - a.e.\ x$ in $\mathbf{T}^\omega$. Observe that

$$\|Z_u^2\|_2^2 = \prod_p \left(1+\frac{1}{p^{2u}}\right) = \frac{\zeta(2u)}{\zeta(4u)}.$$

Since $Z_u(x-e_t) = \mathfrak{z}(x, u+it)$, (5.1) holds with $\mathfrak{z}(x,\sigma+it)$ in place of $\zeta(\sigma+it)$ broadly by putting $k=-1$. Although $Z_u(-e_t) = \zeta(u+it)$ is exceptional, we next show a similar result with the aid of the argument in the preceding sections.

Incidentally, we make a remark on the value of the last integral in (5.3). Since Z_u^k lies in $H^2(\boldsymbol{\sigma}_P)$, Z_u^k is also expressed as

$$Z_u^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^u}\chi_{\log n},$$

for a suitable sequence $\{d_k(n)\}$. When k happens to be in $\mathbf{Z}^+$, $d_k(n)$ coincides with the number of decompositions of n into k factors. Then we have

$$\|Z_u^k\|_2^2 = \int_{\mathbf{T}^\omega} |Z_u^k|^2\, d\boldsymbol{\sigma}_P = \sum_{n=1}^{\infty}\left\{\frac{d_k(n)}{n^u}\right\}^2.$$

For an $\ell > 0$, define the compact rectangle $\mathcal{D}_1$ by

$$\mathcal{D}_1 = \{t+i\sigma\,;\, 0\le t\le \ell,\ 0\le\sigma\le 2-u\}.$$

If $u>1$, then Z_u^k is continuous on $\mathbf{T}^\omega$, so $Z_u^k(-e_z) = \zeta^k(u+iz)$ is completely determined on $\mathbf{R}_+^2$. In case of $1/2 < u \le 1$, $\zeta^k(u+iz+in\ell)$ is analytically continued suitably to $\mathcal{D}_1$, whenever $\zeta(u+iz+in\ell)$ does not vanish on it. Regarding Theorem 4.2, we may consider

$$Z_u^k(-e_t) = \zeta^k(u+it), \quad n\ell \le t < (n+1)\ell$$

for each n in $\mathbf{Z}^+ \setminus J_0$, where J_0 is the subset of $\mathbf{Z}^+$ of density zero.

Let S be a subset of $\mathbf{Z}^+$. Then $\sum_S$ denotes the sum of terms with indices in S.

THEOREM **5.1.** *Let $-\infty < k < \infty$, and let $\ell > 0$. For a fixed $u > 1/2$, there is a subset $J = J(k, \ell, u)$ of $\mathbf{Z}^+$ of density zero such that*

$$\lim_{N\to\infty} \frac{1}{N\ell} \sum_{\mathbf{Z}^+\setminus J}{}_{n=0}^{N-1} \int_{n\ell}^{(n+1)\ell} |\zeta(u+it)|^{2k}\, dt \;=\; \|Z_u^k\|_2^2 \tag{5.4}$$

and

$$\lim_{N\to\infty} \frac{1}{N\ell} \sum_{\mathbf{Z}^+\setminus J}{}_{n=0}^{N-1} \int_{n\ell}^{(n+1)\ell} \zeta(u+it)^{2k}\, dt \;=\; 1\,. \tag{5.5}$$

In order to prove Theorem 5.1, we require two lemmas.

Considering the discrete group Γ^* generated by Γ and $2\pi/\ell$, we may assume that $2\pi/\ell$ lies in Γ as discussed before. Let $(Z_u^k)^\sharp(y,t)$ be the automorphic extension of Z_u^k to $K_{2\pi/\ell} \times \mathbf{R}$. Note that $K_{2\pi/\ell}$ is metrizable. It then follows from Theorem 4.2 that, for $\boldsymbol{\tau} - a.e.\, y$ in $K_{2\pi/\ell}$, $Z_u^\sharp(y,z)$ does not vanish on $\mathcal{D}_1$ and there is a sequence $\{n_j e_\ell\}$ tending to y in $K_{2\pi/\ell}$ such that

$$Z_u^\sharp(n_j e_\ell, z) \;=\; \zeta(u - i(z + n_j\ell))$$

converges uniformly to $Z_u^\sharp(y,z)$ on $\mathcal{D}_1$.

Let $M > 0$, and define

$$I(M) \;=\; \left\{ n \in \mathbf{Z}^+ \,;\, \frac{1}{M} < |(Z_u^k)^\sharp(ne_\ell, z)| < M \text{ for each } z \in \mathcal{D}_1 \right\}.$$

Then the closure of $\{ne_\ell\,;\, n \in I(M)\}$ in $K_{2\pi/\ell}$ is denoted by $E(I(M))$ as before. The following is a minor variation of Lemma 4.1.

LEMMA **5.2.** *Let $\epsilon > 0$ be given. Then there is an $M_0 > 0$ for which*

$$\mathcal{D}_*(I(M)) \;>\; 1 - \epsilon\,, \quad M \ge M_0. \tag{5.6}$$

Consequently, we have

$$\boldsymbol{\tau}(E(I(M))) \longrightarrow 1\,, \quad \text{as} \quad M \to \infty.$$

Proof. It suffices to consider the case where $k = 1$. Let

$$I_1(M) \;=\; \{n \in \mathbf{Z}^+; \|(Z_u)^\sharp(ne_\ell, \cdot)\|_{\mathcal{D}_1} \;\le\; M\}\,.$$

Then we have by Lemma 4.1 that

$$\mathcal{D}^*(\mathbf{Z}^+ \setminus I_1(M)) \longrightarrow 0\,, \quad \text{as} \quad M \to \infty.$$

On the other hand, it follows from Theorem 4.2 that $z \to Z_u^\sharp(y,z)$ is analytic and never vanishes on $\mathcal{D}_1$ for $\boldsymbol{\tau} - a.e.\, y$ in $K_{2\pi/\ell}$. This shows that if we set

$$I_2(M) \;=\; \{n \in \mathbf{Z}^+; \|(Z_u^{-1})^\sharp(ne_\ell, \cdot)\|_{\mathcal{D}_1} \;\le\; M\}$$

then

$$\boldsymbol{\tau}(E(I_2(M))) \;>\; 1-\epsilon\,,$$

for M large enough, since Z_u^{-1} lies in $L^1(\boldsymbol{\sigma}_P)$. Then we have

$$\mathcal{D}^*(\mathbf{Z}^+\setminus I_2(M)) \to 0\,, \quad \text{as} \quad M\to\infty,$$

Therefore, since

$$0 \;\le\; \mathcal{D}^*(\mathbf{Z}^+\setminus I(M)) \;\le\; \mathcal{D}^*(\mathbf{Z}^+\setminus I_1(M)) + \mathcal{D}^*(\mathbf{Z}^+\setminus I_2(M))\,,$$

we obtain the property (5.6) immediately. The last statement follows from (2.6). □

We notice that if $Z_u^\sharp(y,z)$ happens to have a zero in $\mathcal{D}_1$, then

$$\|(Z_u^{-1})^\sharp(n_j e_\ell,\,\cdot\,)\|_{\mathcal{D}_1} \to \infty\,,$$

for any sequence $\{n_j e_\ell\}$ tending to y. Since $(Z_u^k)^\sharp(y,z)$ is determined completely on $K_{2\pi/\ell}\times\{Re\ z>1-u\}$, it follows from a normal families argument that the restriction of $(Z_u^k)^\sharp$ to $E(I(M))\times[0,\ell)$ is continuous on $E(I(M))\times[0,\ell)$.

LEMMA **5.3.** *Let $I(M)$ and $E(I(M))$ be as above. Then there is a divergent increasing sequence $\{M_i\}$ with the following properties:*

(i) *Let $I_1 = I(M_1)$ and $I_i = I(M_i)\setminus I(M_{i-1})$, $i\ge 2$. Then we have $\boldsymbol{\tau}(E(I_i)\cap E(I_j)) = 0$, provided $i\ne j$.*

(ii) *Regarding $E(I_i)\times[0,\ell)$ as a subset of $\mathbf{T}^\omega$, we have*

$$\lim_{N\to\infty}\frac{1}{N\ell}\sum_{n=0}^{N-1}{}_{I_i}\int_0^\ell |(Z_u^k)^\sharp(ne_\ell,t)|^2\,dt \;=\; \int_{E(I_i)\times[0,\ell)} |Z_u^k|^2\,d\boldsymbol{\sigma}_P\,, \tag{5.7}$$

and

$$\lim_{N\to\infty}\frac{1}{N\ell}\sum_{n=0}^{N-1}{}_{I_i}\int_0^\ell (Z_u^{2k})^\sharp(ne_\ell,t)\,dt \;=\; \int_{E(I_i)\times[0,\ell)} Z_u^{2k}\,d\boldsymbol{\sigma}_P\,. \tag{5.8}$$

Proof. (i) Define the increasing function $F(M)$ on $(0,\infty)$ by

$$F(M) \;=\; \int_{E(I(M))\times[0,\ell)} |Z_u^k|^2\,d\boldsymbol{\sigma}_P\,.$$

Recall that $|Z_u^k|$ lies in $L^2(\boldsymbol{\sigma}_P)$. It follows from Lemma 5.2 that

$$\lim_{M\to\infty} F(M) \;=\; \int_{\mathbf{T}^\omega} |Z_u^k|^2\,d\boldsymbol{\sigma}_P \;<\; \infty\,.$$

Since $F(M)$ is increasing, the set of points of $(0,\infty)$ at which $F(M)$ is discontinuous is at most countable. Then we choose a divergent increasing sequence $\{M_i\}$ such that $F(M)$ is continuous at each M_i. It is easy to see that

$$\lim_{\delta\to+0}\boldsymbol{\tau}\left(E(I(M_i+\delta))\right) \;=\; \lim_{\delta\to+0}\boldsymbol{\tau}\left(E(I(M_i-\delta))\right)\,,$$

from which we obtain $\boldsymbol{\tau}\left(E(I_j)\cap E(I_{i+1})\right) = 0$ for $1\le j\le i$.

(ii) Since $\mathcal{D}_*(I(M_j)) > 1-\epsilon$ for large j by Lemma 5.2, if we set $S = \cup\{I_m\,;\, m \neq i\,,\, 1 \leq m \leq j\}$, then $\mathcal{D}^*(\mathbf{Z}^+ \setminus (I_i \cup S)) < \epsilon$. We also see by (i) that $\boldsymbol{\tau}(E(I_i) \cap E(S)) = 0$. Define

$$p_0(y) \;=\; \frac{1}{\ell}\int_0^\ell |(Z_u^k)^\sharp|^2(y,t)\,dt\,, \quad y \in E(I_i)\,,$$

which is continuous on $E(I_i)$. Thus (5.7) follows from Lemma 2.2. The case of (5.8) is similarly obtained. $\square$

We are now in a position to prove the theorem.

Proof of Theorem 5.1. In order to show (5.4), observe that

$$\|Z_u^k\|_2^2 \;=\; \sum_{n=1}^{\infty} \int_{E(I_i)\times[0,\ell)} |Z_u^k|^2\, d\boldsymbol{\sigma}_P\,. \tag{5.9}$$

It is easy to choose a subset J_i of I_i of density zero such that

$$\frac{1}{N\ell}\sum_{n=0}^{N-1}{}_{I_i\setminus J_i} \int_0^\ell |(Z_u^k)^\sharp(ne_\ell,t)|^2\,dt \;\leq\; \int_{E(I_i)\times[0,\ell)} |Z_u^k|^2\, d\boldsymbol{\sigma}_P\,, \tag{5.10}$$

for all N, and (5.7) holds with $I_i \setminus J_i$ in place of I_i. Let

$$J \;=\; \left(\bigcup_{i=1}^{\infty} J_i\right) \cup \left(\mathbf{Z}^+ \setminus \bigcup_{i=1}^{\infty} I_i\right).$$

Since $\cup\{J_m\,;\, m \geq i+1\}$ is contained in $\mathbf{Z}^+ \setminus I(M_i)$, we obtain

$$\mathcal{D}^*(J) \;\leq\; \mathcal{D}^*(\mathbf{Z}^+ \setminus I(M_i)) + \sum_{k=1}^{i} \mathcal{D}^*(J_k) \to 0\,, \quad \text{as} \quad i \to \infty,$$

thus J has density zero. It follows from (5.9), (5.10) and Lemma 5.3 that

$$\lim_{N\to\infty} \frac{1}{N\ell}\sum_{n=0}^{N-1}{}_{\mathbf{Z}^+\setminus J} \int_{n\ell}^{(n+1)\ell} |Z_u^k(e_t)|^2\,dt \;=\; \|Z_u^k\|_2^2\,.$$

Since $|Z_u(e_t)| \;=\; |\overline{\zeta(u+it)}|$, we obtain (5.4). We note that J may be replaced by another subset of $\mathbf{Z}^+$ of density zero containing J.

To show (5.5), we first note that

$$a_0(Z_u^{2k}) \;=\; \int_{\mathbf{T}^\omega} Z_u^{2k}\, d\boldsymbol{\sigma}_P \;=\; 1\,.$$

This implies that

$$\sum_{n=1}^{\infty} \int_{E(I_i)\times[0,\ell)} Z_u^{2k}\, d\boldsymbol{\sigma}_P \;=\; 1\,,$$

which converges absolutely. Choose a subset J_i of I_i of density zero with the property that

$$\frac{1}{N\ell}\left|\sum_{n=0}^{N-1}{}_{I_i\setminus J_i}\int_0^{\ell}(Z_u^{2k})^{\sharp}(ne_\ell,t)\,dt\right| \le \left|\int_{E(I_i)\times[0,\ell)} Z_u^{2k}\,d\boldsymbol{\sigma}_P\right|,$$

for all N. Let J be as above. By the similar way, we then obtain

$$\lim_{N\to\infty}\frac{1}{N\ell}\sum_{n=0}^{N-1}{}_{\mathbf{Z}^+\setminus J}\int_{n\ell}^{(n+1)\ell}\zeta(u-it)^{2k}\,dt = 1.$$

By taking the complex conjugate, (5.5) follows immediately. Replacing J with a larger one, if necessary, we find a set $J = J(k,\ell,u)$ of density zero for which both (5.4) and (5.5) hold at the same time. □

Let us describe a remark on the ordinary mean-value theorems, which is sometimes useful for extending them (compare with [22, Theorem 7.11]).

PROPOSITION **5.4.** *Let $0 \le k < \infty$, and let $\sigma > 1/2$. Suppose that*

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T|\zeta(\sigma+it)|^{2k} = \sum_{n=1}^{\infty}\left\{\frac{d_k(n)}{n^\sigma}\right\}^2. \tag{5.11}$$

If $0 \le \lambda \le k$, then (5.11) holds with k replaced by λ.

Proof. By virtue of Theorem 5.1 we see that (5.11) holds if and only if, for each subset J of $\mathbf{Z}^+$ of density zero,

$$\lim_{N\to\infty}\frac{1}{N\ell}\sum_{n=0}^{N-1}{}_J\int_{n\ell}^{(n+1)\ell}|\zeta(\sigma+it)|^{2k}\,dt = 0. \tag{5.12}$$

Since $0 \le \lambda \le k$, we observe

$$\int_{n\ell}^{(n+1)\ell}|\zeta(\sigma+it)|^{2\lambda}\,dt \le \int_{n\ell}^{(n+1)\ell}\left\{1+|\zeta(\sigma+it)|^{2k}\right\}dt.$$

Thus (5.12) holds with k replaced by λ. □

Under the Reimann hypothesis, Proposition 5.4 extends naturally to the case where $-\infty < k < 0$. By regarding the proof of Proposition 5.4, the set $J = J(k,\ell,u)$ in Theorem 5.1 may be chosen so that $J(k,\ell,u)$ is increasing with respect to $|k|$. This enables J to be independent of k.

6. Complementary remarks.

We discuss briefly a relation between the abscissa of convergence for a Dirichlet series and Hardy spaces $H^q(dt/\pi(1+t^2)), 1 \le q < \infty$. Let $f(s)$ be a Dirichlet series of the form (1.1), and let

$$F(\lambda) = \sum_{\log n \le \lambda} a(n) .$$

If $F(\log n) = O(n^u)$, then $f(s)$ converges for $\sigma > u$, and $t \to f(\sigma - it)$ lies at least in $H^1(dt/\pi(1+t^2))$, since $f(s) = O(|t|^\delta)$ for some $0 < \delta < 1$ (see [21, 9.14 and 9.33] for details).

PROPOSITION **6.1.** *Let $f(s)$ be a Dirichlet series of the form* (1.1) *for which the abscissa of convergence σ_c is finite. Suppose that $f(s)$ extends analytically to $\sigma > \sigma_0\,(\ge 0)$ and*

$$t \to f(\sigma_0 - it) \in H^r(dt/\pi(1+t^2))$$

for some $r > 2$. Then $f(s)$ converges for $\sigma > \sigma_0 + 1/r$, that is, $\sigma_0 + 1/r \ge \sigma_c$.

Proof. For each q with $2 \le q < r$, we choose p such that $1/p + 1/q = 1$. Since

$$|\sigma + it|^p = |\sigma + it|^{\frac{2p}{r}} \cdot |\sigma + it|^{p(1-\frac{2}{r})}$$

and $r/p > 1$, Hölder's inequality implies

$$\int_{-\infty}^{\infty} \left| \frac{f(\sigma+it)}{\sigma+it} \right|^p dt \le \left\{ \int_{-\infty}^{\infty} \frac{|f(\sigma+it)|^r}{|\sigma+it|^2} dt \right\}^{\frac{p}{r}} \cdot \left\{ \int_{-\infty}^{\infty} |\sigma+it|^{\frac{-p(r-2)}{r-p}} dt \right\}^{\frac{r-p}{r}} .$$

Since $r > q = p/(p-1)$, we obtain easily $p(r-2)/(r-p) > 1$. This assures that the last integral converges. Because of our assumption, we also see

$$\int_{-\infty}^{\infty} \frac{|f(\sigma+it)|^r}{|\sigma+it|^2} dt = \frac{\pi}{\sigma} \int_{-\infty}^{\infty} |f(\sigma - it)|^r P_{i\sigma}(t)\, dt < \infty .$$

On the other hand, since

$$\sup_{\sigma > \sigma_0} \int_{-\infty}^{\infty} \left| \frac{f(\sigma+it)}{\sigma+it} \right|^2 dt < \infty ,$$

the Paley-Wiener theorem shows that

$$\frac{f(\sigma+it)}{\sigma+it} = \frac{1}{2\pi} \int_0^{\infty} G(\lambda) e^{-\lambda(\sigma+it)}\, d\lambda$$

for some function $G(\lambda)$ in $L^2(0,\infty)$. Recall that $f(s)$ is absolutely convergent whenever $\sigma > \sigma_c + 1$ (see [21, 9.13]). From this fact, we see easily that $F(\lambda) = G(\lambda)$. Therefore the Fourier transform of $\lambda \to e^{-\lambda\sigma} F(\lambda)$ is $f(\sigma+it)/(\sigma+it)$. It then follows from the Young-Hausdorff theorem that

$$\int_{-\infty}^{\infty} e^{-\lambda\sigma q} |F(\lambda)|^q\, d\lambda < \infty.$$

This can be written as

$$\sum_{n=1}^{\infty} e^{-\sigma q \log(n+1)} \, |F(\log n)|^q \, \{\log(n+1) - \log n\} \;<\; \infty \, .$$

Since the terms of this series tend to 0, we have easily

$$F(\log n) \;=\; 0(n^{\sigma+1/q}) \, ,$$

so $f(s)$ satisfies the desired property. □

Together with Lemmas 2.1 and 3.2, we obtain the following:

COROLLARY **6.2.** *Under the assumptions of Theorem 3.3, each Dirichlet series $\mathfrak{z}(\mathbf{x}, s)$ by (1.4) converges on $\sigma > 1/2$, outside a $\boldsymbol{\sigma}_P$−null subset of $\mathbf{T}^\omega$.*

We know little about the relation between the exceptional null sets in Theorem 3.3 and Corollary 6.2. In this connection, it should be noted that Littlewood's theorem [22, 14.25] states that the convergence of $\zeta(s)^{-1}$ for $\sigma > 1/2$ is equivalent to Riemann hypothesis (so is the convergence of $\mathfrak{z}(\{-1\}, s)$). By virtue of Proposition 6.1, an argument similar to the proof of [22, Theorem 14.2] provides some information:

PROPOSITION **6.3.** *Let $u > 1/2$. If $t \to \mathfrak{z}(x, u - it)$ is an outer function in $H^1(dt/\pi(1+t^2))$, then for each $\epsilon > 0$*

$$\mathfrak{z}(x, \sigma - it) \;=\; O(|t|^\epsilon) \quad \textit{and} \quad \mathfrak{z}(x, \sigma - it)^{-1} \;=\; O(|t|^\epsilon) \, , \qquad \sigma > u \, .$$

Consequently, both $\mathfrak{z}(x, s)$ and $\mathfrak{z}(\{-1\} + x, s)$ converges for $\sigma > u$.

We now describe a few applications of some properties of Rademacher functions. Recall that Rademacher functions $\varphi_1(r), \varphi_2(r), \ldots$ are defined by

$$\varphi_n(r) \;=\; \operatorname{sgn}(\sin(2^n \pi r)) \, , \quad 0 \le r \le 1 .$$

Let $\mathfrak{R}$ be the set of all dyadic rationals in $[0, 1]$, which is countable. Denote by $\{p(n)\}$ the increasing sequence of all primes, and put

$$\epsilon_p \;=\; \epsilon_{p(n)}(r) \;=\; \varphi_n(r) \, .$$

Each r in $[0, 1] \setminus \mathfrak{R}$ determines a sequence $\{\epsilon_p \, ; \, \epsilon_p = \pm 1\}$ in $\mathbf{T}^\omega$ not eventually constant.

For the theory of Rademacher functions, [3, Appendix 1] and [11, Chapter 5] may be consulted, and the following results are largely motivated by [12] and [16].

We give another proof of Lemma 3.2. Let $u > 1/2$. Since

$$\log \zeta(s) \;=\; \sum_{m=1}^{\infty} \sum_{p} \frac{1}{m p^{ms}} \, , \qquad \sigma > 1 \, ,$$

there is an outer function H_u in $A(\mathbf{T}^\omega)$ with which

$$Z_u \;=\; \exp\left\{ \sum_p \frac{1}{p^u} \chi_{\log p} \right\} \cdot H_u \, . \tag{6.1}$$

Define

$$F_u(r,x) \;=\; \sum_p \epsilon_p(r)\frac{1}{p^u}\,\chi_{\log p}(x)\,, \quad (r,x)\in[0,1]\times\mathbf{T}^\omega\,.$$

By [3, Theorem A.1], if x lies in $\mathbf{T}^\omega$, then $F_u(r,x)$ converges for $m_I - a.e.\ r$ in $[0,1]$, where m_I denotes Lebesgue measure on $[0,1]$ as before. Moreover, Khinchin's inequality [3, Theorem A.2] asserts that

$$\int_0^1 |F_u(r,x)|^k\,dr \;\le\; \left(\frac{k}{2}+1\right)^{\frac{k}{2}} A(u)^k\,, \quad k=1,2,\cdots,$$

where

$$A(u) \;=\; \left(\sum_p \frac{1}{p^{2u}}\right)^{\frac{1}{2}} \;<\; \infty\,.$$

Let $-\infty\;<\;\lambda\;<\;\infty$. Then we have

$$\begin{aligned}
\int_{\mathbf{T}^\omega} d\boldsymbol{\sigma}_P \int_0^1 |\exp\{\lambda F_u(r,x)\}|\,dr \;&\le\; \int_{\mathbf{T}^\omega} d\boldsymbol{\sigma}_P \int_0^1 \exp\{|\lambda F_u(r,x)|\}\,dr \\
&=\; \sum_{k=0}^{\infty}\frac{|\lambda|^k}{k!}\int_{\mathbf{T}^\omega} d\boldsymbol{\sigma}_P\int_0^1 |F_u(r,x)|^k\,dr \\
&\le\; \sum_{k=0}^{\infty}\frac{|\lambda|^k}{k!}\left(\frac{k}{2}+1\right)^{\frac{k}{2}} A(u)^k \\
&<\; \infty\,.
\end{aligned}$$

It follows from Fubini's theorem that there is an r in $[0,1]\setminus\mathfrak{N}$ such that

$$\exp\{\lambda F_u(r,x)\} \;=\; \exp\left\{\lambda\sum_p \frac{1}{p^u}\,\chi_{\log p}(x+\{\epsilon_p\})\right\}$$

lies in $L^1(\boldsymbol{\sigma}_P)$. Notice that $\pm F_u(r,x)$ lie in $H^1(\boldsymbol{\sigma}_P)$. Taking $q\;=\;|\lambda|$, we thus see that both Z_u and Z_u^{-1} are outer functions in $H^q(\boldsymbol{\sigma}_P)$, $1\le q<\infty$.

Although $Z_{1/2}$ itself is meaningless, we may consider $z\to Z_{1/2}(x+e_z)$ as a function on $\mathbf{R}^2_+$, outside a $\boldsymbol{\sigma}_P-$null set in $\mathbf{T}^\omega$. Indeed, we can define

$$Z_{1/2}(x+e_z) \;=\; Z_{1/2+\beta}(x+e_{t+i(\sigma-\beta)})\,, \quad z=t+i\sigma\,,$$

where $0\;<\;\beta\;<\sigma$, which is independent of β. Let us show next the nonexistence of

$$\lim_{\sigma\to+0} Z_{1/2}(x+e_{t+i\sigma})$$

$m_{\mathbf{R}} - a.e.\,t$ in $\mathbf{R}$. Therefore, unlike the case of $\zeta(s)$, usually $\mathfrak{z}(x,s)$ may not admit any kind of functional equation (1.2).

Proposition 6.4. *Let $\mathfrak{z}(x,s)$ be a Dirichlet series of the form* (1.4). *Then for $\boldsymbol{\sigma}_P - a.e.\ x$ in $\mathbf{T}^\omega$, $\mathfrak{z}(x,s)$ has the limits,*

$$\lim_{\sigma\to\frac{1}{2}+0} \mathfrak{z}(x,\sigma+it),$$

almost nowhere on $\mathbf{R}$*. In particular, the line* $\sigma = 1/2$ *is the natural boundary of* $\mathfrak{z}(x, s)$*.*

Proof. Since $Z_{1/2}(x+e_{t+i\sigma}) = \mathfrak{z}(x, 1/2+\sigma-it)$, we restrict our attention to the boundary behavior of $z \longrightarrow Z_{1/2}(x+e_z)$ on $\mathbf{R}^2_+$. For $0 \le \sigma < 1$, let

$$Q_\sigma(t_0) \; = \; \{t+i\gamma \,;\, t_0 \le t \le t_0+1\,,\; \sigma \le \gamma \le 1\}\,.$$

We also put

$$g_p(z) \; = \; \frac{1}{p^{\frac{1}{2}}}\,\chi_{\log p}(x+e_z)\,, \quad z \in \mathbf{R}^2_+\,,$$

which is continuous on $Q_0(t_0)$. Then the sequence $\{g_p\}$ satisfies the following two conditions:

(i) If we put

$$G(r, z) \; = \; \sum_p \epsilon_p(r) g_p(z)\,, \quad z \in Q_0(t_0)$$

then, for $m_I - a.e.\, r$ in $[0, 1]$ and $0 < \sigma < 1$, $G(r, z)$ converges uniformly on $Q_\sigma(t_0)$.

(ii) For each N,

$$\sum_{p \ge N} |g_p(t+i\sigma)|^2 \longrightarrow \infty\,, \quad \text{as} \quad \sigma \longrightarrow +0\,,$$

converges uniformly on $[t_0, t_0+1]$.

Indeed, (i) follows immediately follows from Fubini's theorem and a fundamental property of Dirichilet series [21, 9.11], and (ii) is a direct consequence of the fact that

$$\sum_p \frac{1}{p} \; = \; \infty\,.$$

Therefore an argument entirely similar to the proof of [3, Theorem A.4] shows that

$$\lim_{\sigma \to +0} \sum_p \epsilon_p(r) g_p(t+i\sigma)$$

may not exist for $m_I \times m_I - a.e.\,(r, t)$ in $[0, 1] \times [t_0, t_0+1]$. With the aid of Fubini's theorem again, we see easily that

$$\lim_{\sigma \to +0} \exp\left\{\sum_p \frac{1}{p^{\frac{1}{2}}}\,\chi_{\log p}(x+e_{i\sigma})\right\}$$

does not exist for $\boldsymbol{\sigma}_P - a.e.\, x$ in $\mathbf{T}^\omega$. Since $Z_{1/2}$ is represented as

$$Z_{1/2}(x+e_z) \; = \; \exp\left\{\sum_p \frac{1}{p^{\frac{1}{2}}}\,\chi_{\log p}(x+e_z)\right\} \cdot H_{1/2}(x+e_z)$$

for some outer function $H_{1/2}$ in $H^2(\boldsymbol{\sigma}_P)$, the desired conclusion follows. $\square$

REFERENCES

[1] H. Bercovici and C. Foias, *A real variable restatement of Riemann's hypothesis,* Israel J. Math. 48, (1984), 57-68.

[2] A. Beurling, *A closure problem related to the Riemann zeta-function,* Proc. Nat. Acad. Sci. U.S.A. 41, (1955), 312-314.

[3] P. Duren, *Theory of H^p spaces,* Academic Press, New York and London, 1970.

[4] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory,* Princeton University Press, Princeton, New Jersgy, 1981.

[5] T.W. Gamelin, *Uniform algebras,* Prentice-Hall, Englewood Cliffs, New Jersgy, 1969.

[6] J.B. Garnett, *Bounded analytic functions,* Academic Press, New York, 1981.

[7] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers,* Oxford University Press, New York, 1979.

[8] H. Helson, *Compact groups and Dirichlet series,* Ark. För Mat. 8, (1969), 139-143.

[9] H. Helson, *Convergence of Dirichlet series,* Lecture Note in Mathematics 336, Springer-Verlag, Berlin, 1973, 153-160.

[10] H. Helson, *Analyticity on compact abelian groups,* in *Algebras in Analysis,* Academic Press, London, 1975, 1-62.

[11] H. Helson, *Harmonic analysis,* 2nd ed., Hindustan Book Agency, Delhi, 1995.

[12] J.-P. Kahane, *Sur les sèries de Dirichlet* $\sum_1^\infty \pm n^{-s}$, C. R. Acad. Sc. Paris 276, (1973), 739-742.

[13] A. Laurinčikas, *Limit theorems for the Riemann zeta-function,* Kluwer Academic Publisher, Dordrecht, 1996.

[14] P. S. Muhly, *Function algebras and flows,* Acta Sci. Math. (Szeged) 35, (1973), 111-121.

[15] N. Nikolski, *Distance formulae and invariant subspaces, with an application to localization of zeros of the Riemann ζ-function,* Ann. Inst. Fourier, Grenoble 45, (1995), 143-159.

[16] H. Queffélec, *Propriétés presque sûres et quasi-sûres des séries de Dirichilet et des produits d'Euler,* Cand. J. Math. 32, (1980), 531-558.

[17] R. Remmert, *Classical topics in complex function theory,* Springer-Verlag, New York, 1998.

[18] W. Rudin, *Fourier Analysis on Groups,* Interscience, New York, 1962.

[19] K. Petersen, *Ergodic Theory,* Cambridge University Press, Cambridge, 1983.

[20] J. Tanaka, *Extension of almost periodic functions and analyticity on flows,* Sūgaku Expositions, (2001), to appear.

[21] E. C. Titchmarsh, *The theory of functions,* 2nd ed. Oxford University Press, New York, 1939.

[22] E. C. Titchmarsh, *The theory of the Riemann zeta-function,* (revised by D. R. Heath-Brown), Oxford University Press, New York, 1988.

Amer. Math. Soc. Transl.
(2) Vol. **204**, 2001

Extension of Almost Periodic Functions and Analyticity on Flows

Jun-ichi Tanaka

1. Introduction

Let D be the unit disc $\{|z| < 1|\}$ in the complex plane $\mathbf{C}$, and let z_0 be a point on the unit circle $\mathbf{T}$. Suppose that $w = f(z)$ is an analytic function on D. It is then important to investigate the distribution of values as well as the behavior of $f(z)$ around the boundary point z_0. Awareness of such subjects is old in function theory, and it seems to be an origin of value distribution theory and the theory of cluster sets.

In the 1940s, I. M. Gelfand and his coworkers built a theory of commutative Banach algebras. As one branch, theories of function algebras began to develop in the early 1950s, which formed a new link between classical function theory and functional analysis. Thus, for example, the theory of Hardy spaces has been extended to an abstract setting.

In this note, by considering the uniform algebras induced by flows, the boundary behavior of $f(z)$ is investigated in the region between the two circles tangent to $\mathbf{T}$ at z_0, although a certain growth restriction of $|f(z)|$ is required by the Phragmén-Lidelöf principle.

The outline of our method runs as follows. Suppose, for simplicity, that $f(z)$ is bounded in D and $z_0 = 1$. Let $z(w)$ be the conformal map of the upper half-plane $\mathbf{R}^2_+$ onto D by

$$z(w) = \frac{w-i}{w+i}.$$

We then put $g(w) = f(z(w))$. Since $z(w)$ maps ∞ to 1, we want to look into the behavior of $g(w)$ around infinity. Let $\beta\mathbf{Z}$ be the Stone-Čech compactification of the integer group $\mathbf{Z}$. Divide $\mathbf{R}^2_+$ into the countably many strips $[n, n+1) \times (0, \infty)$ for n in $\mathbf{Z}$. Since $\{g(n+w); n \in \mathbf{Z}\}$ is a bounded sequence for each ω in $[0,1) \times (0, \infty)$,

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This article originally appeared in Japanese in Sūgaku **51** (2) (1999), 113–128.

2000 *Mathematics Subject Classification.* Primary 43A17, 46J10, 46J15, 46J20; Secondary 11K70, 11M06, 28D15, 30H05.

Key words and phrases. Uniform algebras, Hardy spaces, outer functions, minimal flows, Dirichlet series.

This research was partially supported by Grant 10640180 from the Japanese Ministry of Education.

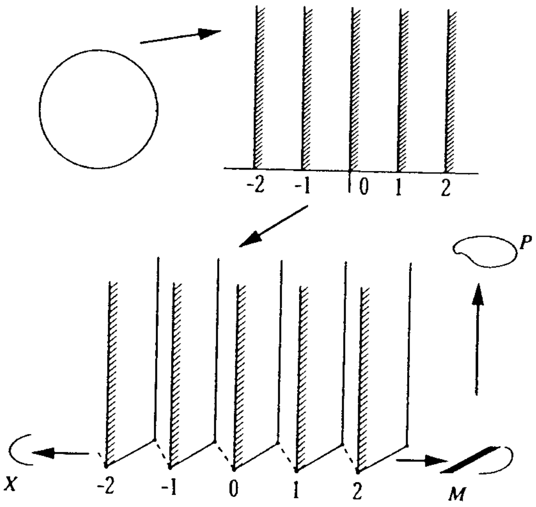

FIGURE 1

$g(w)$ extends to a continuous function on $\beta\mathbf{Z} \times [0, 1) \times (0, \infty)$, which represents concretely a portion of the cluster set of $g(w)$. The shift operator $S_0 n = n + 1$ on $\mathbf{Z}$ extends to a homeomorphism S on $\beta\mathbf{Z}$. We denote by $\mathbf{X}$ the quotient space obtained from $\beta\mathbf{Z} \times [0, 1]$ by identifying $(y, 1)$ with $(Sy, 0)$. By regarding the real line $\mathbf{R}$ as $\mathbf{Z} \times [0, 1]$, the translation on $\mathbf{R}$ induces a continuous flow on $\mathbf{X}$. Thus, using ergodic theory, we try to study the behavior of $g(w)$ on the growth $\mathbf{X} \setminus \mathbf{R}$ (see Figure 1).

In the next section we establish the notation and discuss briefly the principal techniques of Hardy spaces in certain settings. Section 3 treats the uniform algebras induced by flows and, as an application, we deal with a negative answer to a question posed by Forelli [**10**]:

Is the uniform algebra induced by a minimal flow a Dirichlet algebra?

Restricting our attention to the case of almost periodic flows, we discuss in Section 4 analytic functions on a quotient of the Bohr group. After preparing some lemmas, we study Dirichlet series obtained as the limits of translations of the Riemann zeta-function in Section 5, which contains some results published here for the first time.

2. Rudiments of Hardy spaces

We deal with certain subspaces of analytic functions with growth conditions, together with some basic results on boundary behavior. Let $1 \leq p < \infty$. A function $f(z)$ analytic in D is said to belong to the *Hardy space* $H^p(D)$ if

$$\|f\|_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty,$$

while $H^\infty(D)$ is defined to be the space of all bounded analytic functions $f(z)$ on D, normed by

$$\|f\|_\infty = \sup\{|f(z)| : z \in D\}.$$

Let $H^p(d\theta/2\pi)$, $1 \le p \le \infty$, be the space of all functions $f(\theta)$ in $L^p(d\theta/2\pi)$ that have Fourier series of the form

$$f(\theta) \sim \sum_{n=0}^{\infty} a_n e^{in\theta} .$$

Fatou's theorem then enables us to identify $H^p(D)$ and $H^p(d\theta/2\pi)$. The *disc algebra* $A(\mathbf{T})$ is the algebra of all continuous functions in $H^\infty(d\theta/2\pi)$, which is a commutative Banach algebra with the uniform norm. Then $H^p(d\theta/2\pi)$, $1 \le p < \infty$, is also obtained as the closure of $A(\mathbf{T})$ in $L^p(d\theta/2\pi)$, while $H^\infty(d\theta/2\pi)$ is the weak-* closure of $A(\mathbf{T})$ in $L^\infty(d\theta/2\pi)$.

The *Hardy space* $H^p(\mathbf{R}^2_+)$, $1 \le p < \infty$, on $\mathbf{R}^2_+$ is defined similarly to be the space of all analytic functions on $\mathbf{R}^2_+$ such that

$$\|f\|_p^p = \sup_{0<y<\infty} \int_{-\infty}^{\infty} |f(x+iy)|^p \, dy < \infty,$$

while $H^\infty(\mathbf{R}^2_+)$ is the space of all bounded analytic functions on $\mathbf{R}^2_+$. We denote by $H^p(dt)$ the subspace of $L^p(dt)$ of all the boundary-value functions of $H^p(\mathbf{R}^2_+)$, and identify $H^p(\mathbf{R}^2_+)$ with $H^p(dt)$ as usual. A function f in $L^p(dt)$ lies in $H^p(dt)$ if and only if the Fourier transform of f vanishes on $(-\infty, 0)$, in a suitable sense. There is a simple relation between $H^p(D)$ and $H^p(\mathbf{R}^2_+)$. Each function g in $H^p(D)$ is transformed by the linear fractional map $z = (w-i)/(w+i)$ into an analytic function f on $\mathbf{R}^2_+$. Since $t = (e^{i\theta} - i)/(e^{i\theta} + i)$ implies

$$\frac{1}{2\pi} \, d\theta = \frac{1}{\pi} \frac{dt}{1+t^2},$$

we denote by $H^p(dt/\pi(1+t^2))$ the space of all the boundary-value functions of such transformed functions. It is known that $f(t)$ lies in $H^p(dt/\pi(1+t^2))$ if and only if $(1+t^2)^{-2/p} f(t)$ lies in $H^p(dt)$ (see [**25**, Chapter 3]).

The *Poisson kernel* $P_z(t)$ for $\mathbf{R}^2_+$ is defined by

$$P_z(t) = \frac{1}{\pi} \frac{v}{(u-t)^2 + v^2}, \quad z = u + iv, \ v > 0.$$

For $f(t)$ in $H^p(dt/\pi(1+t^2))$, the analytic extension $f(z)$ to $\mathbf{R}^2_+$ is obtained by the convolution

$$f(z) = f * P_{iv}(u) = \int_{-\infty}^{\infty} f(t) P_z(t) \, dt,$$

and we sometimes identify $f(t)$ with its extension $f(z)$ in what follows.

A nonzero function in a Hardy space can be factored into a function of modulus one and a function that has no zeros. We restrict our attention to the case of $H^p(dt/\pi(1+t^2))$. An *inner function* is a function in $H^p(dt/\pi(1+t^2))$ that is of modulus one. If w is a nonnegative function in $L^p(dt/\pi(1+t^2))$ such that

$$\int_{-\infty}^{\infty} \log w(t) \frac{dt}{1+t^2} > -\infty ,$$

and if

$$h(z) = e^{i\gamma} \exp\left\{ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1+tz}{t-z} \log w(t) \frac{dt}{1+t^2} \right\}, \quad z \in \mathbf{R}^2_+ . \tag{2.1}$$

for some γ in $\mathbf{R}$, then $h(z)$ is called an *outer function* in $H^p(dt/\pi(1+t^2))$. It can be shown that $|h(t)| = w(t)$ on $\mathbf{R}$.

Let z_n ($\neq i$) be a sequence in $\mathbf{R}^2_+$ such that

$$\sum_{n=1}^{\infty} \frac{y_n}{1+|z_n|^2} < \infty, \quad z_n = x_n + iy_n .$$

Then the inner function defined by

$$B(z) = \left(\frac{z-i}{z+i}\right)^m \prod_{n=1}^{\infty} \frac{|z_n^2+1|}{z_n^2+1} \frac{z-z_n}{z-\overline{z_n}}$$

is called a *Blaschke product*. If ν is a positive singular measure on $\mathbf{R}$ such that

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2}\, d\nu(t) < \infty,$$

then the inner function defined by

$$S(z) = \exp\left\{i\left(\alpha z + \int_{-\infty}^{\infty} \frac{1+tz}{t-z}\, d\nu(t)\right)\right\}$$

is called a *singular inner function*, where $\alpha \geq 0$.

The following canonical factorization theorem is fundamental to the theory of H^p spaces (as discussed in [**8**], [**13**], [**14**], for instance).

THEOREM 2.1. *Every nonzero function $f(z)$ in $H^p(dt/\pi(1+t^2))$ has a unique factorization of the form*

$$f(z) = B(z)S(z)h(z),$$

where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function, and $h(z)$ is the outer function in $H^p(dt/\pi(1+t^2))$ defined by (2.1) with $w(t) = |f(t)|$. Conversely, every such product $B(z)S(z)h(z)$ belongs to $H^p(dt/\pi(1+t^2))$.

Let A be a commutative Banach algebra with identity 1. Then the class of all maximal ideals of A is called the *maximal ideal space* of A, and denoted by $\mathfrak{M}(A)$. Since there is a one-to-one correspondence between the nonzero complex homomorphisms of A and the maximal ideals in A, it is customary to identify each maximal ideal in $\mathfrak{M}(A)$ with the complex homomorphism that it determines. This allows us to identify $\mathfrak{M}(A)$ with a subset of the unit sphere of the conjugate space A^* of A. We then define the topology of $\mathfrak{M}(A)$ to be the weak-* topology that $\mathfrak{M}(A)$ inherits from A^*, which is the so-called *Gelfand topology* of $\mathfrak{M}(A)$. Then $\mathfrak{M}(A)$ is a compact Hausdorff space. The *Gelfand transform* of f in A is the continuous function $\hat{f}$ on $\mathfrak{M}(A)$ defined by

$$\hat{f}(\xi) = \xi(f), \quad \xi \in \mathfrak{M}(A).$$

Thus the study of certain properties of A reduces to studying the structure of $\mathfrak{M}(A)$.

We see easily that the maximal ideal space $\mathfrak{M}(A(\mathbf{T}))$ of the disc algebra may be considered as the closed unit disc $\overline{D}$. However, when we regard $H^\infty(d\theta/2\pi)$ as a Banach algebra, the structure of $\mathfrak{M}(H^\infty(d\theta/2\pi))$ is very big and complicated. If $z = re^{i\theta}$ lies in D, then each

$$\xi_z(f) = f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})P_r(\theta - t)\, dt, \quad f \in H^\infty(d\theta/2\pi),$$

determines a point in $\mathfrak{M}(H^\infty(d\theta/2\pi))$, where $P_r(\theta)$ is the Poisson kernel for D. This implies that D is naturally embedded as an open subset in $\mathfrak{M}(H^\infty(d\theta/2\pi))$. Carleson's corona theorem [**5**] states that D is dense in $\mathfrak{M}(H^\infty(d\theta/2\pi))$.

For α in $\mathbf{T}$, the *fiber* $\mathfrak{M}_\alpha$ of $\mathfrak{M}(H^\infty(d\theta/2\pi))$ over α is defined to be

$$\mathfrak{M}_\alpha = \{\xi \in \mathfrak{M}(H^\infty(d\theta/2\pi))\,;\xi(z) = \alpha\},$$

where z is the coordinate function. Then we have the decomposition

$$\mathfrak{M}(H^\infty(d\theta/2\pi)) \setminus D = \bigcup_{|\alpha|=1} \mathfrak{M}_\alpha.$$

It is shown that the various fibers $\mathfrak{M}_\alpha$ are homeomorphic to one another. If f lies in $H^\infty(d\theta/2\pi)$, then the *cluster set* of f at α is

$$Cl(f,\alpha) = \bigcap_{r>0} \overline{f(D \cap \Delta(\alpha, r))},$$

where $\Delta(\alpha, r)$ is the open disc with center α and radius r. Then we obtain

$$Cl(f,\alpha) = \hat{f}(\mathfrak{M}_\alpha).$$

Various aspects of H^p theory have been generalized to the abstract setting of uniform algebras, and we give a brief outline. Let Y be a compact Hausdorff space, and let $C(Y)$ be the algebra of all complex-valued continuous functions on Y. We say that a subalgebra A of $C(Y)$ is a *uniform algebra* on Y if A is complete under the uniform norm, contains the constants, and separates the points of Y. A subset E of Y is called a *boundary* for A if

$$\|f\|_\infty = \sup\{|f(x)|\,; x \in E\}, \quad f \in A.$$

There is a smallest closed boundary, which is called the *Šilov boundary* of A. Each complex homomorphism ξ in $\mathfrak{M}(A)$ is represented by a probability measure μ on Y, that is,

$$\hat{f}(\xi) = \int_Y f d\mu, \quad f \in A,$$

which is called a *representing measure* for ξ. Then the Hardy space $H^p(\mu), 1 \le p < \infty$, is defined to be the closure of A in $L^p(\mu)$, while the weak-* closure of A in $L^\infty(\mu)$ is $H^\infty(\mu)$. We recall that, in the case of the disc algebra $A(\mathbf{T})$, $H^p(d\theta/2\pi)$ is also obtained in this manner. To use the tools by functional analysis suitably, we need some conditions that assure a certain size of $H^p(\mu)$ in $L^p(\mu)$. So a uniform algebra A is called a *Dirichlet algebra* on Y, if $\operatorname{Re} A = \{\operatorname{Re} f\,; f \in A\}$ is uniformly dense in $C_{\mathbf{R}}(Y)$, the space of all real-valued continuous functions on Y. We say A is a *logmodular algebra* on Y if

$$\log|A^{-1}| = \{\log|f|; f \in A^{-1}\}$$

is uniformly dense in $C_{\mathbf{R}}(Y)$, where A^{-1} denotes the set of invertible elements of A. As is easily seen, every Dirichlet algebra is a logmodular algebra, and $A(\mathbf{T})$ is a Dirichlet algebra on $\mathbf{T}$. However, the uniform algebra of Gelfand transforms of functions in $H^\infty(d\theta/2\pi)$ is not a Dirichlet algebra, but a logmodular algebra on its Šilov boundary.

More generally, let A be a subalgebra of $L^\infty(\mu)$ containing the constants. Then A is called a *weak-* Dirichlet algebra* if μ is multiplicative on A and $A + \overline{A}$ is dense in $L^\infty(\mu)$ in the weak* topology. Many theorems, such as Szegö's theorem and the factorization theorem, have been extended to the context on weak-* Dirichlet algebras. We refer the reader to [**16**], [**17**], [**40**], [**45**] for further details on weak-* Dirichlet algebras.

Let A be a weak-* Dirichlet algebra in $L^\infty(\mu)$. A closed subspace M of $L^p(\mu)$ is *invariant* if AM is contained in M. An invariant subspace M of $L^p(\mu)$ is *simply invariant* if M is not of the form $I_E L^p(\mu)$, where I_E is the characteristic function of a set E. We denote by A_0 the subspace of A of all functions f such that

$$\hat{f}(\mu) = \int_Y f\, d\mu = 0.$$

If A_0M is not dense in M, then M is simply invariant. It is known that, for each h in $H^p(\mu)$, h satisfies *Jensen's inequality*:

$$\log|\hat{h}(\mu)| \leq \int_Y \log|h|\, d\mu.$$

When equality holds in finite values, h is called an *outer* function in $H^p(\mu)$. It follows from Szegö's theorem that h is outer if and only if the invariant subspace generated by h equals $H^p(\mu)$, that is, the closure of Ah in $L^p(\mu)$ is $H^p(\mu)$. As before, functions in $H^\infty(\mu)$ of unit modulus are called *inner* functions. It is also known that if f is a function in $H^p(\mu)$ such that $\log|f|$ lies in $L^1(\mu)$, then f can be factored into the form $f = qh$, where q is inner and h is outer. However, factorization theorems have not yet been completely developed, especially for functions f such that $\log|f|$ is not in $L^1(\mu)$.

As before, $\mathfrak{M}(A)$ denotes the maximal ideal space of a uniform algebra A. For ξ and ν in $\mathfrak{M}(A)$, we put

$$\xi \sim \nu \iff \|\xi - \nu\| < 2,$$

where $\|\cdot\|$ is the norm on A^* in other words, whenever the distance from ξ to ν in A^* is less than 2. Gleason showed that $\sim$ is an equivalence relation on $\mathfrak{M}(A)$. An equivalence class under $\sim$ is called a *Gleason part* of $\mathfrak{M}(A)$. Wermer [**58**] proved that if A is a logmodular algebra, and if a Gleason part P is not a point, then there is a continuous map $\tau(z)$ of the open unit disc D onto P such that $\hat{f}(\tau(z))$ is a bounded analytic function on D for each f in A.

Let $\{z_n\}$ be a sequence in D. We call $\{z_n\}$ an *interpolating* sequence if for each bounded sequence $\{\alpha_n\}$ in $\mathbf{C}$, there is a function f in $H^\infty(d\theta/2\pi)$ such that $f(z_n) = \alpha_n$, $n = 1, 2, \ldots$. By Carleson's characterization [**4**] of interpolating sequences, they play an important role in unraveling the structure of $\mathfrak{M}(H^\infty(d\theta/2\pi))$. A Blaschke product is called an *interpolating Blaschke product* if it has distinct zeros and if these zeros form an interpolating sequence. Using factorization theorems for Blaschke products, Hoffman [**26**] showed that ξ lies in an analytic disc in $\mathfrak{M}(H^\infty(d\theta/2\pi))\setminus D$ if and only if ξ is in the closure of some interpolating sequence. In view of the size and intractability of $\mathfrak{M}(H^\infty(d\theta/2\pi))$, this would be a great accomplishment.

More information on $\mathfrak{M}(H^\infty(d\theta/2\pi))$ and related results can be found in Izuchi's series of papers [**27**], [**28**], [**29**], [**30**]. In particular, he showed in [**28**] the existence of certain representing measures on a fiber $\mathfrak{M}_\alpha$ under the continuum hypothesis. It would be interesting to investigate cluster sets under this hypothesis.

3. Analytic functions on flows

Let Ω be a compact Hausdorff space on which $\mathbf{R}$ acts as a transformation group. This means that there is a one-parameter group $\{U_t\}_{t\in\mathbf{R}}$ of homeomorphisms on Ω such that the map $(\omega, t) \to U_t\omega$ is continuous on $\Omega \times \mathbf{R}$. The pair $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is referred to as a (*continuous*) *flow*. For simplicity, the translate $U_t\omega$ of ω by t is denoted by $\omega + t$. A subset M of Ω is *invariant* if M contains $U_t M$ for all t in $\mathbf{R}$. Then Zorn's lemma assures that there is a minimal one in the class of all nonempty closed invariant sets. If Ω itself is minimal, then $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is called a *minimal flow*. A flow $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is minimal if and only if the *orbit* of ω,

$$\mathcal{O}(\omega) = \{\omega + t : t \in \mathbf{R}\},$$

is dense in Ω for each ω in Ω. On a minimal flow $(\Omega, \{U_t\}_{t\in\mathbf{R}})$, each function ϕ in $C(\Omega)$ is the extension of the uniformly continuous function $t \to \phi(\omega + t)$ on $\mathbf{R}$; so such a function on $\mathbf{R}$ may be regarded as an almost periodic function in a sense.

By the Markov-Kakutani theorem, there is at least one invariant probability measure on Ω. An invariant measure is said to be *ergodic* if every invariant subset of Ω either is negligible or has negligible complement. We use the symbol $\mathcal{K}$ for the set of all invariant probability measures on Ω. Then $\mathcal{K}$ is a nonempty compact convex set in the weak-* topology. Since a measure in $\mathcal{K}$ is ergodic if and only if it is an extreme point of $\mathcal{K}$, the Krein-Milman theorem assures the existence of an ergodic measure in $\mathcal{K}$. A minimal flow $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is called *strictly ergodic* if there is exactly one invariant probability measure on Ω. A function ϕ in $C(\Omega)$ is *analytic* if $t \to \phi(\omega + t)$ lies in $H^\infty(dt/\pi(1+t^2))$ for each ω in Ω. Let $A(\Omega)$ be the algebra of all analytic functions in $C(\Omega)$. When $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ has no fixed points, $A(\Omega)$ becomes a uniform algebra of Ω, which is called the *uniform algebra induced by* $(\Omega, \{U_t\}_{t\in\mathbf{R}})$. Many results have been obtained about such uniform algebras (see [**9**], [**35**], [**39**], and [**52**]).

Wermer's maximality theorem [**57**] states that the disc algebra $A(\mathbf{T})$ is a maximal closed subalgebra in $C(\mathbf{T})$, that is, every closed subalgebra of $C(\mathbf{T})$ that properly contains $A(\mathbf{T})$ equals $C(\mathbf{T})$. As a generalization of it, Forelli [**11**] proved the following.

THEOREM 3.1. *If $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is minimal, then the induced uniform algebra $A(\Omega)$ is a maximal algebra in $C(\Omega)$.*

In connection with this result, he asked the question stated in the Introduction (see [**36**, §6] for a nice account of related topics). In responce to the question, Muhly [**35**] gave, among other things, a sufficient condition for $A(\Omega)$ to be a Dirichlet algebra.

THEOREM 3.2. *If $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is strictly ergodic, then $A(\Omega)$ is a Dirichlet algebra on Ω.*

THEOREM 3.3. *Let μ be an invariant, ergodic, probability measure on Ω. Then μ is a representing measure for $A(\Omega)$ and $A(\Omega)$ is a weak-* Dirichlet algebra in $L^\infty(\mu)$.*

These results are important. Indeed, under these conditions we can discuss Hardy spaces and invariant subspaces by applying the general theory of uniform algebras. It long remained unknown, however, whether $A(\Omega)$ is a Dirichlet algebra for a minimal flow $(\Omega, \{U_t\}_{t\in\mathbf{R}})$. We finally found in [**48**] an example of $A(\Omega)$ that is not a Dirichlet algebra.

On one hand, the minimal flows that are not strictly ergodic have long been investigated in ergodic theory (see, for example, [**42**]). On the other hand, the difference between Dirichlet algebras and logmodular algebras, just like $A(\mathbf{T})$ and $H^\infty(d\theta/2\pi)$, has been discussed. It may be interesting that these two independent areas are related to each other, via the induced uniform algebras.

Let us mention briefly the maximal ideal space $\mathfrak{M}(A(\Omega))$ of $A(\Omega)$. For ϕ in $A(\Omega)$ and $r > 0$, the convolution of ϕ with the Poisson kernel $P_{ir}(t)$,

$$\phi(x, r) = \phi * P_{ir}(x) = \int_{-\infty}^{\infty} \phi(x+t) P_{ir}(t)\, dt, \quad x \in \Omega,$$

is a complex homomorphism of $A(\Omega)$. This shows that $\Omega \times [0, \infty)$ may be considered as a subset of $\mathfrak{M}(A(\Omega))$. As r tends to ∞, $\{(x, r)\}$ accumulates to points in $\mathfrak{M}(A(\Omega))$ for which representing measures are invariant. In particular, when $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is strictly ergodic, $\mathfrak{M}(A(\Omega))$ is the quotient space obtained from by $\Omega \times [0, \infty]$ by identifying the slice $\Omega \times \{\infty\}$ to a point. Some results relating to the corona theorem have also been obtained. We refer to [**36**], [**47**], and [**59**] for details.

In the rest of this section, we devote ourselves to the construction of an induced uniform algebra that is not a Dirichlet algebra. Let $\beta\mathbf{Z}, S$, and X be as in the Introduction, and let $\{S_t\}_{t\in\mathbf{R}}$ be the one-parameter group of homeomorphisms on X defined by

$$S_t(y, s) = (S^{[t+s]}y, t + s - [t + s]), \quad (y, s) \in X, \tag{3.1}$$

where $[t]$ denotes the largest integer not exceeding t. Then $(X, \{S_t\}_{t\in\mathbf{R}})$ is a flow, which is a special case of the so-called flows built under functions in ergodic theory. Let X_0 be the quotient space obtained from $\mathbf{Z} \times [0, 1]$ by identifying $(n, 1)$ with $(n + 1, 0)$. Then X_0 is a dense invariant set in X that is homeomorphic to $\mathbf{R}$. See [**34**] or [**55**] for Stone-Čech compactifications.

Let $C_{ub}(\mathbf{R})$ be the C^*-algebra of all bounded uniformly continuous functions on $\mathbf{R}$. Then we easily obtain the following proposition.

PROPOSITION 3.4. *Let X be the compact Hausdorff space defined above. Then $C_{ub}(\mathbf{R})$ is isometrically isomorphic to $C(X)$. In particular, X is the maximal ideal space of $C_{ub}(\mathbf{R})$.*

Let us consider the uniform algebra $A(X)$ induced by $(X, \{S_t\}_{t\in\mathbf{R}})$. Let f be a function in $H^\infty(dt/\pi(1 + t^2))$, and let $r > 0$. Since $f(t + ir)$ is the convolution of $f(t)$ with $P_{ir}(t)$, $t \to f(t + ir)$ lies in $C_{ub}(\mathbf{R})$. By Proposition 3.4, this extends to a function $\phi(x, r)$ in $A(X)$. Conversely, since $\mathbf{R}$ is dense in X, each function in $A(X)$ is determined by its restriction to $\mathbf{R}$. Therefore, $A(X)$ is generated by the family $\{\phi(x, r)\,; r > 0\}$. We next regard $\phi(x, r)$ as a function on $X \times (0, \infty)$. Then a function in $H^\infty(dt/\pi(1 + t^2))$ appears on each orbit by Fatou's theorem. Let $H^\infty(X)$ be the space of all boundary functions of $\phi(x, r)$. Then $H^\infty(X)$ is isometrically isomorphic to $H^\infty(dt/\pi(1 + t^2))$. We see also that each point in $X \times (0, \infty)$ determines a complex homomorphism for $H^\infty(X)$. Since $H^\infty(dt/\pi(1 + t^2))$ and $H^\infty(d\theta/2\pi)$ are isomorphic, $X \times (0, \infty)$ corresponds to a portion of the maximal ideal space $\mathfrak{M}(H^\infty(d\theta/2\pi))$. It is easy to see that $X_0 \times (0, \infty)$ and $X \setminus X_0 \times (0, \infty)$ determine D and the union of the accumulation points of the regions between two circles tangent to $\mathbf{T}$ at 1, respectively.

The maximal ideal space $\mathfrak{M}(A(X))$ contains $X \times [0, \infty)$, which is dense in $\mathfrak{M}(A(X))$ by the corona theorem. Each accumulating point of $X \times [0, \infty)$ is a complex homomorphism of $A(X)$ whose representing measure is invariant. This observation suggests a relation between the corona theorem and the individual ergodic theorem. Indeed, it follows from Wiener's Tauberian theorem that the existence of one side of the following limits implies the existence of the other, and the two are equal:

$$\lim_{r\to\infty} \phi * P_{ir}(x) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \phi(x+t)\,dt.$$

Then the individual ergodic theorem assures the existence of the right side. Recall that each invariant ergodic probability measure is a representing measure for $A(X)$ by Theorem 3.3. With the aid of the ergodic theorem, we may show directly that such representing measures are accumulating points of $X_0 \times (0,\infty)$. Let $\mathcal{K}$ be as above. Then such measures consist of the extreme points of $\mathcal{K}$. Thus if Choquet theory could be developed further, there would be another way to prove the corona theorem.

Let M be a minimal set in $(X, \{S_t\}_{t\in\mathbf{R}})$. Restricting S_t to M, we have a minimal flow $(M, \{S_t\}_{t\in\mathbf{R}})$, which has the strongest topology in the class of all minimal flows. There are consequently a great many invariant probability measures on M.

PROPOSITION 3.5. *Let $(M, \{S_t\}_{t\in\mathbf{R}})$ be as above. Then $(M, \{S_t\}_{t\in\mathbf{R}})$ is universal, meaning that every minimal flow is a factor of $(M, \{S_t\}_{t\in\mathbf{R}})$.*

Outline of Proof. Let $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ be a minimal flow, and fix an ω in Ω. The C^*-algebra generated by $\{t \to \phi(\omega+t)\,;\, \phi \in C(\Omega)\}$ is a closed subalgebra of $C_{ub}(\mathbf{R})$. It follows from the Banach-Stone theorem that there is a continuous map τ of X onto Ω such that

$$\tau \circ S_t(x) = U_t \circ \tau(x), \quad x \in X.$$

Then the restriction of τ to M is the desired map of M onto Ω. □

THEOREM 3.6. *Let $A(M)$ be the uniform algebra induced by the above flow $(M, \{S_t\}_{t\in\mathbf{R}})$. Then $A(M)$ is a logmodular algebra that is not a Dirichlet algebra.*

Outline of Proof. As is easily seen by the relation to $H^\infty(dt/\pi(1+t^2))$, $A(X)$ is logmodular on X. Since the restriction $A|_M$ of $A(X)$ to M is a subalgebra of $A(M)$, $A(M)$ is also a logmodular algebra on M. We then see that $\mathfrak{M}(A(M))$ is the closure of $M \times [0,\infty)$ in $\mathfrak{M}(A(M))$. To choose a suitable interpolating sequence, we find by Hoffman's theorem a Gleason part P in $M \times \{\infty\}$ that is homeomorphic to D. Let $\overline{P}$ denote the closure of P in $\mathfrak{M}(A(M))$, and let $\hat{A}|_{\overline{P}}$ be the algebra of all restrictions of Gelfand transforms of functions in $A(M)$. Then $\hat{A}|_{\overline{P}}$ is isometrically isomorphic to $H^\infty(d\theta/2\pi)$. As is stated above, $H^\infty(d\theta/2\pi)$ is not a Dirichlet algebra on its Šilov boundary. This implies that there is a nonzero real measure ν on $\overline{P}\setminus P$ that is orthogonal to $\hat{A}|_{\overline{P}}$. Define

$$L(\phi) = \int_{\overline{P}\setminus P} \hat{\phi}(\xi)\,d\nu(\xi), \quad \phi \in C(M), \tag{3.2}$$

where $\hat{\phi}(\xi)$ denotes the integral with respect to the representing measure for ξ. Then L is a bounded linear functional on $C(M)$. It follows from the Riesz-Kakutani theorem that there is an invariant real measure that is orthogonal to $A(M)$. Thus $\operatorname{Re} A(M)$ cannot be dense in $C_{\mathbf{R}}(M)$. □

We note that the above ν may be chosen as a measure on the Šilov boundary of $A(M)$, each of which has a representing measure that is invariant and ergodic. It therefore might be interesting to study the relation between (3.2) and the ergodic decomposition of invariant measures. Check Figure 1 again for places of M and P, and [**48**] for the detailed proof of Theorem 3.6 and related topics.

4. Analytic functions on compact groups

In this section we will be concerned with Hardy spaces of analytic almost periodic functions of the form

$$f(t) = \sum_j c_j e^{i\lambda_j t}, \quad t \in \mathbf{R},$$

where the λ_j belong to the positive half of a subgroup Γ of $\mathbf{R}$. If Γ is the group $\mathbf{Z}$ of integers, then f may be regarded as an analytic function on D.

We now assume that Γ is a dense subgroup of $\mathbf{R}$ but endowed with the discrete topology. Let K be the dual group of Γ, and let σ be the normalized Haar measure on K. When λ in Γ is considered as a character on K, it is written χ_λ. A function f in $L^1(\sigma)$ is *analytic* if its Fourier coefficients

$$a_\lambda(f) = \int_K \overline{\chi_\lambda}(x) f(x)\, d\sigma(x)$$

vanish for all negative λ in Γ. The *Hardy space* $H^p(\sigma), 1 \le p \le \infty$, is defined to be the space of all analytic functions in $L^p(\sigma)$. Similarly, $A(K)$ denotes the algebra of all analytic functions in $C(K)$. Then $A(K)$ is a Dirichlet algebra on K and σ is a representing measure for $A(K)$. We notice that $H^p(\sigma)$ is also obtained as the closure of $A(K)$ in $L^p(\sigma)$ as in Section 2.

For each t in $\mathbf{R}$, e_t is the element of K defined by $e_t(\lambda) = e^{i\lambda t}$ for any λ in Γ. Then the map $t \to e_t$ embeds $\mathbf{R}$ continuously onto a dense subgroup of K. We then define the one-parameter group $\{T_t\}_{t\in\mathbf{R}}$ of homeomorphisms on K by

$$T_t x = x + e_t, \quad x \in K.$$

The flow $(K, \{T_t\}_{t\in\mathbf{R}})$ is strictly ergodic, and σ is the unique invariant probability measure on it. Since

$$\chi_\lambda(T_t x) = \chi_\lambda(x + e_t) = e^{i\lambda t}\chi_\lambda(x),$$

$\chi_\lambda(x)$ is an eigenfunction with eigenvalue λ. We remark that the discrete spectrum theorem characterizes ergodic flows that are conjugate to $(K, \{T_t\}_{t\in\mathbf{R}})$ (see [**15**], [**56**]).

Fix a positive γ in Γ, and let K_γ be the subgroup consisting of all x in K such that $\chi_\gamma(x) = 1$. Then K may be identified with $K_\gamma \times [0, 2\pi/\gamma)$ via the map $y+s \to (y, s)$. When 2π lies in Γ, we fix a homeomorphism on $K_{2\pi}$ by $Sy = y + e_1$. Then $\{T_t\}_{t\in\mathbf{R}}$ may be represented in the form (3.1).

We see easily that a function f in $L^p(\sigma)$ is analytic if and only if $t \to f(x+t)$ lies in $H^p(dt/\pi(1+t^2))$, for a.e. x in K. Using this fact, we characterize outer functions in $H^2(\sigma)$ as follows.

LEMMA 4.1. *A function h in $H^2(\sigma)$ is outer if and only if $\hat{h}(\sigma) \neq 0$ and $t \to h(x+t)$ is outer in $H^2(dt/\pi(1+t^2))$ for a.e. x in K.*

Outline of Proof. Suppose that $A(K)h$ is not dense in $H^2(\sigma)$. Then there is a nonconstant function g in $H^2(\sigma)$ that is orthogonal to $A(K)h$. Since $t \to \overline{g(x+t)}h(x+t)$ lies in $H^1(dt/\pi(1+t^2))$, $t \to h(x+t)$ cannot be an outer function in $H^2(dt/\pi(1+t^2))$ for a.e. x in K. Conversely, suppose that the set of all x in K such that

$$\log|h * P_{ir}(x)| \;<\; \int_{-\infty}^{\infty} \log|h(x+t)|\, P_{ir}(t)\, dt$$

has positive measure. Since $h * P_{ir}(x)$ lies in $H^2(\sigma)$, we obtain

$$\log|\int_K h * P_{ir}(x) d\sigma(x)| \;\le\; \int_K \log|h * P_{ir}(x)|\, d\sigma(x)$$

by Jensen's inequality. This implies that

$$\log|\hat{h}(\sigma)| \;<\; \int_K \log|h(x)|\, d\sigma(x).$$

Thus h cannot be outer in $H^2(\sigma)$. □

Let f be a function in $C(K)$ with Fourier series

$$f(x) \sim \sum_n a_n \chi_{\lambda_n}(x).$$

Then f is uniformly approximated by a sequence of trigonometric polynomials on X by the Stone-Weierstrass theorem. This implies that $F_x(t) = f(x+t)$ is a uniformly almost periodic function (in the sense of Bohr) with exponents in Γ,

$$F_x(t) \;=\; \sum_n a_n \chi_{\lambda_n}(x) e^{i\lambda_n t}.$$

If $y + e_{t_j}$ tends to x for a fixed y in K, then $F_x(t)$ may be regarded as a kind of limit of translations $F_y(t+t_j)$. The same relation holds between $L^2(\sigma)$-functions and almost periodic functions in the sense of Besicovitch, if we exclude an exceptional null set.

Since the 1950s, $A(K)$ and $H^p(\sigma)$ have been explored as extensions of $A(\mathbf{T})$ and $H^p(d\theta/2\pi)$ to functions of several variables. In particular, the invariant subspace theory has flourished in this setting. Each invariant subspace in $L^2(\sigma)$ corresponds to a function of modulus one on $K \times \mathbf{R}$, which is called a *cocycle*. However, there remain many interesting problems; for example, we do not know whether every simply invariant subspace can be generated by one of its elements.

Let $\mathbf{T}^\omega$ be the complete direct sum of countably many copies of $\mathbf{T}$. If Γ is countable, then K is a closed subgroup of $\mathbf{T}^\omega$ (see [**44**] for details). We now adduce an example that plays an important role in the next section.

EXAMPLE. Let $\mathbf{Z}^\infty$ be the direct sum of countably many copies of $\mathbf{Z}$, that is, the group of all sequences of which all but a finite number of terms vanish. Taking primes p as indices, we denote $\mathbf{Z}^\infty$ by

$$\mathbf{Z}^\infty \;=\; \mathbf{Z}_2 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_5 \oplus \cdots \oplus \mathbf{Z}_p \oplus \cdots,$$

where $\mathbf{Z}_p = \mathbf{Z}$. Then $\mathbf{Z}^\infty$ has a natural order, and its dual group is identified with $\mathbf{T}^\omega$. Indeed, let τ be the isomorphism of $\mathbf{Z}^\infty$ into $\mathbf{R}$ by

$$\tau(\{n_p\}) \;=\; \sum_{p\,;\,prime} n_p \log p, \quad \{n_p\} \in \mathbf{Z}^\infty.$$

We see that $\tau(\mathbf{Z}^\infty)$ is the dense subgroup $\Gamma = \{\log r\,; r \text{ is positive rational}\}$ of $\mathbf{R}$, and that the dual group K of Γ is isomorphic to $\mathbf{T}^\omega$ via the map τ^* defined by

$$\langle \tau^*(x), \{n_p\}\rangle \;=\; \langle x, \tau(\{n_p\})\rangle.$$

We also let

$$\mathbf{T}^\omega = \mathbf{T}_2 \otimes \mathbf{T}_3 \otimes \mathbf{T}_5 \otimes \cdots \otimes \mathbf{T}_p \otimes \cdots,$$

where $\mathbf{T}_p = \mathbf{T}$. Since $e_t(\log p) = e^{it\log p}$, the one-parameter group $\{T_t\}_{t\in\mathbf{R}}$ of homeomorphisms on $\mathbf{T}^\omega$ is defined by

$$T_t(\{e^{i\theta_p}\}) = \{e^{i(\theta_p + t\log p)}\}, \quad \{e^{i\theta_p}\} \in \mathbf{T}^\omega.$$

We note that the normalized Haar measure σ_P on $\mathbf{T}^\omega$ is given by

$$d\sigma_P = \prod_{p;\ prime} \frac{1}{2\pi}\, d\theta_p\,.$$

We identify K with $\mathbf{T}^\omega$ as usual.

Analyticity on compact groups with ordered duals is developed in [**13**, Chapter VII], [**22**], and [**44**, Chapter 8].

5. Extension of the Riemann zeta-function and Dirichlet series with Euler products

By a *Dirichlet series*, we mean a series of the form

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad s = \sigma + it, \tag{5.1}$$

where $\{a_n\}$ is a sequence in $\mathbf{C}$. Then the series converges on a right half-plane $\sigma > \sigma_0$, on which the sum $f(s)$ of the series is an analytic function. The most important Dirichlet series is the *Riemann zeta-function*, which is the case when $a_n = 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p;\ prime} (1 - \frac{1}{p^s})^{-1}, \quad \sigma > 1\,.$$

This infinite product is called the *Euler product*. The function $\zeta(s)$ can be analytically continued over $\mathbf{C}$ except when $s = 1$, where there is a simple pole with residue 1. The *Riemann Hypothesis* amounts to the assertion that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1/2$. If we assume that $\{a_n\}$ is bounded and $a_{mn} = a_m a_n$ in (5.1), then $f(s)$ is also representable as an Euler product:

$$f(s) = \prod_{p;\ prime} (1 - \frac{a_p}{p^s})^{-1}, \quad \sigma > 1. \tag{5.2}$$

In particular, when $|a_n| = 1$, (5.2) is deeply connected with $\zeta(s)$. Indeed, such an $f(s)$ may be obtained as some kind of limit of translations of $\zeta(s)$ in a vague sense.

Using the flow $(\mathbf{T}^\omega, \{T_t\}_{t\in\mathbf{R}})$ in the above example, we first extend $\zeta(s)$ to an analytic function on $\mathbf{T}^\omega$. Let $u > 1/2$. Since

$$\zeta(u + it) = \sum_{n=1}^{\infty} \frac{1}{n^u} e^{-it\log n},$$

$t \to \zeta(u - it)$ can be considered as an analytic almost periodic function on $\mathbf{R}$. So this may be extended to an analytic function on $\mathbf{T}^\omega$ by

$$Z_u(x) = \sum_{n=1}^{\infty} \frac{1}{n^u} \chi_{\log n}(x), \quad x \in \mathbf{T}^\omega. \tag{5.3}$$

Since

$$\|Z_u\|_2^2 = \sum_{n=1}^{\infty} (\frac{1}{n^u})^2 < \infty,$$

$Z_u(x)$ lies in the Hardy space $H^2(\sigma_P)$ defined in Section 4. Restricting $Z_u(x)$ to the orbit $\mathcal{O}(0) = \{e_t\,; t \in \mathbf{R}\}$ and extending it to $\{\operatorname{Im} z > 1/2\}$, we have $\zeta(s)$, $\sigma > 1$. To state things more precisely, we denote by $Z_u(x+z) = Z_u(x+e_z)$ the analytic extension of $t \to Z_u(x+e_t)$ to $\mathbf{R}^2_+$. Then we obtain

$$\begin{aligned} Z_u(e_z) &= \sum_{n=1}^{\infty} \frac{1}{n^u} \chi_{\log n}(e_z) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^u} e^{i(\log n)z} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^u} \frac{1}{n^{-iz}}. \end{aligned}$$

If we put $z = is$ (the $90°$ rotation of the right half-plane), then $Z_u(e_z)$ represents $\zeta(s+u)$. Thus $Z_u(x)$ is regarded as an extension of $\zeta(s)$ to $\mathbf{T}^\omega$, and each $z \to Z_u(x+z), \operatorname{Im} z > 0$, is analogous to $\zeta(s)$ by its almost periodicity. In particular, if $\operatorname{Im} z > 1/2$, then $Z_u(x+z)$ is continuous on $\mathbf{T}^\omega$ as a function of x; so $\zeta(s)$ is determined completely by $z \to Z_u(x+z)$ (see Figure 2).

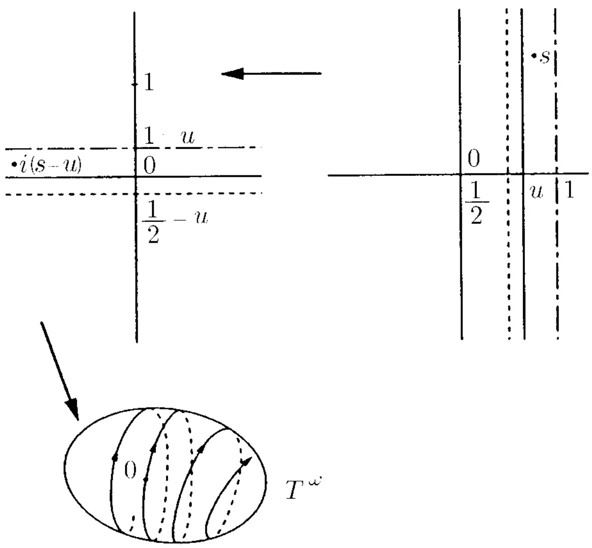

FIGURE 2

LEMMA 5.1. *Let $u > 1/2$, and let $Z_u(x)$ be the function in* (5.3). *Then both $Z_u(x)$ and $Z_u(x)^{-1}$ are outer functions in $H^2(\sigma_P)$.*

Outline of Proof. Recall that the Möbius function $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n=1, \\ (-1)^n, & \text{if } n \text{ is the product of } k \text{ different primes,} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \sigma > 1. \tag{5.4}$$

If we put

$$Z_u(x)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^u} \chi_{\log n}(x), \quad x \in \mathbf{T}^\omega,$$

then $Z_u(x)^{-1}$ lies in $H^2(\sigma_P)$ and $Z_u(x)\,Z_u(x)^{-1} = 1$. From these facts, it follows that both subspaces $H^\infty(\sigma_P)\,Z_u(x)$ and $H^\infty(\sigma_P)\,Z_u(x)^{-1}$ are dense in $H^2(\sigma_P)$. □

By Lemmas 4.1 and 5.1, $z \to Z_u(x+z)$ is an outer function in $H^2(dt/\pi(1+t^2))$ for a.e. x in $\mathbf{T}^\omega$; so it has no zeros. The restriction of $Z_u(x+z)$ to the orbit $\mathcal{O}(0)$ of 0 represents $\zeta(s)$ as mentioned above. Since $\zeta(s)$ has the pole $s = 1$, $\mathcal{O}(0)$ is contained wholly in the exceptional null set in Lemma 4.1.

Let us look into a property of $Z_u(x+z)$ precisely. Since

$$\begin{aligned} Z_u(x+z) &= \sum_{n=1}^{\infty} \frac{1}{n^u} \cdot \chi_{\log n}(x+e_z) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^u} \cdot \chi_{\log n}(x)\, e^{i(\log n)z} \\ &= \sum_{n=1}^{\infty} \frac{a_n}{n^u} \frac{1}{n^{-iz}}, \end{aligned}$$

where we put $a_n = \chi_{\log n}(x)$, it follows that the sequence $\{a_n\}$ satisfies $|a_n| = 1$ and $a_m \cdot a_n = a_{mn}$ by the definition of characters. If $n = p_1^{b_1} \cdot p_2^{b_2} \cdots p_l^{b_l}$ is the factorization of n in prime numbers, then we obtain

$$a_n = a_{p_1}^{b_1} \cdot a_{p_2}^{b_2} \cdots a_{p_l}^{b_l}. \tag{5.5}$$

Conversely, each $x = \{a_p\}$ in $\mathbf{T}^\omega$ decides $\chi_{\log n}(x)$ by (5.5). Thus we have the following.

THEOREM 5.2. *Define a_n by (5.5) with a given $\{a_p\}$ in $\mathbf{T}^\omega$, and let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p;\ prime} \left(1 - \frac{a_p}{p^s}\right)^{-1}, \quad s = \sigma + it, \tag{5.6}$$

which is absolutely convergent on $\sigma > 1$. Then, for a.e. $\{a_p\}$ in $\mathbf{T}^\omega$, $f(s)$ extends analytically to $\sigma > 1/2$ and has no zeros.

In a similar way, we see that $Z_u(x)^{-1}$ has the same property. However, when $\{a_p\}$ in $\mathbf{T}^\omega$ is given concretely, we have no idea for deciding whether $f(s)$ in (5.6) satisfies the conclusion of Theorem 5.2.

Let $x_0 = \{-1\}$ in $\mathbf{T}^\omega$. Since $\chi_{\log p}(x_0) = -1$, we see that

$$\frac{\zeta(s)}{\zeta(2s)} = \prod_{p:\ prime} (1 - \frac{\chi_{\log p}(x_0)}{p^s}), \quad s = \sigma + it.$$

This implies that

$$Z_u(x)\, Z_{2u}(2x)^{-1} = Z_u(x + x_0)^{-1}.$$

Since $Z_{2u}(2x)^{-1}$ is represented as an absolutely convergent Fourier series, $Z_{2u}(2x)^{-1}$ is a continuous outer function on $\mathbf{T}^\omega$. So $Z_u(x)$ acts as if it were self-reciprocal by the translation $x \to x + x_0$. The set of zeros of $Z_u(x+z)$ corresponds to the set of poles of $Z_u(x + x_0 + z)$, although both are empty for a.e. x, by Theorem 5.2. However, using a property of alternating series, we can easily find Dirichlet series of the forms (5.6) that have zeros, poles, or both in $1/2 < \sigma \leq 1$.

The Rademacher functions seem to be useful. Indeed, in a way similar to [**8**, Appendix A], it can be shown that, for a.e. $\{a_p\}$ in $\mathbf{T}^\omega$, the Dirichlet series $f(s)$ of the form (5.6) has the property that the limit

$$\lim_{\sigma \to \frac{1}{2}} f(\sigma + it)$$

does not exist for dt-a.e. t. Thus, unlike the case of $\zeta(s)$, such Dirichlet series cannot satisfy any kind of functional equation and cannot even extend analytically across the line $\sigma = 1/2$. Moreover, a property of the Rademacher functions enables us to show that both $Z_u(x)$ and $Z_u(x)^{-1}$ lie in $H^p(\sigma_P), 1 \leq p < \infty$ (see [**20**] for another proof).

There is a relation between the abscissa of convergence for Dirichlet series and Hardy spaces on $\mathbf{R}$. Let $f(s)$ be a Dirichlet series of the form (5.1). It follows from [**50**, 9.14] that if the function

$$F(\lambda) = \sum_{\log n \leq \lambda} a_n$$

satisfies $F(\log n) = O(n^u)$, then $f(s)$ converges for $\sigma > u$.

THEOREM 5.3. *Let $f(s)$ be a Dirichlet series of the form* (5.1). *Suppose that $f(s)$ extends analytically to $\sigma > \sigma_0\,(\geq 0)$ and*

$$t \to f(\sigma_0 - it) \in H^r(dt/\pi(1+t^2))$$

for $r > 2$. Then $f(s)$ converges for $\sigma > \sigma_0 + 1/r$.

Outline of Proof. For each q with $2 \leq q < r$, we choose p such that $1/q + 1/p = 1$. Since

$$|\sigma + it|^p = |\sigma + it|^{\frac{2p}{r}} \cdot |\sigma + it|^{p(1-\frac{2}{r})},$$

we obtain by Hölder's inequality that

$$\int_{-\infty}^{\infty} \left|\frac{f(\sigma+it)}{\sigma+it}\right|^p dt \leq \left\{\int_{-\infty}^{\infty} \frac{|f(\sigma+it)|^r}{|\sigma+it|^2}\, dt\right\}^{p/r} \left\{\int_{-\infty}^{\infty} |\sigma+it|^{\frac{-p(r-2)}{r-p}}\, dt\right\}^{\frac{r-p}{r}}.$$

Since $r > q = p/(p-1)$, we see that $p(r-2)/(r-p) > 1$. This assures that the last integral converges. On the other hand, we may see that the Fourier transform of $\lambda \to e^{-\lambda\sigma} F(\lambda)$ is

$$\frac{f(\sigma+it)}{\sigma+it}.$$

Then it follows from the Young-Hausdorff theorem that

$$\int_{-\infty}^{\infty} e^{-\lambda\sigma q} |F(\lambda)|^q \, d\lambda \; < \; \infty.$$

This implies that

$$\sum_{n=1}^{\infty} e^{-\sigma q \log(n+1)} |F(\log n)|^q \{\log(n+1) - \log n\} \; < \; \infty .$$

Since the terms of this series tend to 0, we have

$$|F(\log n)| \; = \; O(n^{\sigma+1/q});$$

so $f(s)$ has the desired property. □

As we mentioned above, $Z_u(x)$ lies in $H^p(\sigma_P), 1 \le p < \infty$. Then $t \to Z_u(x+t)$ lies in $H^p(dt/\pi(1+t^2))$. We thus obtain the following.

PROPOSITION 5.4. *Under the hypotheses of Theorem 5.2, the Dirichlet series of the form* (5.6) *converges on* $\sigma > 1/2$, *for a.e.* $\{a_p\}$ *in* $\mathbf{T}^\omega$.

We know little about the relation between the exceptional null sets in Theorem 5.2 and Proposition 5.4. Littlewood's theorem [**51**, 12.25] states that the convergence of (5.4) is equivalent to the Riemann Hypothesis. Regarding this, it would be interesting to investigate such null sets in further detail.

Finally, we would like to make a remark on mean values of negative powers of $\zeta(s)$. Let $0 < p < \infty$ and let $u > 1/2$. Since $Z_u(x)^{-1}$ is outer by Proposition 5.1, we see easily that $Z_u(x)^{-p}$ is defined and lies in $H^1(\sigma_P)$. It then follows from the individual ergodic theorem that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |Z_u(x+t)|^{-p} \, dt \; = \; \int_{\mathbf{T}^\omega} |Z_u(x)|^{-p} \, d\sigma_P(x)$$

holds for a.e. x in $\mathbf{T}^\omega$. Although we do not know whether this holds on the orbit $\mathcal{O}(0)$, it is possible to show a weak version with the aid of a normal family argument and Rouché's theorem. A subset J of the nonnegative integers $\mathbf{Z}^+$ has *density zero* if

$$\frac{1}{N} (J \cap \{0, 1, 2, \cdots, N-1\}) \to 0 \quad (\text{as } N \to \infty).$$

We fix an $\ell > 0$. Then there is a subset J of $\mathbf{Z}^+$ of density zero such that

$$\lim_{N\to\infty} \frac{1}{N\ell} \sum_{\substack{n=0 \\ J \not\ni n}}^{N-1} \int_{n\ell}^{(n+1)\ell} |\zeta(u+it)|^{-p} \, dt \; = \; \int_{\mathbf{T}^\omega} |Z_u(x)|^{-p} \, d\sigma_P(x).$$

Of course, we can derive the counterpart for the case of positive powers of $\zeta(s)$. For the proof, see our subsequent note, *Dirichlet series induced by the Riemann zeta-function.*

In [**1**], [**3**], and [**41**], the Riemann zeta-function is discussed in connection with functional analysis. In Helson's series of notes [**18**], [**19**], [**20**], [**21**], one can find a modern treatment of Dirichlet series by the methods of functional analysis.

The author would like to express his sincere gratitude to Professor Koji Matsumoto for his useful comments. He was kind enough to inform us that methods similar to those of Section 5 are used in the value distribution theory of the Riemann zeta-function, and also to introduce Laurinčikas' book [**33**] as a good reference source in the field. He also gave another proof of Theorem 5.2. Thanks are due to the referee as well, for valuable suggestions that improved the first version of this paper.

References

[1] H. Bercovici and C. Foias, *A real variable restatement of Riemann's hypothesis,* Israel J. Math. **48** (1984), 57-68.
[2] A. Beurling, *On two problems concerning linear transformations in Hilbert space,* Acta Math. **81** (1949), 239-255.
[3] A. Beurling, *A closure problem related to the Riemann Zeta-function,* Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 312-314.
[4] L. Carleson, *An interpolation problem for bounded analytic functions,* Amer. J. Math. **80** (1958), 921-930.
[5] L. Carleson, *Interpolations by bounded analytic functions and the corona problem,* Ann. of Math. **76** (1962), 547-559.
[6] M. Denker and E. Eberlein, *Ergodic flows are strictly ergodic,* Advances in Math. **13** (1974), 437-473.
[7] N. Dunford and J. Schwartz, *Linear Operators, Part I,* Wiley Interscience, New York, 1958.
[8] P. Duren, *Theory of H^p Spaces,* Academic Press, New York and London, 1970.
[9] F. Forelli, *Analytic and quasi-invariant measures,* Acta Math. **118** (1967), 33-59.
[10] F. Forelli, *Fourier theory and flows,* Actes Congrès Internat. Math. (Nice, 1970), vol. 2, Gauthier-Villars, Paris, 1971, pp. 475-476.
[11] F. Forelli, *A maximal algebra,* Math. Scand. **30** (1970), 152-158.
[12] T. Gamelin, *H^p spaces and extremal functions in H^1*, Trans. Amer. Math. Soc. **124** (1966), 158-157.
[13] T. Gamelin, *Uniform Algebras,* Prentice-Hall, Englewood Cliffs, N.J., 1969.
[14] J. Garnett, *Bounded Analytic Functions,* Academic Press, New York, 1981.
[15] P. Halmos, *Lectures on Ergodic Theory,* Mathematical Society of Japan, Tokyo, 1956.
[16] M. Hasumi, *On shift-invariant subspaces,* Sūgaku **17** (1966), 22-32 (in Japanese).
[17] M. Hasumi, *Theory of invariant subspaces,* Sūgaku **28** (1976), 47-59 (in Japanese).
[18] H. Helson, *Convergent Dirichlet series,* Ark. Mat. **4** (1962), 501-510.
[19] H. Helson, *Foundations of the theory of Dirichlet series,* Acta Math. **118** (1967), 61-77.
[20] H. Helson, *Compact groups and Dirichlet series,* Ark. Mat. **8** (1969), 139-143.
[21] H. Helson, *Convergence of Dirichlet series,* Lecture Notes in Mathematics, 336, Springer-Verlag, Berlin, 1973, pp. 153-160.
[22] H. Helson, *Analyticity on compact abelian groups,* in *Algebras in Analysis,* Academic Press, London, 1975, pp. 1-62.
[23] H. Helson, *Compact groups with ordered duals VII,* J. London Math. Soc. (2) **20** (1979), 509-515.
[24] H. Helson and J. Tanaka, *Singular cocycles and the generator problem,* Operator Theoretical Methods (Timişoara, 1998; A. Gheondea et al., eds.), Theta Foundation, Bucharest, 2000, pp. 173-186.
[25] K. Hoffman, *Banach Spaces of Analytic Functions,* Prentice-Hall, Englewood Cliffs, N.J., 1962.
[26] K. Hoffman, *Bounded analytic functions and Gleason parts,* Ann. of Math. **86** (1967), 74-111.
[27] K. Izuchi, *Zero sets of interpolating Blaschke products,* Pacific J. of Math. **119** (1985), 337-342.
[28] K. Izuchi, *QC-level sets and questions of Douglas algebras.* J. Funct. Anal. **65** (1986), 293-308.
[29] K. Izuchi, *Interpolating sequences in a homeomorphic part of H^∞*, Proc. Amer. Math. Soc. **111** (1991), 1057-1065.
[30] K. Izuchi, *Spreading Blaschke products and homeomorhic parts,* in Harmonic/Analytic Function Spaces and Linear Operators, Surikaisekikenkyusho-Kokyuroku No. 1049 (1998), 157-161 (in Japanese). MR **99i:**00018
[31] T. Kano *et al, Riemann Hypothesis,* Nihon-Hyōronsya, 1991 (in Japanese).
[32] N. Kurokawa,*A quarter of a millennium since Euler products*, Gendai-Sūgaku no Ayumi 4, 1992, 13-27 (in Japanese).

[33] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function,* Mathematics and its Applications **352** Kluwer Academic Publishers Group, Dordrecht, 1996.
[34] K. Morita, *Theory of Topological Spaces*, Iwanami-Zensyo, Tokyo, 1981 (in Japanese).
[35] P. Muhly, *Function algebras and flows,* Acta Sci. Math. (Szeged) **35** (1973), 111-121.
[36] P. Muhly, *Function algebras and flows II,* Ark. Mat. **11** (1973), 203-213.
[37] P. Muhly, *Function algebras and flows IV,* Trans. Amer. Math. Soc. **203** (1975), 55-66.
[38] P. Muhly, *Isometries of ergodic Hardy spaces,* Israel J. Math. **36** (1980), 50-74.
[39] P. Muhly, *The function-algebraic ramifications of Wiener's work on prediction theory and random analysis,* in Norbert Wiener, *Collected Works with Commentaries,* Vol. III, MIT Press, Cambridge, MA, 1981, pp. 339-370.
[40] T. Nakazi, *Invariant subspaces of weak-* Dirichlet algebras,* Pacific J. Math. **69** (1977), 151-167.
[41] N. Nikolski, *Distance formulae and invariant subspaces, with an application to localization of zeros of the Riemann ζ-function,* Ann. Inst. Fourier, Grenoble **45** (1995), 143-151.
[42] J. Oxtoby, *Ergodic sets,* Bull. Amer. Math. Soc. **58** (1952), 116-136.
[43] W. Parry, *Compact abelian group extensions of discrete dynamical systems,* Z. Wahrscheinlichkeitstheorie Verw. Geb. **13** (1969), 95-113.
[44] W. Rudin, *Fourier Analysis on Groups,* Interscience, New York, 1962.
[45] T. Srinivasan and J. Wang, *Weak-* Dirichlet algebras,* in *Function Algebras,* Scott-Foresman, Chicago, 1966, pp. 216-249.
[46] O. Takenouchi, *On Banach algebras,* Sūgaku **28** (1976), 1-9 (in Japanese).
[47] J. Tanaka, *Corona problem and flows,* J. Funct. Anal. **102** (1991), 360-378.
[48] J. Tanaka, *Flows in fibers,* Trans. Amer. Math. Soc. **343** (1994), 779-804.
[49] J. Tanaka, *Single generator problem,* Trans. Amer. Math. Soc. **348** (1996), 4113-4129.
[50] E.C. Titchmarsh, *The Theory of Functions,* 2nd ed., Oxford University Press, 1939.
[51] E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function* (revised by D. R. Heath-Brown), Oxford University Press, 1988.
[52] J. Tomiyama, *Function algebras and flows,* Sūgaku **28** (1976), 35-46 (in Japanese).
[53] J. Wada, *Normed Rings*, Kyōritsu-Shyuppan, Tokyo, 1969 (in Japanese).
[54] J. Wada, *Some fundamental concepts of function algebras*, Sūgaku **28** (1976), 9-17 (in Japanese).
[55] R. Walker, *The Stone-Čech Compactification,* Springer-Verlag, Berlin, 1974.
[56] P. Walters, *An Introduction to Ergodic Theory,* Springer-Verlag, New York, 1982.
[57] J. Wermer, *On algebras of continuous functions,* Proc. Amer. Math. Soc. **4** (1953), 866-869.
[58] J. Wermer, *Dirichlet algebras,* Duke Math. J. **27** (1960), 373-382.
[59] J. Xia, *Conditional expectations and the corona problem of ergodic Hardy spaces*, J. Funct. Anal. **64** (1985), 251-274.

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Translated by JUN-ICHI TANAKA

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SINGULAR COCYCLES AND THE GENERATOR PROBLEM

HENRY HELSON and JUN-ICHI TANAKA

ABSTRACT. This is another in a series of papers about function theory on compact abelian groups dual to subgroups of the line. A theorem of Tanaka about cocycles of invariant subspaces generated by one element is reproved; it is shown that singular cocycles come from simply generated subspaces; and a new class of cocycles is shown to intersect every cohomology class of cocycles.

KEYWORDS: *Cocycle, invariant subspace.*

MATHEMATICS SUBJECT CLASSIFICATION: Primary 43A17; Secondary 42A50.

1. INTRODUCTION

Let (X, B, τ_t, μ) be a dynamical system, with μ a probability measure invariant under the flow (τ_t). A function f on X (bounded, or belonging to some appropriate function space) is said to be *analytic* if $f(\tau_t x)$ is analytic in t for a.e. (μ) x (that is, it is the boundary function of an analytic function on the upper halfplane). Function theory on the line can largely be extended to this setting (among many papers we cite [1], [7] and [9].) Since the operator T_t that sends $f(x)$ to $f(\tau_t x)$ is unitary in the space $L^2(X, \mu)$, the spectral theorem can be combined with function theory to study the flow ([3]).

Take for X a compact abelian group K whose dual Γ is contained and dense in the real line R. A natural flow is induced in K by the immersion of Γ in $R : \tau_t x = x + e_t$, where e_t is the element of K defined by: $e_t(\lambda) = \exp(\mathrm{i}t\lambda)$ for λ in Γ. The elements (e_t) form a dense subgroup K_0 of K. The study of analyticity on flows began in this context. The central problem became the extension to K of the description of the closed subspaces of $L^2(T)$ that are invariant under multiplication by $\exp \mathrm{i}x$, where T is the circle group. The analogous subspaces on K are in correspondence with certain functions defined on $R \times K$ called cocycles.

The theory to 1975 is expounded in [3], except for the study of Blaschke cocycles in [2]. One of the problems still unsolved is the question whether every simply invariant subspace of $L^2(K)$ is generated by one of its elements (as is the case, trivially, on the circle group). We know that two elements always suffice. Tanaka ([8]) showed that a subspace is simply generated if and only if its cocycle is cohomologous to a Blaschke cocycle whose zeros are all equidistant from the real axis, and distributed in a regular way. This result makes it interesting to know when two Blaschke cocycles are cohomologous, and this appears to be a difficult problem.

Section 3 of this paper presents the techniques to be used afterwards. In Section 4 the main result of [8] is reproved. The basic Lemma 3.2 is applied in Section 5 to show that every cohomology class of cocycles contains one of a new kind. Finally, in Section 6 the structure of singular analytic cocycles is studied. It is proved that a normalized invariant subspace whose cocycle is a singular cocycle without weight at infinity is simply generated, and that any simply generated subspace has cocycle cohomologous to such a cocycle. In other words, the class of singular cocycles without weight at infinity is identical in cohomology with the class of Blaschke cocycles found in [8].

2. BACKGROUND

This section summarizes some of the definitions and results of [3].

K is a compact abelian group dual to Γ, a dense subgroup of R carrying the discrete topology. The character of K associated with an element λ of Γ is called χ_λ, so that $\chi_\lambda(x) = x(\lambda)$. Normalized Haar measure on K is σ. For each real t, e_t is the element of K defined by $e_t(\lambda) = \exp(it\lambda), \lambda$ in Γ. We construct the Lebesgue spaces $L^p(K)$ $(p \geqslant 1)$ by means of σ. A function f of $L^1(K)$ is *analytic* if its Fourier coefficients

$$a_\lambda = \int_K f(x)\overline{\chi_\lambda(x)}\,\mathrm{d}\sigma(x) \tag{2.1}$$

vanish for all negative λ in Γ. The Hardy space $H^p(K)$ is the subspace of $L^p(K)$ consisting of its analytic functions; $H_0^p(K)$ is the space of such functions for which also $a_0 = 0$.

Let ν be the measure $\frac{dt}{t^2+1}$ on the line. We form the spaces L_ν^p of functions on the line with this measure. H_ν^∞ is the space of boundary functions on R of bounded analytic functions on the upper halfplane, and for $p \geqslant 1$, H_ν^p is the closure in L_ν^p of H_ν^∞. A function f of $L^p(K)$ lies in $H^p(K)$ if and only if $f(x + e_t)$ is in H_ν^p as a function of t for a.e. x.

An *invariant subspace* is a closed subspace M of $L^2(K)$ such that $\chi_\lambda \cdot f$ is in M for each f in M and all positive λ in Γ. That is, the family of subspaces $(\chi_\lambda \cdot M = M_\lambda)$ decreases. If the inclusion is strict (for one, and therefore for all such λ), we say that M is *simply invariant.* Then

$$\bigcap M_\lambda = (0), \quad \bigcup M_\lambda \text{ is dense in } L^2(K). \tag{2.2}$$

Denote the orthogonal projection onto M_λ by P_λ (λ in Γ). For λ not in Γ, define P_λ as a limit. The family $(I - P_\lambda)$ determines a spectral measure in $L^2(K)$, so we can define the unitary group

$$V_t = -\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}P_\lambda. \tag{2.3}$$

This group has a particular form. Define the translation operators: $T_t f(x) = f(x + e_t)$ for any function f on K, and real t. If M is $H^2(K)$, then V_t is T_t.

Otherwise there is a function A_t of modulus 1 such that $V_t = A_t T_t$ for each t, and thus $A_t = V_t \mathbb{1}$ (where $\mathbb{1}$ is the constant function). The mapping from R to $L^2(K)$ that carries t to A_t is continuous. Finally, the family (A_t) satisfies the cocycle identity

$$A_{t+u} = A_t T_t A_u \tag{2.4}$$

for all real t and u. (For each t and u the equality holds a.e. on K.) By adjusting null sets, we can find a measurable function $A(t,x)$ on $R \times K$ such that $A_t(x) = A(t,x)$ a.e. for each t, and

$$A(t+u,x) = A(t,x)A(u,x+e_t) \tag{2.5}$$

for all real t, u and all x in K.

Conversely, every cocycle is obtained in this way from a simply invariant subspace M, but the correspondence is not quite one-one. Define

$$M_+ = \bigcap_{\lambda>0} M_\lambda; \quad M_- = \text{ closure of } \bigcup_{\lambda>0} M_\lambda. \tag{2.6}$$

These are invariant subspaces; either they are identical and equal to M, or they are exactly one dimension apart and one of them equals M. For example, $H^2(K)$ and $H^2_0(K)$ are two versions of this sort. If the versions of M are unequal, they lead to the same spectral measure, unitary group, and cocycle. If M is M_+, then M is *normalized.* The correspondence between normalized subspaces and cocycles is one-one.

If M_+ and M_- are not the same, then $M_+ = q \cdot H^2(K)$ and $M_- = q \cdot H^2_0(K)$ for some unitary (that is, unimodular) function q. We call these *Beurling subspaces.* Their cocycle is $\frac{q(x)}{q(x+e_t)}$; this is a *coboundary.* Otherwise M cannot have this form, and the structure of invariant subspaces is more complicated than on the circle.

We say that the simply invariant subspaces M and N are *equivalent* if $M = q \cdot N$ for some unitary function q. Then their cocycles are cohomologous; the cocycle of M equals the cocycle of N multiplied by the coboundary $\frac{q(x)}{q(x+e_t)}$.

The cocycle associated with a simply invariant subspace was defined by means of the spectral theorem. There is another, function-theoretic connection. On the circle, f belongs to the subspace $q \cdot H^2$ if and only if $\overline{q}f$ is in H^2. On K, f belongs to the subspace M_+ whose cocycle is A if and only if $A(t,x)f(x+e_t)$ is in H^2_ν as a function of t for a.e. x. Further, f generates either M_+ or M_- if and only if this product is an outer function of t (relative to the upper halfplane) for a.e. x.

A function f in $L^2(K)$ generates a simply invariant subspace if and only if

$$\int_{-\infty}^{\infty} \log|f(x+e_t)| \,\mathrm{d}\nu(t) > -\infty \tag{2.7}$$

a.e.. (The integral is finite for a.e. x, or else equals $-\infty$ for a.e. x.) This condition is satisfied if $\log|f|$ is summable, but it is weaker. If f belongs to $L^2(K)$ and $\log|f|$ is summable, then the invariant subspace M generated by f is normalized and is a Beurling subspace $q \cdot H^2(K)$. If $\log|f|$ is not summable, but satisfies (2.7), then $M = M_-$. It is known ([6]) for certain f of this kind that $M_+ = M_-$, but not whether this is the case for all such f. It is not known whether $H_0^2(K)$ has a generator.

A cocycle A is *analytic* if for fixed x, $A(t,x)$ is the boundary function of an inner function $A(z,x)$ on the upper halfplane. The cocycle identity for real t extends to

$$A(z+t,x) = A(t,x)A(z,x+e_t). \tag{2.8}$$

The zeros of $A(z,x)$ can be thought of as a subset of $R_+ \times K$. If t and u are real numbers with u positive, $z = t + iu$, we shall write $x + z$ to mean $(u, x + e_t)$ in the product space.

A *Blaschke cocycle* is an analytic cocycle A such that $A(z,x)$ is a Blaschke inner function for a.e. x. A *singular* cocycle is an analytic cocycle that is a singular inner function for a.e. x. Every analytic cocycle is the product of Blaschke and singular cocycles (which may be trivial).

A function $f(t,x)$ of the special form $g(x+e_t)$ will be called *automorphic*. It obviously has the property that $f(t+u,x) = f(t,x+e_u)$ for real t,u. If f is a measurable function on $R \times K$ such that this equality holds a.e. in x for each real t,u, then there is a measurable function g on K, namely $f(0,x)$, such that $f(t,x) = g(x+e_t)$ a.e. in x for each real t.

3. CONSTRUCTION OF COCYCLES

In this section we deal with cocycles in the strict sense; the defining equation (2.5) should hold everywhere on $R \times K$. And we shall also use *additive* cocycles: real functions $v(t,x)$ defined on $R \times K$ such that

$$v(t+u,x) = v(t,x) + v(u,x+e_t) \tag{3.1}$$

almost everywhere in x for each real t,u. The additive cocycle is *strict* if the equality holds for all t,u,x. v is a *coboundary* if it has the form $w(x+e_t) - w(x)$ for a measurable real function w. Obviously $\exp iv$ is a multiplicative cocycle; and if v is an additive coboundary, the exponential is a multiplicative coboundary.

For notational simplicity we assume henceforth that 2π belongs to Γ. Denote by K_1 the subgroup of K consisting of all x such that $x(2\pi) = 1$. The intersection of K_1 and K_0 consists of all the points (e_n). More generally, K_1 intersects each coset $(x+e_t)$ in a discrete set of points. This copy of the integers has no natural origin, however, unless the coset is K_0 itself, because the flow has no measurable cross-section. Each x in K can be written uniquely as $x_1 + e_u$ where x_1 is in K_1 and $-\frac{1}{2} \leqslant u < \frac{1}{2}$. Thus K is represented as $K_1 \times \left[-\frac{1}{2}, \frac{1}{2}\right)$, and this identification is a Borel isomorphism.

Normalized Haar measure on K_1 will be σ_1; then σ is the product of σ_1 and Lebesgue measure on $\left[-\frac{1}{2}, \frac{1}{2}\right)$.

LEMMA 3.1. *If v is an additive cocycle and $v(1,x) = 0$ for all x in K_1, then v is an additive coboundary.*

Define $w(x + e_t) = v(t,x)$ for x in K_1 and $0 \leqslant t < 1$. Then w is defined unambiguously on all of K, and vanishes on K_1. The hypothesis and (3.1) imply that equality holds for all real t, with x in K_1. Now $v(t,x) = w(x + e_t) - w(x)$ for all real t and x in K_1 because $w(x) = 0$; the same is true for all x in K by the cocycle identity.

The same proof shows for multiplicative cocycles that if $A(1,x) = 1$ for x in K_1, then A is a coboundary.

Let α be a measurable real function on K_1, which will be required to satisfy the condition

$$\sum \frac{|\alpha(x + e_n)|}{n^2 + 1} < \infty, \quad \forall x \in K_1. \tag{3.2}$$

The formula

$$v(t,x) = \sum \alpha(x + e_n) \int_{n-t}^{n} k(u)\,\mathrm{d}u, \quad x \in K_1, \tag{3.3}$$

where k is a real function on the line that is not too large, defines a function on $R \times K_1$ that satisfies

$$v(t + 1, x) = v(1,x) + v(t, x + e_1). \tag{3.4}$$

Any such real function can be extended, in a unique way, to an additive cocycle on $R \times K$.

LEMMA 3.2. *Let $k_n = \int_{n-1}^{n} k(u)\,\mathrm{d}u$ with $|k_n| = \mathrm{O}(n^{-3})$ as $n \to \pm\infty$, and let v be given by* (3.3). *If $\sum k_n = 0$, then v is an additive coboundary. If $\sum k_n$ is a multiple of 2π, and α takes integral values, then* $\exp iv$ *is a multiplicative coboundary.*

Assume first that $\sum k_n = 0$. In (3.3) take $t = 1$:

$$v(1,x) = \sum \alpha(x + e_n)k_n. \tag{3.5}$$

Set $s_n = \sum_{-\infty}^{n} k_j$. Then (3.5) can be written

$$v(1,x) = \sum [\alpha(x + e_n) - \alpha(x + e_{n+1})]s_n. \tag{3.6}$$

This rearrangement is valid because $s_n = \mathrm{O}(n^{-2})$ as n tends to $-\infty$; and since $\sum k_n = 0$, the same holds as n tends to ∞. Let

$$w(x) = \sum \alpha(x + e_n)s_n, \quad x \in K_1. \tag{3.7}$$

Then $v(1,x) = w(x) - w(x + e_1)$ for x in K_1. Extend w to all of K by setting $w(x+e_t) = w(x)$ for x in K_1 and $0 \leqslant t < 1$. Then $w(x) - w(x+e_t)$ is a coboundary that matches v on K_1 for $t = 1$, so by Lemma 3.1 v is a coboundary.

Under the second hypotheses, the sum (3.7) need not converge because s_n is not small for positive n. But if α takes integral values and s_n tends to a multiple of 2π, then the sum converges modulo 2π, so $\exp iw$ is defined, and $\exp iv$ is a coboundary.

A Blaschke cocycle A will be called *special* if for x in K_1 its zeros all lie in the set $K_1 + \mathrm{i}$. Then the cocycle can be written for x in K_1

$$A(t,x) = \prod \left[\frac{n+\mathrm{i}-t}{n-\mathrm{i}-t} \cdot \frac{n-\mathrm{i}}{n+\mathrm{i}}\right]^{\alpha(x+e_n)} \tag{3.8}$$

where α is a non-negative integer-valued function defined on K_1 that satisfies the Blaschke condition (3.2). The cocycle equation determines $A(t,x)$ for all x in K.

If α is summable on K_1 (a stronger condition than (3.2)) then A is a coboundary. To prove this, define

$$u(x+e_t) = \prod \left[\frac{(t-n)^2}{1+(t-n)^2}\right]^{\alpha(x+e_n)}, \quad x \in K_1. \tag{3.9}$$

(We check that the right side is automorphic, so that u is a positive function on K.) Now $\log u$ is summable; hence u generates a Beurling subspace. Also $u(x+e_t)A(t,x)$ is an outer function of t for each x, so A is the cocycle of this subspace. This means that A is a coboundary.

The cocycle (3.8) can be written as the exponential of an additive cocycle: $\exp iv$, where (after some calculation)

$$v(t,x) = 2\sum \alpha(x+e_n) \int_{n-t}^{n} \frac{\mathrm{d}s}{s^2+1}, \quad x \in K_1. \tag{3.10}$$

If α is not integer-valued, (3.10) can serve to define A. Generally, if k is any real function on R that is not too large, and α merely satisfies (3.2), the same formula

$$w(t,x) = 2\sum \alpha(x+e_n) \int_{n-t}^{n} k(s)\,\mathrm{d}s, \quad x \in K_1 \tag{3.11}$$

defines a function on $R \times K_1$ that can be completed to be an additive cocycle.

Lemma 3.2 enables us to replace the function under the integral in (3.10) by a more convenient one k. Let $k_n = \frac{1}{n^2+1}$ for each n except 0; and $k_0 = 1-\gamma$, where γ is chosen so that

$$\sum k_n = \pi. \tag{3.12}$$

Set k equal to k_n on each interval $(n-1,n)$. By Lemma 3.2, the additive cocycle w determined by (3.11) is cohomologous to v. This is true even if α takes non-integral values.

4. SIMPLY GENERATED SUBSPACES

This section reproves a main theorem from [8] in a different way.

THEOREM 4.1. ([8]) *A normalized simply invariant subspace M is simply generated if and only if its cocycle A is cohomologous to a special Blaschke cocycle. If a Blaschke cocycle has zeros with bounded imaginary part, then the corresponding normalized subspace is simply generated.*

Let A be any Blaschke cocycle whose zeros have bounded imaginary part, and M the normalized invariant subspace whose cocycle is A. If M is not equal to M_- there is no problem; M is generated by a unitary function and A is a coboundary. We assume that $M_+ = M_-$. Choose a positive number u and define

$$h(t,x) = \frac{A(t+\mathrm{i}u,x)}{A(t,x)}. \tag{4.1}$$

The cocycle identity shows that h is automorphic. If u is large enough, $A(t+\mathrm{i}u,x)$ has no zeros in the upper halfplane of t. It is smooth for real t, and so is not divisible by a singular inner function, unless perhaps by $\exp \mathrm{i}st$ for some positive s. If this were the case, $A(t,x)$ would be divisible by the singular cocycle $\exp \mathrm{i}st$ (constant in x for each t), contrary to the assumption that A is a Blaschke cocycle. Thus $h(x+e_t)A(t,x)$ is outer in t for a.e. x, proving that h generates M.

A singular cocycle of the form $\exp \mathrm{i}st$ is called a *weight at infinity.*

The proof that every simply generated subspace has cocycle cohomologous to a special Blaschke cocycle requires several steps.

Let M be generated by f. After multiplying f by an outer function (which does not change M), we may assume that f is bounded. Then multiplying f by a unitary function makes f positive, and the product generates a subspace equivalent to M. Thus we may assume that f is positive and bounded by 1.

f must satisfy (2.7). If $\log f$ is summable, then M is a Beurling subspace, its cocycle is a coboundary, and there is nothing to prove. We suppose that $\log f$ is not summable, so that $M_+ = M_-$.

First we replace f by an average. Define a negative function on K_1:

$$\tau(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log f(x+e_u)\,\mathrm{d}u \tag{4.2}$$

and extend τ to all of K by setting $\tau(x+e_t) = \tau(x)$ for $-\frac{1}{2} \leqslant t < \frac{1}{2}$. We say that τ is *constant on intervals centered on* K_1. Set $g = \exp \tau$. Then $0 < g < 1$ and g satisfies (2.7) in place of f, so that g generates a simply invariant subspace N.

LEMMA 4.2. *M and N are equivalent.*

With x fixed, the function $\log f(x+e_t)$ has a conjugate function $v(t,x)$, determined up to an additive constant ([5]). It can be defined, and the constant specified, by the formula

$$v(t,x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \log f(x+e_u) \Big[\frac{1}{t-u} + \frac{1}{u}\Big]\,\mathrm{d}u. \tag{4.3}$$

For every t this integral exists as a principal value for a.e. x. It satisfies the functional equation of a *real additive cocycle*: for all t, u we have

$$v(t+u,x) = v(t,x) + v(u,x+e_t) \tag{4.4}$$

a.e.. By Fubini's theorem, for a.e. x, $v(t,x)$ exists for a.e. t, and this function is conjugate to $\log f(x+e_t)$ in t for a.e. x. Thus $f(x+e_t)\exp \mathrm{i}v(t,x)$ is an outer function of t for a.e. x.

Also $f(x+e_t)A(t,x)$ is outer in t; hence for each x, $A(t,x)$ is a constant times $\exp \mathrm{i}v(t,x)$. At $t=0$ the two expressions are both 1 a.e.; hence the exponential is equal to the cocycle A.

Similarly, $\log g(x+e_t)$ has a conjugate additive cocycle $w(t,x)$, and the cocycle of the invariant subspace N generated by g is $\exp \mathrm{i}w$. In order to show that M and N are equivalent, it will suffice to prove that $\exp \mathrm{i}(v-w)$ is a coboundary; and for this, that $v-w$ is an additive coboundary.

Write $u = \log f - \log g$, so that the additive cocycle conjugate to u is $v-w$. Note that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} u(x+e_u)\,\mathrm{d}u = 0 \tag{4.5}$$

for x in K_1. The proof of Lemma 4.2 is completed by

LEMMA 4.3. *Let k be a real function on K that satisfies*

$$\int |k(x+e_t)|\,\mathrm{d}\nu(t) < \infty \tag{4.6}$$

and (4.5). *Then the additive cocycle conjugate to k is an additive coboundary.*

We need to find a real function h on K such that $h(x+e_t)$ is conjugate to $k(x+e_t)$ in t for each x in K, or equivalently in K_1, because then $h(x+e_t)-h(x)$ is the coboundary conjugate to k. If we can show that the improper integral

$$h(t,x) = \pi^{-1} \lim \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{k(x+e_u)}{t-u}\,\mathrm{d}u \tag{4.7}$$

exists for x in K_1 and a.e. t, then it defines an automorphic function h with the required property.

With N and x fixed, the principal value of the integral exists for a.e. t. Write the integral in (4.7) as

$$\sum_{-N}^{N} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{k(x+e_u)}{t-u}\,\mathrm{d}u. \tag{4.8}$$

Perhaps for one value of n, say for n_0, the integral is singular (we suppose that t is not of the form $n+\frac{1}{2}$); that integral exists as a principal value for a.e. t, and the others can be integrated by parts:

$$(4.9)\qquad \left[k^*(u,x)(t-u)^{-1}\right]_{n-\frac{1}{2}}^{n+\frac{1}{2}} + \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{k^*(u,x)}{(t-u)^2}\,\mathrm{d}u$$

where $k^*(u,x)$ is the indefinite integral of $k(x+e_u)$ that vanishes when u is an integer plus $\frac{1}{2}$. (k^* is well defined on account of (4.5).) The bracket vanishes. The modulus of k^* satisfies (2.7) in place of $\log|f|$ (this requires a little thought), so the sum of the integrals in (4.9) (excluding n_0) converges. That is, (4.7) has a limit $h(t,x)$ for a.e. t as N tends to ∞. It is easy to verify that h has the automorphic property $h(t+1,x) = h(t,x+e_1)$ (x in K_1). If we define $h(x+e_u) = h(u,x)$ for x in K_1, then h is defined on K. It is easy to verify that h is conjugate to u as asserted.

Let $\beta(x)$ be $2\pi\left[\frac{\tau(x)}{2\pi}\right]$, where the bracket is the greatest integer function.

LEMMA 4.4. *The invariant subspace Q generated by $\exp\beta$ is equivalent to N generated by $g=\exp\tau$.*

Since $0 \leqslant \tau-\beta < 2\pi$, $h=\exp(\tau-\beta)$ is bounded from 0 and ∞. Write $h = qk$ where q is unitary and k is outer. Then $\mathrm{e}^\tau = \mathrm{e}^\beta qk$. Since $k\cdot H^2 = H^2$, we have $N = q\cdot Q$, as required.

The original generator f has now been replaced by $\exp\beta$, where $\frac{\beta}{2\pi}$ is a negative integer-valued function constant on intervals centered on K_1. We want to show that the cocycle of Q is cohomologous to a special Blaschke cocycle.

The cocycle of Q is $\exp \mathrm{i}\psi(t,x)$ where ψ is the additive cocycle conjugate in t to $\beta(x+e_t)$. That is,

$$(4.10)\qquad \psi(t,x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \beta(x+e_u)\left[\frac{1}{t-u}+\frac{1}{u}\right]\mathrm{d}u,$$

where the principal value of the integral is meant. Since β is constant on intervals centered on K_1, if x is in K_1 this value exists unless t is an integer plus $\frac{1}{2}$. For any a,b not 0 we have the principal value

$$(4.11)\qquad \int_a^b \frac{1}{u}\,\mathrm{d}u = \log\left|\frac{b}{a}\right|.$$

Thus (4.10) can be written (for x in K_1)

$$(4.12)\qquad \begin{aligned}\psi(t,x) &= \frac{1}{\pi}\sum_{-\infty}^{\infty}\beta(x+e_n)\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\left[\frac{1}{u}-\frac{1}{u-t}\right]\mathrm{d}u\\ &= \frac{1}{\pi}\sum_{-\infty}^{\infty}\beta(x+e_n)\log\left|\frac{n+\frac{1}{2}}{n-\frac{1}{2}}\cdot\frac{n-\frac{1}{2}-t}{n+\frac{1}{2}-t}\right|.\end{aligned}$$

This is to be compared with some special Blaschke cocycle. We choose the same function β to determine the multiplicities of the zeros:

$$A(t,x) = \exp \mathrm{i}\varphi(t,x) = \prod \left[\frac{n+\mathrm{i}-t}{n-\mathrm{i}-t} \cdot \frac{n-\mathrm{i}}{n+\mathrm{i}}\right]^{-\frac{\beta(x+e_n)}{2\pi}}, \quad x \in K_1. \tag{4.13}$$

Thus from (3.10)

$$\varphi(t,x) = -(2\pi)^{-1} \sum \beta(x+e_n) 2 \int_{n-t}^{n} \frac{\mathrm{d}s}{s^2+1}, \tag{4.14}$$

so we have for x in K_1

$$\psi(t,x) - \varphi(t,x) = 2\sum \frac{\beta(x+e_n)}{2\pi}\Big(\log\Big|\frac{n+\frac{1}{2}}{n-\frac{1}{2}} \cdot \frac{n-\frac{1}{2}-t}{n+\frac{1}{2}-t}\Big| + \int_{n-t}^{n} \frac{\mathrm{d}s}{s^2+1}\Big). \tag{4.15}$$

This is an additive cocycle, and we shall show that its exponential is a coboundary.

By Lemma 3.1, it is enough to show that for $t = 1$ the cocycle is a coboundary on K_1. A calculation shows that

$$\log\Big|\frac{n+\frac{1}{2}}{n-\frac{1}{2}} \cdot \frac{n-\frac{1}{2}-1}{n+\frac{1}{2}-1}\Big| = -\frac{1}{n^2} + \mathrm{O}(n^{-3}), \tag{4.16}$$

$$\int_{n-1}^{n} \frac{\mathrm{d}s}{s^2+1} = \frac{1}{n^2} + \mathrm{O}(n^{-3}). \tag{4.17}$$

The sum of the quantities in (4.16) is 0, as we see most easily from (4.12); the sum in (4.17) is π. Lemma 3.2, in the second version, shows that $\exp \mathrm{i}(\varphi - \psi)$ is a coboundary.

We have shown that the cocycle of Q is cohomologous to a special Blaschke cocycle; this completes the proof of Theorem 4.1.

5. ANOTHER EQUIVALENCE THEOREM

The formulas (3.8) and (3.10) define a cocycle even if α takes non-integral values. Of course such cocycles need not be analytic.

THEOREM 5.1. *Every cocycle is cohomologous to a cocycle* (3.8) *with* $0 \leqslant \alpha < 1$.

We know ([4]) that every cocycle is cohomologous to

$$\exp \mathrm{i} \int_0^t m(x+e_u)\,\mathrm{d}u \tag{5.1),}$$

where m is a bounded real function, with bound as small as we please, and smooth on cosets of K_0. Set $g(x) = \int_0^1 m(x+e_u)\,\mathrm{d}u$ for x in K_1. If α is a bounded function on K_1 such that $v(1,x)$, given by (3.10), equals g, then by Lemma 3.1 its exponential is cohomologous to (5.1). We would like to show that for every such function m, there is such an α. This may perhaps not be true; but if (3.10) is modified by means of Lemma 3.2, then it is so.

Let $k_n = \frac{1}{n^2+1}$ for $n \neq 0$; and $k_0 = 1 - \gamma$, where γ is the number such that

$$\sum k_n = \sum \frac{1}{n^2+1} - \gamma = \pi. \tag{5.2}$$

By Lemma 3.2, if we take $t = 1$ in (3.10) and replace the integral by the numbers (k_n), the new function $v(1,x)$ is cohomologous to the old one. To prove the theorem we shall show that if g is any bounded function on K_1, there is a bounded function α on K_1 such that

$$g(x) = 2\sum \alpha(x+e_n)k_n. \tag{5.3}$$

Set $h(\mathrm{e}^{\mathrm{i}t}) = \sum k_n \mathrm{e}^{\mathrm{i}nt}$. A calculation verifies the formula

$$\sum \frac{(-1)^n}{n^2+1}\mathrm{e}^{\mathrm{i}nt} = \frac{\pi}{\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}}(\mathrm{e}^t + \mathrm{e}^{-t}), \quad -\pi \leqslant t \leqslant \pi. \tag{5.4}$$

From this formula, with $t = \pi$, we find that $\gamma = \frac{2\pi}{\mathrm{e}^{2\pi}-1}$, just larger than 0.01.

The minimum of h is found by taking $t = 0$ in (5.4) and subtracting γ, which gives the quantity $\frac{2\pi(\mathrm{e}^{\pi}-1)}{\mathrm{e}^{2\pi}-1}$, equal approximately to 0.26. Thus h is a positive periodic function with absolutely convergent Fourier series. By Wiener's theorem, $\frac{1}{h}$ has absolutely convergent Fourier series, whose coefficients we denote by (r_n).

The function α on K_1 has the Fourier series

$$\alpha(x) \sim \sum \alpha_\lambda \chi_\lambda(x). \tag{5.5}$$

(The characters of K_1 are the characters of K restricted to K_1, with identifications modulo 2π. Thus the sum in (5.5) can be thought of as a sum over the elements of Γ satisfying $0 \leqslant \lambda < 2\pi$.) We see that

$$2\sum_n \alpha(x+e_n)k_n = 2\sum_{n,\lambda} \alpha_\lambda \chi_\lambda(x)\mathrm{e}^{\mathrm{i}n\lambda}k_n = 2\sum_\lambda \alpha_\lambda \chi_\lambda(x)h(\mathrm{e}^{\mathrm{i}\lambda}). \tag{5.6}$$

Thus the transform (5.3), which maps the space of bounded functions on K_1 into itself, is inverted by the analogous transform with coefficients $\frac{r_n}{2}$. This shows that the mapping covers the space, so that each function g is the image of some bounded α.

Null sets can be neglected in this conclusion. In (5.1) only functions m that are smooth on cosets of K_0 need be considered, so that g in (5.3) is defined everywhere; and the inverse transform, applied to a given function g, gives α defined everywhere on K_1 and such that (5.3) holds everywhere.

We want to replace α by a non-negative function less than 1. If in (5.3) α is increased by a positive number c, then the image m is increased by $2\pi c$. If $2\pi c$ belongs to Γ, then $\exp 2\pi \mathrm{i}ct$ is a coboundary (constant in x for each t). Thus α can be made to be positive.

Finally, α can be replaced by its fractional part, by Lemma 3.2.

6. SINGULAR COCYCLES

The structure of Blaschke cocycles was described in [2]. For fixed z, the zeros of the Blaschke product $A(z,x)$ form a set E_x in the upper halfplane; the cocycle identity imposes the relation $E_{x+e_t} = E_x + t$. There are two more constraints on these sets: each must satisfy the Blaschke condition; and they must fit together so that A can be a measurable function of (z,x). The main result of [2] was that any family of sets (E_x) satisfying these conditions is the zero set of a Blaschke cocycle.

Let A be a singular cocycle without weight at infinity. For each x in K, $A(z,x)$ is a singular inner function on the upper halfplane. This leads to the formula

$$A(t,x) = \exp \mathrm{i}\psi(t,x), \quad \psi(t,x) = \int_{-\infty}^{\infty} \frac{t}{u(u-t)}\,\mathrm{d}\mu_x(u), \tag{6.1}$$

where each μ_x is a positive singular measure on the line such that $\frac{\mathrm{d}\mu_x(u)}{u^2+1}$ is a finite measure. The cocycle identity means that for each subset E of the line, $\mu_{x+e_t}(E) = \mu_x(E+t)$. For each real t the integral exists as a principal value for a.e. x. Each family of measures with these properties defines a singular cocycle, provided that A is a measurable mapping from t to $L^2(K)$. For this it is necessary and sufficient that $\mu_x(E)$ be a measurable function of x for each subset E of the line.

It is not possible for the μ_x to be finite measures, but they can be pieced together to define a measure on K that characterizes the cocycle and may be finite. For E a Borel subset of K define

$$\mu(E) = \iint_0^1 \mathbb{1}_E(x+e_u)\,\mathrm{d}\mu_x(u)\,\mathrm{d}\sigma_1(x) \tag{6.2}$$

where $\mathbb{1}_E$ is the indicator function of E. Then μ is a σ-finite measure on K, singular with respect to σ. It is easy to verify that μ determines the measures μ_x, which are generated by a singular cocycle if they satisfy the conditions mentioned above.

The singular cocycle is called *special* if each μ_x is a sum of point masses, which for x in K_1 lie in K_1. We have this analogue of Theorem 4.1.

THEOREM 6.1. *A normalized invariant subspace is simply generated if and only if its cocycle is cohomologous to a special singular cocycle. Any singular cocycle without weight at infinity corresponds to a normalized invariant subspace that is simply generated.*

Let A be a singular cocycle without weight at infinity. If A is a coboundary the corresponding normalized invariant subspace trivially is simply generated, so we assume that A is not a coboundary. The singular inner function $A(z,x)$ has no zeros in the upper halfplane. As in the proof above for Blaschke cocycles, $A(t+\mathrm{i},x)$ is an outer function, and $h(t,x) = \frac{A(t+\mathrm{i},x)}{A(t,x)}$ is automorphic. Thus $h(x+e_t)A(t,x)$ is outer in t for each x, so h is a generator for the invariant subspace whose cocycle is A (unique because A is not a coboundary).

(This argument shows that if μ in (6.2) is a finite measure, then the cocycle is a coboundary. For then $\log|h|$ is summable and generates a Beurling subspace.)

We have to show, in the other direction, that any simply generated subspace has cocycle cohomologous to a special singular cocycle. By Theorem 4.1, it is enough to show that every special Blaschke cocycle B is cohomologous to a special singular cocycle A.

A special Blaschke cocycle (3.8) is, for x in K_1,

$$B(t,x) = \exp \mathrm{i}\varphi(t,x), \quad \varphi(t,x) = 2\sum \alpha(x+e_n)\int_{n-t}^{n} \frac{\mathrm{d}s}{s^2+1} \tag{6.3}$$

where α is non-negative, integer-valued, and satisfies the Blaschke condition.

In (6.1), let μ_x carry point masses at points $n+\frac{1}{2}$ (x in K_1), of magnitudes $2\alpha(x+e_n)$. Then (6.1) can be written, for x in K_1,

$$\psi(t,x) = \sum \frac{2t\alpha(x+e_n)}{(n+\frac{1}{2})(n+\frac{1}{2}-t)}. \tag{6.4}$$

Take $t=1$, and set $\beta_n = \frac{1}{n^2-\frac{1}{4}}$. Then $\sum \beta_n = \sum\left(\frac{1}{n-\frac{1}{2}} - \frac{1}{n+\frac{1}{2}}\right) = 0$, and $|\beta_n - n^{-2}| = O(n^{-3})$. Since α takes integer values, Lemma 3.2 applies to show that $\exp \mathrm{i}\varphi$ and $\exp \mathrm{i}\psi$ are cohomologous.

Translating the point masses from $K_1 + e_{\frac{1}{2}}$ to K_1 gives a cocycle cohomologous to A. This completes the proof.

These results enable us to answer a natural question. Let q be an inner function on K; that is, q is a unitary function in $H^\infty(K)$. For each x in K, $q(x+e_t)$ is an inner function on the line, with a factoring $b(t)s(t)$ into Blaschke and singular parts. Can b and s be chosen globally, to be inner functions on K? The answer is that generally they cannot be. Here is how to construct an example.

Let S be any singular cocycle that is not a coboundary. Any cocycle is cohomologous to a Blaschke cocycle ([3]); let $\overline{S}$ be cohomologous to the Blaschke cocycle B. Thus BS is a coboundary $\frac{q(x+e_t)}{q(x)}$. Then q is an inner function, and its Blaschke and singular factors are not inner functions on K. It remains to find a singular cocycle that is not a coboundary.

There is a positive bounded function f that generates an invariant subspace whose cocycle A is not a coboundary ([6]). By the result of this section, A is cohomologous to a singular analytic cocycle S. This S is not a coboundary, and the proof is complete.

REFERENCES

1. F. FORELLI, Analytic and quasi-invariant measures, *Acta Math.* **118**(1967), 33–59.
2. H. HELSON, Structure of Blaschke cocycles, *Studia Math.* **44**(1972), 493–500.
3. H. HELSON, *Analyticity on Compact Abelian Groups*, in Algebras in Analysis, ed. J. H. Williamson, Academic Press, New York 1975, pp. 1–62.
4. H. HELSON, Compact groups with ordered duals. V, *Bull. London Math. Soc.* **8**(1976), 140–144.

5. H. HELSON, Compact groups with ordered duals. VI, *Bull. London Math. Soc.* **8**(1976), 278–281.
6. H. HELSON, Compact groups with ordered duals. VII, *J. London Math. Soc. Ser. (2)* **20**(1979), 509–515.
7. P.S. MUHLY, Function algebras and flows. I, *Acta Sci. Math. (Szeged)* **35**(1973), 111–121.
8. JUN-ICHI TANAKA, Blaschke cocycles and generators, *Pacific J. Math.* **142**(1990), 357–378.
9. JUN-ICHI TANAKA, Single generator problem, *Trans. Amer. Math. Soc.* **348**(1996), 4113–4129.

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論　説

概周期関数の拡張と流れの上の解析性

田　中　純　一

§1. 序

D を複素平面における単位円板 $\{|z|<1\}$ とし，z_0 を単位円周 $\boldsymbol{T}$ 内の一点とする．いま $w=f(z)$ を D 上の解析関数とするとき，$f(z)$ は境界点 z_0 の付近でどんな値を取り，どのような挙動を示すのだろうか．このような問題意識は関数論の歴史のなかでは古く，解析関数の値分布論を起源とし，さらに集積値集合の理論へと発展してきた．

一方 1930 年代に始まる Banach 環論の一分野として，1950 年代の中頃から，解析関数の作る関数族を抽象化した空間を定義し，関数解析的手法を用いてそれらを考察するという関数環の理論が開発されてきた．そして Hardy 空間や不変部分空間の概念がこの枠内に取り込まれていった．

この論説では位相力学系(連続な流れ)から導入される関数環を考えることにより境界点 z_0 で D に内接する円内での $f(z)$ の動きを具体的に捉える方法と，その応用について述べてみたい．もちろん Phragmén-Lindelöf の原理や，流れの上の関数の持つ概周期性から，境界付近での $|f(z)|$ 増大の仕方や境界での動きにある程度の制約を持った関数 f が対象となる．

まず全体の概略を述べよう．簡便のため $f(z)$ を有界とし，また $z_0=1$ とする．上半平面 $\boldsymbol{R}^2_+$ を D に写す一次写像 $z=(w-i)/(w+i)$ を合成し有界な解析関数 $g(w)$ をつくる．このとき $z(\infty)=1$ より $g(w)$ の無限遠点付近の様子を調べたい．まず $\boldsymbol{R}^2_+$ をおのおのの整数 n に対して $[n,n+1]$ を底辺とする短冊に切り分ける，そしてそれらの短冊を幅 1 の間隔で空間 $\boldsymbol{R}^3$ 内に平行に並べる．つまり $\boldsymbol{Z}$ を整数全体とするとき $\boldsymbol{Z}\times[0,1]\times(0,\infty)$ という集合を考える．短冊の一点 $w\in[0,1]\times(0,\infty)$ に対し，$\{g(n+w)\ ;\ n\in\boldsymbol{Z}\}$ は $\boldsymbol{Z}$ 上の有界列となるから $\boldsymbol{Z}$ の Stone-Čech のコンパクト化 $\beta\boldsymbol{Z}$ 上の連続関数となる．そして詳細は§3で述べるが g は $\beta\boldsymbol{Z}\times[0,1]\times(0,\infty)$ という大きい空間上の連続関数へ拡張される．この拡大された部分での値が $g(z)$ の無限遠点近くでの動きを表してくる．一方 $\boldsymbol{Z}\times[0,1]$ で $(n,1)$ と $(n+1,0)$ を同一視すれば実軸 $\boldsymbol{R}$ と位相同形となる．直感的には $\boldsymbol{R}$ をぐるぐると，無限に長い円柱にらせん状に巻き付けたと思え

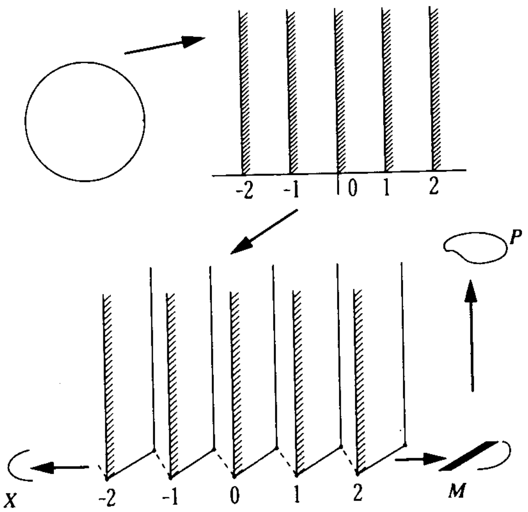

図 1

ばよい．同じ同一視は $\beta\boldsymbol{Z}\times[0,1]$ に対しても考えられ，$\beta\boldsymbol{Z}\times[0,1]$ の商空間 X を得る．商空間 X はらせん状 $\boldsymbol{R}$ のコンパクト化である．これより $\boldsymbol{R}$ での平行移動による流れを $\beta\boldsymbol{Z}\times[0,1]$ の商空間 X へ広げられる．こうして得られた流れを用いてエルゴード理論を利用し，$g(z)$ が拡大された部分での様子を調べようといった試みである．(図1参照.)

次節では古典的な Hardy 空間とそれらの関数環論への一般化について述べ，以後の議論で用いるいくつかの事柄を準備する．

§3では流れから導入される関数環を扱いたい．その中で「極小な流れから導入される関数環は Dirichlet 環となるか？」という問題(1970年 Nice Congress で F.Forelli 氏によって提出)の否定的解答について解説する．

§4では上記の方法を(古典的な)概周期関数に適用するために Bohr 群上の流れとその上の解析関数について概説する．

§5では，§4での準備の下で，一例として Riemann のゼータ関数との関連に触れたい．ゼータ関数に類似した Euler 積を持つ Dirichlet 級数はほとんどの場合帯状領域 $1/2<\mathrm{Re}\,s\leq 1$ で収束して解析的に拡張され零点を持たない，といった結果を示す．ただ私自身この方面には不慣れであり，それらが解析数論などゼータ関数と直接関連する分野にとって有用かどうか分からない．

§2. Hardy 空間の定義といくつかの性質

単位円板 D 上の解析関数は，境界 $\boldsymbol{T}$ 付近での増大度をある程度制限すると，その境界関数の Poisson 積分で再現される．この性質を L^p-ノルムでの評価と考え合わせたものが Hardy 空間である．$1\leq p<\infty$ のとき，Hardy 空間 $H^p(D)$ を D 上の解析的な関数 $f(z)$ で

$$\|f\|_p^p=\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta<\infty$$

をみたすものの全体，また $H^\infty(D)$ を D 上の有界な解析関数の全体と定義する．このとき Fatou の定理より $H^p(D)$ は $\boldsymbol{T}$ 上の境界関数のつくる $L^p(d\theta/2\pi)$ の部分空間 $H^p(d\theta/2\pi)$ と同一視でき，それはさらに $f\in L^p(d\theta/2\pi)$ で負の Fourier 係数を持たない

$$f(\theta)\sim\sum_{n=0}^{\infty}a_n e^{in\theta}$$

という形の Fourier 展開を持つ関数全体の部分空間と一致する．また $\boldsymbol{T}$ 上で連続な解析関数のつくる部分空間を $A(\boldsymbol{T})$ と書く．$A(\boldsymbol{T})$ と $H^\infty(d\theta/2\pi)$ は共に一様ノルムで可換 Banach 環となる．とくに $A(\boldsymbol{T})$ は disc 環と呼ばれる．そして $1\leq p<\infty$ のときは $A(\boldsymbol{T})$ の $L^p(d\theta/2\pi)$ での閉包が $H^p(d\theta/2\pi)$ となり，また $H^\infty(d\theta/2\pi)$ は $A(\boldsymbol{T})$ の $L^\infty(d\theta/2\pi)$ における w^*-閉包となる．したがっておのおのの $H^p(d\theta/2\pi)$ は $H^\infty(d\theta/2\pi)$ に属する関数との積に関して閉じているなど，もとの可換 Banach 環 $H^\infty(d\theta/2\pi)$ に近い性質を持ってくる．

上半平面 $\boldsymbol{R}_+^2$ における Hardy 空間 $H^p(\boldsymbol{R}_+^2)$，$1\leq p<\infty$，は $\boldsymbol{R}_+^2$ 上で解析的な関数で

$$\|f\|_p^p=\sup_{0<y<\infty}\int_{-\infty}^{\infty}|f(x+iy)|^p d\theta<\infty$$

をみたすものの全体とする．さらに $H^\infty(\boldsymbol{R}_+^2)$ で $\boldsymbol{R}_+^2$ 上の有界解析関数全体を表す．そして円周上の場合と同様に $H^p(\boldsymbol{R}_+^2)$ の境界関数全体のつくる $L^p(dt)$ の部分空間を $H^p(dt)$ と書き $H^p(\boldsymbol{R}_+^2)$ と同一視する．$H^p(dt)$ に属する $L^p(dt)$ の関数は $(-\infty,0)$ で0となる Fourier 変換を持つことでも特

徴付けられる．$\boldsymbol{R}_+^2$ を D に写す一次写像 $z=(w-i)/(w+i)$ との合成によって $H^p(D)$ を $\boldsymbol{R}_+^2$ 上の解析関数の族に写す．このとき $t=i(1-e^{i\theta})/(1+e^{i\theta})$ より

$$\frac{1}{2\pi}d\theta=\frac{1}{\pi}\frac{dt}{1+t^2}$$

となり，それらの境界関数のつくる部分空間を $H^p(dt/\pi(1+t^2))$ と記す．$1\leq p<\infty$ のとき $f(t)\in H^p(dt/\pi(1+t^2))$ となる必要十分条件は $(t+i)^{-2/p}\cdot f(t)\in H^p(dt)$ となることが知られている．([25 ; Chapter 3] 参照.)

ここで以後たびたび用いられる $\boldsymbol{R}_+^2$ の Poisson 積分について触れておこう．$\boldsymbol{R}_+^2$ 上の Poisson 核 $P_z(t)$ は

$$P_z(t)=\frac{1}{\pi}\frac{v}{(u-t)^2+v^2},\quad z=u+iv,\quad v>0$$

となる．そして $f(t)\in H^p(dt/\pi(1+t^2))$ と $P_{iv}(t)$ との合成積，

$$f(z)=f*P_{iv}(u)=\int_{-\infty}^{\infty}f(t)P_z(t)\,dt$$

が Poisson 積分となり $f(t)$ の $\boldsymbol{R}_+^2$ への拡張を与える．

Hardy 空間に属する解析関数は，絶対値が境界上で 1 となる内部関数と，境界上での絶対値で形が決まり定義領域内部に零点を持たない外部関数との積で表せる．さらに内部関数は零点だけで定まる Blaschke 積と境界付近で急速に零に収束する部分を持つ特異内部関数と呼ばれる特別な内部関数の積となっている．これらの具体的な形を $H^p(dt/\pi(1+t^2))$ に属する関数について述べておこう．$\boldsymbol{R}_+^2\ni z_n(\neq i)$ となる複素数列が

$$\sum_n\frac{y_n}{1+|z_n|^2}<\infty,\quad z_n=x_n+iy_n$$

をみたすとする．このとき

$$B(z)=\left(\frac{z-i}{z+i}\right)^m\cdot\prod_n\frac{|z_n^2+1|}{z_n^2+1}\cdot\frac{z-z_n}{z-\overline{z_n}}$$

は各 z_n および i で m 次の零点を持つ $\boldsymbol{R}_+^2$ 上の有界解析関数でその境界 $\boldsymbol{R}$ で $|B(t)|=1$ となる．このような内部関数が Blaschke 積である．

次に $d\nu(t)$ を $\boldsymbol{R}$ 上の特異な正値測度で

$$\int_{-\infty}^{\infty}\frac{1}{1+t^2}d\nu(t)<\infty$$

をみたすとし，さらに $\alpha\geq 0$ とする．このとき得られる内部関数，

$$S(z)=\exp\left\{i\left(\alpha t+\int_{-\infty}^{\infty}\frac{1+tz}{t-z}d\nu(t)\right)\right\}$$

が特異内部関数となる．適当な正値関数 $w(t)\in L^p(dt/\pi(1+t^2))$ が

$$\int_{-\infty}^{\infty}\log w(t)\frac{dt}{1+t^2}>-\infty$$

をみたし，ある $\gamma\in\boldsymbol{R}$ に対して

$$h(z)=e^{i\gamma}\cdot\exp\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1+tz}{t-z}\log w(t)\frac{dt}{1+t^2}\right\}$$

と表される関数 $h(z)$ が外部関数である．このとき $|h(t)|=w(t)$ が示される．

次の分解定理は Hardy 空間論では基本的である．([2]，[13]，[14]等参照.)

定理 2.1. f を $H^p(dt/\pi(1+t^2))$ の零でない関数とする．このとき Blaschke 積 $B(z)$，特異内部関数 $S(z)$ および外部関数 $h(z)$ がおのおの一意的に定まり

$$f(z)=B(z)\cdot S(z)\cdot h(z)$$

と分解できる．

A を可換 Banach 環とする．このときその上の(0 でない)複素数値準同形写像の全体 $\mathfrak{M}(A)$ を極大イデアル空間と呼び，A の双対空間 A^* の部分集合として w^*-位相を導入する．とくに A が単位元を持つとき $\mathfrak{M}(A)$ はコンパクト Hausdorff 空間となる．そして $f\in A$ に対し

$$\hat{f}(\xi)=\xi(f),\ \ \xi\in\mathfrak{M}(A)$$

とおき，$\hat{f}$ を $\mathfrak{M}(A)$ 上の連続関数と見なすとき，$\hat{f}$ を f の Gelfand 変換という．したがって可換 Banach 環を扱うときその極大イデアル空間の構造を調べることが重要となる．前述の disc 環 $A(\boldsymbol{T})$ の極大イデアル空間 $\mathfrak{M}(A(\boldsymbol{T}))$ は閉単位円板と同一視されることが簡単に示される．しかし $H^\infty(d\theta/2\pi)$ の極大イデアル空間 $\mathfrak{M}(H^\infty(d\theta/2\pi))$ は極めて複雑で巨大なコンパクト空間になる．いま $z=re^{i\theta}\in D$ を定め，Poisson 核 $P_r(\theta)$ を用い，

$$\xi_z(f)=f(re^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})P_r(\theta-t)\,dt,\ \ f\in H^\infty(d\theta/2\pi)$$

とすると各点 z は $H^\infty(d\theta/2\pi)$ での複素数値準同形写像を定め D が $\mathfrak{M}(H^\infty(d\theta/2\pi))$ に埋め込まれる．そして D が $\mathfrak{M}(H^\infty(d\theta/2\pi))$ の中で稠密となることを主張するのが Carleson のコロナ定理 [5] である．

いま $\alpha\in\boldsymbol{T}$ を定めるとき，

$$M_\alpha=\{\xi\in\mathfrak{M}(H^\infty(d\theta/2\pi))\ ;\ \xi(z)=\alpha\}$$

を α 上のファイバーと呼び，二つのファイバーは互いに位相同形となる．そして D の外側の極大イデアル空間の部分は

$$\mathfrak{M}(H^\infty(d\theta/2\pi))\setminus D=\cup_{|\alpha|=1}M_\alpha$$

と分解される．$f\in H^\infty(d\theta/2\pi)$ に対して f の α における集積値集合 $Cl(f,\alpha)$ を

$$Cl(f,\alpha)=\cap_{r>0}\overline{f(D\cap\Delta(\alpha,r))}$$

と定義する，ここで $\Delta(\alpha,r)$ は α を中心とする半径 r の開円板とする．このとき f の Gelfand 変換 $\hat{f}$ に対して

$$Cl(f,\alpha)=\hat{f}(M_\alpha)$$

となる．

次に Hardy 空間の関数環論への拡張について触れよう．古典的な Hardy 空間の持つ基本的な性質を公理化し，より適用範囲の広い設定の下で理論を展開しようといった試みである．まず Y をコンパクト Hausdorff 空間とし，$C(Y)$ をその上の複素数値連続関数の全体とする．$C(Y)$ の部分環 A が一様ノルム；

$$\|f\|_\infty=\sup\{|f(x)|\ ;\ x\in Y\}$$

について閉じていて，定数関数を含み，Y の点を分離するとき A を Y 上の関数環という．Y の閉集合 E が

$$\|f\|_\infty=\sup\{|f(x)|\ ;\ x\in E\},\ \ f\in A$$

をみたすとき E を関数環 A の境界という．そして境界全体の中に Šilov 境界と呼ばれる最小の境

界の存在が示される．さて A の(0 でない)複素数値準同形を表現する Y 上の確率測度 μ を固定する．このような測度を表現測度という．このとき Hardy 空間 $H^p(\mu)$，$1 \leq p<\infty$，は A の $L^p(\mu)$ でのノルムによる閉包，また $H^\infty(\mu)$ は $L^\infty(\mu)$ における w^*-閉包と定義される．これらの定義が単位円周上の $A(\boldsymbol{T})$ や $H^p(d\theta/2\pi)$ の性質を一般化したものであることは明らかであろう．関数解析的に $H^p(\mu)$ を扱うにはさらに $H^p(\mu)$ のある程度の大きさを保証する条件を仮定するのが通例である．関数環 A の実数部分 $\mathrm{Re}(A)=\{\mathrm{Re}f\,;\, f\in A\}$ が実数値連続関数全体 $C_R(Y)$ の中で稠密となるとき A を Dirichlet 環と呼ぶ．また A^{-1} で A での逆元を持つ要素の集合を表すとき

$$\log|A^{-1}|=\{\log|f|\,;\, f\in A^{-1}\}$$

が $C_R(Y)$ の中で稠密となれば A を logmodular 環という．簡単に分かるように Dirichlet 環なら logmodular 環である．しかし逆は一般には成り立たない．$A(\boldsymbol{T})$ は Dirichlet 環の典型的な例であり，一方 $H^\infty(d\theta/2\pi)$ の関数の Gelfand 変換をその Šilov 境界上に制限して得られる関数環は logmodular 環であるが Dirichlet 環とならない例となる．

さらに条件をゆるめ A を単に $L^\infty(\mu)$ の部分環とし $A+\bar{A}$ が $L^\infty(\mu)$ で w^*-位相で稠密となるとき，A を(表現測度 μ に関して) w^*-Dirichlet 環であるという．これら条件の下では Szegö の定理など多くの結果が成立する．$L^p(\mu)$ の閉部分空間 $\boldsymbol{M}$ が $H^\infty(\mu)\cdot\boldsymbol{M}\subset\boldsymbol{M}$ となるとき不変部分空間と呼ばれる．不変部分空間の研究も w^*-Dirichlet 環の枠内では一般論がかなりできている．より詳しい内容や最近の話題については[16]，[17]，[39]，[43]等が参考になると思う．

A が w^*-Dirichlet 環のとき，$h\in H^p(\mu)$ は Jensen の不等式；

$$\log|\hat{h}(\mu)|\leq\int_Y \log|h(x)|d\mu(x)$$

をみたす．ここで $\hat{h}(\mu)$ は h の μ での積分値を表す．そしてこの不等式が等式となりまた有限な値を取るときに，h を外部関数と定義する．Szegö の定理から h が外部関数となる必要十分条件は h で生成される不変部分空間が $H^p(\mu)$ に一致することが示される．先と同様に $q\in H^p(\mu)$ が $|q|=1$ のとき q を内部関数という．そして $f\in H^p(\mu)$ が $\hat{f}(\mu)\neq 0$ のとき，$f=q\cdot h$ と(定数倍を除いて一意的に)分解される．しかしこれ以上の細かい内容は，たとえば $\hat{f}(\mu)=0$ となるときの分解定理などは，関数環の一般論の枠内ではほとんど手が着けられない．

関数環 A の極大イデアル空間を $\mathfrak{M}(A)$ とする．双対空間 A^* のノルム-位相を $\mathfrak{M}(A)$ に制限し

$$\xi\sim\eta \Longleftrightarrow \|\xi-\eta\|<2$$

という $\mathfrak{M}(A)$ 上の関係を考えると，一つの同値関係となる．この同値関係による同値類 P を Gleason 部分と呼ぶ．そして Wermer[56]等によって A が logmodular 環のとき P が一点でないなら単位円板 D から P 上への連続写像 $\tau(z)$ が存在し，すべての $f\in A$ に対して $\hat{f}(\tau(z))$ は D 上の有界な解析関数となることが示されている．

D 内の点列 $\{z_n\}$ が補間列とは，任意の有界な数列 $\{a_n\}$ に対して $f\in H^\infty(d\theta/2\pi)$ が定まり $f(z_n)=a_n$ となることである．そして Carleson[4]による補間列の特徴付け以後多くの研究が成されてきた．補間列の有用性は，$H^\infty(d\theta/2\pi)$ をその Šilov 境界上の logmodular 環と考えるとき一点でない Gleason 部分に属する複素数値準同形写像はある補間列の集積点となるという Hoffman の定理に代表される．Blaschke 積の零点が補間列となっているとき，それを補間型 Blaschke 積と呼ぶ．この補間型 Blaschke 積の分解定理が Hoffman の定理の証明で本質的な役割を果たしてくる．こ

れらに関連する最近の話題については泉池氏の一連の論文[27]，[28]，[29]，[30]等を参照してほしい．とくに[28]では，ファイバー上のある種の表現測度の存在を連続体仮説の仮定のもとで示している．連続体仮説と集積値集合等の関連を考察する方向はとても興味深い．

§3. 流れの上の解析関数

Ω をコンパクト Hausdorff 空間，また $\{U_t\}_{t\in \boldsymbol{R}}$ を Ω 上の位相同形のつくる 1-助変数変換群とする．$\Omega\times\boldsymbol{R}$ から Ω への写像 $(w,t)\to U_t\omega$ が連続のとき，対 $(\Omega,\{U_t\}_{t\in\boldsymbol{R}})$ を(連続な)流れ，または Ω 上の(連続な) $\boldsymbol{R}$-作用という．簡便のため $U_t\omega$ を $\omega+t$ と略記する．Ω の部分集合 M が $U_t(M)\subset M$，$t\in\boldsymbol{R}$，となるとき M は不変であるという．空でない閉不変集合で包含関係に関し極小のものは極小集合と呼ばれ，Zorn の補題によってその存在が保証される．とくに Ω 自身が極小となる流れ $(\Omega,\{U_t\}_{t\in\boldsymbol{R}})$ を極小な流れという．そして極小な流れとなる必要十分条件はすべての $\omega\in\Omega$ に対しその軌道

$$O(\omega)=\{\omega+t\ ;\ t\in\boldsymbol{R}\}$$

が Ω で稠密となることである．この性質から極小な流れの軌道上に制限して得られる $\boldsymbol{R}$ 上の関数は(広い意味で)概周期性を持つといわれる．

不動点定理と Riesz-角谷の表現定理から閉不変集合上には常に不変な確率測度 μ が存在する．不変な集合 M に対して，$\mu(M)=1$ または 0 となるとき，その確率測度 μ をエルゴード的という．エルゴード的不変確率測度の全体は，不変確率測度全体のつくる w^*-位相でコンパクトな凸集合の端点となるから，Krein-Milman の定理より，エルゴード的な不変確率測度の存在が示される．また極小な流れの上に不変確率測度が唯一存在しているときを強エルゴード的な流れという．

$\phi\in C(\Omega)$ が解析的とは，すべての $\omega\in\Omega$ に対して $t\to\phi(\omega+t)\in H^p(dt/\pi(1+t^2))$ が成立することとする．そして連続な解析関数の全体を $A(\Omega)$ と記す．このとき $(\Omega,\{U_t\}_{t\in\boldsymbol{R}})$ が不動点を持たないなら，$A(\Omega)$ は Ω の点を分離し一つの関数環となる．こうして得られる流れから導入される関数環 $A(\Omega)$ に関しては多くの結果が知られている．基本的ないくつかを紹介しよう．([9]，[11]，[50]等参照．)

先に述べた $A(\boldsymbol{T})$ は $C(\boldsymbol{T})$ での極大環，即ち $A(\boldsymbol{T})$ にその外の連続関数を付け加えて生成される関数環は $C(\boldsymbol{T})$ に一致する，という Wermer の極大定理[55]の一般化として次の定理が示される．

定理 3.1. $(\Omega,\{U_t\}_{t\in\boldsymbol{R}})$ が極小な流れのとき，$A(\Omega)$ は $C(\Omega)$ で極大環となる．

つまり極小な流れの上では $A(\Omega)$ のある程度の大きさが推察される．この結果から§1で述べた Forelli 氏の問題提起[10]へとつながっていった．そしてこの問題に対して，[34]で，Muhly 氏により Dirichlet 環となるための十分条件が与えられた．

定理 3.2. $(\Omega,\{U_t\}_{t\in\boldsymbol{R}})$ が強エルゴード的な流れなら，$A(\Omega)$ は Ω 上の Dirichlet 環となる．

定理 3.3. μ をエルゴード的な不変確率測度とする．このとき μ は $A(\Omega)$ の表現測度となり，$A(\Omega)$ は $L^p(\mu)$ での w^*-Dirichlet 環となる．

これらの結果は重要で，このような仮定の下に $A(\Omega)$ に関数環の一般論が適用でき Hardy 空間や不変部分空間の性質を詳しく論じることが可能となる．しかし極小で強エルゴード的でない流れの場合，導入される関数環が Dirichlet 環かどうかはかなり長い間分からなかった．([35]参照.)

そして[46]でようやく極小な流れから導入される関数環で Dirichlet 環でないものの存在が確認された．

エルゴード理論では，以前より極小で強エルゴード的でない流れについて論議されてきた．(たとえば[41]等参照.)他方関数環論では Dirichlet 環と logmodular 環の差といったことが，たとえば $A(\boldsymbol{T})$ と $H^\infty(d\theta/2\pi)$ の関係のように，たびたび意識されてきた．このように異なる分野の一見独立した事柄が，互いに関連してくるのは面白く，関数論におけるエルゴード理論の有用性が感じられる．

ここで少し $A(\Omega)$ の極大イデアル空間 $\mathfrak{M}(A(\Omega))$ に触れておこう．$\phi\in A(\Omega)$ と Poisson 核 P_{ir}, $r>0$，との合成積，

$$\phi(x,r)=\phi*P_{ir}(x)=\int_{-\infty}^{\infty}\phi(x+t)P_{ir}(t)\,dt,\quad x\in X$$

を考えると $\Omega\times[0,\infty)$ が $\mathfrak{M}(A(\Omega))$ の部分となることが分かる．そして $r\to\infty$ とするときの集積点となる複素数値準同形の表現測度は不変確率測度となる．特に強エルゴード的流れの場合 $\mathfrak{M}(A(\Omega))$ の形は完全に決まり $\Omega\times[0,\infty]$ で $\Omega\times\{\infty\}$ を一点に同一視したコンパクト空間となっている．またさらにコロナ定理との関連等も論じられている．なお，これらの詳細に関しては[35]，[45]，[57]を参照されたい．

以後極小な流れからつくられる関数環で Dirichlet 環でない例の構成を中心に話をすすめたい．

まず§1で概略を述べたコンパクト空間とその上の流れを正確に定義し直し，その中の極小集合を調べよう．S_0 を $\boldsymbol{Z}$ 上のシフト作用素 $S_0n=n+1$，$n\in\boldsymbol{Z}$，とする．S_0 は $\boldsymbol{Z}$ の Stone-Čech のコンパクト化 $\beta\boldsymbol{Z}$ 上の位相同形 S へと拡張できる．X を $\beta\boldsymbol{Z}\times[0,1]$ での各点 $y\in\beta\boldsymbol{Z}$ で $(y,1)$ と $(Sy,0)$ を同一視して得られるコンパクト空間とする．X 上の位相同形のつくる 1-助変数変換群 $\{S_t\}_{t\in\boldsymbol{R}}$ を，Gauss 記号[・]を用いて，

$$S_t(y,s)=(S^{[t+s]}y,\,t+s-[t+s]),\quad(y,s)\in X \tag{3.1}$$

とすると $(X,\{S_t\}_{t\in\boldsymbol{R}})$ は一つの流れとなる．このとき X の中で $\boldsymbol{Z}\times[0,1]$ に対応する部分 X_0 は稠密な不変集合となり $\boldsymbol{R}$ と同一視される．このような流れの構成法は関数の下につくられた流れ (S-flow) といわれるものの簡単な場合である．また Stone-Čech のコンパクト化については[33]，[53]等に詳しい解説がある．

いま $C_{ub}(\boldsymbol{R})$ を $\boldsymbol{R}$ 上の有界な一様連続関数全体のつくる C^*-環とする．このとき次のことが簡単に示される．

命題 3.4．$C_{ub}(\boldsymbol{R})$ と $C(X)$ は(Banach 環として)同形となる．したがって X は $C_{ub}(\boldsymbol{R})$ の極大イデアル空間となる．

ここで流れ $(X,\{S_t\}_{t\in\boldsymbol{R}})$ から導入される関数環 $A(X)$ について考えよう．$f(t)\in H^\infty(dt/\pi(1+t^2))$ を Poisson 積分で $\boldsymbol{R}^2_+$ へ拡張する．そして $r>0$ を固定するとき $t\to f(t+ir)$ は $C_{ub}(\boldsymbol{R})$ に属し X 上の連続な解析関数 $\phi(x,r)$ へ拡張される．一方 X の中には $\boldsymbol{R}$ が稠密に入っているから，$A(X)$ の関数は $\boldsymbol{R}$ に対応する部分の値で決まってしまう．これらより $A(X)$ は関数 $\phi(x,r)$, $r>0$，の全体で生成される関数環となることが分かる．少し定義域を広げて $\phi(x,r)$ を $X\times(0,\infty)$ の関数と見なそう．X の各軌道上に $H^\infty(dt/\pi(1+t^2))$ の関数が広がってくる．Fatou の定理から存在する．これらの関数の境界関数の全体 $H^\infty(X)$ を考えると，これは $H^\infty(dt/\pi(1+t^2))$ と同形となる．

$X\times(0,\infty)$の各点には$H^\infty(X)$の複素数値準同形写像が対応し，また$H^\infty(dt/\pi(1+t^2))$と$H^\infty(d\theta/2\pi)$は同形だから$X\times(0,\infty)$は$H^\infty(d\theta/2\pi)$の極大イデアル空間$\mathfrak{M}(H^\infty(d\theta/2\pi))$の部分に対応する．$X_0\times(0,\infty)$が単位開円板$D$で定まる部分，さらに$(X\backslash X_0)\times(0,\infty)$の$\mathfrak{M}(H^\infty(d\theta/2\pi))$での閉包が内接する円内の集積点として定まる部分となっている．

$A(X)$の極大イデアル空間$\mathfrak{M}(A(X))$は$X\times[0,\infty)$を含む．そしてそれが稠密となっていることがコロナ定理で保証される．このとき$X\times[0,\infty)$の集積点として表現される複素数値準同形写像の表現測度は不変測度となる．この性質からコロナ定理と個別エルゴード定理の関係が期待される．実際，Wiener の Tauber 型定理より，

$$\lim_{r\to\infty}\phi * P_{ir}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\phi(x+t)\,dt$$

という等式が，何れか一方の極限が存在するとき成立し，個別エルゴード定理は右辺の存在を示す．これより少なくともエルゴード的不変確率測度で定まる複素数値準同形は$X_0\times(0,\infty)$の集積点となることが導かれる．エルゴード的不変確率測度は不変確率測度全体のつくるw^*-コンパクトな凸集合の端点となるから，Choquet 理論などが大幅に改善されれば，コロナ定理への別のアプローチが可能かもしれない．

次に$(X,\{S_t\}_{t\in\boldsymbol{R}})$での極小集合$M$を考える．おのおのの$S_t$を$M$に制限して得られる極小な流れ$(M,\{S_t\}_{t\in\boldsymbol{R}})$は最も強い位相を持つ極小の流れとなる．そしてその結果としておびただしい数の確率不変測度がM上に存在してくる．

命題 3.5．このように定義された$(M,\{S_t\}_{t\in\boldsymbol{R}})$は普遍な極小な流れとなる，即ちすべての極小な流れは$(M,\{S_t\}_{t\in\boldsymbol{R}})$の商となる流れ(factor)である．

(証明の概略.)　$(\Omega,\{U_t\}_{t\in\boldsymbol{R}})$を極小な流れとして$\omega\in\Omega$を固定する．$t\to\phi(\omega+t)$，$\phi\in C(\Omega)$の全体の生成する$C^*$-環は$C_{ub}(\boldsymbol{R})$の閉部分環となる．そして命題3.4より$X$から$\Omega$上への連続写像$\tau$で

$$\tau\circ S_t(x)=U_t\circ\tau(x),\quad x\in X$$

となるものが存在する．τをMに制限しても同様の関係は成立してくる．(終わり.)

定理 3.6．$(M,\{S_t\}_{t\in\boldsymbol{R}})$から導入される関数環$A(M)$は Dirichlet 環ではない logmodular 環である．

(証明の概略.)　$H^\infty(dt/\pi(1+t^2))$との関係から$A(X)$が logmodular 環となることが確かめられる．また$A(X)$のMへの制限$A|_M$は$A(M)$の部分環となる．これより$A(M)$もまた logmodular 環となることが分かる．そして$\mathfrak{M}(A(M))$は$M\times[0,\infty)$の$\mathfrak{M}(A(X))$での閉包となることから，Hoffman の定理が適用でき，適当な補間型 Blaschke 積を定め，$M\times[0,\infty)$の集積点の集合$M\times\{\infty\}$中に，ある Gleason 部分Pが存在しDとPが位相同形になっていることがいえる．次に$A(M)$の関数の Gelfand 変換をPの閉包$\bar{P}$に制限して得られる関数環$\hat{A}|_{\bar{P}}$が$H^\infty(d\theta/2\pi)$と同形となることを確かめる．先に述べたように$H^\infty(d\theta/2\pi)$の関数の Gelfand 変換を Šilov 境界に制限して得られる関数環は Dirichlet 環にはならないから$\bar{P}\backslash P$上の零でない実数値測度νで$\hat{A}|_{\bar{P}}$に直交するものが存在する．このとき

$$L(\phi)=\int_{\bar{P}\backslash P}\hat{\phi}(\xi)\,d\nu(\xi),\quad \phi\in C(M) \tag{3.2}$$

は $C(M)$ 上の有界線形汎関数となる，ここで $\hat{\phi}(\xi)$ は ξ の表現測度による ϕ の積分とする．したがって Riesz-角谷の表現定理から M 上の零でない実数値不変測度が定まり，$A(M)$ に直交する．これより $ReA(M)$ は $C_R(M)$ で稠密とはなり得ない．(終わり．)

境界の性質より ν は $\hat{A}|_P$ の Šilov 境界上の測度とできる．Šilov 境界内にはエルゴード的不変確率測度を表現測度とする複素数値準同形写像が稠密にあるから，(3.2)とエルゴード分解との関係を調べると面白いかもしれない．また M や P の位置関係に関しては再度図1で確認してほしい．

§4. コンパクト群上の解析関数

実数 $\boldsymbol{R}$ 上で概周期性を持つ典型的な関数は実数列 $\{\lambda_j\}$ で定まる，

$$f(t)=\sum_j c_j e^{i\lambda_j t},\quad t\in\boldsymbol{R}$$

という形のものであろう．このような関数は右辺の級数の適当な収束条件の下で $\{\lambda_j\}$ から生成される離散群のコンパクトな双対群の上の関数へと拡張される．そして $e^{i\lambda t}$ の拡張された関数(指標)を変換から定まる固有関数と見なして変換を特徴付けたのが von-Neumann の離散スペクトル定理である．([15], [54]等参照．) 以下では $\lambda_j\geq 0$ と制限したとき得られる上半平面 $\boldsymbol{R}^2_+$ における解析関数の拡張について考察する．

Γ を $\boldsymbol{R}$ の稠密な部分群として離散位相を与える．Γ の双対群を K とし σ をその上の正規 Haar 測度とする．$\lambda\in\Gamma$ を K 上の指標(すなわち連続な K から $\boldsymbol{T}$ への準同形写像)と見なすとき，$\chi_\lambda(x)$ と書く．$\boldsymbol{R}$ から Γ に自然に導入される順序を用い，解析関数を次のように定義する．$f\in L^1(\sigma)$ が解析的とは

$$f(x)\sim\sum_{0\leq\lambda\in\Gamma}a_\lambda\chi_\lambda(x)$$

という Fourier 展開を持つこととする．そして $L^p(\sigma)$，$1\leq p\leq\infty$，における解析関数の全体 $H^p(\sigma)$ を K 上の Hardy 空間と呼ぶ．また連続な解析関数からなる関数環を $A(K)$ とおくと，$A(K)$ は K 上の Dirichlet 環となり，さらに σ はその上の表現測度となる．そして $H^p(\sigma)$ は§2で述べた関数環 $A(K)$ と表現測度 σ による Hardy 空間の定義と一致する．

K 上の流れについて考えよう．任意の $t\in\boldsymbol{R}$ に対し $\lambda\to e^{i\lambda t}$ は Γ 上の指標となる．したがって $e_t\in K$ が定まり，$e_t(\lambda)=e^{i\lambda t}$，$\lambda\in\Gamma$，をみたす．このとき $\{e_t\,;\,t\in\boldsymbol{R}\}$ は K における稠密な部分群を形成する．いま K 上の位相同形のつくる1-助変数変換群 $\{T_t\}_{t\in\boldsymbol{R}}$ を

$$T_t x=x+e_t,\quad x\in K$$

とすると流れ $(K,\{T_t\}_{t\in\boldsymbol{R}})$ は σ を唯一の不変確率測度とする強エルゴール的な流れとなる．そしておのおのの軌道には部分群 $\{e_t\,;\,t\in\boldsymbol{R}\}$ の剰余類が対応してくる．簡便のため $2\pi\in\Gamma$ を仮定する，そして $K_{2\pi}$ を $\chi_{2\pi}(x)=1$ となる $x\in K$ の全体のつくる K の閉部分群とする．このとき $x\in K$ は

$$x=y+e_s,\quad (y,s)\in K_{2\pi}\times[0,1)$$

と一意的に書き表され K は $K_{2\pi}\times[0,1)$ と同一視できる．また $Sy=y+e_1$，$y\in K_{2\pi}$ によって与えられる $K_{2\pi}$ 上の位相同形 S を用いると $\{T_t\}_{t\in\boldsymbol{R}}$ は前節(3.1)で定義した1-助変数変換群と同様なものとなる．

この流れを用いると，$f\in L^p(\sigma)$ が解析的となる必要十分条件は，$a.e.\ x\in K$ に対して，$t\to$

$f(x+t)\in H^p(dt/\pi(1+t^2))$ となることが分かる．ここで後に用いる $H^2(\sigma)$ の外部関数の一つの特徴付けを述べよう．

補題 4.1. h が $H^2(\sigma)$ の外部関数となる必要十分条件は，$\hat{h}(\sigma)\neq 0$ となり，また $a.e.\ x\in K$ に対して $t\to h(x+t)$ が $H^2(dt/\pi(1+t^2))$ の外部関数となることである．

(証明の概略.)　$A(K)\cdot h$ が $H^2(\sigma)$ で稠密でないなら，ある定数でない $g\in H^2(\sigma)$ が $A(K)\cdot h$ に直交する．このとき $t\to\overline{g(x+t)}\cdot h(x+t)\in H^2(dt/\pi(1+t^2))$ となり，$t\to h(x+t)$ は $H^2(dt/\pi(1+t^2))$ の外部関数となりえない．

逆に，

$$\log|h*P_{ir}(x)|<\int_{-\infty}^{\infty}\log|h(x+t)|P_{ir}(t)\,dt$$

が正の測度を持つ集合上で成り立つとする．このとき $h*P_{ir}(x)$ は $H^2(\sigma)$ に属するので，Jensen の不等式から，

$$\log\left|\int_K h*P_{ir}(x)\,d\sigma(x)\right|\leq\int_K\log|h*P_{ir}(x)|d\sigma(x)$$

となり，結局

$$\log|\hat{h}(\sigma)|<\int_K\log|h(x)|d\sigma(x)$$

が示され h は $H^2(\sigma)$ の外部関数になり得ない．(終わり.)

任意の $f\in C(K)$ は Stone-Weierstrass の定理から三角多項式の一様極限として表せる．いま $f\sim\sum_n a_n\chi_{\lambda_n}$ を f の Fourier 展開とする．一つの $x\in K$ を定め，$F_x(t)=f(x+e_t)$ とすると $F_x(t)$ は振動数(frequency)が Γ に属する $\boldsymbol{R}$ 上の(Bohr の意味での)概周期関数，

$$F_x(t)=\sum_n a_n\chi_{\lambda_n}(x)\,e^{i\lambda_n t}$$

となる．これらの概周期関数の族 $\{F_x(t)\ ;\ x\in K\}$ は $F_0(t)=\sum_n a_n e^{i\lambda_n t}$ から出発し平行移動を繰り返し，その極限として得られる関数の全体と考えられる．そしてそれらのいずれかを取り K へ拡張したものが f となる．このような促え方は測度零という例外を許せば $L^p(\sigma)$ に属する関数にも可能で，Γ の可算性を仮定し，$L^2(\sigma)$ に適用すれば Besicovitch の意味での概周期関数の K への拡張となる．

このようにして得られた関数環 $A(K)$ や Hardy 空間 $H^p(\sigma)$ は単位円周上の $A(\boldsymbol{T})$ や $H^p(d\theta/2\pi)$ と類似の性質を持つ多変数の関数環の具体例として 1950 年代から詳しく調べられてきた．とりわけ $H^2(\sigma)$ に関する不変部分空間は線形予測理論への応用からコサイクルと呼ばれる $K\times\boldsymbol{R}$ 上の関数と対応させて詳しく研究されてきた．しかし「すべての不変部分空間は単一生成元を持つか？」という単一生成元問題など残されている課題も少なくない．([16], [17], [22], [23], [24], [47] 等参照.)

Γ が $\boldsymbol{R}$ の可算部分群となるとき，K は単位円周 $\boldsymbol{T}$ の可算個の直積 $\boldsymbol{T}^\infty$ の閉部分群となる．次節への準備として，ここで Γ と K のある特殊な具体例を与えておく．

例 4.2. $\boldsymbol{Z}^\infty$ で整数全体のつくる群 $\boldsymbol{Z}$ の可算個の直和，即ち有限個を除いて他は零となる整数列の全体とする．添え字として素数 p を用い，$\boldsymbol{Z}_p=\boldsymbol{Z}$ とし

$$\boldsymbol{Z}^\infty=\boldsymbol{Z}_2\oplus\boldsymbol{Z}_3\oplus\boldsymbol{Z}_5\oplus\cdots\oplus\boldsymbol{Z}_p\oplus\cdots$$

と書くこととする．このとき $\boldsymbol{Z}^\infty$ から $\boldsymbol{R}$ の中への同形写像

$$\tau(\{n_p\})=\sum_{p\,;\,prime} n_p \log p,\quad \{n_p\}\in \boldsymbol{Z}^\infty$$

とすると，$\tau(\boldsymbol{Z}^\infty)$ は $\boldsymbol{R}$ の稠密な部分群 $\Gamma=\{\log r\,;\,r$ は正の有理数$\}$となる．Γ の双対群 K は $\langle\tau^*(x),\{n_p\}\rangle=\langle x,\tau(\{n_p\})\rangle$によって定まる同形写像 τ^* で無限次元トーラス $\boldsymbol{T}^\infty$ と同形となる．先と同様に添え字を素数 p として，$\boldsymbol{T}_p=\boldsymbol{T}$とし

$$\boldsymbol{T}^\infty=\boldsymbol{T}_2\otimes\boldsymbol{T}_3\otimes\boldsymbol{T}_5\otimes\cdots\otimes\boldsymbol{T}_p\otimes\cdots$$

と書く．先の議論から $\boldsymbol{T}^\infty$ 上に定義される流れは，$e_t(\log p)=e^{it\log p}$，となることから

$$T_t(\{e^{i\theta_p}\})=\{e^{i(\theta_p+t\log p)}\},\quad \{e^{i\theta_p}\}\in\boldsymbol{T}^\infty$$

によって与えられる．また $\boldsymbol{T}^\infty$ 上の正規 Haar 測度 σ_P は無限積測度

$$d\sigma_P=\prod_{p\,;\,prime}\frac{1}{2\pi}d\theta_p$$

となる．通常 K と $\boldsymbol{T}^\infty$ を同一視する．

§5. Riemann のゼータ関数の拡張と Euler 積を持つ Dirichlet 級数

$\{a_n\}$を複素数列とするとき，

$$f(s)=\sum_{n=1}^\infty\frac{a_n}{n^s},\quad s=\sigma+it \tag{5.1}$$

を Dirichlet 級数と呼ぶ．そしてこの級数はある右半平面 $\sigma>\sigma_0$ で収束し解析的な関数となる．これらの中で最も重要な例は a_n をすべて 1 として得られる Riemann のゼータ関数，

$$\zeta(s)=\sum_{n=1}^\infty\frac{1}{n^s}=\prod_{p\,;\,prime}\left(1-\frac{1}{p^s}\right)^{-1},\quad \sigma>1$$

であろう．素因数分解の一意性から得られる上式の無限乗積を Euler 積と呼ぶ．$\zeta(s)$は $s=1$ を除いて全平面に解析接続され，$s=1$ で留数 1 の極を持つ．そして帯状領域 $0<\sigma<1$ にある $\zeta(s)$の零点はすべて $\sigma=1/2$ という直線上にあるであろう，というのが有名な Riemann 予想である．さて(5.1)において$\{a_n\}$の有界性と $a_n\cdot a_m=a_{mn}$ を仮定すると，それはまた

$$f(s)=\prod_{p\,;\,prime}\left(1-\frac{a_p}{p^s}\right)^{-1},\quad \sigma>1 \tag{5.2}$$

という Euler 積で表現できる．とくにすべて $|a_n|=1$ となるときは $\zeta(s)$と強い相関を持ち，$\zeta(s)$の平行移動のある種の極限として導かれ，またそのような関数の同様な極限の中に$\zeta(s)$が現れてくる．前節の例 4.2 で扱った $\boldsymbol{T}^\infty$ 上の流れを用い $\zeta(s)$をこの上の関数に拡張してみよう．

まず $u>1/2$ を固定する．このとき

$$\zeta(u+it)=\sum_{n=1}^\infty\frac{1}{n^u}\cdot e^{-it\log n}$$

と書き換えれば $t\to\zeta(u-it)$は $\boldsymbol{R}$ 上の(解析的)概周期関数と見なせ $\boldsymbol{T}^\infty$ 上へ拡張できる．実際，

$$\sum_{n=1}^\infty\left(\frac{1}{n^u}\right)^2<\infty$$

より

$$Z_u(x)=\sum_{n=1}^\infty\frac{1}{n^u}\cdot\chi_{\log n}(x),\quad x\in\boldsymbol{T}^\infty \tag{5.3}$$

とおくと，この関数は前節で定義した $\boldsymbol{T}^\infty$ 上の Hardy 空間 $H^2(\sigma_P)$に属する．そしてこの関数を 0 の軌道$\{e_t\,;\,t\in\boldsymbol{R}\}$に制限し上半平面に拡張して$\{\mathrm{Im}z>1/2\}$での状況を考えると $\zeta(s)$が得られる．

いま

$$Z_u(e_z)=\sum_{n=1}^{\infty}\frac{1}{n^u}\cdot\chi_{\log n}(e_z)$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^u}\cdot e^{i(\log n)z}$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^u}\cdot\frac{1}{n^{-iz}}$$

より $z=is$(右半平面を 90° 回転)とすると $Z_u(e_z)$ は $\zeta(s+u)$ を上半平面で表した形となっている．したがって $Z_u(x)$ は $\zeta(s)$ の $\boldsymbol{T}^\infty$ への拡張となっていて $z\to Z_u(x+z)$, $\mathrm{Im}z>0$, は $\zeta(s)$ の持つ概周期性からそれ自身とかなり類似性の高い関数と見なせる．とくに $\mathrm{Im}z>1/2$ のときは $\boldsymbol{T}^\infty$ 上の連続関数となるから，エルゴード論的には $\zeta(s)$ と同一な関数と考えてよい．(図 2 参照.)

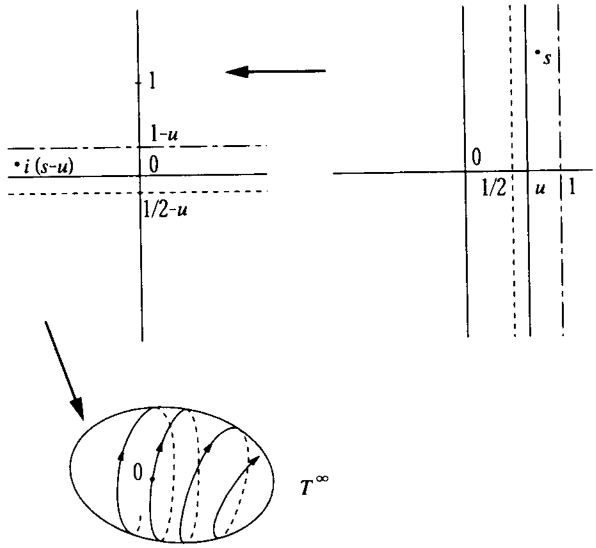

図 2

補題 5.1. いま $u>1/2$ とする．このとき(5.3)による $Z_u(x)$ は $H^2(\sigma_P)$ における外部関数となる．

(証明の概略.) $\sigma>1$ とするとき $\mu(n)$ を Möbius 関数とすれば，

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \tag{5.4}$$

と表せる．ただし Möbius 関数 $\mu(n)$ は次のように定義される：

$$\mu(n)=\begin{cases}(-1)^n & (n \text{ が相異なる } k \text{ 個の素数の積のとき})\\ 1 & (n=1 \text{ のとき})\\ 0 & (\text{その他}).\end{cases}$$

これより

$$Z_u(x)^{-1}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^u}\cdot\chi_{\log n}(x),\quad x\in\boldsymbol{T}^\infty$$

とすると，$Z_u(x)^{-1}\in H^2(\sigma_P)$ となる．そして $Z_u(x)\cdot Z_u(x)^{-1}=1$ が簡単に確かめられ $H^\infty(\sigma_P)\cdot Z_u(x)$ が $H^2(\sigma_P)$ で稠密となる．(終わり.)

この補題 5.1 と補題 4.1 を合わせると，$a.e.\ x\in\boldsymbol{T}^\infty$ に対して $z\to Z_u(x+z)$ は $H^2(dt/\pi(1+t^2))$ の外部関数となり零点を持たない．そして 0 の軌道は上述の様に $\zeta(s)$ を表現するが $s=1$ で極を持つことから $H^2(dt/\pi(1+t^2))$ に属せず補題 4.1 で測度零として除外される部分となる．

ここでもう少し詳しく $Z_u(x)$ を調べてみよう．

$$Z_u(x+z)=\sum_{n=1}^{\infty}\frac{1}{n^u}\cdot\chi_{\log n}(x+e_z)$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^u}\cdot\chi_{\log n}(x)\cdot e^{i(\log n)z}$$
$$=\sum_{n=1}^{\infty}\frac{a_n}{n^{u-iz}}.$$

ただし $a_n=\chi_{\log n}(x)$ とする．このとき指標の性質より数列$\{a_n\}$は $|a_n|=1$ および $a_m\cdot a_n=a_{mn}$ をみたし，いま $n=p_1^{b_1}\cdot p_2^{b_2}\cdots p_l^{b_l}$ を n の素因数分解とすれば，

$$a_n=a_{p_1}^{b_1}\cdot a_{p_2}^{b_2}\cdots a_{p_l}^{b_l} \tag{5.5}$$

によって決まってくる．逆におのおのの $x=\{a_p\}\in \boldsymbol{T}^\infty$ に対して，(5.5)によって $\chi_{\log n}(x)$が定まってくる．以上を Euler 積を持つ Dirichlet 級数に対していい換えると，次の定理を得る．

定理 5.2．$a.e.$ $\{a_p\}\in \boldsymbol{T}^\infty$ に対し a_nを(5.5)で定める．このとき $\sigma>1$ で絶対収束する Dirichlet 級数，

$$f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}=\prod_{p\,:\,prime}\left(1-\frac{a_p}{p^s}\right)^{-1},\quad s=\sigma+it \tag{5.6}$$

は $\sigma>1/2$ まで解析的に拡張され零点を持たない．

証明を少し変更すれば $|a_p|\leq 1$ の場合にも適用できるから，ほぼ同様の主張が $Z_u(x)^{-1}$ についても成り立ってくる．ただ奇妙なのは具体的に$\{a_p\}\in \boldsymbol{T}^\infty$ を定めたとき，(5.6)による f が定理 5.2 の結論をみたすかどうか実際には判定できないことである．

次に $x_0=\{-1\}\in \boldsymbol{T}^\infty$ とおこう．このときよく知られた公式

$$\frac{\zeta(s)}{\zeta(2s)}=\prod_{p\,:\,prime}\left(1-\frac{\chi_{\log p}(x_0)}{p^s}\right),\quad s=\sigma+it$$

から

$$Z_u(x)\cdot Z_{2u}(2x)^{-1}=Z_u(x+x_0)^{-1}$$

が得られる．$Z_{2u}(2x)^{-1}$ は絶対収束する級数で表され，$\boldsymbol{T}^\infty$ 上で連続な外部関数となる．したがって $Z_u(x)$は $x\to x+x_0$ という移動によって自身の逆元に近い性質を持ってくる，つまり $Z_u(x+e_z)$ と $Z_u(x+x_0+e_z)$の間では極は零点に零点は極に移るといった対応がつく．もちろん定理 5.2 からほとんどのxに対しては極も零点も存在してこない．しかし(5.6)で表される Dirichlet 級数の中に有理型関数として解析接続でき $1>\sigma>1/2$ に零点や極を持つ具体例を，交代級数の性質等を用い，つくることができる．

さらに Rademacher 関数の利用も有益かもしれない．たとえば Rademacher 関数の基本的な性質から[8；Appendix A]と類似の方法によって(5.6)による Dirichlet 級数 $f(s)$は $a.e.$ $\{a_p\}\in \boldsymbol{T}^\infty$ に対して

$$\lim_{x\to 1/2} f(\sigma+it)$$

は dt-$a.e.t$ において存在しないことが分かる．したがって一般には関数等式などは持ち得ず，$\sigma=\dfrac{1}{2}$ を超えて解析接続すらできない．

最後に Dirichlet 級数の収束と Hardy 空間の関係について触れておこう．$f(s)$を(5.1)で表示される Dirichlet 級数とする．このとき一般論[48：9.14]より

$$F(\lambda)=\sum_{\log n\leq\lambda}a_n$$

とおいて，$F(\log n)=O(n^u)$をみたせば，$\sigma>u$ という半平面で収束することが示される．

定理 5.3．$f(s)$，$s=\sigma+it$，を(5.1)で表示される Dirichlet 級数とし，$\sigma>\sigma_0(\geq 0)$まで解析的に拡張されるとする．いま $2<r$ に対して，

$$t\to f(\sigma_0-it)\in H^r(dt/\pi(1+t^2))$$

となるなら，$f(s)$ は $\sigma>\sigma_0+1/r$ で収束する．

（証明の概略．） $2\leq q<r$ となる任意の q に対して，$1/q+1/p=1$ となる p を定める．

$$|\sigma+it|^p=|\sigma+it|^{2p/r}\cdot|\sigma+it|^{p(1-2/r)}$$

より Hölder の不等式から

$$\int_{-\infty}^{\infty}\left|\frac{f(\sigma+it)}{\sigma+it}\right|^p dt\leq\left\{\int_{-\infty}^{\infty}\frac{|f(\sigma+it)|^r}{|\sigma+it|^2}dt\right\}^{p/r}\cdot\left\{\int_{-\infty}^{\infty}|\sigma+it|^{-p(r-2)/(r-p)}dt\right\}^{(r-p)/r}$$

となる．そして仮定 $r>q=p/(p-1)$ より，$p(r-2)/(r-p)>1$ となり，最後の積分は収束する．一方 $e^{-\lambda\sigma}F(\lambda)$ の Fourier 変換が

$$\frac{f(\sigma+it)}{\sigma+it}$$

となることより，Young-Hausdorff の定理から

$$\int_{-\infty}^{\infty}e^{-\lambda\sigma q}|F(\lambda)|^q d\lambda<\infty$$

を得る．これより

$$\sum_{n=1}^{\infty}e^{-\sigma q\log(n+1)}|F(\log n)|^q\{\log(n+1)-\log n\}<\infty$$

となり，この級数の項が 0 に収束することから，

$$|F(\log n)|^q=0(n^{\sigma+1/q})$$

となることが分かる．（終わり．）

Dirichlet 級数の積の性質から $Z_u(x)\in H^r(\sigma_P)$，$r\geq 1$，となることが簡単に示されるから，Fubini の定理を用い，定理 5.2 の結論に加えて次が成立する．

命題 5.4．定理 5.2 の仮定の下で，(3.4) の Dirichlet 級数は $\sigma>1/2$ の各点で収束する．

定理 5.2 と命題 5.4 で除外されるおのおのの測度零の不変集合の関係はほとんど分からない．そして (5.4) の $\sigma>1/2$ での収束と Riemann 予想が同値という Littlewood の定理[49；14.25]と見比べると定理 5.3 や命題 5.4 は少しは面白いかもしれない．また定理 5.2 の結論をみたす点は $\boldsymbol{T}^\infty$ の中で稠密となる．そこでそのような $x_n\to 0$ となる点を定め (5.6) による関数列 $f_n(s)$ を考え $n\to\infty$ での状況を調べると正規族の性質と Rouché の定理から，$H^q(dt/\pi(1+t^2))$ でのノルムは発散してしまい $\zeta(s)$ との関係は途切れてしまう．

この他に関数環論との関連でゼータ関数を扱ったものとして[1]，[3]，[40]等がある．またここでの Dirichlet 級数の扱い方については Helson 氏による一連の報告[18]，[19]，[20]，[21]等を参照されたい．

追記

この原稿を提出後，松本耕二氏（名大多元数理）より，§5 の内容について，いくつかのコメントを頂いた：(1) ここで扱った概周期関数の拡張と類似の議論がゼータ関数の確率論的な値分布論にでてくること．(2) 定理 5.2 に関して，Euler 積そのものが直接 $\sigma>1/2$ まで解析的に拡張できること．(3) このようなゼータ関数の値分布論は特に旧ソ連，東欧圏で盛んで，次の本が参考になるだろうこと．

A. Laurinčikas, Limit Theorems for the Riemann Zeta-Function, Kluwer, 1996.

貴重なご意見を頂いたレフェリーの諸先生および松本氏に紙面をお借りし謝辞を表します．

文献

[1] H. Bercovici and C. Foias, A real variable restatement of Riemann's hypothesis, Israel J. Math. **48**, (1984), 57-68.

[2] A. Beurling, On two problems concerning linear transformations in Hilbert space, Acta Math. **81**, (1949), 239-255.

[3] A. Beurling, A closure problem related to the Riemann zeta-function, Proc. Acad. Sci. U.S.A. **41**, (1955), 312-314.

[4] L. Carleson, An interpolation problem for bounded analytic functions, Amer. J. Math. **80**, (1958), 921-930.

[5] L. Carleson, Interpolations by bounded analytic functions and the corona problem, Ann. of Math. **76**, (1962), 547-559.

[6] M. Denker and E. Eberline, Ergodic flows are strictly ergodic, Advances in Math. **13**, (1974), 437-473.

[7] N. Dunford and J. Schwartz, Linear operators, Part I, Wiley Interscience, New York, 1958.

[8] P. Duren, Theory of H^p spaces, Academic Press, New York and London, 1970.

[9] F. Forelli, Analytic and quasi-invariant measures, Acta Math. **118**, (1967), 33-59.

[10] F. Forelli, Fourier theory and flows, Actes Congrès Internat. Math. **2**, (1970), 475-476.

[11] F. Forelli, A maximal algebra, Math. Scand. **30**, (1970), 152-158.

[12] T. Gamelin, H^p spaces and extremal functions in H^1, Trans. Amer. Math. Soc. **124**, (1966), 158-157.

[13] T. Gamelin, Uniform algebras, Prentice-Hall, Englewood Cliffs, N. J., 1969.

[14] J. Garnett, Bounded analytic functions, Academic Press, New York, 1981.

[15] P. Halmos, Lectures on ergodic theory, 日本数学会, 1956.

[16] 荷見守助, Shift-invariant subspaceについて, 数学, 17巻4号, (1966), 22-32.

[17] 荷見守助, 不変部分空間の理論, 数学, 28巻1号, (1976), 47-59.

[18] H. Helson, Convergent Dirichlet series, Ark. För Mat. **4**, (1962), 501-510.

[19] H. Helson, Foundation of the theory of Dirichlet series, Acta Math. **118**, (1967), 61-77.

[20] H. Helson, Compact groups and Dirichlet series, Ark. För Mat. **8**, (1969), 139-143.

[21] H. Helson, Convergence of Dirichlet series, Lecture Note in Mathematics, **336**, Springer-Verlag, Berlin, 1973, 153-160.

[22] H. Helson, Analyticity on compact abelian groups, Algebras in Analysis, Academic Press, London, 1975, 1-62.

[23] H. Helson, Compact groups with orderd duals VII, J. London Math. Soc. **(2)**, **20**, (1979), 509-515.

[24] H. Helson and J. Tanaka, Singular cocycles and the generator problem, to appear in Proc. Internat. Conference on Operator Theory.

[25] K. Hoffman, Banach spaces of analytic functions, Prentice-Hall, Englewood Cliffs N. J., 1962.

[26] K. Hoffman, Bounded analytic functions and Gleason parts, Ann. of Math. **86** (1967), 74-111.

[27] K. Izuchi, Zero sets of interpolating Blaschke products, Pacific J. of Math. **119**, (1985), 337-342.

[28] K. Izuchi, *QC*-level sets and questions of Douglas algebras. J. Funct. Anal. **65**, (1986), 293-308.

[29] K. Izuchi, Ineterpolating sequences in a homeomorphic part of H^∞, Proc. Amer. Math. Soc. **111**, (1991), 1057-1065.

[30] K. Izuchi, Spreading Blaschke products and homeomorphic parts, to appear.

[31] 鹿野健 他, リーマン予想, 日本評論社, 1991.

[32] 黒川信重, オイラー積の250年, 現代数学のあゆみ **4**, 日本評論社, 1992, 13-27.

[33] 森田紀一, 位相空間論, 岩波全書, 岩波書店, 1981.

[34] P. Muhly, Function algebras and flows, Acta Sci. Math. (Szeged) **35**, (1973), 111-121.

[35] P. Muhly, Function algebras and flows II, Ark. Mat. **11**, (1973), 203-213.

[36] P. Muhly, Function algebras and flows IV, Trans. Amer. Math. Soc. **203**, (1975), 55-66.

[37] P. Muhly, Isometries of ergodic Hardy spaces, Israel J. Math. **36**, (1980), 50-74.

[38] P. Muhly, The function-algebraic ramifications of Wiener's work on prediction theory and random analysis, in Collected Works with Commentaries, Vol. III, by Norbert Wiener, MIT Press, Cambridge, MA, 1981, 339-370.

[39] T. Nakazi, Invariant subspaces of weak-* Dirichlet algebras, Pacific J. Math. **69**, (1977), 151-167.

[40] N. Nikolski, Distance formulae and invariant subspaces, with an application to localization of zeros of the Riemann ζ-function, Ann. Inst. Fourier, Grenoble **45**, (1995), 143-151.

[41] J. Oxtoby, Ergodic sets, Bull. Amer. Math. Soc. **58**, (1952), 116-136.

[42] W. Parry, Compact abelian group extensions of discrete dynamical systems, Z. Wahrscheinlichkeitstheorie verw. Geb. **13** (1969), 95-113.

[43] T. Srinivasan and J. Wang, Weak-* Dirichlet algebras, in Function Algebras, Scott-Foresman, Chicago, 1966, 216-249.

[44] 竹之内脩, Banach環概説, 数学, 28巻1号, (1976), 1-9.

[45] J. Tanaka, Corona problem and flows, J. Funct. Anal. **102** (1991), 306-378.

[46] J. Tanaka, Flows in fibers, Trans. Amer.

Math. Soc. **343**, (1994), 779-804.
[47] J. Tanaka, Single generator problem, Trans. Amer. Math. Soc. **348**, (1996), 4113-4129.
[48] E. C. Titchmarsh, The theory of functions, 2nd ed. Oxford UP, 1939.
[49] E. C. Titchmarsh, The theory of the Riemann zeta-function, (revised by D. R. Heath-Brown), Oxford UP, 1988.
[50] 富山淳, 関数環と flow, 数学, 28 巻 1 号, (1976), 35-46.
[51] 和田淳蔵, 関数環におけるいくつかの基本概念, 数学, 28 巻 1 号, (1976), 9-17
[52] 和田淳蔵, ノルム環, 現代の数学 **19**, 共立出版, 1969.
[53] R. Walker, The Stone-Čech compactification, Springer-Verlag, Berlin, 1974.
[54] P. Walters, An introduction to ergodic theory, Springer-Verlag, New York, 1982.
[55] J. Wermer, On algebras of continuous functions, Proc. Amer. Math. Soc. **4**, (1953), 866-869.
[56] J. Wermer, Dirichlet algebras, Duke Math. J. **27**, (1960), 373-382.
[57] J. Xia, Conditional expectations and the corona problem of ergodic Hardy spaces, J. Funct. Anal. **64** (1985), 251-274.

(1998 年 1 月 12 日提出)

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