The Hilbert modular functions for $\sqrt{5}$ via the period mapping for a family of $K3$ surfaces

$K3$ 曲面の周期写像を経由した $\sqrt{5}$ のヒルベルト・モジュラー関数

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Preface

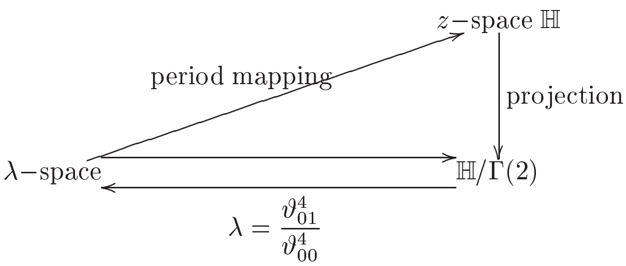
In the classical theory of the elliptic functions, due to Gauss, Jacobi, Schwarz, etc, there exists a close relation among a family of elliptic curves, the Gauss hypergeometric differential equation and the elliptic modular function. This theory is often called the Gauss-Schwarz theory. Set the family $\{S(\lambda)\}$ of the elliptic curves

$$S(\lambda) : y^2 = x(x-1)(x-\lambda),$$

where $\lambda \in \mathbb{C} - \{0,1\}$ is the complex parameter. The period mapping for $\{S(\lambda)\}$ is given by the quotient of period integrals. This is a multivalued analytic mapping on $\mathbb{C} - \{0,1\}$. Now, these period integrals are the linearly independent solutions of the Gauss hypergeometric differential equation ${}_2E_1\left(\frac{1}{2}, \frac{1}{2}, 1; \lambda\right)$, where the projective monodromy group is isomorphic to the principal congruence subgroup $\Gamma(2)$. The period mapping for $\{S(\lambda)\}$ coincides with the Schwarz mapping of ${}_2E_1\left(\frac{1}{2}, \frac{1}{2}, 1; \lambda\right)$. The inverse correspondence of the period mapping defines a modular function for $\Gamma(2)$, that is an meromorphic function on $\mathbb{H}$ given by $z \mapsto \lambda(z)$. Moreover, the modular function $\lambda(z)$ has an explicit theta expression

$$\lambda(z) = \frac{\vartheta_{01}^4(z)}{\vartheta_{00}^4(z)},$$

where $\vartheta_{00}(z)$ and $\vartheta_{01}(z)$ are the Jacobi theta constants.



We can regard $K3$ surfaces as 2-dimensional extension of the elliptic curves, for the canonical bundle of a $K3$ surface is trivial. Several researchers tried to obtain modular functions as the inverse correspondence of period mappings of families of $K3$ surfaces (for example, see Shiga [Shg1] and Matsumoto, Sasaki and Yoshida [MSY]).

In this thesis, we obtain an extension of this classical theory to the Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$ by using a family of $K3$ surfaces with 2 complex parameters. Namely, we study the period mapping for the family $\mathcal{F}$ of $K3$ surfaces with explicit defining equations. The period integrals satisfy a system of partial linear differential equations in

2 variables of rank 4. The inverse correspondence of this period mapping gives a pair of Hilbert modular functions for the field $\mathbb{Q}(\sqrt{5})$. This thesis is organized as follows.

In Chapter 0, we recall the classical elliptic functions without proofs and give a brief survey of basic properties of $K3$ surfaces and elliptic surfaces. Especially, the period mapping for marked $K3$ surfaces and some techniques of the Mordell-Weil latices shall be used in this thesis.

In Chapter 1, we obtain the families $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ of $K3$ surfaces with 2 parameters. Namely, we have the families of $K3$ surfaces derived from 3-dimensional reflexive polytopes with at most terminal singularities with 5 vertices. We give elliptic fibrations for our families. To obtain the period mappings for our families, we need the Néron-Severi lattices and the transcendental lattices. By applying the injectivity of the Torelli theorem for marked $K3$ surfaces, we show that the Picard numbers of our families are equal to 18 (Section 1.3). Moreover, using some techniques of the Mordell-Weil lattice, we determine the lattice structures of the Néron-Severi lattices and the transcendental lattices (Section 1.4). Our period mappings are multivalued analytic mappings on the parameter spaces. Then, we have the projective monodromy groups for our period mappings. In Section 1.5, we determine these projective monodromy groups by applying the surjectivity of the Torelli theorem for marked $K3$ surfaces.

In Chapter 2, we give the systems of linear differential equations which are satisfied by the period integrals for our families of $K3$ surfaces. These differential equations are systems of linear partial differential equation in 2 variables of rank 4. In this thesis, we call them the period differential equation for our families. They give counterparts of the classical Gauss hypergeometric differential equation. In other words, they give the differential equation determined by the Gauss-Manin connection for our families. In Section 2.2, we focus on the family $\mathcal{F}_0$. We show that the period differential equation for $\mathcal{F}_0$ gives the uniformizing differential equation for the symmetric Hilbert modular orbifold for the field $\mathbb{Q}(\sqrt{5})$. This implies that the family $\mathcal{F}_0$ is strongly related to the Hilbert modular function for $\mathbb{Q}(\sqrt{5})$.

In Chapter 3, that is the main part of this thesis, we consider the period mapping for the family $\mathcal{F} = \{S(X,Y)\}$ given by the affine equation

$$S(X,Y) : z^2 = x^3 - 4y^2(4y-5)x^2 + 20Xy^3x + Yy^4.$$

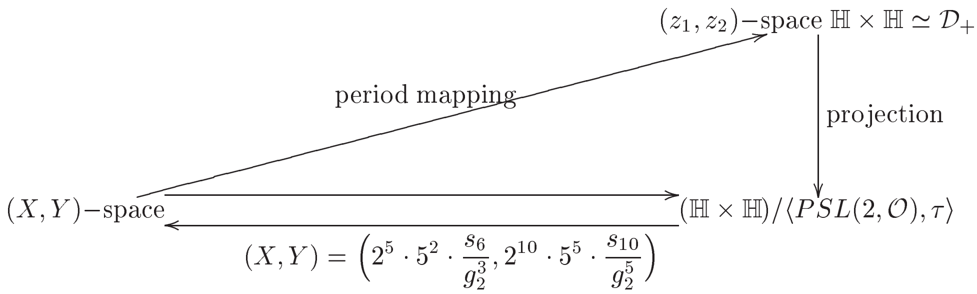
The aim of this chapter is to show that the inverse correspondence of the period mapping for $\mathcal{F}$ gives a pair of the Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$ and to obtain an explicit theta expression of this inverse correspondence. These results give an extension of the classical theory of the elliptic modular functions.

The Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$ have several remarkable properties. There exist various studies on the structure of the field of the Hilbert modular functions or the ring of the Hilbert modular forms (for example Gundlach [Gu], Hirzebruch [Hi] and Müller [Mul]). However, still now, to the best of the author's knowledge, there has not appeared an explicit expression of Hilbert modular functions as an inverse correspondence of the period mapping for a family of algebraic varieties. In this thesis, we give an extension of the above classical story to the Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$ by using the family $\mathcal{F} = \{S(X,Y)\}$.

In Section 3.1, we survey the study of the Hilbert modular orbifold $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}), \tau\rangle$ due to Hirzebruch, where $\mathcal{O} = \mathbb{Z} + \dfrac{1+\sqrt{5}}{2}\mathbb{Z}$ and τ is an involution of $\mathbb{H}\times\mathbb{H}$. In Section 3.2, we study the family $\mathcal{F} = \{S(X,Y)\}$. A generic member $S(X,Y)$ is birationally equivalent to a generic member of the family $\mathcal{F}_0$. We obtain the weighted projective space $\mathbb{P}(1:3:5)$ as a compactification of the parameter space of $\mathcal{F}$. We define the multivalued period mapping $\mathbb{P}(1,3,5) - \{\text{one point}\} \to \mathcal{D}$ for $\mathcal{F}$, where $\mathcal{D}$ is a Hermitian symmetric space of type IV. We have a modular isomorphism between $\mathbb{H} \times \mathbb{H}$ and a connected component $\mathcal{D}_+$ of $\mathcal{D}$. Our multivalued period mapping gives the developing mapping of the Hilbert modular orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2,\mathcal{O}), \tau\rangle$. The inverse correspondence $\mathbb{H} \times \mathbb{H} \to \mathbb{C} \times \mathbb{C}$ given by $(z_1,z_2) \mapsto (X(z_1,z_2), Y(z_1,z_2))$ defines a pair of the Hilbert modular function for $\mathbb{Q}(\sqrt{5})$. In Section 3.3, we consider the subfamily $\mathcal{F}_X = \{S(X,0)\}$. We have an explicit expression of the inverse correspondence of the period mapping for $\mathcal{F}_X$ in the famous elliptic J-function. In Section 3.4, we give explicit expressions of

$$(z_1,z_2) \mapsto (X(z_1,z_2), Y(z_1,z_2)$$

by Müller's modular form. This result gives an extension of the classical elliptic modular λ-function.



Contents

0. Preliminaries

0.1 Classical elliptic modular functions

In the classical theory of elliptic modular functions, there is a closed relation among the elliptic curve, the Gauss hypergeometric differential equation and the elliptic modular functions. In this section, we recall the above classical topics. For detailed proof of the topics in this section, see Griffiths [Gr1], McKean and Moll [MM], Fujiwara [F], Yoshida [Y] and Mumford [Mum].

0.1.1 Elliptic curves and period integrals

An elliptic curve is a compact Riemann surface X of genus 1. The elliptic curve X can be represented by a smooth algebraic curve of degree 3 in $\mathbb{P}^2(\mathbb{C}) = \{(\zeta_0 : \zeta_1 : \zeta_2)\}$. The defining equation of X can be given by

$$\zeta_0\zeta_2^2 = (\zeta_1 - a_1\zeta_0)(\zeta_1 - a_2\zeta_0)(\zeta_1 - a_3\zeta_0), \tag{0.1.1}$$

where a_1, a_2 and a_3 are distinct points in $\mathbb{C}$. The holomorphic mapping

$$X \to \mathbb{C}; (\zeta_0 : \zeta_1 : \zeta_2) \mapsto (\zeta_0 : \zeta_1)$$

gives a 2-sheeted covering of $\mathbb{P}^1(\mathbb{C}) = \{(\zeta_0 : \zeta_1)\}$ with $2(1+1) = 4$ distinct branch points $(\zeta_0 : \zeta_1) = (1 : a_1), (1 : a_2), (1 : a_3)$ and $(0 : 1)$. By a Möbius transformation, we assume $a_1 = 0$, $a_2 = 1$ and $a_3 = \lambda \in \mathbb{C} - \{0, 1\}$. Then, we obtain the following canonical affine equation of a elliptic curve:

$$S(\lambda) : y^2 = x(x-1)(x-\lambda). \tag{0.1.2}$$

The point $\lambda \in \mathbb{C} - \{0, 1\}$ is a complex parameter of the family $\{S(\lambda)\}$ the elliptic curves.

Let $\{\gamma_1, \gamma_2\}$ be a basis of $H_1(S(\lambda), \mathbb{Z})$ such that $(\gamma_1 \cdot \gamma_2) = 1$. See Figure 1.

Let ω be a holomorphic 1-form on $S(\lambda)$. Since $\deg(\omega) = (2 \cdot 1 - 2) = 0$, we have

$$\Omega(S(\lambda)) \simeq \mathbb{C}.$$

The holomorphic 1-form

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

on $S(\lambda)$ is unique up to a constant factor.

Since $d\omega = 0$, the period integrals

$$\int_{\gamma_1} \omega, \quad \int_{\gamma_2} \omega \tag{0.1.3}$$

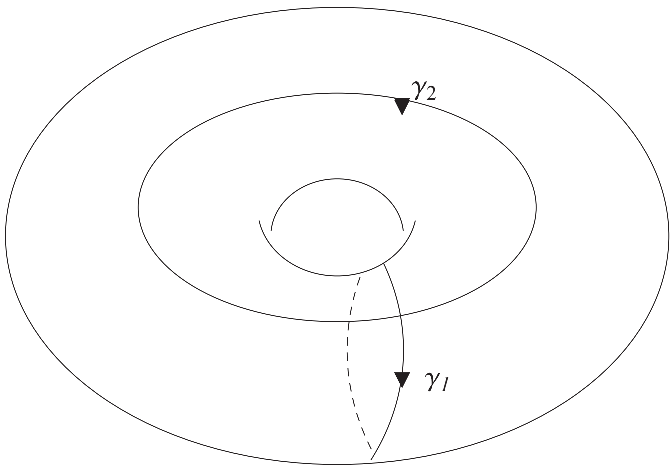


Figure 1: The 1 cycles γ_1 and γ_2 on the complex torus

only depends on the homology class of γ_j $(j = 1, 2)$. So, these integrals are well-defined.

Set

$$\tau(\lambda) = \frac{\displaystyle\int_{\gamma_2} \omega}{\displaystyle\int_{\gamma_1} \omega}.$$

We note that $\tau \in \mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$. Let

$$\Lambda = \{m_1 + m_2\tau | m_1, m_2 \in \mathbb{Z}\} (\subset \mathbb{C}).$$

Then, the elliptic curve $S(\lambda)$ is identified with the complex torus $\mathbb{C}/\Lambda$.

The correspondence $\mathbb{C} - \{0, 1\} \to \mathbb{H}$ given by

$$\Phi : \lambda \mapsto \tau = \tau(\lambda) \tag{0.1.4}$$

is called the period mapping for the family $\{S(\lambda)\}$. We note that Φ is not a single-valued but a multivalued analytic mapping.

Let us treat these period integrals as a integrals on λ-plane. Let $\lambda \in \mathbb{R}$ and $0 < \lambda < 1$. Take the branch of $\sqrt{x(x-1)(x-\lambda)}$ for $x > 1$ such that $\sqrt{x(x-1)(x-\lambda)} > 0$. So,

$$\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} > 0.$$

Similarly, we take $\sqrt{x(x-1)(x-\lambda)} \in i\mathbb{R}_{>0}$ for $\lambda < x < 1$. So,

$$\int_\lambda^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \in -i\mathbb{R}_{>0}.$$

By Figure 2 and considering the analytic continuation, we have

$$\begin{cases} \displaystyle\int_{\gamma_1} \omega = 2\int_\lambda^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \\ \displaystyle\int_{\gamma_2} \omega = 2\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \end{cases} \tag{0.1.5}$$

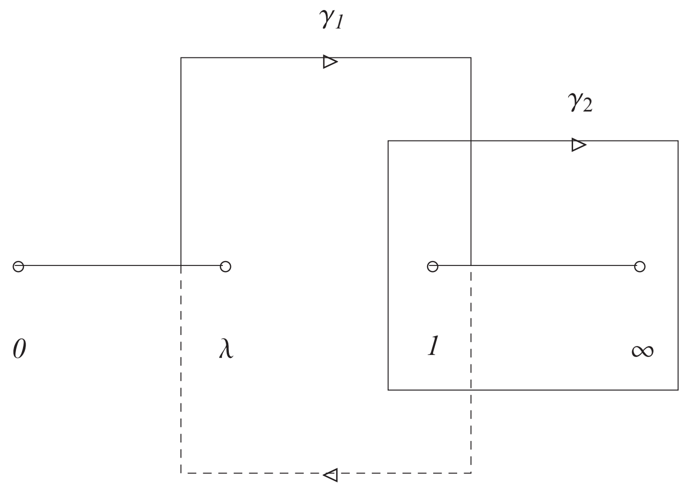


Figure 2: The cycles γ_1 and γ_2 on x-plane.

and

$$\frac{\displaystyle\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}}{\displaystyle\int_\lambda^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}} \in i\mathbb{R}_{>0}$$

for $0 < \lambda < 1$.

0.1.2 The Gauss hypergeometric differential equation

To study the period mapping for $\{S(\lambda)\}$, we consider the Gauss hypergeometric equation.

Let $c \neq 0, -1, -2, \cdots$. The second-order linear differential equation

$$E(a,b,c) : \lambda(1-\lambda)\frac{d^2u}{d\lambda^2} + \big(c-(a+b+1)\lambda\big)\frac{du}{d\lambda} - abu = 0 \tag{0.1.6}$$

is called the Gauss hypergeometric equation. This is a Fuchsian differential equation with 3 regular singular points $0, 1$ and ∞. One solution of (0.1.6) about $\lambda = 0$ is given by the Gauss hypergeometric series

$${}_2F_1(a,b,c;\lambda) = 1 + \frac{ab}{c\cdot 1}\lambda + \frac{a(a+1)b(b+1)}{c(c+1)\cdot 1\cdot 2}\lambda^2 + \cdots .$$

For $\mathrm{Re}(a) > 0, \mathrm{Re}(c-a) > 0$ and $|\lambda| < 1$, we have the Eulerian integral

$${}_2F_1(a,b,c;\lambda) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{-a}(1-t)^{c-a-1}(1-\lambda t)^{-b}dt. \tag{0.1.7}$$

More generally, letting $0 < \mathrm{Re}(a) < \mathrm{Re}(c) < \mathrm{Re}(b) + 1 < 2$ and $p, q \in \left\{0, 1, \infty, \frac{1}{x}\right\}$, the integral

$$F_{pq}(\lambda) = \int_p^q t^{1-a}(1-t)^{c-a-1}(1-\lambda t)^{-b}dt \tag{0.1.8}$$

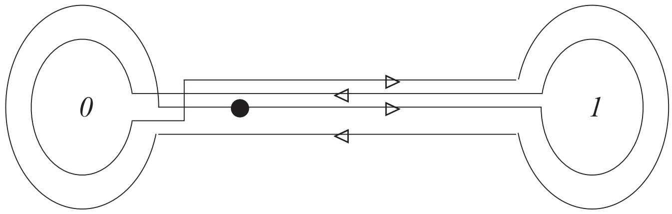


Figure 3: The Pochhammer arc.

is a solution of (0.1.6). Here, the integral arc is given by the Pochhammer arc (see Figure 3).

We have the Riemann scheme of (0.1.6)

$$\left\{\begin{matrix} \lambda=0 & \lambda=1 & \lambda=\infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{matrix}\right\}. \tag{0.1.9}$$

Then, if $c \notin \mathbb{Z}$, we have a system $\{u_1, u_2\}$ of solutions of (0.1.6) around $\lambda = 0$ such that

$$\begin{cases} u_1(\lambda) = (\text{holomorphic}), \\ u_2(\lambda) = \lambda^{1-c}(\text{holomorphic}). \end{cases}$$

If $c \in \mathbb{Z}$, we can find a system $\{u_1, u_2\}$ of solutions of (0.1.6) around $\lambda = 0$ such that

$$\begin{cases} u_1(\lambda) = (\text{holomorphic}), \\ u_2(\lambda) = \log(\lambda) + (\text{holomorphic}), \end{cases}$$

where log stands for the principal value.

Let $\{y_1(\lambda), y_2(\lambda)\}$ be a system of solutions of (0.1.6). We consider the mapping

$$\sigma : \mathbb{H} \to \mathbb{P}^1(\mathbb{C}) : \lambda \mapsto \frac{y_2(\lambda)}{y_1(\lambda)}.$$

This is a multivalued analytic mapping. The image $\sigma(\mathbb{H})$ is a triangle bounded by 3 arcs (i.e. parts of circles). This triangle is called a Schwarz triangle. The image under σ of the union $(\infty, 0) \cup (0, 1) \cup (1, \infty)$ gives the boundary of this Schwarz triangle. Due to the Riemann scheme (0.1.9), we can determine the 3 angles:

$$\begin{cases} \pi|1-c| \quad (\text{at } \sigma(0)) \\ \pi|c-a-b| \quad (\text{at } \sigma(1)) \\ \pi|a-b| \quad (\text{at } \sigma(\infty)). \end{cases}$$

If $|1-c|, |c-a-b|$ and $|a-b| < 1$, the mapping σ sends $\mathbb{H}$ bijectively to a Schwarz triangle.

We apply the Schwarz reflection principle to the mapping σ defined on $\mathbb{H}$ and to the intervals $(-\infty, 0), (0, 1)$ and $(1, \infty)$. The analytic mapping σ is extended to $\mathbb{H}_- = \{z \in \mathbb{C} | \mathrm{Im}(z) < 0\}$ through any of the above 3 intervals. Applying the same principle again on

$\mathbb{H}_-$, we obtain the analytic continuation σ_γ along $\gamma \in \pi_1(\mathbb{C} - \{0,1\}, *)$. There is a matrix $\begin{pmatrix} a, b \\ c, d \end{pmatrix} \in GL(2, \mathbb{C})$ such that

$$\sigma_\gamma = \frac{a\sigma + b}{c\sigma + d}. \tag{0.1.10}$$

Then, we obtain the multivalued analytic mapping

$$\sigma : \mathbb{C} - \{0,1\} \to \mathbb{P}^1(\mathbb{C}); \lambda \mapsto \frac{y_2(\lambda)}{y_1(\lambda)}.$$

This mapping is called the Schwarz mapping for (0.1.6). The image of the multivalued mapping σ is given by conformal reflections of the original Schwarz triangle $\sigma(\mathbb{H})$. By making an even number reflections, we have a linear fractional transformation as (0.1.10). These transformations form the projective monodromy group Γ for (0.1.6).

Set

$$|1 - c| = \frac{1}{p}, \quad |c - a - b| = \frac{1}{q}, \quad |a - b| = \frac{1}{r},$$

where $p, q, r \in \{2, 3, 4, \cdots\} \cup \{\infty\}$.

If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, a finite numbers of the Schwarz triangles cover the whole $\mathbb{P}^1(\mathbb{C})$. If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, the Schwarz triangles cover the plane $\mathbb{C}(\subset \mathbb{P}^1(\mathbb{C}))$. If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, the Schwarz triangles cover the plane $\mathbb{H}(\subset \mathbb{P}^1(\mathbb{C}))$.

To study the period mapping (0.1.4), we consider the case

$$(a, b, c) = \left(\frac{1}{2}, \frac{1}{2}, 1\right).$$

By (0.1.7), the integral

$$\int_p^q t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}(1-\lambda t)^{-\frac{1}{2}} dt$$

is a solution of the Gauss hypergeometric equation ${}_2E_1\left(\frac{1}{2}, \frac{1}{2}, 1\right)$ for $p, q \in \{0, 1, \infty, \frac{1}{\lambda}\}$. Performing a transformation $t = \frac{1}{x}$,

$$\begin{cases} \displaystyle\int_\lambda^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \\ \displaystyle\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \end{cases}$$

are solutions of ${}_2E_1\left(\frac{1}{2}, \frac{1}{2}, 1\right)$, those are period integrals of the elliptic curve $S(\lambda)$. Therefore, we know that the period integrals

$$\int_{\gamma_1} \omega, \quad \int_{\gamma_2} \omega$$

of $S(\lambda)$ gives a system of solutions of ${}_2E_1\left(\frac{1}{2},\frac{1}{2},1\right)$. Hence, the period mapping in (0.1.4)

$$\lambda \mapsto \frac{\int_{\gamma_2}\omega}{\int_{\gamma_1}\omega}$$

for $\{S(\lambda)\}$ gives a Schwarz mapping for ${}_2E_1\left(\frac{1}{2},\frac{1}{2},1\right)$.

In this case, we have $p = q = r = \infty$. The projective monodromy group $\Gamma(\infty,\infty,\infty)$ is isomorphic to the principal congruence subgroup

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) | a \equiv d \equiv 1, b \equiv c \equiv 0 \pmod{2} \right\}$$

of level 2. Therefore, the projective monodromy group of the period mapping for the family $\{S(\lambda)\}$ is $\Gamma(2)$.

0.1.3 The orbifold $\mathbb{H}/\Gamma(2)$

We consider the action of $\Gamma(2)$ on $\mathbb{H} = \{\tau | \mathrm{Im}(\tau) > 0\}$ given by the transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

Since we saw that the projective monodromy group of the period mapping Φ for $\{S(\lambda)\}$ is $\Gamma(2)$, we have the single-valued analytic period mapping $\overline{\Phi} : \mathbb{C} - \{0,1\} \to \mathbb{H}/\Gamma(2)$ given by

$$\lambda \mapsto \overline{\tau} = \overline{\Phi(\lambda)}. \tag{0.1.11}$$

The quotient space $\mathbb{H}/\Gamma(2)$ is not compact. However, adding 3 points $0, 1$ and $\sqrt{-1}\infty$, $\mathbb{H}/\Gamma(2)$ is compactified to

$$\overline{\mathbb{H}/\Gamma(2)} \simeq \mathbb{P}^1(\mathbb{C})$$

(see Figure 4).

The above mentioned 3 points $0, 1$ and $\sqrt{-1}\infty$ are called cusps.

Definition 0.1.1. *Let the holomorphic function f on $\mathbb{H}$ satisfies*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau),$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$ *and the Fourier expansion of f is in the form*

$$f(\tau) = \sum_{n \geq 0} a_n \exp(2\pi\sqrt{-1}\tau).$$

Then we call f is a modular form for $\Gamma(2)$ of weight k.

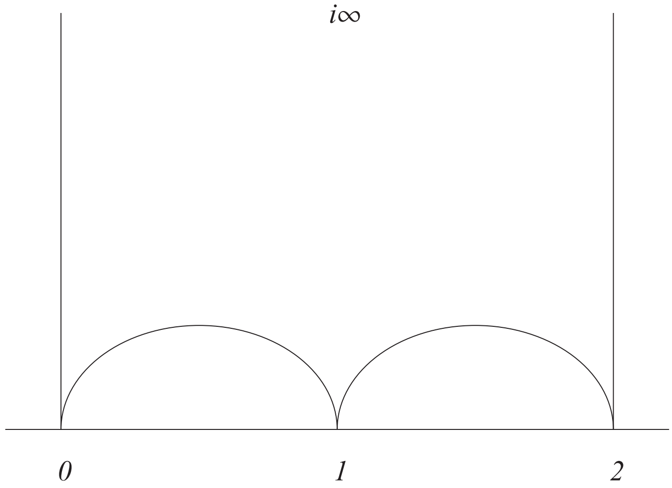


Figure 4: The orbifold $(\mathbb{H})/\Gamma(2)$.

Definition 0.1.2. *The meromorphic function g on $\mathbb{H}$ satisfying*

$$g\Big(\frac{a\tau+b}{c\tau+d}\Big)=g(\tau)$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$ *is called a modular function for* $\Gamma(2)$.

Of course, a modular function is a function on $\mathbb{H}/\Gamma(2)$. If f_1 and f_2 are modular forms of the same weight, then

$$g=\frac{f_1}{f_2}$$

defines the modular function.

0.1.4 The Jacobi theta constants

We consider the ring of modular forms for $\Gamma(2)$. For $z \in \mathbb{H}$ and $(a,b)=(0,0),(0,1)$ or $(1,0)$,

$$\vartheta_{ab}(z)=\sum_{n\in\mathbb{Z}}\exp\Big(\pi\sqrt{-1}\big(n+\frac{a}{2}\big)z+2\pi\sqrt{-1}\big(n+\frac{a}{2}\big)\big(\frac{b}{2}\big)\Big).$$

is called the Jacobi theta constants. This is a holomorphic function on $\mathbb{H}$.

We have the Jacobi identity

$$\vartheta_{00}^4(z)=\vartheta_{01}^4(z)+\vartheta_{10}^4(z). \tag{0.1.12}$$

By the definition of the theta constants, we have

$$\begin{cases} \vartheta_{00}(it)=1+2(\tilde{q}+\tilde{q}^4+\tilde{q}^9+\cdots), \\ \vartheta_{01}(it)=1-2(\tilde{q}-\tilde{q}^4+\tilde{q}^9-\cdots), \\ \vartheta_{10}(it)=2(\tilde{q}^{\frac{1}{4}}+\tilde{q}^{\frac{9}{4}}+\tilde{q}^{\frac{25}{4}}+\cdots), \end{cases} \tag{0.1.13}$$

where $t\in\mathbb{R}$ and $\tilde{q}=e^{-\pi t}$.

The theta constants satisfies the following formulae:

$$\begin{cases} \vartheta_{00}(z+1) = \vartheta_{01}(z), \\ \vartheta_{01}(z+1) = \vartheta_{00}(z), \\ \vartheta_{10}(z+1) = e^{\frac{\pi i}{4}}\vartheta_{01}(z), \end{cases} \tag{0.1.14}$$

and

$$\begin{cases} \vartheta_{00}\Big(-\dfrac{1}{z}\Big) = e^{-\frac{\pi i}{4}}\sqrt{z}\vartheta_{00}(z), \\ \vartheta_{01}\Big(-\dfrac{1}{z}\Big) = e^{-\frac{\pi i}{4}}\sqrt{z}\vartheta_{10}(z), \\ \vartheta_{10}\Big(-\dfrac{1}{z}\Big) = e^{-\frac{\pi i}{4}}\sqrt{z}\vartheta_{01}(z). \end{cases} \tag{0.1.15}$$

Set

$$\vartheta_{ab}(\infty) = \lim_{t\to\infty}\vartheta_{ab}(it).$$

From (0.1.13), we have

$$\vartheta_{00}(\infty) = 1, \vartheta_{01}(\infty) = 1, \vartheta_{10}(\infty) = 0. \tag{0.1.16}$$

Then, from (0.1.14) and (0.1.15), we have

$$\vartheta_{00}(0) : \vartheta_{01}(0) : \vartheta_{10}(0) = 1 : 0 : 1, \tag{0.1.17}$$

and

$$\vartheta_{00}(1) : \vartheta_{01}(1) : \vartheta_{10}(1) = 0 : e^{-\frac{\pi i}{4}} : 1. \tag{0.1.18}$$

By the way, because of (0.1.14) and (0.1.15), $\vartheta_{00}^4, \vartheta_{01}^4$ and ϑ_{10}^4 are modular forms for $\Gamma(2)$ of weight 2. Moreover, the ring of modular forms for $\Gamma(2)$ is given by

$$\mathbb{C}[\vartheta_{00}^4, \vartheta_{01}^4, \vartheta_{10}^4]/(\vartheta_{00}^4(z) = \vartheta_{01}^4(z) + \vartheta_{10}^4(z)) = \mathbb{C}[\vartheta_{00}^4, \vartheta_{01}^4].$$

0.1.5 The theta expression of the inverse correspondence of the period mapping

We saw that the period mapping

$$\Phi : \lambda \mapsto \tau(\lambda) = \frac{\displaystyle\int_{\gamma_2}\omega}{\displaystyle\int_{\gamma_1}\omega} \tag{0.1.19}$$

for $\{S(\lambda)\}$ is a multivalued analytic mapping with the projective monodromy group $\Gamma(2)$. Then, the inverse correspondence $\tau \mapsto \lambda = \lambda(\tau)$ satisfies

$$\lambda(\tau_1) = \lambda(\tau_2)$$

if

$$\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$. Therefore, $\tau \mapsto \lambda(\tau)$ defines a modular function for $\Gamma(2)$.

We consider the integrals in (0.1.19). If $\lambda \to 0$, we have

$$\int_{\gamma_2} \omega = \int_{\lambda}^{0} \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \to 0.$$

So, in this case, $\tau(\lambda) \to 0$. By the same argument, if $\lambda \to 1$, then $\tau(\lambda) \to \sqrt{-1}\infty$. From this, together with the argument principle, we have

$$\begin{cases} \lambda(\sqrt{-1}\infty) = 1, \\ \lambda(0) = 0, \\ \lambda(1) = \infty. \end{cases}$$

On the other hand, by the last subsection, we have

$$\begin{cases} \dfrac{\vartheta_{01}^4}{\vartheta_{00}^4}(\sqrt{-1}\infty) = 1, \\ \dfrac{\vartheta_{01}^4}{\vartheta_{00}^4}(0) = 0, \\ \dfrac{\vartheta_{01}^4}{\vartheta_{00}^4}(1) = \infty. \end{cases}$$

From this we can prove that $\tau \mapsto \lambda(\tau)$ and $\tau \mapsto \dfrac{\vartheta_{01}^4}{\vartheta_{00}^4}(\tau)$ are the same modular functions for $\Gamma(2)$.

So, we have

Theorem 0.1.1. *For $\tau \in \mathbb{H}$,*

$$\lambda(\tau) = \frac{\vartheta_{01}^4(\tau)}{\vartheta_{00}^4(\tau)}. \tag{0.1.20}$$

holds.

Many mathematicians (Picard, Terada [T], Deligne and Mostow [DM], Shiga [Shg1], Matsumoto, Sasaki and Yoshida [MSY], etc) attempted to extend this classical theory of elliptic functions. Especially, [Shg1] and [MSY] studied the moduli of families of $K3$ surfaces and modular functions.

0.2 Complex surfaces

In this thesis, we study the Hilbert modular function for $\mathbb{Q}(\sqrt{5})$ via the moduli of a family of elliptic $K3$ surfaces. We survey the results of complex surfaces we shall apply.

0.2.1 $K3$ surfaces

In this subsection, we recall the definitions and basic properties of $K3$ surfaces. For detailed proof, see [BHPV].

Let X be a compact complex surfaces. Let K_X be the canonical bundle of X. Set

$$\begin{cases} p_g(X) = \dim(H^2(X, \mathcal{O}_X)) = \dim(H^0(X, \mathcal{O}_X(K_X))), \\ q(X) = \dim(H^1(X, \mathcal{O}_X)) = \dim(H^1(X, \mathcal{O}_X(K_X))). \end{cases}$$

For a coherent sheaf $\mathcal{F}$ on X, the Euler characteristic

$$\chi(\mathcal{F}) = \sum_{j=0}^{2} (-1)^j \dim(H^j(X, \mathcal{F}))$$

is well-defined. Especially, we have

$$\chi(\mathcal{O}_X) = 1 - q(X) + p(X).$$

Let $c_1(X)$ and $c_2(X)$ be the Chern classes of the tangent bundle $T(X)$ of X. The cup product

$$H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}) \simeq \mathbb{Z}$$

defines a non-degenerate quadratic form (namely lattice structure) Q. Then, for $D_1, D_2 \in H^2(X, \mathbb{Z})$, we have the intersection number $(D_1 \cdot D_2)$. Letting $b^+(X)$ ($b^-(X)$, resp.) be the number of positive (negative, resp.) eigenvalues of $Q(X)$, we have the index $\tau(X) = b^+(X) - b^-(X)$.

Theorem 0.2.1. (1) (The Riemann-Roch theorem for surfaces) *Let D be a divisor on X. Then It holds that*

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D \cdot (D - K_X)) + \chi(\mathcal{O}_X).$$

(2) (Noether's formula)

$$\chi(X) = \frac{(K_X)^2 + c_2(X)}{12}.$$

(3) (The Hirzebruch index Theorem)

$$\tau(X) = \frac{c_1(X)^2 - 2c_2(X)}{3}.$$

Definition 0.2.1. *Let X be a compact complex surface. If the canonical bundle K_X of X is trivial and $H^1(X, \mathcal{O}_X) = 0$, we call X be a $K3$ surface.*

A $K3$ surface is simply connected. By Noether's formula, we can see that the topological Euler characteristic of X is equal to 24. We have that $c_1(X) = 0$ and $c_2(X) = \chi(X)$. Then, using the Poincaré duality, we have

$$\mathrm{rank}(H^2(X, \mathbb{Z})) = \mathrm{rank}(H_2(X, \mathbb{Z})) = 22.$$

Applying the index theorem, we obtain that the lattice $H_2(X,\mathbb{Z})$ has signature $(3,19)$. Then, by the cup product pairing, we can prove that $H_2(X,\mathbb{Z})$ has the following unimodular lattice structure:

$$H_2(X,\mathbb{Z}) = E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U, \tag{0.2.1}$$

where $E_8(-1)$ and U are given by the intersection matrices

$$E_8(-1) = \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & O & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & 1 & \\ & & & & 1 & -2 & 0 & \\ & & O & & 1 & 0 & -2 & 1 \\ & & & & & & 1 & -2 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Definition 0.2.2. *Let us call*

$$\mathrm{NS}(X) = \mathrm{Div}(X)/\textit{algebraically equivalent}$$

the Néron-Severi lattice of X. This is a sub lattice of $H_2(X.\mathbb{Z})$. The rank of Néron-Severi lattice is called the Picard number. Let us call

$$\mathrm{Tr}(X) = \mathrm{NS}(X)^{\perp}$$

the transcendental lattice of X.

From the exact sequence of the shaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$

we obtain the Chern class mapping

$$\delta^* : H^1(X, \mathcal{O}_X^*) \to H^2(X,\mathbb{Z}).$$

A line bundle over X is given by an image of the above mapping. We also have a canonical homomorphism

$$j^* : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}).$$

Through the Poincaré duality, the image $j^* \circ \delta^*(H^1(X,\mathcal{O}_X^*))$ in $H^2(X,\mathbb{Z})$ is identified with the Néron-Severi lattice. For an algebraic $K3$ surface, note that linear equivalence, algebraic equivalence and numerical equivalence all coincide.

A $K3$ surface is a Kähler manifold. We have a Hodge structure

$$H^2(X,\mathbb{Z}) \otimes \mathbb{C} = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X).$$

We have

$$\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z}).$$

Therefore, we have

$$\int_\gamma \omega = 0$$

where $\gamma \in \mathrm{NS}(X)$ and ω is the unique holomorphic 2-form up to a constant factor.

Theorem 0.2.2. *(The Torelli theorem for $K3$ surfaces) Let S_1 and S_2 are $K3$ surfaces. We suppose that there exists an effective Hodge isometry $\varphi : H_2(S_1,\mathbb{Z}) \to H_2(S_2,\mathbb{Z})$. Then, there exists a biholomorphic mapping $f : S_1 \to S_2$ such that $f_* = \varphi$.*

We shall apply this theorem to our lattice polarized $K3$ surfaces (Theorem 1.3.1, Theorem 1.5.1, Proposition 3.2.3, etc).

0.2.2 Elliptic surfaces

In this thesis, we use some results for elliptic surfaces. In this subsection, we survey them. For detailed proof, see Kodaira [Kod] or Shiga [Shg1], [Shg2].

Definition 0.2.3. *An elliptic surface (S,π,C) is a smooth projective surface S with a proper mapping $\pi : S \to C$ to a smooth projective algebraic curve C such that a generic fibre $\pi^{-1}(p)$ $(p \in C)$ is an ellipic curve. A holomorphic mapping $\varphi : C \to S$ such that $\pi \circ \varphi = id_C$ is called a section of π.*

We will consider the case for $C = \mathbb{P}^1(\mathbb{C})$.

Proposition 0.2.1. *([Shg1]) An elliptic surface (S,π,C) with sections is a $K3$ surface if and only if $C = \mathbb{P}^1(\mathbb{C})$ and the Euler number of X is equal to* 24.

An elliptic surface $(S,\pi,\mathbb{P}^1(\mathbb{C}))$ with sections is given by the compact non-singular model of an affine algebraic surface in $\mathbb{C}^3$. If $\mathbb{P}^1(\mathbb{C}) = (\text{t} - \text{sphere})$, the defining equation of the affine surface is given by the form

$$y^2 = 4x^3 - g_2(t)x - g_3(t), \tag{0.2.2}$$

where $g_2(t)$ and $g_3(t) \in \mathbb{C}[t]$ and π is given by $(x,y,t) \mapsto t$. We call the above defining equation the Kodaira normal form of $(S,\pi,\mathbb{P}^1(\mathbb{C}))$. If S is a $K3$ surface, polynomials g_2 and g_3 satisfy $5 \leq \deg(g_2) \leq 8$ and $7 \leq \deg(g_3) \leq 12$.

For an elliptic surface (S,π,C), a fibre $\pi^{-1}(p)$ $(p \in C)$ is generically a non-singular elliptic curve. But, for some $q \in C$, $\pi^{-1}(q)$ is not a non-singular elliptic curve. In this case, we call $\pi^{-1}(q)$ a singular fibre.

If we have a Kodaira normal form (0.2.2) of $(S,\pi,\mathbb{P}^1(\mathbb{C}))$, we can obtain the singular fibres of $(S,\pi,\mathbb{P}^1(\mathbb{C}))$. See Table 1.

	$\mathrm{ord}_t(g_2)$	$\mathrm{ord}_t(g_3)$	$\mathrm{ord}_t(D)$	The Type of Singular Fibre
(1)	0	0	b	I_b
(2)	≥ 2	≥ 3	$b+6$	I_b^*
(3)	≥ 1	1	2	II
(4)	≥ 2	2	4	IV
(5)	≥ 3	4	8	IV^*
(6)	≥ 4	5	10	II^*
(7)	1	≥ 2	3	III
(8)	3	≥ 5	9	III^*

Table 1: The singular fibres for the elliptic fibration.

Here, the types of singular fibre is due to Kodaira [Kod]. The irreducible components of exceptional curves coming from the canonical resolutions of singular fibres are illustrated in Figure 5, 6, 7.

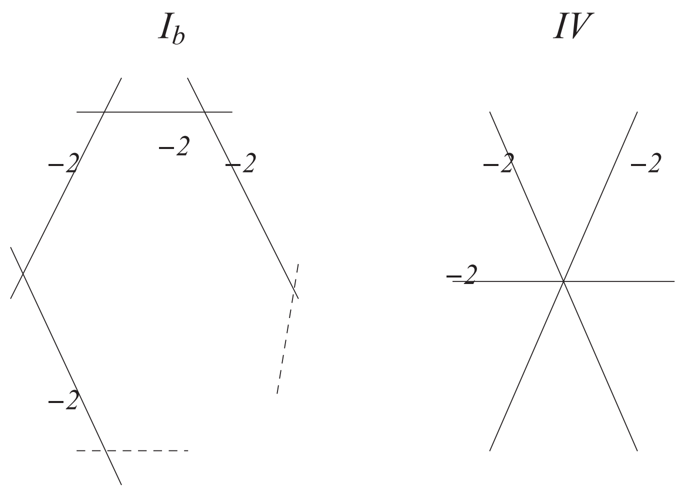


Figure 5: The singular fibres of type I_b and IV.

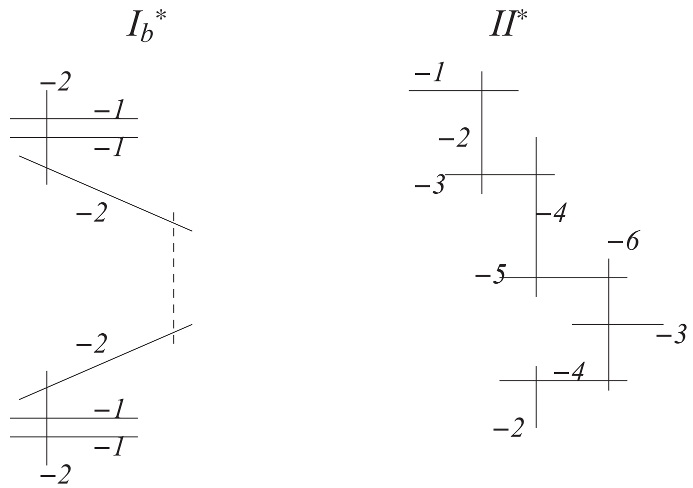


Figure 6: The singular fibres of type I_b^* and II^*.

0.2.3 The Mordell-Weil group of sections

We shall use the theory of the Mordell-Weil lattices due to T. Shioda. For detail, see [Sho1] and [Sho2].

Let S be a compact complex surface and C be a algebraic curve. Let $\pi : S \to C$ be an elliptic fibration with sections. For generic $v \in C$, the fibre $\pi^{-1}(v)$ is an elliptic curve. In the following, we assume that the elliptic fibration $\pi : S \to C$ has singular fibres. $\mathbb{C}(C)$ denotes the field of meromorphic functions on C. If $C = \mathbb{P}^1(\mathbb{C})$, the field $\mathbb{C}(C)$ is isomorphic to the field $\mathbb{C}(t)$ of rational functions.

Here, $E(\mathbb{C}(C))$ denotes the Mordell-Weil group of sections of $\pi : S \to C$. For all $P \in E(\mathbb{C}(C))$ and $v \in C$, we have $(P \cdot \pi^{-1}(v)) = 1$. Note that the section P intersects an irreducible component with multiplicity 1 of every fibre $\pi^{-1}(v)$. Let O be the zero of the group $E(\mathbb{C}(C))$. The section O is given by the set of the points at infinity on every generic fibre.

Set

$$R = \{v \in C | \pi^{-1}(C) \text{ is a singular fibre of } \pi\}.$$

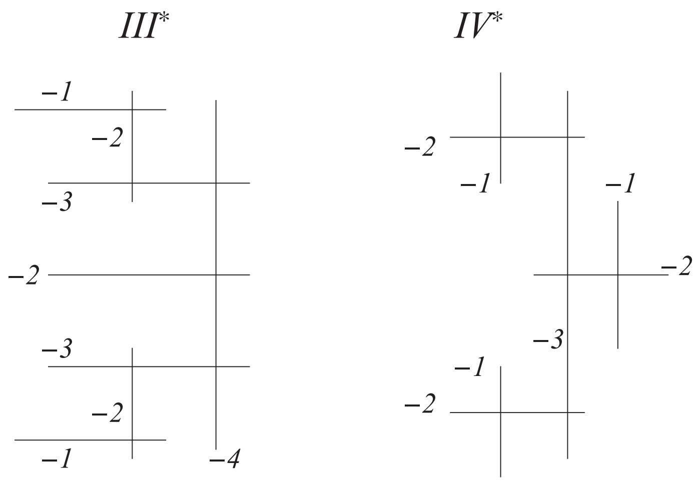


Figure 7: The singular fibres of type III^* and IV^*.

For all $v \in R$, we have

$$\pi^{-1}(v) = \Theta_{v,0} + \sum_{j=1}^{m_v-1} \mu_{v,j}\Theta_{v,j}, \tag{0.2.3}$$

where m_v is the number of irreducible components of $\pi^{-1}(v)$, $\Theta_{v,j}$ $(j = 0, \cdots, m_v - 1)$ are irreducible components with multiplicity $\mu_{v,j}$ of $\pi^{-1}(v)$, and $\Theta_{v,0}$ is the component with $\Theta_{v,0} \cap O \neq \phi$.

Let F be a generic fibre of π. Set

$$T = \langle F, O, \Theta_{v,j} | v \in R, 1 \leq j \leq m_v - 1\rangle_{\mathbb{Z}} \subset \mathrm{NS}(S).$$

We call T the trivial lattice for π. For $P \in E(\mathbb{C}(C))$, $(P) \in \mathrm{NS}(S)$ denotes the corresponding element.

Theorem 0.2.3. (Shioda [Sho1], see also [Sho2] Theorem (3·10))
(1) *The Mordell-Weil group* $E(\mathbb{C}(C))$ *is a finitely generated Abelian group.*
(2) *The Néron-Severi group* $\mathrm{NS}(S)$ *is a finitely generated Abelian group and torsion free.*
(3) *We have the isomorphism of groups* $E(\mathbb{C}(C)) \simeq \mathrm{NS}(S)/T$ *given by*

$$P \mapsto (P) \mod T.$$

We set $\hat{T} = (T \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \mathrm{NS}(S)$ for the trivial lattice T.

Corollary 0.2.1. ([Sho1], see also [Sho2] Proposition (3·11))
(1)

$$\mathrm{rank}(E(\mathbb{C}(C))) = \mathrm{rank}(\mathrm{NS}(S)) - 2 - \sum_{v \in R}(m_v - 1).$$

(2) *Let* $E(\mathbb{C}(C))_{tor}$ *be the torsion part of* $E(\mathbb{C}(C))$*. Then,*

$$E(\mathbb{C}(C))_{tor} \simeq \hat{T}/T.$$

Set

$$E(\mathbb{C}(C))^0 = \{P \in E(\mathbb{C}(C)) | P \cap \Theta_{v,0} \neq \phi \text{ for all } v \in R\}.$$

We have

$$E(\mathbb{C}(C))^0 \subset E(\mathbb{C}(C))/E(\mathbb{C}(C))_{tor} \tag{0.2.4}$$

(see [Sho1], see also [Sho2] Section 5).

Let $v \in R$. Under the notation (0.2.3), we set

$$(\pi^{-1}(v))^\sharp = \bigcup_{0 \leq j \leq m_v - 1,\, \mu_{v,j}=1} \Theta^\sharp_{v,j},$$

where $\Theta^\sharp_{v,j} = \Theta_{v,j} - \{\text{singular points of } \pi^{-1}(v)\}$. Set $m_v^{(1)} = \sharp\{j | 0 \leq j \leq m_v - 1,\, \mu_{v,j} = 1\}$.

Theorem 0.2.4. ([Ne], [Kod], see also [Sho2] Section 7) *Let $v \in R$. The set $(\pi^{-1}(v))^\sharp$ has a canonical group structure.*

Remark 0.2.1. *Especially, for the singular fibre $\pi^{-1}(v)$ of type I_b $(b \geq 1)$, we have*

$$(\pi^{-1}(v))^\sharp \simeq \mathbb{C}^\times \times (\mathbb{Z}/b\mathbb{Z}).$$

For the singular fibre $\pi^{-1}(v)$ of type I_b^* $(b \geq 0)$, we have

$$(\pi^{-1}(v))^\sharp \simeq \begin{cases} \mathbb{C} \times (\mathbb{Z}/4\mathbb{Z}) & (b \in 2\mathbb{Z}+1), \\ \mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})^2 & (b \in 2\mathbb{Z}). \end{cases}$$

For each $v \in C$, we introduce the mapping

$$sp_v : E(\mathbb{C}(C)) \to (\pi^{-1}(v))^\sharp : P \mapsto P \cap \pi^{-1}(v).$$

Note that

$$P \cap \pi^{-1}(v) = (x, a) \in \begin{pmatrix} \mathbb{C}^\times \\ \mathbb{C} \end{pmatrix} \times \{\text{finite group}\}$$

(see [Sho2] Section 7). We call sp_v the specialization mapping.

Theorem 0.2.5. ([Sho2] Section 7) *For all $v \in C$, the specialization mapping*

$$sp_v : P \mapsto (x, a) \in \begin{pmatrix} \mathbb{C}^\times \\ \mathbb{C} \end{pmatrix} \times \{\text{finite group}\}$$

is a homomorphism of groups.

Remark 0.2.2. *Especially for the singular fibre $\pi^{-1}(v)$ of type I_b (I_b^*, resp.), the projection of sp_v*

$$E(\mathbb{C}(C)) \to (\mathbb{Z}/b\mathbb{Z}) \quad ((\mathbb{Z}/4\mathbb{Z}) \text{ or } (\mathbb{Z}/2\mathbb{Z})^2, \text{ resp.})$$

is a homomorphism of groups.

Proposition 0.2.2. ([Sho1] or [Sho2]) *For an elliptic* $K3$ *surface* $(S, \pi, \mathbb{P}^1(\mathbb{C}))$*, let* F *be a general fibre and* P *be a section of* π*. Then,*

$$(F \cdot F) = 0, \qquad (F \cdot P) = 1, \qquad (P \cdot P) = -2.$$

Lemma 0.2.1. *Let* S *be a* $K3$ *surface with the elliptic fibration* $\pi : S \to \mathbb{P}^1(\mathbb{C})$ *and* F *be a fixed general fibre. Then,* π *is the unique elliptic fibration up to* $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ *which has* F *as a general fibre.*

Proof. Note that $\pi \in H^0(S, \mathcal{O}_S(F))$. We shall prove

$$\dim(H^0(S, \mathcal{O}_S(F))) = 2.$$

By Serre's duality,

$$H^2(S, \mathcal{O}_S(F)) \simeq H^0(S, \mathcal{O}_S(K_S - F)) = H^0(S, \mathcal{O}_S(-F)) = 0.$$

So, by the Riemann-Roch Theorem and Proposition 0.2.2, we see that

$$\chi(\mathcal{O}_S(F)) = \chi(\mathcal{O}_S) = 2.$$

Then, we have

$$0 - \dim(H^1(S, \mathcal{O}_S(F))) + \dim(H^0(S, \mathcal{O}_S(F))) = 2.$$

From the exact sequence,

$$0 \to \mathcal{O}_S(-F) \to \mathcal{O}_S \to \mathcal{O}_F \to 0,$$

we obtain the exact sequence

$$\cdots \to H^0(S, \mathcal{O}_S) \to H^0(F, \mathcal{O}_F) \to H^1(S, \mathcal{O}_S(-F)) \to H^1(S, \mathcal{O}_S) \to \cdots.$$

Because S is a $K3$ surface, it holds that $H^1(S, \mathcal{O}_S) = 0$. Moreover, $H^0(S, \mathcal{O}_S) \to H^0(F, \mathcal{O}_F)$ is an onto mapping. Therefore, we have

$$H^1(S, \mathcal{O}_S(F)) = H^1(S, \mathcal{O}_S(-F)) = 0.$$

Hence, we see that $\dim(H^0(S, \mathcal{O}_S(F))) = 2$. □

Chapter 1

Periods for the families of $K3$ surfaces with 2 parameters derived from the reflexive polytopes

To obtain an extension of the theory of classical elliptic functions, we need elliptic $K3$ surfaces with explicit defining equations. In this part, we use 3-dimensional reflexive polytopes with 5 vertices to obtain $K3$ surfaces. We have the families $\mathcal{F}_j$ $(j = 0, 1, 2, 3)$ of $K3$ surfaces with 2 complex parameters from each polytope. We determine the generic Picard numbers (Section 1.3), the Néron-Severi lattices and the transcendental lattices (Section 1.4) of these family $\mathcal{F}_j$ $(j =, 0, 1, 2, 3)$ of $K3$ surfaces. We have the multivalued period mappings for $\mathcal{F}_j$ $(j = 0, 1, 2, 3)$. We determine the projective monodromy groups of these period mappings applying the Torelli theorem for marked $K3$ surfaces (Section 1.5).

1.1 Toric varieties derived from reflexive polytopes

The reflexive polytopes is introduced by Batyrev [Ba] to study the mirror symmetry of Calabi-Yau varieties. In this section, we survey the basic result of the reflexive polytopes. For detail, see [Ba] or [Od].

Set $N = \mathbb{Z}^r$, $N_{\mathbb{R}} = N \otimes \mathbb{R}$, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \simeq \mathbb{Z}^r$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. Let $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ be the canonical $\mathbb{Z}-$bilinear mapping. The pairing $\langle \cdot, \cdot \rangle$ is extended to the $\mathbb{R}$-bilinear mapping $M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$.

If $n_1, \cdots, n_r \in N_{\mathbb{R}}$ are given, we call the set $\sigma = \mathbb{R}_{\geq 0} n_1 + \cdots + \mathbb{R}_{\geq 0} n_r$ a cone. Set $\sigma^{\vee} = \{x \in M_{\mathbb{R}} | \langle x, y \rangle \geq 0, \text{ for all } y \in \sigma\}$. This is called the dual to σ. We call the subset τ of σ a face if $\tau = \{y \in \sigma | \langle m_0, y \rangle = 0\}$ for $m_0 \in \sigma^{\vee}$. If Δ be a set of cones with the two properties
(i) every face of $\sigma \in \Delta$ is contained in Δ,
(ii) if $\sigma_1, \sigma_2 \in \Delta$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2,
then Δ is called a fan.

Letting $\mathcal{S}_\sigma = M \cap \sigma^{\vee}$, set

$$U_\sigma = \{u : \mathcal{S}_\sigma \to \mathbb{C} | u(0) = 1, u(m + n) = u(m)u(n), \text{ for all } m, n \in \mathcal{S}_\sigma\}.$$

Proposition 1.1.1. *The set*

$$T_N emb(\Delta) = \bigcup_{\sigma \in \Delta} U_\sigma$$

gives an irreducible and normal variety of r dimension. Setting $e(m)(u) = u(m)$ for $m \in \mathcal{S}_\sigma$ and $u \in U_\sigma$,

$$(e(m_1), \cdots, e(m_p)) : U_\sigma \to \mathbb{C}^p$$

defines an one to one mapping and U_σ coincides with the set of $\mathbb{C}$-valued points of the affine scheme $\mathrm{Spec}(\mathbb{C}[\mathcal{S}_\sigma])$.

The variety $T_N emb(\Delta)$ is called a toric variety.

If $\sigma_2 \subset \sigma_1$, then $U_{\sigma_2} \subset U_{\sigma_1}$. Especially, any U_σ contains the algebraic torus $T_N = \mathrm{Hom}(M, \mathbb{C}^\times)$.

Proposition 1.1.2. *The toric variety $T_N emb(\Delta)$ associated to a fan Δ is non-singular complex manifold if and only if there exist a $\mathbb{Z}$-basis $\{n_1, \cdots, n_r\}$ of N and $s \le r$ such that $\sigma = \mathbb{R}_{\ge 0} n_1 + \cdots + \mathbb{R}_{\ge 0} n_s$ for any $\sigma \in \Delta$. The toric variety $T_N emb(\Delta)$ is compact if and only if Δ is a finite and complete fan, i.e., Δ is a finite set with the support $|\Delta| = \bigcup_{\sigma \in \Delta} \sigma$ coinciding with the entire $N_\mathbb{R}$.*

If $v \in N_\mathbb{R}$ and $b \in \mathbb{R}$ are given, set $H(v, b) = \{u \in M_\mathbb{R} | \langle u, v \rangle \ge b\}$. We call

$$P = \bigcap_{j=1}^{s} H(v_j, b_s)$$

a polyhedron. A bounded polyhedron is called a polytope.

If r-dimensional polytope $P(\subset M_\mathbb{R})$ is given, take every point $m_0, \cdots, m_s$ of $M \cap P$. We take dual σ_j to the cone $\sum_{k \ne j} \mathbb{R}_{\ge 0}(m_k - m_j)$. Let $\Delta(P)$ be the fan consisting of all faces of $\sigma_0, \cdots, \sigma_s$. Then, we obtain a toric variety $T_n emb(\Delta(P))$.

Definition 1.1.1. *If a polytope*

$$P = \bigcap_{j=1}^{s} H(v_j, -1)$$

contains the origin as a inner point, we call P a reflexive polytope. Moreover, if every vertex of P is a lattice point, the origin is the unique inner lattice point and only the vertices are the lattice points on the boundary, we call P a reflexive polytope at most terminal singularities.

In the following, we consider the toric variety associated to a finite and complete fan. Let $X = T_N emb(\Delta)$ and $\Delta(1)$ be the set of 1-dimensional cones of Δ. For $\rho \in \Delta(1)$, let $n(\rho)$ be the primitive element of ρ.

If a continuous function h on $N_\mathbb{R}$ is linear on $\sigma \in \Delta$ and $h(y) \in \mathbb{Z}$ for $y \in N$, h is called Δ-linear support function. Let $\mathrm{SF}(N, \Delta)$ be the set of Δ-linear support functions. If $h \in \mathrm{SF}(N, \Delta)$, there exists $l_\sigma \in M$ such that $h(n) = \langle l_\sigma, n \rangle$ for any $n \in \sigma$.

Let σ and $\tau \in \Delta$. Since $\sigma \cap \tau \in \Delta$, we have

$$h(n) = \langle l_\sigma, n\rangle = \langle l_{\sigma\cap\tau}, n\rangle = \langle l_\tau, n\rangle$$

for any $n \in \sigma \cap \tau$. So, we obtain

$$\langle l_\sigma - l_\tau, n\rangle = \langle l_\tau - l_\sigma, n\rangle = 0$$

and $l_\sigma - l_\tau$ and $l_\tau - l_\sigma \in \mathcal{S}_{\sigma\cap\tau}$. Then, we have

$$e(l_\sigma - l_\tau) \in \mathcal{O}_X^*(U_\sigma \cap U_\tau).$$

So, $\{e(l_\sigma - l_\tau)\}$ gives a system of transition functions and define a line bundle over X. This line bundle is denoted by L_h.

On the other hand, for $h \in \mathrm{SF}(N, \Delta)$, we define a Weil divisor

$$D_h = -\sum_{\rho\in\Delta(1)} h(n(\rho))V(\rho).$$

This is a divisor given by the defining equation $e(-l_\sigma) = 0$ on U_σ. We note that $[D_h] = L_h$.

For $h \in \mathrm{SF}(N, \Delta)$, we set

$$\Box_h = \{m \in M_\mathbb{R} | \langle m, n\rangle \geq h(n), \text{ for any } n \in N_\mathbb{R}\}.$$

We can prove that $\Box_h$ is a polytope.

Proposition 1.1.3. *The cohomology group $H^0(X, \mathcal{O}_X(D_h))$ is a finitely dimensional vector space. Moreover, a system of generators of $H^0(X, \mathcal{O}_X(D_h))$ is given by $\{e(m)|m \in \Box_h \cap M\}$.*

Proposition 1.1.4. *In $h \in \mathrm{SF}(N, \Delta)$ satisfies $h(n_1) + h(n_2) \leq h(n_1 + n_2)$ for n_1 and $n_2 \in N_\mathbb{R}$. Then*

$$H^q(X, \mathcal{O}_X(D_h)) = 0$$

for $q \geq 1$.

So, we consider the anti-canonical bundle $-K_X$ of X. If a reflexive polytope P with at most terminal singularities is given, there exists $k \in \mathrm{SF}(N, \Delta(P))$ such that D_k coincides with $-K_X$. Moreover, $\Box_k = P$ holds. Therefore, we see

$$H^0(X, \mathcal{O}_X(-K_X)) = \langle e(m)|m \in P \cap M\rangle_\mathbb{C}.$$

From Proposition 1.1.4, we have

Proposition 1.1.5. *For $q \geq 1$,*

$$H^q(X, \mathcal{O}_X(-K_X)) = 0.$$

If the fan $\Delta(P)$ is non-singular, we take $k \in \mathrm{SF}(N, \Delta)$ such that $k(n(\rho)) = -1$ for any $\rho \in \Delta(P)$.

Example 1.1.1. *Let $r = 2$. Set*

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}.$$

We can check that $X = T_N emb(\Delta(P))$ is $\mathbb{P}^2(\mathbb{C})$.

So, we obtain that

$$H^0(X, \mathcal{O}_X(-K_X)) = \langle e(m) | m \in P \cap M \rangle_{\mathbb{C}}$$
$$= \left\{ a_1 + a_2 t_1 + a_3 t_2 + a_4 \frac{t_1^2}{t_2} + a_5 \frac{t_1}{t_2} + a_6 \frac{1}{t_2} + a_7 \frac{1}{t_1 t_2} + a_8 \frac{1}{t_1} + a_9 \frac{t_2}{t_1} + a_{10} \frac{t_2^2}{t_1} | a_j \in \mathbb{C} \right\}$$

This is equal to the set of homogenous equations of order three. Therefore, this coincides with the famous result $K_{\mathbb{P}^2(\mathbb{C})} = -3H$, where H is a hyperplane section of $\mathbb{P}^2(\mathbb{C})$.

1.2 A family of $K3$ surfaces and elliptic fibration

To obtain families of $K3$ surfaces with explicit defining equations, we use the 3-dimensional reflexive polytopes with at most terminal singularities. These polytopes with 5-vertices are given as

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}, \tag{1.2.1}$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}, \tag{1.2.2}$$

$$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \tag{1.2.3}$$

where the column vectors correspond to the coordinates of the vertices (see [Ot] or [KS]).

Among the polytopes in (1.2.1), (1.2.2) and (1.2.3), P_0, P_2, P_3 and P_4 are the Fano polytopes.

Let us start from the polytope P_0 in (1.2.1). We obtain a family of algebraic $K3$ surfaces from P_0 by the following canonical procedure (for detail, see [Od] Chapter 2):

(i) Make a toric 3-fold X from the reflexive polytope P_0. This is a rational variety.

(ii) Take a divisor D on X that is linearly equivalent to $-K_X$.

(iii) Generically, D is represented by a $K3$ surface.

In this case, D is given by

$$a_1 + a_2 t_1 + a_3 t_2 + a_4 t_3 + a_5 \frac{1}{t_3} + a_6 \frac{1}{t_1 t_2 t_3^2} = 0, \tag{1.2.4}$$

with complex parameters $a_1, \cdots, a_6$. Every monomial in the left hand side corresponds to a lattice point in P_0. Setting

$$x = \frac{a_2 t_1}{a_1}, \quad y = \frac{a_3 t_2}{a_1}, \quad z = \frac{a_4 t_3}{a_1}, \quad \lambda = \frac{a_4 a_5}{a_1^2}, \quad \mu = \frac{a_2 a_3 a_4^2 a_6}{a_1^5}, \tag{1.2.5}$$

we obtain a family of $K3$ surfaces $\mathcal{F}_0 = \{S_0(\lambda, \mu)\}$ with two parameters λ, μ with

$$S_0(\lambda, \mu) : F_0(x, y, z) = xyz^2(x + y + z + 1) + \lambda xyz + \mu = 0. \tag{1.2.6}$$

In the same way, we obtain the corresponding families of $K3$ surfaces $\mathcal{F}_j = \{S_j(\lambda, \mu)\}$ for P_j $(j = 1, 2, 3)$ in (1.2.2) given by the affine equations

$$S_1(\lambda, \mu) : F_1(x, y, z) = xyz(x + y + z + 1) + \lambda x + \mu y = 0, \tag{1.2.7}$$
$$S_2(\lambda, \mu) : F_2(x, y, z) = xyz(x + y + z + 1) + \lambda x + \mu = 0, \tag{1.2.8}$$
$$S_3(\lambda, \mu) : F_3(x, y, z) = xyz(x + y + z + 1) + \lambda z + \mu xy = 0. \tag{1.2.9}$$

Remark 1.2.1. *Recently, Ishige [I2] has made a research on the family $\mathcal{F}_4$ derived from the polytope P_4 in (1.2.3). He made a computer aided approximation of a generator of the monodromy group of his differential equation. There, he noticed that his monodromy group is isomorphic to the extended Hilbert modular group for $\mathbb{Q}(\sqrt{2})$.*

In this section, we give elliptic fibrations for our families $\mathcal{F}_j$ $(j = 0, 1, 2, 3)$ of $K3$ surfaces. The singular fibres of these fibration are given as in Table 1.1.

Family	$\mathcal{F}_0$	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$
Singular Fibres	$I_3 + I_{15} + 6I_1$	$I_9 + I_3^* + 6I_1$	$I_1^* + I_{11} + 6I_1$	$I_9 + I_9 + 6I_1$

Table 1.1: The types of singular fibres for our families.

1.2.1 Elliptic fibration for $\mathcal{F}_0$

Proposition 1.2.1. (1) *The surface $S_0(\lambda, \mu)$ is birationally equivalent to the surface defined by the equation*

$$y_1^2 = 4x_0^3 + (\lambda^2 + 2\lambda z + z^2 + 2\lambda z^2 + 2z^3 + z^4)x_0^2 + (-2\lambda\mu z - 2\mu z^2 - 2\mu z^3)x_0 + \mu^2 z^2 \tag{1.2.10}$$

This equation gives an elliptic fibration of $S_0(\lambda, \mu)$ over z-sphere.

(2) *The elliptic surface given by* (1.2.10) *has the holomorphic sections*

$$\begin{cases} Q : z \mapsto (x_0, y_1, z) = (0, \mu z, z), \\ R : z \mapsto (x_0, y_1, z) = (0, -\mu z, z). \end{cases} \tag{1.2.11}$$

Proof. (1) By the birational transformation

$$x = \frac{-\mu}{x_0}, \quad y = \frac{-\lambda x_0 - y_1 + \mu z - x_0 z - x_0 z^2}{2x_0 z},$$

(1.2.6) is transformed to (1.2.10).

(2) This is clear. □

Set

$$\Lambda_0 = \{(\lambda, \mu) \in \mathbb{C}^2 | \lambda\mu(\lambda^2(4\lambda - 1)^3 - 2(2 + 25\lambda(20\lambda - 1))\mu - 3125\mu^2) \neq 0\}. \tag{1.2.12}$$

Proposition 1.2.2. *Suppose $(\lambda,\mu)\in\Lambda_0$. The elliptic surface given by (1.2.10) has the singular fibres of type I_3 over $z=0$, of type I_{15} over $z=\infty$ and other six fibres of type I_1.*

Proof. (1.2.10) is described in the Kodaira normal form

$$y_1^2=4x_1^3-g_2(z)x_1-g_3(z),\ z\neq\infty, \tag{1.2.13}$$

with

$$\begin{cases} g_2(z)=\dfrac{1}{216}(18\lambda^4+432\lambda\mu z+72\lambda^3 z(1+z)+108\lambda^2z^2(1+z)^2\\ \qquad\qquad +72\lambda z^3(1+z)^3+18z^2(1+z)(24\mu+z^2(1+z)^3)),\\ g_3(z)=\dfrac{-1}{216}(\lambda^6+36\lambda^3\mu z+6\lambda^5z(1+z)+108\lambda^2\mu z^2(1+z)+15\lambda^4z^2(1+z)^2\\ \qquad +108\lambda\mu z^3(1+z)^2+20\lambda^3z^3(1+z)^3+15\lambda^2z^4(1+z)^4+6\lambda z^5(1+z)^5\\ \qquad\qquad +z^2(216\mu^2+36\mu z^2(1+z)^3+z^4(1+z)^6)),\end{cases}$$

and

$$y_2^2=4x_2^3-h_2(z_1)x_2-h_3(z_1),\ z_1\neq\infty, \tag{1.2.14}$$

with

$$\begin{cases} h_2(z_1)=2\mu z_1^5(1+z_1+\lambda z_1^2)+\dfrac{1}{12}(1+z_1+\lambda z_1^2)^4,\\ h_3(z_1)=-(\dfrac{1}{6}\mu z_1^5(1+z_1+\lambda z_1^2)^3+\dfrac{1}{216}(1+z_1+\lambda z_1^2)^6+\mu^2z_1^{10}),\end{cases}$$

where $z_1=1/z$. We have the discriminant of the right hand side of (1.2.13) for x_1 ((1.2.14) for x_2, resp.):

$$\begin{cases} D_0=64\mu^3z^3(\lambda^3+3\lambda^2z+27\mu z+3\lambda z^2+3\lambda^2z^2+z^3+6\lambda z^3+3z^4+3\lambda z^4+3z^5+z^6),\\ D_\infty=64\mu^3z_1^{15}(1+3z_1+3z_1^2+3\lambda z_1^2+z_1^3+6\lambda z_1^3+3\lambda z_1^4+3\lambda^2z_1^4+3\lambda^2z_1^5+27\mu z_1^5+\lambda^3z_1^6),\end{cases} \tag{1.2.15}$$

respectively.

From these data, we obtain the required statement (see [Kod]). □

Remark 1.2.2. *We have a parametrization*

$$\lambda(a)=\frac{(a-1)(a+1)}{5},\quad \mu(a)=\frac{(2a-3)^3(a+1)^2}{3125}$$

of the locus $\lambda^2(4\lambda-1)^3-2(2+25\lambda(20\lambda-1))\mu-3125\mu^2=0$. It is a rational curve. In Section 2.1, we shall obtain the above Λ_0 as the complement of the singular locus of the period differential equation for $\mathcal{F}_0$ in the (λ,μ)-space.

Remark 1.2.3. *Let χ denote the Euler characteristic. According to* [Kod] Theorem 12.1 *(see also* [Shg2]*), an elliptic fibred algebraic surface S over $\mathbb{P}^1(\mathbb{C})$ is a $K3$ surface if and only if $\chi(S)=24$ provided S is given in the Kodaira normal form. Due to this criterion and* Proposition 1.2.2, *we can check directly that $S_0(\lambda,\mu)$ is a $K3$ surface for $(\lambda,\mu)\in\Lambda_0$.*

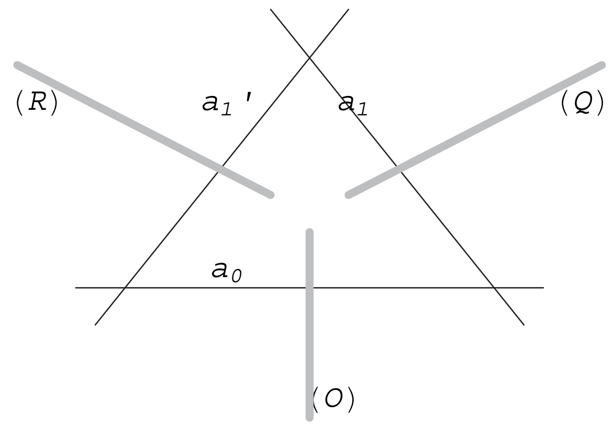


Figure 1.1: The singular fibre at $z = 0$.

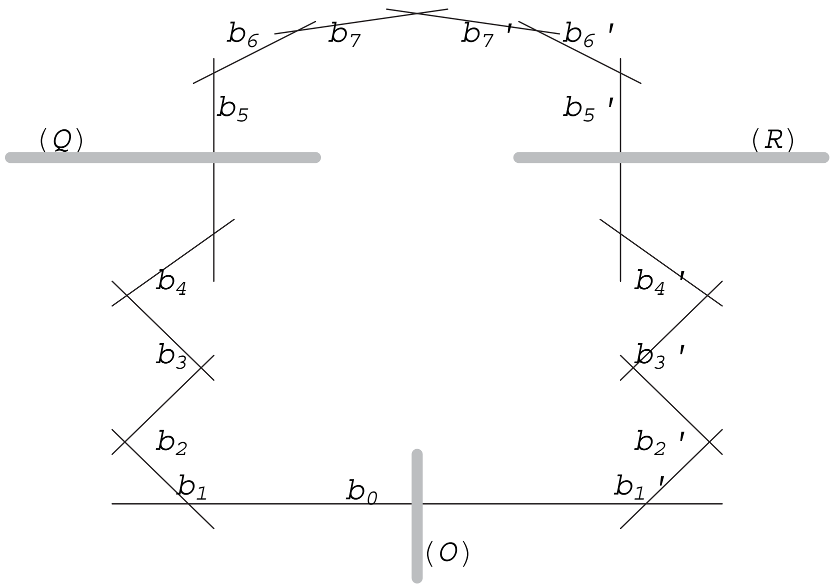


Figure 1.2: The singular fibre at $z = \infty$.

For $(\lambda, \mu) \in \Lambda_0$, let O be the zero of the Mordell-Weil group of sections of the elliptic fibration given by (1.2.10) over $\mathbb{C}(z)$. O is given by the set of the points at infinity on every fibre. Let Q and R be the sections in (1.2.11). R is the inverse element of Q in the Mordell-Weil group. Let F be a general fibre of this fibration. Let $I_3 = a_0 + a_1 + a_1'$ be the irreducible decomposition of the fibre at $z = 0$ given as in Figure 1.1. We may suppose $O \cap a_0 \neq \phi, Q \cap a_1 \neq \phi$ and $R \cap a_1' \neq \phi$. By the same way, let $I_{15} = b_0 + b_1 + \cdots + b_7 + b_1' + \cdots + b_7'$ be the irreducible decomposition of the fibre at $z = \infty$ given as in Figure 1.2. We may suppose $O \cap b_0 \neq \phi, Q \cap b_5 \neq \phi$ and $R \cap b_5' \neq \phi$.

We set a sublattice $L_0 = L_0(\lambda, \mu) \subset H_2(S_0(\lambda, \mu), \mathbb{Z})$ for $(\lambda, \mu) \in \Lambda_0$ by

$$L_0(\lambda, \mu) = \langle b_1, b_2, b_3, b_4, b_5, Q, b_6, b_7, b_1', b_2', b_3', b_4', b_5', R, b_6', b_7', F, O \rangle_{\mathbb{Z}}. \qquad (1.2.16)$$

Set

$$A_{18}(-1) = \underbrace{\begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & \ddots & & & \\ & & & \ddots & & & \\ & & & \ddots & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{pmatrix}}_{18}.$$

Let $E_{i,j}$ $(1 \leq i, j \leq 18)$ be the matrix unit. We obtain the corresponding intersection matrix M_0 for L_0:

$$\begin{aligned} M_0 =& A_{18}(-1) + 2E_{17,17} - (E_{6,7} + E_{7,6}) + (E_{5,7} + E_{7,5}) - (E_{14,15} + E_{15,14}) + (E_{13,15} + E_{15,13}) \\ & - (E_{8,9} + E_{9,8}) - (E_{16,17} + E_{17,16}) + (E_{6,17} + E_{17,6}) + (E_{8,16} + E_{16,8}) + (E_{14,17} + E_{17,14}). \end{aligned} \tag{1.2.17}$$

We have

$$\det(M_0) = -5. \tag{1.2.18}$$

Therefore, the generators of L_0 are independent.

1.2.2 Elliptic fibration for $\mathcal{F}_1$

Proposition 1.2.3. *The surface $S_1(\lambda, \mu)$ is birationally equivalent to the surface defined by the equation*

$$z_1^2 = y_1^3 + (\mu^2 + 2\mu x_1 + x_1^2 - 4x_1^3)y_1^2 + (-8\lambda\mu x_1^3 - 8\lambda x_1^4)y_1 + 16\lambda^2 x_1^6. \tag{1.2.19}$$

This equation gives an elliptic fibration of $S_1(\lambda, \mu)$ with the holomorphic section

$$Q : x_1 \mapsto (x_1, y_1, z_1) = (x_1, 0, 4\lambda x_1^3). \tag{1.2.20}$$

Proof. By the birational transformation

$$x = -\frac{2x_1^2 y_1}{-4\lambda x_1^3 + \mu y_1 + x_1 y_1 + z_1}, y = \frac{y_1^2}{2x_1(-4\lambda x_1^3 + \mu y_1 + x_1 y_1 + z_1)},$$
$$z = -\frac{-4\lambda x_1^3 + \mu y_1 + x_1 y_1 + z_1}{2x_1 y_1},$$

(1.2.7) is transformed to (1.2.19). □

(1.2.19) gives an elliptic fibration for the surface $S_1(\lambda, \mu)$. Set

$$\Lambda_1 = \{(\lambda, \mu) \in \mathbb{C}^2 | \lambda\mu(729\lambda^2 - 54\lambda(27\mu - 1) + (1 + 27\mu)^2 \neq 0)\}. \tag{1.2.21}$$

Proposition 1.2.4. *Suppose $(\lambda, \mu) \in \Lambda_1$. The elliptic surface given by* (1.2.19) *has the singular fibres of type I_9 over $x_1 = 0$, of type I_3^* over $x_1 = \infty$ and other six fibres of type I_1.*

Proof. (1.2.19) is described in the Kodaira normal form

$$z_2^2 = 4y_2^3 - g_2(x_1)y_2 - g_3(x_1), \quad x_1 \neq \infty, \tag{1.2.22}$$

with

$$\begin{cases} g_2(x_1) = -4\Big(-\frac{\mu^4}{3} - \frac{4\mu^3 x_1}{3} - 2\mu^2 x_1^2 - \frac{4\mu x_1^3}{3} - 8\lambda\mu x_1^3 \\ \qquad + \frac{8\mu^2 x_1^3}{3} - \frac{x_1^4}{3} - 8\lambda x_1^4 + \frac{16\mu x_1^4}{3} + \frac{8x_1^5}{3} - \frac{16x_1^6}{3}\Big), \\ g_3(x_1) = -4\Big(\frac{2\mu^6}{27} + \frac{4\mu^5 x_1}{9} + \frac{10\mu^4 x_1^2}{9} + \frac{40\mu^3 x_1^3}{27} + \frac{8\lambda\mu^3 x_1^3}{3} - \frac{8\mu^4 x_1^3}{9} + \frac{10\mu^2 x_1^4}{9} + 8\lambda\mu^2 x_1^4 \\ \qquad + \frac{4\mu x_1^5}{9} + 8\lambda\mu x_1^5 - \frac{16\mu^2 x_1^5}{3} + \frac{2x_1^6}{27} + \frac{8\lambda x_1^6}{3} + 16\lambda^2 x_1^6 - \frac{32\mu x_1^6}{9} - \frac{32\lambda\mu x_1^6}{3} \\ \qquad - \frac{32\mu^3 x_1^4}{9} + \frac{32\mu^2 x_1^6}{9} - \frac{8x_1^7}{9} - \frac{32\lambda x_1^7}{3} + \frac{64\mu x_1^7}{9} + \frac{32x_1^8}{9} - \frac{128x_1^9}{27}\Big), \end{cases}$$

and

$$z_3^2 = 4y_3^3 - h_2(x_2)y_3 - h_3(x_2), \quad x_2 \neq \infty, \tag{1.2.23}$$

with

$$\begin{cases} h_2(x_2) = -4\Big(-\frac{16x_2^2}{3} + \frac{8x_2^3}{3} - \frac{x_2^4}{3} - 8\lambda x_2^4 + \frac{16\mu x_2^4}{3} - \frac{4\mu x_2^5}{3} \\ \qquad -8\lambda\mu x_2^5 + \frac{8\mu^2 x_2^5}{3} - 2\mu^2 x_2^6 - \frac{4\mu^3 x_2^7}{3} - \frac{\mu^4 x_2^8}{3}\Big), \\ h_3(x_2) = -4\Big(-\frac{128x_2^3}{27} + \frac{32x_2^4}{9} - \frac{8x_2^5}{9} - \frac{32\lambda x_2^5}{3} + \frac{64\mu x_2^5}{9} + \frac{2x_2^6}{27} \\ \qquad +\frac{8\lambda x_2^6}{3} + 16\lambda^2 x_2^6 - \frac{32\mu x_2^6}{9} - \frac{32\lambda\mu x_2^6}{3} + \frac{32\mu^2 x_2^6}{9} + \frac{4\mu x_2^7}{9} \\ \qquad -\frac{16\mu^2 x_2^7}{3} + 8\lambda\mu^2 x_2^8 - \frac{32\mu^4 x_2^8}{9} + 8\lambda\mu x_2^7 \\ \qquad -\frac{16\mu^2 x_2^7}{3} + \frac{10\mu^2 x_2^8}{9} + 8\lambda\mu^2 x_2^8 - \frac{32\mu^4 x_2^8}{9} - \frac{32\mu^4 x_2^8}{9} \\ \qquad +\frac{10\mu^2 x_2^8}{9} + \frac{40\mu^3 x_2^{11}}{27} + \frac{8\lambda\mu^3 x_2^9}{3} - \frac{8\mu^4 x_2^9}{9} + \frac{10\mu^4 x_2^{10}}{9} + \frac{4\mu^5 x_2^{11}}{9} + \frac{2\mu^6 x_2^{12}}{27}\Big), \end{cases}$$

where $x_1 = 1/x_2$. We have the discriminant of the right hand side of (1.2.22) for y_1 ((1.2.23) for y_2, resp.):

$$\begin{cases} D_0 = 256\lambda^2 x_1^9(\lambda\mu^3 - \mu^4 + 3\lambda\mu^2 x_1 - 4\mu^3 x_1 + 3\lambda\mu x_1^2 - 6\mu^2 x_1^2 + \lambda x_1^3 + 27\lambda^2 x_1^3 \\ \qquad -4\mu x_1^3 - 36\lambda\mu x_1^3 + 8\mu^2 x_1^3 - x_1^4 - 36\lambda x_1^4 + 16\mu x_1^4 + 8x_1^5 - 16x_1^6), \\ D_\infty = 256\lambda^2 x_2^9(-16 + 8x_2 - x_2^2 - 36\lambda x_2^2 + 16\mu x_2^2 + \lambda x_2^3 + 27\lambda^2 x_2^3 - 4\mu x_2^3 - 36\lambda\mu x_2^3 \\ \qquad 8\mu^2 x_2^3 + 3\lambda\mu x_2^4 - 6\mu^2 x_2^4 + 3\lambda\mu^2 x_2^5 - 4\mu^3 x_2^5 + \lambda\mu^3 x_2^6 - \mu^4 x_2^6). \end{cases}$$

From these deta, we obtain the required statement. $\square$

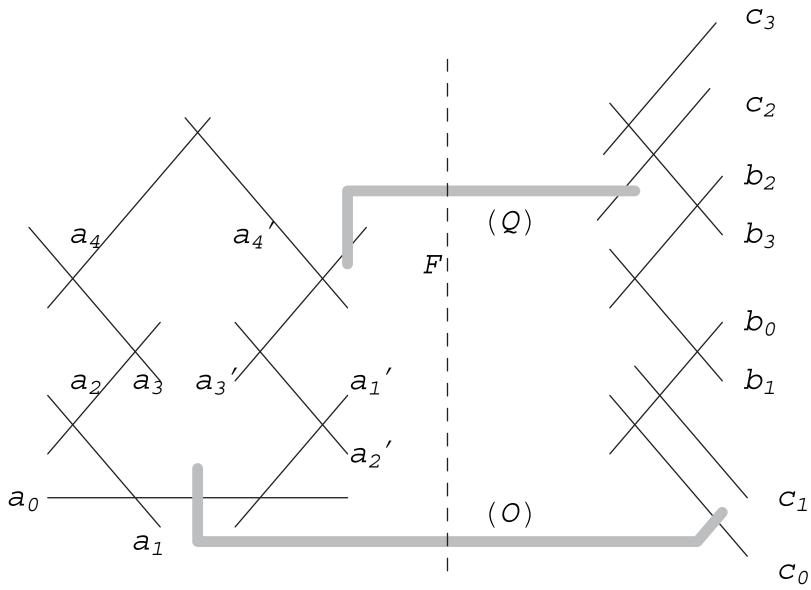


Figure 1.3: An elliptic fibration for P_1.

The elliptic fibration given by (1.2.19) is illustrated in Figure 1.3.

For this fibration, let O be the zero of the Mordell-Weil group, Q be the section in (1.2.20) and F be a general fibre. Note that $Q \cap a_3 \neq \phi$ at $x_1 = 0$ and $Q \cap c_2 \neq \phi$ at $x_1 = \infty$. Set

$$L_1 = \langle a_1, a_2, a_3, a_4, a'_4, a'_3, a'_2, a'_1, c_1, b_0, b_1, b_2, b_3, c_2, c_3, O, Q, F \rangle_{\mathbb{Z}}. \tag{1.2.24}$$

We have the following intersection matrix M_1 for L_1:

$$\begin{aligned} M_1 = &A_{18}(-1) - (E_{8,9} + E_{9,8}) - (E_{14,15} + E_{15,14}) + (E_{13,15} + E_{15,13}) + (E_{3,17} + E_{17,3}) \\ &+ (E_{14,17} + E_{17,14}) - (E_{16,17} + E_{17,16}) + (E_{16,18} + E_{18,16}) - (E_{15,16} + E_{16,15}) + 2E_{18,18}. \end{aligned} \tag{1.2.25}$$

We have $\det(M_1) = -9$. Therefore, the generators of L_1 are independent.

1.2.3 Elliptic fibration for $\mathcal{F}_2$

Proposition 1.2.5. *The surface $S_2(\lambda, \mu)$ is birationally equivalent to the surface defined by the equation*

$$z_1^2 = x_1^3 + (-4\lambda y + y^2 + 2y^3 + y^4)x_1^2 + (-8\mu y^3 - 8\mu y^4)x_1 + 16\mu^2 y^4. \tag{1.2.26}$$

This equation gives an elliptic fibration of $S_2(\lambda, \mu)$ with the holomorphic section

$$Q : y \mapsto (x_1, y, z_1) = (0, y, 4\mu y^2) \tag{1.2.27}$$

Proof. By the birational transformation

$$x = \frac{x_1^2}{2y(x_1 y - 4\mu y^2 + x_1 y + z_1)}, z = -\frac{x_1 y - 4\mu y^2 + x_1 y + z_1}{2x_1 y},$$

(1.2.8) is transformed to (1.2.26). □

(1.2.26) gives an elliptic fibration for $S_2(\lambda,\mu)$. Set

$$\Lambda_2 = \{(\lambda,\mu)\in\mathbb{C}^2 | \lambda\mu(\lambda^2(1+27\lambda)^2 - 2\lambda\mu(1+189\lambda) + (1+576\lambda)\mu^2 - 256\mu^3) \neq 0\}. \quad (1.2.28)$$

Proposition 1.2.6. *Suppose $(\lambda,\mu)\in\Lambda_2$. The elliptic surface given by (1.2.26) has the singular fibres of type I_1^* over $y=0$, of type I_{11} over $y=\infty$ and other six fibres of type I_1.*

Proof. (1.2.26) is described in the Kodaira normal form

$$z_2^2 = 4x_2^3 - g_2(y)x_2 - g_3(y), \quad y\neq\infty, \quad (1.2.29)$$

with

$$\begin{cases} g_2(y) = -4\Big(-\dfrac{16\lambda^2y^2}{3} + \dfrac{8\lambda y^3}{3} - 8\mu y^3 - \dfrac{y^4}{3} \\ \qquad + \dfrac{16\lambda y^4}{3} - 8\mu y^4 - \dfrac{4y^5}{3} + \dfrac{8\lambda y^5}{3} - 2y^6 - \dfrac{4y^7}{3} - \dfrac{y^8}{3}\Big), \\ g_3(y) = -4\Big(-\dfrac{128\lambda^3y^3}{27} + \dfrac{32\lambda^2y^4}{9} - \dfrac{32\lambda\mu y^4}{3} + 16\mu^2y^4 \\ \qquad -\dfrac{8\lambda y^5}{9} + \dfrac{64\lambda^2y^5}{9} + \dfrac{8\mu y^5}{3} - \dfrac{32\lambda\mu y^5}{3} + \dfrac{2y^6}{27} - \dfrac{32\lambda y^6}{9} \\ \qquad + \dfrac{32\lambda^2y^6}{9} + 8\mu y^6 + \dfrac{4y^7}{9} - \dfrac{16\lambda y^7}{3} + 8\mu y^7 + \dfrac{10y^8}{9} - \dfrac{32\lambda y^8}{9} \\ \qquad + \dfrac{8\mu y^8}{3} + \dfrac{40y^9}{27} - \dfrac{8\lambda y^9}{9} + \dfrac{10y^{10}}{9} + \dfrac{4y^{11}}{9} + \dfrac{2y^{12}}{27}\Big), \end{cases}$$

and

$$z_3^2 = 4x_3^3 - h_2(y_1)x_3 - h_3(y_1), \quad y_1\neq\infty, \quad (1.2.30)$$

with

$$\begin{cases} h_2(y_1) = -4\Big(-\dfrac{1}{3} - \dfrac{4y_1}{3} - 2y_1^2 - \dfrac{4y_1^2}{3} + \dfrac{8\lambda y_1^3}{3} \\ \qquad -\dfrac{y_1^4}{3} + \dfrac{16\lambda y_1^4}{3} - 8\mu y_1^4 + \dfrac{8\lambda y_1^5}{3} - 8\mu y_1^5 - \dfrac{16\lambda^2y_1^6}{3}\Big), \\ h_3(y_1) = -4\Big(\dfrac{2}{27} + \dfrac{4y_1}{9} + \dfrac{10y_1^2}{9} + \dfrac{40y_1^3}{27} - \dfrac{8\lambda y_1^3}{9} \\ \qquad + \dfrac{10y_1^4}{9} - \dfrac{32\lambda y_1^4}{9} + \dfrac{8\mu y_1^4}{3} + \dfrac{4y_1^5}{9} - \dfrac{16\lambda y_1^5}{3} + 8\mu y_1^5 \\ \qquad + \dfrac{2y_1^6}{27} - \dfrac{32\lambda y_1^6}{9} + \dfrac{32\lambda^2y_1^6}{9} + 8\mu y_1^6 - \dfrac{8\lambda y_1^7}{9} + \dfrac{64\lambda^2y_1^7}{9} + \dfrac{8\mu y_1^9}{3} \\ \qquad - \dfrac{32\lambda\mu y_1^7}{3} + \dfrac{32\lambda^2y_1^8}{9} - \dfrac{32\lambda\mu y_1^8}{3} + 16\mu^2y_1^8 - \dfrac{128\lambda^3y_1^9}{27}\Big), \end{cases}$$

where $y = 1/y_1$. We have the discriminant of the right hand side of (1.2.29) for x_2((1.2.30) for x_3, resp.):

$$\begin{cases} D_0 = -256\mu^2y^7(16\lambda^3 - 8\lambda^2y + 36\lambda\mu y - 27\mu^2y + \lambda y^2 - 16\lambda^2y^2 - \mu y^2 + 36\lambda\mu y^2 + 4\lambda y^3 \\ \qquad -8\lambda^2y^3 - 3\mu y^3 + 6\lambda y^4 - 3\mu y^4 + 4\lambda y^5 - \mu y^5 + \lambda y^6), \\ D_\infty = -256\mu^2y_1^{11}(\lambda + 4\lambda y_1 - \mu y_1 + 6\lambda y_1^2 - 3\mu y_1^2 + 4\lambda y_1^3 - 8\lambda^2y_1^3 - 3\mu y_1^3 + \lambda y_1^4 - 16\lambda^2y_1^4 \\ \qquad -\mu y_1^4 + 36\lambda\mu y_1^4 - 8\lambda^2y_1^5 + 36\lambda\mu y_1^5 - 27\mu^2y_1^5 + 16\lambda^3y_1^6). \end{cases}$$

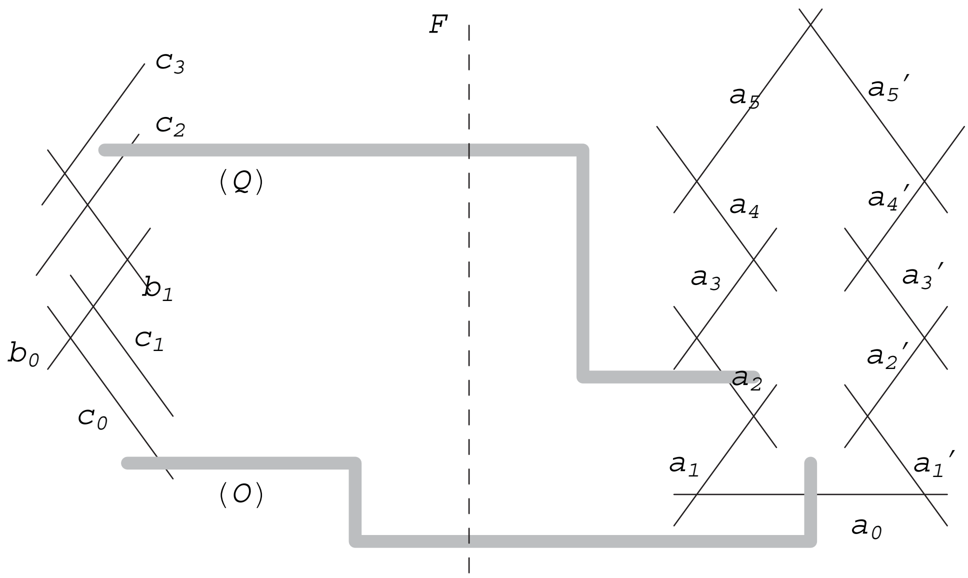


Figure 1.4: An elliptic fibration for P_2

From these data, we obtain the required statement. □

The elliptic fibration given by (1.2.26) is illustrated in Figure 1.4.

For this fibration, let O be the Mordell-Weil group, Q be the section in (1.2.27) and F be a general fibre. Note $Q \cap a_2 \neq \phi$ and $Q \cap c_2 \neq \phi$. Set

$$L_2 = \langle a_1, a_2, a_3, a_4, a_5, a_5', a_4', a_3', a_2', a_1', c_1, b_0, b_1, c_2, c_3, O, Q, F \rangle_{\mathbb{Z}}. \tag{1.2.31}$$

We have the following intersection matrix M_2 for L_2:

$$\begin{aligned} M_2 =& A_{18}(-1) - (E_{10,11} + E_{11,10}) - (E_{15,16} + E_{16,15}) + (E_{4,17} + E_{17,4}) + (E_{14,17} + E_{17,4}) \\ &- (E_{14,15} + E_{15,14}) + (E_{13,15} + E_{15,13}) - (E_{16,17} + E_{17,16}) + (E_{16,18} + E_{18,16}) + 2E_{18,18}. \end{aligned} \tag{1.2.32}$$

We have $\det(M_2) = -9$.

1.2.4 Elliptic fibrations for $\mathcal{F}_3$

Proposition 1.2.7. *The surface $S_3(\lambda, \mu)$ is birationally equivalent to the surface defined by the equation*

$$y_1^2 = z_1^3 + (\lambda^2 + 2\lambda x_1 + x_1^2 - 4\mu x_1^2 - 4x_1^3) z_1^2 + 16\mu x_1^5 z_1. \tag{1.2.33}$$

This equation gives an elliptic fibration of $S_3(\lambda, \mu)$ with the holomorphic sections

$$\begin{cases} Q : z_1 \mapsto (x_1, y_1, z_1) = (x_1, 4\mu x_1^2(x_1 + \lambda), 4x_1^2 \mu), \\ O' : z_1 \mapsto (x_1, y_1, z_1) = (x_1, 0, 0). \end{cases} \tag{1.2.34}$$

The section O' satisfies $2O' = O$.

Proof. By the birational transformation

$$x = \frac{2x_1^2(4\mu x_1^2 - z_1)}{y_1 + \lambda z_1 + x_1 z_2}, y = \frac{y_1 + \lambda z_1 + x_1 z_1}{2x_1(4\mu x_1^2 - z_1)}, z = -\frac{z_1(4\mu x_1^2 - z_1)}{2x_1(y_1 + \lambda z_1 + x_1 z_1)},$$

(1.2.9) is transformed to (1.2.33). $\square$

(1.2.33) gives an elliptic fibration for $S_3(\lambda, \mu)$. Set

$$\Lambda_3 = \{(\lambda, \mu) \in \mathbb{C}^2 | \lambda\mu(729\lambda^2 - (4\mu - 1)^3 + 54\lambda(1 + 12\mu)) \neq 0\}. \tag{1.2.35}$$

Proposition 1.2.8. *Suppose* $(\lambda, \mu) \in \Lambda_3$. *The elliptic surface given by* (1.2.33) *has the singular fibres of type* I_{10} *over* $z = 0$, *of type* I_2^* *over* $z = \infty$ *and other six fibres of type* I_1.

Proof. (1.2.33) is described in the Kodaira normal form

$$y_2^2 = 4z_2^3 - g_2(x_1)z_2 - g_3(x_1), \quad x_1 \neq \infty, \tag{1.2.36}$$

with

$$\begin{cases} g_2(x_1) = -4\Big(-\frac{\lambda^4}{3} - \frac{4\lambda^3 x_1}{3} - 2\lambda^2 x_1^2 + \frac{8\lambda^2\mu x_1^2}{3} - \frac{4\lambda x_1^3}{3} + \frac{8\lambda^2 x_1^3}{3} + \frac{16\lambda\mu x_1^3}{3} \\ \qquad -\frac{x_1^4}{3} + \frac{16\lambda x_1^4}{3} + \frac{8\mu x_1^4}{3} - \frac{16\mu^2 x_1^4}{3} + \frac{8x_1^5}{3} + \frac{16\mu x_1^5}{3} - \frac{16x_1^6}{3}\Big), \\ g_3(x_1) = -4\Big(\frac{2\lambda^6}{27} + \frac{4\lambda^5 x_1}{9} + \frac{10\lambda^4 x_1^2}{9} - \frac{8\lambda^4\mu x_1^2}{9} \\ \qquad +\frac{40\lambda^3 x_1^3}{27} - \frac{8\lambda^4 x_1^3}{9} - \frac{32\lambda^3\mu x_1^3}{9} + \frac{10\lambda^2 x_1^4}{9} - \frac{32\lambda^3 x_1^4}{9} \\ \qquad -\frac{16\lambda^2\mu x_1^4}{3} + \frac{32\lambda^2\mu^2 x_1^4}{9} + \frac{4\lambda x_1^5}{9} - \frac{16\lambda^2 x_1^5}{3} - \frac{32\lambda\mu x_1^5}{9} \\ \qquad +\frac{16\lambda^2\mu x_1^5}{9} + \frac{64\lambda\mu^2 x_1^5}{9} + \frac{2x_1^6}{27} - \frac{32\lambda x_1^6}{9} + \frac{32\lambda^2 x_1^6}{9} \\ \qquad -\frac{8\mu x_1^6}{9} + \frac{32\lambda\mu x_1^6}{9} + \frac{32\mu^2 x_1^6}{9} - \frac{128\mu^3 x_1^6}{27} - \frac{8x_1^7}{9} \\ \qquad +\frac{64\lambda x_1^7}{9} + \frac{16\mu x_1^7}{9} + \frac{64\mu^2 x_1^7}{9} + \frac{32x_1^8}{9} + \frac{64\mu x_1^8}{9} - \frac{128x_1^9}{27}\Big), \end{cases}$$

and

$$y_3^2 = 4z_3^3 - h_2(x_2)z_3 - h_3(x_2), \quad x_2 \neq \infty, \tag{1.2.37}$$

with

$$\begin{cases} h_2(x_2) = -4\Big(-\frac{16x_2^2}{3} + \frac{8x_2^3}{3} + \frac{16\mu x_2^3}{3} - \frac{x_2^4}{3} + \frac{16\lambda x_2^4}{3} + \frac{8\mu x_2^4}{3} - \frac{16\mu^2 x_2^4}{3} - \frac{4\lambda x_2^5}{3} \\ \qquad + \frac{8\lambda^2 x_2^5}{3} + \frac{16\lambda\mu x_2^5}{3} - 2\lambda^2 x_2^6 + \frac{8\lambda^2\mu x_2^6}{3} - \frac{4\lambda^3 x_2^7}{3} - \frac{\lambda^4 x_2^8}{3}\Big), \\ h_3(x_2) = -4\Big(-\frac{128x_2^3}{27} + \frac{32x_2^4}{9} + \frac{64\mu x_2^4}{9} - \frac{8x_2^5}{9} + \frac{64\lambda x_2^5}{9} + \frac{16\mu x_2^5}{9} \\ \qquad + \frac{64\mu^2 x_2^5}{9} + \frac{2x_2^6}{27} - \frac{32\lambda x_2^6}{9} + \frac{32\lambda^2 x_2^6}{9} - \frac{8\mu x_2^6}{9} \\ \qquad + \frac{32\lambda\mu x_2^6}{9} + \frac{32\mu^2 x_2^6}{9} - \frac{128\mu^3 x_2^6}{27} + \frac{4\lambda x_2^7}{9} - \frac{16\lambda^2 x_2^7}{3} \\ \qquad - \frac{32\lambda^2\mu^2 x_2^7}{9} + \frac{16\lambda^2\mu x_2^7}{9} - \frac{32\lambda^3 x_2^8}{9} + \frac{64\lambda\mu^2 x_2^7}{9} \\ \qquad + \frac{10\lambda^2 x_2^8}{9} - \frac{16\lambda^2\mu x_2^8}{3} + \frac{40\lambda^3 x_2^9}{27} - \frac{8\lambda^4 x_2^9}{9} - \frac{32\lambda^3\mu x_2^9}{9} \\ \qquad + \frac{10\lambda^4 x_2^{10}}{9} - \frac{8\lambda^4\mu x_2^{10}}{9} + \frac{4\lambda^5 x_2^{11}}{9} + \frac{2\lambda^6 x_2^{12}}{9}\Big), \end{cases}$$

where $x_1 = 1/x_2$. We have the discriminant of the right hand side of (1.2.36) for z_2 ((1.2.37) for z_3 resp):

$$\begin{cases} D_0 = -256\mu^3 x_1^{10}(\lambda^4 + 4\lambda^3 x_1 + 6\lambda^2 x_1^2 - 8\lambda^2\mu x_1^2 + 4\lambda x_1^3 - 8\lambda^2 x_1^3 - 16\lambda\mu x_1^3 \\ \qquad + x_1^4 - 16\lambda x_1^4 - 8\mu x_1^4 + 16\mu^2 x_1^4 - 8x_1^5 - 32\mu x_1^5 + 16x_1^6), \\ D_\infty = -256\mu^2 x_2^8(16 - 8x_2 - 32\mu x_2 + x_2^2 - 16\lambda x_2^2 - 8\mu x_2^2 + 16\mu^2 x_2^2 \\ \qquad + 4\lambda x_2^3 - 8\lambda^2 x_2^3 - 16\lambda\mu x_2^3 + 6\lambda^2 x_2^4 - 8\lambda^2\mu x_2^4 + 4\lambda^3 x_2^5 + \lambda^4 x_2^6). \end{cases}$$

From these data, we obtain the required statement. □

The elliptic fibration given by (1.2.33) is illustrated in Figure 1.5.

For this fibration, let O be the zero of the Mordell-Weil group, Q be the section in (1.2.34) and F be a general fibre. Set

$$L_3' = \langle a_1, a_2, a_3, a_4, a_0', a_4', a_3', a_2', a_1', c_1, b_0, b_1, b_2, c_2, c_3, O, F, Q\rangle_{\mathbb{Z}}. \tag{1.2.38}$$

We have $\det(L_3') = -36$.

We need another elliptic fibration.

Proposition 1.2.9. *The surface $S_3(\lambda,\mu)$ is birationally equivalent to the surface defined by the equation*

$$y_1^2 = x_1^3 + (\mu^2 + 2\mu z + z^2 + 2\mu z^2 + 2z^3 + z^4)x_1^2 + (-8\lambda\mu z^3 - 8\lambda z^4 - 8\lambda z^5)x_1 + 16\lambda^2 z^6. \tag{1.2.39}$$

This equation gives an elliptic fibration of $S_3(\lambda,\mu)$ with the holomorphic sections

$$\begin{cases} Q_0 : z \mapsto (x_1, y_1, z) = (0, 4\lambda z^3, z), \\ R_0 : z \mapsto (x_1, y_1, z) = (0, -4\lambda z^3, z). \end{cases} \tag{1.2.40}$$

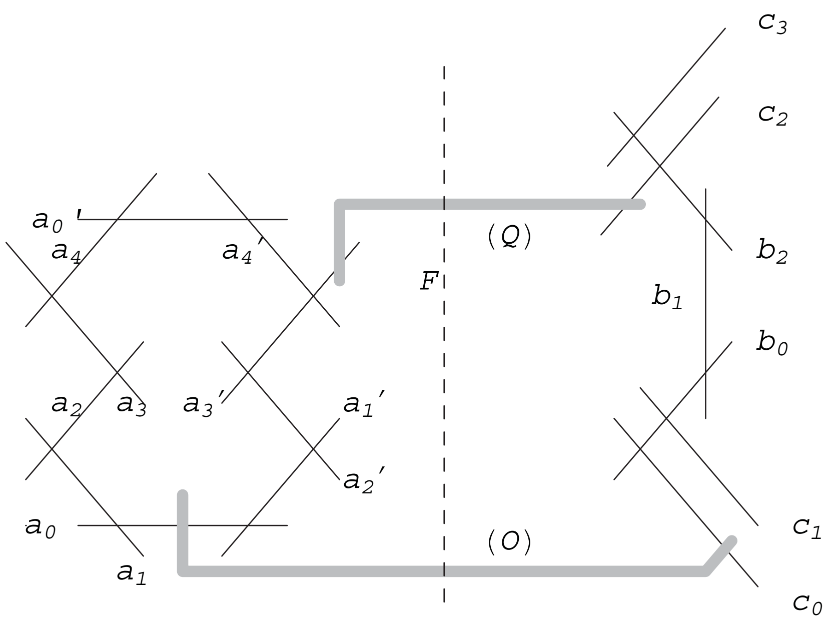


Figure 1.5: An elliptic fibration for P_3

Proof. By the birational transformation

$$x = -\frac{4\lambda z^2}{x_1'}, y = \frac{-\mu x_1' - y_1' - x_1' z - x_1' z^2 + 4\lambda z^3}{2x_1' z},$$

(1.2.9) is transformed to (1.2.39). □

Proposition 1.2.10. *Suppose* $(\lambda, \mu) \in \Lambda_3$*. The elliptic surface given by* (1.2.39) *has the singular fibres of type* I_9 *over* $z = 0$*, of type* I_9 *over* $z = \infty$ *and other six fibres of type* I_1*.*

Proof. (1.2.39) is described in the Kodaira normal form

$$y_2^2 = 4x_2^3 - g_2(z)x_2 - g_3(z), \quad z \neq \infty, \tag{1.2.41}$$

with

$$
\begin{cases}
g_2(z) = -4\Big(-\dfrac{\mu^4}{3} - \dfrac{4\mu^3 z}{3} - 2\mu^2 z^2 - \dfrac{4\mu^3 z^2}{3} - \dfrac{4\mu z^3}{3} - 8\lambda\mu z^3 - 4\mu^2 z^3 - \dfrac{z^4}{3} - 8\lambda z^4 \\
\qquad -4\mu z^4 - 2\mu^2 z^4 - \dfrac{4z^5}{3} - 8\lambda z^5 - 4\mu z^5 - 2z^6 - \dfrac{4\mu z^6}{3} - \dfrac{4z^7}{3} - \dfrac{z^8}{3}\Big), \\
g_3(z) = -4\Big(\dfrac{2\mu^6}{27} + \dfrac{4\mu^5 z}{9} + \dfrac{10\mu^4 z^2}{9} + \dfrac{4\mu^5 z^2}{9} \\
\qquad +\dfrac{40\mu^3 z^3}{27} + \dfrac{8\lambda\mu^3 z^3}{3} + \dfrac{20\mu^4 z^3}{9} + \dfrac{10\mu^2 z^4}{9} \\
\qquad +8\lambda\mu^2 z^4 + \dfrac{40\mu^3 z^4}{9} + \dfrac{10\mu^4 z^4}{9} + \dfrac{4\mu z^5}{9} \\
\qquad +8\lambda\mu z^5 + \dfrac{40\mu^2 z^5}{9} + 8\lambda\mu^2 z^5 + \dfrac{40\mu^3 z^5}{9} + \dfrac{2z^6}{27} \\
\qquad +\dfrac{8\lambda z^6}{3} + 16\lambda^2 z^6 + \dfrac{20\mu z^6}{9} + 16\lambda\mu z^6 + \dfrac{20\mu^2 z^6}{3} \\
\qquad +\dfrac{40\mu^3 z^6}{27} + \dfrac{4z^7}{9} + 8\lambda z^7 + \dfrac{40\mu z^7}{9} + 8\lambda\mu z^7 \\
\qquad +\dfrac{10z^8}{9} + 8\lambda z^8 + \dfrac{40\mu z^8}{9} + \dfrac{10\mu^2 z^8}{9} + \dfrac{40 z^9}{27} + \dfrac{8\lambda z^9}{3} \\
\qquad +\dfrac{20\mu z^9}{9} + \dfrac{10 z^{10}}{9} + \dfrac{4\mu z^{10}}{9} + \dfrac{4z^{11}}{9} + \dfrac{2z^{12}}{27}\Big),
\end{cases}
$$

and

$$
y_3^2 = 4x_3^3 - h_2(z_1)x_3 - h_3(z_1), \quad z_1 \neq \infty, \tag{1.2.42}
$$

with

$$
\begin{cases}
h_2(z_1) = -4\Big(-\dfrac{1}{3} - \dfrac{4z_1}{3} - 2z_1 - \dfrac{4\mu z_1^2}{3} - \dfrac{4z_1^3}{3} - 8\lambda z_1^3 - 4z_1^3 - \dfrac{z_1^4}{3} - 8\lambda z_1^4 \\
\qquad -4\mu z_1^4 - 2\mu^2 z_1^2 - \dfrac{4\mu z_1^5}{3} - 8\lambda\mu z_1^5 - 4\mu^2 z_1^5 \\
\qquad -2\mu^2 z_1^6 - \dfrac{4\mu^3 z_1^6}{3} - \dfrac{4\mu^3 z_1^7}{3} - \dfrac{\mu^4 z_1^8}{3}\Big), \\
h_3(z_1) = -4\Big(\dfrac{2}{27} + \dfrac{4z_1}{9} + \dfrac{10z_1^2}{9} + \dfrac{4\mu z_1^2}{9} + \dfrac{40z_1^3}{27} + \dfrac{8\lambda z_1^3}{3} + \dfrac{20\mu z_1^3}{9} + \dfrac{10z_1^4}{9} + 8\lambda z_1^4 \\
\qquad +\dfrac{40\mu z_1^4}{9} + \dfrac{10\mu^2 z_1^4}{9} + \dfrac{4z_1^5}{9} + 8\lambda z_1^5 + \dfrac{40\mu z_1^5}{9} + 8\lambda\mu z_1^5 + \dfrac{40\mu^2 z_1^5}{9} \\
\qquad +\dfrac{2z_1^6}{27} + \dfrac{8\lambda z_1^6}{3} + 16\lambda^2 z_1^6 + \dfrac{20\mu z_1^6}{9} + 16\lambda\mu z_1^6 + \dfrac{20\mu^2 z_1^6}{3} + \dfrac{40\mu^3 z_1^6}{27} \\
\qquad +\dfrac{4\mu z_1^7}{9} + 8\lambda\mu^2 z_1^7 + \dfrac{40\mu^3 z_1^7}{9} + \dfrac{10\mu^2 z_1^8}{9} + 8\lambda\mu^2 z_1^8 + \dfrac{10\mu^4 z_1^8}{9} + \dfrac{40\mu^3 z_1^8}{9} \\
\qquad +\dfrac{40\mu^3 z_1^9}{27} + \dfrac{8\lambda\mu^3 z_1^9}{3} + \dfrac{20\mu^4 z_1^9}{9} + \dfrac{10\mu^4 z_1^{10}}{9} + \dfrac{4\mu^5 z_1^{10}}{9} + \dfrac{4\mu^5 z_1^{11}}{9} + \dfrac{2\mu^6 z_1^{12}}{27}\Big),
\end{cases}
$$

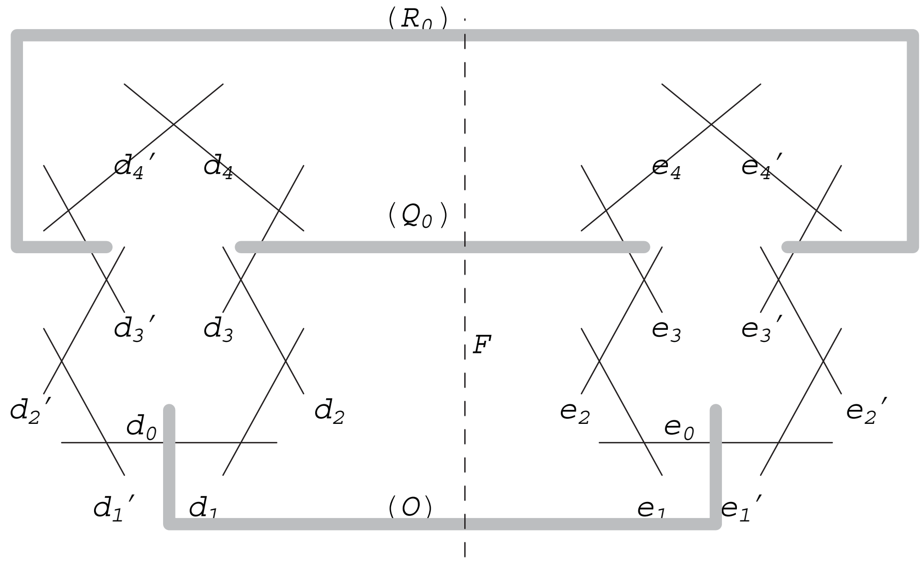


Figure 1.6: Another elliptic fibration for P_3

where $z = 1/z_1$ We have the discriminant of the right hand side of (1.2.41) for x'_2((1.2.42) for x'_3, resp.):

$$\begin{cases} D_0 = 256\lambda^3 z^9(\mu^3 + 3\mu^2 z + 3\mu z^2 + 3\mu^2 z^2 + z^3 + 27\lambda z^3 + 6\mu z^3 + 3z^4 + 3\mu z^4 + 3z^5 + z^6), \\ D_\infty = 256\lambda^3 z_1^9(1 + 3z_1 + 3z_1^2 + 3\mu z_1^2 + z_1^3 + 27\lambda z_1^3 + 6\mu z_1^3 + 3\mu z_1^4 + 3\mu^2 z_1^4 + 3\mu^2 z_1^5 + \mu^3 z_1^6). \end{cases}$$

From these data, we obtain the required statement. □

This fibration is illustrated in Figure 1.6.

For this fibration, let O be the zero of the Mordell-Weil group, Q_0 and R_0 be the sections in (1.2.40) and F be a general fibre. Set

$$L_3 = \langle d_1, d_2, d_3, d_4, d'_4, d'_3, d'_2, d'_1, e_1, e_2, e_3, e_4, e'_3, e'_2, O, Q_0, R_0, F\rangle_{\mathbb{Z}}. \quad (1.2.43)$$

We have the following intersection matrix M_3 for L_3:

$$\begin{aligned} M_3 =& A_{18}(-1) + 2E_{18,18} - (E_{8,9} + E_{9,8}) - (E_{14,15} + E_{15,14}) - (E_{12,13} + E_{13,12}) \\ &+ (E_{3,16} + E_{16,3}) + (E_{6,17} + E_{17,6}) + (E_{11,16} + E_{16,11}) + (E_{13,17} + E_{17,13}) \\ &- (E_{15,16} + E_{16,15}) + (E_{15,18} + E_{18,15}) + (E_{16,18} + E_{18,16}) - (E_{16,17} + E_{17,16}) \end{aligned} \quad (1.2.44)$$

We have $\det(M_3) = -9$.

1.3 The Picard numbers

In this section, we define the period mappings and determine the Picard numbers for our families. We state the precise argument only for the case of the family $\mathcal{F}_0$ of the $K3$ surfaces $S_0(\lambda, \mu)$.

1.3.1 S-marked $K3$ surfaces

The lattice $L := L_0$ in (1.2.16) is contained in $\mathrm{NS}(S_0(\lambda,\mu))$ and of rank 18. So we have

Proposition 1.3.1.

$$\mathrm{rank}\ \mathrm{NS}(S(\lambda,\mu)) \geq 18.$$

We have also

Proposition 1.3.2. *L is a primitive sublattice of $H_2(S_0(\lambda,\mu),\mathbb{Z})$.*

Proof. By (1.2.18), we have $\det(L) = -5$. It does not contain any square factor. So L is primitive. □

Definition 1.3.1. *For a $K3$ surface $S_0(\lambda,\mu)$ ($(\lambda,\mu)\in\Lambda$), set*

$$\begin{cases} \gamma_5 = b_1,\ \gamma_6 = b_2,\ \gamma_7 = b_3,\ \gamma_8 = b_4,\ \gamma_9 = b_5,\ \gamma_{10} = Q, \\ \gamma_{11} = b_6,\ \gamma_{12} = b_7,\ \gamma_{13} = b_1',\ \gamma_{14} = b_2',\ \gamma_{15} = b_3',\ \gamma_{16} = b_4', \\ \gamma_{17} = b_5',\ \gamma_{18} = R,\ \gamma_{19} = b_6',\ \gamma_{20} = b_7',\ \gamma_{21} = O,\ \gamma_{22} = F, \end{cases}$$

given by (1.2.16). *Let $\check{S}_0 = S_0(\lambda_0,\mu_0)$ be a reference surface for a fixed point $(\lambda_0,\mu_0)\in\Lambda=\Lambda_0$. Set $\check{L} = L(\lambda_0,\mu_0) \subset H_2(\check{S},\mathbb{Z})$. We define a S-marking ψ of $S_0(\lambda,\mu)$ to be an isomorphism $\psi : H_2(S(\lambda,\mu),\mathbb{Z}) \to \check{L}$ with the property that $\psi^{-1}(\gamma_j) = \gamma_j$ for $5 \leq j \leq 22$. We call the pair $(S_0(\lambda,\mu),\psi)$ an S-marked $K3$ surface.*

By the above definition, a S-marking ψ has the property:

$$\begin{aligned} &\psi^{-1}(F) = F,\ \psi^{-1}(O) = O,\ \psi^{-1}(Q) = Q,\ \psi^{-1}(R) = R, \\ &\psi^{-1}(b_j) = b_j,\ \psi^{-1}(b_j') = b_j' \quad (1 \leq j \leq 7). \end{aligned}$$

Definition 1.3.2. *Two S-marked $K3$ surfaces (S,ψ) and (S',ψ') are said to be isomorphic if there is a biholomorphic mapping $f : S \to S'$ with*

$$\psi' \circ f_* \circ \psi^{-1} = \mathrm{id}_{H_2(\check{S},\mathbb{Z})}.$$

Two S-marked $K3$ surfaces (S,ψ) and (S',ψ') are said to be equivalent if there is a biholomorphic mapping $f : S \to S'$ with

$$\psi' \circ f_* \circ \psi^{-1}|_{\check{L}} = \mathrm{id}_{\check{L}}.$$

By Proposition 1.3.2, the basis $\{\gamma_5,\cdots,\gamma_{22}\}$ of $L(\subset H_2(S_0(\lambda,\mu),\mathbb{Z}))$ is extended to a basis

$$\{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\cdots,\gamma_{22}\} \tag{1.3.1}$$

of $H_2(S_0(\lambda,\mu),\mathbb{Z})$. Let $\{\gamma_1^*,\cdots,\gamma_{22}^*\}$ be the dual basis of $\{\gamma_1,\cdots,\gamma_{22}\}$ with respect to the intersection form (0.2.1). Set

$$L_t = \langle \gamma_1^*,\gamma_2^*,\gamma_3^*,\gamma_4^* \rangle_{\mathbb{Z}} \subset H_2(S_0(\lambda,\mu),\mathbb{Z}). \tag{1.3.2}$$

We have $L_t = L^{\perp}$.

1.3.2 Period mapping

First, we state the definition of the period mapping for general $K3$ surfaces.

For a $K3$ surface S, there exists unique holomorphic 2-form ω up to a constant factor. Let $\{\gamma_1, \cdots \gamma_{22}\}$ be a basis of $H_2(S, \mathbb{Z})$.

$$\eta' = \Big(\int_{\gamma_1} \omega : \cdots : \int_{\gamma_{22}} \omega \Big) \in \mathbb{P}^{21}(\mathbb{C})$$

is called a period of S. Let $\{\gamma_1, \cdots, \gamma_r\}$ be a basis of $\mathrm{Tr}(S)$. Note that

$$\int_{\gamma} \omega = 0, \qquad ({}^\forall \gamma \in \mathrm{NS}(S)). \tag{1.3.3}$$

The period η' is reduced to

$$\eta = \Big(\int_{\gamma_1} \omega : \cdots : \int_{\gamma_r} \omega \Big) \in \mathbb{P}^{r-1}(\mathbb{C}).$$

We note that $\mathrm{NS}(S)$ is a lattice of signature $(1, \cdot)$ and $\mathrm{Tr}(S)$ is a lattice of the signature $(2, \cdot)$.

Definition 1.3.3. *Let $\check{S}_0 = S_0(\lambda_0, \mu_0)$ be the reference surface. Take a small neighborhood δ of (λ_0, μ_0) in Λ so that we have a local topological trivialization*

$$\tau : \{S_0(\lambda, \mu) | (\lambda, \mu) \in \delta\} \to \check{S}_0 \times \delta.$$

Let $p : \check{S}_0 \times \delta \to \check{S}_0$ be the canonical projection, and set $r = p \circ \tau$. Then,

$$r'(\lambda, \mu) = r|_{S_0(\lambda, \mu)}$$

gives a deformation of surfaces. We note that r' preserves the lattice L. Take an S-marking $\check{\psi}$ of $\check{S}_0$. We obtain the S-markings of $S_0(\lambda, \mu)$ by $\psi = \check{\psi} \circ r'_*$ for $(\lambda, \mu) \in \delta$. We define the local period mapping $\Phi = \Phi_0 : \delta \to \mathbb{P}^3(\mathbb{C})$ by

$$\Phi((\lambda, \mu)) = \Big(\int_{\psi^{-1}(\gamma_1)} \omega : \ldots : \int_{\psi^{-1}(\gamma_4)} \omega \Big), \tag{1.3.4}$$

where $\gamma_1, \cdots, \gamma_4 \in L$ are given by (1.3.1). We define the multivalued period mapping $\Lambda \to \mathbb{P}^3(\mathbb{C})$ by making the analytic continuation of Φ along any arc starting from (λ_0, μ_0) in Λ.

In general, we have the Riemann-Hodge relation for the period:

$$\eta' M {}^t\eta' = 0, \qquad \eta' M {}^t\bar{\eta'} > 0,$$

where M is the intersection matrix $(\gamma_j^* \cdot \gamma_k^*)_{1 \le j,k \le 22}$.

For our case, according to the relation (1.3.3), the Riemann-Hodge relation is reduced to

$$\eta A {}^t\eta = 0, \qquad \eta A {}^t\bar{\eta} > 0, \tag{1.3.5}$$

where

$$A = (\gamma_j^* \cdot \gamma_k^*)_{1 \le j,k \le 4}$$

and

$$\eta = \Big(\int_{\psi^{-1}(\gamma_1)} \omega : \int_{\psi^{-1}(\gamma_2)} \omega : \int_{\psi^{-1}(\gamma_3)} \omega : \int_{\psi^{-1}(\gamma_4)} \omega \Big).$$

Remark 1.3.1. *In* Theorem 1.4.1, *we shall show that the above matrix A is given by*

$$A = A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Set

$$\mathcal{D} = \mathcal{D}_0 = \{\xi = (\xi_1 : \xi_2 : \xi_3 : \xi_4) \in \mathbb{P}^3(\mathbb{C}) \mid \xi A {}^t\xi = 0, \xi A {}^t\bar{\xi} > 0\}.$$

We have $\Phi(\Lambda) \subset \mathcal{D}$. Note that $\mathcal{D}$ is composed of two connected components. Let $\mathcal{D}^+$ be the component where $(1 : 1 : -\sqrt{-1} : 0)$ is a point of $\mathcal{D}^+$. And let $\mathcal{D}^-$ be the other component.

Definition 1.3.4. *The fundamental group $\pi_1(\Lambda, *)$ acts on the $\mathbb{Z}$-module $\langle \psi^{-1}(\gamma_1), \cdots, \psi^{-1}(\gamma_4)\rangle_{\mathbb{Z}}$. So, it induces the action on $\mathcal{D}$. This action induces a group of projective linear transformations which is a subgroup of $PGL(4, \mathbb{Z})$. We call it the projective monodromy group of the period mapping $\Phi : \Lambda \to \mathcal{D}$.*

1.3.3 The Picard number

Definition 1.3.5. *Let $(S_1, \pi_1, \mathbb{P}^1(\mathbb{C}))$ and $(S_2, \pi_2, \mathbb{P}^1(\mathbb{C}))$ be two elliptic surfaces. If there exist a biholomorphic mapping $f : S_1 \to S_2$ and $\varphi \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ such that $\varphi \circ \pi_1 = \pi_2 \circ f$, we say $(S_1, \pi_1, \mathbb{P}^1(\mathbb{C}))$ and $(S_2, \pi_2, \mathbb{P}^1(\mathbb{C}))$ are isomorphic as elliptic surfaces.*

For an elliptic surface given by the Kodaira normal form $y^2 = 4x^3 - g_2(z)x - g_3(z)$, we define the j-invariant (see [Kod] Section 7):

$$j(z) = \frac{g_2^3(z)}{g_2^3(z) - 27g_3^2(z)} \in \mathbb{C}(z). \tag{1.3.6}$$

From the definition, we have

Proposition 1.3.3. *Let $(S_1, \pi_1, \mathbb{P}^1(\mathbb{C}))$ and $(S_2, \pi_2, \mathbb{P}^1(\mathbb{C}))$ be two elliptic surfaces given by the Kodaira normal forms. Let $j_1(z)$ and $j_2(z)$ be the corresponding j-invariants of the Kodaira normal forms, respectively. If $(S_1, \pi_1, \mathbb{P}^1(\mathbb{C}))$ and $(S_2, \pi_2, \mathbb{P}^1(\mathbb{C}))$ are isomorphic, then there exists $\varphi \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ such that $\pi_1^{-1}(p)$ and $\pi_2^{-1}(\varphi(p))$ are the fibres of the same type for any $p \in \mathbb{P}^1(\mathbb{C})$ and $j_2 \circ \varphi = j_1$.*

For $(\lambda, \mu) \in \Lambda$, let

$$\pi : S_0(\lambda, \mu) \to \mathbb{P}^1(\mathbb{C}) = (z\text{-sphere})$$

be the canonical elliptic fibration given by (1.2.10).

Lemma 1.3.1. *Suppose* $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in\Lambda$. *If* $(S(\lambda_1,\mu_1),\pi_1,\mathbb{P}^1(\mathbb{C}))$ *is isomorphic to* $(S(\lambda_2,\mu_2),\pi_2,\mathbb{P}^1(\mathbb{C}))$ *as elliptic surfaces, then it holds* $(\lambda_1,\mu_1)=(\lambda_2,\mu_2)$.

Proof. Let $f: S_1\to S_2$ be the biholomorphic mapping which gives the equivalence of elliptic surfaces. According to Proposition 1.3.3, there exists $\varphi\in\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ which satisfies $\varphi\circ\pi_1=\pi_2\circ f$. By Proposition 1.2.2, we have $\pi_j^{-1}(0)=I_3$ and $\pi_j^{-1}(\infty)=I_{15}$ $(j=1,2)$. So, φ has the form $\varphi: z\mapsto az$ with some $a\in\mathbb{C}-0$. Let $D_0(z;\lambda_j,\mu_j)$ $(j=1,2)$ be the discriminant. From (1.2.15), we have

$$\frac{D_0(z;\lambda_j,\mu_j)}{64\mu_j^3z^3} = \lambda_j^3+3\lambda_j^2z+27\mu_jz+3\lambda_jz^2+3\lambda_j^2z^2+z^3+6\lambda_jz^3+3z^4+3\lambda_jz^4+3z^5+z^6 \quad (j=1,2).$$

The six roots of $D_0(z;\lambda_1,\mu_1)/64\mu_1^3z^3$ ($D_0(z;\lambda_2,\mu_2)/64\mu_2^3z^3$, resp.) give the six images of singular fibres of type I_1 of $S(\lambda_1,\mu_1)$ ($S(\lambda_2,\mu_2)$, resp.). The roots of $D_0(z;\lambda_1,\mu_1)/64\mu_1^3z^3$ are sent by φ to those of $D_0(z;\lambda_2,\mu_2)/64\mu_2^3z^3$. Observing the coefficients of $D_0(z;\lambda_1,\mu_1)$ and $D_0(z;\lambda_2,\mu_2)$, we obtain that $a=1$. Therefore, we have $(\lambda_1,\mu_1)=(\lambda_2,\mu_2)$. □

Proposition 1.3.4. *Two S-marked* $K3$ *surfaces* $(S(\lambda_1,\mu_1),\psi_1)$ *and* $(S(\lambda_2,\mu_2),\psi_2)$ *are equivalent if and only if there exists an isomorphism of elliptic surfaces between* $(S(\lambda_1,\mu_1),\pi_1,\mathbb{P}^1(\mathbb{C}))$ *and* $(S(\lambda_2,\mu_2),\pi_2,\mathbb{P}^1(\mathbb{C}))$.

Proof. The sufficiency is clear. We prove the necessity. Let $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in\Lambda$. Suppose the equivalence of S-marked $K3$ surfaces

$$(S(\lambda_1,\mu_1),\psi_1)\simeq(S(\lambda_2,\mu_2),\psi_2).$$

Then, there exists a biholomorphic mapping $f: S(\lambda_1,\mu_1)\to S(\lambda_2,\mu_2)$ such that $\psi_2\circ f_*\circ\psi_1^{-1}|_L=id_L$. Especially, for general fibres $F_1\in\mathrm{Div}(S_1)$ and $F_2\in\mathrm{Div}(S_2)$, we have $f_*(F_1)=F_2$.

So, $S(\lambda_2,\mu_2)$ has two elliptic fibrations π_2 and $\pi_1\circ f^{-1}$ which have a general fibre F_2. According to Lemma 0.2.1, it holds

$$\pi_2=\pi_1\circ f^{-1}$$

up to $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$. □

Corollary 1.3.1. *Let* (λ_1,μ_1) *and* (λ_2,μ_2) *be in* Λ. *Two S-marked* $K3$ *surfaces* $(S(\lambda_1,\mu_1),\psi_1)$ *and* $(S(\lambda_2,\mu_2),\psi_2)$ *are equivalent if and only if* $(\lambda_1,\mu_1)=(\lambda_2,\mu_2)$.

Proof. From the proposition and Lemma 1.3.1, we obtain the required statement. □

Theorem 1.3.1. (The local Torelli theorem for S-marked $K3$ surfaces) *Let* $\delta\subset\Lambda$ *be a sufficiently small neighborhood of* (λ_0,μ_0), *and* (λ_1,μ_1), $(\lambda_2,\mu_2)\in\delta$. *Suppose* $\Phi(\lambda_1,\mu_1)=\Phi(\lambda_2,\mu_2)$, *then there exists an isomorphism of S-marked* $K3$ *surfaces* $(S(\lambda_1,\mu_1),\psi_1)\simeq(S(\lambda_2,\mu_2),\psi_2)$.

We have

Theorem 1.3.2. *For a generic point* $(\lambda,\mu)\in\Lambda$, *we have*

$$\mathrm{rank}\ \mathrm{NS}(S(\lambda,\mu))=18.$$

Proof. By Proposition 1.3.1, we already have rank $\mathrm{NS}(S_0(\lambda,\mu)) \geq 18$. Let δ be a small neighborhood of (λ,μ). Suppose we have rank $\mathrm{NS}(S(\lambda',\mu')) > 18$ for all $(\lambda',\mu') \in \delta$. Then, $\Phi(\delta)$ cannot contain any open set of $\mathcal{D}$. By Corollary 1.3.1 and Theorem 1.3.1, the period mapping is injective. This is a contradiction. □

Corollary 1.3.2. *The $\mathbb{C}$-vector space generated by the germs of holomorphic functions*

$$\int_{\psi^{-1}(\gamma_1)} \omega, \cdots, \int_{\psi^{-1}(\gamma_4)} \omega$$

is 4-dimensional.

Proof. It is clear, for the rank of the transcendental lattice $\mathrm{Tr}(S_0(\lambda,\mu))$ is $22-18=4$. □

We can determine the Picard number of the family $\mathcal{F}_j$ $(j=1,2,3)$ by the same method. Recall the lattice L_1 (L_2, L_3, resp.) in (1.2.24) ((1.2.31), (1.2.43), resp.) for $\mathcal{F}_1$ ($\mathcal{F}_2, \mathcal{F}_3$, *resp.*). Set $j \in \{1,2,3\}$. Let $\{\gamma_1^*, \cdots, \gamma_{22}^*\}$ be a basis of $H_2(S_j(\lambda,\mu),\mathbb{Z})$ such that we have $\langle \gamma_1^*, \cdots, \gamma_4^* \rangle_{\mathbb{Z}} = L_j^{\perp}$. Take a dual basis $\{\gamma_1, \cdots, \gamma_{22}\}$ of $H_2(S_j(\lambda,\mu),\mathbb{Z})$, namely it holds $(\gamma_j \cdot \gamma_k^*) = \delta_{jk}$ $(1 \leq j,k \leq 22)$. By the same procedure as for $\mathcal{F}_0$, we define the multivalued analytic period mapping $\Phi_j : \Lambda_j \to \mathcal{D}_j$ given by

$$(\lambda,\mu) \mapsto \Big(\int_{\gamma_1} \omega_j : \cdots : \int_{\gamma_4} \omega_j \Big),$$

where ω_j is the unique holomorphic 2-form on $S_j(\lambda,\mu)$ up to a constant factor and $\mathcal{D}_j$ is the domain of type IV defined by the intersection matrix $(\gamma_j^* \cdot \gamma_k^*)_{1\leq j,k \leq 4}$. Moreover, we have the Kodaira normal forms of the elliptic fibrations (1.2.19), (1.2.26) and (1.2.39) (these forms appear in the proofs of Proposition 1.2.4, 1.2.6 and 1.2.10). Observing the coefficients of these forms, we can prove the lemmas corresponding to Lemma 1.3.1. Therefore, we obtain the following theorem.

Theorem 1.3.3. *The Picard number of a generic member of the families $\mathcal{F}_j$ $(j=1,2,3)$ are equal to 18.*

1.4 The Néron-Severi lattices

For our further study, we need the explicit lattice structures of the Néron-Severi lattices and those of the transcendental lattices. In this section, we show the following theorem.

Theorem 1.4.1. *The intersection matrices of Néron-Severi lattices* NS *and the transcendental lattices* Tr *of a generic member of $\mathcal{F}_j$ $(j=0,1,2,3)$ are given as in* Table 1.2.

Remark 1.4.1. *Koike* [Koi] *made a research on the families of $K3$ surfaces derived from the dual polytopes of 3-dimensional Fano polytopes. The polytopes P_0, P_2 and P_3 in our notation are the Fano polytopes. Due to Koike, we have Néron-Severi lattices for the dual polytopes P_0°, P_2° and P_3° (given by* Table 1.3*).*

Table 1.3 *and* Table 1.2 *support the mirror symmetry conjecture for the reflexive polytopes P_0, P_2 and P_3.*

Polytope	Family	NS	Tr
P_0	$\mathcal{F}_0$	$E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$	$U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} =: A_0$
P_1	$\mathcal{F}_1$	$E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	$U \oplus \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} =: A_1$
P_2	$\mathcal{F}_2$	$E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$	$U \oplus \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix} =: A_2$
P_3	$\mathcal{F}_3$	$E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}$	$U \oplus \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix} =: A_3$

Table 1.2: The Néron-Severi lattices and the transcendental lattices for the polytopes P_0, P_1, P_2 and P_3.

Dual Polytope	NS	Tr
P_0°	$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$	$U \oplus E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$
P_2°	$\begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}$	$U \oplus E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$
P_3°	$\begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$	$U \oplus E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}$

Table 1.3: The Néron-Severi lattices and the transcendental lattices for the dual polytopes.

Remark 1.4.2. *According to the above theorem, a generic member of $\mathcal{F}_j$ $(j = 0, 1, 2, 3)$ has the Shioda-Inose structure.* (*see Morrison* [Mo], Theorem 6.3).

Remark 1.4.3. *The Néron-Severi lattices of $K3$ surfaces with non-symplectic involutions are studied by Nikulin* [Ni]. *All of our cases are not contained in his results. The lattice structures of* 95 *weighted projective $K3$ surfaces given by M. Reid are studied by Belcastro* [Be]. *Our lattice of $\mathcal{F}_0$ coincides with No.30 and No.86 in her list. Our lattices of $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ are not contained in her results, neither.*

1.4.1 Proof for the case P_0

We prove Theorem 1.4.1 for the case P_0 in a naive way. Recall the lattice L_0 in (1.2.16). By Theorem 1.3.2, for generic $(\lambda, \mu) \in \Lambda_0$,

$$\dim(\mathrm{NS}(S_0(\lambda, \mu))) = 18 = \dim(L_0).$$

According to Proposition 1.3.2, we have $(L \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \mathrm{NS}(S_0(\lambda, \mu)) = L_0$. Hence, we have

$$\mathrm{NS}(S_0(\lambda, \mu)) = L_0$$

for generic $(\lambda, \mu) \in \Lambda_0$.

Lemma 1.4.1. *The lattice L_0 is isomorphic to the lattice given by the intersection matrix*

$$M_0' = E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},$$

and its orthogonal complement is given by

$$A_0 = U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

Proof. Let M_0 be the matrix given in (1.2.17). Set

$$r_j = {}^t(0, \cdots, 0, \overbrace{1}^{\text{j−th}}, 0, \cdots, 0) \qquad (1 \leq j \leq 18) \tag{1.4.1}$$

and

$$\begin{cases} v_{16} = {}^t(-1,-2,-3,-4,-5,-2,-4,-3,-1,-2,-3,-4,-5,-2,-4,-2,1,1), \\ v_{17} = {}^t(5,10,15,20,25,13,17,9,1,2,3,4,5,3,3,1,1,-3), \\ v_{18} = {}^t(-2,-4,-6,-8,-10,-6,-6,-2,0,0,0,0,0,-1,1,2,-2,1). \end{cases}$$

Set

$$U = (r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, v_{16}, v_{17}, v_{18}).$$

This is an unimodular matrix. Then, we have ${}^tUM_0U = M_0'$. By observing $E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$ and M_0', we obtain the matrix A_0. □

Therefore, we obtain Theorem 1.4.1 for P_0.

1.4.2 Proof for the case P_1

Recall the elliptic fibration given by (1.2.19) and Figure 1.3.

The trivial lattice for this fibration is

$$T_1 = \langle a_1, a_2, a_3, a_4, a_4', a_3', a_2', a_1', c_1, b_0, b_1, b_2, b_3, c_2, c_3, O, F \rangle_{\mathbb{Z}}.$$

Let Q be the section in (1.2.20). From (1.2.24), we have

$$L_1 = \langle Q, T_1 \rangle_{\mathbb{Z}}.$$

This is a subgroup of $\mathrm{NS}(S_1(\lambda,\mu))$. According to Theorem 1.3.3 and Theorem 0.2.3 (3), we obtain

$$\mathrm{NS}(S_1(\lambda,\mu)) \otimes_{\mathbb{Z}} \mathbb{Q} = L_1 \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We obtain also

$$\mathrm{NS}(S_1(\lambda,\mu)) = (\langle Q \rangle_{\mathbb{Q}} \cap \mathrm{NS}(S_1(\lambda,\mu))) + \hat{T}_1 \tag{1.4.2}$$

for generic $(\lambda,\mu) \in \Lambda_1$. Since $\det(L_1) = -9$, we deduce that

$$[\mathrm{NS}(S_1(\lambda,\mu)) : L_1] = 1 \quad \text{or} \quad [\mathrm{NS}(S_1(\lambda,\mu)) : L_1] = 3. \tag{1.4.3}$$

In the following, we prove

$$[\mathrm{NS}(S_1(\lambda,\mu)) : L_1] = 1.$$

Lemma 1.4.2. *For generic* $(\lambda,\mu)\in\Lambda_1$, $\hat{T_1}=T_1$.

Proof. From (1.4.2) and (1.4.3), we have $\hat{T_1}=T_1$ or $[\hat{T_1}:T_1]=3$. We assume $[\hat{T_1}:T_1]=3$. Then, according to Corollary 0.2.1 (2),

$$E(\mathbb{C}(x_1))_{tor}\simeq \hat{T_1}/T_1\simeq \mathbb{Z}/3\mathbb{Z}. \qquad (1.4.4)$$

Therefore there exists $R_0\in E(\mathbb{C}(x_1))_{tor}$ such that $3R_0=O$. By Remark 0.2.2 and (0.2.4), we suppose that $R_0\cap a_3\neq\phi$ at $x_1=0$ and $R_0\cap c_0\neq\phi$ at $x_1=\infty$. Put $(R_0\cdot O)=k\in\mathbb{Z}$. Set $\bar{T_1}=\langle T_1,R_1\rangle_{\mathbb{Z}}$. By calculating the intersection matrix, we have

$$\det(\bar{T_1})=-72(1+k+k^2)\neq 0. \qquad (1.4.5)$$

On the other hand, due to (1.4.4), we have $\mathrm{rank}(\bar{T_1})=17$. So it follows $\det(\bar{T_1})=0$. This contradicts (1.4.5). □

By the above lamma, we have

$$\mathrm{NS}(S_1(\lambda,\mu))=(\langle Q\rangle_{\mathbb{Q}}\cap \mathrm{NS}(S_1(\lambda,\mu)))+T_1. \qquad (1.4.6)$$

Lemma 1.4.3. *For generic* $(\lambda,\mu)\in\Lambda_1$, $\mathrm{NS}(S_1(\lambda,\mu))=L_1$.

Proof. It is sufficient to prove $[\mathrm{NS}(S_1(\lambda,\mu)):L_1]=1$. We assume $[\mathrm{NS}(S_1(\lambda,\mu)):L_1]=3$. By (1.4.6), there exists $R_1\in E(\mathbb{C}(x_1))$ such that $3R_1=Q$. According to Remark 0.2.2,

$$(R_1\cdot c_3)=1, \quad \text{at } x_1=\infty$$

and

$$\begin{cases}(R_1\cdot a_1)=1,\\ \text{or}\\ (R_1\cdot a_4)=1, & \text{at } x_1=0.\\ \text{or}\\ (R_1\cdot a_7)=1,\end{cases}$$

We assume $(R_1\cdot O)=0$, for Q in (1.2.20) does not intersect O. By the addition theorem for elliptic curves, we have $2Q$ and we can check $2Q$ does not intersect O. If we have $p\in R_1\cap Q$, then it holds $R_1|_p=Q|_p$. By the assumption, we have $(3R_1)|_p=Q|_p$. It means that $2Q\cap O\neq\phi$. But, it is not the case. So, we suppose $(R_1\cdot Q)=0$ also. Set $\tilde{L_1}=\langle L_1,R_1\rangle_{\mathbb{Z}}$. By calculating the intersection matrix, we have

$$\det(\tilde{L_1})=\begin{cases}12 & (\text{if } (R_1\cdot a_1)=1),\\ -30 & (\text{if } (R_1\cdot a_4)=1),\\ 6 & (\text{if } (R_1\cdot a_7)=1).\end{cases} \qquad (1.4.7)$$

On the other hand, we have $\mathrm{rank}(\tilde{L_1})=18$ from Theorem 1.3.3. Hence, we obtain $\det(\tilde{L_1})=0$. This contradicts (1.4.7). Therefore, we have $[\mathrm{NS}(S_1(\lambda,\mu)):L_1]=1$. □

Lemma 1.4.4. *The lattice L_1 is isomorphic to the lattice given by the intersection matrix*

$$E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix},$$

and its orthogonal complement is given by the intersection matrix

$$A_1 = U \oplus \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.$$

Proof. Let M_1 be the intersection matrix in (1.2.25). Set

$$\begin{cases} v_{15}^{(1)} = {}^t(0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,-1,0,-1), \\ v_{16}^{(1)} = {}^t(11,22,33,26,19,12,5,-2,2,4,6,8,10,7,5,1,18,-4), \\ v_{17}^{(1)} = {}^t(8,16,24,19,14,9,4,-1,2,4,6,8,10,7,5,-1,13,-5), \\ v_{18}^{(1)} = {}^t(91,182,273,214,155,96,37,-22,18,36,54,72,90,63,45,0,150,-36). \end{cases}$$

Recall the vectors in (1.4.1). Set

$$U_1 = (r_7, r_6, r_5, r_4, r_3, r_{17}, r_2, r_1, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, v_{15}^{(1)}, v_{16}^{(1)}, v_{17}^{(1)}, v_{18}^{(1)}).$$

This is an unimodular matrix. We have

$${}^tU_1 M_1 U_1 = E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.$$

□

Therefore, we obtain Theorem 1.4.1 for P_1.

1.4.3 Proof for the case P_2

The elliptic fibration given by (1.2.26) is illustrated in Figure 1.4.

The trivial lattice for this fibration is

$$T_2 = \langle a_1, a_2, a_3, a_4, a_5, a_5', a_4', a_3', a_2', a_1', c_1, b_0, b_1, c_2, c_3, O, F \rangle_{\mathbb{Z}}.$$

Let Q be the section in (1.2.27). From (1.2.31), we have

$$L_2 = \langle Q, T_2 \rangle_{\mathbb{Z}}.$$

This is a subgroup of $\mathrm{NS}(S_2(\lambda,\mu))$. As in the case $\mathcal{F}_1$, so we obtain

$$\mathrm{NS}(S_2(\lambda,\mu)) = (\langle Q \rangle_{\mathbb{Q}} \cap \mathrm{NS}(S_2(\lambda,\mu))) + \hat{T}_2$$

for generic $(\lambda,\mu) \in \Lambda_2$. Since $\det(L_2) = -9$, we have

$$[\mathrm{NS}(S_2(\lambda,\mu)) : L_2] = 1 \text{ or } [\mathrm{NS}(S_2(\lambda,\mu)) : L_2] = 3. \tag{1.4.8}$$

In the following, we prove $[\mathrm{NS}(S_2(\lambda,\mu)) : L_2] = 1$.

Lemma 1.4.5. *For generic $(\lambda,\mu) \in \Lambda_2$, $\hat{T}_2 = T_2$.*

Proof. Because we have $\det(T_2) = -44$ and (1.4.8), it follows $\hat{T_2} = T_2$. □

Therefore, we obtain

$$\mathrm{NS}(S_2(\lambda,\mu)) = (\langle Q\rangle_{\mathbb{Q}} \cap \mathrm{NS}(S_2(\lambda,\mu))) + T_2. \tag{1.4.9}$$

Lemma 1.4.6. *For generic* $(\lambda,\mu) \in \Lambda_2$, $\mathrm{NS}(S_2(\lambda,\mu)) = L_2$.

Proof. We assume $[\mathrm{NS}(S_2(\lambda,\mu)) : L_2] = 3$. From (1.4.9), there exists $R_1 \in E(\mathbb{C}(y))$ such that $3R_1 = Q$. According to Remark 0.2.2, we obtain $(R_1 \cdot a_3') = 1$ and $(R_1 \cdot c_3) = 1$. Because the section Q in (1.2.27) and the section $2Q$ do not intersect O, we have $(R_1 \cdot O) = 0$ and $(R_1 \cdot Q) = 0$. Set $\tilde{L_2} = \langle L_2, R_1\rangle_{\mathbb{Z}}$. Calculating its intersection matrix, we have $\det(\tilde{L_2}) = -38$. As in the proof of Lemma 1.4.3, this contradicts to Theorem 1.3.3. □

Lemma 1.4.7. *The lattice L_2 is isomorphic to the lattice given by the following intersection matrix*

$$E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix},$$

and its orthogonal complement is given by the intersection matrix

$$A_2 = U \oplus \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}.$$

Proof. Let M_2 be the intersection matrix in (1.2.32). Set

$$\begin{cases} v_{14}^{(2)} = {}^t(5,4,15,26,13,10,8,6,4,2,12,24,36,30,18,-4,24,-8), \\ v_{15}^{(2)} = {}^t(1,-2,3,8,1,0,0,0,0,0,6,12,18,15,9,0,12,1), \\ v_{17}^{(2)} = {}^t(56,13,162,311,120,100,80,60,40,20,170,340,510,425,255,-28,340,-56), \\ v_{18}^{(2)} = {}^t(27,6,80,154,60,50,40,30,20,10,84,168,252,210,126,-14,168,-28). \end{cases}$$

Recall the vectors in (1.4.1). Set

$$U_2 = (r_3, r_4, r_{17}, r_{14}, r_{13}, r_{15}, r_{12}, r_{11}, r_{10}, r_9, r_8, r_7, r_6, v_{14}^{(2)}, v_{15}^{(2)}, r_{16}, v_{17}^{(2)}, v_{18}^{(2)}).$$

This is an unimodular matrix. We have

$${}^tU_2 M_2 U_2 = E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}.$$

□

Therefore, we obtain Theorem 1.4.1 for P_2.

1.4.4 Proof for the case P_3

The elliptic fibration given by (1.2.33) is illustrated in Figure 1.5.

The trivial lattice for this fibration is

$$T_3 = \langle a_1, a_2, a_3, a_4, a_0', a_4', a_3', a_2', a_1', c_1, b_0, b_1, b_2, c_2, c_3, O, F\rangle_{\mathbb{Z}}.$$

Let Q be the section in (1.2.34). From (1.2.38), we see

$$L_3' = \langle Q, T_3\rangle_{\mathbb{Z}}.$$

This is a subgroup of $\mathrm{NS}(S_3(\lambda,\mu))$ and we have $\det(L_3') = -36$. Moreover, the section O' in (1.2.34) is a 2-torsion section for this elliptic fibretion. Due to Corollary 0.2.1, $[\hat{T_3} : T_3]$ is divided by 2. Hence, we have

$$[\mathrm{NS}(S_3(\lambda,\mu)) : L_3'] = 2 \text{ or } [\mathrm{NS}(S_3(\lambda,\mu)) : L_3'] = 6. \tag{1.4.10}$$

Lemma 1.4.8. *For generic* $(\lambda,\mu) \in \Lambda_3$*,* $[\hat{T_3} : T_3] = 2$.

Proof. We have $\det(T_3) = -40$. From (1.4.10), we obtain $[\hat{T_3} : T_3] = 2$. □

Lemma 1.4.9. *For generic* $(\lambda,\mu) \in \Lambda_3$*,* $[\mathrm{NS}(S_3(\lambda,\mu)) : L_3'] = 2$.

Proof. We shall show that $[\mathrm{NS}(S_3(\lambda,\mu)) : L_3'] = 2$. We assume $[\mathrm{NS}(S_3(\lambda,\mu)) : L_3'] = 6$. From Lemma 1.4.8, there exists $R_1 \in E(\mathbb{C}(x_1))$ such that $3R_1 = Q$. According to Remark 0.2.2, $(R_1 \cdot c_2) = 1$ and $(R_1 \cdot a_4) = 1$. Also we have $(R_1 \cdot O) = 0$, for Q in (1.2.34) does not intersect O. Moreover, we assume that $(R_1 \cdot Q) = 0$ or 1, for the section $2P$ does not intersect O at $x_1 \neq \infty$. Set $\tilde{L_3'} = \langle L_3', R\rangle_{\mathbb{Z}}$. Calculating the intersection matrix, we have

$$\det(\tilde{L_3'}) = \begin{cases} -16 & (\text{if } (R_1 \cdot Q) = 0) \\ -112 & (\text{if } (R_1 \cdot Q) = 1) \end{cases}. \tag{1.4.11}$$

On the other hand, Theorem 1.3.3 implies that $\mathrm{rank}(\tilde{L_3'}) = 18$ and $\det(\tilde{L_3'}) = 0$. This is a contradiction to (1.4.11). □

Due to the above lemma, we have

$$|\det(\mathrm{NS}(S_3(\lambda,\mu)))| = 9$$

for generic $(\lambda,\mu) \in \Lambda_3$.

To determine the explicit lattice structure for $\mathcal{F}_3$, we use another elliptic fibration defined by (1.2.39). This fibration is illustrated in Figure 1.6.

Let Q_0 and R_0 be the sections in (1.2.40) for this elliptic fibration. Recall

$$L_3 = \langle d_1, d_2, d_3, d_4, d_4', d_3', d_2', d_1', e_1, e_2, e_3, e_4, e_3', e_2', O, Q_0, R_0, F\rangle_{\mathbb{Z}}.$$

in (1.2.43). For generic $(\lambda,\mu) \in \Lambda_3$, since

$$L_3 \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{NS}(S_3(\lambda,\mu)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and $\det(L_3') = -9$, we deduce that

$$L_3 = \mathrm{NS}(S_3(\lambda,\mu)).$$

Lemma 1.4.10. *The lattice L_3 is isomorphic to the lattice given by the intersection matrix*

$$E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix},$$

and its orthogonal complement is given by the intersection matrix

$$A_3 = U \oplus \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}.$$

Proof. Let M_3 be the intersection matrix in (1.2.44). Set

$$\begin{cases} v_9^{(3)} &= {}^t(28, 56, 84, 27, 21, 15, 10, 5, 34, 68, 102, 51, -1, -1, 1, 85, -1, -16), \\ v_{17}^{(3)} &= {}^t(5, 10, 15, 5, 4, 3, 2, 1, 6, 12, 18, 9, 0, 0, 0, 15, 0, -3), \\ v_{18}^{(3)} &= {}^t(468, 936, 1404, 432, 378, 324, 216, 108, 576, 1152, 1728, 864, 36, 18, 35, 1440, 54, -252). \end{cases}$$

Recall the vectors in (1.4.1). Set

$$U_3 = (r_1, r_2, r_3, r_{16}, r_{11}, r_{12}, r_{10}, r_9, v_9^{(3)}, r_{14}, r_{13}, r_{17}, r_6, r_5, r_7, r_8, v_{17}^{(3)}, v_{18}^{(3)}).$$

This is an unimodular matrix. We have

$${}^tU_3 M_3 U_3 = E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}.$$

□

Therefore, we obtain Theorem 1.4.1 for P_3.

1.5 Monodromy groups

We defined the projective monodromy groups of our period mappings in Section 1.3. Those are nothing but the projective monodromy groups of the period differential equations determined by the previous section. We determine them in this section. We make a precise argument only for the period mapping $\Phi : \Lambda_0 \to \mathcal{D}_0$ for $\mathcal{F}_0$. In this section, we set $\Lambda := \Lambda_0, L := L_0, A := A_0$ and $\mathcal{D} := \mathcal{D}_0$.

First, take a generic point $(\lambda_0, \mu_0) \in \Lambda$. Let $\check{S}_0 = S_0(\lambda_0, \mu_0)$ be a reference surface. Set $\check{L} = \mathrm{NS}(\check{S}_0)$ which is generated by the system (1.2.16). Recalling the argument of Section 1.3 and 1.4, we have a $\mathbb{Z}$-basis $\{\gamma_1, \cdots, \gamma_{22}\}$ of $H_2(\check{S}_0, \mathbb{Z})$ with $\langle \gamma_5, \cdots, \gamma_{22} \rangle_{\mathbb{Z}} = \check{L}$.

$A(= A_0)$ is the intersection matrix of the transcendental lattice given in Theorem 1.4.1. Set

$$PO(A, \mathbb{Z}) = \{g \in GL(4, \mathbb{Z}) | {}^tgAg = A\}. \tag{1.5.1}$$

It acts on $\mathcal{D}$ by

$${}^t\xi \mapsto g\,{}^t\xi \quad (\xi \in \mathcal{D}, g \in PO(A, \mathbb{Z})).$$

Recall that $\mathcal{D}$ is composed of two connected components:

$$\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-.$$

Definition 1.5.1. *Let* $PO^+(A,\mathbb{Z})$ *denote the subgroup of* $PO(A,\mathbb{Z})$ *given by*

$$\{g \in PO(A,\mathbb{Z}) | g(\mathcal{D}_\pm) = \mathcal{D}_\pm\}.$$

Remark 1.5.1. $PO(A,\mathbb{Z})$ *is generated by the system:*

$$\begin{cases} G_1 = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, G_2 = \begin{pmatrix} 1 & -1 & -2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, G_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \\ H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{cases}$$

G_1, G_2, G_3, H_2 *generate* $PO^+(A,\mathbb{Z})$ *(see* [I1] *or* [Ma]*).*

In the following, we show that the projective monodromy group of our period mapping is isomorphic to the group $PO^+(A,\mathbb{Z})$. To prove this, we apply the Torelli type theorem for polarized $K3$ surfaces.

1.5.1 The Torelli theorem for P-marked $K3$ surfaces

First, we state necessary properties of polarized $K3$ surfaces.

Definition 1.5.2. *Let* S *be an algebraic* $K3$ *surface. An isomorphism* $\psi : H_2(S,\mathbb{Z}) \to H_2(\check{S}_0,\mathbb{Z})$ *is said to be a P-marking if we have*

(i) $\psi^{-1}(\check{L}) \subset \mathrm{NS}(S)$,

(ii) $\psi^{-1}(F), \psi^{-1}(O), \psi^{-1}(Q), \psi^{-1}(R), \psi^{-1}(b_j)$ *and* $\psi^{-1}(b'_j)$ $(1 \le j \le 7)$ *are all effective divisors,*

(iii) $\psi^{-1}(F)$ *is nef. Namely,* $(\psi^{-1}(F) \cdot C) \ge 0$ *for any effective class* C.

A pair (S,ψ) of a $K3$ surface and a P-marking is called a P-marked $K3$ surface. A S-marked $K3$ surface $(S_0(\lambda,\mu),\psi)$ is a P-marked $K3$ surface.

Definition 1.5.3. *Two P-marked* $K3$ *surfaces* (S_1,ψ_1) *and* (S_2,ψ_2) *are said to be isomorphic if there is a biholomorphic mapping* $f : S_1 \to S_2$ *with*

$$\psi_2 \circ f_* \circ \psi_1^{-1} = id_{H_2(\check{S}_0,\mathbb{Z})}.$$

Two P-marked $K3$ *surfaces* (S_1,ψ_1) *and* (S_2,ψ_2) *are said to be equivalent if there is a biholomorphic mapping* $f : S_1 \to S_2$ *with*

$$\psi_2 \circ f_* \circ \psi_1^{-1}|_{\check{L}} = id_{\check{L}}.$$

The period of a P-marked $K3$ surface (S,ψ) is defined by

$$\Phi(S,\psi) = \Big(\int_{\psi^{-1}(\gamma_1)} \omega : \cdots : \int_{\psi^{-1}(\gamma_4)} \omega \Big). \tag{1.5.2}$$

We use some general facts. These are exposed in [KSTT].

Proposition 1.5.1. (Pjateckiĭ-Šapiro and Šafarevič [PS]) *Let S be a $K3$ surface.*

(1) *Suppose $C \in \mathrm{NS}(S)$ satisfies $(C \cdot C) = 0$ and $C \neq 0$. Then there exists an isometry γ of $\mathrm{NS}(S)$ such that $\gamma(C)$ becomes to be effective and nef.*

(2) *Suppose $C \in \mathrm{NS}(S)$ is effective, nef and $(C \cdot C) = 0$. Then, for certain $m \in \mathbb{N}$ and an elliptic curve $E \in S$, we have $C = m[E]$.*

(3) *A linear system of an elliptic curve E on S determines an elliptic fibration $S \to \mathbb{P}^1(\mathbb{C})$.*

Proposition 1.5.2. *A P-marked $K3$ surface (S, ψ) is realized as an elliptic $K3$ surface which has $\psi^{-1}(F)$ as a general fibre. Especially, if S is realized as a $K3$ surface $S_0(\lambda, \mu)$ by the Kodaira normal form for some $(\lambda, \mu) \in \Lambda$, it is a S-marked $K3$ surface.*

Proof. Set $C = \psi^{-1}(F) \in \mathrm{Div}(S)$. By Definition 1.5.3, C is effective, nef and $(C \cdot C) = 0$. According to Proposition 1.5.1 (2), there exists a positive integer m and an elliptic curve E such that $C = m[E]$. Since

$$m(E \cdot \psi^{-1}(O)) = (C \cdot \psi^{-1}(O)) = (F \cdot O) = 1,$$

we deduce that $m = 1$. Proposition 1.5.1 (3) says that there is an elliptic fibration $\pi : S \to \mathbb{P}^1(\mathbb{C})$ which has $C = \psi^{-1}(F)$ as a general fibre. □

Let $\mathbb{X}$ be the isomorphic classes of P-marked $K3$ surfaces and set

$$[\mathbb{X}] = \mathbb{X}/ \text{ P-marked equivalence.}$$

By (1.5.2), we obtain our period mapping $\Phi : X \to \mathbb{P}^3(\mathbb{C})$.

Theorem 1.5.1. (The Torelli theorem for polarized $K3$ surfaces)

(1) $\Phi(\mathbb{X}) \subset \mathcal{D}$.

(2) *$\Phi : \mathbb{X} \to \mathcal{D}$ is a bijective correspondence.*

(3) *Let S_1 and S_2 be algebraic $K3$ surfaces. Suppose an isometry $\varphi : H_2(S_1, \mathbb{Z}) \to H_2(S_2, \mathbb{Z})$ preserves ample classes. Then there exists a biholomorphic map $f : S_1 \to S_2$ such that $\varphi = f_*$.*

Here, we prove the following two key lemmas.

Lemma 1.5.1. *A P-marked $K3$ surface (S, ψ) is equivalent to the P-marked reference surface $(\check{S}_0, \check{\psi})$ if and only if $\Phi(S, \psi) = g \circ \Phi(\check{S}_0, \check{\psi})$ for some $g \in PO(A, \mathbb{Z})$.*

Proof. The necessity is clear. We prove the sufficiency. Suppose $\Phi(\check{S}_0, \check{\psi}) = p \in \mathcal{D}$. Take $g \in PO(A, \mathbb{Z})$. According to Theorem 1.5.1 (2), we take a P-marked $K3$ surface (S_g, ψ_g) such that $\Phi(S_g, \psi_g) = g \circ \Phi(\check{S}_0, \check{\psi})$. Let L_t be the transcendental lattice given by (1.3.2). Note $g \in \mathrm{Aut}(L_t) = PO(A, \mathbb{Z})$. Due to Nikulin [Ni], $g : L_t \to L_t$ is extended to an isomorphism $\hat{g} : H_2(\check{S}_0, \mathbb{Z}) \to H_2(S_g, \mathbb{Z})$ which preserves the Néron-Severi lattice L. Then, by Theorem 1.5.1 (3), there is a biholomorphic mapping $f : \check{S}_0 \to S_g$ such that $f_* = \hat{g}$. Therefore, two P-marked $K3$ surfaces $(\check{S}_0, \check{\psi})$ and (S_g, ψ_g) are equivalent. □

Remark 1.5.2. *$PO(A, \mathbb{Z})$ is a reflection group (see* [Ma]*).*

According to the Torelli theorem and Lemma 1.5.1, we identify $[\mathbb{X}]$ with $\mathcal{D}/PO(A, \mathbb{Z})$.

Lemma 1.5.2. *Let (S, ψ) be a P-marked $K3$ surface which is equivalent to $(\check{S}_0, \check{\psi})$. Then (S, ψ) has a unique canonical elliptic fibration $(S, \pi, \mathbb{P}^1(\mathbb{C}))$ that is given by the Kodaira normal form of $\check{S}_0 = S_0(\lambda_0, \mu_0)$ not coming from any other $(\lambda, \mu) \in \Lambda$.*

Proof. From Proposition 1.5.2, (S, ψ) ($(\check{S}_0, \check{\psi})$, resp.) has an elliptic fibration $(S, \pi, \mathbb{P}^1(\mathbb{C}))$ ($(\check{S}_0, \check{\pi}, \mathbb{P}^1(\mathbb{C}))$, resp.) with a general fibre $\psi^{-1}(F)$ ($\check{\psi}^{-1}(F)$, resp.). Because (S, ψ) and $(\check{S}_0, \check{\psi})$ are equivalent as P-marked $K3$ surfaces, we have a biholomorphic mapping $f : S \to \check{S}_0$ such that

$$\check{\psi} \circ f_* = \psi \quad (f_* : H_2(S, \mathbb{Z}) \simeq H_2(\check{S}_0, \mathbb{Z})).$$

So, we have

$$f_* = \psi.$$

It means that f preserves general fibres of S and $\check{S}_0$. According to the uniqueness of the fibration (Lemma 0.2.1), $(S, \pi, \mathbb{P}^1(\mathbb{C}))$ and $(\check{S}_0, \check{\pi}, \mathbb{P}^1(\mathbb{C}))$ are isomorphic as elliptic surfaces. Therefore, there exists $\varphi \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ such that $\varphi \circ \pi = \pi_0 \circ f$.

Let $y^2 = 4x^3 - g_2(z)x - g_3(z)$ ($y^2 = 4x^3 - \check{g}_2(z)x - \check{g}_3(z)$, resp.) be the Kodaira normal form of $(S, \pi, \mathbb{P}^1(\mathbb{C}))$ ($(\check{S}_0, \check{\pi}, \mathbb{P}^1(\mathbb{C}))$, resp.). According to Proposition 1.3.3, we assume $\pi^{-1}(0) = I_3$ and $\pi^{-1}(\infty) = I_{15}$. So as in the proof of Lemma 1.3.1, φ is given by $z \mapsto az$ ($a \in \mathbb{C} - 0$). Let j ($\check{j}$, resp.) be the j-invariant and D ($\check{D}$, resp.) be the discriminant of S ($\check{S}_0$, resp.). By Proposition 1.3.3, we have $D = \check{D} \circ \varphi$ and $j = \check{j} \circ \varphi$. Observing the expressions (1.2.14), (1.2.15) around $z = \infty$ and the definition of j-function (1.3.6), we have $a^3 = 1$. By the transformation $z \mapsto \omega z$ or $z \mapsto \bar{\omega} z$ (where ω is a cubic root of unity), we assume $a = 1$. Comparing j with $\check{j}$ and D with $\check{D}$, we have $g_2^3 = \check{g}_2^3$ and $g_3^2 = \check{g}_3^2$. By the transformations in the form $x \mapsto \omega x$ or $x \mapsto \bar{\omega} x$ or $y \mapsto -y$, we obtain $g_2 = \check{g}_2$ and $g_3 = \check{g}_3$. Hence, as in the proof of Lemma 1.3.1, we have the required statement. □

Remark 1.5.3. *According to the above two lemmas, $\Lambda = \Lambda_0$ is embedded in $[\mathbb{X}]$.*

1.5.2 Projective monodromy groups

Theorem 1.5.2. *The projective monodromy group of the period mapping $\Phi : \Lambda \to \mathcal{D}$ is isomorphic to $PO^+(A, \mathbb{Z})$.*

Proof. Let $* = (\lambda_0, \mu_0)$ be a generic point of Λ. Set $\check{S}_0 = S_0(\lambda_0, \mu_0)$. Note that $\mathrm{NS}(\check{S}_0) \simeq L$. Let G be the projective monodromy group induced from the fundamental group $\pi_1(\Lambda, *)$ (see Definition 1.3.4). We have clearly the inclusion $G \subset PO^+(A, \mathbb{Z})$.

Therefore, we prove the converse inclusion $PO^+(A, \mathbb{Z}) \subset G$. Take an element $g \in PO^+(A, \mathbb{Z})$, and let $p = \Phi(\check{S}_0, \check{\psi}) \in \mathcal{D}$ and let $q = g(p) \in \mathcal{D}$. p, q are in the same connected component of $\mathcal{D}$. So we suppose that $p, q \in \mathcal{D}^+$. Let α be an arc connecting p and q in $\mathcal{D}^+$. By the Torelli theorem, we obtain $[\Phi^{-1}(\alpha)] \subset [\mathbb{X}]$. By Lemma 1.5.1 and Lemma 1.5.2, we have $q = \Phi(\check{S}_0, \psi)$ so that $(\check{S}_0, \psi)$ is equivalent to $(\check{S}, \check{\psi})$. Hence, the end point of $[\Phi^{-1}(\alpha)]$ is (λ_0, μ_0).

Next, we show that there is α such that $[\Phi^{-1}(\alpha)] \subset \Lambda$. For this purpose, it is enough to show that Λ is a Zariski open set in some compactification K of $[\mathbb{X}]$. Here, we note that the compact (λ, μ) space $\mathbb{P}^2(\mathbb{C})$ and K are birationally equivalent and they contain Λ as a common open set. Λ is a Zariski open set in $\mathbb{P}^2(\mathbb{C})$. Hence, Λ is Zariski open in K also. Therefore, we obtain the required inclusion. □

We have the elliptic fibration (1.2.19) ((1.2.26), (1.2.39), resp.) for $\mathcal{F}_1$ ($\mathcal{F}_2$, $\mathcal{F}_3$, resp.). Using these fibrations, we can define the P-markings for $\mathcal{F}_j$ ($j = 1, 2, 3$). Moreover, as we prove Lemma 1.5.2, so we can prove the corresponding lemmas through observations of the coefficients of the Kodaira normal forms of elliptic fibrations for $\mathcal{F}_j$ ($j = 1, 2, 3$). Therefore, we have

Theorem 1.5.3. *Let $j \in \{1, 2, 3\}$. The projective monodromy group of the period mapping for the family $\mathcal{F}_j$ is equal to $PO^+(A_j, \mathbb{Z})$.*

Remark 1.5.4. *This is essentially noticed in the research of Ishige* [I2] *on the family of $K3$ surfaces coming from the polytope P_4. He found this result by a computer-aided approximation of a generator system of the monodromy group. However, it is not given an exact error estimation there. So, for our cases P_0, P_1, P_2 and P_3, we give here a proof based on the Torelli theorem for polarized $K3$ surfaces.*

Chapter 2

Period differential equations

In this chapter, we obtain the differential equations satisfied by the period integrals for the family $\mathcal{F}_j$ ($j = 0, 1, 2, 3$) (Section 2.1). To obtain them, we need the power series expansions of the period integrals and the theory of the GKZ hypergeometric equations. Then, we have a remarkable fact for the differential equation for $\mathcal{F}_0$. Namely, this equation gives an uniformizing differential equation for the symmetric Hilbert modular orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau\rangle$ for $\mathbb{Q}(\sqrt{5})$ (Section 2.2).

2.1 Period differential equations for the families $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$

Recall F_j ($j = 0, 1, 2, 3$) in (1.2.6), (1.2.7), (1.2.8) and (1.2.9). The unique holomorphic 2-form on $S_j(\lambda, \mu)$ ($j = 0, 1, 2, 3$) is given by

$$\omega_0 = \frac{z dz \wedge dx}{\partial F_0/\partial y}, \qquad \omega_j = \frac{dz \wedge dx}{\partial F_j/\partial y} \quad (j = 1, 2, 3), \tag{2.1.1}$$

up to a constant factor.

Proposition 2.1.1. *Let $j \in \{0, 1, 2, 3\}$. There is a 2-cycle Γ_j on $S_j(\lambda, \mu)$ such that the period integral $\iint_{\Gamma_j} \omega_j$ has the following power series expansion, which is valid in a sufficiently small neighborhood of $(\lambda, \mu) = (0, 0)$.*

(0) (*Periods for $\mathcal{F}_0$*)

$$\eta_0(\lambda, \mu) = \iint_{\Gamma_0} \omega = (2\pi i)^2 \sum_{n,m=0}^{\infty} (-1)^m \frac{(5m + 2n)!}{n!(m!)^3(2m + n)!} \lambda^n \mu^m.$$

(1) (*Periods for $\mathcal{F}_1$*)

$$\eta_1(\lambda, \mu) = \iint_{\Gamma_1} \omega_1 = (2\pi i)^2 \sum \frac{(3m + 3n)!}{(n!)^2(m!)^2(m + n)!} \lambda^n \mu^m.$$

(2) (*Periods for $\mathcal{F}_2$*)

$$\eta_2(\lambda, \mu) = \iint_{\Gamma_2} \omega_2 = (2\pi i)^2 \sum_{n,m=0}^{\infty} (-1)^n \frac{(4m + 3n)!}{(m!)^2 n!((m + n)!)^2} \lambda^n \mu^m.$$

(3) (*Periods for $\mathcal{F}_3$*)

$$\eta_3(\lambda,\mu) = \iint_{\Gamma_3} \omega_3 = (2\pi i)^2 \sum_{n,m=0}^{\infty} (-1)^n \frac{(3m+2n)!}{(m!)^2(n!)^3} \lambda^n \mu^m.$$

Proof. Here, we state the detailed proof only for the case (0).

When (λ,μ) is sufficiently small, $S_0(\lambda,\mu)$ in (1.2.6) is regarded as a double cover by the projection

$$p : (x,y,z) \mapsto (x,z).$$

Let $\xi_1(x,z), \xi_2(x,z)$ be the two roots of $F_0(x,y,z) = 0$ in y. Then, we have

$$F_0(x,y,z) = xz^2(y-\xi_1(x,z))(y-\xi_2(x,z)).$$

and

$$\frac{\partial F_0}{\partial y}(x,y,z) = xz^2((y-\xi_1(x,z)) + (y-\xi_2(x,z))).$$

Therefore, at $(x,\xi_1(x,z),z) \in S_0(\lambda,\mu)$,

$$\frac{\partial F_0}{\partial y}(x,\xi_1(x,z),z) = xz^2(\xi_1(x,z)-\xi_2(x,y)).$$

We have a local inverse mapping of p

$$q : (x,z) \mapsto (x,\xi_1(x,z),z).$$

Let γ_1 (γ_2, γ_3, resp.) be a cycle in x-plane (y-plane, z-plane, resp.) which goes around the origin once in the positive direction. We suppose that there exists $\delta > 0$ such that it holds

$$|\xi_1(x,z)| - |\xi_2(x,z)| \geq \delta$$

for any $(x,z) \in \gamma_1 \times \gamma_3$. We assume that $x = -1$ stays outside of γ_1, $z = -1-x$ stays outside of γ_3 for any $x \in \gamma_1$, and that $y = \xi_1(x,z)$ stays inside of γ_2 and $y = \xi_2(x,z)$ and $-1-x-z$ stay outside of γ_2 for any $(x,z) \in \gamma_1 \times \gamma_3$. Moreover, by taking a neighborhood U of the origin sufficiently small, we assume

$$|\lambda xyz + \mu| \leq |xyz^2(x+y+z+1)|$$

for any $(x,y,z) \in \gamma_1 \times \gamma_2 \times \gamma_3$ and $(\lambda,\mu) \in U$. So, $q(\gamma_1 \times \gamma_3)$ is a 2-cycle on $S_0(\lambda,\mu)$.

Let us calculate the period integral on the 2-cycle $q(\gamma_1 \times \gamma_3)$ on $S_0(\lambda,\mu)$. Let ω be the holomorphic 2-form given in (2.1.1). By the residue theorem,

$$\begin{aligned} \iint_{q(\gamma_1\times\gamma_3)} \omega &= \iint_{\gamma_3\times\gamma_1} \frac{zdz\wedge dx}{xz^2(\xi_1(x,z)-\xi_2(x,z))} \\ &= \frac{1}{2\pi\sqrt{-1}} \iiint_{\gamma_3\times\gamma_1\times\gamma_2} \frac{zdz\wedge dx\wedge dy}{xz^2(y-\xi_1(x,z))(y-\xi_2(x,z))} \\ &= \frac{1}{2\pi\sqrt{-1}} \iiint_{\gamma_3\times\gamma_1\times\gamma_2} \frac{zdz\wedge dx\wedge dy}{xyz^2(x+y+z+1)+\lambda xyz+\mu}. \end{aligned} \tag{2.1.2}$$

By the residue theorem and the binomial theorem, we have

$$
\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\iiint_{\gamma_3\times\gamma_1\times\gamma_2}\frac{zdz\wedge dx\wedge dy}{xyz^2(x+y+z+1)+\lambda xyz+\mu}\\
&=\frac{1}{2\pi\sqrt{-1}}\iiint_{\gamma_3\times\gamma_1\times\gamma_2}\frac{1}{xyz^2(x+y+z+1)}\frac{zdz\wedge dx\wedge dy}{1+\frac{\lambda xyz+\mu}{xyz^2(x+y+z+1)}}\\
&=\frac{1}{2\pi\sqrt{-1}}\sum_{l=0}^{\infty}\iiint_{\gamma_3\times\gamma_1\times\gamma_2}\frac{z(-\lambda xyz-\mu)^l}{(xyz^2(x+y+z+1))^{l+1}}dz\wedge dx\wedge dy\\
&=\frac{1}{2\pi\sqrt{-1}}\sum_{m,n=0}^{\infty}\iiint_{\gamma_3\times\gamma_1\times\gamma_2}\binom{m+n}{m}\frac{x^ny^nz^{n+1}dz\wedge dx\wedge dy}{(xyz^2(x+y+z+1))^{m+n+1}}(-\lambda)^n(-\mu)^m\\
&=\frac{1}{2\pi\sqrt{-1}}\sum_{m,n=0}^{\infty}\iiint_{\gamma_3\times\gamma_1\times\gamma_2}\frac{(m+n)!}{n!m!}\frac{dz\wedge dx\wedge dy}{x^{m+1}y^{m+1}z^{2m+n+1}(x+y+z+1)^{m+n+1}}(-\lambda)^n(-\mu)^m\\
&=\sum_{n,m=0}^{\infty}\iint_{\gamma_3\times\gamma_1}\frac{(2m+n)!}{(m!)^2n!}(-1)^m\frac{dz\wedge dx}{x^{m+1}z^{2m+n+1}(x+z+1)^{2m+n+1}}(-\lambda)^n(-\mu)^m\\
&=(2\pi\sqrt{-1})\sum_{n,m=0}^{\infty}\int_{\gamma_3}\frac{(3m+n)!}{(m!)^3n!}\frac{dz}{z^{2m+n+1}(z+1)^{3m+n+1}}(-\lambda)^n(-\mu)^m\\
&=(2\pi\sqrt{-1})^2\sum_{n,m=0}^{\infty}(-1)^m\frac{(5m+2n)!}{(m!)^3n!(2m+n)!}\lambda^n\mu^m.
\end{aligned}
$$

The above power series is holomorphic on U. □

Remark 2.1.1. *In the case* (1)*, our period reduces to the Appell F_4(see* [Koi] *):*

$$\eta_1(\lambda,\mu)=F_4\Big(\frac{1}{3},\frac{2}{3},1,1;27\lambda,27\mu\Big)=F\Big(\frac{1}{3},\frac{2}{3},1;x\Big)F\Big(\frac{1}{3},\frac{2}{3},1;y\Big),$$

where F is the Gauss hypergeometric function and $x(1-y)=27\lambda, y(1-x)=27\mu$.

From the divisor in (1.2.4), let us obtain the GKZ system of equations for the periods. In the following, we use the notation

$$\theta_\lambda=\lambda\frac{\partial}{\partial\lambda},\quad \theta_\mu=\mu\frac{\partial}{\partial\mu}.$$

Proposition 2.1.2. *Let $\eta_j(\lambda,\mu)$ $(j=0,1,2,3)$ be the periods given in* Proposition 2.1.1. *Then,*

$$D_1^{(j)}\eta_j(\lambda,\mu)=D_2^{(j)}\eta_j(\lambda,\mu)=0\qquad (j=0,1,2,3),$$

where $D_1^{(j)}$ and $D_2^{(j)}$ are given as follows.

(0) (*The GKZ system of equations for $\mathcal{F}_0$*)

$$\begin{cases} D_1^{(0)} &=\theta_\lambda(\theta_\lambda+2\theta_\mu)-\lambda(2\theta_\lambda+5\theta_\mu+1)(2\theta_\lambda+5\theta_\mu+2),\\ D_2^{(0)} &=\lambda^2\theta_\mu^3+\mu\theta_\lambda(\theta_\lambda-1)(2\theta_\lambda+5\theta_\mu+1).\end{cases}$$

(1) (*The GKZ system of equations for* $\mathcal{F}_1$)

$$\begin{cases} D_1^{(1)} = \lambda\theta_\mu^2 - \mu\theta_\lambda^2, \\ D_2^{(1)} = \lambda(3\theta_\lambda + 3\theta_\mu)(3\theta_\lambda + 3\theta_\mu - 1)(3\theta_\lambda + 3\theta_\mu - 2). \end{cases}$$

(2) (*The GKZ system of equations for* $\mathcal{F}_2$)

$$\begin{cases} D_1^{(2)} &= \lambda\theta_\mu^2 + \mu\theta_\lambda(3\theta_\lambda + 4\theta_\mu + 1), \\ D_2^{(2)} &= \theta_\lambda(\theta_\lambda + \theta_\mu)^2 + \lambda(3\theta_\lambda + 4\theta_\mu + 1)(3\theta_\lambda + 4\theta_\mu + 2)(3\theta_\lambda + 4\theta_\mu + 3). \end{cases}$$

(3) (*The GKZ system of equations for* $\mathcal{F}_3$)

$$\begin{cases} D_1^{(3)} &= \theta_\lambda^2 - \mu(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2), \\ D_2^{(3)} &= \theta_\lambda^3 + \lambda(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2)(3\theta_\lambda + 2\theta_\mu + 3). \end{cases}$$

Proof. Extending the matrix P_j $(j = 0, 1, 2, 3)$, set

$$\mathcal{A}_0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -2 \end{pmatrix}, \qquad \mathcal{A}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix},$$

$$\mathcal{A}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \qquad \mathcal{A}_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix},$$

and $\beta = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. From the matrix $\mathcal{A}_j$ $(j = 0, 1, 2, 3)$ and the vector β, we have the GKZ system for $\eta_j(\lambda, \mu)$ $(j = 0, 1, 2, 3)$. In the following, we state the detailed proof only for $\mathcal{F}_0$.

The GKZ system of equations defined by $\mathcal{A}_0$ and β has a solution

$$\begin{aligned} &\iiint_\Delta R_0^{-1} t_1^{-1} t_2^{-1} t_3^{-1} dt_1 \wedge dt_2 \wedge dt_3 \\ &= \iiint_\Delta \frac{t_3 dt_1 \wedge dt_2 \wedge dt_3}{(t_1 t_2 t_3^2 (a_1 + a_2 t_1 + a_3 t_2 + a_4 t_3) + a_5 t_1 t_2 t_3 + a_6)}, \end{aligned} \tag{2.1.3}$$

where

$$R_0 = a_1 + a_2 t_1 + a_3 t_2 + a_4 t_3 + a_5 \frac{1}{t_3} + a_6 \frac{1}{t_1 t_2 t_3^2},$$

and Δ is a twisted cycle. By the parameter transformation (1.2.5), (2.1.3) is transformed to

$$\frac{1}{a_1} \iiint_\Delta \frac{z dx \wedge dy \wedge dz}{xyz^2(x + y + z + 1) + \lambda xyz + \mu} = \frac{1}{a_1} \eta(\lambda, \mu).$$

Set $\theta_j = a_j \dfrac{\partial}{\partial a_j}$. The above mentioned GKZ system is given by the following equations (2.1.4),(2.1.5) and (2.1.6) :

$$\begin{cases} (\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6)\eta = -\eta, \\ (\theta_2 - \theta_6)\eta = 0, \\ (\theta_3 - \theta_6)\eta = 0, \\ (\theta_4 - \theta_5 - 2\theta_6)\eta = \eta, \end{cases} \tag{2.1.4}$$

$$\frac{\partial^2}{\partial a_4 \partial a_5}\eta = \frac{\partial^2}{\partial a_1^2}\eta, \tag{2.1.5}$$

$$\frac{\partial^3}{\partial a_2 \partial a_3 \partial a_6}\eta = \frac{\partial^3}{\partial a_1 \partial a_5^2}\eta. \tag{2.1.6}$$

By (1.2.5), we have

$$\theta_\lambda = \theta_5, \quad \theta_\mu = \theta_6.$$

So, from (2.1.4) we have

$$\begin{cases} \theta_2\eta = \theta_\mu\eta, \\ \theta_3\eta = \theta_\mu\eta, \\ \theta_4\eta = (\theta_\lambda + 2\theta_\mu)\eta, \\ \theta_1\eta = (-2\theta_\lambda - 5\theta_\mu - 1)\eta. \end{cases}$$

From (2.1.5), we have

$$\begin{cases} \dfrac{\partial^2}{\partial a_4 \partial a_5}\eta = \dfrac{1}{a_4 a_5}\theta_4\theta_5\eta = \dfrac{1}{a_4 a_5}(\theta_\lambda + 2\theta\_\mu)\theta_\lambda\eta, \\ \dfrac{\partial^3}{\partial a_1^2}\eta = \dfrac{1}{a_1^2}\theta_1(\theta_1 - 1)\eta = \dfrac{1}{a_1^2}(2\theta_\lambda + 5\theta_\mu + 1)(2\theta_\lambda + 5\theta_\mu + 2)\eta. \end{cases}$$

Hence, we obtain

$$(\theta_\lambda + 2\theta_\mu)\eta = \lambda(2\theta_\lambda + 5\theta_\mu + 1)(2\theta_\lambda + 5\theta_\mu + 2)\eta.$$

Similarly, from (2.1.6), we have

$$\begin{cases} \dfrac{\partial^3}{\partial a_2 \partial a_3 \partial a_6}\eta = \dfrac{1}{a_2 a_3 a_6}\theta_2\theta_3\theta_6\eta = \dfrac{1}{a_2 a_3 a_6}\theta_\mu^3\eta, \\ \dfrac{\partial^3}{\partial a_1 \partial a_5^2}\eta = \dfrac{1}{a_5^2}\theta_1\theta_5(\theta_5 - 1)\eta = \dfrac{1}{a_1 a_5^2}(-2\theta_\lambda - 5\theta_\mu - 1)\theta_\lambda(\theta_\lambda - 1)\eta, \end{cases}$$

hence

$$\lambda^2\theta_\mu^3\eta = -\mu(2\theta_\lambda + 5\theta_\mu + 1)\theta_\lambda(\theta_\lambda - 1)\eta.$$

□

We obtain 6×6 Pfaffian systems from the above GKZ systems $D_1^{(j)}u = D_2^{(j)}u = 0$ $(j = 0, 1, 2, 3)$. These systems are integrable. Therefore, each system has a 6-dimensional space of solutions. However, as we remarked in Corollary 1.3.2, we expect the systems of differential equations with 4-dimensional space of solutions. It suggests that the above systems are reducible. So, using the above $D_1^{(j)}$ $(j = 0, 1, 2, 3)$, we determine the period differential equation for $\mathcal{F}_j$ $(j = 0, 1, 2, 3)$ with 4-dimensional spaces of solutions.

Theorem 2.1.1. *Let $j \in \{0, 1, 2, 3\}$. Set the system of differential equations $D_1^{(j)}u = D_3^{(j)}u = 0$ as follows. Then,*

$$D_1^{(j)}\eta_j(\lambda, \mu) = D_3^{(j)}\eta_j(\lambda, \mu) = 0,$$

where $\eta_j(\lambda, \mu)$ is given in Proposition 2.1.1. *The space of solutions of this system is 4-dimensional.*

(0) (*The period differential equation for $\mathcal{F}_0$*)

$$\begin{cases} D_1^{(0)} &= \theta_\lambda(\theta_\lambda + 2\theta_\mu) - \lambda(2\theta_\lambda + 5\theta_\mu + 1)(2\theta_\lambda + 5\theta_\mu + 2), \\ D_3^{(0)} &= \lambda^2(4\theta_\lambda^2 - 2\theta_\lambda\theta_\mu + 5\theta_\mu^2) \\ & \qquad -8\lambda^3(1 + 3\theta_\lambda + 5\theta_\mu + 2\theta_\lambda^2 + 5\theta_\lambda\theta_\mu) + 25\mu\theta_\lambda(\theta_\lambda - 1). \end{cases} \tag{2.1.7}$$

(1) (*The period differential equation for $\mathcal{F}_1$*)

$$\begin{cases} D_1^{(1)} &= \lambda\theta_\mu^2 + \mu\theta_\lambda(3\theta_\lambda + 4\theta_\mu + 1), \\ D_3^{(1)} &= \lambda\theta_\lambda(3\theta_\lambda + 2\theta_\mu) \\ & \qquad +\mu\theta_\lambda(1 - \theta_\lambda) + 9\lambda^2(3\theta_\lambda + 4\theta_\mu + 1)(3\theta_\lambda + 4\theta_\mu + 2). \end{cases} \tag{2.1.8}$$

(2) (*The period differential equation for $\mathcal{F}_2$*)

$$\begin{cases} D_1^{(2)} &= \lambda\theta_\mu^2 + \mu\theta_\lambda(3\theta_\lambda + 4\theta_\mu + 1), \\ D_3^{(2)} &= \lambda\theta_\lambda(3\theta_\lambda + 2\theta_\mu) + \mu\theta_\lambda(1 - \theta_\lambda) \\ & \qquad +9\lambda^2(3\theta_\lambda + 4\theta_\mu + 1)(3\theta_\lambda + 4\theta_\mu + 2). \end{cases} \tag{2.1.9}$$

(3) (*The period differential equation for $\mathcal{F}_3$*)

$$\begin{cases} D_1^{(3)} &= \theta_\lambda^2 - \mu(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2), \\ D_3^{(3)} &= \theta_\lambda(3\theta_\lambda - 2\theta_\mu) + 4\mu\theta_\lambda(3\theta_\lambda + 2\theta_\mu + 1) \\ & \qquad +9\lambda(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2). \end{cases} \tag{2.1.10}$$

Proof. We determine $D_3^{(j)}$ $(j = 0, 1, 2, 3)$ by the method of indeterminate coefficients. Set $D = f_1 + f_2\theta_\lambda + f_3\theta_\mu + f_4\theta_\lambda^2 + f_5\theta_\lambda\theta_\mu + f_6\theta_\mu^2$, where $f_1 \cdots f_6 \in \mathbb{C}[\lambda, \mu]$. Let $j \in \{0, 1, 2, 3\}$. We can determine the polynomials $f_1, \cdots, f_6$ so that D satisfies $D\eta_j = 0$ (η_j is given in Proposition 2.1.1) and is independent of $D_1^{(j)}$. Thus, we obtain the above $D_3^{(j)}$.

In the following, we prove that the spaces of solutions are 4-dimensional. Let $j \in \{0, 1, 2, 3\}$. By making up the Pfaffian system of $D_1^{(j)}u = D_3^{(j)}u = 0$, we shall show the required statement. Set $\varphi = {}^t(1, \theta_\lambda, \theta_\mu, \theta_\lambda^2)$. We obtain the Pfaffian system $\Omega_j = \alpha_j d\lambda + \beta_j d\mu$ with $d\varphi = \Omega_j\varphi$ as follows. We can check that

$$d\Omega_j = \Omega_j \wedge \Omega_j.$$

Therefore, each system $D_1^{(j)}u = D_3^{(j)}u = 0$ has the 4-dimensional space of solution.
(0) (The Pfaffian system for $\mathcal{F}_0$)

Setting

$$\begin{cases} t = \lambda^2(4\lambda-1)^3 - 2(2+25\lambda(20\lambda-1))\mu - 3125\mu^2, \\ s = 1 - 15\lambda - 100\lambda^2, \end{cases}$$

we have

$$\alpha_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{11}/s & a_{12}/(2\lambda s) & a_{13}/(2s) & a_{14}/(2\lambda s) \\ a_{21}/(st) & a_{22}/(2st) & a_{23}/(2st) & a_{24}/(2st) \end{pmatrix}$$

with

$$\begin{cases} a_{11} = \lambda(1+20\lambda), \qquad a_{12} = 6\lambda^2 + 120\lambda^3 + 125\mu, \\ a_{13} = 5\lambda(3+40\lambda), \qquad a_{14} = -(\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu), \\ a_{21} = -\lambda^3(2 + 2125\mu + \lambda(-17 + 616\lambda - 2320\lambda^2 + 2500(9+80\lambda)\mu)), \\ a_{22} = -(-2\lambda^3(-1+4\lambda)(8+5\lambda(-13+4\lambda(83+40\lambda))) \\ \qquad +(-16+5\lambda(94+5\lambda(59+10\lambda(-73+20\lambda(37+160\lambda)))))\mu \\ \qquad +3125(-4+5\lambda(21+200\lambda))\mu^2), \\ a_{23} = -\lambda^3(22 + 26875\mu + \lambda(-47 + 300000\mu + 100\lambda(51 + 4\lambda(-49+20\lambda) + 20000\mu))), \\ a_{24} = 12ts + 3s(15\lambda - 2) + 2t(-3(1-4\lambda)^2\lambda^2(-1+10\lambda) + 75\lambda(-1+40\lambda)\mu), \end{cases}$$

and

$$\beta_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ b_{11}/s & b_{12}/(2\lambda s) & b_{13}/(2s) & b_{14}/(2\lambda s) \\ b_{21}/(s) & b_{22}/(\lambda^2 s) & b_{23}/(s) & b_{24}/(\lambda^2 s) \\ b_{31}/(ts) & b_{32}/(2\lambda ts) & b_{33}/(2ts) & b_{34}/(2\lambda ts) \end{pmatrix}$$

with

$$\begin{cases} b_{11} = \lambda(1+20\lambda), \qquad b_{12} = 6\lambda^2 + 120\lambda^3 + 125\mu, \\ b_{13} = 5\lambda(3+40\lambda), \qquad b_{14} = -(\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu), \\ b_{21} = -2\lambda(-1+4\lambda), \qquad b_{22} = -(6\lambda^3(-1+4\lambda) - 5\mu + 50\lambda\mu), \\ b_{23} = -\lambda(-11+20\lambda), \qquad b_{24} = -((1-4\lambda)^2\lambda^2 - (5-50\lambda)\mu), \\ b_{31} = -(4(1-4\lambda)^2\lambda^4(7+20\lambda) \\ \qquad -\lambda(-4+25\lambda(-3+2\lambda(-7+20\lambda(1+80\lambda))))\mu + 3125\lambda(1+20\lambda)\mu^2), \\ b_{32} = -(24(1-4\lambda)^2\lambda^5(7+20\lambda) \\ \qquad -2\lambda(-4+5\lambda(8+\lambda(-43+10\lambda(-57+20\lambda(7+160\lambda)))))\mu \\ \qquad -125(-4+25\lambda(-3+32\lambda(1+10\lambda)))\mu^2 + 390625\mu^3)), \\ b_{33} = -(4\lambda^3(-1+4\lambda)(-1+2\lambda(-32+25\lambda(1+12\lambda))) + 15625\lambda(3+40\lambda)\mu^2 \\ \qquad -5\lambda(-12+5\lambda(-1+10\lambda)(33+20\lambda(23+160\lambda)))\mu), \\ b_{34} = -(4\lambda^4(-1+4\lambda)^3(7+20\lambda) + 3\lambda(-4+\lambda(31-490\lambda+76000\lambda^3))\mu \\ \qquad +250(-2+25\lambda(-2+\lambda(11+260\lambda)))\mu^2 - 390625\mu^3). \end{cases}$$

(1) (The Pfaffian system for $\mathcal{F}_1$)
Setting
$$t_1 = 729\lambda^2 - 54\lambda(27\mu - 1) + (1 + 27\mu)^2,$$
we have
$$\alpha_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1/9 & -1/2 & -1/2 & -(1+27\lambda+27\mu)/(54\lambda) \\ a_{11}/t_1 & a_{12}/(2t_1) & a_{23}/(2t_1) & a_{24}/(2t_1) \end{pmatrix}$$
with
$$\begin{cases} a_{11} = 3\lambda(1 - 27\lambda + 27\mu), & a_{12} = 3\lambda(5 - 351\lambda + 135\mu), \\ a_{13} = 27\lambda(1 - 3\lambda + 27\mu), & a_{14} = 3(-729\lambda^2 + (1 + 27\mu)^2), \end{cases}$$
and
$$\beta_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1/9 & -1/2 & -1/2 & -(1+27\lambda+27\mu)/(54\lambda) \\ 0 & 0 & 0 & \mu/\lambda \\ b_{11}/t_1 & b_{12}/(2t_1) & b_{13}/(2t_1) & b_{14}/(2t_1) \end{pmatrix}$$
with
$$\begin{cases} b_{11} = 3\lambda(1 + 27\lambda - 27\mu), & b_{12} = 27\lambda(1 + 27\lambda - 3\mu), \\ b_{13} = 3\lambda(5 + 135\lambda - 351\mu), & b_{14} = (1 + 27\lambda)^2 + 108(27\lambda - 1)\mu - 3645\mu^2. \end{cases}$$

(2) (The Pfaffian system for $\mathcal{F}_2$)
Setting
$$\begin{cases} t_2 = \lambda^2(1 + 27\lambda)^2 - 2\lambda\mu(1 + 189\lambda) + (1 + 576\lambda)\mu^2 - 256\mu^3, \\ s_2 = 1 + 108\lambda - 288\mu, \end{cases}$$
we have
$$\alpha_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{11}/s_2 & a_{12}/(2\lambda s_2) & a_{13}/(s_2) & a_{14}/(2\lambda s_2) \\ a_{21}/(t_2 s_2) & a_{22}/(t_2 s_2) & a_{23}/(t_2 s_2) & a_{24}/(t_2 s_2) \end{pmatrix}$$
with
$$\begin{cases} a_{11} = -9\lambda, & a_{12} = -(81\lambda^2 + \mu - 144\lambda\mu), \\ a_{13} = -54\lambda, & a_{14} = -3\lambda(1 + 27\lambda - 144\mu) + \mu, \\ a_{21} = -6\lambda^3(1 + 1458\lambda^2 - 2592\lambda\mu + 6\mu(-55 + 4608\mu)), \\ a_{22} = -3\lambda^2(11 + 54\lambda(5 + 351\lambda)) + \lambda(1 + 4\lambda(61 + 810\lambda(5 + 72\lambda)))\mu + 64(17 + 2808\lambda)\mu^3 \\ \qquad -147456\mu^4 - 2(1 + 9\lambda(53 + 32\lambda(131 + 864\lambda)))\mu^2, \\ a_{23} = -8\lambda^3((2 - 27\lambda)^2 + 9(-133 + 2160\lambda)\mu + 82944\mu^2), \\ a_{24} = 3r_2 s_2 + 162\lambda r_2 - 3\lambda s_2(\lambda + 81\lambda^2 + 1458\lambda^3 - 378\lambda\mu + \mu(-1 + 288\mu)), \end{cases}$$

and

$$\beta_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ b_{11}/s_2 & b_{12}/(2\lambda s_2) & b_{13}/s_2 & b_{14}/(2\lambda s_2) \\ b_{21}/(s_2) & b_{22}/(\lambda^2 s_2) & b_{23}/s_2 & b_{24}/(\lambda^2 s_2) \\ b_{31}/(t_2 s_2) & b_{32}/(2\lambda t_2 s_2) & b_{33}/(t_2 s_2) & b_{34}/(2\lambda t_2 s_2) \end{pmatrix}$$

with

$$\begin{cases} b_{11} = -9\lambda, & b_{12} = -(81\lambda^2 + \mu - 144\lambda\mu), \\ b_{13} = -54\lambda, & b_{14} = -3\lambda(1 + 27\lambda - 144\mu) + \mu, \\ b_{21} = 36\mu, & b_{22} = \mu(\lambda(-1 + 54\lambda) + 2\mu), \\ b_{23} = 216\mu, & b_{24} = (3(1 - 54\lambda)\lambda - 2\mu)\mu, \\ b_{31} = 3\lambda(81\lambda^3(1 + 27\lambda) + \lambda(-1 + 36\lambda)(-5 + 108\lambda)\mu + 3(-1 + 32\lambda)(1 + 432\lambda)\mu^2 + 768\mu^3, \\ b_{32} = 2187\lambda^5(1 + 27\lambda) - (1 + 192\lambda(11 + 1164\lambda))\mu^3 + 256(1 + 864\lambda)\mu^4 \\ \qquad -\lambda^2(2 + 27\lambda(4 + 9\lambda(77 + 864\lambda)))\mu + \lambda(5 + \lambda(1279 + 864\lambda(85 + 864\lambda)))\mu^2, \\ b_{33} = 2\lambda(3\lambda^2(1 + 27\lambda)(-1 + 135\lambda) + 2\lambda(23 + 54\lambda(-11 + 972\lambda))\mu \\ \qquad +9(-3 + 64\lambda)(1 + 432\lambda)\mu^2 + 6912\mu^3, \\ b_{34} = -(-81\lambda^4(1 + 27\lambda)^2 + \lambda^2(-7 + 9\lambda(-58 + 27\lambda(-125 + 3456\lambda)))\mu \\ \qquad +\lambda(8 + 9\lambda(425 + 24192\lambda))\mu^2 - (1 + 3456\lambda(1 + 162\lambda))\mu^3 + 256(1 + 1440\lambda)\mu^4. \end{cases}$$

(3) (The Pfaffian system for $\mathcal{F}_3$)
Setting

$$\begin{cases} t_3 = 729\lambda^2 - (4\mu - 1)^3 + 54\lambda(1 + 12\mu), \\ s_3 = -54\lambda + (1 - 4\mu)^2, \end{cases}$$

we have

$$\alpha_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{11}/s_3 & a_{12}/(2s_3) & a_{13}/s_3 & a_{14}/(2s_3) \\ a_{21}/(t_3 s_3) & a_{22}/(t_3 s_3) & a_{23}/(t_3 s_3) & a_{24}/(t_3 s_3) \end{pmatrix}$$

with

$$\begin{cases} a_{11} = 9\lambda, & a_{12} = 81\lambda + 4(1 - 4\mu)\mu, \\ a_{13} = 27\lambda, & a_{14} = 3 + 81\lambda - 48\mu^2, \\ a_{21} = -2\lambda(-2187\lambda^2 + 27\lambda(4\mu - 9)(4\mu - 1) - (-1 + 4\mu)^3(3 + 8\mu)), \\ a_{22} = 3\lambda(9477\lambda^2 + (1 - 4\mu)^2(-11 + 4\mu(-9 + 16\mu)) - 27\lambda(25 + 4\mu(-31 + 40\mu))), \\ a_{23} = 2\lambda(729\lambda^2 + (-1 + 4\mu)^3(11 + 16\mu) + 27\lambda(-1 + 4\mu)(19 + 20\mu)), \\ a_{24} = 81\lambda(-2 + 27\lambda + 8\mu)(1 + 27\lambda - 16\mu^2), \end{cases}$$

and

$$\beta_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ b_{11}/s_3 & b_{12}/(2s_3) & b_{13}/s_3 & b_{14}/(2s_3) \\ b_{21}/s_3 & b_{22}/s_3 & b_{23}/s_3 & b_{24}/s_3 \\ b_{31}/(t_3 s_3) & b_{32}/(2t_3 s_3) & b_{33}/(t_3 s_3) & b_{34}/(2t_3 s_3) \end{pmatrix}$$

with

$$\begin{cases} b_{11} = 9\lambda, & b_{12} = 81\lambda + 4(1-4\mu)\mu, \\ b_{13} = 27\lambda, & b_{14} = 3 + 81\lambda - 48\mu^2, \\ b_{21} = -2\mu(-1+4\mu), & b_{22} = -3\mu(-3+4\mu), \\ b_{23} = -6\mu(-1+4\mu), & b_{24} = 9\mu(3+4\mu), \\ b_{31} = -3\lambda(2187\lambda^2 + 32(1-4\mu)^2\mu(1+\mu) + 27\lambda(3+16\mu(2+\mu))) \\ b_{32} = -9\lambda(6561\lambda^2 - 81\lambda(-3+4\mu)(1+8\mu) + 4\mu(-1+4\mu)(-33+4\mu(-3+16\mu))), \\ b_{33} = -3\lambda(3645\lambda^2 + 2(1-4\mu)^2(1+16\mu(3+2\mu)) + 27\lambda(7+16\mu(5+9\mu))), \\ b_{34} = -r_3s_3 + r_3(-8+351\lambda+32\mu) + s_3(9(729\lambda^2 + (1-4\mu)^2 + 54\lambda(1+8\mu)). \end{cases}$$

□

Remark 2.1.2. *By changing the system* $\varphi = {}^t(1, \theta_\lambda, \theta_\mu, \theta_\lambda^2)$ *to other ones, we see that* $s = 0$ *is not a singularity. Together with the singularities of* θ_λ *and* θ_μ*, we obtain the singular locus of the system* (2.1.7)*:*

$$\lambda = 0, \quad \mu = 0, \quad \lambda^2(4\lambda-1)^3 - 2(2+25\lambda(20\lambda-1))\mu - 3125\mu^2 = 0. \tag{2.1.11}$$

This is the locus mentioned in Remark 1.2.2.

By the same way, from the Puffian systems in the above proof, we obtain the singular locus of the system (2.1.8)*:*

$$\lambda = 0, \quad \mu = 0, \quad 729\lambda^2 - 54\lambda(27\mu-1) + (1+27\mu)^2 = 0,$$

the singular locus of the system (2.1.9)*:*

$$\lambda = 0, \quad \mu = 0, \quad \lambda^2(1+27\lambda)^2 - 2\lambda\mu(1+189\lambda) + (1+576\lambda)\mu^2 - 256\mu^3 = 0,$$

and the singular locus of the system (2.1.10)*:*

$$\lambda = 0, \quad \mu = 0, \quad 729\lambda^2 - (4\mu-1)^3 + 54\lambda(1+12\mu) = 0.$$

Omitting these locus from $\mathbb{C}^2$ *we have the domain* Λ_j $(j = 1, 2, 3)$ *in* (1.2.21), (1.2.28) *and* (1.2.35).

Remark 2.1.3. *Takayama and Nakayama* [TN] *determined the systems of differential equations for the Fano polytopes with* 6 *vertices by their new approximation method, that is a special use of D-module algorithm.*

2.2 Period differential equation and the Hilbert modular orbifold for the field $\mathbb{Q}(\sqrt{5})$

Let $\mathcal{O}$ be the ring of integers in the real quadratic field $\mathbb{Q}(\sqrt{5})$. Set $\mathbb{H}_\pm = \{z \in \mathbb{C} | \pm \mathrm{Im}(z) > 0\}$. The Hilbert modular group $PSL(2, \mathcal{O})$ acts on $(\mathbb{H}_+ \times \mathbb{H}_+) \cup (\mathbb{H}_- \times \mathbb{H}_-)$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : (z_1, z_2) \mapsto \Big(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \Big),$$

for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2, \mathcal{O})$, where $'$ means the conjugate in $\mathbb{Q}(\sqrt{5})$.

Set

$$W = \begin{pmatrix} 1 & 1 \\ \dfrac{1-\sqrt{5}}{2} & \dfrac{1+\sqrt{5}}{2} \end{pmatrix}.$$

It holds

$$A = U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = U \oplus WU^t W.$$

The correspondence

$$j : (z_1, z_2) \to (z_1 z_2 : -1 : z_1 : z_2)(I_2 \oplus {}^t W^{-1})$$

defines a biholomorphic mapping

$$(\mathbb{H}_+ \times \mathbb{H}_+) \cup (\mathbb{H}_- \times \mathbb{H}_-) \to \mathcal{D}.$$

The group $PSL(2, \mathcal{O})$ is generated by three elements

$$g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & \dfrac{1+\sqrt{5}}{2} \\ 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Set

$$\begin{cases} \tau : (z_1, z_2) \to (z_2, z_1), \\ \tau' : (z_1, z_2) \to \left(\dfrac{1}{z_1}, \dfrac{1}{z_2}\right). \end{cases}$$

We have an isomorphism

$$\begin{array}{ccccc} \tilde{j} & : & \langle PSL(2, \mathcal{O}), \tau \rangle & \to & PO^+(A, \mathbb{Z}) \\ & ; & g & \mapsto & j \circ g \circ j^{-1} = \tilde{j}(g) =: \tilde{g}. \end{array}$$

Especially,

$$\begin{cases} \tilde{g_1} = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{g_2} = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \\ \tilde{g_3} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{\tau} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{cases} \tag{2.2.1}$$

The above j gives a modular isomorphism

$$((\mathbb{H}_+ \times \mathbb{H}_+) \cup (\mathbb{H}_- \times \mathbb{H}_-), \langle PSL(2, \mathcal{O}), \tau, \tau' \rangle) \simeq (\mathcal{D}_+, PO^+(A, \mathbb{Z})).$$

Especially, we have

$$j : (\mathbb{H} \times \mathbb{H}, \langle PSL(2, \mathcal{O}), \tau \rangle) \simeq (\mathcal{D}_+, PO^+(A, \mathbb{Z})). \tag{2.2.2}$$

The mapping $j^{-1} \circ \Phi : \Lambda \to \mathbb{H} \times \mathbb{H}$ gives an explicit transcendental correspondence between Λ and $\mathbb{H} \times \mathbb{H}$.

There are several researches on the Hilbert modular orbifolds for the field $\mathbb{Q}(\sqrt{5})$. Hirzebruch [Hi] studied the orbifold $(\mathbb{H} \times \mathbb{H})/\langle \Gamma, \tau \rangle$ (the group Γ is given in (2.2.4)). There, he used Klein's icosahedral polynomials. Kobayashi, Kushibiki and Naruki [KKN] studied the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$ and determined its branch divisor in terms of the icosahedral invariants. Sato [Sa] gave the uniformizing differential equation (see Definition 2.2.3) of the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$.

Because of the modular isomorphism (2.2.2) and Theorem 1.5.2, our period differential equation (2.1.7) for the family $\mathcal{F}_0 = \{S_0(\lambda, \mu)\}$ should be connected to the uniformizing differential equation of the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$.

In this section, we realize the explicit relation between our period differential equation and the uniformizing differential equation of the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$. We give the exact birational transformation (2.2.12) from our (λ, μ)-space to (x, y)-space, where (x, y) are affine coordinates expressed by Klein's icosahedral polynomials in (2.2.6). Moreover, we show that the uniformizing differential equation with the normalization factor (2.2.16) coincides with our period differential equation (2.1.7).

2.2.1 Linear differential equations in 2 variables of rank 4

First, we survey the study of Sasaki and Yoshida [SY]. It supplies a fundamental tool for the research on uniformizing differential equations of the Hilbelt modular orbifolds.

We consider a system of linear differential equations

$$\begin{cases} Z_{XX} = lZ_{XY} + aZ_X + bZ_Y + pZ, \\ Z_{YY} = mZ_{XY} + cZ_X + dZ_Y + qZ, \end{cases} \tag{2.2.3}$$

where (X, Y) are independent variables and Z is the unknown. We assume its space of solutions is 4-dimensional.

Definition 2.2.1. *We call the symmetric* 2*-tensor*

$$l(dX)^2 + 2(dX)(dY) + m(dY)^2$$

the holomorphic conformal structure of (2.2.3) *.*

Remark 2.2.1. *The above symmetric* 2*-tensor is equal to the holomorphic conformal structure of the complex surface patch embedded in* $\mathbb{P}^3(\mathbb{C})$ *defined by the projective solution of* (2.2.3)*.*

Definition 2.2.2. *Let* Z_0, Z_1, Z_2 *and* Z_3 *be linearly independent solutions of* (2.2.3)*. Put* $Z = {}^t(Z_0, Z_1, Z_2, Z_3)$*. The function*

$$e^{2\theta} = \det(Z, Z_X, Z_Y, Z_{XY})$$

is called the normalization factor of (2.2.3)*.*

Proposition 2.2.1. ([SY] Proposition 4.1, see also [Sa] p.181) *The surface patch by the projective solution of* (2.2.3) *is a part of non degenerate quadratic surface in* $\mathbb{P}^3(\mathbb{C})$ *if and only if*

$$\begin{cases} a = \dfrac{\partial}{\partial X}\Big(\dfrac{1}{4}\xi + \theta\Big) - \dfrac{l}{2}\dfrac{\partial}{\partial Y}\Big(\log(l) - \dfrac{1}{4}\xi + \theta\Big), \\ b = \dfrac{l}{2}\dfrac{\partial}{\partial X}\Big(\log(l) - \dfrac{3}{4}\xi - \theta\Big), \\ c = \dfrac{m}{2}\dfrac{\partial}{\partial Y}\Big(\log(m) - \dfrac{3}{4}\xi - \theta\Big), \\ d = \dfrac{\partial}{\partial Y}\Big(\dfrac{1}{4}\xi + \theta\Big) - \dfrac{m}{2}\dfrac{\partial}{\partial X}\Big(\log(m) - \dfrac{1}{4}\xi + \theta\Big), \end{cases}$$

where $\xi = \log(1 - lm)$.

Proposition 2.2.2. ([SY] Section 3) *Perform a coordinate change of the equation* (2.2.3) *from* (X,Y) *to* (U,V) *and denote the coefficients of the transformed equation by the same letter with bars. Then*

$$\begin{cases} \bar{l} = -\lambda/\nu, \quad \overline{m} = -\mu/\nu, \\ \bar{a} = (R(U)\beta - S(U)\alpha)/\nu, \quad \bar{b} = (R(V)\beta - S(V)\alpha)/\nu, \\ \bar{c} = (S(U)\gamma - R(U)\delta))/\nu, \quad \bar{d} = (S(V)\gamma - R(V)\delta)/\nu, \\ \bar{p} = (\alpha q - \beta p)/\nu, \quad \bar{q} = (\delta p - \gamma q)/\nu, \end{cases}$$

where

$$\begin{cases} \Delta = U_X V_Y - U_Y V_X, \\ \lambda = l V_Y^2 - 2 V_X V_Y + m V_X^2, \\ \mu = l U_Y^2 - 2 U_X U_Y + m U_X^2, \\ \nu = l U_Y V_Y - U_X V_Y - U_Y V_X + m U_X V_X, \end{cases}$$

and

$$\begin{cases} \alpha = (V_X^2 - l V_X V_Y)/\Delta, \quad \beta = (V_Y^2 - m V_X V_Y)/\Delta, \\ \gamma = (U_X^2 - l U_X U_Y)/\Delta, \quad \delta = (U_Y^2 - m U_X U_Y)/\Delta, \\ R(U) = U_{XX} - (l U_{XY} + a U_X + b U_Y), \\ S(U) = U_{YY} - (m U_{XY} + c U_X + d U_Y), \\ R(V) = V_{XX} - (l V_{XY} + a V_X + b V_Y), \\ S(V) = V_{YY} - (m V_{XY} + c V_X + d V_Y). \end{cases}$$

2.2.2 Uniformizing differential equation of the Hilbert modular orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2,\mathcal{O}), \tau\rangle$

The quotient space $(\mathbb{H} \times \mathbb{H})/\langle PSL(2,\mathcal{O}), \tau\rangle$ carries the structure of an orbifold. Let us sum up the facts about the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2,\mathcal{O}), \tau\rangle$ and the result of Sato [Sa] on the uniformizing differential equation.

Remark 2.2.2. *The results about this orbifold shall be stated more detailed in Section 3.1.*

Set

$$\Gamma(\sqrt{5}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2,\mathcal{O}) \middle| \, \alpha \equiv \delta \equiv 1, \ \beta \equiv \gamma \equiv 0 \pmod{\sqrt{5}} \right\}. \tag{2.2.4}$$

Γ is a normal subgroup of $PSL(2,\mathcal{O})$. The quotient group $PSL(2,\mathcal{O})/\Gamma(\sqrt{5})$ is isomorphic to the alternating group $\mathcal{A}_5$ of degree 5. $\mathcal{A}_5$ is isomorphic to the icosahedral group I. Let $\overline{M}$ be a compactification of an orbifold M. Hirzebruch [Hi] showed that $\overline{\mathbb{H}\times\mathbb{H}/\langle\Gamma,\tau\rangle}$ is isomorphic to $\mathbb{P}^2(\mathbb{C})$. Therefore, $\mathbb{P}^2(\mathbb{C})$ admits an action of the alternating group $\mathcal{A}_5$. This action is equal to the action of the icosahedral group I on $\mathbb{P}^2(\mathbb{C})$ introduced by F. Klein. We list Klein's I-invariant polynomials on $\mathbb{P}^2(\mathbb{C}) = \{(\zeta_0 : \zeta_1 : \zeta_2)\}$:

$$\begin{cases} \mathfrak{A}(\zeta_0:\zeta_1:\zeta_2) = \zeta_0^2 + \zeta_1\zeta_2, \\ \mathfrak{B}(\zeta_0:\zeta_1:\zeta_2) = 8\zeta_0^4\zeta_1\zeta_2 - 2\zeta_0^2\zeta_1^2\zeta_2^2 + \zeta_1^3\zeta_2^3 - \zeta_0(\zeta_1^5+\zeta_2^5), \\ \mathfrak{C}(\zeta_0:\zeta_1:\zeta_2) = 320\zeta_0^6\zeta_1^2\zeta_2^2 - 160\zeta_0^4\zeta_1^3\zeta_2^3 + 20\zeta_0^2\zeta_1^4\zeta_2^4 + 6\zeta_1^5\zeta_2^5 \\ \qquad\qquad -4\zeta_0(\zeta_1^5+\zeta_2^5)(32\zeta_0^4 - 20\zeta_0^2\zeta_1\zeta_5 + 5\zeta_1^2\zeta_2^2) + \zeta_1^{10} + \zeta_2^{10}, \\ 12\mathfrak{D}(\zeta_0:\zeta_1:\zeta_2) = (\zeta_1^5 - \zeta_2^5)(-1024\zeta_0^{10} + 3840\zeta_0^8\zeta_1\zeta_2 - 3840\zeta_0^6\zeta_1^2\zeta_2^2 \\ \qquad\qquad +1200\zeta_0^4\zeta_1^3\zeta_2^3 - 100\zeta_0^2\zeta_1^4\zeta_2^4 + \zeta_1^5\zeta_2^5) \\ \qquad\qquad +\zeta_0(\zeta_1^{10} - \zeta_2^{10})(352\zeta_0^4 - 160\zeta_0^2\zeta_1\zeta_2 + 10\zeta_1^2\zeta_2^2) + (\zeta_1^{15} - \zeta_2^{15}). \end{cases}$$

We have the following relation:

$$144\mathfrak{D}^2 = -1728\mathfrak{B}^5 + 720\mathfrak{A}\mathfrak{C}\mathfrak{B}^3 - 80\mathfrak{A}^2\mathfrak{C}^2\mathfrak{B} + 64\mathfrak{A}^3(5\mathfrak{B}^2 - \mathfrak{A}\mathfrak{C})^2 + \mathfrak{C}^3. \tag{2.2.5}$$

Kobayashi, Kushibiki and Naruki [KKN] showed that a compactification $\overline{(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle}$ is birationally equivalent to $\mathbb{P}^2(\mathbb{C})$. Let

$$\varphi : \mathbb{P}^2(\mathbb{C}) = \overline{(\mathbb{H}\times\mathbb{H})/\langle\Gamma,\tau\rangle} \to \overline{(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle} = \mathbb{P}^2(\mathbb{C})$$

be a rational mapping defined by

$$(\zeta_0:\zeta_1:\zeta_2) \mapsto (\mathfrak{A}^5 : \mathfrak{A}^2\mathfrak{B} : \mathfrak{C}).$$

φ is a holomorphic mapping of $\mathbb{P}^2(\mathbb{C}) - \{A = 0\}$ to $\mathbb{P}^2(\mathbb{C}) -$ (a line at infinity L_∞) $\subset$ $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$. Set

$$X = \frac{\mathfrak{B}}{\mathfrak{A}^3}, \quad Y = \frac{\mathfrak{C}}{\mathfrak{A}^5}. \tag{2.2.6}$$

X and Y are the affine coordinates identifying $(1 : X : Y) \in \mathbb{P}^2(\mathbb{C}) - L_\infty$ with $(X,Y) \in \mathbb{C}^2$ (These properties of the Hilbert modular orbifold shall be stated in detail in Section 3.1).

Proposition 2.2.3. ([KKN]) *The branch locus of the orbifold* $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$ *in* $\mathbb{P}^2(\mathbb{C}) - L_\infty = \mathbb{C}^2$ *is, using the affine coordinates* (2.2.6),

$$D = Y(1728X^5 - 720X^3Y + 80XY^2 - 64(5X^2 - Y)^2 - Y^3) = 0$$

of index 2. *The orbifold structure on* $\overline{(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle}$ *is given by* $(\mathbb{P}^2(\mathbb{C}), 2D + \infty L_\infty)$.

We note that $\mathbb{H}\times\mathbb{H}$ is embedded in $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ which is isomorphic to a non-degenerate quadric surface in $\mathbb{P}^3(\mathbb{C})$. Let $\pi:\mathbb{H}\times\mathbb{H}\to(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$ be the canonical projection. The multivalued inverse mapping π^{-1} is called the developing map of the orbifold $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$.

Definition 2.2.3. *Let us consider a system of linear differential equations on the orbifold $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$ with 4-dimensional space of solutions. Let z_0,z_1,z_2,z_3 be linearly independent solutions of the system. If*

$$(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle\to\mathbb{P}^3(\mathbb{C}):p\mapsto(z_0(p):z_1(p):z_2(p):z_3(p))$$

gives the developing map of the orbifold $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$, we call this system the uniformizing differential equation of the orbifold.

From Proposition 2.2.3, T. Sato obtained the following result.

Theorem 2.2.1. ([Sa] Example. 4) *The holomorphic conformal structure of the uniformizing differential equation of the orbifold $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$ is*

$$\frac{-20(4X^2+3XY-4Y)}{36X^2-32X-Y}(dX)^2+2(dX)(dY)+\frac{-2(54X^3-50X^2-3XY+2Y)}{5Y(36X^2-32X-Y)}(dY)^2, \tag{2.2.7}$$

where (X,Y) is the affine coordinates in (2.2.6) .

Let

$$\begin{cases} z_{XX}=lz_{XY}+az_X+bz_Y+pz,\\ z_{YY}=mz_{XY}+cz_X+dz_Y+qz \end{cases} \tag{2.2.8}$$

be the uniformizing differential equation of $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$, where (x,y) is the affine coordinates in (2.2.6). We already obtained the coefficients l and m (see Definition 2.2.1 and Theorem 2.2.1). If the normalization factor of (2.2.8) is given, the coefficients a,b,c and d are determined by Proposition 2.2.1. The other coefficients p and q are determined by the integrability condition of (2.2.8).

Remark 2.2.3. *Sato* [Sa] *determined the uniformizing differential equation of* $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$

$$\begin{cases} z_{XX}=lz_{XY}+a_sz_X+b_sz_Y+p_sz,\\ z_{YY}=mz_{XY}+c_sz_X+d_sz_Y+q_sz \end{cases}$$

with

$$\begin{cases} a_s(X,Y)=\dfrac{-20(3X-2)}{36X^2-32X-Y}, & b_s(X,Y)=\dfrac{-10(8X+3Y)}{36X^2-32X-Y},\\ c_s(X,Y)=\dfrac{3X-2}{5y(36X^2-32X-Y)}, & d_s(X,Y)=\dfrac{-198X^2+180X+7Y}{5Y(36X^2-32X-Y)},\\ p_s(X,Y)=\dfrac{-3}{(36X^2-32X-Y)}, & q_s(X,Y)=\dfrac{3}{100Y(36X^2-32X-Y)}. \end{cases}$$

Here, the normalization factor

$$e^{2\theta} = \frac{-36X^2 + 32X + Y}{Y^{1/2}(1728X^5 - 720X^3Y + 80XY^2 - 64(5X^2 - Y)^2 - Y^3))^{3/2}}. \tag{2.2.9}$$

exactly corresponds to the above data a_s, b_s, c_s, d_s, p_s and q_s. It should coincides with the original normalization factor in [Sa] *p.185, because Sato used the above data. However, it is not the case. We suppose there would be contained some typos in the original one.*

2.2.3 Exact relation between period differential equation and unifomizing differential equation

The modular isomorphism (2.2.2) implies that our period differential equation (2.1.7) should be related to the uniformizing differential equation of the orbifold $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$. In this subsection, we show that the holomorphic conformal structure of (2.1.7) is transformed to (2.2.7) in Theorem 2.2.1 by an explicit birational transformation. Moreover, we determine a normalization factor which is different from that of Sato's (2.2.9). The uniformizing differential equation of the orbifold $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$ with our normalizing factor corresponds to the period differential equation (2.1.7).

Proposition 2.2.4. *The period differential equation* (2.1.7) *is represented in the form*

$$\begin{cases} z_{\lambda\lambda} = l_0 z_{\lambda\mu} + a_0 z_\lambda + b_0 z_\mu + p_0 z, \\ z_{\mu\mu} = m_0 z_{\lambda\mu} + c_0 z_\lambda + d_0 z_\mu + q_0 z \end{cases} \tag{2.2.10}$$

with

$$\begin{cases} l_0 = \dfrac{2\mu(-1+15\lambda+100\lambda^2)}{\lambda+16\lambda^2-80\lambda^3+125\mu}, & m_0 = \dfrac{2(\lambda^2-8\lambda^3+16\lambda^4+5\mu-50\lambda\mu)}{\mu(\lambda+16\lambda^2-80\lambda^3+125\mu)}, \\ a_0 = \dfrac{(-1+10\lambda)(1+20\lambda)}{\lambda+16\lambda^2-80\lambda^3+125\mu}, & b_0 = \dfrac{5\mu(3+40\lambda)}{\lambda+16\lambda^2-80\lambda^3+125\mu}, \\ c_0 = -\dfrac{5(-1+10\lambda)}{\mu(\lambda+16\lambda^2-80\lambda^3+125\mu)}, & d_0 = \dfrac{-\lambda-20\lambda^2+96\lambda^3-200\mu}{\mu(\lambda+16\lambda^2-80\lambda^3+125\mu)}, \\ p_0 = \dfrac{2(1+20\lambda)}{\lambda+16\lambda^2-80\lambda^3+125\mu}, & q_0 = -\dfrac{10}{\mu(\lambda+16\lambda^2-80\lambda^3+125\mu)}. \end{cases}$$

Proof. Straightforward calculation. □

Especially, the holomorphic conformal structure of the period differential equation (2.1.7) is

$$\frac{2\mu(-1+15\lambda+100\lambda^2)}{\lambda+16\lambda^2-80\lambda^3+125\mu}(d\lambda)^2 + 2(d\lambda)(d\mu) + \frac{2(\lambda^2-8\lambda^3+16\lambda^4+5\mu-50\lambda\mu)}{\mu(\lambda+16\lambda^2-80\lambda^3+125\mu)}(d\mu)^2. \tag{2.2.11}$$

Theorem 2.2.2. *Set a birational transformation*

$$f : (\lambda,\mu) \mapsto (X,Y) = \Big(\frac{25\mu}{2(\lambda-1/4)^3}, -\frac{3125\mu^2}{(\lambda-1/4)^5}\Big) \tag{2.2.12}$$

from (λ,μ)-space to (x,y)-space. The holomorphic conformal structure (2.2.11) *is transformed to the holomorphic conformal structure* (2.2.7) *by f.*

Proof. The inverse f^{-1} is given by

$$\lambda(X,Y) = \frac{1}{4} - \frac{Y}{20X^2}, \qquad \mu(X,Y) = -\frac{Y^3}{10^5X^5}. \tag{2.2.13}$$

We have

$$\begin{cases} l_0(\lambda(X,Y),\mu(X,Y)) = \dfrac{-Y^2(4X^2-Y)(9X^2-Y)}{250X^3(240X^4-88X^2Y+8Y^2-XY^2)}, \\ m_0(\lambda(X,Y),\mu(X,Y)) = \dfrac{-4000X^3(100X^4-40X^2Y+3X^3Y+4Y^2-XY^2)}{Y^2(240X^4-88X^2Y+8Y^2-XY^2)}. \end{cases} \tag{2.2.14}$$

By (2.2.12) and (2.2.13), we have

$$X_\lambda = \frac{60X^3}{Y}, \quad Y_\lambda = 100Y^2, \quad X_\mu = -\frac{10^5X^6}{Y^3}, \quad Y_\mu = -\frac{2\cdot 10^5X^5}{Y^2}. \tag{2.2.15}$$

From (2.2.14) and (2.2.15) and Proposition 2.2.2, by the birational transformation $f : (\lambda,\mu) \mapsto (X,Y)$, the coefficients l_0 and m_0 are transformed to

$$\overline{l_0} = \frac{-20(4X^2+3XY-4Y)}{36X^2-32X-Y}, \quad \overline{m_0} = \frac{-2(54X^3-50X^2-3XY+2Y)}{5Y(36X^2-32X-Y)}.$$

These are equal to the coefficients of the holomorphic conformal structure (2.2.7). Therefore, the holomorphic conformal structure (2.2.11) is transformed to (2.2.7). □

Remark 2.2.4. *The birational transformation* (2.2.12) *is obtained as the composition of certain birational transformations. First, blow up at* $(\lambda,\mu) = (1/4,0) \in ((\lambda,\mu)$*-space) three times:* $(\lambda,\mu) \mapsto (\lambda,u_1) = \Big(\lambda, \dfrac{\mu}{\lambda-1/4}\Big)$, $(\lambda,u_1) \mapsto (\lambda,u_2) = \Big(\lambda, \dfrac{u_1}{\lambda-1/4}\Big)$, $(\lambda,u_2) \mapsto (\lambda,u_3) = \Big(\lambda, \dfrac{u_2}{\lambda-1/4}\Big)$*. Cancel* λ *by* $\lambda = \dfrac{u_2}{u_3} + \dfrac{1}{4}$*. Then, we have the following birational transformation:*

$$\psi_0 : (\lambda,\mu) \mapsto (u_2,u_3) = \Big(\frac{\mu}{(\lambda-1/4)^2}, \frac{\mu}{(\lambda-1/4)^3}\Big).$$

(Its inverse is given by

$$\psi_0^{-1} : (u_2,u_3) \mapsto (\lambda,\mu) = \Big(\frac{u_2}{u_3} + \frac{1}{4}, \frac{u_2^3}{u_3^2}\Big).)$$

On the other hand, blow up at $(X,Y) = (0,0) \in ((x,y)$*-space):*

$$\psi_1 : (X,Y) \mapsto (X,s) = \Big(X, \frac{Y}{X}\Big).$$

(Its inverse is given by

$$\psi_1^{-1} : (X,s) \mapsto (X,Y) = (X,Xs).)$$

Moreover, we define the holomorphic mapping

$$\chi : (u_2,u_3) \mapsto (x,s) = \Big(\frac{25}{2}u_3, -250u_2\Big).$$

We have $f = \psi_1^{-1} \circ \chi \circ \psi_0$.

Instead the normalization factor (2.2.9) used by Sato, that is referred in Remark 2.2.3, we need a new normalization factor (2.2.16). Together with the conformal structure coming from $(l_1, m_1) = (l, m)$, we obtain the new uniformizing differential equation which we are looking for.

Proposition 2.2.5. *The uniformizing differential equation of the orbifold* $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$ *with the normalization factor*

$$e^{2\theta} = \frac{X^4(-36X^2+32X+Y)}{Y^{5/2}(1728X^5-720X^3Y+80XY^2-64(5X^2-Y)^2-Y^3)^{3/2}} \tag{2.2.16}$$

is

$$\begin{cases} z_{XX} = l_1 z_{XY} + a_1 z_X + b_1 z_Y + p_1 z, \\ z_{YY} = m_1 z_{XY} + c_1 z_X + d_1 z_Y + q_1 z \end{cases} \tag{2.2.17}$$

with

$$\begin{cases} l_1 = \dfrac{-20(4X^2+3XY-4Y)}{36X^2-32X-Y}, \quad m_1 = \dfrac{-2(54X^3-50X^2-3XY+2Y)}{5Y(36X^2-32X-Y)}, \\ a_1 = \dfrac{-2(20X^3-8XY+9X^2Y+Y^2)}{XY(36X^2-32X-Y)}, \quad b_1 = \dfrac{10Y(-8+3X)}{X(36X^2-32X-Y)}, \\ c_1 = \dfrac{-2(-25X^2+27X^3+2Y-3XY)}{5Y^2(36X^2-32X-Y)}, \quad d_1 = \dfrac{-2(-120X^2+135X^3-2Y-3XY)}{5XY(36X^2-32X-Y)} \\ p_1 = \dfrac{-2(8X-Y)}{X^2(36X^2-32X-Y)}, \quad q_1 = \dfrac{-2(-10+9X)}{25XY(36X^2-32X-Y)}. \end{cases}$$

Proof. l_1 and m_1 are given in Theorem 2.2.1. According to Proposition 2.2.1, the other coefficients are determined by l_1, m_1 and θ in (2.2.16). □

Theorem 2.2.3. *By the birational transformation f in* (2.2.12)*, our period differential equation* (2.2.10) *is transformed to the uniformizing differential equation* (2.2.17) *of the orbifold* $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$.

Proof. We have

$$\begin{cases} a_0(\lambda(X,Y),\mu(X,Y)) = \dfrac{400X^2(3X^2-Y)(6X^2-Y)}{Y(240X^4-88X^2Y+8Y^2-XY^2)}, \\ b_0(\lambda(X,Y),\mu(X,Y)) = \dfrac{-Y^2(13X^2-2Y)}{25X(240X^4-88X^2Y+8Y^2-XY^2)}, \\ c_0(\lambda(X,Y),\mu(X,Y)) = \dfrac{2\cdot 10^8 X^9(3X^2-Y)}{Y^4(240X^4-88X^2Y+8Y^2-XY^2),} \\ d_0(\lambda(X,Y),\mu(X,Y)) = \dfrac{160000X^5(175X^4-65X^2Y+6Y^2-XY^2)}{Y^3(240X^4-88X^2Y+8Y^2-XY^2)}, \\ p_0(\lambda(X,Y),\mu(X,Y)) = \dfrac{1600X^4(6X^2-Y)}{Y(240X^4-88X^2Y+8Y^2-XY^2)}, \\ q_0(\lambda(X,Y),\mu(X,Y)) = \dfrac{8\cdot 10^8 X^{11}}{Y^4(240X^4-88X^2Y+8Y^2-XY^2)}. \end{cases} \tag{2.2.18}$$

By (2.2.12) and (2.2.13), we have

$$\begin{cases} X_{\lambda\lambda} = \dfrac{4800X^5}{Y^2}, \quad Y_{\lambda\lambda} = \dfrac{12000X^4}{Y}, \quad X_{\mu\mu} = 0, \\ Y_{\mu\mu} = \dfrac{2\cdot 10^{10}X^{10}}{Y^5}, \quad X_{\lambda\mu} = \dfrac{-6\cdot 10^6 X^8}{Y^4}, \quad Y_{\lambda\mu} = \dfrac{-2\cdot 10^7 X^7}{Y^3}. \end{cases} \tag{2.2.19}$$

From (2.2.14), (2.2.15), (2.2.18) and (2.2.19) and Proposition 2.2.2, by the birational transformation $f : (\lambda,\mu) \mapsto (X, Y)$, the coefficients a_0, b_0, c_0, d_0, p_0 and q_0 are transformed to

$$\begin{cases} \overline{a_0} = \dfrac{-2(20X^3 - 8XY + 9X^2Y + Y^2)}{XY(36X^2 - 32X - Y)}, \quad \overline{b_0} = \dfrac{10Y(-8+3X)}{X(36X^2 - 32X - Y)}, \\ \overline{c_0} = \dfrac{-2(-25X^2 + 27X^3 + 2Y - 3XY)}{5Y^2(36X^2 - 32X - Y)}, \quad \overline{d_0} = \dfrac{-2(-120X^2 + 135X^3 - 2Y - 3XY)}{5XY(36X^2 - 32X - Y)}, \\ \overline{p_0} = \dfrac{-2(8X - Y)}{X^2(36X^2 - 32X - Y)}, \quad \overline{q_0} = \dfrac{-2(-10+9X)}{25XY(36X^2 - 32X - Y)}. \end{cases}$$

These are equal to the coefficients of (2.2.17). □

Therefore, the uniformizing differential equation of the orbifold $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$ with the normalization factor (2.2.16) is connected to our family $\mathcal{F}_0 = \{S_0(\lambda,\mu)\}$ of $K3$ surfaces.

Chapter 3

A theta expression of the Hilbert modular functions for $\sqrt{5}$ via the period mapping for a family of $K3$ surfaces

According to Section 2.2, the period mapping for the family $\mathcal{F}_0$ of $K3$ surfaces

$$S_0(\lambda,\mu) : x_0 y_0 z_0^2 (x_0 + y_0 + z_0 + 1) + \lambda x_0 y_0 z_0 + \mu = 0, \tag{3.0.1}$$

is strongly related to the Hilbert modular function for $\mathbb{Q}(\sqrt{5})$.

In this chapter, we consider the family $\mathcal{F} = \{S(X,Y)\}$ of $K3$ surfaces over $\mathbb{P}(1:3:5)$. Note that a member $S(X,Y)$ is birationally equivalent to a member $S_0(\lambda,\mu)$ of $\mathcal{F}_0$. Using the results of Hirzebruch [Hi] and Müller [Mul], we prove that the inverse correspondence of the multivalued period mapping for our family $\mathcal{F}$ gives a pair of Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$.

3.1 The Hilbert modular orbifold $\overline{(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle}$

Here, we recall the action of the Hilbert modular group on $\mathbb{H}\times\mathbb{H}$.

Let $\mathcal{O}$ be the ring of integers in the real quadratic field $\mathbb{Q}(\sqrt{5})$. Set $\mathbb{H}_\pm = \{z \in \mathbb{C} \mid \pm \mathrm{Im}(z) > 0\}$. The Hilbert modular group $PSL(2,\mathcal{O})$ acts on $(\mathbb{H}_+\times\mathbb{H}_+)\cup(\mathbb{H}_-\times\mathbb{H}_-)$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : (z_1,z_2) \mapsto \Big(\frac{\alpha z_1+\beta}{\gamma z_1+\delta}, \frac{\alpha' z_2+\beta'}{\gamma' z_2+\delta'}\Big),$$

for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2,\mathcal{O})$, where $'$ means the conjugate in $\mathbb{Q}(\sqrt{5})$. We consider the involution

$$\tau : (z_1,z_2) \mapsto (z_2,z_1)$$

also.

Definition 3.1.1. *If a holomorphic function g on $\mathbb{H}\times\mathbb{H}$ satisfies the transformation law*

$$g\Big(\frac{az_1+b}{cz_1+d},\frac{a'z_2+b'}{c'z_2+d'}\Big)=(cz_1+d)^k(c'z_2+d')^kg(z_1,z_2)$$

for any $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in PSL(2,\mathcal{O})$, *we call g a Hilbert modular form of weight k for $\mathbb{Q}(\sqrt{5})$. If $g(z_2,z_1)=g(z_1,z_2)$, g is called a symmetric modular form. If $g(z_2,z_1)=-g(z_1,z_2)$, g is called an alternating modular form.*

If a meromorphic function f on $\mathbb{H}\times\mathbb{H}$ satisfies

$$f\Big(\frac{az_1+b}{cz_1+d},\frac{a'z_2+b'}{c'z_2+d'}\Big)=f(z_1,z_2)$$

for any $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in PSL(2,\mathcal{O})$, *we call f a Hilbert modular function for $\mathbb{Q}(\sqrt{5})$.*

Hirzebruch [Hi] studied the Hilbert modular orbifold $\overline{(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle}$. Here, we survey his results.

Recall

$$\Gamma(\sqrt{5})=\Big\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\Big|\alpha\equiv\delta\equiv 1,\beta\equiv\delta\equiv 0 \pmod{\sqrt{5}}\Big\}.$$

in Section 2.2.2. The group $PSL(2,\mathcal{O})/\Gamma(\sqrt{5})$ is isomorphic to the alternating group $\mathcal{A}_5$. Hirzebruch [Hi] studied the canonical bundle of the orbifold $\overline{(\mathbb{H}\times\mathbb{H})/\Gamma(\sqrt{5})}$ by an algebrogeometric method. He proved

Proposition 3.1.1. ([Hi] pp.307-310) (1) *The non-singular model of $\overline{(\mathbb{H}\times\mathbb{H})/\langle\Gamma(\sqrt{5}),\tau\rangle}$ is $\mathbb{P}^2(\mathbb{C})=\{(\zeta_0;\zeta_1;\zeta_2)\}$ by adding six points. A homogeneous polynomial of degree k in ζ_0,ζ_1 and ζ_2 defines a modular form for $\Gamma(\sqrt{5})$ of weight k.*

(2) *The ring of symmetric modular forms for $PSL(2,\mathcal{O})$ is isomorphic to the ring*

$$\mathbb{C}[\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D}]/(R(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})=0),$$

where $R(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})$ is the Klein relation (2.2.5). *$\mathfrak{A}$ ($\mathfrak{B},\mathfrak{C},\mathfrak{D}$, resp.) gives a symmetric modular form for $PSL(2,\mathcal{O})$ of weight* 2 $(6,10,15$, *resp.*).

(3) *There exists an alternating modular form $\mathfrak{c}$ of weight 5 such that $\mathfrak{c}^2=\mathfrak{C}$. The ring of Hilbert modular forms for $PSL(2,\mathcal{O})$ is isomorphic to the ring*

$$\mathbb{C}[\mathfrak{A},\mathfrak{B},\mathfrak{c},\mathfrak{D}]/(R(\mathfrak{A},\mathfrak{B},\mathfrak{c}^2,\mathfrak{D})=0).$$

For our further study, we need the weighted projective space $\mathbb{P}(1,3,5)$. Let $c\in\mathbb{C}-\{0\}$.

$$(a_0,a_1,a_2)\sim(ca_0,c^3a_1,c^5a_2)$$

gives an equivalence relation on $\mathbb{C}^3-\{(0,0,0)\}$. We call $\mathbb{P}(1,3,5):=(\mathbb{C}^3-\{(0,0,0)\})/\sim$ the weighted projective space of weight $(1,3,5)$. This is a 2-dimensional algebraic variety.

Let $c'\in\mathbb{C}-\{0\}$. We consider the action $(\zeta_0,\zeta_1,\zeta_2)\mapsto(c'\zeta_0,c'\zeta_1,c'\zeta_2)$. Because $\mathfrak{A}$ ($\mathfrak{B},\mathfrak{C}$, resp.) is a homogeneous polynomial of degree 2 $(6,10$, resp) in ζ_0,ζ_1 and ζ_2, we have the action $(\mathfrak{A},\mathfrak{B},\mathfrak{C}) \mapsto (c'^2\mathfrak{A}, c'^6\mathfrak{B}, c'^{10}\mathfrak{C})$. Therefore, we regard $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$-space as the weighted projective space $\mathbb{P}(1,3,5)$. Especially,

$$(X,Y) = \Big(\frac{\mathfrak{B}}{\mathfrak{A}^3}, \frac{\mathfrak{C}}{\mathfrak{A}^5}\Big)$$

in (2.2.6) gives a system of affine coordinates on $\{\mathfrak{A} \neq 0\}$.

By the arguments of Klein [Kl], [Hi] and Kobayashi, Kushibiki and Naruki [KKN], we know the following properties of the action of $\mathcal{A}_5$ on $\overline{(\mathbb{H}\times\mathbb{H})/\langle\Gamma(\sqrt{5}),\tau\rangle} = \mathbb{P}^2(\mathbb{C}) = \{\zeta_0 : \zeta_1 : \zeta_2\}$.

Proposition 3.1.2. (1) *The correspondence* $(\zeta_0:\zeta_1:\zeta_2) \mapsto \big(\mathfrak{A}(\zeta_0:\zeta_1:\zeta_2) : \mathfrak{B}(\zeta_0:\zeta_1:\zeta_2) : \mathfrak{C}(\zeta_0:\zeta_1:\zeta_2)\big)$ *gives an identification between* $\overline{\mathbb{P}^2(\mathbb{C})/\mathcal{A}_5}$ *and* $\mathbb{P}(1,3,5)$. *Then, the Hilbert modular orbifold* $\overline{(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle}$ *is identified with* $\mathbb{P}(1,3,5)$. *The cusp* $(\sqrt{-1}\infty,\sqrt{-1}\infty) \in \overline{(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle}$ *is given by the point* $(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) = (1:0:0)$. *So, the quotient space* $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle$ *corresponds to* $\mathbb{P}(1,3,5) - \{(1:0:0)\}$.

(2) *The divisor* $\{\mathfrak{D} = 0\}$ *consists of fifteen lines in* $\mathbb{P}^2(\mathbb{C})$. *These fifteen lines of* $\{\mathfrak{D} = 0\}$ *are the reflection lines of fifteen involutions of* $\mathcal{A}_5$ (*note that* $\mathcal{A}_5$ *is generated by three involutions*).

(3) *The involution* τ *induces an involution on the orbifold* $\overline{(\mathbb{H}\times\mathbb{H})/PSL(2,\mathcal{O})}$. *The branch locus of the canonical projection* $\overline{(\mathbb{H}\times\mathbb{H})/PSL(2,\mathcal{O})} \to \mathbb{P}(1,3,5)$ *is given by* $\{\mathfrak{C} = 0\}$.

Set

$$\mathfrak{X} = \{(X,Y)\in\mathbb{C}^2 | Y(1728X^5 - 720X^3Y + 80XY^2 - 64(5X^2-Y)^2 - Y^3) \neq 0\}. \quad (3.1.1)$$

3.2 The period of the family $\mathcal{F}$

3.2.1 The family $\mathcal{F}$ of $K3$ surfaces

By a birational transformation, we obtain a new family of $K3$ surfaces with explicit defining equations from the family $\mathcal{F}_0 = \{S_0(\lambda,\mu)\}$ in (3.0.1).

Proposition 3.2.1. *The family of* $K3$ *surfaces* $\mathcal{F}_0 = \{S_0(\lambda,\mu)\}$ *for* $(\lambda,\mu)\in\Lambda$ *is transformed to the family* $\mathcal{F} = \{S(X,Y)\}$ *for* $(X,Y)\in\mathfrak{X}$:

$$S(X,Y) : z^2 = x^3 - 4y^2(4y-5)x^2 + 20Xy^3x + Yy^4. \quad (3.2.1)$$

Proof. By the transformation (2.2.12) and the birational transformation given by

$$\begin{cases} x_0 = \dfrac{Yy}{10Xx_1}, \\ y_0 = \dfrac{4Y^2x_1y_1^2}{-50X^2Yx_1y_1 - 5XY^2y_1^2 + 5XYz_1}, \\ z_0 = -\dfrac{10XYx_1y_1 + Y^2y_1^2 - Yz_1}{20XYx_1y_1}, \end{cases}$$

the family $\mathcal{F}_0 = \{S_0(\lambda,\mu)\}$ is transformed to the family $\mathcal{F}_1 = \{S_1(X,Y)\}$ given by

$$S_1(X,Y) : z_1^2 = Y(x_1^3 - 4y_1^2(4y_1-5)x_1^2 + 20Xy_1^3x_1 + Yy_1^4)$$

over $\mathfrak{X}$. Then, by the correspondence $(x_1,y_1,z_1) \mapsto (x,y,z) = \Big(x_1, y_1, \dfrac{1}{\sqrt{Y}} z_1\Big)$, we have the family $\mathcal{F} = \{S(X,Y)\}$ given by (3.2.1). □

Because we have the biholomorphic mapping (2.2.12) and $\check{S}(\lambda,\mu)$ is birationally equivalent to $S(X,Y)$, we obtain the multivalued analytic period mapping

$$\Phi_1 : \mathfrak{X} \to \mathcal{D}_+ ; (X,Y) \mapsto \Big(\int_{\Gamma_1} \omega : \int_{\Gamma_2} \omega : \int_{\Gamma_3} \omega : \int_{\Gamma_4} \omega \Big), \tag{3.2.2}$$

where $\omega = \dfrac{dx \wedge dy}{z}$ is the unique holomorphic 2-form on $S(X,Y)$ up to a constant factor and $\Gamma_1, \cdots, \Gamma_4$ are certain 2-cycles on $S(X,Y)$ (this period mapping is stated in detail at the beginning of Section 3.2.2).

Remark 3.2.1. *The correspondence $(x_1,y_1,z_1) \mapsto (x,y,z) = (x_1, y_1, \frac{1}{\sqrt{Y}} z_1)$ in the proof of Proposition 3.2.1 induces the double covering $\mathfrak{X}' \to \mathfrak{X}$ given by $(X,Y') \mapsto (X,Y) = (X, Y'^2)$. However, (X,Y') and $(X,-Y') \in \mathfrak{X}'$ define mutually isomorphic P-marked $K3$ surfaces (see* Definition 3.2.1 *). So, we obtain the above period mapping Φ_1 on $\mathfrak{X}$.*

Due to Theorem 1.3.2, Theorem 1.4.1, we have clearly

Theorem 3.2.1. (1) *For a generic point $(X,Y) \in \mathfrak{X}$,*

$$\mathrm{rank}(\mathrm{NS}(S(X,Y))) = 18.$$

(2) *For a generic point $(X,Y) \in \mathfrak{X}$, the intersection matrix of the transcendental lattice* $\mathrm{Tr}(S(X,Y))$ *is given by*

$$A = U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}. \tag{3.2.3}$$

(3) *The projective monodromy group of the multivalued analytic period mapping $\Phi : \mathfrak{X} \to \mathcal{D}_+$ is isomorphic to $PO^+(A,\mathbb{Z})$.*

(4) *The period differential equation for the family $\mathcal{F} = \{S(X,Y)\}$ is given by* (2.2.17).

Proposition 3.2.2. *Under the correspondence* (2.2.6)*, the surface $S(X,Y)$ is birationally equivalent to*

$$S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) : z^2 = x^3 - 4(4y^3 - 5\mathfrak{A}y^2)x^2 + 20\mathfrak{B}y^3x + \mathfrak{C}y^4. \tag{3.2.4}$$

Proof. Putting $X = \dfrac{\mathfrak{B}}{\mathfrak{A}^3}, Y = \dfrac{\mathfrak{C}}{\mathfrak{A}^5}$ to (3.2.1), we have

$$\mathfrak{A}^5 z^2 = \mathfrak{A}^5 x^3 + (20y^2 - 16y^3)\mathfrak{A}^5 x^2 + 20\mathfrak{A}^2\mathfrak{B}y^3x + \mathfrak{C}y^4.$$

Then, by the correspondence

$$x \mapsto \frac{x}{\mathfrak{A}^3}, \qquad y \mapsto \frac{y}{\mathfrak{A}}, \qquad z \mapsto \frac{z}{\sqrt{\mathfrak{A}^9}},$$

we obtain (3.2.4). □

Remark 3.2.2. *For two surfaces*

$$\begin{cases} S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) : z^2 = x^3 - 4(4y^3 - 5\mathfrak{A}y^2)x^2 + 20\mathfrak{B}y^3x + \mathfrak{C}y^4, \\ S(k^2\mathfrak{A}:k^6\mathfrak{B}:k^{10}\mathfrak{C}) : z^2 = x^3 - 4(4y^3 - 5k^2\mathfrak{A}y^2)x^2 + 20k^6\mathfrak{B}y^3x + k^{10}\mathfrak{C}y^4, \end{cases}$$

we have an isomorphism $S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \to S(k^2\mathfrak{A}:k^6\mathfrak{B}:k^{10}\mathfrak{C})$ *given by* $(x,y,z) \mapsto (k^6x, k^2y, k^9z)$ *as elliptic surfaces. Therefore,* $(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \in \mathbb{P}(1:3:5)$ *gives an isomorphism class of these elliptic* $K3$ *surfaces.*

We set $K_1 = \{Y = 0\}$ and $K_2 = \{1728X^5 - 720X^3Y + 80XY^2 - 64(5X^2 - Y)^2 - Y^3 = 0\}$.

Theorem 3.2.2. *The* $(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$*-space* $\mathbb{P}(1,3,5)$ *gives a compactification of the parameter space* $\mathfrak{X}$ *of the family* $\mathcal{F} = \{S(X,Y)\}$ *of* $K3$ *surfaces given by* (3.2.1). *Namely, if* $(1:0:0) \neq (\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \in \mathbb{P}(1,3,5)$*, then the corresponding surface* $S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$ *is a* $K3$ *surface. On the other hand,* $S(1:0:0)$ *is a rational surface.*

Proof. First, we prove the case $\mathfrak{A} \neq 0$. In this case, we consider $S(X,Y)$ in (3.2.1). We have the Kodaira normal form of (3.2.1):

$$z_1^2 = x_1^3 - g_2(y)x - g_3(y) \qquad (y \neq \infty), \tag{3.2.5}$$

with

$$\begin{cases} g_2(y) = -\Big(20Xy^3 - \dfrac{16}{3}y^4(4y-5)^2\Big) \\ g_3(y) = -\Big(Yy^4 + \dfrac{80}{3}y^5(4y-5)X - \dfrac{128}{27}y^6(4y-5)^3\Big), \end{cases}$$

and

$$z_2^2 = x_2^3 - h_2(y_1)x_2 - h_3(y_1) \qquad (y \neq 0), \tag{3.2.6}$$

with

$$\begin{cases} h_2(y_1) = -\Big(20Xy_1^5 - \dfrac{256}{3}y_1^2 + \dfrac{640}{3}y_1^3 - \dfrac{400}{3}y_1^4\Big), \\ h_3(y_1) = -\Big(Yy_1^8 + \dfrac{320}{3}Xy_1^6 - \dfrac{400}{3}Xy_1^7 - \dfrac{8192}{27}y_1^3 + \dfrac{10240}{9}y_1^4 - \dfrac{12800}{9}y_1^5 + \dfrac{16000}{27}y_1^6\Big), \end{cases}$$

where $y_1 = \dfrac{1}{y}$. The discriminant D_0 (D_∞, resp.) of the right hand side of (3.2.5) ((3.2.6), resp.) is given by

$$\begin{cases} D_0 = y^8(27Y^2 + 32000X^3y - 7200XYy - 160000X^2y^2 + 32000Yy^2 \\ \qquad +5760XYy^2 + 256000X^2y^3 - 76800Yy^3 - 102400X^2y^4 + 61440Yy^4 - 16384Yy^5), \\ D_\infty = y_1^{11}(-16384Y - 102400X^2y_1 + 61440Yy_1 + 256000X^2y_1^2 - 76800Yy_1^2 \\ \qquad -160000X^2y_1^3 + 32000Yy_1^3 + 5760XYy_1^3 + 32000X^3y_1^4 - 7200XYy_1^4 + 27Y^2y_1^5). \end{cases}$$

If $(X,Y) \in \mathfrak{X}$, then we have

$$\mathrm{ord}_y(D_0) = 8, \qquad \mathrm{ord}_y(g_2) = 3, \qquad \mathrm{ord}_y(g_3) = 4,$$

so $\pi^{-1}(0)$ is the singular fibre of type IV^*. Similarly, we have

$$\mathrm{ord}_y(D_\infty) = 11, \qquad \mathrm{ord}_y(h_2) = 2, \qquad \mathrm{ord}_y(h_3) = 3,$$

so $\pi^{-1}(\infty) = I_5^*$. We have other 5 singular fibres of type I_1. Therefore, for $(X, Y) \in \mathfrak{X}$, $S(X, Y)$ is an elliptic $K3$ surface whose singular fibres are of type $IV^* + 5I_1 + I_5^*$.

By the same way, we know the structure of the elliptic surface $S(X, Y)$ for $(X, Y) \notin \mathfrak{X}$. If $X \neq 0$ and $Y = 0$ (namely, $(X, Y) \in K_1 - \{(0, 0)\}$), then $S(X, 0)$ is an elliptic $K3$ surface with the singular fibres of type $III^* + 3I_1 + I_6^*$. If $(X, Y) \in K_2 - \{(0, 0)\}$, $S(X, Y)$ is an elliptic K3 surface with the singular fibres of type $IV^* + 3I_1 + I_2 + I_5^*$. However, if $(X, Y) = (0, 0)$, we can check that $S(0, 0)$ is birationally equivalent to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. So, $S(0, 0)$ is not a $K3$ surface, but a rational surface.

Next, we consider the case $\mathfrak{A} = 0$. In this case, note that $(\mathfrak{B}, \mathfrak{C}) \neq (0, 0)$. We have the equation of $S(0 : \mathfrak{B} : \mathfrak{C})$: $z^2 = x^3 - 16y^3x^2 + 20\mathfrak{B}y^3x + \mathfrak{C}y^4$. On $\{\mathfrak{A} = 0\} \subset \mathbb{P}(1, 3, 5)$, we use the parameter $l = \dfrac{\mathfrak{C}^3}{\mathfrak{B}^5}$. By the transformation $x = \dfrac{\mathfrak{C}^3}{\mathfrak{B}^4}x', y = \dfrac{\mathfrak{C}^2}{\mathfrak{B}^3}y', z = \dfrac{\sqrt{\mathfrak{C}^9}}{\mathfrak{B}^6}z'$, we have

$$S(l) : z'^2 = x'^3 - 16ly'^3x'^2 + 20y'^3x' + y'^4.$$

The discriminant of the right hand side is given by $y'^8(27+32000y'+5760ly'^2-102400l^2y'^4-16384l^3y'^5)$. From this, we can see that $S(l)$ is an elliptic $K3$ surface with the singular fibres of type $IV^* + 5I_1 + I_5^*$. □

Hence, we obtain the extended family $\{S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) | (\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1, 3, 5) - \{(1 : 0 : 0)\}\}$ of $K3$ surfaces. For simplicity, let $\mathcal{F}$ denotes this extended family.

Remark 3.2.3. *In Section 1.5, we proved that the parameter space* Λ *of the family* $\mathcal{F}_0 = \{S_0(\lambda, \mu)\}$ *is birationally equivalent to the symmetric Hilbert modular orbifold. However, it is difficult to obtain an exact compactification of the parameter space* Λ*. For example, the period* $j^{-1} \circ \Phi_0(\lambda, \mu)$ *for* $\mathcal{F}_0$ *on* Λ *does not give the point in the diagonal* $\Delta = \{(z, z) \in \mathbb{H} \times \mathbb{H}\}$*, for the set* $(j^{-1} \circ \Phi)^{-1}(\Delta)$ *is blowed down to one point in* (λ, μ)*-space.*

For a precise study of the period mapping, we need the new family $\mathcal{F} = \{S(X, Y)\}$ *on the orbifold* $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau\rangle$*. By the birational transformation (2.2.12),* Λ *is birationally equivalent to this Hilbert modular orbifold. As in Section 3.1, this orbifold has an exact compactification by adding one point (namely the cusp). Moreover, for example, we can see that the image of the divisor* $\{Y = 0\}$ *gives the diagonal* Δ*. Therefore, this new family* $\mathcal{F}$ *is suitable to study the modular property.*

3.2.2 The extension of the period mapping

Set $c_0 = (1 : 0 : 0) \in \mathbb{P}(1, 3, 5)$. In this subsection, we extend the period mapping $\Phi_1 : \mathfrak{X} \to \mathcal{D}_+$ in (3.2.2) to $\mathbb{P}(1, 3, 5) - \{c_0\} \to \mathcal{D}_+$.

First, we recall the S-marking on $\mathfrak{X}$. According to Theorem 3.2.2 and its proof, we have the elliptic $K3$ surface

$$\pi_{(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})} : S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \to \mathbb{P}^1(\mathbb{C}) = (y\text{−sphere})$$

for any $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1, 3, 5) - \{c_0\}$.

Take a generic point $(X_0, Y_0) \in \mathfrak{X}$. The elliptic $K3$ surface $\check{S} = S(X_0, Y_0)$ given by (3.2.5) and (3.2.6) has the singular fibres of type $IV^* + 5I_1 + I_5^*$. Let F be a general fibre

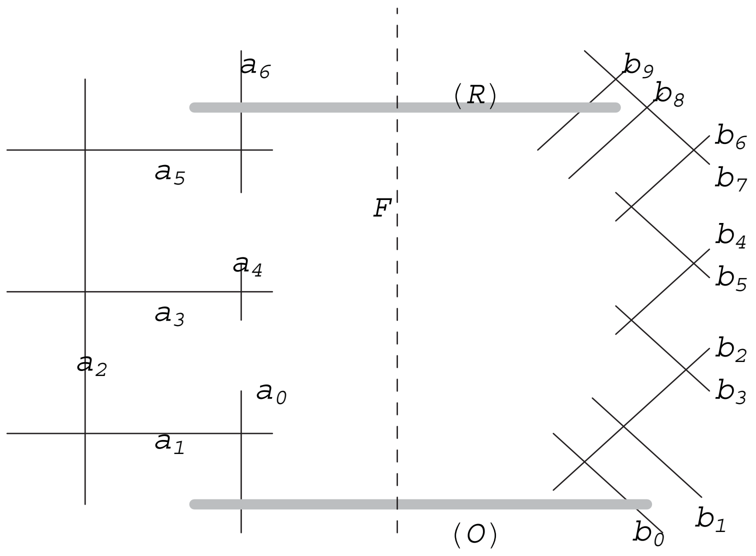


Figure 3.1: An elliptic fibration for $S(X, Y)$.

of this elliptic fibration and O be the zero of the Mordell-Weil group of sections. We have two irreducible components of the divisor C given by $\{x = 0, z^2 = Yy^4\}$. We take the section R given by $y \mapsto (x, y, z) = (0, y, \sqrt{Y} y^2)$. This gives a component of the divisor C. Let us consider the irreducible decomposition $\bigcup_{j=0}^{6} a_j$ ($\bigcup_{j=0}^{9} b_j$, resp.) of the singular fibre $\pi_{(X,Y)}^{-1}(0)$ ($\pi_{(X,Y)}^{-1}(\infty)$, resp.) of type IV^* (I_5^*, resp.). These curves are illustrated in Figure 3.1. Note that $a_0 \cap O \neq \phi$, $b_0 \cap O \neq \phi$, $a_6 \cap R \neq \phi$ and $b_9 \cap R \neq \phi$.

As we stated in Remark 3.2.3, we need the improved family $\mathcal{F}$ for a precise study of the period mapping. So, we define the S-marking and P-marking for $\mathcal{F}$ as in Section 1.5 to consider the period mapping exactly.

We set $\Gamma_5 = F, \Gamma_6 = O, \Gamma_7 = R$, $\Gamma_{8+k} = a_{k+1}$ $(0 \leq k \leq 5)$, $\Gamma_{14+l} = b_{l+1}$ $(0 \leq l \leq 8)$. We have the lattice $\check{L} = \langle \Gamma_5, \cdots, \Gamma_{22} \rangle_{\mathbb{Z}} \subset H_2(\check{S}, \mathbb{Z})$. We can check that $|\det(\check{L})| = 5$. Hence, from Theorem 3.2.1 (2), we have

$$\check{L} = \mathrm{NS}(\check{S}).$$

Since $\check{L}$ is a primitive lattice, there exists $\Gamma_1, \cdots, \Gamma_4 \in H_2(\check{S}, \mathbb{Z})$ such that

$$\langle \Gamma_1, \cdots, \Gamma_4, \Gamma_5, \cdots, \Gamma_{22} \rangle_{\mathbb{Z}} = H_2(\check{S}, \mathbb{Z}).$$

Let $\{\Gamma_1^*, \cdots, \Gamma_{22}^*\}$ be the dual basis of $\{\Gamma_1, \cdots, \Gamma_{22}\}$ in $H_2(\check{S}, \mathbb{Z})$. Then, $\langle \Gamma_1^*, \cdots, \Gamma_4^* \rangle_{\mathbb{Z}}$ is the transcendental lattice. We may assume that its intersection matrix is

$$(\Gamma_j^* \cdot \Gamma_k^*)_{1 \leq j,k \leq 4} = A \qquad (3.2.7)$$

where A is given by (3.2.3). We define the period of $\check{S}$ by

$$\Phi_1(X_0, Y_0) = \Big(\int_{\Gamma_1} \omega : \cdots : \int_{\Gamma_4} \omega \Big).$$

Take a small connected neighborhood U_0 of (X_0, Y_0) in $\mathfrak{X}$ so that we have a local topological trivialization

$$\tau : \{S(p) | p \in U_0\} \to \check{S} \times U_0. \tag{3.2.8}$$

Let $\varpi : \check{S} \times U_0 \to \check{S}$ be the canonical projection. Set $r = \varpi \circ \tau$. Then,

$$r'_p = r|_{S(p)}$$

gives a deformation of surfaces. For any $p \in U_0$, we have an isometry $\psi_p : H_2(S(p), \mathbb{Z}) \to H_2(\check{S}, \mathbb{Z})$ given by

$$\psi_p = {r_p'}_*.$$

We call this isometry the S-marking on U_0. By an analytic continuation along an arc $\alpha \subset \mathfrak{X}$, we define the S-marking on $\mathfrak{X}$. This depends on the choice of α. The S-mariking preserves the Néron-Severi lattice. We define the period mapping $\Phi_1 : \mathfrak{X} \to \mathcal{D}_+$ by

$$p \mapsto \Big(\int_{\psi_p^{-1}(\Gamma_1)} \omega : \cdots : \int_{\psi_p^{-1}(\Gamma_4)} \omega \Big).$$

This is equal to the period mapping in (3.2.2).

Definition 3.2.1. *Let S be an algebraic $K3$ surface. An isometry*

$$\psi : H_2(S, \mathbb{Z}) \to H_2(\check{S}, \mathbb{Z})$$

is called the P-marking if

(i) $\psi^{-1}(\mathrm{NS}(\check{S})) \subset \mathrm{NS}(S)$,

(ii) $\psi^{-1}(F), \psi^{-1}(O), \psi^{-1}(R), \psi^{-1}(a_j)$ $(1 \leq j \leq 6)$ *and* $\psi^{-1}(b_j)$ $(1 \leq j \leq 9)$ *are all effective divisors,*

(iii) $(\psi^{-1}(F) \cdot C) \geq 0$ *for any effective class C. Namely, $\psi^{-1}(F)$ is nef. A pair (S, ψ) is called a P-marked $K3$ surface.*

Definition 3.2.2. *Two P-marked $K3$ surfaces (S_1, ψ_1) and (S_2, ψ_2) are said to be isomorphic if there is a biholomorphic mapping $f : S_1 \to S_2$ with*

$$\psi_2 \circ f_* \circ \psi_1^{-1} = \mathrm{id}_{H_2(\check{S}, \mathbb{Z})}.$$

Two P-marked $K3$ surfaces (S_1, ψ_1) and (S_2, ψ_2) are said to be equivalent if there is a biholomorphic mapping $f : S_1 \to S_2$ with

$$(\psi_2 \circ f_* \circ \psi_1^{-1})|_{\mathrm{NS}(\check{S})} = \mathrm{id}_{\mathrm{NS}(\check{S})}.$$

Remark 3.2.4. *The other connected component R' of the divisor C given by the section $y \mapsto (x, y, -\sqrt{Y} y^2)$ intersects a_4 (b_8, resp.) at $y = 0$ ($y = \infty$, resp.). Letting q be the involution of $S(X, Y)$ given by $(x, y, z) \mapsto (x, y, -z)$, we have $q_*(R') = R$, $q_*(a_4) = a_6$, $q_*(a_3) = a_5$ and $q_*(b_8) = b_9$. Then, we can see that P-marked $K3$ surfaces $(\check{S}, id)$ and $(\check{S}, q_*)$ are isomorphic by q. This shows that our argument does not depend on the choice of the curves R or R'.*

The period of a P-marked $K3$ surface (S,ψ) is given by

$$\tilde{\Phi}'(S,\psi) = \Big(\int_{\psi^{-1}(\Gamma_1)} \omega : \cdots : \int_{\psi^{-1}(\Gamma_4)} \omega \Big). \tag{3.2.9}$$

It is a point in $\mathcal{D}$. Let $\mathbb{X}$ be the isomorphism classes of P-marked $K3$ surfaces and let

$$[\mathbb{X}] = \mathbb{X}/(P\text{−marked equivalence}).$$

By the Torelli theorem for $K3$ surfaces, the period mapping $\tilde{\Phi}' : \mathbb{X} \to \mathcal{D}$ for P-marked $K3$ surfaces defined by (3.2.9) gives an identification between $\mathbb{X}$ and $\mathcal{D}$. Moreover, a P-marked $K3$ surface (S_1,ψ_1) is equivalent to a P-marked $K3$ surface (S_2,ψ_2) if and only if

$$\tilde{\Phi}'(S_1,\psi_1) = g \circ \tilde{\Phi}'(S_2,\psi_2)$$

for some $g \in PO(A,\mathbb{Z})$ (see [Na2] Lemma 5.1). Therefore, we identify $[\mathbb{X}]$ with

$$\mathcal{D}/PO(A,\mathbb{Z}) = \mathcal{D}_+/PO^+(A,\mathbb{Z}) \simeq (\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle. \tag{3.2.10}$$

Recall that the above isomorphism is given by the modular isomorphism j in (2.2.2).

We note that $\mathfrak{X}$ is embedded in $[\mathbb{X}]$ (see Section 1.5.1). Then, an S-marked $K3$ surface is a P-marked $K3$ surface and the period mapping for P-marked $K3$ surfaces is an extension of the period mapping for S-marked $K3$ surfaces. From $\tilde{\Phi}' : \mathbb{X} \to \mathcal{D}$, we obtain a multivalued mapping $\Phi' : [\mathbb{X}] \to \mathcal{D}_+$. We have

$$\Phi'|_{\mathfrak{X}} = \Phi_1, \tag{3.2.11}$$

where Φ is the period mapping in (3.2.2) for S-marked $K3$ surfaces.

Now, we extend the period mapping $\Phi_1 : \mathfrak{X} \to \mathcal{D}_+$ in (3.2.2) to $\Phi^{ext} : \mathbb{P}(1,3,5) - \{c_0\} \to \mathcal{D}_+$. We recall that $(\mathbb{P}(1,3,5) - \{c_0\}) - \mathfrak{X} = (K_1 \cup K_2 \cup \{\mathfrak{A} = 0\}) - \{c_0\}$.

First, since the local topological trivialization on $\mathfrak{X}$ in (3.2.8) is naturally extended to $\{\mathfrak{A} = 0\}$, there exist S-markings on $\{\mathfrak{A} = 0\}$ and the period mapping (3.2.2) on $\mathfrak{X}$ is extended to $\mathbb{P}(1,3,5) - (K_1 \cup K_2 \cup \{c_0\}) \to \mathcal{D}_+$.

According to (3.2.10), Theorem 3.2.1 (3) and Proposition 3.1.2 (3) (Proposition 3.1.2 (2), resp.), the local monodromy of the period mapping Φ_1 in (3.2.2) around K_1 (K_2, resp.) is locally finite. Hence, the period mapping $\mathbb{P}(1,3,5) - (K_1 \cup K_2 \cup \{c_0\}) \to \mathcal{D}_+$ can be extended to $\mathbb{P}(1,3,5) - \{c_0\} \to \mathcal{D}_+$. We note that this extension is assured by Theorem (9.5) in Griffiths [Gr2].

Therefore, we have the extended period mapping

$$\Phi^{ext} : \mathbb{P}(1,3,5) - \{c_0\} \to \mathcal{D}_+ \tag{3.2.12}$$

with

$$\Phi^{ext}|_{\mathfrak{X}} = \Phi_1. \tag{3.2.13}$$

Since we have (3.2.10) and Proposition 3.1.2 (1), the P-marked equivalence classes $[\mathbb{X}]$ is identified with $\mathbb{P}(1,3,5) - \{c_0\}$. Because we have (3.2.11), (3.2.13) and $\mathfrak{X}$ is a Zariski open set in $\mathbb{P}(1,3,5) - \{c_0\}$, Φ^{ext} in (3.2.12) is equal to the period mapping Φ' on $[\mathbb{X}]$.

Let $[\Phi^{ext}(p)] \in \mathcal{D}_+/PO^+(A,\mathbb{Z})$ be the equivalence class of $\Phi^{ext}(p) \in \mathcal{D}_+$. From the above argument, we have

Proposition 3.2.3. *The period mapping $\Phi' : [\mathbb{X}] \to \mathcal{D}_+$ for P-marked $K3$ surfaces is given by the period mapping Φ^{ext} in* (3.2.12) *for the family $\mathcal{F} = \{S(p) | p \in \mathbb{P}(1,3,5) - \{c_0\}\}$ of $K3$ surfaces. This is an extension of the period mapping in* (3.2.2) *for S-marked $K3$ surfaces. Especially, if $[\Phi^{ext}(p_1)] = [\Phi^{ext}(p_2)]$ in $\mathcal{D}_+/PO^+(A,\mathbb{Z})$, then $p_1 = p_2$.*

In the following, Φ denotes the above extended period mapping Φ^{ext} in (3.2.12). For $p \in \mathbb{P}(1,3,5) - \{c_0\}$, let

$$\psi_p : H_2(S(p),\mathbb{Z}) \to H_2(\check{S},\mathbb{Z})$$

be a P-marking naturally induced by the above proposition. The period of $S(p)$ is given by

$$\Phi(p) = \Big(\int_{\psi_p^{-1}(\Gamma_1)} \omega : \int_{\psi_p^{-1}(\Gamma_2)} \omega : \int_{\psi_p^{-1}(\Gamma_3)} \omega : \int_{\psi_p^{-1}(\Gamma_4)} \omega \Big). \tag{3.2.14}$$

According to Theorem 3.2.1 (3) (or Theorem 2.2.3), the multivalued analytic mapping $(j^{-1} \circ \Phi)|_{\mathfrak{X}} : \mathfrak{X} \to \mathbb{H} \times \mathbb{H}$ gives a developing map of the canonical projection $\Pi : \mathbb{H} \times \mathbb{H} \to (\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}), \tau\rangle$. Here, by Proposition 3.2.3, $(j^{-1} \circ \Phi)|_{\mathfrak{X}}$ is extended to the analytic mapping

$$j^{-1} \circ \Phi : \mathbb{P}(1,3,5) - \{c_0\} \to \mathbb{H} \times \mathbb{H}.$$

This gives a developing map of Π.

Remark 3.2.5. *Sato* [Sa] *showed that the system of differential equations on $\mathfrak{X}$*

$$\begin{cases} u_{XX} = L u_{XY} + A u_X + B u_Y + P u, \\ u_{YY} = M u_{XY} + C u_X + D u_Y + Q u \end{cases}$$

with $L = \dfrac{-20(4X^2 + 3XY - 4Y)}{36X^2 - 32X - Y}, M = \dfrac{-2(54X^3 - 50X^2 - 3XY + 2Y)}{5Y(36X^2 - 32X - Y)}$ *is an uniformizing differential equation of $\overline{(\mathbb{H} \times \mathbb{H})/\langle PSL(2,\mathcal{O}), \tau\rangle}$. Namely, taking linearly independent solutions y_0, y_1, y_2 and y_3, the mapping $p \mapsto (y_0(p) : \cdots : y_3(p))$ gives a developing map $\mathfrak{X} \to \mathcal{D}_+$. Of course, our equation* (2.2.17) *is also an unifomizing differential equation in this sense. But, note that we do not know whether we can extend it to the singular locus applying the theory of the uniformizing differential equations. Since we regard $\mathbb{P}(1,3,5) - \{c_0\}$ as the parameter space of $\mathcal{F}$ and $p \mapsto (y_0(p) : \cdots : y_3(p))$ is the period mapping for $\mathcal{F}$, we obtain the extension of the solutions of* (2.2.17) *to the singular locus.*

Hence, we obtain the following theorem.

Theorem 3.2.3. *The mapping $j^{-1} \circ \Phi : \mathbb{P}(1,3,5) - \{c_0\} \to \mathbb{H} \times \mathbb{H}$ gives the developing map of Π. Namely, the inverse mapping of $\Pi : \mathbb{H} \times \mathbb{H} \to (\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}), \tau\rangle$ is given by $j^{-1} \circ \Phi$ through the identification $(\mathbb{H} \times \mathbb{H})/\langle PSL(2,\mathcal{O}), \tau\rangle \simeq \mathbb{P}(1,3,5) - \{c_0\}$ given by* Proposition 3.1.2 (1).

Let $\mathit{\Delta}$ be the diagonal:

$$\mathit{\Delta} = \{(z_1, z_2) \in \mathbb{H} \times \mathbb{H} | z_1 = z_2\}.$$

From the above theorem and Proposition 3.1.2 (3), we have

Corollary 3.2.1.

$$\Pi(\varDelta) = \{(\mathfrak{A}:\mathfrak{B}:0)\} - \{c_0\}$$

through the identification $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle \simeq \mathbb{P}(1,3,5)-\{c_0\}$ *given by* Proposition 3.1.2 (1).

Due to Theorem 3.2.3, we obtain the system of coordinates (z_1, z_2) of $\mathbb{H}\times\mathbb{H}$ coming from the period (3.2.14) of $K3$ surface $S(p)$:

$$\big(z_1(p), z_2(p)\big) = \left(-\frac{\displaystyle\int_{\Gamma_3}\omega + \frac{1-\sqrt{5}}{2}\int_{\Gamma_4}\omega}{\displaystyle\int_{\Gamma_2}\omega}, -\frac{\displaystyle\int_{\Gamma_3}\omega + \frac{1+\sqrt{5}}{2}\int_{\Gamma_4}\omega}{\displaystyle\int_{\Gamma_2}\omega} \right). \tag{3.2.15}$$

Here, for simplicity, let Γ_j denotes the 2-cycle $\psi_p^{-1}(\Gamma_j)$ on $S(p)$ for $j=1,2,3,4$.

According to Proposition 3.1.2 (1), by adding one cusp, we have the compactification $\overline{(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle}$. Then, putting $\Pi\circ j^{-1}\circ\Phi(c_0) = \overline{(\sqrt{-1}\infty,\sqrt{-1}\infty)}$, we obtain an extended mapping

$$\Pi\circ j^{-1}\circ\Phi : \mathbb{P}(1,3,5) \to \overline{(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle}, \tag{3.2.16}$$

where $\overline{(\sqrt{-1}\infty,\sqrt{-1}\infty)}$ stands for the $\langle PSL(2,\mathcal{O}),\tau\rangle$ orbit of $(\sqrt{-1}\infty,\sqrt{-1}\infty)$.

3.3 The family $\mathcal{F}_X$ and the period differential equation

In this section, we consider the familly $\mathcal{F}_X = \{S(X,0)\}$. The period mapping for $\mathcal{F}_X$ gives a multivalued mapping to the diagonal

$$\varDelta = \{(z_1,z_2)\in\mathbb{H}\times\mathbb{H} | z_1 = z_2\}.$$

The inverse correspondence of the period mapping for $\mathcal{F}_X$ is expressed in terms of the elliptic J-function.

3.3.1 The family $\mathcal{F}_X$

In Section 3.2, we have the $K3$ surfaces $S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$ for $(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})\in\mathbb{P}(1,3,5)-\{c_0\}$ and the period mapping (3.2.14). Restricting them to $\{\mathfrak{C}=0\}$, we obtain the familly $\{S(\mathfrak{A}:\mathfrak{B}:0) | (\mathfrak{A}:\mathfrak{B}:0)\neq c_0\}$ of $K3$ surfaces with $S(\mathfrak{A}:\mathfrak{B}:0) : z^2 = x^3 - 4y^2(4y - 5\mathfrak{A})x^2 + 20\mathfrak{B}y^3x$. Then, we have the family $\mathcal{F}_X = \{S(X,0)\}$ of $K3$ surfaces with

$$S(X,0) : z^2 = x^3 - 4y^2(4y-5)x^2 + 20Xy^3x,$$

where $X\Big(=\dfrac{\mathfrak{B}}{\mathfrak{A}^3}\Big)\in\mathbb{P}^1(\mathbb{C})-\{0\}$. In this section, we consider the family $\mathcal{F}_X$ and the period mapping for $\mathcal{F}_X$.

Set $\Sigma = (X-\text{sphere}\mathbb{P}^1(\mathbb{C})) - \{0, \frac{25}{27}, \infty\}$. Because we have Proppsition 3.2.3, we can prove the following theorem for the subfamily $\mathcal{F}'_X = \Big\{S(X,0)\Big| X\in\Sigma\Big\}$ as in [Na1].

Theorem 3.3.1. (1) *For a generic point* $X \in \Sigma$, $\mathrm{rank}(\mathrm{NS}(S(X,0))) = 19$.

(2) *For a generic point* $X \in \Sigma$, *the intersection matrix of the Néron-Severi lattice* $\mathrm{NS}(S(X,0))$ *is given by*

$$E_8(-1) \oplus E_8(-1) \oplus U \oplus \langle -2 \rangle$$

and that of transcendental lattice $\mathrm{Tr}(S(X,0))$ *is given by*

$$U \oplus \langle 2 \rangle =: A_X.$$

(3) *The projective monodromy group of the multivalued period mapping for* $\mathcal{F}'_X$ *is isomorphic to* $PO^+(A_X, \mathbb{Z})$.

From the period mapping Φ in (3.2.14), the system of coordinates (z_1, z_2) in (3.2.15), Corollary 3.2.1 and the above theorem, we obtain a multivalued period mapping Φ_X for $\mathcal{F}_X$ such that

$$j^{-1} \circ \Phi_X : \{X | X \in \mathbb{P}^1(\mathbb{C}) - \{0\}\} \to \mathit{\Delta}, \tag{3.3.1}$$

where Φ_X is given by $X \mapsto (\xi_1 : \xi_2 : \xi_3 : \xi_4) = \left(\int_{\Gamma_1} \omega : \int_{\Gamma_2} \omega : \int_{\Gamma_3} \omega : 0\right) \in \mathcal{D}_+$ with the Riemann-Hodge relation $\left(\int_{\Gamma_1} \omega\right)\left(\int_{\Gamma_2} \omega\right) + \left(\int_{\Gamma_3} \omega\right)^2 = 0$. Set $\Sigma = \big(X\text{−sphere } \mathbb{P}^1(\mathbb{C})\big) - \left\{0, \frac{25}{27}, \infty\right\}$. The fundamental group $\pi_1(\Sigma, *)$ induces the projective monodromy group M_X for Φ_X. According to the above theorem (3), M_X is isomorphic to $PO^+(A_X, \mathbb{Z})$. From (3.2.15), we have the coordinate z of $\mathit{\Delta} \simeq \mathbb{H}$:

$$z = -\frac{\int_{\Gamma_3} \omega}{\int_{\Gamma_2} \omega}. \tag{3.3.2}$$

Recalling (3.2.16), we obtain an extended mapping $\Pi \circ j^{-1} \circ \Phi_X : \mathbb{P}^1(\mathbb{C}) \to \overline{\mathit{\Delta}/M_X}$. We note $\Pi \circ j^{-1} \circ \Phi_X(0)$ is the M_X orbit of $(\sqrt{-1}\infty, \sqrt{-1}\infty)$. The action of M_X on $\mathit{\Delta}(\subset \mathbb{H} \times \mathbb{H})$ induces the action of $PSL(2,\mathbb{Z})$ on $\mathbb{H}$, for we have the coordinate z in (3.3.2). Namely, there exist $\gamma_1, \gamma_2 \in \pi_1(\Sigma, *)$ such that

$$\gamma_1(z) = z + 1, \qquad \gamma_2(z) = -\frac{1}{z}. \tag{3.3.3}$$

So, $\overline{\mathit{\Delta}/M_X}$ is identified with the orbifold $\overline{\mathbb{H}/PSL(2,\mathbb{Z})} \simeq \mathbb{P}^1(\mathbb{C})$.

Remark 3.3.1. *The projective monodromy group* $M_X \simeq PO^+(A_X, \mathbb{Z})$ *of the period mapping* Φ_X *is generated by two elements:*

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.3.4}$$

These are induced by the monodromy matrices in (2.2.1).

3.3.2 The Gauss hypergeometric equation ${}_2E_1\Big(\frac{1}{12},\frac{5}{12},1;t\Big)$

We recall the Gauss hypergeometric equation

$$ {}_2E_1\Big(\frac{1}{12},\frac{5}{12},1;t\Big) : t(1-t)\frac{d^2}{dt^2}u+(1-\frac{3}{2}t)\frac{d}{dt}u-\frac{5}{144}u=0. \tag{3.3.5}$$

The Riemann scheme of ${}_2E_1\Big(\frac{1}{12},\frac{5}{12},1;t\Big)$ is given by

$$\left\{\begin{matrix} t=0 & t=1 & t=\infty \\ 0 & 0 & 1/12 \\ 0 & 1/2 & 5/12 \end{matrix}\right\}.$$

We can take the solutions $y_1(t)$ and $y_2(t)$ of ${}_2E_1\Big(\frac{1}{12},\frac{5}{12},1;t\Big)$ such that the inverse mapping of the Schwarz mapping

$$\sigma : \quad t\mapsto \frac{y_2(t)}{y_1(t)}=\sigma(t)=z_0 \quad \in \mathbb{H} \tag{3.3.6}$$

is given by

$$z_0 \mapsto \frac{1}{J(z_0)}, \tag{3.3.7}$$

where $J(z)$ is the elliptic J function with $J\Big(\frac{1+\sqrt{-3}}{2}\Big)=0$, $J(\sqrt{-1})=1$ and $J(\sqrt{-1}\infty)=\infty$.

Remark 3.3.2. *The above J function is given by*

$$J(z)=\frac{1}{1728}\Big(\frac{1}{q}+744+196884q+\cdots\Big), \tag{3.3.8}$$

where $q=e^{2\pi\sqrt{-1}z}$.

Note that the Schwarz mapping σ is a multivalued analytic mapping. We can choose the single-valued branch of the Schwarz mapping σ on $(0,1)\subset\mathbb{R}$ such that $\sigma(t)\in\sqrt{-1}\mathbb{R}$ and

$$\lim_{t\to+0}\sigma(t)=\sqrt{-1}\infty, \qquad \lim_{t\to 1-0}\sigma(t)=\sqrt{-1}. \tag{3.3.9}$$

Then, the single-valued branch of the solutions $y_1(t)$ and $y_2(t)$ near $(0,1)(\subset\mathbb{R})$ is in the form

$$\left\{\begin{aligned} & y_1(t)=u_{11}(t), \\ & y_2(t)=\log(t)\cdot u_{21}(t)+u_{22}(t), \end{aligned}\right. \tag{3.3.10}$$

where $u_{jk}(t)$ are unit holomorphic functions around $t=0$ and log stands for the principal value.

The projective monodromy group of ${}_2E_1\Big(\frac{1}{12}, \frac{5}{12}, 1; t\Big)$ is isomorphic to $PSL(2,\mathbb{Z})$. In other words, the action of the fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}, *)$ on $\mathbb{H} = \Big\{z_0 = \frac{y_2}{y_1}\Big\}$ is generated by the two actions

$$z_0 \mapsto z_0 + 1 \qquad z_0 \mapsto -\frac{1}{z_0}, \tag{3.3.11}$$

if we normalize a basis y_1, y_2 of the solutions of ${}_2E_1(\frac{1}{12}, \frac{5}{12}, 1; t)$ around a base point.

Remark 3.3.3. *The projective monodromy group for the system* $(y_2^2(t); -y_1^2(t); y_1(t)y_2(t))$ *is* $\langle B_1, B_2\rangle$ *where*

$$B_1 = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

These matrices are equal to those of (3.3.4).

3.3.3 The period differential equation for the diagonal Δ

In this subsection, we determine the period differential equation for $\mathcal{F}_X$. Considering the solutions of this period differential equation, we obtain the expression of X using the coordinate z in (3.3.2).

Proposition 3.3.1. *On the locus* $\{Y = 0\}$, *the period differential equation* (2.2.17) *is restricted to the following ordinary differential equation of rank* 4*:*

$$\begin{aligned} &\frac{d^4}{dX^4}u + \frac{3(243X^2 - 4060X + 2000)}{2X(81X^2 - 1155X + 1000)}\frac{d^3}{dX^3}u \\ &\quad + \frac{2034X^2 - 40680X + 8000}{8X^2(81X^2 - 1155X + 1000)}\frac{d^2}{dX^2}u + \frac{15(3X - 80)}{8X^2(81X^2 - 1155X + 1000)}\frac{d}{dX}u = 0. \end{aligned} \tag{3.3.12}$$

Proof. Recalling the period differential equation (2.2.17), set

$$\begin{cases} E_1 u = L_1 u_{XY} + A_1 u_X + B_1 u_Y + P_1 u, \\ E_2 u = M_1 u_{XY} + C_1 u_X + D_1 u_Y + Q_1 u. \end{cases}$$

Deriving these equations, we have the system of equations

$$\begin{cases} u_{XX} = E_1 u, \quad u_{XXX} = \dfrac{\partial}{\partial X}E_1 u, \quad u_{XXY} = \dfrac{\partial}{\partial Y}E_1 u, \\ u_{XXXX} = \dfrac{\partial^2}{\partial X^2}E_1 u, \quad u_{XXXY} = \dfrac{\partial^2}{\partial X \partial Y}E_1 u, \\ u_{YY} = E_2 u, \quad u_{XYY} = \dfrac{\partial}{\partial X}E_2 u, \quad u_{YYY} = \dfrac{\partial}{\partial Y}E_2 u, \quad u_{XXYY} = \dfrac{\partial^2}{\partial Y^2}E_1 u = \dfrac{\partial^2}{\partial X^2}E_2 u. \end{cases}$$

Our periods satisfy this system. From this system, canceling the terms $u_Y, u_{XY}, u_{YY}, u_{XXY}$, $u_{XYY}, u_{YYY}, u_{XXXY}$ and u_{XXYY}, we can obtain the differential equation

$$a_4(X,Y)u_{XXXX} + a_3(X,Y)u_{XXX} + a_2(X,Y)u_{XX} + a_1(X,Y)u_X + a_0(X,Y)u = 0,$$

where $a_j(X,Y)$ $(j = 1,2,3,4)$ is a polynomial in X and Y. Putting $Y = 0$, we have (3.3.12). □

Set

$$\check{\eta}_j(X) = \int_{\Gamma_j} \omega \qquad (j \in 1,2,3).$$

The equation (3.3.12) has the 4-dimensional space of solutions generated by $\check{\eta}_1(X), \check{\eta}_2(X), \check{\eta}_3(X)$ and 1. The Riemann scheme of (3.3.12) is geven by

$$\begin{Bmatrix} X=0 & X=25/27 & X=40/3 & X=\infty \\ 0 & 0 & 0 & 0 \\ 1 & 1/2 & 1 & -5/6 \\ 1 & 1 & 2 & -1/2 \\ 1 & 2 & 4 & -1/6 \end{Bmatrix}.$$

Setting $X = \frac{25}{27}t$, the equation (3.3.12) is transposed to

$$W_4 u = 0,$$

where

$$W_4 = \frac{d^4}{dt^4} + \frac{1620t^3 - 29232t^2 + 15552t}{72t^2(t-1)(5t-72)} \frac{d^3}{dt^3} + \frac{565t^2 - 12204t + 2592}{36t^2(t-1)(5t-72)} \frac{d^2}{dt^2} + \frac{25t - 720}{72t^2(t-1)(5t-72)} \frac{d}{dt}.$$

Straightforward calculation shows the following.

Proposition 3.3.2. *Set*

$$W_3 = \frac{d^3}{dt^3} + \frac{3}{2(t-1)} \frac{d^2}{dt^2} + \frac{5t-36}{36t^2(t-1)} \frac{d}{dt} + \frac{72-5t}{72t^3(t-1)}.$$

Then,

$$W_4 = \left(\frac{d}{dt} + \frac{15t^2 - 298t + 216}{t(t-1)(5t-72)} \right) \circ W_3. \tag{3.3.13}$$

Set $\eta_j(t) = \check{\eta}_j\left(\frac{25}{27}t\right)$ for $j \in \{1,2,3\}$.

Proposition 3.3.3. *The periods $\eta_1(t), \eta_2(t)$ and $\eta_3(t)$ are the solutions of*

$$W_3 u = 0$$

satisfying

$$\eta_1 \eta_2 + \eta_3^2 = 0. \tag{3.3.14}$$

Proof. Set

$$W_1 = \frac{d}{dt} + \frac{15t^2 - 298t + 216}{t(t-1)(5t-72)}.$$

Let $V = \langle \eta_1, \eta_2, \eta_3 \rangle_{\mathbb{C}}$ and $V' = \langle W_3\eta_1, W_3\eta_2, W_3\eta_3 \rangle_{\mathbb{C}}$. Since the linear mapping given by $f \mapsto W_3 f$ is monodromy-equivalent and V is an irreducible representation, according to Schur's lemma, we have $V \simeq V'$ or $V' = \{0\}$. It follows from (3.3.13) that $V' \subset \mathrm{Ker}(W_1)$. Because $\dim(\mathrm{W}_1(W_1)) = 1$, we have $V' = \{0\}$. □

Proposition 3.3.4. *If u_1 and u_2 are solutions of ${}_2E_1\Big(\dfrac{1}{12}, \dfrac{5}{12}, 1; t\Big)$, then $tu_1^2(t), tu_2^2(t)$ and $tu_1(t)u_2(t)$ are solutions of the period differential equation $W_3 u = 0$.*

Proof. Take any solutions of ${}_2E_1\Big(\dfrac{1}{12}, \dfrac{5}{12}, 1; t\Big)$ $u_1(t)$ and $u_2(t)$. For $j \in \{1, 2\}$,

$$u_j'' = \frac{1 - 3t/2}{t(t-1)} u_j' - \frac{5}{144t(t-1)} u_j, \tag{3.3.15}$$

then

$$u_j^{(3)} = \frac{535t^2 - 715t + 288}{144t^2(t-1)^2} u_j' + \frac{5(7t-4)}{288t^2(t-1)^2} u_j. \tag{3.3.16}$$

Here, by a straightforward calculation, we have

$$\begin{aligned} W_3(tu_1u_2) = {}& \frac{5}{72t(t-1)} u_1u_2 + \frac{113t - 36}{36t(t-1)}(u_1'u_2 + u_1u_2') + \frac{3(3t-2)}{t-1} u_1'u_2' \\ & + \frac{3(3t-2)}{2(t-1)}(u_1''u_2 + u_1u_2'') + 3t(u_1'u_2'' + u_1''u_2') + t(u_1^{(3)}u_2 + u_1u_2^{(3)}). \end{aligned} \tag{3.3.17}$$

Substituting (3.3.15) and (3.3.16) for (3.3.17), we have $W_3(tu_1u_2) = 0$. □

Remark 3.3.4. *According to* (3.3.12), *the derivation $\dfrac{d}{dt}\eta_j$ $(j = 1, 2, 3)$ of the period is a solution of the equation*

$$\begin{aligned} \frac{d^3}{dt^3}v + {}& \frac{1620t^3 - 29232t^2 + 15552t}{72t^2(t-1)(5t-72)} \frac{d^2}{dt^2}v \\ & + \frac{1130t^2 - 24408t + 5184}{72t^2(t-1)(5t-72)} \frac{d}{dt}v + \frac{25t - 720}{72t^2(t-1)(5t-72)} v = 0. \end{aligned} \tag{3.3.18}$$

Then, set

$$S(t) = {}_3F_2\Big(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}; 1, 1; t\Big) + \frac{1}{5}{}_3F_2\Big(\frac{7}{6}, \frac{1}{2}, \frac{5}{6}; 1, 1; t\Big),$$

where ${}_3F_2$ is the generalized hypergeometric series:

$${}_3F_2(a_1, a_2, a_3; b_1, b_2; t) = \sum_{t=0}^{\infty} \frac{(a_1, n)(a_2, n)(a_3, n)}{(b_1, n)(b_2, n)n!} t^n.$$

We see that $S(t)$ is a holomorphic solution of (3.3.18) around $t = 0$. The indefinite integral of $S(t)$ with the integral constant 0 is given by

$$\begin{aligned} &t \cdot {}_3F_2\Big(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}; 1, 2; t\Big) + \frac{1}{5} t \cdot {}_3F_2\Big(\frac{7}{6}, \frac{1}{2}, \frac{5}{6}; 1, 2; t\Big) \\ &= \frac{6}{5} t \cdot {}_3F_2\Big(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}; 1, 1; t\Big) = \frac{6}{5} t \cdot \Big({}_2F_1\Big(\frac{1}{12}, \frac{5}{12}, 1; t\Big)\Big)^2. \end{aligned}$$

Here, we apply Clausen's formula. From the above proposition, this gives a holomorphic solution of $W_3 u = 0$ around $t = 0$.

Let $y_1(t)$ and $y_2(t)$ are the single-valued branch of the solutions of ${}_2E_1\Big(\frac{1}{12}, \frac{5}{12}, 1; t\Big)$ near $(0,1) \subset \mathbb{R}$ given in (3.3.9). Let

$$s_1(t) = t y_1^2(t), \quad s_2(t) = t y_1(t) y_2(t), \quad s_3(t) = t y_2^2(t).$$

Note that, if $t \in (0,1) \subset \mathbb{R}$, we have

$$\begin{cases} s_1(t) = t \cdot v_{11}(t), \\ s_2(t) = t \cdot (\log(t) v_{21}(t) + v_{22}(t)), \\ s_3(t) = t \cdot (\log^2(t) v_{31}(t) + \log(t) v_{32}(t) + v_{33}(t)), \end{cases} \tag{3.3.19}$$

where $v_{jk}(t)$ are unit holomorphic functions around $t = 0$. Moreover, they satisfy

$$\big(-s_1(t)\big) \cdot s_3(t) + s_2^2(t) = 0. \tag{3.3.20}$$

Lemma 3.3.1. *Taking a branch of the multivalued analytic mapping $t \mapsto (\eta_1(t) : \eta_2(t) : \eta_3(t))$,*

$$\big(\eta_1(t) : \eta_2(t) : \eta_3(t)\big) = \big(s_3(t) : -s_1(t) : s_2(t)\big) \in \mathbb{P}^2(\mathbb{C}).$$

Proof. Because we have Proposition 3.1.2 (1) and the coordinate z in (3.3.2), we take the single-valued branch of the multivalued period mapping $t \mapsto \big(\eta_1(t) : \eta_2(t) : \eta_3(t)\big)$ on $t \in (0,1) \subset \mathbb{R}$ such that

$$\lim_{t \to +0} -\frac{\eta_3(t)}{\eta_2(t)} = \sqrt{-1}\infty. \tag{3.3.21}$$

In this proof, we consider $\eta_1(t), \eta_2(t)$ and $\eta_3(t)$ near $(0,1) (\subset \mathbb{R})$.

According to Proposition 3.3.4, we have

$$\eta_j(t) = \sum_{k=1}^{3} a_{jk} s_k(t) \qquad (j = 1,2,3),$$

where a_{jk} $(j,k = 1,2,3)$ are constants. Since we have (3.3.21), we obtain $a_{23} = 0$. So, $\eta_2(t) = a_{21} s_1(t) + a_{22} s_2(t)$. From (3.3.19), we see that $\eta_1(t)\eta_2(t)$ does not contain $\log^4(t)$. Then, from (3.3.14), we have $a_{33} = 0$. Recalling (3.3.21) again, we obtain $a_{22} = 0$. Because we consider $y \mapsto \big(\eta_1(t) : \eta_2(t) : \eta_3(t)\big) \in \mathbb{P}^2(\mathbb{C})$, we assume that $a_{21} = -1$. Then, the single-valued branches $\eta_j(t)$ $(j = 1,2,3)$ are in the form

$$\begin{cases} \eta_1(t) = a_{11} s_1(t) + a_{12} s_2(t) + a_{13} s_3(t), \\ \eta_2(t) = -s_1(t), \\ \eta_3(t) = a_{31} s_1(t) + a_{32} s_2(t). \end{cases}$$

Hence, using (3.3.6), the coordinate z in (3.3.2) is given by

$$z = a_{32}\frac{s_2(z)}{s_1(z)} + a_{31} = a_{32}z_0 + a_{31}.$$

Considering the actions of $\pi_1(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\})$ on $z = -\frac{\eta_3}{\eta_2}$ -space in (3.3.3) and $z_0 = \frac{y_2}{y_1}$ -space in (3.3.11), $a_{31} = 0$ and $a_{32} = 1$ follows.

Therefore, using (3.3.14) again, we obtain

$$\eta_1(t) = s_3(t), \qquad \eta_2(t) = -s_1(t), \qquad \eta_3(t) = s_2(t).$$

□

Corollary 3.3.1. *A coordinate z in* (3.3.2) *of the diagonal $\Delta(\simeq \mathbb{H})$ is equal to*

$$z = \frac{y_2(t)}{y_1(t)}.$$

Proof. From the above lemma, this is clear. □

Theorem 3.3.2. *The inverse of the multivalued period mapping $j^{-1} \circ \Phi_X : X \mapsto (z,z)$ is given by*

$$X(z,z) = \frac{25}{27} \cdot \frac{1}{J(z)},$$

where $z \in \mathbb{H}$ is given in (3.3.2).

Proof. From the above Corollary and the inverse Schwarz mapping (3.3.7), we have $t(z) = \frac{1}{J(z)}$. Therefore,

$$X(z,z) = \frac{25}{27} \cdot t(z) = \frac{25}{27} \cdot \frac{1}{J(z)}$$

□

3.4 The theta expressions of X and Y

In this section, we obtain the explicit theta expression of the multivalued period mapping for $\mathcal{F} = \{S(X,Y)\}$ of $K3$ surfaces.

3.4.1 The classical elliptic modular forms

First, we recall the classical elliptic forms. Let $z \in \mathbb{H}$.

The classical Eisenstein series are given by

$$G_2(z) = 60 \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(mz+n)^4}, \qquad G_3(z) = 140 \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(mz+n)^6}.$$

$G_2(z)$ ($G_3(z)$, resp.) is a modular form of weight 4 (6, resp.) for $PSL(2,\mathbb{Z})$. The ring of modular forms for $PSL(2,\mathbb{Z})$ is $\mathbb{C}[G_2, G_3]$. We have $G_2(\sqrt{-1}\infty) = \frac{4\pi^4}{3}$ and $G_3(\sqrt{-1}\infty) = \frac{8\pi^6}{27}$. Let $E_4(z) = \frac{3}{4\pi^4}G_2(z)$ and $E_6(z) = \frac{27}{8\pi^6}G_3(z)$ be the normalized Eisenstein series. The discriminant form is

$$\Delta(z) = G_2^3(z) - 27G_3^2(z).$$

We have $\Delta(\sqrt{-1}\infty) = 0$. This is a cusp form of weight 12. The cusp form of weight 12 is Δ up to a constant factor. The J function in (3.3.8) is given by

$$J(z) = \frac{G_2^3(z)}{G_2^3(z) - 27G_3^2(z)} = \frac{G_2^3(z)}{\Delta(z)}. \tag{3.4.1}$$

The field of modular functions for the modular group $PSL(2,\mathbb{Z})$ is $\mathbb{C}(J(z))$.

For $a, b \in \{0,1\}$, the Jacobi theta constants are defined by

$$\vartheta_{ab}(z) = \sum_{n\in\mathbb{Z}} \exp\Big(\sqrt{-1}\pi\Big(n + \frac{a}{2}\Big)^2 z + 2\sqrt{-1}\pi\Big(n + \frac{a}{2}\Big)\frac{b}{2}\Big)$$

for $(a,b) = (0,0), (0,1)$ and $(1,0)$. $\vartheta_{00}^4, \vartheta_{01}^4$ and ϑ_{10}^4 are the modular form of weight 2 for the principal congruence subgroup $\Gamma(2) = \Big\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} | \alpha \equiv \delta \equiv 1, \beta \equiv \gamma \equiv 0 \pmod 2\Big\}$. The ring of modular forms for $\Gamma(2)$ is

$$\mathbb{C}[\vartheta_{00}^4, \vartheta_{01}^4, \vartheta_{10}^4]/(\vartheta_{01}^4 + \vartheta_{10}^4 = \vartheta_{00}^4) = \mathbb{C}[\vartheta_{00}^4, \vartheta_{01}^4].$$

We note that

$$\frac{1}{1728}\Big(\frac{3}{4\pi^4}\Big)^3 \Delta(z) = \frac{1}{2^8}\vartheta_{00}^8(z)\vartheta_{01}^8(z)\vartheta_{10}^8(z).$$

3.4.2 Müller's modular forms

Next, we survey the theta functions for Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$. They are introduced by Müller [Mul].

Set

$$\mathfrak{S}_2 = \{Z \in \mathrm{Mat}(2,2) | {}^tZ = Z, \mathrm{Im}(Z) > 0\}.$$

This is the Siegel upper half plane consisting of 2×2 complex matrices. For $a, b \in \{0,1\}^2$ with ${}^tab \equiv 0$ (mod2), set

$$\vartheta(Z; a, b) = \sum_{g\in\mathbb{Z}^2} \exp\Big(\pi\sqrt{-1}\big({}^t\big(g + \frac{1}{2}a\big)Z\big(g + \frac{1}{2}a\big) + {}^tgb\big)\Big).$$

We use the mapping $\psi : \mathbb{H}\times\mathbb{H} \to \mathfrak{S}_2$ given by

$$(z_1, z_2) = \zeta \mapsto \begin{pmatrix} \mathrm{Tr}\Big(\frac{\varepsilon\zeta}{\sqrt{5}}\Big) & \mathrm{Tr}\Big(\frac{\zeta}{\sqrt{5}}\Big) \\ \mathrm{Tr}\Big(\frac{\zeta}{\sqrt{5}}\Big) & \mathrm{Tr}\Big(-\frac{\varepsilon'\zeta}{\sqrt{5}}\Big) \end{pmatrix}$$
$$= \frac{1}{2\sqrt{5}}\begin{pmatrix} (1+\sqrt{5})z_1 - (1-\sqrt{5})z_2 & 2(z_1 - z_2) \\ 2(z_1 - z_2) & (-1+\sqrt{5})z_1 + (1+\sqrt{5})z_2 \end{pmatrix},$$

j	0	1	2	3	4	5	6	7	8	9
${}^t a$	$(0,0)$	$(1,1)$	$(0,0)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$	$(1,0)$	$(0,0)$	$(0,1)$
${}^t b$	$(0,0)$	$(0,0)$	$(1,1)$	$(1,1)$	$(0,0)$	$(0,0)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$

Table 3.1: The correspondence j, a and b.

where $\varepsilon = \dfrac{1+\sqrt{5}}{2}$.

For $j \in \{0, 1, \cdots, 9\}$, we set

$$\theta_j(z_1, z_2) = \vartheta(\psi(z_1, z_2); a, b),$$

where the correspondence between j and (a, b) is given by Table 3.1: These theta constants are the holomorphic functions on $\mathbb{H} \times \mathbb{H}$.

Let $a \in \mathbb{Z}$ and $j_1, \cdots, j_r \in \{0, \cdots, 9\}$. We set $\theta^a_{j_1,\cdots,j_r} = \theta^a_{j_1} \cdots \theta^a_{j_r}$.

Set $s_5 = 2^{-6}\theta_{0123456789}$. This is an alternating modular form of weight 5. The following g_2 (s_6, s_{10}, s_{15}, resp.) is a symmetric Hilbert modular form of weight 2 ($6, 10, 15$, resp.) for $\mathbb{Q}(\sqrt{5})$.

$$\begin{cases} g_2 = \theta_{0145} - \theta_{1279} - \theta_{3478} + \theta_{0268} + \theta_{3569}, \\ s_6 = 2^{-8}(\theta^2_{012478} + \theta^2_{012569} + \theta^2_{034568} + \theta^2_{236789} + \theta^2_{134579}), \\ s_{10} = s_5^2 = 2^{-12}\theta^2_{0123456789}, \\ s_{15} = -2^{-18}(\theta^9_{07}\theta^5_{18}\theta_{24} - \theta^9_{25}\theta^5_{16}\theta_{09} + \theta^9_{58}\theta^5_{03}\theta_{46} - \theta^9_{09}\theta^5_{25}\theta_{16} + \theta^9_{09}\theta^5_{16}\theta_{25} - \theta^9_{67}\theta^5_{23}\theta_{89} \\ \qquad +\theta^9_{18}\theta^5_{24}\theta_{07} - \theta^9_{24}\theta^5_{18}\theta_{07} - \theta^9_{46}\theta^5_{03}\theta_{58} - \theta^9_{24}\theta^5_{07}\theta_{18} - \theta^9_{89}\theta^5_{67}\theta_{23} - \theta^9_{07}\theta^5_{24}\theta_{18} \\ \qquad +\theta^9_{89}\theta^5_{23}\theta_{67} - \theta^9_{49}\theta^5_{13}\theta_{57} + \theta^9_{16}\theta^5_{09}\theta_{25} - \theta^9_{03}\theta^5_{46}\theta_{58} + \theta^9_{16}\theta^5_{25}\theta_{09} - \theta^9_{46}\theta^5_{58}\theta_{03} \\ \qquad -\theta^9_{25}\theta^5_{09}\theta_{16} - \theta^9_{57}\theta^5_{49}\theta_{13} + \theta^9_{67}\theta^5_{89}\theta_{23} + \theta^9_{58}\theta^5_{46}\theta_{03} + \theta^9_{57}\theta^5_{13}\theta_{49} - \theta^9_{23}\theta^5_{89}\theta_{67} \\ \qquad +\theta^9_{18}\theta^5_{07}\theta_{24} + \theta^9_{03}\theta^5_{58}\theta_{46} + \theta^9_{23}\theta^5_{67}\theta_{89} + \theta^9_{49}\theta^5_{57}\theta_{13} - \theta^9_{13}\theta^5_{57}\theta_{49} + \theta^9_{13}\theta^5_{49}\theta_{57}). \end{cases} \tag{3.4.2}$$

Proposition 3.4.1. ([Mul] Satz 1) (1) *The ring of the symmetric Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ is given by*

$$\mathbb{C}[g_2, s_6, s_{10}, s_{15}]/(M(g_2, s_6, s_{10}, s_{15}) = 0),$$

where

$$\begin{aligned} &M(g_2, s_6, s_{10}, s_{15}) \\ &= s_{15}^2 - \Big(5^5 s_{10}^3 - \frac{5^3}{2} g_2^2 s_6 s_{10}^2 + \frac{1}{2^4} g_2^5 s_{10}^2 + \frac{3^2 \cdot 5^2}{2} g_2 s_6^3 s_{10} - \frac{1}{2^3} g_2^4 s_6^2 s_{10} - 2 \cdot 3^3 s_6^5 + \frac{1}{2^4} g_2^3 s_6^4\Big). \end{aligned} \tag{3.4.3}$$

(2) *The ring of the Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ is given by*

$$\mathbb{C}[g_2, s_5, s_6, s_{15}]/(M(g_2, s_5^2, s_6, s_{15}) = 0).$$

Müller's modular forms have the following properties:

Proposition 3.4.2. ([Mul] pp.244-245)

$$\begin{cases} g_2(i\infty, i\infty) = 1, \\ s_6(z,z) = \dfrac{2}{1728}\left(\dfrac{3}{4\pi^4}\right)^3 \Delta(z) = \dfrac{1}{2^7}\vartheta_{00}^8(z)\vartheta_{01}^8(z)\vartheta_{10}^8(z), \\ s_{10}(z,z) = 0. \end{cases}$$

Especially,

$$\begin{cases} \dfrac{4\pi^4}{3} g_2(z,z) = \dfrac{4\pi^4}{3} E_4(z) = G_2(z), \\ 2^{11}\pi^{12} s_6(z,z) = G_2^3(z) - 27G_3^2(z) = \Delta(z). \end{cases}$$

3.4.3 The theta expression of X-function and Y-function

Now, we obtain the theta expressions of the parameters X and Y. According to Proposition 3.1.1, $X = \dfrac{\mathfrak{B}}{\mathfrak{A}^3}$ and $Y = \dfrac{\mathfrak{C}}{\mathfrak{A}^5}$ define the Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$. From Theorem 3.2.3, via the period mapping for $\mathcal{F}$, we can regard X and Y as the functions of variables z_1 and z_2 in (3.2.15). Here, using this system of coordinates (z_1, z_2), we represent X and Y as the quotients of Müller's modular forms.

For our argument, we set $Z = \dfrac{\mathfrak{D}^2}{\mathfrak{A}^{15}}$. This defines a Hilbert modular function for $\mathbb{Q}(\sqrt{5})$ also.

Lemma 3.4.1. *The modular functions $X(z_1,z_2), Y(z_1,z_2)$ and $Z(z_1,z_2)$ have the expressions*

$$\begin{cases} X(z_1,z_2) = k_1 \dfrac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}, \\ Y(z_1,z_2) = k_2 \dfrac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}, \\ Z(z_1,z_2) = k_3 \dfrac{s_{15}^2(z_1,z_2)}{g_2^{15}(z_1,z_2)}, \end{cases} \tag{3.4.4}$$

for some k_1, k_2 and $k_3 \in \mathbb{C}$.

Proof. Since $X = \dfrac{\mathfrak{B}}{\mathfrak{A}^3}$, X is given by the quotient of Hilbert modular forms of weight 6 and its denominator is the cube of a Hilbert modular form of weight 2. Note that, a Hilbert modular form of weight 2 is equal to g_2 up to a constant factor. Then, we have

$$X(z_1,z_2) = \frac{k_{11}s_6(z_1,z_2) + k_{12}g_2^3(z_1,z_2)}{k_{13}g_2^3(z_1,z_2)},$$

where k_{11}, k_{12} and k_{13} are constants. Recalling Proposition 3.1.2 (1), we have $X(\sqrt{-1}\infty, \sqrt{-1}\infty) = 0$. Then, from Proposition 3.4.2, we obtain $k_{12} = 0$, so

$$X(z_1,z_2) = k_1 \frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}.$$

Since $Y = \dfrac{\mathfrak{C}}{\mathfrak{A}^5}$, Y is given by the quotient of Hilbert modular forms of weight 10. Its denominator is the 5-th power of a modular form of weight 2. Then,

$$Y(z_1, z_2) = \frac{k_{21}s_{10}(z_1, z_2) + k_{22}g_2^5(z_1, z_2) + k_{23}g_2^2(z_1, z_2)s_6(z_1, z_2)}{k_{24}g_2^5(z_1, z_2)},$$

where k_{21}, k_{22}, k_{23} and k_{24} are constants. By Proposition 3.1.2 (3), we have $Y(z, z) = 0$. According to (3.4.2) and Proposition 3.4.2, if a modular form g of weight 10 vanishes on the diagonal Δ, then we have $g = \text{const} \cdot s_{10}$. So, we obtain $k_{22} = k_{23} = 0$. Therefore,

$$Y(z_1, z_2) = k_2 \frac{s_{10}(z_1, z_2)}{g_2^5(z_1, z_2)}.$$

Recalling Proposition 3.1.1 (2), we note that $\mathfrak{D}$ defines a symmetric Hilbert modular form of weight 15. Since $Z = \dfrac{\mathfrak{D}^2}{\mathfrak{A}^{15}}$, Z is given by the quotient of modular forms of weight 30. Its denominator is the 15-th power of a modular form of weight 2 and its numerator is given by the square of a symmetric modular form of weight 15. According to Proposition 3.4.1 (2), a symmetric modular form of weight 15 is given by $\text{const} \cdot s_{15}$. Then, we have

$$Z(z_1, z_2) = k_3 \frac{s_{15}^2(z_1, z_2)}{g_2^{15}(z_1, z_2)}.$$

□

Theorem 3.4.1. *The inverse correspondence of the multivalued mapping* $j^{-1} \circ \Phi : (X, Y) \mapsto (z_1, z_2)$ *for the family* $\mathcal{F}$ *is given by the quotient of Müller's modular forms:*

$$\begin{cases} X(z_1, z_2) = 2^5 \cdot 5^2 \cdot \dfrac{s_6(z_1, z_2)}{g_2^3(z_1, z_2)}, \\ Y(z_1, z_2) = 2^{10} \cdot 5^5 \cdot \dfrac{s_{10}(z_1, z_2)}{g_2^5(z_1, z_2)}, \end{cases}$$

where (z_1, z_2) *is the system of coordinates given by* (3.2.15).

Proof. First, we obtain the expression of X. To obtain it, we determine the constant k_1 in (3.4.4). Due to Theorem 3.3.2, (3.4.1) and Proposition 3.4.2, we have

$$X(z, z) = \frac{25}{27} \cdot \frac{1}{J(z)} = \frac{25}{27} \cdot \frac{2^{11}\pi^{12}s_6(z, z)}{\left(\dfrac{4\pi^4}{3}\right)^3 g_2^3(z, z)} = 2^5 \cdot 5^2 \cdot \frac{s_6(z, z)}{g_2^3(z, z)}.$$

So, we obtain $k_1 = 2^5 \cdot 5^2$.

Next, we determine the constant k_3 in (3.4.4). By (2.2.5), we have

$$\begin{aligned} 144Z(z_1, z_2) = &-1728X^5(z_1, z_2) + 720X^3(z_1, z_2)Y(z_1, z_2) \\ &- 80X(z_1, z_2)Y^2(z_1, z_2) + 64(5X^2(z_1, z_2) - Y(z_1, z_2))^2 + Y^3(z_1, z_2). \end{aligned} \quad (3.4.5)$$

Recalling that $Y(z, z) = 0$, we have

$$\begin{aligned} 144Z(z, z) &= -1728X^5(z, z) + 64 \cdot 25 \cdot X^4(z, z) \\ &= -2^{26} \cdot 5^{10} \cdot \left(2^5 \cdot 3^3 \cdot \frac{s_6(z, z)}{g_2^3(z, z)} - 1\right)\left(\frac{s_6(z, z)}{g_2^3(z, z)}\right)^4. \end{aligned} \quad (3.4.6)$$

On the other hand, from (3.4.3), we have

$$\begin{aligned}\frac{s_{15}^2(z_1,z_2)}{g_2^{15}(z_1,z_2)} &= 5^5\Big(\frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}\Big)^3 - \frac{5^3}{2}\Big(\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}\Big)\Big(\frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}\Big)^2\\ &+ \frac{3^2\cdot 5^2}{2}\Big(\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}\Big)^2\Big(\frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}\Big) + \frac{1}{2^4}\Big(\frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}\Big)^2\\ &- \frac{1}{2^3}\Big(\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}\Big)^2\Big(\frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}\Big) - 2\cdot 3^3\Big(\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}\Big)^5 + \frac{1}{2^4}\Big(\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}\Big)^4.\end{aligned} \tag{3.4.7}$$

So, because $s_{10}(z,z)=0$, we have

$$\Big(\frac{s_{15}^2(z,z)}{g_2^{15}(z,z)}\Big) = \frac{1}{2^4}\Big(-2^5\cdot 3^3\frac{s_6(z,z)}{g_2^3(z,z)}+1\Big)\Big(\frac{s_6(z,z)}{g_2^3(z,z)}\Big)^4. \tag{3.4.8}$$

Since

$$Z(z,z) = k_3\frac{s_{15}^2(z,z)}{g_2^{15}(z,z)},$$

comparing (3.4.6), (3.4.8), we have $k_3 = 2^{26}\cdot 5^{10}\cdot 3^{-2}$.

Finally, from (3.4.5), (3.4.7), $k_1 = 2^5\cdot 5^2$ and $k_3 = 2^{26}\cdot 5^{10}\cdot 3^{-2}$, we have

$$k_2 = 2^{10}\cdot 5^5.$$

□

Thus, we have an expression of the pair of the Hilbert modular functions X and Y as the pair of the quotients of Müller's modular forms via the period mapping for our family $\mathcal{F}$ of $K3$ surfaces.

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