Studies on verified computation for solutions to systems of linear and semilinear elliptic partial differential equations

楕円型連立線形-半線形偏微分方程式の解の
精度保証付き数値計算に関する研究

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Callico cat cafe in Shinjuku, Tokyo on 28th January 2014

Kouta Sekine

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Chapter 1

Introduction

In this thesis, we are concerned with a computer-assisted proof method for existence and local uniqueness of solutions to elliptic systems:

$$\begin{cases} -\varepsilon^2\Delta u = f(u) - \delta v & \text{in } \Omega, \\ -\Delta v = u - \gamma v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, Ω is a bounded polygonal domain with arbitrary shape in $\mathbb{R}^2$. $\varepsilon \neq 0$, γ and δ are real parameters. A mapping $f : H_0^1(\Omega) \to L^2(\Omega)$ is assumed to be Fréchet differentiable, where H_0^1 and L^2 are function spaces defined in Section 2.1, respectively.

Our goal is that, using Newton-Kantorovich like theorem, we prove existence and local uniqueness of solutions satisfying the system (1). As an introduction, we first introduce previous works of the system (1) and Newton-Kantorovich like theorem. Next, more detail for our purpose is stated.

1.1. Previous works

Solutions of partial differential equations are applied to understand various phenomena in physics, engineering, biology etc. In general, because it is difficult to find the exact solution of a nonlinear partial differential equation, its approximate solution is given by numerical computations. However, various errors occur in numerical computations, e.g. discretization error, rounding error etc. Therefore, results of numerical computations are not necessarily exact solutions of partial differential equations. To overcome these difficulties, it is necessary to discuss how to prove the existence and local uniqueness of exact solution from the approximate solution. That is the aim of numerical verification method.

The basic approach of numerical verification method is interval arithmetic, which have been proposed by T. Sunaga [**31**]. Interval arithmetic treat a rounding error which occur to use floating-point numbers conforming IEEE 754 standard [**2, 3**]. M.T. Nakao has presented an infinite-dimensional interval method [**18, 19, 20, 21, 22,**

23]. Nakao's theory treat partial differential equations. M. Plum has also presented another numerical verification method [**25, 27**] and other works. In [**27**], M. Plum shows the Newton-Kantorovich like theorem by using Banach's fixed point theorem. Let X and Y be Banach spaces. To state a little bit more detail, we are concerned with a problem of finding a solution $u \in X$ satisfying the following nonlinear equation:

$$\mathcal{F}(u) = 0, \tag{2}$$

where the nonlinear operator $\mathcal{F} : X \to Y$ is assumed to be Fréchet differentiable. M. Plum has proved the following theorem;

Theorem 1.1 (M. Plum [**27**]). *Let $\hat{u} \in X$ be an approximate solution of (2). Let $W \subset X$ be a convex closed ball centered zero with radius ρ:*

$$W := \{w \in X : \|w\|_X \leq \rho\}.$$

Assuming that the Fréchet derivative $\mathcal{F}'[\hat{u}]$ is nonsingular and satisfies

$$\|\mathcal{F}'[\hat{u}]^{-1}\|_{L(Y,X)} \leq C_1 \tag{3}$$

for a certain positive constant C_1. Let C_2 be a positive constant satisfying

$$\|\mathcal{F}(\hat{u})\|_Y \leq C_2. \tag{4}$$

Let $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$, where $\mathbb{R}$ is the set of real numbers. We assume that there exist nonlinear functions $C_3 : \mathbb{R}^+ \to \mathbb{R}^+$ and $C_4 : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\sup_{w \in W} \left\| \int_0^1 (\mathcal{F}'[\hat{u} + tw] - \mathcal{F}'[\hat{u}])\, w dt \right\|_Y \leq C_3(\rho) \tag{5}$$

and

$$\sup_{w \in W} \|\mathcal{F}'[\hat{u} + w] - \mathcal{F}'[\hat{u}]\|_{L(X,Y)} \leq C_4(\rho), \tag{6}$$

respectively. If a constant ρ *satisfies*

$$C_2 \leq \frac{\rho}{C_1} - C_3(\rho) \text{ and } C_1 C_4(\rho) < 1, \tag{7}$$

then there exists a solution $u^* \in \hat{u} + W$ *of* $\mathcal{F}(u) = 0$ *and unique in* $\hat{u} + W$*.*

This theorem states that if there exists a ρ satisfying (7), then there exists a solution u^* of $\mathcal{F}(u) = 0$ in $\hat{u} + W$ and u^* is unique in $\hat{u} + W$.

In this thesis, we consider a verification method for solutions to systems of elliptic partial differential equations (1). The system (1) is derived from the FitzHugh-Nagumo model [**32**]. The original one-dimensional parabolic differential equation, which is called FitzHugh-Nagumo equation, was derived to serve as a prototype simplification of nerve conduction equations [**7, 17**]. The system (1) has been well studied from theoretical and numerical sides [**28, 29, 32**]. A numerical verification theory for (1) on bounded convex domain has been proposed by Y. Watanabe [**35**]. Here, following [**35**], we briefly sketch decoupling technique for the system. If γ is not point spectrum of the Laplace operator and u is a known function, the boundary value problem:

$$\begin{cases} -\Delta v = u - \gamma v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{8}$$

has a unique solution by Riesz's representation theorem. Then, v is presented by $v = Bu$, where $B : L^2(\Omega) \to H_0^1(\Omega)$ is a solution operator of (2). Substituting this for (1), it follows that

$$\begin{cases} -\Delta u = \dfrac{1}{\varepsilon^2}\left(f(u) - \delta B u\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Decoupling the original problem (1) into the linear Dirichlet problem (8) and the nonlinear Dirichlet problem (9), according to [**35**] one can verify the existence and local uniqueness of solutions for (8) and (9) using Nakao's theory, which is based on fixed-point theorems, respectively.

1.2. Purpose

Our goal is to prove existence and local uniqueness of solutions satisfying the system (1) on a bounded polygonal domain with arbitrary shape. In particular, for (9), we apply Theorem 1.1.

In the following, we will present the methods of calculating

- a constant C_1 of norm estimation (3) for inverse of linearized operator.
- an upper bound C_2 of a residual norm (4).
- nonlinear functions C_3 and C_4 satisfying (5) and (6), respectively.
- an error bound for the linear Dirichlet problem (8).
- an inner inclusion of a region defined by (7).

The aim of this thesis is to treat a numerical verification method of (8) and (9) on bounded nonconvex domains. On the nonconvex domain, calculating C_2 of residual norm for (9) is one of the most important tasks because exact solutions u^* and v^* of (8) and (9) do not have H^2-regularity, respectively. For the Dirichlet problem of semilinear elliptic partial differential equations, several methods for calculation an upper bound C_2 are proposed in [**36**]. In [**33**], A. Takayasu, X. Liu and S. Oishi have presented how to derive an upper bound C_2 using the Raviart-Thomas mixed finite element on a bounded polygonal domain. However, an upper bound C_2 of (9) including the solution operator B seem to have not yet be published. In this thesis, we present the method of calculating for an upper bound C_2 of (9) including a solution operator B based on the Raviart-Thomas mixed finite element and properties of B. In addition to this, an error bound of (8) is also shown by the Raviart-Thomas mixed finite element.

Next, in order to apply Theorem 1.1, to find ρ satisfying (7) is one of the most important tasks. By a series of papers M. Plum [**25, 26, 27**] has shown that such ρ

can be found for various interesting nonlinear partial differential equations. However, a systematic way of finding ρ seems to have not yet be published. Therefore, we present an algorithm of constructing an inner inclusion of a region defined by (7) based on Moore-Jones's algorithm of finding all solutions of one dimensional nonlinear equations proposed in [**16**], which is based on Krawczyk's operator [**12, 13**]. Here, inclusion means a subset of a region defined by (7). One of the features of our algorithm is that if a region of the solution for (7) is empty, we can prove that there is no solution of (7).

1.3. Outline

The outline of this paper is as follows: In Chapter 2, we provide a verification theory for (1) on a bounded polygonal domain with arbitrary shape in $\mathbb{R}^2$. A main point of this chapter is to develop a method of calculating for C_2 of (9) involving a solution operator B. In Chapter 3, we will present an algorithm of constructing an inner inclusion of a region defined by (7). In Chapter 4, we provide some numerical examples. We first demonstrate the algorithm of constructing an inner inclusion of a region defined by (7). We also present results of numerical verifications for (1).

Chapter 2

Verification theory

2.1. Notation and basic theorems

Let $L^p(\Omega)$, $p \in [1, \infty)$ denote the functional space of the p-th power Lebesgue integrable functions. For $p = 2$, let us define the inner product

$$(u, w)_{L^2} := \int_\Omega u(x)w(x)dx$$

and the norm

$$\|u\|_{L^2} := \sqrt{(u, u)_{L^2}}.$$

For s being a fixed positive real number, let $H^s(\Omega)$ denote the L^2 Sobolev space of order s. Then, the function space $H_0^1(\Omega)$ is defined by

$$H_0^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}$$

with the inner product $(u, w)_{H_0^1} := (\nabla u, \nabla w)_{L^2}$ and the norm $\|u\|_{H_0^1} := \|\nabla u\|_{L^2}$. Let $H^{-1}(\Omega)$ be the topological dual space of $H_0^1(\Omega)$. We denote $Tu \in \mathbb{R}$ by $\langle T, u\rangle$, where $T \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$. The norm of $T \in H^{-1}(\Omega)$ is defined by

$$\|T\|_{H^{-1}} := \sup_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{|\langle T, u\rangle|}{\|u\|_{H_0^1}}.$$

Let $L^\infty(\Omega)$ be the space of functions that are essentially bounded on Ω with the norm

$$\|u\|_{L^\infty} := \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|.$$

X and Y are assumed to be Banach spaces. We denote the set of bounded linear operators by $L(X, Y)$ with the operator norm

$$\|\mathcal{T}\|_{L(X,Y)} = \sup_{u \in X\setminus\{0\}} \frac{\|\mathcal{T}u\|_Y}{\|u\|_X}.$$

Furthermore, there exists the constant $C_{e,p}$ satisfying

$$\|u\|_{L^p} \leq C_{e,p}\|u\|_{H_0^1} \text{ for } u \in H_0^1(\Omega)$$

from Sobolev's embedding theorem. For concrete value of $C_{e,p}$, see Appendix A.

Let X_h be the finite-dimensional subspace spanned by linearly independent $H_0^1(\Omega)$ conforming finite element base functions. Depending on the mesh size h ($0 < h < 1$), the orthogonal projection $\mathcal{P}_h : H_0^1(\Omega) \to X_h$ is defined by

$$(\nabla(u - \mathcal{P}_h u), \nabla\phi_h)_{L^2} = 0,\ \forall \phi_h \in X_h. \tag{10}$$

For $\mathcal{P}_h$, we have the constant C_h satisfying

$$\|u - \mathcal{P}_h u\|_{H_0^1} \le C_h \|\Delta u\|_{L^2}. \tag{11}$$

For piecewise linear finite elements, the detailed method of calculating the constant C_h is well studied by F. Kikuchi and X. Liu [**10**], and K. Kobayashi [**11**] on convex domains. On nonconvex domains, C_h is computable by the method of X. Liu and S. Oishi [**15**]. From Aubin-Nitsche's trick for (11), we have $\|u - \mathcal{P}_h u\|_{L^2} \le C_h \|\nabla(u - \mathcal{P}_h u)\|_{L^2}$.

2.2. Verification framework of semilinear elliptic equation

Setting $g(u) := (f(u) - \delta Bu)/\varepsilon^2$, we rewrite (9) as

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where the mapping $g : H_0^1(\Omega) \to L^2(\Omega)$ is Fréchet differentiable. In fact, we have

$$\begin{aligned} & \|g(u+w) - g(u) - \frac{1}{\varepsilon^2}(f'[u]w - \delta Bw)\|_{L^2} \\ = \ & \frac{1}{\varepsilon^2} \|(f(u+w) - \delta B(u+w)) - (f(u) - \delta Bu) - (f'[u]w - \delta Bw)\|_{L^2} \\ = \ & \frac{1}{\varepsilon^2} \|f(u+w) - f(u) - f'[u]w)\|_{L^2},\ \forall w \in H_0^1(\Omega). \end{aligned}$$

From the Fréchet differentiability of $f : H_0^1(\Omega) \to L^2(\Omega)$, we have

$$\frac{\|f(u+w) - f(u) - f'[u]w)\|_{L^2}}{\|w\|_{H_0^1}} = o(\|w\|_{H_0^1}),$$

where $o(\|w\|_{H_0^1})$ means faster convergence than $\|w\|_{H_0^1} \to 0$. Let $\hat{u}$ be a finite element approximation of the weak solution of (12). We assume $\|f'[\hat{u}]\|_{L^\infty} < +\infty$. We consider the weak form of (12). The weak formulation of (12) is to find $u \in H_0^1(\Omega)$ satisfying

$$(\nabla u, \nabla w)_{L^2} = (g(u), w)_{L^2},\ \forall w \in H_0^1(\Omega). \tag{13}$$

For $u \in H_0^1(\Omega)$, we define the linear operator $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ and the nonlinear operator $\mathcal{N} : H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$\langle \mathcal{A}u, w\rangle \ := \ (\nabla u, \nabla w)_{L^2},\ \forall w \in H_0^1(\Omega) \tag{14}$$

and

$$\langle \mathcal{N}(u), w\rangle \ := \ (g(u), w)_{L^2},\ \forall w \in H_0^1(\Omega),$$

respectively. The weak form is denoted by the following nonlinear operator equation:

$$\mathcal{A}u = \mathcal{N}(u) \text{ in } H^{-1}(\Omega).$$

Let $\mathcal{F}$ be the nonlinear operator mapping from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ described by

$$\mathcal{F}(u) := \mathcal{A}u - \mathcal{N}(u).$$

The solution of

$$\mathcal{F}(u) = 0 \tag{15}$$

is equivalent to (13). For the nonlinear operator equation, we calculate verified error bounds by using Theorem 1.1. For the verification method using Newton-Kantorovich's theorem of (15), see Appendix C. From the Fréchet differentiability of g, the operator $\mathcal{F}$ is Fréchet differentiable. The Fréchet derivative of $\mathcal{F}$ at $\hat{u} \in H_0^1(\Omega)$ is given by

$$\mathcal{F}'[\hat{u}] = \mathcal{A} - \mathcal{N}'[\hat{u}],$$

where

$$\langle \mathcal{N}'[\hat{u}]u, w\rangle = (g'[\hat{u}]u, w), \ \forall w \in H_0^1(\Omega). \tag{16}$$

2.3. Solution operator of linear equation and its properties

In this section, we discuss the solution operator of (8). For a given $u \in H_0^1(\Omega)$, the solution operator $B : L^2(\Omega) \to H_0^1(\Omega)$ gives the weak solution $v \in H_0^1(\Omega)$ satisfying

$$(\nabla v, \nabla w)_{L^2} + \gamma(v, w)_{L^2} = (u, w)_{L^2}, \ \forall w \in H_0^1(\Omega). \tag{17}$$

For $u, v \in H_0^1(\Omega)$, we define the operator $\mathcal{L} : H_0^1(\Omega) \to H^{-1}(\Omega)$ and the identity embedding operator $\mathcal{I} : L^2(\Omega) \to H^{-1}(\Omega)$ by

$$\langle \mathcal{L}v, w\rangle := (\nabla v, \nabla w)_{L^2} + \gamma(v, w)_{L^2}, \ \forall w \in H_0^1(\Omega) \tag{18}$$

and

$$\langle \mathcal{I}u, w\rangle := (u, w)_{L^2}, \ \forall w \in H_0^1(\Omega), \tag{19}$$

respectively. Then, the equation (17) is denoted by $\mathcal{L}v = \mathcal{I}u$ in $H^{-1}(\Omega)$. If the operator $\mathcal{L}$ is invertible, the solution operator B is denoted the composite operator:

$$B := \mathcal{L}^{-1}\mathcal{I}. \tag{20}$$

Remark 2.1. *Let us define the elliptic operator* $A : H^{3/2+r}(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ *by*

$$(Av, w)_{L^2} := (\nabla v, \nabla w)_{L^2},\ \forall w \in H_0^1(\Omega),$$

where $0 < r \leq 1/2$. *This operator* A *has same spectrum as the Laplace operator. A compact operator* $A^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ *is defined by a inverse operator of* A *and a embedding operator from* $H^{3/2+r}(\Omega) \cap H_0^1(\Omega)$ *to* $H_0^1(\Omega)$. *Using the operator* A, *(17) is rewritten by* $(A+\gamma)v = u$ *in* $L^2(\Omega)$. *We can also describe* $B = (A+\gamma)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$. *A note that* $B\phi$ *also is in* $H^{3/2+r}(\Omega) \cap H_0^1(\Omega)$ *for any* $\phi \in L^2(\Omega)$ [**1**].

Next, we consider the operator norm $\|\mathcal{L}^{-1}\|_{H^{-1},H_0^1}$ using eigenvalues for the elliptic operator A. Let us discuss whether the operator $\mathcal{L}$ is invertible or not. As a property of the isometric operator $\mathcal{A}$, we have

$$\|\mathcal{A}v\|_{H^{-1}} = \sup_{w \in H_0^1(\Omega)\backslash\{0\}} \frac{|\langle \mathcal{A}v, w\rangle|}{\|w\|_{H_0^1}} = \sup_{w \in H_0^1(\Omega)\backslash\{0\}} \frac{|(\nabla v, \nabla w)_{L^2}|}{\|w\|_{H_0^1}} = \|v\|_{H_0^1}.$$

Then, we find $K \geq 0$ satisfying

$$\sup_{v \in H_0^1(\Omega)\backslash\{0\}} \frac{\|v\|_{H_0^1}}{\|\mathcal{L}v\|_{H^{-1}}} = \sup_{v \in H_0^1(\Omega)\backslash\{0\}} \frac{\|v\|_{H_0^1}}{\|\mathcal{A}^{-1}\mathcal{L}v\|_{H_0^1}} \leq K.$$

Let us consider the following eigenvalue problem: Find $v \in H_0^1(\Omega)$ and $\hat{\lambda} \in \mathbb{R}$ such that

$$(\nabla v, \nabla w)_{L^2} + \gamma(v, w)_{L^2} = \hat{\lambda}(\nabla v, \nabla w)_{L^2},\ \forall w \in H_0^1(\Omega). \tag{21}$$

(21) is equivalent to the eigenvalue problem for the elliptic operator: Find $v \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$(\nabla v, \nabla w)_{L^2} = \lambda(v, w)_{L^2},\ \forall w \in H_0^1(\Omega),$$

where we set $\lambda = -\gamma/(1-\hat{\lambda})$. Note that the spectrum of the elliptic operator is discrete. We denote spectrum of A by $\mathrm{Spec}(A)$. Moreover, each eigenvalue is evaluated explicitly by Theorem A.1 in Appendix A. If $\lambda + \gamma \neq 0$ holds, then the operator $\mathcal{L}$ is invertible and the following inequality holds for eigenvalues λ,

$$\|\mathcal{L}^{-1}\|_{L(H^{-1},H_0^1)} = \sup_{v\in H_0^1(\Omega)\backslash\{0\}} \frac{\|v\|_{H_0^1}}{\|\mathcal{A}^{-1}\mathcal{L}v\|_{H_0^1}} \le \max_{\lambda\in \mathrm{Spec}(A)} \left|\frac{\lambda}{\lambda+\gamma}\right| = K. \tag{22}$$

Since $\mathcal{I}$ is a compact operator, if $\mathcal{L}$ is bounded, the solution operator $B : L^2(\Omega) \to H_0^1(\Omega)$ becomes compact and self-adjoint.

For a given $u_h \in X_h$, the operator $B_h : X_h \to X_h$ gives the discretized solution $v_h \in X_h$ satisfying

$$(\nabla v_h, \nabla w_h)_{L^2} + \gamma(v_h, w_h)_{L^2} = (u_h, w_h)_{L^2},\ \forall w_h \in X_h.$$

Thus, we have

$$v_h = B_h u_h. \tag{23}$$

Another orthogonal projection $\mathcal{P}_{h,\gamma} : H_0^1(\Omega) \to X_h$ is also defined by

$$(\nabla(u - \mathcal{P}_{h,\gamma}u), \nabla\phi_h)_{L^2} + \gamma(u - \mathcal{P}_{h,\gamma}u, \phi_h)_{L^2} = 0,\ \forall \phi_h \in X_h.$$

Using this projection, we have

$$\mathcal{P}_{h,\gamma} B|_{X_h} = B_h,$$

where $|_{X_h}$ means that it restricts the domain $L^2(\Omega)$ to X_h.

2.4. Verification theories for semilinear elliptic problem

2.4.1. Norm estimation for inverse of linearized operator. This part estimates the norm of inverse operator (3). In [**24**], S. Oishi shows a numerical method that proves the existence of the inverse operator of $\mathcal{F}'[\hat{u}]$. C_1 is assumed to satisfy

$$\|u\|_{H_0^1} \le C_1 \|\mathcal{F}'[\hat{u}]u\|_{H^{-1}},\ \forall u \in H_0^1(\Omega). \tag{24}$$

From the isometric operator $\mathcal{A}$, we have

$$\|\mathcal{F}'[\hat{u}]u\|_{H^{-1}} = \|\mathcal{A}^{-1}\mathcal{F}'[\hat{u}]u\|_{H_0^1}, \ \forall u \in H_0^1(\Omega). \tag{25}$$

Using $\mathcal{N}'[\hat{u}] = \mathcal{I}g'[\hat{u}]$, the operator $\mathcal{A}^{-1}\mathcal{F}'[\hat{u}]$ is transformed into

$$\mathcal{A}^{-1}\mathcal{F}'[\hat{u}] = I - \mathcal{A}^{-1}\mathcal{N}'[\hat{u}] = I - (\mathcal{I}A)^{-1}\mathcal{I}g'[\hat{u}] = I - A^{-1}g'[\hat{u}].$$

From the compactness of A^{-1} and boundedness of $g'[\hat{u}]$, the operator $A^{-1}g'[\hat{u}]$ is compact. Then, $\mathcal{A}^{-1}\mathcal{F}'[\hat{u}]$ becomes the Fredholm operator. From (24) and (25), $\mathcal{A}^{-1}\mathcal{F}'[\hat{u}]$ is one to one. Then, $\mathcal{A}^{-1}\mathcal{F}'[\hat{u}]$ is bijection as per the Fredholm alternative theorem. Since $\mathcal{A}$ is bijection, then $\mathcal{F}'[\hat{u}]$ is also bijection. Thus, it is proved that $\mathcal{F}'[\hat{u}]$ is invertible.

By using spectrum of linearized operator, we estimate C_1 explicitly. For a different method of calculating the upper bound C_1 for (15) using Theorem B.1, see Appendix B. Let Ψ be the linear operator mapping from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$,

$$\langle \Psi u, w\rangle := (\nabla u, \nabla w)_{L^2} + \sigma(u, w)_{L^2}, \ \forall w \in H_0^1(\Omega),$$

where positive number σ is assumed to be satisfying (28). From Riesz's representation theorem, the inverse of Ψ exists. Then, we define the σ-inner product and the σ-norm by

$$(u, w)_\sigma \ := \ (\nabla u, \nabla w)_{L^2} + \sigma(u, w)_{L^2}, \ \forall w \in H_0^1(\Omega)$$

and

$$\|u\|_\sigma \ = \ \sqrt{(u, u)_\sigma},$$

respectively. For $u \in H_0^1(\Omega)$, we have a property,

$$\begin{aligned}\|\Psi u\|_{H^{-1}} &= \sup_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{|\langle \Psi u, w\rangle|}{\|w\|_{H_0^1}} \\ &\geq \sup_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{|(u,w)_\sigma|}{\|w\|_\sigma} = \|u\|_\sigma.\end{aligned}$$

Before we introduce the procedure for estimating C_1, let us prepare the following lemma.

LEMMA 2.2. *Let λ be each eigenvalue of the elliptic operator A. We consider the eigenvalue problem: Find $u \in H_0^1(\Omega)$ and $\eta \in \mathbb{R}$ such that*

$$(\nabla u, \nabla w)_{L^2} + \gamma(u,w)_{L^2} = \eta(u,w)_{L^2},\ \forall w \in H_0^1(\Omega). \tag{26}$$

We define the constant by

$$K_1 := \max\left\{\frac{1}{|\eta|} : \eta \text{ is each eigenvalue of (26) with } \eta \neq 0\right\}.$$

Then, K_1 gives an upper bound of the following quantity for $\forall u \in H_0^1(\Omega)$,

$$\frac{|(Bu,u)_{L^2}|}{(u,u)_{L^2}} \leq K_1.$$

PROOF. Since the operator B is compact, the spectrum of B is point spectrum and $\{0\}$. From the definition $B = \mathcal{L}^{-1}\mathcal{I}$,

$$Bu = \frac{1}{\eta}u \text{ in } L^2(\Omega) \iff \mathcal{L}u = \eta\mathcal{I}u \text{ in } H^{-1}(\Omega).$$

It follows that

$$\frac{|(Bu,u)_{L^2}|}{(u,u)_{L^2}} \leq \frac{\|Bu\|_{L^2}}{\|u\|_{L^2}} \leq \max\left\{\frac{1}{|\eta|} : \frac{1}{\eta} \in \mathrm{Spec}(B)\right\} = \max_{\eta \in \mathrm{Spec}(\mathcal{I}^{-1}\mathcal{L})} \frac{1}{|\eta|}.$$

□

We use spectrum of the linearized operator for calculating C_1, which is related to the invertibility of the operator. We check whether spectrum of $\Psi^{-1}\mathcal{F}'[\hat{u}]$ include $\{0\}$

or not with computer-assistance. Furthermore, the norm estimation of the inverse operator is given by the minimal absolute value of spectrum. From the definition of $\mathcal{F}'[\hat{u}]$ and the property of Ψ, it follows that

$$\begin{aligned}
\|\mathcal{F}'[\hat{u}]^{-1}\|_{L(H^{-1},H_0^1)} &= \sup_{u\in H_0^1(\Omega)\backslash\{0\}} \frac{\|u\|_{H_0^1}}{\|\mathcal{F}'[\hat{u}]u\|_{H^{-1}}} \\
&= \sup_{u\in H_0^1(\Omega)\backslash\{0\}} \frac{\|u\|_{H_0^1}}{\|\mathcal{A}u-\mathcal{N}'[\hat{u}]u\|_{H^{-1}}} \\
&\le \sup_{u\in H_0^1(\Omega)\backslash\{0\}} \frac{\|u\|_{\sigma}}{\|\Psi^{-1}(\mathcal{A}u-\mathcal{N}'[\hat{u}]u)\|_{\sigma}} \\
&\le \sup_{\mu\in \mathrm{Spec}(\Psi^{-1}(\mathcal{A}-\mathcal{N}'[\hat{u}]))} \frac{1}{|\mu|} =: C_1.
\end{aligned}$$

Let us consider the following problem: Find $u \in H_0^1(\Omega)$ and $\mu \in \mathbb{R}$ such that

$$(\nabla u, \nabla w)_{L^2} - (g'[\hat{u}]u, w)_{L^2} = \mu((\nabla u, \nabla w)_{L^2} + \sigma(u, w)_{L^2}), \ \forall w \in H_0^1(\Omega),$$

which is transformed into

$$\begin{aligned}
& (\nabla u, \nabla w)_{L^2} - (g'[\hat{u}]u, w)_{L^2} = \mu(\nabla u, \nabla w)_{L^2} + \mu\sigma(u, w)_{L^2}, \\
\iff\ & (\nabla u, \nabla w)_{L^2} + \sigma(u, w)_{L^2} - (g'[\hat{u}]u, w)_{L^2} = \mu(\nabla u, \nabla w)_{L^2} + \mu\sigma(u, w)_{L^2} + \sigma(u, w)_{L^2} \\
\iff\ & (1-\mu)((\nabla u, \nabla w)_{L^2} + \sigma(u, w)_{L^2}) = \sigma(u, w)_{L^2} + (g'[\hat{u}]u, w)_{L^2} \\
\iff\ & (u, w)_{\sigma} = \frac{1}{1-\mu}\left(\sigma(u, w)_{L^2} + (g'[\hat{u}]u, w)_{L^2}\right), \ \forall w \in H_0^1(\Omega). \qquad (27)
\end{aligned}$$

If the condition

$$\sigma(u, u)_{L^2} + (g'[\hat{u}]u, u)_{L^2} \ge 0, \ \forall u \in H_0^1(\Omega)$$

holds, the term in (27): $\sigma(u,w)_{L^2} + (g'[\hat{u}]u,w)_{L^2}$ becomes the inner product. From Lemma 2.2, it follows that, for $\forall u \in H_0^1(\Omega)$,

$$
\begin{aligned}
\sigma(u,u)_{L^2} + (g'[\hat{u}]u,u)_{L^2} &= \sigma(u,u)_{L^2} + \frac{1}{\varepsilon^2}(f'[\hat{u}]u,u)_{L^2} - \frac{\delta}{\varepsilon^2}(Bu,u)_{L^2} \\
&\geq ((\sigma + \frac{1}{\varepsilon^2}f'[\hat{u}])u,u)_{L^2} - \frac{|\delta|}{\varepsilon^2}\frac{|(Bu,u)_{L^2}|}{(u,u)_{L^2}}(u,u)_{L^2} \\
&\geq ((\sigma + \frac{1}{\varepsilon^2}f'[\hat{u}])u,u)_{L^2} - \frac{|\delta|}{\varepsilon^2}K_1(u,u)_{L^2}.
\end{aligned}
$$

Thus, the assumption of σ is given as follows:

$$
\sigma \geq -\frac{1}{\varepsilon^2}\left(\operatorname*{ess\,inf}_{x\in\Omega}\ f'[\hat{u}] - |\delta|K_1\right). \tag{28}
$$

Let us define the d-inner product and the d-norm by

$$
(u,w)_d \ := \ \sigma(u,w)_{L^2} + (g'[\hat{u}]u,w)_{L^2}, \ \forall w \in H_0^1(\Omega)
$$

and

$$
\|u\|_d \ = \ \sqrt{(u,u)_d},
$$

respectively.

REMARK 2.3. *Let $\mathcal{D}$ be the linear operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$,*

$$
\langle \mathcal{D}u, w\rangle := \sigma(u,w)_{L^2} + (g'[\hat{u}]u,w)_{L^2}, \ \forall w \in H_0^1(\Omega).
$$

Since $\mathcal{I}$ is compact operator and $(\sigma + g'[\hat{u}])$ is bounded, $\mathcal{D}$ is bounded. The boundedness of the linear operator Ψ^{-1} yields that the composite operator $\Psi^{-1}\mathcal{D}$ is compact. Then, the problem (27) is rewritten by

$$
\Psi u = \frac{1}{1-\mu}\mathcal{D}u \iff \Psi^{-1}\mathcal{D}u = (1-\mu)u \ \textit{in} \ H^{-1}(\Omega).
$$

Therefore, the spectrum $1-\mu$ becomes point spectrum and $\{0\}$. The spectrum μ is point spectrum and $\{1\}$.

By setting $\hat{\mu} := 1/(1-\mu)$ for (27), we consider how to evaluate each eigenvalue $\hat{\mu}$. To enclose eigenvalues of (27), we extend Liu and Oishi's procedure [**15**] described in Theorem A.1. Let us give the following theorem which encloses each eigenvalue of (27).

THEOREM 2.4. *Let $\{\hat{\mu}_k\}$ be each eigenvalue of (27). $\hat{\mu}_k^h$ is assumed to be approximate eigenvalues of the discrete problem of (27). $C_{h,\sigma}$ is computable constant which is available from the constant C_h. We define the constant K_2 by*

$$K_2 := \sqrt{\left\|\sigma + \frac{f'[\hat{u}]}{\varepsilon^2}\right\|_{L^\infty} + K_1\left|\frac{\delta}{\varepsilon^2}\right|}.$$

If

$$C_{h,\sigma}^2 K_2^2 \hat{\mu}_i^h < 1$$

holds, then we have the verified enclosure of each $\hat{\mu}_k$:

$$\frac{\hat{\mu}_k^h}{1 + C_{h,\sigma}^2 K_2^2 \hat{\mu}_k^h} \leq \hat{\mu}_k \leq \hat{\mu}_k^h.$$

REMARK 2.5. *Theorem 2.4 is the extension of theorem A.1 by X. Liu and S. Oishi* [**15**] *for the linearized problem of (9). The main feature is that our theorem can treat the eigenvalue problem (27) whose the right-hand side contains the compact self-adjoint operator $B : L^2(\Omega) \to H_0^1(\Omega)$.*

PROOF. Let u_k be each eigenfunction for eigenvalues $\hat{\mu}_k$. Let E_k be the linear space which consists of base functions $\{u_j\}_{j=1}^k$. We define the space $E_k^1 := \{u_k \in E_k : \|u_k\|_d = 1\}$. It follows that

$$\hat{\mu}_k = \frac{(\nabla u_k, \nabla u_k)_{L^2} + \sigma(u_k, u_k)_{L^2}}{\sigma(u_k, u_k)_{L^2} + (g'[\hat{u}]u_k, u_k)_{L^2}} = \frac{\|u_k\|_\sigma^2}{\|u_k\|_d^2} = \|u_k\|_\sigma.$$

The orthogonal projection $\mathcal{P}_{h,\sigma} : H_0^1(\Omega) \to X_h$ is defined by

$$(u - \mathcal{P}_{h,\sigma}u, w_h)_\sigma = 0, \ \forall w_h \in X_h.$$

We consider how to calculate the constant $C_{h,\sigma}$ that satisfies $\|u - \mathcal{P}_{h,\sigma}u\|_\sigma \leq C_{h,\sigma}\|(A+\sigma)u\|_{L^2}$. From the minimalization principle, we obtain

$$\begin{aligned}
\|u - \mathcal{P}_{h,\sigma}u\|_\sigma^2 &\leq \|u - \mathcal{P}_h u\|_\sigma^2 \\
&= \|\nabla(u - \mathcal{P}_h u)\|_{L^2}^2 + \sigma\|u - \mathcal{P}_h u\|_{L^2}^2 \\
&\leq \|\nabla(u - \mathcal{P}_h u)\|_{L^2}^2 + \sigma C_h^2\|\nabla(u - \mathcal{P}_h u)\|_{L^2}^2 \\
&= (1 + \sigma C_h^2)\|u - \mathcal{P}_h u\|_{H_0^1}^2 \\
&\leq (1 + \sigma C_h^2)C_h^2\|Au\|_{L^2}^2 \\
&\leq (1 + \sigma C_h^2)C_h^2\|(A + \sigma)u\|_{L^2}^2.
\end{aligned}$$

Namely, $C_{h,\sigma}$ is estimated by $C_h\sqrt{(1 + \sigma C_h^2)}$. Aubin-Nitsche's trick yields $\|u - \mathcal{P}_{h,\sigma}u\|_{L^2} \leq C_{h,\sigma}\|u - \mathcal{P}_{h,\sigma}u\|_\sigma$. From the min-max principle, we have

$$\begin{aligned}
\hat{\mu}_k^h &\leq \max_{u\in E_k^1} \frac{\|\mathcal{P}_{h,\sigma}u\|_\sigma^2}{\|\mathcal{P}_{h,\sigma}u\|_d^2} \\
&= \max_{u\in E_k^1} \frac{\|u\|_\sigma^2 - \|u - \mathcal{P}_{h,\sigma}u\|_\sigma^2}{\|u + \mathcal{P}_{h,\sigma}u - u\|_d^2} \\
&= \max_{u\in E_k^1} \frac{\|u\|_\sigma^2 - \|u - \mathcal{P}_{h,\sigma}u\|_\sigma^2}{\|u\|_d^2 + 2(u, \mathcal{P}_{h,\sigma}u - u)_d + \|P_{h_\sigma}u - u\|_d^2} \\
&\leq \max_{u\in E_k^1} \frac{\hat{\mu}_k - \|u - \mathcal{P}_{h,\sigma}u\|_\sigma^2}{1 + 2(u, \mathcal{P}_{h,\sigma}u - u)_d + \|\mathcal{P}_{h,\sigma}u - u\|_d^2} \\
&\leq \max_{u\in E_k^1} \frac{\hat{\mu}_k - \|u - \mathcal{P}_{h,\sigma}u\|_\sigma^2}{1 - 2\|u\|_d\|\mathcal{P}_{h,\sigma}u - u\|_d + \|\mathcal{P}_{h,\sigma}u - u\|_d^2} \\
&= \max_{u\in E_k^1} \frac{\hat{\mu}_k - \|u - \mathcal{P}_{h,\sigma}u\|_\sigma^2}{1 - 2\|\mathcal{P}_{h,\sigma}u - u\|_d + \|\mathcal{P}_{h,\sigma}u - u\|_d^2} \\
&= \max_{u\in E_k^1} \frac{\hat{\mu}_k - \|u - \mathcal{P}_{h,\sigma}u\|_\sigma^2}{(1 - \|u - \mathcal{P}_{h,\sigma}u\|_d)^2}.
\end{aligned} \tag{29}$$

We have the following inequality between $\|u\|_d$ and $\|u\|_{L^2}$,

$$
\begin{aligned}
\|u\|_d^2 &= (u,u)_d \\
&= \sigma(u,u)_{L^2} + (g'[\hat{u}]u,u)_{L^2} \\
&= \sigma(u,u)_{L^2} + \left(\frac{f'[\hat{u}]}{\varepsilon^2}u,u\right)_{L^2} - \left(\frac{\delta}{\varepsilon^2}Bu,u\right)_{L^2} \\
&\le \left|\left(\left(\sigma + \frac{f'[\hat{u}]}{\varepsilon^2}\right)u,u\right)_{L^2}\right| + \frac{\left|\left(\frac{\delta}{\varepsilon^2}Bu,u\right)_{L^2}\right|}{(u,u)_{L^2}}\|u\|_{L^2}^2 \\
&\le \left(\left\|\left(\sigma + \frac{f'[\hat{u}]}{\varepsilon^2}\right)u\right\|_{L^2}\|u\|_{L^2} + K_1\left|\frac{\delta}{\varepsilon^2}\right|\|u\|_{L^2}^2\right) \\
&\le \left(\left\|\sigma + \frac{f'[\hat{u}]}{\varepsilon^2}\right\|_{L^\infty} + K_1\left|\frac{\delta}{\varepsilon^2}\right|\right)\|u\|_{L^2}^2
\end{aligned}
$$

Since $K_2 = \sqrt{\|\sigma + \frac{f'[\hat{u}]}{\varepsilon^2}\|_{L^\infty} + K_1\left|\frac{\delta}{\varepsilon^2}\right|}$, we have

$$\|u\|_d \le K_2\|u\|_{L^2}.$$

Then, we give the following estimation

$$
\begin{aligned}
\|u - \mathcal{P}_{h,\sigma}u\|_d &\le K_2\|u - \mathcal{P}_{h,\sigma}u\|_{L^2} \\
&\le C_{h,\sigma}K_2\|u - \mathcal{P}_{h,\sigma}u\|_\sigma.
\end{aligned}
$$

Substituting this for (29), we have

$$\hat{\mu}_k^h \le \max_{u \in E_k^1} \frac{\hat{\mu}_k - \|u - \mathcal{P}_{h,\sigma}u\|_\sigma^2}{(1 - C_{h,\sigma}K_2\|u - \mathcal{P}_{h,\sigma}u\|_\sigma)^2}. \tag{30}$$

Letting $g(t) := (\hat{\mu}_k - t^2)/(1 - C_{h,\sigma}K_2t)^2$, $g(t)$ is a monotone increasing function on the range: $t \le C_{h,\sigma}K_2\hat{\mu}_k$ and $t < 1/(C_{h,\sigma}K_2)$. From the assumption $\hat{\mu}_k C_{h,\sigma}^2 K_2^2 < 1$, it follows that

$$C_{h,\sigma}K_2\hat{\mu}_k < \frac{1}{C_{h,\sigma}K_2}.$$

If $t \leq C_{h,\sigma} K_2 \hat{\mu}_k$ holds, then $g(t)$ becomes a monotone increasing function. For $u = \sum_{j=1}^{k} c_j u_j \in E_k^1$, it satisfies

$$
\begin{aligned}
\|u - \mathcal{P}_{h,\sigma} u\|_\sigma &\leq C_{h,\sigma} \|(A+\sigma)u\|_{L^2} \\
&= C_{h,\sigma} \left\| \sum_{j=1}^{k} c_j (A+\sigma) u_j \right\|_{L^2} \\
&= C_{h,\sigma} \left\| \sum_{j=1}^{k} c_j \hat{\mu}_j (\sigma + g'[\hat{u}]) u_j \right\|_{L^2} \\
&\leq C_{h,\sigma} \left\| \hat{\mu}_k (\sigma + g'[\hat{u}]) \sum_{j=1}^{k} c_j u_j \right\|_{L^2} \\
&= C_{h,\sigma} \|\hat{\mu}_k (\sigma + g'[\hat{u}]) u\|_{L^2} \\
&= \hat{\mu}_k C_{h,\sigma} \sup_{w \in L^2(\Omega)\setminus\{0\}} \frac{|((\sigma + g'[\hat{u}])u, w)_{L^2}|}{\|w\|_{L^2}} \\
&= \hat{\mu}_k C_{h,\sigma} \sup_{w \in L^2(\Omega)\setminus\{0\}} \frac{|(u, w)_d|}{\|w\|_{L^2}} \\
&\leq \hat{\mu}_k C_{h,\sigma} \sup_{w \in L^2(\Omega)\setminus\{0\}} \frac{\|u\|_d \|w\|_d}{\|w\|_{L^2}}.
\end{aligned}
$$

By using $\|w\|_d \leq K_2 \|w\|_{L^2}$ and $\|u\|_d = 1$, we have

$$
\|u - \mathcal{P}_{h,\sigma} u\|_\sigma \leq \hat{\mu}_k C_{h,\sigma} K_2. \tag{31}
$$

Substituting (31) for (30), we obtain

$$
\begin{aligned}
\hat{\mu}_k^h &\leq \frac{\hat{\mu}_k - \hat{\mu}_k^2 C_{h,\sigma}^2 K_2^2}{(1 - \hat{\mu}_k C_{h,\sigma}^2 K_2^2)^2} \\
&= \frac{\hat{\mu}_k}{1 - \hat{\mu}_k C_{h,\sigma}^2 K_2^2}.
\end{aligned}
$$

Finally, this means that

$$
\frac{\hat{\mu}_k^h}{\hat{\mu}_k^h C_{h,\sigma}^2 K_2^2 + 1} \leq \hat{\mu}_k.
$$

□

2.4.2. Residual norm. In this part, we discuss the constant C_2 of residual norm (4). A method of calculating C_2 has been proposed by A. Takayasu, X. Liu and S. Oishi [**33**]. This method gives a sharp bound of residual evaluation using the Raviart-Thomas mixed finite element. We apply the method to our residual formulation. We put the operator $\hat{g} : H_0^1(\Omega) \to L^2(\Omega)$ by $\hat{g}(u) = (f(u) - \delta B_h)/\varepsilon^2$. Let $H(\mathrm{div}, \Omega)$ denote the space of vector functions such that $H(\mathrm{div}, \Omega) := \{\psi \in (L^2(\Omega))^2 : \mathrm{div}\ \psi \in L^2(\Omega)\}$. Let K_h be a triangle element in the triangulation of Ω. We define

$P_k(K_h)$: the space of polynomials of degree less than or equal to k on K_h.

The Raviart-Thomas finite element space RT_k is given by

$$RT_k := \left\{ p_h \in (L^2(\Omega))^2 : p_h|_{K_h} = \begin{pmatrix} a_k \\ b_k \end{pmatrix} + c_k \cdot \begin{pmatrix} x \\ y \end{pmatrix}, a_k, b_k, c_k \in P_k(K_h), \right.$$
$$\left. p_h \cdot \overrightarrow{n} \text{ is continuous on the inter-element boundaries.} \right\},$$

where $\overrightarrow{n}$ is an outward unit normal vector on each K_h. This is a finite dimensional subspace of $H(\mathrm{div}, \Omega)$. We define

$$M_h := \{v \in L^2(\Omega) : v|_{K_h} \in P_k(K_h)\}.$$

From the property $\mathrm{div}(RT_k) = M_h$, see [**5**], we define a subspace of RT_k for given $\hat{g}_h \in M_h$,

$$W_{\hat{g}_h} := \{p_h \in RT_k : \mathrm{div}\ p_h + \hat{g}_h = 0 \text{ for } \hat{g}_h \in M_h\}.$$

Let another orthogonal projection $\mathcal{P}_{h,k} : L^2(\Omega) \to M_h$ satisfy

$$((I - \mathcal{P}_{h,k})\phi, w_h)_{L^2} = 0, \ \forall w_h \in M_h$$

for $\phi \in L^2(\Omega)$. Let $\hat{g}_h(\hat{u}) := \mathcal{P}_{h,k}\hat{g}(\hat{u})$. We assume that the constant $C_{h,k}$ satisfies $\|w - \mathcal{P}_{h,k}w\|_{L^2} \le C_{h,k}\|w\|_{H_0^1}$ for $w \in H_0^1(\Omega)$. If $1 + \gamma C_{e,2}^2 > 0$ holds, we define the

γ-inner product by

$$(v, w)_\gamma := (\nabla v, \nabla w)_{L^2} + \gamma(v, w)_{L^2},\ \forall w \in H_0^1(\Omega)$$

and the γ-norm by

$$\|v\|_\gamma = \sqrt{(v, v)_\gamma},\ \forall v \in H_0^1(\Omega).$$

If we obtain the constant $C_{h,\gamma}$ satisfying $\|v - \mathcal{P}_{h,\gamma} v\|_\gamma \leq C_{h,\gamma}\|(A + \gamma)v\|_{L^2}$, $\|v - \mathcal{P}_{h,\gamma} v\|_{L^2} \leq C_{h,\gamma}\|v - \mathcal{P}_{h,\gamma} v\|_\gamma$ holds from Aubin-Nitsche's trick. The detailed method of calculating $C_{h,\gamma}$ is described in the last part of this section. For $p_h \in W_{\hat{g}_h(\hat{u})}$, it follows that

$$
\begin{aligned}
& \|\mathcal{F}(\hat{u})\|_{H^{-1}} \\
= & \sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\nabla\hat{u},\nabla w)_{L^2}-(g(\hat{u}),w)_{L^2}|}{\|w\|_{H_0^1}} \\
= & \sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\nabla\hat{u},\nabla w)_{L^2}-(\frac{f(\hat{u})}{\varepsilon^2},w)_{L^2}+\frac{\delta}{\varepsilon^2}(B\hat{u},w)_{L^2}|}{\|w\|_{H_0^1}} \\
= & \sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\nabla\hat{u},\nabla w)_{L^2}-(\frac{f(\hat{u})}{\varepsilon^2},w)_{L^2}+\frac{\delta}{\varepsilon^2}(B_h\hat{u},w)_{L^2}+\frac{\delta}{\varepsilon^2}((B-B_h)\hat{u},w)_{L^2}|}{\|w\|_{H_0^1}} \\
\leq & \sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\nabla\hat{u},\nabla w)_{L^2}-(\hat{g}(\hat{u}),w)_{L^2}|+|\frac{\delta}{\varepsilon^2}||((I-\mathcal{P}_{h,\gamma})B\hat{u},w)_{L^2}|}{\|w\|_{H_0^1}} \\
\leq & \sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\nabla\hat{u}-p_h,\nabla w)_{L^2}+(p_h,\nabla w)_{L^2}-(\hat{g}(\hat{u}),w)_{L^2}|}{\|w\|_{H_0^1}} \\
& + \sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|\frac{\delta}{\varepsilon^2}|\|(I-\mathcal{P}_{h,\gamma})B\hat{u}\|_{L^2}\|w\|_{L^2}}{\|w\|_{H_0^1}} \\
\leq & \|\nabla\hat{u}-p_h\|_{L^2}+\sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\mathrm{div}\ p_h+\hat{g}(\hat{u}),w)_{L^2}|}{\|w\|_{H_0^1}}+C_{h,\gamma}C_{e,2}\left|\frac{\delta}{\varepsilon^2}\right|\|(I-\mathcal{P}_{h,\gamma})B\hat{u}\|_{\gamma} \\
\leq & \|\nabla\hat{u}-p_h\|_{L^2}+\sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\mathrm{div}\ p_h+\hat{g}_h(\hat{u})+\hat{g}(\hat{u})-\hat{g}_h(\hat{u}),w)_{L^2}|}{\|w\|_{H_0^1}} \\
& +C_{h,\gamma}^2C_{e,2}\left|\frac{\delta}{\varepsilon^2}\right|\|(A+\gamma)B\hat{u}\|_{L^2} \\
\leq & \|\nabla\hat{u}-p_h\|_{L^2}+\sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\hat{g}(\hat{u})-\hat{g}_h(\hat{u}),(I-\mathcal{P}_{h,k})w)_{L^2}|}{\|w\|_{H_0^1}}+C_{h,\gamma}^2C_{e,2}\left|\frac{\delta}{\varepsilon^2}\right|\|\hat{u}\|_{L^2} \\
\leq & \|\nabla\hat{u}-p_h\|_{L^2}+C_{h,k}\|\hat{g}(\hat{u})-\hat{g}_h(\hat{u})\|_{L^2}+C_{h,\gamma}^2C_{e,2}\left|\frac{\delta}{\varepsilon^2}\right|\|\hat{u}\|_{L^2}.
\end{aligned}
$$

Next, we consider how to calculate the constant $C_{h,\gamma}$. From the minimization principle, we obtain

$$
\begin{aligned}
\|u-\mathcal{P}_{h,\gamma}u\|_\gamma^2 &\leq \|u-\mathcal{P}_hu\|_\gamma^2 \\
&= \|\nabla(u-\mathcal{P}_hu)\|_{L^2}^2+\gamma\|u-\mathcal{P}_hu\|_{L^2}^2.
\end{aligned}
$$

If $\gamma \geq 0$, then the error is given by

$$
\begin{aligned}
\|u - \mathcal{P}_{h,\gamma} u\|_\gamma^2 &\leq \|\nabla(u - \mathcal{P}_h u)\|_{L^2}^2 + \gamma C_h^2 \|\nabla(u - \mathcal{P}_h u)\|_{L^2}^2 \\
&= (1 + \gamma C_h^2)\|u - \mathcal{P}_h u\|_{H_0^1}^2 \\
&\leq (1 + \gamma C_h^2) C_h^2 \|Au\|_{L^2}^2 \\
&\leq (1 + \gamma C_h^2) C_h^2 \|(A + \gamma)u\|_{L^2}^2.
\end{aligned}
$$

The constant $C_{h,\gamma}$ is given by $C_{h,\gamma} = C_h\sqrt{1 + \gamma C_h^2}$. In the case of $\gamma < 0$, from the eigenvalue problem (21), estimation of $C_{h,\gamma}$ has changed,

$$
\begin{aligned}
\|v - \mathcal{P}_{h,\gamma} v\|_\gamma^2 &\leq \|\nabla(v - \mathcal{P}_h v)\|_{L^2}^2 \\
&= \|u - \mathcal{P}_h v\|_{H_0^1}^2 \\
&\leq C_h^2 \|Av\|_{L^2}^2 \\
&\leq C_h^2 K^2 \|(A + \gamma)v\|_{L^2}^2.
\end{aligned}
$$

The constant $C_{h,\gamma}$ is given by $C_{h,\gamma} = C_h K$.

2.4.3. The nonlinear function $C_3(\rho)$. In this part, we consider the nonlinear function of (5). It follows that

$$
\begin{aligned}
& \sup_{w\in W}\left\|\int_0^1 \left(\mathcal{F}'[\hat{u}+tw]-\mathcal{F}'[\hat{u}]\right) w dt\right\|_{H^{-1}} \\
\le\ & \sup_{w\in W}\int_0^1 \left\|\left(\mathcal{F}'[\hat{u}+tw]-\mathcal{F}'[\hat{u}]\right) w\right\|_{H^{-1}} dt \\
\le\ & \sup_{w\in W}\int_0^1 \left\|\mathcal{F}'[\hat{u}+tw]-\mathcal{F}'[\hat{u}]\right\|_{L(H_0^1,H^{-1})} dt\|w\|_{H_0^1} \\
=\ & \sup_{w\in W}\int_0^1 \sup_{\phi,\psi\in H_0^1(\Omega)\backslash\{0\}} \frac{|\langle(\mathcal{N}'[\hat{u}+tw]-\mathcal{N}'[\hat{u}])\phi,\psi\rangle|}{\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}} dt\|w\|_{H_0^1} \\
=\ & \sup_{w\in W}\int_0^1 \sup_{\phi,\psi\in H_0^1(\Omega)\backslash\{0\}} \frac{|((g'[\hat{u}+tw]-g'[\hat{u}])\phi,\psi)_{L^2}|}{\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}} dt\|w\|_{H_0^1} \\
=\ & \sup_{w\in W}\int_0^1 \sup_{\phi,\psi\in H_0^1(\Omega)\backslash\{0\}} \frac{|((f'[\hat{u}+tw]-\delta B-f'[\hat{u}]+\delta B)\phi,\psi)_{L^2}|}{\varepsilon^2\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}} dt\|w\|_{H_0^1} \\
=\ & \sup_{w\in W}\int_0^1 \sup_{\phi,\psi\in H_0^1(\Omega)\backslash\{0\}} \frac{|((f'[\hat{u}+tw]-f'[\hat{u}])\phi,\psi)_{L^2}|}{\varepsilon^2\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}} dt\|w\|_{H_0^1}.
\end{aligned}
$$

For example, we put $f(u)=\alpha u+\beta u^2+\gamma u^3$, where $\alpha,\ \beta,\ \gamma\in\mathbb{R}$. From Hölder's inequality and Sobolev's embedding theorem, $C_3(\rho)$ is derived by

$$
\begin{aligned}
& \sup_{w\in W}\left\|\int_0^1 \left(\mathcal{F}'[\hat{u}+tw]-\mathcal{F}'[\hat{u}]\right) w dt\right\|_{H^{-1}} \\
\le\ & \sup_{w\in W}\int_0^1 \sup_{\phi,\psi\in H_0^1(\Omega)\backslash\{0\}} \frac{|((\alpha+2\beta(\hat{u}+tw)+3\gamma(\hat{u}+tw)^2-(\alpha+2\beta\hat{u}+3\gamma\hat{u}^2))\phi,\psi)_{L^2}|}{\varepsilon^2\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}} dt\|w\|_{H_0^1} \\
=\ & \sup_{w\in W}\int_0^1 \sup_{\phi,\psi\in H_0^1(\Omega)\backslash\{0\}} \frac{|((2\beta tw+3\gamma(2\hat{u}tw+t^2w^2))\phi,\psi)_{L^2}|}{\varepsilon^2\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}} dt\|w\|_{H_0^1} \\
\le\ & \sup_{w\in W}\int_0^1 \sup_{\phi,\psi\in H_0^1(\Omega)\backslash\{0\}} \frac{|(2\beta tw\phi,\psi)_{L^2}|+|(3\gamma(2\hat{u}t+t^2w)w\phi,\psi)_{L^2}|}{\varepsilon^2\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}} dt\|w\|_{H_0^1} \\
\le\ & \frac{1}{\varepsilon^2}\sup_{w\in W}\int_0^1 \left(\left(C_{e,3}^3|\beta|+C_{e,4}^4|3\gamma|\|\hat{u}\|_{H_0^1}\right)|2t|\|w\|_{H_0^1}+C_{e,4}^4|\gamma||3t^2|\|w\|_{H_0^1}^2\right) dt\|w\|_{H_0^1} \\
\le\ & \frac{1}{\varepsilon^2}\sup_{w\in W}\left(\left(C_{e,3}^3|\beta|+3C_{e,4}^4|\gamma|\|\hat{u}\|_{H_0^1}\right)\|w\|_{H_0^1}+C_{e,4}^4|\gamma|\|w\|_{H_0^1}^2\right)\|w\|_{H_0^1} \\
\le\ & \frac{1}{\varepsilon^2}\left(\left(C_{e,3}^3|\beta|+3C_{e,4}^4|\gamma|\|\hat{u}\|_{H_0^1}\right)\rho+C_{e,4}^4|\gamma|\rho^2\right)\rho =: C_3(\rho). \qquad (32)
\end{aligned}
$$

2.4.4. The nonlinear function $C_4(\rho)$. In this part, we consider the nonlinear function of (6). It follows that

$$\begin{aligned}
&\sup_{w\in W}\|\mathcal{F}'[\hat{u}+w]-\mathcal{F}'[\hat{u}]\|_{L(H_0^1,H^{-1})}\\
=\;&\sup_{w\in W}\sup_{\phi\in H_0^1(\Omega)\backslash\{0\}}\frac{\|(\mathcal{F}'[\hat{u}+w]-\mathcal{F}'[\hat{u}])\phi\|_{H^{-1}}}{\|\phi\|_{H_0^1}}\\
=\;&\sup_{w\in W}\sup_{\phi,\psi\in H_0^1(\Omega)\backslash\{0\}}\frac{|((g'[\hat{u}+w]-g'[\hat{u}])\phi,\psi)_{L^2}|}{\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}}\\
=\;&\sup_{w\in W}\sup_{\phi,\psi\in H_0^1(\Omega)\backslash\{0\}}\frac{|((f'[\hat{u}+w]-\delta B-f'[\hat{u}]+\delta B)\phi,\psi)_{L^2}|}{\varepsilon^2\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}}\\
=\;&\sup_{w\in W}\sup_{\phi,\psi\in H_0^1(\Omega)\backslash\{0\}}\frac{|((f'[\hat{u}+w]-f'[\hat{u}])\phi,\psi)_{L^2}|}{\varepsilon^2\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}}
\end{aligned}$$

For example, we put $f(u)=\alpha u+\beta u^2+\gamma u^3$, where $\alpha,\ \beta,\ \gamma\in\mathbb{R}$. We have an upper bound using the same transformation in (32). The nonlinear function $C_4(\rho)$ is described by

$$\begin{aligned}
&\sup_{w\in W}\left\|\mathcal{F}'[\hat{u}+w]-\mathcal{F}'[\hat{u}]\right\|_{L(H_0^1,H^{-1})}\\
\le\;&\sup_{w\in W}\sup_{\phi,\psi\in H_0^1(\Omega)\backslash\{0\}}\frac{|((\alpha+2\beta(\hat{u}+w)+3\gamma(\hat{u}+w)^2-((\alpha+2\beta\hat{u}+3\gamma\hat{u}^2)\phi,\psi)_{L^2}|}{\varepsilon^2\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}}\\
=\;&\sup_{w\in W}\sup_{\phi,\psi\in H_0^1(\Omega)\backslash\{0\}}\frac{|((2\beta w+3\gamma(2\hat{u}w+w^2))\phi,\psi)_{L^2}|}{\varepsilon^2\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}}\\
\le\;&\frac{2}{\varepsilon^2}(C_{e,3}^3|\beta|+3C_{e,4}^4|\gamma|\|\hat{u}\|_{H_0^1})\rho+\frac{3}{\varepsilon^2}C_{e,4}^4|\gamma|\rho^2=:C_4(\rho).
\end{aligned}$$

2.5. Error estimate for linear equation

In this section, we consider a method of calculating the error estimate $\|v^*-\hat{v}\|_{H_0^1}$. Let $\hat{u},\ \hat{v}\in X_h$ be computable approximate solutions of (8) and (9). We define $\kappa(\hat{v}):=\hat{u}-\gamma\hat{v}\in L^2(\Omega)$ and $\kappa_h(\hat{v}):=\mathcal{P}_{h,k}\kappa(\hat{v})\in M_h$. By using the Raviart-Thomas mixed finite element in the same way as in Subsection 2.4.2, for $q_h\in W_{\kappa_h(\hat{v})}$, we obtain from (22),

$$
\begin{aligned}
& \|v^* - \hat{v}\|_{H_0^1} \\
\leq\ & \|\mathcal{L}^{-1}\|_{L(H^{-1},H_0^1)} \|\mathcal{L}v^* - \mathcal{L}\hat{v}\|_{H^{-1}} \\
\leq\ & K\|\mathcal{I}u^* - \mathcal{I}\hat{u} + \mathcal{I}\hat{u} - \mathcal{L}\hat{v}\|_{H^{-1}} \\
\leq\ & K(\|\mathcal{I}u^* - \mathcal{I}\hat{u}\|_{H^{-1}} + \|\mathcal{I}\hat{u} - \mathcal{L}\hat{v}\|_{H^{-1}}) \\
=\ & K\left(\sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|\langle \mathcal{I}u^* - \mathcal{I}\hat{u}, w\rangle|}{\|w\|_{H_0^1}} + \sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|\langle \mathcal{I}\hat{u} - \mathcal{L}\hat{v}, w\rangle|}{\|w\|_{H_0^1}} \right) \\
=\ & K\left(\sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(u^* - \hat{u}, w)_{L^2}|}{\|w\|_{H_0^1}} + \sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\nabla\hat{v}, \nabla w)_{L^2} + \gamma(\hat{v}, w)_{L^2} - (\hat{u}, w)_{L^2}|}{\|w\|_{H_0^1}} \right) \\
\leq\ & K\Bigg(C_{e,2}^2\|u^* - \hat{u}\|_{H_0^1} + \sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\nabla\hat{v} - q_h, \nabla w)_{L^2}|}{\|w\|_{H_0^1}} \\
& \quad + \sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\mathrm{div}\ q_h + \kappa(\hat{v}), w)_{L^2}|}{\|w\|_{H_0^1}} \Bigg) \\
\leq\ & K\left(C_{e,2}^2\rho + \|\nabla\hat{v} - q_h\|_{L^2} + \sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\mathrm{div}\ q_h + \kappa_h(\hat{v}) + \kappa(\hat{v}) - \kappa_h(\hat{v}), w)_{L^2}|}{\|w\|_{H_0^1}} \right) \\
\leq\ & K\left(C_{e,2}^2\rho + \|\nabla\hat{v} - q_h\|_{L^2} + \sup_{w\in H_0^1(\Omega)\backslash\{0\}} \frac{|(\kappa(\hat{v}) - \kappa_h(\hat{v}), w - \mathcal{P}_{h,k}w)_{L^2}|}{\|w\|_{H_0^1}} \right) \\
\leq\ & C_{e,2}^2 K\rho + K\|\nabla\hat{v} - q_h\|_{L^2} + C_{h,k}K\|\kappa(\hat{v}) - \kappa_h(\hat{v})\|_{L^2}.
\end{aligned}
$$

Chapter 3

An algorithm of identifying parameters satisfying a sufficient condition

3.1. Notation and Definitions

For $a, b \in \mathbb{R}$ satisfying $-\infty < a < b < \infty$, $[a, b]$ denotes an interval $[a, b] := \{x \in \mathbb{R} : a \le x \le b\}$. Let $\mathbb{IR}$ be the set of intervals in $\mathbb{R}$. For $\mathbf{x} \in \mathbb{IR}$, $\sup(\mathbf{x}) \in \mathbb{R}$ denotes $y \in \mathbf{x}$ satisfying $x \le y$ for all $x \in \mathbf{x}$. Similarly, $\inf(\mathbf{x}) \in \mathbb{R}$ denotes $y \in \mathbf{x}$ satisfying $y \le x$ for all $x \in \mathbf{x}$. Let $\mathbb{F}$ be a set of floating-point numbers conforming IEEE 754 standard [**2, 3**]. The floating-point predecessor and successor of a real number $x \in \mathbb{R}$ are defined by $\mathrm{pred}(x) := \max\{f \in \mathbb{F} : f < x\}$ and $\mathrm{succ}(x) := \min\{f \in \mathbb{F} : f > x\}$, respectively.

Let us define two nonlinear functions $g_1, g_2 : \mathbb{R}^+ \to \mathbb{R}$ by

$$g_1(\rho) := C_1 C_3(\rho) - \rho + C_1 C_2$$

and

$$g_2(\rho) := C_1 C_4(\rho) - 1,$$

respectively. We will propose an algorithm of identifying inner inclusions of regions Γ^e and Γ^u, where

$$\Gamma^e := \left\{\rho \in \mathbb{R}^+ : g_1(\rho) \le 0\right\}$$

and

$$\Gamma^u := \left\{\rho \in \mathbb{R}^+ : g_2(\rho) < 0\right\}$$

respectively. By definition, if $\rho \in \Gamma^e \cap \Gamma^u$, ρ satisfies the sufficient condition (1.1) of Theorem 1.1. It is obvious that $\underline{\rho} = \min\{\rho : \rho \in \Gamma^e \cap \Gamma^u\}$ and $\overline{\rho} = \max\{\rho : \rho \in \Gamma^e \cap \Gamma^u\}$.

3.2. Algorithm of generating inner inclusions of Γ^e and Γ^u

In this section, we propose an algorithm of obtaining such inner inclusions of Γ^e and Γ^u based on Moore-Jones's algorithm of finding all solutions of one dimensional nonlinear equations proposed in [**16**], which is based on Krawczyk's operator [**12, 13**].

For Moore-Jones's algorithm and Krawczyk's operator, see Appendix D. Figure 3.1 (a) illustrates a behavior of the function g_1 in case that there exist ρ's satisfying $g_1(\rho) = 0$. Let r be a positive real number. On the interval $[0, r]$, we first identify all solution of $g_1(\rho) = 0$ by Moore-Jones's algorithm.[1] For example, we usually put $r = \|\hat{u}\|_{L^\infty}$ because $\Gamma^e \cap \Gamma^u$ relates to a maximum value of an approximate solution $\hat{u}$. Let $I_i (i = 1, 2, \cdots, n)$ be intervals such that each I_i contains one and only one positive solution ρ_i of $g_1(\rho) = 0$. We assume that $I_i \cap I_j = \phi$ for $1 \leq i < j \leq n$. If $g_1(\rho) = 0$ have no solutions, then there does not exist any $\rho > 0$ satisfying (7) provided that Γ^e is included in $[0, r]$. Figure 3.1 (b) shows an example of this case. Let $\bar{g}_1 : \mathbb{IR} \to \mathbb{IR}$ be an interval extension of the nonlinear function g_1. Put $I_0 = [0, 0] \in \mathbb{IR}$ and $I_{n+1} = [r, r] \in \mathbb{IR}$. We show now how to construct an inner inclusion of $\Gamma^e := \{\rho \in [0, r] : g_1(\rho) \leq 0\}$. We first calculate

$$d_i := \bar{g}_1 \left(\frac{\inf(I_{i+1}) + \sup(I_i)}{2} \right) \in \mathbb{IR},$$

for $i = 0, 1, \cdots, n$. Put

$$I_i^e = \begin{cases} [\sup(I_i), \inf(I_{i+1})] \text{ if } \sup(d_i) \leq 0, \\ \phi \text{ if } \sup(d_i) > 0. \end{cases}$$

An inner inclusion $\tilde{\Gamma}^e$ of Γ^e is given as

$$\tilde{\Gamma}^e = \bigcup_{i=0}^{n} I_i^e.$$

Algorithm 1 summaries this procedure of calculating $\tilde{\Gamma}^e$.

[1] In this thesis, we assume that all solutions of $g_1(\rho) = 0$ can be obtained by Moore-Jones's algorithm. For the propose, we assume $C_3(\rho)$ is C^2 with respect to ρ. In this case, by Sard's lemma, with probability one we can choose a small negative ε which is a regular value of g_1 [**37**]. If Moore-Jones's algorithm fails to find all solutions, then instead of $g_1(\rho) = 0$ considering $g_1(\rho) = \varepsilon$, without loss of generality, we can assume that all solutions of $g_1(\rho) = 0$ can be obtained by Moore-Jones's algorithm.

On the interval $[\inf(\tilde{\Gamma}^e), r]$, we then identify all solution of $g_2(\rho) = 0$ by Moore-Jones's algorithm.[2] Let $I_i (i = 1, 2, \cdots, n)$ be intervals such that each I_i contains one and only one positive solution ρ_i of $g_2(\rho) = 0$. We assume that $I_i \cap I_j = \phi$ for $1 \leq i < j \leq n$. If $g_2(\rho) = 0$ have no solutions, then there does not exist any $\rho > 0$ satisfying (7) provided that Γ^u is included in $[\inf(\tilde{\Gamma}^e), r]$. Let $\bar{g}_2 : \mathbb{IR} \to \mathbb{IR}$ be an interval extension of nonlinear functions g_2. Put $I_0 = [\inf(\tilde{\Gamma}^e), \inf(\tilde{\Gamma}^e)] \in \mathbb{IR}$ and $I_{n+1} = [r, r] \in \mathbb{IR}$. We show now how to construct an inner inclusion of $\Gamma^u := \{\rho \in [\inf(\tilde{\Gamma}^e), r] : g_2(\rho) < 0\}$. We first calculate

$$d_i := \bar{g}_2 \left(\frac{\inf(I_{i+1}) + \sup(I_i)}{2} \right) \in \mathbb{IR},$$

for $i = 0, 1, \cdots, n$. Put

$$I_i^u = \begin{cases} [\mathrm{succ}(\sup(I_i)), \mathrm{pred}(\inf(I_{i+1}))] \text{ if } \sup(d_i) < 0, \\ \phi \text{ if } \sup(d_i) \geq 0. \end{cases}$$

An inner inclusion $\tilde{\Gamma}^u$ of Γ^u is given as

$$\tilde{\Gamma}^u = \bigcup_{i=0}^{n} I_i^u.$$

Algorithm 2 summaries this procedure of calculating $\tilde{\Gamma}^u$.

It is clear that if $\rho \in \tilde{\Gamma}^e \cap \tilde{\Gamma}^u$, then ρ satisfies the sufficient condition (7) of Theorem 1.1. Finally, calculate

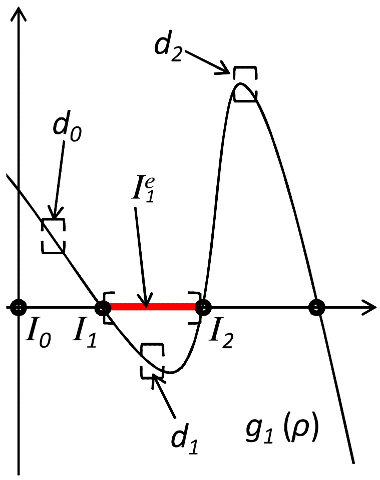
$$\rho_e = \min\{\rho : \rho \in \tilde{\Gamma}^e \cap \tilde{\Gamma}^u\}$$

and

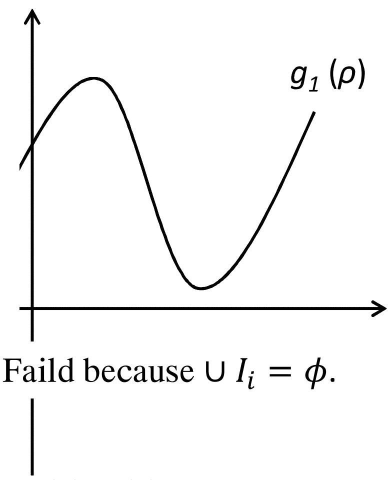
$$\rho_u = \max\{\rho : \rho \in \tilde{\Gamma}^e \cap \tilde{\Gamma}^u\}.$$

[2]Similarly, we also assume that all solutions of $g_2(\rho) = 0$ can be obtained by Moore-Jones's algorithm. For this end, we assume $C_4(\rho)$ is C^2 with respect to ρ. In this case, by Sard's lemma, with probability one we can choose small negative ε which is a regular value of g_2. If Moore-Jones's algorithm fails to find all solutions, then instead of $g_2(\rho) = 0$ considering $g_2(\rho) = \varepsilon$, without loss of generality, we can assume that all solutions of $g_2(\rho) = 0$ can be obtained by Moore-Jones's algorithm.

Then ρ_e and ρ_u become approximating of $\underline{\rho}$ and $\overline{\rho}$, respectively. If we search tighter inclusions of all solutions of $g_1(\rho) = 0$ and $g_2(\rho) = 0$ and if $\Gamma^e \cap \Gamma^u$ is include in $[0, r]$, then it is obvious ρ_e and ρ_u approach to $\underline{\rho}$ and $\overline{\rho}$, respectively.



(a) The function g_1 has a ρ satisfying $g_1(\rho) = 0$.



(b) $g_1(\rho) = 0$ have no solutions.

FIGURE 3.1. Example of the proposed algorithm.

Algorithm 1 of obtaining $\tilde{\Gamma}^e = \{\rho \in [0, r] : g_1(\rho) \le 0\}$.

Find $I_i \in \mathbb{IR}$ $(i = 1, 2, \cdots, n)$ that contain $\rho_i \in \mathbb{R}^+$ of $g_1(\rho_i) = 0$ in $[0, r]$.
if $\cup I_i$ is empty **then**
  error('Failure in verification');
end if
$I_0 = 0$;
$I_{n+1} = r$;
for $i = 0 : n$ **do**
  $d_i = \bar{g}_1((\inf(I_{i+1}) + \sup(I_i))/2)$;
  if $\sup(d_i) \le 0$ **then**
    $I_i^e = [\sup(I_i), \inf(I_{i+1})]$;
  else
    $I_i^e = \phi$
  end if
end for
Put $\tilde{\Gamma}^e = \cup_{i=1}^n I_i^e$
if $\tilde{\Gamma}^e$ is empty **then**
  error('Failure in verification');
end if

Algorithm 2 of obtaining $\tilde{\Gamma}^u = \{\rho \in [\inf(\tilde{\Gamma}^e), r] : g_2(\rho) < 0\}$.

Find $I_i \in \mathbb{IR}$ $(i = 1, 2, \cdots, n)$ that contain $\rho_i \in \mathbb{R}^+$ of $g_2(\rho_i) = 0$ in $[\inf(\tilde{\Gamma}^e), r]$.
if $\cup I_i$ is empty **then**
  error('Failure in verification');
end if
$I_0 = \inf(\tilde{\Gamma}^e)$;
$I_{n+1} = r$;
for $i = 0 : n$ **do**
  $d_i = \bar{g}_2((\inf(I_{i+1}) + \sup(I_i))/2)$;
  if $\sup(d_i) < 0$ **then**
    $I_i^u = [\mathrm{succ}(\sup(I_i)), \mathrm{pred}(\inf(I_{i+1}))]$;
  else
    $I_i^u = \phi$
  end if
end for
Put $\tilde{\Gamma}^e = \cup_{i=1}^n I_i^e$
if $\tilde{\Gamma}^u$ is empty **then**
  error('Failure in verification');
end if

Chapter 4

Numerical results

In this chapter, we present some examples of numerical verification. All computations are carried on PC with 3.10 GHz Intel Xeon E5-2687W CPU, 128 G Byte RAM and Cent OS 6.3. We use MATLAB2012a with INTLAB version 6 [**30**], a toolbox for verified numerical computations. Gmsh [**6**] (`http://geuz.org/gmsh/`) is used for obtaining triangular mesh. Table 4.1 shows several notations which are used throughout this thesis.

Table 4.1. Explanation of variables.

C_1	Norm estimation for inverse of linearized operator in (3).
C_2	Residual norm in (4).
ρ_e	By Theorem 1.1, the existence of exact solution is proved in $\|u - \hat{u}\|_{H_0^1} \le \rho_e$.
ρ_u	By Theorem 1.1, the uniqueness of exact solution is proved in $\|u - \hat{u}\|_{H_0^1} \le \rho_u$.
ρ_v	An error bound $\|v - \hat{v}\|_{H_0^1} \le \rho_v$ via Theorem 1.1.
ζ_1	By Newton-Kantorovich's theorem, existence of a solution is proved in $\|u - \hat{u}\|_{H_0^1} \le \zeta_1$.
ζ_2	By Newton-Kantorovich's theorem, uniqueness of a solution is proved in $\|u - \hat{u}\|_{H_0^1} \le \zeta_2$.
N-K C	The verification condition of Newton Kantorovich's theorem. (To prove the existence, the parameter describing the verified condition should be less than or equal to 1/2.)

4.1. Illustrative examples of Algorithms 1 and 2

We first present results of numerical verifications of applying Theorem 1.1 with Algorithms 1 and 2 for some nonlinear elliptic Dirichlet boundary value problems. We also compare these results with those obtained by Newton-Kantorovich's theorem [**33**]. Let us consider the following Dirichlet problem of a nonlinear elliptic partial differential equation:

$$\begin{cases} -\Delta u = \lambda(u + u^2 - u^3) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{33}$$

where Ω is a bounded polygonal domain. From the discussion in [**33**], a weak formula for (33) can be represented as

$$\mathcal{F}(u) = 0, \tag{34}$$

where $\mathcal{F}$ is a nonlinear mapping from $H_0^1(\Omega) \to H^{-1}(\Omega)$. We now consider to apply Theorem 1.1 to (34). Several methods for calculating the constant C_1 and C_2 are proposed in [**24, 18, 19, 20, 25, 33**] and so on. In this Section 4.1, for the constant C_1, we use methods of calculating in [**24, 19**]. We also use a method presented in [**33**] for the calculation of C_2. M. Plum [**25, 27**] has presented how to construct functions $C_3(\rho)$ and $C_4(\rho)$.

We have calculated all approximate solutions by the finite element method with piecewise quadratic base functions on a regular triangulation. Here, we denote a mesh size by the second longest length of each side on a triangle element. In the following, we present verified results of (33) on convex and nonconvex domains.

4.1.1. Example 1. We first consider the semilinear elliptic equation (33) on the square domain $(0,1) \times (0,1)$ with an uniform mesh. The approximate solution $\hat{u}$ for $\lambda = 36$ is presented in Figure 4.1. Table 4.2 displays verification results of (33) for $\lambda = 35.7$, 36, 37.6 and 37.8. In the cases of $\lambda = 36$, 37.6 and 37.8, Newton-Kantorovich's theorem is failed. On the other hand, Theorem 1.1 succeeded to verify the existence of exact solutions in the cases of $\lambda = 36$ and 37.6. For $\lambda = 37.8$, both methods are failed. Algorithms 1 and 2 proved that there is no solution satisfying (7) for $\lambda = 37.8$. In Figure 4.2, we show shapes of nonlinear functions g_1 and g_2 for $\lambda = 36$. By Algorithms 1 and 2, we obtained $\tilde{\Gamma}^e = [2.782\times10^{-2}, 7.206\times10^{-2}]$ and $\tilde{\Gamma}^u = [2.782\times10^{-2}, 4.332\times10^{-2}]$ so that $\tilde{\Gamma}^e \cap \tilde{\Gamma}^u = [2.782\times10^{-2}, 4.332\times10^{-2}]$. If $\rho \in \tilde{\Gamma}^e \cap \tilde{\Gamma}^u$, ρ satisfies the sufficient condition (7) of Theorem 1.1. Table 4.3 shows computational time needed for verification. As seen from this table, the computational time needed for Algorithms 1 and 2 is less than 0.5 [sec] which is negligible compared with that for calculating C_1 and C_2 which is more than 200 [sec]. Effect of introducing Algorithms 1 and 2 is seen from the fact that Theorem 1.1 with Algorithms 1 and 2 yields better estimates than those of Newton-Kantorovich's theorem.

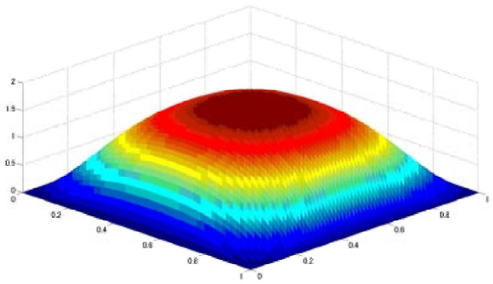


FIGURE 4.1. Approximate solution $\hat{u}$ for (33).

TABLE 4.2. Verification results of (33) ($h = 2^{-5}$).

λ	C_1	C_2	ρ_e	ρ_u	N-K C	ζ_1	ζ_2
35.7	2.120	9.289×10^{-3}	2.664×10^{-2}	4.398×10^{-2}	0.4858	3.370×10^{-2}	3.936×10^{-2}
36	2.138	9.406×10^{-3}	2.782×10^{-2}	4.332×10^{-2}	0.5063	Failed	Failed
37.6	2.237	1.004×10^{-2}	3.986×10^{-2}	3.991×10^{-2}	0.6299	Failed	Failed
37.8	2.251	1.012×10^{-2}	Failed	Failed	0.6472	Failed	Failed

TABLE 4.3. Computational time needed for verification of (33) $h = 2^{-5}$ with ([sec]).

λ	C_1	C_2	Algorithms 1 and 2
35.7	17.77	184.0	0.5000
36	20.27	183.0	0.1001
37.6	19.81	183.6	0.1258
37.8	18.17	184.3	0.0503

4.1.2. Example 2. Next, let us consider the case that Ω is a bounded nonconvex polygonal domain whose vertices are given by

$$\{(0.5, 0), (1, 0.5), (1, 1), (0.5, 0.75), (0, 1), (0, 0.5)\}.$$

Figure 4.3 shows this bounded nonconvex polygonal domain Ω. It is well-known that the solution does not have H^2-regularity at a reentrant corner. It causes slow convergence. In order to improve the accuracy of approximate solutions, we use a nonuniform mesh centered at that corner as shown in Figure 4.3. We consider the Dirichlet problem of the semilinear elliptic equation (33) on this nonconvex domain Ω using this nonuniform mesh. For $\lambda = 35$ and 45, approximate solutions $\hat{u}$ are shown in Figure 4.4. In the cases of $\lambda = 35$ and 45, verification results of (33) are shown in

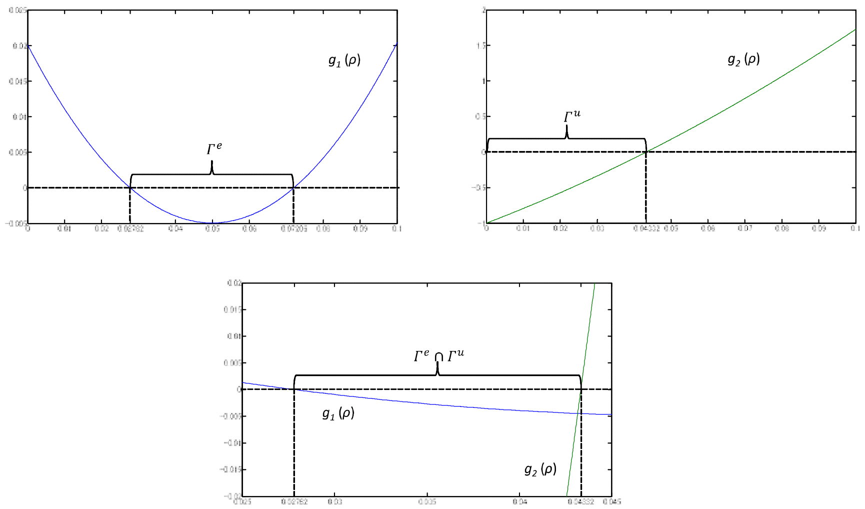


FIGURE 4.2. Nonlinear functions g_1 and g_2 for $\lambda = 36$.

Table 4.4. Although Theorem 1.1 proved the existence of a solution, verification by Newton-Kantorovich's theorem for $\lambda = 45$ was failed. Table 4.5 shows computational time needed for verification. As seen from this table, computational time needed for Algorithms 1 and 2 is less than 0.11 [sec] which is negligible compared with that for calculating C_1 and C_2 which is more than 10^4 [sec]. As the previous example, effect of introducing Algorithms 1 and 2 is seen from the fact that Theorem 1.1 with Algorithms 1 and 2 yields better estimates than those of Newton-Kantorovich's theorem.

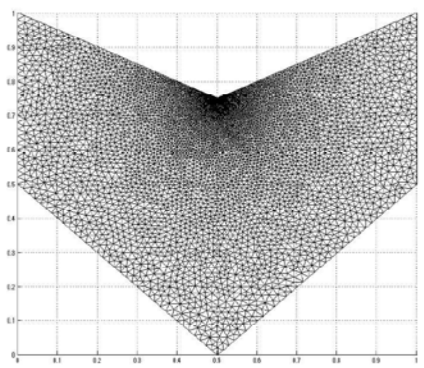
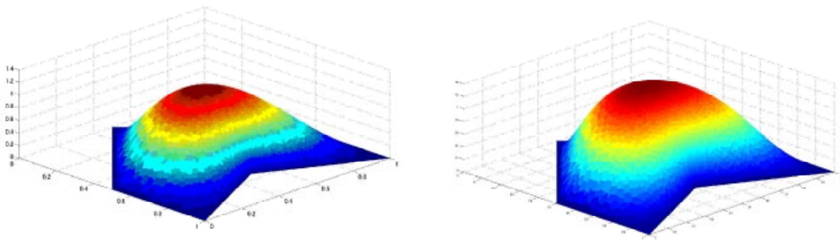


FIGURE 4.3. Bounded nonconvex domain Ω ($1.646 \times 10^{-3} \leq h \leq 2.644 \times 10^{-2}$).



(a) $\lambda = 35$ (b) $\lambda = 45$

FIGURE 4.4. Approximate solution $\hat{u}$ of (33).

TABLE 4.4. Verification result of (33).

λ	C_1	C_2	ρ_e	ρ_u	N-K C	ζ_1	ζ_2
35	2.747	7.909×10^{-3}	2.510×10^{-2}	6.319×10^{-2}	0.1159	2.597×10^{-2}	4.346×10^{-2}
45	2.474	1.173×10^{-2}	4.218×10^{-2}	5.105×10^{-2}	0.5527	Failed	Failed

TABLE 4.5. Computational time needed for verification of (33) with ([sec]).

λ	C_1	C_2	Algorithms 1 and 2
35	2.468×10^3	1.438×10^4	0.0994
45	2.483×10^3	1.437×10^4	0.1095

4.2. NUMERICAL EXAMPLE FOR SYSTEMS OF ELLIPTIC PDEs (CASE OF A NONLINEAR TERM $f(u) = u - u^3$)

Let us consider the following Dirichlet boundary value problem of a system of nonlinear elliptic partial differential equations:

$$\begin{cases} -\Delta u = 100(u - u^3 - v) & \text{in } \Omega, \\ -\Delta v = u - \gamma v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{35}$$

where Ω is bounded polygonal domains. From the discussion in Chapter 2, we can test a numerical existence of solutions for (35).

We have calculated all approximate solutions by the finite element method with piecewise quadratic base functions on a regular triangulation. Here, we denote a mesh size by the second longest length of each side on a triangle element. In the following, we present verified results of (35) on nonconvex domains.

4.2.1. Example 1. Let us consider the case that Ω is a bounded nonconvex polygonal domain whose vertices are given by

$$\{(0.5, 0), (1, 0.5), (1, 1), (0.5, 0.75), (0, 1), (0, 0.5)\}.$$

Figure 4.5 shows this bounded nonconvex polygonal domain Ω. It is also known that an exact solution does not have H^2-regularity at a reentrant corner. It causes slow convergence. In order to improve the accuracy of approximate solutions, we used a nonuniform mesh centered at that corner as shown in Figure 4.5. In Figure 4.6,

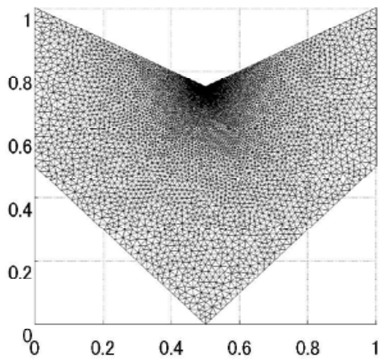


FIGURE 4.5. Bounded nonconvex domain Ω ($7.270 \times 10^{-4} \leq h \leq 2.696 \times 10^{-2}$).

we show approximate solutions tin the case of $\gamma = 1.2$. In Figure 4.7, we also show approximate solutions tin the case of $\gamma = -1.2$. Using Theorem 1.1 with Algorithms 1 and 2, verification results for (35) on Ω such as Figure 4.5 are given in Table 4.6. From Table 4.6, we succeed to prove the existence and local uniqueness of solutions which are located in neighborhood of these approximate solutions. Verification results for (35) using Newton-Kantorovich's theorem are also shown in Table 4.7. Comparing these two tables, Theorem 1.1 with Algorithms 1 and 2 gives shaper error bounds and larger regions of uniqueness. Table 4.8 shows computational time needed for verification. As seen from this table, computational time needed for Algorithms 1 and 2 is less than 0.31 [sec] which is negligible compared with that for calculating C_1 and C_2 which is more than 27800 [sec].

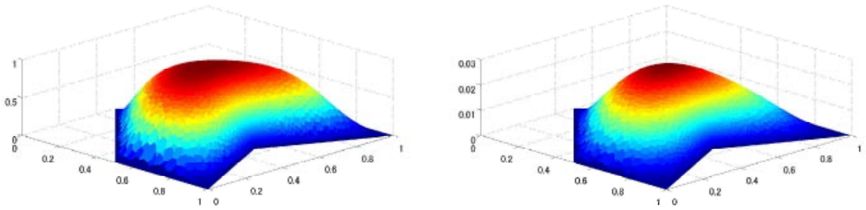


FIGURE 4.6. Approximate solution $\hat{u}$ (left) and $\hat{v}$ (right) of (35) ($\gamma = 1.2$).

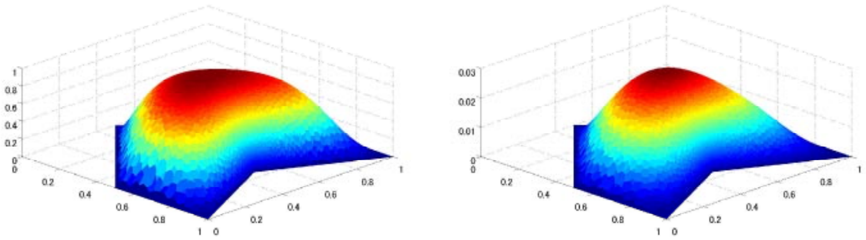


FIGURE 4.7. Approximate solution $\hat{u}$ (left) and $\hat{v}$ (right) of (35) ($\gamma = -1.2$).

TABLE 4.6. Verification results of (35) using Theorem 1.1 with Algorithms 1 and 2.

γ	C_1	C_2	ρ_e	ρ_u	ρ_v
1.2	1.855	9.932×10^{-3}	2.146×10^{-2}	4.823×10^{-2}	7.248×10^{-4}
-1.2	1.853	9.918×10^{-3}	2.139×10^{-2}	4.829×10^{-2}	7.349×10^{-4}

TABLE 4.7. Verification results of (35) using Newton-Kantorovich's theorem.

γ	N-K C	ζ_1	ζ_2
1.2	0.3794	2.471×10^{-2}	3.684×10^{-2}
-1.2	0.3777	2.459×10^{-2}	3.676×10^{-2}

4.2.2. Example 2. Next, let us consider the case that Ω is a bounded nonconvex polygonal domain whose vertices are given by

$$\{(0,0), (0.2,0), (0.2,0.4), (0.8,0.4), (0.8,0), (1,0), (1,0.5), (0.5,1), (0.1,0)\}.$$

TABLE 4.8. Computational time needed for verification of (35) with ([sec]).

γ	C_1	C_2	Algorithms 1 and 2
1.2	2.038×10^3	2.174×10^4	0.3032
-1.2	2.056×10^3	2.174×10^4	0.1155

Figure 4.8 shows this bounded nonconvex polygonal domain Ω. It is also known that an exact solution does not have H^2-regularity at a reentrant corner. It causes slow convergence. In order to improve the accuracy of approximate solutions, we used a nonuniform mesh centered at that corner as shown in Figure 4.8. In Figure 4.9, we

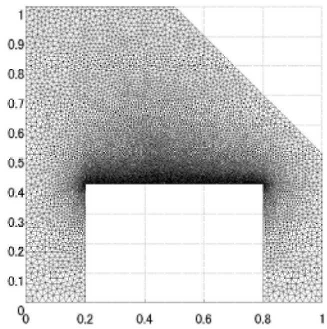


FIGURE 4.8. Bounded nonconvex domain Ω ($1.625 \times 10^{-3} \leq h \leq 2.686 \times 10^{-2}$).

show approximate solutions tin the case of $\gamma = 1.2$. In Figure 4.10, we also show approximate solutions tin the case of $\gamma = -1.2$. Using Theorem 1.1 with Algorithms 1 and 2, verification results for (35) on Ω such as Figure 4.8 are given in Table 4.9. From Table 4.9, we succeed to prove the existence and local uniqueness of solutions which are located in neighborhood of these approximate solutions. Verification results for (35) using Newton-Kantorovich's theorem are also shown in Table 4.10. Although Theorem 1.1 can prove the existence of a solution, verification by Newton-Kantorovich's theorem is failed. Table 4.11 shows computational time needed for verification. As seen from this table, computational time needed for Algorithms 1 and 2 is less than 0.33 [sec] which is negligible compared with that for calculating C_1 and C_2 which is more than 85200 [sec].

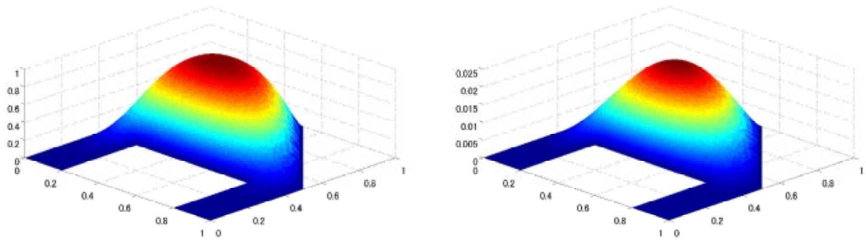


FIGURE 4.9. Approximate solution $\hat{u}$ (left) and $\hat{v}$ (right) of (35) ($\gamma = 1.2$).

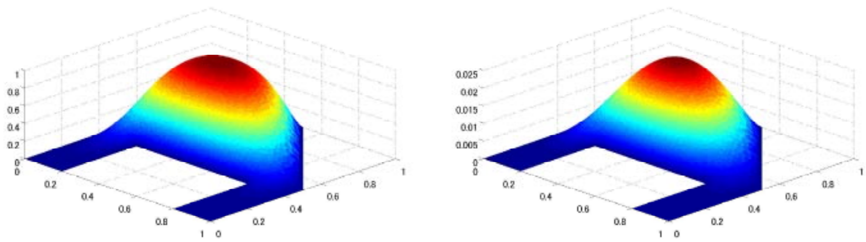


FIGURE 4.10. Approximate solution $\hat{u}$ (left) and $\hat{v}$ (right) of (35) ($\gamma = -1.2$).

TABLE 4.9. Verification results of (35) using Theorem 1.1 with Algorithms 1 and 2.

γ	C_1	C_2	ρ_e	ρ_u	ρ_v
1.2	2.048	1.245×10^{-2}	3.219×10^{-2}	4.735×10^{-2}	8.961×10^{-4}
-1.2	2.047	1.244×10^{-2}	3.210×10^{-2}	4.738×10^{-2}	9.060×10^{-4}

TABLE 4.10. Verification results of (35) using Newton-Kantorovich's theorem.

γ	N-K C	ζ_1	ζ_2
1.2	0.5133	Failed	Failed
-1.2	0.5114	Failed	Failed

4.3. NUMERICAL EXAMPLE FOR SYSTEMS OF ELLIPTIC PDEs (CASE OF A NONLINEAR TERM $f(u) = u + u^2 - u^3$)

Now, let us consider the following Dirichlet boundary value problem of a system of nonlinear elliptic partial differential equations:

$$\begin{cases} -\Delta u = \lambda(u + u^2 - u^3 - v) & \text{in } \Omega, \\ -\Delta v = u - 1.2v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{36}$$

TABLE 4.11. Computational time needed for verification of (35) with ([sec]).

γ	C_1	C_2	Algorithms 1 and 2
1.2	8.258×10^3	7.697×10^4	0.1248
-1.2	8.277×10^3	7.714×10^4	0.3291

where Ω is an unit square, $(0,1) \times (0,1)$ or a bounded nonconvex polygonal domain. For (9), we apply Theorem 1.1 with Algorithms 1 and 2. From the discussion in Chapter 2, we can test a numerical existence of solutions for (36).

We have calculated all approximate solutions by the finite element method with piecewise quadratic base functions on a regular triangulation. Here, we denote a mesh size by the second longest length of each side on a triangle element. In the following, we present verification results for (36) on convex and nonconvex domains.

4.3.1. In case of square domain. We first consider the elliptic system (36) on the square domain $(0,1) \times (0,1)$. In Figure 4.11 and Figure 4.12, we show two approximate solutions, say Type I : $(\hat{u}_1, \hat{v}_1)$ and Type II : $(\hat{u}_2, \hat{v}_2)$, in the case of $\lambda = 17$. Verification results for (36) using Theorem 1.1 with the proposed algorithm are given in Table 4.12. Verification results for (36) using Newton-Kantorovich's theorem are also shown in Table 4.13. The mesh size is taken as $h = 2^{-5}$. Comparing these two tables, Theorem 1.1 with Algorithms 1 and 2 gives shaper error bounds and larger regions of uniqueness. Table 4.14 shows computational time needed for verification. As seen from this table, computational time needed for Algorithms 1 and 2 is less than 0.16 [sec] which is negligible compared with that for calculating C_1 and C_2 which is more than 850 [sec]. Also in this case, the effect of introducing Algorithms 1 and 2 is seen from the fact that Theorem 1.1 with Algorithms 1 and 2 yields better estimates than those of Newton-Kantorovich's theorem.

TABLE 4.12. Verification results of (36) using Theorem 1.1 with Algorithms 1 and 2 ($h = 2^{-5}$).

Type	C_1	C_2	ρ_e	ρ_u	ρ_v
I	6.993	2.101×10^{-3}	1.766×10^{-2}	4.206×10^{-2}	9.899×10^{-4}
II	12.41	1.101×10^{-3}	1.688×10^{-2}	3.403×10^{-2}	9.059×10^{-4}

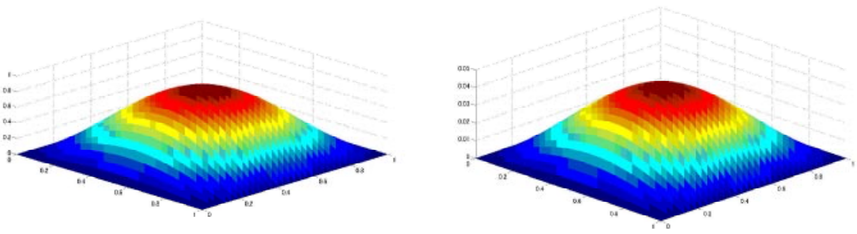


FIGURE 4.11. Type I : $\hat{u}_1$ (left) and $\hat{v}_1$ (right) of (36).

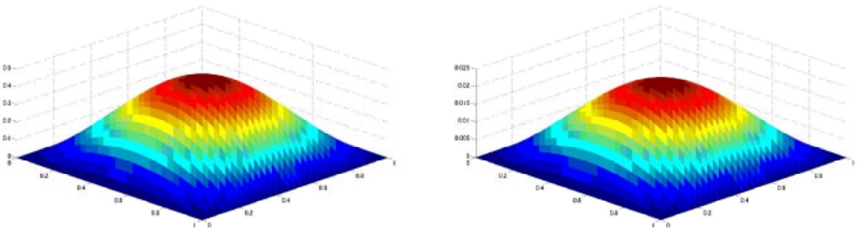


FIGURE 4.12. Type II : $\hat{u}_2$ (left) and $\hat{v}_2$ (right) of (36).

TABLE 4.13. Verification results of (36) using the method proposed in [**33**] ($h = 2^{-5}$).

Type	N-K C	ζ_1	ζ_2
I	0.2841	1.773×10^{-2}	2.938×10^{-2}
II	0.4858	1.692×10^{-2}	2.732×10^{-2}

TABLE 4.14. Computational time needed for verification of (36) $h = 2^{-5}$ with ([sec]).

Type	C_1	C_2	Algorithms 1 and 2
I	781.2	171.6	0.1588
II	738.8	169.8	0.1493

4.3.2. In case of a bounded nonconvex polygonal domain. Next, let us consider the case that Ω is a bounded nonconvex polygonal domain whose vertices are given by

$$\{(0.5, 0), (1, 0.5), (1, 1), (0.5, 0.75), (0, 1), (0, 0.5)\}.$$

Figure 4.13 shows this bounded nonconvex polygonal domain Ω. It is also known that an exact solution does not have H^2-regularity at a reentrant corner. It causes slow convergence. In order to improve the accuracy of approximate solutions, we used a nonuniform mesh centered at that corner as shown in Figure 4.13.

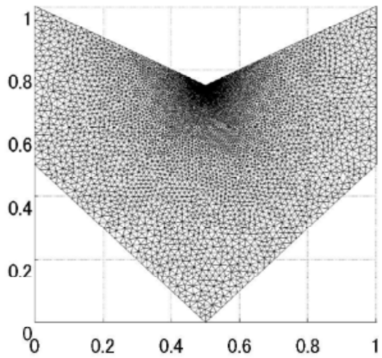


FIGURE 4.13. Bounded nonconvex domain Ω ($7.270 \times 10^{-4} \le h \le 2.696 \times 10^{-2}$).

We consider the nonlinear elliptic system (36) on this nonconvex domain Ω. For $\lambda = 35$, Figure 4.14 shows approximate solutions $\hat{u}$ and $\hat{v}$. Table 4.15 presents the result of verification based on Theorem 1.1 with Algorithms 1 and 2. We got $C_1 = 3.136$, $C_2 = 6.121 \times 10^{-3}$ for this nonuniform mesh triangulation. Based on Table 4.15, the verification parameter describing the sufficient condition of Newton-Kantorovich's theorem was calculated as 0.2642. For each case, Theorem 1.1 with Algorithms 1 and 2 yields better estimate than that by Newton-Kantorovich's theorem. Table 4.16 shows computational time needed for verification. As seen from this table, computational time needed for Algorithms 1 and 2 is less than 0.1 [sec] which is negligible compared with that for calculating C_1 and C_2 which is more than 10^4 [sec].

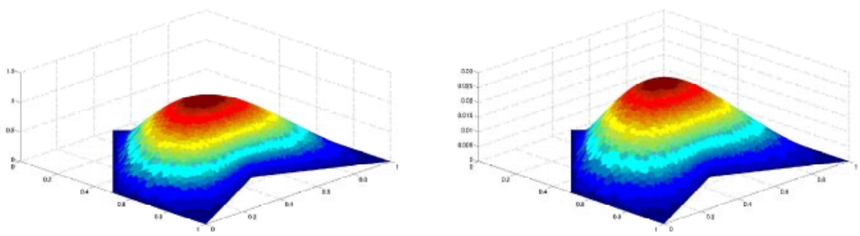


FIGURE 4.14. Approximate solution $\hat{u}$ (left) and $\hat{v}$ (right) of (36).

Table 4.15. Verification result for (36).

ρ_e	ρ_u	ρ_v	ζ_1	ζ_2
2.207×10^{-2}	5.814×10^{-2}	7.165×10^{-4}	2.276×10^{-2}	3.838×10^{-2}

Table 4.16. Computational time needed for verification of (36) with ([sec]).

C_1	C_2	Algorithms 1 and 2
1.790×10^4	1.776×10^4	0.0974

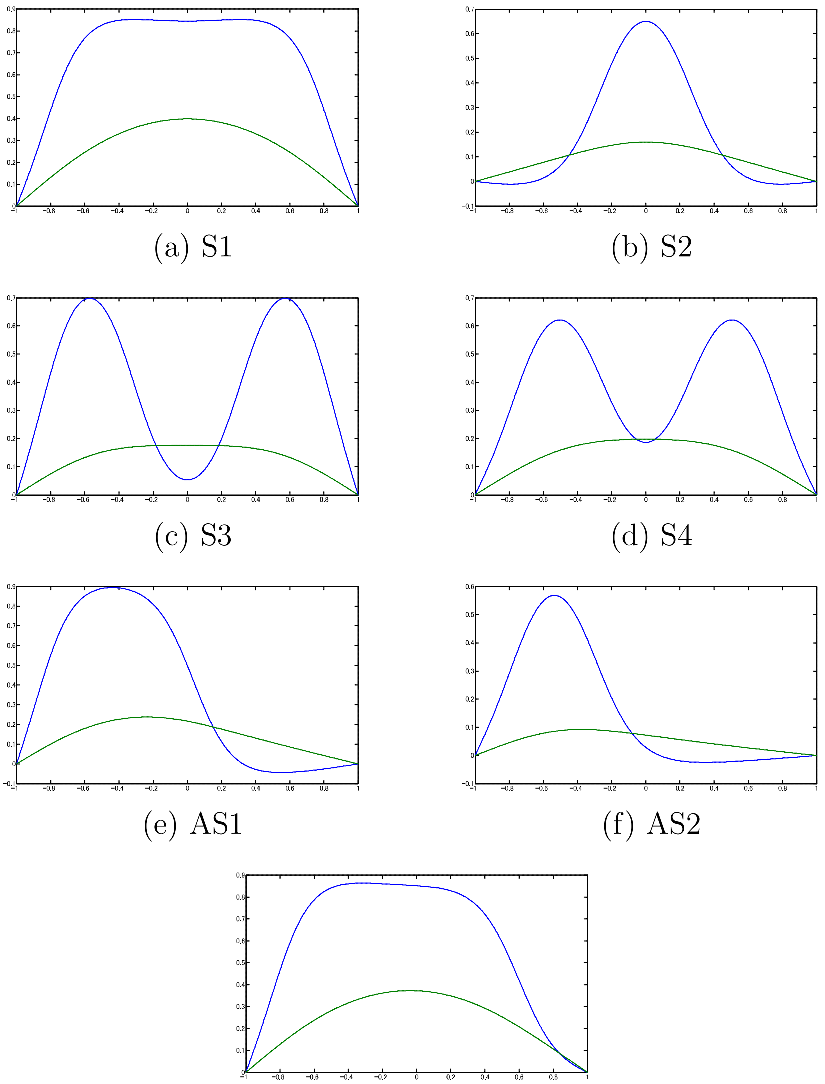
4.4. Numerical results for system of ordinary differential equations

Let us consider the following system of nonlinear ordinary differential equations:

$$\begin{cases} -\dfrac{d^2u}{dx^2} = \dfrac{1}{0.08^2}\left(-\dfrac{1}{4}u + \dfrac{5}{4}u^2 - u^3 - 0.2v\right) & \text{in } -1 < x < 1, \\ -\dfrac{d^2v}{dx^2} = u & \text{in } -1 < x < 1, \\ u(-1) = u(1) = v(-1) = v(1) = 0. \end{cases} \tag{37}$$

M.T. Nakao and Y. Watanabe have studied numerical verification methods, which are called FN-Int and IN-Linz, of the above system in [**21**]. For (37), we also apply the proposal numerical verification method in Chapter 2. We calculated approximate solutions by the finite element method with piecewise quadratic base functions. Let N be a number of equidistant partitions for the interval [-1,1].

Approximate solutions $\hat{u}$ and $\hat{v}$ are shown in Figure 4.15. Verification results for (37) are provided in Table 4.17. In Table 4.18, we present verification results through Newton-Kantorovich's theorem. For five approximate solutions S1, S3, AS1, AS2 and AS3, Theorem 1.1 with Algorithms 1 and 2 proved the existence and local uniqueness of each solution. However, for two approximate solutions S2 and S4, both methods failed. For all approximate solutions, M.T. Nakao and Y. Watanabe [**21**] have succeeded to prove the existence of solutions which are located in neighborhood of these approximate solutions using Nakao's IN-Linz method, which is based on Schauder's fixed point theory. Although M.T. Nakao and Y. Watanabe [**21**] do not give the uniqueness results for AS1 and AS3, for these problems, the local uniqueness of each solution can be proved using Theorem 1.1 with Algorithms 1 and 2. Table 4.19 shows computational time needed for verification. As seen from this table, computational time needed for Algorithms 1 and 2 is less than 0.18 [sec] which is negligible compared with that for calculating C_1 and C_2 which is more than 10 [sec].



(a) S1 (b) S2

(c) S3 (d) S4

(e) AS1 (f) AS2

(g) AS3

FIGURE 4.15. Approximate solutions $\hat{u}$ (blue) and $\hat{v}$ (green) for (37).

TABLE 4.17. Verification result of (37) using Theorem 1.1 with Algorithms 1 and 2.

Type	N	C_1	C_2	ρ_e	ρ_u	ρ_v
S1	512	3.282	4.410×10^{-5}	1.648×10^{-4}	6.764×10^{-4}	7.545×10^{-5}
S2	1024	13.90	2.034×10^{-5}	Failed	Failed	Failed
S3	512	4.162	4.552×10^{-5}	2.650×10^{-4}	4.646×10^{-4}	1.117×10^{-4}
S4	1024	7.521	2.803×10^{-5}	Failed	Failed	Failed
AS1	1024	4.857	3.436×10^{-5}	2.236×10^{-4}	4.404×10^{-4}	1.002×10^{-4}
AS2	512	5.202	2.621×10^{-5}	1.585×10^{-4}	5.661×10^{-4}	6.613×10^{-5}
AS3	512	3.643	4.179×10^{-5}	1.765×10^{-4}	6.413×10^{-4}	7.945×10^{-5}

TABLE 4.18. Verification result of (37) using the method proposed in [**33**].

Type	N-K C	ζ_1	ζ_2
S1	0.2139	1.648×10^{-4}	2.894×10^{-4}
S2	1.3967	Failed	Failed
S3	0.4077	2.650×10^{-4}	3.789×10^{-4}
S4	0.6733	Failed	Failed
AS1	0.3787	2.236×10^{-4}	3.337×10^{-4}
AS2	0.2407	1.585×10^{-4}	2.726×10^{-4}
AS3	0.2373	1.765×10^{-4}	3.045×10^{-4}

TABLE 4.19. Computational time needed for verification of (37) with ([sec]).

Type	C_1	C_2	Algorithms 1 and 2
S1	8.128	3.627	0.1470
S2	76.72	18.82	0.0809
S3	10.38	3.602	0.1640
S4	72.59	18.75	0.0945
AS1	50.56	18.87	0.1532
AS2	11.29	3.656	0.1727
AS3	8.100	3.645	0.1343

Chapter 5

Conclusion

We presented a numerical verification method for solutions to systems of linear and semilinear elliptic partial differential equations on a bounded polygonal domain with arbitrary shape in $\mathbb{R}^2$.

In Chapter 2, we provided a numerical verification method for solutions to systems of linear and semilinear elliptic partial differential equations. The aim of this chapter is to treat a numerical verification method of the equations on bounded nonconvex domains. In particular, we presented the methods of calculating an upper bound of a residual norm for equations including a solution operator. In Chapter 3, we proposed an algorithm of constructing an inner inclusion of a region defined by a sufficient condition of the Newton-Kantorovich like theorem. The algorithm bases on Moore-Jones's algorithm, which is based on Krawczyk's operator, of finding all solutions of one dimensional nonlinear equations. One of the features of our algorithm is that if a region defined by a sufficient condition of the Newton-Kantorovich like theorem is empty, we can prove that there is no solution satisfying a sufficient condition of the Newton-Kantorovich like theorem. In Chapter 4, we provided some numerical examples. We presented results of a numerical verification method for solutions to systems of linear and semilinear elliptic partial differential equations on bounded nonconvex domains.

Finally, we would like to prove existence and local uniqueness for solutions to systems of semilinear and semilinear elliptic partial deferential equations as our future work.

Chapter A

Enclose method for eigenvalues of the Laplace operator and methods of calculating for the embedding constant $C_{e,p}$

There is a lot of study considering enclose methods for eigenvalue of the Laplace operator. For example, using Katou's bound and Lehmann-Goerisch's theorem [**4, 8, 14**], the lower bound of nth eigenvalue is proved by the lower bound of $n+1$th eigenvalue. Verified evaluation methods for eigenvalues of the Laplace operator have been studied by M. Plum [**26**], M.T. Nakao et al. [**23**], and X. Liu et al [**15**].

The eigenvalue problem is transformed into the variational problem:

$$\text{Find } \lambda \in \mathbb{R} \text{ and } v \in H_0^1(\Omega) \text{ s.t. } (\nabla v, \nabla w) = \lambda(v, w),\ w \in H_0^1(\Omega). \tag{38}$$

Denoting the eigenpairs of (38) by $\{\lambda_i, v_i\}$, these eigenpairs are simply the stationary value and critical points of the Rayleigh quotient in $H_0^1(\Omega)$:

$$R(\psi) := \frac{(\nabla\psi, \nabla\psi)}{(\psi, \psi)}.$$

Let each eigenfunction be orthogonally normalized under L^2 norm. Let us introduce the following theorem which encloses each eigenvalue of the Laplace operator.

THEOREM A.1 (Liu-Oishi [**15**]). *Let $\{\lambda_i\}$ be eigenvalues of the Laplace operator. $\{\lambda_i^h\}$ are assumed to be approximate eigenvalues of the discrete problem of (38). C_h is computable error constant of the orthogonal projection defined in (11). If*

$$C_h^2 \lambda_i^h < 1$$

holds, then we have the verified enclosure of each eigenvalue λ_i,

$$\frac{\lambda_i^h}{1 + C_h^2 \lambda_i^h} \le \lambda_i \le \lambda_i^h.$$

Next, we introduce methods of calculation for the embedding constant $C_{e,p}$. For $p = 2$, the embedding constant $C_{e,2}$ is given by the minimal eigenvalue λ_1 of the Laplace operator as follows:

$$C_{e,2} = \sup_{u \in H_0^1(\Omega)\backslash\{0\}} \frac{\|u\|_{L^2}}{\|u\|_{H_0^1}} = \sup_{u \in H_0^1(\Omega)\backslash\{0\}} \frac{\sqrt{(u,u)_{L^2}}}{\sqrt{(u,u)_{H_0^1}}} = \frac{1}{\sqrt{\lambda_1}},\ \forall u \in H_0^1(\Omega).$$

Furthermore, we can estimate the minimal eigenvalue λ_1 using Theorem A.1 and we get the embedding constant $C_{e,2}$.

For the embedding constant $C_{e,p}$ of $p \in [2, \infty)$, M. Plum has been proposed following lemma:

LEMMA A.2 (M. Plum [**27**]). *Let* $\sigma := 1/\lambda_1$. *Let* $p \in [2, \infty)$. *With* ν *denoting the longest integer* $\leq p/2$, $C_{e,p}$ *holds for*

$$C_{e,p} := \left(\frac{1}{2}\right)^{\frac{1}{2}+\frac{2\nu-3}{p}} \left[\frac{p}{2}\left(\frac{p}{2}-1\right)\cdots\left(\frac{p}{2}-\nu+2\right)\right]^{\frac{2}{p}} \sigma^{-\frac{1}{p}}$$

where if $\nu = 1$, *then we have* $\left[\frac{p}{2}\left(\frac{p}{2}-1\right)\cdots\left(\frac{p}{2}-\nu+2\right)\right] = 1$.

M.T. Nakao and N. Yamamoto have also presented another method of calculating for $C_{e,p}$ [**22**], which is based on Talenti's best constant in Sobolev inequality [**34**].

LEMMA A.3 (Nakao-Yamamoto [**22**]). *Let* $|\Omega|$ *be a measure of* Ω. *Let* $p \in [3, \infty)$. *Let* $q = 2p/(p+2)$ *be a real number satisfying* $1 < q < 2$. *We have Talenti's best constant* C_T *in Sobolev inequality satisfying*

$$C_T = \pi^{-\frac{1}{2}} 2^{-\frac{1}{q}} \left(\frac{q-1}{2-q}\right)^{1-\frac{1}{q}} \left[\Gamma\left(\frac{2}{q}\right)\Gamma\left(3-\frac{2}{q}\right)\right]^{-\frac{1}{2}},$$

where $\Gamma(\cdot)$ *means a Gamma function. Then,* $C_{e,p}$ *holds for*

$$C_{e,p} = C_T |\Omega|^{\frac{2-q}{2q}}.$$

Chapter B

Norm estimation for inverse of linearized operator using Nakao-Hashimoto-Watanabe's theorem

In Section 2.4.1, we present the method of calculating the upper bound C_1 for (15) using Theorem 2.4. M.T. Nakao, K. Hashimoto and Y. Watanabe [**19**] have proposed a different type of calculating C_1 for a nonlinear elliptic Dirichlet boundary value problem as below.

THEOREM B.1 (Nakao-Hashimoto-Watanabe [**19**]). *Let* $\mathcal{N}'[\hat{u}] : H_0^1(\Omega) \to H^{-1}(\Omega)$ *be the linear compact operator defined in (16) and* X_h *be the finite dimensional subspace of* $H_0^1(\Omega)$ *spanned by finite element base functions. Let* $\mathcal{P}_h : H_0^1(\Omega) \to H^{-1}(\Omega)$ *be the orthogonal projection defined in (10). For three constants* K_3*,* K_4 *and* K_5*, we assume*

$$\begin{aligned} \|g'[\hat{u}]u\|_{L^2} &\leq K_3\|u\|_{H_0^1}, \ \forall u \in H_0^1(\Omega), \\ \|g'[\hat{u}](I-\mathcal{P}_h)u\|_{L^2} &\leq K_4\|(I-\mathcal{P}_h)u\|_{H_0^1}, \ \forall u \in H_0^1(\Omega), \end{aligned}$$

and

$$\|\mathcal{P}_h\mathcal{A}^{-1}\mathcal{N}'[\hat{u}](I-\mathcal{P}_h)u\|_{H_0^1} \leq K_5\|(I-\mathcal{P}_h)u\|_{H_0^1}, \ \forall u \in H_0^1(\Omega).$$

Assuming that finite dimensional operator $\mathcal{P}_h(I-\mathcal{A}^{-1}\mathcal{N}'[\hat{u}])|_{X_h} : X_h \to X_h$ *is invertible with*

$$\|(\mathcal{P}_h(I-\mathcal{A}^{-1}\mathcal{N}'[\hat{u}])|_{X_h})^{-1}\|_{L(H_0^1,H_0^1)} \leq \tau.$$

Here, $\mathcal{P}_h(I-\mathcal{A}^{-1}\mathcal{N}'[\hat{u}])|_{X_h}$ *is the restriction of* $\mathcal{P}_h(I-\mathcal{A}^{-1}\mathcal{N}'[\hat{u}]) : H_0^1(\Omega) \to X_h$ *to* X_h*. We put* $\kappa := C_h(K_3\tau K_5 + K_4)$*. If* $\kappa < 1$*, then the linear operator* $\mathcal{F}'[\hat{u}]$ *is invertible and we estimate*

$$\|\mathcal{F}'[\hat{u}]^{-1}\|_{L(H^{-1},H_0^1)} \leq C_1.$$

Here,

$$C_1 := \left\| \begin{pmatrix} \tau\left(1+\frac{C_h K_3 \tau K_5}{1-\kappa}\right) & \frac{\tau K_5}{1-\kappa} \\ \frac{C_h \tau K_3}{1-\kappa} & \frac{1}{1-\kappa} \end{pmatrix} \right\|_2,$$

where $\|\cdot\|_2$ *means Euclid norm.*

We introduce the method of calculating the upper bound $K_3,\ K_4,\ K_5$ and τ for (15).

B.1. Calculating method of constants K_3, K_4 and K_5

From the definition of K_3, for $u \in H_0^1(\Omega)$, we have

$$\begin{aligned} \|g'[\hat{u}]u\|_{L^2} &= \left\| \frac{1}{\varepsilon^2}\left(f'[\hat{u}]u - \delta Bu\right)\right\|_{L^2} \\ &\le \frac{1}{\varepsilon^2}\left(\|f'[\hat{u}]u\|_{L^2} + \|\delta Bu\|_{L^2}\right) \\ &\le \frac{C_{e,2}}{\varepsilon^2}\left(\|f'[\hat{u}]\|_{L^\infty} + |\delta|\,\|B\|_{L(L^2,H_0^1)}\right)\|u\|_{H_0^1}. \end{aligned}$$

A upper bound of the operator norm of B are estimated by $C_{e,2}$ and K, which is defined by (22), as follow:

$$\|B\|_{L(L^2,H_0^1)} \le C_{e,2}K. \tag{39}$$

Furthermore, we put

$$K_3 = \frac{C_{e,2}}{\varepsilon^2}\left(\|f'[\hat{u}]\|_{L^\infty} + C_{e,2}K|\delta|\right).$$

For $u \in H_0^1(\Omega)$, we can also estimate using (39) and Aubin-Nitsche's trick for (11) as follow:

$$\begin{aligned}
\|g'[\hat{u}](I-\mathcal{P}_h)u\|_{L^2} &= \left\| \frac{1}{\varepsilon^2}\left(f'[\hat{u}](I-\mathcal{P}_h)u - \delta B(I-\mathcal{P}_h)u\right)\right\|_{L^2} \\
&\le \frac{1}{\varepsilon^2}\left(\|f'[\hat{u}](I-\mathcal{P}_h)u\|_{L^2} + \|\delta B(I-\mathcal{P}_h)u\|_{L^2}\right) \\
&\le \frac{C_h}{\varepsilon^2}\left(\|f'[\hat{u}]\|_{L^\infty} + |\delta|\,\|B\|_{L(L^2,H_0^1)}\right)\|(I-\mathcal{P}_h)u\|_{H_0^1} \\
&\le \frac{C_h}{\varepsilon^2}\left(\|f'[\hat{u}]\|_{L^\infty} + C_{e,2}K|\delta|\right)\|(I-\mathcal{P}_h)u\|_{H_0^1}.
\end{aligned}$$

Furthermore, we put

$$K_4 = \frac{C_h}{\varepsilon^2}\left(\|f'[\hat{u}]\|_{L^\infty} + C_{e,2}K|\delta|\right).$$

For $u \in H_0^1(\Omega)$, we have

$$\begin{aligned}
&\|\mathcal{P}_h\mathcal{A}^{-1}\mathcal{N}'[\hat{u}](I-\mathcal{P}_h)u\|_{H_0^1}^2 \\
= \;& (\mathcal{P}_h\mathcal{A}^{-1}\mathcal{N}'[\hat{u}](I-\mathcal{P}_h)u, \mathcal{P}_h\mathcal{A}^{-1}\mathcal{N}'[\hat{u}](I-\mathcal{P}_h)u)_{H_0^1} \\
= \;& (\nabla\mathcal{A}^{-1}\mathcal{N}'[\hat{u}](I-\mathcal{P}_h)u, \nabla\mathcal{P}_h\mathcal{A}^{-1}\mathcal{N}'[\hat{u}](I-\mathcal{P}_h)u)_{H_0^1} \\
= \;& (f'[\hat{u}](I-\mathcal{P}_h)u, \mathcal{P}_h\mathcal{A}^{-1}\mathcal{N}'[\hat{u}](I-\mathcal{P}_h)u)_{H_0^1} \\
\le \;& \|f'[\hat{u}](I-\mathcal{P}_h)u\|_{L^2}\|\mathcal{P}_h\mathcal{A}^{-1}\mathcal{N}'[\hat{u}](I-\mathcal{P}_h)u\|_{L^2} \\
\le \;& K_4\|(I-\mathcal{P}_h)u\|_{H_0^1}C_{e,2}\|\mathcal{P}_h\mathcal{A}^{-1}\mathcal{N}'[\hat{u}](I-\mathcal{P}_h)u\|_{H_0^1}.
\end{aligned}$$

Furthermore, we put

$$K_5 = C_{e,2}K_4.$$

B.2. Calculating method of the constant τ

Let ϕ_i be piecewise base functions satisfying

$$X_h = \mathrm{span}\{\phi_1, \phi_2, \cdots, \phi_n\} \subset H_0^1(\Omega),$$

where n is the number of node points. Let $\mathbf{u}, \mathbf{w} \in \mathbb{R}^n$ be real vector and $u_h, w_h \in X_h$be the elements satisfying

$$\mathbf{u} := (u_1, u_2, \cdots, u_n)^T \quad , \quad u_h = (\phi_1, \phi_2, \cdots, \phi_n) \cdot \mathbf{u},$$
$$\mathbf{w} := (w_1, w_2, \cdots, w_n)^T \quad , \quad w_h = (\phi_1, \phi_2, \cdots, \phi_n) \cdot \mathbf{w},$$

respectively. Let G and D be real $n \times n$ matrices whose elements are given by

$$\begin{aligned} G_{ij} &:= (\nabla \phi_j, \nabla \phi_i)_{L^2} - (g'[\hat{u}]\phi_j, \phi_i)_{L^2}, \\ D_{ij} &:= (\nabla \phi_j, \nabla \phi_i)_{L^2}, \end{aligned}$$

for $1 \le i \le n$ and $1 \le j \le n$. Since D is positive definite, D has Cholesky factorization $D = HH^T$. For any $u_h \in X_h$, we have a property

$$\|u_h\|_{H_0^1} = \mathbf{u}^T D \mathbf{u} = \mathbf{u}^T H H^T \mathbf{u} = (H^T \mathbf{u})^T (H^T \mathbf{u}) = \|H^T \mathbf{u}\|_2. \tag{40}$$

If G is nonsingular, then $\mathcal{P}_h(I - \mathcal{A}^{-1}\mathcal{N}'[\hat{u}])|_{X_h}$ is nonsingular. Putting $w_h \in Xh$ as

$$(u_h, \phi_i)_{H_0^1} = ((\mathcal{P}_h(I - \mathcal{A}^{-1}\mathcal{N}'[\hat{u}])|_{X_h})^{-1} w_h, \phi_i)_{H_0^1}, \ 1 \le i \le n. \tag{41}$$

(41) is transformed into

$$((I - \mathcal{A}^{-1}\mathcal{N}'[\hat{u}])u_h, \phi_i)_{H_0^1} = (w_h, \phi_i)_{H_0^1}, \ 1 \le i \le n. \tag{42}$$

For the left-hand side of (42), we have

$$\begin{aligned}
((I-\mathcal{A}^{-1}\mathcal{N}'[\hat{u}])u_h,\phi_i)_{H_0^1} &= ((I-A^{-1}g'[\hat{u}])u_h,\phi_i)_{H_0^1}\\
&= (u_h,\phi_i)_{H_0^1}-(A^{-1}g'[\hat{u}]u_h,\phi_i)_{H_0^1}\\
&= (\nabla u_h,\nabla\phi_i)_{H_0^1}-(g'[\hat{u}]u_h,\phi_i)_{L^2}\\
&= \sum_{j=1}^{n}\Big((\nabla\phi_j,\nabla\phi_i)_{H_0^1}-(g'[\hat{u}]\phi_j,\phi_i)_{L^2}\Big)u_j\\
&= \sum_{j=1}^{n}G_{ij}u_j.
\end{aligned}$$

For the right-hand side of (42), we have

$$\begin{aligned}
(w_h,\phi_i)_{H_0^1} &= \sum_{j=1}^{n}(\phi_j,\phi_i)_{H_0^1}w_j\\
&= \sum_{j=1}^{n}D_{ij}w_j.
\end{aligned}$$

Furthermore, we obtain

$$\mathbf{u}=G^{-1}D\mathbf{w}. \tag{43}$$

Finally, from (40) and (43), we can estimate

$$\begin{aligned}
\|(\mathcal{P}_h(I-\mathcal{A}^{-1}\mathcal{N}'[\hat{u}])|_{X_h})^{-1}\|_{L(H_0^1,H_0^1)} &= \sup_{w_h\in X_h\setminus\{0\}}\frac{\|(\mathcal{P}_h(I-\mathcal{A}^{-1}\mathcal{N}'[\hat{u}])|_{X_h})^{-1}w_h\|_{H_0^1}}{\|w_h\|_{H_0^1}}\\
&= \sup_{w_h\in X_h\setminus\{0\}}\frac{\|u_h\|_{H_0^1}}{\|w_h\|_{H_0^1}}\\
&= \sup_{w_h\in X_h\setminus\{0\}}\frac{\|H^T\mathbf{u}\|_2}{\|H^T\mathbf{w}\|_2}\\
&= \sup_{w_h\in X_h\setminus\{0\}}\frac{\|H^TG^{-1}D\mathbf{w}\|_2}{\|H^T\mathbf{w}\|_2}\\
&= \sup_{w_h\in X_h\setminus\{0\}}\frac{\|H^TG^{-1}HH^T\mathbf{w}\|_2}{\|H^T\mathbf{w}\|_2}\\
&\le \|H^TG^{-1}H\|_2=\tau.
\end{aligned}$$

Chapter C

Verification theory for solutions to system of elliptic PDEs using Newton-Kantorovich's theorem

In this appendix, we consider a verification method of (15) using Newton-Kantorovich's theorem. The numerical verification method of elliptic boundary value problems using Newton-Kantorovich's theorem has been studied by A. Takayasu, X. Liu and S. Oishi [**33**]. We apply the frame work to our verification procedure for (15).

We first introduce Newton-Kantorovich's theorem.

Theorem C.1 (Newton-Kantorovich [**9**]). *Let $\hat{u} \in X$ be an approximate solution of (2). Assuming that the Fréchet derivative $\mathcal{F}'[\hat{u}]$ is nonsingular and satisfies*

$$\|\mathcal{F}'[\hat{u}]^{-1}\mathcal{F}(\hat{u})\|_{H_0^1} \leq \alpha$$

for a certain positive α. Then, let $\bar{B}(\hat{u}, 2\alpha) := \{v \in H_0^1(\Omega) : \|v - \hat{u}\|_{H_0^1} \leq 2\alpha\}$ be a closed ball and $D \supset \bar{B}(\hat{u}, 2\alpha)$ be an open ball. We assume that the following holds for a certain positive ω,

$$\|\mathcal{F}'[\hat{u}]^{-1}(\mathcal{F}'[w] - \mathcal{F}'[m])\|_{L(H_0^1, H_0^1)} \leq \omega\|w - m\|_{H_0^1}, \ \forall w, m \in D.$$

If $\alpha\omega \leq 1/2$ holds, then there exists a solution $u \in H_0^1(\Omega)$ of $\mathcal{F}(u) = 0$ satisfying

$$\|u - \hat{u}\|_{H_0^1} \leq \rho := \frac{1 - \sqrt{1 - 2\alpha\omega}}{\omega}. \tag{44}$$

Furthermore, the solution u is unique in D.

Corollary C.2. *To apply Theorem C.1, we will calculate three constants C_1, C_2 and C_5. These satisfy the following inequalities*

$$\begin{aligned} \|\mathcal{F}'[\hat{u}]^{-1}\|_{L(H^{-1}, H_0^1)} &\leq C_1, \\ \|\mathcal{F}(\hat{u})\|_{H^{-1}} &\leq C_2, \\ \|\mathcal{F}'[w] - \mathcal{F}'[m]\|_{L(H_0^1, H^{-1})} &\leq C_5\|w - m\|_{H_0^1}, \ \forall w, m \in D. \end{aligned} \tag{45}$$

If $C_1^2 C_2 C_5 < 1/2$ holds, then there is a solution $u \in H_0^1(\Omega)$ of $\mathcal{F}(u) = 0$ satisfying

$$\|u - \hat{u}\|_{H_0^1} \leq \rho := \frac{1 - \sqrt{1 - 2C_1^2 C_2 C_5}}{C_1 C_5}.$$

Furthermore, the solution u is unique in $\bar{B}(\hat{u}, 2C_1C_2) := \{v \in H_0^1(\Omega) : \|u - \hat{u}\|_{H_0^1} \leq 2C_1C_2\} \subset D$.

For constants C_1 and C_2, see Section 2.4. We consider the Lipschitz constant of $\mathcal{F}' : H_0^1(\Omega) \to L(H_0^1, H^{-1})$. Here, we assume that $g' : H_0^1(\Omega) \to L(H_0^1, L^2)$ is the Lipschitz continuous on the open ball $D \supset \bar{B}(\hat{u}, 2\alpha)$. There exists the positive constant C_L satisfying

$$\begin{aligned} & |((f'[w] - f'[m])\phi, \psi)_{L^2}| \leq C_L \|w - m\|_{H_0^1} \|\phi\|_{H_0^1} \|\psi\|_{H_0^1}, \\ & \forall w, \forall m \in D \text{ and } \forall \phi, \forall \psi \in H_0^1(\Omega). \end{aligned} \tag{46}$$

For $w, m \in D$, we have

$$\begin{aligned} & \|\mathcal{F}'[w] - \mathcal{F}'[m]\|_{L(H_0^1, H^{-1})} \\ = \; & \|\mathcal{N}'[w] - \mathcal{N}'[m]\|_{L(H_0^1, H^{-1})} \\ = \; & \sup_{\phi \in H_0^1(\Omega)\backslash\{0\}} \frac{\|(\mathcal{N}'[w] - \mathcal{N}'[m])\phi\|_{H^{-1}}}{\|\phi\|_{H_0^1}} \\ = \; & \sup_{\phi \in H_0^1(\Omega)\backslash\{0\}} \sup_{\psi \in H_0^1(\Omega)\backslash\{0\}} \frac{|\langle(\mathcal{N}'[w] - \mathcal{N}'[m])\phi, \psi\rangle|}{\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}} \\ = \; & \sup_{\phi \in H_0^1(\Omega)\backslash\{0\}} \sup_{\psi \in H_0^1(\Omega)\backslash\{0\}} \frac{|((g'[w] - g'[m])\phi, \psi)_{L^2}|}{\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}} \\ = \; & \sup_{\phi \in H_0^1(\Omega)\backslash\{0\}} \sup_{\psi \in H_0^1(\Omega)\backslash\{0\}} \frac{|((f'[w] - \delta B - f'[m] + \delta B)\phi, \psi)_{L^2}|}{\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}} \\ = \; & \sup_{\phi \in H_0^1(\Omega)\backslash\{0\}} \sup_{\psi \in H_0^1(\Omega)\backslash\{0\}} \frac{|((f'[w] - f'[m])\phi, \psi)_{L^2}|}{\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}} \\ \leq \; & C_L \|w - m\|_{H_0^1} \end{aligned} \tag{47}$$

Therefore, one can put $C_5 := C_L$.

For example, we put $f(u) = \alpha u + \beta u^2 + \gamma u^3$, where $\alpha,\ \beta,\ \gamma \in \mathbb{R}$. From Hölder's inequality and Sobolev's embedding theorem, for $w, m \in D$ and $\phi, \psi \in H_0^1(\Omega)$, C_5 is derived by

$$
\begin{aligned}
& |((f'[w] - f'[m])\phi, \psi)_{L^2}| \\
= \ & |((\alpha + 2\beta w + 3\gamma w^2) - (\alpha + 2\beta m + 3\gamma m^2)\phi, \psi)_{L^2}| \\
= \ & |((\alpha + 2\beta w + 3\gamma w^2) - (\alpha + 2\beta m + 3\gamma m^2)\phi, \psi)_{L^2}| \\
\leq \ & |(2\beta(w - m)\phi, \psi)_{L^2}| + |(3\gamma(w + m)(w - m)\phi, \psi)_{L^2}| \\
\leq \ & |2\beta| \|w - m\|_{L^3} \|\phi\|_{L^3} \|\psi\|_{L^3} + |3\gamma| \|w + m\|_{L^4} \|w - m\|_{L^4} \|\phi\|_{L^4} \|\psi\|_{L^4} \\
\leq \ & 2|\beta| C_{e,3}^3 \|w - m\|_{H_0^1} \|\phi\|_{H_0^1} \|\psi\|_{H_0^1} + 3|\gamma| C_{e,4}^4 \|w + m\|_{H_0^1} \|w - m\|_{H_0^1} \|\phi\|_{H_0^1} \|\psi\|_{H_0^1}.
\end{aligned}
$$

Since $w, m \in D$, it follows that

$$\|w + m\|_{H_0^1} < 2\|\hat{u}\|_{H_0^1} + 4(C_1 C_2 + \epsilon).$$

Therefore, we have

$$C_5 = 2|\beta| C_{e,3}^3 + 6|\gamma| C_{e,4}^4 (\|\hat{u}\|_{H_0^1} + 2(C_1 C_2 + \epsilon)).$$

Chapter D

Moore-Jones's algorithm

In this appendix, we introduce Moore-Jones's algorithm. Moore-Jones's algorithm, which is based on Krawczyk's operator [**12, 13**], find all solutions of nonlinear equations [**16**]. We first introduce Krawczyk's operator for one dimensional nonlinear equations.

D.1. Krawczyk's operator for one dimensional nonlinear equations

Krawczyk's operator yields a numerical existence test of solutions for finite dimensional nonlinear equations. For $\mathbf{x} \in \mathbb{IR}$, $\mathrm{mid}(\mathbf{x}) \in \mathbb{R}$ denotes $(\sup(\mathbf{x}) + \inf(\mathbf{x}))/2$. Let $g_1 : \mathbb{R} \to \mathbb{R}$ be a C^1-function. We are concerned with a problem of finding a solution $\rho \in \mathbb{R}$ satisfying the following nonlinear equation:

$$g_1(\rho) = 0.$$

Let $l \in \mathbb{R} \setminus \{0\}$. For $\mathbf{x} \in \mathbb{IR}$, we define Krawczyk's operator $K : \mathbb{IR} \to \mathbb{IR}$ by

$$K(\mathbf{x}) := \mathrm{mid}(\mathbf{x}) - l^{-1} g_1(\mathrm{mid}(\mathbf{x})) + (1 - l^{-1} g_1'(\mathbf{x}))(\mathbf{x} - \mathrm{mid}(\mathbf{x})), \tag{48}$$

where g_1' is an interval extension of derivative of the nonlinear function g_1. If

$$K(\mathbf{x}) \subseteq \mathbf{x} \text{ and } \sup(|1 - l^{-1} g_1'(\mathbf{x})|) < 1$$

holds, there exists a solution $\rho \in \mathbf{x}$ of $g_1(u) = 0$ and unique in $\mathbf{x}$. Algorithm 3 summaries this procedure of calculating $\mathbf{x}$.

Algorithm 3 of obtaining $\mathbf{x} \in \mathbb{IR}$, which include $\rho \in \mathbb{R}$ satisfying $g_1(\rho) = 0$.

$\mathbf{Kx} = \mathrm{mid}(\mathbf{x}) - l^{-1} g_1(\mathrm{mid}(\mathbf{x})) + (1 - l^{-1} g_1'(\mathbf{x})(\mathbf{x} - \mathrm{mid}(\mathbf{x}))$
$uni = \sup(|1 - l^{-1} g_1'(\mathbf{x})|)$
if $\mathbf{Kx} \subseteq \mathbf{x}$ and $uni < 1$ **then**
  loop
    $\mathbf{rx} = \mathbf{x}$
    $\mathbf{x} = \mathbf{Kx}$
    $\mathbf{Kx} = \mathrm{mid}(\mathbf{x}) - l^{-1} g_1(\mathrm{mid}(\mathbf{x})) + (1 - l^{-1} g_1'(\mathbf{x})(\mathbf{x} - \mathrm{mid}(\mathbf{x}))$
    if $\mathbf{Kx} \supseteq \mathbf{x}$ **then**
      return $\mathbf{rx}$
    end if
  end loop
else
  error('Failure in verification');
  $\mathbf{rx} = NaN$
  return $\mathbf{rx}$
end if

D.2. Moore-Jones's algorithm

Following [**16**], we briefly sketch Moore-Jones's algorithm for one dimensional nonlinear equations. Let $B \in \mathbb{IR}$ be an starting interval. List T is the list of subregions of B yet to be tested and $T_i \in \mathbb{IR}$ is the i-th element of T. List P is the list of subregions of B contain a solution ρ to $g_1(\rho) = 0$.

For any $\rho \in T_i$, we first check whether $g_1(\rho) = 0$ have no solutions or not. Put $g = g_1(T_i) \in \mathbb{IR}$. If $0 \notin g$, then $g_1(\rho) = 0$ have no solutions for any $\rho \in T_i$ and we delete T_i. Next, as a interval $\mathbf{x} = T_i$, we try to a numerical existence test using Algorithm 3. If Algorithm 3 succeed, we put $P_j = \mathbf{rx}$ and we delete T_i. If $\mathbf{rx} = NaN$, we put

$$T_{i+1} = [\mathrm{mid}(T_i), \sup(T_i)]$$

and

$$T_i = [\inf(T_i), \mathrm{mid}(T_i)].$$

Algorithm 4 illustrates Moore-Jones's algorithm.

Algorithm 4 Moore-Jones's algorithm.

```
Set list P to empty
T_1 = B
i = 1
j = 1
while i ≠ 0 do
   g = g_1(T_i)
   if 0 ∉ g then
      flag = 0 // There is no solution in T_i
   else
      Compute rx using Algorithm 3 as starting value x = T_i
      if rx == NaN then
         flag = −1
      else
         flag = 1 //g_1(ρ) = 0 have a exact solution in T_i
      end if
   end if
   if flag == 0 then
      Delete T_i
      i = i − 1
   else if flag == 1 then
      P_j = rx
      Delete T_i
      i = i − 1
      j = j + 1
   else
      T_{i+1} = [mid(T_i), sup(T_i)]
      T_i = [inf(T_i), mid(T_i)]
      i = i + 1
   end if
end while
```

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