Free boundary problems of the incompressible Navier-Stokes equations in some unbounded domains

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CHAPTER 1

Introduction

1.1. Background

In this doctoral thesis, we are concerned with free boundary problems of the incompressible Navier-Stokes equations in some unbounded domains. Such problems arise from mathematical analysis of incompressible flows of viscous fluids with a free surface, and the problems are mathematically to find the velocity field $\mathbf{v} = \mathbf{v}(x,t) = (v_1(x,t), \dots, v_N(x,t))^T$[1] of the fluids, the pressure filed $\pi = \pi(x,t)$, and the free boundary $\Gamma = \Gamma(t)$ satisfying the following system:

$$
\begin{cases}
\rho(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) = \operatorname{Div}\mathbf{T} - \rho c_g \mathbf{e}_N & \text{in } \Omega(t),\ t>0, \\
\operatorname{div}\mathbf{v} = 0 & \text{in } \Omega(t),\ t>0, \\
-[\![\mathbf{T}\mathbf{n}_\Gamma]\!] = c_\sigma \kappa_\Gamma \mathbf{n}_\Gamma & \text{on } \Gamma(t),\ t>0, \\
[\![\mathbf{v}]\!] = 0 & \text{on } \Gamma(t),\ t>0, \\
V_\Gamma = \mathbf{v}\cdot\mathbf{n}_\Gamma & \text{on } \Gamma(t),\ t>0, \\
\mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega_0, \\
\Gamma|_{t=0} = \Gamma_0, &
\end{cases}
\tag{1.1.1}
$$

where the stress tensor $\mathbf{T}$ is decomposed as $\mathbf{T} = -\pi\mathbf{I} + \boldsymbol{\tau}$ by the identity $\mathbf{I}$ and some shear stress $\boldsymbol{\tau}$, and besides, $\mathbf{e}_N = (0,\dots,0,1)^T$. Let T_{ij} be the (i,j)-th entry of $\mathbf{T}$ and $D_j = \partial/\partial x_j$ $(j = 1,\dots,N)$, and then

$$
\operatorname{Div}\mathbf{T} = \left(\sum_{j=1}^N D_j T_{1j}, \dots, \sum_{j=1}^N D_j T_{Nj}\right)^T, \quad \operatorname{div}\mathbf{v} = \sum_{j=1}^N D_j v_j,
$$

$$
(\mathbf{v}\cdot\nabla)\mathbf{v} = \left(\sum_{j=1}^N v_j D_j v_1, \dots, \sum_{j=1}^N v_j D_j v_N\right)^T.
$$

Here $\mathbf{v}_0$ and Γ_0 denote initial data for the velocity field $\mathbf{v}$ and the free boundary Γ, respectively. In this thesis, we consider the case where Γ_0 is given by the graph of some scalar function h_0, that is,

$$
\Gamma_0 = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1},\ x_N = h_0(x')\},
$$

and then $\Gamma(t)$ is the position of Γ_0 at time t. We furthermore suppose that the unknown free boundary $\Gamma(t)$ has the form:

$$
\Gamma(t) = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1},\ x_N = h(x',t)\} \quad (t>0)
\tag{1.1.2}
$$

[1] $\mathbf{M}^T$ denotes the transposed $\mathbf{M}$, and N the dimension which is a positive integer greater than or equal to 2 throughout this thesis.

through some scalar function $h(x',t)$. On the other hand, $\Omega_0 = \Omega_{10} \cup \Omega_{20}$ with

$$\Omega_{i0} = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1},\ (-1)^i(x_N - h_0(x')) > 0\}$$

for $i = 1, 2$, where viscous fluids, *fluid*$_1$ and *fluid*$_2$, occupy Ω_{10} and Ω_{20}, respectively. Similarly $\Omega(t) = \Omega_1(t) \cup \Omega_2(t)$ for $\Omega_i(t)$ $(i = 1, 2)$ describing the regions occupied by the *fluid*$_i$ at time t. We then denote the unit normal field on Γ_0, pointing from Ω_{10} to Ω_{20}, by $\mathbf{n}_0$, and also the unit normal field on $\Gamma(t)$ by $\mathbf{n}_\Gamma$ analogously.

Let ρ_1 and ρ_2 be positive constants which describe the density of the *fluid*$_1$ and *fluid*$_2$, respectively, and then $\rho = \rho_1\chi_{\Omega_1(t)} + \rho_2\chi_{\Omega_2(t)}$, where χ_D is the indicator function of sets $D \subset \mathbf{R}^N$. The non-negative parameters c_g and c_σ are the gravitational acceleration and the surface tension coefficient, respectively. In addition, κ_Γ denotes the mean curvature of $\Gamma(t)$, and V_Γ the normal velocity of $\Gamma(t)$ with respect to $\mathbf{n}_\Gamma$. $[\![f]\!] = [\![f]\!](x,t)$ is the jump of the quantity f, defined on $\Omega(t)$, across the free boundary $\Gamma(t)$ as

$$[\![f]\!](x,t) = \lim_{\varepsilon\to 0+} (f(x + \varepsilon\mathbf{n}_\Gamma, t) - f(x - \varepsilon\mathbf{n}_\Gamma, t)) \quad \text{for } x \in \Gamma(t).$$

Under the assumption (1.1.2), we can reduce (1.1.1) to

$$(1.1.3)\qquad \begin{cases} \rho(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) = \operatorname{Div}\mathbf{T} - \rho c_g\mathbf{e}_N & \text{in } \Omega(t),\ t>0, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega(t),\ t>0, \\ -[\![\mathbf{T}\mathbf{n}_\Gamma]\!] = c_\sigma\kappa_\Gamma\mathbf{n}_\Gamma & \text{on } \Gamma(t),\ t>0, \\ [\![\mathbf{v}]\!] = 0 & \text{on } \Gamma(t),\ t>0, \\ \partial_t h + \mathbf{v}'\cdot\nabla' h - \mathbf{v}\cdot\mathbf{e}_N = 0 & \text{on } \Gamma(t),\ t>0, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega_0, \\ h|_{t=0} = h_0 & \text{on } \mathbf{R}^{N-1}, \end{cases}$$

where $\mathbf{v}' = (v_1, \ldots, v_{N-1})^T, \nabla' = (D_1, \ldots, D_{N-1})^T$, and $\mathbf{v}'\cdot\nabla' h = \sum_{j=1}^{N-1} v_j D_j h$ respectively, and besides, $\mathbf{n}_\Gamma$ and κ_Γ are given by

$$\mathbf{n}_\Gamma = \frac{1}{\sqrt{1+|\nabla' h|^2}}\begin{pmatrix} -\nabla' h \\ 1 \end{pmatrix}, \quad \kappa_\Gamma = -\nabla'\cdot\left(\frac{-\nabla' h}{\sqrt{1+|\nabla' h|^2}}\right) = \Delta' h - G_\kappa(h)$$

with $\Delta' h = \sum_{j=1}^{N-1} D_j^2 h$ and

$$G_\kappa(h) = \frac{|\nabla' h|^2\Delta' h}{(1+\sqrt{1+|\nabla' h|^2})\sqrt{1+|\nabla' h|^2}} + \sum_{j,k=1}^{N-1}\frac{D_j h D_k h D_j D_k h}{(1+|\nabla' h|^2)^{3/2}}.$$

We are interested in three types related to equations (1.1.3) as follows:

One-phase flows of Newtonian fluids: Layer type Let Ω_{20} be empty, and then note that $\Omega_2(t)$ is also empty for any $t > 0$. In addition, we suppose that the domains Ω_{10} and $\Omega_1(t)$ have flat bottoms, which means that

$$\begin{aligned} \Omega_{10} &= \{(x', x_N) \mid x' \in \mathbf{R}^{N-1},\ -b < x_N < h_0(x')\}, \\ \Omega_1(t) &= \{(x', x_N) \mid x' \in \mathbf{R}^{N-1}, -b < x_N < h(x', t)\} \end{aligned}$$

for some positive number b. Such a case is said to be *layer type* in this thesis.

In Chapter 2, we consider some linearized system of (1.1.3) of the layer type with $c_\sigma = 0$ and $c_g = 0$ in the case where $\mathbf{v}_0 = 0$. We here assume that Ω_{10} is occupied by Newtonian fluids, that is, the shear stress $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau} = \mu \mathbf{D}(\mathbf{v}), \quad \mathbf{D}(\mathbf{v}) = \nabla \mathbf{v} + (\nabla \mathbf{v})^T \tag{1.1.4}$$

for a positive constant μ describing the viscosity coefficient of *fluid*$_1$. Then the linearized system is given by

$$\begin{cases} \partial_t \mathbf{v} - \mu \operatorname{Div} \mathbf{D}(\mathbf{v}) + \nabla \pi = \mathbf{f} & \text{in } \Omega,\ t > 0, \\ \operatorname{div} \mathbf{v} = f_d & \text{in } \Omega,\ t > 0, \\ (\mu \mathbf{D}(\mathbf{v}) - \pi \mathbf{I})\mathbf{e}_N = \mathbf{g} & \text{on } \Gamma_0,\ t > 0, \\ \mathbf{v} = 0 & \text{on } \Gamma_{-b},\ t > 0, \\ \mathbf{v}|_{t=0} = 0 & \text{in } \Omega, \end{cases} \tag{1.1.5}$$

where we have set $\rho_1 = 1$ without loss of generality. In addition,

$$\begin{aligned} \Omega &= \{(x', x_N) \in \mathbf{R}^N \mid x' \in \mathbf{R}^{N-1}, -b < x_N < 0\}, \\ \Gamma_0 &= \{(x', x_N) \in \mathbf{R}^N \mid x' \in \mathbf{R}^{N-1}, x_N = 0\}, \\ \Gamma_{-b} &= \{(x', x_N) \in \mathbf{R}^N \mid x' \in \mathbf{R}^{N-1}, x_N = -b\}. \end{aligned}$$

Our approach to (1.1.5) is essentially based on analysis of *generalized* resolvent equations, associated with (1.1.5), given by

$$\begin{cases} \lambda \mathbf{v} - \mu \operatorname{Div} \mathbf{D}(\mathbf{v}) + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = f_d & \text{in } \Omega, \\ (\mu \mathbf{D}(\mathbf{v}) - \pi \mathbf{I})\mathbf{e}_N = \mathbf{g} & \text{on } \Gamma_0, \\ \mathbf{v} = 0 & \text{on } \Gamma_{-b}, \end{cases} \tag{1.1.6}$$

where each term is independent of time t, and λ is the resolvent parameter contained in $\Sigma_{\varepsilon,\gamma_0}$, which is defined as

$$\Sigma_{\varepsilon,\gamma_0} = \{\lambda \in \mathbf{C} \mid |\arg \lambda| \leq \pi - \varepsilon,\ |\lambda| \geq \gamma_0\} \quad (0 < \varepsilon < \pi/2,\ \gamma_0 > 0).$$

Here *generalized* means that we deal with the inhomogeneous divergence equation: $\operatorname{div} \mathbf{v} = f_d$ instead of $\operatorname{div} \mathbf{v} = 0$.

Resolvent estimates concerning (1.1.6) were proved by [**Abe04**], [**Abe05b**], [**Abe06**], and [**Shi13**], while the maximal regularity theorem of (1.1.5) was proved by [**Abe05a**] in $L_q((0,T), L_q(\Omega))$-spaces for any $T > 0$ and $q > 3/2$.

For equations (1.1.6), we show the $\mathcal{R}$-boundedness of families of solution operators defined on $\Sigma_{\varepsilon,\gamma_0}$. Since the $\mathcal{R}$-boundedness implies the uniform boundedness, our results especially cover the above resolvent estimates due to Abe, Abels, and Shibata. Furthermore, in $L_p((0,\infty), L_q(\Omega))$-spaces for $1 < p, q < \infty$, the maximal regularity theorem of (1.1.5) will be proved as an application of the $\mathcal{R}$-bounded solution operator families with the help of [**Wei01**, Theorem 3.4]. These are main objects in Chapter 2.

We here introduce the history related to the original nonlinear problem of (1.1.5), which is given by the following system with $c_\sigma = 0$ and $c_g = 0$:

$$
\left\{
\begin{aligned}
\rho_1(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) &= \mathrm{Div}\,\mathbf{T} - \rho_1 c_g \mathbf{e}_N && \text{in } \Omega_1(t),\ t>0,\\
\mathrm{div}\,\mathbf{v} &= 0 && \text{in } \Omega_1(t),\ t>0,\\
\mathbf{T}\mathbf{n}_\Gamma &= c_\sigma \kappa_\Gamma \mathbf{n}_\Gamma, && \text{on } \Gamma(t),\ t>0,\\
\mathbf{v} &= 0 && \text{on } \Gamma_{-b},\ t>0,\\
\partial_t h + \mathbf{v}'\cdot\nabla' h - \mathbf{v}\cdot\mathbf{e}_N &= 0 && \text{on } \Gamma(t),\ t>0,\\
\mathbf{v}|_{t=0} &= \mathbf{v}_0 && \text{in } \Omega_{10},\\
h|_{t=0} &= h_0 && \text{on } \mathbf{R}^{N-1}.
\end{aligned}
\right.
\tag{1.1.7}
$$

Beale is the pioneer who had dealt with (1.1.7) mathematically. He proved the local well-posedness in the case where $c_\sigma = 0$ and $c_g > 0$ in [**Bea81**], and furthermore, the global well-posedness for sufficiently small initial data in the case where $c_\sigma > 0$ and $c_g > 0$ in [**Bea84**]. In addition, [**BN85**] showed large-time behavior of the solution obtained in the study of [**Bea84**]. These results were proved by using function spaces based on L_2 in both time and space. Along with these studies in such function spaces, there were several results due to [**All87**], [**TT95**], [**Tan96**], [**HK09**], [**Hat11**], and [**Bae11**].

As another approach, there was the study of [**Abe05a**], which showed the local well-posedness of (1.1.7) in the case where $c_\sigma = 0$ and $c_g > 0$ by using function spaces based on L_p in both time and space. In such function spaces, [**DGH$^+$11**] and [**Göt12**] showed the local well-posedness of more complicated systems, containing rotational effects, of the layer type.

One-phase flows of Newtonian fluids: Half space type Let Ω_{20} be empty, and then such a case is said to be *half space type* in this thesis. On the other hand, we call a case *whole space type* if both Ω_{10} and Ω_{20} are occupied by viscous fluids, respectively.

The aim of Chapter 3 and Chapter 4 is to show the global well-posedness of (1.1.3) of the half space type with $c_\sigma > 0$ and $c_g > 0$ for suitable initial data $\mathbf{v}_0$ and h_0. We here assume that Ω_{10} is occupied by Newtonian fluids, that is, the shear stress $\boldsymbol{\tau}$ is given by (1.1.4).

Prüss and Simonett showed the local well-posedness of (1.1.3) of the whole space type with $c_\sigma > 0$ and $c_g \geq 0$ for Newtonian fluids in [**PS10a**], [**PS10b**], and [**PS11**]. We note that these settings due to Prüss and Simonett contain our situation described above, and that they used function spaces based on L_p in both time and space to show the local well-posedness.

On the other hand, there was another approach due to Shibata and Shimizu [**SS12**] by using more general function spaces. In [**SS12**], they considered some resolvent problem and linearized problem associated with (1.1.3) of the half space type with $c_\sigma > 0$ and $c_g > 0$, which is the same situation as ours. They showed the $\mathcal{R}$-boundedness of solution operator families of the resolvent problem with $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ for any $0 < \varepsilon < \pi/2$ and some large positive number γ_0 depending on ε, and furthermore, the maximal regularity theorem of the linearized problem in function spaces based on L_p in time and L_q in space for $1 < p, q < \infty$ by combining the $\mathcal{R}$-bounded solution operator families and [**Wei01**, Theorem 3.4]. In the sequel, a framework of such function spaces is said to be L_p-L_q framework.

Although the linear theory of [**SS12**] is enough to show the local well-posedness of our situation, it seems that we need to improve the linear theory to show the global well-posedness, because decay properties of solutions to the linearized problem are not obtained in [**SS12**]. The works due to Prüss and Simonett also do not contain such objects.

From viewpoint of this, in Chapter 3, we show decay properties of the Stokes semi-groups associated with the linearized problem considered in [**SS12**].

In Chapter 4, the global well-posedness and large-time behavior of solutions will be proved. Main ideas to show them are to use the decay properties obtained in Chapter 3 and the L_p-L_q framework with suitable assumptions of exponents p and q.

Two-phase flows of generalized Newtonian fluids: Whole space type In Chapter 5, we consider (1.1.3) of the whole space type with $c_\sigma > 0$ and $c_g \geq 0$ for suitable initial data $\mathbf{v}_0$ and h_0. We here assume that both Ω_{10} and Ω_{20} are occupied by generalized Newtonian fluids, that is, the shear stress $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau} = \chi_{\Omega_1(t)}\boldsymbol{\tau}_1 + \chi_{\Omega_2(t)}\boldsymbol{\tau}_2, \quad \boldsymbol{\tau}_i = \mu_i(|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}) \quad (i = 1, 2) \tag{1.1.8}$$

for given scalar functions μ_1, μ_2 defined on $[0, \infty)$. The scalar functions are called viscosity functions, and also $|\mathbf{D}(\mathbf{v})|^2 = \sum_{i,j=1}^N D_{ij}(\mathbf{v})^2$, where $D_{ij}(\mathbf{v})$ denotes the (i, j)-th entry of $\mathbf{D}(\mathbf{v})$. We note that if μ_1 and μ_2 are just positive constants, then the fluids are Newtonian fluids.

A typical example of viscosity functions μ is given by

$$\mu(|\mathbf{D}(\mathbf{v})|^2) = \alpha + \beta|\mathbf{D}(\mathbf{v})|^{d-2} \quad \text{for some } d \geq 1$$

with $\alpha \geq 0$ and $\beta > 0$. If $d < 2$, then the fluid is called shear thinning fluids, while it is called shear thickening fluids if $d > 2$. Fluids of the type (1.1.8) are some special case of the so called *Stokesian fluids*, which were investigated mathematically for fixed domains by [**Ama94**] and [**Ama96**].

Bothe and Prüss gave in [**BP07**] the local well-posedness of fixed domain problems in the case of the generalized Newtonian fluids with

$$\mu \in C^1((0, \infty)), \quad \mu(s) > 0, \quad \mu(s) + 2(d\mu/ds)(s) > 0 \quad \text{for } s \geq 0. \tag{1.1.9}$$

Note that our assumptions, introduced in Chapter 5, of viscosity functions μ are different from (1.1.9). Concerning mathematical results for certain classes of the generalized Newtonian fluids on fixed domains, we refer e.g. to the articles [**DR05**], [**FMS03**], [**MNR01**], and [**PR01**].

There were several studies of two-phase flows of Newtonian fluids on domains different from the whole space type. [**Den94**] showed the local well-posedness in the L_2-L_2 framework, and also [**Tan95**] the local well-posedness in the case including thermo-capillary convection. They used *Lagrangian coordinates* to show the local well-posedness.

As another approach, in [**PS10a**], [**PS10b**], and [**PS11**], Prüss and Simonett used *Hanzawa transform* to show the local well-posedness of (1.1.3) of the whole space type with $c_\sigma > 0$ and $c_g \geq 0$ for Newtonian fluids in the L_p-L_p framework.

Shibata and Shimizu [**SS11**] considered some resolvent problem and linearized problem associated with (1.1.3) of the whole space type with $c_\sigma > 0$ and $c_g > 0$ for Newtonian fluids. In the paper, they showed resolvent estimates and the maximal regularity theorem for such problems, respectively, in the L_p-L_q framework.

In the case of the whole space type for the generalized Newtonian fluids, [**Abe07**] showed the existence theorem in the context of measure-valued varifold solutions with $c_\sigma > 0$ and $c_g = 0$. His result covers in particular situations where $\mu_i(s) = \nu_i s^{(d-2)/2}$ for $d \geq 1$ and $\nu_i > 0$ $(i = 1, 2)$. Note, however, that his approach does not give the uniqueness of solutions.

On the other hand, in the case of the generalized Newtonian fluids, we show the unique existence theorem of strong solutions for the whole space type with $c_\sigma > 0$ and $c_g \geq 0$ under some assumptions of viscosity functions μ_1 and μ_2 in Chapter 5.

1.2. Notation

We here introduce notation used throughout this doctoral dissertation. Let $\mathbf{N}$ be the set of all natural numbers and $\mathbf{C}$ the set of all complex numbers, and put $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. We then define a sector $\Sigma_{\varepsilon,\lambda_0}$ as

$$\Sigma_{\varepsilon,\lambda_0} = \{\lambda \in \mathbf{C} \mid |\arg \lambda| \leq \pi - \varepsilon, \ |\lambda| \geq \lambda_0\}$$

for $0 < \varepsilon < \pi/2$ and $\lambda_0 > 0$, and besides,

$$\Sigma_\varepsilon = \Sigma_{\varepsilon,0} = \{\lambda \in \mathbf{C} \mid |\arg \lambda| < \pi - \varepsilon, \ \lambda \neq 0\}.$$

In addition, we set

$$\begin{aligned} \mathbf{R}_+^N &= \{(x', x_N) \mid x' \in \mathbf{R}^{N-1}, \ x_N > 0\}, \\ \mathbf{R}_-^N &= \{(x', x_N) \mid x' \in \mathbf{R}^{N-1}, \ x_N < 0\}, \\ \mathbf{R}_0^N &= \{(x', x_N) \mid x' \in \mathbf{R}^{N-1}, \ x_N = 0\}, \\ \dot{\mathbf{R}}^N &= \mathbf{R}_+^N \cup \mathbf{R}_-^N. \end{aligned}$$

The letter C denotes a generic constant and $C(a, b, c, \dots)$ a generic constant depending on the quantities $a, b, c, \dots$. The value of C and $C(a, b, c, \dots)$ may change from line to line.

Let $\mathbf{u}$ and $\mathbf{M}$ be N-component vectors and $N \times N$ matrices, respectively Then u_i denotes ith component of $\mathbf{u}$, $\mathbf{u}'$ the tangential component of $\mathbf{u}$, and M_{ij} the (i, j)-th entry of $\mathbf{M}$, that is,

$$\mathbf{u} = (u_1, \dots, u_N)^T, \quad \mathbf{u}' = (u_1, \dots, u_{N-1})^T, \quad \mathbf{M} = (M_{ij}).$$

Let $m \geq 1$ be an integer and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{N}_0^m$ a multi-index whose length is $|\alpha|$, and then

$$D_x^\alpha f(x) = D_1^{\alpha_1} \dots D_m^{\alpha_m} f(x) = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_m}^{\alpha_m}} f(x_1, \dots, x_m).$$

If there is no confusion, then we omit the subscript x of D_x^α.

Suppose that Ω are domains of $\mathbf{R}^N$. Then, for N-vector functions $\mathbf{u}$ and scalar functions θ defined on Ω,

$$\begin{aligned} &\nabla \mathbf{u} = (D_i u_j), \quad \nabla^2 \mathbf{u} = \{D_i D_j u_k \mid i, j, k = 1, \dots, N\}, \\ &\nabla \theta = (D_1 \theta, \dots, D_N \theta)^T, \end{aligned}$$

and besides, for scalar functions h defined on $\mathbf{R}^{N-1}$,

$$\nabla' h = (D_1 h, \dots, D_{N-1} h)^T.$$

Let I be an interval of $\mathbf{R}$ additionally, and then for N-vector functions $\mathbf{u}(x)$, $\mathbf{v}(x)$, $\mathbf{U}(x,t)$, and $\mathbf{V}(x,t)$

$$(\mathbf{u}, \mathbf{v})_\Omega = \int_\Omega \mathbf{u}(x) \cdot \mathbf{v}(x)\, dx, \quad (\mathbf{U}, \mathbf{V})_{\Omega\times I} = \int_{\Omega\times I} \mathbf{U}(x,t) \cdot \mathbf{V}(x,t)\, dxdt.$$

Let X be Banach spaces and $\|\cdot\|_X$ its norm. Then X^m denotes the m-product space of X with $m \in \mathbf{N}$, while we use the symbol $\|\cdot\|_X$ to denote its norm for short, that is,

$$\|\mathbf{u}\|_X = \sum_{j=1}^m \|u_j\|_X \quad \text{for } \mathbf{u} = (u_1, \dots, u_m)^T \in X^m.$$

In addition, let Y be another Banach space endowed with $\|\cdot\|_Y$, and then the set of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. For simplicity, we set $\mathcal{L}(X) = \mathcal{L}(X,X)$. Here the definition of the $\mathcal{R}$-boundedness is introduced as follows:

Definition 1.2.1. *A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded, if there exist a constant $C > 0$ and $p \in [1,\infty)$ such that for every $m \in \mathbf{N}$, $\{T_j\}_{j=1}^m \subset \mathcal{T}$, and $\{x_j\}_{j=1}^m \subset X$, and for all sequences $\{r_j(u)\}_{j=1}^m$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$ the following inequality holds:*

$$\left\{\int_0^1 \|\sum_{j=1}^m r_j(u)T_jx_j\|_Y^p du\right\}^{1/p} \le C\left\{\int_0^1 \|\sum_{j=1}^m r_j(u)x_j\|_X^p du\right\}^{1/p}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$ or simply by $\mathcal{R}(\mathcal{T})$.

Remark 1.2.2. It is well-known that $\mathcal{T}$ is $\mathcal{R}$-bounded for any p $(1 \le p < \infty)$, provided that $\mathcal{T}$ is $\mathcal{R}$-bounded for some p $(1 \le p < \infty)$. This fact is proved by Kahane's inequality (cf. [**KW04**, Theorem 2.4]).

We see that the $\mathcal{R}$-bound behaves like the norm by the following proposition.

Proposition 1.2.3. (1) *Let X and Y be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families on $\mathcal{L}(X,Y)$. Then $\mathcal{T} + \mathcal{S} = \{T + S \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

(2) *Let X, Y, and Z be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y,Z)$, respectively. Then $\mathcal{S}\mathcal{T} = \{ST \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(X,Z)$, and*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T}) \le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S}).$$

Since it is difficult to check the definition of the $\mathcal{R}$-boundedness, we often use the following proposition, which was proved by [**DHP03**, Proposition 3.3], to show the $\mathcal{R}$-boundedness of operator families

Proposition 1.2.4. *Let Ω be domains of $\mathbf{R}^N$, and let Λ be an index set and $1 < q < \infty$. Consider a family $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda\} \subset \mathcal{L}(L_q(\Omega))$ of kernel operators given by*

$$(T_\lambda f)(x) = \int_\Omega k_\lambda(x,y) f(y)dy \quad (x \in \Omega,\ f \in L_q(\Omega)).$$

Assume that the kernels k_λ *are dominated by a kernel* k_0*, that is,* $|k_\lambda(x,y)| \leq k_0(x,y)$ $(x, y \in \Omega,\ \lambda \in \Lambda)$*, and let* T_0 *be a kernel operator and its kernel* k_0*. Then* $\mathcal{T}$ *is* $\mathcal{R}$*-bounded on* $\mathcal{L}(L_q(\Omega))$ *and* $\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\mathcal{T}) \leq C(N,q)\|T_0\|_{\mathcal{L}(L_q(\Omega))}$ *for some positive constant* $C(N,q)$ *depending only on* N *and* q*, provided that* $T_0 \in \mathcal{L}(L_q(\Omega))$*.*

We introduce the following symbols for $0 < \varepsilon < \pi/2$:

$$
\begin{aligned}
&A = |\xi'|, \quad B = \sqrt{\lambda + \mu^{-1}|\xi'|^2} \quad (\xi' \in \mathbf{R}^{N-1},\ \lambda \in \Sigma_\varepsilon),\\
&D(A,B) = B^3 + AB^2 + 3A^2B - A^3,\\
&L(A,B) = (B-A)D(A,B) + A(c_g + c_\sigma A^2),
\end{aligned}
\tag{1.2.1}
$$

which are used to give exact formulas of solutions to linearized problems. Here we have chosen a brunch such that $\operatorname{Re} B \geq 0$, and also μ, c_g, and c_σ are positive constants. In addition, we set

$$
\mathcal{M}(a) = \frac{e^{-Ba} - e^{-Aa}}{B - A} \quad \text{for } a \geq 0.
\tag{1.2.2}
$$

Then note that for $l = 1, 2$

$$
\begin{aligned}
&\frac{\partial^l}{\partial a^l}\mathcal{M}(a) = (-1)^l\left((B+A)^{l-1}e^{-Ba} + A^l\mathcal{M}(a)\right),\\
&\mathcal{M}(a) = -a\int_0^1 e^{-(B\theta + A(1-\theta))a}\,d\theta.
\end{aligned}
\tag{1.2.3}
$$

We here define two classes of symbols as follows: Let $\varepsilon \in (0, \pi/2)$ and $\gamma_0 \geq 0$, and let $m(\xi', \lambda)$ and $\ell(\xi', \lambda)$ $(\lambda = \gamma + i\tau)$ be functions defined on $(\mathbf{R}^{N-1}\setminus\{0\}) \times \Sigma_{\varepsilon,\gamma_0}$, which are infinitely many times differentiable with respect to ξ' and holomorphic with respect to λ. If there exist a real number s such that for any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$ and $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times \Sigma_{\varepsilon,\gamma_0}$,

$$
\begin{aligned}
&|D_{\xi'}^{\alpha'} m(\xi',\lambda)| \leq C(|\lambda|^{1/2} + A)^{s-|\alpha'|},\ |D_{\xi'}^{\alpha'}(\tau\partial_\tau m(\xi',\lambda))| \leq C(|\lambda|^{1/2} + A)^{s-|\alpha'|},\\
&|D_{\xi'}^{\alpha'} \ell(\xi',\lambda)| \leq C(|\lambda|^{1/2} + A)^{s}A^{-|\alpha'|},\ |D_{\xi'}^{\alpha'}(\tau\partial_\tau \ell(\xi',\lambda))| \leq C(|\lambda|^{1/2} + A)^{s}A^{-|\alpha'|}
\end{aligned}
$$

for some positive constant C independent of ξ' and λ, then $m(\xi', \lambda)$ is called a symbol of order s with type 1 and $\ell(\xi', \lambda)$ is called a symbol of order s with type 2. In what follows, we denote the set of all symbols defined on $(\mathbf{R}^{N-1}\setminus\{0\}) \times \Sigma_{\varepsilon,\gamma_0}$ of order s with type i $(i = 1, 2)$ by $\mathbb{M}_{s,i,\varepsilon,\gamma_0}$. In particular, it follows that $\mathbb{M}_{s,1,\varepsilon,\gamma_0} \subset \mathbb{M}_{s,2,\varepsilon,\gamma_0}$ for any $s \in \mathbf{R}$ by the definition of $\mathbb{M}_{s,i,\varepsilon,\gamma_0}$ $(i = 1, 2)$, and also the following lemma holds (cf. [**SS12**, Lemma 5.1] and [**Sai15**, Lemma 5.1]).

Lemma 1.2.5. *Let* $0 < \varepsilon < \pi/2$*,* $\gamma_0 \geq 0$*, and* $s_1, s_2 \in \mathbf{R}$*.*

(1) $\mathbb{M}_{s_1,i,\varepsilon,\gamma_0} \subset \mathbb{M}_{s_2,i,\varepsilon,\gamma_0}$ $(i = 1, 2)$ *for any* $s_1 \leq s_2$*, provided that* $\gamma_0 > 0$*.*
(2) *Given* $m_i \in \mathbb{M}_{s_i,1,\varepsilon,\gamma_0}$ $(i = 1, 2)$*, we have* $m_1 m_2 \in \mathbb{M}_{s_1+s_2,1,\varepsilon,\gamma_0}$*.*
(3) *Given* $\ell_i \in \mathbb{M}_{s_i,i,\varepsilon,\gamma_0}$ $(i = 1, 2)$*, we have* $\ell_1\ell_2 \in \mathbb{M}_{s_1+s_2,2,\varepsilon,\gamma_0}$*.*
(4) *Given* $n_i \in \mathbb{M}_{s_i,2,\varepsilon,\gamma_0}$ $(i = 1, 2)$*, we have* $n_1 n_2 \in \mathbb{M}_{s_1+s_2,2,\varepsilon,\gamma_0}$*.*

The symbols defined as (1.2.1) satisfy the following properties (cf. [**SS12**, Lemma 5.2, Lemma 5.3, Lemma 7.2] and [**Sai15**, Lemma 5.2]).

Lemma 1.2.6. *Let* $0 < \varepsilon < \pi/2$ *and* $\lambda = \gamma + i\tau$*.*

(1) *Let a, a_1, and a_2 be non-negative numbers and $s \in \mathbf{R}$. Then, for any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$, $l = 0, 1$, and $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times \Sigma_\varepsilon$, we have*

$$b_{\varepsilon,\mu}(|\lambda|^{1/2} + A) \leq \mathrm{Re}B \leq |B| \leq \max\{\mu^{-1/2}, 1\}(|\lambda|^{1/2} + A)$$

with $b_{\varepsilon,\mu} = (1/\sqrt{2})\{\sin(\varepsilon/2)\}^{3/2} \min\{\mu^{-1/2}, 1\}$, and furthermore,

$$|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^l A^s\}| \leq CA^{s-|\alpha'|}, \quad |D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^l B^s\}| \leq C(|\lambda|^{1/2} + A)^{s-|\alpha'|},$$
$$|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^l D(A,B)^s\}| \leq C(|\lambda|^{1/2} + A)^{3s} A^{-|\alpha'|}$$
$$|D_{\xi}'^{\alpha'}\{(\tau\partial_\tau)^l e^{-Aa}\}| \leq CA^{-|\alpha'|}e^{-(1/2)Aa},$$
$$|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^l e^{-Ba}\}| \leq C(|\lambda|^{1/2} + A)^{-|\alpha'|}e^{-(1/4)b_{\varepsilon,\mu}(|\lambda|^{1/2}+A)a},$$
$$|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^l (e^{-Aa_1}e^{-Ba_2})\}| \leq CA^{-|\alpha'|}e^{-(1/4)b_{\varepsilon,\mu}\{A(a_1+a_2)+|\lambda|^{1/2}a_2\}}$$
$$|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^l \mathcal{M}(a)\}| \leq CaA^{-|\alpha'|}e^{-(1/4)b_{\varepsilon,\mu}Aa},$$
$$|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^l \mathcal{M}(a)\}| \leq C|\lambda|^{-1/2}A^{-|\alpha'|}e^{-(1/4)b_{\varepsilon,\mu}Aa},$$

where $C = C(\alpha', s, \varepsilon)$ is a positive constant. In particular, the constant C is independent of ξ', λ, a, a_1, and a_2, and also

$$A^s \in \mathbb{M}_{s,2,\varepsilon,0} \ (s \geq 0), \quad B^s \in \mathbb{M}_{s,1,\varepsilon,0}, \quad D(A,B)^s \in \mathbb{M}_{3s,2,\varepsilon,0},$$
$$e^{-Aa}, e^{-Aa_1}e^{-Ba_1} \in \mathbb{M}_{0,2,\varepsilon,0}, \quad e^{-Ba} \in \mathbb{M}_{0,1,\varepsilon,0}.$$

(2) *There exists a positive constant $\lambda_0 = \lambda_0(\varepsilon) \geq 1$, depending on ε, such that for any $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times \Sigma_{\varepsilon,\lambda_0}$, multi-index $\alpha' \in \mathbf{N}_0^{N-1}$, and $l = 0, 1$,*

$$|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^l L(A,B)^{-1}\}| \leq C\left(|\lambda|(|\lambda|^{1/2} + A)^2 + A(c_g + c_\sigma A^2)\right)^{-1} A^{-|\alpha'|}$$

for some positive constant $C = C(\alpha', \varepsilon, \lambda_0)$.

Let $f(x)$ and $g(\xi)$ be functions defined on $\mathbf{R}^N$, and then the Fourier transform of f and inverse Fourier transform of $g(\xi)$ are defined by

$$\mathcal{F}_x[f](\xi) = \int_{\mathbf{R}^N} e^{-ix\cdot\xi} f(x)\, dx, \quad \mathcal{F}_\xi^{-1}[g](x) = \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} e^{ix\cdot\xi} g(\xi)\, d\xi.$$

In addition, we define the partial Fourier transform of $f(x)$ and inverse partial Fourier transform of $g(\xi)$ with respect to tangential variables $x' = (x_1, \ldots, x_{N-1})$ and $\xi' = (\xi_1, \ldots, \xi_{N-1})$, respectively, as follows:

$$\mathcal{F}_{x'}[f](\xi', x_N) = \widehat{f}(\xi', x_N) = \int_{\mathbf{R}^{N-1}} e^{-ix'\cdot\xi'} f(x', x_N)\, dx', \tag{1.2.4}$$
$$\mathcal{F}_{\xi'}^{-1}[g](x', \xi_N) = \frac{1}{(2\pi)^{N-1}} \int_{\mathbf{R}^{N-1}} e^{ix'\cdot\xi'} g(\xi', \xi_N)\, d\xi'.$$

If there is no confusion, then we will omit the subscripts x, ξ, x', and ξ' from the definitions above. In order to obtain some special formulas, we use the following lemma, which is proved by the residue theorem.

Lemma 1.2.7. *Let $a \in \mathbf{R} \setminus \{0\}$ and $\xi = (\xi_1, \ldots, \xi_N) \in \mathbf{R}^N$. Then we have*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ia\xi_N}}{|\xi|^2} d\xi_N = \frac{e^{-|a|A}}{2A}, \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\xi_N e^{ia\xi_N}}{|\xi|^2} d\xi_N = -\mathrm{sign}(a)\frac{e^{-|a|A}}{2},$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ia\xi_N}}{\lambda + |\xi|^2} d\xi_N = \frac{e^{-B|a|}}{2B}, \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\xi_N e^{ia\xi_N}}{\lambda + |\xi|^2} d\xi_N = \mathrm{sign}(a)\frac{i}{2}e^{-B|a|},$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ia\xi_N}}{|\xi|^2(\lambda+\mu|\xi|^2)}d\xi_N=\frac{1}{2\lambda}(\frac{e^{-|a|A}}{A}-\frac{e^{-|a|B}}{B}),$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i\xi_N e^{ia\xi_N}}{|\xi|^2(\lambda+\mu|\xi|^2)}d\xi_N=-\frac{1}{2\lambda}\mathrm{sign}(a)(e^{-|a|A}-e^{-|a|B}),$$

where sign(a) defined by the formula: $\mathrm{sign}(a)=1$ *when* $a>0$ *and* $\mathrm{sign}(a)=-1$ *when* $a<0$.

The following proposition was proved by [**SS01**, Theorem 2.3] and [**KS12**, Theorem 2.4.4]

PROPOSITION 1.2.8. *Let* X *be a Banach space and* $\|\cdot\|_X$ *its norm. Suppose that* L *and* n *be a non-negative integer and positive integer, respectively. Let* $0<\sigma\leq 1$ *and* $s=L+\sigma-n$*, and set*

$$l(\sigma)=\begin{cases}1 & \sigma=1,\\ 0 & 0<\sigma<1.\end{cases}$$

Let $f(\xi)$ *be a function of* $C^{L+l(\sigma)+1}(\mathbf{R}^n\setminus\{0\},X)$ *which satisfies the following two conditions:*

(1) $D_\xi^\gamma f\in L_1(\mathbf{R}^n,X)$ *for any multi-index* $\gamma\in\mathbf{N}_0^n$ *with* $|\gamma|\leq L$.
(2) *For any multi-index* $\gamma\in\mathbf{N}_0^n$ *with* $|\gamma|\leq L+l(\sigma)+1$ *there exists a positive number* $C(\gamma)$ *such that*

$$\|D_\xi^\gamma f(\xi)\|_X\leq C(\gamma)|\xi|^{s-|\gamma|}\quad(\xi\in\mathbf{R}^n\setminus\{0\}).$$

Then there exists a positive constant $C(n,s)$ *such that for* $x\neq 0$

$$\|\mathcal{F}^{-1}[f](x)\|_X\leq C(n,s)\left(\max_{|\gamma|\leq L+l(\sigma)+1}C(\gamma)\right)|x|^{-(n+s)}.$$

1.3. Function spaces

Let X be Banach spaces and Ω domains of $\mathbf{R}^m$ for $m\in\mathbf{N}$. We then denote, for $1\leq p\leq\infty$ and $k\in\mathbf{N}$, the X-valued Lebesgue and Sobolev spaces on Ω by $L_p(\Omega,X)$ and $W_p^k(\Omega,X)$, respectively, and set $W_p^0(\Omega,X)=L_p(\Omega,X)$. In addition, let $W_p^s(\Omega,X)$ be the X-valued Sobolev-Slobodeckij spaces on Ω for $1\leq p<\infty$ and $s\in(0,\infty)\setminus\mathbf{N}$, and also its norm

$$\|u\|_{W_p^s(\Omega,X)}=\|u\|_{W_p^{[s]}(\Omega,X)}+\sum_{|\alpha|=s}\left(\int_\Omega\int_\Omega\frac{\|D_x^\alpha u(x)-D_y^\alpha u(y)\|_X^p}{|x-y|^{m+(s-[s])p}}\,dxdy\right)^{1/p},$$

where $[s]=\max\{n\in\mathbf{N}_0\mid n<s\}$. On the other hand, for $1<p<\infty$ and $s\geq 0$, $H_p^s(\mathbf{R}^m,X)$ denote the X-valued Bessel potential spaces of order s on $\mathbf{R}^m$, that is,

$$H_p^s(\mathbf{R}^m,X)=\{u\in L_p(\mathbf{R}^m,X)\mid \|u\|_{H_p^s(\mathbf{R}^m,X)}<\infty\},$$
$$\|u\|_{H_p^s(\mathbf{R}^m,X)}=\|\mathcal{F}_\xi^{-1}[(1+|\xi|^2)^{s/2}\mathcal{F}[u](\xi)]\|_{L_p(\mathbf{R}^m,X)}.$$

We especially set

$$H_{p,0}^s(\mathbf{R},X)=\{u\in H_p^s(\mathbf{R},X)\mid u(t)=0\text{ for }t<0\},$$
$$\|u\|_{H_{p,0}^s(\mathbf{R},X)}=\|u\|_{H_p^s(\mathbf{R},X)}\ \text{ for }u\in H_{p,0}^s(\mathbf{R},X),$$

and also, in the case of Ω, $H^s_p(\Omega, X)$ are defined as

$$H^s_p(\Omega, X) = \{u \mid \exists v \in H^s_p(\mathbf{R}^m, X) \text{ such that } u = v \text{ on } \Omega\},$$
$$\|u\|_{H^s_p(\Omega,X)} = \inf\{\|v\|_{H^s_p(\mathbf{R}^m,X)} \mid v \in H^s_p(\mathbf{R}^m, X),\ u = v \text{ on } \Omega\}.$$

Furthermore, let $C(\Omega, X)$ be the set of all X-valued continuous functions on Ω, and $BUC(\Omega, X)$ the set of all X-valued uniformly continuous and bounded functions on Ω, respectively. Then, for $k \in \mathbf{N}$,

$$C^k(\Omega, X) = \{u \in C(\Omega, X) \mid D^\alpha_x u \in C(\Omega, X) \text{ for } |\alpha| = 1, \dots, k\},$$
$$BUC^k(\Omega, X) = \{u \in BUC(\Omega, X) \mid D^\alpha_x u \in BUC(\Omega, X) \text{ for } |\alpha| = 1, \dots, k\}.$$

Remark 1.3.1. If $X = \mathbf{R}$ or $\mathbf{C}$ in the definitions above, then X is abbreviated in this thesis. For examples, $L_p(\Omega, \mathbf{C}) = L_p(\Omega)$, $C(\Omega, \mathbf{R}) = C(\Omega)$, and so on.

We here introduce the Besov spaces and the homogeneous Sobolev spaces. Suppose that $1 < p, q < \infty$ and $\Omega \in \{\mathbf{R}^N, \mathbf{R}^N_+, \mathbf{R}^N_-\}$. Let $(\cdot,\cdot)_{\theta,p}$ be the real interpolation functor for $0 < \theta < 1$ (cf. [**Tri83**, Definition 2.4.1]), and then we defined the Besov spaces as

$$B^s_{q,p}(\Omega) = (W^{s_1}_q(\Omega), W^{s_2}_q(\Omega))_{\theta,p},$$

for $0 \leq s_1 < s_2$ and $s = (1-\theta)s_1 + \theta s_2$, while $\|\cdot\|_{B^s_{q,p}(\Omega)}$ denotes its norm. The homogeneous Sobolev spaces are defined as

$$\widehat{W}^1_q(\Omega) = \{u \in L_{1,\mathrm{loc}}(\Omega) \mid \|D^\alpha u\|_{L_q(\Omega)} < \infty \text{ for } |\alpha| = 1\}.$$

Moreover, in the case of $\mathbf{R}^N$, the dual spaces of $\widehat{W}^1_{q'}(\mathbf{R}^N)$ are denoted by $\widehat{W}^{-1}_q(\mathbf{R}^N)$ for $1/q + 1/q' = 1$. To introduce $\widehat{W}^{-1}_q(\mathbf{R}^N_-)$, let ι be the extension operator given by [**AF03**, Theorem 5.19]. Then we note the following properties of ι.

Lemma 1.3.2. *Let ι be as mentioned above. Then the following assertions hold.*

(1) *Let $1 < q < \infty$ and $f \in W^1_q(\mathbf{R}^N_-)$. It then holds that*

$$\iota f = f \text{ in } \mathbf{R}^N_-, \quad \iota f \in W^1_q(\mathbf{R}^N_-), \quad \|D^\alpha_x(\iota f)\|_{L_q(\mathbf{R}^N_-)} \leq C(q)\|D^\alpha_x f\|_{L_q(\mathbf{R}^N_-)}$$

for any multi-index $\alpha \in \mathbf{N}^N_0$ with $|\alpha| \leq 1$ and a positive constant $C(q)$.

(2) *Let $1 < q < \infty$ and $f \in W^1_q(\mathbf{R}^N_-)$. It then holds that*

$$\|(1-\Delta)^{-1/2}\iota(D^\alpha_x f)\|_{L_q(\mathbf{R}^N_-)} \leq C(q)\|f\|_{L_q(\mathbf{R}^N_-)}$$

for any multi-index $\alpha \in \mathbf{N}^N_0$ with $|\alpha| \leq 1$ and a positive constant $C(q)$, where

$$(1-\Delta)^{-1/2} g = \mathcal{F}^{-1}_\xi[(1+|\xi|^2)^{-1/2}\mathcal{F}[g](\xi)](x).$$

By using the extension operator ι, we define $\widehat{W}^{-1}_q(\mathbf{R}^N_-)$ as

$$\widehat{W}^{-1}_q(\mathbf{R}^N_-) = \{f \in L_{1,\mathrm{loc}}(\mathbf{R}^N_-) \mid (1-\Delta)^{-1/2}\iota f \in L_q(\mathbf{R}^N)\},$$
$$\|f\|_{\widehat{W}^{-1}_q(\mathbf{R}^N_-)} = \|(1-\Delta)^{-1/2}\iota f\|_{L_q(\mathbf{R}^N)},$$

while $\widehat{W}^{-1}_q(\mathbf{R}^N_+)$ are defined similarly. The following lemmas will be used to control some nonlinear terms in Chapter 4.

Lemma 1.3.3. *Let $1 < p, q < \infty$ and ι be the extension operator of Lemma 1.3.2. Suppose that*

$$f \in W_p^1(\mathbf{R}, \widehat{W}_q^{-1}(\mathbf{R}_-^N)) \cap L_p(\mathbf{R}, W_q^1(\mathbf{R}_-^N))$$

and $f = 0$ for $t < 0$. Then there exists a positive constant $C = C(p, q)$ such that

$$\|f\|_{H_{q,p,0}^{1,1/2}(\mathbf{R}_-^N)} \leq C\left(\|\partial_t(1-\Delta)^{-1/2}\iota f\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^N))} + \|f\|_{L_p(\mathbf{R}_+, W_q^1(\mathbf{R}_-^N))}\right).$$

Proof. It can be proved in the same way as [**Shi15**, Lemma A.1]. □

Lemma 1.3.4. *Let $1 < p, q < \infty$ and ι be the extension operator of Lemma 1.3.2. Suppose that $\alpha \in \mathbf{N}_0^N$ with $|\alpha| \leq 1$. Then we have*

$$\begin{aligned}(1.3.1)\qquad &\|\partial_t(1-\Delta)^{-1/2}\iota((D_x^\alpha f)g)\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^N))} \\ &\leq C(p,q)\Big(\|(\partial_t f)g\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}_-^N))} + \|(\partial_t f)D_x^\alpha g\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}_-^N))} \\ &+ \|(D_x^\alpha f)\partial_t g\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}_-^N))}\Big)\end{aligned}$$

with some positive constant $C(p, q)$, provided that the right-hand side is valid. Especially, if $N < q < \infty$, then

$$\begin{aligned}(1.3.2)\qquad &\|\partial_t(1-\Delta)^{-1/2}\iota((D_x^\alpha f)g)\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^N))} \\ &\leq C(p,q)\Big(\|\partial_t f\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}_-^N))}\|g\|_{L_\infty(\mathbf{R}_+, L_\infty(\mathbf{R}_-^N))} \\ &+ \|\partial_t f\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}_-^N))}\|D_x^\alpha g\|_{L_\infty(\mathbf{R}_+, L_q(\mathbf{R}_-^N))} \\ &+ \|D_x^\alpha f\|_{L_\infty(\mathbf{R}_+, L_q(\mathbf{R}_-^N))}\|\partial_t g\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}_-^N))}\Big).\end{aligned}$$

Proof. We note that

$$\partial_t((D_x^\alpha f)g) = D_x^\alpha((\partial_t f)g) - (\partial_t f)D_x^\alpha g + (D_x^\alpha f)\partial_t g.$$

Then, by Lemma 1.3.2 and 1.3.3, it holds that

(1.3.3)

$$\begin{aligned}&\|\partial_t(1-\Delta)^{-1/2}\iota((D_x^\alpha f)g)\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^N))} \\ &\leq C(p,q)\Big(\|(\partial_t f)g\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}_-^N))} + \|(1-\Delta)^{-1/2}\iota((\partial_t f)D_x^\alpha g)\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^N))} \\ &+ \|(1-\Delta)^{-1/2}\iota((D_x^\alpha f)\partial_t g)\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^N))}\Big)\end{aligned}$$

with a positive constant $C(p, q)$. Since $\|(1-\Delta)^{-1/2}F\|_{L_q(\mathbf{R}^N)} \leq C(q)\|F\|_{L_q(\mathbf{R}^N)}$ by Fourier multiplier theorem of Mhiklin, it follows from (1.3.3) and Lemma 1.3.2 that (1.3.1) holds. Concerning (1.3.2), we use the following inequality:

$$\|(1-\Delta)^{-1/2}\iota(uv)\|_{L_q(\mathbf{R}^N)} \leq C(q)\|u\|_{L_q(\mathbf{R}_-^N)}\|v\|_{L_q(\mathbf{R}_-^N)},$$

which is obtained in the proof of [**Shi15**, Lemma 3.3]. By combining this inequality with (1.3.3), we have (1.3.2). □

Next we consider some embedding properties. Here and hereafter, for Banach spaces X and Y endowed with $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, $X \hookrightarrow Y$ means that X is continuously embedded into Y, that is, there exists a constant $C > 0$ such that

$$\|u\|_Y \leq C\|u\|_X \quad \text{for every } u \in X.$$

Let I be intervals of $\mathbf{R}$ and Ω domains of $\mathbf{R}^N$, and then we set

$$\begin{aligned}
&W^{2,1}_{q,p}(\Omega\times I) = W^1_p(I, L_q(\Omega)) \cap L_p(I, W^2_q(\Omega)),\\
&W^{2,1}_{q,p,0}(\Omega\times \mathbf{R}) = \{u \in W^{2,1}_{q,p}(\Omega\times\mathbf{R}) \mid u(x,t) = 0 \text{ for } t<0\},\\
&H^{1,1/2}_{q,p}(\Omega\times I) = H^{1/2}_p(I, L_q(\Omega)) \cap L_p(I, W^1_q(\Omega)),\\
&H^{1,1/2}_{q,p,0}(\Omega\times \mathbf{R}) = \{u \in H^{1,1/2}_{q,p}(\Omega\times\mathbf{R}) \mid u(x,t) = 0 \text{ for } t<0\}.
\end{aligned}$$

Concerning the function spaces introduced above, we see that the following embedding properties hold.

LEMMA 1.3.5. *Let* $\Omega \in \{\mathbf{R}^N, \mathbf{R}^N_+, \mathbf{R}^N_-\}$, *and suppose that* $T \in (0,\infty)$ *or* $T=\infty$. *Then the following assertions hold.*

(1) $W^{2,1}_{q,p}(\Omega\times(0,T)) \hookrightarrow BUC((0,T), BUC^1(\Omega))$, *provided that* $1<p,q<\infty$ *and* $2/p+N/q<1$.
(2) $W^{2,1}_{q,p}(\Omega\times(0,T)) \hookrightarrow BUC((0,T), W^1_q(\Omega))$ *for* $2<p<\infty$ *and* $1<q<\infty$.
(3) $W^{2,1}_{q,p,0}(\Omega\times\mathbf{R}) \hookrightarrow H^{1/2}_{p,0}(\mathbf{R}, W^1_q(\Omega))$ *for* $2<p<\infty$ *and* $1<q<\infty$.

PROOF. (1) By [**MS12**, Proposition 3.2, Remark 3.3], we have

$$W^{2,1}_{q,p}(\Omega\times(0,T)) \hookrightarrow W^\sigma_p((0,T), W^{2(1-\sigma)}_q(\Omega)) \tag{1.3.4}$$

for $1<p,q<\infty$ and $0<\sigma<1$. By taking σ in such a way that

$$\sigma > \frac{1}{p}, \quad 2(1-\sigma) > 1 + \frac{N}{q}, \tag{1.3.5}$$

and using Sobolev's embedding theorem, we obtain

$$\begin{aligned}
W^\sigma_p((0,T), W^{2(1-\sigma)}_q(\Omega)) &\hookrightarrow BUC((0,T), W^{2(1-\sigma)}_q(\Omega))\\
&\hookrightarrow BUC((0,T), BUC^1(\Omega)),
\end{aligned}$$

where we note that (1.3.5) is equivalent to the assumptions: $1<p,q<\infty$ and $2/p+N/q<1$. This completes the proof of (1).
(2) Let $\sigma = 1/2$ in (1.3.4). We then obtain the required property by Sobolev's embedding theorem, noting $2<p<\infty$.
(3) It follows from [**SS08**, Proposition 2.8] that $W^{2,1}_{q,p}(\Omega\times\mathbf{R}) \hookrightarrow H^{1/2}_p(\mathbf{R}, W^1_q(\Omega))$ for $1<p,q<\infty$, so that the required property holds clearly. □

Finally we introduce weighted spaces with respect to time t. To this end, let X be Banach spaces, and besides,

$$\begin{aligned}
L_{p,\mathrm{loc},0}(\mathbf{R}, X) &= \{u \in L_{p,\mathrm{loc}}(\mathbf{R},X) \mid u(t) = 0 \text{ for } t<0\},\\
W^l_{p,\mathrm{loc},0}(\mathbf{R}, X) &= \{u \in W^l_{p,\mathrm{loc}}(\mathbf{R},X) \mid u(t) = 0 \text{ for } t<0\},\\
L_{p,\gamma}(\mathbf{R}, X) &= \{u \in L_{p,\mathrm{loc}}(\mathbf{R},X) \mid e^{-\gamma t}u \in L_p(\mathbf{R},X)\}
\end{aligned}$$

for $l\in\mathbf{N}$ and $1\le p<\infty$. We then set, for $\gamma>0$,

$$\begin{aligned}
L_{p,\gamma,0}(\mathbf{R}, X) &= \{u \in L_{p,\mathrm{loc},0}(\mathbf{R},X) \mid e^{-\gamma t}u \in L_p(\mathbf{R},X)\},\\
W^l_{p,\gamma,0}(\mathbf{R}, X) &= \{u \in W^l_{p,\mathrm{loc},0}(\mathbf{R},X) \mid e^{-\gamma t}\partial_t^k u \in L_p(\mathbf{R},X) \text{ for } k=0,1,\dots,l\}.
\end{aligned}$$

In addition, in order to show weighted Bessel potential spaces, we define the Laplace transform and its inverse as follows: For $\lambda = \gamma + i\tau$,

$$\mathcal{L}_t[f](\tau) = \int_{\mathbf{R}} e^{-\lambda t} f(t)dt = \mathcal{F}_t[e^{-\gamma t} f](\tau),$$
$$\mathcal{L}_\lambda^{-1}[g](t) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{\lambda t} g(\tau)d\tau = e^{\gamma t}\mathcal{F}_\tau^{-1}[g](t).$$

If there is no confusion, we denote $\mathcal{L}_t$ and $\mathcal{L}_\lambda^{-1}$ by $\mathcal{L}$ and $\mathcal{L}^{-1}$ for short. Let $\gamma > 0$, and then we set

$$\begin{aligned}(\Lambda_\gamma^{1/2} f)(x,t) &= \mathcal{L}_\lambda^{-1}[|\lambda|^{1/2}\mathcal{L}_t[f(x,t)](\tau)](t)\\ &= e^{\gamma t}\mathcal{F}_\tau^{-1}[(\gamma^2+\tau^2)^{1/4}\mathcal{F}_t[e^{-\gamma t} f(x,t)](\tau)](t) \quad (\lambda = \gamma + i\tau).\end{aligned}$$

The weighted Bessel potential spaces are defined as

$$\begin{aligned}&H_{p,\gamma}^{1/2}(\mathbf{R}, X) = \{u \in L_{p,\gamma}(\mathbf{R}, X) \mid e^{-\gamma t}\Lambda_\gamma^{1/2} f \in L_p(\mathbf{R}, X)\},\\ &H_{q,p,\gamma}^{1,1/2}(\mathbf{R}\times\Omega) = H_{p,\gamma}^{1/2}(\mathbf{R}, L_q(\Omega)) \cap L_{p,\gamma}(\mathbf{R}, W_q^1(\Omega)),\\ &H_{q,p,\gamma,0}^{1,1/2}(\mathbf{R}\times\Omega) = \{f \in H_{q,p,\gamma}^{1,1/2}(\mathbf{R}\times\Omega) \mid f(t) = 0 \text{ for } t < 0\}\end{aligned}$$

for $\gamma > 0$ and $1 < p, q < \infty$ with norms:

$$\begin{aligned}&\|u\|_{H_{p,\gamma}^{1/2}(\mathbf{R},X)} = \|e^{-\gamma t}\Lambda_\gamma^{1/2} u\|_{L_p(\mathbf{R},X)},\\ &\|u\|_{H_{q,p,\gamma}^{1,1/2}(\mathbf{R}\times\Omega)} = \|u\|_{H_{p,\gamma}^{1/2}(\mathbf{R},L_q(\Omega))} + \|u\|_{L_{p,\gamma}(\mathbf{R},W_q^1(\Omega))}.\end{aligned}$$

REMARK 1.3.6. Let $\gamma_0 > 0$ and $1 < p < \infty$. Then, by vector-valued Fourier multiplier theorem of [**Zim89**, Proposition 3], we see that there exists a positive constant $C = C(\gamma_0, p)$ such that for every $u \in H_{p,\gamma_0}^{1/2}(\mathbf{R}, X)$

$$C^{-1}\|e^{-\gamma_0 t}\Lambda_{\gamma_0}^{1/2} u\|_{L_p(\mathbf{R},X)} \le \|e^{-\gamma_0 t} u\|_{H_p^{1/2}(\mathbf{R},X)} \le C\|e^{-\gamma_0 t}\Lambda_{\gamma_0}^{1/2} u\|_{L_p(\mathbf{R},X)},$$

if X is a UMD-space. Concerning UMD-spaces, see e.g. [**KW04**].

Part 1

One-phase flows of Newtonian fluids: Layer type

CHAPTER 2

$\mathcal{R}$-boundedness of solution operator families of some generalized Stokes resolvent equations in an infinite layer

2.1. Main results

Let $\Omega \subset \mathbf{R}^N$ be an infinite layer, that is,

$$\Omega = \{(x', x_N) \in \mathbf{R}^N \mid x' \in \mathbf{R}^{N-1},\ 0 < x_N < \delta\} \quad (\delta > 0),$$

while Γ_δ and Γ_0 denote its boundaries:

$$\Gamma_\delta = \{(x', x_N) \in \mathbf{R}^N \mid x' \in \mathbf{R}^{N-1},\ x_N = \delta\},$$
$$\Gamma_0 = \{(x', x_N) \in \mathbf{R}^N \mid x' \in \mathbf{R}^{N-1},\ x_N = 0\}.$$

This chapter is concerned with the following generalized Stokes resolvent equations in Ω:

$$\begin{cases} \lambda\mathbf{u} - \operatorname{Div}\mathbf{S}(\mathbf{u}, \theta) = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = f_d & \text{in } \Omega, \\ \mathbf{S}(\mathbf{u}, \theta)\mathbf{e}_N = \mathbf{g} & \text{on } \Gamma_\delta, \\ \mathbf{u} = 0 & \text{on } \Gamma_0. \end{cases} \tag{2.1.1}$$

Here unknowns $\mathbf{u} = \mathbf{u}(x) = (u_1(x), \dots, u_N(x))^T$ and $\theta = \theta(x)$ are the N-component velocity field and the pressure field, respectively; the right members $\mathbf{f} = \mathbf{f}(x) = (f_1(x) \dots, f_N(x))^T$ and $\mathbf{g} = \mathbf{g}(x) = (g_1(x), \dots, g_N(x))^T$ are given N-component vectors, and $f_d = f_d(x)$ is a given scalar function; stress tensor $\mathbf{S}(\mathbf{u}, \theta)$ is defined as

$$\mathbf{S}(\mathbf{u}, \theta) = -\theta\mathbf{I} + \mu\mathbf{D}(\mathbf{u}), \quad \mathbf{D}(\mathbf{u}) = \nabla\mathbf{u} + (\nabla\mathbf{u})^T,$$

where μ is a positive constant describing viscosity coefficients. Then $\operatorname{Div}\mathbf{S}(\mathbf{u}, \theta)$ are N-component vectors whose i-th component is given by

$$\sum_{j=1}^N D_j\{\mu(D_j u_i + D_i u_j) - \delta_{ij}\theta\} = \mu(\Delta u_i + D_i \operatorname{div}\mathbf{u}) - D_i\theta$$

for $i = 1, \dots, N$.

We here set $W^1_{q,\Gamma_\delta}(\Omega) = \{\theta \in W^1_q(\Omega) \mid \theta|_{\Gamma_\delta} = 0\}$ for $1 < q < \infty$, and besides,

$$\widehat{W}^1_{q,\Gamma_\delta}(\Omega) = \{\theta \in L_{q,\mathrm{loc}}(\Omega) \mid \nabla\theta \in L_q(\Omega)^N,\ \theta|_{\Gamma_\delta} = 0,$$
$$\exists\{\theta_j\}_{j=1}^\infty \subset W^1_{q,\Gamma_\delta}(\Omega) \text{ such that } \lim_{j\to\infty}\|\nabla(\theta_j - \theta)\|_{L_q(\Omega)} = 0\}.$$

Remark 2.1.1. If we set $\dot{W}^1_{q,\Gamma_\delta}(\Omega) = \{\theta \in L_{q,\mathrm{loc}}(\Omega) \mid \nabla\theta \in L_q(\Omega)^N,\ \theta|_{\Gamma_\delta} = 0\}$, then there holds $\widehat{W}^1_{q,\Gamma_\delta}(\Omega) = \dot{W}^1_{q,\Gamma_\delta}(\Omega)$. which was proved in [**Shi13**, Theorem A.3].

Let $\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)$ be the dual space of $\widehat{W}^1_{q',\Gamma_\delta}(\Omega)$ with $1 < q < \infty$ and $1/q + 1/q' = 1$. In addition, in the same manner as [**FS94**, Section 1], we set

$$\begin{aligned} &W^1_q(\Omega) \cap \widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega) \\ &\qquad = \{f \in W^1_q(\Omega) \mid (f, \varphi)_\Omega \text{ is continuous on } W^1_{q',\Gamma_\delta}(\Omega) \text{ under } \|\nabla \cdot \|_{L'_q(\Omega)}\}. \end{aligned}$$

Thus we see that for $f \in W^1_q(\Omega) \cap \widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)$

$$\|f\|_{\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)} = \sup\{|(f, \varphi)_\Omega| \mid \varphi \in W^1_{q',\Gamma_\delta}(\Omega),\ \|\nabla\varphi\|_{L_{q'}(\Omega)} = 1\}.$$

Let $\varphi(x_N) \in C^\infty(\mathbf{R})$ be a cut-cut function such that $0 \le \varphi(x_N) \le 1$ and

$$\varphi(x_N) = \begin{cases} 1 & x_N \le 1/3, \\ 0 & x_N \ge 2/3. \end{cases} \tag{2.1.2}$$

We set $\varphi_0(x_N) = \varphi(x_N/\delta)$ and $\varphi_\delta(x_N) = 1 - \varphi(x_N/\delta)$, and besides,

$$\Phi_{\xi'}(x_N) = \varphi_\delta(x_N)e^{-A(d_1(x_N)+d_2(\delta))} - (\varphi_\delta(x_N) + 2\varphi_0(x_N))e^{-Ad_2(x_N)} \tag{2.1.3}$$

for $\xi' \in \mathbf{R}^{N-1}$ by using (1.2.1), where $d_1(x_N)$ and $d_2(x_N)$ are defined as

$$d_1(x_N) = \delta - x_N, \quad d_2(x_N) = x_N. \tag{2.1.4}$$

Then one of our main results is stated as follows:

THEOREM 2.1.2. *Let $1 < q < \infty$ and $f_d \in W^1_q(\Omega) \cap \widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)$. Then, for every $J = 1, \dots .N$, there exist operators $K_J \in \mathcal{L}(W^1_q(\Omega) \cap \widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega), W^2_q(\Omega))$ such that*

$$\mathbf{u} = Kf_d = (K_1 f_d, \dots, K_N f_d)^T$$

solves the divergence equation: $\operatorname{div} \mathbf{u} = f_d$ *in* Ω *and satisfies*

$$\begin{aligned} \|Kf_d\|_{L_q(\Omega)} &\le C(N, q, \delta)\|f_d\|_{\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)}, \\ \|\nabla K f_d\|_{L_q(\Omega)} &\le C(N, q, \delta)\|f_d\|_{L_q(\Omega)}, \\ \|\nabla^2 K f_d\|_{L_q(\Omega)} &\le C(N, q, \delta)\|\nabla f_d\|_{L_q(\Omega)} \end{aligned}$$

with some positive constant $C(N, q, \delta)$. In addition, $(K_N f_d)(x', 0)$ has the following special formula:

$$(K_N f_d)(x', 0) = \frac{1}{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\Phi_{\xi'}(y_N)\widehat{f}_d(\xi', y_N)\right](x')\,dy_N. \tag{2.1.5}$$

For simplicity we denote $\mathcal{L}(L_q(\Omega)^n, L_q(\Omega)^m)$ with $n, m \in \mathbf{N}$ by $\mathcal{L}(L_q(\Omega))$ in the sequel. Theorem 2.1.2 enables us to show the $\mathcal{R}$-boundedness of solution operator families to the resolvent equations (2.1.1) in the following theorem.

THEOREM 2.1.3. *Let $0 < \varepsilon < \pi/2$, $\gamma_0 > 0$, and $1 < q < \infty$. Suppose that*

$$\mathbf{f} \in L_q(\Omega)^N, \quad f_d \in W^1_q(\Omega) \cap \widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega), \quad \mathbf{g} \in W^1_q(\Omega)^N.$$

Then, for every $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ and $j = 1, \dots, N-1$, there exist operators

$$\begin{aligned} \mathcal{U}_j(\lambda) &\in \mathcal{L}(L_q(\Omega)^{N^3+2N^2+5N+2}, W^2_q(\Omega)), \\ \mathcal{U}_N(\lambda) &\in \mathcal{L}(L_q(\Omega)^{N^3+2N^2+3N}, W^2_q(\Omega)), \\ \mathcal{P}(\lambda) &\in \mathcal{L}(L_q(\Omega)^{N^3+2N^2+4N}, W^1_q(\Omega)), \\ \mathcal{V}_N(\lambda) &\in \mathcal{L}(W^1_q(\Omega) \cap \widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega), W^2_q(\Omega)), \end{aligned}$$

$$\mathcal{Q}(\lambda) \in \mathcal{L}(W^1_q(\Omega) \cap \widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega), W^1_q(\Omega))$$

such that the following assertions hold.

(1) *Setting* $\mathbf{u} = (u_1, \ldots, u_N)^T$ *and* θ *as*

$$\begin{aligned}
u_j =& \mathcal{U}_j(\lambda)\Big(\mathbf{f}, \nabla f_d, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, K(\lambda f_d), \nabla K(\lambda^{1/2} f_d), \nabla^2 K f_d,\\
&\quad D_j\mathcal{V}_N(\lambda)(\lambda^{1/2} f_d), \nabla\mathcal{V}_N(\lambda)(D_j f_d), D_j\mathcal{Q}(\lambda)(\lambda f_d)\Big) + K_j f_d,\\
u_N =& \mathcal{U}_N(\lambda)\Big(\mathbf{f}, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, K(\lambda f_d), \nabla K(\lambda^{1/2} f_d), \nabla^2 K f_d\Big)\\
&\quad + \mathcal{V}_N(\lambda) f_d + K_N f_d,\\
\theta =& \mathcal{P}(\lambda)\Big(\mathbf{f}, \nabla f_d, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, K(\lambda f_d), \nabla K(\lambda^{1/2} f_d), \nabla^2 K f_d\Big) + \mathcal{Q}(\lambda)(\lambda f_d)
\end{aligned}$$

for $j = 1, \ldots, N-1$*, where* K *is the operator obtained in Theorem 2.1.2, we see that* $(\mathbf{u}, \theta)$ *solves uniquely (2.1.1) for* $\lambda \in \Sigma_{\varepsilon,\gamma_0}$*. In addition, there exists a positive constant* $M = M(N, q, \varepsilon, \gamma_0, \mu, \delta)$*, depending only on* $N, q, \varepsilon, \gamma_0, \mu$*, and* δ*, such that*

$$\begin{aligned}
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda\mathcal{U}_J(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le M,\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\gamma\mathcal{U}_J(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le M,\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla\mathcal{U}_J(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le M,\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla^2\mathcal{U}_J(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le M,\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell\mathcal{P}(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le M,\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{P}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le M,\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{V}_N(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le M
\end{aligned}$$

for $\ell = 0, 1$ *and* $J = 1, \ldots, N$*. Concerning the second spatial derivatives of* $\mathcal{V}_N(\lambda) f_d$*, we have*

$$D_j D_k \mathcal{V}_N(\lambda) f_d = \begin{cases} D_j\mathcal{V}_N(\lambda)(D_k f_d) & \text{if } k \neq N,\\ D_k\mathcal{V}_N(\lambda)(D_j f_d) & \text{if } j \neq N,\end{cases}$$

and besides, for every $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ *there exists* $\widetilde{\mathcal{V}}_N(\lambda) \in \mathcal{L}(L_q(\Omega)^{N+1}, L_q(\Omega))$ *such that* $D_N^2\mathcal{V}_N(\lambda) f_d = \widetilde{\mathcal{V}}_N(\lambda)(\lambda^{1/2} f_d, \nabla f_d)$ *and*

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell\widetilde{\mathcal{V}}_N(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) \le M$$

for $\ell = 0, 1$ *and some positive constant* $M = M(N, q, \varepsilon, \mu, \gamma_0, \delta)$*.*

(2) *Let* $q' = q/(q-1)$*. Then, for every* $\lambda \in \Sigma_{\varepsilon,\gamma_0}$*, there exist bounded linear operators*

$$\mathcal{V}_N^*(\lambda), \mathcal{Q}^*(\lambda), \mathcal{Q}_1^*(\lambda), \ldots, \mathcal{Q}_N^*(\lambda) : L_{q'}(\Omega) \to \widehat{W}^1_{q',\Gamma_\delta}(\Omega)$$

such that for $J = 1, \ldots, N$*,* $\psi \in W^1_q(\Omega) \cap \widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)$*, and* $\varphi \in L_{q'}(\Omega)$

$$\begin{aligned}
(\mathcal{V}_N(\lambda)\psi, \varphi)_\Omega &= (\psi, \mathcal{V}_N^*(\lambda)\varphi)_\Omega,\\
(\mathcal{Q}(\lambda)\psi, \varphi)_\Omega &= (\psi, \mathcal{Q}^*(\lambda)\varphi)_\Omega,\\
(D_J\mathcal{Q}(\lambda)\psi, \varphi)_\Omega &= (\psi, \mathcal{Q}_J^*(\lambda)\varphi)_\Omega.
\end{aligned}$$

In addition, there exists a positive constant $M = M(N, q, \varepsilon, \mu, \gamma_0, \delta)$ such that for $\ell = 0, 1$ and $J = 0, 1 \ldots, N$

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{V}_N^*(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq M,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{Q}^*(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq M,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{Q}_J^*(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq M.$$

The original paper of this chapter is [**Sai15**], and also this chapter is organized as follows: In the next section, we prove the maximal L_p-L_q regularity theorem of the Stokes equations, corresponding to (2.1.1), with help of Theorem 2.1.2, Theorem 2.1.3, and Weis's operator valued Fourier multiplier theorem [**Wei01**, Theorem 3.4]. In addition, we consider some resolvent problem, in the whole space $\mathbf{R}^N$, which plays an important role in later sections. In Section 2.3, we prove Theorem 2.1.2. In Section 2.4, we give exact solution formulas for some reduced system of the equations (2.1.1). The methods to obtain the formulas are based on solving ordinary differential equations, with respect to x_N variable in the Fourier space, which are obtained by using the partial Fourier transform with respect to x' variable. In Section 2.5, we prove some technical lemmas which are used to show the $\mathcal{R}$-boundedness of solution operator families. In Section 2.6, applying the technical lemmas in Section 2.5 to the exact solution formulas obtained in Section 2.4, we show the $\mathcal{R}$-boundedness of solution operator families of the equations (2.1.1). In Section 2.7, we show that solutions to the Stokes equations (2.2.1) introduced in the Section 2.2 satisfy the uniqueness and initial conditions.

2.2. Maximal L_p-L_q regularity theorem

In this section, we first show the maximal L_p-L_q theorem for the following Stokes equations as an application of Theorem 2.1.3:

$$(2.2.1)\qquad \begin{cases} \partial_t\mathbf{U} - \operatorname{Div}\mathbf{S}(\mathbf{U},\Theta) = \mathbf{F}, \quad \operatorname{div}\mathbf{U} = F_d & \text{in } \Omega\times(0,\infty), \\ \mathbf{S}(\mathbf{U},\Theta)\mathbf{e}_N = \mathbf{G} & \text{on } \Gamma_\delta\times(0,\infty), \\ \mathbf{U} = 0 & \text{on } \Gamma_0\times(0,\infty), \end{cases}$$

subject to the initial condition: $\mathbf{U}(x,0) = 0$ in Ω. Here $\mathbf{U} = \mathbf{U}(x,t), \Theta = \Theta(x,t), \mathbf{F} = \mathbf{F}(x,t), \mathbf{G} = \mathbf{G}(x,t)$, and $F_d = F_d(x,t)$ denote time-dependent functions corresponding to $\mathbf{u}, \theta, \mathbf{f}, \mathbf{g}$, and f_d in (2.1.1). For simplicity, we set

$$W^{\pm1,1}_{q,p,\gamma,0}(\Omega\times\mathbf{R}) = W^1_{p,\gamma,0}(\mathbf{R}, \widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)) \cap L_{p,\gamma,0}(\mathbf{R}, W^1_q(\Omega)).$$

We then obtain the maximal L_p-L_q regularity theorem as follows:

THEOREM 2.2.1. *Let $1 < p, q < \infty$ and $\gamma_0 > 0$. Then, for every*

$$\mathbf{F} \in L_{p,\gamma_0,0}(\mathbf{R}, L_q(\Omega))^N, \quad F_d \in W^{\pm1,1}_{q,p,\gamma_0,0}(\Omega\times\mathbf{R}), \quad \mathbf{G} \in H^{1,1/2}_{q,p,\gamma_0,0}(\Omega\times\mathbf{R})^N,$$

equation (2.2.1) *admits a unique solution* $(\mathbf{U}, \Theta)$ *with*

$$\mathbf{U} \in (W^1_{p,\mathrm{loc},0}(\mathbf{R}, L_q(\Omega)) \cap L_{p,\mathrm{loc},0}(\mathbf{R}, W^2_q(\Omega))^N, \quad \Theta \in L_{p,\mathrm{loc},0}(\mathbf{R}, W^1_q(\Omega)).$$

In addition, the solution $(\mathbf{U},\Theta)$ *satisfies the estimate:*

$$
\begin{aligned}
&\|e^{-\gamma_0 t}(\partial_t\mathbf{U},\mathbf{U},\Lambda_{\gamma_0}^{1/2}\nabla\mathbf{U},\nabla^2\mathbf{U})\|_{L_p(\mathbf{R},L_q(\Omega))}+\|e^{-\gamma_0 t}\Theta\|_{L_p(\mathbf{R},W_q^1(\Omega))}\\
&\le C\Big(\|e^{-\gamma_0 t}(\mathbf{F},\Lambda_{\gamma_0}^{1/2}F_d,\nabla F_d,\Lambda_{\gamma_0}^{1/2}\mathbf{G},\nabla\mathbf{G})\|_{L_p(\mathbf{R},L_q(\Omega))}\\
&\quad+\|e^{-\gamma_0 t}(\partial_t F_d,F_d)\|_{L_p(\mathbf{R},\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega))}\Big)
\end{aligned}
\tag{2.2.2}
$$

for some positive constant $C=C(N,p,q,\gamma_0,\mu,\delta)$ *depending only on* $N,p,q,\gamma_0,\mu,$ *and* δ*. If* F_d*=0 and* $\mathbf{G}=0$*, then the solution* $\mathbf{U}$ *satisfies*

$$
\gamma\|e^{-\gamma t}\mathbf{U}\|_{L_p(\mathbf{R},L_q(\Omega))}\le C\|e^{-\gamma t}\mathbf{F}\|_{L_p(\mathbf{R},L_q(\Omega))}
\tag{2.2.3}
$$

for any $\gamma\ge\gamma_0$ *with some positive constant* C *independent of* γ*.*

Remark 2.2.2. Although trace spaces for the boundary data $\mathbf{G}$ does not appear explicitly since $\mathbf{G}$ is always defined on Ω, it is given by the so-called *Triebel-Lizorkin spaces* (cf. e.g. [**DHP07**, Theorem 2.3]).

Proof. We only show (2.2.2) here, and note that the uniqueness and initial condition will be discussed in Section 2.7.

By applying the Laplace transform to (2.2.1) and setting

$$
\mathbf{u}=\mathcal{L}[\mathbf{U}](\lambda),\ \ \theta=\mathcal{L}[\Theta](\lambda),\ \ \mathbf{f}=\mathcal{L}[\mathbf{F}](\lambda),\ \ f_d=\mathcal{L}[F_d](\lambda),\ \ \mathbf{g}=\mathcal{L}[\mathbf{G}](\lambda),
$$

we see that $(\mathbf{u},\theta)$ satisfies (2.1.1) with right members $\mathbf{f},f_d,$ and $\mathbf{g}$. Therefore, by Theorem 2.1.3, solutions $(\mathbf{U},\Theta)$ of equations (2.2.1) are given by

(2.2.4)

$$
\begin{aligned}
&U_j(t)=K_j(F_d(t))+\\
&\mathcal{L}^{-1}_{\lambda_0}\Big[\mathcal{U}_j(\lambda_0)\mathcal{L}\Big[\Big(\mathbf{F},\nabla F_d,\widetilde{\mathbf{G}},\nabla\mathbf{G},K(\partial_t F_d),\nabla K\widetilde{F}_d,\nabla^2KF_d,F_d^1,F_d^2,F_d^3\Big)\Big](\lambda_0)\Big](t),\\
&U_N(t)=K_N(F_d(t))+\mathcal{L}^{-1}_{\lambda_0}\left[\mathcal{V}_N(\lambda_0)\mathcal{L}[F_d](\lambda_0)\right](t)\\
&+\mathcal{L}^{-1}_{\lambda_0}\Big[\mathcal{U}_N(\lambda_0)\mathcal{L}\Big[\Big(\mathbf{F},\widetilde{\mathbf{G}},\nabla\mathbf{G},K(\partial_t F_d),\nabla K\widetilde{F}_d,\nabla^2KF_d\Big)\Big](\lambda_0)\Big](t),\\
&\Theta(t)=\mathcal{L}^{-1}_{\lambda_0}\Big[\mathcal{P}(\lambda_0)\mathcal{L}\Big[\Big(\mathbf{F},\nabla F_d,\widetilde{\mathbf{G}},\nabla\mathbf{G},K(\partial_t F_d),\nabla K\widetilde{F}_d,\nabla^2KF_d\Big)\Big](\lambda_0)\Big](t)\\
&+\mathcal{L}^{-1}_{\lambda_0}\left[\mathcal{Q}(\lambda_0)\lambda_0\mathcal{L}[F_d](\lambda_0)\right](t)
\end{aligned}
$$

for $j=1,\dots,N-1$, where $\lambda_0=\gamma_0+i\tau$ and K_J $(J=1,\dots,N)$ is the operator obtained in Theorem 2.1.2. Here F_d^k $(k=1,2,3)$ are defined as

$$
\begin{aligned}
&F_d^1=\mathcal{L}^{-1}_{\lambda_0}[D_j\mathcal{V}_N(\lambda_0)\widetilde{F}_d](t),\quad F_d^2=\mathcal{L}^{-1}_{\lambda_0}[\nabla\mathcal{V}_N(\lambda_0)\mathcal{L}[D_jF_d](\lambda_0)](t),\\
&F_d^3=\mathcal{L}^{-1}_{\lambda_0}[D_j\mathcal{Q}(\lambda_0)\lambda_0\mathcal{L}[F_d](\lambda_0)](t),
\end{aligned}
$$

and also the symbol $\widetilde{X}$ for $X\in\{\mathbf{G},F_d\}$ denotes $\widetilde{X}=\mathcal{L}^{-1}_{\lambda_0}[\lambda_0^{1/2}\mathcal{L}[X](\lambda_0)](t)$. Then, by the Fourier multiplier theorem of Zimmermann [**Zim89**, Proposition 3], it holds that

$$
\|e^{-\gamma_0 t}\widetilde{X}\|_{L_p(\mathbf{R},L_q(\Omega))}\le C(N,q)\|e^{-\gamma_0 t}\Lambda_{\gamma_0}^{1/2}X\|_{L_p(\mathbf{R},L_q(\Omega))}.
\tag{2.2.5}
$$

By formulas (2.2.4), Theorem 2.1.2, Theorem 2.1.3 (1), and [**Wei01**, Theorem 3.4], we obtain

$$
\begin{aligned}
(2.2.6)\qquad & \|e^{-\gamma_0 t}(\partial_t \mathbf{U}, \mathbf{U}, \Lambda_{\gamma_0}^{1/2}\nabla \mathbf{U}, \nabla^2 \mathbf{U})\|_{L_p(\mathbf{R}, L_q(\Omega))} + \|e^{-\gamma_0 t}\Theta\|_{L_p(\mathbf{R}, W_q^1(\Omega))} \\
& \le C\Big(\|e^{-\gamma_0 t}(\mathbf{F}, \Lambda_{\gamma_0}^{1/2}F_d, \nabla F_d, \Lambda_{\gamma_0}^{1/2}\mathbf{G}, \nabla \mathbf{G})\|_{L_p(\mathbf{R}, L_q(\Omega))} \\
& \quad + \|e^{-\gamma_0 t}(F_d, \partial_t F_d)\|_{L_p(\mathbf{R}, \widehat{W}_{q,\Gamma_\delta}^{-1}(\Omega))} \\
& \quad + \sum_{k=1}^{3}\|e^{-\gamma_0 t}F_d^k\|_{L_p(\mathbf{R}, L_q(\Omega))} + \sum_{\ell=1}^{6}\|e^{-\gamma_0 t}I_\ell\|_{L_p(\mathbf{R}, L_q(\Omega))}\Big)
\end{aligned}
$$

for a positive constant $C = C(N, p, q, \gamma_0, \mu, \delta)$, where I_ℓ $(\ell = 1, \dots, 6)$ are given by

$$
\begin{aligned}
I_1(t) &= \mathcal{L}_{\lambda_0}^{-1}[\lambda_0 \mathcal{V}_N(\lambda_0)\mathcal{L}[F_d](\lambda_0)](t), & I_2(t) &= \mathcal{L}_{\lambda_0}^{-1}[\gamma_0 \mathcal{V}_N(\lambda_0)\mathcal{L}[F_d](\lambda_0)](t), \\
I_3(t) &= \mathcal{L}_{\lambda_0}^{-1}[\nabla \mathcal{V}_N(\lambda_0)\mathcal{L}[\widetilde{F}_d](\lambda_0)](t), & I_4(t) &= \mathcal{L}_{\lambda_0}^{-1}[\nabla^2 \mathcal{V}_N(\lambda_0)\mathcal{L}[F_d](\lambda_0)](t), \\
I_5(t) &= \mathcal{L}_{\lambda_0}^{-1}[\mathcal{Q}(\lambda_0)\lambda_0\mathcal{L}[F_d](\lambda_0)](t), & I_6(t) &= \mathcal{L}_{\lambda_0}^{-1}[\nabla \mathcal{Q}(\lambda_0)\lambda_0\mathcal{L}[F_d](\lambda_0)](t).
\end{aligned}
$$

It is clear that

$$
(2.2.7)\qquad \|e^{-\gamma_0 t}F_d^3\|_{L_p(\mathbf{R}, L_q(\Omega))} \le \|e^{-\gamma_0 t}I_6\|_{L_p(\mathbf{R}, L_q(\Omega))},
$$

and furthermore, by (2.2.5)

(2.2.8)

$$
\begin{aligned}
& \|e^{-\gamma_0 t}F_d^1\|_{L_p(\mathbf{R}, L_q(\Omega))} \le C\|e^{-\gamma_0 t}\widetilde{F}_d\|_{L_p(\mathbf{R}, L_q(\Omega))} \le C\|e^{-\gamma_0 t}\Lambda_{\gamma_0}^{1/2}F_d\|_{L_p(\mathbf{R}, L_q(\Omega))}, \\
& \|e^{-\gamma_0 t}F_d^2\|_{L_p(\mathbf{R}, L_q(\Omega))} \le C\|e^{-\gamma_0 t}\nabla F_d\|_{L_p(\mathbf{R}, L_q(\Omega))}
\end{aligned}
$$

in the same way as we have obtained (2.2.6). Therefore, it is sufficient to consider estimates of $I_\ell(t)$ $(\ell = 1, \dots, 6)$.

LEMMA 2.2.3. *Let $1 < p, q < \infty$ and $\gamma_0 > 0$, and let $F_d \in W_{q,p,\gamma_0,0}^{\pm 1,1}(\mathbf{R} \times \Omega)$. Then there exists a positive constant $C = C(N, p, q, \gamma_0, \mu, \delta)$ such that*

$$
\begin{aligned}
\|e^{-\gamma_0 t}(I_1, I_5, I_6)\|_{L_p(\mathbf{R}, L_q(\Omega))} &\le C\|e^{-\gamma_0 t}\partial_t F_d\|_{L_p(\mathbf{R}, \widehat{W}_{q,\Gamma_\delta}^{-1}(\Omega))}, \\
\|e^{-\gamma_0 t}I_2\|_{L_p(\mathbf{R}, L_q(\Omega))} &\le C\|e^{-\gamma_0 t}F_d\|_{L_p(\mathbf{R}, \widehat{W}_{q,\Gamma_\delta}^{-1}(\Omega))}, \\
\|e^{-\gamma_0 t}(I_3, I_4)\|_{L_p(\mathbf{R}, L_q(\Omega))} &\le C\|e^{-\gamma_0 t}(\Lambda_{\gamma_0}^{1/2}F_d, \nabla F_d)\|_{L_p(\mathbf{R}, L_q(\Omega))}.
\end{aligned}
$$

PROOF. Here we consider only $I_1(t)$, because we can show the required estimates of $I_2(t)$, $I_5(t)$, and $I_6(t)$ similarly. The estimates of $I_3(t)$ and $I_4(t)$ are proved by combining Theorem 2.1.3 (1) with [**Wei01**, Theorem 3.4] and using (2.2.5).

Let $\varphi \in C_0^\infty(\mathbf{R}^{N+1})$, and note $e^{-\gamma_0 t}I_1(t) = \mathcal{F}_\tau^{-1}[\lambda_0 \mathcal{V}_N(\lambda_0)\mathcal{F}[e^{-\gamma_0 t}F_d](\lambda_0)](t)$ for $\lambda_0 = \gamma_0 + i\tau$. Then, by Theorem 2.1.3 (2), there holds

$$
\begin{aligned}
(e^{-\gamma_0 t}I_1, \varphi)_{\mathbf{R}\times\Omega} &= (F_d, e^{-\gamma_0 t}\mathcal{F}_\tau[\lambda_0 \mathcal{V}_N^*(\lambda_0)\mathcal{F}_t^{-1}[\varphi](\tau)])_{\mathbf{R}\times\Omega} \\
&= -(F_d, \partial_t(e^{-\gamma_0 t}\mathcal{F}_\tau[\mathcal{V}_N^*(\lambda_0)\mathcal{F}_t^{-1}[\varphi](\tau)]))_{\mathbf{R}\times\Omega} \\
&= (e^{-\gamma_0 t}\partial_t F_d, \mathcal{F}_\tau[\mathcal{V}_N^*(\lambda_0)\mathcal{F}_t^{-1}[\varphi](\tau)])_{\mathbf{R}\times\Omega},
\end{aligned}
$$

which, combined with Theorem 2.1.3 (2) and [**Wei01**, Theorem 3.4], furnishes that

$$
\begin{aligned}
&|(e^{-\gamma_0 t} I_1, \varphi)_{\mathbf{R}\times\Omega}| \\
&\le \int_{\mathbf{R}} \|e^{-\gamma_0 t}\partial_t F_d(t)\|_{\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)} \|\mathcal{F}_\tau[\nabla \mathcal{V}_N^*(\lambda_0)\mathcal{F}_t^{-1}[\varphi](\tau)](t)\|_{L_{q'}(\Omega)}\,dt \\
&\le C\|e^{-\gamma_0 t}\partial_t F_d\|_{L_p(\mathbf{R},\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega))}\|\varphi\|_{L_{p'}(\mathbf{R},L_{q'}(\Omega))}
\end{aligned}
$$

for $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$ with $C = C(N, p, q, \gamma_0, \mu, \delta)$. This implies that the required inequalities hold. □

Combining Lemma 2.2.3, (2.2.7), and (2.2.8) with (2.2.6), we complete the proof of Theorem 2.2.1. □

In the proof of Theorem 2.1.3, we deal with the following resolvent problem in the whole space:

$$
\lambda u - \mu\Delta u = f \quad \text{in } \mathbf{R}^N, \tag{2.2.9}
$$

where $u = u(x)$ is a scalar unknown function. Concerning (2.2.9), we have the following lemma.

LEMMA 2.2.4. *Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Then, for every $\lambda \in \Sigma_\varepsilon$, there exist operators $H(\lambda) \in \mathcal{L}(L_q(\mathbf{R}^N), W_q^2(\mathbf{R}^N))$ such that $u = H(\lambda)f$ uniquely solves (2.2.9) for $f \in L_q(\mathbf{R}^N)$, and also there hold*

$$
\begin{aligned}
\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N))}(\{(\tau\partial_\tau)^\ell(\lambda H(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_\varepsilon\}) &\le M, \\
\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N))}(\{(\tau\partial_\tau)^\ell(\gamma H(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_\varepsilon\}) &\le M, \\
\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N))}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla H(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_\varepsilon\}) &\le M, \\
\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N))}(\{(\tau\partial_\tau)^\ell(\nabla^2 H(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_\varepsilon\}) &\le M
\end{aligned} \tag{2.2.10}
$$

for $\ell = 0, 1$ with some positive constant M depending only on N, q, ε, and μ.

PROOF. For $\lambda \in \Sigma_\varepsilon$, we set $H(\lambda)f = \mathcal{F}_\xi^{-1}[(\lambda + \mu|\xi|^2)^{-1}]\mathcal{F}[f](\xi)$, and then $u(x) = H(\lambda)f$ are solutions to (2.2.9). We can show (2.2.10) by the solution formula above and [**ES13**, Theorem 3.3]. □

In the remaining of this section, for functions $\mathbf{f}$ defined on Ω, we consider the following resolvent problem in the whole space:

$$
\lambda\mathbf{u} - \operatorname{Div}\mathbf{S}(\mathbf{u}, \theta) = E\mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \mathbf{R}^N, \tag{2.2.11}
$$

where $\mathbf{u} = (u_1(x), \dots, u_N(x))^T$ and $\theta = \theta(x)$ are unknown functions. Here $E\mathbf{f}$ denotes some extension of $\mathbf{f}$ from Ω to $\mathbf{R}^N$ as follows: First, let $g(x)$ be functions defined on Ω, and set $g_0(x) = \varphi_0(x_N)g(x)$ and $g_\delta(x) = \varphi_\delta(x_N)g(x)$ for the cut-off functions φ_0 and φ_δ defined as the next line of (2.1.2). Second, we set

$$
\begin{aligned}
g_0^e(x) &= \begin{cases} g_0(x', x_N) & x_N > 0, \\ g_0(x', -x_N) & x_N < 0, \end{cases} \qquad
g_0^o(x) = \begin{cases} g_0(x', x_N) & x_N > 0, \\ -g_0(x', -x_N) & x_N < 0, \end{cases} \\
g_\delta^e(x) &= \begin{cases} g_\delta(x', x_N) & x_N < \delta, \\ g_\delta(x', 2\delta - x_N) & x_N > \delta, \end{cases} \\
g_\delta^o(x) &= \begin{cases} g_\delta(x', x_N) & x_N < \delta, \\ -g_\delta(x', 2\delta - x_N) & x_N > \delta. \end{cases}
\end{aligned} \tag{2.2.12}
$$

Third, $E\mathbf{f} = (\bar{f}_1, \dots, \bar{f}_N)$ with $\bar{f}_j(x) = f^e_{j0}(x) + f^o_{j\delta}(x)$ $(j = 1, \dots, N-1)$ and $\bar{f}_N(x) = f^o_{N0}(x) + f^e_{N\delta}(x)$.

LEMMA 2.2.5. *Let $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Then, for every $\lambda \in \Sigma_\varepsilon$, there exist operators $S_0(\lambda) \in \mathcal{L}(L_q(\Omega)^N, W^2_q(\Omega)^N)$ and $T_0 \in \mathcal{L}(L_q(\Omega)^N, W^1_q(\Omega))$ such that $(\mathbf{u}, \theta) = (S_0(\lambda)\mathbf{f}, T_0\mathbf{f})$ uniquely solves (2.2.11) for $\mathbf{f} \in L_q(\Omega)^N$. In addition, there hold*

$$
\begin{aligned}
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda S_0(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_\varepsilon\}) \leq M,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\gamma S_0(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_\varepsilon\}) \leq M,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla S_0(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_\varepsilon\}) \leq M,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla^2 S_0(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_\varepsilon\}) \leq M,\\
&\|T_0\|_{\mathcal{L}(L_q(\Omega)^N, L_q(\Omega))} + \|\nabla T_0\|_{\mathcal{L}(L_q(\Omega)^N)} \leq M
\end{aligned}
\tag{2.2.13}
$$

for $\ell = 0, 1$ with some positive constant M depending only on N, q, ε, μ, and δ.

PROOF. The uniqueness is guaranteed by [**SS12**, Theorem 3.1]. We here show properties (2.2.13) only. For any $\lambda \in \Sigma_\varepsilon$, representation formulas of the solution $(\mathbf{u}, \theta)$ to equations (2.2.11) are given by

$$
\begin{aligned}
\mathbf{u} &= \bar{S}_0(\lambda)(E\mathbf{f}) = \mathcal{F}^{-1}_\xi\left[\frac{P(\xi)\mathcal{F}[E\mathbf{f}](\xi)}{\lambda + \mu|\xi|^2}\right](x),\\
\theta &= \bar{T}_0 E\mathbf{f} = -\mathcal{F}^{-1}_\xi\left[\frac{i\xi \cdot \mathcal{F}[E\mathbf{f}](\xi)}{|\xi|^2}\right](x),
\end{aligned}
\tag{2.2.14}
$$

where $P(\xi)$ is an $N \times N$ matrix whose (j, k) component $P_{jk}(\xi)$ is given by $P_{jk} = \delta_{jk} - \xi_j\xi_k|\xi|^{-2}$ (cf. [**SS12**, Section 3]). The properties: $\|\nabla \bar{T}_0\|_{\mathcal{L}(L_q(\mathbf{R}^N)^N)} \leq C$ and

$$
\begin{aligned}
&\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N))}(\{(\tau\partial_\tau)^\ell(\lambda \bar{S}_0(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N))}(\{(\tau\partial_\tau)^\ell(\gamma \bar{S}_0(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N))}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla \bar{S}_0(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N))}(\{(\tau\partial_\tau)^\ell(\nabla^2 \bar{S}_0(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \leq C
\end{aligned}
$$

for some positive constant C, follow from [**ES13**, Theorem 3.3]. Thus, setting $S_0(\lambda) = \bar{S}_0(\lambda)E$ and $T_0 = \bar{T}_0 E$, we have (2.2.13) by Proposition 1.2.3 except for the boundedness of T_0. From now on, we show the required estimate concerning $\|T_0\|_{\mathcal{L}(L_q(\Omega)^N, L_q(\Omega))}$. By the definition of $E\mathbf{f}$, we have

$$
\begin{aligned}
&\mathcal{F}_{x'}[T_0\mathbf{f}](\xi', x_N)\\
&= -\sum_{j=1}^{N-1} i\xi_j \int_0^\delta \varphi_0(y_N)\mathcal{F}^{-1}_{\xi_N}\left[\frac{e^{-iy_N\xi_N} + e^{iy_N\xi_N}}{|\xi|^2}\right](x_N)\widehat{f}_j(\xi', y_N)\,dy_N\\
&- \sum_{j=1}^{N-1} i\xi_j \int_0^\delta \varphi_\delta(y_N)\mathcal{F}^{-1}_{\xi_N}\left[\frac{e^{-iy_N\xi_N} - e^{-i(2\delta - y_N)\xi_N}}{|\xi|^2}\right](x_N)\widehat{f}_j(\xi', y_N)\,dy_N\\
&- \int_0^\delta \varphi_0(y_N)\mathcal{F}^{-1}_{\xi_N}\left[\frac{i\xi_N(e^{-iy_N\xi_N} - e^{iy_N\xi_N})}{|\xi|^2}\right](x_N)\widehat{f}_N(\xi', y_N)\,dy_N\\
&- \int_0^\delta \varphi_\delta(y_N)\mathcal{F}^{-1}_{\xi_N}\left[\frac{i\xi_N(e^{-iy_N\xi_N} + e^{-i(2\delta - y_N)\xi_N})}{|\xi|^2}\right](x_N)\widehat{f}_N(\xi', y_N)\,dy_N.
\end{aligned}
$$

Inserting the identities in Lemma 1.2.7 into the formula above, we have, by the inverse partial Fourier transform with respect to ξ',

(2.2.15)

$$\begin{aligned}
T_0\mathbf{f} = &-\sum_{j=1}^{N-1}\int_0^\delta \frac{\varphi_0(y_N)}{2}\mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_j}{A}\left(e^{-|x_N-y_N|A}+e^{-(x_N+y_N)A}\right)\widehat{f}_j(y_N)\right]dy_N\\
&-\sum_{j=1}^{N-1}\int_0^\delta \frac{\varphi_\delta(y_N)}{2}\mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_j}{A}\left(e^{-|x_N-y_N|A}-e^{-(2\delta-x_N-y_N)A}\right)\widehat{f}_j(y_N)\right]dy_N\\
&+\int_0^\delta \frac{\varphi_0(y_N)}{2}\mathcal{F}_{\xi'}^{-1}\left[\left(\mathrm{sign}(x_N-y_N)e^{-|x_N-y_N|A}-e^{-(x_N+y_N)A}\right)\widehat{f}_N(y_N)\right]dy_N\\
&+\int_0^\delta \frac{\varphi_\delta(y_N)}{2}\mathcal{F}_{\xi'}^{-1}\left[\left(\mathrm{sign}(x_N-y_N)e^{-|x_N-y_N|A}-e^{-(2\delta-x_N-y_N)A}\right)\widehat{f}_N(y_N)\right]dy_N
\end{aligned}$$

for $0\leq x_N\leq\delta$, where $\mathcal{F}_{\xi'}^{-1}[g]=\mathcal{F}_{\xi'}^{-1}[g](x')$ and $\widehat{h}(y_N)=\widehat{h}(\xi',y_N)$. By Lemma 1.2.6 and Leibniz's rule, it holds that for $s=0,1$, $j=1,\ldots,N-1$, and $0\leq x_N,y_N\leq\delta$,

$$\begin{aligned}
&\left|D_{\xi'}^{\alpha'}\left(\frac{i\xi_j}{A}e^{-|x_N-y_N|A}\right)\right|\leq CA^{-|\alpha'|},\quad \left|D_{\xi'}^{\alpha'}\left(\left(\frac{i\xi_j}{A}\right)^s e^{-(x_N+y_N)A}\right)\right|\leq CA^{-|\alpha'|},\\
&\left|D_{\xi'}^{\alpha'}\left(\left(\frac{i\xi_j}{A}\right)^s e^{-(2d-x_N-y_N)A}\right)\right|\leq CA^{-|\alpha'|},\\
&\left|D_{\xi'}^{\alpha'}\left(\mathrm{sign}(x_N-y_N)e^{-|x_N-y_N|A}\right)\right|\leq CA^{-|\alpha'|}
\end{aligned}$$

with some positive constant $C=C(\alpha')$. Especially, C is independent of x_N,y_N,ξ', and δ. Thus, using the Fourier multiplier theorem of Mikhlin-Hörmander type (cf. [**Mik65**, Appendix Theorem 2]) with respect to x' and Hölder's inequality to the formal (2.2.15), we have

$$\begin{aligned}
&\|(T_0f)(\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\leq C(N,q)\int_0^\delta \|f(\cdot,y_N)\|_{L_q(\mathbf{R}^{N-1})}dy_N\leq C(N,q)\delta^{1-(1/q)}\|f\|_{L_q(\Omega)},
\end{aligned}$$

which implies that $\|T_0f\|_{L_q(\Omega)}\leq C(N,q,\delta)\|f\|_{L_q(\Omega)}$ for some positive constant $C(N,q,\delta)$. This completes the proof. □

REMARK 2.2.6. Let $S_0(\lambda)\mathbf{f}=(S_{01}(\lambda)\mathbf{f},\ldots,S_{0N}(\lambda)\mathbf{f})^T$. Then, by the definition of $E\mathbf{f}$ and (2.2.14), we have

$$\begin{aligned}
&\mathcal{F}_{x'}[S_{0N}(\lambda)\mathbf{f}](\xi',0)\\
&=\sum_{k=1}^{N-1}\frac{i\xi_k}{2\pi}\int_0^\delta \varphi_0(y_N)\left\{\int_{-\infty}^{\infty}\frac{i\xi_N(e^{-iy_N\xi_N}+e^{iy_N\xi_N})}{|\xi|^2(\lambda+\mu|\xi|^2)}d\xi_N\right\}\widehat{f}_k\xi',y_N)dy_N\\
&+\sum_{k=1}^{N-1}\frac{i\xi_k}{2\pi}\int_0^\delta \varphi_\delta(y_N)\left\{\int_{-\infty}^{\infty}\frac{i\xi_N(e^{-iy_N\xi_N}-e^{-i(2\delta-y_N)\xi_N})}{|\xi|^2(\lambda+\mu|\xi|^2)}d\xi_N\right\}\widehat{f}_k(\xi',y_N)dy_N\\
&+\frac{A^2}{2\pi}\int_0^\delta \varphi_0(y_N)\left\{\int_{-\infty}^{\infty}\frac{e^{-iy_N\xi_N}-e^{iy_N\xi_N}}{|\xi|^2(\lambda+\mu|\xi|^2)}d\xi_N\right\}\widehat{f}_N(\xi',y_N)dy_N
\end{aligned}$$

$$+\frac{A^2}{2\pi}\int_0^\delta \varphi_\delta(y_N)\left\{\int_{-\infty}^{\infty}\frac{e^{-iy_N\xi_N}+e^{-i(2\delta-y_N)\xi_N}}{|\xi|^2(\lambda+\mu|\xi|^2)}d\xi_N\right\}\widehat{f}_N(\xi',y_N)dy_N.$$

Inserting the identities in Lemma 1.2.7 into the above formula furnishes that (2.2.16)

$$\begin{aligned}&\mathcal{F}_{x'}[S_{0N}(\lambda)\mathbf{f}](\xi',0)\\&=\sum_{k=1}^{N-1}\sum_{n=1}^{2}\frac{i\xi_k(-1)^n}{2\lambda}\int_0^\delta \varphi_\delta(y_N)(e^{-A(d_n(0)+d_n(y_N))}-e^{-B(d_n(0)+d_n(y_N))})\widehat{f}_k(y_N)dy_N\\&+\frac{A^2}{2\lambda}\sum_{n=1}^{2}\int_0^\delta \varphi_\delta(y_N)(\frac{e^{-A(d_n(0)+d_n(y_N))}}{A}-\frac{e^{-B(d_n(0)+d_n(y_N))}}{B})\widehat{f}_N(y_N)dy_N,\end{aligned}$$

where we have used the abbreviation: $\widehat{g}(y_N)=\widehat{g}(\xi',y_N)$.

2.3. Proof of Theorem 2.1.2

In this section, we shall prove Theorem 2.1.2. Let $f_d\in W^1_q(\Omega)\cap\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)$, and set $f_d^*(x)=f^o_{d0}(x)+f^o_{d\delta}(x)$ by using (2.2.12). We first show the estimate: $\|f_d^*\|_{\widehat{W}^{-1}_q(\mathbf{R}^N)}\leq C\|f_d\|_{\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)}$ with a positive constant C independent of f_d and f_d^*. By the definition of f_d^*, we have

$$\begin{aligned}&(f_d^*,\psi)_{\mathbf{R}^N}\\&\qquad=\int_\Omega(-\varphi_\delta(x_N)\psi(x',2\delta-x_N)+\psi(x',x_N)-\varphi_0(x_N)\psi(x',-x_N))f_d(x)\,dx\end{aligned}$$

for any $\psi\in C_0^\infty(\mathbf{R}^N)$. We thus see that

$$\begin{aligned}&|(f_d^*,\psi)_{\mathbf{R}^N}|\leq\|f_d\|_{\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)}\\&\quad\times\Big\|\nabla\Big(-\varphi_\delta(x_N)\psi(x',2\delta-x_N)+\psi(x',x_N)-\varphi_0(x_N)\psi(x',-x_N)\Big)\Big\|_{L_{q'}(\Omega)}\\&\leq C\|f_d\|_{\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)}\Big(\|\psi(x',2\delta-x_N)-\psi(x',-x_N)\|_{L_{q'}(\Omega)}+\|\nabla\psi\|_{L_{q'}(\mathbf{R}^N)}\Big)\\&\leq C\|f_d\|_{\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)}\|\nabla\psi\|_{L_{q'}(\mathbf{R}^N)}\end{aligned}$$

with some positive constant $C=C(q,\delta)$, since $-\varphi_\delta(x_N)\psi(x',2\delta-x_N)+\psi(x',x_N)-\varphi_0(x_N)\psi(x',-x_N)\in\widehat{W}^1_{q',\Gamma_\delta}(\Omega)$ and

$$\psi(x',2\delta-x_N)-\psi(x',-x_N)=\int_{-x_N}^{2\delta-x_N}(D_N\psi)(x',s)\,ds.$$

The last inequality implies that the required estimate holds.

We here set

$$u_J(x)=-\mathcal{F}_\xi^{-1}\left[\frac{i\xi_J}{|\xi|^2}\mathcal{F}_x[f_d^*](\xi)\right](x)\quad(J=1,\ldots,N).\tag{2.3.1}$$

Then it is clear that $\mathbf{u}=(u_1,\ldots,u_N)^T$ satisfies the divergence equation: $\operatorname{div}\mathbf{u}=f_d$ in Ω. By [**SS12**, Lemma 3.2] and the inequality obtained above, we have

$$\|u_J\|_{L_q(\Omega)}\leq\|u_J\|_{L_q(\mathbf{R}^N)}\leq C\|f_d^*\|_{\widehat{W}^{-1}_q(\mathbf{R}^N)}\leq C\|f_d\|_{\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)}$$

for $J=1,\ldots,N$ and some positive constant $C=C(N,q,\delta)$.

Next we show the estimates of ∇u_J and $\nabla^2 u_J$. By the Fourier multiplier theorem of Mikhlin-Hörmander type, we obtain, for $j = 1, \dots, N-1$ and $J, K, L = 1, \dots, N$,

$$\begin{aligned} \text{(2.3.2)} \quad & \|\nabla u_J\|_{L_q(\Omega)} \le \|u_J\|_{L_q(\mathbf{R}^N)} \le C\|f_d^*\|_{L_q(\mathbf{R}^N)} \le C\|f_d\|_{L_q(\Omega)}, \\ & \|D_K D_L u_j\|_{L_q(\Omega)} \le \|D_K D_L u_j\|_{L_q(\mathbf{R}^N)} \le C\|D_j f_d^*\|_{L_q(\mathbf{R}^N)} \le C\|D_j f_d\|_{L_q(\Omega)}, \\ & \|D_j D_K u_N\|_{L_q(\Omega)} \le \|D_j D_K u_N\|_{L_q(\mathbf{R}^N)} \le C\|D_j f_d^*\|_{L_q(\mathbf{R}^N)} \le C\|D_j f_d\|_{L_q(\Omega)} \end{aligned}$$

with some positive constant $C = C(N, q)$. On the other hand, we have the relation: $D_N^2 u_N = D_N f_d - \sum_{j=1}^{N-1} D_N D_j u_j$ in Ω, because $\mathbf{u}$ satisfies the divergence equation in Ω. This relation combined with (2.3.2) furnishes that $\|D_N^2 u_N\|_{L_q(\Omega)} \le C\|\nabla f_d\|_{L_q(\Omega)}$. Concerning the representation formula (2.1.5), we have

$$\begin{aligned} \widehat{u}_N(\xi', x_N) = -\int_0^\delta \Big\{ \frac{1}{2\pi} \int_{-\infty}^\infty \frac{i\xi_N}{|\xi|^2} \Big(& -\varphi_\delta(y_N) e^{i(x_N + y_N - 2\delta)\xi_N} \\ & + e^{i(x_N - y_N)\xi_N} - \varphi_0(y_N) e^{i(x_N + y_N)\xi_N} \Big) \, d\xi_N \Big\} \, \widehat{f}_d(\xi', y_N) \, dy_N \end{aligned}$$

by (2.3.1) and the definition of f_d^*, which, combined with Lemma **??**, furnishes that

$$\widehat{u}_N(\xi', 0) = -\frac{1}{2} \int_0^\delta \Big(-\varphi_\delta(y_N) e^{-A(2\delta - y_N)} + e^{-Ay_N} + \varphi_0(y_N) e^{-Ay_N} \Big) \widehat{f}_d(\xi', y_N) \, dy_N.$$

Taking the inverse partial Fourier transform, with respect to ξ', of the above formula implies that (2.1.5) hold.

2.4. Solution formulas

In this section, we give exact solution formulas of the equations:

$$\text{(2.4.1)} \qquad \begin{cases} \lambda \mathbf{u} - \operatorname{Div} \mathbf{S}(\mathbf{u}, \theta) = \mathbf{f}, & \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{S}(\mathbf{u}, \theta)\mathbf{e}_N = \mathbf{g} & \text{on } \Gamma_\delta, \\ \mathbf{u} = -K f_d & \text{on } \Gamma_0, \end{cases}$$

where K is the same operator as in Theorem 2.1.2.

We first reduce equations (2.4.1) to the case where $\mathbf{f} = 0$. To this end, we set $\mathbf{u} = S_0(\lambda)\mathbf{f} + \mathbf{v}$ and $\theta = T_0\mathbf{f} + \pi$ for $S_0(\lambda)$ and T_0 obtained in Lemma 2.2.5. Then equations (2.4.1) are reduced to

$$\text{(2.4.2)} \qquad \begin{cases} \lambda \mathbf{v} - \operatorname{Div} \mathbf{S}(\mathbf{v}, \pi) = 0, & \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \\ \mu(D_N v_j + D_j v_N) = a_j & \text{on } \Gamma_\delta, \\ 2\mu D_N v_N - \pi = a_N + T_0\mathbf{f} & \text{on } \Gamma_\delta, \\ \mathbf{v} = \mathbf{b} & \text{on } \Gamma_0 \end{cases}$$

for $j = 1, \dots, N-1$, where a_J $(J = 1, \dots, N)$ and $\mathbf{b}$ are defined by

$$\text{(2.4.3)} \qquad a_J = g_J - \mu(D_J S_{0N}(\lambda)\mathbf{f} + D_N S_{0J}(\lambda)\mathbf{f}), \quad \mathbf{b} = -(K f_d + S_0(\lambda)\mathbf{f}).$$

Next we shall give solution formulas of (2.4.2). Applying the partial Fourier transform with respect to x' variable to (2.4.2), we have

$$\text{(2.4.4)} \quad \begin{cases} \mu(B^2 - D_N^2)\widehat{v}_j(\xi', x_N) + i\xi_j\widehat{\pi}(\xi', x_N) = 0 & (0 < x_N < \delta), \\ \mu(B^2 - D_N^2)\widehat{v}_N(\xi', x_N) + D_N\widehat{\pi}(\xi', x_N) = 0 & (0 < x_N < \delta), \\ \sum_{j=1}^{N-1} i\xi_j\widehat{v}_j(\xi', x_N) + D_N\widehat{v}_N(\xi', x_N) = 0 & (0 < x_N < \delta), \\ \mu(D_N\widehat{v}_j(\xi', \delta) + i\xi_j\widehat{v}_N(\xi', \delta)) = \widehat{a}_j(\xi', \delta), \\ 2\mu D_N\widehat{v}_N(\xi', \delta) - \widehat{\pi}(\xi', \delta) = \widehat{a}_N(\xi', \delta) + \widehat{T_0 f}(\xi', \delta), \\ \widehat{v}_J(\xi', 0) = \widehat{b}_J(\xi', 0) \end{cases}$$

for $j = 1, \ldots, N-1$ and $J = 1, \ldots, N$, where b_J denotes the J-th component of $\mathbf{b}$.

From now on, we solve (2.4.4) as the ordinary differential equations with respect to x_N. In (2.4.4), multiplying the first equation by $-i\xi_j$ and adding the resultant formulas to the third equation multiplied by $\mu(B^2 - D_N^2)$ yield that

$$\text{(2.4.5)} \quad \mu(B^2 - D_N^2)D_N\widehat{v}_N(\xi', x_N) + A^2\widehat{\pi}(\xi', x_N) = 0 \quad (0 < x_N < \delta).$$

Apply D_N to the second equation of (2.4.4), and thus the resultant formula combined with (2.4.5) implies that

$$\text{(2.4.6)} \quad (D_N^2 - A^2)\widehat{\pi}(\xi', x_N) = 0 \quad (0 < x_N < \delta).$$

On the other hand, if we apply $-(D_N^2 - A^2)$ to the first and second equation of (2.4.4), then we obtain, by using (2.4.6),

$$\mu(D_N^2 - A^2)(D_N^2 - B^2)\widehat{v}_J(\xi', x_N) = 0 \quad (0 < x_N < \delta)$$

for $J = 1, \ldots, N$. From viewpoint of this, we set

$$\text{(2.4.7)} \quad \begin{aligned} \widehat{v}_J(\xi', x_N) =& \alpha_{\delta J}(e^{-B(\delta - x_N)} - e^{-A(\delta - x_N)}) \\ &+ \beta_{\delta J}e^{-B(\delta - x_N)} + \alpha_{0J}(e^{-Bx_N} - e^{-Ax_N}) + \beta_{0J}e^{-Bx_N}, \\ \widehat{\pi}(\xi', x_N) =& \gamma_\delta e^{-A(\delta - x_N)} + \gamma_0 e^{-Ax_N}, \end{aligned}$$

where $\alpha_{\delta J}, \beta_{\delta J}, \alpha_{0J}, \beta_{0J}, \gamma_\delta$, and γ_0 denote constants, depending on λ and ξ', which are determined by the boundary conditions of (2.4.4). In the sequel, we write $\alpha_\delta = (\alpha_{\delta 1}, \ldots, \alpha_{\delta N})$, and also β_δ, α_0, and β_0 are defined similarly. Inserting (2.4.7) into (2.4.4), for $j = 1, \ldots, N-1$, we obtain the relations:

$$\text{(2.4.8)} \quad \begin{aligned} -\mu(B^2 - A^2)\alpha_{\delta j} + i\xi_j\gamma_\delta &= 0, & -\mu(B^2 - A^2)\alpha_{0j} + i\xi_j\gamma_0 &= 0, \\ -\mu(B^2 - A^2)\alpha_{\delta N} + A\gamma_\delta &= 0, & -\mu(B^2 - A^2)\alpha_{0N} - A\gamma_0 &= 0, \\ i\xi' \cdot \alpha'_\delta + i\xi' \cdot \beta'_\delta + B\alpha_{\delta N} + B\beta_{\delta N} &= 0, & -i\xi' \cdot \alpha'_\delta - A\alpha_{\delta N} &= 0, \\ i\xi' \cdot \alpha'_0 + i\xi' \cdot \beta'_0 - B\alpha_{0N} - B\beta_{0N} &= 0, & -i\xi' \cdot \alpha'_0 + A\alpha_{0N} &= 0, \end{aligned}$$

which are corresponding to the first, second, and third equations, and furthermore, the boundary conditions yield that

(2.4.9)
$$\begin{aligned}
&\mu\left\{(B-A)\alpha_{\delta j}+B\beta_{\delta j}-(Be^{-B\delta}-Ae^{-A\delta})\alpha_{0j}-Be^{-B\delta}\beta_{0j}\right.\\
&\qquad\left.+i\xi_j\left(\beta_{\delta N}+(e^{-B\delta}-e^{-A\delta})\alpha_{0N}+e^{-B\delta}\beta_{0N}\right)\right\}=\widehat{a}_j(\xi',\delta),\\
&2\mu\left\{(B-A)\alpha_{\delta N}+B\beta_{\delta N}-(Be^{-B\delta}-Ae^{-A\delta})\alpha_{0N}-Be^{-B\delta}\beta_{0N}\right\}\\
&\qquad-(\gamma_\delta+\gamma_0e^{-A\delta})=\widehat{a}_N(\xi',\delta)+\widehat{T_0f}(\xi',\delta),\\
&(e^{-B\delta}-e^{-A\delta})\alpha_{\delta j}+e^{-B\delta}\beta_{\delta j}+\beta_{0j}=\widehat{b}_j(\xi',0),\\
&(e^{-B\delta}-e^{-A\delta})\alpha_{\delta N}+e^{-B\delta}\beta_{\delta N}+\beta_{0N}=\widehat{b}_N(\xi',0).
\end{aligned}$$

We can write $i\xi'\cdot\alpha'_\delta,\alpha_{\delta N},i\xi'\cdot\alpha'_0,\alpha_{0N},\gamma_\delta$, and γ_0 by using $i\xi'\cdot\beta'_\delta,\beta_{\delta N},i\xi'\cdot\beta'_0$, and β_{0N}. In fact, by the relations (2.4.8), we obtain the formulas:

(2.4.10)
$$\begin{aligned}
\alpha_{\delta N}&=-\frac{1}{B-A}\left(i\xi'\cdot\beta'_\delta+B\beta_{\delta N}\right), & i\xi'\cdot\alpha'_\delta&=\frac{A}{B-A}\left(i\xi'\cdot\beta'_\delta+B\beta_{\delta N}\right),\\
\alpha_{0N}&=\frac{1}{B-A}\left(i\xi'\cdot\beta'_0-B\beta_{0N}\right), & i\xi'\cdot\alpha'_0&=\frac{A}{B-A}\left(i\xi'\cdot\beta'_0-B\beta_{0N}\right),\\
\gamma_\delta&=-\mu\frac{B+A}{A}\left(i\xi'\cdot\beta'_\delta+B\beta_{\delta N}\right), & \gamma_0&=-\mu\frac{B+A}{A}\left(i\xi'\cdot\beta'_0-B\beta_{0N}\right).
\end{aligned}$$

Combining (2.4.9) with (2.4.10), we achieve the following simultaneous linear equations with respect to $i\xi'\cdot\beta'_\delta,\beta_{\delta N},i\xi'\cdot\beta'_0$, and β_{0N}:

(2.4.11)
$$\begin{aligned}
&L\begin{bmatrix}i\xi'\cdot\beta'_\delta & \beta_{\delta N} & i\xi'\cdot\beta'_0 & \beta_{0N}\end{bmatrix}^T\\
&\quad=\begin{bmatrix}\mu^{-1}i\xi'\cdot\widehat{a}' & \mu^{-1}A(\widehat{a}_N+\widehat{T_0f}) & i\xi'\cdot\widehat{b}' & \widehat{b}_N\end{bmatrix}^T,
\end{aligned}$$

where L is a 4×4 matrix whose (i,j)-th components L_{ij} are given by

$$\begin{aligned}
&L_{11}=B+A, && L_{12}=A(B-A),\\
&L_{13}=-(B^2+A^2)\mathcal{M}(\delta)-(B+A)e^{-A\delta},\\
&L_{14}=A\left((B^2+A^2)\mathcal{M}(\delta)+(B-A)e^{-A\delta}\right),\\
&L_{21}=B-A, && L_{22}=B(B+A),\\
&L_{23}=-2AB\mathcal{M}(\delta)+(B-A)e^{-A\delta},\\
&L_{24}=B\left(2A^2\mathcal{M}(\delta)-(B+A)e^{-A\delta}\right),\\
&L_{31}=B\mathcal{M}(\delta)+e^{-A\delta}, && L_{32}=AB\mathcal{M}(\delta),\\
&L_{33}=1, && L_{34}=0,\\
&L_{41}=-\mathcal{M}(\delta), && L_{42}=-A\mathcal{M}(\delta)+e^{-A\delta},\\
&L_{43}=0, && L_{44}=1.
\end{aligned}$$

The determinant $\det L$ is represented as $\det L=(B-A)^{-2}\ell_1(\lambda,\xi')$ with

(2.4.12)
$$\begin{aligned}
\ell_1(\lambda,\xi')=&(B^5+2A^2B^3+5A^4B)(1+e^{-2A\delta})(1+e^{-2B\delta})\\
&-(16A^2B^3+16A^4B)e^{-A\delta}e^{-B\delta}\\
&-(AB^4+6A^3B^2+A^5)(1-e^{-2A\delta})(1-e^{-2B\delta}).
\end{aligned}$$

The following lemma was proved in [**Abe04**, Lemma 2.2].

LEMMA 2.4.1. *If $\lambda \in \mathbf{C} \setminus (-\infty, 0]$ and $\xi' \neq 0$, then $\ell_1(\lambda, \xi') \neq 0$.*

In view of Lemma 2.4.1 and (2.4.10), if $\lambda \in \mathbf{C} \setminus (-\infty, 0]$ and $\xi' \neq 0$, then the solutions $\widehat{v}_N, \widehat{\pi}$ of (2.4.4) and the solutions v_N, π of (2.4.2) are represented as

$$\widehat{v}_N(\xi', x_N) = \sum_{k=1}^{4}\sum_{\ell=1}^{2}\left(\frac{\widetilde{L}_{k,2\ell}}{\det L}\mathcal{M}(d_\ell(x_N)) + \frac{L_{k,2\ell}}{\det L}e^{-Bd_\ell(x_N)}\right) r_k, \tag{2.4.13}$$

$$\widehat{\pi}(\xi', x_N) = \mu\left(\frac{B+A}{A}\right)\sum_{k=1}^{4}\sum_{\ell=1}^{2}(-1)^{\ell+1}\frac{\widetilde{L}_{k,2\ell}}{\det L}e^{-Ad_\ell(x_N)}r_k,$$

$$v_N(x) = \mathcal{F}_{\xi'}^{-1}[\widehat{v}_N(\xi', x_N)](x'), \quad \pi(x) = \mathcal{F}_{\xi'}^{-1}[\widehat{\pi}(\xi', x_N)](x'),$$

where $L_{i,j}$ denote the (i,j)-cofactors of L and

$$\widetilde{L}_{k,2\ell} = (-1)^\ell L_{k,2\ell-1} - BL_{k,2\ell} \quad (k = 1, \ldots, 4,\ \ell = 1, 2). \tag{2.4.14}$$

Here we have set, for $\mathbf{a} = (a_1, \ldots, a_N)^T$ and $\mathbf{b}$ given by (2.4.3),

$$r_1 = \mu^{-1}i\xi' \cdot \widehat{\mathbf{a}}'(\xi', \delta), \qquad r_2 = \mu^{-1}A(\widehat{a}_N(\xi', \delta) + \widehat{T_0\mathbf{f}}(\xi', \delta)), \tag{2.4.15}$$

$$r_3 = i\xi' \cdot \widehat{\mathbf{b}}'(\xi', 0), \qquad r_4 = \widehat{b}_N(\xi', 0),$$

and also the cofactors $L_{i,j}$ are given by

(2.4.16)

$$L_{1,1} = B(B+A)\left\{1 + e^{-2A\delta} - 2Ae^{-A\delta}\mathcal{M}(\delta) + 2A^2\mathcal{M}(\delta)^2\right\},$$

$$L_{1,2} = -(B-A)(1 - e^{-2A\delta}) + 2B(B-A)e^{-A\delta}\mathcal{M}(\delta) - 2AB(B+A)\mathcal{M}(\delta)^2,$$

$$\begin{aligned} L_{1,3} = B\{&-(B+A)(1+e^{-2A\delta})e^{-A\delta} \\ &-(B^2 + A^2 + (B^2 - 3A^2)e^{-2A\delta})\mathcal{M}(\delta) + 2A^2(B-A)e^{-A\delta}\mathcal{M}(\delta)^2\},\end{aligned}$$

$$\begin{aligned} L_{1,4} = &(B-A)(1 - e^{-2A\delta})e^{-A\delta} \\ &+ (B^2 + A^2 - (B^2 - 4AB + A^2)e^{-2A\delta})\mathcal{M}(\delta) + 2AB(B-A)e^{-A\delta}\mathcal{M}(\delta)^2,\end{aligned}$$

$$\begin{aligned} L_{2,1} = A\{&-(B-A)(1 - e^{-2A\delta}) \\ &-2A(B-A)e^{-A\delta}\mathcal{M}(\delta) - (B^2 + A^2)(B+A)\mathcal{M}(\delta)^2\},\end{aligned}$$

$$L_{2,2} = (B+A)\left\{1 + e^{-2A\delta} + 2Be^{-A\delta}\mathcal{M}(\delta) + (B^2 + A^2)\mathcal{M}(\delta)^2\right\},$$

$$\begin{aligned} L_{2,3} = A\{&(B-A)(1 - e^{-2A\delta})e^{-A\delta} \\ &-2(AB + (B^2 - AB + A^2)e^{-2A\delta})\mathcal{M}(\delta) - (B-A)(B^2 + A^2)e^{-A\delta}\mathcal{M}(\delta)^2\},\end{aligned}$$

$$\begin{aligned} L_{2,4} = &-(B+A)(1 + e^{-2A\delta})e^{-A\delta} \\ &- 2(-A^2 + B^2e^{-2A\delta}) - (B-A)(B^2 + A^2)e^{-A\delta}\mathcal{M}(\delta)^2,\end{aligned}$$

$$\begin{aligned} L_{3,1} = &(D_1(A,B) + D_2(A,B)e^{-2A\delta})e^{-A\delta} \\ &+ (BD_3(A,B) + (B - 2A)D_2(A,B)e^{-2A\delta})\mathcal{M}(\delta) \\ &- A(B-A)D_2(A,B)e^{-A\delta}\mathcal{M}(\delta)^2,\end{aligned}$$

$$\begin{aligned} L_{3,2} = &-2(B^2 - A^2)e^{-A\delta} \\ &+ (-D_1(-A,B) + D_2(A,B)e^{-2A\delta})\mathcal{M}(\delta) + (B-A)D_2(A,B)e^{-A\delta}\mathcal{M}(\delta)^2,\end{aligned}$$

$$\begin{aligned} L_{3,3} = &D_2(A,B) + D_1(A,B)e^{-2A\delta} \\ &+ 2AD_1(-A,B)e^{-A\delta}\mathcal{M}(\delta) + A(B+A)D_2(-A,B)\mathcal{M}(\delta)^2,\end{aligned}$$

$$
\begin{aligned}
L_{3,4} =&(B+A)\left\{2(B-A)e^{-2A\delta}+2(B-A)^2e^{-A\delta}\mathcal{M}(\delta)+D_2(-A,B)\mathcal{M}(\delta)^2\right\},\\
L_{4,1} =&AB\left\{-2(B^2-A^2)e^{-A\delta}\right.\\
&\left.-(D_3(A,B)+D_2(A,B)e^{-2A\delta})\mathcal{M}(\delta)-(B-A)D_2(A,B)e^{-A\delta}\mathcal{M}(\delta)^2\right\},\\
L_{4,2} =&(D_1(A,B)+D_2(A,B)e^{-2A\delta})e^{-A\delta}\\
&+(AD_1(-A,B)+(2B-A)D_2(A,B)e^{-2A\delta})\mathcal{M}(\delta)\\
&+B(B-A)D_2(A,B)e^{-A\delta}\mathcal{M}(\delta)^2,\\
L_{4,3} =&AB(B+A)\left\{2(B-A)e^{-2A\delta}+2(B-A)^2e^{-A\delta}\mathcal{M}(\delta)+D_2(-A,B)\mathcal{M}(\delta)^2\right\},\\
L_{4,4} =&D_2(A,B)+D_1(A,B)e^{-2A\delta}\\
&+2BD_3(A,B)e^{-A\delta}\mathcal{M}(\delta)+B(B+A)D_2(-A,B)\mathcal{M}(\delta)^2,
\end{aligned}
$$

where $D_i(A,B)$ $(i=1,2,3)$ are defined as

$$
\begin{aligned}
D_1(A,B)&=B^3+3AB^2-A^2B+A^3,\\
D_2(A,B)&=B^3+AB^2+3A^2B-A^3,\\
D_3(A,B)&=B^3+AB^2-A^2B+3A^3.
\end{aligned}
\tag{2.4.17}
$$

In the remaining part of this section, we give the representation formula of j-th component of $\mathbf{v}$ for $j=1,\dots,N-1$ in (2.4.2). For a function f defined on Ω, we define an extension operator E_0 as follows:

$$
E_0f(x)=\begin{cases} f(x) & 0\le x_N\le\delta,\\ 0 & x_N\notin[0,\delta].\end{cases}
$$

Putting $v_j=-H(\lambda)E_0D_j\pi+w_j$ in (2.4.2) for $j=1,\dots,N-1$, where $H(\lambda)$ is the same operator as in Lemma 2.2.4, yields that

$$
\begin{cases}
\lambda w_j-\mu\Delta w_j=0 & \text{in }\Omega,\\
D_Nw_j=\mu^{-1}a_j-D_jv_N+D_NH(\lambda)E_0D_j\pi & \text{on }\Gamma_\delta,\\
w_j=b_j+H(\lambda)E_0D_j\pi & \text{on }\Gamma_0,
\end{cases}
$$

which, applied the partial Fourier transform with respect to x', furnishes that

$$
\begin{cases}
(D_N^2-B^2)\widehat{w}_j(\xi',x_N)=0 & (0<x_N<\delta),\\
D_N\widehat{w}_j(\xi',\delta)=\widehat{h}_j^1(\xi',\delta),\\
\widehat{w}_j(\xi',0)=\widehat{h}_j^2(\xi',0),
\end{cases}
\tag{2.4.18}
$$

where the right members are given by

$$
h_j^1(x)=\mu^{-1}a_j-D_jv_N+D_NH(\lambda)E_0D_j\pi,\quad h_j^2(x)=b_j+H(\lambda)E_0D_j\pi.
\tag{2.4.19}
$$

We thus obtain $\widehat{w}_j(\xi',x_N)=\widehat{w}_j^1(\xi',x_N)+\widehat{w}_j^2(\xi',x_N)$ with

$$
\begin{aligned}
\widehat{w}_j^1(\xi',x_N)&=\sum_{\ell=1}^2(-1)^{\ell+1}\frac{e^{-B(d_\ell(x_N)+d_\ell(\delta))}}{B(1+e^{-2B\delta})}\widehat{h}_j^1(\xi',\delta),\\
\widehat{w}_j^2(\xi',x_N)&=\sum_{\ell=1}^2\frac{e^{-B(d_\ell(x_N)+d_\ell(0))}}{1+e^{-2B\delta}}\widehat{h}_j^2(\xi',0)
\end{aligned}
$$

by solving the ordinary differential equations (2.4.18) with respect to x_N. Thus, for the j-th component of $\mathbf{v}$ in (2.4.2), we have the following solution formula:

$$v_j(x) = -H(\lambda)E_0 D_j \pi + w_j^1(x) + w_j^2(x), \quad w_j^k(x) = \mathcal{F}_{\xi'}^{-1}[\widehat{w}_j^k(\xi', x_N)](x') \tag{2.4.20}$$

for $j = 1, \ldots, N-1$ and $k = 1, 2$.

2.5. Technical lemmas

In this section, we show several estimates of Fourier multipliers, which will be used to estimate solutions obtained in Section 2.4. To this end, we set, by using the symbols (2.1.2),

$$E_i(y_N) = \begin{cases} \varphi_\delta(y_N) & i = 1, \\ \varphi'(y_N) & i = 2, \\ \varphi_0(y_N) & i = 3, \end{cases}$$

where $\varphi'(y_N) = (d\varphi/dy_N)(y_N)$. The symbol defined as (2.1.4) is used below, and then we have the following lemma.

LEMMA 2.5.1. *Let $0 < \varepsilon < \pi/2$, $1 < q < \infty$, $\gamma_0 > 0$, and $m_k \in \mathbb{M}_{0,k,\varepsilon,\gamma_0}$ $(k = 1, 2)$. For $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ and $X, Y \in \{A, B\}$, we define $K_{j,\ell,n}^i(\lambda)$ $(i = 1, 2, 3,\ j = 1, \ldots, 5,\ \ell, n = 1, 2)$ by*

$$[K_{1,\ell,n}^i(\lambda)f](x) = \int_0^\delta \mathcal{F}_{\xi'}^{-1}\left[E_i(y_N)m_1\lambda^{1/2}e^{-B(d_\ell(x_N)+d_n(y_N))}\widehat{f}(y_N)\right](x')dy_N,$$

$$[K_{2,\ell,n}^i(\lambda)f](x) = \int_0^\delta \mathcal{F}_{\xi'}^{-1}\left[E_i(y_N)m_2Ae^{-Xd_\ell(x_N)}e^{-Yd_n(y_N)}\widehat{f}(y_N)\right](x')dy_N,$$

$$[K_{3,\ell,n}^i(\lambda)f](x) = \int_0^\delta \mathcal{F}_{\xi'}^{-1}\left[E_i(y_N)m_2A^2\mathcal{M}(d_\ell(x_N))e^{-Xd_n(y_N)}\widehat{f}(y_N)\right](x')dy_N,$$

$$[K_{4,\ell,n}^i(\lambda)f](x) = \int_0^\delta \mathcal{F}_{\xi'}^{-1}\left[E_i(y_N)m_2A\lambda^{1/2}\mathcal{M}(d_\ell(x_N))e^{-Xd_n(y_N)}\widehat{f}(y_N)\right](x')dy_N,$$

$$[K_{5,\ell,n}^i(\lambda)f](x) = \int_0^\delta \mathcal{F}_{\xi'}^{-1}\left[E_i(y_N)m_2AB\mathcal{M}(d_\ell(x_N))e^{-Xd_n(y_N)}\widehat{f}(y_N)\right](x')dy_N,$$

where $\widehat{f}(y_N) = \widehat{f}(\xi', y_N)$. Then, for $i = 1, 2, 3$, $j = 1, \ldots, 5$, $\ell, n = 1, 2$, and $s = 0, 1$, the sets $\{(\tau\partial_\tau)^s K_{j,\ell,n}^i(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}$ are $\mathcal{R}$-bounded families, in $\mathcal{L}(L_q(\Omega))$, whose $\mathcal{R}$ bounds do not exceed some positive constant $C(N, q, \varepsilon, \gamma_0, \mu, \delta)$.

PROOF. In what follows, we say that the family of operator $\{\mathcal{T}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ has the required properties if $\{(\tau\partial_\tau)^s\mathcal{T}(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}$ $(s = 0, 1)$ are $\mathcal{R}$-bounded families, in $\mathcal{L}(L_q(\Omega))$, whose $\mathcal{R}$-bound do not exceed some positive constant $C(N, q, \varepsilon, \gamma_0, \mu, \delta)$.

We first consider $K_{1,\ell,n}^i(\lambda)$ for $i = 1, 2, 3$ and $\ell, n = 1, 2$. For $i = 1, 2, 3$, setting

$$k_{1,\lambda}^i(x, y) = \mathcal{F}_{\xi'}^{-1}[E_i(y_N)m_1(\lambda, \xi')\lambda^{1/2}e^{-B(d_\ell(x_N)+d_n(y_N))}](x' - y'),$$

we have

$$[K_{1,\ell,n}^i(\lambda)f](x) = \int_\Omega k_{1,\lambda}^i(x, y)f(y)dy.$$

We prove that there exists a positive constant $C_{N,\varepsilon,\gamma_0,\mu}$ such that

$$|k^i_{1,\lambda}(x,y)| \leq \frac{C_{N,\varepsilon,\gamma_0,\mu}E_i(y_N)}{\{|x'-y'|^2+(d_\ell(x_N)+d_n(y_N))^2\}^{N/2}}, \tag{2.5.1}$$

$$|\tau\partial_\tau k^i_{1,\lambda}(x,y)| \leq \frac{C_{N,\varepsilon,\gamma_0,\mu}E_i(y_N)}{\{|x'-y'|^2+(d_\ell(x_N)+d_n(y_N))^2\}^{N/2}} \tag{2.5.2}$$

for any $\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}$ and $i=1,2,3$. By the assumption of m_1, Leibniz's rule and Lemma 1.2.6, for any multi-index $\alpha'\in\mathbb{N}_0^{N-1}$ we have

$$\begin{aligned}&|D^{\alpha'}_{\xi'}\{m_1(\lambda,\xi')\lambda^{1/2}e^{-B(d_\ell(x_N)+d_n(y_N))}\}| \\ &\leq C_{\alpha',\varepsilon,\gamma_0,\mu}|\lambda|^{1/2}(|\lambda|^{1/2}+A)^{-|\alpha'|}e^{-(1/4)b_{\varepsilon,\mu}(|\lambda|^{1/2}+A)(d_\ell(x_N)+d_n(y_N))}.\end{aligned} \tag{2.5.3}$$

Using the identity:

$$e^{ix'\cdot\xi'}=\sum_{|\alpha'|=k}\left(\frac{-ix'}{|x'|^2}\right)^{\alpha'}D^{\alpha'}_{\xi'}e^{ix'\cdot\xi'},$$

for any $k\in\mathbb{N}_0$, we see that $k^i_{1,\lambda}(x,y)$ can be written in the form:

$$\begin{aligned}k^i_{1,\lambda}(x,y)=&\sum_{|\alpha'|=N}\left(\frac{i(x'-y')}{|x'-y'|^2}\right)^{\alpha'}\left(\frac{1}{2\pi}\right)^{N-1} \\ &\times\int_{\mathbf{R}^{N-1}}e^{i(x'-y')}D^{\alpha'}_{\xi'}\{E_i(y_N)m_1\lambda^{1/2}e^{-B(d_\ell(x_N)+d_n(y_N))}\}d\xi'.\end{aligned} \tag{2.5.4}$$

Applying (2.5.3) to (2.5.4) and using the change of variables: $\xi'=|\lambda|^{1/2}\eta'$ furnish that

$$\begin{aligned}|k^i_{1,\lambda}(x,y)|&\leq C_{N,\varepsilon,\gamma_0,\mu}E_i(y_N)|x'-y'|^{-N}|\lambda|^{1/2}\int_{\mathbf{R}^{N-1}}(|\lambda|^{1/2}+A)^{-N}d\xi' \\ &=C_{N,\varepsilon,\gamma_0,\mu}E_i(y_N)|x'-y'|^{-N}\int_{\mathbf{R}^{N-1}}(1+|\eta'|)^{-N}d\eta'.\end{aligned} \tag{2.5.5}$$

Moreover, by (2.5.3) with $\alpha'=0$, we have

$$\begin{aligned}&|k^i_{1,\lambda}(x,y)| \\ &\leq CE_i(y_N)|\lambda|^{1/2}\int_{\mathbb{R}^{N-1}}e^{-(1/4)b_{\varepsilon,\mu}(|\lambda|^{1/2}+A)(d_\ell(x_N)+d_n(y_N))}d\xi' \\ &\leq CE_i(y_N)|\lambda|^{1/2}N!\int_{\mathbb{R}^{N-1}}\left\{\frac14 b_{\varepsilon,\mu}(|\lambda|^{1/2}+A)(d_\ell(x_N)+d_n(y_N))\right\}^{-N}d\xi' \\ &\leq CE_i(y_N)(d_\ell(x_N)+d_n(y_N))^{-N}\int_{\mathbb{R}^{N-1}}(1+|\eta'|)^{-N}d\eta'\end{aligned} \tag{2.5.6}$$

for some positive constant $C=C(N,\varepsilon,\gamma_0,\mu)$. Combining (2.5.5) and (2.5.6) implies (2.5.1). Since

$$\begin{aligned}&\tau\partial_\tau(m_1(\lambda,\xi')\lambda^{1/2}e^{-B(d_\ell(x_N)+d_n(y_N))}) \\ =&(\tau\partial_\tau m_1(\lambda,\xi'))\lambda^{1/2}e^{-B(d_\ell(x_N)+d_n(y_N))}+m_1(\lambda,\xi')\frac{i\tau}{2\lambda^{1/2}}e^{-B(d_\ell(x_N)+d_n(y_N))} \\ &+m_1(\lambda,\xi')\lambda^{1/2}(\tau\partial_\tau e^{-B(d_\ell(x_N)+d_n(y_N))}),\end{aligned}$$

we have, by Leibniz's rule, the assumption of m_1, and Lemma 1.2.6,

$$\begin{aligned}(2.5.7)\qquad &|D^{\alpha'}_{\xi'}\{\tau\partial_\tau(E_i(y_N)m_1(\lambda,\xi')\lambda^{1/2}e^{-B(d_\ell(x_N)+d_n(y_N))})\}| \\ &\leq CE_i(y_N)|\lambda|^{1/2}(|\lambda|^{1/2}+A)^{-|\alpha'|}e^{-(1/4)b_{\varepsilon,\mu}(|\lambda|^{1/2}+A)(d_\ell(x_N)+d_n(y_N))}\end{aligned}$$

for a positive constant $C = C(\alpha', \varepsilon, \gamma_0, \mu)$ Employing the same argument as in proving (2.5.1) by (2.5.3), we have (2.5.2) by (2.5.7). Now, using Proposition 1.2.4, we prove that $K^i_{1,\ell,n}(\lambda)$ has the required properties. For this purpose, in view of (2.5.1) and (2.5.2), we set $k_0(x) = C(N, \varepsilon, \gamma_0, \mu)|x|^{-N}$ and define the operator K^i_0 by the formula:

$$[K^i_0 f](x) = \int_\Omega E_i(y_N)k_0(x'-y', d_\ell(x_N)+d_n(y_N))f(y)dy.$$

We prove that K^i_0 is a bounded linear operator on $L_q(\Omega)$, for $i = 1, 2, 3$, whose bound does not exceed a constant $C(N, q, \varepsilon, \gamma_0, \mu, \delta)$. By Young's inequality, we have

$$\begin{aligned}(2.5.8)\qquad &\|[K^i_0 f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\ &\leq \int_0^\delta E_i(y_N)\|k_0(\cdot, d_\ell(x_N)+d_n(y_N))\|_{L_1(\mathbf{R}^{N-1})}\|f(\cdot,y_N)\|_{L_q(\mathbf{R}^{N-1})}dy_N \\ &\leq C\int_0^\delta E_i(y_N)\frac{\|f(\cdot,y_N)\|_{L_q(\mathbf{R}^{N-1})}}{d_\ell(x_N)+d_n(y_N)}dy_N.\end{aligned}$$

We shall prove in the case of $i = 1$. For $\ell, n = 1$ in (2.5.8), using the change of variables: $y_N = \delta - (\delta - x_N)t$, we obtain

$$\begin{aligned}&\|[K^1_0 f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\ &\leq C\int_{-\infty}^\delta \varphi_\delta(y_N)\frac{\|f(\cdot,y_N)\|_{L_q(\mathbf{R}^{N-1})}}{(\delta-x_N)+(\delta-y_N)}dy_N \\ &= C\int_0^\infty \varphi_\delta(\delta-(\delta-x_N)t)\frac{\|f(\cdot,\delta-(\delta-x_N)t)\|_{L_q(\mathbf{R}^{N-1})}}{1+t}dt.\end{aligned}$$

By the last inequality, Minkowski's integral inequality, and the change of variables: $s = \delta - (\delta - x_N)t$, it holds that

$$\begin{aligned}&\|K^1_0 f\|_{L_q(\mathbf{R}^{N-1}\times(-\infty,\delta))} \\ &\leq C\int_0^\infty \frac{1}{1+t}\Big[\int_{-\infty}^\delta\{\varphi_\delta(\delta-(\delta-x_N)t)\|f(\cdot,\delta-(\delta-x_N)t)\|_{L_q(\mathbf{R}^{N-1})}\}^q dx_N\Big]^{1/q}dt \\ &= C\int_0^\infty \frac{1}{1+t}\Big[\int_{-\infty}^\delta\{\varphi_\delta(s)\|f(\cdot,s)\|_{L_q(\mathbf{R}^{N-1})}\}^q\frac{ds}{t}\Big]^{1/q}dt \\ &\leq C\|f\|_{L_q(\Omega)}\int_0^\infty\frac{1}{(1+t)t^{1/q}}dt\end{aligned}$$

for some positive constant $C = C(N, \varepsilon, \gamma_0, \mu)$. We thus have the inequality:

$$\|K^1_0 f\|_{L_q(\Omega)} \leq C(N, q, \varepsilon, \gamma_0, \mu)\|f\|_{L_q(\Omega)}.$$

If $\ell = 2$ and $n = 1$ in (2.5.8), using the change of variables: $y_N = \delta - x_N t$, we have

$$\begin{aligned}
&\|[K_0^1 f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\leq C\int_{-\infty}^{\delta} \varphi_\delta(y_N)\frac{\|f(\cdot, y_N)\|_{L_q(\mathbf{R}^{N-1})}}{x_N + (\delta - y_N)}dy_N \\
&= C\int_0^\infty \varphi_\delta(\delta - x_N t)\frac{\|f(\cdot, \delta - x_N t)\|_{L_q(\mathbf{R}^{N-1})}}{1+t}dt.
\end{aligned}$$

By the last inequality, Minkowski's integral inequality, and the change of variables: $s = \delta - x_N t$, we obtain

$$\begin{aligned}
&\|K_0^1 f\|_{L_q(\mathbf{R}^{N-1}\times(0,\infty))} \\
&\leq C\int_0^\infty \frac{1}{1+t}\Big[\int_0^\infty \{\varphi_\delta(\delta - x_N t)\|f(\cdot, \delta - x_N t)\|_{L_q(\mathbf{R}^{N-1})}\}^q dx_N\Big]^{1/q} dt \\
&= C\int_0^\infty \frac{1}{1+t}\Big[\int_{-\infty}^{\delta} \{\varphi_\delta(s)\|f(\cdot, s)\|_{L_q(\mathbb{R}^{N-1})}\}^q \frac{ds}{t}\Big]^{1/q} dt \\
&\leq C\|f\|_{L_q(\Omega)}\int_0^\infty \frac{1}{(1+t)t^{1/q}}dt.
\end{aligned}$$

Namely we have the inequality: $\|K_0^1 f\|_{L_q(\Omega)} \leq C(N, q, \varepsilon, \gamma_0, \mu)\|f\|_{L_q(\Omega)}$. In the case of $n = 2$ in (2.5.8), using Hölder's inequality, we have

$$\begin{aligned}
\|[K_0^1 f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} &\leq C\int_{\delta/3}^{\delta} \varphi_\delta(y_N)\frac{\|f(\cdot, y_N)\|_{L_q(\mathbb{R}^{N-1})}}{y_N}dy_N \\
&\leq C\left(\frac{3}{\delta}\right)\int_{\delta/3}^{\delta} \varphi_\delta(y_N)\|f(\cdot, y_N)\|_{L_q(\mathbf{R}^{N-1})}dy_N \\
&\leq C\|\varphi_\delta\|_{L_{q'}(0,\delta)}\|f\|_{L_q(\Omega)},
\end{aligned}$$

which shows that $\|K_0^1 f\|_{L_q(\Omega)} \leq C(N, q, \varepsilon, \gamma_0, \mu, \delta)\|f\|_{L_q(\Omega)}$. This completes the proof in the case of $i = 1$ in (2.5.8).

Next we consider the case of $i = 2$ in (2.5.8). By Hölder's inequality, we have

$$\begin{aligned}
\|[K_0^2 f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} &\leq C\int_{\delta/3}^{(2/3)\delta} \varphi'(y_N)\frac{\|f(\cdot, y_N)\|_{L_q(\mathbf{R}^{N-1})}}{d_n(y_N)}dy_N \\
&\leq C\left(\frac{3}{\delta}\right)\int_{\delta/3}^{(2/3)\delta} \varphi'(y_N)\|f(\cdot, y_N)\|_{L_q(\mathbf{R}^{N-1})}dy_N \\
&\leq C\|\varphi'\|_{L_{q'}(0,\delta)}\|f\|_{L_q(\Omega)},
\end{aligned}$$

which shows that $\|K_0^2 f\|_{L_q(\Omega)} \leq C(N, q, \varepsilon, \gamma_0, \mu, \delta)\|f\|_{L_q(\Omega)}$. Concerning K_0^3, by the same argument as in the case of K_0^1, we can prove the inequality:

$$\|K_0^3 f\|_{L_q(\Omega)} \leq C(N, q, \varepsilon, \gamma_0, \mu, \delta)\|f\|_{L_q(\Omega)}.$$

We finished proving that $K_0^i \in \mathcal{L}(L_q(\Omega))$ for $i = 1, 2, 3$. Thus, using Proposition 1.2.4, we obtain

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^s K_{1,\ell,n}^i(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C(N, q, \varepsilon, \gamma_0, \mu, \delta),$$

so that $\{K_{1,\ell,n}^i(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ have the required properties.

Secondly, we consider $K^i_{2,\ell,n}(\lambda)$ for $i = 1, 2, 3$ and $\ell, n = 1, 2$. The case where $X = B$ and $Y = A$ is only considered here, since we can show similarly the other cases. If we set

$$k^i_{2,\lambda}(x, y) = \mathcal{F}^{-1}_{\xi'}[E_i(y_N)m_2(\lambda, \xi')Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)}](x' - y'),$$

then the operator $K^i_{2,\ell,n}(\lambda)$ is given by the formula:

$$[K^i_{2,\ell,n}(\lambda)f](x) = \int_\Omega k^i_{2,\lambda}(x, y)f(y)dy.$$

As we proved that $\{K^i_{1,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ has the required properties by using (2.5.1) and (2.5.2), to prove that $\{K^i_{2,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ has the required properties, it is sufficient to prove that for any $\lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}$ there hold the estimates:

$$\begin{aligned} |k^i_{2,\lambda}(x, y)| &\le \frac{C(N, \varepsilon, \gamma_0, \mu)E_i(y_N)}{\{|x' - y'|^2 + (d_\ell(x_N) + d_n(y_N))^2\}^{N/2}}, \\ |\tau\partial_\tau k^i_{2,\lambda}(x, y)| &\le \frac{C(N, \varepsilon, \gamma_0, \mu)E_i(y_N)}{\{|x' - y'|^2 + (d_\ell(x_N) + d_n(y_N))^2\}^{N/2}}. \end{aligned} \tag{2.5.9}$$

By the assumption of m_2, Leibniz's rule and Lemma 1.2.6, we have

$$\begin{aligned} &|D^{\alpha'}_{\xi'}\{m_2(\lambda, \xi')Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)}\}| \\ &\le C \sum_{\beta'+\gamma'+\delta'=\alpha'} |D^{\beta'}_{\xi'}m_2(\lambda, \xi')||D^{\gamma'}_{\xi'}A||D^{\delta'}_{\xi'}(e^{-Bd_\ell(x_N)}e^{-Ad_\ell(x_N)})| \\ &\le C \sum_{\beta'+\gamma'+\delta'=\alpha'} A^{-|\beta'|}A^{1-|\gamma'|}A^{-|\delta'|}e^{-(1/4)b_{\varepsilon,\mu}A(d_\ell(x_N)+d_n(y_N))} \\ &\le CA^{1-|\alpha'|}e^{-(1/4)b_{\varepsilon,\mu}A(d_\ell(x_N)+d_n(y_N))} \end{aligned} \tag{2.5.10}$$

for a positive constant $C = C(\alpha', \varepsilon, \gamma_0, \mu)$. Since

$$\begin{aligned} &\tau\partial_\tau(m_2(\lambda, \xi')Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)}) \\ &= \{\tau\partial_\tau m_2(\lambda, \xi')\}Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)} + m_2(\lambda, \xi')A\{\tau\partial_\tau(e^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)})\}, \end{aligned}$$

as discussed in (2.5.10), we also have

$$\begin{aligned} &|D^{\alpha'}_{\xi'}\{\tau\partial_\tau(m_2(\lambda, \xi')Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)})\}| \\ &\quad \le C(\alpha', \varepsilon, \gamma_0, \mu)A^{1-|\alpha'|}e^{-(1/4)b_{\varepsilon,\mu}A(d_\ell(x_N)+d_n(y_N))}. \end{aligned} \tag{2.5.11}$$

In view of (2.5.10) and (2.5.11), we apply Proposition 1.2.8 with $n = N - 1$ and $\sigma = 1$ to obtain

$$\begin{aligned} |k^i_{2,\lambda}(x, y)| &\le C(N, \varepsilon, \gamma_0, \mu)E_i(y_N)|x' - y'|^{-N}, \\ |\tau\partial_\tau k^i_{2,\lambda}(x, y)| &\le C(N, \varepsilon, \gamma_0, \mu)E_i(y_N)|x' - y'|^{-N}. \end{aligned} \tag{2.5.12}$$

On the other hand, using (2.5.10) and (2.5.11) with $\alpha' = 0$ and the change of variables: $4^{-1}b_{\varepsilon,\mu}(d_\ell(x_N)+d_n(y_N))\xi' = \eta'$, we have

$$
\begin{aligned}
&|(\tau\partial)^s k^i_{2,\lambda}(x,y)| \\
&\leq CE_i(y_N)\Big(\frac{1}{2\pi}\Big)^{N-1}\int_{\mathbb{R}^{N-1}} Ae^{-4^{-1}b_{\varepsilon,\mu}A(d_\ell(x_N)+d_n(y_N))}d\xi' \\
&= CE_i(y_N)\Big(\frac{1}{2\pi}\Big)^{N-1}\Big\{\frac{1}{4}b_{\varepsilon,\mu}(d_\ell(x_N)+d_n(y_N))\Big\}^{-N}\int_{\mathbb{R}^{N-1}}|\eta'|e^{-|\eta'|}d\eta' \\
&\leq CE_i(y_N)(d_\ell(x_N)+d_n(y_N))^{-N}
\end{aligned}
$$

for $s = \{0,1\}$ with some positive constant $C = C(N,\varepsilon,\gamma_0,\mu)$, which, combined with (2.5.12), furnishes (2.5.9), and therefore $\{K^i_{2,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ has required properties. Analogously we can show that $\{K^i_{j,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ have the required properties in the case of $j = 3,4$ (cf. [**SS12**, Lemma 5.4]).

Finally we consider $K^i_{5,\ell,n}(\lambda)$ for $i = 1,2,3$ and $\ell = 1,2$ with $X = A$. The case where $X = B$ can be shown similarly. By $B = B^2/B = (\lambda/\mu B^2) + A^2/B$, we write $[K^i_{5,\ell,n}(\lambda)f](x)$ to

$$
\begin{aligned}
&[K^i_{5,\ell,n}(\lambda)f](x) \\
&= \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[E_i(y_N)m_2\frac{\lambda^{1/2}}{\mu B}A\lambda^{1/2}\mathcal{M}(d_\ell(x_N))e^{-Ad_n(y_N)}\widehat{f}(\xi',y_N)\right](x')dy_N \\
&+ \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[E_i(y_N)m_2\frac{A}{B}A^2\mathcal{M}(d_\ell(x_N))e^{-Ad_n(y_N)}\widehat{f}(\xi',y_N)\right](x')dy_N,
\end{aligned}
$$

and then we see the fact that $\{K^i_{5,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ have the required properties by the above results of $K^i_{3,\ell,n}$, $K^i_{4,\ell,n}$ and $\lambda^{1/2}/(\mu B)$, $A/B \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}$. □

By using Lemma 2.5.1, we have the lemmas as follows:

Lemma 2.5.2. *Let $0 < \varepsilon < \pi/2$, $1 < q < \infty$, $\gamma_0 > 0$, and $k_i \in \mathbb{M}_{-i,i,\varepsilon,\gamma_0}$ ($i = 1,2$). For $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ and $X, Y \in \{A, B\}$, we define $L^i_{j,\ell,n}(\lambda)$ ($i,j = 1,2,3$, $\ell, n = 1,2$) by*

$$
\begin{aligned}
[L^i_{1,\ell,n}(\lambda)f](x) &= \int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)k_1e^{-B(d_\ell(x_N)+d_n(y_N))}\widehat{f}(\xi',y_N)\Big](x')dy_N, \\
[L^i_{2,\ell,n}(\lambda)f](x) &= \int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)k_2Ae^{-Xd_\ell(x_N)}e^{-Yd_n(y_N)}\widehat{f}(\xi',y_N)\Big](x')dy_N, \\
[L^i_{3,\ell,n}(\lambda)f](x) &= \int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)k_2AB\mathcal{M}(d_\ell(x_N))e^{-Xd_n(y_N)}\widehat{f}(\xi',y_N)\Big](x')dy_N.
\end{aligned}
$$

Then, for $i = 1,2,3$, $j = 1,\dots,7$, $\ell, n = 1,2$, $s = 0,1$, and $k,m = 1,\dots,N$, the sets:

$$
\begin{aligned}
&\{(\tau\partial_\tau)^s(\lambda L^i_{j,\ell,n}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}, \\
&\{(\tau\partial_\tau)^s(\gamma L^i_{j,\ell,n}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}, \\
&\{(\tau\partial_\tau)^s(\lambda^{1/2}D_kL^i_{j,\ell,n}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}, \\
&\{(\tau\partial_\tau)^s(D_kD_mL^i_{j,\ell,n}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}
\end{aligned}
$$

are $\mathcal{R}$-bounded families, in $\mathcal{L}(L_q(\Omega))$, whose $\mathcal{R}$-bounds do not exceed some positive constant $C(N, q, \varepsilon, \gamma_0, \mu, \delta)$.

PROOF. In what follows, we say that the set $\{\mathcal{T}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ has the required property if for $s = 0, 1$ $\{(\tau\partial_\tau)^s\mathcal{T}(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}$ are $\mathcal{R}$-bounded families, in $\mathcal{L}(L_q(\Omega))$, whose $\mathcal{R}$-bounds do not exceed some positive constant $C_(N, q, \varepsilon, \gamma_0, \mu, \delta)$.

We first consider the operators $L^i_{1,\ell,n}(\lambda)$ for $i = 1, 2, 3$ and $\ell, n = 1, 2$. We write

$$\begin{aligned}
&\lambda[L^i_{1,\ell,n}(\lambda)f](x)\\
&\quad = \int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)\lambda^{1/2}k_1\lambda^{1/2}e^{-B(d_\ell(x_N)+d_n(y_N))}\widehat{f}(\xi', y_N)\Big](x')dy_N,\\
&\gamma[L^i_{1,\ell,n}(\lambda)f](x)\\
&\quad = \int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)\gamma\lambda^{-1/2}k_1\lambda^{1/2}e^{-B(d_\ell(x_N)+d_n(y_N))}\widehat{f}(\xi', y_N)\Big](x')dy_N,\\
&\lambda^{1/2}D_N[L^i_{1,\ell,n}(\lambda)f](x)\\
&\quad = (-1)^{\ell-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)Bk_1\lambda^{1/2}e^{-B(d_\ell(x_N)+d_n(y_N))}\widehat{f}(\xi', y_N)\Big](x')dy_N,
\end{aligned}$$

where $\lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}$. Since $\lambda^{1/2}, \gamma\lambda^{-1/2}, B \in \mathbb{M}_{1,1,\varepsilon,\gamma_0}$, by Lemma 1.2.5 $\lambda^{1/2}k_1, \gamma\lambda^{-1/2}k_1, Bk_1 \in \mathbb{M}_{0,1,\varepsilon,\gamma_0}$, so that Lemma 2.5.1 furnishes that the sets: $\{\lambda L^i_{1,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$, $\{\gamma L^i_{1,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$, and $\{\lambda^{1/2}D_N L^i_{1,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ have the required properties. For $k, m = 1, \ldots, N-1$, we write

$$\begin{aligned}
&\lambda^{1/2}D_k[L^i_{1,\ell,n}(\lambda)f](x)\\
&\quad = \int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)\lambda^{1/2}(i\xi_kA^{-1})k_1Ae^{-B(d_\ell(x_N)+d_n(y_N))}\widehat{f}(\xi', y_N)\Big](x')dy_N,\\
&D_mD_k[L^i_{1,\ell,n}(\lambda)f](x)\\
&\quad = \int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)(i\xi_m)(i\xi_kA^{-1})k_1Ae^{-B(d_\ell(x_N)+d_n(y_N))}\widehat{f}(\xi', y_N)\Big](x')dy_N,\\
&D_ND_k[L^i_{1,\ell,n}(\lambda)f](x)\\
&\quad = (-1)^{\ell-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)B(i\xi_kA^{-1})k_1Ae^{-B(d_\ell(x_N)+d_n(y_N))}\widehat{f}(\xi', y_N)\Big](x')dy_N.
\end{aligned}$$

Since $\lambda^{1/2}$, $i\xi_m$, $B \in \mathbb{M}_{1,1,\varepsilon,\gamma_0}$ and $i\xi_kA^{-1} \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}$, by Lemma 1.2.5

$$\lambda^{1/2}(i\xi_kA^{-1})k_1, (i\xi_m)(i\xi_kA^{-1})k_1, B(i\xi_kA^{-1})k_1 \in \mathbb{M}_{0,2,\varepsilon,\gamma_0},$$

so that Lemma 2.5.1 furnishes that for $k, m = 1, \ldots, N-1$ the sets:

$$\begin{aligned}
&\{\lambda^{1/2}D_kL^i_{1,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\},\\
&\{D_mD_kL^i_{1,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\},\\
&\{D_mD_kL^i_{1,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}
\end{aligned}$$

have the required properties. Furthermore, since

$$D_N^2L^i_{1,\ell,n}(\lambda) = \lambda\mu^{-1}L^i_{1,\ell,n}(\lambda) + \sum_{k=1}^{N-1} D_k^2L^i_{1,\ell,n}(\lambda),$$

we see easily that the sets $\{D_N^2L^i_{1,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ have the required properties.

Secondly, we consider the operators $L^i_{2,\ell,n}(\lambda)$ for $i = 1,2,3$ and $\ell, n = 1,2$. We show only the case that $X = B$ and $Y = A$ here. The other cases can be shown similarly. For $k, m = 1, \dots, N-1$, we write

$$\begin{aligned}&\Big(\lambda, \gamma, \lambda^{1/2}D_k, D_mD_k, \lambda^{1/2}D_N, D_ND_k, D_N^2\Big)\,[L^i_{2,\ell,n}(\lambda)f](x)\\&= \int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)\Big(\lambda, \gamma, \lambda^{1/2}(i\xi_k), (i\xi_m)(i\xi_k), (-1)^{\ell-1}\lambda^{1/2}B, (-1)^{\ell-1}B(i\xi_k), B^2\Big)\\&\times k_2Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)}\widehat{f}(\xi', y_N)\Big](x')\,dy_N.\end{aligned}$$

By Lemma 1.2.5, the symbols:

$$\lambda k_2, \gamma k_2, \lambda^{1/2}(i\xi_k)k_2(i\xi_m)(i\xi_k)k_2, (-1)^{\ell-1}\lambda^{1/2}Bk_2, (-1)^{\ell-1}B(i\xi_k)k_2, B^2k_2$$

belong to $\mathbb{M}_{0,2,\varepsilon,\gamma_0}$, so that Lemma 2.5.1 furnishes that for $k, m = 1, \dots, N$ the sets:

$$\{\lambda L^i_{2,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}, \qquad \{\gamma L^i_{2,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\},$$
$$\{\lambda^{1/2}D_kL^i_{2,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}, \qquad \{D_mD_kL^i_{2,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$$

have the required properties.

Finally, we consider the operators $L^i_{3,\ell,n}(\lambda)$ for $i = 1,2,3$ and $\ell, n = 1,2$. We show only the case where $X = A$ here. For $k, m = 1, \dots, N-1$, we write

$$\begin{aligned}&\Big(\lambda, \gamma, \lambda^{1/2}D_k, D_mD_k\Big)\,[L^i_{3,\ell,n}(\lambda)f](x) =\\&\qquad= \int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)\Big(\lambda, \gamma, \lambda^{1/2}(i\xi_k), (i\xi_m)(i\xi_k)\Big)\\&\qquad\qquad\times k_2AB\mathcal{M}(d_\ell(x_N))e^{-Ad_n(y_N)}\widehat{f}(\xi', y_N)\Big](x')\,dy_N.\end{aligned}$$

By Lemma 1.2.5, λk_2, γk_2, $\lambda^{1/2}(i\xi_k)k_2$ and $(i\xi_m)(i\xi_k)k_2$ belong to $\mathbb{M}_{0,2,\varepsilon,\gamma_0}$, so that Lemma 2.5.1 furnishes that the sets: $\{\lambda L^i_{3,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$, $\{\gamma L^i_{3,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$, $\{\lambda^{1/2}D_kL^i_{3,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$, and $\{D_mD_kL^i_{3,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ have the required properties for $k, m = 1, \dots, N-1$. By (1.2.3), we have

$$\begin{aligned}&\Big(\lambda^{1/2}D_N, D_ND_k\Big)\,[L^i_{3,\ell,n}(\lambda)f](x)\\&= (-1)^{\ell-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)\Big(\lambda^{1/2}, i\xi_k\Big)Bk_2Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)}\widehat{f}(y_N)\Big](x')\,dy_N\\&+ (-1)^{\ell-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)\Big(\lambda^{1/2}, i\xi_k\Big)Ak_2AB\mathcal{M}(d_\ell(x_N))e^{-Ad_n(y_N)}\widehat{f}(y_N)\Big](x')\,dy_N\end{aligned}$$

for $k = 1, \dots, N-1$, where $\widehat{f}(y_N) = \widehat{f}(\xi', y_N)$ By Lemma 1.2.5, $\lambda^{1/2}Bk_2$ and $(i\xi_k)Bk_2$ belong to $\mathbb{M}_{0,2,\varepsilon,\gamma_0}$, so that Lemma 2.5.1 furnishes that for $k = 1, \dots, N-1$ the sets: $\{\lambda^{1/2}D_NL^i_{3,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ and $\{D_ND_kL^i_{3,\ell,n}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ have the required properties. Concerning $D_N^2\mathcal{M}(d_\ell(x_N))$, we have, by (1.2.3),

$$\begin{aligned}&D_N^2[L^i_{3,\ell,n}(\lambda)f](x)\\&\quad= \int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)(A+B)Bk_2(\lambda, \xi')Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)}\widehat{f}(\xi', y_N)\Big](x')\,dy_N\\&\qquad+ \int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)A^2k_2(\lambda, \xi')AB\mathcal{M}(d_\ell(x_N))e^{-Ad_n(y_N)}\widehat{f}(\xi', y_N)\Big](x')\,dy_N.\end{aligned}$$

By Lemma 1.2.5 $(A+B)Bk_2$ and A^2k_2 belong to $\mathbb{M}_{0,2,\varepsilon,\gamma_0}$, so that Lemma 2.5.1 furnishes that the sets: $\{D_N^2 L^i_{3,\ell,n}(\lambda)\,|\,\lambda\in\Sigma_{\varepsilon,\gamma_0}\}$ have the required properties. □

LEMMA 2.5.3. *Let $0<\varepsilon<\pi/2$, $1<q<\infty$, $\gamma_0>0$, and $m_0\in\mathbb{M}_{0,2,\varepsilon,\gamma_0}$. For $\lambda\in\Sigma_{\varepsilon,\gamma_0}$, we define $L^i_{4,\ell,n}(\lambda)$ $(i=1,2,3,\ \ell,n=1,2)$ by*

$$[L^i_{4,\ell,n}(\lambda)f](x)=\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[E_i(y_N)m_0e^{-A(d_\ell(x_N)+d_n(y_N))}\widehat{f}(\xi',y_N)\Big](x')dy_N.$$

Then, for $i=1,2,3$, $\ell,n=1,2$, $s=0,1$, and $j=1,\dots,N$, the sets:

$$\{(\tau\partial_\tau)^s L^i_{4,\ell,n}(\lambda)\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\},$$
$$\{(\tau\partial_\tau)^s(D_jL^i_{4,\ell,n}(\lambda))\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}$$

are $\mathcal{R}$-bounded families, in $\mathcal{L}(L_q(\Omega))$, whose $\mathcal{R}$-bounds do not exceed some positive constant $C(N,q,\varepsilon,\gamma_0,\mu,\delta)$.

PROOF. We first show that $\{(\tau\partial_\tau)^s L^i_{4,\ell,n}(\lambda)\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}$ are $\mathcal{R}$-bounded families for $i=1,2,3$, $\ell,n=1,2$ and $s=0,1$. For this purpose, we consider the operators $\widetilde{L}^i_{4,\ell,n}(\lambda,x_N,y_N)$ given by the formulas:

$$\Big[\widetilde{L}^i_{4,\ell,n}(\lambda,x_N,y_N)f\Big](x')$$
$$=\mathcal{F}^{-1}_{\xi'}[E_i(y_N)m_0(\xi',\lambda)e^{-A(d_\ell(x_N)+d_n(y_N))}\mathcal{F}_{x'}[f](\xi')](x').$$

By the definition of k_3, Lemma 1.2.6, Leibniz's rule, and [**ES13**, Theorem 3.3], there hold

(2.5.13)
$$\mathcal{R}_{L_q(\mathbf{R}^{N-1})}\big(\{(\tau\partial_\tau)^s\widetilde{L}^i_{4,\ell,n}(\lambda,x_N,y_N)\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0},\ 0\le x_N,y_N\le\delta\}\big)$$
$$\le C(N,q,\varepsilon,\gamma_0,\mu).$$

From now on, we prove that for $i=1,2,3$, $\ell,n=1,2$ and $s=0,1$,

$$\{(\tau\partial_\tau)^s L^i_{4,\ell,n}(\lambda)\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}$$

are $\mathcal{R}$-bounded families in $\mathcal{L}(L_q(\Omega))$ by checking the definition of $\mathcal{R}$-boundedness. Noting that for $f\in L_q(\Omega)$

$$[L^i_{4,\ell,n}(\lambda)f](x)=\int_0^\delta[\widetilde{L}^i_{4,\ell,n}(\lambda,x_N,y_N)f](x')dy_N,$$

we have, by Minkowski's integral inequality, Hölder's inequality, and (2.5.13),

$$\int_0^1\|\sum_{j=1}^m r_j(u)(\tau\partial_\tau)^s L^i_{4,\ell,n}(\lambda_j)f_j\|^q_{L_q(\Omega)}du$$
$$=\int_0^1\|\sum_{j=1}^m r_j(u)\int_0^\delta(\tau\partial_\tau)^s\widetilde{L}^i_{4,\ell,n}(\lambda_j,x_N,y_N)f_jdy_N\|^q_{L_q(\Omega)}du$$
$$\le\delta^{q-1}\int_0^\delta\int_0^\delta\int_0^1\|\sum_{j=1}^m r_j(u)(\tau\partial_\tau)^s\widetilde{L}^i_{4,\ell,n}(\lambda_j,x_N,y_N)f_j\|^q_{L_q(\mathbb{R}^{N-1})}du\,dx_Ndy_N$$
$$\le\delta^{q-1}C(N,q,\varepsilon,\gamma_0,\mu)\int_0^\delta\int_0^\delta\int_0^1\|\sum_{j=1}^m r_j(u)f_j\|^q_{L_q(\mathbb{R}^{N-1})}du\,dx_Ndy_N$$

$$\leq C(N,q,\varepsilon,\gamma_0,\mu,\delta)\int_0^1 \|\sum_{j=1}^m r_j(u)f_j\|_{L_q(\Omega)}^q du.$$

We thus obtain the inequality:

$$\mathcal{R}_{L_q(\Omega)}\big(\{(\tau\partial_\tau)^s L^i_{4,\ell,n}(\lambda) \mid \lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}\big)\leq C(N,q,\varepsilon,\gamma_0,\mu,\delta).$$

Secondly, we consider $D_j L^i_{4,\ell,n}(\lambda)$ for $i=1,2,3$, $\ell,n=1,2$ and $j=1,\dots,N$. For $k=1,\dots,N-1$, we have

$$\begin{aligned}&(D_k, D_N)\,[L^i_{4,\ell,n}(\lambda)f](x)\\&=\int_0^\delta \mathcal{F}^{-1}_{\xi'}[E_i(y_N)(i\xi_k A^{-1},(-1)^{\ell-1})m_0 A e^{-A(d_\ell(x_N)+d_n(y_N))}\widehat{f}(\xi',y_N)](x')dy_N,\end{aligned}$$

which, combined with Lemma 2.5.1, furnishes that the sets: $\{D_j L^i_{4,\ell,n}(\lambda)\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}$ are $\mathcal{R}$-bounded families, in $\mathcal{L}(L_q(\Omega))$, whose $\mathcal{R}$-bounds do not exceed some positive constant $C(N,q,\varepsilon,\gamma_0,\mu,\delta)$, because $i\xi_k A^{-1}m_0$ and $(-1)^{\ell-1}m_0$ belong to $\mathbb{M}_{0,2,\varepsilon,\gamma_0}$ by Lemma 1.2.5 and Lemma 1.2.6. This complete the proof of the lemma. □

We use the following lemma for terms arising from the solution of the divergence equation.

Lemma 2.5.4. *Let $0<\varepsilon<\pi/2$, $1<q<\infty$, $\gamma_0>0$, and $m_0\in\mathbb{M}_{0,2,\varepsilon,\gamma_0}$. For $\lambda\in\Sigma_{\varepsilon,\gamma_0}$ and $X\in\{A,B\}$, we define $L^i_{j,\ell,n}(\lambda)$ ($i=1,2,3$, $j=5,6,7$, $\ell,n=1,2$) by*

$$\begin{aligned}[L^i_{5,\ell,n}(\lambda)f](y)&=\int_0^\delta \mathcal{F}_{\xi'}\left[m_0 e^{-Xd_\ell(x_N)}e^{-Ad_n(y_N)}\widehat{f}(\xi',x_N)\right](y')\,dx_N,\\ [L^i_{6,\ell,n}(\lambda)f](y)&=\int_0^\delta \mathcal{F}_{\xi'}\left[m_0 B\mathcal{M}(d_\ell(x_N))e^{-Ad_n(y_N)}\widehat{f}(\xi',x_N)\right](y')\,dx_N,\\ [L^i_{7,\ell,n}(\lambda)f](y)&=\int_0^\delta \mathcal{F}_{\xi'}\left[m_0 A^{-1}e^{-A(d_\ell(x_N)+d_n(y_N))}\widehat{f}(\xi',x_N)\right](y')\,dx_N.\end{aligned}$$

Then, for $i=1,2,3$, $j=5,6,7$, $\ell,n=1,2$, $s=0,1$, and $k=1,\dots,N$, the sets $\{(\tau\partial_\tau)^s(D_k L^i_{j,\ell,n}(\lambda))\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}$ are $\mathcal{R}$-bounded families, in $\mathcal{L}(L_q(\Omega))$, whose $\mathcal{R}$-bound do not exceed some positive constant $C(N,q,\varepsilon,\gamma_0,\mu,\delta)$.

Proof. We consider only $L^i_{6,\ell,n}(\lambda)$ here, since we can show the case of $L^i_{5,\ell,n}(\lambda)$ similarly. Concerning $L^i_{7,\ell,n}(\lambda)$, use the same argumentation as in Lemma 2.5.3. For $k=1,\dots,N-1$, there holds

$$\begin{aligned}&(D_k,D_N)[L_{6,\ell,n}(\lambda)f](y)=\\&\int_0^\delta\mathcal{F}_{\xi'}\left[\varphi_0(x_N)m_0(-i\xi_k A^{-1},(-1)^{n-1})AB\mathcal{M}(d_\ell(x_N))e^{-Ad_n(y_N)}\widehat{f}(x_N)\right](y')\,dx_N,\end{aligned}$$

where $\widehat{f}(y_N)=\widehat{f}(\xi',y_N)$, so that we have the required properties in the same manner as in the proof of Lemma 2.5.1. □

2.6. Proof of Theorem 2.1.3

In this section, we prove Theorem 2.1.3. To this end, we set

$$K'f_d = (K_1 f_d, \ldots, K_{N-1} f_d)^T, \quad S_0'(\lambda)f = (S_{01}(\lambda)f, \ldots, S_{0N-1}(\lambda)f)^T,$$

where K and $S_0(\lambda)$ are in Theorem 2.1.2, Lemma 2.2.5 and Remark 2.2.6. A crucial part of the proof of Theorem 2.1.3 is to show the following theorem.

THEOREM 2.6.1. *Let $0 < \varepsilon < \pi/2$, $\gamma_0 > 0$, and $1 < q < \infty$, and let $q' = q/(q-1)$. Suppose that*

$$\mathbf{f} \in L_q(\Omega)^N, \quad f_d \in W_q^1(\Omega) \cap \widehat{W}_{q,\Gamma_\delta}^{-1}(\Omega), \quad \mathbf{g} \in W_q^1(\Omega)^N.$$

(1) *For any $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ and $j = 1, \ldots, N-1$, there exist the operators*

$$\begin{aligned} \mathcal{S}_N(\lambda) &\in \mathcal{L}(L_q(\Omega)^{N^3+N^2+2N-1}, W_q^2(\Omega)), \quad I_1(\lambda) \in \mathcal{L}(W_q^1(\Omega) \cap \widehat{W}_{q,\Gamma_\delta}^{-1}(\Omega), W_q^2(\Omega)),\\ \mathcal{T}(\lambda) &\in \mathcal{L}(L_q(\Omega)^{N^3+N^2+3N-1}, W_q^1(\Omega)), \quad I_2(\lambda) \in \mathcal{L}(W_q^1(\Omega) \cap \widehat{W}_{q,\Gamma_\delta}^{-1}(\Omega), W_q^1(\Omega)),\\ \mathcal{S}_j(\lambda) &\in \mathcal{L}(L_q(\Omega)^{N^3+N^2+4N+1}, W_q^2(\Omega)) \end{aligned}$$

such that $\mathbf{u} = (u_1, \ldots, u_N)^T$ and θ are given by

$$\begin{aligned} u_N =& \mathcal{S}_N(\lambda)(\mathbf{f}, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, K'(\lambda f_d), \nabla K'(\lambda^{1/2} f_d), \nabla^2 K' f_d) + I_1(\lambda) f_d,\\ \theta =& \mathcal{T}(\lambda)(\mathbf{f}, \nabla f_d, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, K'(\lambda f_d), \nabla K'(\lambda^{1/2} f_d), \nabla^2 K' f_d) + I_2(\lambda)(\lambda f_d),\\ u_j =& \mathcal{S}_j(\lambda)(\mathbf{f}, \nabla f_d, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, K'(\lambda f_d), \nabla K'(\lambda^{1/2} f_d), \nabla^2 K' f_d,\\ & D_j I_1(\lambda)(\lambda^{1/2} f_d), \nabla D_j I_1(\lambda) f_d, D_j I_2(\lambda)(\lambda f_d)) \end{aligned}$$

solves equations (2.4.1). In addition, there hold

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda\mathcal{S}_J(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C,\\ \mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\gamma\mathcal{S}_J(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C,\\ \mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla\mathcal{S}_J(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C,\\ \mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla^2\mathcal{S}_J(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C,\\ \mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell\mathcal{T}(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C,\\ \mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{T}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C,\\ \mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla I_1(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C \end{aligned}$$

for $\ell = 0, 1$ and $J = 1, \ldots .N$ with a positive constant $C = C(N, q, \varepsilon, \mu, \gamma_0, \delta)$. Concerning the second spatial derivatives of $I_1(\lambda) f_d$, we have

$$D_j D_k I_1(\lambda) f_d = \begin{cases} D_j I_1(\lambda)(D_k f_d) & \text{if} \quad k \neq N,\\ D_k I_1(\lambda)(D_j f_d) & \text{if} \quad j \neq N, \end{cases}$$

and also for any $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ there exists an operator $\widetilde{I}_1(\lambda) \in \mathcal{L}(L_q(\Omega)^{N+1}, L_q(\Omega))$ such that $D_N^2 I_1(\lambda) f_d = \widetilde{I}_1(\lambda)(\lambda^{1/2} f_d, \nabla f_d)$. Moreover, there holds

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \widetilde{I}_1(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \le C(N, q, \varepsilon, \mu, \gamma_0, \delta)$$

for $\ell = 0, 1$ with some positive constant $C(N, q, \varepsilon, \mu, \gamma_0, \delta)$.

(2) *For any* $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ *and* $J = 1, \ldots, N$, *there exist the operators* $I_1^*(\lambda)$, $I_2^*(\lambda)$, *and* $I_{3J}^*(\lambda)$, *from* $L_{q'}(\Omega)$ *to* $\widehat{W}^1_{q',\Gamma_\delta}(\Omega)$, *such that*

$$\begin{aligned}(I_1(\lambda)\psi, \varphi)_\Omega &= (\psi, I_1^*(\lambda)\varphi)_\Omega,\\ (I_2(\lambda)\psi, \varphi)_\Omega &= (\psi, I_2^*(\lambda)\varphi)_\Omega,\\ (D_J I_2(\lambda)\psi, \varphi)_\Omega &= (\psi, I_{3J}^*(\lambda)\varphi)_\Omega\end{aligned}$$

for any $\psi \in W^1_q(\Omega) \cap \widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega)$ *and* $\varphi \in L_{q'}(\Omega)$. *In addition, there hold*

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla X^*(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \le C(N, q, \varepsilon, \mu, \gamma_0, \delta)$$

for $\ell = 0, 1$, $J = 1, \ldots, N$, *and* $X \in \{I_1, I_2, I_{31}, \ldots, I_{3N}\}$ *with some positive constant* $C(N, q, \varepsilon, \mu, \gamma_0, \delta)$.

If Theorem 2.6.1 holds, then we have Theorem 2.1.3 easily. In fact, solutions of equations (2.1.1) are given by

$$\begin{aligned}u_j =&K_j f_d + \mathcal{S}_j(\lambda)(\mathbf{F}, \nabla f_d, \lambda^{1/2}\mathbf{G}, \nabla\mathbf{G}, K'(\lambda f_d), \nabla K'(\lambda^{1/2} f_d), \nabla^2 K' f_d,\\ &D_j I_1(\lambda)(\lambda^{1/2} f_d), \nabla I_1(\lambda)(D_j f_d), D_j I_2(\lambda)(\lambda f_d))\\ u_N =&K_N f_d + \mathcal{S}_N(\lambda)(\mathbf{F}, \lambda^{1/2}\mathbf{G}, \nabla\mathbf{G}, K'(\lambda f_d), \nabla K'(\lambda^{1/2} f_d), \nabla^2 K' f_d) + I_1(\lambda) f_d,\\ \theta =&\mathcal{T}(\lambda)(\mathbf{F}, \nabla f_d, \lambda^{1/2}\mathbf{G}, \nabla\mathbf{G}, K'(\lambda f_d), \nabla K'(\lambda^{1/2} f_d), \nabla^2 K' f_d) + I_2(\lambda)(\lambda f_d)\end{aligned}$$

with $\mathbf{F} = \mathbf{f} - K(\lambda f_d) + 2\mu \mathrm{Div}\mathbf{D}(K f_d)$ and $\mathbf{G} = \mathbf{g} - 2\mu\mathbf{D}(K f_d)\mathbf{e}_N$ for $j = 1, \ldots, N-1$. Therefore, setting

$$\begin{aligned}&\mathcal{U}_j(\lambda)(\mathbf{f}, \nabla f_d, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, K(\lambda f_d), \nabla K(\lambda^{1/2} f_d), \nabla^2 K f_d,\\ &\quad D_j I_1(\lambda)(\lambda^{1/2} f_d), \nabla I_1(\lambda)(D_j f_d), D_j I_2(\lambda)(\lambda f_d))\\ &= \mathcal{S}_j(\lambda)(\mathbf{F}, \nabla f_d, \lambda^{1/2}\mathbf{G}, \nabla\mathbf{G}, K'(\lambda f_d), \nabla K'(\lambda^{1/2} f_d), \nabla^2 K' f_d,\\ &\quad D_j I_1(\lambda)(\lambda^{1/2} f_d), \nabla I_1(\lambda)(D_j f_d), D_j I_2(\lambda)(\lambda f_d)),\\ &\mathcal{U}_N(\lambda)(\mathbf{f}, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, K(\lambda f_d), \nabla K(\lambda^{1/2} f_d), \nabla^2 K f_d)\\ &= \mathcal{S}_N(\lambda)(\mathbf{F}, \lambda^{1/2}\mathbf{G}, \nabla\mathbf{G}, K'(\lambda f_d), \nabla K'(\lambda^{1/2} f_d), \nabla^2 K' f_d),\\ &\mathcal{P}(\lambda)(\mathbf{f}, \nabla f_d, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, K(\lambda f_d), \nabla K(\lambda^{1/2} f_d), \nabla^2 K f_d)\\ &= \mathcal{T}(\lambda)(\mathbf{F}, \nabla f_d, \lambda^{1/2}\mathbf{G}, \nabla\mathbf{G}, K'(\lambda f_d), \nabla K'(\lambda^{1/2} f_d), \nabla^2 K' f_d),\\ &\mathcal{V}_N(\lambda) = I_1(\lambda), \quad \widetilde{\mathcal{V}}_N(\lambda) = \widetilde{I}_1(\lambda), \quad \mathcal{V}_N^*(\lambda) = I_1^*(\lambda),\\ &\mathcal{Q}(\lambda) = I_2(\lambda), \qquad \mathcal{Q}^*(\lambda) = I_2^*(\lambda), \quad \mathcal{Q}_J^*(\lambda) = I_{3J}^*(\lambda)\end{aligned}$$

for $j = 1, \ldots, N-1$ and $J = 1, \ldots, N$, we have the required operators in Theorem 2.1.3, noting Lemma 1.2.3.

From now on, we prove Theorem 2.6.1. Let $\zeta \in C_0^\infty(\mathbb{R}^{N-1})$ be a function which satisfies $0 \le \zeta \le 1$ and

$$\zeta(\xi') = \begin{cases} 1 & |\xi'| \le \dfrac{1}{2},\\[2mm] 0 & |\xi'| \ge \dfrac{3}{4},\end{cases}$$

and also we define the cut-off functions $\zeta_1(\xi',\lambda),\dots,\zeta_4(\xi',\lambda)$ by

$$\begin{aligned}
\zeta_1(\xi',\lambda) &= \zeta\left(\frac{\xi'}{\sigma_1}\right), \quad \zeta_2(\xi',\lambda) = \left\{1-\zeta\left(\frac{\xi'}{\sigma_1}\right)\right\}\zeta\left(\frac{\xi'}{(\sigma_2\lambda)^{1/2}}\right), \\
\zeta_3(\xi',\lambda) &= \left\{1-\zeta\left(\frac{\xi'}{\sigma_1}\right)\right\}\left\{1-\zeta\left(\frac{\xi'}{(\sigma_2\lambda)^{1/2}}\right)\right\}\zeta\left(\frac{\xi'}{\sigma_3}\right), \\
\zeta_4(\xi',\lambda) &= \left\{1-\zeta\left(\frac{\xi'}{\sigma_1}\right)\right\}\left\{1-\zeta\left(\frac{\xi'}{(\sigma_2\lambda)^{1/2}}\right)\right\}\left\{1-\zeta\left(\frac{\xi'}{\sigma_3}\right)\right\},
\end{aligned} \tag{2.6.1}$$

where σ_1, σ_2 and σ_3 are some positive constants, depending only on ε, γ_0, μ and δ, which will be given in Appendix A. In the present section, we often use the relations:

$$\begin{aligned}
B &= \frac{B^2}{B} = \frac{\lambda}{\mu B} - \sum_{j=1}^{N-1}\frac{(i\xi_j)^2}{B} & 1 &= \frac{B^2}{B^2} = \frac{\lambda}{\mu B^2} - \sum_{j=1}^{N-1}\frac{(i\xi_j)^2}{B^2}, \\
A &= \frac{A^2}{A} = \sum_{j=1}^{N-1}\frac{(i\xi_j)^2}{A}, & 1 &= \frac{A^2}{A^2} = -\sum_{j=1}^{N-1}\frac{(i\xi_j)^2}{A^2}.
\end{aligned} \tag{2.6.2}$$

We first construct the operators $\mathcal{S}_N(\lambda), I_1(\lambda), \widetilde{I}_1(\lambda)$, and $I_1^*(\lambda)$. For the purpose, we consider the normal velocity $v_N(x)$ defined as (2.4.13). By the cut-off functions $\zeta_j(\xi',\lambda)$, we have $v_N(x) = \sum_{j,k=1}^4\sum_{\ell=1}^2 V_{k,\ell}^j(x)$, where

$$V_{k,\ell}^j(x) = \mathcal{F}_{\xi'}^{-1}\left[\zeta_j(\xi',\lambda)\left(\frac{\widetilde{L}_{k,2\ell}}{\det L}\mathcal{M}(d_\ell(x_N)) + \frac{L_{k,2\ell}}{\det L}e^{-Bd_\ell(x_N)}\right)r_k\right](x'). \tag{2.6.3}$$

Here $r_k, \widetilde{L}_{k,2\ell}, L_{k,2\ell}$, and $\det L$ are give in Section 2.4. In order to use the lemmas obtained in Section 2.5, we write (2.6.3) by integrals. For $X\in\{A,B\}$, there hold the identities:

$$\begin{aligned}
\widehat{f}(\xi',\delta) &= \int_0^\delta \frac{d}{dy_N}\left[\varphi_\delta(y_N)e^{-X(\delta-y_N)}\widehat{f}(\xi',y_N)\right]dy_N \\
&= \int_0^\delta \varphi_\delta'(y_N)e^{-Xd_1(y_N)}\widehat{f}(\xi',y_N)dy_N \\
&\quad + \int_0^\delta \varphi_\delta(y_N)Xe^{-Xd_1(y_N)}\widehat{f}(\xi',y_N)dy_N \\
&\quad + \int_0^\delta \varphi_\delta(y_N)e^{-Xd_1(y_N)}\widehat{D_N f}(\xi',y_N)dy_N, \\
\widehat{g}(\xi',0) &= -\int_0^\delta \frac{d}{dy_N}\left[\varphi_0(y_N)e^{-Xy_N}\widehat{g}(\xi',y_N)\right]dy_N \\
&= -\int_0^\delta \varphi_0'(y_N)e^{-Xd_2(y_N)}\widehat{g}(\xi',y_N)dy_N \\
&\quad + \int_0^\delta \varphi_0(y_N)Xe^{-Xd_2(y_N)}\widehat{g}(\xi',y_N)dy_N \\
&\quad + \int_0^\delta \varphi_0(y_N)e^{-Xd_2(y_N)}\widehat{D_N g}(\xi',y_N)dy_N,
\end{aligned} \tag{2.6.4}$$

where $\varphi_a'(y_N) = (d\varphi_a/dy_N)(y_N)$ for $a \in \{0, \delta\}$. Applying (2.6.4) with $X = A$ to $V^j_{k,\ell}(x)$ and using (2.4.15), (2.6.2) yield that

(2.6.5)

$$\begin{aligned}
V^j_{1,\ell}(x) &= \frac{1}{\mu}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi_\delta'\frac{i\xi_k}{AB^3}\frac{\widetilde{L}_{1,2\ell}}{\det \mathrm{L}}AB\mathcal{M}(d_\ell(x_N))e^{-A_1(y_N)} \\
&\quad\times\left(\frac{\lambda^{1/2}}{\mu}\widehat{\lambda^{1/2}a_k}(y_N) - i\xi'\cdot\widehat{\nabla' a_k}(y_N)\right)\Bigg](x')\,dy_N \\
&\quad+\frac{1}{\mu}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta\frac{1}{B}\frac{\widetilde{L}_{1,2\ell}}{\det \mathrm{L}}AB\mathcal{M}(d_\ell(x_N))e^{-A_1(y_N)}\widehat{D_k a_k}(y_N)\right](x')\,dy_N \\
&\quad+\frac{1}{\mu}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta\frac{i\xi_k}{AB}\frac{\widetilde{L}_{1,2\ell}}{\det \mathrm{L}}AB\mathcal{M}(d_\ell(x_N))e^{-A_1(y_N)}\widehat{D_N a_k}(y_N)\right](x')\,dy_N \\
&\quad+\frac{1}{\mu}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi_\delta'\frac{i\xi_k}{AB^2}\frac{L_{1,2\ell}}{\det \mathrm{L}}Ae^{-Bd_\ell(x_N)}e^{-A_1(y_N)} \\
&\quad\times\left(\frac{\lambda^{1/2}}{\mu}\widehat{\lambda^{1/2}a_k}(y_N) - i\xi'\cdot\widehat{\nabla' a_k}(y_N)\right)\Bigg](x')\,dy_N \\
&\quad+\frac{1}{\mu}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta\frac{L_{1,2\ell}}{\det \mathrm{L}}Ae^{-Bd_\ell(x_N)}e^{-A_1(y_N)}\widehat{D_k a_k}(y_N)\right](x')\,dy_N \\
&\quad+\frac{1}{\mu}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta\frac{i\xi_k}{A}\frac{L_{1,2\ell}}{\det \mathrm{L}}Ae^{-Bd_\ell(x_N)}e^{-A_1(y_N)}\widehat{D_N a_k}(y_N)\right](x')\,dy_N, \\
V^j_{2,\ell}(x) &= \frac{1}{\mu}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi_\delta'\frac{1}{B^3}\frac{\widetilde{L}_{2,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_1(y_N)} \\
&\quad\times\left(\frac{\lambda^{1/2}}{\mu}\widehat{\lambda^{1/2}a_N}(y_N) - i\xi'\cdot\widehat{\nabla' a_N}(y_N)\right)\Bigg](x')\,dy_N \\
&\quad-\frac{1}{\mu}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta\frac{i\xi_k}{AB}\frac{\widetilde{L}_{2,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_1(y_N)}\widehat{D_k a_N}(y_N)\right](x')\,dy_N \\
&\quad+\frac{1}{\mu}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta\frac{1}{B}\frac{\widetilde{L}_{2,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_1(y_N)}\widehat{D_N a_N}(y_N)\right](x')\,dy_N \\
&\quad+\frac{1}{\mu}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi_\delta'\frac{1}{B^2}\frac{L_{2,2\ell}}{\det L}Ae^{-Bd_\ell(y_N)}e^{-Ad_1(y_N)} \\
&\quad\times\left(\frac{\lambda^{1/2}}{\mu}\widehat{\lambda^{1/2}a_N}(y_N) - i\xi'\cdot\widehat{\nabla' a_N}(y_N)\right)\Bigg](x')\,dy_N \\
&\quad-\frac{1}{\mu}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta\frac{i\xi_k}{A}\frac{L_{2,2\ell}}{\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_1(y_N)}\widehat{D_k a_N}(y_N)\right](x')\,dy_N \\
&\quad+\frac{1}{\mu}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta\frac{L_{2,2\ell}}{\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_1(y_N)}\widehat{D_N a_N}(y_N)\right](x')\,dy_N
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\mu}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi'_\delta\frac{1}{B}\frac{\widetilde{L}_{2,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_1(y_N)}\widehat{T_0 f}(\xi',y_N)\right](x')\,dy_N \\
&- \frac{1}{\mu}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta\frac{i\xi_k}{AB}\frac{\widetilde{L}_{2,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_1(y_N)}\widehat{D_k T_0 f}(y_N)\right](x')\,dy_N \\
&+ \frac{1}{\mu}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta\frac{1}{B}\frac{\widetilde{L}_{2,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_1(y_N)}\widehat{D_N T_0 f}(y_N)\right](x')\,dy_N \\
&+ \frac{1}{\mu}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi'_\delta\frac{L_{2,2\ell}}{\det L}Ae^{-Bd_\ell(y_N)}e^{-Ad_1(y_N)}\widehat{T_0 f}(\xi',y_N)\right](x')\,dy_N \\
&- \frac{1}{\mu}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta\frac{i\xi_k}{A}\frac{L_{2,2\ell}}{\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_1(y_N)}\widehat{D_k T_0 f}(y_N)\right](x')\,dy_N \\
&+ \frac{1}{\mu}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta\frac{L_{2,2\ell}}{\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_1(y_N)}\widehat{D_N T_0 f}(y_N)\right](x')\,dy_N,
\end{aligned}
$$

$$
\begin{aligned}
V^j_{3,\ell}(x) = &-\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi'_0\frac{i\xi_k}{AB^3}\frac{\widetilde{L}_{3,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_2(y_N)} \\
&\times\left(\mu^{-1}\widehat{\lambda b_k}(y_N) - \widehat{\Delta' b_k}(y_N)\right)\Bigg](x')\,dy_N \\
&+\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi_0\frac{i\xi_k}{B^3}\frac{\widetilde{L}_{3,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_2(y_N)} \\
&\times\left(\mu^{-1}\widehat{\lambda b_k}(y_N) - \widehat{\Delta' b_k}(y_N)\right)\Bigg](x')\,dy_N \\
&+\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi_0\frac{i\xi_k}{AB^3}\frac{\widetilde{L}_{3,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_2(y_N)} \\
&\times\left(\mu^{-1}\lambda^{1/2}\mathcal{F}_{x'}[\lambda^{1/2}D_N b_k](\xi',y_N) - i\xi'\cdot\mathcal{F}_{x'}[\nabla' D_N b_k](\xi',y_N)\right)\Bigg](x')\,dy_N \\
&-\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi'_0\frac{i\xi_k}{AB^2}\frac{L_{3,2\ell}}{\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_2(y_N)} \\
&\times\left(\mu^{-1}\widehat{\lambda b_k}(y_N) - \widehat{\Delta' b_k}(y_N)\right)\Bigg](x')\,dy_N \\
&+\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi_0\frac{i\xi_k}{B^2}\frac{L_{3,2\ell}}{\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_2(y_N)} \\
&\times\left(\mu^{-1}\widehat{\lambda b_k}(y_N) - \widehat{\Delta' b_k}(y_N)\right)\Bigg](x')\,dy_N \\
&+\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi_0\frac{i\xi_k}{AB^2}\frac{L_{3,2\ell}}{\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_2(y_N)} \\
&\times\left(\mu^{-1}\lambda^{1/2}\mathcal{F}_{x'}[\lambda^{1/2}D_N b_k](\xi',y_N) - i\xi'\cdot\mathcal{F}_{x'}[\nabla' D_N b_k](\xi',y_N)\right)\Bigg](x')\,dy_N
\end{aligned}
$$

for $j = 1,2,3,4$ and $\ell = 1,2$, where we have used the abbreviations:

$$
\zeta_j = \zeta_j(\xi',\lambda), \quad \varphi_\delta = \varphi_\delta(y_N), \quad \varphi_0 = \varphi_0(y_N), \quad \widehat{f}(y_N) = \widehat{f}(\xi',y_N), \tag{2.6.6}
$$

$$i\xi' \cdot \widehat{\nabla' f}(\xi', x_N) = \sum_{j=1}^{N-1} i\xi_j \widehat{D_j f}(\xi', x_N), \quad \Delta' g(x) = \sum_{j=1}^{N-1} D_j^2 g(x)$$

for scalar functions $f(x)$ and $g(x)$. We will discuss $V^j_{4,\ell}(x)$ later. The following lemma will be proved in Appendix A.

Lemma 2.6.2. *Let $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Then we have the properties:*

$$\zeta_j(\lambda, \xi') \frac{\widetilde{L}_{3,2\ell}}{\det L} \in \mathbb{M}_{0,2,\varepsilon,\gamma_0},$$
$$\zeta_j(\lambda, \xi') \frac{\widetilde{L}_{1,2\ell}}{\det L}, \;\; \zeta_j(\lambda, \xi') \frac{\widetilde{L}_{2,2\ell}}{\det L}, \;\; \zeta_j(\lambda, \xi') \frac{L_{3,2\ell}}{\det L} \in \mathbb{M}_{-1,2,\varepsilon,\gamma_0},$$
$$\zeta_j(\lambda, \xi') \frac{L_{1,2\ell}}{\det L}, \;\; \zeta_j(\lambda, \xi') \frac{L_{2,2\ell}}{\det L} \in \mathbb{M}_{-2,2,\varepsilon,\gamma_0}$$

for $j = 1, \ldots, 4$ and $\ell = 1, 2$.

From viewpoint of (2.6.5) and Lemma 2.5.2 , we define solution operators as follows:

$$S^j_{1,\ell}(\lambda)(\lambda^{1/2}\mathbf{a}', \nabla \mathbf{a}') = V^j_{1,\ell}(x),$$
$$S^j_{2,\ell}(\lambda)(\lambda^{1/2}a_N, \nabla a_N) + \widetilde{S}^j_{2,\ell}(\lambda)(T_0\mathbf{f}, \nabla T_0\mathbf{f}) = V^j_{2,\ell}(x),$$
$$S^j_{3,\ell}(\lambda)(\lambda\mathbf{b}', \lambda^{1/2}\nabla\mathbf{b}', \nabla^2\mathbf{b}') = V^j_{3,\ell}(x) \tag{2.6.7}$$

for $\lambda \in \Sigma_{\varepsilon,\gamma_0}$. Since there hold, for $k = 1, \ldots, N-1$,

$$\frac{i\xi_k}{A}, \;\; \frac{i\xi_k}{B}, \;\; \frac{\lambda^{1/2}}{B} \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}, \tag{2.6.8}$$

we have, by Lemma 1.2.5, Lemma 1.2.6, Lemma 2.5.2, and Lemma 2.6.2,

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\lambda S^j_{k,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \le C,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\gamma S^j_{k,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \le C,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\lambda^{1/2}\nabla S^j_{k,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \le C,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\nabla^2 S^j_{k,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \le C$$
$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\lambda \widetilde{S}^j_{2,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \le C,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\gamma \widetilde{S}^j_{2,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \le C,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\lambda^{1/2}\nabla \widetilde{S}^j_{2,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \le C,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\nabla^2 \widetilde{S}^j_{2,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \le C \tag{2.6.9}$$

for $j = 1, \ldots, 4$, $k = 1, 2, 3$, $\ell = 1, 2$, and $m = 0, 1$ with some positive constant $C = C(N, q, \varepsilon, \mu, \gamma_0, \delta)$.

Next we consider the term $V^j_{4,\ell}(x)$. By (2.1.5) and (2.2.16), we have $V^j_{4,\ell}(x) = I^j_{1,\ell}(\lambda) f_d + S^j_{4,\ell}(\lambda)\mathbf{f}$ such that for $j = 1, \ldots, 4$, $\ell = 1, 2$, and $m = 1, 2, 3$

$$I^j_{1,\ell}(\lambda) f_d = -\frac{1}{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j \frac{1}{B}\frac{\widetilde{L}_{4,2\ell}}{\det L} B\mathcal{M}(d_\ell(x_N))\Phi(\xi', y_N)\widehat{f}_d(y_N)\right](x')\,dy_N$$
$$-\frac{1}{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j \frac{L_{4,2\ell}}{\det L} e^{-Bd_\ell(x_N)}\Phi(\xi', y_N)\widehat{f}_d(y_N)\right](x')\,dy_N,$$

$$
\begin{aligned}
&S^m_{4,\ell}(\lambda)\mathbf{f} = \sum_{k=1}^{N-1}\sum_{n=1}^{2}(-1)^n\Big\{ \\
&- \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_m\varphi_\delta\frac{i\xi_k e^{-Ad_n(0)}}{2AB}\frac{\widetilde{L}_{4,2\ell}}{\lambda\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_n(y_N)}\widehat{f}_k(y_N)\right](x')\,dy_N \\
&+ \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_m\varphi_\delta\frac{i\xi_k e^{-Bd_n(0)}}{2AB}\frac{\widetilde{L}_{4,2\ell}}{\lambda\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Bd_n(y_N)}\widehat{f}_k(y_N)\right](x')\,dy_N \\
&- \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_m\varphi_\delta\frac{i\xi_k e^{-Ad_n(0)}}{2A}\frac{L_{4,2\ell}}{\lambda\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)}\widehat{f}_k(y_N)\right](x')\,dy_N \\
&+ \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_m\varphi_\delta\frac{i\xi_k e^{-Bd_n(0)}}{2A}\frac{L_{4,2\ell}}{\lambda\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)}\widehat{f}_k(y_N)\right](x')\,dy_N\Big\} \\
&- \sum_{n=1}^{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_m\varphi_\delta\frac{e^{-Ad_n(0)}}{2B}\frac{\widetilde{L}_{4,2\ell}}{\lambda\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_n(y_N)}\widehat{f}_N(y_N)\right](x')\,dy_N \\
&+ \sum_{n=1}^{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_m\varphi_\delta\frac{Ae^{-Bd_n(0)}}{2B^2}\frac{\widetilde{L}_{4,2\ell}}{\lambda\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_n(y_N)}\widehat{f}_N(y_N)\right](x')\,dy_N \\
&- \sum_{n=1}^{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_m\varphi_\delta\frac{e^{-Ad_n(0)}}{2}\frac{L_{4,2\ell}}{\lambda\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)}\widehat{f}_N(y_N)\right](x')\,dy_N \\
&+ \sum_{n=1}^{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_m\varphi_\delta\frac{Ae^{-Bd_n(0)}}{2B}\frac{L_{4,2\ell}}{\lambda\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_n(y_N)}\widehat{f}_N(y_N)\right](x')\,dy_N,
\end{aligned}
$$

$$
\begin{aligned}
&S^4_{4,\ell}(\lambda)\mathbf{f} = \\
&- \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\frac{\zeta_4\varphi_0'}{A^3B}\frac{\widetilde{L}_{4,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_2(y_N)}\mathcal{F}_{x'}[\Delta' S_{0N}(\lambda)\mathbf{f}](\xi',y_N)\right](x')\,dy_N \\
&+ \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\frac{\zeta_4\varphi_0}{A^2B}\frac{\widetilde{L}_{4,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_2(y_N)}\mathcal{F}_{x'}[\Delta' S_{0N}(\lambda)\mathbf{f}](\xi',y_N)\right](x')\,dy_N \\
&+ \sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[\zeta_4\varphi_0\frac{i\xi_k}{A^3B}\frac{\widetilde{L}_{4,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))e^{-Ad_2(y_N)} \\
&\qquad \times \mathcal{F}_{x'}[D_kD_NS_{0N}(\lambda)\mathbf{f}](\xi',y_N)\Big](x')\,dy_N \\
&- \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_4\varphi_0'\frac{1}{A^3}\frac{L_{4,2\ell}}{\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_2(y_N)}\mathcal{F}_{x'}[\Delta' S_{0N}(\lambda)\mathbf{f}](\xi',y_N)\right](x')\,dy_N \\
&+ \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_4\varphi_0\frac{1}{A^2}\frac{L_{4,2\ell}}{\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_2(y_N)}\mathcal{F}_{x'}[\Delta' S_{0N}(\lambda)\mathbf{f}](\xi',y_N)\right](x')\,dy_N \\
&+ \sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[\zeta_4\varphi_0\frac{i\xi_k}{A^3}\frac{L_{4,2\ell}}{\det L}Ae^{-Bd_\ell(x_N)}e^{-Ad_2(y_N)} \\
&\qquad \times \mathcal{F}_{x'}[D_kD_NS_{0N}(\lambda)\mathbf{f}](\xi',y_N)\Big](x')\,dy_N,
\end{aligned}
$$

where we have used (2.6.6) and (2.6.4) with $X = A$ for $S^4_{4,\ell}(\lambda)$. Note that by (2.4.14) and (1.2.3) there hold

$$\begin{aligned}
&D_k I^j_{1,\ell}(\lambda) f_d \\
&\quad = -\frac{1}{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j(\lambda,\xi')\frac{i\xi_k}{AB}\frac{\widetilde{L}_{4,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))\Phi(\xi',y_N)\widehat{f}_d(\xi',y_N)\right](x')\,dy_N \\
&\qquad -\frac{1}{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j(\lambda,\xi')\frac{i\xi_k}{A}\frac{L_{4,2\ell}}{\det L}Ae^{-Bd_\ell(x_N)}\Phi(\xi',y_N)\widehat{f}_d(\xi',y_N)\right](x')\,dy_N, \\
&D_N I^j_{1,\ell}(\lambda) f_d \\
&\quad = \frac{(-1)^\ell}{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j(\lambda,\xi')\frac{1}{B}\frac{\widetilde{L}_{4,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))\Phi(\xi',y_N)\widehat{f}_d(\xi',y_N)\right](x')\,dy_N \\
&\qquad +\frac{1}{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j(\lambda,\xi')\frac{A^{-1}L_{4,2\ell-1}}{\det L}Ae^{-Bd_\ell(x_N)}\Phi(\xi',y_N)\widehat{f}_d(\xi',y_N)\right](x')\,dy_N
\end{aligned}$$

for $j = 1,\dots,4$, $k = 1,\dots,N-1$, and $\ell = 1,2$, and also by (1.2.3) and (2.6.2)

$$\begin{aligned}
D_k D_J I^j_{1,\ell}(\lambda) f_d &= D_J I^j_{1,\ell}(\lambda)(D_k f_d), \\
D_N^2 I^j_{1,\ell}(\lambda)(f_d) &= \widetilde{I}^j_{1,\ell}(\lambda)(\lambda^{1/2} f_d, \nabla f_d)
\end{aligned} \tag{2.6.10}$$

for $J = 1,\dots,N$, where

$$\begin{aligned}
&\widetilde{I}^j_{1,\ell}(\lambda)(\lambda^{1/2} f_d, \nabla f_d) = \\
&\quad \frac{1}{2}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\frac{i\xi_k}{AB}\frac{\widetilde{L}_{4,2\ell}}{\det L}AB\mathcal{M}(d_\ell(x_N))\Phi_{\xi'}(y_N)\widehat{D_k f_d}(\xi',y_N)\right](x')\,dy_N \\
&\quad -\frac{1}{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\frac{1}{B^2}\frac{\widetilde{L}_{4,2\ell}}{\det L}Ae^{-Bd_\ell(x_N)}\Phi_{\xi'}(y_N)\right. \\
&\qquad \left.\times\left(\frac{\lambda^{1/2}}{\mu}\widehat{\lambda^{1/2} f_d}(\xi',y_N) - i\xi'\cdot\widehat{\nabla' f_d}(\xi',y_N)\right)\right](x')\,dy_N \\
&\quad -\frac{(-1)^\ell}{2\mu}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\frac{\lambda^{1/2}}{B}\frac{L_{4,2\ell-1}}{A\det L}Ae^{-Bd_\ell(x_N)}\Phi_{\xi'}(y_N)\widehat{\lambda^{1/2} f_d}(\xi',y_N)\right](x')\,dy_N \\
&\quad +\frac{(-1)^\ell}{2}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\frac{i\xi_k}{B}\frac{L_{4,2\ell-1}}{A\det L}Ae^{-Bd_\ell(x_N)}\Phi_{\xi'}(y_N)\widehat{D_k f_d}(\xi',y_N)\right](x')\,dy_N.
\end{aligned}$$

We then have the following lemma which will be proved in Appendix A.

Lemma 2.6.3. *Let $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Then we have the properties:*

$$\begin{aligned}
&\zeta_j(\lambda,\xi')\frac{\widetilde{L}_{4,2\ell}}{\det L} \in \mathbb{M}_{1,2,\varepsilon,\gamma_0}, \qquad \zeta_j(\lambda,\xi')\frac{A^{-1}L_{4,2\ell-1}}{\det L},\ \zeta_j(\lambda,\xi')\frac{L_{4,2\ell}}{\det L} \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}, \\
&\zeta_m(\lambda,\xi')\frac{\widetilde{L}_{4,2\ell}}{\lambda\det L} \in \mathbb{M}_{-1,2,\varepsilon,\gamma_0}, \quad \zeta_m(\lambda,\xi')\frac{L_{4,2\ell}}{\lambda\det L} \in \mathbb{M}_{-2,2,\varepsilon,\gamma_0}
\end{aligned}$$

for $j = 1,\dots,4$, $\ell = 1,2$, and $m = 1,2,3$.

In the same manner as in the proof of (2.6.9), we obtain, by Lemma 1.2.5, 1.2.6, 2.5.2, and 2.6.3,

$$\begin{aligned}
(2.6.11)\qquad &\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\lambda S^j_{4,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\gamma S^j_{4,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\lambda^{1/2}\nabla S^j_{4,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\nabla^2 S^j_{4,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\nabla I^j_{1,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\widetilde{I}^j_{1,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C
\end{aligned}$$

for $j = 1, 2, 3, 4$, $\ell = 1, 2$, $m = 0, 1$, and a positive constant $C = C(N, q, \varepsilon, \mu, \gamma_0, \delta)$, where we also have used Lemma 2.2.5, 1.2.3, and A.3 (2).

We here consider the operator $I^{j*}_{1,\ell}(\lambda)$:

$$\begin{aligned}
&[I^{j*}_{1,\ell}(\lambda)\varphi](y)\\
&= -\frac{1}{2}\int_0^\delta \mathcal{F}_{\xi'}\left[\zeta_j(\lambda,\xi')\frac{1}{B}\frac{\widetilde{L}_{4,2\ell}}{\det L}B\mathcal{M}(d_\ell(x_N))\Phi(\xi',y_N)\mathcal{F}^{-1}_{x'}[\varphi](\xi',x_N)\right](y')\,dx_N\\
&-\frac{1}{2}\int_0^\delta \mathcal{F}_{\xi'}\left[\zeta_j(\lambda,\xi')\frac{L_{4,2\ell}}{\det L}e^{-Bd_\ell(x_N)}\Phi(\xi',y_N)\mathcal{F}^{-1}_{x'}[\varphi](\xi',x_N)\right](y')\,dx_N
\end{aligned}$$

for $\varphi \in L_{q'}(\Omega)$ with $1/q + 1/q' = 1$. Then, for any $\psi \in W^1_q(\Omega_y) \cap \widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega_y)$ and $\varphi \in L_{q'}(\Omega_x)$, where subscript x and y denote their variables, there hold

$$(2.6.12)\qquad (I^j_{1,\ell}(\lambda)\psi,\varphi)_{\Omega_x} = (\psi, I^{j*}_{1,\ell}(\lambda)\varphi)_{\Omega_y},\quad [I^{j*}_{1,\ell}(\lambda)\varphi]|_{y_N=\delta} = 0,$$

since $\Phi(\xi',\delta) = 0$. Moreover, we have, by Lemma 2.5.4 and Lemma 2.6.3,

$$(2.6.13)\qquad \mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m(\nabla_y I^{j*}_{1,\ell}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C$$

for $j = 1, \ldots, 4$, $\ell = 1, 2$, and $m = 0, 1$ with a positive constant $C = (N, q, \varepsilon, \gamma_0, \mu, \delta)$.

Summing up the above argumentation, we set

$$\begin{aligned}
&S_1(\lambda)(\lambda^{1/2}\mathbf{a}, \nabla\mathbf{a}, \lambda\mathbf{b}', \lambda^{1/2}\nabla\mathbf{b}', \nabla^2\mathbf{b}') =\\
&\sum_{j=1}^4\sum_{\ell=1}^2\left(S^j_{1,\ell}(\lambda)(\lambda\mathbf{a}', \nabla\mathbf{a}') + S^j_{2,\ell}(\lambda)(\lambda^{\frac12}a_N, \nabla a_N) + S^j_{3,\ell}(\lambda)(\lambda\mathbf{b}', \lambda^{\frac12}\nabla\mathbf{b}', \nabla^2\mathbf{b}')\right),\\
&\widetilde{S}_1(\lambda)\mathbf{f} = \sum_{j=1}^4\sum_{\ell=1}^2\left(\widetilde{S}^j_{2,\ell}(\lambda)(T_0\mathbf{f}, \nabla T_0\mathbf{f}) + S^j_{4,\ell}(\lambda)\mathbf{f}\right),\quad I_1(\lambda)f_d = \sum_{j=1}^4\sum_{\ell=1}^2 I^j_{1,\ell}(\lambda)f_d,
\end{aligned}$$

and then $v_N(x)$ in (2.4.13) is given by

$$(2.6.14)\qquad v_N(x) = S_1(\lambda)(\lambda^{1/2}\mathbf{a}, \nabla\mathbf{a}, \lambda\mathbf{b}', \lambda^{1/2}\nabla\mathbf{b}', \nabla^2\mathbf{b}') + \widetilde{S}_1(\lambda)\mathbf{f} + I_1(\lambda)f_d.$$

By Lemma 1.2.3, (2.6.9), (2.6.11), and (2.6.13), we have

$$\begin{aligned}
(2.6.15)\qquad &\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda X_1(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\gamma X_1(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla X_1(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla^2 X_1(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,
\end{aligned}$$

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla I_1(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\widetilde{I}_1(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla I_1^*(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C$$

for $\ell = 1, 2$ and $X \in \{S, \widetilde{S}\}$, with some positive constant $C = C(N, q, \varepsilon, \mu, \gamma_0, \delta)$, where $\widetilde{I}_1(\lambda)$ and $I_1^*(\lambda)$ are defined as

$$\widetilde{I}_1(\lambda)(\lambda^{1/2}f_d, \nabla f_d) = \sum_{j=1}^4\sum_{\ell=1}^2 \widetilde{I}_{1,\ell}^j(\lambda)(\lambda^{1/2}f_d, \nabla f_d),$$
$$I_1^*(\lambda)\varphi = \sum_{j=1}^4\sum_{\ell=1}^2 I_{1,\ell}^{j*}(\lambda)\varphi \quad \text{for } \varphi \in L_{q'}(\Omega),$$

and also by (2.6.10) and (2.6.12) there hold

$$\begin{aligned}
&D_J D_k I_1(\lambda)f_d = D_J I_1(\lambda)(D_k f_d),\\
&D_N^2 I_1(\lambda)f_d = \widetilde{I}_1(\lambda)(\lambda^{1/2}f_d, \nabla f_d)\\
&(I_1(\lambda)\psi, \varphi)_{\Omega_x} = (\psi, I_1^*(\lambda)\varphi)_{\Omega_y}, \quad [I_1^*(\lambda)\varphi]|_{y_N=\delta} = 0
\end{aligned} \tag{2.6.16}$$

for $J = 1, \ldots, N$, $k = 1, \ldots, N-1$ and any $\psi \in W_q^1(\Omega_y) \cap \widehat{W}_{q,\Gamma_\delta}^{-1}(\Omega_y)$, $\varphi \in L_{q'}(\Omega_x)$. Thus, setting

$$u_N = S_{0N}(\lambda)f + S_1(\lambda)(\lambda^{1/2}\mathbf{a}, \nabla\mathbf{a}, \lambda\mathbf{b}', \lambda^{1/2}\nabla\mathbf{b}', \nabla^2\mathbf{b}') + \widetilde{S}_1(\lambda)\mathbf{f} + I_1(\lambda)f_d,$$

we see that u_N is the N-th component of the velocity $\mathbf{u}$ to equations (2.4.1), and also the operator $\mathcal{S}_N(\lambda)$ in Theorem 2.6.1 is given by

$$\begin{aligned}
&\mathcal{S}_N(\lambda)(\mathbf{f}, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, K'(\lambda f_d), \nabla K'(\lambda^{1/2}f_d), \nabla^2 K' f_d)\\
&= S_{0N}(\lambda)\mathbf{f} + S_1(\lambda)(\lambda^{1/2}\mathbf{a}, \nabla\mathbf{a}, \lambda\mathbf{b}', \lambda^{1/2}\nabla\mathbf{b}', \nabla^2\mathbf{b}') + \widetilde{S}_1(\lambda)\mathbf{f}\\
&= S_{0N}(\lambda)\mathbf{f} + S_1(\lambda)\left(\lambda^{1/2}\mathbf{a}, \nabla\mathbf{a}, -\lambda S_0'(\lambda)\mathbf{f}, -\lambda^{1/2}\nabla S_0'(\lambda)\mathbf{f}, -\nabla^2 S_0'(\lambda)\mathbf{f}\right)\\
&\quad + S_1(\lambda)\left(0, 0, -K'(\lambda f_d), -\nabla K'(\lambda^{1/2}f_d), -\nabla^2 K' f_d\right) + \widetilde{S}_1(\lambda)\mathbf{f}.
\end{aligned}$$

$\mathcal{S}_N(\lambda), I_1(\lambda), \widetilde{I}_1(\lambda)$, and $I_1^*(\lambda)$ satisfy the required properties in Theorem 2.6.1 by (2.6.15), (2.6.16), Lemma 2.2.5 and Lemma 1.2.3.

Next we construct the operators $\mathcal{T}(\lambda), I_2(\lambda), I_2^*(\lambda)$, and $I_{2J}^*(\lambda)$ for $J = 1, \ldots, N$. For the purpose, we consider the pressure $\pi(x)$ in (2.4.13) in the same manner as $v_N(x)$. By the cut-off functions $\zeta_j(\xi', \lambda)$ defined as (2.6.1), we have $\pi(x) = \sum_{j,k=1}^4 \sum_{\ell=1}^2 (-1)^{\ell+1}\Pi_{k,\ell}^j(x)$ with

$$\Pi_{k,\ell}^j(x) = \mu\mathcal{F}_{\xi'}^{-1}\left[\zeta_j(\lambda, \xi')\frac{B+A}{A}\frac{\widetilde{L}_{k,2\ell}}{\det L}e^{-Ad_\ell(x_N)}r_k\right](x').$$

It follows from (2.6.4) with $X = A$ and (2.6.2) that

$$
\begin{aligned}
&\Pi^j_{1,\ell}(x)\\
&\quad= \sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi'_\delta \frac{i\xi_k(B+A)}{AB^2}\frac{\widetilde{L}_{1,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_1(y_N))}\\
&\qquad\times\left(\frac{\lambda^{1/2}}{\mu}\widehat{\lambda^{1/2}a_k}(y_N) - i\xi'\cdot\widehat{\nabla' a_k}(y_N)\right)\Bigg](x')\,dy_N\\
&\quad+ \sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta(B+A)\frac{\widetilde{L}_{1,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_1(y_N))}\widehat{D_k a_k}(y_N)\right](x')\,dy_N\\
&\quad+ \sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta \frac{i\xi_k(B+A)}{A}\frac{\widetilde{L}_{1,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_1(y_N))}\widehat{D_N a_k}(y_N)\right](x')\,dy_N,\\
&\Pi^j_{2,\ell}(x)\\
&\quad= \int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi'_\delta \frac{B+A}{B^2}\frac{\widetilde{L}_{2,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_1(y_N))}\\
&\qquad\times\left(\frac{\lambda^{1/2}}{\mu}\widehat{\lambda^{1/2}a_N}(y_N) - i\xi'\cdot\widehat{\nabla' a_N}(y_N)\right)\Bigg](x')\,dy_N\\
&\quad- \sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta \frac{i\xi_k(B+A)}{A}\frac{\widetilde{L}_{2,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_1(y_N))}\widehat{D_k a_N}(y_N)\right](x')\,dy_N\\
&\quad+ \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta(B+A)\frac{\widetilde{L}_{2,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_1(y_N))}\widehat{D_N a_N}(y_N)\right](x')\,dy_N\\
&\quad+ \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi'_\delta(B+A)\frac{\widetilde{L}_{2,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_1(y_N))}\widehat{T_0 f}(y_N)\right](x')\,dy_N\\
&\quad- \sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta \frac{i\xi_k(B+A)}{A}\frac{\widetilde{L}_{2,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_1(y_N))}\widehat{D_k T_0 f}(y_N)\right](x')\,dy_N\\
&\quad+ \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\varphi_\delta(B+A)\frac{\widetilde{L}_{2,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_1(y_N))}\widehat{D_N T_0 f}(y_N)\right](x')\,dy_N,\\
&\Pi^j_{3,\ell}(x) =\\
&\quad- \mu\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi'_0 \frac{i\xi_k(B+A)}{AB^2}\frac{\widetilde{L}_{3,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_2(y_N))}\\
&\qquad\times\left(\mu^{-1}\widehat{\lambda b_k}(y_N) - \widehat{\Delta' b_k}(y_N)\right)\Bigg](x')\,dy_N\\
&\quad+ \mu\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi_0 \frac{i\xi_k(B+A)}{B^2}\frac{\widetilde{L}_{3,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_2(y_N))}\\
&\qquad\times\left(\mu^{-1}\widehat{\lambda b_k}(y_N) - \widehat{\Delta' b_k}(y_N)\right)\Bigg](x')\,dy_N\\
&\quad+ \mu\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Bigg[\zeta_j\varphi_0 \frac{i\xi_k(B+A)}{AB^2}\frac{\widetilde{L}_{3,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_2(y_N))}
\end{aligned}
$$

$$\times\Big(\mu^{-1}\lambda^{1/2}\mathcal{F}_{x'}[\lambda^{1/2}D_Nb_k](\xi',y_N) - i\xi'\cdot\mathcal{F}_{x'}[\nabla' D_Nb_k](\xi',y_N)\Big)\Big](x')\,dy_N,$$

for $j=1,\dots,4$ and $\ell=1,2$, where we have used the abbreviations (2.6.6). From viewpoint of the above formulas and Lemma 2.5.3, we define solution operators as follows:

$$\begin{aligned}
T^j_{1,\ell}(\lambda)(\lambda^{1/2}\mathbf{a}',\nabla\mathbf{a}') &= \Pi^j_{1,\ell}(x),\\
T^j_{2,\ell}(\lambda)(\lambda^{1/2}a_N,\nabla a_N) + \widetilde{T}^j_{2,\ell}(\lambda)(T_0\mathbf{f},\nabla T_0\mathbf{f}) &= \Pi^j_{2,\ell}(x),\\
T^j_{3,\ell}(\lambda)(\lambda\mathbf{b}',\lambda^{1/2}\nabla\mathbf{b}',\nabla^2\mathbf{b}') &= \Pi^j_{3,\ell}(x).
\end{aligned}$$

Then, by Lemma 1.2.5, Lemma 1.2.6, Lemma 2.5.3, and Lemma 2.6.2, we obtain

$$\begin{aligned}
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m T^j_{k,\ell}(\lambda)\mid \lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}) &\le C,\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m (\nabla T^j_{k,\ell}(\lambda))\mid \lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}) &\le C,\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m \widetilde{T}^j_{2,\ell}(\lambda)\mid \lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}) &\le C,\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m (\nabla \widetilde{T}^j_{2,\ell}(\lambda))\mid \lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}) &\le C
\end{aligned}\tag{2.6.17}$$

for $j=1,\dots,4$, $k=1,2,3$, $\ell=1,2$, and $m=0,1$ with some positive constant $C=C(N,q,\varepsilon,\mu,\gamma_0,\delta)$, noting that $(B+A)/B\in\mathbb{M}_{0,2,\varepsilon,\gamma_0}$ and (2.6.8). Concerning the term $\Pi^j_{4,\ell}(x)$, by (2.6.2), the special formulas (2.1.5), and (2.2.16), we have $\Pi^j_{4,\ell}(x) = I^j_{2,\ell}(\lambda)(\lambda f_d) + \widetilde{I}^j_{2,\ell}(\lambda)(\nabla f_d) + T^j_{4,\ell}(\lambda)\mathbf{f}$ with

$$\begin{aligned}
&I^j_{2,\ell}(\lambda)(\lambda f_d)\\
&= -\frac{1}{2}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\frac{(B+A)}{AB^2}\frac{\widetilde{L}_{4,2\ell}}{\det L}e^{-Ad_\ell(x_N)}\Phi(\xi',y_N)\widehat{\lambda f_d}(y_N)\right](x')\,dy_N,\\
&\widetilde{I}^j_{2,\ell}(\lambda)(\nabla f_d)\\
&= \frac{\mu}{2}\sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_j\frac{i\xi_k(B+A)}{AB^2}\frac{\widetilde{L}_{4,2\ell}}{\det L}e^{-Ad_\ell(x_N)}\Phi(\xi',y_N)\widehat{D_kf_d}(y_N)\right](x')\,dy_N,\\
&T^m_{4,\ell}(\lambda)\mathbf{f} = \frac{\mu}{2}\sum_{k=1}^{N-1}\sum_{n=1}^2(-1)^n\Bigg\{\\
&-\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_m\varphi_\delta\frac{i\xi_k(B+A)}{A}\frac{e^{-Ad_n(0)}\widetilde{L}_{4,2\ell}}{\lambda\det L}e^{-A(d_\ell(x_N)+d_n(y_N))}\widehat{f}_k(y_N)\right](x')\,dy_N\\
&+\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_m\varphi_\delta\frac{i\xi_k(B+A)}{A}\frac{e^{-Bd_n(0)}\widetilde{L}_{4,2\ell}}{\lambda\det L}e^{-Ad_\ell(x_N)}e^{-Bd_n(y_N)}\widehat{f}_k(y_N)\right](x')\,dy_N\Bigg\}\\
&+\frac{\mu}{2}\sum_{n=1}^2\Bigg\{\\
&-\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_m\varphi_\delta(B+A)\frac{e^{-Ad_n(0)}\widetilde{L}_{4,2\ell}}{\lambda\det L}e^{-A(d_\ell(x_N)+d_n(y_N))}\widehat{f}_N(y_N)\right](x')\,dy_N\\
&+\int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_m\varphi_\delta\frac{A(B+A)}{B}\frac{e^{-Bd_n(0)}\widetilde{L}_{4,2\ell}}{\lambda\det L}e^{-Ad_\ell(x_N)}e^{-Bd_n(y_N)}\widehat{f}_N(y_N)\right](x')\,dy_N\Bigg\},
\end{aligned}$$

$$
\begin{aligned}
T^4_{4,\ell}(\lambda)\mathbf{f} = \mu\Big\{ \\
&- \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_4\varphi_0'\frac{B+A}{A^3}\frac{\widetilde{L}_{4,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_2(y_N))}\mathcal{F}_{x'}[\Delta' S_{0N}(\lambda)\mathbf{f}](\xi',y_N)\right](x')\,dy_N \\
&+ \int_0^\delta \mathcal{F}^{-1}_{\xi'}\left[\zeta_4\varphi_0\frac{B+A}{A^2}\frac{\widetilde{L}_{4,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_2(y_N))}\mathcal{F}_{x'}[\Delta' S_{0N}(\lambda)\mathbf{f}](\xi',y_N)\right](x')\,dy_N \\
&+ \sum_{k=1}^{N-1}\int_0^\delta \mathcal{F}^{-1}_{\xi'}\Big[\zeta_4\varphi_0\frac{i\xi_k(B+A)}{A^3}\frac{\widetilde{L}_{4,2\ell}}{\det L}e^{-A(d_\ell(x_N)+d_2(y_N))} \\
&\quad\times\mathcal{F}_{x'}[D_kD_NS_{0N}(\lambda)\mathbf{f}](\xi',y_N)\Big](x')\,dy_N\Big\}
\end{aligned}
$$

for $j=1,\dots,4$, $\ell=1,2$, and $m=1,2,3$, where we have used (2.6.4) with $X=A$ for $T^4_{4,\ell}(\lambda)$.

Here we set the operators $I^{j*}_{2,\ell}(\lambda)$ and $I^{j*}_{3J,\ell}(\lambda)$ as

$$
\begin{aligned}
&[I^{j*}_{2,\ell}(\lambda)\varphi](y) \\
&\quad= -\frac{1}{2}\int_0^\delta \mathcal{F}_{\xi'}\left[\zeta_j\frac{B+A}{AB^2}\frac{\widetilde{L}_{4,2\ell}}{\det L}e^{-Ad_\ell(x_N)}\Phi(\xi',y_N)\mathcal{F}_{x'}[\varphi](\xi',x_N)\right](y')\,dx_N, \\
&[I^{j*}_{3J,\ell}(\lambda)\varphi](y) \\
&\quad= -\frac{1}{2}\int_0^\delta \mathcal{F}_{\xi'}\left[\zeta_j\frac{X_J(B+A)}{AB^2}\frac{\widetilde{L}_{4,2\ell}}{\det L}e^{-Ad_\ell(x_N)}\Phi(\xi',y_N)\mathcal{F}_{x'}[\varphi](\xi',x_N)\right](y')\,dx_N,
\end{aligned}
$$

for $\varphi\in L_{q'}(\Omega)$ with $1/q+1/q'=1$, where $X_J=i\xi_J$ if $J=1,\dots,N-1$ and $X_N=(-1)^{\ell+1}A$. Then there hold

$$
\begin{aligned}
(2.6.18)\qquad &(I^j_{2,\ell}(\lambda)\psi,\varphi)_{\Omega_x} = (\psi, I^{j*}_{2,\ell}(\lambda)\varphi)_{\Omega_y}, \quad [I^{j*}_{2,\ell}(\lambda)\varphi]|_{y_N=\delta}=0, \\
&(D_JI^j_{2,\ell}(\lambda)\psi,\varphi)_{\Omega_x} = (\psi, I^{j*}_{3J,\ell}(\lambda)\varphi)_{\Omega_y}, \quad [I^{j*}_{3J,\ell}(\lambda)\varphi]|_{y_N=\delta}=0
\end{aligned}
$$

for any $\psi\in W^1_q(\Omega_y)\cap\widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega_y)$ and $\varphi\in L_{q'}(\Omega_x)$. By Lemma 1.2.5, 1.2.6, 2.5.3, 2.5.4, and 2.6.3, we have

$$
\begin{aligned}
(2.6.19)\qquad &\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m T^j_{4,\ell}(\lambda)\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\})\le C, \\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m (\nabla T^j_{4,\ell}(\lambda))\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\})\le C, \\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m \widetilde{I}^j_{2,\ell}(\lambda)\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\})\le C, \\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m (\nabla\widetilde{I}^j_{2,\ell}(\lambda))\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\})\le C, \\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m (\nabla I^{j*}_{2,\ell}(\lambda))\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\})\le C, \\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^m (\nabla I^{j*}_{3J,\ell}(\lambda))\mid\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\})\le C
\end{aligned}
$$

for $j=1,\dots,4$, $\ell=1,2$, $m=0,1$, and $J=1,\dots,N$ with some positive constant $C=C(N,q,\varepsilon,\mu,\gamma_0,\delta)$, where we also have used Lemma 2.2.5, 1.2.3, and A.3 (2).

Summing up the above argumentation, we set

$$
T_1(\lambda)(\lambda^{1/2}\mathbf{a},\nabla\mathbf{a},\lambda\mathbf{b}',\lambda^{1/2}\nabla\mathbf{b}',\nabla^2\mathbf{b}') = \sum_{j=1}^4\sum_{\ell=1}^2(-1)^{\ell+1}
$$

$$\times \left(T^j_{1,\ell}(\lambda)(\lambda^{1/2}\mathbf{a}', \nabla\mathbf{a}') + T^j_{2,\ell}(\lambda)(\lambda^{1/2}a_N, \nabla a_N) + T^j_{3,\ell}(\lambda)(\lambda\mathbf{b}', \lambda^{1/2}\nabla\mathbf{b}', \nabla^2\mathbf{b}')\right),$$

$$\widetilde{T}_1(\lambda)(\mathbf{f}, \nabla f_d) = \sum_{j=1}^4\sum_{\ell=1}^2(-1)^{\ell+1}\left(\widetilde{T}^j_{2,\ell}(\lambda)(T_0\mathbf{f}, \nabla T_0\mathbf{f}) + T^j_{4,\ell}(\lambda)\mathbf{f} + \widetilde{I}^j_{2,\ell}(\lambda)(\nabla f_d)\right),$$

$$I_2(\lambda)(\lambda f_d) = \sum_{j=1}^4\sum_{\ell=1}^2(-1)^{\ell+1}I^j_{2,\ell}(\lambda)(\lambda f_d),$$

and then $\pi(x)$ in (2.4.13) is given by

$$\pi(x) = T_1(\lambda)(\lambda^{1/2}\mathbf{a}, \nabla\mathbf{a}, \lambda\mathbf{b}', \lambda^{1/2}\nabla\mathbf{b}', \nabla^2\mathbf{b}') + \widetilde{T}_1(\lambda)(\mathbf{f}, \nabla f_d) + I_2(\lambda)(\lambda f_d). \tag{2.6.20}$$

Moreover, by Lemma 1.2.3, (2.6.17), and (2.6.19), we have

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell X_1(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \nabla X_1(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \nabla I_2^*(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,\\
&\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \nabla I_{3J}^*(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C
\end{aligned} \tag{2.6.21}$$

for $\ell = 0, 1$, $J = 1, \ldots, N$, and $X \in \{T, \widetilde{T}\}$ with some positive constant $C = C(N, q, \varepsilon, \mu, \gamma_0, \delta)$, where

$$I_2^*(\lambda)\varphi = \sum_{j=1}^4\sum_{\ell=1}^2(-1)^{\ell+1}I^{j*}_{2,\ell}(\lambda)\varphi,$$

$$I_{3J}^*(\lambda)\varphi = \sum_{j=1}^4\sum_{\ell=1}^2(-1)^{\ell+1}I^{j*}_{3J,\ell}(\lambda)\varphi \quad \text{for } \varphi \in L_{q'}(\Omega),$$

and also by (2.6.18) there hold, for any $\psi \in W^1_q(\Omega_y) \cap \widehat{W}^{-1}_{q,\Gamma_\delta}(\Omega_y)$ and $\varphi \in L_{q'}(\Omega_x)$,

$$\begin{aligned}
(I_2(\lambda)\psi, \varphi)_{\Omega_x} &= (\psi, I_2^*(\lambda)\varphi)_{\Omega_y}, \quad [I_2^*(\lambda)\varphi]|_{y_N=\delta} = 0,\\
(D_J I_2(\lambda)\psi, \varphi)_{\Omega_x} &= (\psi, I_{3J}^*(\lambda)\varphi)_{\Omega_y}, \quad [I_{3J}^*(\lambda)\varphi]|_{y_N=\delta} = 0.
\end{aligned} \tag{2.6.22}$$

Therefore. setting

$$\theta = T_0\mathbf{f} + T_1(\lambda)(\lambda^{1/2}\mathbf{a}, \nabla\mathbf{a}, \lambda\mathbf{b}', \lambda^{1/2}\nabla\mathbf{b}', \nabla^2\mathbf{b}') + \widetilde{T}_1(\lambda)(\mathbf{f}, \nabla f_d) + I_2(\lambda)(\lambda f_d),$$

we see that θ is the pressure of the equation (2.4.1), and the operator $\mathcal{T}(\lambda)$ in Theorem 2.6.1 is given by

$$\begin{aligned}
&\mathcal{T}(\lambda)(\mathbf{f}, \nabla f_d, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, K'(\lambda f_d), \nabla K'(\lambda^{1/2}f_d), \nabla^2 K' f_d)\\
&\quad = T_0\mathbf{f} + T_1(\lambda)(\lambda^{1/2}\mathbf{a}, \nabla\mathbf{a}, \lambda\mathbf{b}', \lambda^{1/2}\nabla\mathbf{b}', \nabla^2\mathbf{b}') + \widetilde{T}_1(\lambda)(\mathbf{f}, \nabla f_d)\\
&\quad = T_0\mathbf{f} + T_1(\lambda)\left(\lambda^{1/2}\mathbf{a}, \nabla\mathbf{a}, -\lambda S_0'(\lambda)\mathbf{f}, -\lambda^{1/2}\nabla S_0'(\lambda)\mathbf{f}, -\nabla^2 S_0'(\lambda)\mathbf{f}\right)\\
&\qquad + T_1(\lambda)\left(0, 0, -K'(\lambda f_d), -\nabla K'(\lambda^{1/2}f_d), -\nabla^2 K' f_d\right) + \widetilde{T}_1(\lambda)(\mathbf{f}, \nabla f_d).
\end{aligned}$$

$\mathcal{T}(\lambda), I_2(\lambda), I_2^*(\lambda)$, and $I_{3J}^*(\lambda)$ satisfy the required properties in Theorem 2.6.1 by (2.6.21), (2.6.22), Lemma 2.2.5, and Lemma 1.2.3.

Finally we construct the operator $\mathcal{S}_j(\lambda)$. For the purpose, we consider the horizontal velocities v_j ($j = 1, \ldots, N-1$) defined as (2.4.20). Set

$$p_\ell(\xi', \lambda) = (-1)^{\ell+1}\frac{e^{-Bd_\ell(\delta)}}{1 + e^{-2B\delta}}. \quad q_\ell(\xi', \lambda) = \frac{e^{-Bd_\ell(0)}}{1 + e^{-2B\delta}}.$$

Then we have the following lemma.

LEMMA 2.6.4. *Let* $0 < \varepsilon < \pi/2$ *and* $\gamma_0 > 0$. *Then, for* $\ell = 1, 2$, *we have*

$$p_\ell(\xi', \lambda), \quad q_\ell(\xi', \lambda) \in \mathbb{M}_{0,1,\varepsilon,\gamma_0}.$$

PROOF. Since it holds that for any multi-index $\alpha \in \mathbb{N}_0^{N-1}$ and $m = 0, 1$

$$\left| D_{\xi'}^{\alpha'} \left\{ (\tau\partial_\tau)^m (1 + e^{-2B\delta})^{-1} \right\} \right| \leq C(\alpha', \varepsilon, \mu, \gamma_0, \delta)(|\lambda|^{1/2} + A)^{-|\alpha'|}$$

with some positive constant $C(\alpha', \varepsilon, \mu, \gamma_0, \delta)$, we see that $(1 + e^{-2B\delta})^{-1}$ belongs to $\mathbb{M}_{0,1,\varepsilon,\gamma_0}$. By this fact, Lemma 1.2.5, and Lemma 1.2.6, $p_\ell(\xi', \lambda)$ and $q_\ell(\xi', \lambda)$ belong to $\mathbb{M}_{0,1,\varepsilon,\gamma_0}$ for $\ell = 1, 2$. □

By (2.6.2) and (2.6.4) with $X = B$, w_j^1 and w_j^2 ($j = 1, \ldots, N-1$), defined as (2.4.20), are given by

$$\begin{aligned}
w_j^1(x) &= \sum_{\ell=1}^2 \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\varphi_\delta' \frac{\lambda^{-1/2} p_\ell(\xi', \lambda)}{B} e^{-B(d_\ell(x_N)+d_1(y_N))} \widehat{\lambda^{1/2} h_j^1}(y_N)](x')\, dy_N \\
&+ \sum_{\ell=1}^2 \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\varphi_\delta \frac{\lambda^{1/2} p_\ell(\xi', \lambda)}{\mu B^2} e^{-B(d_\ell(x_N)+d_1(y_N))} \widehat{\lambda^{1/2} h_j^1}(y_N)](x')\, dy_N \\
&- \sum_{k=1}^{N-1} \sum_{\ell=1}^2 \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\varphi_\delta \frac{i\xi_k p_\ell(\xi', \lambda)}{B^2} e^{-B(d_\ell(x_N)+d_1(y_N))} \widehat{D_k h_j^1}(y_N)](x')\, dy_N \\
&+ \sum_{\ell=1}^2 \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\varphi_\delta \frac{p_\ell(\xi', \lambda)}{B} e^{-B(d_\ell(x_N)+d_1(y_N))} \widehat{D_N h_j^1}(\xi', y_N)](x')\, dy_N, \\
w_j^2(x) &= -\sum_{\ell=1}^2 \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\varphi_0' \frac{q_\ell(\xi', \lambda)}{\mu B^2} e^{-B(d_\ell(x_N)+d_2(y_N))} \widehat{\lambda h_j^2}(y_N)](x')\, dy_N \\
&+ \sum_{\ell=1}^2 \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\varphi_0' \frac{q_\ell(\xi', \lambda)}{B^2} e^{-B(d_\ell(x_N)+d_2(y_N))} \widehat{\Delta' h_j^2}(y_N)](x')\, dy_N \\
&+ \sum_{\ell=1}^2 \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\varphi_0 \frac{q_\ell(\xi', \lambda)}{\mu B} e^{-B(d_\ell(x_N)+d_2(y_N))} \widehat{\lambda h_j^2}(y_N)](x')\, dy_N \\
&- \sum_{\ell=1}^2 \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\varphi_0 \frac{q_\ell(\xi', \lambda)}{B} e^{-B(d_\ell(x_N)+d_2(y_N))} \widehat{\Delta' h_j^2}(y_N)](x')\, dy_N \\
&- \sum_{\ell=1}^2 \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\varphi_0 \frac{\lambda^{1/2} q_\ell(\xi', \lambda)}{\mu B^2} e^{-B(d_\ell(x_N)+d_2(y_N))} \mathcal{F}_{x'}[\lambda^{1/2} D_N h_j^2](\xi', y_N)](x')\, dy_N \\
&+ \sum_{k=1}^{N-1} \sum_{\ell=1}^2 \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\varphi_0 \frac{i\xi_k q_\ell(\xi', \lambda)}{B^2} e^{-B(d_\ell(x_N)+d_2(y_N))} \mathcal{F}_{x'}[D_k D_N h_j^2](\xi', y_N)](x')\, dy_N
\end{aligned}$$

for $j = 1, \ldots, N-1$, where we have used (2.6.6). We define the solution operators as follows:

$$S_2(\lambda)(\lambda^{1/2} h_j^1, \nabla h_j^1) = w_j^1(x), \quad S_3(\lambda)(\lambda h_j^2, \lambda^{1/2} \nabla h_j^2, \nabla^2 h_j^2) = w_j^2(x).$$

Since there hold, by Lemma 1.2.5, Lemma 1.2.6, and Lemma 2.6.4,

$$\frac{\lambda^{-1/2}p_\ell(\xi',\lambda)}{B}, \quad \frac{\lambda^{1/2}p_\ell(\xi',\lambda)}{B^2}, \quad \frac{i\xi_k p_\ell(\xi',\lambda)}{B^2}, \quad \frac{p_\ell(\xi',\lambda)}{B},$$
$$\frac{q_\ell(\xi',\lambda)}{B^2}, \quad \frac{q_\ell(\xi',\lambda)}{B}, \quad \frac{\lambda^{1/2}q_\ell(\xi',\lambda)}{B^2}, \quad \frac{i\xi_k q_\ell(\xi',\lambda)}{B^2}$$

belong to $\mathbb{M}_{-1,1,\varepsilon,\gamma_0}$, we obtain, by Lemma 2.5.2,

$$\begin{aligned}
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda S_m(\lambda)) \mid \lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}) &\le C,\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\gamma S_m(\lambda)) \mid \lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}) &\le C,\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla S_m(\lambda)) \mid \lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}) &\le C,\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\nabla^2 S_m(\lambda)) \mid \lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\gamma_0}\}) &\le C
\end{aligned} \tag{2.6.23}$$

for $\ell=0,1$ and $m=1,2$ with some positive constant $C=C(N,q,\varepsilon,\mu,\gamma_0,\delta)$. Then $v_j(x)$ is given by

$$v_j = -H(\lambda)E_0D_j\pi + S_2(\lambda)(\lambda^{1/2}h_j^1, \nabla h_j^1) + S_3(\lambda)(\lambda h_j^2, \lambda^{1/2}\nabla h_j^2, \nabla^2 h_j^2).$$

Therefore, setting

$$u_j = S_{0j}(\lambda)\mathbf{f} - H(\lambda)E_0D_j\pi + S_2(\lambda)(\lambda^{1/2}h_j^1, \nabla h_j^1) + S_3(\lambda)(\lambda h_j^2, \lambda^{1/2}\nabla h_j^2, \nabla^2 h_j^2),$$

we see that, for $j=1,\dots,N-1$, u_j is the j-th component of the velocity $\mathbf{u}$ to the equation (2.4.1), and the operator $\mathcal{S}_j(\lambda)$ in Theorem 2.6.1 is given by the right hand side in the above setting through the relations: (2.4.19), (2.6.14), and (2.6.20). Note that $\mathcal{S}_j(\lambda)$ satisfies the required properties in Theorem 2.6.1 by Lemma 2.2.4, Lemma 2.2.5, Lemma 1.2.3, (2.6.15), (2.6.21), and (2.6.23). This completes the proof of Theorem 2.6.1.

2.7. Initial condition and uniqueness

Let $(\mathbf{U},\Theta)$ be a solution satisfying the estimates (2.2.2) and (2.2.3) to equations (2.2.1). In this section, we prove that the solution $(\mathbf{U},\Theta)$ vanishes for $t<0$ if $\mathbf{F}$, F_d, and $\mathbf{G}$ vanish for $t<0$, and also we show the uniqueness of solutions to equations (2.2.1). We first have the following lemma.

Lemma 2.7.1. *Let $1<q<\infty$ and q' be its dual exponent. Let $\mathbf{u}\in W_q^2(\Omega)^N$, $\mathbf{v}\in W_{q'}^2(\Omega)^N$, $\theta\in W_q^1(\Omega)$, and $\pi\in W_{q'}^1(\Omega)$. Then, for the unit outer normal ν to $\Gamma_\delta\cup\Gamma_0$, we have the formula:*

$$\begin{aligned}
(\mathbf{u}, \mathrm{Div}\mathbf{S}(\mathbf{v},\pi))_\Omega &= (\mathrm{Div}\mathbf{S}(\mathbf{u},\theta),\mathbf{v})_\Omega\\
&\quad + (\mathbf{u},\mathbf{S}(\mathbf{v},\pi)\nu)_{\Gamma_\delta\cup\Gamma_0} - (\mathbf{S}(\mathbf{u},\theta)\nu,\mathbf{v})_{\Gamma_\delta\cup\Gamma_0} + (\mathrm{div}\mathbf{u},\pi)_\Omega - (\theta,\mathrm{div}\mathbf{v})_\Omega.
\end{aligned}$$

2.7.1. Initial condition.

We first consider the case where $F_d=0$ and $\mathbf{G}=0$ in (2.2.1). In this case, by (2.2.3) there holds

$$\begin{aligned}
\gamma\|\mathbf{U}\|_{L_p((-\infty,0),L_q(\Omega))} &\le \gamma\|e^{-\gamma t}\mathbf{U}\|_{L_p((-\infty,0),L_q(\Omega))} \le \gamma\|e^{-\gamma t}\mathbf{U}\|_{L_p(\mathbb{R},L_q(\Omega))}\\
&\le C\|e^{-\gamma t}\mathbf{F}\|_{L_p(\mathbb{R},L_q(\Omega))} = C\|e^{-\gamma t}\mathbf{F}\|_{L_p((0,\infty),L_q(\Omega))}\\
&\le C\|e^{-\gamma_0 t}\mathbf{F}\|_{L_p((0,\infty),L_q(\Omega))}
\end{aligned}$$

with a positive constant $C=C(N,p,q,\gamma_0,\mu,\delta)$, so that

$$\|\mathbf{U}\|_{L_p((-\infty,0),L_q(\Omega))} \le \gamma^{-1}C\|e^{-\gamma_0 t}\mathbf{F}\|_{L_p((0,\infty),L_q(\Omega))}$$

holds for any $\gamma \geq \gamma_0$. In the last inequality, taking the limit: $\gamma \to \infty$, we obtain $\mathbf{U} = 0$ $(t < 0)$, which furnishes that $\mathbf{U}|_{t=0} = 0$. In addition, $\nabla\Theta(t) = 0$ $(t < 0)$ by the first equation of (2.2.1), so that $\Theta(t)$ depends only on time variable t when $t < 0$. But now, $\Theta(t)|_{\Gamma_\delta} = 0$ $(t < 0)$ by the boundary condition on Γ_δ, which means that $\Theta(t) = 0$ for $t < 0$. This completes the proof of the case where $F_d = 0$ and $\mathbf{G} = 0$ in (2.2.1).

Next we consider the case where $F_d \neq 0$ or $\mathbf{G} \neq 0$ in (2.2.1). Let $\mathbf{\Phi} \in C_0^\infty(\Omega \times (-\infty, 0))^N$ and set $\mathbf{\Psi}(x,t) = \mathbf{\Phi}(x,-t)$. Since $\mathbf{\Psi} \in L_{p',\gamma_0,0}(\mathbf{R}, L_{q'}(\Omega))^N$, as was seen in the case where $F_d = 0$ and $\mathbf{G} = 0$, there exists a solution $(\mathbf{V}, \Pi)$, which satisfies $\mathbf{V} \in (L_{p',\gamma_0,0}(\mathbf{R}, W^2_{q'}(\Omega)) \cap W^1_{p',\gamma_0,0}(\mathbf{R}, L_{q'}(\Omega)))^N$ and $\Pi \in L_{p',\gamma_0,0}(\mathbf{R}, W^1_{q'}(\Omega))$ with $1/p = 1/p' = 1$ and $1/q + 1/q' = 1$, to the equations:

$$\begin{cases} \partial_t \mathbf{V} - \mathrm{Div}\mathbf{S}(\mathbf{V}, \Pi) = \mathbf{\Psi}, \quad \mathrm{div}\,\mathbf{V} = 0 & \text{in} \quad \Omega \times \mathbf{R}, \\ \mathbf{S}(\mathbf{V}, \Pi)\mathbf{e}_N = 0 & \text{on} \quad \Gamma_\delta \times \mathbf{R}, \\ \mathbf{V} = 0 & \text{on} \quad \Gamma_0 \times \mathbf{R}. \end{cases}$$

Setting $\mathbf{W}(x,t) = \mathbf{V}(x,-t)$ and $P(x,t) = \Pi(x,-t)$, we see that $(\mathbf{W}, P)$ satisfies

$$\begin{cases} \partial_t \mathbf{W} + \mathrm{Div}\mathbf{S}(\mathbf{W}, P) = -\mathbf{\Phi} \quad \mathrm{div}\mathbf{W} = 0 & \text{in} \quad \Omega \times \mathbf{R}, \\ \mathbf{S}(\mathbf{W}, P)\mathbf{e}_N = 0 & \text{on} \quad \Gamma_\delta \times \mathbf{R}, \\ \mathbf{W} = 0 & \text{on} \quad \Gamma_0 \times \mathbf{R}, \end{cases} \tag{2.7.1}$$

and the conditions: $\mathbf{W}(t) = 0$ and $P(t) = 0$ for $t > 0$. By (2.7.1), integration by parts and Lemma 2.7.1,

$$\begin{aligned} (\mathbf{U}, \Phi)_{\Omega\times(-\infty,0)} &= -(\mathbf{U}, \partial_t \mathbf{W})_{\Omega\times(-\infty,0)} - (\mathbf{U}, \mathrm{Div}\mathbf{S}(\mathbf{W}, P))_{\Omega\times(-\infty,0)} \\ &= (\partial_t \mathbf{U}, \mathbf{W})_{\Omega\times(-\infty,0)} - (\mathrm{Div}\mathbf{S}(\mathbf{U}, \Theta), \mathbf{W})_{\Omega\times(-\infty,0)} = (\mathbf{F}, \mathbf{W})_{\Omega\times(-\infty,0)} = 0 \end{aligned}$$

because $\mathbf{F}(t) = 0$ when $t < 0$, which furnishes that $\mathbf{U}(t) = 0$ $(t < 0)$. We also have $\Theta(t) = 0$ $(t < 0)$ in the same manner as in the case where $F_d = 0$ and $\mathbf{G} = 0$.

2.7.2. Uniqueness. We prove the uniqueness of solutions to equations (2.2.1). Suppose that

$$\mathbf{U} \in (W^1_{p,\mathrm{loc},0}(\mathbf{R}, L_q(\Omega)) \cap L_{p,\mathrm{loc},0}(\mathbf{R}, W^2_q(\Omega)))^N, \quad \Theta \in L_{p,\mathrm{loc},0}(\mathbf{R}, W^1_q(\Omega))$$

satisfy the following homogeneous equations:

$$\begin{cases} \partial_t \mathbf{U} - \mathrm{Div}\,\mathbf{S}(\mathbf{U}, \Theta) = 0, \quad \mathrm{div}\,\mathbf{U} = 0 & \text{in } \Omega \times \mathbf{R}, \\ \mathbf{S}(\mathbf{U}, \Theta)\mathbf{e}_N = 0 & \text{on } \Gamma_\delta \times \mathbf{R}, \\ \mathbf{U} = 0 & \text{on } \Gamma_0 \times \mathbf{R}. \end{cases} \tag{2.7.2}$$

Let $\mathbf{\Phi}$ be any function in $C_0^\infty(\Omega \times \mathbf{R})^N$, and let T_0 and T_1 be positive constants satisfying the condition: $\mathrm{supp}\,\mathbf{\Phi} \subset \Omega \times (-T_0, T_1)$. Setting $\mathbf{\Phi}_{T_1}(x,t) = \mathbf{\Phi}(x, T_1 - t)$, we see that $\mathrm{supp}\,\mathbf{\Phi}_{T_1} \subset \Omega \times (0, T_0 + T_1)$. Since $\mathbf{\Phi}_{T_1} \in L_{p',\gamma_0,0}(\mathbf{R}, L_{q'}(\Omega))^N$, there exist $\mathbf{V} \in (W^1_{p'\gamma_0,0}(\mathbf{R}, L_{q'}(\Omega)) \cap L_{p',\gamma_0,0}(\mathbf{R}, W^2_{q'}(\Omega)))^N$ and $\Pi \in L_{p',\gamma_0,0}(\mathbf{R}, W^1_{q'}(\Omega))$, where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, satisfying the equations:

$$\begin{cases} \partial_t \mathbf{V} - \mathrm{Div}\,\mathbf{S}(\mathbf{V}, \Pi) = \mathbf{\Phi}_{T_1}, \quad \mathrm{div}\,\mathbf{V} = 0 & \text{in } \Omega \times \mathbf{R}, \\ \mathbf{S}(\mathbf{V}, \Pi)\mathbf{e}_N = 0 & \text{on } \Gamma_\delta \times \mathbf{R}, \\ \mathbf{V} = 0 & \text{on } \Gamma_0 \times \mathbf{R}. \end{cases}$$

Set $\mathbf{W}(x,t) = \mathbf{V}(x, T_1 - t)$ and $P(x,t) = \Pi(x, T_1 - t)$. Then $(\mathbf{W}, P)$ satisfies

$$\begin{aligned}\{\partial_t \mathbf{W} + \mathrm{Div}\mathbf{S}(\mathbf{W}, P)\}(x,t) &= -\{\partial_t \mathbf{W} - \mathrm{Div}\mathbf{S}(\mathbf{W}, P)\}(x, T_1 - t)\\ &= -\mathbf{\Phi}_{T_1}(x, T_1 - t) = -\mathbf{\Phi}(x,t),\end{aligned}$$

and $\mathbf{W}(t) = 0$ and $P(t) = 0$ for $t > T_1$. Thus, using integration by parts and Lemma 2.7.1, we obtain

$$0 = (\partial_t \mathbf{U} - \mathrm{Div}\mathbf{S}(\mathbf{U}, \Theta), \mathbf{W})_{\Omega\times\mathbf{R}} = -(\mathbf{U}, \partial_t \mathbf{W} + \mathrm{Div}\mathbf{S}(\mathbf{W}, P))_{\Omega\times\mathbf{R}} = (\mathbf{U}, \mathbf{\Phi})_{\Omega\times\mathbf{R}},$$

which furnishes that that $\mathbf{U} = 0$. Employing the same argumentation as in the proof of the initial condition, we also have $\Theta = 0$. This completes the proof of Theorem 2.2.1.

Part 2

One-phase flows of Newtonian fluids: Half space type

CHAPTER 3

L_q-L_r estimates of Stokes semigroups with surface tension and gravity in $\mathbf{R}^N_+$

3.1. Main results

In this chapter, we show L_q-L_r estimates of the Stokes semigroups associated with the following Stokes equations in $\mathbf{R}^N_+$ ($N \geq 2$):

$$
\left\{
\begin{aligned}
\partial_t \mathbf{u} - \operatorname{Div} \mathbf{S}(\mathbf{u}, \theta) &= 0 && \text{in } \mathbf{R}^N_+,\ t > 0, \\
\operatorname{div} \mathbf{u} &= 0 && \text{in } \mathbf{R}^N_+,\ t > 0, \\
\partial_t h - \mathbf{u} \cdot \mathbf{n} &= 0 && \text{on } \mathbf{R}^N_0,\ t > 0, \\
\mathbf{S}(\mathbf{u}, \theta)\mathbf{n} + (c_g - c_\sigma \Delta') h \mathbf{n} &= 0 && \text{on } \mathbf{R}^N_0,\ t > 0, \\
\mathbf{u}|_{t=0} &= \mathbf{f} && \text{in } \mathbf{R}^N_+, \\
h|_{t=0} &= g && \text{on } \mathbf{R}^{N-1},
\end{aligned}
\right.
\tag{3.1.1}
$$

where $c_g > 0$ denotes the gravitational acceleration, $c_\sigma > 0$ the surface tension coefficient, and $\mathbf{n} = (0, \ldots, 0, -1)^T$ the unit outer normal field on $\mathbf{R}^N_0$. In addition, the stress tensor $\mathbf{S}(\mathbf{u}, \theta)$ is given by

$$
\mathbf{S}(\mathbf{u}, \theta) = -\theta \mathbf{I} + \mathbf{D}(\mathbf{u}), \quad \mathbf{D}(\mathbf{u}) = \nabla \mathbf{u} + (\nabla \mathbf{u})^T.
$$

We first define a suitable solenoidal space $J_q(\mathbf{R}^N_+)$ to construct the Stokes semigroup. To this end, we set, for $1 < q < \infty$,

$$
\begin{aligned}
W^1_{q,0}(\mathbf{R}^N_+) &= \{\theta \in W^1_q(\mathbf{R}^N_+) \mid \theta = 0 \ \text{ on } \mathbf{R}^N_0\}, \\
\widehat{W}^1_{q,0}(\mathbf{R}^N_+) &= \{\theta \in \widehat{W}^1_q(\mathbf{R}^N_+) \mid \theta = 0 \ \text{ on } \mathbf{R}^N_0\}.
\end{aligned}
$$

As was seen in [**Shi13**, Theorem A.3], $W^1_{q,0}(\mathbf{R}^N_+)$ is dense in $\widehat{W}^1_{q,0}(\mathbf{R}^N_+)$ with the gradient norm $\|\nabla \cdot\|_{L_q(\mathbf{R}^N_+)}$. The solenoidal space $J_q(\mathbf{R}^N_+)$ is then defined as

$$
J_q(\mathbf{R}^N_+) = \left\{ \mathbf{u} \in L_q(\mathbf{R}^N_+)^N \mid (\mathbf{u}, \nabla \varphi)_{\mathbf{R}^N_+} = 0 \ \text{ for any } \varphi \in \widehat{W}^1_{q',0}(\mathbf{R}^N_+) \right\},
$$

where $1/q + 1/q' = 1$, and also we set

$$
\begin{aligned}
X_q &= J_q(\mathbf{R}^N_+) \times W^{2-1/q}_q(\mathbf{R}^{N-1}), \\
X^0_q &= L_q(\mathbf{R}^N_+) \times L_q(\mathbf{R}^{N-1}), \\
X^i_q &= L_q(\mathbf{R}^N_+) \times W^{i-1/q}_q(\mathbf{R}^{N-1}) \quad (i = 1, 2).
\end{aligned}
\tag{3.1.2}
$$

Now we have the following theorem.

THEOREM 3.1.1. *Let $1 < q < \infty$. Then, for every $t > 0$, there exist operators*

$S(t) \in \mathcal{L}(X^2_q, W^2_q(\mathbf{R}^N_+)^N)$, $\Pi(t) \in \mathcal{L}(X^2_q, \widehat{W}^1_q(\mathbf{R}^N_+))$, $T(t) \in \mathcal{L}(X^2_q, W^{3-1/q}_q(\mathbf{R}^{N-1}))$

such that for $\mathbf{F} = (\mathbf{f}, g) \in X_q$

$$
\begin{aligned}
&S(\cdot)\mathbf{F} \in C^1((0,\infty), J_q(\mathbf{R}^N_+)) \cap C^0((0,\infty), W^2_q(\mathbf{R}^N_+)^N),\\
&\Pi(\cdot)\mathbf{F} \in C^0((0,\infty), \widehat{W}^1_q(\mathbf{R}^N_+)),\\
&T(\cdot)\mathbf{F} \in C^1((0,\infty), W^{2-1/q}_q(\mathbf{R}^{N-1})) \cap C^0((0,\infty), W^{3-1/q}_q(\mathbf{R}^{N-1})),
\end{aligned}
$$

and $(\mathbf{u}, \theta, h) = (S(t)\mathbf{F}, \Pi(t)\mathbf{F}, T(t)\mathbf{F})$ *solves uniquely* (3.1.1) *with*

$$\lim_{t\to 0} \|(\mathbf{u}(t), h(t)) - (\mathbf{f}, g)\|_{X_q} = 0.$$

REMARK 3.1.2. If we consider the operator

$$\mathcal{S}(t) : X_q \ni \mathbf{F} \mapsto (S(t)\mathbf{F}, T(t)\mathbf{F}) \in X_q,$$

then $\{\mathcal{S}(t)\}_{t\geq 0}$ is the required Stokes semigroups (see Section 3.3 for the detail).

Here we extend $T(t)\mathbf{F}$ to $\mathcal{E}(T(t)\mathbf{F})$, defined on $\mathbf{R}^N_+$, by

$$\mathcal{E}(T(t)\mathbf{F}) = \mathcal{F}^{-1}_{\xi'}[e^{-Ax_N}\widehat{T(t)\mathbf{F}}(\xi')](x'), \;\; \mathcal{E} : W^{1-1/q}_q(\mathbf{R}^{N-1}) \to \widehat{W}^1_q(\mathbf{R}^N_+), \tag{3.1.3}$$

which is the so-called *harmonic extension*. The L_q-L_r estimates of the operators $S(t)$, $\Pi(t)$, $T(t)$, and $\mathcal{E}T(t)$ then are showed in the following theorem.

THEOREM 3.1.3. *Let* $1 < p < \infty$ *and* $\mathbf{F} = (\mathbf{f}, g) \in X^2_p$. *Moreover, let* $S(t), \Pi(t)$, *and* $T(t)$ *be given by Theorem* 3.1.1. *Then they are decomposed as follows:*

$$
\begin{aligned}
S(t)\mathbf{F} &= S_0(t)\mathbf{F} + S_\infty(t)\mathbf{F} + R(t)\mathbf{f},\\
\Pi(t)\mathbf{F} &= \Pi_0(t)\mathbf{F} + \Pi_\infty(t)\mathbf{F} + P(t)\mathbf{f},\\
T(t)\mathbf{F} &= T_0(t)\mathbf{F} + T_\infty(t)\mathbf{F},
\end{aligned}
\tag{3.1.4}
$$

and also the right members satisfy the following estimates.

(1) *Let* $1 \leq r \leq 2 \leq q \leq \infty$, $k = 1, 2$, *and* $l = 0, 1, 2$, *and set*

$$
\begin{aligned}
m(r,q) &= \frac{N-1}{2}\left(\frac{1}{r} - \frac{1}{q}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right),\\
n(r,q) &= \frac{N-1}{2}\left(\frac{1}{r} - \frac{1}{q}\right) + \min\left\{\frac{1}{2}\left(\frac{1}{r} - \frac{1}{q}\right), \frac{1}{8}\left(2 - \frac{1}{q}\right)\right\}.
\end{aligned}
$$

In addition, let $\mathbf{F} \in X^0_r$. *Then there exists a positive constant* $C = C(N, q, r)$ *such that for* $t \geq 1$

$$
\begin{aligned}
\|(\partial_t S_0(t)\mathbf{F}, \nabla\Pi_0(t)\mathbf{F})\|_{L_q(\mathbf{R}^N_+)} &\leq C(t+1)^{-m(q,r)-\frac{1}{4}}\|\mathbf{F}\|_{X^0_r},\\
\|(S_0(t)\mathbf{F}, \partial_t\mathcal{E}(T_0(t)\mathbf{F}))\|_{L_q(\mathbf{R}^N_+)} &\leq C(t+1)^{-m(q,r)}\|\mathbf{F}\|_{X^0_r} \quad ((q,r) \neq (2,2)),\\
\|\nabla^k S_0(t)\mathbf{F}\|_{L_q(\mathbf{R}^N_+)} &\leq C(t+1)^{-n(q,r)-\frac{k}{8}}\|\mathbf{F}\|_{X^0_r},\\
\|\nabla^k\partial_t\mathcal{E}(T_0(t)\mathbf{F})\|_{L_q(\mathbf{R}^N_+)} &\leq C(t+1)^{-m(q,r)-\frac{k}{2}}\|\mathbf{F}\|_{X^0_r},\\
\|\nabla^{1+l}\mathcal{E}(T_0(t)\mathbf{F})\|_{L_q(\mathbf{R}^N_+)} &\leq C(t+1)^{-m(q,r)-\frac{1}{4}-\frac{l}{2}}\|\mathbf{F}\|_{X^0_r},\\
\|T_0(t)\mathbf{F}\|_{L_q(\mathbf{R}^{N-1})} &\leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\|\mathbf{F}\|_{X^0_r} \quad (r \neq 2).
\end{aligned}
$$

On the other hand, let $0 < t \leq 1$, *and then*

$$
\begin{aligned}
&\|(\partial_t S_0(t)\mathbf{F}, \nabla\Pi_0(t)\mathbf{F})\|_{L_q(\mathbf{R}^N_+)} + \|(\nabla S_0(t)\mathbf{F}, \nabla\partial_t\mathcal{E}(T_0(t)\mathbf{F}))\|_{W^1_q(\mathbf{R}^N_+)}\\
&+ \|\nabla\mathcal{E}(T_0(t)\mathbf{F})\|_{W^2_q(\mathbf{R}^N_+)} \leq Ct^{-\alpha}\|\mathbf{F}\|_{X^0_r},
\end{aligned}
$$

$$\|(S_0(t)\mathbf{F}, \partial_t \mathcal{E}(T_0(t)\mathbf{F}))\|_{L_q(\mathbf{R}^N_+)} \leq Ct^{-\alpha}\|\mathbf{F}\|_{X^0_r} \quad ((q,r) \neq (2,2)),$$
$$\|T_0(t)\mathbf{F}\|_{L_q(\mathbf{R}^{N-1})} \leq Ct^{-\alpha}\|\mathbf{F}\|_{X^0_r} \quad (r \neq 2)$$

for any $\alpha > 0$ with a positive constant $C = C(N,q,r,\alpha)$.

(2) *Let $\alpha > 0$. Then there exist positive constants δ and $C = C(N,p,\alpha)$ such that for any $t > 0$*

$$\|S_\infty(t)\mathbf{F}\|_{W^1_p(\mathbf{R}^N_+)} + \|\mathcal{E}(T_\infty(t)\mathbf{F})\|_{W^2_p(\mathbf{R}^N_+)} + \|\partial_t \mathcal{E}(T_\infty(t)\mathbf{F})\|_{L_p(\mathbf{R}^N_+)}$$
$$\leq Ct^{-\alpha}e^{-\delta t}\|\mathbf{F}\|_{X^2_p},$$
$$\|(\partial_t S_\infty(t)\mathbf{F}, \nabla\partial_t \mathcal{E}(T_\infty(t)\mathbf{F}))\|_{L_p(\mathbf{R}^N_+)} \leq Ct^{-1/2}e^{-\delta t}\|\mathbf{F}\|_{X^2_p},$$
$$\|(\nabla^2 S_\infty(t)\mathbf{F}, \nabla\Pi_\infty(t)\mathbf{F}, \nabla^3\mathcal{E}(T_\infty(t)\mathbf{F}), \nabla^2\partial_t\mathcal{E}(T_\infty(t)\mathbf{F}))\|_{L_p(\mathbf{R}^N_+)}$$
$$\leq Ct^{-1}e^{-\delta t}\|\mathbf{F}\|_{X^2_p}.$$

It especially holds that

$$\|T_\infty(t)\mathbf{F}\|_{L_p(\mathbf{R}^{N-1})} \leq Ct^{-\alpha}e^{-\delta t}\|\mathbf{F}\|_{X^2_p} \quad (t > 0)$$

with some positive constant $C = C(N,p,\alpha)$.

(3) *There exists a positive constant $C = C(N,p)$ such that for every $t > 0$ and $l = 0,1,2$*

$$\|(\partial_t R(t)\mathbf{f}, \nabla P(t)\mathbf{f})\|_{L_p(\mathbf{R}^N_+)} \leq Ct^{-1}\|\mathbf{f}\|_{L_p(\mathbf{R}^N_+)},$$
$$\|\nabla^l R(t)\mathbf{f}\|_{L_p(\mathbf{R}^N_+)} \leq Ct^{-l/2}\|\mathbf{f}\|_{L_p(\mathbf{R}^N_+)}.$$

The original paper of this chapter is [**SS15**], and also this chapter is organized as follows: In the next section, we consider some resolvent problem arising from (3.1.1) with $c_g = c_\sigma = 0$, and then we show resolvent estimates and special representation formulas of the solution. In Section 3.3, we show Theorem 3.1.1 by using the standard theory of analytic semigroups. In Section 3.4, we give the decompositions (3.1.4) of $S(t)\mathbf{F}$, $\Pi(t)\mathbf{F}$, and $T(t)\mathbf{F}$. In Section 3.5, we prove pointwise estimates with respect to time t for the low frequency parts of $S(t)\mathbf{F}$, $\Pi(t)\mathbf{F}$, $T(t)\mathbf{F}$, and $\mathcal{E}(T(t)\mathbf{F})$, that is, we prove Theorem 3.1.3 (1). Finally, Section 3.6 shows Theorem 3.1.3 (2).

3.2. Preliminaries

In this section, we consider the resolvent problem:

$$\begin{cases} \lambda\mathbf{w} - \operatorname{Div}\mathbf{S}(\mathbf{w},p) = \mathbf{f}, \quad \operatorname{div}\mathbf{w} = 0 & \text{in } \mathbf{R}^N_+, \\ \mathbf{S}(\mathbf{w},p)\mathbf{n} = 0 & \text{on } \mathbf{R}^N_0 \end{cases} \tag{3.2.1}$$

in order to obtain the following lemma.

Lemma 3.2.1. *Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Then, for every $\lambda \in \Sigma_\varepsilon$ and $\mathbf{f} \in L_q(\mathbf{R}^N_+)^N$, equations (3.2.1) admits a unique solution $(\mathbf{w},p) \in W^2_q(\mathbf{R}^N_+)^N \times \widehat{W}^1_q(\mathbf{R}^N_+)$, and also the solution $(\mathbf{w},p)$ satisfies*

$$\|(\lambda\mathbf{w}, \lambda^{1/2}\nabla\mathbf{w}, \nabla^2\mathbf{w}, \nabla p)\|_{L_q(\mathbf{R}^N_+)} \leq C(N,\varepsilon,q)\|\mathbf{f}\|_{L_q(\mathbf{R}^N_+)} \tag{3.2.2}$$

with a positive constant $C(N,\varepsilon,q)$. In addition, the N-th component $\widehat{w}_N(\xi',0,\lambda)$ of $\widehat{\mathbf{w}}(\xi',0,\lambda)$ is given by

$$
\begin{aligned}
\widehat{w}_N(\xi',0,\lambda) &= \sum_{k=1}^{N-1}\frac{i\xi_k(B-A)}{D(A,B)}\int_0^\infty e^{-By_N}\widehat{f}_k(\xi',y_N)\,dy_N \\
&\quad + \frac{A(B+A)}{D(A,B)}\int_0^\infty e^{-By_N}\widehat{f}_N(\xi',y_N)\,dy_N \\
&\quad - \sum_{k=1}^{N-1}\frac{i\xi_k(B^2+A^2)}{D(A,B)}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_k(\xi',y_N)\,dy_N \\
&\quad - \frac{A(B^2+A^2)}{D(A,B)}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_N(\xi',y_N)\,dy_N \qquad (3.2.3)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{N-1}\frac{i\xi_k(B-A)}{D(A,B)}\int_0^\infty e^{-Ay_N}\widehat{f}_k(\xi',y_N)\,dy_N \qquad (3.2.4)\\
&\quad + \frac{A(B+A)}{D(A,B)}\int_0^\infty e^{-Ay_N}\widehat{f}_N(\xi',y_N)\,dy_N \\
&\quad - \sum_{k=1}^{N-1}\frac{2i\xi_k AB}{D(A,B)}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_k(\xi',y_N)\,dy_N \\
&\quad - \frac{2A^3}{D(A,B)}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_N(\xi',y_N)\,dy_N,
\end{aligned}
$$

where $A, B, D(A,B)$, and $\mathcal{M}(y_N)$ are defined as (1.2.1) *with $\mu = 1$ and* (1.2.2).

PROOF. The lemma was proved by [**SS03**, Theorem 4.1] except for (3.2.3) and (3.2.4) essentially, but we here prove the estimate (3.2.2) again by another approach for the sake of Chapter 4. Let $j = 1,\dots,N-1$ and $J = 1,\dots,N$ below.

For given functions $g(x)$ defined on $\mathbf{R}^N_+$, we set their even extensions $g^e(x)$ and odd extensions $g^o(x)$ as follows:

$$
g^e(x) = \begin{cases} g(x',x_N) & \text{in } \mathbf{R}^N_+, \\ g(x',-x_N) & \text{in } \mathbf{R}^N_-, \end{cases} \quad g^o(x) = \begin{cases} g(x',x_N) & \text{in } \mathbf{R}^N_+, \\ -g(x',-x_N) & \text{in } \mathbf{R}^N_-. \end{cases} \tag{3.2.5}
$$

In addition, for the right member $\mathbf{f} = (f_1,\dots,f_N)^T$ of (3.2.1), we set

$$
E\mathbf{f} = (f_1^o,\dots,f_{N-1}^o, f_N^e)^T, \tag{3.2.6}
$$

and let $(\mathbf{w}^1, p^1)$ be the solution to the resolvent equations in $\mathbf{R}^N$:

$$
\lambda\mathbf{w}^1 - \operatorname{Div}\mathbf{S}(\mathbf{w}^1,p^1) = E\mathbf{f}, \quad \operatorname{div}\mathbf{w}^1 = 0 \quad \text{in } \mathbf{R}^N. \tag{3.2.7}
$$

Let $(E\mathbf{f})_J$ be the J-th component of $E\mathbf{f}$, and then we have the following solution formulas (cf. [**SS12**, Section 3]):

$$
\begin{aligned}
w_J^1(x,\lambda) &= \mathcal{F}_\xi^{-1}\left[\frac{\widehat{(E\mathbf{f})}_J(\xi)}{\lambda+|\xi|^2}\right](x) - \sum_{K=1}^N \mathcal{F}_\xi^{-1}\left[\frac{\xi_J\xi_K\widehat{(E\mathbf{f})}_K(\xi)}{|\xi|^2(\lambda+|\xi|^2)}\right](x). \qquad (3.2.8)\\
p^1(x,\lambda) &= -\mathcal{F}_\xi^{-1}\left[\frac{i\xi}{|\xi|^2}\cdot\widehat{E\mathbf{f}}(\xi)\right](x).
\end{aligned}
$$

By Fourier multiplier theorem of Hörmander-Mikhlin type, we obtain

$$
\begin{aligned}
(3.2.9)\quad &\|(\lambda\mathbf{w}^1,\lambda^{1/2}\nabla\mathbf{w}^1,\nabla^2\mathbf{w}^1,\nabla p^1)\|_{L_q(\mathbf{R}^N_+)}\\
&\leq \|(\lambda\mathbf{w}^1,\lambda^{1/2}\nabla\mathbf{w}^1,\nabla^2\mathbf{w}^1,\nabla p^1)\|_{L_q(\mathbf{R}^N)} \leq C\|E\mathbf{f}\|_{L_q(\mathbf{R}^N)} \leq C\|\mathbf{f}\|_{L_q(\mathbf{R}^N_+)}
\end{aligned}
$$

with some positive constant $C = C(N,q,\varepsilon)$. As was seen in [**SS03**, Section 4], we have the fact that by the definition of the extension $E\mathbf{f}$

$$
(3.2.10)\qquad D_N w^1_N(x',0,\lambda) = 0, \quad p^1(x',0,\lambda) = 0.
$$

In addition, since there hold

$$
\begin{aligned}
\mathcal{F}[f^o_j](\xi) &= \int_0^\infty (e^{-iy_N\xi_N} - e^{iy_N\xi_N})\widehat{f}_j(\xi',y_N)\,dy_N,\\
\mathcal{F}[f^e_N](\xi) &= \int_0^\infty (e^{-iy_N\xi_N} + e^{iy_N\xi_N})\widehat{f}_N(\xi',y_N)\,dy_N,
\end{aligned}
$$

applying the partial Fourier transform and Lemma 1.2.7 to (3.2.8) yields that

$$
\begin{aligned}
\widehat{w}^1_N(\xi',0,\lambda) &= \sum_{k=1}^{N-1}\frac{i\xi_k}{\lambda}\int_0^\infty (e^{-Ay_N} - e^{-By_N})\widehat{f}_k(\xi',y_N)\,dy_N\\
&\quad + \int_0^\infty \frac{e^{-By_N}}{B}\widehat{f}_N(\xi',y_N)\,dy_N\\
&\quad + \frac{1}{\lambda}\int_0^\infty (Ae^{-Ay_N} - Be^{-By_N})\widehat{f}_N(\xi',y_N)\,dy_N\\
\widehat{D_N w^1_j}(\xi',0,\lambda) &= -\sum_{k=1}^{N-1}\frac{\xi_j\xi_k}{\lambda}\int_0^\infty (e^{-Ay_N} - e^{-By_N})\widehat{f}_k(\xi',y_N)\,dy_N\\
&\quad + \int_0^\infty e^{-By_N}\widehat{f}_j(\xi',y_N)\,dy_N\\
&\quad + \frac{i\xi_j}{\lambda}\int_0^\infty (Ae^{-Ay_N} - Be^{-By_N})\widehat{f}_N(\xi',y_N)\,dy_N.
\end{aligned}
$$

We thus see that by $\lambda = B^2 - A^2$ and $e^{-By_N} - e^{-Ay_N} = (B-A)\mathcal{M}(y_N)$

$$
\begin{aligned}
(3.2.11)\qquad \widehat{w}^1_N(\xi',0,\lambda) =& \frac{A}{B(B+A)}\int_0^\infty e^{-Bx_N}\widehat{f}_N(\xi',y_N)\,dy_N\\
&- \sum_{k=1}^{N-1}\frac{i\xi_k}{B+A}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_k(\xi',y_N)\,dy_N\\
&- \frac{A}{B+A}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_N(\xi',y_N)\,dy_N,\\
\widehat{D_N w^1_j}(\xi',0,\lambda) =& \int_0^\infty e^{-By_N}\widehat{f}_j(\xi',y_N)\,dy_N\\
&- \frac{i\xi_j}{B+A}\int_0^\infty e^{-By_N}\widehat{f}_N(\xi',y_N)\,dy_N\\
&+ \sum_{k=1}^{N-1}\frac{\xi_j\xi_k}{B+A}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_k(\xi',y_N)\,dy_N
\end{aligned}
$$

$$- \frac{i\xi_j A}{B+A} \int_0^\infty \mathcal{M}(y_N) \widehat{f}_N(\xi', y_N)\, dy_N.$$

Next we consider the following equations:

$$\text{(3.2.12)} \quad \begin{cases} \lambda \mathbf{w}^2 - \operatorname{Div} S(\mathbf{w}^2, p^2) = 0, \quad \operatorname{div} \mathbf{w}^2 = 0 & \text{in } \mathbf{R}^N_+, \\ D_j w^2_N + D_N w^2_j = -g_j & \text{on } \mathbf{R}^N_0, \\ -p^2 + 2D_N w^2_N = 0 & \text{on } \mathbf{R}^N_0 \end{cases}$$

with $g_j = D_j w^1_N + D_N w^1_j$, which are obtained as follows: First we set $\mathbf{w} = \mathbf{w}^1 + \mathbf{w}^2$ and $p = p^1 + p^2$ in (3.2.1). Second we use (3.2.10) for the third line.

We then obtain the formulas (cf. [**SS12**, Section 4]):

$$\text{(3.2.13)} \quad w^2_J(x, \lambda) = \mathcal{F}^{-1}_{\xi'}[\widehat{w}^2_J(\xi', x_N, \lambda)](x'), \quad p^2(x, \lambda) = \mathcal{F}^{-1}_{\xi'}[\widehat{p}^2(\xi', x_N, \lambda)](x'),$$

$$\widehat{w}^2_j(\xi', x_N, \lambda) = \left(\frac{i\xi_j(3B - A)}{BD(A,B)} e^{-Bx_N} - \frac{2i\xi_j B}{D(A,B)} \mathcal{M}(x_N) \right) \sum_{k=1}^{N-1} i\xi_k \widehat{g}_k(\xi', 0, \lambda)$$
$$+ \frac{1}{B} e^{-Bx_N} \widehat{g}_j(\xi', 0, \lambda),$$

$$\widehat{w}^2_N(\xi', x_N, \lambda) = \left(\frac{B - A}{D(A,B)} e^{-Bx_N} + \frac{2AB}{D(A,B)} \mathcal{M}(x_N) \right) \sum_{j=1}^{N-1} i\xi_k \widehat{g}_k(\xi', 0, \lambda).$$

$$\widehat{p}^2(\xi', x_N, \lambda) = - \frac{2B(B+A)}{D(A,B)} e^{-Ax_N} \sum_{k=1}^{N-1} i\xi_k \widehat{g}_k(\xi', 0, \lambda).$$

It holds that by (3.2.11)

$$\begin{aligned} \widehat{g}_k(\xi', 0, \lambda) &= i\xi_k \widehat{w}^1_N(\xi', 0, \lambda) + \widehat{D_N w^1_k}(\xi', 0, \lambda) \\ &= \sum_{l=1}^{N-1} \frac{2\xi_k \xi_l}{B+A} \int_0^\infty \mathcal{M}(y_N) \widehat{f}_l(\xi', y_N)\, dy_N \\ &- \frac{2i\xi_k A}{B+A} \int_0^\infty \mathcal{M}(y_N) \widehat{f}_N(\xi', y_N)\, dy_N \\ &- \frac{i\xi_k(B-A)}{B(B+A)} \int_0^\infty e^{-By_N} \widehat{f}_N(\xi', y_N)\, dy_N + \int_0^\infty e^{-By_N} \widehat{f}_k(\xi', y_N)\, dy_N, \end{aligned}$$

which, inserted into (3.2.13), furnishes that

(3.2.14)

$$\begin{aligned} &w^2_j(x, \lambda) \\ &= \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \left[\frac{2\xi_j \xi_k B}{A(B+A)D(A,B)} A^3 \mathcal{M}(x_N) \mathcal{M}(y_N) \widehat{f}_k(\xi', y_N) \right] (x')\, dy_N \\ &- \int_0^\infty \mathcal{F}^{-1}_{\xi'} \left[\frac{4i\xi_j B}{(B+A)D(A,B)} A^3 \mathcal{M}(x_N) \mathcal{M}(y_N) \widehat{f}_N(\xi', y_N) \right] (x')\, dy_N \\ &+ \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \left[\frac{2\xi_j \xi_k B}{A^2 D(A,B)} A^2 \mathcal{M}(x_N) e^{-By_N} \widehat{f}_k(\xi', y_N) \right] (x')\, dy_N \end{aligned}$$

$$
\begin{aligned}
&- \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{2i\xi_j(B-A)}{(B+A)D(A,B)} A^2 \mathcal{M}(x_N) e^{-By_N} \widehat{f}_N(\xi', y_N) \right] (x')\, dy_N \\
&- \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\xi_j\xi_k(3B-A)}{B(B+A)D(A,B)} A^2 e^{-Bx_N} \mathcal{M}(y_N) \widehat{f}_k(\xi', y_N) \right] (x')\, dy_N \\
&+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{2i\xi_j A(3B-A)}{B(B+A)D(A,B)} A^2 e^{-Bx_N} \mathcal{M}(y_N) \widehat{f}_N(\xi', y_N) \right] (x')\, dy_N \\
&- \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\xi_j\xi_k(3B-A)}{ABD(A,B)} A e^{-B(x_N+y_N)} \widehat{f}_k(\xi', y_N) \right] (x')\, dy_N \\
&+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_j A(3B-A)(B-A)}{B^2(B+A)D(A,B)} A e^{-B(x_N+y_N)} \widehat{f}_N(\xi', y_N) \right] (x')\, dy_N \\
&+ \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\xi_j\xi_k}{A^2B(B+A)} A^2 e^{-Bx_N} \mathcal{M}(y_N) \widehat{f}_k(\xi', y_N) \right] (x')\, dy_N \\
&- \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{2i\xi_j}{AB(B+A)} A^2 e^{-Bx_N} \mathcal{M}(y_N) \widehat{f}_N(\xi', y_N) \right] (x')\, dy_N \\
&- \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_j(B-A)}{AB^2(B+A)} A e^{-B(x_N+y_N)} \widehat{f}_N(\xi', y_N) \right] (x')\, dy_N \\
&+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{1}{B} e^{-B(x_N+y_N)} \widehat{f}_j(\xi', y_N) \right] (x')\, dy_N,
\end{aligned}
$$

$$
\begin{aligned}
&w_N^2(x,\lambda) \\
&= \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{4i\xi_k B}{(B+A)D(A,B)} A^3 \mathcal{M}(x_N)\mathcal{M}(y_N) \widehat{f}_k(\xi', y_N) \right] (x')\, dy_N \\
&+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{4AB}{(B+A)D(A,B)} A^3 \mathcal{M}(x_N)\mathcal{M}(y_N) \widehat{f}_N(\xi', y_N) \right] (x')\, dy_N \\
&+ \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{2i\xi_k B}{AD(A,B)} A^2 \mathcal{M}(x_N) e^{-By_N} \widehat{f}_k(\xi', y_N) \right] (x')\, dy_N \\
&+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{2A(B-A)}{(B+A)D(A,B)} A^2 \mathcal{M}(x_N) e^{-By_N} \widehat{f}_N(\xi', y_N) \right] (x')\, dy_N \\
&+ \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{2i\xi_k(B-A)}{(B+A)D(A,B)} A^2 e^{-Bx_N} \mathcal{M}(y_N) \widehat{f}_k(\xi', y_N) \right] (x')\, dy_N \\
&+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{2A(B-A)}{(B+A)D(A,B)} A^2 e^{-Bx_N} \mathcal{M}(y_N) \widehat{f}_N(\xi', y_N) \right] (x')\, dy_N \\
&+ \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{i\xi_k(B-A)}{AD(A,B)} A e^{-B(x_N+y_N)} \widehat{f}_k(\xi', y_N) \right] (x')\, dy_N \\
&+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{A(B-A)^2}{B(B+A)D(A,B)} A e^{-B(x_N+y_N)} \widehat{f}_N(\xi', y_N) \right] (x')\, dy_N,
\end{aligned}
$$

$$\begin{aligned}
&p^2(x,\lambda)\\
&\quad= -\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{4i\xi_k B}{D(A,B)}A^2e^{-Ax_N}\mathcal{M}(y_N)\widehat{f}_k(\xi',y_N)\right](x')\,dy_N\\
&\quad-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{4AB}{D(A,B)}A^2e^{-Ax_N}\mathcal{M}(y_N)\widehat{f}_N(\xi',y_N)\right](x')\,dy_N\\
&\quad-\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{2i\xi_k B(B+A)}{AD(A,B)}Ae^{-Ax_N}e^{-By_N}\widehat{f}_k(\xi',y_N)\right](x')\,dy_N\\
&\quad-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{2A(B-A)}{D(A,B)}Ae^{-Ax_N}e^{-By_N}\widehat{f}_N(\xi',y_N)\right](x')\,dy_N.
\end{aligned}$$

Since it holds that by Lemma 1.2.5 and 1.2.6

$$\begin{aligned}
&\frac{1}{D(A,B)}\biggl(\frac{2\xi_j\xi_k B}{A(B+A)},\frac{4i\xi_j B}{B+A},\frac{2\xi_j\xi_k B}{A^2},\frac{2i\xi_j(B-A)}{B+A},\frac{\xi_j\xi_k(3B-A)}{B(B+A)},\\
&\frac{2i\xi_j A(3B-A)}{B(B+A)},\frac{\xi_j\xi_k(3B-A)}{AB},\frac{i\xi_j A(3B-A)(B-A)}{B^2(B+A)},\frac{4i\xi_k B}{B+A},\frac{4AB}{B+A},\\
&\frac{2i\xi_k B}{A},\frac{2A(B-A)}{B+A},\frac{2i\xi_k(B-A)}{B+A},\frac{2A(B-A)}{B+A},\frac{i\xi_k(B-A)}{A},\frac{A(B-A)^2}{B(B+A)}\biggr)
\end{aligned}$$

belong to $\mathbb{M}_{-2,2,\varepsilon,0}$ for $j,k=1,\ldots,N-1$, and besides,

$$\begin{aligned}
&\frac{\xi_j\xi_k}{A^2B(B+A)},\frac{2i\xi_j}{AB(B+A)},\frac{i\xi_j(B-A)}{AB^2(B+A)}\in\mathbb{M}_{-2,2,\varepsilon,0},\quad \frac{1}{B}\in\mathbb{M}_{-1,1,\varepsilon,0},\\
&\frac{4i\xi_k B}{D(A,B)},\frac{4AB}{D(A,B)},\frac{2i\xi_k B(B+A)}{AD(A,B)},\frac{2A(B-A)}{D(A,B)}\in\mathbb{M}_{-1,2,\varepsilon,\gamma_0},
\end{aligned}$$

we obtain, by applying Corollary B.3 (2) to the formulas (3.2.14),

$$\|(\lambda\mathbf{w}^2,\lambda^{1/2}\nabla\mathbf{w}^2,\nabla^2\mathbf{w}^2,\nabla p^2)\|_{L_q(\mathbf{R}^N_+)}\le C(N,q,\varepsilon)\|\mathbf{f}\|_{L_q(\mathbf{R}^N_+)} \tag{3.2.15}$$

for any $\lambda\in\Sigma_\varepsilon$ with a positive constant $C(N,q,\varepsilon)$. Combining (3.2.9) with (3.2.15) furnishes the estimate (3.2.2).

On the other hand, setting $x_N=0$ in (3.2.14), we have

$$\begin{aligned}
\widehat{w}^2_N(\xi',0,\lambda)&=\sum_{k=1}^{N-1}\frac{i\xi_k(B-A)}{D(A,B)}\int_0^\infty e^{-By_N}\widehat{f}_k(\xi',y_N)\,dy_N\\
&+\frac{A^2(B-A)^2}{B(B+A)D(A,B)}\int_0^\infty e^{-By_N}\widehat{f}_N(\xi',y_N)\,dy_N\\
&+\sum_{k=1}^{N-1}\frac{2i\xi_k A^2(B-A)}{(B+A)D(A,B)}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_k(\xi',y_N)\,dy_N\\
&+\frac{2A^3(B-A)}{(B+A)D(A,B)}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_N(\xi',y_N)\,dy_N,
\end{aligned}$$

which, combined with (3.2.11), furnishes (3.2.3) since $\widehat{w}_N(\xi',0,\lambda)=\widehat{w}^1_N(\xi',0,\lambda)+\widehat{w}^2_N(\xi',0,\lambda)$. We finally obtain (3.2.4) by using the relation: $e^{-By_N}=e^{-Ay_N}+(B-A)\mathcal{M}(y_N)$ in (3.2.3). This completes the proof of the lemma. □

We devote the last part of this section to introduce the following fundamental lemmas.

LEMMA 3.2.2. *Let $s_i \geq 0$ $(i = 1, 2, 3, 4)$. Then there exists a positive constant $C = C(s_1, s_2, s_3, s_4)$ such that for every $a > 0$, $\tau > 0$, and $Z > 0$*

$$e^{-s_1 Z^2 \tau} Z^{s_2} e^{-s_3 Z^{(s_4)} a} \leq C(\tau^{s_2/2} + a^{s_2/s_4})^{-1}.$$

LEMMA 3.2.3. *Let $1 \leq q, r \leq \infty$, $a > 0$, $b_1 > 0$, and $b_2 > 0$.*

(1) *Set $g(x_N, \tau) = \{\tau^a + (x_N)^{b_1}\}^{-1}$ for $x_N > 0$ and $\tau > 0$. Then there exists a positive constant $C(q, a, b_1)$ such that for every $\tau > 0$*

$$\|g(\tau)\|_{L_q(0,\infty)} \leq C(q, a, b_1)\tau^{-a\left(1-\frac{1}{b_1 q}\right)},$$

provided that $b_1 q > 1$.

(2) *Let $f \in L_r(0, \infty)$, and also we set, for $x_N > 0$ and $\tau > 0$,*

$$g(x_N, \tau) = \int_0^\infty \frac{f(y_N)}{\tau^a + (x_N)^{b_1} + (y_N)^{b_2}}\, dy_N.$$

Then there exists a positive constant $C = C(q, r, a, b_1, b_2)$ such that for every $\tau > 0$

$$\|g(\tau)\|_{L_q(0,\infty)} \leq C\tau^{-a\left(1-\frac{1}{b_1 q}-\frac{1}{b_2}+\frac{1}{b_2 r}\right)}\|f\|_{L_r(0,\infty)},$$

provided that for $r' = r/(r-1)$

$$b_1 q > 1, \quad b_2\left(1 - \frac{1}{b_1 q}\right)r' > 1.$$

3.3. Generation of the Stokes semigroup

In this section, we prove Theorem 3.1.1 by the theory of analytic semigroups. We start with the resolvent problem:

$$\text{(3.3.1)} \qquad \begin{cases} \lambda \mathbf{u} - \operatorname{Div} \mathbf{S}(\mathbf{u}, \theta) = \mathbf{f} & \text{in } \mathbf{R}_+^N, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \mathbf{R}_+^N, \\ \lambda h - \mathbf{u} \cdot \mathbf{n} = g & \text{on } \mathbf{R}_0^N, \\ \mathbf{S}(\mathbf{u}, \theta)\mathbf{n} + (c_g - c_\sigma \Delta')h\mathbf{n} = 0 & \text{on } \mathbf{R}_0^N. \end{cases}$$

The following lemma was proved by [**SS09**, Theorem 1.1].

LEMMA 3.3.1. *Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Then there exists a positive number $\lambda_0 = \lambda_0(\varepsilon) \geq 1$ such that equations (3.3.1) admits a unique solution*

$$(\mathbf{u}, \theta, h) \in W_q^2(\mathbf{R}_+^N)^N \times \widehat{W}_q^1(\mathbf{R}_+^N) \times W_q^{3-1/q}(\mathbf{R}^{N-1})$$

for every $\lambda \in \Sigma_{\varepsilon,\lambda_0}$, $\mathbf{f} \in L_q(\mathbf{R}_+^N)^N$, and $g \in W_q^{2-1/q}(\mathbf{R}^{N-1})$. In addition, the solution $(\mathbf{u}, \theta, h)$ satisfies

$$\begin{aligned} &\|(\lambda \mathbf{u}, \lambda^{1/2}\nabla \mathbf{u}, \nabla^2 \mathbf{u}, \nabla\theta)\|_{L_q(\mathbf{R}_+^N)} + \|(\lambda h, \nabla' h)\|_{W_q^{2-1/q}(\mathbf{R}^{N-1})} \\ &\qquad \leq C(N, q, \varepsilon, \lambda_0)\left(\|\mathbf{f}\|_{L_q(\mathbf{R}_+^N)} + \|g\|_{W_q^{2-1/q}(\mathbf{R}^{N-1})}\right) \end{aligned}$$

for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ with some positive constant $C(N, q, \varepsilon, \lambda_0)$.

The following proposition is used to eliminate the pressure θ from (3.3.1).

PROPOSITION 3.3.2. *Let $1 < q < \infty$ and $q' = q/(q-1)$. Then, for every $\mathbf{f} \in L_q(\mathbf{R}^N_+)^N$, there exists a unique $\theta \in \widehat{W}^1_{q,0}(\mathbf{R}^N_+)$ satisfying the variational equation:*

$$(\nabla\theta, \nabla\varphi)_{\mathbf{R}^N_+} = (\mathbf{f}, \nabla\varphi)_{\mathbf{R}^N_+} \quad \text{for any } \varphi \in \widehat{W}^1_{q',0}(\mathbf{R}^N_+), \tag{3.3.2}$$

and furthermore, there holds the estimate: $\|\nabla\theta\|_{L_q(\mathbf{R}^N_+)} \le C\|\mathbf{f}\|_{L_q(\mathbf{R}^N_+)}$ with some positive constant C independent of f, θ and φ.

PROOF. Since $C^\infty_0(\mathbf{R}^N_+)$ is dense in $L_q(\mathbf{R}^N_+)$, it suffices to show the existence of a solution $\theta \in \widehat{W}^1_{q,0}(\mathbf{R}^N_+)$ to (3.3.2) with $f \in C^\infty_0(\mathbf{R}^N_+)^N$. Let $g^o(x)$ and $g^e(x)$ be the odd and even extensions defined as (3.2.5), and put $\mathbf{F} = (f^o_1, \dots, f^o_{N-1}, f^e_N)^T$. Then, setting $\theta(x) = \mathcal{F}^{-1}_\xi[|\xi|^{-2}\mathcal{F}[\operatorname{div}\mathbf{F}](\xi)](x)$, we easily see that by $\operatorname{div}\mathbf{F} = (\operatorname{div}\mathbf{f})^o$ and Lemma 1.2.7

$$\Delta\theta = \operatorname{div}\mathbf{f} \quad \text{in } \mathbf{R}^N_+, \quad \theta|_{\mathbf{R}^N_0} = 0,$$

and besides, Fourier multiplier theorem of Hörmander-Mikhlin type yields that

$$\|\nabla\theta\|_{L_q(\mathbf{R}^N_+)} \le \|\nabla\theta\|_{L_q(\mathbf{R}^N)} \le C\|\mathbf{F}\|_{L_q(\mathbf{R}^N)} \le C\|\mathbf{f}\|_{L_q(\mathbf{R}^N_+)}$$

with some positive constant $C = C(N, q)$. These facts implies that $\theta \in \widehat{W}^1_{q,0}(\mathbf{R}^N_+)$, and that θ satisfies (3.3.2). Finally, the uniqueness follows from the the existence of a solution to the dual problem of (3.3.2), which completes the proof of the proposition. □

By Proposition 3.3.2 we see that for $\mathbf{f} \in L_q(\mathbf{R}^N_+)^N$ and $g \in W^{1-1/q}_q(\mathbf{R}^{N-1})$ there exists a unique solution $\theta \in \widehat{W}^1_{q,0}(\mathbf{R}^N_+)$ to the variational equation:

$$(\nabla\theta, \nabla\varphi)_{\mathbf{R}^N_+} = (\mathbf{f} - \nabla\widetilde{g}, \nabla\varphi)_{\mathbf{R}^N_+} \quad \text{for any } \varphi \in \widehat{W}^1_{q',0}(\mathbf{R}^N_+), \tag{3.3.3}$$

where $\widetilde{g}$ is an extension of g satisfying $\|\widetilde{g}\|_{W^1_q(\mathbf{R}^N_+)} \le C\|g\|_{W^{1-1/q}_q(\mathbf{R}^{N-1})}$ with some positive constant C independent of g and $\widetilde{g}$. Furthermore, the solution θ satisfies $\|\nabla\theta\|_{L_q(\mathbf{R}^N_+)} \le C(N, q)(\|f\|_{L_q(\mathbf{R}^N_+)} + \|\nabla\widetilde{g}\|_{L_q(\mathbf{R}^N_+)})$ for a positive constant $C(N, q)$. Thus, setting $\psi = \theta + \widetilde{g}$ in (3.3.3) furnishes that

$$(\nabla\psi, \nabla\varphi)_{\mathbf{R}^N_+} = (f, \nabla\varphi)_{\mathbf{R}^N_+} \text{ for any } \varphi \in \widehat{W}^1_{q',0}(\mathbf{R}^N_+), \ \ \psi|_{\mathbf{R}^N_0} = g,$$

and $\|\psi\|_{L_q(\mathbf{R}^N_+)} \le C(N, q)\left(\|f\|_{L_q(\mathbf{R}^N_+)} + \|g\|_{W^{1-1/q}_q(\mathbf{R}^{N-1})}\right)$.

As mentioned above, for $\mathbf{u} \in W^2_q(\mathbf{R}^N_+)^N$ and $h \in W^{3-1/q}_q(\mathbf{R}^{N-1})$, we define $K(\mathbf{u})$ and $\widetilde{K}(h)$ as the solutions to

$$\begin{cases} (\nabla K(\mathbf{u}), \nabla\varphi)_{\mathbf{R}^N_+} = (\Delta\mathbf{u}, \nabla\varphi)_{\mathbf{R}^N_+}, \\ K(\mathbf{u})|_{\mathbf{R}^N_0} = 2D_N u_N|_{\mathbf{R}^N_0}, \end{cases} \qquad \begin{cases} (\nabla\widetilde{K}(h), \nabla\varphi)_{\mathbf{R}^N_+} = 0, \\ \widetilde{K}(h)|_{\mathbf{R}^N_0} = (c_g - c_\sigma\Delta')h \end{cases}$$

for any $\varphi \in \widehat{W}^1_{q',0}(\mathbf{R}^N_+)$, respectively. Here $K(\mathbf{u})$ and $\widetilde{K}(h)$ also satisfy

$$\|\nabla K(\mathbf{u})\|_{L_q(\mathbf{R}^N_+)} \le C\|\mathbf{u}\|_{W^2_q(\mathbf{R}^N_+)}, \ \ \|\nabla\widetilde{K}(h)\|_{L_q(\mathbf{R}^N_+)} \le C\|h\|_{W^{3-1/q}_q(\mathbf{R}^{N-1})}$$

with some positive constant $C = C(N, q)$.

Now we consider the equations:

$$
\begin{cases} \lambda \mathbf{u} - \mathrm{Div}\, \mathbf{S}(\mathbf{u}, K(\mathbf{u}) + \widetilde{K}(h)) = \mathbf{f} & \text{in } \mathbf{R}^N_+, \\ \lambda h - \mathbf{u}\cdot\mathbf{n} = g & \text{on } \mathbf{R}^N_0, \\ \mathbf{S}(\mathbf{u}, K(\mathbf{u}) + \widetilde{K}(h))\mathbf{n} + (c_g - c_\sigma \Delta')h\mathbf{n} = 0 & \text{on } \mathbf{R}^N_0 \end{cases} \tag{3.3.4}
$$

in the function space X_q defined as (3.1.2). The following proposition then holds.

Proposition 3.3.3. *Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Then there exists a positive number $\lambda_0 = \lambda_0(\varepsilon) \geq 1$ such that* (3.3.1) *is equivalent to* (3.3.4) *for every $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ and $(\mathbf{f}, g) \in X_q$, which means that the following assertion holds: $(\mathbf{u}, \theta, h) = (\mathbf{u}, K(\mathbf{u}) + \widetilde{K}(h), h)$ is a unique solution of* (3.3.1) *if $(\mathbf{u}, h) \in W^2_q(\mathbf{R}^N_+)^N \times W^{3-1/q}_q(\mathbf{R}^{N-1})$ is a unique solution of* (3.3.4)*, and conversely $(\mathbf{u}, h)$ is a unique solution of* (3.3.4) *provided that*

$$
(\mathbf{u}, \theta, h) \in W^2_q(\mathbf{R}^N_+)^N \times \widehat{W}^1_q(\mathbf{R}^N_+) \times W^{3-1/q}_q(\mathbf{R}^{N-1})
$$

is a unique solution of (3.3.1).

Proof. We first assume that $(\mathbf{u}, h) \in W^2_q(\mathbf{R}^N_+)^N \times W^{3-1/q}_q(\mathbf{R}^{N-1})$ is a unique solution of (3.3.4). We shall check the divergence free condition. Let $\varphi \in \widehat{W}^1_{q',0}(\mathbf{R}^N_+)$ with $q' = q/(q-1)$, and then by the first equation of (3.3.4) and the definitions of $K(\mathbf{u})$ and $\widetilde{K}(h)$

$$
0 = (\mathbf{f}, \nabla\varphi)_{\mathbf{R}^N_+} = (\lambda\mathbf{u}, \nabla\varphi)_{\mathbf{R}^N_+} = -(\lambda\, \mathrm{div}\, \mathbf{u}, \varphi)_{\mathbf{R}^N_+},
$$

which furnishes that $\mathrm{div}\, \mathbf{u} = 0$ in $\mathbf{R}^N_+$. Setting $\theta = K(\mathbf{u}) + \widetilde{K}(h)$, we easily see that $(\mathbf{u}, \theta, h)$ solves (3.3.1). By Lemma 3.3.1 the uniqueness of (3.3.1) holds for $\lambda \in \Sigma_{\varepsilon,\lambda_0}$, where λ_0 is the same number as in the lemma.

Next we show the opposite direction. Let $(\mathbf{u}, \theta, h) \in W^2_q(\mathbf{R}^N_+)^N \times \widehat{W}^1_q(\mathbf{R}^N_+) \times W^{3-1/q}_q(\mathbf{R}^{N-1})$ be a unique solution of (3.3.1). By the definitions of $K(\mathbf{u})$ and $\widetilde{K}(h)$

$$
0 = (\mathbf{f}, \nabla\varphi)_{\mathbf{R}^N_+} = (\nabla(\theta - K(\mathbf{u}) - \widetilde{K}(h)), \nabla\varphi)_{\mathbf{R}^N_+}, \quad 0 = (\theta - K(\mathbf{u}) - \widetilde{K}(h))|_{\mathbf{R}^N_0}
$$

for any $\varphi \in \widehat{W}^1_{q',0}(\mathbf{R}^N_+)$, so that $\theta = K(\mathbf{u}) + \widetilde{K}(h)$ by Proposition 3.3.2. Therefore $(\mathbf{u}, h)$ is a solution of (3.3.4), and furthermore, the uniqueness of (3.3.4) for $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ follows from the uniqueness of (3.3.1). This completes the proof. □

In view of (3.3.4), we here set the operator $\mathcal{A}_q$ as

$$
\mathcal{A}_q\mathbf{U} = (\mathrm{Div}\, \mathbf{S}(\mathbf{u}, K(\mathbf{u}) + \widetilde{K}(h)), -\mathbf{u}\cdot\mathbf{n}) \quad \text{for } \mathbf{U} = (\mathbf{u}, h) \in \mathcal{D}(\mathcal{A}_q)
$$

with domain $\mathcal{D}(\mathcal{A}_q)$ defined by

$$
\mathcal{D}(\mathcal{A}_q) = \{(\mathbf{u}, h) \in (W^2_q(\mathbf{R}^N_+)^N \cap J_q(\mathbf{R}^N_+)) \times W^{3-1/q}_q(\mathbf{R}^{N-1}) \mid \\ \mathbf{D}(\mathbf{u})\mathbf{n} - (\mathbf{n}\cdot\mathbf{D}(\mathbf{u})\mathbf{n})\mathbf{n} = 0 \quad \text{on } \mathbf{R}^N_0\}.
$$

The equations (3.3.4) then is written by $\lambda\mathbf{U} - \mathcal{A}_q\mathbf{U} = \mathbf{F}$ for $\mathbf{F} = (\mathbf{f}, g)$.

Lemma 3.3.4. *Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Then there exist a positive number $\lambda_0 = \lambda_0(\varepsilon) \geq 1$ such that*

$$
\|\lambda(\lambda I - \mathcal{A}_q)^{-1}\|_{\mathcal{L}(X_q)} \leq C \quad (\lambda \in \Sigma_{\varepsilon,\lambda_0}) \tag{3.3.5}
$$

with some positive constant $C = C(N, q, \varepsilon, \lambda_0)$. In addition, $\mathcal{A}_q$ is a densely defined closed operator on X_q.

PROOF. By Lemma 3.3.1 and Proposition 3.3.3, we have (3.3.5), so that our main task is to prove that $\mathcal{A}_q$ is a densely defined closed operator on X_q.

First of all, we note that the range of $\mathcal{A}_q$ is contained in X_q. In fact, for $\mathbf{U} = (\mathbf{u}, h) \in \mathcal{D}(\mathcal{A}_q)$ and $\varphi \in \widehat{W}^1_{q',0}(\mathbf{R}^N_+)$ with $q' = q/(q-1)$

$$(\operatorname{Div}\mathbf{S}(\mathbf{u}, K(\mathbf{u}) + \widetilde{K}(h)), \nabla\varphi)_{\mathbf{R}^N_+} = (-\Delta\mathbf{u} + \nabla K(\mathbf{u}), \nabla\varphi)_{\mathbf{R}^N_+} = 0,$$

which implies that $\operatorname{Div}\mathbf{S}(\mathbf{u}, K(\mathbf{u}) + \widetilde{K}(h)) \in J_q(\mathbf{R}^N_+)$. On the other hand, it is clear that $\mathcal{A}_q\mathbf{U} \in L_q(\mathbf{R}^N_+)^N \times W^{2-1/q}_q(\mathbf{R}^{N-1})$. The range of $\mathcal{A}_q$ thus is contained in X_q.

Next we show that $\mathcal{D}(\mathcal{A}_q)$ is dense in X_q. Let $\mathbf{F} = (\mathbf{f}, g) \in X_q$. By Lemma 3.3.1 and Proposition 3.3.3, there exists a positive integer $m_0 \geq 1$ such that for any $m \in \mathbf{N}$ with $m \geq m_0$ there exists $(\mathbf{u}^m, h^m) \in \mathcal{D}(\mathcal{A}_q)$ satisfying the equations:

$$\begin{cases} m\mathbf{u}^m - \operatorname{Div}\mathbf{S}(\mathbf{u}^m, K(\mathbf{u}^m) + \widetilde{K}(h^m)) = \mathbf{f} & \text{in } \mathbf{R}^N_+, \\ mh^m - \mathbf{u}^m \cdot \mathbf{n} = g & \text{on } \mathbf{R}^N_0, \\ \mathbf{S}(\mathbf{u}^m, K(\mathbf{u}^m) + \widetilde{K}(h^m)) + (c_g - c_\sigma\Delta')h^m = 0 & \text{on } \mathbf{R}^N_0 \end{cases} \tag{3.3.6}$$

and the estimates:

$$\begin{aligned} \|(m\mathbf{u}^m, m^{1/2}\nabla\mathbf{u}^m, \nabla^2\mathbf{u}^m)\|_{L_q(\mathbf{R}^N_+)} + \|(mh^m, \nabla' h^m)\|_{W^{2-1/q}_q(\mathbf{R}^{N-1})} \\ + \|\nabla(K(\mathbf{u}^m) + \widetilde{K}(h^m))\|_{L_q(\mathbf{R}^N_+)} \leq C(N, q, m_0)\|\mathbf{F}\|_{X_q}. \end{aligned} \tag{3.3.7}$$

In particular, (3.3.7) implies that for $j, k = 1, \ldots, N$

$$\begin{aligned} &\mathbf{u}^m \to 0 \quad \text{in } W^1_q(\mathbf{R}^N_+)^N \quad \text{as } m \to \infty, \\ &D_jD_k\mathbf{u}^{m(l)} \to 0 \quad \text{weakly in } L_q(\mathbf{R}^N_+)^N \quad \text{as } l \to \infty, \end{aligned} \tag{3.3.8}$$

where $\{m(l)\}_{l=1}^\infty$ is a subsequence of $\{m\}_{m=m_0}^\infty$. We set $\mathbf{v}^m = m\mathbf{u}^m$, $\eta^m = mh^m$, and $\theta^m = K(\mathbf{u}^m) + K(h^m)$. Then by (3.3.7) there exist functions $\widetilde{\mathbf{f}} \in L_q(\mathbf{R}^N_+)^N$, $\widetilde{g} \in W^{2-1/q}_q(\mathbf{R}^{N-1})$, and subsequences $\{\mathbf{v}^{m(l)}\}_{l=1}^\infty$, $\{\eta^{m(l)}\}_{l=1}^\infty$ such that

$$\begin{aligned} &\mathbf{v}^{m(l)} \to \widetilde{\mathbf{f}} \quad \text{weakly in } L_q(\mathbf{R}^N_+)^N \quad \text{as } l \to \infty, \\ &\eta^{m(l)} \to \widetilde{g} \quad \text{weakly in } W^{2-1/q}_q(\mathbf{R}^{N-1}) \quad \text{as } l \to \infty, \end{aligned} \tag{3.3.9}$$

and especially $\widetilde{\mathbf{f}} \in J_q(\mathbf{R}^N_+)$ by (3.3.6), (3.3.9), and the definitions of K and $\widetilde{K}$. In addition, there exist a function $\theta \in \widehat{W}^1_{q,0}(\mathbf{R}^N_+)$ and subsequence $\{\theta^{m(l)}\}_{l=1}^\infty$ such that

$$\nabla\theta^{m(l)} \to \nabla\theta \quad \text{weakly in } L_q(\mathbf{R}^N_+)^N \quad \text{as } l \to \infty, \tag{3.3.10}$$

because $\widehat{W}^1_{q,0}(\mathbf{R}^N_+)$ is a reflexive Banach space (cf. [**Gal11**, Exercise II.6.2]). Passing a subsequence of $\{m\}_{m=m_0}^\infty$ in (3.3.6) if necessary, we have, by (3.3.8), (3.3.9), and (3.3.10),

$$\begin{aligned} (\widetilde{\mathbf{f}} + \nabla\theta, \varphi)_{\mathbf{R}^N_+} &= (\mathbf{f}, \varphi)_{\mathbf{R}^N_+} \quad \text{for any } \varphi \in C^\infty_0(\mathbf{R}^N_+)^N, \\ (\widetilde{g}, \psi)_{\mathbf{R}^{N-1}} &= (g, \psi)_{\mathbf{R}^{N-1}} \quad \text{for any } \psi \in C^\infty_0(\mathbf{R}^{N-1}). \end{aligned}$$

This implies that $\mathbf{f} = \widetilde{\mathbf{f}}$ and $g = \widetilde{g}$ since $L_q(\mathbf{R}^N_+)^N = J_q(\mathbf{R}^N_+) \oplus G_q(\mathbf{R}^N_+)$ with $G_q(\mathbf{R}^N_+) = \{\nabla\theta \in L_q(\mathbf{R}^N_+)^N \mid \theta \in \widehat{W}^1_{q,0}(\mathbf{R}^N_+)\}$. We thus can replace $(\widetilde{\mathbf{f}}, \widetilde{g})$ by $(\mathbf{f}, g)$

in (3.3.9), which, combined with Mazur's theorem, furnishes that for any $\varepsilon > 0$ there exists a positive number n_0 and non-negative numbers c_l and d_l $(l = 1, \dots, n_0)$ such that $\sum_{l=1}^{n_0} c_l = 1$, $\sum_{l=1}^{n_0} d_l = 1$, and

$$\Big\| \mathbf{f} - \sum_{l=1}^{n_0} c_l \mathbf{v}^{m(l)} \Big\|_{L_q(\mathbf{R}^N_+)} < \varepsilon, \qquad \Big\| g - \sum_{l=1}^{n_0} d_l \eta^{m(l)} \Big\|_{W_q^{2-(1/q)}(\mathbf{R}^{N-1})} < \varepsilon.$$

Since $(\sum_{l=1}^{n_0} c_l v^{m(l)}, \sum_{l=1}^{n_0} d_l \eta^{m(l)}) \in \mathcal{D}(\mathcal{A}_q)$, we see that $\mathcal{D}(\mathcal{A}_q)$ is dense in X_q.

We finally show that $\mathcal{A}_q$ is a closed operator on X_q. To this end, it is sufficient to prove that $\mathbf{U} = (\mathbf{u}, h) \in \mathcal{D}(\mathcal{A}_q)$ and $\mathbf{V} = \mathcal{A}_q \mathbf{U}$ for any sequences $\{\mathbf{U}_j\}_{j=1}^\infty \subset \mathcal{D}(\mathcal{A}_q)$ with

$$\mathbf{U}_j \to \mathbf{U} \quad \text{in } X_q, \quad \mathcal{A}_q \mathbf{U}_j \to \mathbf{V} \quad \text{in } X_q \quad \text{as } j \to \infty.$$

Setting $\mathbf{F}_j = \lambda_0 \mathbf{U}_j - \mathcal{A}_q \mathbf{U}_j$, where λ_0 is the same constant as in Lemma 3.3.1, we have $\mathbf{F}_j \to \lambda_0 \mathbf{U} - \mathbf{V}$ in X_q as $j \to \infty$, and therefore by Proposition 3.3.3 and Lemma 3.3.1 with $\lambda = \lambda_0$

$$\begin{aligned} &\|\mathbf{U}_j - \mathbf{U}_k\|_{W_q^2(\mathbf{R}^N_+) \times W_q^{3-1/q}(\mathbf{R}^{N-1})} \\ &\quad \le C(N, q, \varepsilon, \lambda_0) \|\mathbf{F}_j - \mathbf{F}_k\|_{X_q} \to 0 \quad \text{as } j, k \to \infty. \end{aligned}$$

This shows that there exists a function $\widetilde{\mathbf{U}} \in W_q^2(\mathbf{R}^N_+)^N \times W_q^{3-1/q}(\mathbf{R}^{N-1})$ such that $\mathbf{U}_j \to \widetilde{\mathbf{U}}$ in $W_q^2(\mathbf{R}^N_+)^N \times W_q^{3-1/q}(\mathbf{R}^{N-1})$ as $j \to \infty$. Since $\mathbf{U} = \widetilde{\mathbf{U}}$, it is clear that $\mathbf{U} \in \mathcal{D}(\mathcal{A}_q)$, and also that

$$\begin{aligned} \|\mathbf{V} - \mathcal{A}_q \mathbf{U}\|_{X_q} &\le \|\mathbf{V} - \mathcal{A}_q \mathbf{U}_j\|_{X_q} + \|\mathcal{A}_q \mathbf{U}_j - \mathcal{A}_q \mathbf{U}\|_{X_q} \\ &\le \|\mathbf{V} - \mathcal{A}_q \mathbf{U}_j\|_{X_q} + \|\mathbf{U}_j - \mathbf{U}\|_{W_q^2(\mathbf{R}^N_+) \times W_q^{3-1/q}(\mathbf{R}^{N-1})} \to 0 \end{aligned}$$

as $j, k \to \infty$, which completes the proof of the lemma. □

By Lemma 3.3.4, we obtain the following proposition.

Proposition 3.3.5. *Let $1 < q < \infty$. Then $\mathcal{A}_q$ generates an analytic semigroup $\{\mathcal{S}(t)\}_{t \ge 0}$ on X_q. In addition, there exist a positive constants γ_0 and $C = C(N, q, \gamma_0)$ such that for any $t > 0$*

$$\begin{aligned} \|\mathcal{S}(t)\mathbf{F}\|_{X_q} &\le C e^{\gamma_0 t} \|\mathbf{F}\|_{X_q}, \\ \|\partial_t \mathcal{S}(t)\mathbf{F}\|_{X_q} &\le C t^{-1} e^{\gamma_0 t} \|\mathbf{F}\|_{X_q}, \\ \|\partial_t \mathcal{S}(t)\mathbf{F}\|_{X_q} &\le C e^{\gamma_0 t} \|\mathbf{F}\|_{\mathcal{D}(\mathcal{A}_q)}. \end{aligned}$$

Let P_1 and P_2 be projections from X_q to $J_q(\mathbf{R}^N_+)$ and to $W_q^{2-1/q}(\mathbf{R}^{N-1})$, respectively. If we set for $\mathbf{F} \in X_q$

$$\begin{aligned} S(t)\mathbf{F} &= P_1 \mathcal{S}(t)\mathbf{F}, \\ T(t)\mathbf{F} &= P_2 \mathcal{S}(t)\mathbf{F}, \\ \Pi(t)\mathbf{F} &= K(P_1 \mathcal{S}(t)\mathbf{F}) + \widetilde{K}(P_2 \mathcal{S}(t)\mathbf{F}), \end{aligned} \tag{3.3.11}$$

then the standard theory of analytic semigroups tells us that

$$(\mathbf{u}, \theta, h) = (S(t)\mathbf{F}, \Pi(t)\mathbf{F}, T(t)\mathbf{F})$$

solves (3.1.1) uniquely and satisfies the required properties in Theorem 3.1.1.

3.4. Decompositions of $S(t), \Pi(t)$, and $T(t)$

In this section, we give decompositions of $S(t), \Pi(t)$, and $T(t)$ obtained in Theorem 3.1.1. To this end, we first calculate exact formulas of solutions to (3.3.1). Let $(\mathbf{w}^1, p^1)$ and $(\mathbf{w}^2, p^2)$ be solutions, given by (3.2.8) and (3.2.13), to the equations (3.2.7) and (3.2.12), respectively, and then note that $(\mathbf{w}, p) = (\mathbf{w}^1 + \mathbf{w}^2, p^1 + p^2)$ solves (3.2.1) uniquely. In addition, let $(\mathbf{v}, \pi, h)$ be solutions to

$$(3.4.1)\qquad \begin{cases} \lambda \mathbf{v} - \Delta \mathbf{v} + \nabla \pi = 0, & \text{in } \mathbf{R}^N_+ \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \mathbf{R}^N_+, \\ \lambda h - \mathbf{v}\cdot\mathbf{n} = \mathbf{w}\cdot\mathbf{n} + g & \text{on } \mathbf{R}^N_0, \\ \mathbf{S}(\mathbf{v}, \pi)\mathbf{n} + (c_g - c_\sigma \Delta')h\mathbf{n} = 0 & \text{on } \mathbf{R}^N_0. \end{cases}$$

Then $(\mathbf{u}, \theta, h) = (\mathbf{v} + \mathbf{w}, \pi + p, h)$ is the solution to the equations (3.3.1). Let $j, k = 1, \dots, N-1$ and $J = 1, \dots, N$ in the present section. Representation formulas of $(\mathbf{v}, \pi, h)$ are given by

$$(3.4.2)\qquad \begin{aligned} &v_J(x, \lambda) = \mathcal{F}^{-1}_{\xi'}[\widehat{v}_J(\xi', x_N, \lambda)](x'), \quad \pi(x, \lambda) = \mathcal{F}^{-1}_{\xi'}[\widehat{\pi}(\xi', x_N, \lambda)](x'), \\ &h(x', \lambda) = \mathcal{F}^{-1}_{\xi'}\left[\frac{D(A,B)}{(B+A)L(A,B)}(-\widehat{w}_N(\xi', 0, \lambda) + \widehat{g}(\xi'))\right](x') \end{aligned}$$

(cf. [**SS12**, Section 7]), where

$$\begin{aligned} &\widehat{v}_j(\xi', x_N, \lambda) \\ &\quad= \left(-\frac{i\xi_j(B-A)}{D(A,B)}e^{-Bx_N} + \frac{i\xi_j(B^2+A^2)}{D(A,B)}\mathcal{M}(x_N)\right)(c_g + c_\sigma A^2)\widehat{h}(\xi', \lambda), \\ &\widehat{v}_N(\xi', x_N, \lambda) = \left(\frac{A(B+A)}{D(A,B)}e^{-Bx_N} - \frac{A(B^2+A^2)}{D(A,B)}\mathcal{M}(x_N)\right)(c_g + c_\sigma A^2)\widehat{h}(\xi', \lambda), \\ &\widehat{\pi}(\xi', x_N, \lambda) = \frac{(B+A)(B^2+A^2)}{D(A,B)}e^{-Ax_N}(c_g + c_\sigma A^2)\widehat{h}(\xi', \lambda), \end{aligned}$$

for $A, B, D(A,B), L(A,B)$, and $\mathcal{M}(x_N)$ defined as (1.2.1) with $\mu = 1$ and (1.2.2). Inserting (3.2.3) into $h(x', \lambda)$, we have the following decompositions:

$$\widehat{v}_J(\xi', x_N, \lambda) = \sum_{d\in\{\mathbf{f}, g\}} \widehat{v}^d_J(\xi', x_N, \lambda), \quad \widehat{\pi}(\xi', x_N, \lambda) = \sum_{d\in\{\mathbf{f}, g\}} \widehat{\pi}^d(\xi', x_N, \lambda),$$

where the right-hand sides are given by

(3.4.3)

$$\begin{aligned} \widehat{v}^{\mathbf{f}}_J(\xi', x_N, \lambda) &= \sum_{K=1}^N \frac{\mathcal{V}^{BB}_{JK}(\xi', \lambda)(c_g + c_\sigma A^2)}{L(A,B)} \int_0^\infty e^{-B(x_N+y_N)}\widehat{f}_K(\xi', y_N)\, dy_N \\ &+ \sum_{K=1}^N \frac{\mathcal{V}^{B\mathcal{M}}_{JK}(\xi', \lambda)(c_g + c_\sigma A^2)}{L(A,B)} \int_0^\infty e^{-Bx_N}\mathcal{M}(y_N)\widehat{f}_K(\xi', y_N)\, dy_N \\ &+ \sum_{K=1}^N \frac{\mathcal{V}^{\mathcal{M}B}_{JK}(\xi', \lambda)(c_g + c_\sigma A^2)}{L(A,B)} \int_0^\infty \mathcal{M}(x_N)e^{-By_N}\widehat{f}_K(\xi', y_N)\, dy_N \end{aligned}$$

$$+\sum_{K=1}^{N}\frac{\mathcal{V}_{JK}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)(c_g+c_\sigma A^2)}{L(A,B)}\int_0^\infty \mathcal{M}(x_N)\mathcal{M}(y_N)\widehat{f}_K(\xi',y_N)\,dy_N$$

$$\widehat{v}_j^{g}(\xi',x_N,\lambda)=\frac{i\xi_j(c_g+c_\sigma A^2)}{(B+A)L(A,B)}\left(-(B-A)e^{-Bx_N}+(B^2+A^2)\mathcal{M}(x_N)\right)\widehat{g}(\xi'),$$

$$\widehat{v}_N^{g}(\xi',x_N,\lambda)=\frac{A(c_g+c_\sigma A^2)}{(B+A)L(A,B)}\left((B+A)e^{-Bx_N}-(B^2+A^2)\mathcal{M}(x_N)\right)\widehat{g}(\xi'),$$

$$\widehat{\pi}^{\mathbf{f}}(\xi',x_N,\lambda)=\sum_{K=1}^{N}\frac{\mathcal{P}_K^{AA}(\xi',\lambda)(c_g+c_\sigma A^2)}{L(A,B)}\int_0^\infty e^{-A(x_N+y_N)}\widehat{f}_K(\xi',y_N)\,dy_N$$

$$+\sum_{K=1}^{N}\frac{\mathcal{P}_k^{A\mathcal{M}}(\xi',\lambda)(c_g+c_\sigma A^2)}{L(A,B)}\int_0^\infty e^{-Ax_N}\mathcal{M}(y_N)\widehat{f}_K(\xi',y_N)\,dy_N,$$

$$\widehat{\pi}^{g}(\xi',x_N,\lambda)=\frac{(B^2+A^2)(c_g+c_\sigma A^2)}{L(A,B)}e^{-Ax_N}\widehat{g}(\xi'),$$

and furthermore,

(3.4.4)

$$\mathcal{V}_{jk}^{BB}(\xi',\lambda)=-\frac{\xi_j\xi_k(B-A)^2}{(B+A)D(A,B)},\qquad \mathcal{V}_{jN}^{BB}(\xi',\lambda)=\frac{i\xi_jA(B-A)}{D(A,B)},$$

$$\mathcal{V}_{Nk}^{BB}(\xi',\lambda)=-\frac{i\xi_kA(B-A)}{D(A,B)},\qquad \mathcal{V}_{NN}^{BB}(\xi',\lambda)=-\frac{A^2(B+A)}{D(A,B)},$$

$$\mathcal{V}_{jk}^{B\mathcal{M}}(\xi',\lambda)=\frac{\xi_j\xi_k(B-A)(B^2+A^2)}{(B+A)D(A,B)},\qquad \mathcal{V}_{jN}^{B\mathcal{M}}(\xi',\lambda)=-\frac{i\xi_jA(B-A)(B^2+A^2)}{(B+A)D(A,B)},$$

$$\mathcal{V}_{Nk}^{B\mathcal{M}}(\xi',\lambda)=\frac{i\xi_kA(B^2+A^2)}{D(A,B)},\qquad \mathcal{V}_{NN}^{B\mathcal{M}}(\xi',\lambda)=\frac{A^2(B^2+A^2)}{D(A,B)},$$

$$\mathcal{V}_{jk}^{\mathcal{M}B}(\xi',\lambda)=\frac{\xi_j\xi_k(B-A)(B^2+A^2)}{(B+A)D(A,B)},\qquad \mathcal{V}_{jN}^{\mathcal{M}B}(\xi',\lambda)=-\frac{i\xi_jA(B^2+A^2)}{D(A,B)},$$

$$\mathcal{V}_{Nk}^{\mathcal{M}B}(\xi',\lambda)=\frac{i\xi_kA(B-A)(B^2+A^2)}{(B+A)D(A,B)},\qquad \mathcal{V}_{NN}^{\mathcal{M}B}(\xi',\lambda)=\frac{A^2(B^2+A^2)}{D(A,B)},$$

$$\mathcal{V}_{jk}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)=-\frac{\xi_j\xi_k(B^2+A^2)^2}{(B+A)D(A,B)},\qquad \mathcal{V}_{jN}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)=\frac{i\xi_jA(B^2+A^2)^2}{(B+A)D(A,B)},$$

$$\mathcal{V}_{Nk}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)=-\frac{i\xi_kA(B^2+A^2)^2}{(B+A)D(A,B)},\qquad \mathcal{V}_{NN}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)=-\frac{A^2(B^2+A^2)^2}{(B+A)D(A,B)},$$

$$\mathcal{P}_k^{AA}(\xi',\lambda)=-\frac{i\xi_k(B-A)(B^2+A^2)}{D(A,B)},\qquad \mathcal{P}_N^{AA}(\xi',\lambda)=-\frac{A(B+A)(B^2+A^2)}{D(A,B)},$$

$$\mathcal{P}_k^{A\mathcal{M}}(\xi',\lambda)=\frac{2i\xi_kAB(B^2+A^2)}{D(A,B)},\qquad \mathcal{P}_N^{A\mathcal{M}}(\xi',\lambda)=\frac{2A^3(B^2+A^2)}{D(A,B)}.$$

In addition, we see that, by inserting (3.2.4) into $\widehat{h}(\xi',\lambda)$, $\widehat{h}(\xi',\lambda)=\widehat{h}^{\mathbf{f}}(\xi,\lambda)+\widehat{h}^{g}(\xi,\lambda)$ with

$$\widehat{h}^{\mathbf{f}}(\xi',\lambda)=-\sum_{k=1}^{N-1}\frac{i\xi_k(B-A)}{(B+A)L(A,B)}\int_0^\infty e^{-Ay_N}\widehat{f}_k(\xi',y_N)\,dy_N \tag{3.4.5}$$

$$
\begin{aligned}
&-\frac{A}{L(A,B)}\int_0^\infty e^{-Ay_N}\widehat{f}_N(\xi',y_N)\,dy_N\\
&+\sum_{k=1}^{N-1}\frac{2i\xi_k AB}{(B+A)L(A,B)}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_k(\xi',y_N)\,dy_N\\
&+\frac{2A^3}{(B+A)L(A,B)}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_N(\xi',y_N)\,dy_N,\\
\widehat{h}^g(\xi',\lambda)=&\frac{D(A,B)}{(B+A)L(A,B)}\widehat{g}(\xi').
\end{aligned}
$$

Next we shall construct cut-off functions. Let $\varphi\in C_0^\infty(\mathbf{R}^{N-1})$ be a function such that $0\le\varphi(\xi')\le 1$, $\varphi(\xi')=1$ for $|\xi'|\le 1/3$, and $\varphi(\xi')=0$ for $|\xi'|\ge 2/3$. Let $A_0\in(0,1)$ be a sufficiently small number, which is determined in Section 3.5. We then define φ_0 and φ_∞ as

$$\varphi_0(\xi')=\varphi(\xi'/A_0),\ \ \varphi_\infty(\xi')=1-\varphi(\xi'/A_0), \tag{3.4.6}$$

and besides, we set, for $t>0$,

$$
\begin{aligned}
S_a^d(t;A_0)\mathbf{F}&=\frac{1}{2\pi i}\int_{\Gamma(\varepsilon)}e^{\lambda t}\mathcal{F}_{\xi'}^{-1}[\varphi_a(\xi')\widehat{\mathbf{v}}^d(\xi',x_N,\lambda)](x')\,d\lambda,\\
\Pi_a^d(t;A_0)\mathbf{F}&=\frac{1}{2\pi i}\int_{\Gamma(\varepsilon)}e^{\lambda t}\mathcal{F}_{\xi'}^{-1}[\varphi_a(\xi')\widehat{\pi}^d(\xi',x_N,\lambda)](x')\,d\lambda,\\
T_a^d(t;A_0)\mathbf{F}&=\frac{1}{2\pi i}\int_{\Gamma(\varepsilon)}e^{\lambda t}\mathcal{F}_{\xi'}^{-1}[\varphi_a(\xi')\widehat{h}^d(\xi',\lambda)](x')\,d\lambda,\\
R_1(t)E\mathbf{f}&=\frac{1}{2\pi i}\int_{\Gamma(\varepsilon)}e^{\lambda t}\mathbf{w}^1(x,\lambda)\,d\lambda,\ \ R_2(t)\mathbf{f}=\frac{1}{2\pi i}\int_{\Gamma(\varepsilon)}e^{\lambda t}\mathbf{w}^2(x,\lambda)\,d\lambda,\\
P_1(t)E\mathbf{f}&=\frac{1}{2\pi i}\int_{\Gamma(\varepsilon)}e^{\lambda t}p^1(x,\lambda)\,d\lambda,\ \ P_2(t)\mathbf{f}=\frac{1}{2\pi i}\int_{\Gamma(\varepsilon)}e^{\lambda t}p^2(x,\lambda)\,d\lambda
\end{aligned} \tag{3.4.7}
$$

with $a\in\{0,\infty\}$ and $d\in\{\mathbf{f},g\}$, where E is the extension operator defined as (3.2.6). Here we have taken the integral path $\Gamma(\varepsilon)=\Gamma^+(\varepsilon)\cup\Gamma^-(\varepsilon)$ as follows:

$$\Gamma^\pm(\varepsilon)=\{\lambda\in\mathbf{C}\mid \lambda=\widetilde{\lambda}_0(\varepsilon)+ue^{\pm i(\pi-\varepsilon)},\ u\in(0,\infty)\} \tag{3.4.8}$$

for some $\varepsilon\in(0,\pi/2)$ with $\widetilde{\lambda}_0(\varepsilon)=2\lambda_0(\varepsilon)/\sin\varepsilon$, where $\lambda_0(\varepsilon)$ is the same number as in Lemma 3.3.1 (cf. Figure 3.4.1 below). By (3.4.7), the operators in Theorem 3.1.3 are defined as

$$
\begin{aligned}
&S_a(t)\mathbf{F}=\sum_{d\in\{\mathbf{f},g\}}S_a^d(t;A_0)\mathbf{F},\quad \Pi_a(t)\mathbf{F}=\sum_{d\in\{\mathbf{f},g\}}\Pi_a^d(t;A_0)\mathbf{F},\\
&T_a(t)\mathbf{F}=\sum_{d\in\{\mathbf{f},g\}}T_a^d(t;A_0)\mathbf{F},\\
&R(t)\mathbf{f}=R_1(t)E\mathbf{f}+R_2(t)\mathbf{f},\quad P(t)=P_1(t)E\mathbf{f}+P_2(t)\mathbf{f}
\end{aligned} \tag{3.4.9}
$$

for $a\in\{0,\infty\}$.

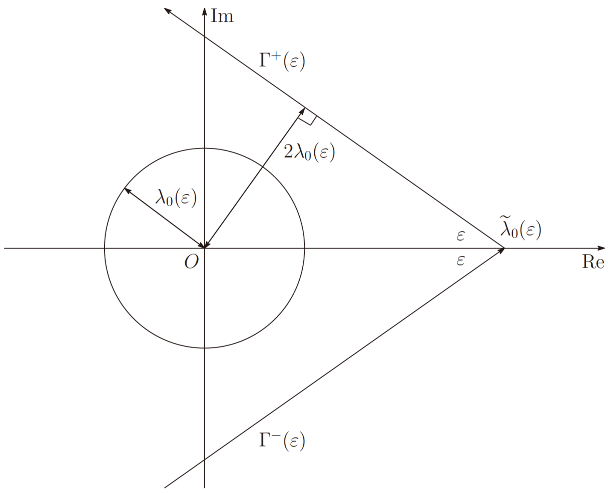

FIGURE 3.4.1. $\Gamma(\varepsilon) = \Gamma^+(\varepsilon) \cup \Gamma^-(\varepsilon)$

REMARK 3.4.1. (1) Let $1 < p < \infty$. Then, by Proposition 3.3.3 and (3.3.11), we see that for $\mathbf{F} \in X_p$, defined as (3.1.2),

$$\begin{aligned} S(t)\mathbf{F} &= \sum_{a\in\{0,\infty\}} S_a(t)\mathbf{F} + R(t)\mathbf{f}, \\ \Pi(t)\mathbf{F} &= \sum_{a\in\{0,\infty\}} \Pi_a(t)\mathbf{F} + P(t)\mathbf{f}, \\ T(t)\mathbf{F} &= \sum_{a\in\{0,\infty\}} T_a(t)\mathbf{F}. \end{aligned}$$

Since the right-hand sides are valid for $\mathbf{F} \in X_p^2$ by Lemma 3.3.1, we extend $S(t), \Pi(t)$, and $T(t)$ to the operators defined on X_p^2 by the relations above. For simplicity, such extended operators are denoted by $S(t)$, $\Pi(t)$, and $T(t)$ again.

(2) Let $1 < p < \infty$ and $\mathbf{f} \in L_p(\mathbf{R}_+^N)^N$. Then, by (3.2.9) and (3.2.15), we see that for $i = 1, 2$, $l = 0, 1, 2$, and $t > 0$

$$\begin{aligned} &\|\nabla^l R_i(t)\mathbf{f}\|_{L_p(\mathbf{R}_+^N)} \leq C(N,p)t^{-l/2}\|\mathbf{f}\|_{L_p(\mathbf{R}_+^N)}, \\ &\|(\partial_t R_i(t)\mathbf{f}, \nabla P_i(t)\mathbf{f})\|_{L_p(\mathbf{R}_+^N)} \leq C(N,p)t^{-1}\|\mathbf{f}\|_{L_p(\mathbf{R}_+^N)} \end{aligned}$$

with some positive constant $C(N,p)$, which furnishes that Theorem 3.1.3 (3) holds.

(3) The extension $\mathcal{E}(T(t)\mathbf{F})$, defined as (3.1.3), is decomposed into

$$\begin{aligned} \mathcal{E}(T(t)\mathbf{F}) &= \sum_{a\in\{0,\infty\}} \sum_{d\in\{\mathbf{f},g\}} \mathcal{E}(T_a^d(t;A_0)\mathbf{F}) \\ &= \sum_{a\in\{0,\infty\}} \sum_{d\in\{\mathbf{f},g\}} \frac{1}{2\pi i}\int_{\Gamma(\varepsilon)} e^{\lambda t}\mathcal{F}_{\xi'}^{-1}\left[\varphi_a(\xi')e^{-Ax_N}\widehat{h}^d(\xi',\lambda)\right](x')\,d\lambda. \end{aligned} \tag{3.4.10}$$

We devote the last part of this section to the proof of the following lemma.

LEMMA 3.4.2. *Let $\xi' \in \mathbf{R}^{N-1} \setminus \{0\}$ and $\lambda \in \{z \in \mathbf{C} \mid \mathrm{Re} z \geq 0\}$. Then it holds that $L(A, B) \neq 0$.*

PROOF. Applying the partial Fourier transform with respect to tangential variable $x' \in \mathbf{R}^{N-1}$ to the equations (3.3.1) with $\mathbf{f} = 0$ and $g = 0$ yields that

(3.4.11)
$$\begin{aligned}
&\lambda\widehat{u}_j(x_N) - \sum_{k=1}^{N-1} i\xi_k(i\xi_j\widehat{u}_k(x_N) + i\xi_k\widehat{u}_j(x_N)) \\
&\qquad\qquad -D_N(D_N\widehat{u}_j(x_N) + i\xi_j\widehat{u}_N(x_N)) + i\xi_j\widehat{\theta}(x_N) = 0, \\
&\lambda\widehat{u}_N(x_N) - \sum_{k=1}^{N-1} i\xi_k(D_N\widehat{u}_k(x_N) + i\xi_k\widehat{u}_N(x_N)) - 2D_N^2\widehat{u}_N(x_N) + D_N\widehat{\theta}(x_N) = 0, \\
&\sum_{k=1}^{N-1} i\xi_k\widehat{u}_k(x_N) + D_N\widehat{u}_N(x_N) = 0, \\
&\lambda\widehat{h} + \widehat{u}_N(0) = 0, \\
&D_N\widehat{u}_j(0) + i\xi_j\widehat{u}_N(0) = 0, \quad -\widehat{\theta}(0) + 2D_N\widehat{u}_N(0) + (c_g + c_\sigma A^2)\widehat{h} = 0
\end{aligned}$$

for $x_N > 0$, where

$$\widehat{u}_J(x_N) = \widehat{u}_J(\xi', x_N), \quad \widehat{\theta}(x_N) = \widehat{\theta}(\xi', x_N), \quad \widehat{h} = \widehat{h}(\xi').$$

Here we set

$$\widehat{\mathbf{u}}(x_N) = (\widehat{u}_1(x_N), \dots, \widehat{u}_N(x_N))^T, \quad \|\mathbf{f}\|^2 = \int_0^\infty \mathbf{f}(x_N) \cdot \overline{\mathbf{f}(x_N)}\, dx_N$$

for m-vector functions $\mathbf{f}$, where $m \in \mathbf{N}$, and show that $L(A, B) \neq 0$ by contradiction from now. Suppose that $L(A, B) = 0$, and then we know that (3.4.11) admits a solution $(\widehat{u}(x_N), \widehat{\theta}(x_N), \widehat{h}) \neq 0$ which decays exponentially as $x_N \to \infty$ (see e.g. [**SS12**, Section 4]). On the other hand, we obtain

$$\begin{aligned}
0 = \lambda\|\widehat{\mathbf{u}}\|^2 + 2\|D_N\widehat{u}_N\|^2 + \sum_{j,k=1}^{N-1} \|i\xi_k\widehat{u}_j\|^2 + \|\sum_{j=1}^{N-1} i\xi_j\widehat{u}_j\|^2 \\
+ \sum_{j=1}^{N-1} \|D_N\widehat{u}_j + i\xi_j\widehat{u}_N\|^2 + \overline{\lambda}(c_g + c_\sigma A^2)|\widehat{h}|^2
\end{aligned}$$

through the following two steps: First multiply the first equation by $\overline{\widehat{u}_j(x_N)}$ and the second equation by $\overline{\widehat{u}_N(x_N)}$ in (3.4.11), and integrate the resultant formulas with respect to $x_N \in (0, \infty)$. Second use integration by parts with the third to sixth equations of (3.4.11). Taking real parts in the obtained identity above yields that

$$D_N\widehat{u}_N(x_N) = 0, \quad D_N\widehat{u}_j(x_N) + i\xi_j\widehat{u}_N = 0 \quad \text{for } \mathrm{Re}\,\lambda \geq 0.$$

In particular, $\widehat{u}_N$ is a constant, but $\widehat{u}_N = 0$ since $\lim_{x_N\to\infty} \widehat{u}_N = 0$. We thus obtain $D_N\widehat{u}_j = 0$, which furnishes that $\widehat{u}_j = 0$ by $\lim_{x_N\to\infty} \widehat{u}_j = 0$. Combining $\widehat{u}_j = 0$ and the first equation of (3.4.11) yields that $i\xi_j\widehat{\theta} = 0$. This implies that $\widehat{\theta} = 0$ because $\xi' \neq 0$. In addition, by the sixth equation of (3.4.11), we have $(c_g + c_\sigma A^2)\widehat{h} = 0$. Therefore, since $c_g + c_\sigma A^2 \neq 0$, we see that $\widehat{h} = 0$. Summing up the argument above, it holds that $\widehat{u} = 0$, $\widehat{\theta} = 0$, and $\widehat{h} = 0$, which contradicts to $L(A, B) = 0$. This completes the proof of the lemma. □

3.5. Analysis of low frequency parts

In this section, we show Theorem 3.1.3 (1). If we consider the Lopatinskii determinant $L(A, B)$, defined as (1.2.1) with $\mu = 1$, as a polynomial with respect to B, then it has the following four roots:

$$B_j^{\pm} = e^{\pm i(2j-1)(\pi/4)} c_g^{1/4} A^{1/4} - \frac{A^{7/4}}{2e^{\pm i(2j-1)(\pi/4)} c_g^{1/4}} - \frac{c_\sigma A^{9/4}}{e^{\pm i(2j-1)(3\pi/4)} c_g^{3/4}} + O(A^{10/4}) \quad \text{as } A \to 0 \tag{3.5.1}$$

for $j = 1, 2$. Set $\lambda_\pm = (B_1^{\pm})^2 - A^2$, and then

$$\lambda_\pm = \pm i c_g^{1/2} A^{1/2} - 2A^2 \mp \frac{2c_\sigma}{i c_g^{1/2}} A^{10/4} + O(A^{11/4}) \quad \text{as } A \to 0. \tag{3.5.2}$$

REMARK 3.5.1. Let $0 < \varepsilon < \pi/2$ and $\xi' \in \mathbf{R}^{N-1} \setminus \{0\}$. Then, for $\lambda \in \Sigma_\varepsilon$, we choose a brunch such that $\mathrm{Re} B = \mathrm{Re}\sqrt{\lambda + A^2} > 0$. Note that $\lambda_\pm \in \Sigma_\varepsilon$ and $\mathrm{Re}(\lambda_\pm + A^2) < 0$.

We set $\varepsilon_0 = \tan^{-1}\{(A^2/8)/A^2\} = \tan^{-1}(1/8)$, and

$$\Gamma_0^{\pm} = \{\lambda \in \mathbf{C} \mid \lambda = \lambda_\pm + (c_g^{1/2}/4)A^{1/2} e^{\pm iu},\ u : 0 \to 2\pi\},$$
$$\Gamma_1^{\pm} = \{\lambda \in \mathbf{C} \mid \lambda = -A^2 + (A^2/4)e^{\pm iu},\ u : 0 \to \pi/2\},$$
$$\Gamma_2^{\pm} = \{\lambda \in \mathbf{C} \mid \lambda = -(A^2(1-u) + \gamma_0 u) \pm i((A^2/4)(1-u) + \widetilde{\gamma}_0 u),\ u : 0 \to 1\},$$
$$\Gamma_3^{\pm} = \{\lambda \in \mathbf{C} \mid \lambda = -(\gamma_0 \pm i\widetilde{\gamma}_0) + u e^{\pm i(\pi - \varepsilon_0)},\ u : 0 \to \infty\}$$

(cf. Figure 3.5.1 below), where $\gamma_0 = \lambda_0(\varepsilon_0)$ given by Lemma 3.3.1 and

$$\widetilde{\gamma}_0 = \frac{1}{8}\left(\lambda_0(\varepsilon_0) + \widetilde{\lambda}_0(\varepsilon_0)\right) = \frac{1}{8}\left(1 + \frac{2}{\sin\varepsilon_0}\right)\lambda_0(\varepsilon_0) = \frac{(1+2\sqrt{65})\gamma_0}{8} \tag{3.5.3}$$

for the same $\widetilde{\lambda}_0(\varepsilon_0)$ as in (3.4.8) with $\varepsilon = \varepsilon_0$. By Cauchy's integral theorem, we

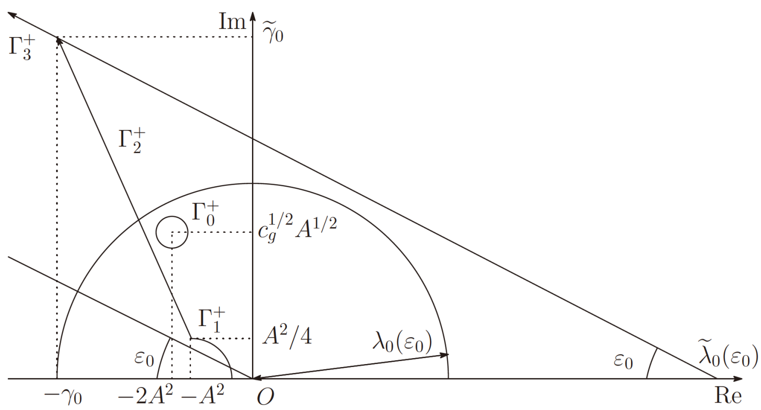

FIGURE 3.5.1. Γ_σ^+ $(\sigma = 0, 1, 2, 3)$

then decompose $S_0^d(t;A_0)\mathbf{F}$, $\Pi_0^d(t;A_0)\mathbf{F}$, $T_0^d(t;A_0)\mathbf{F}$, and $\mathcal{E}(T_0^d(t;A_0)\mathbf{F})$, defined as (3.4.7) and (3.4.10), into

$$
\begin{aligned}
&(3.5.4) \quad S_0^d(t;A_0)\mathbf{F} = \sum_{\sigma=0}^{3} S_0^{d,\sigma}(t;A_0)\mathbf{F}, \qquad \Pi_0^d(t;A_0)\mathbf{F} = \sum_{\sigma=0}^{3} \Pi_0^{d,\sigma}(t;A_0)\mathbf{F}, \\
&T_0^d(t;A_0)\mathbf{F} = \sum_{\sigma=0}^{3} T_0^{d,\sigma}(t;A_0)\mathbf{F}, \quad \mathcal{E}(T_0^d(t;A_0)\mathbf{F}) = \sum_{\sigma=0}^{3} \mathcal{E}(T_0^{d,\sigma}(t;A_0)\mathbf{F})
\end{aligned}
$$

for $d \in \{\mathbf{f}, g\}$, where

$$
\begin{aligned}
(3.5.5) \quad S_0^{d,\sigma}(t;A_0)\mathbf{F} &= \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_\sigma^+\cup\Gamma_\sigma^-} e^{\lambda t}\varphi_0(\xi')\widehat{\mathbf{v}}^d(\xi',x_N,\lambda)\,d\lambda\right](x'), \\
\Pi_0^{d,\sigma}(t;A_0)\mathbf{F} &= \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_\sigma^+\cup\Gamma_\sigma^-} e^{\lambda t}\varphi_0(\xi')\widehat{\pi}^d(\xi',x_N,\lambda)\,d\lambda\right](x'), \\
T_0^{d,\sigma}(t;A_0)\mathbf{F} &= \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_\sigma^+\cup\Gamma_\sigma^-} e^{\lambda t}\varphi_0(\xi')\widehat{h}^d(\xi',\lambda)\,d\lambda\right](x'), \\
\mathcal{E}(T_0^{d,\sigma}(t;A_0)\mathbf{F}) &= \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_\sigma^+\cup\Gamma_\sigma^-} e^{\lambda t}\varphi_0(\xi')e^{-Ax_N}\widehat{h}^d(\xi',\lambda)\,d\lambda\right](x')
\end{aligned}
$$

with $\varphi_0(\xi')$ defined as (3.4.6). In order to estimate each term in (3.5.5), we here introduce the operators $K_n^{\pm,\sigma}(t;A_0)$ and $L_n^{\pm,\sigma}(t;A_0)$ defined by

$$
\begin{aligned}
(3.5.6) \quad &[K_n^{\pm,\sigma}(t;A_0)f](x) \\
&= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\int_{\Gamma_\sigma^\pm} e^{\lambda t}\varphi_0(\xi')k_n(\xi',\lambda)\mathcal{X}_n(x_N,y_N)\,d\lambda\,\widehat{f}(\xi',y_N)\right](x'), \\
&[L_n^{\pm,\sigma}(t;A_0)g](x) \\
&= \mathcal{F}_{\xi'}^{-1}\left[\int_{\Gamma_\sigma^\pm} e^{\lambda t}\varphi_0(\xi')l_n(\xi',\lambda)\mathcal{Y}_n(x_N)\,d\lambda\,\widehat{g}(\xi')\right](x')
\end{aligned}
$$

with some multipliers $k_n(\xi',\lambda)$ and $l_n(\xi',\lambda)$, where

$$
\mathcal{X}_n(x_N,y_N) = \begin{cases} e^{-A(x_N+y_N)} & (n=1), \\ e^{-Ax_N}\mathcal{M}(y_N) & (n=2), \\ e^{-B(x_N+y_N)} & (n=3), \\ e^{-Bx_N}\mathcal{M}(y_N) & (n=4), \\ \mathcal{M}(x_N)e^{-By_N} & (n=5), \\ \mathcal{M}(x_N)\mathcal{M}(y_N) & (n=6), \end{cases} \quad \mathcal{Y}_n(x_N) = \begin{cases} e^{-Ax_N} & (n=1), \\ e^{-Bx_N} & (n=2), \\ \mathcal{M}(x_N) & (n=3). \end{cases}
$$

REMARK 3.5.2. Estimates of $\partial_t\mathcal{E}(T_0^g(t;A_0)\mathbf{F})$ will be showed in the last part of this section.

3.5.1. Analysis on $\Gamma_0^\pm$. Our aim here is to show the following theorem for the operators defined as (3.5.5) with $\sigma = 0$.

THEOREM 3.5.3. *Let $1 \le r \le 2 \le q \le \infty$ and $\mathbf{F} = (\mathbf{f}, g) \in X_r^0$. Then there exists an $A_0 \in (0,1)$ such that the following assertions hold.*

(1) *Let* $k = 0, 1$, $l = 0, 1, 2$, *and* $\alpha' \in \mathbf{N}_0^{N-1}$. *Then there exists a positive constant* $C = C(N, q, r, \alpha')$ *such that for any* $t > 0$

$$\|\partial_t^k D_{x'}^{\alpha'} D_N^l S_0^{\mathbf{f},0}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{q})-\frac{k}{4}-\frac{|\alpha'|}{2}-\frac{l}{8}}\|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$

$$\|\partial_t^k D_{x'}^{\alpha'} D_N^l S_0^{g,0}(t; A_0)\mathbf{F}\|_{L_r(\mathbf{R}_+^N)} \leq C\begin{cases} (t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{k}{4}-\frac{|\alpha'|}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})} & (l = 0), \\ (t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{8}(2-\frac{1}{q})-\frac{k}{4}-\frac{|\alpha'|}{2}-\frac{l}{8}}\|g\|_{L_r(\mathbf{R}^{N-1})} & (l = 1, 2). \end{cases}$$

(2) *There exists a positive constant* $C = C(N, q, r)$ *such that for any* $t > 0$

$$\|\nabla\Pi_0^{\mathbf{f},0}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{4}}\|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$

$$\|\nabla\Pi_0^{g,0}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{1}{4}}\|g\|_{L_r(\mathbf{R}^{N-1})}.$$

(3) *Let* $\alpha \in \mathbf{N}_0^N$. *Then there exists a positive constant* $C = C(N, q, r, \alpha)$ *such that for any* $t > 0$

$$\|D_x^\alpha \nabla\mathcal{E}(T_0^{\mathbf{f},0}(t; A_0)\mathbf{F})\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{4}-\frac{|\alpha|}{2}}\|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$

$$\|D_x^\alpha \partial_t\mathcal{E}(T_0^{\mathbf{f},0}(t; A_0)\mathbf{F})\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{q})-\frac{|\alpha|}{2}}\|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$

$$\|D_x^\alpha \nabla\mathcal{E}(T_0^{g,0}(t; A_0)\mathbf{F})\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{1}{4}-\frac{|\alpha|}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}.$$

(4) *There exists a positive constant* $C = C(N, q, r)$ *such that for any* $t > 0$

$$\|T_0^{\mathbf{f},0}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}^{N-1})} \leq C(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{r}-\frac{1}{2})}\|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)} \quad (r \neq 2),$$

$$\|T_0^{g,0}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}^{N-1})} \leq C(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})}\|g\|_{L_r(\mathbf{R}^{N-1})}.$$

We start with the following lemma in order to show Theorem 3.5.3.

LEMMA 3.5.4. *Let* $1 \leq r \leq 2 \leq q \leq \infty$, *and let* $f \in L_r(\mathbf{R}_+^N)$ *and* $g \in L_r(\mathbf{R}^{N-1})$. *We use the symbols defined as* (3.5.6) *with*

$$k_n(\xi', \lambda) = \frac{\kappa_n(\xi', \lambda)}{L(A, B)}, \quad l_n(\xi', \lambda) = \frac{m_n(\xi', \lambda)}{L(A, B)}.$$

(1) *Let* $s \geq 0$ *and suppose that there exist positive constants* $A_1 \in (0, 1)$ *and* $C = C(s)$ *such that for any* $A \in (0, A_1)$

$$|\kappa_1(\xi', \lambda_\pm)| \leq CA^{\frac{6}{4}+s}, \quad |\kappa_2(\xi', \lambda_\pm)| \leq CA^{\frac{7}{4}+s}, \quad |\kappa_3(\xi', \lambda_\pm)| \leq CA^{\frac{6}{4}+s},$$

$$|\kappa_4(\xi', \lambda_\pm)| \leq CA^{\frac{7}{4}+s}, \quad |\kappa_5(\xi', \lambda_\pm)| \leq CA^{\frac{7}{4}+s}, \quad |\kappa_6(\xi', \lambda_\pm)| \leq CA^{\frac{8}{4}+s}.$$

Then there exist positive constants $A_0 \in (0, A_1)$ *and* $C = C(N, q, r, s)$ *such that for any* $t > 0$

$$\|K_n^{\pm,0}(t; A_0)f\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{q})-\frac{s}{2}}\|f\|_{L_r(\mathbf{R}_+^N)} \quad (n = 1, 2, 6),$$

$$\|K_3^{\pm,0}(t; A_0)f\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-(\frac{N-1}{2}+\frac{1}{8})(\frac{1}{r}-\frac{1}{q})-\frac{3}{8}-\frac{s}{2}}\|f\|_{L_r(\mathbf{R}_+^N)},$$

$$\|K_4^{\pm,0}(t; A_0)f\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-(\frac{N-1}{2}+\frac{1}{8})(\frac{1}{r}-\frac{1}{q})-\frac{3}{8r}-\frac{s}{2}}\|f\|_{L_r(\mathbf{R}_+^N)},$$

$$\|K_5^{\pm,0}(t; A_0)f\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-(\frac{N-1}{2}+\frac{1}{8})(\frac{1}{r}-\frac{1}{q})-\frac{3}{8}(1-\frac{1}{q})-\frac{s}{2}}\|f\|_{L_r(\mathbf{R}_+^N)}.$$

(2) *Let $s \geq 0$ and suppose that there exist positive constants $A_1 \in (0,1)$ and $C = C(s)$ such that for any $A \in (0, A_1)$*

$$|m_1(\xi', \lambda_\pm)| \leq CA^{1+s}, \ \ |m_2(\xi', \lambda_\pm)| \leq CA^{1+s}, \ \ |m_3(\xi', \lambda_\pm)| \leq CA^{\frac{5}{4}+s}.$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(N, q, r, s)$ such that for any $t > 0$ and $n = 1, 3$

$$\|L_n^{\pm,0}(t; A_0)g\|_{L_q(\mathbf{R}^N_+)} \leq C(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{s}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})},$$
$$\|L_2^{\pm,0}(t; A_0)g\|_{L_q(\mathbf{R}^N_+)} \leq C(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{8}(2-\frac{1}{q})-\frac{s}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}.$$

(3) *Suppose that there exist positive constants $A_1 \in (0,1)$ and C such that for any $A \in (0, A_1)$*

$$|\kappa_1(\xi', \lambda_\pm)| \leq CA, \ \ |\kappa_2(\xi', \lambda_\pm)| \leq CA^{\frac{5}{4}}, \ \ |l_1(\xi', \lambda_\pm)| \leq CA^{\frac{2}{4}}.$$

Then there exists a positive constants $A_0 \in (0, A_1)$ and $C = C(N, q, r)$ such that for any $t > 0$ and $n = 1, 2$

$$\|[K_n^{\pm,0}(t; A_0)f]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})} \leq C(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{r}-\frac{1}{2})}\|f\|_{L_r(\mathbf{R}^N_+)} \ \ (r \neq 2),$$
$$\|[L_1^{\pm,0}(t; A_0)g]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})} \leq C(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})}\|g\|_{L_r(\mathbf{R}^{N-1})}.$$

PROOF. We use the abbreviations: $\|\cdot\|_2 = \|\cdot\|_{L_2(\mathbf{R}^{N-1})}$, $\widehat{f}(y_N) = \widehat{f}(\xi', y_N)$, and $\widetilde{t} = t+1$ for $t > 0$ in this proof, and consider only estimates on Γ_0^+ since the case of Γ_0^- can be proved similarly.
(1) We first show the inequality for $K_1^{+,0}(t; A_0)$. Noting that

$$B^2 - (B_1^+)^2 = \lambda - \lambda_+, \ \ L(A,B) = (B - B_1^+)(B - B_1^-)(B - B_2^+)(B - B_2^-), \tag{3.5.7}$$

we have, by the residue theorem,

(3.5.8)

$$\begin{aligned}
&[K_1^{+,0}(t; A_0)f](x) \\
&= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\int_{\Gamma_0^+} e^{\lambda t}\frac{\varphi_0(\xi')\kappa_1(\xi', \lambda)(B + B_1^+)e^{-A(x_N+y_N)}}{(\lambda - \lambda_+)(B - B_1^-)(B - B_2^+)(B - B_2^-)}\,d\lambda \widehat{f}(y_N)\right](x')\,dy_N \\
&= 4\pi i \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[e^{\lambda_+ t}\frac{\varphi_0(\xi')\kappa_1(\xi', \lambda_+)B_1^+e^{-A(x_N+y_N)}}{(B_1^+ - B_1^-)(B_1^+ - B_2^+)(B_1^+ - B_2^-)}\widehat{f}(y_N)\right](x')\,dy_N.
\end{aligned}$$

In view of (3.5.1) and (3.5.2), we can choose $A_0 \in (0, A_1)$ in such a way that

$$|e^{\lambda_+ t}| \leq Ce^{-A^2\widetilde{t}}, \ \ |B_1^+ - B_1^-| \geq CA^{\frac{1}{4}}, \ \ |B_1^+ - B_2^\pm| \geq CA^{\frac{1}{4}} \tag{3.5.9}$$

for any $A \in (0, A_0)$ and $t > 0$ with some positive constant C. Thus, by L_q-L_r estimates of the $(N-1)$-dimensional heat kernel and Parseval's theorem, we have

$$\begin{aligned}(3.5.10)\qquad &\|[K_1^{+,0}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\ &\le C\widetilde{t}^{-\frac{N-1}{2}(\frac12-\frac1q)-\frac{s}{2}}\int_0^\infty \left\|e^{-(A^2/2)\widetilde{t}}Ae^{-A(x_N+y_N)}\widehat{f}(y_N)\right\|_2 dy_N\\ &\le C\widetilde{t}^{-\frac{N-1}{2}(\frac12-\frac1q)-\frac{s}{2}}\int_0^\infty \frac{\|e^{-(A^2/4)\widetilde{t}}\,\widehat{f}(y_N)\|_{L_2(\mathbf{R}^{N-1})}}{\widetilde{t}^{1/2}+x_N+y_N}\,dy_N\\ &\le C\widetilde{t}^{-\frac{N-1}{2}(\frac1r-\frac1q)-\frac{s}{2}}\int_0^\infty \frac{\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\widetilde{t}^{1/2}+x_N+y_N}\,dy_N,\end{aligned}$$

where we have used Lemma 3.2.2 with $s_1 = 1/4$, $s_i = 1$ $(i = 2,3,4)$, $\tau = \widetilde{t}$, $a = x_N + y_N$, and $Z = A$ to calculate from the second line to third line. If $q > 2$, then applying Lemma 3.2.3 (2) with $a = 1/2$, $b_1 = b_2 = 1$, and $\tau = \widetilde{t}$ to (3.5.10) implies that the required inequality holds. In the case of $(q,r) = (2,2)$, by (3.5.10)

$$\|[K_1^{+,0}(t;A_0)f](\cdot\,,x_N)\|_2 \le C\widetilde{t}^{-\frac{s}{2}}\int_0^\infty \left\|\mathcal{F}_{\xi'}^{-1}\left[Ae^{-A(x_N+y_N)}\widehat{f}(y_N)\right]\right\|_2 dy_N,$$

and then it follows from Corollary B.3 (1) that

$$\|K_1^{+,0}(t;A_0)f\|_{L_2(\mathbf{R}^N_+)} \le C(t+1)^{-\frac{s}{2}}\|f\|_{L_2(\mathbf{R}^N_+)}.$$

On the other hand, in the case of $1 \le r < 2$ and $q = 2$, by the second lien of (3.5.10), Lemma 3.2.2, and Hölder's inequality

$$\begin{aligned}\left\|K_1^{+,0}(t;A_0)f\right\|_{L_2(\mathbf{R}^N_+)} &\le C\widetilde{t}^{-\frac{s}{2}}\int_0^\infty \left\|e^{-(A^2/2)\widetilde{t}}A^{1/2}e^{-Ay_N}\widehat{f}(y_N)\right\|_2 dy_N\\ &\le C\widetilde{t}^{-\frac{N-1}{2}(\frac1r-\frac12)-\frac{s}{2}}\int_0^\infty \frac{\|f(\cdot,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\widetilde{t}^{1/4}+(y_N)^{1/2}}\,dy_N\\ &\le C\widetilde{t}^{-\frac{N}{2}(\frac1r-\frac12)-\frac{s}{2}}\|f\|_{L_r(\mathbf{R}^N_+)},\end{aligned}$$

which complete the proof for the case of $K_1^{+,0}(t;A_0)$. We here summarize the argumentation above to the following lemma.

Lemma 3.5.5. *Let $1 \le r \le 2 \le q \le \infty$, $\tau > 0$, and $s_i > 0$ $(i = 1,2)$. For $x_N > 0$ and $f \in L_r(\mathbf{R}^N_+)$, we set*

$$F(x_N,\tau) = \int_0^\infty \left\|e^{-s_1A^2\tau}Ae^{-s_2A(x_N+y_N)}\widehat{f}(\xi',y_N)\right\|_{L_2(\mathbf{R}^{N-1})} dy_N.$$

Then there exists a positive constant $C = C(N,q,r)$ such that for any $\tau > 0$

$$\|F(\tau)\|_{L_q((0,\infty))} \le C\tau^{-\frac{N-1}{2}(\frac1r-\frac12)-\frac12(\frac1r-\frac1q)}\|f\|_{L_r(\mathbf{R}^N_+)}.$$

Secondly we show the inequality for $K_{0,2}^{+,0}(t;A_0)$. We here set

$$\mathcal{M}_\pm(a) = \frac{e^{-B_1^\pm a}-e^{-Aa}}{B_1^\pm - A} \quad \text{for } a > 0.$$

In view of (3.5.1) and (3.5.2), we can choose $A_0 \in (0, A_1)$ in such a way that for any $A \in (0,A_0)$ and $a > 0$

$$(3.5.11)\qquad |\mathcal{M}_\pm(a)| = \frac{|e^{-B_1^\pm a}-e^{-Aa}|}{|B_1^\pm - A|} \le CA^{-1/4}e^{-Aa}$$

with some positive constant C. Thus, by the same calculations as in (3.5.8) and (3.5.10), we obtain

$$\begin{aligned}&\|[K_2^{+,0}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac12-\frac1q\right)-\frac{s}{2}}\int_0^\infty\left\|e^{-(A^2/2)\widetilde{t}}Ae^{-A(x_N+y_N)}\widehat{f}(y_N)\right\|_2dy_N,\end{aligned}$$

which, combined with Lemma 3.5.5, furnishes that the required inequality holds.

Thirdly we show the inequality for $K_3^{+,0}(t;A_0)$. In view of (3.5.1) and (3.5.2), we can choose $A_0\in(0,1)$ such that

$$|e^{-B_1^+(x_N+y_N)}|\leq e^{-cA^{1/4}(x_N+y_N)}\quad\text{for any }A\in(0,A_0)\tag{3.5.12}$$

with some positive constant c, so that we easily see that by Lemma 3.2.2

$$\begin{aligned}&\|[K_3^{+,0}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac12-\frac1q\right)-\frac{s}{2}}\int_0^\infty\left\|e^{-(A^2/2)\widetilde{t}}Ae^{-cA^{1/4}(x_N+y_N)}\widehat{f}(y_N)\right\|_2dy_N\\&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac1r-\frac1q\right)-\frac{s}{2}}\int_0^\infty\frac{\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\widetilde{t}^{1/2}+(x_N)^4+(y_N)^4}dy_N.\end{aligned}$$

This yields the required inequality together with Lemma 3.2.3 for $a=1/2$ and $b_1=b_2=4$.

Finally we show the inequalities for $K_n^{+,0}(t;A_0)$ ($n=4,5,6$). Using similar argumentations to the above cases, we have for $n=4,5$

$$\|[K_4^{+,0}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac1r-\frac1q\right)-\frac{s}{2}}\int_0^\infty\frac{\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\widetilde{t}^{1/2}+(x_N)^4+y_N}dy_N,$$

$$\|[K_5^{+,0}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac1r-\frac1q\right)-\frac{s}{2}}\int_0^\infty\frac{\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\widetilde{t}^{1/2}+x_N+(y_N)^4}dy_N,$$

which, combined with Lemma 3.2.3 (2), completes the proof for the cases of $n=4,5$. In addition, for $n=6$, we have

$$\begin{aligned}&\|[K_6^{+,0}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac12-\frac1q\right)-\frac{s}{2}}\int_0^\infty\left\|e^{-(A^2/2)\widetilde{t}}Ae^{-CA(x_N+y_N)}\widehat{f}(y_N)\right\|_2dy_N,\end{aligned}$$

which furnishes the required inequality of $K_6^{+,0}(t;A_0)$ by Lemma 3.5.5.

(2) Noting (3.5.7), we have, by the residue theorem,

$$[L_n^{+,0}(t;A_0)g](x)=4\pi i\mathcal{F}_{\xi'}^{-1}\left[\frac{e^{\lambda_+t}\varphi_0(\xi')m_n(\xi',\lambda)B_1^+}{(B_1^+-B_1^-)(B_1^+-B_2^-)(B_1^+-B_2^-)}\mathcal{Y}_n(x_N)\widehat{g}(\xi')\right](x').$$

Thus, by (3.5.9), (3.5.11), (3.5.12), Lemma 3.2.2, L_q-L_r estimates of the $(N-1)$-dimensional heat kernel, and Parseval's theorem, we have

$$\begin{aligned}&\|[L_n^{+,0}(t;A_0)g](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac12-\frac1q\right)-\frac{s}{2}}\|e^{-(A^2/2)\widetilde{t}}A^{1/2}e^{-Ax_N}\widehat{g}(\xi')\|_2\\&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac1r-\frac1q\right)-\frac{s}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}/(\widetilde{t}^{1/4}+(x_N)^{1/2}),\\&\|[L_2^{+,0}(t;A_0)g](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac1r-\frac1q\right)-\frac{s}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}/(\widetilde{t}^{1/4}+(x_N)^2).\end{aligned}\tag{3.5.13}$$

For $L_2^{+,0}(t;A_0)$, it follows from Lemma 3.2.3 (1) that the required inequality holds. We consider $L_n^{+,0}(t;A_0)$ $(n=1,3)$ below. If $q>2$, then by Lemma 3.2.3 (1) we obtain the required inequality. In the case of $q=2$, we see that by the first inequality of (3.5.13)

$$\begin{aligned}\|L_n^{+,0}(t;A_0)g\|_{L_2(\mathbf{R}^N_+)} &\le C\widetilde{t}^{-\frac{s}{2}}\|e^{-(A^2/2)\widetilde{t}}\widehat{g}(\xi')\|_{L_2(\mathbf{R}^{N-1})}\\ &\le C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{2})-\frac{s}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}\end{aligned}$$

with some positive constant C, which completes the proof.
(3) As mentioned above, we have for $n=1,2$

$$\begin{aligned}&\|[K_n^{+,0}(t;A_0)f]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})}\\ &\quad\le C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\int_0^\infty \left\|e^{-(A^2/2)\widetilde{t}}A^{1/2}e^{-Ay_N}\widehat{f}(y_N)\right\|_2\,dy_N,\\ &\quad\le C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})}\int_0^\infty \|f(\cdot,y_N)\|_{L_r(\mathbf{R}^{N-1})}/(\widetilde{t}^{1/4}+(y_N)^{1/2})\,dy_N\\ &\|[L_1^{+,0}(t;A_0)g]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})}\\ &\quad\le C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\|e^{-(A^2/2)\widetilde{t}}\widehat{g}(\xi')\|_2\le C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})}\|g\|_{L_r(\mathbf{R}^{N-1})},\end{aligned}$$

which, combined with Hölder's inequality, furnishes the required inequalities. □

COROLLARY 3.5.6. *Let $1\le r\le 2\le q\le\infty$, and let $f\in L_r(\mathbf{R}^N_+)^N$ and $g\in L_r(\mathbf{R}^{N-1})$. We use the symbols defined in* (3.5.6) *with $\sigma=0$ and*

$$k_n(\xi',\lambda)=\frac{\kappa_n(\xi',\lambda)}{L(A,B)},\quad l_n(\xi',\lambda)=\frac{m_n(\xi',\lambda)}{L(A,B)}.$$

(1) *Let $\alpha\in\mathbf{N}_0^N$ and suppose that there exist positive constants $A_1\in(0,1)$ and C such that for any $A\in(0,A_1)$*

$$|\kappa_1(\xi',\lambda_\pm)|\le CA,\quad |\kappa_2(\xi',\lambda_\pm)|\le CA^{\frac{5}{4}},\quad |m_1(\xi',\lambda_\pm)|\le CA^{\frac{2}{4}}.$$

Then there exist positive constants $A_0\in(0,A_1)$ and $C=C(N,q,r,\alpha)$ such that for any $t>0$ and $n=1,2$

$$\begin{aligned}&\|D_x^\alpha\nabla K_n^{\pm,0}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)}\le C(t+1)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{4}-\frac{|\alpha|}{2}}\|f\|_{L_r(\mathbf{R}^N_+)},\\ &\|D_x^\alpha\partial_t K_n^{\pm,0}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)}\le C(t+1)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{q})-\frac{|\alpha|}{2}}\|f\|_{L_r(\mathbf{R}^N_+)},\\ &\|D_x^\alpha\nabla L_1^{\pm,0}(t;A_0)g\|_{L_q(\mathbf{R}^N_+)}\le C(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{1}{4}-\frac{|\alpha|}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}.\end{aligned}$$

(2) *Let $k=0,1$, $l=0,1,2$, and $\alpha'\in\mathbf{N}_0^{N-1}$. We suppose that there exist positive constants $A_1\in(0,1)$ and C such that for any $A\in(0,A_1)$*

$$\begin{aligned}&|\kappa_3(\xi',\lambda_\pm)|\le CA^{\frac{6}{4}},\quad |\kappa_4(\xi',\lambda_\pm)|\le CA^{\frac{7}{4}},\quad |\kappa_5(\xi',\lambda_\pm)|\le CA^{\frac{7}{4}},\\ &|\kappa_6(\xi',\lambda_\pm)|\le CA^{\frac{8}{4}},\quad |m_2(\xi',\lambda_\pm)|\le CA,\quad |m_3(\xi',\lambda_\pm)|\le CA^{\frac{5}{4}}.\end{aligned}$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(N, q, r, \alpha')$ such that for any $t > 0$ and $n = 3, 4, 5, 6$

$$\|\partial_t^k D_{x'}^{\alpha'} D_N^l K_n^{\pm,0}(t; A_0) f\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{k}{4}-\frac{|\alpha'|}{2}-\frac{l}{8}} \|f\|_{L_q(\mathbf{R}_+^N)},$$

$$\sum_{n=2}^{3} \|\partial_t^k D_{x'}^{\alpha'} D_N^l L_n^{\pm,0}(t; A_0) g\|_{L_q(\mathbf{R}_+^N)}$$
$$\leq C \begin{cases} (t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{k}{4}-\frac{|\alpha'|}{2}} \|g\|_{L_r(\mathbf{R}^{N-1})} & (l = 0), \\ (t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{8}\left(2-\frac{1}{q}\right)-\frac{k}{4}-\frac{|\alpha'|}{2}-\frac{l}{8}} \|g\|_{L_r(\mathbf{R}^{N-1})} & (l = 1, 2). \end{cases}$$

PROOF. We consider the cases of $K_5^{\pm,0}(t; A_0), K_6^{\pm,0}(t; A_0)$, and $L_3^{\pm,0}(t; A_0)$. The other inequalities can be proved by Lemma 3.5.4 directly. Let $n = 5, 6$ and $t > 0$ below.

By using (3.5.6), we have

$$\partial_t^k D_{x'}^{\alpha'} [K_n^{\pm,0}(t; A_0) f](x)$$
$$= \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\int_{\Gamma_0^\pm} e^{\lambda t} \varphi_0(\xi') \lambda^k (i\xi')^{\alpha'} \frac{\kappa_n(\xi', \lambda)}{L(A, B)} \mathcal{X}_n(x_N, y_N)\, d\lambda\, \widehat{f}(\xi', y_N) \right] (x'),$$

$$\partial_t^k D_{x'}^{\alpha'} [L_3^{\pm,0}(t; A_0) g](x)$$
$$= \mathcal{F}_{\xi'}^{-1} \left[\int_{\Gamma_0^\pm} e^{\lambda t} \varphi_0(\xi') \lambda^k (i\xi')^{\alpha'} \frac{m_3(\xi', \lambda)}{L(A, B)} \mathcal{M}(x_N)\, d\lambda\, \widehat{g}(\xi') \right] (x')$$

for $k = 0, 1$ and any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$. Since it holds that

$$|(\lambda_\pm)^k (i\xi')^{\alpha'} \kappa_5(\xi', \lambda_\pm)| \leq C A^{\frac{7}{4}+\frac{k}{2}+|\alpha'|},$$
$$|(\lambda_\pm)^k (i\xi')^{\alpha'} \kappa_6(\xi', \lambda_\pm)| \leq C A^{\frac{8}{4}+\frac{k}{2}+|\alpha'|},$$
$$|(\lambda_\pm)^k (i\xi')^{\alpha'} m_3(\xi', \lambda_\pm)| \leq C A^{\frac{5}{4}+\frac{k}{2}+|\alpha'|}$$

for any $A \in (0, A_0)$ by choosing some $A_0 \in (0, A_1)$, we have, by Lemma 3.5.4,

$$\|\partial_t^k D_{x'}^{\alpha'} K_n^{\pm,0}(t; A_0) f\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{k}{4}-\frac{|\alpha'|}{2}} \|f\|_{L_r(\mathbf{R}_+^N)},$$
$$\|\partial_t^k D_{x'}^{\alpha'} L_3^{\pm,0}(t; A_0) g\|_{L_q(\mathbf{R}_-^3)} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{k}{4}-\frac{|\alpha'|}{2}} \|g\|_{L_r(\mathbf{R}^{N-1})}$$

for $k = 0, 1$ and any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$ with some positive constant C.

On the other hand, we have, by (1.2.3) and (3.5.6),

$$\partial_t^k D_{x'}^{\alpha'} D_N^l [K_5^{\pm,0}(t) f](x) =$$
$$(-1)^l \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\int_{\Gamma_0^\pm} e^{\lambda t} \varphi_0 \lambda^k (i\xi')^{\alpha'} \frac{(B+A)^{l-1} \kappa_5}{L(A, B)} e^{-B(x_N + y_N)}\, d\lambda\, \widehat{f}(y_N) \right] (x')\, dy_N$$
$$+ (-1)^l \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\int_{\Gamma_0^\pm} e^{\lambda t} \varphi_0 \lambda^k (i\xi')^{\alpha'} \frac{A^l \kappa_5}{L(A, B)} \mathcal{M}(x_N) e^{-B y_N}\, d\lambda\, \widehat{f}(y_N) \right] (x')\, dy_N,$$

$$\partial_t^k D_{x'}^{\alpha'} D_N^l [K_6^{\pm,0}(t; A_0) f](x) =$$
$$(-1)^l \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\int_{\Gamma_0^\pm} e^{\lambda t} \varphi_0 \lambda^k (i\xi')^{\alpha'} \frac{(B+A)^{l-1} \kappa_6}{L(A, B)} e^{-B x_N} \mathcal{M}(y_N)\, d\lambda\, \widehat{f}(y_N) \right] (x')\, dy_N$$

$$+ (-1)^l \int_0^\infty \mathcal{F}^{-1}_{\xi'} \left[\int_{\Gamma_0^\pm} e^{\lambda t} \varphi_0 \lambda^k (i\xi')^{\alpha'} \frac{A^l \kappa_6}{L(A,B)} \mathcal{M}(x_N)\mathcal{M}(y_N)\, d\lambda\, \widehat{f}(y_N) \right] (x')\, dy_N,$$

$$\partial_t^k D_{x'}^{\alpha'} D_N^l [L_3^{\pm,0}(t;A_0)g](x)$$

$$= (-1)^l \mathcal{F}^{-1}_{\xi'} \left[\int_{\Gamma_0^\pm} e^{\lambda t} \varphi_0 \lambda^k (i\xi')^{\alpha'} \frac{(B+A)^{l-1} m_3(\xi',\lambda)}{L(A,B)} e^{-Bx_N}\, d\lambda\, \widehat{g}(\xi') \right] (x')$$

$$+ (-1)^l \mathcal{F}^{-1}_{\xi'} \left[\int_{\Gamma_0^\pm} e^{\lambda t} \varphi_0 \lambda^k (i\xi')^{\alpha'} \frac{A^l m_3(\xi',\lambda)}{L(A,B)} \mathcal{M}(x_N)\, d\lambda\, \widehat{g}(\xi') \right] (x')$$

for $k = 0, 1, l = 1, 2$, and any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$, where we have set

$$\varphi_0 = \varphi_0(\xi'), \quad \kappa_n = \kappa_n(\xi',\lambda) \ (n = 5, 6), \quad \widehat{f}(y_N) = \widehat{f}(\xi', y_N).$$

Since it holds that

$$|(\lambda_\pm)^k (i\xi')^{\alpha'} (B_1^\pm + A)^{l-1} \kappa_5(\xi',\lambda_\pm)| \le CA^{\frac{7}{4}+\frac{k}{2}+|\alpha'|+\frac{l-1}{4}} = CA^{\frac{6}{4}+\frac{k}{2}+|\alpha'|+\frac{l}{4}},$$
$$|(\lambda_\pm)^k (i\xi')^{\alpha'} A^l \kappa_5(\xi',\lambda_\pm)| \le CA^{\frac{7}{4}+\frac{k}{2}+|\alpha'|+l},$$
$$|(\lambda_\pm)^k (i\xi')^{\alpha'} (B_1^\pm + A)^{l-1} \kappa_6(\xi',\lambda_\pm)| \le CA^{\frac{8}{4}+\frac{k}{2}+|\alpha'|+\frac{l-1}{4}} = CA^{\frac{7}{4}+\frac{k}{2}+|\alpha'|+\frac{l}{4}},$$
$$|(\lambda_\pm)^k (i\xi')^{\alpha'} A^l \kappa_6(\xi',\lambda_\pm)| \le CA^{\frac{8}{4}+\frac{k}{2}+|\alpha'|+l},$$
$$|(\lambda_\pm)^k (i\xi')^{\alpha'} (B_1^\pm + A)^{l-1} m_3(\xi',\lambda_\pm)| \le CA^{\frac{5}{4}+\frac{k}{2}+|\alpha'|+\frac{l-1}{4}} = CA^{1+\frac{k}{2}+|\alpha'|+\frac{l}{4}}$$
$$|(\lambda_\pm)^k (i\xi')^{\alpha'} A^l m_3(\xi',\lambda_\pm)| \le CA^{\frac{5}{4}+\frac{k}{2}+|\alpha'|+l}$$

for any $A \in (0, A_0)$ with some positive constant $A_0 \in (0, A_1)$, we have, by using Lemma 3.5.4,

$$\|\partial_t^k D_{x'}^{\alpha'} D_N^l K_5^{\pm,0}(t) f\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-\left(\frac{N-1}{2}+\frac{1}{8}\right)\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{k}{4}-\frac{|\alpha'|}{2}-\frac{l}{8}} \|f\|_{L_r(\mathbf{R}^N_+)}$$
$$\times \left((t+1)^{-\frac{3}{8}-\frac{l}{8}} + (t+1)^{-\frac{3}{8}\left(1-\frac{1}{q}\right)-l}\right)$$
$$\le C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{k}{4}-\frac{|\alpha'|}{2}-\frac{l}{8}} \|f\|_{L_r(\mathbf{R}^N_+)},$$
$$\|\partial_t^k D_{x'}^{\alpha'} D_N^l K_6^{\pm,0}(t) f\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{k}{4}-\frac{|\alpha'|}{2}} \|f\|_{L_r(\mathbf{R}^N_+)}$$
$$\times \left((t+1)^{-\frac{1}{8}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{3}{8r}-\frac{l}{8}} + (t+1)^{-\frac{1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-l}\right)$$
$$\le C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{k}{4}-\frac{|\alpha'|}{2}-\frac{l}{8}} \|f\|_{L_r(\mathbf{R}^N_+)},$$
$$\|\partial_t^k D_{x'}^{\alpha'} D_N^l L_3^{\pm,0}(t;A_0) g\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{k}{4}-\frac{|\alpha'|}{2}} \|g\|_{L_r(\mathbf{R}^{N-1})}$$
$$\times \left((t+1)^{-\frac{1}{8}\left(2-\frac{1}{q}\right)-\frac{l}{8}} + (t+1)^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-l}\right)$$
$$\le C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{8}\left(2-\frac{1}{q}\right)-\frac{k}{4}-\frac{|\alpha'|}{2}-\frac{l}{8}} \|g\|_{L_r(\mathbf{R}^{N-1})}$$

for $k = 0, 1$, $l = 1, 2$, and any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$ with some positive constant C. This completes the proof of the corollary. □

Noting that for some $A_2 \in (0, 1)$ and $C > 0$ there holds $|D(A, B_1^\pm)| \ge CA^{3/4}$ for any $A \in (0, A_2)$, we see that there exist positive constants $A_1 \in (0, A_2)$ and C

such that for any $A \in (0, A_1)$ and $j, k = 1, \dots, N$

$$|\mathcal{V}_{jk}^{BB}(\xi', \lambda_\pm)| \leq CA^{\frac{6}{4}}, \quad |\mathcal{V}_{jk}^{B\mathcal{M}}(\xi', \lambda_\pm)| \leq CA^{\frac{7}{4}}, \quad |\mathcal{V}_{jk}^{\mathcal{M}B}(\xi', \lambda_\pm)| \leq CA^{\frac{7}{4}},$$
$$|\mathcal{V}_{jk}^{\mathcal{M}\mathcal{M}}(\xi', \lambda_\pm)| \leq CA^{\frac{8}{4}}, \quad |\mathcal{P}_{j}^{AA}(\xi', \lambda_\pm)| \leq CA, \quad |\mathcal{P}_{j}^{A\mathcal{M}}(\xi', \lambda_\pm)| \leq CA^{\frac{5}{4}}.$$

Therefore, recalling the formulas (3.4.3), (3.4.4), (3.4.5), (3.5.5) with $\sigma = 0$, we obtain the required estimates of Theorem 3.5.3 (1), (2), and (3) by Corollary 3.5.6. In addition, Theorem 3.5.3 (4) follows from Lemma 3.5.4 (3) directly.

3.5.2. Analysis on $\Gamma_1^\pm$. Our aim here is to show the following theorem for the operators defined as (3.5.5) with $\sigma = 1$.

THEOREM 3.5.7. *Let $1 \leq r \leq 2 \leq q \leq \infty$ and $\mathbf{F} = (\mathbf{f}, g) \in X_r^0$. Then there exists an $A_0 \in (0,1)$ such that the following assertions hold.*

(1) *Let $k = 0, 1$, $l = 0, 1, 2$, and $\alpha' \in \mathbf{N}_0^{N-1}$. Then there exists a positive constant $C = C(N, q, r, \alpha')$ such that for any $t > 0$*

$$\|\partial_t^k D_{x'}^{\alpha'} D_N^l S_0^{\mathbf{f},1}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{2k+|\alpha'|+l}{2}} \|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$
$$\|\partial_t^k D_{x'}^{\alpha'} D_N^l S_0^{g,1}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)}$$
$$\leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{3}{4}-\frac{2k+|\alpha'|+l}{2}} \|g\|_{L_r(\mathbf{R}^{N-1})}.$$

(2) *There exists a positive constant $C = C(N, q, r)$ such that for any $t > 0$*

$$\|\nabla \Pi_0^{\mathbf{f},1}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-1} \|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$
$$\|\nabla \Pi_0^{g,1}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{7}{4}} \|g\|_{L_r(\mathbf{R}^{N-1})}.$$

(3) *Let $\alpha \in \mathbf{N}_0^N$. Then there exists a positive constant $C = C(N, q, r, \alpha)$ such that for any $t > 0$*

$$\|D_x^\alpha \nabla \mathcal{E}(T_0^{\mathbf{f},1}(t; A_0)\mathbf{F})\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-1-\frac{|\alpha|}{2}} \|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$
$$\|D_x^\alpha \partial_t \mathcal{E}(T_0^{\mathbf{f},1}(t; A_0)\mathbf{F})\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{3}{2}-\frac{|\alpha|}{2}} \|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$
$$\|D_x^\alpha \nabla \mathcal{E}(T_0^{g,1}(t; A_0)\mathbf{F})\|_{L_q(\mathbf{R}_+^N)}$$
$$\leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{7}{4}-\frac{|\alpha|}{2}} \|g\|_{L_r(\mathbf{R}^{N-1})}.$$

(4) *There exists a positive constant C such that for any $t > 0$*

$$\|T_0^{\mathbf{f},1}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}^{N-1})} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{3}{4}} \|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$
$$\|T_0^{g,1}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}^{N-1})} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-1} \|g\|_{L_r(\mathbf{R}^{N-1})}.$$

We start with the following lemmas in order to show Theorem 3.5.7.

LEMMA 3.5.8. *Let $f(z) = z^3 + 2z^2 + 12z - 8$. Then $f(z) \neq 0$ for $z \in \{\omega \in \mathbf{C} \mid \mathrm{Re}\,\omega \geq 0\} \setminus (0,1)$.*

PROOF. We note that $f(z)$ has only one real root α because $f(0) = -8$, $f(1) = 7$ and $f'(z) = 3z^2+4z+12 > 0$ for $z \in \mathbf{R}$, and it is clear that α is in $(0,1)$. Let β and $\bar{\beta}$ be the other roots of $f(z)$. Then since $\alpha+\beta+\bar{\beta} = -2$, we have $2\mathrm{Re}\beta = -2-\alpha < 0$. This completes the proof. $\square$

LEMMA 3.5.9. *Let $\lambda \in \Gamma_1^{\pm}$ and $\xi' \in \mathbf{R}^{N-1}$. Then*

$$\frac{A}{4} \leq \mathrm{Re} B \leq |B| \leq \frac{A}{2}, \quad |D(A,B)| \geq CA^3$$

for some positive constant C independent of ξ' and λ. In addition, there exist positive constants $A_1 \in (0,1)$ and C such that $|L(A,B)| \geq CA$ for any $A \in (0, A_1)$.

PROOF. We first show the inequalities for B and $D(A,B)$. Note that

$$B = \sqrt{\lambda + A^2} = \frac{A}{2} e^{\pm i(u/2)},$$

since $\lambda = -A^2 + (A^2/4)e^{\pm iu}$ for $u \in [0, \pi/2]$ on $\Gamma_1^{\pm}$. Therefore, it is clear that the required inequalities of B hold. For $D(A,B)$ inserting the above identity into $D(A,B)$ furnishes that

$$D(A,B) = \frac{A^3}{8}\left((e^{\pm i(u/2)})^3 + 2(e^{\pm i(u/2)})^2 + 12(e^{\pm i(u/2)}) - 8\right),$$

which, combined with Lemma 3.5.8, implies that $|D(A,B)| \geq CA^3$ for some positive constant C independent of ξ' and λ.

Finally we show the last inequality. By (3.5.2)

$$B^2 - (B_1^{\pm})^2 = \mp i c_g^{1/2} A^{1/2} + A^2\left(1 + \frac{e^{\pm iu}}{4}\right) + O(A^{10/4}) \quad \text{as } A \to 0,$$

so that there exist positive constants $A_1 \in (0,1)$ and C such that

$$|B^2 - (B_1^{\pm})^2| \geq CA^{1/2} \quad \text{for any } A \in (0, A_1).$$

On the other hand, we have $|B + B_1^{\pm}| \leq CA^{1/4}$ on $\Gamma_1^{\pm}$ when A is sufficiently small, which, combined with the inequality above, yields that

$$|B - B_1^{\pm}| = \frac{|B^2 - (B_1^{\pm})^2|}{|B + B_1^{\pm}|} \geq CA^{1/4} \quad \text{for any } A \in (0, A_1).$$

Since $|B - B_1^{\pm}| \leq |B - B_2^{\pm}|$ follows from $\mathrm{Re}B \geq 0$ and (3.5.1), we obtain

$$|L(A,B)| = |(B - B_1^+)(B - B_1^-)(B - B_2^+)(B - B_2^-)| \geq CA$$

for any $A \in (0, A_1)$, $\lambda \in \Gamma_1^{\pm}$, and a positive constant C independent of ξ' and λ. □

Next we show some multiplier theorem on $\Gamma_1^{\pm}$.

LEMMA 3.5.10. *Let $1 \leq r \leq 2 \leq q \leq \infty$, and let $f \in L_r(\mathbf{R}_+^N)$ and $g \in L_r(\mathbf{R}^{N-1})$. We use the symbols defined as (3.5.6) with $\sigma = 1$.*

(1) *Let $s \geq 0$ and suppose that there exist positive constants $A_1 \in (0,1)$ and $C = C(s)$ such that for any $\lambda \in \Gamma_1^{\pm}$ and $A \in (0, A_1)$*

$$|k_n(\xi', \lambda)| \leq CA^{-1+s} \quad (n = 1, 3),$$
$$|k_n(\xi', \lambda)| \leq CA^{s} \quad (n = 2, 4, 5),$$
$$|k_6(\xi', \lambda)| \leq CA^{1+s}.$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(N, q, r, s)$ such that for any $t > 0$ and $n = 1, \ldots, 6$

$$\|K_n^{\pm,1}(t; A_0) f\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r} - \frac{1}{q}\right) - \frac{s}{2}} \|f\|_{L_r(\mathbf{R}_+^N)}.$$

(2) *Let $s \geq 0$ and suppose that there exist positive constants $A_1 \in (0,1)$ and $C = C(s)$ such that for any $\lambda \in \Gamma_1^{\pm}$ and $A \in (0, A_1)$*

$$|l_n(\xi', \lambda)| \leq CA^s \quad (n = 1, 2), \quad |l_3(\xi', \lambda)| \leq CA^{1+s}.$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(N, a, r, s)$ such that for any $t > 0$ and $n = 1, 2, 3$

$$\|L_n^{\pm,1}(t; A_0)g\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{3}{4}-\frac{s}{2}}.\|g\|_{L_r(\mathbf{R}^{N-1})}$$

(3) *Suppose that there exist positive constants $A_1 \in (0,1)$ and C such that for any $\lambda \in \Gamma_1^{\pm}$ and $A \in (0, A_1)$*

$$|k_1(\xi', \lambda)| \leq C, \quad |k_2(\xi', \lambda)| \leq CA, \quad |l_1(\xi', \lambda)| \leq C.$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(N, q, r)$ such that for any $t > 0$ and $n = 1, 2$

$$\|[K_n^{\pm,1}(t; A_0)f]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)-\frac{3}{4}}\|f\|_{L_r(\mathbf{R}_+^N)},$$

$$\|[L_1^{\pm,1}(t; A_0)g]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-1}\|g\|_{L_r(\mathbf{R}^{N-1})}.$$

PROOF. We use the abbreviations: $\|\cdot\|_2 = \|\cdot\|_{L_2(\mathbf{R}^{N-1})}$, $\widehat{f}(y_N) = \widehat{f}(\xi', y_N)$, and $\widetilde{t} = t+1$ for $t > 0$ in this proof, and consider only estimates on Γ_1^+ since estimates on Γ_1^- can be shown similarly.

(1) Since $\lambda = -A^2 + (A^2/4)e^{iu}$ with $u \in [0, \pi/2]$ on Γ_1^+, we have

$$[K_n^{+,1}(t; A_0)f](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\int_0^{\frac{\pi}{2}} e^{\left(-A^2+(A^2/4)e^{iu}\right)t}\varphi_0(\xi')\right.$$
$$\left.\times k_1(\xi', \lambda)\mathcal{X}_n(x_N, y_N)\left(i\frac{A^2}{4}e^{iu}\right)du\widehat{f}(y_N)\right](x')\,dy_N.$$

Noting that $|e^{\left(-A^2+(A^2/4)e^{iu}\right)t}| \leq Ce^{-(3/4)A^2\widetilde{t}}$ for some positive constant C independent of ξ', u, and t, we see that by Lemma 3.2.2, L_q-L_r estimates of the $(N-1)$-dimensional heat kernel, and Parseval's theorem

$$\begin{aligned}
&\|[K_{0,1}^{+,1}(t; A_0)f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\int_0^\infty\int_0^{\frac{\pi}{2}}\left\|e^{-(A^2/2)\widetilde{t}}A^{1+s}e^{-A(x_N+y_N)}\widehat{f}(y_N)\right\|_2 du dy_N \\
&\leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{s}{2}}\int_0^\infty\left\|e^{-(A^2/4)\widetilde{t}}Ae^{-A(x_N+y_N)}\widehat{f}(y_N)\right\|_2 dy_N
\end{aligned}$$

with some positive constant C, and besides, for $n = 2, \ldots, 6$

$$\begin{aligned}
&\|[K_n^{+,1}(t; A_0)f](\cdot, x_N)\|_{L_q(\mathbf{R}_+^N)} \\
&\leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{s}{2}}\int_0^\infty\left\|e^{-(A^2/4)\widetilde{t}}Ae^{-C(x_N+y_N)}\widehat{f}(y_N)\right\|_2 dy_N
\end{aligned}$$

similarly, where we have used Lemma 3.5.9 and the fact that for $a > 0$ and $\lambda \in \Gamma_1^{\pm}$

$$(3.5.14) \qquad |\mathcal{M}(a)| \leq a\int_0^1 |e^{-(B\theta+A(1-\theta))y_N}|\,d\theta \leq ae^{-(A/4)a} \leq 8A^{-1}e^{-(A/8)a}.$$

We thus obtain the required inequalities concerning $K_n^{+,1}(t; A_0)$ $(n = 1, \ldots, 6)$ by using Lemma 3.5.5.

(2) Since $\lambda = -A^2 + (A^2/4)e^{iu}$ for $u \in [0, \pi/2]$ on Γ_1^+, we have

$$[L_n^{+,1}(t; A_0)g](x) = \mathcal{F}_{\xi'}^{-1}\left[\int_0^{\frac{\pi}{2}} e^{\left(-A^2+(A^2/4)e^{iu}\right)t}\varphi_0(\xi') \times l_1(\xi',\lambda)\mathcal{Y}_n(x_N)\left(i\frac{A^2}{4}e^{iu}\right) du\,\widehat{g}(\xi')\right](x').$$

By calculations similar to (1) and Lemma 3.2.2, we have

$$\begin{aligned}
&\|[L_1^{+,1}(t; A_0)g](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad \leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2}-\frac{s}{2}}\left\|e^{-(A^2/2)\widetilde{t}}Ae^{-Ax_N}\widehat{g}(\xi')\right\|_2 \\
&\quad \leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}-\frac{s}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}/(\widetilde{t}^{1/2}+x_N)
\end{aligned}$$

with some positive constant C, and also for $n = 2, 3$ we have, by Lemma 3.5.9 and (3.5.14),

$$\|[L_n^{+,1}(t; A_0)g](\cdot, x_N)\|_{L_q(\mathbf{R}_+^N)} \leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}-\frac{s}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}/(\widetilde{t}^{1/2}+x_N).$$

We thus obtain the required inequalities for $L_n^{+,1}(t; A_0)$ $(n = 1, 2, 3)$ by using Lemma 3.2.3 (1).

(3) As stated above, for $n = 1, 2$, we have, by (3.5.14),

$$\begin{aligned}
&\|[K_n^{+,1}(t; A_0)f]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad \leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\int_0^\infty \left\|e^{-(A^2/2)\widetilde{t}}A^2e^{-CAy_N}\widehat{f}(y_N)\right\|_2 dy_N \\
&\quad \leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\int_0^\infty \frac{\|f(\cdot, y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\widetilde{t}+(y_N)^2}dy_N, \\
&\|[L_1^{+,1}(t; A_0)g]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad \leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|e^{-(A^2/2)\widetilde{t}}A^2\,\widehat{g}(\xi')\|_2 \leq C\widetilde{t}^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-1}\|g\|_{L_r(\mathbf{R}^{N-1})}
\end{aligned}$$

with some positive constant C, which, combined with Hölder's inequality, furnishes that the required inequalities hold. □

By Lemma 3.5.9 we see that there exist positive constants $A_1 \in (0, 1)$ and C such that for any $\lambda \in \Gamma_1^{\pm}$, $A \in (0, A_1)$, and $j, k = 1, \ldots, N$

$$\begin{aligned}
&|\mathcal{V}_{jk}^{BB}(\xi',\lambda)/L(A,B)| \leq CA^{-1}, \quad &&|\mathcal{V}_{jk}^{B\mathcal{M}}(\xi',\lambda)/L(A,B)| \leq C, \\
&|\mathcal{V}_{jk}^{\mathcal{M}B}(\xi',\lambda)/L(A,B)| \leq C, &&|\mathcal{V}_{jk}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)/L(A,B)| \leq CA, \\
&|\mathcal{P}_j^{AA}(\xi',\lambda)/L(A,B)| \leq C, &&|\mathcal{P}_j(\xi',\lambda)/L(A,B)| \leq CA.
\end{aligned}$$

Therefore, recalling the formulas: (3.4.3), (3.4.4), (3.4.5), (3.5.5) with $\sigma = 1$ and using (1.2.3), we obtain the required inequalities of Theorem 3.5.7 by Lemma 3.5.9 and Lemma 3.5.10.

3.5.3. Analysis on $\Gamma_2^{\pm}$.

Our aim here is to show th following theorem for the operators defined as (3.5.5) with $\sigma = 2$.

THEOREM 3.5.11. *Let $1 \leq r \leq 2 \leq q \leq \infty$ and $\mathbf{F} = (\mathbf{f}, g) \in X_r^0$. Then there exists an $A_0 \in (0, 1)$ such that the following assertions hold.*

(1) *Let $k = 0, 1$, $l = 0, 1, 2$, and $\alpha' \in \mathbf{N}_0^{N-1}$. Then there exists a positive constant $C = C(N, q, r, \alpha')$ such that for any $t > 0$*

$$\|\partial_t^k D_{x'}^{\alpha'} D_N^l S_0^{\mathbf{f},2}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-k-\frac{|\alpha'|+l}{2}}\|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$

$$\|\partial_t^k D_{x'}^{\alpha'} D_N^l S_0^{g,2}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-k-\frac{|\alpha'|+l}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})},$$

provided that $k + l + |\alpha'| \neq 0$. In addition, if $(q, r) \neq (2, 2)$, then

$$\|S_0^{\mathbf{f},2}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$

$$\|S_0^{g,2}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|g\|_{L_r(\mathbf{R}^{N-1})}.$$

(2) *There exists a positive constant $C = C(N, q, r)$ such that for any $t > 0$*

$$\|\nabla \Pi_0^{\mathbf{f},2}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{4}}\|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$

$$\|\nabla \Pi_0^{g,2}(t; A_0)\mathbf{F}\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-1}\|g\|_{L_r(\mathbf{R}^{N-1})}.$$

(3) *Let $\alpha \in \mathbf{N}_0^N$. Then there exists a positive constant $C = C(N, q, r, \alpha)$ such that for any $t > 0$*

$$\|D_x^\alpha \nabla \mathcal{E}(T_0^{\mathbf{f},2}(t)\mathbf{F})\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{4}-\frac{|\alpha|}{2}}\|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)},$$

$$\|D_x^\alpha \partial_t \mathcal{E}(T_0^{\mathbf{f},2}(t)\mathbf{F})\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}}\|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)} \quad \text{if } |\alpha| \neq 0,$$

$$\|D_x^\alpha \nabla \mathcal{E}(T_0^{g,2}(t)\mathbf{F})\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-1-\frac{|\alpha|}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}.$$

In addition, if $(q, r) \neq (2, 2)$, then

$$\|\partial_t \mathcal{E}(T_0^{\mathbf{f},2}(t; A_0)\mathbf{F})\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)}.$$

(4) *There exists a positive constant $C = C(N, q, r)$ such that for any $t > 0$*

$$\|T_0^{\mathbf{f},2}(t; A_0)\mathbf{f}\|_{L_q(\mathbf{R}^{N-1})} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}\|\mathbf{f}\|_{L_r(\mathbf{R}_+^N)} \quad (r \neq 2),$$

$$\|T_0^{g,2}(t; A_0)g\|_{L_q(\mathbf{R}^{N-1})} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\|g\|_{L_r(\mathbf{R}^{N-1})} \quad (r \neq 2).$$

We start with the following lemma in order to show Theorem 3.5.11.

LEMMA 3.5.12. *There exist positive constants $A_1 \in (0, 1)$, $b_0 \geq 1$, and C such that for any $\lambda \in \Gamma_2^\pm$ and $A \in (0, A_1)$*

$$b_0^{-1}(A\sqrt{1-u} + \sqrt{u} + A) \leq \mathrm{Re}B \leq |B| \leq b_0(A\sqrt{1-u} + \sqrt{u} + A),$$

$$|D(A, B)| \geq C(A\sqrt{1-u} + \sqrt{u} + A)^3,$$

$$|L(A, B)| \geq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^4.$$

PROOF. We first show the inequalities for B. Set $\sigma = \lambda + A^2$ and $\theta = \arg \sigma$. Noting that

$$\lambda = -(A^2(1-u) + \gamma_0 u) \pm i((A^2/4)(1-u) + \widetilde{\gamma}_0 u)$$

for $u \in [0, 1]$ on $\Gamma_2^\pm$, we have

$$\begin{aligned}|\sigma| + A^2(1-u) + \gamma_0 u - A^2 &\leq 2(A^2(1-u) + \gamma_0 u + A^2) + \frac{A^2}{4}(1-u) + \widetilde{\gamma}_0 u \\ &\leq 3\max\{\gamma_0, \widetilde{\gamma}_0\}(A^2(1-u) + u + A^2) \leq 3\max\{\gamma_0, \widetilde{\gamma}_0\}(A\sqrt{1-u} + \sqrt{u} + A)^2,\end{aligned}$$

which is used to obtain

$$\begin{aligned}
\mathrm{Re}B &= |\sigma|^{1/2}\cos\frac{\theta}{2} = \frac{|\sigma|^{1/2}}{\sqrt{2}}(1+\cos\theta)^{1/2} = \frac{1}{\sqrt{2}}\left(\frac{|\sigma|^2-(\mathrm{Re}\sigma)^2}{|\sigma|-\mathrm{Re}\sigma}\right)^{1/2} \\
&= \frac{(A^2/4)(1-u)+\gamma_0 u+(A^2/8)-(A^2/8)(1-u)-(A^2/8)u}{\sqrt{2}(|\sigma|+A^2(1-u)+\gamma_0 u-A^2)^{1/2}} \\
&\geq \frac{(A^2/8)(1-u)+\gamma_0 u+(A^2/8)-(A_1^2/8)u}{\sqrt{6}\max\{\gamma_0^{1/2},\widetilde{\gamma}_0^{1/2}\}(A\sqrt{1-u}+\sqrt{u}+A)} \\
&\geq \frac{(1/8)\{A^2(1-u)+\gamma_0 u+A^2\}}{\sqrt{6}\max\{\gamma_0^{1/2},\widetilde{\gamma}_0^{1/2}\}(A\sqrt{1-u}+\sqrt{u}+A)} \geq \frac{A\sqrt{1-u}+u+A}{24\sqrt{6}\max\{\gamma_0^{1/2},\widetilde{\gamma}_0^{1/2}\}}
\end{aligned}$$

for any $A \in (0, A_1)$, provided that $A_1^2 \leq 7\gamma_0$. It is clear that the other inequalities concerning B hold.

Next we consider $D(A,B)$. Noting that $\lambda \in \Gamma_2^{\pm} \subset \Sigma_{\varepsilon_0}$ and using Lemma 1.2.6 (1), we obtain

$$|D(A,B)| \geq C(\varepsilon_0)(|\lambda|^{\frac{1}{2}}+A)^3 \geq C(\varepsilon_0)(A\sqrt{1-u}+\sqrt{u}+A)^3.$$

Finally we show the inequality for $L(A,B)$. By (3.5.2)

$$\begin{aligned}
&B^2-(B_1^{\pm})^2 \\
&= -(A^2(1-u)+\gamma_0 u) \pm i\left(\frac{A^2}{4}(1-u)+\widetilde{\gamma}_0 u - c_g^{1/2}A^{1/2}\right)+2A^2+O(A^{10/4})
\end{aligned}$$

as $A \to 0$, and also we have

$$\begin{aligned}
&\left|-(A^2(1-u)+\gamma_0 u) \pm i\left(\frac{A^2}{4}(1-u)+\widetilde{\gamma}_0 u - c_g^{1/2}A^{1/2}\right)\right|^2 \\
&= (A^2(1-u)+\gamma_0 u)^2 + \left(\frac{A^2}{4}(1-u)+\widetilde{\gamma}_0 u\right)^2 \\
&\quad + c_g A - 2c_g^{1/2}A^{1/2}\left(\frac{A^2}{4}(1-u)+\widetilde{\gamma}_0 u\right) \\
&\geq \left(A^2(1-u)+\frac{\widetilde{\gamma}_0}{3}u\right)^2 + \frac{1}{11}c_g A - \frac{1}{10}\left(\frac{A^2}{4}(1-u)+\widetilde{\gamma}_0 u\right)^2 \\
&\geq \frac{1}{90}(A^2(1-u)+\widetilde{\gamma}_0 u)^2 + \frac{1}{11}c_g A \geq C(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^4.
\end{aligned}$$

We thus see that, by the inequality for B obtained above, there exist positive constants $A_1 \in (0,1)$ and C such that for any $A \in (0, A_1)$ and $\lambda \in \Gamma_2^{\pm}$

$$\begin{aligned}
|B-B_1^{\pm}| &= \frac{|B^2-(B_1^{\pm})^2|}{|B+B_1^{\pm}|} \\
&\geq \frac{C\left(A\sqrt{1-u}+\sqrt{u}+A^{1/4}\right)^2}{b_0(A\sqrt{1-u}+\sqrt{u}+A)+c_g^{1/4}A^{1/4}} \geq C(A\sqrt{1-u}+\sqrt{u}+A^{1/4}).
\end{aligned}$$

Since $|B-B_1^{\pm}| \leq |B-B_2^{\pm}|$ follows from $\mathrm{Re}B \geq 0$ and (3.5.1), we have the required inequality of $L(A,B)$, which completes the proof of the lemma. □

LEMMA 3.5.13. *Let* $1 \leq r \leq 2 \leq q \leq \infty$, *and let* $f \in L_r(\mathbf{R}_+^N)$ *and* $g \in L_r(\mathbf{R}^{N-1})$. *We use the symbols defined as* (3.5.6) *with* $\sigma = 2$.

(1) *Let $s \geq 0$ and suppose that there exist positive constants $A_1 \in (0,1)$ and $C = C(s)$ such that for any $\lambda \in \Gamma_2^{\pm}$ and $A \in (0, A_1)$*

$$\begin{aligned}
&|k_1(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A)^{-2} A|B|^s \quad (n = 1, 3),\\
&|k_2(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A)^{-2} A^2|B|^s,\\
&|k_n(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A)^{-1} A|B|^s \quad (n = 4, 5),\\
&|k_6(\xi', \lambda)| \leq CA|B|^s.
\end{aligned}$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(N, q, r, s)$ such that for any $t > 0$ and $n = 1, \dots, 6$

$$\|K_n^{\pm,2}(t; A_0)f\|_{L_q(\mathbf{R}^N_+)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{s}{2}}\|f\|_{L_r(\mathbf{R}^N_+)},$$

provided that $s > 0$. In the case of $s = 0$,

$$\|K_n^{\pm,2}(t; A_0)f\|_{L_q(\mathbf{R}^N_+)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\|f\|_{L_r(\mathbf{R}^N_+)} \quad \text{if } (q, r) \neq (2, 2).$$

(2) *Let $s \geq 0$ and suppose that there exist positive constants $A_1 \in (0,1)$ and $C = C(s)$ such that for any $\lambda \in \Gamma_2^{\pm}$ and $A \in (0, A_1)$*

$$\begin{aligned}
&|l_n(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A|B|^s \quad (n = 1, 2),\\
&|l_3(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-3} A|B|^s.
\end{aligned}$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(N, q, r, s)$ such that for any $t > 0$ and $n = 1, 3$

$$\begin{aligned}
&\|L_n^{\pm,2}(t; A_0)g\|_{L_q(\mathbf{R}^N_+)}\\
&\quad \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{s}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})} \quad (s > 0),\\
&\|L_2^{\pm,2}(t; A_0)g\|_{L_q(\mathbf{R}^N_+)}\\
&\quad \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{3}{4}-\frac{s}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})} \quad (s \geq 0).
\end{aligned}$$

For $n = 1, 3$, in the case of $s = 0$,

$$\|L_n^{\pm,2}(t; A_0)g\|_{L_q(\mathbf{R}^N_+)} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|g\|_{L_r(\mathbf{R}^{N-1})},$$

provided that $(q, r) \neq (2, 2)$.

(3) *Suppose that there exist positive constants $A_1 \in (0,1)$ and C such that for any $\lambda \in \Gamma_2^{\pm}$ and $A \in (0, A_1)$*

$$\begin{aligned}
&|k_1(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A,\\
&|k_2(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A^2,\\
&|l_1(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-2}.
\end{aligned}$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(N, q, r)$ such that for any $t > 0$, $n = 1, 2$, and $r \neq 2$

$$\begin{aligned}
&\|[K_n^{+,2}(t; A_0)f]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}\|f\|_{L_r(\mathbf{R}^N_+)},\\
&\|[L_1^{+,2}(t; A_0)g]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\|g\|_{L_r(\mathbf{R}^{N-1})}.
\end{aligned}$$

PROOF. We often use the abbreviations: $\|\cdot\|_2 = \|\cdot\|_{L_2(\mathbf{R}^{N-1})}$, $\widehat{f}(y_N) = \widehat{f}(\xi', y_N)$, $\varphi_0 = \varphi_0(\xi')$, and $\widetilde{t} = t+1$ for $t > 0$ in this proof, and consider only estimates on Γ_3^+ since estimates on Γ_3^- can be shown similarly.

(1) We first show the inequality for $K_1^{+,2}(t; A_0)$. Recalling that on Γ_2^+

$$\lambda = -(A^2(1-u) + \gamma_0 u) + i((A^2/4)(1-u) + \widetilde{\gamma}_0 u) \quad \text{for } u \in [0,1], \tag{3.5.15}$$

we have by (3.5.6)

$$[K_1^{+,2}(t; A_0)f](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\int_0^1 e^{\{-(A^2(1-u)+\gamma_0 u)+i((A^2/4)(1-u)+\widetilde{\gamma}_0 u)\}t}\varphi_0 \times k_1(\xi',\lambda)e^{-A(x_N+y_N)}\left\{-(\gamma_0 - A^2) + i\left(\widetilde{\gamma}_0 - \frac{A^2}{4}\right)\right\} du\,\widehat{f}(y_N)\right](x')\,dy_N.$$

Since it follows from Lemma 3.5.12 that

$$|e^{\{-(A^2(1-u)+\gamma_0 u)\pm i((A^2/4)(1-u)+\widetilde{\gamma}_0 u)\}t}| \leq e^{-\frac{3}{4}A^2 t}e^{-\frac{1}{4}(A^2(1-u)+\gamma_0 u)t} \leq Ce^{-\frac{3}{4}A^2\widetilde{t}}e^{-C|B|^2\widetilde{t}} \tag{3.5.16}$$

with some positive constant C, independent of ξ', λ, and t, for any $A \in (0, A_0)$ by choosing a suitable $A_0 \in (0, A_1)$, we have, by Lemma 3.5.12, L_q-L_r estimates of the $(N-1)$-dimensional heat kernel, and Parseval's theorem,

(3.5.17)

$$\begin{aligned}
&\|[K_1^{+,2}(t; A_0)f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\int_0^\infty \left\|\int_0^1 \frac{e^{-(A^2/2)\widetilde{t}}e^{-C|B|^2\widetilde{t}}A|B|^s e^{-A(x_N+y_N)}\varphi_0}{(A\sqrt{1-u}+\sqrt{u}+A)^2}\,du\,\widehat{f}(y_N)\right\|_2 dy_N \\
&\leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\int_0^\infty \left\|\int_0^1 \frac{e^{-C|B|^2\widetilde{t}}|B|^{s-\delta}\varphi_0}{(\sqrt{u})^{2-\delta}}\,du\,e^{-\frac{A^2}{2}\widetilde{t}}Ae^{-A(x_N+y_N)}\widehat{f}(y_N)\right\|_2 dy_N
\end{aligned}$$

for a sufficiently small $\delta > 0$. If $s > 0$, then by Lemma 3.5.12 we have

$$\int_0^1 \frac{e^{-C|B|^2\widetilde{t}}|B|^{s-\delta}\varphi_0}{(\sqrt{u})^{2-\delta}}\,du \leq C\widetilde{t}^{-\frac{s-\delta}{2}}\int_0^1 \frac{e^{-Cu\widetilde{t}}}{(\sqrt{u})^{2-\delta}}\,du \leq C\widetilde{t}^{-\frac{s}{2}} \tag{3.5.18}$$

for a positive constant C. We thus obtain

$$\begin{aligned}
&\|[K_1^{+,2}(t; A_0)f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{s}{2}}\int_0^\infty \left\|e^{-(A^2/2)\widetilde{t}}Ae^{-A(x_N+y_N)}\widehat{f}(y_N)\right\|_2 dy_N,
\end{aligned}$$

which, combined with Lemma 3.5.5, furnishes the required inequality. In the case of $s = 0$, by Lemma 3.2.2 and (3.5.17)

$$\begin{aligned}
&\|[K_1^{+,2}(t; A_0)f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\int_0^\infty \left\|\int_0^1 \frac{e^{-Cu\widetilde{t}}}{(\sqrt{u})^{2-\delta}}\,du\,e^{-(A^2/2)\widetilde{t}}A^{1-\delta}e^{-A(x_N+y_N)}\widehat{f}(y_N)\right\|_2 dy_N \\
&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{\delta}{2}}\int_0^\infty \frac{\|f(\cdot, y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\widetilde{t}^{(1-\delta)/2} + (x_N)^{1-\delta} + (y_N)^{1-\delta}}\,dy_N,
\end{aligned}$$

which furnishes that the required inequality holds by Lemma 3.2.3 (2) and choosing a sufficiently small $\delta > 0$ when $(q, r) \neq (2, 2)$. Concerning $K_2^{+,2}(t; A_0)$, we can show the estimate in a similar way to the case of $K_1^{+,2}(t; A_0)$, noting that by (1.2.3) and Lemma 3.5.12

$$|\mathcal{M}(a)| \leq a \int_0^1 e^{-\{(\mathrm{Re}B)\theta + A(1-\theta)\}a}\, d\theta \leq a e^{-b_0^{-1}Aa} \leq 2b_0 A^{-1} e^{-(b_0^{-1}/2)Aa} \tag{3.5.19}$$

for $a > 0$ and any $A \in (0, A_0)$ by choosing a suitable $A_0 \in (0, A_1)$.

Next we show the inequalities for $K_n^{+,2}(t; A_0)$ with $n = 3, 4, 5$. Since $|\lambda| \geq C|B|^2$ on $\Gamma_2^{\pm}$ with some positive constant C by Lemma 3.5.12, we have

$$|\mathcal{M}(a)| = \frac{|e^{-Ba} - e^{-Aa}|}{|B - A|} \leq e^{-CAa}\frac{|B + A|}{|\lambda|} \leq C\frac{e^{-CAa}}{|B|} \leq C\frac{e^{-CAa}}{\sqrt{u}} \tag{3.5.20}$$

with $a > 0$ and some positive constant C for any $A \in (0, A_0)$ by choosing a suitable $A_0 \in (0, A_1)$. Thus, by (3.5.16) and Lemma 3.5.12,

$$\begin{aligned}
&\|[K_n^{+,2}(t)f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\leq C\tilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})} \int_0^\infty \left\| \int_0^1 \frac{e^{-(A^2/2)\tilde{t}} e^{-C|B|^2\tilde{t}} A|B|^s e^{-CA(x_N+y_N)}\varphi_0}{(A\sqrt{1-u} + \sqrt{u} + A)\sqrt{u}}\, du \widehat{f}(y_N) \right\|_2 dy_N,
\end{aligned}$$

which furnishes that the required inequalities of $K_n^{+,2}(t; A_0)$ $(n = 3, 4, 5)$ hold in the same manner as we have obtained the estimate of $K_1^{+,2}(t; A_0)$ from (3.5.17).

Finally, we consider $K_6^{+,2}(t; A_0)f$. By (3.5.18) and (3.5.20), we have, for $s > 0$,

$$\begin{aligned}
&\|[K_6^{+,2}(t; A_0)f](\cdot\,, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\leq C\tilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})} \int_0^\infty \left\| \int_0^1 e^{-\frac{A^2}{2}\tilde{t}} e^{-C|B|^2\tilde{t}} \varphi_0 A|B|^s \mathcal{M}(x_N)\mathcal{M}(y_N)\, du\, \widehat{f}(y_N) \right\|_2 dy_N \\
&\leq C\tilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})} \int_0^\infty \left\| \int_0^1 e^{-\frac{A^2}{2}\tilde{t}} e^{-C|B|^2\tilde{t}} \varphi_0 A|B|^{s-2} e^{-CA(x_N+y_N)}\, du\, \widehat{f}(y_N) \right\|_2 dy_N \\
&\leq C\tilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})} \int_0^\infty \left\| \int_0^1 \frac{e^{-C|B|^2\tilde{t}}|B|^{s-\delta}\varphi_0}{(\sqrt{u})^{2-\delta}}\, du\, e^{-\frac{A^2}{2}\tilde{t}} A e^{-CA(x_N+y_N)}\, \widehat{f}(y_N) \right\|_2 dy_N \\
&\leq C\tilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{s}{2}} \int_0^\infty \left\| e^{-\frac{A^2}{2}\tilde{t}} A e^{-CA(x_N+y_N)}\, du\, \widehat{f}(y_N) \right\|_2 dy_N
\end{aligned}$$

for a positive constant C by choosing a sufficiently small $\delta > 0$. We thus obtain the required inequality of $K_6^{+,2}(t; A_0)$ by Lemma 3.5.5 if $s > 0$. In the case of $s = 0$, since it follows that by (1.2.3) and Lemma 3.5.12

$$\begin{aligned}
|\mathcal{M}(a)| &\leq a \int_0^1 e^{-\{(\mathrm{Re}B)\theta + A(1-\theta)\}a}\, d\theta \leq a \int_0^1 e^{-\{(b_0^{-1}(\sqrt{u}+A))\theta + A(1-\theta)\}a}\, d\theta \\
&\leq a e^{-b_0^{-1}Aa} \int_0^1 e^{-b_0^{-1}\sqrt{u}\theta a}\, d\theta \quad (a > 0,\ \lambda \in \Gamma_2^+)
\end{aligned}$$

for any $A \in (0, A_0)$ by choosing an $A_0 \in (0, A_1)$, we obtain easily, by Lemma 3.2.2,

$$\begin{aligned}
&\|[K_6^{+,2}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\le C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\int_0^\infty \left\|\int_0^1 e^{-(A^2/2)\widetilde{t}}\varphi_0 A\mathcal{M}(x_N)\mathcal{M}(y_N)\,du\,\widehat{f}(y_N)\right\|_2 dy_N \\
&\le C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\int_0^\infty x_N y_N \left\| e^{-(A^2/2)\widetilde{t}}Ae^{-CA(x_N+y_N)}\,\widehat{f}(y_N)\right\|_2 \\
&\quad\times \iiint_{[0,1]^3} e^{-Cu\widetilde{t}}e^{-C\sqrt{u}\varphi x_N}e^{-C\sqrt{u}\psi y_N}\,du d\varphi d\psi dy_N \\
&\le C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})}\int_0^\infty \frac{x_N y_N\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\widetilde{t}^{1/2}+x_N+y_N}\iint_{[0,1]^2}\frac{d\varphi d\psi dy_N}{\widetilde{t}+(\varphi x_N)^2+(\psi y_N)^2}
\end{aligned}$$

with some positive constant C. The change of variable: $\psi y_N = \{\widetilde{t}+(\varphi x_N)^2\}^{1/2}l$ yields that

$$\begin{aligned}
\int_0^1 \frac{d\psi}{\widetilde{t}+(\varphi x_N)^2+(\psi y_N)^2} &\le \frac{1}{\widetilde{t}+(\varphi x_N)^2}\int_0^\infty \frac{1}{1+l^2}\frac{\{\widetilde{t}+(\varphi x_N)^2\}^{1/2}}{y_N}\,dl \\
&\le \frac{C}{y_N(\widetilde{t}^{1/2}+\varphi x_N)}
\end{aligned}$$

for a positive constant C, so that

$$\begin{aligned}
&\|[K_6^{+,2}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\le C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})}\int_0^\infty \frac{x_N\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\widetilde{t}^{1/2}+x_N+y_N}\int_0^1\frac{d\varphi}{\widetilde{t}^{1/2}+\varphi x_N}\,dy_N \\
&= C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})}\int_0^\infty \frac{x_N\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{(\widetilde{t}^{1/2}+x_N+y_N)^{1-\delta}}\int_0^1\frac{d\varphi dy_N}{(\widetilde{t}^{1/2}+x_N+y_N)^{\delta}(\widetilde{t}^{1/2}+\varphi x_N)}
\end{aligned}$$

for any $0<\delta<1$. We then see that by the change of variable: $\varphi x_N = \widetilde{t}^{1/2}l$

$$\begin{aligned}
&\int_0^1 \frac{d\varphi}{(\widetilde{t}^{1/2}+x_N+y_N)^{\delta}(\widetilde{t}^{1/2}+\varphi x_N)} \le \int_0^1 \frac{d\varphi}{(\widetilde{t}^{1/2}+\varphi x_N)^{\delta}(\widetilde{t}^{1/2}+\varphi x_N)} \\
&\quad\le C\int_0^1\frac{d\varphi}{\widetilde{t}^{(1+\delta)/2}+(\varphi x_N)^{1+\delta}} \le \frac{C}{\widetilde{t}^{(1+\delta)/2}}\int_0^\infty \frac{1}{1+l^{1+\delta}}\frac{\widetilde{t}^{1/2}}{x_N}\,dl \le \frac{C}{x_N\widetilde{t}^{\delta/2}}
\end{aligned}$$

with a positive constant C, which furnishes that

$$\begin{aligned}
&\|[K_6^{+,2}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad\le C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{\delta}{2}}\int_0^\infty \frac{\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\widetilde{t}^{(1-\delta)/2}+x_N^{1-\delta}+y_N^{1-\delta}}\,dy_N.
\end{aligned}$$

Together with Lemma 3.2.3 (2), we obtain the required inequality by choosing a sufficiently $\delta>0$ if $(q,r)\ne(2,2)$.

(2) We first show the inequality for $L_1^{+,2}(t;A_0)$. By (3.5.6) and (3.5.15), we have

$$\begin{aligned}
[L_1^{+,2}(t;A_0)g](x) = \mathcal{F}_{\xi'}^{-1}\Bigg[\int_0^1 & e^{\{-(A^2(1-u)+\gamma_0 u)+i((A^2/4)(1-u)+\widetilde{\gamma}_0 u)\}t}\varphi_0 \\
&\times l_1(\xi',\lambda)e^{-A(x_N+y_N)}\left\{-(\gamma_0-A^2)\pm i\left(\widetilde{\gamma}_0-\frac{A^2}{4}\right)\right\}\widehat{g}(\xi')\Bigg](x').
\end{aligned}$$

In a similar way to the case of $K_1^{+,2}(t;A_0)$, we have by (3.5.18) and Lemma 3.2.2 (3.5.21)

$$\begin{aligned}
&\|[L_1^{\pm,2}(t;A_0)g](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\le C\tilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\left\|\int_0^1\frac{e^{-(A^2/2)\tilde{t}}e^{-C|B|^2\tilde{t}}\varphi_0A^{1/2}|B|^se^{-Ax_N}}{(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^2}\,du\,\widehat{g}(\xi')\right\|_2\\
&\quad\le C\tilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\left\|\int_0^1\frac{e^{-C|B|^2\tilde{t}}\varphi_0|B|^{s-\delta}}{(\sqrt{u})^{2-\delta}}\,du\,e^{-(A^2/2)\tilde{t}}A^{1/2}\,e^{-Ax_N}\,\widehat{g}(\xi')\right\|_2\\
&\quad\le C\tilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{s}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}/(\tilde{t}^{1/4-\delta/8}+(x_N)^{1/2-\delta/4})
\end{aligned}$$

for a sufficiently small $\delta>0$ with some positive constant C, provided that $s>0$. We thus obtain the required inequality by Lemma 3.2.3 (1) when $s>0$ and $q>2$. In the case of $s>0$ and $q=2$, by (3.5.18) and (3.5.21), we have

$$\begin{aligned}
\|L_1^{+,2}(t;A_0)g\|_{L_2(\mathbf{R}^N_+)}&\le C\left\|\int_0^1\frac{e^{-C|B|^2\tilde{t}}\varphi_0|B|^{s-\delta}}{(\sqrt{u})^{2-\delta}}\,du\,e^{-(A^2/2)\tilde{t}}\widehat{g}(\xi')\right\|_2\\
&\le C\tilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{2})-\frac{s}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}.
\end{aligned}$$

If $s=0$, then we have, by Lemma 3.2.2, Lemma 3.5.12, and (3.5.21),

$$\begin{aligned}
&\|[L_1^{+,2}(t;A_0)g](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\le C\tilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\left\|\int_0^1\frac{e^{-(A^2/2)\tilde{t}}e^{-C|B|^2\tilde{t}}\varphi_0A^{1/2}e^{-Ax_N}}{(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^2}\,du\,\widehat{g}(\xi')\right\|_2\\
&\quad\le C\tilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\left\|\int_0^1\frac{e^{-Cu\tilde{t}}}{(\sqrt{u})^{2-\delta}}\,du\,e^{-(A^2/2)\tilde{t}}A^{1/2-\delta/4}e^{-Ax_N}\widehat{g}(\xi')\right\|_2\\
&\quad\le C\tilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{\delta}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}/(\tilde{t}^{1/4-\delta/8}+(x_N)^{1/2-\delta/4})
\end{aligned}\tag{3.5.22}$$

which, combined with Lemma 3.2.2, furnishes that the required inequality holds for $q>2$ by choosing a sufficiently small $\delta>0$. In the case of $s=0$ and $q=2$, by (3.5.22) and Young's inequality with $1+1/2=1/p+1/r$ for $1\le r<2$, we have

$$\begin{aligned}
\|L_1^{+,2}(t;A_0)g\|_{L_2(\mathbf{R}^N_+)}&\le C\tilde{t}^{-\delta/2}\|e^{-(A^2/2)\tilde{t}}A^{-\delta/4}\,\widehat{g}(\xi')\|_2\\
&\le C\tilde{t}^{-\delta/2}\|\mathcal{F}^{-1}_{\xi'}[e^{-(A^2/2)\tilde{t}}A^{-\delta/4}]\|_{L_p(\mathbf{R}^{N-1})}\|g\|_{L_r(\mathbf{R}^{N-1})}.
\end{aligned}\tag{3.5.23}$$

On the other hand, by Proposition 1.2.8 with $n=N-1$, $L=N-2$, and $\sigma=1-\delta/4$, we have

$$|\mathcal{F}^{-1}_{\xi'}[e^{-(A^2/2)\tilde{t}}A^{-\delta/4}](x')|\le C|x'|^{-(N-1-\delta/4)}$$

for a positive constant C, and furthermore, by direct calculations

$$|\mathcal{F}^{-1}_{\xi'}[e^{-(A^2/2)\tilde{t}}A^{-\delta/4}](x')|\le C\tilde{t}^{-(1/2)(N-1-\delta/4)}.$$

We thus obtain

$$|\mathcal{F}^{-1}_{\xi'}[e^{-(A^2/2)\tilde{t}}A^{-\delta/4}](x')|\le\frac{C}{\tilde{t}^{(1/2)(N-1-\delta/4)}+|x'|^{N-1-\delta/4}}$$

for a positive constant C. Therefore, by choosing a sufficiently small $\delta > 0$, we see that

$$\|\mathcal{F}^{-1}_{\xi'}[e^{-(A^2/2)\widetilde{t}}A^{-\delta/4}]\|_{L_p(\mathbf{R}^N_+)} \leq C\widetilde{t}^{-\frac{N-1}{2}(1-\frac{1}{p})+\frac{\delta}{8}} = C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{2})+\frac{\delta}{8}}, \tag{3.5.24}$$

noting that $p > 1$ because $1 \leq r < 2$, which, combined with (3.5.23), furnishes that the required inequality holds. Summing up the case of $s = 0$, we have obtained

$$\|L^{+,2}_1(t; A_0)g\|_{L_q(\mathbf{R}^N_+)} \leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\|g\|_{L_r(\mathbf{R}^{N-1})}$$

for some positive constant C and $1 \leq r \leq 2 \leq q \leq \infty$ if $(q, r) \neq (2, 2)$.

Concerning $L^{+,2}_2(t; A_0)$, we see that by Lemma 3.2.2

$$\begin{aligned}
&\|[L^{+,2}_2(t; A_0)g](\cdot\,, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{s}{2}}\left\|\int_0^1 e^{-(A^2/2)\widetilde{t}}e^{-C|B|^2\widetilde{t}}\varphi_0\, e^{-(\mathrm{Re}B)x_N}\, du\,\widehat{g}(\xi')\right\|_2 \\
&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{s}{2}}\left\|\int_0^1 e^{-Cu\widetilde{t}}e^{-C\sqrt{u}x_N}\, du\, e^{-(A^2/2)\widetilde{t}}\widehat{g}(\xi')\right\|_2 \\
&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{s}{2}}\frac{\left\|e^{-(A^2/2)\widetilde{t}}\widehat{g}(\xi')\right\|_2}{\widetilde{t}+(x_N)^2} \leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{s}{2}}\frac{\|g\|_{L_r(\mathbf{R}^{N-1})}}{\widetilde{t}+(x_N)^2},
\end{aligned}$$

which, combined with Lemma 3.2.3 (1), furnishes that the required inequality holds for $L^{+,2}_2(t; A_0)$.

Finally, we show the inequality for $L^{+,2}_3(t; A_0)$. We have easily, by (3.5.20),

$$\begin{aligned}
&\|[L^{\pm,2}_3(t; A_0)g](\cdot\,, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\left\|\int_0^1 \frac{e^{-(A^2/2)\widetilde{t}}e^{-C|B|^2\widetilde{t}}\varphi_0 A^{1/2}|B|^s e^{-CAx_N}}{(A\sqrt{1-u}+\sqrt{u}+A^{1/4})\sqrt{u}}\, du\,\widehat{g}(\xi')\right\|_2
\end{aligned}$$

for a positive constant C. We thus obtain the required inequality in the same manner as we have obtained the estimate of $L^{+,2}_1(t; A_0)$ from (3.5.21).

(3) As mentioned above, for $n = 1, 2$, we have, by (3.5.19),

$$\begin{aligned}
&\|[K^{+,2}_n(t; A_0)f]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\int_0^\infty\left\|\int_0^1 \frac{e^{-(A^2/2)\widetilde{t}}e^{-C|B|^2\widetilde{t}}\varphi_0 A^{1/2}e^{-CAy_N}}{(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^2}\, du\,\widehat{f}(y_N)\right\|_2 dy_N, \\
&\|[L^{+,2}_1(t; A_0)g]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad\leq C\widetilde{t}^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\left\|\int_0^1 \frac{e^{-(A^2/2)\widetilde{t}}e^{-C|B|^2\widetilde{t}}\varphi_0}{(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^2}\, du\,\widehat{g}(\xi')\right\|_2
\end{aligned}$$

with some positive constant C. Concerning $K_n^{+,2}(t;A_0)$ $(n=1,2)$, by Lemma 3.2.2 and Hölder's inequality, we see that for a sufficiently small $\delta>0$

$$
\begin{aligned}
&\|[K_n^{+,2}(t;A_0)f]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})}\\
&\leq C\tilde{t}^{-\frac{N-1}{2}(\frac12-\frac1q)}\int_0^\infty\left\|\int_0^1\frac{e^{-Cu\tilde{t}}}{(\sqrt{u})^{2-\delta}}\,du\,e^{-(A^2/2)\tilde{t}}A^{1/2-\delta/4}e^{-CAy_N}\widehat{f}(y_N)\right\|_2 dy_N\\
&\leq C\tilde{t}^{-\frac{N-1}{2}(\frac1r-\frac1q)-\frac{\delta}{2}}\int_0^\infty\frac{\|f(\,\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\tilde{t}^{(1/2)(1/2-\delta/4)}+(y_N)^{1/2-\delta/4}}\,dy_N\\
&\leq C\tilde{t}^{-\frac{N-1}{2}(\frac1r-\frac1q)-\frac12(\frac1r-\frac12)-\frac38\delta}\|f\|_{L_r(\mathbf{R}^N_+)}
\end{aligned}
$$

for $1\leq r<2$ with some positive constant C. On the other hand, by using (3.5.23) and (3.5.24)

$$
\begin{aligned}
&\|[L_1^{+,2}(t;A_0)g]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})}\\
&\leq C\tilde{t}^{-\frac{N-1}{2}(\frac12-\frac1q)}\left\|\int_0^1\frac{e^{-Cu\tilde{t}}}{(\sqrt{u})^{2-\delta}}\,du\,e^{-(A^2/2)\tilde{t}}A^{-\delta/4}\widehat{g}(\xi')\right\|_2\\
&\leq C\tilde{t}^{-\frac{N-1}{2}(\frac12-\frac1q)-\frac{\delta}{2}}\|\mathcal{F}^{-1}_{\xi'}[e^{-(A^2/2)\tilde{t}}A^{-\delta/4}\widehat{g}(\xi')]\|_2\\
&\leq C\tilde{t}^{-\frac{N-1}{2}(\frac1r-\frac1q)-\frac38\delta}\|g\|_{L_r(\mathbf{R}^{N-1})}
\end{aligned}
$$

for $1\leq r<2$ with some positive constant C, which completes the proof of the lemma. $\square$

Corollary 3.5.14. *Let $1\leq r\leq 2\leq q\leq\infty$, and let $f\in L_r(\mathbf{R}^N_+)^N$ and $g\in L_r(\mathbf{R}^{N-1})$. We use the symbols defined as (3.5.6) with $\sigma=2$.*

(1) *Let $\alpha\in\mathbf{N}_0^N$ and we assume that there exist positive constants $A_1\in(0,1)$ and C such that for any $\lambda\in\Gamma_2^\pm$ and $A\in(0,A_1)$*

$$
\begin{aligned}
|k_1(\xi',\lambda)|&\leq C(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^{-4}A,\\
|k_2(\xi',\lambda)|&\leq C(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^{-4}A^2,\\
|l_1(\xi',\lambda)|&\leq C(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^{-4}|B|^2.
\end{aligned}
$$

Then there exist positive constants $A_0\in(0,A_1)$ and $C=C(N,q,r,\alpha)$ such that for any $t>0$ and $n=1,2$

$$
\begin{aligned}
\|D_x^\alpha\nabla K_n^{\pm,2}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)}&\leq C(t+1)^{-\frac N2(\frac1r-\frac1q)-\frac14-k-\frac{|\alpha|}{2}}\|f\|_{L_r(\mathbf{R}^N_+)},\\
\|D_x^\alpha\partial_t K_n^{\pm,2}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)}&\leq C(t+1)^{-\frac N2(\frac1r-\frac1q)-\frac{|\alpha|}{2}}\|f\|_{L_r(\mathbf{R}^N_+)}\quad\text{if } |\alpha|\neq 0,\\
\|D_x^\alpha\nabla L_1^{\pm,2}(t;A_0)g\|_{L_q(\mathbf{R}^N_+)}&\leq C(t+1)^{-\frac{N-1}{2}(\frac1r-\frac1q)-\frac12(\frac12-\frac1q)-1-\frac{|\alpha|}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}.
\end{aligned}
$$

In addition, if $(q,r)\neq(2,2)$, then we have, for any $t>0$ and $n=1,2$,

$$
\|\partial_t K_n^{\pm,2}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)}\leq C(t+1)^{-\frac N2(\frac1r-\frac1q)}\|f\|_{L_r(\mathbf{R}^N_+)}.
$$

(2) *Let $k = 0, 1$, $l = 0, 1, 2$, and $\alpha' \in \mathbf{N}_0^{N-1}$. We suppose that there exist positive constants $A_1 \in (0, 1)$ and C such that for any $\lambda \in \Gamma_2^\pm$ and $A \in (0, A_1)$*

$$\begin{aligned}
|k_3(\xi', \lambda)| &\leq C(A\sqrt{1-u} + \sqrt{u} + A)^{-2} A, \\
|k_n(\xi', \lambda)| &\leq C(A\sqrt{1-u} + \sqrt{u} + A)^{-2} A|B| \quad (n = 4, 5), \\
|k_6(\xi', \lambda)| &\leq C(A\sqrt{1-u} + \sqrt{u} + A)^{-2} A|B|^2, \\
|l_2(\xi', \lambda)| &\leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A, \\
|l_3(A, B)| &\leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A|B|.
\end{aligned}$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(N, q, r, \alpha')$ such that for any $t > 0$

$$\begin{aligned}
&\|\partial_t^k D_{x'}^{\alpha'} D_N^l K_n^{\pm,2}(t; A_0) f\|_{L_q(\mathbf{R}_+^N)} \\
&\quad \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-k-\frac{|\alpha'|+\ell}{2}} \|f\|_{L_r(\mathbf{R}_+^N)} \quad (n = 3, 4, 5, 6), \\
&\|\partial_t^k D_{x'}^{\alpha'} D_N^\ell L_n^{\pm,2}(t; A_0) g\|_{L_q(\mathbf{R}_+^N)} \\
&\quad \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-k-\frac{|\alpha'|+\ell}{2}} \|g\|_{L_r(\mathbf{R}^{N-1})} \quad (n = 2, 3),
\end{aligned}$$

provided that $k + l + |\alpha'| \neq 0$. In addition, if $(q, r) \neq (2, 2)$, then there hold

$$\begin{aligned}
&\|K_n^{\pm,2}(t; A_0) f\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} \|f\|_{L_r(\mathbf{R}_+^N)} \quad (n = 3, 4, 5, 6), \\
&\|L_n^{\pm,2}(t; A_0) g\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \|g\|_{L_r(\mathbf{R}^{N-1})} \quad (n = 2, 3)
\end{aligned}$$

for any $t > 0$ with some positive constant $C = C(N, q, r)$.

Proof. We consider only the cases of $K_5^{\pm,2}(t; A_0)$, $K_6^{\pm,2}(t; A_0)$ and $L_3^{\pm,2}(t; A_0)$. The other inequalities can be proves by Lemma 3.5.13 directly. By (3.5.6)

$$\begin{aligned}
&\partial_t^k D_{x'}^{\alpha'} [K_n^{\pm,2}(t; A_0) f](x) \\
&\quad = \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\int_{\Gamma_2^\pm} e^{\lambda t} \varphi_0(\xi') \lambda^k (i\xi')^{\alpha'} k_n(\xi', \lambda) \mathcal{X}_n(x_N, y_N) \, d\lambda \, \widehat{f}(\xi', y_N) \right] (x'), \\
&\partial_t^k D_{x'}^{\alpha'} [L_3^{\pm,2}(t; A_0) g](x) \\
&\quad = \mathcal{F}_{\xi'}^{-1} \left[\int_{\Gamma_2^\pm} e^{\lambda t} \varphi_0(\xi') \lambda^k (i\xi')^{\alpha'} \ell_3(\xi', \lambda) \mathcal{M}(x_N) \, d\lambda \, \widehat{g}(\xi') \right] (x')
\end{aligned}$$

for $n = 5, 6$, $k = 0, 1$, and any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$. Since by Lemma 3.5.12

$$\begin{aligned}
&|\lambda^k (i\xi')^{\alpha'} k_n(\xi', \lambda)| \leq C \begin{cases} (A\sqrt{1-u} + \sqrt{u} + A)^{-2} A|B|^{1+2k+|\alpha'|} & (n = 5), \\ CA|B|^{2k+|\alpha'|} & (n = 6), \end{cases} \\
&|\lambda^k (i\xi')^{\alpha'} l_3(\xi', \lambda)| \leq (A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-3} A|B|^{2k+|\alpha'|}
\end{aligned}$$

for $\lambda \in \Gamma_2^\pm$ and $A \in (0, A_0)$ by choosing some $A_0 \in (0, A_1)$, we obtain, by using Lemma 3.5.13,

(3.5.25)
$$\|\partial_t^k D_{x'}^{\alpha'} K_n^{\pm,2}(t; A_0)f\|_{L_q(\mathbf{R}_+^N)} \le C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-k-\frac{|\alpha'|}{2}}\|f\|_{L_r(\mathbf{R}_+^N)} \quad (n=5,6),$$
$$\|\partial_t^k D_{x'}^{\alpha'} L_3^{\pm,2}(t; A_0)g\|_{L_q(\mathbf{R}_+^N)} \le C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-k-\frac{|\alpha'|}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}$$

for any $t > 0$ with some positive constant C, provided that $k + |\alpha'| \ne 0$. In the case of $k + |\alpha'| = 0$, we have, by Lemma 3.5.13,

(3.5.26)
$$\|K_n^{\pm,2}(t; A_0)f\|_{L_q(\mathbf{R}_+^N)} \le C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\|f\|_{L_r(\mathbf{R}_+^N)} \quad (n=5,6),$$
$$\|L_3^{\pm,2}(t; A_0)g\|_{L_q(\mathbf{R}_+^N)} \le C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|g\|_{L_r(\mathbf{R}^{N-1})}$$

if $(q, r) \ne (2, 2)$. On the other hand, by (1.2.3)

$$\begin{aligned}
&\partial_t^k D_{x'}^{\alpha'} D_N^l [K_{0,5}^{\pm,2}(t)f](x)\\
&= (-1)^l \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\int_{\Gamma_2^\pm} e^{\lambda t}\varphi_0 \lambda^k (i\xi')^{\alpha'}(B+A)^{l-1}k_5 e^{-B(x_N+y_N)}\,d\lambda\,\widehat{f}(y_N)\right](x')\,dy_N\\
&+ (-1)^l \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\int_{\Gamma_2^\pm} e^{\lambda t}\varphi_0 \lambda^k (i\xi')^{\alpha'} A^l k_5 \mathcal{M}(x_N) e^{-By_N}\,d\lambda\,\widehat{f}(y_N)\right](x')\,dy_N,\\
&\partial_t^k D_{x'}^{\alpha'} D_N^l [K_6^{\pm,2}(t; A_0)f](x) =\\
&(-1)^l \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\int_{\Gamma_2^\pm} e^{\lambda t}\varphi_0 \lambda^k (i\xi')^{\alpha'}(B+A)^{l-1}k_6 e^{-Bx_N}\mathcal{M}(y_N)\,d\lambda\,\widehat{f}(y_N)\right](x')\,dy_N\\
&+ (-1)^l \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\int_{\Gamma_2^\pm} e^{\lambda t}\varphi_0 \lambda^k (i\xi')^{\alpha'} A^l k_6 \mathcal{M}(x_N)\mathcal{M}(y_N)\,d\lambda\,\widehat{f}(y_N)\right](x')\,dy_N,\\
&\partial_t^k D_{x'}^{\alpha'} D_N^l [L_3^{\pm,2}(t; A_0)g](x)\\
&= (-1)^l \mathcal{F}_{\xi'}^{-1}\left[\int_{\Gamma_2^\pm} e^{\lambda t}\varphi_0 \lambda^k (i\xi')^{\alpha'}(B+A)^{l-1}l_3(\xi',\lambda)e^{-Bx_N}\,d\lambda\,\widehat{g}(\xi')\right](x')\\
&+ (-1)^l \mathcal{F}_{\xi'}^{-1}\left[\int_{\Gamma_2^\pm} e^{\lambda t}\varphi_0 \lambda^k (i\xi')^{\alpha'} A^l l_3(\xi',\lambda)\mathcal{M}(x_N)\,d\lambda\,\widehat{g}(\xi')\right](x')
\end{aligned}$$

for $l = 1, 2$, $k = 0, 1$, and $\alpha' \in \mathbf{N}_0^{N-1}$, where we have set

$$\varphi_0 = \varphi_0(\xi'), \quad k_n = k_n(\xi', \lambda) \ (n = 5, 6), \quad \widehat{f}(y_N) = \widehat{f}(\xi', y_N).$$

Since by Lemma 3.5.12

$$\begin{aligned}
|\lambda^k (i\xi')^{\alpha'}(B+A)^{l-1}k_5(\xi',\lambda)| &\le C(A\sqrt{1-u}+\sqrt{u}+A)^{-2}A|B|^{2k+|\alpha'|+l}\\
|\lambda^k (i\xi')^{\alpha'}A^l k_5(\xi',\lambda)| &\le C(A\sqrt{1-u}+\sqrt{u}+A)^{-1}A|B|^{2k+|\alpha'|+l},\\
|\lambda^k (i\xi')^{\alpha'}(B+A)^{l-1}k_6(\xi',\lambda)| &\le C(A\sqrt{1-u}+\sqrt{u}+A)^{-1}A|B|^{2k+|\alpha'|+l},\\
|\lambda^k (i\xi')^{\alpha'}A^l k_6(\xi',\lambda)| &\le CA|B|^{2k+|\alpha'|+l}
\end{aligned}$$

for any $\lambda \in \Gamma_2^\pm$ and $A \in (0, A_0)$ by choosing a suitable $A_0 \in (0, A_1)$, we have, by using Lemma 3.5.13,

$$\|\partial_t^k D_{x'}^{\alpha'} D_N^l K_n^{\pm,2}(t; A_0) f\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-k-\frac{|\alpha'|+l}{2}} \|f\|_{L_r(\mathbf{R}_+^N)} \tag{3.5.27}$$

for any $t > 0$, $k = 0, 1$, $l = 1, 2$, $\alpha' \in \mathbf{N}_0^{N-1}$, and $n = 5, 6$ with some positive constant C. In addition,

$$|\lambda^k (i\xi')^{\alpha'} (B+A)^{l-1} l_3(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A|B|^{2k+|\alpha'|+l},$$
$$|\lambda^k (i\xi')^{\alpha'} A^l l_3(\xi', \lambda)| \leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-3} A|B|^{2k+|\alpha'|+l}$$

for any $\lambda \in \Gamma_2^\pm$ and $A \in (0, A_0)$, and therefore by Lemma 3.5.13

$$\begin{aligned} &\|\partial_t^k D_{x'}^{\alpha'} D_N^l L_3^{\pm,2}(t; A_0) g\|_{L_q(\mathbf{R}_+^N)} \\ &\quad \leq C(t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-k-\frac{|\alpha'|+\ell}{2}} \|g\|_{L_r(\mathbf{R}^{N-1})} \end{aligned}$$

for any $t > 0$, $k = 0, 1$, $l = 1, 2$, and $\alpha' \in \mathbf{N}_0^{N-1}$ with a positive constant C, which, combined with (3.5.25), (3.5.26), and (3.5.27), furnishes the required estimates for $K_5^{\pm,2}(t; A_0)$, $K_6^{\pm,2}(t; A_0)$, and $L_3^{\pm,2}(t; A_0)$. This completes the proof. □

We see that by Lemma 3.5.12 there exist positive constants $A_1 \in (0, 1)$ and C such that for $j, k = 1, \ldots, N$, $\lambda \in \Gamma_2^\pm$, and $A \in (0, A_1)$

$$\left|\frac{\mathcal{V}_{jk}^{BB}(\xi', \lambda)}{L(A, B)}\right| \leq \frac{CA}{(A\sqrt{1-u} + \sqrt{u} + A)^2}, \quad \left|\frac{\mathcal{V}_{jk}^{B\mathcal{M}}(\xi', \lambda)}{L(A, B)}\right| \leq \frac{CA|B|}{(A\sqrt{1-u} + +A)^2},$$
$$\left|\frac{\mathcal{V}_{jk}^{\mathcal{M}B}(\xi', \lambda)}{L(A, B)}\right| \leq \frac{CA|B|}{(A\sqrt{1-u} + \sqrt{u} + A)^2}, \quad \left|\frac{\mathcal{V}_{jk}^{\mathcal{M}\mathcal{M}}(\xi', \lambda)}{L(A, B)}\right| \leq \frac{CA|B|^2}{(A\sqrt{1-u} + \sqrt{u} + A)^2},$$
$$\left|\frac{\mathcal{P}_j^{AA}(\xi', \lambda)}{L(A, B)}\right| \leq \frac{CA}{(A\sqrt{1-u}\sqrt{u} + A^{\frac{1}{4}})^4}, \quad \left|\frac{\mathcal{P}_j^{A\mathcal{M}}(\xi', \lambda)}{L(A, B)}\right| \leq \frac{CA^2}{(A\sqrt{1-u} + \sqrt{u} + A^{\frac{1}{4}})^4},$$

and furthermore,

$$\begin{aligned} \left|\frac{A}{L(A, B)}\right| &\leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A, \\ \left|\frac{A(B^2 + A^2)}{(B+A)L(A, B)}\right| &\leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A|B|, \\ \left|\frac{D(A, B)}{(B+A)L(A, B)}\right| &\leq C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} |B|^2. \end{aligned}$$

Therefore, remembering the formulas: (3.4.3), (3.4.4), (3.4.5), (3.5.5) with $\sigma = 2$ and using Corollary 3.5.14, we have Theorem 3.5.11 (1)-(3). Theorem 3.5.11 (4) follows from Lemma 3.5.13 (3) directly.

3.5.4. Analysis on $\Gamma_3^\pm$. Our aim here is to show the following theorem for the operators defined as (3.5.5) with $\sigma = 3$.

THEOREM 3.5.15. *Let $1 \leq r \leq 2 \leq q \leq \infty$, $(\alpha', \alpha) \in \mathbf{N}_0^{N-1} \times \mathbf{N}_0^N$, and $\mathbf{F} = (\mathbf{f}, g) \in X_r^0$. Then there exist positive constants δ_0, $A_0 \in (0, 1)$, and $C =$*

$C(N,q,r,\alpha',\alpha)$ *such that for any* $t>0$

$$\begin{aligned}
&\|(\partial_t S_0^{\mathbf{f},3}(t;A_0)\mathbf{F}, \nabla\Pi_0^{\mathbf{f},3}(t;A_0)\mathbf{F})\|_{L_q(\mathbf{R}^N_+)} + \|D^{\alpha'}_{x'}S_0^{\mathbf{f},3}(t;A_0)\mathbf{F}\|_{W^2_q(\mathbf{R}^N_+)}\\
&\quad + \|D^\alpha_x\nabla\mathcal{E}(T_0^{\mathbf{f},3}(t;A_0)\mathbf{F})\|_{L_q(\mathbf{R}^N_+)} + \|D^\alpha_x\partial_t\mathcal{E}(T_0^{\mathbf{f},3}(t;A_0)\mathbf{F})\|_{L_q(\mathbf{R}^N_+)}\\
&\quad + \|T_0^{\mathbf{f},3}(t;A_0)\mathbf{F}\|_{L_q(\mathbf{R}^{N-1})} \le C(|\log t|+1)e^{-\delta_0 t}\|\mathbf{f}\|_{L_r(\mathbf{R}^N_+)},\\
&\|(\partial_t S_0^{g,3}(t;A_0)\mathbf{F}, \nabla\Pi_0^{g,3}(t;A_0)\mathbf{F})\|_{L_q(\mathbf{R}^N_+)} + \|D^{\alpha'}_{x'}S_0^{g,3}(t;A_0)\mathbf{F}\|_{W^2_q(\mathbf{R}^N_+)}\\
&\quad + \|D^\alpha_x\nabla\mathcal{E}(T_0^{g,3}(t;A_0)\mathbf{F})\|_{L_q(\mathbf{R}^N_+)}\\
&\quad + \|T_0^{g,3}(t;A_0)\mathbf{F}\|_{L_q(\mathbf{R}^{N-1})} \le C(|\log t|+1)e^{-\delta_0 t}\|g\|_{L_r(\mathbf{R}^{N-1})}.
\end{aligned}$$

In order to show Theorem 3.5.15, we start with the following lemma.

LEMMA 3.5.16. *Let* $1\le r\le 2\le q\le\infty$, *and let* $f\in L_r(\mathbf{R}^N_+)$ *and* $g\in L_r(\mathbf{R}^{N-1})$. *We use the operators defined as* (3.5.6) *with* $\sigma=3$ *and*

$$k_n(\xi',\lambda)=\kappa_n(\xi',\lambda)/L(A,B),\quad l_n(\xi',\lambda)=m_n(\xi',\lambda)/L(A,B).$$

(1) *Let* $s\ge 0$ *and suppose that there exist positive constants* $A_1\in(0,1)$ *and* $C=C(s)$ *such that for any* $\lambda\in\Gamma_3^\pm$ *and* $A\in(0,A_1)$

$$\begin{aligned}
&|\kappa_n(\xi',\lambda)|\le C(|\lambda|^{1/2}+A)^2A^{1+s}\ \ (n=1,2,4,5,6),\\
&|\kappa_3(\xi',\lambda)|\le C(|\lambda|^{1/2}+A)^2A^s.
\end{aligned}$$

Then there exist positive constants δ_0, $A_0\in(0,A_1)$, $C=C(N,q,r,s)$ *such that for any* $t>0$ *and* $n=1,\ldots,6$

$$\|K_n^{\pm,3}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)}\le C(|\log t|+1)e^{-\delta_0 t}\|f\|_{L_r(\mathbf{R}^N_+)}$$

(2) *Let* $s\ge 0$ *and suppose that there exist positive constants* $A_1\in(0,1)$ *and* C *such that for any* $\lambda\in\Gamma_3^\pm$ *and* $A\in(0,A_1)$

$$\begin{aligned}
&|m_1(\xi',\lambda)|\le C(|\lambda|^{1/2}+A)^2A^{1+s},\\
&|m_2(\xi',\lambda)|\le C(|\lambda|^{1/2}+A)^2A^s,\\
&|m_3(\xi',\lambda)|\le C|\lambda|^{1/2}(|\lambda|^{1/2}+A)^2A^{1+s}.
\end{aligned}$$

Then there exist positive constants δ_0, $A_0\in(0,A_1)$, *and* $C=C(N,q,r)$ *such that for any* $t>0$ *and* $n=1,2,3$

$$\|L_n^{\pm,3}(t;A_0)g\|_{L_q(\mathbf{R}^N_+)}\le C(|\log t|+1)e^{-\delta_0 t}\|g\|_{L_r(\mathbf{R}^{N-1})}.$$

(3) *Suppose that there exist positive constants* $A_1\in(0,1)$ *and* C *such that for any* $\lambda\in\Gamma_3^\pm$ *and* $A\in(0,A_1)$

$$\begin{aligned}
&|\kappa_n(\xi',\lambda)|\le C(|\lambda|^{1/2}+A)^2A\quad(n=1,2),\\
&|m_1(\xi',\lambda)|\le C(|\lambda|^{1/2}+A)^2.
\end{aligned}$$

Then there exist positive constants δ_0, $A_0\in(0,1)$, *and* C *such that for any* $t>0$ *and* $n=1,2$

$$\begin{aligned}
&\|[K_n^{\pm,3}(t;A_0)f]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})}\le C(|\log t|+1)e^{-\delta_0 t}\|f\|_{L_r(\mathbf{R}^N_+)},\\
&\|[L_1^{\pm,3}(t;A_0)g]|_{x_N=0}\|_{L_q(\mathbf{R}^{N-1})}\le C(|\log t|+1)e^{-\delta_0 t}\|g\|_{L_r(\mathbf{R}^{N-1})}.
\end{aligned}$$

PROOF. We use the abbreviations: $\|\cdot\|_2 = \|\cdot\|_{L_2(\mathbf{R}^{N-1})}$, $\widehat{f}(y_N) = \widehat{f}(\xi', y_N)$, $\varphi_0 = \varphi_0(\xi')$, and $\widetilde{t} = t+1$ for $t > 0$ in this proof, and consider only estimates on Γ_3^+ since estimates on Γ_3^- can be shown similarly.
(1) First we show the inequality for $K_1^{+,3}(t; A_0)$. Noting that $\lambda = -\gamma_0 + i\widetilde{\gamma}_0 + ue^{i(\pi-\varepsilon_0)}$ for $u \in [0,\infty)$ on Γ_3^+, we have, by (3.5.6),

$$[K_1^{+,3}(t; A_0)f](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\int_0^\infty e^{\{-\gamma_0+i\widetilde{\gamma}_0+ue^{i(\pi-\varepsilon_0)}\}t}\varphi_0 \times \frac{\kappa_1(\xi',\lambda)}{L(A,B)} e^{-A(x_N+y_N)} e^{i(\pi-\varepsilon_0)}\,du\,\widehat{f}(y_N)\Big](x')\,dy_N.$$

Since $e^{-(\gamma_0/2)t}e^{A^2\widetilde{t}} \le Ce^{-A^2\widetilde{t}}$ for any $A \in (0, A_0)$ by choosing a suitable $A_0 \in (0, A_1)$, we obtain, by Lemma 1.2.6 (2), L_q-L_r estimates of the $(N-1)$-dimensional heat kernel, and Parseval's theorem,

$$\begin{aligned}&\|[K_1^{+,3}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\ &\le C\widetilde{t}^{-\frac{N-1}{2}\left(\frac12-\frac1q\right)}\int_0^\infty \left\|\int_0^\infty \varphi_0 e^{A^2\widetilde{t}}e^{-(\gamma_0+u\cos\varepsilon_0)t}\frac{A^{1+s}}{|\lambda|}e^{-A(x_N+y_N)}\,du\,\widehat{f}(y_N)\right\|_2 dy_N\\ &\le C\widetilde{t}^{-\frac{N-1}{2}\left(\frac12-\frac1q\right)}e^{-\frac{\gamma_0}{2}t}\int_0^\infty \frac{e^{-u(\cos\varepsilon_0)t}}{|\lambda|}\,du\int_0^\infty \left\|e^{-A^2\widetilde{t}}Ae^{-A(x_N+y_N)}\widehat{f}(y_N)\right\|_2 dy_N,\end{aligned}$$

which, combined with Lemma 3.5.5, furnishes that

$$\|K_1^{+,3}(t;A_0)f\|_{L_q(\mathbf{R}_+^N)} \le Ce^{-(\gamma_0/4)t}\|f\|_{L_r(\mathbf{R}_+^N)}\int_0^\infty \frac{e^{-u(\cos\varepsilon_0)t}}{|\lambda|}\,du \tag{3.5.28}$$

for some positive constant C. We here calculate the integral on the right-hand side. It holds that

$$\begin{aligned}\int_0^\infty \frac{e^{-u(\cos\varepsilon_0)t}}{|\lambda|}\,du &\le \int_0^\infty \frac{e^{-u(\cos\varepsilon_0)t}}{\gamma_0+u\cos\varepsilon_0}\,du = \frac{1}{\cos\varepsilon_0}\int_0^\infty \frac{e^{-\ell}}{\gamma_0 t+\ell}\,d\ell\\ &\le \frac{1}{\cos\varepsilon_0}\int_0^\infty \frac{e^{-\ell}}{t+\ell}\,d\ell,\end{aligned}$$

which, combined with

$$\begin{aligned}\int_0^\infty \frac{e^{-\ell}}{t+\ell}\,d\ell &= -\log t + \int_0^\infty e^{-\ell}\log(t+\ell)\,d\ell\\ &= -\log t - t\log t - 1 + \int_0^\infty e^{-\ell}(t+\ell)\log(t+\ell)\,d\ell,\end{aligned}$$

furnishes that $\int_0^\infty e^{-\ell}/(t+\ell)\,d\ell = O(\log t)$ as $t \to 0$. Thus the required inequality follows from (3.5.28). Analogously, we can show the cases of $n = 2,4,5,6$ by using the fact that by Lemma 1.2.6 (1)

$$|e^{-Ba}| \le Ce^{-Ca}, \quad |\mathcal{M}(a)| \le C|\lambda|^{-1/2}e^{-Ca} \le Ce^{-Ca}$$

for any $a > 0$ and $\lambda \in \Gamma_3^\pm$ with some positive constant C.

We finally consider the inequality for $K_3^{+,3}(t;A_0)$. By Lemma 1.2.6 (1), Hölder's inequality, and calculations similar to the case of $K_1^{+,3}(t;A_0)$, we easily see, for

$r' = r/(r-1)$, that

$$
\begin{aligned}
&\|[K_3^{+,3}(t;A_0)f](\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\leq Ce^{-(\gamma_0/4)t}\int_0^\infty \left\|\int_0^\infty \frac{e^{-u(\cos\varepsilon_0)t}}{|\lambda|}e^{-C|\lambda|^{\frac12}(x_N+y_N)}\,du\,e^{-A^2\tilde{t}}\widehat{f}(y_n)\right\|_2 dy_N\\
&\leq Ce^{-(\gamma_0/8)t}\int_0^\infty \frac{e^{-u(\cos\varepsilon_0)t}e^{-C|\lambda|^{\frac12}x_N}}{|\lambda|}\left(\int_0^\infty e^{-r'C|\lambda|^{\frac12}y_N}\,dy_N\right)^{\frac{1}{r'}}du\|f\|_{L_r(\mathbf{R}^N_+)}\\
&\leq Ce^{-(\gamma_0/8)t}\|f\|_{L_r(\mathbf{R}^N_+)}\int_0^\infty \frac{e^{-u(\cos\varepsilon_0)t}e^{-C|\lambda|^{\frac12}x_N}}{|\lambda|^{1+1/(2r')}}\,du,
\end{aligned}
$$

which furnishes that

$$
\begin{aligned}
\|K_3^{\pm,3}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)} &\leq Ce^{-(\gamma_0/8)t}\|f\|_{L_r(\mathbf{R}^N_+)}\int_0^\infty \frac{e^{-u(\cos\varepsilon_0)t}}{|\lambda|^{1+1/(2q)+1/(2r')}}\,du\\
&\leq Ce^{-(\gamma_0/8)t}\|f\|_{L_r(\mathbf{R}^N_+)}\int_0^\infty \frac{e^{-u(\cos\varepsilon_0)t}}{|\lambda|}\,du
\end{aligned}
$$

with some positive constant C. This inequality implies that the required inequality for $K_3^{+,3}(t;A_0)$ as mentioned above.
(2), (3) We can prove in a similar way to (1), so that we may omit the proof. □

We see that by Lemma 1.2.6 there exist positive constants $A_1 \in (0,1)$ and C such that for any $\lambda \in \Gamma_3^\pm$ and $A \in (0, A_1)$

$$
\begin{aligned}
&|\mathcal{V}_{jk}^{BB}(\xi',\lambda)| \leq C, \quad |\mathcal{V}_{jk}^{B\mathcal{M}}(\xi',\lambda)| \leq CA, \quad |\mathcal{V}_{jk}^{\mathcal{M}B}(\xi',\lambda)| \leq CA,\\
&|\mathcal{V}_{jk}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)| \leq CA, \quad |\mathcal{P}_{j}^{AA}(\xi',\lambda)| \leq CA, \quad |\mathcal{P}_{j}^{A\mathcal{M}}(\xi',\lambda)| \leq CA
\end{aligned}
$$

for $j,k = 1,\ldots,N$. Therefore, remembering the formulas: (3.4.3), (3.4.4), (3.4.5), (3.5.5) with $\sigma = 3$ and using (1.2.3), we have Theorem 3.5.15 by Lemma 3.5.16.

Finally we consider $\partial_t\mathcal{E}(T_0^g(t;A_0)F)$ given by

$$
\begin{aligned}
\partial_t\mathcal{E}(T_0^d(t;A_0)F) &= \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma(\varepsilon_0)} e^{\lambda t}\frac{\varphi_0(\xi')\lambda D(A,B)}{(B+A)L(A,B)}\,d\lambda\,e^{-Ax_N}\widehat{g}(\xi')\right](x')\\
&= \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma(\varepsilon_0)} e^{\lambda t}\,d\lambda\,\varphi_0(\xi')e^{-Ax_N}\widehat{g}(\xi')\right](x')\\
&\quad - \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma(\varepsilon_0)} e^{\lambda t}\frac{\varphi_0(\xi')A(c_g+c_\sigma A^2)}{L(A,B)}\,d\lambda\,e^{-Ax_N}\widehat{g}(\xi')\right](x'),
\end{aligned}
$$

where we have used the relations: $D(A,B) = (B-A)^{-1}\{L(A,B) - A(c_g + c_\sigma A^2)\}$ and $B^2 - A^2 = \lambda$. Note that the first term vanishes by Cauchy's integral theorem, so that it suffices to consider the second term only. Set for $\sigma = 0,1,2,3$

$$
I_\sigma^\pm(t;A_0) = \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_\sigma^\pm} e^{\lambda t}\frac{\varphi_0(\xi')A(c_g+c_\sigma A^2)}{L(A,B)}\,d\lambda\,e^{-Ax_N}\widehat{g}(\xi')\right](x').
$$

Since by Lemma 3.5.12 there exist positive constants $A_1 \in (0,1)$ and C such that for any $\lambda \in \Gamma_2^\pm$ and $A \in (0, A_1)$

$$
|\varphi_0(\xi')A(c_g+c_\sigma A^2)/L(A,B)| \leq C(A\sqrt{1-u}+\sqrt{u}+A)^{-4}A,
$$

we have, by Lemma 3.5.13 and for any $t > 0$, $\alpha \in \mathbf{N}_0^N$ with $|\alpha| \neq 0$, and $1 \leq r \leq 2 \leq q \leq \infty$,

$$\|D_x^\alpha I_2^\pm(t; A_0)\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{|\alpha|}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})}$$

with some $A_0 \in (0, A_1)$ and a positive constant C. In addition, if $(q, r) \neq (2, 2)$, then we have

$$\|I_2^\pm(t; A_0)\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\|g\|_{L_r(\mathbf{R}^{N-1})}.$$

On the other hand, in the other cases, it follows from Lemma 3.5.4, 3.5.10, and 3.5.16 that

$$\|D_x^\alpha I_n^\pm(t; A_0)\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{|\alpha|}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})} \quad (n = 0, 1),$$
$$\|D_x^\alpha I_3^\pm(t; A_0)\|_{L_q(\mathbf{R}_+^N)} \leq C(|\log t| + 1)e^{-\delta_0 t}\|g\|_{L_r(\mathbf{R}^{N-1})}$$

for any $t > 0$, $\alpha \in \mathbf{N}_0^N$, and $1 \leq r \leq 2 \leq q \leq \infty$ with some positive constant C.

Summing up the argumentation above, for any $t > 0$, $\alpha \in \mathbf{N}_0^N$, and $1 \leq r \leq 2 \leq q \leq \infty$, we have obtained

$$\begin{aligned}
&\|D_x^\alpha \partial_t \mathcal{E}(T_0^g(t; A_0)\mathbf{F})\|_{L_q(\mathbf{R}_+^N)} \\
&\quad \leq C(|\log t| + 1)(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{|\alpha|}{2}}\|g\|_{L_r(\mathbf{R}^{N-1})} \quad (|\alpha| \neq 0), \\
&\|\partial_t \mathcal{E}(T_0^g(t; A_0)\mathbf{F})\|_{L_q(\mathbf{R}_+^N)} \\
&\quad \leq C(|\log t| + 1)(t+1)^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\|g\|_{L_r(\mathbf{R}^{N-1})} \quad ((q, r) \neq (2, 2))
\end{aligned}$$

with some positive constant C, which, combined with Theorem 3.5.3, 3.5.7, 3.5.11, 3.5.15, and the formulas: (3.4.9), (3.5.4), completes the proof of Theorem 3.1.3 (1).

3.6. Analysis of high frequency parts

In this section, we show Theorem 3.1.3 (2). If we consider the Lopatinskii determinant $L(A, B)$, defined as (1.2.1) with $\mu = 1$, as a polynomial with respect to B, it then has the following four roots:

$$B_j = a_j A + \frac{c_\sigma}{4(1 - a_j - a_j^3)} + \frac{(1 + 3a_j^2)c_\sigma^2}{32(1 - a_j - a_j^3)^3}\frac{1}{A} + O\left(\frac{1}{A^2}\right) \quad \text{as } A \to \infty, \tag{3.6.1}$$

where a_j $(j = 1, \ldots, 4)$ are the solutions to $x^4 + 2x^2 - 4x + 1 = 0$. More precisely, we have the following information about a_j: a_1 and a_2 are real numbers such that $a_1 = 1$ and $0 < a_2 < 1/2$, while a_3, a_4 are complex numbers satisfying $\operatorname{Re} a_j < 0$ for $j = 3, 4$. We define λ_j by $\lambda_j = B_j^2 - A^2$ for $j = 1, 2$, and then

$$\begin{aligned}
\lambda_1 &= -\frac{c_\sigma}{2}A - \frac{3}{16}c_\sigma^2 + O(\frac{1}{A}), \\
\lambda_2 &= -(1 - a_2^2)A^2 + \frac{a_2 c_\sigma}{2(1 - a_2 - a_2^3)}A + O(1) \quad \text{as } A \to \infty.
\end{aligned} \tag{3.6.2}$$

Let $L_0 = \{\lambda \in \mathbf{C} \mid L(A, B) = 0,\ \operatorname{Re} B \geq 0,\ A \in \operatorname{supp} \varphi_\infty\}$, where φ_∞ is defined as (3.4.6), and then the expansions (3.5.2), (3.6.2) and Lemma 3.4.2 implies that there exist positive numbers $0 < \varepsilon_\infty < \pi/2$ and $\lambda_\infty > 0$ such that $L_0 \subset \Sigma_{\varepsilon_\infty} \cap \{z \in \mathbf{C} \mid \operatorname{Re} z < -\lambda_\infty\}$. Let $0 \leq \gamma_\infty \leq \min\{\lambda_\infty, 4^{-1} \times (A_0/6)^2\}$ for A_0, which is the same number as in (3.4.6), and also we set, for $d \in \{\mathbf{f}, g\}$ and (3.4.7),

$$S_\infty^d(t) = S_\infty^d(t; A_0), \quad \Pi_\infty^d(t) = \Pi_\infty^d(t; A_0), \quad T_\infty^d(t) = T_\infty^d(t; A_0) \tag{3.6.3}$$

for short. In order to estimate each term above, we use the following integral paths

$$\Gamma_4^\pm = \{\lambda \in \mathbf{C} \mid \lambda = -\gamma_\infty \pm iu,\ u : 0 \to \widetilde{\gamma}_\infty\},$$
$$\Gamma_5^\pm = \{\lambda \in \mathbf{C} \mid \lambda = -\gamma_\infty \pm i\widetilde{\gamma}_\infty + ue^{\pm i(\pi-\varepsilon_\infty)},\ u : 0 \to \infty\},$$

where $\widetilde{\gamma}_\infty = (\tan\varepsilon_\infty)(\widetilde{\lambda}_0(\varepsilon_\infty) + \gamma_\infty)$ and $\widetilde{\lambda}_0(\varepsilon_\infty) = 2\lambda_0(\varepsilon_\infty)/\sin\varepsilon_\infty$ is the same number as in (3.4.8) with $\varepsilon = \varepsilon_\infty$. Furthermore, for $d \in \{\mathbf{f}, g\}$, setting

$$\mathbf{v}_\infty^d(x,\lambda) = (v_{\infty 1}^d(x,\lambda), \ldots, v_{\infty N}^d(x,\lambda))^T,$$
$$v_{\infty j}^d(x,\lambda) = \mathcal{F}_{\xi'}^{-1}[\varphi_\infty(\xi')\widehat{v}_j^d(\xi', x_N, \lambda)](x') \quad (j = 1, \ldots, N),$$
$$\pi_\infty^d(x,\lambda) = \mathcal{F}_{\xi'}^{-1}[\varphi_\infty(\xi')\widehat{\pi}^d(\xi', x_N, \lambda)](x'),$$
$$h_\infty^d(x,\lambda) = \mathcal{F}_{\xi'}^{-1}[\varphi_\infty(\xi')e^{-Ax_N}\widehat{h}^d(\xi', \lambda)](x')$$

with (3.4.3) and (3.4.5), we have, by Cauchy's integral theorem, the following decompositions:

(3.6.4)

$$S_\infty^d(t)\mathbf{F} = \sum_{\sigma=4}^5 S_\infty^{d,\sigma}(t)\mathbf{F},\ \ \Pi_\infty^d(t)\mathbf{F} = \sum_{\sigma=4}^5 \Pi_\infty^{d,\sigma}(t)\mathbf{F},\ \ \mathcal{E}(T_\infty^d(t)\mathbf{F}) = \sum_{\sigma=4}^5 \mathcal{E}(T_\infty^{d,\sigma}(t)\mathbf{F}),$$

where the right-hand sides are given by

(3.6.5)

$$S_\infty^{d,\sigma}(t)\mathbf{F} = \frac{1}{2\pi i}\int_{\Gamma_\sigma^+\cup\Gamma_\sigma^-} e^{\lambda t}\mathbf{v}_\infty^d(x,\lambda)\,d\lambda, \quad \Pi_\infty^{d,\sigma}(t)\mathbf{F} = \frac{1}{2\pi i}\int_{\Gamma_\sigma^+\cup\Gamma_\sigma^-} e^{\lambda t}\pi_\infty^d(x,\lambda)\,d\lambda,$$
$$\mathcal{E}(T_\infty^{d,\sigma}(t)\mathbf{F}) = \frac{1}{2\pi i}\int_{\Gamma_\sigma^+\cup\Gamma_\sigma^-} e^{\lambda t}h_\infty^d(x,\lambda)\,d\lambda.$$

To progress our argumentation, by using the relation $1 = B^2/B^2 = (\lambda + A^2)/B^2$, we write $\mathbf{v}_\infty^{\mathbf{f}}$, $\pi_\infty^{\mathbf{f}}$, and $h_\infty^{\mathbf{f}}$ as follows: for $j = 1, \ldots, N$

(3.6.6)

$$\begin{aligned}
&v_{\infty j}^{\mathbf{f}}(x,\lambda)\\
&= \sum_{k=1}^N \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty \mathcal{V}_{jk}^{BB}(\xi',\lambda)(c_g + c_\sigma A^2)}{AL(A,B)} Ae^{-B(x_N+y_N)}\widehat{f}_k(y_N)\right](x')\,dy_N\\
&+ \sum_{k=1}^N \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty \lambda^{\frac12}\mathcal{V}_{jk}^{B\mathcal{M}}(\xi',\lambda)(c_g + c_\sigma A^2)}{AB^2L(A,B)} A\lambda^{\frac12}e^{-Bx_N}\mathcal{M}(y_N)\widehat{f}_k(y_N)\right](x')\,dy_N\\
&+ \sum_{k=1}^N \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty \mathcal{V}_{jk}^{B\mathcal{M}}(\xi',\lambda)(c_g + c_\sigma A^2)}{B^2L(A,B)} A^2e^{-Bx_N}\mathcal{M}(y_N)\widehat{f}_k(y_N)\right](x')\,dy_N\\
&+ \sum_{k=1}^N \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty \lambda^{\frac12}\mathcal{V}_{jk}^{\mathcal{M}B}(\xi',\lambda)(c_g + c_\sigma A^2)}{AB^2L(A,B)} A\lambda^{\frac12}\mathcal{M}(x_N)e^{-By_N}\widehat{f}_k(y_N)\right](x')\,dy_N\\
&+ \sum_{k=1}^N \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty \mathcal{V}_{jk}^{\mathcal{M}B}(\xi',\lambda)(c_g + c_\sigma A^2)}{B^2L(A,B)} A^2\mathcal{M}(x_N)e^{-By_N}\widehat{f}_k(y_N)\right](x')\,dy_N\\
&+ \sum_{k=1}^N \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty \mathcal{V}_{jk}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)(c_g + c_\sigma A^2)}{AB^2L(A,B)} A\lambda\mathcal{M}(x_N)\mathcal{M}(y_N)\widehat{f}_k(y_N)\right](x')\,dy_N
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{N}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty \mathcal{V}^{\mathcal{M}\mathcal{M}}_{jk}(\xi',\lambda)(c_g+c_\sigma A^2)}{AB^2L(A,B)}A^3\mathcal{M}(x_N)\mathcal{M}(y_N)\widehat{f}_k(y_N)\right](x')\,dy_N,\\
&\pi^{\mathbf{f}}_\infty(x,\lambda)\\
&=\sum_{k=1}^{N}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty \mathcal{P}^{AA}_{k}(\xi',\lambda)(c_g+c_\sigma A^2)}{AL(A,B)}Ae^{-A(x_N+y_N)}\widehat{f}_k(y_N)\right](x')\,dy_N\\
&+\sum_{k=1}^{N}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty \mathcal{P}^{A\mathcal{M}}_{k}(\xi',\lambda)(c_g+c_\sigma A^2)}{A^2L(A,B)}A^2e^{-Ax_N}\mathcal{M}(y_N)\widehat{f}_k(y_N)\right](x')\,dy_N,\\
&h^{\mathbf{f}}_\infty(x,\lambda)\\
&=-\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty i\xi_k(B-A)}{A(B+A)L(A,B)}Ae^{-A(x_N+y_N)}\widehat{f}_k(y_N)\right](x')\,dy_N\\
&-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty}{L(A,B)}Ae^{-A(x_N+y_N)}\widehat{f}_N(y_N)\right](x')\,dy_N\\
&+\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty 2i\xi_k B}{A(B+A)L(A,B)}A^2e^{-Ax_N}\mathcal{M}(y_N)\widehat{f}_k(y_N)\right](x')\,dy_N\\
&+\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty 2A}{(B+A)L(A,B)}A^2e^{-Ax_N}\mathcal{M}(y_N)\widehat{f}_N(y_N)\right](x')\,dy_N
\end{aligned}
$$

with (3.4.4), where we have set $\varphi_\infty=\varphi_\infty(\xi')$ and $\widehat{f}(y_N)=\widehat{f}(\xi',y_N)$. In addition, using the relations:

(3.6.7)

$$
\begin{aligned}
e^{-Bx_N}\widehat{g}(0)&=\int_0^\infty Be^{-B(x_N+y_N)}\widehat{g}(y_N)\,dy_N-\int_0^\infty e^{-B(x_N+y_N)}\widehat{D_Ng}(y_N)\,dy_N,\\
\mathcal{M}(x_N)\widehat{g}(0)&=\int_0^\infty\left(e^{-B(x_N+y_N)}+A\mathcal{M}(x_N+y_N)\right)\widehat{g}(y_N)\,dy_N\\
&\quad+\int_0^\infty \mathcal{M}(x_N+y_N)\widehat{D_Ng}(y_N)\,dy_N
\end{aligned}
$$

and the identity: $1=A^2/A^2=-\sum_{k=1}^{N-1}(i\xi_k)^2/A^2$, we write $\mathbf{v}^g_\infty$, π^g_∞, and h^g_∞ as follows: for $j=1,\ldots,N-1$

(3.6.8)

$$
\begin{aligned}
&v^g_{\infty j}(x,\lambda)\\
&=-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty i\xi_j(c_g+c_\sigma A^2)}{A^2L(A,B)}Ae^{-B(x_N+y_N)}\widehat{\Delta' g}(y_N)\right](x')\,dy_N\\
&+\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty \xi_j\xi_k(B-A)(c_g+c_\sigma A^2)}{A^3(B+A)L(A,B)}Ae^{-B(x_N+y_N)}\widehat{D_kD_Ng}(y_N)\right](x')\,dy_N\\
&-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty i\xi_j(B^2+A^2)(c_g+c_\sigma A^2)}{A^3(B+A)L(A,B)}A^2\mathcal{M}(x_N+y_N)\widehat{\Delta' g}(y_N)\right](x')\,dy_N
\end{aligned}
$$

$$- \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\varphi_\infty \xi_j \xi_k (B^2+A^2)(c_g + c_\sigma A^2)}{A^4(B+A)L(A,B)} \right.$$
$$\left. \times A^2 \mathcal{M}(x_N+y_N) \widehat{D_k D_N g}(y_N) \right] (x')\, dy_N,$$

$v^g_{\infty N}(x,\lambda)$

$$= - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\varphi_\infty (B-A)(c_g + c_\sigma A^2)}{A(B+A)L(A,B)} A e^{-B(x_N+y_N)} \widehat{\Delta' g}(y_N) \right] (x')\, dy_N$$
$$+ \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\varphi_\infty i\xi_k (c_g + c_\sigma A^2)}{A^2 L(A,B)} A e^{-B(x_N+y_N)} \widehat{D_k D_N g}(y_N) \right] (x')\, dy_N$$
$$+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\varphi_\infty (B^2+A^2)(c_g + c_\sigma A^2)}{A^2(B+A)L(A,B)} A^2 \mathcal{M}(x_N+y_N) \widehat{\Delta' g}(y_N) \right] (x')\, dy_N$$
$$- \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\varphi_\infty i\xi_k (B^2+A^2)(c_g + c_\sigma A^2)}{A^3(B+A)L(A,B)} \right.$$
$$\left. \times A^2 \mathcal{M}(x_N+y_N) \widehat{D_k D_N g}(y_N) \right] (x')\, dy_N,$$

$\pi^g_\infty(x,\lambda)$

$$= - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\varphi_\infty (B^2+A^2)(c_g + c_\sigma A^2)}{A^2 L(A,B)} A e^{-A(x_N+y_N)} \widehat{\Delta' g}(y_N) \right] (x')\, dy_N$$
$$+ \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\varphi_\infty i\xi_k (B^2+A^2)(c_g + c_\sigma A^2)}{A^3 L(A,B)} \right.$$
$$\left. \times A e^{-A(x_N+y_N)} \widehat{D_k D_N g}(y_N) \right] (x')\, dy_N,$$

$h^g_\infty(x,\lambda)$

$$= - \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\varphi_\infty D(A,B)}{A^2(B+A)L(A,B)} A e^{-A(x_N+y_N)} \widehat{\Delta' g}(y_N) \right] (x')\, dy_N$$
$$+ \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\frac{\varphi_\infty i\xi_k D(A,B)}{A^3(B+A)L(A,B)} A e^{-A(x_N+y_N)} \widehat{D_k D_N g}(y_N) \right] (x')\, dy_N.$$

REMARK 3.6.1. We extend $g \in W_p^{2-1/p}(\mathbf{R}^{N-1})$ to a function $\widetilde{g}$, which is defined on $\mathbf{R}^N_+$ and satisfies $\|\widetilde{g}\|_{W_p^2(\mathbf{R}^N_+)} \leq C\|g\|_{W_p^{2-1/p}(\mathbf{R}^{N-1})}$ for some positive constant C independent of g and $\widetilde{g}$. For simplicity, such $\widetilde{g}$ is denoted by g again in the present section.

3.6.1. Analysis on $\Gamma_4^\pm$. We first show the following lemma concerning estimates of the symbols defined as (1.2.1) with $\mu = 1$.

LEMMA 3.6.2. *Let A_0 be a positive number defined as in* (3.4.6).

(1) *There exists a positive constant $A_\infty \geq 1$ such that for any $A \geq A_\infty$ and $\lambda \in \Gamma_4^\pm$*

$$2^{-1}A \leq \mathrm{Re}B \leq |B| \leq 2A, \ \ |D(A,B)| \geq A^3, \ \ |L(A,B)| \geq (c_\sigma/16)(8^{-1}A)^3.$$

(2) *There exist positive constants C_1, C_2, and C such that for any $A \in [A_0/6, 2A_\infty]$ and $\lambda \in \Gamma_4^\pm$*

$$C_1 A \leq \mathrm{Re}B \leq |B| \leq C_2 A, \ \ |D(A,B)| \geq CA^3, \ \ |L(A,B)| \geq CA^3,$$

where A_∞ is the same constant as in (1).

(3) *Let $\alpha' \in \mathbf{N}_0^{N-1}$, $s \in \mathbf{R}$, and $a > 0$. Then there exist positive constants c and C, independent of a, such that for any $\lambda \in \Gamma_4^\pm$ and $A \geq A_0/6$*

$$|D_{\xi'}^{\alpha'} B^s| \leq C A^{s-|\alpha'|}, \quad |D_{\xi'}^{\alpha'} D(A,B)^s| \leq C A^{3s-|\alpha'|},$$
$$|D_{\xi'}^{\alpha'} e^{-Ba}| \leq C A^{-|\alpha'|} e^{-cAa}, \quad |D_{\xi'}^{\alpha'} L(A,B)^{-1}| \leq C A^{-3-|\alpha'|},$$
$$|D_{\xi'}^{\alpha'} \mathcal{M}(a)| \leq C A^{-1-|\alpha'|} e^{-cAa}.$$

Proof. (1) We first consider the estimates of B. For $\lambda \in \Gamma_4^\pm$, set $\sigma = \lambda + A^2 = -\gamma_\infty + A^2 \pm iu$ ($u \in [0, \widetilde{\gamma}_\infty]$) and $\theta = \arg \sigma$. Then we have

$$\mathrm{Re}B = |\sigma|^{\frac12} \cos\frac{\theta}{2} = \frac{|\sigma|^{\frac12}}{\sqrt2}(1+\cos\theta)^{\frac12} = \frac{1}{\sqrt2}(|\sigma| + A^2 - \gamma_\infty)^{\frac12},$$

so that

$$\mathrm{Re}B \geq \frac{1}{\sqrt2}\left(2A^2 - 2\gamma_\infty - \widetilde{\gamma}_\infty\right)^{\frac12} \geq \frac{A}{\sqrt2}$$

for any $A \geq A_\infty$, provided that A_∞ satisfies $A_\infty^2 \geq 2\gamma_\infty + \widetilde{\gamma}_\infty$. On the other hand, it clearly holds that $|B| \leq 2A$.

Next we show the inequality for $D(A,B)$. Since

$$\begin{aligned} D(A,B) &= B(B^2+3A^2) + A(B^2 - A^2) = B(\lambda + 4A^2) + \lambda A \\ &= 4A^2 B + (B+A)(-\gamma_\infty \pm iu), \end{aligned}$$

we see that by the inequality for B obtained above

$$\begin{aligned} |D(A,B)| &\geq 4A^2|B| - |B+A||-\gamma_\infty \pm iu| \\ &\geq 4A^2(\mathrm{Re}B) - (|B|+A)(\gamma_\infty + \widetilde{\gamma}_\infty) \\ &\geq 2A^3 - 3(\gamma_\infty + \widetilde{\gamma}_\infty)A \geq A^3 \end{aligned}$$

for any $A \geq A_\infty$, provided that A_∞ satisfies $A_\infty^2 \geq 3(\gamma_\infty + \widetilde{\gamma}_\infty)$.

Finally we show the inequality for $L(A,B)$. Since

$$B_1^2 - B^2 = -\frac{c_\sigma}{2}A - \frac{3}{16}c_\sigma^2 - (-\gamma_\infty \pm iu) + O(\frac{1}{A}) \quad \text{as } A \to \infty,$$

there exist positive constants A_∞ and C such that for any $A \geq A_\infty$ and $\lambda \in \Gamma_4^\pm$ we have $|B_1^2 - B^2| \geq (c_\sigma/4)A$, which, combined with the inequality for B and (3.6.1), furnishes that

$$|B_1 - B| \geq \frac{|B_1^2 - B^2|}{|B_1 + B|} \geq \frac{(c_\sigma/4)A}{4A} \geq \frac{c_\sigma}{16} \quad (A \geq A_\infty, \ \lambda \in \Gamma_4^\pm).$$

In addition, we have

$$B_2^2 - B^2 = -(1-a_2^2)A^2 + O(A) \quad \text{as } A \to \infty,$$

so that there exists a positive number A_∞ such that for any $A \geq A_\infty$ and $\lambda \in \Gamma_4^\pm$ we have $|B_2^2 - B^2| \geq (A^2/2)$, from which it follows that

$$|B_2 - B| = \frac{|B_2^2 - B^2|}{|B_2 + B|} \geq \frac{(A^2/2)}{4A} = \frac{A}{8}.$$

Since $|B - B_2| \leq |B - B_j|$ ($j = 3,4$), we thus obtain

$$|L(A,B)| \geq (c_\sigma/16)(8^{-1}A)^3 \quad (A \geq A_\infty, \ \lambda \in \Gamma_4^\pm).$$

(2) It sufficient to show the existence of positive constants C_1, C_2, and C such that for any $A \in [A_0/6, 2A_\infty]$ and $\lambda \in \Gamma_4^\pm$

$$C_1 \leq \mathrm{Re} B \leq |B| \leq C_2, \quad |D(A,B)| \geq C, \quad |L(A,B)| \geq C.$$

It is obvious that the inequalities for B holds, so that we here consider $D(A,B)$ and $L(A,B)$ only.

First we show the inequality for $D(A,B)$. Set

$$\widetilde{A} = \frac{A}{2}, \quad \widetilde{\lambda} = -\gamma_\infty + 3\widetilde{A}^2 \pm iu \quad \text{for } u \in [0, \widetilde{\gamma}_\infty],$$

and then note that $B = (\widetilde{\lambda} + \widetilde{A}^2)^{1/2}$. In addition, we see that

$$\{B/\widetilde{A} \in \mathbf{C} \mid \lambda \in \Gamma_4^\pm \text{ and } A \in [A_0/6, 2A_\infty]\} \subset \{z \in \mathbf{C} \mid 1 \leq \mathrm{Re} z\}.$$

In fact, setting $\sigma = 1 - (\gamma_\infty/A^2) \pm i(u/A^2)$ and $\theta = \arg \sigma$, we have

$$\begin{aligned}\mathrm{Re}\frac{B}{\widetilde{A}} &= 2|\sigma|^{1/2}\cos\frac{\theta}{2} = 2|\sigma|^{1/2}\left(\frac{1+\cos\theta}{2}\right)^{1/2} = \sqrt{2}(|\sigma| + \mathrm{Re}\sigma)^{1/2} \\ &\geq 2(\mathrm{Re}\sigma)^{1/2} = 2\left(1 - \frac{\gamma_\infty}{A^2}\right)^{1/2} \geq 2\left(1 - \frac{4^{-1}\times(A_0/6)^2}{(A_0/6)^2}\right)^{1/2} = \sqrt{3}.\end{aligned}$$

Together with Lemma 3.5.8 and the formula:

$$\begin{aligned}D(A,B) &= B^3 + 2\widetilde{A}B^2 + 12\widetilde{A}^2 B - 8\widetilde{A}^3 \\ &= \widetilde{A}^3\left\{\left(\frac{B}{\widetilde{A}}\right)^3 + 2\left(\frac{B}{\widetilde{A}}\right)^2 + 12\left(\frac{B}{\widetilde{A}}\right) - 8\right\},\end{aligned}$$

there exists a positive constant C such that $|D(A,B)| \geq C$ for any $A \in [A_0/6, 2A_\infty]$ and $\lambda \in \Gamma_4^\pm$. The inequality for $L(A,B)$ follows from the definition of the integral path $\Gamma_4^\pm$.

(3) We see that by (1) and (2) there exist positive constants C_1, C_2, and C such that for any $\lambda \in \Gamma_4^\pm$ and $A \geq A_0/6$

$$C_1 A \leq \mathrm{Re} B \leq |B| \leq C_2 A, \ \ |D(A,B)| \geq CA^3, \ \ |L(A,B)| \geq CA^3. \tag{3.6.9}$$

We thus obtain the required inequalities by using Leibniz's rule and Bell's formula (cf. e.g. [**SS12**, Lemma 5.2, Lemma 5.3, Lemma 7.2]), because by (3.6.9)

$$\begin{aligned}|D_{\xi'}^{\alpha'} D(A,B)| &= |D_{\xi'}^{\alpha'}(B^3 + AB^2 + 3A^2 B - A^3)| \leq CA^3, \\ |D_{\xi'}^{\alpha'} L(A,B)| &= \left|D_{\xi'}^{\alpha'}\left(\frac{\lambda}{B+A}D(A,B) + A(c_g + c_\sigma A^2)\right)\right| \leq CA^3\end{aligned}$$

for any $\alpha' \in \mathbf{N}_0^{N-1}$, $\lambda \in \Gamma_4^\pm$, and $A \geq A_0/6$. □

Now we have a multiplier theorem on $\Gamma_4^\pm$.

Lemma 3.6.3. *Let $1 < p < \infty$ and $f \in L_p(\mathbf{R}_+^N)$. We use the symbols defined as* (B.1) *with $k_n(\xi', \lambda) = \varphi_\infty(\xi')\kappa_n(\xi', \lambda)$ $(n = 1, \dots, 10)$, and suppose that for any $\alpha' \in \mathbf{N}_0^{N-1}$ there exists a positive constant $C(\alpha')$ such that*

$$|D_{\xi'}^{\alpha'}\kappa_n(\xi', \lambda)| \leq C(\alpha')A^{-|\alpha'|} \quad (n = 1, \dots, 10)$$

for any $\lambda \in \Gamma_4^\pm$ and $A \geq A_0/6$, where A_0 is defined as in (3.4.6)*. Then there exists a positive constant $C(N,p)$ such that for any $\lambda \in \Gamma_4^\pm$*

$$\|K_n(\lambda)f\|_{L_p(\mathbf{R}_+^N)} \leq C(N,p)\|f\|_{L_p(\mathbf{R}_+^N)} \quad (n = 1, \dots, 10).$$

PROOF. Since $|\lambda|^{\frac{1}{2}} \leq CA$ for $\lambda \in \Gamma_4^{\pm}$ and $A \geq A_0/6$ with some positive constant C, we can deal with $K_5(\lambda), K_7(\lambda)$, and $K_{10}(\lambda)$ in the same manner as $K_4(\lambda), K_6(\lambda)$, and $K_9(\lambda)$, respectively.

As mentioned above, it is sufficient to consider $K_i(\lambda)$ for $i = 1, 2, 3, 4, 6, 8, 9$. We can show such cases by Lemma 3.6.2 and calculations similar to the proof of Lemma B.1, so that we may omit the detailed proof. □

COROLLARY 3.6.4. *Let $1 < p < \infty$ and let $f \in L_p(\mathbf{R}_+^N)$. We use the symbols defined as* (B.1) *with $k_n(\xi', \lambda) = \varphi_\infty(\xi')\kappa_n(\xi', \lambda)$ $(n = 1, \ldots, 10)$.*

(1) *Suppose that for any $\alpha' \in \mathbf{N}_0^{N-1}$ there exists a positive constant $C(\alpha')$ such that for any $\lambda \in \Gamma_4^{\pm}$ and $A \geq A_0/6$*

$$|D_{\xi'}^{\alpha'} \kappa_n(\xi', \lambda)| \leq C(\alpha') A^{-l-|\alpha'|} \quad (i = 1, 2, 3, \;\; n = 1, 2).$$

Then there exists a positive constant $C(N, p)$ such that for any $\lambda \in \Gamma_4^{\pm}$

$$\|(\lambda K_n(\lambda) f, K_n(\lambda) f)\|_{W_p^l(\mathbf{R}_+^N)} \leq C(N, p) \|f\|_{L_p(\mathbf{R}_+^N)}$$

with $l = 1, 2, 3$ and $n = 1, 2$.

(2) *Suppose that for any $\alpha' \in \mathbf{N}_0^{N-1}$ there exists a positive constant $C(\alpha')$ such that for any $\lambda \in \Gamma_4^{\pm}$ and $A \geq A_0/6$*

$$|D_{\xi'}^{\alpha'} \kappa_n(\xi', \lambda)| \leq C(\alpha') A^{-2-|\alpha'|} \quad (n = 3, \ldots, 10). \tag{3.6.10}$$

Then there exists a positive constant $C(N, p)$ such that for any $\lambda \in \Gamma_4^{\pm}$ and $n = 3, \ldots, 10$

$$\|\lambda K_n(\lambda) f\|_{L_p(\mathbf{R}_+^N)} + \|K_n(\lambda) f\|_{W_p^2(\mathbf{R}_+^N)} \leq C(N, p) \|f\|_{L_p(\mathbf{R}_+^N)}.$$

Thanks to Corollary 3.6.4, the following lemma holds.

LEMMA 3.6.5. *Let $1 < p < \infty, \mathbf{f} \in L_p(\mathbf{R}_+^N)^N$, and $g \in W_p^2(\mathbf{R}_+^N)$. In addition, let $\mathbf{v}_\infty^d = (v_{\infty 1}^d, \ldots, v_{\infty N}^d)^T, \pi_\infty^d$, and h_∞^d be given by* (3.6.6) *and* (3.6.8) *for $d \in \{\mathbf{f}, g\}$. Then there exist a positive constant $C = C(N, p)$ such that for any $\lambda \in \Gamma_4^{\pm}$*

$$\|(\lambda \mathbf{v}_\infty^{\mathbf{f}}, \nabla \pi_\infty^{\mathbf{f}})\|_{L_p(\mathbf{R}_+^N)} + \|(\mathbf{v}_\infty^{\mathbf{f}}, \lambda h_\infty^{\mathbf{f}})\|_{W_p^2(\mathbf{R}_+^N)} + \|h_\infty^{\mathbf{f}}\|_{W_p^3} \leq C \|\mathbf{f}\|_{L_p(\mathbf{R}_+^N)},$$
$$\|(\lambda \mathbf{v}_\infty^g, \nabla \pi_\infty^g)\|_{L_p(\mathbf{R}_+^N)} + \|(\mathbf{v}_\infty^g, \lambda h_\infty^g)\|_{W_p^2(\mathbf{R}_+^N)} + \|h_\infty^g\|_{W_p^3} \leq C \|g\|_{W_p^2(\mathbf{R}_+^N)}.$$

PROOF. By (3.4.4), Lemma 3.6.2, and Leibniz's rule, we see that for $j, k = 1, \ldots, N-1$ and $J, K = 1, \ldots, N$

(3.6.11)

$$\frac{\mathcal{V}_{JK}^{BB}(\xi', \lambda)(c_g + c_\sigma A^2)}{AL(A, B)}, \quad \frac{\lambda^{\frac{1}{2}} \mathcal{V}_{JK}^{B\mathcal{M}}(\xi', \lambda)(c_g + c_\sigma A^2)}{AB^2 L(A, B)}, \quad \frac{\mathcal{V}_{JK}^{B\mathcal{M}}(\xi', \lambda)(c_g + c_\sigma A^2)}{B^2 L(A, B)},$$
$$\frac{\lambda^{\frac{1}{2}} \mathcal{V}_{JK}^{\mathcal{M}B}(\xi', \lambda)(c_g + c_\sigma A^2)}{AB^2 L(A, B)}, \quad \frac{\mathcal{V}_{JK}^{\mathcal{M}B}(\xi', \lambda)(c_g + c_\sigma A^2)}{B^2 L(A, B)}, \quad \frac{\mathcal{V}_{JK}^{\mathcal{M}\mathcal{M}}(\xi', \lambda)(c_g + c_\sigma A^2)}{AB^2 L(A, B)},$$
$$\frac{\mathcal{V}_{JK}^{\mathcal{M}\mathcal{M}}(\xi', \lambda)(c_g + c_\sigma A^2)}{AB^2 L(A, B)}, \quad \frac{i\xi_j(c_g + c_\sigma A^2)}{A^2 L(A, B)}, \quad \frac{\xi_j \xi_k (B - A)(c_g + c_\sigma A^2)}{A^3 (B + A) L(A, B)},$$
$$\frac{i\xi_j (B^2 + A^2)(c_g + c_\sigma A^2)}{A^3 (B + A) L(A, B)}, \quad \frac{\xi_j \xi_k (B^2 + A^2)}{A^4 (B + A) L(A, B)}, \quad \frac{(B - A)(c_g + c_\sigma A^2)}{A(B + A) L(A, B)},$$
$$\frac{i\xi_k(c_g + c_\sigma A^2)}{A^2 L(A, B)}, \quad \frac{(B^2 + A^2)(c_g + c_\sigma A^2)}{A^2 (B + A) L(A, B)}, \quad \frac{i\xi_k (B^2 + A^2)(c_g + c_\sigma A^2)}{A^3 (B + A) L(A, B)}$$

satisfy the condition (3.6.10) for$\lambda \in \Gamma_4^\pm$ and $A \geq A_0/6$, so that Corollary 3.6.4 (2) yields that

$$\|\lambda \mathbf{v}_\infty^{\mathbf{f}}\|_{L_q(\mathbf{R}_+^N)} + \|\mathbf{v}_\infty^{\mathbf{f}}\|_{W_p^2(\mathbf{R}_+^N)} \leq C(N,p)\|\mathbf{f}\|_{L_p(\mathbf{R}_+^N)},$$
$$\|\lambda \mathbf{v}_\infty^{g}\|_{L_q(\mathbf{R}_+^N)} + \|\mathbf{v}_\infty^{g}\|_{W_p^2(\mathbf{R}_+^N)} \leq C(N,p)\|g\|_{W_p^2(\mathbf{R}_+^N)}$$

for any $\lambda \in \Gamma_4^\pm$ with some positive constant $C(N,p)$. The other assertions follow from Corollary 3.6.4 (1) analogously. □

Applying Lemma 3.6.5 to the formulas (3.6.5), we have

(3.6.12)
$$\begin{aligned}
&\|(\partial_t S_\infty^{\mathbf{f},4}(t)\mathbf{F}, \nabla \Pi_\infty^{\mathbf{f},4}(t)\mathbf{F})\|_{L_p(\mathbf{R}_+^N)} + \|S_\infty^{\mathbf{f},4}(t)\mathbf{F}\|_{W_p^2(\mathbf{R}_+^N)} \\
&\quad + \|\partial_t \mathcal{E}(T_\infty^{\mathbf{f},4}(t)\mathbf{F})\|_{W_p^2(\mathbf{R}_+^N)} + \|\mathcal{E}(T_\infty^{\mathbf{f},4}(t)\mathbf{F})\|_{W_p^3(\mathbf{R}_+^N)} \leq Ce^{-\gamma_\infty t}\|\mathbf{f}\|_{L_p(\mathbf{R}_+^N)}, \\
&\|(\partial_t S_\infty^{g,4}(t)\mathbf{F}, \nabla \Pi_\infty^{g,4}(t)\mathbf{F})\|_{L_p(\mathbf{R}_+^N)} + \|S_\infty^{g,4}(t)\mathbf{F}\|_{W_p^2(\mathbf{R}_+^N)} \\
&\quad + \|\mathcal{E}(T_\infty^{g,4}(t)\mathbf{F})\|_{W_p^3(\mathbf{R}_+^N)} \leq Ce^{-\gamma_\infty t}\|g\|_{W_p^2(\mathbf{R}_+^N)}
\end{aligned}$$

for any $t > 0$ with some positive constant $C = C(N,p)$.

REMARK 3.6.6. We will show estimates concerning $\partial_t \mathcal{E}(T_\infty^g(t)\mathbf{F})$ in the last part of this section.

3.6.2. Analysis on $\Gamma_5^\pm$. We start with the following lemma.

LEMMA 3.6.7. *Let $1 < p < \infty$ and $f \in L_p(\mathbf{R}_+^N)$. We use the symbols defined as* (B.1) *with $k_n(\xi',\lambda) = \varphi_\infty(\xi')\kappa_n(\xi',\lambda)$ $(n = 1,\dots,10)$.*

(1) *Let $\alpha' \in \mathbf{N}_0^{N-1}$, and suppose that there exists a positive constant $C(\alpha')$ such that for any $\lambda \in \Gamma_5^\pm$, $A \geq A_0/6$, and $n = 1, 2$*
$$|D_{\xi'}^{\alpha'}\kappa_n(\xi',\lambda)| \leq \frac{C(\alpha')(|\lambda|^{1/2}+A)^2 A^{-|\alpha'|}}{|\lambda|(|\lambda|^{1/2}+A)^2 + A(c_g + c_\sigma A^2)}.$$
Then there exists a positive constant $C(N,p)$ such that for any $\lambda \in \Gamma_5^\pm$
$$\|\nabla K_n(\lambda)f\|_{L_p(\mathbf{R}_+^N)} \leq C(N,p)\|f\|_{L_p(\mathbf{R}_+^N)} \quad (n = 1, 2).$$

(2) *Let $\alpha' \in \mathbf{N}_0^{N-1}$, and suppose that there exists a positive constant $C(\alpha')$ such that for any $\lambda \in \Gamma_5^\pm$, $A \geq A_0/6$, and $n = 1, 2$*
$$|D_{\xi'}^{\alpha'}\kappa_n(\xi',\lambda)| \leq \frac{C(\alpha')A^{-|\alpha'|}}{|\lambda|(|\lambda|^{1/2}+A)^2 + A(c_g + c_\sigma A^2)}.$$
Then there exists a positive constant $C = C(N,p)$ such that for every $\lambda \in \Gamma_5^\pm$ and $n = 1, 2$
$$\|(\lambda^2 K_n(\lambda)f, \lambda^{3/2}\nabla K_n(\lambda)f, \lambda\nabla^2 K_n(\lambda)f, \nabla^3 K_n(\lambda)f)\|_{L_p(\mathbf{R}_+^N)} \leq C\|f\|_{L_p(\mathbf{R}_+^N)}.$$

(3) *Let $\alpha' \in \mathbf{N}_0^{N-1}$, and suppose that there exists a positive constant $C(\alpha')$ such that for any $\lambda \in \Gamma_5^\pm$, $A \geq A_0/6$, and $n = 1, 2$*
$$|D_{\xi'}^{\alpha'}\kappa_n(\xi',\lambda)| \leq \frac{C(\alpha')(|\lambda|^{1/2}+A)^2 A^{-|\alpha'|}}{A^2\{|\lambda|(|\lambda|^{1/2}+A)^2 + A(c_g + c_\sigma A^2)\}}.$$

Then there exists a positive constant $C(N,p)$ *such that for every* $\lambda \in \Gamma_5^{\pm}$ *and* $n = 1, 2$

$$\|\lambda K_n(\lambda) f\|_{W_p^2(\mathbf{R}_+^N)} + \|K_n(\lambda) f\|_{W_p^3(\mathbf{R}_+^N)} \leq C(N,p)\|f\|_{L_p(\mathbf{R}_+^N)}.$$

(4) *Let* $\alpha' \in \mathbf{N}_0^{N-1}$, *and suppose that there exists a positive constant* $C(\alpha')$ *such that for any* $\lambda \in \Gamma_5^{\pm}$, $A \geq A_0/6$, *and* $n = 3, \ldots, 10$

$$|D_{\xi'}^{\alpha'} \kappa_n(\xi', \lambda)| \leq \frac{C(\alpha')(|\lambda|^{1/2} + A)A^{-|\alpha'|}}{|\lambda|(|\lambda|^{1/2} + A)^2 + A(c_g + c_\sigma A^2)}. \tag{3.6.13}$$

Then there exists a positive constant $C(N,p)$ *such that for any* $\lambda \in \Gamma_5^{\pm}$ *and* $n = 3, \ldots, 10$

$$\|(\lambda^{3/2} K_n(\lambda) f, \lambda \nabla K_n(\lambda) f, \nabla^2 K_n(\lambda) f)\|_{L_p(\mathbf{R}_+^N)} \leq C(N,p)\|f\|_{L_p(\mathbf{R}_+^N)}.$$

PROOF. Noting that $\Gamma_5^{\pm} \subset \Sigma_{\varepsilon_\infty, \lambda_0(\varepsilon_\infty)}$ (see the beginning of this section concerning ε_∞ and $\lambda_0(\varepsilon_\infty)$), we have the required properties by Corollary B.3 (1). □

Now we have the following lemma.

LEMMA 3.6.8. *Let* $1 < p < \infty$, $\mathbf{f} \in L_p(\mathbf{R}_+^N)^N$, *and* $g \in W_p^2(\mathbf{R}_+^N)$. *In addition, let* $\mathbf{v}_\infty^d = (v_{\infty 1}^d, \ldots, v_{\infty N}^d)^T, \pi_\infty^d$, *and* h_∞^d *be given by* (3.6.6) *and* (3.6.8) *for* $d \in \{\mathbf{f}, g\}$. *Then there exists a positive constant* $C(N,p)$ *such that for any* $\lambda \in \Gamma_5^{\pm}$

$$\|(\lambda^{3/2}\mathbf{v}_\infty^{\mathbf{f}}, \lambda\nabla\mathbf{v}_\infty^{\mathbf{f}}, \nabla^2\mathbf{v}_\infty^{\mathbf{f}}, \nabla\pi_\infty^{\mathbf{f}})\|_{L_p(\mathbf{R}_+^N)} \leq C(N,p)\|\mathbf{f}\|_{L_p(\mathbf{R}_+^N)},$$
$$\|(\lambda^2 h_\infty^{\mathbf{f}}, \lambda^{3/2}\nabla h_\infty^{\mathbf{f}}, \lambda\nabla^2 h_\infty^{\mathbf{f}}, \nabla^3 h_\infty^{\mathbf{f}})\|_{L_p(\mathbf{R}_+^N)} \leq C(N,p)\|\mathbf{f}\|_{L_p(\mathbf{R}_+^N)},$$

and besides,

$$\|(\lambda^{3/2}\mathbf{v}_\infty^{g}, \lambda\nabla\mathbf{v}_\infty^{g}, \nabla^2\mathbf{v}_\infty^{g}, \nabla\pi_\infty^{g})\|_{L_p(\mathbf{R}_+^N)} \leq C(N,p)\|g\|_{W_p^2(\mathbf{R}_+^N)},$$
$$\|\lambda h_\infty^g\|_{W_p^2(\mathbf{R}_+^N)} + \|h_\infty^g\|_{W_p^3(\mathbf{R}_+^N)} \leq C(N,p)\|g\|_{W_p^2(\mathbf{R}_+^N)}.$$

PROOF. Since $\Gamma_5^{\pm} \subset \Sigma_{\varepsilon_\infty, \lambda_0(\varepsilon_\infty)}$ and the symbols (3.6.11) satisfy the condition (3.6.13) by Lemma 1.2.6 and Leibniz's rule, we have, by Lemma 3.6.7 (4),

$$\|(\lambda^{3/2}\mathbf{v}_\infty^{\mathbf{f}}, \lambda\nabla\mathbf{v}_\infty^{\mathbf{f}}, \nabla^2\mathbf{v}_\infty^{\mathbf{f}})\|_{L_p(\mathbf{R}_+^N)} \leq C(N,p)\|\mathbf{f}\|_{L_p(\mathbf{R}_+^N)},$$
$$\|(\lambda^{3/2}\mathbf{v}_\infty^{g}, \lambda\nabla\mathbf{v}_\infty^{g}, \nabla^2\mathbf{v}_\infty^{g})\|_{L_p(\mathbf{R}_+^N)} \leq C(N,p)\|g\|_{W_p^2(\mathbf{R}_+^N)}$$

with some positive constant $C(N,p)$. Concerning the other estimates, we similarly have the following observation: First Lemma 3.6.7 (1) yields the estimates of $\nabla\pi_\infty^d$ for $d \in \{\mathbf{f}, g\}$. Second Lemma 3.6.7 (2) yields the estimates of $h_\infty^{\mathbf{f}}$. Finally Lemma 3.6.7 (3) yields the estimates of h_∞^g. □

As was seen in the proof of Lemma 3.5.16, it holds that for $\lambda \in \Gamma_5^{\pm}$

$$\int_0^\infty \frac{e^{(\mathrm{Re}\lambda)t}}{|\lambda|^{1-s}}\, du \leq C \begin{cases} (|\log t| + 1)e^{-\gamma_\infty t} & (s = 0), \\ t^{-s}e^{-\gamma_\infty t} & (s > 0). \end{cases}$$

Then, applying Lemma 3.6.8 to the formulas (3.6.5), we have

(3.6.14)
$$\begin{aligned}&\|(\nabla^2 S^{\mathbf{f},5}_\infty(t)\mathbf{F}, \nabla\Pi^{\mathbf{f},5}_\infty(t)\mathbf{F}, \nabla^2\partial_t\mathcal{E}(T^{\mathbf{f},5}_\infty(t)\mathbf{F}), \nabla^3\mathcal{E}(T^{\mathbf{f},5}_\infty(t)F))\|_{L_p(\mathbf{R}^N_+)}\\ &\quad\le Ct^{-1}e^{-\gamma_\infty t}\|\mathbf{f}\|_{L_p(\mathbf{R}^N_+)},\\ &\|(\partial_t S^{\mathbf{f},5}_\infty(t)\mathbf{F}, \nabla\partial_t\mathcal{E}(T^{\mathbf{f},5}_\infty(t)\mathbf{F}))\|_{L_p(\mathbf{R}^N_+)} \le Ct^{-1/2}e^{-\gamma_\infty t}\|\mathbf{f}\|_{L_p(\mathbf{R}^N_+)},\\ &\|S^{\mathbf{f},5}_\infty(t)\mathbf{F}\|_{W^1_p(\mathbf{R}^N_+)} + \|\mathcal{E}(T^{\mathbf{f},5}_\infty(t)\mathbf{F})\|_{W^2_p(\mathbf{R}^N_+)} + \|\partial_t\mathcal{E}(T^{\mathbf{f},5}_\infty(t)\mathbf{F})\|_{L_p(\mathbf{R}^N_+)}\\ &\quad\le C(|\log t|+1)e^{-\gamma_\infty t}\|\mathbf{f}\|_{L_p(\mathbf{R}^N_+)},\end{aligned}$$

and furthermore,

(3.6.15)
$$\begin{aligned}&\|(\nabla^2 S^{g,5}_\infty(t)\mathbf{F}, \nabla\Pi^{g,5}_\infty(t)\mathbf{F}, \nabla^3\mathcal{E}(T^{g,5}_\infty(t)\mathbf{F}))\|_{L_p(\mathbf{R}^N_+)} \le Ct^{-1}e^{-\gamma_\infty t}\|g\|_{W^2_p(\mathbf{R}^N_+)},\\ &\|\partial_t S^{g,5}_\infty(t)\mathbf{F}\|_{L_p(\mathbf{R}^N_+)} \le Ct^{-1/2}e^{-\gamma_\infty t}\|g\|_{W^2_p(\mathbf{R}^N_+)},\\ &\|S^{g,5}_\infty(t)\mathbf{F}\|_{W^1_p(\mathbf{R}^N_+)} + \|\mathcal{E}(T^{g,5}_\infty(t)\mathbf{F})\|_{W^2_p(\mathbf{R}^N_+)} \le C(|\log t|+1)e^{-\gamma_\infty t}\|g\|_{W^2_p(\mathbf{R}^N_+)}\end{aligned}$$

for any $t>0$ with some positive constant $C=C(N,p)$.

Finally we consider $\partial_t\mathcal{E}(T^g_\infty(t)\mathbf{F})$. By (3.6.7) and the same manner as in the last part of Section 3.5, we have

$$\begin{aligned}&\partial_t\mathcal{E}(T^g_\infty(t)\mathbf{F})\\ &=\frac{1}{2\pi i}\int_{\Gamma(\varepsilon_\infty)} e^{\lambda t}\Big\{\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{\varphi_\infty(c_g+c_\sigma A^2)}{AL(A,B)}Ae^{-A(x_N+y_N)}\widehat{\Delta' g}(y_N)\Big](x')\,dy_N\\ &+\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{\varphi_\infty i\xi_k(c_g+c_\sigma A^2)}{A^2L(A,B)}Ae^{-A(x_N+y_N)}\widehat{D_kD_N g}(y_N)\Big](x')\,dy_N\Big\}\,d\lambda.\end{aligned}$$

Thanks to the analysis on $\Gamma^\pm_4$ and $\Gamma^\pm_5$, we obtain

(3.6.16)
$$\begin{aligned}&\|\partial_t\mathcal{E}(T^g_\infty(t)\mathbf{F})\|_{W^1_p(\mathbf{R}^N_+)} \le C(|\log t|+1)e^{-\gamma_\infty t}\|g\|_{W^2_p(\mathbf{R}^N_+)},\\ &\|\nabla^2\partial_t\mathcal{E}(T^g_\infty(t)\mathbf{F})\|_{L_p(\mathbf{R}^N_+)} \le Ct^{-1}e^{-\gamma_\infty t}\|g\|_{W^2_p(\mathbf{R}^N_+)}\end{aligned}$$

for every $t>0$ and a positive constant $C=C(N,p)$.

Summing up (3.6.12), (3.6.14), (3.6.15), and (3.6.16), we have obtained the following theorem, noting the relations (3.4.7), (3.4.9), (3.6.3), and (3.6.5).

THEOREM 3.6.9. *Let $1<p<\infty$ and $\mathbf{F}\in X^2_p$ defined as* (3.1.2). *Then there exist positive constants δ and $C=C(N,p)$ such that for every $t>0$*

$$\begin{aligned}&\|(\nabla^2 S_\infty(t)\mathbf{F}, \nabla\Pi_\infty(t)\mathbf{F}, \nabla^3\mathcal{E}(T_\infty(t)\mathbf{F}), \nabla^2\partial_t\mathcal{E}(T_\infty(t)\mathbf{F}))\|_{L_p(\mathbf{R}^N_+)}\\ &\quad\le Ct^{-1}e^{-\delta t}\|\mathbf{F}\|_{X^2_p},\\ &\|(\partial_t S_\infty(t)\mathbf{F}, \nabla\partial_t\mathcal{E}(T_\infty(t)\mathbf{F}))\|_{L_p(\mathbf{R}^N_+)} \le Ct^{-1/2}e^{-\delta t}\|\mathbf{F}\|_{X^2_p},\\ &\|S_\infty(t)\mathbf{F}\|_{W^1_p(\mathbf{R}^N_+)} + \|\mathcal{E}(T_\infty(t)\mathbf{F})\|_{W^2_p(\mathbf{R}^N_+)} + \|\partial_t\mathcal{E}(T_\infty(t)\mathbf{F})\|_{L_p(\mathbf{R}^N_+)}\\ &\quad\le C(|\log t|+1)e^{-\delta t}\|\mathbf{F}\|_{X^2_p}.\end{aligned}$$

In particular, Theorem 3.6.9 completes the proof of Theorem 3.1.3 (2).

CHAPTER 4

Global well-posedness of a free boundary problem for the incompressible Navier-Stokes equations in some unbounded domain

4.1. Main results

In this chapter, we show the global well-posedess and large-time behavior of solutions for the following incompressible Navier-Stokes equations:

$$
\begin{cases}
\rho(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) = \operatorname{Div}\mathbf{S}(\mathbf{v},p) - \rho c_g \mathbf{e}_3 & \text{in } \Omega(t),\ t>0,\\
\operatorname{div}\mathbf{v} = 0 & \text{in } \Omega(t),\ t>0,\\
\mathbf{S}(\mathbf{v},p)\mathbf{n}_\Gamma = c_\sigma \kappa_\Gamma \mathbf{n}_\Gamma & \text{on } \Gamma(t),\ t>0,\\
V_\Gamma = \mathbf{v}\cdot\mathbf{n}_\Gamma & \text{on } \Gamma(t),\ t>0,\\
\mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega_0,\\
\Gamma|_{t=0} = \Gamma_0. &
\end{cases}
\tag{4.1.1}
$$

Here $\mathbf{v} = \mathbf{v}(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))^T$ and $p = p(x,t)$ denote the velocity field of a fluid occupying $\Omega(t)$ and the pressure field at $x\in\Omega(t)$ for $t>0$, respectively. In addition, ρ, c_g, and c_σ are positive constants which describe the density of the fluid, gravitational acceleration, and surface tension coefficient, respectively. The stress tensor $\mathbf{S}(\mathbf{v},p)$ is given by

$$
\mathbf{S}(\mathbf{v},p) = -p\mathbf{I} + \mu\mathbf{D}(\mathbf{v}), \quad \mathbf{D}(\mathbf{v}) = \nabla\mathbf{v} + (\nabla\mathbf{v})^T,
$$

where $\mu>0$ is the viscosity coefficient of the fluid.

We suppose that the unknown free surface $\Gamma(t)$ and domain $\Omega(t)$ are given by a scalar function $h = h(x',t)$ as follows:

$$
\begin{aligned}
\Gamma(t) &= \{(x', x_3) \mid x'\in\mathbf{R}^2,\ x_3 = h(x',t)\},\\
\Omega(t) &= \{(x', x_3) \mid x'\in\mathbf{R}^2,\ x_3 < h(x',t)\}.
\end{aligned}
$$

In addition, we denote the outer unit normal filed on $\Gamma(t)$ by $\mathbf{n}_\Gamma$, while the evolution velocity of $\Gamma(t)$ with respect to $\mathbf{n}_\Gamma$ by V_Γ and the mean curvature of $\Gamma(t)$ by κ_Γ, respectively. It then holds that

$$
\begin{aligned}
\mathbf{n}_\Gamma &= \frac{1}{\sqrt{1+|\nabla' h(x',t)|^2}}\begin{pmatrix} -\nabla' h(x',t)\\ 1\end{pmatrix}, \quad V_\Gamma = \frac{\partial_t h(x',t)}{\sqrt{1+|\nabla' h(x',t)|^2}},\\
\kappa_\Gamma &= \nabla'\cdot\left(\frac{\nabla' h(x',t)}{\sqrt{1+|\nabla' h(x',t)|^2}}\right) = \Delta' h(x',t) - G_\kappa(h),
\end{aligned}
$$

where for $\partial_j = \partial/\partial x_j$ $(j = 1, 2, 3)$

$$G_\kappa(h) = \frac{|\nabla' h(x',t)|^2 \Delta' h(x',t)}{(1+\sqrt{1+|\nabla' h(x',t)|^2})\sqrt{1+|\nabla' h(x',t)|^2}} + \sum_{j,k=1}^{2} \frac{\partial_j h(x',t)\partial_k h(x',t)\partial_j\partial_k h(x',t)}{(1+|\nabla' h(x',t)|^2)^{3/2}}.$$

The initial data are given by $\mathbf{v}_0 = \mathbf{v}_0(x)$ for the velocity field and $h_0 = h_0(x')$ for the free surface, which means that

$$\Gamma_0 = \{(x', x_3) \mid x' \in \mathbf{R}^2,\ x_3 = h_0(x')\},$$
$$\Omega_0 = \{(x', x_3) \mid x' \in \mathbf{R}^2,\ x_3 < h_0(x')\},$$

and besides, the outer unit normal filed on Γ_0 is denoted by $\mathbf{n}_0$. Noting $\mathbf{e}_3 = \nabla x_3$ and setting $\pi = p + \rho c_g x_3$, we reduce (4.1.1) to

$$(4.1.2)\qquad \begin{cases} \rho(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) = \mu\Delta\mathbf{v} - \nabla\pi & \text{in } \Omega(t),\ t>0, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega(t),\ t>0, \\ \mathbf{S}(\mathbf{v},\pi)\mathbf{n}_\Gamma + (\rho c_g h - c_\sigma \Delta' h)\mathbf{n}_\Gamma = -c_\sigma G_\kappa(h)\mathbf{n}_\Gamma & \text{on } \Gamma(t),\ t>0, \\ \partial_t h + \mathbf{v}'\cdot\nabla' h - \mathbf{v}\cdot\mathbf{e}_3 = 0 & \text{on } \Gamma(t),\ t>0, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega_0, \\ h|_{t=0} = h_0 & \text{on } \mathbf{R}^2. \end{cases}$$

To solve the equations (4.1.2), we consider the following auxiliary problem:

$$(4.1.3)\qquad \begin{cases} \Delta H = 0 & \text{in } \mathbf{R}^3_-,\ t \geq 0, \\ H = h & \text{on } \mathbf{R}^3_0,\ t \geq 0. \end{cases}$$

We shall solve (4.1.2) with (4.1.3) instead of the equations (4.1.1) in this chapter, that is, we find $(\mathbf{v}, \pi, h, H)$ satisfying (4.1.2) and (4.1.3). Then our main result is stated as follows:

THEOREM 4.1.1. *Let ρ, μ, c_g, and c_σ be positive constants, and suppose that exponents p, q satisfy*

$$(4.1.4)\qquad 2 < p < \infty, \quad 3 < q < \frac{16}{5}, \quad \frac{2}{p} + \frac{3}{q} < 1.$$

Then there exists a positive number ε_0 such that for every initial data

$$\mathbf{v}_0 \in (B^{2(1-1/p)}_{q,p}(\Omega_0) \cap B^{2(1-1/p)}_{q/2,p}(\Omega_0))^3,$$
$$h_0 \in B^{3-1/p-1/q}_{q,p}(\mathbf{R}^2) \cap B^{3-1/p-1/2}_{2,p}(\mathbf{R}^2) \cap L_{q/2}(\mathbf{R}^2)$$

satisfying the smallness condition:

$$\|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega_0)\cap B^{2(1-1/p)}_{q/2,p}(\Omega_0)} + \|h_0\|_{B^{3-1/p-1/q}_{q,p}(\mathbf{R}^2)\cap B^{3-1/p-1/2}_{2,p}(\mathbf{R}^2)\cap L_{q/2}(\mathbf{R}^2)} < \varepsilon_0$$

and the compatibility conditions:

$$\operatorname{div}\mathbf{v}_0 = 0 \ \text{ in } \Omega_0, \quad \mathbf{D}(\mathbf{v}_0)\mathbf{n}_0 - (\mathbf{n}_0\cdot\mathbf{D}(\mathbf{v}_0)\mathbf{n}_0)\mathbf{n}_0 = 0 \ \text{ on } \Gamma_0,$$

the equations (4.1.2) *and* (4.1.3) *admit a unique solution $(\mathbf{v}, \pi, h, H)$ in a function space defined as* (4.1.5).

REMARK 4.1.2. (1) In Theorem 4.1.1, the uniqueness holds in

$$\{(\mathbf{v}, \pi, h, H) \mid (\Theta^*\mathbf{v}, \Theta^*\pi, h, H) \in X_{\delta_0}\}, \tag{4.1.5}$$

where Θ^* is a transform defined as (4.2.1) and (4.2.3), and also X_{δ_0} is given by Theorem 4.5.1.

(2) Since it holds, by Theorem 4.5.1 and Remark 4.5.2, that

$$\begin{aligned} &H \in C(\mathbf{R}_+, C^2(\mathbf{R}^3_-)) \cap C^1(\mathbf{R}_+, C^1(\mathbf{R}^3_-)), \\ &\nabla H \in BUC(\mathbf{R}_+, BUC^1(\mathbf{R}^3_-)) \end{aligned} \tag{4.1.6}$$

and also that

$$\|\nabla H\|_{L_\infty(\mathbf{R}_+, L_\infty(\mathbf{R}^3_-))} \leq \frac{1}{2},$$

we see that Θ given by (4.2.1) is a C^1-diffeomorphism. In fact, it holds that

$$\frac{\partial x_3}{\partial \xi_3} = 1 - \frac{\partial}{\partial \xi_3} H(\xi, t) \geq \frac{1}{2},$$

which furnishes that Θ is a bijection. On the other hand, by (4.1.6) and the inverse function theorem, the inverse Θ^{-1} of Θ is a C^1-function.

(3) The solution H of (4.1.3) is given by

$$H(x, t) = \mathcal{F}^{-1}_{\xi'}[e^{|\xi'|x_3}\widehat{h}(\xi', t)](x'). \tag{4.1.7}$$

Let $1 < q < \infty$ and η be an extension of h, which is defined on $\mathbf{R}^3_-$ and satisfies $\eta|_{\mathbf{R}^3_0} = h$ and $\|\eta(t)\|_{W^1_q(\mathbf{R}^3_-)} \leq C\|h(t)\|_{W^{1-1/q}_q(\mathbf{R}^2)}$ with some positive constant C independent of h, η, and t. Since it holds that

$$\begin{aligned} &e^{|\xi'|x_3}\widehat{h}(\xi', t) = e^{|\xi'|x_3}\widehat{\eta}(\xi', 0, t) \\ &= \int_{-\infty}^0 \frac{d}{dy_3}\left(e^{|\xi'|(x_3+y_3)}\widehat{\eta}(\xi', y_3, t)\right) dy_3 \\ &= \int_{-\infty}^0 |\xi'|e^{|\xi'|(x_3+y_3)}\widehat{\eta}(\xi', y_3, t)\, dy_3 + \int_{-\infty}^0 e^{|\xi'|(x_3+y_3)}\widehat{\partial_3\eta}(\xi', y_3, t)\, dy_3, \end{aligned}$$

we have

$$\begin{aligned} H(x, t) = &\int_{-\infty}^0 \mathcal{F}^{-1}_{\xi'}\left[|\xi'|e^{|\xi'|(x_3+y_3)}\widehat{\eta}(\xi', y_3, t)\right](x')\, dy_3 \\ &+ \int_{-\infty}^0 \mathcal{F}^{-1}_{\xi'}\left[e^{|\xi'|(x_3+y_3)}\widehat{\partial_3\eta}(\xi', y_3, t)\right](x')\, dy_3. \end{aligned}$$

By $|\xi'|^2 = -\sum_{k=1}^2 (i\xi_k)^2$, we obtain, for $j = 1, 2$,

$$\begin{aligned} \partial_j H(x, t) = &\sum_{k=1}^2 \int_{-\infty}^0 \mathcal{F}^{-1}_{\xi'}\left[\frac{\xi_j\xi_k}{|\xi'|^2}|\xi'|e^{|\xi'|(x_3+y_3)}\widehat{\partial_k\eta}(\xi', y_3, t)\right](x')\, dy_3 \\ &+ \int_{-\infty}^0 \mathcal{F}^{-1}_{\xi'}\left[\frac{i\xi_j}{|\xi'|}|\xi'|e^{|\xi'|(x_3+y_3)}\widehat{\partial_3\eta}(\xi', y_3, t)\right](x')\, dy_3, \\ \partial_3 H(x, t) = &-\sum_{k=1}^2 \int_{-\infty}^0 \mathcal{F}^{-1}_{\xi'}\left[\frac{i\xi_k}{|\xi'|}|\xi'|e^{|\xi'|(x_3+y_3)}\widehat{\partial_k\eta}(\xi', y_3, t)\right](x')\, dy_3 \\ &+ \int_{-\infty}^0 \mathcal{F}^{-1}_{\xi'}\left[|\xi'|e^{|\xi'|(x_3+y_3)}\widehat{\partial_3\eta}(\xi', y_3, t)\right](x')\, dy_3, \end{aligned}$$

which, combined with Corollary B.3 (1), furnishes that

$$\|\nabla H(t)\|_{L_q(\mathbf{R}^3_-)} \leq C(q)\|\nabla\eta(t)\|_{L_q(\mathbf{R}^3_-)} \leq C(q)\|h(t)\|_{W_q^{1-1/q}(\mathbf{R}^2)}$$

with some positive constant C. Analogously, we see that

$$\begin{aligned}
\|\nabla^2 H(t)\|_{L_q(\mathbf{R}^3_-)} &\leq C(q)\|\nabla' h(t)\|_{W_q^{1-1/q}(\mathbf{R}^2)}, \\
\|\nabla^3 H(t)\|_{L_q(\mathbf{R}^3_-)} &\leq C(q)\|(\nabla')^2 h(t)\|_{W_q^{1-1/q}(\mathbf{R}^2)}, \\
\|\nabla\partial_t H(t)\|_{L_q(\mathbf{R}^3_-)} &\leq C(q)\|\partial_t h(t)\|_{W_q^{1-1/q}(\mathbf{R}^2)}, \\
\|\nabla^2\partial_t H(t)\|_{L_q(\mathbf{R}^3_-)} &\leq C(q)\|\nabla'\partial_t h(t)\|_{W_q^{1-1/q}(\mathbf{R}^2)}.
\end{aligned}$$

(4) Let $m(s,r)$ and $n(s,r)$ be

$$m(s,r) = \left(\frac{1}{s} - \frac{1}{r}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{r}\right), \quad n(s,r) = \frac{3}{2}\left(\frac{1}{s} - \frac{1}{r}\right).$$

For $r \in \{q, 2\}$ satisfying (4.1.4), we see, by Theorem 4.5.1, that

$$\begin{aligned}
&\|\mathbf{v}(t)\|_{L_r(\Omega(t))} = O(t^{-m(\frac{q}{2},r)}), && \|\nabla\mathbf{v}(t)\|_{L_r(\Omega(t))} = O(t^{-n(\frac{q}{2},r)-\frac{1}{8}}), \\
&\|h(t)\|_{L_r(\mathbf{R}^2)} = O(t^{-(\frac{2}{q}-\frac{1}{r})}), && \|\nabla' h(t)\|_{L_r(\mathbf{R}^2)} = O(t^{-m(\frac{q}{2},r)-\frac{1}{4}}), \\
&\|\partial_t h(t)\|_{L_r(\mathbf{R}^2)} = O(t^{-m(\frac{q}{2},r)})
\end{aligned}$$

as $t \to \infty$, since it follows from the trace theorem that

$$\|\nabla' h(t)\|_{L_r(\mathbf{R}^2)} = \|\nabla' H(t)\|_{L_r(\mathbf{R}^3_0)} \leq C(r)\|\nabla' H(t)\|_{W_r^1(\mathbf{R}^3_-)}$$

with some positive constant $C(r)$ independent of h, H, and t.

This chapter is organized as follows: In the next section, we reduce the equations (4.1.2) to some problem on a fixed domain by using the so-called *Hanzawa transform*. In Section 4.3, we consider some linear problems and show estimates of solutions to the linear problems. Section 4.4 completes estimates of solutions to the linearized equations of the fixed-domain problem. In Section 4.5, we first show the unique global existence of solutions for the fixed-domain problem. Next the global well-posedness of the equations (4.1.2) will be proved.

4.2. Reduction to a fixed domain problem

In this section, we reduce (4.1.2) to a fixed domain problem. Let $(\mathbf{v}, \pi, h, H)$ be solutions to (4.1.2) and (4.1.3). We here introduce the following transformation:

$$\text{(4.2.1)} \qquad \begin{aligned}
&\Theta : \mathbf{R}^3_- \times (0,\infty) \ni (\xi, t) \mapsto (x,t) \in \bigcup_{s\in(0,\infty)} \Omega(s) \times \{s\}, \\
&\Theta(\xi,t) = (\xi', \xi_3 + H(\xi,t), t).
\end{aligned}$$

In the sequel, we note that $\partial_j = \partial/\partial x_j$ and $D_j = \partial/\partial\xi_j$ $(j = 1,2,3)$. The inverse of Jacobian matrix J_Θ of Θ is given by

$$\text{(4.2.2)} \qquad J_\Theta^{-1} = \frac{\partial(\xi,t)}{\partial(x,t)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-D_1 H}{1+D_3 H} & \frac{-D_2 H}{1+D_3 H} & \frac{1}{1+D_3 H} & \frac{-\partial_t H}{1+D_3 H} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let Ω and Ω^* be

$$\Omega = \bigcup_{s\in(0,\infty)} \Omega(s)\times\{s\}, \quad \Omega^* = \mathbf{R}^3_- \times (0,\infty),$$

and furthermore,

$$\Theta^* f(x,t) = f(\Theta(\xi,t)), \quad \Theta_* g(\xi,t) = g(\Theta^{-1}(x,t)) \tag{4.2.3}$$

for functions f defined on Ω and g defined on Ω^*. We then have the following properties concerning the transformation (4.2.1).

LEMMA 4.2.1. *Let $f = f(x,t)$ be sufficiently regular functions defined on Ω, and let $\bar f(\xi,t) = \Theta^* f(x,t)$. Then the following properties hold.*

(1) *For the first derivatives of f, we have*

$$\Theta^*\partial_t f = \left(\partial_t - \left(\frac{\partial_t H}{1+D_3H}\right) D_3\right)\bar f,$$
$$\Theta^*\partial_j f = \left(D_j - \left(\frac{D_j H}{1+D_3H}\right) D_3\right)\bar f \quad (j=1,2,3).$$

(2) *For the second spatial derivatives of f, we have*

$$\Theta^*\partial_j\partial_k f = (D_jD_k - \mathcal{F}_{jk}(H))\,\bar f \quad (j,k=1,2,3),$$

where

$$\begin{aligned}\mathcal{F}_{jk}(H) =& \frac{1}{(1+D_3H)^3}\big\{(D_jD_kH)(1+D_3H)^2 - (D_kH)(D_jD_3H)(1+D_3H)\\ &- (D_jH)(D_3D_kH)(1+D_3H) + (D_jH)(D_kH)(D_3^2H)\big\}D_3\\ &+ \left(\frac{D_kH}{1+D_3H}\right) D_jD_3 + \left(\frac{D_jH}{1+D_3H}\right) D_3D_k - \frac{(D_jH)(D_kH)}{(1+D_3H)^2}D_3^2.\end{aligned}$$

PROOF. (1) By (4.2.2) it is clear that

$$\begin{aligned}(\Theta^*\partial_t f)(x,t) &= (\partial_t f)(\Theta(\xi,t)) = (\partial_t f)(x,t) = \partial_t(f(\Theta(\xi,t)))\\ &= \partial_t(\bar f(\xi,t)) = \partial_t\bar f + (\partial_t\xi_3)D_3\bar f = \partial_t\bar f - \frac{\partial_t H}{1+D_3H}D_3\bar f,\\ \Theta^*\partial_j f &= D_j\bar f + (\partial_j\xi_3)D_3\bar f = D_j\bar f - \frac{D_jH}{1+D_3H}D_3\bar f \quad (j=1,2),\\ \Theta^*\partial_3 f &= (\partial_3\xi_3)D_3\bar f = \frac{D_3\bar f}{1+D_3H} = D_3\bar f - \frac{D_3H}{1+D_3H}D_3\bar f.\end{aligned}$$

(2) Calculate

$$\Theta^*\partial_j\partial_k f = \left(D_j - \left(\frac{D_jH}{1+D_3H}\right)D_3\right)\left(D_k - \left(\frac{D_kH}{1+D_3H}\right)D_3\right)\bar f.$$

□

To progress our argumentation, we define matrices $\mathbf{M}_i(H)$ $(i = 1, 2, 3)$ as

$$\mathbf{M}_1(H) = \begin{pmatrix} D_3H & 0 & 0 \\ 0 & D_3H & 0 \\ -D_1H & -D_2H & 0 \end{pmatrix}, \tag{4.2.4}$$
$$\mathbf{M}_2(H) = \begin{pmatrix} 0 & 0 & -D_1H \\ 0 & 0 & -D_2H \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}_3(H) = \begin{pmatrix} 0 & 0 & D_1H \\ 0 & 0 & D_2H \\ 0 & 0 & D_3H \end{pmatrix}.$$

Then $\mathbf{D}(\mathbf{v})$ and $\operatorname{div}\mathbf{v}$ are transformed by (4.2.1) as follows:

LEMMA 4.2.2. *Let $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))^T$ be sufficiently regular functions defined on Ω, and let $\bar{\mathbf{u}}(\xi,t) = \Theta^*\mathbf{u}(x,t)$. Then*

$$\Theta^*\mathbf{D}_x(\mathbf{u}) = \mathbf{D}_\xi(\bar{\mathbf{u}}) - \frac{1}{1+D_3H}\left\{(\nabla_\xi H \otimes D_3\bar{\mathbf{u}}) + (\nabla_\xi H \otimes D_3\bar{\mathbf{u}})^T\right\},$$
$$\Theta^*(\operatorname{div}_x \mathbf{u}) = \operatorname{div}_\xi \bar{\mathbf{u}} - \frac{\nabla_\xi H \cdot D_3\bar{\mathbf{u}}}{1+D_3H} = \frac{\operatorname{div}_\xi\{(\mathbf{I}+\mathbf{M}_1(H))\bar{\mathbf{u}}\}}{1+D_3H},$$

where subscripts x and ξ denote the derivatives of their coordinates.

PROOF. We only show the second identity of $\Theta^*(\operatorname{div}_x \mathbf{u})$. Let $\psi = \psi(x,t) \in C_0^\infty(\Omega)$ and $\bar{\psi}(\xi,t) = \Theta^*\psi(x,t)$, and then by Lemma 4.2.1

$$\begin{aligned}
&-(\Theta^*(\operatorname{div}_x \mathbf{u}), \psi)_\Omega = (\Theta^*\mathbf{u}, \nabla_x\psi)_\Omega \\
&= \int_{\Omega^*} \bar{\mathbf{u}}(\xi,t)\cdot\left((\nabla_\xi\bar{\psi})(\xi,t) - \frac{(D_3\bar{\psi})(\xi,t)\nabla_\xi H(\xi,t)}{1+D_3H}\right)(1+D_3H(\xi,t))\,d\xi dt \\
&= \int_{\Omega^*} \bar{\mathbf{u}}(\xi,t)\cdot(\mathbf{I}+\mathbf{M}_1(H)^T)(\nabla_\xi\bar{\psi})(\xi,t)\,d\xi dt = -\left(\operatorname{div}_\xi\{(\mathbf{I}+\mathbf{M}_1(H))\bar{\mathbf{u}}\}, \bar{\psi}\right)_{\Omega^*} \\
&= -\left((1+D_3H)^{-1}\operatorname{div}_\xi\{(\mathbf{I}+\mathbf{M}_1(H))\bar{\mathbf{u}}\}, \psi\right)_\Omega.
\end{aligned}$$

This furnishes that the required property holds. □

Let $\bar{\mathbf{v}}(\xi,t) = \Theta^*\mathbf{v}(x,t)$ and $\bar{\pi}(\xi,t) = \Theta^*\pi(x,t)$. Then, by Lemma 4.2.2,

$$\begin{aligned}\Theta^*\mathbf{S}(\mathbf{v},\pi) = &-\bar{\pi} + \mu\mathbf{D}_\xi(\bar{\mathbf{v}}) \\ &- \frac{\mu}{1+D_3H}\left\{(\nabla_\xi H \otimes D_3\bar{\mathbf{v}}) + (\nabla_\xi H \otimes D_3\bar{\mathbf{v}})^T\right\},\end{aligned} \tag{4.2.5}$$

and also we note that $(\mathbf{I}+\mathbf{M}_2(H))^{-1} = \mathbf{I} - \mathbf{M}_2(H)$ and that

$$(1+|\nabla'_\xi H|^2)^{-1/2}(\mathbf{I}+\mathbf{M}_2(H))\mathbf{e}_3 = \mathbf{n}_\Gamma \quad \text{for } \xi \in \mathbf{R}_0^3 \tag{4.2.6}$$

by (4.1.7). In addition, applying Θ^* to the third equation of (4.1.2), we have, by Lemma 4.2.1,

$$\Theta^*(\mathbf{S}(\mathbf{v},\pi)\mathbf{n}_\Gamma) + (\rho c_g h - c_\sigma\Delta'_\xi h)\mathbf{n}_\Gamma = -c_\sigma G_\kappa(h)\mathbf{n}_\Gamma \quad \text{for } \xi \in \mathbf{R}_0^3. \tag{4.2.7}$$

Thus multiplying (4.2.7) by $(1+|\nabla'_\xi H|^2)^{1/2}(\mathbf{I}+\mathbf{M}_2(H))^{-1}$ from the left-hand side, we see, by (4.2.5) and (4.2.6), that

$$
\begin{aligned}
&- c_\sigma G_\kappa(h)\mathbf{e}_3 - (\rho c_g - c_\sigma\Delta'_\xi)h\mathbf{e}_3 \\
&= (1+|\nabla'_\xi H|^2)^{1/2}(\mathbf{I}+\mathbf{M}_2(H))^{-1}\Theta^*(\mathbf{S}(\mathbf{v},\pi)\mathbf{n}_\Gamma) \\
&= -\bar\pi\mathbf{e}_3 + \mu(1+|\nabla'_\xi H|^2)^{1/2}(\mathbf{I}-\mathbf{M}_2(H))\mathbf{D}_\xi(\bar{\mathbf{v}})\mathbf{n}_\Gamma \\
&- \frac{\mu(1+|\nabla'_\xi H|^2)^{1/2}(\mathbf{I}-\mathbf{M}_2(H))}{1+D_3H}\left\{(\nabla_\xi H\otimes D_3\bar{\mathbf{v}}) + (\nabla_\xi H\otimes D_3\bar{\mathbf{v}})^T\right\}\mathbf{n}_\Gamma \\
&= \mathbf{S}_\xi(\bar{\mathbf{v}},\bar\pi)\mathbf{e}_3 + \mu\mathbf{D}_\xi(\bar{\mathbf{v}})((1+|\nabla'_\xi H|^2)^{1/2}\mathbf{n}_\Gamma - \mathbf{e}_3) \\
&- \mu\mathbf{M}_2(H)\mathbf{D}_\xi(\bar{\mathbf{v}})(1+|\nabla'_\xi H|^2)^{1/2}\mathbf{n}_\Gamma \\
&- \frac{\mu(\mathbf{I}-\mathbf{M}_2(H))}{1+D_3H}\left\{(\nabla_\xi H\otimes D_3\bar{\mathbf{v}}) + (\nabla_\xi H\otimes D_3\bar{\mathbf{v}})^T\right\}(1+|\nabla'_\xi H|^2)^{1/2}\mathbf{n}_\Gamma
\end{aligned}
\tag{4.2.8}
$$

for $\xi\in\mathbf{R}^3_0$. Finally, applying Θ^* to the first equation of (4.1.2) and multiplying the resultant formula by $\mathbf{I}+\mathbf{M}_3(H)$ from the left-hand side, we have achieved, by Lemma 4.2.1, Lemma 4.2.2, and (4.2.8), the following equations:

$$
\begin{cases}
\partial_t\bar{\mathbf{v}} - \Delta\bar{\mathbf{v}} + \nabla\bar\pi = \mathbf{F}(\bar{\mathbf{v}},H) & \text{in } \mathbf{R}^3_-,\ t>0, \\
\operatorname{div}\bar{\mathbf{v}} = \operatorname{div}\mathbf{F}_d(\bar{\mathbf{v}},H) = F_d(\bar{\mathbf{v}},H) & \text{in } \mathbf{R}^3_-,\ t>0, \\
\mathbf{S}(\bar{\mathbf{v}},\bar\pi)\mathbf{e}_3 + (c_g - c_\sigma\Delta')h\mathbf{e}_3 = \mathbf{G}(\bar{\mathbf{v}},H) & \text{on } \mathbf{R}^3_0,\ t>0, \\
\partial_t h - \bar{\mathbf{v}}\cdot\mathbf{e}_3 = G_h(\bar{\mathbf{v}},H) & \text{on } \mathbf{R}^3_0,\ t>0, \\
\bar{\mathbf{v}}|_{t=0} = \bar{\mathbf{v}}_0 & \text{in } \mathbf{R}^3_-, \\
h|_{t=0} = h_0 & \text{on } \mathbf{R}^2,
\end{cases}
\tag{4.2.9}
$$

where we have set $\rho=\mu=1$ without loss of generality, $\bar{\mathbf{v}}_0(\xi) = \Theta_0^*\mathbf{v}_0(x) = \mathbf{v}_0(\Theta_0(\xi))$ with $\Theta_0(\xi) = \xi + H_0(\xi)$ for $\xi\in\mathbf{R}^3_-$ and

$$
H_0(\xi) = \mathcal{F}^{-1}_{y'}[e^{|y'|\xi_3}\widehat{h}_0(y')](\xi') \quad (\xi_3<0),
\tag{4.2.10}
$$

and also the right members are given by

$$
\mathbf{F}(\bar{\mathbf{v}},H) = \mathbf{F}_1(\bar{\mathbf{v}},H) + \mathbf{F}_2(\bar{\mathbf{v}},H),\ \ \mathbf{F}_1(\bar{\mathbf{v}},H) = (\mathbf{I}+\mathbf{M}_3(H))\Big(\frac{\partial_t H D_3\bar{\mathbf{v}}}{1+D_3H} - (\bar{\mathbf{v}}\cdot\nabla)\bar{\mathbf{v}}\Big)
$$

$$
\mathbf{F}_2(\bar{\mathbf{v}},H) = (-\partial_t\bar v_3 + \Delta\bar v_3)\nabla H + (\mathbf{I}+\mathbf{M}_3(H))\Big(\sum_{j=1}^3\mathcal{F}_{jj}(H)\bar{\mathbf{v}} + \frac{(\bar{\mathbf{v}}\cdot\nabla H)D_3\bar{\mathbf{v}}}{1+D_3H}\Big)
$$

$$
\begin{aligned}
\mathbf{G}(\bar{\mathbf{v}},H) &= -\sigma G_\kappa(H)\mathbf{e}_3 + \mathbf{D}(\bar{\mathbf{v}})(D_1H,D_2H,0)^T \\
&- \mathbf{M}_2(H)\mathbf{D}(\bar{\mathbf{v}})(D_1H,D_2H,-1)^T \\
&- \frac{\mathbf{I}-\mathbf{M}_2(H)}{1+D_3H}\left\{(\nabla H\otimes D_3\bar{\mathbf{v}}) + (\nabla H\otimes D_3\bar{\mathbf{v}})^T\right\}(D_1H,D_2H,-1)^T,
\end{aligned}
$$

$$
\mathbf{F}_d(\bar{\mathbf{v}},H) = -\mathbf{M}_1(H)\bar{\mathbf{v}},\ \ F_d(\bar{\mathbf{v}},H) = \frac{\nabla H\cdot D_3\bar{\mathbf{v}}}{1+D_3H},\ \ G_h(\bar{\mathbf{v}},H) = -\bar{\mathbf{v}}'\cdot\nabla' H.
$$

Here we have used the identities: $\nabla'h = \nabla'H$ and $G_\kappa(h) = G_\kappa(H)$ on $\mathbf{R}^3_0$ by (4.1.7) in order to derive $\mathbf{G}(\bar{\mathbf{v}},H)$ and $G_h(\bar{\mathbf{v}},H)$.

Remark 4.2.3. (1) Θ_0 is a C^2-diffeomorphism if $h_0\in B^{3-1/p-1/q}_{q,p}(\mathbf{R}^2)$ is sufficiently small and the assumption (4.1.4) holds. In fact, we have the following observation: First, setting $\Omega_L = \mathbf{R}^2 \times (-L, 0)$ for $L > 0$, we have, by the Fourier multiplier theorem of Hörmander-Mikhlin type,

$$\|H_0\|_{L_q(\Omega_L)} \leq C(q) \left(\int_{-L}^{0} d\xi_3 \right)^{1/q} \|h_0\|_{L_q(\mathbf{R}^2)} \leq C(q, L)\|h_0\|_{L_q(\mathbf{R}^2)} \tag{4.2.11}$$

with some positive constant $C(q, L)$. Secondly, in the same manner as in Remark 4.1.2 (3), it holds, by (4.2.11), that

$$\begin{aligned} &\|H_0\|_{W_q^2(\Omega_L)} \leq C_1\|h_0\|_{W_q^{2-1/q}(\mathbf{R}^2)}, \quad \|H_0\|_{W_q^3(\Omega_L)} \leq C_1\|h_0\|_{W_q^{3-1/q}(\mathbf{R}^2)}, \\ &\|\nabla H_0\|_{W_q^1(\mathbf{R}^3_-)} \leq C_2\|h_0\|_{W_q^{2-1/q}(\mathbf{R}^2)}, \quad \|\nabla H_0\|_{W_q^2(\mathbf{R}^3_-)} \leq C_2\|h_0\|_{W_q^{3-1/q}(\mathbf{R}^2)} \end{aligned} \tag{4.2.12}$$

with some positive constant $C_1 = C_1(q, L)$ and $C_2 = C_2(q)$. Thirdly, we obtain, by (4.2.12) and the real interpolation theorem,

$$\begin{aligned} &\|H_0\|_{W_q^{3-1/p}(\Omega_L)} \leq C_1(p, q, L)\|h_0\|_{B_{q,p}^{3-1/p-1/q}(\mathbf{R}^2)} \quad \text{for every } L > 0, \\ &\|\nabla H_0\|_{W_q^{2-1/p}(\mathbf{R}^3_-)} \leq C_2(p, q)\|h_0\|_{B_{q,p}^{3-1/p-1/q}(\mathbf{R}^2)} \end{aligned}$$

which, combined with Sobolev's embedding theorem, furnishes that

$$H_0 \in C^2(\mathbf{R}^3_-), \quad \nabla H_0 \in BUC^1(\mathbf{R}^3_-),$$

and besides,

$$\|\nabla H_0\|_{BUC^1(\mathbf{R}^3_-)} \leq M(p, q)\|h_0\|_{B_{q,p}^{3-1/p-1/q}(\mathbf{R}^2)} \tag{4.2.13}$$

with some positive constant $M(p, q)$. Thus Θ_0 is a C^2-diffeomorphism from $\mathbf{R}^3_-$ to Ω_0 similarly to Remark 4.1.2 (2), provided that h_0 satisfies the smallness condition: $M(p, q)\|h_0\|_{B_{q,p}^{3-1/p-1/q}(\mathbf{R}^2)} \leq 1/2$.

(2) Let exponents p, q satisfy (4.1.4) and $r \in \{q, q/2\}$. Then there exists a positive number ε_0 such that for any H_0 satisfying $\|\nabla H_0\|_{BUC^1(\mathbf{R}^3_-)} \leq \varepsilon_0$,

$$C(\varepsilon_0)^{-1}\|\mathbf{v}_0\|_{B_{r,p}^{2(1-1/p)}(\Omega_0)} \leq \|\bar{\mathbf{v}}_0\|_{B_{r,p}^{2(1-1/p)}(\mathbf{R}^3_-)} \leq C(\varepsilon_0)\|\mathbf{v}_0\|_{B_{r,p}^{2(1-1/p)}(\Omega_0)} \tag{4.2.14}$$

with some positive constant $C(\varepsilon_0)$. Especially, by (4.2.13), if $\|h_0\|_{B_{q,p}^{3-1/p-1/q}(\mathbf{R}^2)}$ is sufficiently small, then (4.2.14) holds.

In the last part of this section, we introduce notation and several function spaces, which will be used in the following sections. Let $1 < p, q < \infty$, and then the natural norm associated with the maximal regularity theorem of (4.2.9) and (4.1.3) (cf. [**SS12**, Theorem 1.4]) is defined as

$$\begin{aligned} \mathbb{M}_{q,p}(\mathbf{u}, \theta, h, \partial_t h, H) =& \|(\partial_t \mathbf{u}, \mathbf{u}, \nabla \mathbf{u}, \nabla^2 \mathbf{u}, \nabla \theta)\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \\ &+ \|h\|_{L_p(\mathbf{R}_+, W_q^{3-1/q}(\mathbf{R}^2))} + \|\partial_t h\|_{L_p(\mathbf{R}_+, W_q^{2-1/q}(\mathbf{R}^2))} \\ &+ \|\nabla H\|_{L_p(\mathbf{R}_+, W_q^2(\mathbf{R}^3_-))} + \|\nabla \partial_t H\|_{L_p(\mathbf{R}_+, W_q^1(\mathbf{R}^3_-))}. \end{aligned}$$

In addition, let X be a Banach space and its norm $\| \cdot \|_X$. Then, for $s > 0$ and $1 \leq p \leq \infty$, we set

$$\begin{aligned} &L_p^s(\mathbf{R}_+, X) = \{u \in L_p(\mathbf{R}_+, X) \mid \|u\|_{L_p^s(\mathbf{R}_+, X)} < \infty\}, \\ &\|u\|_{L_p^s(\mathbf{R}_+, X)} = \|(t+2)^s u\|_{L_p(\mathbf{R}_+, X)}, \\ &W_p^{1,s}(\mathbf{R}_+, X) = \{u \in W_p^1(\mathbf{R}_+, X) \mid \|u\|_{W_p^{1,s}(\mathbf{R}_+, X)} < \infty\}, \\ &\|u\|_{W_p^{1,s}(\mathbf{R}_+, X)} = \|\partial_t((t+2)^s u)\|_{L_p(\mathbf{R}_+, X)}. \end{aligned}$$

Key ideas, in the proof of Theorem 4.1.1, are as follows: First, for the highest order derivatives, we use weighted L_p-norms given by

$$\begin{aligned}&\mathbb{W}_{q,p}(\mathbf{u}, H; \delta_1, \delta_2)\\&\quad = \|(\partial_t \mathbf{u}, \nabla^2 \mathbf{u})\|_{L_p^{\delta_1}(\mathbf{R}_+, L_q(\mathbf{R}_-^3))} + \|(\nabla^2 \partial_t H, \nabla^3 H)\|_{L_p^{\delta_2}(\mathbf{R}_+, L_q(\mathbf{R}_-^3))}\end{aligned}$$

with positive numbers δ_1, δ_2. Secondly, for the lower order terms, we use weighted L_∞-norms given by

$$\begin{aligned}\mathbb{W}_{r,\infty}(\mathbf{u}, h, \partial_t h, H) &= \|\mathbf{u}\|_{L_\infty^{m(q/2,r)}(\mathbf{R}_+, L_r(\mathbf{R}_-^3))} + \|\nabla \mathbf{u}\|_{L_\infty^{n(q/2,r)+1/8}(\mathbf{R}_+, L_r(\mathbf{R}_-^3))}\\&+ \|h\|_{L_\infty^{2/q-1/r}(\mathbf{R}_+, L_r(\mathbf{R}^2))} + \|\partial_t h\|_{L_\infty^{m(q/2,r)}(\mathbf{R}_+, L_r(\mathbf{R}^2))}\\&+ \|\nabla H\|_{L_\infty^{m(q/2,r)+1/4}(\mathbf{R}_+, W_r^1(\mathbf{R}_-^3))} + \|\nabla \partial_t H\|_{L_\infty^{m(q/2,r)+1/2}(\mathbf{R}_+, L_r(\mathbf{R}_-^3))},\end{aligned}$$

where $m(q/2, r)$ and $n(q/2, r)$ are defined as in Remark 4.1.2 (4).

Let exponents p, q satisfy (4.1.4), and then we set

$$\begin{aligned}&\mathbb{F}_1 = \mathbb{F}_2 = \bigcap_{r \in \{q,2\}} L_p(\mathbf{R}_+, L_r(\mathbf{R}_-^3))^3, \quad \mathbb{G}_h = \bigcap_{r \in \{q,2\}} W_{r,p}^{2,1}(\mathbf{R}_-^3 \times \mathbf{R}_+),\\&\mathbb{F}_{d1} = \bigcap_{r \in \{q,2\}} W_p^1(\mathbf{R}_+, L_r(\mathbf{R}_-^3))^3, \qquad \mathbb{F}_{d2} = \bigcap_{r \in \{q,2\}} L_p(\mathbf{R}_+, W_r^1(\mathbf{R}_-^3)),\\&\mathbb{G} = \bigcap_{r \in \{q,2\}} W_p^1(\mathbf{R}_+, \widehat{W}_r^{-1}(\mathbf{R}_-^3))^3 \cap L_p(\mathbf{R}_+, W_r^1(\mathbf{R}_-^3))^3,\end{aligned}$$

and besides, for $\delta > 0$ and $\varepsilon > 0$

$$\begin{aligned}\widetilde{\mathbb{F}}_1(\delta, \varepsilon) &= L_p^\delta(\mathbf{R}_+, L_q(\mathbf{R}_-^3))^3 \cap L_\infty^\varepsilon(\mathbf{R}_+, L_{q/2}(\mathbf{R}_-^3))^3,\\\widetilde{\mathbb{F}}_2(\delta, \varepsilon) &= L_p^\delta(\mathbf{R}_+, L_q(\mathbf{R}_-^3))^3 \cap L_p^\varepsilon(\mathbf{R}_+, L_{q/2}(\mathbf{R}_-^3))^3,\\\widetilde{\mathbb{G}}_h(\delta, \varepsilon) &= L_p^\delta(\mathbf{R}_+, W_q^2(\mathbf{R}_-^3)) \cap L_p^\varepsilon(\mathbf{R}_+, W_{q/2}^2(\mathbf{R}_-^3)),\\\widetilde{\mathbb{F}}_{d1}(\delta, \varepsilon) &= W_p^{1,\delta}(\mathbf{R}_+, L_q(\mathbf{R}_-^3))^3 \cap W_p^{1,\varepsilon}(\mathbf{R}_+, L_{q/2}(\mathbf{R}_-^3))^3,\\\widetilde{\mathbb{F}}_{d2}(\delta, \varepsilon) &= L_p^\delta(\mathbf{R}_+, W_q^1(\mathbf{R}_-^3)) \cap L_p^\varepsilon(\mathbf{R}_+, W_{q/2}^1(\mathbf{R}_-^3)),\\\widetilde{\mathbb{G}}(\delta, \varepsilon) &= W_p^{1,\delta}(\mathbf{R}_+, \widehat{W}_q^{-1}(\mathbf{R}_-^3))^3 \cap L_p^\delta(\mathbf{R}_+, W_q^1(\mathbf{R}_-^3))^3\\&\quad \cap W_p^{1,\varepsilon}(\mathbf{R}_+, \widehat{W}_{q/2}^{-1}(\mathbf{R}_-^3))^3 \cap L_p^\varepsilon(\mathbf{R}_+, W_{q/2}^1(\mathbf{R}_-^3))^3.\end{aligned}$$

Moreover, we define additional function spaces as

$$\begin{aligned}\mathbb{A}_1 &= L_\infty^{m(q/2,q)}(\mathbf{R}_+, L_q(\mathbf{R}_-^3)) \cap L_\infty^{m(q/2,2)}(\mathbf{R}_+, L_2(\mathbf{R}_-^3)),\\\mathbb{A}_2 &= L_\infty^{m(q/2,q)+1/2}(\mathbf{R}_+, L_q(\mathbf{R}_-^3)) \cap L_\infty^{m(q/2,2)+1/2}(\mathbf{R}_+, L_2(\mathbf{R}_-^3)),\\\widehat{\mathbb{A}}_2 &= L_\infty^{m(q/2,q)+1/2}(\mathbf{R}_+, \widehat{W}_q^1(\mathbf{R}_-^3)) \cap L_\infty^{m(q/2,2)+1/2}(\mathbf{R}_+, \widehat{W}_2^1(\mathbf{R}_-^3)),\\\mathbb{A}_3 &= L_p^{m(q/2,q)+1/2}(\mathbf{R}_+, W_q^1(\mathbf{R}_-^3)) \cap L_p^{m(q/2,2)+1/2}(\mathbf{R}_+, W_2^1(\mathbf{R}_-^3)).\end{aligned}$$

For the initial data, we set

$$\begin{aligned}\mathbb{I}_1 &= B_{q,p}^{2(1-1/p)}(\mathbf{R}_-^3)^3 \cap B_{q/2,p}^{2(1-1/p)}(\mathbf{R}_-^3)^3,\\\mathbb{I}_2 &= B_{q,p}^{3-1/p-1/q}(\mathbf{R}^2) \cap B_{2,p}^{3-1/p-1/2}(\mathbf{R}^2) \cap L_{q/2}(\mathbf{R}^2).\end{aligned}$$

4.3. Linear theory I

In this section, we consider the following linear system:

$$
\begin{cases} \partial_t \mathbf{u} + \mathbf{u} - \Delta \mathbf{u} + \nabla \theta = 0 & \text{in } \mathbf{R}^3_+,\ t > 0, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \mathbf{R}^3_+,\ t > 0, \\ \mathbf{S}(\mathbf{u}, \theta)\mathbf{n} = \mathbf{g} & \text{on } \mathbf{R}^3_0,\ t > 0, \\ \mathbf{u}|_{t=0} = 0 & \text{in } \mathbf{R}^3_+ \end{cases} \tag{4.3.1}
$$

for $\mathbf{n} = (0, 0, -1)^T$, and furthermore,

$$
\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \theta = \mathbf{f}_1 + \mathbf{f}_2 & \text{in } \mathbf{R}^3_+,\ t > 0, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \mathbf{R}^3_+,\ t > 0, \\ \mathbf{S}(\mathbf{u}, \theta)\mathbf{n} + (c_g - c_\sigma \Delta')h\mathbf{n} = 0 & \text{on } \mathbf{R}^3_0,\ t > 0, \\ \partial_t h - \mathbf{u} \cdot \mathbf{n} = g_h & \text{on } \mathbf{R}^3_0,\ t > 0, \\ \mathbf{u}|_{t=0} = 0 & \text{in } \mathbf{R}^3_+, \\ h|_{t=0} = 0 & \text{in } \mathbf{R}^3_0 \end{cases} \tag{4.3.2}
$$

with the auxiliary problem:

$$
\begin{cases} \Delta H = 0 & \text{in } \mathbf{R}^3_+,\ t \geq 0, \\ H = h & \text{on } \mathbf{R}^3_0,\ t \geq 0. \end{cases} \tag{4.3.3}
$$

Remark 4.3.1. Although we consider (4.3.1)-(4.3.3) in $\mathbf{R}^3_+$ to use some results obtained in [**SS08**], [**SS12**], and Chapter 3, the case of $\mathbf{R}^3_-$ can be treated by using a suitable transformation. In addition, we use the symbols: $A, B, D(A, B), L(A, B)$, and $\mathcal{M}(a)$ $(a > 0)$ defined as (1.2.1) with $\mu = 1$ and (1.2.2) in this section.

4.3.1. Analysis of Equations (4.3.1)**.** We start with the following theorem to analyze the equations (4.3.1).

Theorem 4.3.2. *Let $1 < p, q < \infty$ and $\mathbf{g} \in H^{1,1/2}_{q,p,0}(\mathbf{R}^3_+ \times \mathbf{R})^3$. Then the equations* (4.3.1) *admits a unique solution* $(\mathbf{u}, \theta)$ *with*

$$
\mathbf{u} \in W^{2,1}_{q,p,0}(\mathbf{R}_+ \times \mathbf{R})^3, \quad \theta \in L_{p,0}(\mathbf{R}, \widehat{W}^1_q(\mathbf{R}^3_+)),
$$

and also the solution satisfies the estimate:

$$
\|(\partial_t \mathbf{u}, \mathbf{u}, \nabla \mathbf{u}, \nabla^2 \mathbf{u}, \nabla \theta)\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_+))} \leq C(p, q)\|\mathbf{g}\|_{H^{1,1/2}_{q,p,0}(\mathbf{R}_+ \times \mathbf{R})} \tag{4.3.4}
$$

with a positive constant $C(p, q)$. In addition, the solution $\mathbf{u}$ is represented as

$$
\mathbf{u}(x, t) = \int_0^t [\mathcal{B}(t - s)\mathbf{g}(\cdot, 0, s)](x)\, ds \quad (t > 0)
$$

by an operator $\mathcal{B}(\tau) \in \mathcal{L}(L_q(\mathbf{R}^2)^3, W^1_q(\mathbf{R}^3_+)^3)$ satisfying

$$
\|\nabla^l \mathcal{B}(\tau)\mathbf{a}\|_{L_q(\mathbf{R}^3_+)} \leq C(q)\, \tau^{-\frac{1+l}{2} + \frac{1}{2q}} e^{-\tau} \|\mathbf{a}\|_{L_q(\mathbf{R}^2)} \quad (\tau > 0,\ l = 0, 1)
$$

for any $\mathbf{a} \in L_q(\mathbf{R}^2)^3$ and a positive constant $C(q)$ independent of τ and $\mathbf{a}$.

The unique existence of solutions and (4.3.4) were proved by [**SS08**, Theorem 5.1] and [**SS12**, Theorem 1.2]. Our aim here is to show the remainders. Let $\widetilde{f}(\xi',\lambda)$ be the Fourier-Laplace transform of $f(x',t)$ defined on $\mathbf{R}^2\times\mathbf{R}$, that is,

$$\widetilde{f}(\xi',\lambda)=\mathcal{F}_{x'}\mathcal{L}_t[f](\xi',\lambda)=\int_{\mathbf{R}^2\times\mathbf{R}}e^{-(ix'\cdot\xi'+\lambda t)}f(x',t)\,dx'dt\quad(\lambda=\gamma+i\tau),\tag{4.3.5}$$

and then we obtain the following lemma.

Lemma 4.3.3. *Let $1<q<\infty$, $0<\varepsilon<\pi/2$, and $f(x',t)\in C_0^\infty(\mathbf{R}^2\times\mathbf{R}_+)$. Suppose that $m(\xi',\lambda)\in\mathbb{M}_{-1,2,\varepsilon,0}$ and set, for $\lambda=\gamma+i\tau$ $(\gamma\ge 0)$,*

$$I(x,t)=\mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[m(\xi',\lambda)e^{-Bx_3}\widetilde{f}(\xi',\lambda)\right](x',t)\qquad(x_3>0),$$
$$J(x,t)=\mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[m(\xi',\lambda)A\mathcal{M}(x_3)\widetilde{f}(\xi',\lambda)\right](x',t)\quad(x_3>0).$$

Then, for every $t>0$, there exist operators $\mathcal{I}(t),\mathcal{J}(t)\in\mathcal{L}(L_q(\mathbf{R}^2),W_q^1(\mathbf{R}^3_+))$ such that for any $g\in L_q(\mathbf{R}^2)$ and $l=0,1$

$$\|(\nabla^l\mathcal{I}(t)g,\nabla^l\mathcal{J}(t)g)\|_{L_q(\mathbf{R}^3_+)}\le C(q)\,t^{-\frac{1+l}{2}+\frac{1}{2q}}\|g\|_{L_q(\mathbf{R}^2)}\quad(t>0)$$

with a positive constant $C(q)$ independent of t and g, and besides,

$$I(x,t)=\int_0^t[\mathcal{I}(t-s)f(\cdot,s)](x)\,ds,\;\;J(x,t)=\int_0^t[\mathcal{J}(t-s)f(\cdot,s)](x)\,ds\;\;(t>0).$$

Proof. For $g\in L_q(\mathbf{R}^2)$, setting

$$[\mathcal{I}(t)g](x)=\mathcal{F}_{\xi'}^{-1}\left[\mathcal{L}_\lambda^{-1}\left[m(\xi',\lambda)e^{-Bx_3}\right](t)\,\widehat{g}(\xi')\right](x'),$$
$$[\mathcal{J}(t)g](x)=\int_0^1\mathcal{F}_{\xi'}^{-1}\left[\mathcal{L}_\lambda^{-1}\left[m(\xi',\lambda)x_3Ae^{-(B\theta+A(1-\theta))x_3}\right](t)\,\widehat{g}(\xi')\right](x')\,d\theta,$$

we see, by (1.2.3) and Proposition C.1 with $s=-1$ and $\gamma_0=0$, that

$$I(x,t)=\int_0^t[\mathcal{I}(t-s)f(\cdot,s)](x)\,ds,\quad J(x,t)=\int_0^t[\mathcal{J}(t-s)f(\cdot,s)](x)\,ds.$$

We first show the required estimates of $\mathcal{I}(t)g$ by applying Proposition 1.2.8 with $X=\mathbf{R}$, $L=0$, $n=1$, and $\sigma=1/2$. Let $\lambda=i\tau$ for $\tau\in\mathbf{R}\setminus\{0\}$, and then

$$D_{\xi'}^{\alpha'}\mathcal{L}_\lambda^{-1}[m(\xi',\lambda)e^{-Bx_3}](t)=\mathcal{F}_\tau^{-1}[D_{\xi'}^{\alpha'}(m(\xi',i\tau)e^{-Bx_3})](t)\tag{4.3.6}$$

for any $\alpha'\in\mathbf{N}_0^2$. Since, by Lemma 1.2.6, $m\in\mathbb{M}_{-1,2,\varepsilon,0}$, and Leibniz's rule,

$$\begin{aligned}&|(\tau\partial_\tau)^lD_{\xi'}^{\alpha'}(m(\xi',i\tau)e^{-Bx_3})|\\&\quad\le C(\alpha')(|\tau|^{1/2}+A)^{-1}e^{-c(|\tau|^{1/2}+A)x_3}A^{-|\alpha'|}\\&\quad\le C(\alpha')A^{-|\alpha'|}|\tau|^{-1/2}\end{aligned}\tag{4.3.7}$$

for $l=0,1$ with positive constants $C(\alpha')$ and c, which, combined with Proposition 1.2.8, furnishes that for $t\in\mathbf{R}\setminus\{0\}$

$$|\mathcal{F}_\tau^{-1}[D_{\xi'}^{\alpha'}(m(\xi',i\tau)e^{-Bx_3})](t)|\le C|t|^{-1/2}A^{-|\alpha'|}.$$

On the other hand, by using (4.3.6) and (4.3.7) with $l=0$, we have

$$
\begin{aligned}
|\mathcal{F}_\tau^{-1}[D_{\xi'}^{\alpha'}(m(\xi',i\tau)e^{-Bx_3})](t)| &\le C(\alpha')A^{-|\alpha'|}\int_{\mathbf{R}}|\tau|^{-1/2}e^{-b|\tau|^{1/2}x_3}\,d\tau \\
&\le C(\alpha')A^{-|\alpha'|}x_3^{-1},
\end{aligned}
$$

which, combined with the inequality above, furnishes that

$$
|\mathcal{F}_\tau^{-1}[D_{\xi'}^{\alpha'}(m(\xi',i\tau)e^{-Bx_3})](t)| \le \frac{C(\alpha')}{|t|^{1/2}+x_3}A^{-|\alpha'|}.
$$

Thus, applying Fourier multiplier theorem of Hölmander-Mikhlin type to (4.3.6), we have

$$
\|[\mathcal{I}(t)g](\cdot,x_3)\|_{L_q(\mathbf{R}^2)} \le \frac{C(q)}{|t|^{1/2}+x_3}\|g\|_{L_q(\mathbf{R}^2)},
$$

and therefore $\|\mathcal{I}(t)g\|_{L_q(\mathbf{R}_+^3)} \le C(q)\,t^{-1/(2q)}\|g\|_{L_q(\mathbf{R}^2)}$ for $t>0$ with a positive constant $C(q)$. Analogously, we have

$$
\|\nabla[\mathcal{I}(t)g](\cdot,x_3)\|_{L_q(\mathbf{R}^2)} \le \frac{C(q)}{|t|+(x_3)^2}\|g\|_{L_q(\mathbf{R}^2)} \quad (t\in\mathbf{R}\setminus\{0\}),
$$

which furnishes that the required estimate of $\nabla\mathcal{I}(t)g$ holds.

We next show the estimate of $\mathcal{J}(t)g$. Let $\lambda=i\tau$ for $\tau\in\mathbf{R}\setminus\{0\}$, and then by Young's inequality

$$
\begin{aligned}
&\|[\mathcal{J}(t)g](\cdot,x_3)\|_{L_q(\mathbf{R}^2)} \\
&\le \int_0^1\left\|\mathcal{F}_\tau^{-1}\left[\mathcal{F}_{\xi'}^{-1}\left[m(\xi',i\tau)x_3Ae^{-(B\theta+A(1-\theta))x_3}\right](\cdot)\right](t)\right\|_{L_1(\mathbf{R}^2)}d\theta\|g\|_{L_q(\mathbf{R}^2)}.
\end{aligned}
$$

We thus obtain, by setting $X=L_q((0,\infty),L_1(\mathbf{R}^2))$,

$$
\begin{aligned}
(4.3.8)\quad &\|\mathcal{J}(t)g\|_{L_q(\mathbf{R}_+^3)} \\
&\le \int_0^1\left\|\mathcal{F}_\tau^{-1}\left(\mathcal{F}_{\xi'}^{-1}\left[m(\xi',\lambda)x_3Ae^{-(B\theta+A(1-\theta))x_3}\right](\cdot)\right)(t)\right\|_X d\theta\|g\|_{L_q(\mathbf{R}^2)}.
\end{aligned}
$$

To continue the proof, we apply Proposition 1.2.8 with $X=L_q((0,\infty),L_1(\mathbf{R}^2))$, $L=0$, $n=1$, and $\sigma=1/2-1/(2q)$ to the right-hand side of the aforementioned inequality. For $\alpha'\in\mathbf{N}_0^2$, $l=0,1$, and $0<\varepsilon<1$, we have, by Lemma 1.2.6, $m\in\mathbb{M}_{-1,2,\varepsilon,0}$, and Leibniz's rule,

$$
\begin{aligned}
(4.3.9)\quad &\left|D_{\xi'}^{\alpha'}\left\{(\tau\partial_\tau)^l\left(x_3Am(\xi',i\tau)e^{-(B\theta+A(1-\theta)x_3)}\right)\right\}\right| \\
&\le C(\alpha')\frac{x_3A}{|\tau|^{1/2}+A}e^{-c(|\tau|^{1/2}\theta+A)x_3}A^{-|\alpha'|} \\
&\le C(\alpha')\,x_3^\varepsilon|\tau|^{-1/2}e^{-c|\tau|^{1/2}\theta x_3}(x_3A)^{1-\varepsilon}e^{-cAx_3}A^{\varepsilon-|\alpha'|} \\
&\le C(\alpha')\,x_3^\varepsilon|\tau|^{-1/2}e^{-c|\tau|^{1/2}\theta x_3}e^{-(c/2)Ax_3}A^{\varepsilon-|\alpha'|}
\end{aligned}
$$

for some positive constants $C(\alpha')$ and c, which, combined with Proposition 1.2.8 with $X=\mathbf{R}$, $L=2$, $n=2$, and $\sigma=\varepsilon$, furnishes that

$$
\begin{aligned}
&\left|\mathcal{F}_{\xi'}^{-1}\left[(\tau\partial_\tau)^l\left(x_3Am(\xi',i\tau)e^{-(B\theta+A(1-\theta)x_3)}\right)\right](x')\right| \\
&\le Cx_3^\varepsilon|\tau|^{-1/2}e^{-c|\tau|^{1/2}\theta x_3}|x'|^{-(2+\varepsilon)}.
\end{aligned}
$$

On the other hand, we use (4.3.9) again with $\alpha' = 0$, and then

$$\begin{aligned}
&\left|\mathcal{F}^{-1}_{\xi'}\left[(\tau\partial_\tau)^l\left(x_3 Am(\xi', i\tau)e^{-(B\theta+A(1-\theta)x_3)}\right)\right](x')\right| \\
&\quad\le Cx_3^\varepsilon|\tau|^{-1/2}e^{-c|\tau|^{1/2}\theta x_3}\int_{\mathbf{R}^2} A^\varepsilon e^{-(c/2)Ax_3}\,d\xi' \\
&\quad\le Cx_3^\varepsilon|\tau|^{-1/2}e^{-c|\tau|^{1/2}\theta x_3}(x_3)^{-(2+\varepsilon)}
\end{aligned}$$

for $l = 0, 1$ with a positive constant C. We thus obtain

$$\left|\mathcal{F}^{-1}_{\xi'}\left[(\tau\partial_\tau)^l\left(x_3 Am(\xi', i\tau)e^{-(B\theta+A(1-\theta)x_3)}\right)\right](x')\right| \le \frac{Cx_3^\varepsilon|\tau|^{-1/2}e^{-c|\tau|^{1/2}\theta x_3}}{|x'|^{2+\varepsilon}+(x_3)^{2+\varepsilon}},$$

which furnishes that

$$\left\|(\tau\partial_\tau)^l\mathcal{F}^{-1}_{\xi'}\left[x_3 Am(\xi',\lambda)e^{-(B\theta+A(1-\theta))x_3}\right]\right\|_X \le C\theta^{-\frac{1}{q}}|\tau|^{-\frac12-\frac{1}{2q}}$$

for $l = 0, 1$ with a positive constant C. Then Proposition 1.2.8 and (4.3.8) implies, for any $t > 0$, that

$$\|\mathcal{J}(t)g\|_{L_q(\mathbf{R}^3_+)} \le C\int_0^1 \theta^{-\frac1q}\,d\theta\, t^{-\frac12+\frac{1}{2q}}\|g\|_{L_q(\mathbf{R}^2)} \le Ct^{-\frac12+\frac{1}{2q}}\|g\|_{L_q(\mathbf{R}^2)}$$

with some positive constant C independent of t and g. The estimate of $\nabla\mathcal{J}(t)g$ can be proved analogously, so that we may omit the proof. □

We next consider the equations:

$$\begin{cases}
\partial_t\mathbf{v} - \Delta\mathbf{v} + \nabla\pi = 0 & \text{in } \mathbf{R}^3_+,\ t>0,\\
\operatorname{div}\mathbf{v} = 0 & \text{in } \mathbf{R}^3_+,\ t>0,\\
\mathbf{S}(\mathbf{v},\pi)\mathbf{n} = \mathbf{h} & \text{on } \mathbf{R}^3_0,\ t>0,\\
\mathbf{v}|_{t=0} = 0 & \text{in } \mathbf{R}^3_+.
\end{cases}$$

Let $\mathbf{h} \in H^{1,1/2}_{q,p,1,0}(\mathbf{R}^3_+\times\mathbf{R})$ for $1 < p, q < \infty$, and then the equations above admits a unique solution $(\mathbf{v},\pi)$ with

$$\mathbf{v} \in W^{2,1}_{q,p,1,0}(\mathbf{R}^3_+\times\mathbf{R})^3, \quad \pi \in L_{p,1,0}(\mathbf{R}, \widehat{W}^1_q(\mathbf{R}^3_+))$$

by [**SS12**, Theorem 1.2]. Furthermore, in [**SS12**, Section 4], the exact formula of $\mathbf{v}$ is given by

$$\begin{aligned}
v_j(x,t) = &-\sum_{k=1}^2\mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}\left[\frac{2\xi_j\xi_k B}{AD(A,B)}A\mathcal{M}(x_3)\widetilde{h}_k(\xi',0,\lambda)\right](x',t)\\
&+\mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}\left[\frac{2i\xi_j(B^2-3AB+A^2)}{AD(A,B)}A\mathcal{M}(x_3)\widetilde{h}_3(\xi',0,\lambda)\right](x',t)\\
&+\sum_{k=1}^2\mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}\left[\frac{\xi_j\xi_k(3B-A)}{BD(A,B)}e^{-Bx_3}\widetilde{h}_k(\xi',0,\lambda)\right](x',t)\\
&-\mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}\left[\frac{2i\xi_j(B-A)}{D(A,B)}e^{-Bx_3}\widetilde{h}_3(\xi',0,\lambda)\right](x',t)\\
&-\mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}\left[\frac{1}{B}e^{-Bx_3}\widetilde{h}_j(\xi',0,\lambda)\right](x',t),
\end{aligned}$$

$$
\begin{aligned}
v_3(x,t) = &-\sum_{k=1}^{2}\mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{2i\xi_k B}{D(A,B)}A\mathcal{M}(x_3)\widetilde{h}_k(\xi',0,\lambda)\right](x',t)\\
&+\mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{B^2+A^2}{D(A,B)}A\mathcal{M}(x_3)\widetilde{h}_3(\xi',0,\lambda)\right](x',t)\\
&-\sum_{k=1}^{2}\mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_k(B-A)}{D(A,B)}e^{-Bx_3}\widetilde{h}_k(\xi',0,\lambda)\right](x',t)\\
&-\mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{A(B+A)}{D(A,B)}e^{-Bx_3}\widetilde{h}_3(\xi',0,\lambda)\right](x',t)
\end{aligned}
$$

for $j=1,2$, where we have used the relation: $e^{-Ax_3}=e^{-Bx_3}-(B-A)\mathcal{M}(x_3)$. Since the symbols:

$$
\begin{aligned}
&\frac{\xi_j\xi_k B}{AD(A,B)},\ \frac{2i\xi_j(B^2-3AB+A^2)}{AD(A,B)},\ \frac{\xi_j\xi_k(3B-A)}{BD(A,B)},\ \frac{2i\xi_j(B-A)}{D(A,B)},\ \frac{1}{B}\\
&\frac{2i\xi_k B}{D(A,B)},\ \frac{B^2+A^2}{D(A,B)},\ \frac{i\xi_k(B-A)}{D(A,B)},\ \frac{A(B+A)}{D(A,B)}\quad (j,k=1,2)
\end{aligned}
$$

belong to $\mathbb{M}_{-1,2,\varepsilon,0}$ for any $0<\varepsilon<\pi/2$ by Lemma 1.2.5 and Lemma 1.2.6, it follows from Lemma 4.3.3 that there exists an operator

$$\mathcal{C}(\tau)\in\mathcal{L}(L_q(\mathbf{R}^2)^3, W_q^1(\mathbf{R}^3_+)^3)\quad(\tau>0)$$

such that the solution $\mathbf{v}$ is represented as

$$\mathbf{v}(x,t)=\int_0^t[\mathcal{C}(t-s)\mathbf{h}(\cdot,0,s)](x)\,ds\quad(t>0).$$

In addition, for any $\mathbf{a}\in L_q(\mathbf{R}^2)^3$, we have

$$\|\nabla^l\mathcal{C}(\tau)\mathbf{a}\|_{L_q(\mathbf{R}^3_+)}\le C(q)\,\tau^{-\frac{1+l}{2}+\frac{1}{2q}}\|\mathbf{a}\|_{L_q(\mathbf{R}^2)}\quad(\tau>0)\tag{4.3.10}$$

for $l=0,1$ with some positive constant $C(q)$.

Proof of Theorem 4.3.2. Let $(\mathbf{u},\theta)$ be the solution to (4.3.1). Then multiplying the equations (4.3.1) by e^t yields that

$$
\begin{cases}
\partial_t(e^t\mathbf{u})-\Delta(e^t\mathbf{u})+\nabla(e^t\theta)=0 & \text{in } \mathbf{R}^3_+,\ t>0,\\
\mathrm{div}(e^t\mathbf{u})=0 & \text{in } \mathbf{R}^3_+,\ t>0,\\
\mathbf{S}(e^t\mathbf{u},e^t\theta)\mathbf{n}=e^t\mathbf{g} & \text{on } \mathbf{R}^3_0,\ t>0,\\
e^t\mathbf{u}|_{t=0}=0 & \text{in } \mathbf{R}^3_+.
\end{cases}
$$

Since $e^t\mathbf{g}\in H^{1,1/2}_{q,p,1,0}(\mathbf{R}^3_+\times\mathbf{R})$, we obtain, by using $\mathcal{C}(\tau)$ introduced above,

$$e^t\mathbf{u}(x,t)=\int_0^t[\mathcal{C}(t-s)(e^s\mathbf{g}(\cdot,0,s))](x)\,ds,$$

and therefore

$$\mathbf{u}(x,t)=\int_0^t e^{-(t-s)}[\mathcal{C}(t-s)\mathbf{g}(\cdot,0,s)](x)\,ds.$$

Setting $\mathcal{B}(\tau)=e^{-\tau}\mathcal{C}(\tau)$ and (4.3.10) completes the proof of Theorem 4.3.2.

THEOREM 4.3.4. *Let exponents p, q satisfy (4.1.4), and let $1 < r \le q$. Suppose that $\mathbf{g} \in H^{1,1/2}_{r,p,0}(\mathbf{R}^3_+ \times \mathbf{R})$. Then the equations (4.3.1) admits a unique solution $(\mathbf{u}, \theta)$ with*

$$\mathbf{u} \in W^{2,1}_{r,p,0}(\mathbf{R}^3_+ \times \mathbf{R})^3, \quad \theta \in L_{p,0}(\mathbf{R}, \widehat{W}^1_r(\mathbf{R}^3_+)),$$

and also the solution satisfies

$$\|(\partial_t \mathbf{u}, \mathbf{u}, \nabla \mathbf{u}, \nabla^2 \mathbf{u}, \nabla \theta)\|_{L_p(\mathbf{R}_+, L_r(\mathbf{R}^3_+))} \le C \|\mathbf{g}\|_{H^{1,1/2}_{r,p,0}(\mathbf{R}^3_+ \times \mathbf{R})}$$

for a positive constant $C = C(p, r)$. If we additionally assume that

$$\mathbf{g} \in L^{\alpha(r)}_p(\mathbf{R}_+, W^1_r(\mathbf{R}^3_+))^3, \quad (t+2)^{\beta(r)} \mathbf{g} \in H^{1,1/2}_{r,p,0}(\mathbf{R}^3_+ \times \mathbf{R})^3$$

for non-negative numbers $\alpha(r)$ and $\beta(r)$, then the following assertions hold.

(1) *There exists a positive constant $C = C(p, r)$ such that for any $t > 0$*

$$\|\mathbf{u}(t)\|_{W^1_r(\mathbf{R}^3_+)} \le C(t+2)^{-\alpha(r)} \|\mathbf{g}\|_{L^{\alpha(r)}_p(\mathbf{R}_+, W^1_r(\mathbf{R}^3_+))}.$$

(2) *There exists a positive constant $C = C(p, r)$ such that*

$$\begin{aligned} &\|(\partial_t \mathbf{u}, \mathbf{u}, \nabla \mathbf{u}, \nabla^2 \mathbf{u}, \nabla \theta)\|_{L^{\beta(r)}_p(\mathbf{R}_+, L_r(\mathbf{R}^3_+))} \\ &\quad \le C \left(\|\mathbf{g}\|_{L^{\alpha(r)}_p(\mathbf{R}_+, W^1_r(\mathbf{R}^3_+))} + \|(t+2)^{\beta(r)} \mathbf{g}\|_{H^{1,1/2}_{r,p,0}(\mathbf{R}^3_+ \times \mathbf{R})} \right), \end{aligned}$$

provided that $p(1 + \alpha(r) - \beta(r)) > 1$.

PROOF. The assertions except for the last two inequalities were already given in Theorem 4.3.2, so that we here show the inequalities only.

We first consider the estimates of $\|\nabla^l \mathbf{u}(t)\|_{L_r(\mathbf{R}^3_+)}$ for $l = 0, 1$ and $t > 0$ by using Theorem 4.3.2. We then have

$$\mathbf{u}(x, t) = \left(\int_0^{t/2} + \int_{t/2}^t \right) [\mathcal{B}(t-s) \mathbf{g}(\cdot, 0, s)](x) \, ds =: \mathbf{u}_1(x, t) + \mathbf{u}_2(x, t).$$

Concerning $\mathbf{u}_1(x, t)$, it follows from the trace theorem that for $p' = p/(p-1)$

$$\begin{aligned} \|\nabla^l \mathbf{u}_1(t)\|_{L_r(\mathbf{R}^3_+)} &\le C \int_0^{t/2} e^{-(t-s)} (t-s)^{-\frac{1+l}{2} + \frac{1}{2r}} \|\mathbf{g}(s)\|_{L_r(\mathbf{R}^3_0)} \, ds \\ &\le C e^{-t/2} t^{-\frac{1+l}{2} + \frac{1}{2r}} \left(\int_0^{t/2} ds \right)^{1/p'} \|\mathbf{g}\|_{L_p(\mathbf{R}_+, W^1_r(\mathbf{R}^3_+))} \\ &\le C e^{-t/4} \|\mathbf{g}\|_{L_p(\mathbf{R}_+, W^1_r(\mathbf{R}^3_+))} \end{aligned}$$

with some positive constant $C = C(p, r)$, because by (4.1.4)

$$\frac{1}{p'} - \frac{1+l}{2} + \frac{1}{2r} \ge \frac{1}{p'} - 1 + \frac{1}{2q} = -\frac{1}{p} + \frac{1}{2q} > \frac{2}{q} - \frac{1}{2} > 0. \tag{4.3.11}$$

On the other hand, we have, by the trace theorem and (4.3.11),

$$
\begin{aligned}
&\|\nabla^l \mathbf{u}_2(t)\|_{L_r(\mathbf{R}^3_+)} \leq C\int_{t/2}^t e^{-(t-s)}(t-s)^{-\frac{1+l}{2}+\frac{1}{2r}}\|\mathbf{g}(s)\|_{W^1_r(\mathbf{R}^3_+)}\,ds \\
&\leq C(t+2)^{-\alpha(r)}\int_{t/2}^t e^{-(t-s)}(t-s)^{-\frac{l+1}{2}+\frac{1}{2r}}(s+2)^{\alpha(r)}\|\mathbf{g}(s)\|_{W^1_r(\mathbf{R}^3_+)}\,ds \\
&\leq C(t+2)^{-\alpha(r)} \\
&\quad\times\left(\int_{t/2}^t e^{-p'(t-s)}(t-s)^{-p'\left(\frac{l+1}{2}-\frac{1}{2r}\right)}\,ds\right)^{1/p'}\|\mathbf{g}\|_{L_p^{\alpha(r)}(\mathbf{R}_+,W^1_r(\mathbf{R}^3_+))} \\
&\leq C(t+2)^{-\alpha(r)}\|\mathbf{g}\|_{L_p^{\alpha(r)}(\mathbf{R}_+,W^1_r(\mathbf{R}^3_+))}
\end{aligned}
$$

with a positive constant $C = C(p,r)$, which, combined with the estimates of $\mathbf{u}_1(x,t)$, completes the required inequality in (1).

Next, we shall prove the inequality of (2). Since $(\mathbf{u},\theta)$ satisfies the equations (4.3.1), we see, by setting $\mathbf{U} = (t+2)^{\beta(r)}\mathbf{u}$ and $\Theta = (t+2)^{\beta(r)}\theta$, that

$$
\begin{cases}
\partial_t\mathbf{U} + \mathbf{U} - \Delta\mathbf{U} + \nabla\Theta = -\beta(r)(t+2)^{-1+\beta(r)}\mathbf{u} & \text{in } \mathbf{R}^3_+,\ t>0, \\
\operatorname{div}\mathbf{U} = 0 & \text{in } \mathbf{R}^3_+,\ t>0, \\
\mathbf{S}(\mathbf{U},\Theta)\mathbf{n} = (t+2)^{\beta(r)}\mathbf{g} & \text{on } \mathbf{R}^3_0,\ t>0, \\
\mathbf{U}|_{t=0} = 0 & \text{in } \mathbf{R}^3_+.
\end{cases}
$$

It then follows from $p(1+\alpha(r)-\beta(r)) > 1$ and the estimate of (1) that

$$
\begin{aligned}
&\|(t+2)^{-1+\beta(r)}\mathbf{u}\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_+))} \\
&\quad\leq C\|(t+2)^{-(1+\alpha(r)-\beta(r))}\|_{L_p(\mathbf{R}_+)}\|\mathbf{g}\|_{L_p^{\alpha(r)}(\mathbf{R}_+,W^1_r(\mathbf{R}^3_+))} \\
&\quad\leq C\|\mathbf{g}\|_{L_p^{\alpha(r)}(\mathbf{R}_+,W^1_r(\mathbf{R}^3_+))},
\end{aligned}
\tag{4.3.12}
$$

so that by Theorem 4.3.2

$$
\begin{aligned}
&\|(\partial_t\mathbf{U},\mathbf{U},\nabla\mathbf{U},\nabla^2\mathbf{U},\nabla\Theta)\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_+))} \\
&\quad\leq C\left(\|\mathbf{g}\|_{L_p^{\alpha(r)}(\mathbf{R}_+,W^1_r(\mathbf{R}^3_+))} + \|(t+2)^{\beta(r)}\mathbf{g}\|_{H^{1,1/2}_{r,p,0}(\mathbf{R}^3_+\times\mathbf{R})}\right)
\end{aligned}
\tag{4.3.13}
$$

with some positive constant $C = C(p,r)$. Noting that

$$
(t+2)^{\beta(r)}\partial_t\mathbf{u} = \partial_t\mathbf{U} - \beta(r)(t+2)^{-1+\beta(r)}\mathbf{u},
$$

we obtain, by (4.3.12) and (4.3.13),

$$
\begin{aligned}
&\|(t+2)^{\beta(r)}\partial_t\mathbf{u}\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_+))} \\
&\quad\leq C\left(\|\mathbf{g}\|_{L_p^{\alpha(r)}(\mathbf{R}_+,W^1_r(\mathbf{R}^3_+))} + \|(t+2)^{\beta(r)}\mathbf{g}\|_{H^{1,1/2}_{r,p,0}(\mathbf{R}^3_+\times\mathbf{R})}\right),
\end{aligned}
$$

which, combined with (4.3.13), completes the proof of the theorem. □

4.3.2. Analysis of Equations (4.3.2) **and** (4.3.3)**.** We first remind the following proposition, which was proved by [**SS12**, Theorem 1.4] essentially.

PROPOSITION 4.3.5. *Let $1 < p, q < \infty$. Then there exists a number $\gamma_0 \geq 1$ such that for every*

$$
\mathbf{f} \in L_{p,\gamma_0,0}(\mathbf{R}, L_q(\mathbf{R}^3_+))^3, \quad g_h \in L_{p,\gamma_0,0}(\mathbf{R}, W^2_q(\mathbf{R}^3_+)),
$$

the equations (4.3.2) *and* (4.3.3) *admit a unique solution* $(\mathbf{u}, \theta, h, H)$ *with*

$$\begin{aligned} &\mathbf{u} \in W^{2,1}_{q,p,\gamma_0,0}(\mathbf{R}^3_+ \times \mathbf{R})^3, \quad \theta \in L_{p,\gamma_0,0}(\mathbf{R}, \widehat{W}^1_q(\mathbf{R}^3_+)), \\ &h \in W^1_{p,\gamma_0,0}(\mathbf{R}, W^{2-1/q}_q(\mathbf{R}^2)) \cap L_{p,\gamma_0,0}(\mathbf{R}, W^{3-1/q}_q(\mathbf{R}^2)), \\ &H \in W^1_{p,\gamma_0,0}(\mathbf{R}, W^2_q(\Omega_L)) \cap L_{p,\gamma_0,0}(\mathbf{R}, W^3_q(\Omega_L)), \end{aligned} \tag{4.3.14}$$

where $\Omega_L = \mathbf{R}^2 \times (0, L)$ *for* $L > 0$, *and besides,*

$$\nabla H \in L_{p,\gamma_0,0}(\mathbf{R}, W^2_q(\mathbf{R}^3_+))^3, \quad \nabla \partial_t H \in L_{p,\gamma_0,0}(\mathbf{R}, W^1_q(\mathbf{R}^3_+))^3.$$

The solution $(\mathbf{u}, \theta, h, H)$ *satisfies the estimate:*

$$\begin{aligned} &\|e^{-\gamma_0 t}(\partial_t \mathbf{u}, \mathbf{u}, \nabla \mathbf{u}, \nabla^2 \mathbf{u}, \nabla \theta)\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_+))} \\ &\quad + \|e^{-\gamma_0 t}\partial_t h\|_{L_p(\mathbf{R}_+, W^{2-1/q}_q(\mathbf{R}^2))} + \|e^{-\gamma_0 t} h\|_{L_p(\mathbf{R}_+, W^{3-1/q}_q(\mathbf{R}^2))} \\ &\quad + \|e^{-\gamma_0 t}\nabla H\|_{L_p(\mathbf{R}_+, W^2_q(\mathbf{R}^3_+))} + \|e^{-\gamma_0 t}\nabla \partial_t H\|_{L_p(\mathbf{R}_+, W^1_q(\mathbf{R}^3_+))} \\ &\leq C(p, q, \gamma_0)\left(\|e^{-\gamma_0 t}\mathbf{f}\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_+))} + \|e^{-\gamma_0 t} g_h\|_{L_p(\mathbf{R}_+, W^2_q(\mathbf{R}^3_+))}\right) \end{aligned} \tag{4.3.15}$$

for a positive constant $C(p, q, \gamma_0)$ *depending only on* p, q, *and* γ_0. *If we assume that* $g_h \in H^{1/2}_{p,\gamma_0,0}(\mathbf{R}, W^1_q(\mathbf{R}^3_+))$ *additionally, then for every* $L > 0$

$$H \in H^{3/2}_{p,\gamma_0,0}(\mathbf{R}, W^1_q(\Omega_L)), \quad \nabla \partial_t H \in H^{1/2}_{p,\gamma_0,0}(\mathbf{R}, L_q(\mathbf{R}^3_+))^3, \tag{4.3.16}$$

and furthermore,

$$\begin{aligned} \|e^{-\gamma_0 t}\Lambda^{1/2}_{\gamma_0}\nabla \partial_t H\|_{L_p(\mathbf{R}, L_q(\mathbf{R}^3_+))} &\leq C(p, q, \gamma_0)\Big(\|e^{-\gamma_0 t}\mathbf{f}\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_+))} \\ &\quad + \|e^{-\gamma_0 t} g_h\|_{L_p(\mathbf{R}_+, W^2_q(\mathbf{R}^3_+))} + \|e^{-\gamma_0 t}\Lambda^{1/2}_{\gamma_0} g_h\|_{L_p(\mathbf{R}, W^1_q(\mathbf{R}^3_+))}\Big) \end{aligned} \tag{4.3.17}$$

with some positive constant $C(p, q, \gamma_0)$.

REMARK 4.3.6. (1) Let $1 < p, q < \infty$, and suppose that $\mathbf{f} \in L_{p,0}(\mathbf{R}, L_q(\mathbf{R}^3_+))^3$ and $g_h \in W^{2,1}_{q,p,0}(\mathbf{R}^3_+ \times \mathbf{R})$. Then, by Lemma 1.3.5 (3) and Remark 1.3.6, we see that

$$\begin{aligned} &\|e^{-\gamma_0 t}\Lambda^{1/2}_{\gamma_0} g_h\|_{L_p(\mathbf{R}, W^1_q(\mathbf{R}^3_+))} \leq C(p)\|e^{-\gamma_0 t} g_h\|_{H^{1/2}_{p,0}(\mathbf{R}, W^1_q(\mathbf{R}^3_+))} \\ &\quad \leq C(p, q)\|e^{-\gamma_0 t} g_h\|_{W^{2,1}_{q,p,0}(\mathbf{R}^3_+ \times \mathbf{R})} \leq C(p, q, \gamma_0)\|g_h\|_{W^{2,1}_{q,p}(\mathbf{R}^3_+ \times \mathbf{R}_+)}, \end{aligned}$$

which, combined with (4.3.17), furnishes that

$$\begin{aligned} &\|e^{-\gamma_0 t}\Lambda^{1/2}_{\gamma_0}\nabla \partial_t H\|_{L_p(\mathbf{R}, L_q(\mathbf{R}^3_+))} \\ &\qquad \leq C(p, q, \gamma_0)\left(\|\mathbf{f}\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_+))} + \|g_h\|_{W^{2,1}_{q,p}(\mathbf{R}^3_+ \times \mathbf{R}_+)}\right). \end{aligned} \tag{4.3.18}$$

(2) Let $2 < p < \infty$ and $1 < q < \infty$, and suppose that $\mathbf{f} \in L_{p,0}(\mathbf{R}, L_q(\mathbf{R}^3_+))^3$ and $g_h \in W^{2,1}_{q,p,0}(\mathbf{R}^3_+ \times \mathbf{R})$. It then follows from Lemma 1.3.5 (2) and (4.3.15) that for every $T > 0$

$$\begin{aligned} &\|(\mathbf{u}, \nabla H)\|_{BUC((0,T), W^1_q(\mathbf{R}^3_+))} \leq C\|(\mathbf{u}, \nabla H)\|_{W^{2,1}_{q,p}(\mathbf{R}^3_+ \times (0,T))} \\ &\leq Ce^{\gamma_0 T}\Big(\|e^{-\gamma_0 t}(\partial_t \mathbf{u}, \nabla \partial_t H)\|_{L_p((0,T), L_q(\mathbf{R}^3_+))} \\ &\qquad + \|e^{-\gamma_0 t}(\mathbf{u}, \nabla H)\|_{L_p((0,T), W^2_q(\mathbf{R}^3_+))}\Big) \\ &\leq C\left(\|\mathbf{f}\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_+))} + \|g_h\|_{L_p(\mathbf{R}_+, W^2_q(\mathbf{R}^3_+))}\right) \end{aligned}$$

with some positive constant $C = C(p, q, \gamma_0, T)$. Additionally, we obtain, by Lemma 1.3.5 (3), Remark 1.3.6, and (4.3.18),

$$
\begin{aligned}
&\|\nabla \partial_t H\|_{BUC((0,T),L_q(\mathbf{R}^3_+))} \leq e^{\gamma_0 T}\|e^{-\gamma_0 t}\nabla\partial_t H\|_{BUC((0,T),L_q(\mathbf{R}^3_+))} \\
&\quad\leq C\|e^{-\gamma_0 t}\nabla\partial_t H\|_{H^{1/2}_p((0,T),L_q(\mathbf{R}^3_+))} \leq C\|e^{-\gamma_0 t}\nabla\partial_t H\|_{H^{1/2}_{p,0}(\mathbf{R},L_q(\mathbf{R}^3_+))} \\
&\quad\leq C\|e^{-\gamma_0 t}\Lambda^{1/2}_{\gamma_0}\nabla\partial_t H\|_{L_p(\mathbf{R},L_q(\mathbf{R}^3_+))} \\
&\quad\leq C\left(\|\mathbf{f}\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_+))} + \|g_h\|_{W^{2,1}_{q,p}(\mathbf{R}^3_+\times\mathbf{R}_+)}\right)
\end{aligned}
$$

with some positive constant $C = C(p, q, \gamma_0, T)$.

(3) Since it follows from Sobolev embedding theorem and [**MS12**, Proposition 3.2] with $s = 0, r = 2, \alpha = 1$, and $\beta = 1$ that for every $L > 0$ and $0 < \sigma < 1$

$$
\begin{aligned}
&W^1_{p,0}(\mathbf{R}, W^2_q(\Omega_L)) \cap L_{p,0}(\mathbf{R}, W^3_q(\Omega_L)) \\
&\quad\hookrightarrow W^{\sigma}_{p,0}(\mathbf{R}, W^{3-\sigma}_q(\Omega_L)) \hookrightarrow BUC(\mathbf{R}_+, W^{3-\sigma}(\Omega_L)) \\
&\quad\hookrightarrow BUC(\mathbf{R}_+, BUC^2(\Omega_L)),
\end{aligned}
$$

provided that $\sigma > 1/p$ and $3 - \sigma > 2 + 3/q$. By [**MS12**, Proposition 2.10, 3.2], we similarly see, for every $L > 0$, that

$$
\begin{aligned}
&H^{3/2}_{p,0}(\mathbf{R}, W^1_q(\Omega_L)) \cap L_{p,0}(\mathbf{R}, W^3_q(\Omega_L)) \\
&\quad\hookrightarrow H^{(3/2)\sigma}_{p,0}(\mathbf{R}, H^{3-2\sigma}_q(\Omega_L)) \hookrightarrow BUC^1(\mathbf{R}_+, BUC^1(\Omega_L)),
\end{aligned}
$$

if $(3/2)\sigma > 1 + 1/p$ and $3 - 2\sigma > 1 + 3/q$. We thus obtain, by using (4.3.14) and (4.3.16),

$$e^{-\gamma_0 t}H \in BUC(\mathbf{R}_+, BUC^2(\Omega_L)) \cap BUC^1(\mathbf{R}_+, BUC^1(\Omega_L))$$

for every $L > 0$ under the assumption (4.1.4). The last property implies that

$$H \in C(\mathbf{R}_+, C^2(\mathbf{R}^3_+)) \cap C^1(\mathbf{R}_+, C^1(\mathbf{R}^3_+)).$$

We use the function spaces given at the end of Section 4.2 below, and then the following theorem holds.

THEOREM 4.3.7. *Let exponents p, q satisfy (4.1.4). Suppose that $\varepsilon_1 > 1$ and $\varepsilon_2, \varepsilon_3 \geq 1$, and that $0 < \delta_1, \delta_2 \leq 1$ satisfy*

$$
\begin{aligned}
&p(\min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\} - \delta_1) > 1, \qquad p\left(m\left(\frac{q}{2}, q\right) + \frac{1}{4} - \delta_1\right) > 1, \\
&p(\min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\} - \delta_2) > 1, \qquad p\left(m\left(\frac{q}{2}, 2\right) + 1 - \delta_2\right) > 1.
\end{aligned}
\tag{4.3.19}
$$

In addition, we set $\delta_0 = \max\{\delta_1, \delta_2\}$, and let the right members $\mathbf{f}_1$, $\mathbf{f}_2$, and g_h of the equations (4.3.2) satisfy the conditions as follows:

(1) *Let $\mathbf{f}_1 \in \mathbb{F}_1 \cap \widetilde{\mathbb{F}}_1(\delta_0, \varepsilon_1)$;*
(2) *Let $\mathbf{f}_2 \in \mathbb{F}_2 \cap \widetilde{\mathbb{F}}_2(\delta_0, \varepsilon_2)$;*
(3) *Let $g_h \in \mathbb{G}_h \cap \widetilde{\mathbb{G}}_h(\delta_0, \varepsilon_3) \cap \mathbb{A}_1 \cap \widehat{\mathbb{A}}_2$.*

Then the solution $(\mathbf{u},\theta,h,H)$ *obtained in Proposition* 4.3.5 *to the equations* (4.3.2) *and* (4.3.3) *possesses the estimate:*

$$\sum_{r\in\{q,2\}}\Big(\mathbb{W}_{r,\infty}(\mathbf{u},h,\partial_t h,H)+\mathbb{M}_{r,p}(\mathbf{u},\theta,h,\partial_t h,H)\Big)+\mathbb{W}_{q,p}(\mathbf{u},H;\delta_1,\delta_2)$$
$$\leq C(p,q)\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}+\|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\mathbb{A}_1\cap\widehat{\mathbb{A}}_2}\right)$$

with some positive constant $C(p,q)$.

Proof. In the proof, we shall show the following inequality:

$$\sum_{r\in\{q,2\}}\Big(\mathbb{W}_{r,\infty}(\mathbf{u},h,0,H)+\mathbb{M}_{r,p}(\mathbf{u},0,h,0,H)\Big)+\mathbb{W}_{q,p}(\mathbf{u},H;\delta_1,\delta_2)$$
$$\leq C(p,q)\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}+\|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\mathbb{A}_1\cap\widehat{\mathbb{A}}_2}\right) \tag{4.3.20}$$

with some positive constant $C(p,q)$. If we obtain (4.3.20), then the required estimate of θ and $\partial_t h$ follows from the fact that θ and h satisfy the equations (4.3.2). In fact, it holds, by the trace theorem, that

$$\|\nabla\theta\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_+))}\leq\|(\partial_t\mathbf{u},\Delta\mathbf{u},\mathbf{f}_1,\mathbf{f}_2)\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_+))},$$
$$\|\partial_t h\|_{L_p(\mathbf{R}_+,W^{2-1/r}_r(\mathbf{R}^2))}\leq C\|(\mathbf{u},g_h)\|_{L_p(\mathbf{R}_+,W^2_r(\mathbf{R}^3_+))},$$
$$\|\partial_t h\|_{L^{m(q/2,r)}_\infty(\mathbf{R}_+,L_r(\mathbf{R}^2))}\leq C\|(\mathbf{u},g_h)\|_{L^{m(q/2,r)}_\infty(\mathbf{R}_+,W^1_r(\mathbf{R}^3_+))}$$

for $r\in\{q,2\}$, which, combined with (4.3.20), furnishes the required estimate of Theorem 4.3.7.

To show the required decay properties and weighted estimates, we first give the exact formulas of the solution to the equations (4.3.2). Let $(\bar{\mathbf{w}},\bar{p})$ be the solution to the following resolvent equations:

$$\begin{cases}\lambda\bar{\mathbf{w}}-\Delta\bar{\mathbf{w}}+\nabla\bar{p}=\mathcal{L}[\mathbf{f}](\lambda) & \text{in } \mathbf{R}^3_+\\ \operatorname{div}\bar{\mathbf{w}}=0 & \text{in } \mathbf{R}^3_+,\\ \mathbf{S}(\bar{\mathbf{w}},\bar{p})\mathbf{n}=0 & \text{on } \mathbf{R}^3_0,\end{cases}$$

where we have set $\mathbf{f}=\mathbf{f}_1+\mathbf{f}_2$, and $(\bar{\mathbf{v}},\bar{\pi},\bar{h})$ the solution to the following resolvent equations:

$$\begin{cases}\lambda\bar{\mathbf{v}}-\Delta\bar{\mathbf{v}}+\nabla\bar{\pi}=0 & \text{in } \mathbf{R}^3_+,\\ \operatorname{div}\bar{\mathbf{v}}=0 & \text{in } \mathbf{R}^3_+,\\ \mathbf{S}(\bar{\mathbf{v}},\bar{\pi})\mathbf{n}+(c_g-c_\sigma\Delta')\bar{h}\mathbf{n}=0 & \text{on } \mathbf{R}^3_0,\\ \partial_t\bar{h}-\bar{\mathbf{v}}\cdot\mathbf{n}=\mathcal{L}[g_h](\lambda)+\bar{\mathbf{w}}\cdot\mathbf{n} & \text{on } \mathbf{R}^3_0,\end{cases}$$

We then see that

$$\mathbf{u}=\mathcal{L}^{-1}_\lambda[\bar{\mathbf{v}}+\bar{\mathbf{w}}](t),\quad \theta=\mathcal{L}^{-1}_\lambda[\bar{\rho}+\bar{\pi}](t),\quad h=\mathcal{L}^{-1}_\lambda[\bar{h}](t),$$
$$H=\mathcal{F}^{-1}_{\xi'}[e^{-|\xi'|x_3}\mathcal{F}_{x'}[\bar{h}](\xi',t)](x')\quad(x_3>0)$$

solve the equations (4.3.2) and (4.3.3) uniquely. Here $\bar{\mathbf{w}}=\mathbf{w}^1+\mathbf{w}^2$, where $\mathbf{w}^i$ $(i=1,2)$ are given by (3.2.8) and (3.2.13), respectively, and also $(\bar{\mathbf{v}},\bar{h})$ is the same function as $(\mathbf{v},h)$ of (3.4.2). By (3.4.6), setting, for $a\in\{0,\infty\}$,

$$\mathbf{v}_a=\mathbf{v}_a(x,t)=\mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}[\varphi_a(\xi')\mathcal{F}_{x'}[\bar{\mathbf{v}}](\xi',x_3,\lambda)](x',t), \tag{4.3.21}$$

$$h_a = h_a(x', t) = \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}[\varphi_a(\xi')\mathcal{F}_{x'}[\bar{h}](\xi', \lambda)](x', t),$$
$$H_a = H_a(x, t) = \mathcal{F}_{\xi'}^{-1}[\varphi_a(\xi')e^{-|\xi'|x_3}\mathcal{F}_{x'}[h](\xi', t)](x'),$$
$$\mathbf{w} = \mathbf{w}(x, t) = \mathcal{L}_\lambda^{-1}[\bar{\mathbf{w}}(x, \lambda)](t),$$

we see that

$$\mathbf{u} = \mathbf{v}_0 + \mathbf{v}_\infty + \mathbf{w}, \quad h = h_0 + h_\infty, \quad H = H_0 + H_\infty.$$

In addition, it follows from Proposition C.2, (3.4.7), and (3.4.9) that

$$\mathbf{v}_a(t) = \int_0^t S_a(t-s)\mathbf{F}(s)\,ds, \qquad \mathbf{w}(t) = \int_0^t R(t-s)\mathbf{f}(s)\,ds$$
$$h_a(t) = \int_0^t T_a(t-s)\mathbf{F}(s)\,ds, \qquad H_a(t) = \int_0^t \mathcal{E}(T_a(t-s)\mathbf{F}(s))\,ds \tag{4.3.22}$$

for $a \in \{0, \infty\}$, $\mathbf{F} = (\mathbf{f}, g_h) = (\mathbf{f}_1 + \mathbf{f}_2, g_h)$, and the extension operator $\mathcal{E}$ defined as (3.1.3). By Proposition C.2 again, we also have, for $a \in \{0, \infty\}$,

$$\partial_t \mathbf{v}_0(t) = \int_0^t \partial_t S_0(t-s)\mathbf{F}(s)\,ds, \tag{4.3.23}$$
$$\partial_t H_a(t) = \int_0^t \partial_t \mathcal{E}(T_a(t-s)\mathbf{F}(s))\,ds + \mathcal{F}_{\xi'}^{-1}[\varphi_a(\xi')e^{-|\xi'|x_3}\widehat{g}_h(\xi', 0, t)](x').$$

Remark 4.3.8. Let $a \in \{0, \infty\}$, and then $H_a(t) = H_a^1(t) + H_a^2(t)$ with

$$
\begin{aligned}
&H_a^1(x, t)\\
&= -\sum_{k=1}^2 \int_0^\infty \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_a(\xi')i\xi_k(B-A)}{(B+A)L(A,B)}e^{-A(x_3+y_3)}\widetilde{f}_k(\xi', y_3, \lambda)\right](x', t)\,dy_3\\
&- \int_0^\infty \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_a(\xi')A}{L(A,B)}e^{-A(x_3+y_3)}\widetilde{f}_3(\xi', y_3, \lambda)\right](x', t)\,dy_3\\
&+ \sum_{k=1}^2 \int_0^\infty \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_a(\xi')2i\xi_k AB}{(B+A)L(A,B)}e^{-Ax_3}\mathcal{M}(y_3)\widetilde{f}_k(\xi', y_3, \lambda)\right](x', t)\,dy_3\\
&+ \int_0^\infty \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_a(\xi')2A^3}{(B+A)L(A,B)}e^{-Ax_3}\mathcal{M}(y_3)\widetilde{f}_3(\xi', y_3, \lambda)\right](x', t)\,dy_3,\\
&H_a^2(x, t)\\
&= \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_a(\xi')D(A,B)}{(B+A)L(A,B)}e^{-Ax_3}\widetilde{g}_h(\xi', 0, \lambda)\right](x', t),
\end{aligned}
$$

where $\widetilde{f}$ is defined as (4.3.5). Since $\lambda = B^2 - A^2$ and

$$\frac{\lambda D(A,B)}{(B+A)L(A,B)} = \frac{(B-A)D(A,B)}{L(A,B)} = 1 - \frac{A(c_g + c_\sigma A^2)}{L(A,B)},$$

we obtain

$$
\begin{aligned}
\partial_t H_a^2(x, t) &= \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_a(\xi')\lambda D(A,B)}{(B+A)L(A,B)}e^{-Ax_3}\widetilde{g}_h(\xi', 0, \lambda)\right](x', t)\\
&= -\mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_a(\xi')A(c_g + c_\sigma A^2)}{L(A,B)}\widetilde{g}_h(\xi', 0, \lambda)\right](x', t)\\
&+ \mathcal{F}_{\xi'}^{-1}\left[\varphi_a(\xi')e^{-Ax_3}\widehat{g}_h(\xi', 0, t)\right](x').
\end{aligned} \tag{4.3.24}
$$

Note that the last term of (4.3.24) did not appear when we considered the initial value problem (3.1.1), because g was independent of time t in the case of (3.1.1) (see the end of Section 3.5 and 3.6). We thus obtain the second identity of (4.3.23) by Proposition C.2 and (4.3.24).

We here summarize properties of the operators used in (4.3.22) and (4.3.23).

PROPOSITION 4.3.9. *Let exponents p, q satisfy* (4.1.4), *and* $2 \le r \le q$. *We set* $\mathbf{F} = (\mathbf{f}, g)$, *and suppose that*

$$\mathbf{f} \in L_r(\mathbf{R}^3_+)^3 \cap L_s(\mathbf{R}^3_+)^3, \quad g \in W_r^{2-1/r}(\mathbf{R}^2) \cap L_s(\mathbf{R}^2) \quad (s = q/2).$$

Then the following assertions hold.

(1) *Let $k = 1, 2$ and $l = 0, 1, 2$. Then there exists a positive constant $C = C(q, r)$ such that for any $t \ge 1$*

$$\begin{aligned}
\|\partial_t S_0(t)\mathbf{F}\|_{L_r(\mathbf{R}^3_+)} &\le C(t+2)^{-m(s,r)-\frac{1}{4}}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times L_s(\mathbf{R}^2)},\\
\|S_0(t)\mathbf{F}\|_{L_r(\mathbf{R}^3_+)} &\le C(t+2)^{-m(s,r)}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times L_s(\mathbf{R}^2)},\\
\|\nabla^k S_0(t)\mathbf{F}\|_{L_r(\mathbf{R}^3_+)} &\le C(t+2)^{-n(s,r)-\frac{k}{8}}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times L_s(\mathbf{R}^2)},\\
\|\nabla^l \partial_t \mathcal{E}(T_0(t)\mathbf{F})\|_{L_r(\mathbf{R}^3_+)} &\le C(t+2)^{-m(s,r)-\frac{l}{2}}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times L_s(\mathbf{R}^2)},\\
\|\nabla^{1+l} \mathcal{E}(T_0(t)\mathbf{F})\|_{L_r(\mathbf{R}^3_+)} &\le C(t+2)^{-m(s,r)-\frac{1}{4}-\frac{l}{2}}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times L_s(\mathbf{R}^2)},\\
\|T_0(t)\mathbf{F}\|_{L_r(\mathbf{R}^2)} &\le C(t+2)^{-\left(\frac{1}{s}-\frac{1}{r}\right)}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times L_s(\mathbf{R}^2)}.
\end{aligned}$$

On the other hand, let $0 < t \le 1$, and then

$$\begin{aligned}
&\|\partial_t S_0(t)\mathbf{F}\|_{L_r(\mathbf{R}^3_+)} + \|(S_0(t)\mathbf{F}, \partial_t\mathcal{E}(T_0(t)\mathbf{F}), \nabla\mathcal{E}(T_0(t)\mathbf{F}))\|_{W_r^2(\mathbf{R}^3_+)}\\
&\qquad + \|T_0(t)\mathbf{F}\|_{L_r(\mathbf{R}^2)} \le C(\alpha, q, r)\, t^{-\alpha}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times L_s(\mathbf{R}^2)}
\end{aligned}$$

for any $\alpha > 0$ with some positive constant $C(\alpha, q, r)$.

(2) *There exist positive constants σ_0 and $C = C(q, r)$ such that for any $t \ge 1$*

$$\begin{aligned}
&\|(S_\infty(t)\mathbf{F}, \partial_t\mathcal{E}(T_\infty(t)\mathbf{F}), \nabla\mathcal{E}(T_\infty(t)\mathbf{F}))\|_{W_r^1(\mathbf{R}^3_+)}\\
&\qquad + \|T_\infty(t)\mathbf{F}\|_{L_r(\mathbf{R}^2)} \le Ce^{-\sigma_0 t}\|\mathbf{F}\|_{L_r(\mathbf{R}^3_+)^3\times W_r^{2-1/r}(\mathbf{R}^2)}.
\end{aligned}$$

(3) *If $g \in W_s^{2-1/s}(\mathbf{R}^2)$ $(s = q/2)$ additionally, then there exist positive constants σ_0 and $C = C(q, r)$ such that for any $t > 0$*

$$\begin{aligned}
&\|(S_\infty(t)\mathbf{F}, \partial_t\mathcal{E}(T_\infty(t)\mathbf{F}), \nabla\mathcal{E}(T_\infty(t)\mathbf{F}))\|_{W_r^1(\mathbf{R}^3_+)}\\
&\qquad + \|T_\infty(t)\mathbf{F}\|_{L_r(\mathbf{R}^2)} \le Ct^{-\frac{3}{2}\left(\frac{1}{s}-\frac{1}{r}\right)-\frac{1}{2}}e^{-\sigma_0 t}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times W_s^{2-1/s}(\mathbf{R}^2)}.
\end{aligned}$$

(4) *There exists a positive constant $C = C(q, r)$ such that for any $t > 0$*

$$\|\nabla^k R(t)\mathbf{f}\|_{L_r(\mathbf{R}^3_+)} \le Ct^{-\frac{3}{2}\left(\frac{1}{s}-\frac{1}{r}\right)-\frac{k}{2}}\|\mathbf{f}\|_{L_s(\mathbf{R}^3_+)} \quad (k = 0, 1).$$

PROOF. Noting the assumption (4.1.4), we obtain the estimates of (1) and (2) by Theorem 3.1.3 directly.

We show the inequality of (3). By Theorem 3.1.3 (2), there exist a positive constants δ_0 and $C(q)$ such that for any $t>0$

$$\begin{aligned}(4.3.25)\qquad &\|(S_\infty(t)\mathbf{F}, \partial_t\mathcal{E}(T_\infty(t)\mathbf{F}), \nabla\mathcal{E}(T_\infty(t)\mathbf{F}))\|_{W^1_s(\mathbf{R}^3_+)}\\ &\quad\le C(q)\,t^{-1/2}e^{-\delta_0 t}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times W^{2-1/s}_s(\mathbf{R}^2)},\\ &\|(\nabla^2 S_\infty(t)\mathbf{F}, \nabla^3\mathcal{E}(T_\infty(t)\mathbf{F}), \nabla^2\partial_t\mathcal{E}(T_\infty(t)\mathbf{F}))\|_{L_s(\mathbf{R}^3_+)}\\ &\quad\le C(q)\,t^{-1}e^{-\delta_0 t}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times W^{2-1/s}_s(\mathbf{R}^2)}.\end{aligned}$$

In addition, we see, by the trace theorem, that

$$\begin{aligned}(4.3.26)\qquad \|T_\infty(t)\mathbf{F}\|_{W^1_s(\mathbf{R}^2)} &= \|T_\infty(t)\mathbf{F}\|_{L_s(\mathbf{R}^2)} + \|\nabla'\mathcal{E}(T_\infty(t)\mathbf{F})\|_{Ls(\mathbf{R}^3_0)}\\ &\le C(q)\left(\|T_\infty(t)\mathbf{F}\|_{L_s(\mathbf{R}^2)} + \|\nabla\mathcal{E}(T_\infty(t)\mathbf{F})\|_{W^1_s(\mathbf{R}^3_+)}\right)\\ &\le C(q)\,t^{-\frac12}e^{-\delta_0 t}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times W^{2-1/s}_{q/2}(\mathbf{R}^2)}.\end{aligned}$$

Combining (4.3.25) and (4.3.26) with Sobolev's embedding theorem:

$$\begin{aligned}(4.3.27)\qquad &\|u\|_{L_r(\mathbf{R}^3_+)} \le C(q,r)\|u\|^{1-3\left(\frac1s-\frac1r\right)}_{L_s(\mathbf{R}^3_+)}\|\nabla u\|^{3\left(\frac1s-\frac1r\right)}_{L_s(\mathbf{R}^3_+)},\\ &\|v\|_{L_r(\mathbf{R}^2)} \le C(q,r)\|v\|^{1-2\left(\frac1s-\frac1r\right)}_{L_s(\mathbf{R}^2)}\|\nabla v\|^{2\left(\frac1s-\frac1r\right)}_{L_s(\mathbf{R}^2)},\end{aligned}$$

we obtain, for any $t>0$,

$$\begin{aligned}&\|(S_\infty(t)\mathbf{F}, \partial_t\mathcal{E}(T_\infty(t)\mathbf{F}, \nabla\mathcal{E}(T_\infty(t)\mathbf{F}))\|_{L_r(\mathbf{R}^3_+)}\\ &\quad\le Ct^{-\frac12}e^{-\delta_0 t}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times W^{2-1/s}_s(\mathbf{R}^2)}\\ &\quad\le Ct^{-\frac32\left(\frac2q-\frac1r\right)-\frac12}e^{-(\delta_0/2)t}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times W^{2-1/s}_s(\mathbf{R}^2)},\\ &\|(\nabla S_\infty(t)\mathbf{F}, \nabla\partial_t\mathcal{E}(T_\infty(t)\mathbf{F}, \nabla^2\mathcal{E}(T_\infty(t)\mathbf{F}))\|_{L_r(\mathbf{R}^3_+)}\\ &\quad\le Ct^{-\frac32\left(\frac1s-\frac1r\right)-\frac12}e^{-\delta_0 t}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times W^{2-1/s}_s(\mathbf{R}^2)},\\ &\|T_\infty(t)\mathbf{F}\|_{L_r(\mathbf{R}^2)} \le Ct^{-\frac32\left(\frac1s-\frac1r\right)-\frac12}e^{-(\delta_0/2)t}\|\mathbf{F}\|_{L_s(\mathbf{R}^3_+)^3\times W^{2-1/s}_s(\mathbf{R}^2)}\end{aligned}$$

with some positive constant $C=C(q,r)$. The inequality of (4) is also proved by (4.3.27) and Theorem 3.1.3 (3), which completes the proof of the proposition. □

We suppose that $r\in\{q,2\}$ below, and also set $\varepsilon_0=\min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$.

Step 1 We here consider the estimates of $\mathbf{w}$. It follows from [**SS12**, Theorem 1.2] that for some $\gamma_0>0$

$$\begin{aligned}(4.3.28)\qquad &\|e^{-\gamma_0 t}(\partial_t\mathbf{w}, \mathbf{w}, \nabla\mathbf{w}, \nabla^2\mathbf{w})\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_+))}\\ &\quad\le C\|e^{-\gamma_0 t}(\mathbf{f}_1,\mathbf{f}_2)\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_+))} \le C\|(\mathbf{f}_1,\mathbf{f}_2)\|_{\mathbb{F}_1\times\mathbb{F}_2}\end{aligned}$$

with a positive constant $C=C(p,q,\gamma_0)$, and also that

$$(4.3.29)\qquad \|(\partial_t\mathbf{w}, \nabla^2\mathbf{w})\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_+))} \le C\|\mathbf{f}\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_+))} \le C\|(\mathbf{f}_1,\mathbf{f}_2)\|_{\mathbb{F}_1\times\mathbb{F}_2}$$

with a positive constant $C=C(p,q)$. By using (4.3.22), we set

$$\mathbf{w}(t) = \left(\int_0^{t/2} + \int_{t/2}^t\right) R(t-s)\mathbf{f}(s)\,ds =: \mathbf{I}_1(t) + \mathbf{I}_2(t) \quad (t>0).$$

Then, for $l=0,1$ and $t>0$, we see, by Proposition 4.3.9 (4), that

$$
\begin{aligned}
\|\nabla^l \mathbf{I}_1(t)\|_{L_r(\mathbf{R}^3_+)} &\le C\int_0^{t/2}(t-s)^{-\frac{3}{2}\left(\frac{2}{q}-\frac{1}{r}\right)-\frac{l}{2}}\|\mathbf{f}(s)\|_{L_{q/2}(\mathbf{R}^3_+)}\,ds\\
&\le Ct^{-\frac{3}{2}\left(\frac{2}{q}-\frac{1}{r}\right)-\frac{l}{2}}\left\{\int_0^{t/2}(s+2)^{-\varepsilon_1}\,ds\,\|\mathbf{f}_1\|_{L^{\varepsilon_1}_\infty(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))}\right.\\
&\quad\left.+\Big(\int_0^{t/2}(s+2)^{-p'\varepsilon_2}\,ds\Big)^{1/p'}\|\mathbf{f}_2\|_{L^{\varepsilon_2}_p(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))}\right\}\\
&\le Ct^{-\frac{3}{2}\left(\frac{2}{q}-\frac{1}{r}\right)-\frac{l}{2}}\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}\right),\\
\|\nabla^l \mathbf{I}_2(t)\|_{L_r(\mathbf{R}^3_+)} &\le C(t+2)^{-\varepsilon_0}\int_{t/2}^{t}(t-s)^{-\frac{3}{2}\left(\frac{2}{q}-\frac{1}{r}\right)-\frac{l}{2}}\\
&\quad\times\left((s+2)^{\varepsilon_1}\|\mathbf{f}_1(s)\|_{L_{q/2}(\mathbf{R}^3_+)}+(s+2)^{\varepsilon_2}\|\mathbf{f}_2(s)\|_{L_{q/2}(\mathbf{R}^3_+)}\right)ds\\
&\le C(t+2)^{-\varepsilon_0}\left\{\int_{t/2}^{t}(t-s)^{-\frac{3}{2}\left(\frac{2}{q}-\frac{1}{r}\right)-\frac{l}{2}}\,ds\,\|\mathbf{f}_1\|_{L^{\varepsilon_1}_\infty(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))}\right.\\
&\quad\left.+\Big(\int_{t/2}^{t}(t-s)^{-p'\left\{\frac{3}{2}\left(\frac{2}{q}-\frac{1}{r}\right)+\frac{l}{2}\right\}}\,ds\Big)^{1/p'}\|\mathbf{f}_2\|_{L^{\varepsilon_2}_p(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))}\right\}\\
&\le C(t+2)^{-\varepsilon_0+1-\frac{3}{2}\left(\frac{2}{q}-\frac{1}{r}\right)-\frac{l}{2}}\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}\right)\\
&\le C(t+2)^{-\frac{3}{2}\left(\frac{2}{q}-\frac{1}{r}\right)-\frac{l}{2}}\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}\right)
\end{aligned}
$$

with some positive constant $C=C(p,q)$, where by (4.1.4)

$$
p'\left\{\frac{3}{2}\left(\frac{2}{q}-\frac{1}{r}\right)+\frac{l}{2}\right\}<p'\left(\frac{3}{2q}+\frac{1}{2}\right)<1 \tag{4.3.30}
$$

for $1/p+1/p'=1$. We thus obtain for $t\ge 1$ and $l=0,1$

$$
\|\nabla^l \mathbf{w}(t)\|_{L_r(\mathbf{R}^3_+)}\le C(t+2)^{-n\left(\frac{q}{2},r\right)-\frac{l}{2}}\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}\right). \tag{4.3.31}
$$

On the other hand, by Lemma 1.3.5 (2) and (4.3.28),

$$
\sup_{0<t<2}\|\mathbf{w}(t)\|_{W^1_r(\mathbf{R}^3_+)}\le C\|\mathbf{w}\|_{W^{2,1}_{r,p}(\mathbf{R}^3_+\times(0,2))}\le C\|(\mathbf{f}_1,\mathbf{f}_2)\|_{\mathbb{F}_1\times\mathbb{F}_2},
$$

which, combined with (4.3.31), furnishes that

$$
\begin{aligned}
\sum_{r\in\{q,2\}}\mathbb{W}_{r,\infty}(\mathbf{w},0,0,0)&\le\sum_{r\in\{q,2\}}\sum_{l=0,1}\|\nabla^l\mathbf{w}\|_{L^{n(q/2,r)+l/2}_\infty(\mathbf{R}_+,L_r(\mathbf{R}^3_+))}\\
&\le C\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}\right).
\end{aligned} \tag{4.3.32}
$$

Finally, by (4.3.32) and the assumption (4.1.4),

$$
\begin{aligned}
&\|\nabla^l\mathbf{w}\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_+))}\le\|(t+2)^{-n(q/2,r)-l/2}\|_{L_p(\mathbf{R}_+)}\|\nabla^l\mathbf{w}\|_{L^{n(q/2,r)+l/2}_\infty(\mathbf{R}_+,L_r(\mathbf{R}^3_+))}\\
&\le C\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}\right)
\end{aligned}
$$

for $l = 0, 1$, we obtain

$$\sum_{r\in\{q,2\}} \mathbb{M}_{r,p}(\mathbf{w}, 0, 0, 0, 0) \le C\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}\right), \tag{4.3.33}$$

where we have used the inequality (4.3.29).

Step 2 We here consider the estimates of $\mathbf{v}_\infty$, h_∞, and H_∞. By (B.7), we have the maximal regularity property:

$$\begin{aligned}
&\|(\partial_t\mathbf{v}_\infty, \mathbf{v}_\infty, \nabla\mathbf{v}_\infty, \nabla^2\mathbf{v}_\infty)\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_+))} \\
&\quad + \|(\partial_t H_\infty)\|_{L_p(\mathbf{R}_+,W^2_r(\mathbf{R}^3_+))} + \|H_\infty\|_{L_p(\mathbf{R}_+,W^3_r(\mathbf{R}^3_+))} \\
&\le C\left(\|\mathbf{f}\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_+))} + \|g_h\|_{L_p(\mathbf{R}_+,W^{2-1/r}_r(\mathbf{R}^3_0))}\right)
\end{aligned} \tag{4.3.34}$$

with $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$, so that we show the other estimates below. By (4.3.22), we set

$$\begin{aligned}
\mathbf{v}_\infty(t) &= \left(\int_0^{t/2} + \int_{t/2}^t\right) S_\infty(t-s)\mathbf{F}(s)\,ds =: \mathbf{I}_1(t) + \mathbf{I}_2(t) \quad (t>0), \\
h_\infty(t) &= \left(\int_0^{t/2} + \int_{t/2}^t\right) T_\infty(t-s)\mathbf{F}(s)\,ds =: J_1(t) + J_2(t) \quad (t>0).
\end{aligned}$$

By Proposition 4.3.9 (3), (4.3.30), and the trace theorem, we have, for $t > 0$,

$$\begin{aligned}
&\|\mathbf{I}_1(t)\|_{W^1_r(\mathbf{R}^3_+)} + \|\nabla\mathcal{E}(J_1(t))\|_{W^1_r(\mathbf{R}^3_+)} + \|J_1(t)\|_{L_r(\mathbf{R}^2)} \\
&\le Ce^{-(\sigma_0/2)t}\left\{\int_0^{t/2}(t-s)^{-n(\frac{q}{2},r)-\frac12}(s+2)^{-\varepsilon_1}\,ds\,\|\mathbf{f}_1\|_{L^{\varepsilon_1}_\infty(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))}\right. \\
&+\left(\int_0^{t/2}(t-s)^{-p'\left(n(\frac{q}{2},r)+\frac12\right)}(s+2)^{-p'\varepsilon_2}\,ds\right)^{1/p'}\|\mathbf{f}_2\|_{L^{\varepsilon_2}_p(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))} \\
&\left.+\left(\int_0^{t/2}(t-s)^{-p'\left(n(\frac{q}{2},r)+\frac12\right)}(s+2)^{-p'\varepsilon_3}\right)^{1/p'}\|g_h\|_{L^{\varepsilon_3}_p(\mathbf{R}^3_+,W^2_{q/2}(\mathbf{R}^3_+))}\right\} \\
&\le Ce^{-(\sigma_0/4)t}\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\mathbb{G}_h(\delta_0,\varepsilon_3)}\right), \\
&\|\mathbf{I}_2(t)\|_{W^1_r(\mathbf{R}^3_+)} + \|\nabla\mathcal{E}(J_2(t))\|_{W^1_r(\mathbf{R}^3_+)} + \|J_2(t)\|_{L_r(\mathbf{R}^2)} \\
&\le C(t+2)^{-\varepsilon_0}\left\{\int_{t/2}^t(t-s)^{-n(\frac{q}{2},r)-\frac12}e^{-\sigma_0(t-s)}\,ds\,\|\mathbf{f}_1\|_{L^{\varepsilon_1}_\infty(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))}\right. \\
&+\left(\int_{t/2}^t(t-s)^{-p'\left(n(\frac{q}{2},r)+\frac12\right)}e^{-p'\sigma_0(t-s)}\,ds\right)^{1/p'}\|\mathbf{f}_2\|_{L^{\varepsilon_2}_p(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))} \\
&\left.+\left(\int_{t/2}^t(t-s)^{-p'\left(n(\frac{q}{2},r)+\frac12\right)}e^{-p'\sigma_0(t-s)}\,ds\right)^{1/p'}\|g_h\|_{L^{\varepsilon_3}_p(\mathbf{R}_+,W^2_{q/2}(\mathbf{R}^3_+))}\right\} \\
&\le C(t+2)^{-\varepsilon_0}\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right)
\end{aligned}$$

for some positive constant $C = C(p,q)$, which furnishes that

$$(4.3.35)\quad \|(\mathbf{v}_\infty, \nabla H_\infty)\|_{L^{\varepsilon_0}_\infty(\mathbf{R}_+, W^1_r(\mathbf{R}^3_+))} + \|h_\infty\|_{L^{\varepsilon_0}_\infty(\mathbf{R}_+, L_r(\mathbf{R}^2))} \leq C\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right).$$

We next consider the estimates of $\nabla\partial_t H_\infty$. As mentioned above, we easily see that for every $t>0$

$$(4.3.36)\quad \left\|\nabla\int_0^t \partial_t\mathcal{E}(T_\infty(t-s)\mathbf{F}(s))\,ds\right\|_{L_r(\mathbf{R}^3_+)} \leq C(t+2)^{-\varepsilon_0}\Big(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\Big).$$

On the other hand, by the relation: $A^2 = -\sum_{j=1}^2 (i\xi_j)^2$, we have

$$\begin{aligned}
&\mathcal{F}^{-1}_{\xi'}[\varphi_a(\xi')e^{-Ax_3}\widehat{g}_h(\xi',0,t)](x') \\
&= -\sum_{j=1}^2\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\varphi_a(\xi')\frac{i\xi_j}{A}e^{-Ax_3}\widehat{D_j g_h}(\xi',y_3,t)\right](x')\,dy_3 \\
&\quad - \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\varphi_a(\xi')e^{-Ax_3}\widehat{D_3 g_h}(\xi',y_3,t)\right](x')\,dy_3
\end{aligned}$$

for $a\in\{0,\infty\}$, so that by Corollary B.3 (1)

$$(4.3.37)\quad \|\nabla\mathcal{F}^{-1}_{\xi'}[\varphi_a(\xi')e^{-Ax_3}\widehat{g}_h(\xi',0,t)](x')\|_{L_r(\mathbf{R}^N_+)} \leq C\|\nabla g_h(t)\|_{L_r(\mathbf{R}^N_+)}\quad (t>0)$$

with some positive constant C. By (4.3.36) and (4.3.37), we have

$$(4.3.38)\quad \|\nabla\partial_t H_\infty\|_{L^{m(q/2,r)+1/2}_\infty(\mathbf{R}_+, L_r(\mathbf{R}^3_+))} \leq C\Big(\|\mathbf{f}_1\|_{\mathbb{F}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\widetilde{\mathbf{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\Big)$$

with some positive constant C, since $0 < m(q/2,r)+1/2 < 1$ for $r\in\{q,2\}$. Summing up (4.3.34), (4.3.35), and (4.3.38), we have obtained

$$(4.3.39)\quad \sum_{r\in\{q,2\}}\Big(\mathbb{W}_{r,\infty}(\mathbf{v}_\infty, h_\infty, 0, H_\infty) + \mathbb{M}_{r,p}(\mathbf{v}_\infty, 0, h_\infty, 0, H_\infty)\Big) \leq C\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\right).$$

Step 3 We consider the estimates of $\mathbf{v}_0$, h_0, and H_0. To show the estimates, we here introduce the following lemma.

LEMMA 4.3.10. *Let exponents p,q satisfy* (4.1.4)*, and $2\leq r\leq q$. Let $\varepsilon_1>1$ and $\varepsilon_2,\varepsilon_3\geq 1$. Suppose that*

$$\mathbf{f}_1\in L^{\varepsilon_1}_\infty(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_+)^3),\quad \mathbf{f}_2\in L^{\varepsilon_2}_p(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_+)^3),$$
$$g_h\in L^{\varepsilon_3}_p(\mathbf{R}_+, W^2_{q/2}(\mathbf{R}^3_+)).$$

In addition, let $X = L_{q/2}(\mathbf{R}^3_+)^3\times W^2_{q/2}(\mathbf{R}^3_+)$ and

$$\mathcal{S}(t)\in\mathcal{L}(X, L_r(\mathbf{R}^3_+)^3),\quad \mathcal{T}(t)\in\mathcal{L}(X, L_r(\mathbf{R}^2))\quad (t>0)$$

be operators satisfying

$$\|\mathcal{S}(t)\mathbf{G}\|_{L_r(\mathbf{R}^3_+)} \leq C(t+2)^{-a}\|\mathbf{G}\|_X, \quad \|\mathcal{T}(t)\mathbf{G}\|_{L_r(\mathbf{R}^2)} \leq C(t+2)^{-b}\|\mathbf{G}\|_X$$

for $t \geq 1$ *and some* $0 < a, b < 1$ *with a positive constant* C *independent of time* t *and* $\mathbf{G}$*, while*

$$\|\mathcal{S}(t)\mathbf{G}\|_{L_r(\mathbf{R}^3_+)} \leq Ct^{-\alpha}\|\mathbf{G}\|_X, \quad \|\mathcal{T}(t)\mathbf{G}\|_{L_r(\mathbf{R}^2)} \leq Ct^{-\beta}\|\mathbf{G}\|_X$$

for $0 < t < 1$ *and some* $\alpha, \beta > 0$*, which satisfy* $p'\alpha < 1$ *and* $p'\beta < 1$ *with* $p' = p/(p-1)$*. Then, setting*

$$\mathcal{G} = \|\mathbf{f}_1\|_{L^{\varepsilon_1}_\infty(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_+))} + \|\mathbf{f}_2\|_{L^{\varepsilon_2}_p(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_+))} + \|g_h\|_{L^{\varepsilon_3}_p(\mathbf{R}_+, W^2_{q/2}(\mathbf{R}^3_+))},$$

we see that the following assertions hold.

(1) *Let* $\mathbf{F} = (\mathbf{f}, g_h)$ *with* $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$*. Then there exists a positive constant* $C(p, r)$ *such that for any* $t \geq 2$

$$\Big\|\int_0^t \mathcal{S}(t-s)\mathbf{F}(s)\,ds\Big\|_{L_r(\mathbf{R}^3_+)} \leq C(p,r)(t+2)^{-a}\mathcal{G},$$
$$\Big\|\int_0^t \mathcal{T}(t-s)\mathbf{F}(s)\,ds\Big\|_{L_r(\mathbf{R}^2)} \leq C(p,r)(t+2)^{-b}\mathcal{G}.$$

(2) *Let* $\mathbf{F} = (\mathbf{f}, g_h)$ *with* $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$*. If* $pa > 1$ *and* $pb > 1$*, then there exists a positive constant* $C(p, r)$ *such that*

$$\Big\|\int_0^t \mathcal{S}(t-s)\mathbf{F}(s)\,ds\Big\|_{L_p((2,\infty), L_r(\mathbf{R}^3_+))} \leq C(p,r)\mathcal{G},$$
$$\Big\|\int_0^t \mathcal{T}(t-s)\mathbf{F}(s)\,ds\Big\|_{L_p((2,\infty), L_r(\mathbf{R}^2))} \leq C(p,r)\mathcal{G}.$$

Proof. We here prove the case of $\mathcal{S}(t)$ only. Let $t \geq 2$, and set

$$\int_0^t \mathcal{S}(t-s)\mathbf{F}(s)\,ds = \left(\int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^t\right)\mathcal{S}(t-s)\mathbf{F}(s)\,ds$$
$$=: \mathbf{I}_1(t) + \mathbf{I}_2(t) + \mathbf{I}_3(t).$$

Setting $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, we have $\varepsilon_0 \geq 1$, and then

$$\begin{aligned}\|\mathbf{I}_1(t)\|_{L_r(\mathbf{R}^3_+)} &\leq C(t+2)^{-a}\Big\{\int_0^{t/2}(s+2)^{-\varepsilon_1}\,ds\,\|\mathbf{f}_1\|_{L^{\varepsilon_1}_\infty(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_+))}\\
&+ \Big(\int_0^{t/2}(s+2)^{-p'\varepsilon_2}\,ds\Big)^{1/p'}\|\mathbf{f}_2\|_{L^{\varepsilon_2}_p(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_+))}\\
&+ \Big(\int_0^{t/2}(s+2)^{-p'\varepsilon_3}\,ds\Big)^{1/p'}\|g_h\|_{L^{\varepsilon_3}_p(\mathbf{R}_+, W^2_{q/2}(\mathbf{R}^3_+))}\Big\}\\
&\leq C(t+2)^{-a}\mathcal{G},\end{aligned}$$

$$\begin{aligned}
\|\mathbf{I}_2(t)\|_{L_r(\mathbf{R}^3_+)} &\le C(t+2)^{-\varepsilon_0}\bigg\{\int_{t/2}^{t-1}(t+2-s)^{-a}\,ds\,\|\mathbf{f}_1\|_{L^{\varepsilon_1}_\infty(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))}\\
&+\Big(\int_{t/2}^{t-1}(t+2-s)^{-p'a}\,ds\Big)^{1/p'}\|\mathbf{f}_2\|_{L^{\varepsilon_2}_p(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))}\\
&+\Big(\int_{t/2}^{t-1}(t+2-s)^{-p'a}\,ds\Big)^{1/p'}\|g_h\|_{L^{\varepsilon_3}_p(\mathbf{R}_+,W^2_{q/2}(\mathbf{R}^3_+))}\bigg\}\\
&\le C(t+2)^{-\varepsilon_0+1-a}\mathcal{G}\le C(t+2)^{-a}\mathcal{G},\\
\|\mathbf{I}_3(t)\|_{L_r(\mathbf{R}^3_+)} &\le C(t+2)^{-\varepsilon_0}\bigg\{\int_{t-1}^{t}(t-s)^{-\alpha}\,ds\,\|\mathbf{f}_1\|_{L^{\varepsilon_1}_\infty(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))}\\
&+\Big(\int_{t-1}^{t}(t-s)^{-p'\alpha}\,ds\Big)^{1/p'}\|\mathbf{f}_2\|_{L^{\varepsilon_2}_p(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))}\\
&+\Big(\int_{t-1}^{t}(t-s)^{-p'\alpha}\,ds\Big)^{1/p'}\|g_h\|_{L^{\varepsilon_3}_p(\mathbf{R}_+,W^2_{q/2}(\mathbf{R}^3_+))}\bigg\}\\
&\le C(t+2)^{-\varepsilon_0}\mathcal{G}.
\end{aligned}$$

We thus obtain the required estimates of $\mathcal{S}(t)$ in (1), since $a<1\le\varepsilon_0$. Then, taking L_p-norm with respect to time t for the inequality obtained above implies that the inequality of $\mathcal{S}(t)$ in (2) holds. □

By the formulas (4.3.22), (4.3.23), Proposition 4.3.9 (1), and Lemma 4.3.10

$$\begin{aligned}
\text{(4.3.40)}\qquad &\sup_{t\ge2}\Big((t+2)^{m(\frac{q}{2},r)}\|\mathbf{v}_0(t)\|_{L_r(\mathbf{R}^3_+)}+(t+2)^{n(\frac{q}{2},r)+\frac18}\|\nabla\mathbf{v}_0(t)\|_{L_r(\mathbf{R}^3_+)}\\
&\quad+(t+2)^{m(\frac{q}{2},r)+\frac14}\|\nabla H_0(t)\|_{W^1_r(\mathbf{R}^3_+)}+(t+2)^{\left(\frac{2}{q}-\frac{1}{r}\right)}\|h_0(t)\|_{L_r(\mathbf{R}^2)}\Big)\\
&\le C\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}+\|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right)
\end{aligned}$$

with some positive constant C, and besides,

$$\begin{aligned}
\text{(4.3.41)}\qquad &\|(\partial_t\mathbf{v}_0,\mathbf{v}_0,\nabla\mathbf{v}_0,\nabla^2\mathbf{v}_0)\|_{L_p((2,\infty),L_r(\mathbf{R}^3_+))}+\|\nabla H_0\|_{L_p((2,\infty),W^2_r(\mathbf{R}^3_+))}\\
&\quad+\|h_0\|_{L_p((2,\infty),L_r(\mathbf{R}^2))}\\
&\le C\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}+\|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right).
\end{aligned}$$

On the other hand, noting $\mathbf{v}_0=\mathbf{u}-(\mathbf{v}_\infty+\mathbf{w})$ and $h_0=h-h_\infty$, we have, by Remark 4.3.6 (2), (4.3.15), (4.3.32), and (4.3.39),

$$\begin{aligned}
\text{(4.3.42)}\qquad &\sup_{0<t<3}\Big(\|\mathbf{v}_0(t)\|_{W^1_r(\mathbf{R}^3_+)}+\|\nabla H_0(t)\|_{W^1_r}+\|h_0(t)\|_{L_r(\mathbf{R}^2)}\Big)\\
&\quad\le C\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}+\|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\right),\\
&\|(\partial_t\mathbf{v}_0,\mathbf{v}_0,\nabla\mathbf{v}_0,\nabla^2\mathbf{v}_0)\|_{L_p((0,3),L_r(\mathbf{R}^3_+))}+\|\nabla H_0\|_{L_p((0,3),W^2_r(\mathbf{R}^3_+))}\\
&\qquad+\|h_0\|_{L_p((0,3),L_r(\mathbf{R}^2))}\\
&\quad\le C\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}+\|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\right)
\end{aligned}$$

with some positive constant C.

Here we consider $\nabla\partial_t H_0$. By Proposition 4.3.9 (1) and Lemma 4.3.10, we have, for any $t \geq 2$,

$$\begin{aligned}
&\left\|\nabla\int_0^t \partial_t\mathcal{E}(T_0(t-s)\mathbf{F}(s))\,ds\right\|_{L_r(\mathbf{R}^3_+)} \\
&\quad\leq C(t+2)^{-m(\frac{q}{2},r)-\frac12}\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_1)}+\|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right), \\
&\left\|\nabla\int_0^t \partial_t\mathcal{E}(T_0(t-s)\mathbf{F}(s))\,ds\right\|_{L_p((2,\infty),W^1_r(\mathbf{R}^3_+))} \\
&\quad\leq C\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_1)}+\|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right),
\end{aligned}\tag{4.3.43}$$

which, combined with (4.3.37), furnishes that for $t \geq 2$

$$\begin{aligned}
&\|\nabla\partial_t H_0(t)\|_{L_r(\mathbf{R}^3_+)} \\
&\quad\leq C(t+2)^{-m(\frac{q}{2},r)-\frac12}\Big(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_1)}+\|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\Big), \\
&\|\nabla\partial_t H_0\|_{L_p((2,\infty),W^1_r(\mathbf{R}^3_+))} \\
&\quad\leq C\Big(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_1)}+\|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\Big).
\end{aligned}\tag{4.3.44}$$

On the other hand, by $H_0 = H - H_\infty$, Remark 4.3.6 (2), (4.3.15), and (4.3.39),

$$\begin{aligned}
&\sup_{0<t<3}\|\nabla\partial_t H_0(t)\|_{L_r(\mathbf{R}^3_+)} \\
&\quad\leq C\Big(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_1)}+\|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\Big), \\
&\|\nabla\partial_t H_0\|_{L_p((0,3),W^1_r(\mathbf{R}^3_+))} \\
&\quad\leq C\Big(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_1)}+\|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\Big)
\end{aligned}\tag{4.3.45}$$

with some positive constant C.

Summing up (4.3.40)-(4.3.42), (4.3.44), and (4.3.45), we have obtained

$$\begin{aligned}
&\sum_{r\in\{q,2\}}(\mathbb{W}_{r,\infty}(\mathbf{v}_0,h_0,0,H_0)+\mathbb{M}_{r,p}(\mathbf{v}_0,0,h_0,0,H_0)) \\
&\quad\leq C\Big(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_1)}+\|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\Big).
\end{aligned}\tag{4.3.46}$$

Step 4 We here show the estimate of $\mathbb{W}_{q,p}(\mathbf{u},H;\delta_1,\delta_2)$. By setting

$$\mathbf{u}^\delta = (t+2)^\delta\mathbf{u},\ \ \theta^\delta = (t+2)^\delta\theta,\ \ z^\delta = (t+2)^\delta h,\ \text{ and } Z^\delta = (t+2)^\delta H$$

for $\delta > 0$ and $t > 0$ in the equations (4.3.2) and (4.3.3), we see that

$$\begin{cases}
\partial_t\mathbf{u}^\delta - \Delta\mathbf{u}^\delta + \nabla\theta^\delta = \mathbf{f}^\delta & \text{in } \mathbf{R}^3_+,\ t>0, \\
\operatorname{div}\mathbf{u}^\delta = 0 & \text{in } \mathbf{R}^3_+,\ t>0, \\
\mathbf{S}(\mathbf{u}^\delta,\theta^\delta)\mathbf{n} + (c_g - c_\sigma\Delta')z^\delta\mathbf{n} = 0 & \text{on } \mathbf{R}^3_0,\ t>0, \\
\partial_t z^\delta - \mathbf{u}^\delta\cdot\mathbf{n} = g^\delta_h & \text{on } \mathbf{R}^3_0,\ t>0, \\
\mathbf{u}^\delta|_{t=0} = 0 & \text{in } \mathbf{R}^3_+, \\
z^\delta|_{t=0} = 0 & \text{on } \mathbf{R}^2,
\end{cases}\tag{4.3.47}$$

where we have set

$$\mathbf{f}^\delta = (t+2)^\delta \mathbf{f}_1 + (t+2)^\delta \mathbf{f}_2 - \delta(t+2)^{-1+\delta}\mathbf{u},$$
$$g_h^\delta = (t+2)^\delta g_h - \delta(t+2)^{-1+\delta} h,$$

and furthermore,

$$\begin{cases} \Delta Z^\delta = 0 & \text{on } \mathbf{R}^3_+,\ t \geq 0, \\ Z^\delta = z^\delta & \text{on } \mathbf{R}^3_0,\ t \geq 0. \end{cases}$$

Let $\mathbf{F}^\delta = (\mathbf{f}^\delta, g_h^\delta)$ below. In the same manner as in (4.3.22) and (4.3.23), we decompose $\mathbf{u}^\delta$ and Z^δ as follows:

$$\mathbf{u}^\delta = \mathbf{v}_0^\delta + \mathbf{v}_\infty^\delta + \mathbf{w}^\delta, \quad Z^\delta = Z_0^\delta + Z_\infty^\delta$$

with

$$\mathbf{v}_a^\delta(t) = \int_0^t S_a(t-s)\mathbf{F}^\delta(s)\,ds, \quad \mathbf{w}^\delta(t) = \int_0^t R(t-s)\mathbf{F}^\delta(s)\,ds,$$
$$Z_a^\delta = \int_0^t \mathcal{E}(T_a(t-s)\mathbf{F}^\delta(s))\,ds \quad (a \in \{0,\infty\}),$$

and also

$$\partial_t \mathbf{v}_0^\delta(t) = \int_0^t \partial_t S_0(t-s)\mathbf{F}^\delta(s)\,ds,$$
$$\partial_t Z_a^\delta(t) = \int_0^t \partial_t \mathcal{E}(T_a(t-s)\mathbf{F}^\delta(s))\,ds + \mathcal{F}^{-1}_{\xi'}\left[\varphi_a(\xi')e^{-Ax_3}\widehat{g}_h^\delta(\xi',0,t)\right](x').$$

We first show the required estimates of $\mathbf{v}_\infty^\delta$, $\mathbf{w}^\delta$, and Z_∞^δ. In Step 1-Step 3, by combining (4.3.32), (4.3.33), (4.3.39), and (4.3.46), we have obtained

$$\sum_{r\in\{q,2\}} \Big(\mathbb{W}_{r,\infty}(\mathbf{u},h,0,H) + \mathbb{M}_{r,p}(\mathbf{u},0,h,0,H)\Big) \leq C(p,q)\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\right) \tag{4.3.48}$$

with some positive constant $C(p,q)$. Especially, (4.3.48) yields that

$$\|\mathbf{f}^{\delta_0}\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_+))} + \|g_h^{\delta_0}\|_{L_p(\mathbf{R}_+,W_q^{2-1/q}(\mathbf{R}^3_0))} \leq C(p,q)\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\mathbb{A}_2}\right)$$

with some positive constant $C(p,q)$, since it follows from the assumption (4.1.4), $0 < \delta_0 \leq 1$, and the trace theorem that

$$\begin{aligned} &\|(t+2)^{-1+\delta_0}\mathbf{u}\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_+))} \\ &\leq \|(t+2)^{-1+\delta_0-m(q/2,q)}\|_{L_p(\mathbf{R}_+)}\|\mathbf{u}\|_{L_\infty^{m(q/2,q)}(\mathbf{R}_+,L_q(\mathbf{R}^3_+))} \\ &\leq C\|\mathbf{u}\|_{L_\infty^{m(q/2,q)}(\mathbf{R}_+,L_q(\mathbf{R}^3_+))}, \end{aligned} \tag{4.3.49}$$

$$
\begin{aligned}
&\|(t+2)^{-1+\delta_0} h\|_{L_p(\mathbf{R}_+, W_q^{2-1/q}(\mathbf{R}^2))} \\
&\quad \le C\left(\|(t+2)^{-1+\delta_0} h\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^2))} + \|(t+2)^{-1+\delta_0}\nabla' H\|_{L_p(\mathbf{R}_+, W_q^{1-1/q}(\mathbf{R}^3_0))}\right) \\
&\quad \le C\Big(\|(t+2)^{-1+\delta_0-1/q}\|_{L_p(\mathbf{R}_+)}\|h\|_{L_\infty^{1/q}(\mathbf{R}_+, L_q(\mathbf{R}^2))} \\
&\quad + \|(t+2)^{-1+\delta_0-m(q/2,q)-1/4}\|_{L_p(\mathbf{R}_+)}\|\nabla H\|_{L_\infty^{m(q/2,q)+1/4}(\mathbf{R}_+, W_q^1(\mathbf{R}^3_+))}\Big) \\
&\quad \le C\left(\|h\|_{L_\infty^{1/q}(\mathbf{R}_+, L_q(\mathbf{R}^2))} + \|\nabla H\|_{L_\infty^{m(q/2,q)+1/4}(\mathbf{R}_+, W_q^1(\mathbf{R}^3_+))}\right)
\end{aligned}
$$

with some positive constant $C = C(p,q)$. We thus obtain, by (4.3.29) and (4.3.34) with $\mathbf{f} = \mathbf{f}^\delta$ and $g_h = g_h^\delta$ for $\delta \in \{\delta_1, \delta_2\}$,

$$
\begin{aligned}
&\|(\partial_t \mathbf{v}_\infty^{\delta_1}, \partial_t \mathbf{w}^{\delta_1}, \nabla^2 \mathbf{v}_\infty^{\delta_1}, \nabla^2 \mathbf{w}^{\delta_1})\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_+))} \\
&\quad \le C\left(\|\mathbf{f}^{\delta_0}\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_+))} + \|g_h^{\delta_0}\|_{L_p(\mathbf{R}_+, W_q^{2-1/q}(\mathbf{R}^3_0))}\right) \\
&\quad \le C\left(\|\mathbf{f}_1\|_{\mathbb{F}_1 \cap \widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\mathbb{F}_1 \cap \widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\mathbb{G}_h \cap \widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3) \cap \widehat{\mathbb{A}}_2}\right), \\
&\|(\nabla^2 \partial_t Z_\infty^{\delta_2}, \nabla^3 Z_\infty^{\delta_2})\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_+))} \\
&\quad \le C\left(\|\mathbf{f}^{\delta_0}\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_+))} + \|g_h^{\delta_0}\|_{L_p(\mathbf{R}_+, W_q^{2-1/q}(\mathbf{R}^3_0))}\right) \\
&\quad \le C\left(\|\mathbf{f}_1\|_{\mathbb{F}_1 \cap \widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\mathbb{F}_1 \cap \widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\mathbb{G}_h \cap \widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3) \cap \widehat{\mathbb{A}}_2}\right).
\end{aligned}
\tag{4.3.50}
$$

Secondly, we consider the estimates of $\partial_t \mathbf{v}_0^{\delta_1}$. To this end, we set

$$
\mathbf{F}_1^\delta = (t+2)^\delta(\mathbf{f}_1 + \mathbf{f}_2, g_h), \quad \mathbf{F}_2^\delta = -\delta(t+2)^{-1+\delta}(\mathbf{u}, h) \quad (\delta > 0,\ t > 0).
$$

Let $t \ge 2$, and then we have

$$
\begin{aligned}
\partial_t \mathbf{v}_0^{\delta_1}(t) &= \left(\int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^t\right) \partial_t S_0(t-s)\left(\mathbf{F}_1^{\delta_1}(s) + \mathbf{F}_2^{\delta_1}(s)\right) ds \\
&=: \mathbf{I}_1(t) + \mathbf{I}_2(t) + \mathbf{I}_3(t) + \mathbf{J}_1(t) + \mathbf{J}_2(t) + \mathbf{J}_3(t).
\end{aligned}
$$

By Proposition 4.3.9 (1), we see that

$$
\begin{aligned}
&\|\mathbf{I}_1(t)\|_{L_q(\mathbf{R}^3_+)} \le C(t+2)^{-m(\frac{q}{2},q)-\frac14}\Big\{\int_0^{t/2}(s+2)^{-(\varepsilon_1-\delta_1)}\,ds\|\mathbf{f}_1\|_{L_\infty^{\varepsilon_1}(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_+))} \\
&\quad + \left(\int_0^{t/2}(s+2)^{-(\varepsilon_2-\delta_1)p'}\,ds\right)^{1/p'}\|\mathbf{f}_2\|_{L_p^{\varepsilon_2}(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_+))} \\
&\quad + \left(\int_0^{t/2}(s+2)^{-(\varepsilon_3-\delta_1)p'}\,ds\right)^{1/p'}\|g_h\|_{L_p^{\varepsilon_3}(\mathbf{R}_+, W_{q/2}^2(\mathbf{R}^3_+))}\Big\} \\
&\quad \le C(t+2)^{-m(\frac{q}{2},q)-\frac14}\Big\{(t+2)^{1-(\varepsilon_1-\delta_1)}\|\mathbf{f}_1\|_{L_\infty^{\varepsilon_1}(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_+))} \\
&\quad + (t+2)^{\frac{1}{p'}-(\varepsilon_2-\delta_1)}\|\mathbf{f}_2\|_{L_p^{\varepsilon_2}(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_+))} \\
&\quad + (t+2)^{\frac{1}{p'}-(\varepsilon_3-\delta_1)}\|g_h\|_{L_p^{\varepsilon_3}(\mathbf{R}_+, W_{q/2}^2(\mathbf{R}^3_+))}\Big\} \\
&\quad \le C(t+2)^{-m(\frac{q}{2},q)-\frac14+\delta_1}\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right),
\end{aligned}
$$

$$\|\mathbf{I}_2(t)\|_{L_q(\mathbf{R}^3_+)} \leq C(t+2)^{-\varepsilon_0+\delta_1}\Big\{\int_{t/2}^{t-1}(t+2-s)^{-m(\frac{q}{2},q)-\frac14}\,ds\|\mathbf{f}_1\|_{L^{\varepsilon_1}_\infty(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))}$$
$$+\Big(\int_{t/2}^{t-1}(t+2-s)^{-p'\left(m(\frac{q}{2},q)+\frac14\right)}ds\Big)^{1/p'}\|\mathbf{f}_2\|_{L^{\varepsilon_2}_p(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_+))}$$
$$+\Big(\int_{t/2}^{t-1}(t+2-s)^{-p'\left(m(\frac{q}{2},q)+\frac14\right)}ds\Big)^{1/p'}\|g_h\|_{L^{\varepsilon_3}_p(\mathbf{R}_+,W^2_{q/2}(\mathbf{R}^3_+))}\Big\}$$
$$\leq C(t+2)^{-\varepsilon_0+\delta_1+1-m(\frac{q}{2},q)-\frac14}\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}+\|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right)$$
$$\leq C(t+2)^{-m(\frac{q}{2},q)-\frac14+\delta_1}\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}+\|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right),$$
$$\|\mathbf{I}_3(t)\|_{L_q(\mathbf{R}^3_+)} \leq C(t+2)^{-\varepsilon_0+\delta_1}\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}+\|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right),$$

since $\varepsilon_0 = \min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\} \geq 1$, so that we have, by the assumption (4.3.19), (4.3.51)

$$\sum_{i=1}^3\|\mathbf{I}_i\|_{L_p((2,\infty),L_q(\mathbf{R}^3_+))} \leq C\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}+\|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right).$$

Concerning $\mathbf{J}_i(t)$ $(i=1,2,3)$, by Proposition 4.3.9 (1) and (4.3.48),

$$\|\mathbf{J}_1(t)\|_{L_q(\mathbf{R}^3_+)} \leq C(t+2)^{-m(2,q)-\frac14}\int_0^{t/2}(s+2)^{-1+\delta_1-m(q/2,2)}\,ds$$
$$\times\left(\|\mathbf{u}\|_{L^{m(q/2,2)}_\infty(\mathbf{R}_+,L_2(\mathbf{R}^3_+))}+\|h\|_{L^{m(q/2,2)}_\infty(\mathbf{R}_+,L_2(\mathbf{R}^2))}\right)$$
$$\leq C(t+2)^{-m(2,q)-\frac14+\delta_1-m(\frac{q}{2},2)}\left(\|\mathbf{u}\|_{L^{m(q/2,2)}_\infty(\mathbf{R}_+,L_2(\mathbf{R}^3_+))}+\|h\|_{L^{m(q/2,2)}_\infty(\mathbf{R}_+,L_2(\mathbf{R}^2))}\right)$$
$$= C(t+2)^{-m(\frac{q}{2},q)-\frac14+\delta_1}\left(\|\mathbf{u}\|_{L^{m(q/2,2)}_\infty(\mathbf{R}_+,L_2(\mathbf{R}^3_+))}+\|h\|_{L^{m(q/2,2)}_\infty(\mathbf{R}_+,L_2(\mathbf{R}^2))}\right),$$
$$\|\mathbf{J}_2(t)\|_{L_q(\mathbf{R}^3_+)} \leq C(t+2)^{-1+\delta_1-m(\frac{q}{2},2)}\int_{t/2}^{t-1}(t+2-s)^{-m(2,q)-\frac14}\,ds$$
$$\times\left(\|\mathbf{u}\|_{L^{m(q/2,2)}_\infty(\mathbf{R}_+,L_2(\mathbf{R}^3_+))}+\|h\|_{L^{m(q/2,2)}_\infty(\mathbf{R}_+,L_2(\mathbf{R}^2))}\right)$$
$$\leq C(t+2)^{-m(\frac{q}{2},q)-\frac14+\delta_1}\left(\|\mathbf{u}\|_{L^{m(q/2,2)}_\infty(\mathbf{R}_+,L_2(\mathbf{R}^3_+))}+\|h\|_{L^{m(q/2,2)}_\infty(\mathbf{R}_+,L_2(\mathbf{R}^2))}\right),$$
$$\|\mathbf{J}_3(t)\|_{L_q(\mathbf{R}^3_+)}$$
$$\leq C(t+2)^{-1+\delta_1-m(\frac{q}{2},2)}\left(\|\mathbf{u}\|_{L^{m(q/2,2)}_\infty(\mathbf{R}_+,L_2(\mathbf{R}^3_+))}+\|h\|_{L^{m(q/2,2)}_\infty(\mathbf{R}_+,L_2(\mathbf{R}^2))}\right).$$

Since it holds, by the assumptions (4.1.4) and (4.3.19), that

$$p\left(m\left(\frac{q}{2},q\right)+\frac14-\delta_1\right)>1,\quad p\left(1-\delta_1+m\left(\frac{q}{2},2\right)\right)\geq pm\left(\frac{q}{2},2\right)>1,$$

we have, by (4.3.48),

$$\|\mathbf{J}_i\|_{L_p((2,\infty),L_q(\mathbf{R}^3_+))} \leq C\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}+\|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\right)$$

for $i=1,2,3$, which, combined with (4.3.50) and (4.3.51), furnishes that

$$\|\partial_t((t+2)^{\delta_1}\mathbf{u})\|_{L_p((2,\infty),L_q(\mathbf{R}^3_+))}$$
$$\leq C\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)}+\|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}+\|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\right).$$

Noting that

$$(t+2)^{\delta_1}\partial_t\mathbf{u} = \partial_t((t+2)^{\delta_1}\mathbf{u}) - \delta_1(t+2)^{-1+\delta_1}\mathbf{u},$$

we have, by the last inequality, (4.3.48), and (4.3.49),

$$\begin{aligned}&\|(t+2)^{\delta_1}\partial_t\mathbf{u}\|_{L_p((2,\infty),L_q(\mathbf{R}^3_+))}\\ &\quad\le C\left(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\widehat{\mathbb{A}}_2}\right),\end{aligned}$$

which completes the required weighted estimates of $\partial_t\mathbf{u}$. Analogously, we can show the weighted estimates of $\nabla^2\mathbf{u}$.

We next show the weighted estimate of $\nabla^3 H$. Let $t \ge 2$, and then

$$\begin{aligned}\nabla^3 Z_0^{\delta_2}(t) &= \left(\int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^{t}\right)\mathcal{E}\{T_0(t-s)(\mathbf{F}_1^{\delta_2}(s) + \mathbf{F}_2^{\delta_2}(s))\}\,ds\\ &=: I_1(t) + I_2(t) + I_3(t) + J_1(t) + J_2(t) + J_3(t).\end{aligned}$$

In the same manner as in the case of $\partial_t\mathbf{v}_0^{\delta_1}$, we have

$$\begin{aligned}&\|I_1(t)\|_{L_q(\mathbf{R}^3_+)} \le C(t+2)^{-m(q/2,q)-5/4+\delta_2}\\ &\qquad\times\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right),\\ &\|I_2(t)\|_{L_q(\mathbf{R}^3_+)} + \|I_3(t)\|_{L_q(\mathbf{R}^3_+)} \le C(t+2)^{-(\varepsilon_0-\delta_2)}\\ &\qquad\times\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right),\\ &\|J_1(t)\|_{L_q(\mathbf{R}^3_+)} \le C(t+2)^{-m(q/2,q)-5/4+\delta_2}\\ &\qquad\times\left(\|\mathbf{u}\|_{L_\infty^{m(q/2,2)}(\mathbf{R}_+,L_2(\mathbf{R}^3_+))} + \|h\|_{L_\infty^{m(q/2,2)}(\mathbf{R}_+,L_2(\mathbf{R}^2))}\right),\\ &\|J_2(t)\|_{L_q(\mathbf{R}^3_+)} + \|J_3(t)\|_{L_q(\mathbf{R}^3_+)} \le C(t+2)^{-1+\delta_2-m(2/q,2)}\\ &\qquad\times\left(\|\mathbf{u}\|_{L_\infty^{m(q/2,2)}(\mathbf{R}_+,L_2(\mathbf{R}^3_+))} + \|h\|_{L_\infty^{m(q/2,2)}(\mathbf{R}_+,L_2(\mathbf{R}^2))}\right).\end{aligned}$$

Noting the assumption (4.3.19):

$$p(1-\delta_2+m(q/2,2)) > 1, \quad p(\varepsilon_0-\delta_2) > 1,$$

we thus obtain the required weighted estimate of $\nabla^3 H$ by (4.3.48) and (4.3.50).

Finally, we show the weighted estimate of $\nabla^2\partial_t H_0$. Now, by (4.3.37), we have

$$\begin{aligned}&\left\|\nabla^2\mathcal{F}^{-1}_{\xi'}\left[\varphi_0(\xi')e^{-Ax_3}\widehat{g}_h^{\delta_2}(\xi',0,t)\right]\right\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_+))}\\ &\quad\le C\|(t+2)^{\delta_2}\nabla^2 g_h\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_+))} \le C\|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)},\end{aligned}$$

while

$$\begin{aligned}&\left\|\nabla^2\int_0^t \partial_t\mathcal{E}\{T_0(t-s)(\mathbf{F}_1^{\delta_2}(s) + \mathbf{F}_2^{\delta_2}(s))\}\,ds\right\|_{L_p((2,\infty),L_q(\mathbf{R}^3_+))}\\ &\quad\le C\left(\|\mathbf{f}_1\|_{\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|g_h\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)}\right.\\ &\qquad\left.+\|\mathbf{u}\|_{L_\infty^{m(q/2,2)}(\mathbf{R}_+,L_2(\mathbf{R}^3_+))} + \|h\|_{L_\infty^{m(q/2,2)}(\mathbf{R}_+,L_2(\mathbf{R}^2))}\right)\end{aligned}$$

with some positive constant $C = C(p,q)$ similarly to the case of $\nabla^3 H$. We thus obtain the weighted estimate of $\nabla^2\partial_t H_0$ by the last two inequalities together with (4.3.48) and (4.3.50), which completes the proof of the theorem. □

4.4. Linear theory II

In this section, we consider the full linear system:

$$
\begin{cases}
\partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla\theta = \mathbf{f}_1 + \mathbf{f}_2 & \text{in } \mathbf{R}^3_-,\ t>0,\\
\operatorname{div}\mathbf{u} = f_d = \operatorname{div}\mathbf{f}_d & \text{in } \mathbf{R}^3_-,\ t>0,\\
\mathbf{S}(\mathbf{u},\theta)\mathbf{e}_3 + (c_g - c_\sigma\Delta')h\mathbf{e}_3 = \mathbf{g} & \text{on } \mathbf{R}^3_0,\ t>0,\\
\partial_t h - \mathbf{u}\cdot\mathbf{e}_3 = g_h & \text{on } \mathbf{R}^3_0,\ t>0,\\
\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \mathbf{R}^3_- \quad h|_{t=0} = h_0 & \text{on } \mathbf{R}^2
\end{cases}
\tag{4.4.1}
$$

with the auxiliary problem:

$$
\begin{cases}
\Delta H = 0 & \text{in } \mathbf{R}^3_-,\ t\geq 0,\\
H = h & \text{on } \mathbf{R}^3_0,\ t \geq 0.
\end{cases}
\tag{4.4.2}
$$

Here we set

$$[\mathbf{g}]_{\tan} = \mathbf{g} - (\mathbf{g}\cdot\mathbf{e}_3)\mathbf{e}_3,$$

and then the main theorem concerning the equations (4.4.1) and (4.4.2) is stated as follows:

THEOREM 4.4.1. *Let exponents p,q satisfy* (4.1.4). *Let $\varepsilon_1 > 1$ and $\varepsilon_2, \varepsilon_3 \geq 1$, and also $0 < \delta_1, \delta_2 \leq 1$ satisfy the assumption* (4.3.19). *Then we set $\delta_0 = \max\{\delta_1,\delta_2\}$, and also suppose that the right members of the equations* (4.4.1) *satisfy the following conditions:*

(1) *Let* $\mathbf{f}_1 \in \mathbb{F}_1 \cap \widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)$;
(2) *Let* $\mathbf{f}_2 \in \mathbb{F}_2 \cap \widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)$;
(3) *Let* $g_h \in \mathbb{G}_h \cap \widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3) \cap \mathbb{A}_1 \cap \widehat{\mathbb{A}}_2$;
(4) *Let* $\mathbf{f}_d \in \mathbb{F}_{d1} \cap \widetilde{\mathbb{F}}_{d1}(\delta_0,\varepsilon_2) \cap \widetilde{\mathbb{F}}_{d1}(\delta_0,\varepsilon_3) \cap \mathbb{A}_1$;
(5) *Let* $f_d \in \mathbb{F}_{d2} \cap \widetilde{\mathbb{F}}_{d2}(\delta_0,\varepsilon_2) \cap \widetilde{\mathbb{F}}_{d2}(\delta_0,\varepsilon_3) \cap \mathbb{A}_2 \cap \mathbb{A}_3$;
(6) *Let* $\mathbf{g} \in \mathbb{G} \cap \widetilde{\mathbb{G}}(\delta_0,\varepsilon_2) \cap \widetilde{\mathbb{G}}(\delta_0,\varepsilon_3) \cap \mathbb{A}_3$;
(7) *Let f_d and $\mathbf{g}$ satisfy additionally*

$$(f_d,\mathbf{g}) \in L^{\alpha(q)}_p(\mathbf{R}_+, W^1_q(\mathbf{R}^3_-))^4 \cap L^{\alpha(q/2)}_p(\mathbf{R}_+, W^1_{q/2}(\mathbf{R}^3_-))^4$$

for positive numbers $\alpha(q)$ and $\alpha(q/2)$ satisfying

$$p(1+\alpha(q)-\delta_0) > 1, \quad p(1+\alpha(q/2) - \max\{\varepsilon_2,\varepsilon_3\}) > 1; \tag{4.4.3}$$

(8) *Let $(\mathbf{u}_0, h_0) \in \mathbb{I}_1 \times \mathbb{I}_2$ and satisfy the compatibility conditions:*

$$f_d|_{t=0} = \operatorname{div}\mathbf{u}_0 \quad \text{in } \mathbf{R}^3_-, \quad [\mathbf{g}]_{\tan}|_{t=0} = [\mathbf{D}(\mathbf{u}_0)\mathbf{e}_3]_{\tan} \quad \text{on } \mathbf{R}^3_0.$$

Then there exists a unique solution $(\mathbf{u},\theta,h,H)$ to the equations (4.4.1) *and* (4.4.2), *and also the solution satisfies the following estimate:*

$$
\begin{aligned}
&\sum_{r\in\{q,2\}} \Big(\mathbb{W}_{r,\infty}(\mathbf{u},h,\partial_t h,H) + \mathbb{M}_{r,p}(\mathbf{u},\theta,h,\partial_t h,H)\Big) + \mathbb{W}_{q,p}(\mathbf{u},H;\delta_1,\delta_2)\\
&\quad \leq C\Big(\|(\mathbf{u}_0,h_0)\|_{\mathbb{I}_1\times\mathbb{I}_2} + \|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)}\\
&\quad + \|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\mathbb{A}_1\cap\widehat{\mathbb{A}}_2} + \|\mathbf{f}_d\|_{\mathbb{F}^1_d\cap\widetilde{\mathbb{F}}_{d1}(\delta_0,\varepsilon_2)\cap\widetilde{\mathbb{F}}_{d1}(\delta_0,\varepsilon_3)\cap\mathbb{A}_1}\\
&\quad + \|f_d\|_{\mathbb{F}_{d2}\cap\widetilde{\mathbb{F}}_{d2}(\delta_0,\varepsilon_2)\cap\widetilde{\mathbb{F}}_{d2}(\delta_0,\varepsilon_3)\cap\mathbb{A}_2\cap\mathbb{A}_3} + \|\mathbf{g}\|_{\mathbb{G}\cap\widetilde{\mathbb{G}}(\delta_0,\varepsilon_2)\cap\widetilde{\mathbb{G}}(\delta_0,\varepsilon_3)\cap\mathbb{A}_3}\\
&\quad + \|(f_d,\mathbf{g})\|_{L^{\alpha(q)}_p(\mathbf{R}_+,W^1_q(\mathbf{R}^3_-))\cap L^{\alpha(q/2)}_p(\mathbf{R}_+,W^1_{q/2}(\mathbf{R}^3_-))}\Big).
\end{aligned}
\tag{4.4.4}
$$

In addition,

$$H \in C^1(\mathbf{R}_+, C^1(\mathbf{R}^3_-)) \cap C(\mathbf{R}_+, C^2(\mathbf{R}^3_-)). \tag{4.4.5}$$

Proof. In this proof, we suppose that $r \in \{q, 2\}$.

Step 1 We first construct initial flows. Let $(\mathbf{V}, \Pi, Y, Z)$ be the solution to the following equations:

$$\begin{cases} \partial_t \mathbf{V} - \Delta \mathbf{V} + \nabla \Pi = 0 & \text{in } \mathbf{R}^3_-,\ t > 0, \\ \operatorname{div} \mathbf{V} = 0 & \text{in } \mathbf{R}^3_-,\ t > 0, \\ \mathbf{S}(\mathbf{V}, \Pi)\mathbf{e}_3 + (c_g - c_\sigma \Delta')Y\mathbf{e}_3 = 0 & \text{on } \mathbf{R}^3_0,\ t > 0, \\ \partial_t Y - \mathbf{V} \cdot \mathbf{e}_3 = 0 & \text{on } \mathbf{R}^3_0,\ t > 0, \\ \mathbf{V}|_{t=0} = 0 \quad \text{in } \mathbf{R}^3_-, \quad Y|_{t=0} = h_0 & \text{on } \mathbf{R}^2, \end{cases}$$

and besides,

$$\begin{cases} \Delta Z = 0 & \text{in } \mathbf{R}^3_-,\ t \geq 0, \\ Z = Y & \text{on } \mathbf{R}^3_0,\ t \geq 0. \end{cases}$$

On the other hand, let $\mathbf{W}$ be the solution to

$$\begin{cases} \partial_t \mathbf{W} + \mathbf{W} - \Delta \mathbf{W} = 0 & \text{in } \mathbf{R}^3,\ t > 0, \\ \mathbf{W}|_{t=0} = \widetilde{\mathbf{u}}_0 & \text{in } \mathbf{R}^3, \end{cases} \tag{4.4.6}$$

where $\widetilde{\mathbf{u}}_0$ is an extension such that $\widetilde{\mathbf{u}}_0|_{\mathbf{R}^3_-} = \mathbf{u}_0$ and

$$\|\widetilde{\mathbf{u}}_0\|_{B^{2(1-1/p)}_{s,p}(\mathbf{R}^3)} \leq C(p,q)\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{s,p}(\mathbf{R}^3_-)} \quad \text{for } s \in \{q, q/2\} \tag{4.4.7}$$

(cf. [**Tri83**, Theorem 2.9.2]) with a positive constant $C(p,q)$.

Remark 4.4.2. Let exponents p, q satisfy (4.1.4), and $q/2 < s < q$. Then, by (4.4.7) and [**Tri83**, Theorem 2.4.7] with

$$s_0 = s_1 = 2\left(1 - \frac{1}{p}\right), \quad p_0 = \frac{q}{2}, \quad p_1 = q, \quad q_0 = q_1 = p, \quad \Theta = 2 - \frac{q}{s},$$

we obtain

$$\begin{aligned} \|\widetilde{\mathbf{u}}_0\|_{B^{2(1-1/p)}_{s,p}} &\leq C\|\widetilde{\mathbf{u}}_0\|^{1-\Theta}_{B^{2(1-1/p)}_{q/2,p}(\mathbf{R}^3)}\|\widetilde{\mathbf{u}}_0\|^{\Theta}_{B^{2(1-1/p)}_{q,p}(\mathbf{R}^3)} \\ &\leq C\|\mathbf{u}_0\|^{1-\Theta}_{B^{2(1-1/p)}_{q/2,p}(\mathbf{R}^3_-)}\|\mathbf{u}_0\|^{\Theta}_{B^{2(1-1/p)}_{q,p}(\mathbf{R}^3_-)} \leq C\|\mathbf{u}_0\|_{\mathbb{I}_1} \end{aligned}$$

with some positive constant $C = C(p,q)$.

Secondly, we consider locally-in-time estimates of $(\mathbf{V}, \Pi, Y, Z)$. For every $T > 0$, in the same manner as [**SS08**, Theorem 3.9], we have, by Proposition 3.3.5,

$$\begin{aligned} &\|\mathbf{V}\|_{W^{2,1}_{r,p}(\mathbf{R}^3_- \times (0,T))} + \|\partial_t Y\|_{L_p((0,T), W^{2-1/r}_r(\mathbf{R}^2))} \\ &\quad + \|Y\|_{L_p((0,T), W^{3-1/r}_r(\mathbf{R}^2))} \leq C\|h_0\|_{B^{3-1/p-1/r}_{r,p}(\mathbf{R}^2)} \leq C\|h_0\|_{\mathbb{I}_2} \end{aligned} \tag{4.4.8}$$

with some positive constant $C = C(p,q,T)$. In addition, by Remark 4.1.2 (2),

$$\begin{aligned} &\|\nabla \partial_t Z\|_{L_p((0,T), W^1_r(\mathbf{R}^3_-))} + \|\nabla Z\|_{L_p((0,T), W^2_r(\mathbf{R}^3_-))} \\ &\leq C(p,q,T)\left(\|\partial_t Y\|_{L_p((0,T), W^{2-1/r}_r(\mathbf{R}^2))} + \|Y\|_{L_p((0,T), W^{3-1/r}_r(\mathbf{R}^2))}\right) \\ &\leq C(p,q,T)\|h_0\|_{\mathbb{I}_2}. \end{aligned} \tag{4.4.9}$$

Concerning $\mathbf{W}$, the following lemma holds by (4.4.7) and Remark 4.4.2.

LEMMA 4.4.3. *Let exponents p, q satisfy* (4.1.4), *and $q/2 \le s \le q$. Let $\mathbf{W}$ be the solution obtained above to the equations* (4.4.6). *Then, for any $\alpha \ge 0$, there exists a positive constant $C = C(p, q, s, \alpha)$ such that*

$$\|(\partial_t \mathbf{W}, \mathbf{W}, \nabla \mathbf{W}, \nabla^2 \mathbf{W})\|_{L_p^\alpha(\mathbf{R}_+, L_s(\mathbf{R}^3_-))} \le C\|\mathbf{u}_0\|_{\mathbb{I}_1},$$
$$\|(\mathbf{W}, \nabla \mathbf{W})\|_{L_\infty^\alpha(\mathbf{R}_+, L_s(\mathbf{R}^3_-))} \le C\|\mathbf{u}_0\|_{\mathbb{I}_1}.$$

By Proposition 4.3.9, (4.4.8), (4.4.9), and Lemma 4.4.3, we obtain

$$\sum_{r\in\{q,2\}} \Big(\mathbb{W}_{r,\infty}(\mathbf{V}+\mathbf{W}, Y, 0, Z) + \mathbb{M}_{r,p}(\mathbf{V}+\mathbf{W}, 0, Y, 0, Z)\Big) + \mathbb{W}_{q,p}(\mathbf{V}+\mathbf{W}, Z; \delta_1, \delta_2) \le C(p,q)\|(\mathbf{u}_0, h_0)\|_{\mathbb{I}_1\times\mathbb{I}_2} \tag{4.4.10}$$

with some positive constant $C(p, q)$.

Step 2 We here consider the divergence equations:

$$\operatorname{div} \mathbf{u}_d = f_d - \operatorname{div} \mathbf{W} = \operatorname{div}(\mathbf{f}_d - \mathbf{W}) \quad \text{in } \mathbf{R}^3_-,\ t > 0. \tag{4.4.11}$$

Noting $(f_d - \operatorname{div} \mathbf{W})|_{t=0} = 0$ in $\mathbf{R}^3_-$ by the compatibility condition, we obtain, by [**SS12**, Lemma 4.1 (2)], $\mathbf{u}_d|_{t=0} = 0$ in $\mathbf{R}^3_-$ and

$$\|\mathbf{u}_d(t)\|_{L_s(\mathbf{R}^3_-)} \le C(q)\Big(\|\mathbf{f}_d(t)\|_{L_s(\mathbf{R}^3_-)} + \|\mathbf{W}(t)\|_{L_s(\mathbf{R}^3_-)}\Big),$$
$$\|\nabla \mathbf{u}_d(t)\|_{L_s(\mathbf{R}^3_-)}, \le C(q)\Big(\|f_d(t)\|_{L_s(\mathbf{R}^3_-)} + \|\nabla \mathbf{W}(t)\|_{L_s(\mathbf{R}^3_-)}\Big),$$
$$\|\nabla^2 \mathbf{u}_d(t)\|_{L_s(\mathbf{R}^3_-)} \le C(q)\Big(\|\nabla f_d(t)\|_{L_s(\mathbf{R}^3_-)} + \|\nabla^2 \mathbf{W}(t)\|_{L_s(\mathbf{R}^3_-)}\Big),$$
$$\|\partial_t \mathbf{u}_d(t)\|_{L_s(\mathbf{R}^3_-)} \le C(q)\Big(\|\partial_t \mathbf{f}_d(t)\|_{L_s(\mathbf{R}^3_-)} + \|\partial_t \mathbf{W}(t)\|_{L_s(\mathbf{R}^3_-)}\Big)$$

for any $t > 0$ and $q/2 \le s \le q$ with some positive $C(q)$. By using Lemma 4.4.3, we thus obtain the following lemma.

LEMMA 4.4.4. *Let exponents p, q satisfy the assumption* (4.1.4), *and $q/2 \le s \le q$. Let $\mathbf{u}_d$ is the solution obtained above to the equations* (4.4.11). *Then, for any $\alpha \ge 0$, there exists a positive constant $C = C(p, q, s, \alpha)$ such that*

$$\|(\partial_t \mathbf{u}_d, \mathbf{u}_d, \nabla \mathbf{u}_d, \nabla^2 \mathbf{u}_d)\|_{L_p^\alpha(\mathbf{R}_+, L_s(\mathbf{R}^3_-))}$$
$$\le C(\|(\partial_t \mathbf{f}_d, \mathbf{f}_d, f_d, \nabla f_d)\|_{L_p^\alpha(\mathbf{R}_+, L_s(\mathbf{R}^3_-))} + \|\mathbf{u}_0\|_{\mathbb{I}_1}),$$
$$\|\mathbf{u}_d\|_{L_\infty^\alpha(\mathbf{R}_+, L_s(\mathbf{R}^3_-))} \le C(\|\mathbf{f}_d\|_{L_\infty^\alpha(\mathbf{R}_+, L_s(\mathbf{R}^3_-))} + \|\mathbf{u}_0\|_{\mathbb{I}_1}),$$
$$\|\nabla \mathbf{u}_d\|_{L_\infty^\alpha(\mathbf{R}_+, L_s(\mathbf{R}^3_-))} \le C(\|f_d\|_{L_\infty^\alpha(\mathbf{R}_+, L_s(\mathbf{R}^3_-))} + \|\mathbf{u}_0\|_{\mathbb{I}_1}).$$

It then follows from Lemma 4.4.4 that

$$\sum_{r\in\{q,2\}} \Big(\mathbb{W}_{r,\infty}(\mathbf{u}_d, 0, 0, 0) + \mathbb{M}_{r,p}(\mathbf{u}_d, 0, 0, 0, 0)\Big) + \|(\partial_t \mathbf{u}_d, \nabla^2 \mathbf{u}_d)\|_{L_p^{\delta_1}(\mathbf{R}_+, L_q(\mathbf{R}^3_+))}$$
$$\le C(p,q)\Big(\|\mathbf{f}_d\|_{\mathbb{F}_{d1}\cap\widetilde{\mathbb{F}}_{d1}(\delta_0,\varepsilon_0)} + \|f_d\|_{\mathbb{F}_{d2}\cap\widetilde{\mathbb{F}}_{d2}(\delta_0,\varepsilon_0)} + \|\mathbf{u}_0\|_{\mathbb{I}_1}\Big) \tag{4.4.12}$$

with a positive constant $C = C(p, q)$.

Step 3 By using $(\mathbf{V}, \Pi, Y, Z)$, $\mathbf{W}$, and $\mathbf{u}_d$ obtained in Step 1 and 2, we set

$$\mathbf{u} = \widetilde{\mathbf{u}} + \mathbf{u}_d + \mathbf{V} + \mathbf{W}, \quad \theta = \widetilde{\theta} + \Pi, \quad h = \kappa + Y, \quad H = K + Z$$

in the equations (4.4.1) and (4.4.2), and then we have

$$\begin{cases} \partial_t \widetilde{\mathbf{u}} - \Delta \widetilde{\mathbf{u}} + \nabla \widetilde{\theta} = \mathbf{f}_1 + \mathbf{F} & \text{in } \mathbf{R}^3_-, \ t > 0, \\ \operatorname{div} \widetilde{\mathbf{u}} = 0 & \text{in } \mathbf{R}^3_-, \ t > 0, \\ \mathbf{S}(\widetilde{\mathbf{u}}, \widetilde{\theta})\mathbf{e}_3 + (c_g - c_\sigma \Delta')\kappa \mathbf{e}_3 = \mathbf{G} & \text{on } \mathbf{R}^3_0, \ t > 0, \\ \partial_t \kappa - \widetilde{\mathbf{u}} \cdot \mathbf{e}_3 = G_h & \text{on } \mathbf{R}^3_0, \ t > 0, \\ \widetilde{\mathbf{u}}|_{t=0} = 0 \quad \text{in } \mathbf{R}^3_-, \quad \kappa|_{t=0} = 0 & \text{on } \mathbf{R}^2, \end{cases}$$

where we have set

$$\mathbf{F} = \mathbf{f}_2 + \mathbf{W} - \partial_t \mathbf{u}_d + \Delta \mathbf{u}_d$$
$$G_h = g_h + (\mathbf{W} + \mathbf{u}_d) \cdot \mathbf{e}_3, \quad \mathbf{G} = \mathbf{g} - \mathbf{D}(\mathbf{W} + \mathbf{u}_d)\mathbf{e}_3,$$

and besides,

$$\begin{cases} \Delta K = 0 & \text{in } \mathbf{R}^3_-, \ t \geq 0, \\ K = \kappa & \text{on } \mathbf{R}^3_0, \ t \geq 0. \end{cases}$$

Let $(\mathbf{v}, \pi)$ and $(\mathbf{w}, \rho, \kappa, K)$ be the solutions to

$$(4.4.13) \qquad \begin{cases} \partial_t \mathbf{v} + \mathbf{v} - \Delta \mathbf{v} + \nabla \pi = 0 & \text{in } \mathbf{R}^3_-, \ t > 0, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \mathbf{R}^3_-, \ t > 0, \\ \mathbf{S}(\mathbf{v}, \pi)\mathbf{e}_3 = \mathbf{G} & \text{on } \mathbf{R}^3_0, \ t > 0, \\ \mathbf{v}|_{t=0} = 0 & \text{in } \mathbf{R}^3_-, \end{cases}$$

$$\begin{cases} \partial_t \mathbf{w} - \Delta \mathbf{w} + \nabla \rho = \mathbf{f}_1 + \mathbf{F} + \mathbf{v} & \text{in } \mathbf{R}^3_-, \ t > 0, \\ \operatorname{div} \mathbf{w} = 0 & \text{in } \mathbf{R}^3_-, \ t > 0, \\ \mathbf{S}(\mathbf{w}, \rho)\mathbf{e}_3 + (c_g - c_\sigma \Delta')\kappa \mathbf{e}_3 = 0 & \text{on } \mathbf{R}^3_0, \ t > 0, \\ \partial_t \kappa - \mathbf{w} \cdot \mathbf{e}_3 = G_h + \mathbf{v} \cdot \mathbf{e}_3 & \text{on } \mathbf{R}^3_0, \ t > 0, \\ \mathbf{w}|_{t=0} = 0 \quad \text{in } \mathbf{R}^3_-, \quad \kappa|_{t=0} = 0 & \text{on } \mathbf{R}^2, \end{cases}$$

and then $\widetilde{\mathbf{u}} = \mathbf{v} + \mathbf{w}$ and $\widetilde{\pi} = \pi + \rho$. Consequently, the solution $(\mathbf{u}, \theta, h, H)$ of the equations of (4.4.1) and (4.4.2) have been decomposed into

$$\mathbf{u} = \mathbf{v} + \mathbf{w} + \mathbf{u}_d + \mathbf{V} + \mathbf{W},$$
$$\theta = \pi + \rho + \Pi,$$
$$h = \kappa + Y, \quad H = K + Z.$$

Step 4 We here estimate the solution $\mathbf{v}$ to the equations (4.4.13).

LEMMA 4.4.5. *Let exponents p, q satisfy the condition* (4.1.4), *and $r \in \{q, 2\}$. Then the solution $\mathbf{v}$ obtained above to the equations* (4.4.13) *satisfies the following properties.*

(1) *There exists a positive constant C such that*

$$\|\mathbf{v}\|_{L^{n(q/2,r)+1/8}_\infty(\mathbf{R}_+, W^1_r(\mathbf{R}^3_-))} \leq C\Big(\|(\mathbf{g}, f_d)\|_{\mathbb{A}_3} + \|\mathbf{u}_0\|_{\mathbb{I}_1}\Big),$$
$$\|\mathbf{v}\|_{\mathbb{A}_1 \cap \widehat{\mathbb{A}}_2} \leq C\Big(\|(\mathbf{g}, f_d)\|_{\mathbb{A}_3} + \|\mathbf{u}_0\|_{\mathbb{I}_1}\Big),$$

$$\|\mathbf{v}\|_{\mathbb{F}_2} + \|\mathbf{v}\|_{\mathbb{G}_h} \le C\Big(\|\mathbf{g}\|_{\mathbb{G}} + \|\mathbf{f}_d\|_{\mathbb{F}_{d1}} + \|f_d\|_{\mathbb{F}_{d2}} + \|\mathbf{u}_0\|_{\mathbb{I}_1}\Big).$$

(2) *Let $\alpha(q)$ and $\alpha(q/2)$ be positive numbers satisfying*

$$p(1+\alpha(q)-\delta_0) > 1, \quad p(1+\alpha(q/2)-\max\{\varepsilon_2,\varepsilon_3\}) > 1.$$

Then there exists a positive constant C such that

$$\|\mathbf{v}\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} \le C\Big(\|(\mathbf{g}, f_d)\|_{L_p^{\alpha(q)}(\mathbf{R}_+, W_q^1(\mathbf{R}_+^3))\cap L_p^{\alpha(q/2)}(\mathbf{R}_+, W_{q/2}^1(\mathbf{R}_+^3))} + \|\mathbf{u}_0\|_{\mathbb{I}_1}$$
$$+ \|\mathbf{g}\|_{\widetilde{\mathbb{G}}(\delta_0,\varepsilon_2)} + \|\mathbf{f}_d\|_{\widetilde{\mathbb{F}}_{d1}(\delta_0,\varepsilon_2)} + \|f_d\|_{\widetilde{\mathbb{F}}_{d2}(\delta_0,\varepsilon_2)}\Big),$$

$$\|\mathbf{v}\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)} \le C\Big(\|(\mathbf{g}, f_d)\|_{L_p^{\alpha(q)}(\mathbf{R}_+, W_q^1(\mathbf{R}_+^3))\cap L_p^{\alpha(q/2)}(\mathbf{R}_+, W_{q/2}^1(\mathbf{R}_+^3))} + \|\mathbf{u}_0\|_{\mathbb{I}_1}$$
$$+ \|\mathbf{g}\|_{\widetilde{\mathbb{G}}(\delta_0,\varepsilon_3)} + \|\mathbf{f}_d\|_{\widetilde{\mathbb{F}}_{d1}(\delta_0,\varepsilon_3)} + \|f_d\|_{\widetilde{\mathbb{F}}_{d2}(\delta_0,\varepsilon_3)}\Big).$$

Proof. (1) By Theorem 4.3.4, Lemma 4.4.3, and Lemma 4.4.4, we have

$$\|\mathbf{v}\|_{L_\infty^{n(q/2,r)+1/8}(\mathbf{R}_+, W_r^1(\mathbf{R}_-^3))} \le C\|\mathbf{G}\|_{L_p^{n(q/2,r)+1/8}(\mathbf{R}_+, W_r^1(\mathbf{R}_-^3))}$$
$$\le C\Big(\|(\mathbf{g}, f_d)\|_{L_p^{n(q/2,r)+1/8}(\mathbf{R}_+, W_r^1(\mathbf{R}_-^3))} + \|\mathbf{u}_0\|_{\mathbb{I}_1}\Big)$$
$$\le C\Big(\|(\mathbf{g}, f_d)\|_{\mathbb{A}_3} + \|\mathbf{u}_0\|_{\mathbb{I}_1}\Big),$$
$$\|\mathbf{v}\|_{\mathbb{F}_2} + \|\mathbf{v}\|_{\mathbb{G}_h} \le 2\|\mathbf{v}\|_{\mathbb{G}_h} \le C\|\mathbf{G}\|_{H^{1,1/2}_{q,p,0}(\mathbf{R}_-^3\times\mathbf{R})\cap H^{1,1/2}_{q/2,p,0}(\mathbf{R}_-^3\times\mathbf{R})}$$
$$\le C\Big(\|\mathbf{g}\|_{\mathbb{G}} + \|\mathbf{f}_d\|_{\mathbb{F}_{d1}} + \|f_d\|_{\mathbb{F}_{d2}} + \|\mathbf{u}_0\|_{\mathbb{I}_1}\Big)$$

for $r \in \{q, 2\}$ with some positive constant C, where we have also used Lemma 1.3.3. These inequalities furnish the required inequalities.

(2) In the same manner as (1), we see that the required inequalities hold. □

Step 5 We here consider the estimates of $\mathbf{w}$, κ, and K. By using Lemma 4.3.7, we have

$$\sum_{r\in\{q,2\}}\Big(\mathbb{W}_{r,\infty}(\mathbf{w},\kappa,0,K) + \mathbb{M}_{r,p}(\mathbf{w},0,\kappa,0,K)\Big) + \mathbb{W}_{q,p}(\mathbf{w},K;\delta_1,\delta_2)$$
$$\le C\Big(\|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{F}+\mathbf{v}\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} + \|G_h + \mathbf{v}\cdot\mathbf{e}_3\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\mathbb{A}_1\cap\widehat{\mathbb{A}}_2}\Big)$$

with some positive constant $C = C(p,q)$. We then see, by Lemma 4.4.4 and Lemma 4.4.5, that

$$\|\mathbf{F}+\mathbf{v}\|_{\mathbb{F}_2} \le C\Big(\|\mathbf{f}_2\|_{\mathbb{F}_2} + \|\mathbf{f}_d\|_{\mathbb{F}_{1d}} + \|f_d\|_{\mathbb{F}_{d2}} + \|\mathbf{g}\|_{\mathbb{G}} + \|\mathbf{u}_0\|_{\mathbb{I}_1}\Big),$$
$$\|G_h + \mathbf{v}\cdot\mathbf{e}_3\|_{\mathbb{G}_h} \le C\Big(\|g_h\|_{\mathbb{G}_h} + \|\mathbf{f}_d\|_{\mathbb{F}_{1d}} + \|f_d\|_{\mathbb{F}_{d2}} + \|\mathbf{g}\|_{\mathbb{G}} + \|\mathbf{u}_0\|_{\mathbb{I}_1}\Big)$$

with some positive constant C. In addition,

$$\|\mathbf{F}+\mathbf{v}\|_{\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} \le C\Big(\|(\mathbf{g}, f_d)\|_{L_p^{\alpha(q)}(\mathbf{R}_+, W_q^1(\mathbf{R}_+^3))\cap L_p^{\alpha(q/2)}(\mathbf{R}_+, W_{q/2}^1(\mathbf{R}_+^3))} + \|\mathbf{u}_0\|_{\mathbb{I}_1}$$
$$+ \|\mathbf{g}\|_{\widetilde{\mathbb{G}}(\delta_0,\varepsilon_2)} + \|\mathbf{f}_d\|_{\widetilde{\mathbb{F}}_{d1}(\delta_0,\varepsilon_2)} + \|f_d\|_{\widetilde{\mathbb{F}}_{d2}(\delta_0,\varepsilon_2)}\Big),$$
$$\|G_h + \mathbf{v}\cdot\mathbf{e}_3\|_{\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)} \le C\Big(\|(\mathbf{g}, f_d)\|_{L_p^{\alpha(q)}(\mathbf{R}_+, W_q^1(\mathbf{R}_+^3))\cap L_p^{\alpha(q/2)}(\mathbf{R}_+, W_{q/2}^1(\mathbf{R}_+^3))}$$
$$+ \|\mathbf{u}_0\|_{\mathbb{I}_1} + \|\mathbf{g}\|_{\widetilde{\mathbb{G}}(\delta_0,\varepsilon_3)} + \|\mathbf{f}_d\|_{\widetilde{\mathbb{F}}_{d1}(\delta_0,\varepsilon_3)} + \|f_d\|_{\widetilde{\mathbb{F}}_{d2}(\delta_0,\varepsilon_3)}\Big),$$

and furthermore,

$$\|G_h + \mathbf{v}\cdot\mathbf{e}_3\|_{\mathbb{A}_1\cap\widehat{\mathbb{A}}_2} \le C\Big(\|g_h\|_{\mathbb{A}_1\cap\widehat{\mathbb{A}}_2} + \|\mathbf{f}_d\|_{\mathbb{A}_1} + \|f_d\|_{\mathbb{A}_2} + \|(\mathbf{g}, f_d)\|_{\mathbb{A}_3} + \|\mathbf{u}_0\|_{\mathbb{I}_1}\Big),$$

since $\|\mathbf{u}_d\|_{\widehat{\mathbb{A}}_2} \le C\|f_d\|_{\mathbb{A}_2}$. Thus we obtain the required estimate (4.4.4) with $\theta = 0$ and $\partial_t h = 0$ by (4.4.10), (4.4.12), and Lemma 4.4.5. For the estimates of θ and $\partial_t h$, use the first and fourth equations of (4.4.1).

Finally, we show (4.4.5). Since the initial flow is given by an analytic semi-group (cf. Theorem 3.1.1), $Z \in C^1(\mathbf{R}_+, W_q^2(\Omega_L)) \cap C(\mathbf{R}_+, W_q^3(\Omega_L))$ for $\Omega_L = \mathbf{R}^2\times(-L,0)$ $(L>0)$. We then see, by Sobolev's embedding theorem, that

$$Z \in C^1(\mathbf{R}_+, C^1(\mathbf{R}^3_-)) \cap C(\mathbf{R}_+, C^2(\mathbf{R}^3_-)),$$

which, combined with Remark 4.3.6 (3), furnishes (4.4.5). This completes the proof of the theorem. □

4.5. Nonlinear problem

In this section, we solve the nonlinear problem (4.2.9) with (4.1.3) by using the contraction mapping theorem. We here write again the system:

$$\text{(4.5.1)} \qquad \begin{cases} \partial_t\mathbf{u} - \Delta\mathbf{u} + \nabla\theta = \mathbf{F}(\mathbf{u}, H) & \text{in } \mathbf{R}^3_-,\ t>0,\\ \operatorname{div}\mathbf{u} = F_d(\mathbf{u}, H) = \operatorname{div}\mathbf{F}_d(\mathbf{u}, H) & \text{in } \mathbf{R}^3_-,\ t>0,\\ \mathbf{S}(\mathbf{u},\theta)\mathbf{e}_3 + (c_g - c_\sigma\Delta')h\mathbf{e}_3 = \mathbf{G}(\mathbf{u}, H) & \text{on } \mathbf{R}^3_0,\ t>0,\\ \partial_t h - \mathbf{u}\cdot\mathbf{e}_3 = G_h(\mathbf{u}, H) & \text{on } \mathbf{R}^3_0,\ t>0,\\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \mathbf{R}^3_-,\\ h|_{t=0} = h_0 & \text{on } \mathbf{R}^2, \end{cases}$$

and besides,

$$\text{(4.5.2)} \qquad \begin{cases} \Delta H = 0 & \text{in } \mathbf{R}^3_-,\ t\ge 0,\\ H = h & \text{on } \mathbf{R}^3_0,\ t \ge 0, \end{cases}$$

where the nonlinear terms on the right-hand sides of (4.5.1) are given in Section 4.2. In order to use the contraction mapping theorem, for $R>0$, we set the following function space:

$$\text{(4.5.3)} \quad X_R = \Big\{\mathbf{z} = (\mathbf{u},\theta,h,H) \mid \|\mathbf{z}\|_X = \sum_{r\in\{q,2\}}\Big(\mathbb{W}_{r,\infty}(\mathbf{u},h,\partial_t h,H) + \mathbb{M}_{r,p}(\mathbf{u},\theta,h,\partial_t h,H)\Big) + \mathbb{W}_{q,p}(\mathbf{u},H;1/2,3/4) < R\Big\}.$$

Then, reminding $[\mathbf{u}]_{\tan} = \mathbf{u} - (\mathbf{u}\cdot\mathbf{e}_3)\mathbf{e}_3$, we obtain the following theorem.

Theorem 4.5.1. *Let exponents p, q satisfy* (4.1.4), *and $c_g, c_\sigma > 0$. Suppose that $(\mathbf{u}_0, h_0) \in \mathbb{I}_1\times\mathbb{I}_2$ and H_0 is given by* (4.2.10). *Then there exist positive constants ε_0 and δ_0, depending only on p, q, c_g, and c_σ, such that the equations* (4.5.1) *and* (4.5.2) *admits a unique solution $(\mathbf{u},\theta,h,H)$ in X_{δ_0} if the initial data $(\mathbf{u}_0,h_0)$ satisfies the smallness condition: $\|(\mathbf{u}_0,h_0)\|_{\mathbb{I}_1\times\mathbb{I}_2} < \varepsilon_0$ and the compatibility conditions:*

$$\text{(4.5.4)} \quad \operatorname{div}\mathbf{u}_0 = F_d(\mathbf{u}_0, H_0) \quad \text{in } \mathbf{R}^3_-, \qquad [\mathbf{D}(\mathbf{u}_0)\mathbf{e}_3]_{\tan} = [\mathbf{G}(\mathbf{u}_0,H_0)]_{\tan} \quad \text{on } \mathbf{R}^3_0.$$

Remark 4.5.2. It holds that $H \in C^1(\mathbf{R}_+, C^1(\mathbf{R}^3_-)) \cap C(\mathbf{R}_+, C^2(\mathbf{R}^3_-))$ by Theorem 4.4.1.

PROOF. We prove the theorem by using the contraction mapping theorem and Theorem 4.4.1 with

$$\varepsilon_1 = m\left(\frac{q}{2}, q\right) + n\left(\frac{q}{2}, q\right) + \frac{1}{8} = \frac{2}{q} + \frac{3}{8}, \quad \varepsilon_2 = \varepsilon_3 = 1, \tag{4.5.5}$$
$$\delta_1 = \frac{1}{2}, \quad \delta_2 = \frac{3}{4}, \quad \alpha(q) = 0, \quad \alpha\left(\frac{q}{2}\right) = \frac{1}{4}.$$

Then we note as follows: First, the assumption $3 < q < 16/5$ implies that $\varepsilon_1 > 1$. Secondly, we see, by (4.1.4), that

$$p > 32, \quad p\left(1 + \alpha(q) - \frac{3}{4}\right) = p\left(1 + \alpha\left(\frac{q}{2}\right) - 1\right) = \frac{p}{4} > 1. \tag{4.5.6}$$

Thirdly, by Lemma 1.3.5 (1), (2), and the assumption (4.1.4), we have

$$\begin{aligned} \|(\mathbf{u}, \nabla H)\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))} &\le M\|\mathbf{z}\|_X, \\ \|(\mathbf{u}, \nabla H)\|_{L_\infty(\mathbf{R}_+, W^1_q(\mathbf{R}^3_-))} &\le M\|\mathbf{z}\|_X, \\ \|(\mathbf{u}, \nabla H)\|_{L_\infty(\mathbf{R}_+, W^1_2(\mathbf{R}^3_-))} &\le M\|\mathbf{z}\|_X \end{aligned} \tag{4.5.7}$$

for some positive constant M, depending only on p and q, and $\mathbf{z} = (\mathbf{u}, \theta, h, H) \in X_{\delta_0}$ with $\delta_0 > 0$.

From now on, to use Theorem 4.4.1 under the condition (4.5.5), we show that there exists a positive number δ_0 such that for $\mathbf{z} = (\mathbf{u}, \theta, h, H) \in X_{\delta_0}$

$$\begin{aligned} &\mathbf{F}_1(\mathbf{u}, H) \in \mathbb{F}_1 \cap \widetilde{\mathbb{F}}_1(3/4, 2/q + 3/8), \quad \mathbf{F}_2(\mathbf{u}, H) \in \mathbb{F}_2 \cap \widetilde{\mathbb{F}}_2(3/4, 1), \\ &G_h(\mathbf{u}, H) \in \mathbb{G}_h \cap \widetilde{\mathbb{G}}_h(3/4, 1) \cap \mathbb{A}_1 \cap \widehat{\mathbb{A}}_2, \\ &\mathbf{F}_d(\mathbf{u}, H) \in \mathbb{F}_{d1} \cap \widetilde{\mathbb{F}}_{d1}(3/4, 1) \cap \mathbb{A}_1, \\ &F_d(\mathbf{u}, H) \in \mathbb{F}_{d2} \cap \widetilde{\mathbb{F}}_{d2}(3/4, 1) \cap \mathbb{A}_2 \cap \mathbb{A}_3, \\ &\mathbf{G}(\mathbf{u}, H) \in \mathbb{G} \cap \widetilde{\mathbb{G}}(3/4, 1) \cap \mathbb{A}_3, \\ &(F_d(\mathbf{u}, H), \mathbf{G}(\mathbf{u}, H)) \in L_p(\mathbf{R}_+, W^1_q(\mathbf{R}^3_-))^4 \cap L_p^{1/4}(\mathbf{R}_+, W^1_{q/2}(\mathbf{R}^3_-))^4 \end{aligned} \tag{4.5.8}$$

with the inequality:

$$\begin{aligned} &\|\mathbf{F}_1(\mathbf{u}, H)\|_{\mathbb{F}_1 \cap \widetilde{\mathbb{F}}_1(3/4, 2/q+3/8)} + \|\mathbf{F}_2(\mathbf{u}, H)\|_{\mathbb{F}_2 \cap \widetilde{\mathbb{F}}_2(3/4,1)} \\ &\quad + \|G_h(\mathbf{u}, H)\|_{\mathbb{G}_h \cap \widetilde{\mathbb{G}}_h(3/4,1) \cap \mathbb{A}_1 \cap \widehat{\mathbb{A}}_2} + \|\mathbf{F}_d(\mathbf{u}, H)\|_{\mathbb{F}_{d1} \cap \widetilde{\mathbb{F}}_{d1}(3/4,1) \cap \mathbb{A}_1} \\ &\quad + \|F_d(\mathbf{u}, H)\|_{\mathbb{F}_{d2} \cap \widetilde{\mathbb{F}}_{d2}(3/4,1) \cap \mathbb{A}_2 \cap \mathbb{A}_3} + \|\mathbf{G}(\mathbf{u}, H)\|_{\mathbb{G} \cap \widetilde{\mathbb{G}}(3/4,1) \cap \mathbb{A}_3} \\ &\quad + \|(F_d(\mathbf{u}, H), \mathbf{G}(\mathbf{u}, H))\|_{L_p(\mathbf{R}_+, W^1_q(\mathbf{R}^3_-))^4 \cap L_p^{1/4}(\mathbf{R}_+, W^1_{q/2}(\mathbf{R}^3_-))} \\ &\le C(p, q)\|\mathbf{z}\|_X^2 \end{aligned} \tag{4.5.9}$$

for a positive constant $C(p, q)$.

Case $\mathbf{F}_1(\mathbf{u}, H)$ By (4.5.7), it is clear that for $r \in \{q, 2\}$

$$\begin{aligned} \|(\mathbf{u} \cdot \nabla)\mathbf{u}\|_{L_p(\mathbf{R}_+, L_r(\mathbf{R}^3_-))} &\le \|\mathbf{u}\|_{L_\infty(\mathbf{R}_+, L_\infty(\mathbf{R}^3_-))} \|\nabla \mathbf{u}\|_{L_p(\mathbf{R}_+, L_r(\mathbf{R}^3_-))} \\ &\le C(p, q)\|\mathbf{z}\|_X^2 \end{aligned} \tag{4.5.10}$$

with some positive constant $C(p,q)$, and besides, by Sobolev's embedding theorem and Hölder's inequality

$$\begin{aligned}
&\|(\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \le \|\mathbf{u}(t)\|_{L_\infty(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\\
&\quad\le C(q)\|\mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\\
&\quad\le C(q)(t+2)^{-(2/q+3/8)}\|\mathbf{z}\|_X^2,\\
&\|(\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)\|_{L_{q/2}(\mathbf{R}^3_-)} \le \|\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\\
&\quad\le (t+2)^{-(2/q+3/8)}\|\mathbf{z}\|_X^2
\end{aligned}$$

for every $t>0$. Then, noting that $p(2/q+3/8-3/4) > p/4 > 1$ by the assumption: $3<q<16/5$ and (4.5.6), we have

$$\begin{aligned}
&\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L_p^{3/4}(\mathbf{R}_+,L_q(\mathbf{R}^3_-))} \le C(p,q)\|\mathbf{z}\|_X^2,\\
&\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L_\infty^{2/q+3/8}(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_-))} \le C(p,q)\|\mathbf{z}\|_X^2
\end{aligned}$$

for a positive constant $C(p,q)$, which, combined with (4.5.10), furnishes that

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(3/4,2/q+3/8)} \le C(p,q)\|\mathbf{z}\|_X^2. \tag{4.5.11}$$

Concerning $\partial_t H D_3\mathbf{u}$, we use Sobolev's inequality (cf. [**AF03**, Theorem 4.31]):

$$\|f\|_{L_6(\mathbf{R}^3_-)} \le C\|\nabla f\|_{L_2(\mathbf{R}^3_-)}$$

with a positive constant C. By Sobolev's inequality, Hölder's inequality, and Sobolev's embedding theorem, we have, for any $t>0$,

$$\begin{aligned}
&\|\partial_t H(t)D_3\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\\
&\quad\le \|\partial_t H(t)\|_{L_6(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_r(\mathbf{R}^3_-)} \quad (1/6+1/r=1/q)\\
&\quad\le C\|\nabla\partial_t H(t)\|_{L_2(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)}\\
&\|\partial_t H(t)D_3\mathbf{u}(t)\|_{L_2(\mathbf{R}^3_-)}\\
&\quad\le \|\partial_t H(t)\|_{L_6(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_3(\mathbf{R}^3_-)}\\
&\quad\le C\|\nabla\partial_t H(t)\|_{L_2(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|^{\alpha}_{L_2(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|^{1-\alpha}_{L_q(\mathbf{R}^3_-)},\\
&\|\partial_t H(t)D_3\mathbf{u}(t)\|_{L_{q/2}(\mathbf{R}^3_-)}\\
&\quad\le \|\partial_t H(t)\|_{L_6(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_s(\mathbf{R}^3_-)} \quad (1/6+1/s=2/q)\\
&\quad\le C\|\nabla\partial_t H(t)\|_{L_2(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|^{\beta}_{L_2(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|^{1-\beta}_{L_q(\mathbf{R}^3_-)},
\end{aligned} \tag{4.5.12}$$

where we note that $0<\alpha,\beta<1$ and

$$3\left(\frac{1}{q}-\frac{1}{r}\right)=\frac{1}{2}<1, \qquad \frac{1}{3}=\frac{\alpha}{2}+\frac{1-\alpha}{q}, \qquad \frac{1}{s}=\frac{\beta}{2}+\frac{1-\beta}{q}. \tag{4.5.13}$$

By (4.5.7) and (4.5.12), we obtain

$$\begin{aligned}
&\|\partial_t H D_3\mathbf{u}\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\\
&\quad\le C\|\nabla\partial_t H\|_{L_\infty(\mathbf{R}_+,L_2(\mathbf{R}^3_-))}\|\nabla\mathbf{u}\|_{L_p(\mathbf{R}_+,W^1_q(\mathbf{R}^3_-))} \le C\|\mathbf{z}\|_X^2,\\
&\|\partial_t H D_3\mathbf{u}\|_{L_p(\mathbf{R}_+,L_2(\mathbf{R}^3_-))}\\
&\quad\le C\|\nabla\partial_t H\|_{L_p(\mathbf{R}_+,L_2(\mathbf{R}^3_-))}\|\nabla\mathbf{u}\|^{\alpha}_{L_\infty(\mathbf{R}_+,L_2(\mathbf{R}^3_-))}\|\nabla\mathbf{u}\|^{1-\alpha}_{L_\infty(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\\
&\quad\le C\|\mathbf{z}\|_X^2
\end{aligned} \tag{4.5.14}$$

with some positive constant $C = C(p,q)$. In addition, it follows from (4.5.12) that for any $t > 0$

$$\|\partial_t H(t) D_3 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \leq C(t+2)^{-m(q/2,2)-1/2}\|\mathbf{z}\|_X$$
$$\times \Big((t+2)^{-n(q/2,q)-1/8}\|\mathbf{z}\|_X + (t+2)^{-1/2}\{(t+2)^{1/2}\|\nabla^2\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\}\Big),$$
$$\|\partial_t H(t) D_3 \mathbf{u}(t)\|_{L_{q/2}(\mathbf{R}^3_-)} \leq C(t+2)^{-m(q/2,2)-1/2}\|\mathbf{z}\|_X$$
$$\times \Big((t+2)^{-n(q/2,2)-1/8}\|\mathbf{z}\|_X\Big)^{\beta}\Big((t+2)^{-n(q/2,q)-1/8}\|\mathbf{z}\|_X\Big)^{1-\beta}$$
$$= C(t+2)^{-(2/q+3/8)}\|\mathbf{z}\|_X^2,$$

because $m(q/2,2) + 1/2 = 2/q$ and by (4.5.13)

$$\beta n\left(\frac{q}{2},2\right) + (1-\beta) n\left(\frac{q}{2},q\right) = \frac{3\beta}{2}\left(\frac{2}{q}-\frac{1}{2}\right) + \frac{3(1-\beta)}{2q}$$
$$= \frac{3}{2q} - \frac{3\beta}{2}\left(\frac{1}{2}+\frac{1}{q}-\frac{2}{q}\right) = \frac{3}{2q} - \frac{3}{2}\cdot\frac{6-q}{3(q-2)}\cdot\frac{q-2}{2q} = \frac{1}{4}.$$

Then, noting that by (4.5.6)

$$p\left(m\left(\frac{q}{2},2\right) + \frac{1}{2} + n\left(\frac{q}{2},q\right) + \frac{1}{8} - \frac{3}{4}\right) = p\left(\frac{7}{2q}-\frac{5}{8}\right) > p\left(\frac{7}{8}-\frac{5}{8}\right) = \frac{p}{4} > 1,$$
$$m\left(\frac{q}{2},2\right) + \frac{1}{2} + \frac{1}{2} - \frac{3}{4} = \frac{2}{q} - \frac{1}{4} > \frac{2}{4} - \frac{1}{4} > 0$$

since $q < 16/5 < 4$, we see that

$$\|\partial_t H D_3\mathbf{u}\|_{L_p^{3/4}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))}$$
$$\leq C(p,q)\|\mathbf{z}\|_X\Big(\|(t+2)^{-(m(q/2,2)+1/2+n(q/2,q)+1/8-3/4)}\|_{L_p(\mathbf{R}_+)}\|\mathbf{z}\|_X$$
$$+ \|(t+2)^{-(m(q/2,2)+1/2+1/2-3/4)}\|_{L_\infty(\mathbf{R}_+)}\|\nabla^2\mathbf{u}\|_{L_p^{1/2}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))}\Big)$$
$$\leq C(p,q)\|\mathbf{z}\|_X^2,$$
$$\|\partial_t H D_3\mathbf{u}\|_{L_\infty^{2/q+3/8}(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_-))} \leq C(p,q)\|\mathbf{z}\|_X^2,$$

which, combined with (4.5.14), furnishes that

$$\|\partial_t H D_3\mathbf{u}\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(3/4,2/q+3/8)} \leq C(p,q)\|\mathbf{z}\|_X^2. \tag{4.5.15}$$

Summing up (4.5.7), (4.5.11), and (4.5.15), we have

$$\|\mathbf{F}_1(\mathbf{u},H)\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(3/4,2/q+3/8)}$$
$$\leq C(p,q)\left(\frac{\|\partial_t H D_3\mathbf{u}\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(3/4,2/q+3/8)}}{1-\|\nabla H\|_{L_\infty(\mathbf{R}_+, L_\infty(\mathbf{R}^3_-))}} + \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(3/4,2/q+3/8)}\right)$$
$$\leq C(p,q)\left(\frac{1}{1-\delta_0 M} + 1\right)\|\mathbf{z}\|_X^2.$$

Here and hereafter, we suppose that δ_0 satisfies the condition: $\delta_0 M < 1/2$, and then we complete the required estimate of $\mathbf{F}_1(\mathbf{u},H)$ in (4.5.9) by the last inequality.

Case $\mathbf{F}_2(\mathbf{u},H)$ By (4.5.7), it is clear that for $r \in \{q,2\}$ and $j = 1,2,3$

$$\|(\partial_t u_3 \nabla H, \Delta u_3 \nabla H, \mathcal{F}_{jj}(H)\mathbf{u}, (\mathbf{u}\cdot\nabla H) D_3\mathbf{u})\|_{L_p(\mathbf{R}_+, L_r(\mathbf{R}^3_-))} \leq C\|\mathbf{z}\|_X^2 \tag{4.5.16}$$

with some positive constant $C = C(p,q)$, where $\mathcal{F}_{jj}(H)$ are given in Lemma 4.2.1. In addition, it follows from Hölder's inequality and Sobolev's embedding theorem that for any $t > 0$ and $j = 1, 2, 3$

(4.5.17)
$$
\begin{aligned}
&\|\partial_t u_3(t) \nabla H(t)\|_{L_q(\mathbf{R}^3_-)} \le \|\partial_t \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)} \\
&\quad \le C(q) \|\partial_t \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)} \\
&\quad \le C(q)(t+2)^{-m(q/2,q)-3/4} \|\mathbf{z}\|_X \big\{(t+2)^{1/2} \|\partial_t \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\big\}, \\
&\|\Delta u_3(t) \nabla H(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\quad \le C(q)(t+2)^{-m(q/2,q)-3/4} \|\mathbf{z}\|_X \big\{(t+2)^{1/2} \|\nabla^2 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\big\}, \\
&\|\mathcal{F}_{jj}(H(t)) \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\quad \le C(q) \Big(\|\nabla \mathbf{u}(t)\|_{L_\infty(\mathbf{R}^3_-)} \|\nabla^2 H(t)\|_{L_q(\mathbf{R}^3_-)} + \|\nabla^2 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)} \Big) \\
&\quad \le C(q) \Big(\|\nabla \mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)} \|\nabla^2 H(t)\|_{L_q(\mathbf{R}^3_-)} + \|\nabla^2 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)} \Big) \\
&\quad \le C(q)(t+2)^{-m(q/2,q)-1/4} \|\mathbf{z}\|_X \Big((t+2)^{-n(q/2,q)-1/8} \|\mathbf{z}\|_X \\
&\quad + (t+2)^{-1/2} \big\{(t+2)^{1/2} \|\nabla^2 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\big\} \Big), \\
&\|(\mathbf{u}(t) \cdot \nabla H(t)) D_3 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \le \|\mathbf{u}(t)\|_{L_\infty(\mathbf{R}^3_-)} \|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)} \|\nabla \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\quad \le C(q) \|\mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)} \|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)} \|\nabla \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\quad \le C(q)(t+2)^{-m(q/2,q)-m(q/2,q)-1/4-n(q/2,q)-1/8} \|\mathbf{z}\|_X^2.
\end{aligned}
$$

We then obtain, for $j = 1, 2, 3$,

(4.5.18)
$$
\begin{aligned}
&\|\partial_t u_3 \nabla H\|_{L_p^{3/4}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q) \|(t+2)^{-m(q/2,q)}\|_{L_\infty(\mathbf{R}_+)} \|\mathbf{z}\|_X \|\partial_t \mathbf{u}\|_{L_p^{1/2}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q) \|\mathbf{z}\|_X^2, \\
&\|\Delta u_3 \nabla H\|_{L_p^{3/4}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q) \|(t+2)^{-m(q/2,q)}\|_{L_\infty(\mathbf{R}_+)} \|\mathbf{z}\|_X \|\nabla^2 \mathbf{u}\|_{L_p^{1/2}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q) \|\mathbf{z}\|_X^2, \\
&\|\mathcal{F}_{jj}(H) \mathbf{u}\|_{L_p^{3/4}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q) \|\mathbf{z}\|_X \Big(\|(t+2)^{-(m(q/2,q)+1/4+n(q/2,q)+1/8-3/4)}\|_{L_p(\mathbf{R}_+)} \|\mathbf{z}\|_X \\
&\quad + \|(t+2)^{-m(q/2,q)}\|_{L_\infty(\mathbf{R}_+)} \|\nabla^2 \mathbf{u}\|_{L_p^{1/2}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \Big) \\
&\quad \le C(p,q) \|\mathbf{z}\|_X^2, \\
&\|(\mathbf{u} \cdot \nabla H) D_3 \mathbf{u}\|_{L_p^{3/4}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q) \|(t+2)^{-(m(q/2,q)+m(q/2,q)+1/4+n(q/2,q)+1/8-3/4)}\|_{L_p(\mathbf{R}_+)} \|\mathbf{z}\|_X^2 \\
&\quad \le C(p,q) \|\mathbf{z}\|_X^2,
\end{aligned}
$$

because by (4.5.6) and the assumption: $3 < q < 16/5$

$$
\begin{aligned}
&p\left(m\left(\frac{q}{2},q\right)+m\left(\frac{q}{2},q\right)+\frac{1}{4}+n\left(\frac{q}{2},q\right)+\frac{1}{8}-\frac{3}{4}\right)\\
&\quad> p\left(m\left(\frac{q}{2},q\right)+\frac{1}{4}+n\left(\frac{q}{2},q\right)+\frac{1}{8}-\frac{3}{4}\right)=p\left(\frac{2}{q}-\frac{1}{8}\right)>\frac{p}{2}>1.
\end{aligned}
$$

Analogously it holds that for $j = 1, 2, 3$

$$
\|(\partial_t u_3\Delta H, \Delta u_3\nabla H, \mathcal{F}_{jj}(F)\mathbf{u}, (\mathbf{u}\cdot\nabla H)D_3\mathbf{u})\|_{L^1_p(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_-))} \leq C\|\mathbf{z}\|_X^2 \tag{4.5.19}
$$

with a positive constant $C = C(p, q)$ by using the inequalities, which are obtained in a similar way to (4.5.17), as follows: For every $t > 0$ and $j = 1, 2, 3$

$$
\begin{aligned}
&\|\partial_t u_3(t)\nabla H(t)\|_{L_{q/2}(\mathbf{R}^3_-)}\\
&\quad\leq C(p,q)(t+2)^{-m(q/2,q)-3/4}\|\mathbf{z}\|_X\big\{(t+2)^{1/2}\|\partial_t\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\big\},\\
&\|\Delta u_3(t)\nabla H(t)\|_{L_{q/2}(\mathbf{R}^3_-)}\\
&\quad\leq C(p,q)(t+2)^{-m(q/2,q)-3/4}\|\mathbf{z}\|_X\big\{(t+2)^{1/2}\|\nabla^2\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\big\},\\
&\|\mathcal{F}_{jj}(H(t))\mathbf{u}(t)\|_{L_{q/2}(\mathbf{R}^3_-)}\\
&\quad\leq C(p,q)(t+2)^{-m(q/2,q)-1/4}\|\mathbf{z}\|_X\Big((t+2)^{-n(q/2,q)-1/8}\|\mathbf{z}\|_X\\
&\quad+(t+2)^{-1/2}\big\{(t+2)^{1/2}\|\nabla^2\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\big\},\\
&\|(\mathbf{u}(t)\cdot\nabla H(t))D_3\mathbf{u}(t)\|_{L_{q/2}(\mathbf{R}^3_-)}\\
&\quad\leq C(p,q)(t+2)^{-m(q/2,q)-m(q/2,q)-1/4-n(q/2,q)-1/8}\|\mathbf{z}\|_X^2.
\end{aligned}
$$

Here we note that $m(q/2, q) + 3/4 - 1 = 1/(2q) > 0$ and by (4.5.6)

$$
\begin{aligned}
&p\left(m\left(\frac{q}{2},q\right)+m\left(\frac{q}{2},q\right)+\frac{1}{4}+n\left(\frac{q}{2},q\right)+\frac{1}{8}-1\right)\\
&\quad> p\left(m\left(\frac{q}{2},q\right)+\frac{1}{4}+n\left(\frac{q}{2},q\right)+\frac{1}{8}-1\right)=p\left(\frac{2}{q}-\frac{3}{8}\right)>\frac{p}{4}>1
\end{aligned}
$$

since $3 < q < 16/5$. By combining (4.5.7) with (4.5.16), (4.5.18), and (4.5.19), we obtain the required inequality of $\mathbf{F}_2(\mathbf{u}, H)$ in (4.5.9).

Case $G_h(\mathbf{u}, H)$ By (4.5.7), it is clear that for $r \in \{q, 2\}$

$$
\|\mathbf{u}'\cdot\nabla' H\|_{W^{2,1}_{r,p}(\mathbf{R}^3_-\times\mathbf{R}_+)} \leq C(p,q)\|\mathbf{z}\|_X^2 \tag{4.5.20}
$$

with some positive constant $C(p, q)$. On the other hand, it follows from Hölder's inequality and Sobolev's embedding theorem that for any $t > 0$

$$
\begin{aligned}
&\|\mathbf{u}'(t)\cdot\nabla' H(t)\|_{W^2_q(\mathbf{R}^3_-)} \leq C(q)\Big(\|\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)}\\
&\quad+\|\nabla\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)}+\|\nabla\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla^2 H(t)\|_{L_\infty(\mathbf{R}^3_-)}\\
&\quad+\|\nabla^2\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)}+\|\mathbf{u}(t)\|_{L_\infty(\mathbf{R}^3_-)}\|\nabla^3 H(t)\|_{L_q(\mathbf{R}^3_-)}\Big)\\
&\leq C(q)\Big(\|\mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)}\|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)}+\|\mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)}\|\nabla^3 H(t)\|_{L_q(\mathbf{R}^3_-)}\\
&\quad+\|\nabla^2\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C(q)\Big((t+2)^{-m(q/2,q)-m(q/2,q)-1/4}\|\mathbf{z}\|_X^2 \\
&\quad + (t+2)^{-m(q/2,q)-3/4}\|\mathbf{z}\|_X\big\{(t+2)^{1/2}\|\nabla^2\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\big\} \\
&\quad + (t+2)^{-m(q/2,q)-3/4}\|\mathbf{z}\|_X\big\{(t+2)^{3/4}\|\nabla^3 H(t)\|_{L_q(\mathbf{R}^3_-)}\big\}\Big),
\end{aligned}
$$

and furthermore, it similarly holds that for any $t > 0$

$$
\begin{aligned}
\|\mathbf{u}'(t)\cdot\nabla' H(t)\|_{W^2_{q/2}(\mathbf{R}^3_-)} &\leq C(q)\Big((t+2)^{-m(q/2,q)-m(q/2,q)-1/4}\|\mathbf{z}\|_X^2 \\
&+ (t+2)^{-m(q/2,q)-3/4}\|\mathbf{z}\|_X\big\{(t+2)^{1/2}\|\nabla^2\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\big\} \\
&+ (t+2)^{-m(q/2,q)-3/4}\|\mathbf{z}\|_X\big\{(t+2)^{3/4}\|\nabla^3 H(t)\|_{L_q(\mathbf{R}^3_-)}\big\}\Big).
\end{aligned}
$$

Then, noting that by (4.5.6) and the assumption: $3 < q < 16/5$

$$
p\left(m\left(\frac{q}{2},q\right)+m\left(\frac{q}{2},q\right)+\frac{1}{4}-\frac{3}{4}\right)=\frac{p}{q}>\frac{5p}{16}>1,
$$

we have

$$
\begin{aligned}
&\|\mathbf{u}'\cdot\nabla' H\|_{L^{3/4}_p(\mathbf{R}_+,W^2_q(\mathbf{R}^3_-))} \\
&\leq C(p,q)\Big(\|(t+2)^{-(m(q/2,q)+m(q/2,q)+1/4-3/4)}\|_{L_p(\mathbf{R}_+)}\|\mathbf{z}\|_X^2 \\
&\quad + \|(t+2)^{-m(q/2,q)}\|_{L_\infty(\mathbf{R}_+)}\|\nabla^2\mathbf{u}\|_{L^{1/2}_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))} \\
&\quad + \|(t+2)^{-m(q/2,q)}\|_{L_\infty(\mathbf{R}_+)}\|\nabla^3 H\|_{L^{3/4}_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\Big) \\
&\leq C(p,q)\|\mathbf{z}\|_X^2
\end{aligned}
$$

with some positive constant $C(p,q)$, and also it similarly holds that

$$
\|\mathbf{u}'\cdot\nabla' H\|_{L^1_p(\mathbf{R}_+,W^2_{q/2}(\mathbf{R}^3_-))}\leq C(p,q)\|\mathbf{z}\|_X^2
$$

since $m(q/2,q)+3/4-1 = 1/(2q) > 0$ and

$$
p\left(m\left(\frac{q}{2},q\right)+m\left(\frac{q}{2},q\right)+\frac{1}{4}-1\right)=p\left(\frac{1}{q}-\frac{1}{4}\right)>\frac{p}{16}>1
$$

by (4.5.6) and the assumption: $3 < q < 16/5$. By combining the inequalities above with (4.5.20), we obtain the required estimate of $G_h(\mathbf{u},H)$ in (4.5.9).

Case $\mathbf{F}_d(\mathbf{u},H)$ By (4.5.7), it is clear that for $r\in\{q,2\}$

(4.5.21)
$$
\begin{aligned}
\|\mathbf{M}_1(H)\mathbf{u}\|_{W^1_p(\mathbf{R}_+,L_r(\mathbf{R}^3_-))} &\leq \|\nabla H\|_{L_\infty(\mathbf{R}_+,L_\infty(\mathbf{R}^3_-))}\|\mathbf{u}\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_-))} \\
&+ \|\partial_t\nabla H\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_-))}\|\mathbf{u}\|_{L_\infty(\mathbf{R}_+,L_\infty(\mathbf{R}^3_-))} \\
&+ \|\nabla H\|_{L_\infty(\mathbf{R}_+,L_\infty(\mathbf{R}^3_-))}\|\partial_t\mathbf{u}\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_-))} \\
&\leq C(p,q)\|\mathbf{z}\|_X^2
\end{aligned}
$$

with some positive constant $C(p,q)$, where $\mathbf{M}_1(H)$ is defined as (4.2.4). On the other hand, it follows from Hölder's inequality and Sobolev's embedding theorem that for any $t > 0$

$$
\begin{aligned}
&\|(\partial_t \mathbf{M}_1(H(t)))\mathbf{u}\|_{L_q(\mathbf{R}^3_-)} \le \|\partial_t \nabla H(t)\|_{L_q(\mathbf{R}^3_-)} \|\mathbf{u}(t)\|_{L_\infty(\mathbf{R}^3_-)} \\
&\quad \le C(q)\|\partial_t \nabla H(t)\|_{L_q(\mathbf{R}^3_-)} \|\mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)} \\
&\quad \le C(q)(t+2)^{-m(q/2,q)-1/2-m(q/2,q)} \|\mathbf{z}\|_X^2, \\
&\|\mathbf{M}_1(H(t))\partial_t \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \le \|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)} \|\partial_t \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\quad \le C(q)\|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)} \|\partial_t \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\quad \le C(q)(t+2)^{-m(q/2,q)-3/4} \|\mathbf{z}\|_X \big\{(t+2)^{1/2} \|\partial_t \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\big\},
\end{aligned}
$$

and furthermore,

$$
\begin{aligned}
&\|(\partial_t \mathbf{M}_1(H(t)))\mathbf{u}(t)\|_{L_{q/2}(\mathbf{R}^3_-)} \le \|\partial_t \nabla H(t)\|_{L_q(\mathbf{R}^3_-)} \|\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\quad \le (t+2)^{-m(q/2,q)-1/2-m(q/2,q)} \|\mathbf{z}\|_X^2, \\
&\|\mathbf{M}_1(H(t))\partial_t \mathbf{u}(t)\|_{L_{q/2}(\mathbf{R}^3_-)} \le \|\nabla H(t)\|_{L_q(\mathbf{R}^3_-)} \|\partial_t \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\quad \le (t+2)^{-m(q/2,q)-3/4} \|\mathbf{z}\|_X \big\{(t+2)^{1/2} \|\partial_t \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\big\}.
\end{aligned}
$$

Then, noting that by (4.5.6)

$$
p\left(m\left(\frac{q}{2},q\right)+\frac{1}{2}+m\left(\frac{q}{2},q\right)-\frac{3}{4}\right) = p\left(\frac{1}{q}+\frac{1}{4}\right) > 1,
$$

we have

$$
\begin{aligned}
&\|(\partial_t \mathbf{M}_1(H))\mathbf{u}\|_{L_p^{3/4}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q)\|(t+2)^{-(m(q/2,q)+1/2+m(q/2,q)-3/4)}\|_{L_p(\mathbf{R}_+)} \|\mathbf{z}\|_X^2 \\
&\quad \le C(p,q)\|\mathbf{z}\|_X^2, \\
&\|\mathbf{M}_1(H)\partial_t \mathbf{u}\|_{L_p^{3/4}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q)\|(t+2)^{-m(q/2,q)}\|_{L_\infty(\mathbf{R}_+)} \|\mathbf{z}\|_X \|\partial_t \mathbf{u}\|_{L_p^{1/2}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q)\|\mathbf{z}\|_X^2
\end{aligned}
$$

with some positive constant $C(p,q)$, and besides, it similarly holds that

$$
\begin{aligned}
\|(\partial_t \mathbf{M}_1(H))\mathbf{u}\|_{L_p^1(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_-))} &\le C(p,q)\|\mathbf{z}\|_X^2, \\
\|\mathbf{M}_1(H)\partial_t \mathbf{u}\|_{L_p^1(\mathbf{R}_+, L_{q/2}(\mathbf{R}^3_-))} &\le C(p,q)\|\mathbf{z}\|_X^2
\end{aligned}
$$

since $m(q/2,q) + 3/4 - 1 = 1/(2q) > 0$ and

$$
p\left(m\left(\frac{q}{2},q\right)+\frac{1}{2}+m\left(\frac{q}{2},q\right)-1\right) = \frac{p}{q} > \frac{5p}{16} > 1
$$

by (4.5.6) and the assumption: $3 < q < 16/5$. We thus obtain, by Hölder's inequality and (4.5.7),

$$
\begin{aligned}
\text{(4.5.22)}\quad &\|\mathbf{M}_1(H)\mathbf{u}\|_{W_p^{1,3/4}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} = \|\partial_t((t+2)^{3/4}\mathbf{M}_1(H)\mathbf{u})\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q)\Big(\|\nabla H\|_{L_\infty(\mathbf{R}_+, L_\infty(\mathbf{R}^3_-))} \|\mathbf{u}\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} \\
&\quad + \|(\partial_t \mathbf{M}_1(H))\mathbf{u}\|_{L_p^{3/4}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))} + \|\mathbf{M}_1(H)\partial_t \mathbf{u}\|_{L_p^{3/4}(\mathbf{R}_+, L_q(\mathbf{R}^3_-))}\Big) \\
&\quad \le C(p,q)\|\mathbf{z}\|_X^2,
\end{aligned}
$$

$$
\begin{aligned}
&\|\mathbf{M}_1(H)\mathbf{u}\|_{W^{1,1}_p(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_-))} = \|\partial_t((t+2)\mathbf{M}_1(H)\mathbf{u})\|_{L_p(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_-))}\\
&\quad\le C(p,q)\Big(\|\nabla H\|_{L_\infty(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\|\mathbf{u}\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\\
&\quad+\|(\partial_t\mathbf{M}_1(H))\mathbf{u}\|_{L^1_p(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_-))}+\|\mathbf{M}_1(H)\partial_t\mathbf{u}\|_{L^1_p(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_-))}\Big)\\
&\le C(p,q)\|\mathbf{z}\|_X^2
\end{aligned}
$$

with some positive constant $C(p,q)$. In addition, it is clear that by (4.5.7) and Hölder's inequality

$$
\begin{aligned}
&\|\mathbf{M}_1(H(t))\mathbf{u}(t)\|_{L_r(\mathbf{R}^3_-)} \le \|\nabla H(t)\|_{L_r(\mathbf{R}^3_-)}\|\mathbf{u}(t)\|_{L_\infty(\mathbf{R}^3_-)}\\
&\quad\le C(p,q)(t+2)^{-m(q/2,r)-1/4}\|\mathbf{z}\|_X^2
\end{aligned}
$$

for $r\in\{q,2\}$ and any $t>0$, which furnishes that $\|\mathbf{M}_1(H)\mathbf{u}\|_{\mathbb{A}_1}\le C(p,q)\|\mathbf{z}\|_X^2$. By combining the inequality with (4.5.21) and (4.5.22), we obtain the required inequality of $\mathbf{F}_d(\mathbf{u},H)$ in (4.5.9).

Case $F_d(\mathbf{u},H)$ By (4.5.7), it is clear that for $r\in\{q,2\}$

$$
\|F_d(\mathbf{u},H)\|_{L_p(\mathbf{R}_+,W^1_r(\mathbf{R}^3_-))}\le C(p,q)\|\mathbf{z}\|_X^2 \tag{4.5.23}
$$

with some positive constant $C(p,q)$. On the other hand, it holds that for any $t>0$

$$
\begin{aligned}
&\|\nabla H(t)\cdot D_3\mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)}\le C(q)\|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)}\\
&\quad\le C(q)(t+2)^{-m(q/2,q)-1/4}\|\mathbf{z}\|_X\\
&\quad\times\Big((t+2)^{-n(q/2,q)-1/8}\|\mathbf{z}\|_X+(t+2)^{-1/2}\big\{(t+2)^{1/2}\|\nabla^2\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\big\}\Big)
\end{aligned}
$$

since $W^1_q(\mathbf{R}^3_-)$ is a Banach algebra, and also that by Hölder's inequality

$$
\begin{aligned}
&\|\nabla H(t)\cdot D_3\mathbf{u}(t)\|_{W^1_{q/2}(\mathbf{R}^3_-)}\le C(q)\|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)}\\
&\quad\le C(q)(t+2)^{-m(q/2,q)-1/4}\|\mathbf{z}\|_X\\
&\quad\times\Big((t+2)^{-n(q/2,q)-1/8}\|\mathbf{z}\|_X+(t+2)^{-1/2}\big\{(t+2)^{1/2}\|\nabla^2\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\big\}\Big).
\end{aligned}
$$

Then, noting that $m(q/2,q)+3/4-1=1/(2q)>0$ and

$$
p\left(m\left(\frac{q}{2},q\right)+\frac14+n\left(\frac{q}{2},q\right)+\frac18-1\right)=p\left(\frac2q-\frac38\right)>\frac p4>1
$$

by (4.5.6) and the assumption: $3<q<16/5$, we have

$$
\begin{aligned}
&\|\nabla H\cdot D_3\mathbf{u}\|_{L^1_p(\mathbf{R}_+,W^1_q(\mathbf{R}^3_-))}+\|\nabla H\cdot D_3\mathbf{u}\|_{L^1_p(\mathbf{R}_+,W^1_{q/2}(\mathbf{R}^3_-))}\\
&\quad\le C(p,q)\|\mathbf{z}\|_X\Big(\|(t+2)^{-(m(q/2,q)+1/4+n(q/2,q)+1/8-1)}\|_{L_p(\mathbf{R}_+)}\|\mathbf{z}\|_X\\
&\quad+\|(t+2)^{-(m(q/2,q)+3/4-1)}\|_{L_\infty(\mathbf{R}_+)}\|\nabla^2\mathbf{u}\|_{L^{1/2}_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\Big)\\
&\quad\le C(p,q)\|\mathbf{z}\|_X^2
\end{aligned} \tag{4.5.24}
$$

with some positive constant $C(p,q)$. In addition, since for $r\in\{q,2\}$ and any $t>0$

$$
\begin{aligned}
&\|\nabla H(t)\cdot D_3\mathbf{u}(t)\|_{L_r(\mathbf{R}^3_-)}\le\|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_r(\mathbf{r}^3_-)}\\
&\quad\le C(q)\|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_r(\mathbf{r}^3_-)}\\
&\quad\le C(q)(t+2)^{-m(q/2,q)-1/4-n(q/2,r)-1/8}\|\mathbf{z}\|_X^2
\end{aligned}
$$

by Hölder's inequality and Sobolev's embedding theorem, which furnishes that

$$\|\nabla H \cdot D_3\mathbf{u}\|_{\mathbb{A}_2} \leq C(p,q)\|\mathbf{z}\|_X^2, \tag{4.5.25}$$

where we have used

$$m\left(\frac{q}{2},q\right)+\frac{1}{4}+n\left(\frac{q}{2},r\right)+\frac{1}{8} > m\left(\frac{q}{2},r\right)+\frac{1}{2}.$$

Concerning $\mathbb{A}_3$-norm, we shall calculate as follows: First, by Hölder's inequality

$$\begin{aligned}
&\|\nabla H(t)\cdot D_3\mathbf{u}(t)\|_{W_2^1(\mathbf{R}^3_-)} \leq \|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_2(\mathbf{R}^3_-)}\\
&\quad + \|\nabla^2 H(t)\|_{L_\infty(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_2(\mathbf{R}^3_-)} + \|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)}\|\nabla^2\mathbf{u}(t)\|_{L_2(\mathbf{R}^3_-)}\\
&\leq C(q)\Big(\|\nabla H(t)\|_{W_q^2(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{W_2^1(\mathbf{R}^3_-)} + \|\nabla^2 H(t)\|_{W_q^1(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_2(\mathbf{R}^3_-)}\Big)\\
&\leq C(q)\Big((t+2)^{-m(q/2,q)-1/4-n(q/2,2)-1/8}\|\mathbf{z}\|_X^2\\
&\quad + (t+2)^{-m(q/2,q)-1/4}\|\mathbf{z}\|_X\|\nabla^2\mathbf{u}(t)\|_{L_2(\mathbf{R}^3_-)}\\
&\quad + (t+2)^{-3/4-n(q/2,2)-1/8}\|\mathbf{z}\|_X\big\{(t+2)^{3/4}\|\nabla H(t)\|_{L_q(\mathbf{R}^3_-)}\big\}\Big)
\end{aligned}$$

with some positive constant $C(q)$, which furnishes that

(4.5.26)

$$\begin{aligned}
&\|\nabla H\cdot D_3\mathbf{u}\|_{L_p^{m(q/2,2)+1/2}(\mathbf{R}_+,W_2^1(\mathbf{R}^3_-))}\\
&\leq C(p,q)\Big(\|(t+2)^{-(m(q/2,q)+1/4+n(q/2,2)+1/8-m(q/2,2)-1/2)}\|_{L_p(\mathbf{R}_+)}\|\mathbf{z}\|_X^2\\
&\quad + \|(t+2)^{-(m(q/2,q)+1/4-m(q/2,2)-1/2)}\|_{L_\infty(\mathbf{R}_+)}\|\mathbf{z}\|_X\|\nabla^2\mathbf{u}\|_{L_p(\mathbf{R}_+,L_2(\mathbf{R}^3_-))}\\
&\quad + \|(t+2)^{-(3/4+n(q/2,2)+1/8-m(q/2,2)-1/2)}\|_{L_\infty(\mathbf{R}_+)}\|\mathbf{z}\|_X\|\nabla^3 H\|_{L_p^{3/4}(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\Big)\\
&\leq C(p,q)\|\mathbf{z}\|_X^2,
\end{aligned}$$

since by the assumption: $q < 3 < 16/5$ and (4.5.6)

$$\begin{aligned}
&m\left(\frac{q}{2},2\right)+\frac{1}{4}-m\left(\frac{q}{2},2\right)-\frac{1}{2}=\frac{1}{2}\left(1-\frac{3}{q}\right)>0,\\
&\frac{3}{4}+n\left(\frac{q}{2},2\right)+\frac{1}{8}-m\left(\frac{q}{2},2\right)-\frac{1}{2}=\frac{1}{q}+\frac{1}{8}>0,\\
&p\left(m\left(\frac{q}{2},q\right)+\frac{1}{4}+n\left(\frac{q}{2},2\right)+\frac{1}{8}-m\left(\frac{q}{2},2\right)-\frac{1}{2}\right)\\
&\quad > p\left(m\left(\frac{q}{2},q\right)+\frac{1}{4}+n\left(\frac{q}{2},2\right)+\frac{1}{8}-n\left(\frac{q}{2},2\right)-\frac{1}{2}\right)=p\left(\frac{1}{2q}+\frac{1}{8}\right)>1.
\end{aligned}$$

Secondly, noting that $1-m(q/2,q)-1/2 = 1/4-1/(2q) > 0$ by $q > 3$, we have

$$\begin{aligned}
\|\nabla H\cdot D_3\mathbf{u}\|_{\mathbb{A}_3} &= \|\nabla H\cdot D_3\mathbf{u}\|_{L_p^{m(q/2,q)+1/2}(\mathbf{R}_+,W_q^1(\mathbf{R}^3_-))}\\
&\quad + \|\nabla H\cdot D_3\mathbf{u}\|_{L_p^{m(q/2,2)+1/2}(\mathbf{R}_+,W_2^1(\mathbf{R}^3_-))}\\
&\leq \|(t+2)^{1-m(q/2,q)-1/2}\|_{L_\infty(\mathbf{R}_+)}\|\nabla H\cdot D_3\mathbf{u}\|_{L_p^1(\mathbf{R}_+,W_q^1(\mathbf{R}^3_-))}\\
&\quad + \|\nabla H\cdot D_3\mathbf{u}\|_{L_p^{m(q/2,2)+1/2}(\mathbf{R}_+,W_2^1(\mathbf{R}^3_-))}\\
&\leq C(p,q)\|\mathbf{z}\|_X^2
\end{aligned} \tag{4.5.27}$$

by using (4.5.24) and (4.5.26).

Summing up, (4.5.7), (4.5.23), (4.5.24), (4.5.25), and (4.5.27), we have

$$\begin{aligned}&\|F_d(\mathbf{u},H)\|_{\mathbb{F}_{d2}\cap\widetilde{\mathbb{F}}_{d2}(3/4,1)\cap\mathbb{A}_2\cap\mathbb{A}_3}\\&\leq C(p,q)\left\|\frac{1}{1+D_3H}\right\|_{W^1_\infty(\mathbf{R}^3_-)}\|\nabla H\cdot D_3\mathbf{u}(t)\|_{\mathbb{F}_{d2}\cap\widetilde{\mathbb{F}}_{d2}(3/4,1)\cap\mathbb{A}_2\cap\mathbb{A}_3}\\&\leq C(p,q)\|\mathbf{z}\|_X^2,\end{aligned}$$

which completes the required estimates of $F_d(\mathbf{u},H)$ in (4.5.9).

Case $\mathbf{G}(\mathbf{u},H)$ In the sequel, we suppose that $f\in\{u_1,u_2,u_3,D_1H,D_2H,D_3H\}$, and let g be any of the following terms:

$$\frac{|\nabla'H|^2}{(1+\sqrt{1+|\nabla'H|^2})\sqrt{1+|\nabla'H|^2}},\quad\frac{D_iHD_jH}{(1+|\nabla'H|^2)^{3/2}},\quad D_iH,\quad D_iHD_jH$$

for $i,j=1,2,3$. Then we have the lemma as follows.

Lemma 4.5.3. *Let f and g be as mentioned above, and exponents p,q satisfy (4.1.4). Then there exists a positive number $0<r_0<1$ such that for any $\mathbf{z}=(\mathbf{u},\theta,h,H)\in X_{r_0}$ the following assertions hold.*

(1) *There exists a $C(p,q)>0$, depending only on p and q, such that*

$$\begin{aligned}&\|g\|_{L_\infty(\mathbf{R}_+,W^1_\infty(\mathbf{R}^3_-))}\leq C(p,q)\|\mathbf{z}\|_X,\\&\|g\|_{L_\infty^{m(q/2,q)+1/4}(\mathbf{R}_+,W^1_q(\mathbf{R}^3_-))}\leq C(p,q)\|\mathbf{z}\|_X,\\&\|\nabla f\|_{L_\infty^{n(q/2,q)+1/8}(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\leq C(p,q)\|\mathbf{z}\|_X.\end{aligned}$$

(2) *There exists a $C(p,q)>0$, depending only on p and q, such that*

$$\begin{aligned}&\|(t+2)^{1/2}\partial_tf\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\leq C(p,q)\|\mathbf{z}\|_X,\\&\|(t+2)^{1/2}\partial_tg\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\leq C(p,q)\|\mathbf{z}\|_X.\end{aligned}$$

It especially holds that

$$\begin{aligned}&\|\partial_t((t+2)^{1/2}f)\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\leq C(p,q)\|\mathbf{z}\|_X,\\&\|\partial_t((t+2)^{1/2}g)\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\leq C(p,q)\|\mathbf{z}\|_X,\\&\|\partial_t((t+2)^{1/4}g)\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\leq C(p,q)\|\mathbf{z}\|_X.\end{aligned}$$

(3) *There exists a $C(p,q)>0$, depending only on p and q, such that*

$$\begin{aligned}&\|(\nabla f)g\|_{L^1_p(\mathbf{R}_+,W^1_q(\mathbf{R}^3_-))}\leq C(p,q)\|\mathbf{z}\|_X^2,\\&\|(\nabla f)g\|_{L^1_p(\mathbf{R}_+,W^1_{q/2}(\mathbf{R}^3_-))}\leq C(p,q)\|\mathbf{z}\|_X^2,\\&\|(\nabla f)g\|_{L_p^{m(q/2,2)+1/2}(\mathbf{R}_+,W^1_2(\mathbf{R}^3_-))}\leq C(p,q)\|\mathbf{z}\|_X^2.\end{aligned}$$

Proof. (1) We use the following expansions:

$$\begin{aligned}&\frac{1}{\sqrt{1+x}}=1-\frac{x}{2}+O(x^2)\quad\text{as }|x|\to0,\\&\frac{1}{1+\sqrt{1+x}}=\frac12-\frac{x}{16}+O(x^2)\quad\text{as }|x|\to0,\\&\frac{1}{(1+x)^{3/2}}=1-\frac32x+O(x^2)\quad\text{as }|x|\to0.\end{aligned}$$

Combining the expansions above with (4.5.7), we see that there exist positive constants C and $0 < r_1 < 1$ such that

$$\begin{aligned}(4.5.28)\qquad &\Big\|\frac{1}{\sqrt{1+|\nabla' H|^2}}\Big\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))} \le C,\\ &\Big\|\frac{1}{1+\sqrt{1+|\nabla' H|^2}}\Big\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))} \le C,\\ &\Big\|\frac{1}{(1+|\nabla' H|^2)^{3/2}}\Big\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))} \le C\end{aligned}$$

if $H(x,t)$ satisfies $\|\nabla H\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))} \le r_1$. We then take a positive number $0 < r_0 < 1$ in such a way that for $\mathbf{z} = (\mathbf{u}, \theta, h, H) \in X_{r_0}$

$$\|\nabla H\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))} \le M\|\mathbf{z}\|_X \le Mr_0 \le r_1,$$

where we have used (4.5.7). We thus obtain, by using (4.5.7) again,

$$\begin{aligned}&\Big\|\frac{|\nabla' H|^2}{(1+\sqrt{1+|\nabla' H|^2})\sqrt{1+|\nabla' H|^2}}\Big\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))}\\ &\le \Big\|\frac{1}{1+\sqrt{1+|\nabla' H|^2}}\Big\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))}\Big\|\frac{1}{\sqrt{1+|\nabla' H|^2}}\Big\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))}\\ &\quad\times \|\nabla H\|^2_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))}\\ &\le C(M\|\mathbf{z}\|_X)^2 \le C(p,q)\|\mathbf{z}\|_X\end{aligned}$$

for any $\mathbf{z} = (\mathbf{u}, \theta, h, H) \in X_{r_0}$ with some positive constant $C(p,q)$, and also it similarly holds that for $i, j = 1, 2, 3$

$$\Big\|\Big(\frac{D_iHD_jH}{(1+|\nabla' H|^2)^{3/2}}, D_iH, D_iHD_jH\Big)\Big\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))} \le C(p,q)\|\mathbf{z}\|_X,$$

which completes the first required inequality of (1).

Next we show the second required inequality. By using (4.5.7) and (4.5.28), we have, for any $t > 0$ and $\mathbf{z} = (\mathbf{u}, \theta, h, H) \in X_{r_0}$,

$$\begin{aligned}&\Big\|\frac{|\nabla' H(t)|^2}{(1+\sqrt{1+|\nabla' H(t)|^2})\sqrt{1+|\nabla' H(t)|^2}}\Big\|_{W^1_q(\mathbf{R}^3_-)}\\ &\le \Big\|\frac{1}{1+\sqrt{1+|\nabla' H|^2}}\Big\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))}\Big\|\frac{1}{\sqrt{1+|\nabla' H|^2}}\Big\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))}\\ &\quad\times \|\nabla H\|_{L_\infty(\mathbf{R}_+, W^1_\infty(\mathbf{R}^3_-))}\|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)}\\ &\le C(M\|\mathbf{z}\|_X)(t+2)^{-m(q/2,q)-1/4}\big\{(t+2)^{m(q/2,q)+1/4}\|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)}\big\}\\ &\le C(p,q)(t+2)^{-m(q/2,q)-1/4}\|\mathbf{z}\|_X\end{aligned}$$

with some positive constant $C(p,q)$, which furnishes that

$$\left\|\frac{|\nabla' H|^2}{(1+\sqrt{1+|\nabla' H|^2})\sqrt{1+|\nabla' H|^2}}\right\|_{L_\infty^{m(q/2,q)+1/4}(\mathbf{R}_+, W_q^1(\mathbf{R}_-^3))} \le C(p,q)\|\mathbf{z}\|_X.$$

Analogously we have

$$\left\|\left(\frac{D_iHD_jH}{(1+|\nabla' H|^2)^{3/2}}, D_iH, D_iHD_jH\right)\right\|_{L_\infty^{m(q/2,q)+1/4}(\mathbf{R}_+, W_q^1(\mathbf{R}_-^3))} \le C(p,q)\|\mathbf{z}\|_X$$

for $i,j=1,2,3$ and $\mathbf{z}=(\mathbf{u},\theta,h,H)\in X_{r_0}$, which completes the second required inequality of (1). We obtain the last required inequality by the definition of X-norm, noting that $n(q/2,q)+1/8 < m(q/2,q)+1/4$ under the assumption (4.1.4).
(2) By direct calculations, we can show the required estimates, so that we may omit the proof here.
(3) Since $W_q^1(\mathbf{R}_-^3)$ is a Banach algebra, we obtain, by using the inequalities which are obtained in (1),

$$\begin{aligned}
&\|(\nabla f(t))g(t)\|_{W_q^1(\mathbf{R}_-^3)} \le C(q)\|\nabla f(t)\|_{W_q^1(\mathbf{R}_-^3)}\|g(t)\|_{W_q^1(\mathbf{R}_-^3)}\\
&\le C(q)(t+2)^{-m(q/2,q)-1/4}\|g\|_{L_\infty^{m(q/2,q)+1/4}(\mathbf{R}_+, W_q^1(\mathbf{R}_-^3))}\\
&\quad\times\Big((t+2)^{-n(q/2,q)-1/8}\|\nabla f\|_{L_\infty^{n(q/2,q)+1/8}(\mathbf{R}_+, L_q(\mathbf{R}_-^3))}\\
&\quad+(t+2)^{1/2}\{(t+2)^{1/2}\|\nabla^2 f(t)\|_{L_q(\mathbf{R}_-^3)}\}\Big)\\
&\le C(q)(t+2)^{-m(q/2,q)-1/4}\|\mathbf{z}\|_X\Big((t+2)^{-n(q/2,q)-1/8}\|\mathbf{z}\|_X\\
&\quad+(t+2)^{1/2}\{(t+2)^{1/2}\|\nabla^2 f(t)\|_{L_q(\mathbf{R}_-^3)}\}\Big)
\end{aligned}$$

for any $t>0$, and also it similarly holds that by Hölder's inequality

$$\begin{aligned}
&\|(\nabla f(t))g(t)\|_{W_{q/2}^1(\mathbf{R}_-^3)}\\
&\le C(q)(t+2)^{-m(q/2,q)-1/4}\|\mathbf{z}\|_X\Big((t+2)^{-n(q/2,q)-1/8}\|\mathbf{z}\|_X\\
&\quad+(t+2)^{1/2}\{(t+2)^{1/2}\|\nabla^2 f(t)\|_{L_q(\mathbf{R}_-^3)}\}\Big).
\end{aligned}$$

We thus obtain

$$\|(\nabla f)g\|_{L_p^1(\mathbf{R}_+, W_q^1(\mathbf{R}_-^3))\cap L_p^1(\mathbf{R}_+, W_{q/2}^1(\mathbf{R}_-^3))} \le C(p,q)\|\mathbf{z}\|_X^2.$$

In addition, we see that by Hölder's inequality

$$\begin{aligned}
&\|(\nabla f(t))g(t)\|_{W_2^1(\mathbf{R}_-^3)} \le \|\nabla f(t)\|_{L_2(\mathbf{R}_-^3)}\|g(t)\|_{L_\infty(\mathbf{R}_-^3)}\\
&\quad+\|\nabla^2 f(t)\|_{L_2(\mathbf{R}_-^3)}\|g(t)\|_{L_\infty(\mathbf{R}_-^3)}+\|\nabla f(t)\|_{L_\infty(\mathbf{R}_-^3)}\|\nabla g(t)\|_{L_2(\mathbf{R}_-^3)}\\
&\le C(q)\Big((t+2)^{-n(q/2,2)-1/8-m(q/2,q)-1/4}\|\mathbf{z}\|_X^2\\
&\quad+(t+2)^{-m(q/2,q)-1/4}\|\mathbf{z}\|_X\|\nabla^2 f(t)\|_{L_2(\mathbf{R}_-^3)}\\
&\quad+(t+2)^{-n(q/2,q)-1/8-m(q/2,2)-1/4}\|\mathbf{z}\|_X^2\\
&\quad+(t+2)^{-m(q/2,2)-1/4-1/2}\|\mathbf{z}\|_X\{(t+2)^{1/2}\|\nabla^2 f(t)\|_{L_q(\mathbf{R}_-^3)}\}\Big),
\end{aligned}$$

which furnishes that

$$\|(\nabla f)g\|_{L_p^{m(q/2,2)+1/2}(\mathbf{R}_+,W_2^1(\mathbf{R}^3_-))} \leq C(p,q)\|\mathbf{z}\|_X^2.$$

This completes the proof of the lemma. □

By using (1.3.1), Lemma 1.3.2, (4.5.7), and Lemma 4.5.3, we have

(4.5.29)

$$\begin{aligned}
&\|(D_if)g\|_{W_p^1(\mathbf{R}_+,\widehat{W}_r^{-1}(\mathbf{R}^3_-))} = \sum_{k=0,1} \|\partial_t^k(1-\Delta)^{-1/2}\iota((D_if)g)\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_-))}\\
&\leq C(p,q)\Big(\|D_if\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_-))}\|g\|_{L_\infty(\mathbf{R}_+,L_\infty(\mathbf{R}^3_-))}\\
&\quad + \|\partial_t f\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_-))}\|g\|_{L_\infty(\mathbf{R}_+,L_\infty(\mathbf{R}^3_-))}\\
&\quad + \|\partial_t f\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_-))}\|D_ig\|_{L_\infty(\mathbf{R}_+,L_\infty(\mathbf{R}^3_-))}\\
&\quad + \|D_if\|_{L_\infty(\mathbf{R}_+,L_\infty(\mathbf{R}^3_-))}\|\partial_t g\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_-))}\Big)\\
&\leq C(p,q)\|\mathbf{z}\|_X^2
\end{aligned}$$

for $r \in \{q,2\}$ and $i = 1,2,3$ with a positive constant $C(p,q)$, where we have used $\|(1-\Delta)^{-1/2}F\|_{L_q(\mathbf{R}^3)} \leq C(q)\|F\|_{L_q(\mathbf{R}^3)}$ to obtain the second line. On the other hand, it follows from Lemma 4.5.3 that

$$\begin{aligned}
&\|(\nabla f)g\|_{L_p(\mathbf{R}_+,W_q^1(\mathbf{R}^3_-))\cap L_p(\mathbf{R}_+,W_2^1(\mathbf{R}^3_-))}\\
&\quad\leq C(p,q)\|\nabla f\|_{L_p(\mathbf{R}_+,W_q^1(\mathbf{R}^3_-))\cap L_p(\mathbf{R}_+,W_2^1(\mathbf{R}^3_-))}\|g\|_{L_\infty(\mathbf{R}_+,W_\infty^1(\mathbf{R}^3_-))}\\
&\quad\leq C(p,q)\|\mathbf{z}\|_X^2,
\end{aligned}$$

which, combined with (4.5.29), furnishes that $\|\mathbf{G}(\mathbf{u},H)\|_{\mathbb{G}} \leq C(p,q)\|\mathbf{z}\|_X^2$.

Next we consider the estimate of $\widetilde{\mathbb{G}}(3/4,1)$ and $\mathbb{A}_3$-norm. By (1.3.2), we have

$$\begin{aligned}
&\|(D_if)g\|_{W_p^{1,3/4}(\mathbf{R}_+,\widehat{W}_q^{-1}(\mathbf{R}^3_-))}\\
&\quad= \|\partial_t(1-\Delta)^{-1/2}\iota\{(t+2)^{1/2}(D_if)\cdot(t+2)^{1/4}g\}\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\\
&\quad\leq C(p,q)\Big(\|\partial_t((t+2)^{1/2}f)\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\|g\|_{L_\infty^{1/4}(\mathbf{R}_+,L_\infty(\mathbf{R}^3_-))}\\
&\quad+ \|\partial_t((t+2)^{1/2}f)\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\|D_ig\|_{L_\infty^{1/4}(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\\
&\quad+ \|D_if\|_{L_\infty^{1/2}(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\|\partial_t((t+2)^{1/4}g)\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\Big)\\
&\quad\leq C(p,q)\|\mathbf{z}\|_X^2,\\
&\|(D_if)g\|_{W_p^{1,1}(\mathbf{R}_+,\widehat{W}_{q/2}^{-1}(\mathbf{R}^3_-))}\\
&\quad= \|\partial_t(1-\Delta)^{-1/2}\iota\{(t+2)^{1/2}(D_if)\cdot(t+2)^{1/2}g\}\|_{L_p(\mathbf{R}_+,L_{q/2}(\mathbf{R}^3_-))}\\
&\quad\leq C(p,q)\Big(\|\partial_t((t+2)^{1/2}f)\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\|g\|_{L_\infty^{1/2}(\mathbf{R}_+,L_\infty(\mathbf{R}^3_-))}\\
&\quad+ \|\partial_t((t+2)^{1/2}f)\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\|D_ig\|_{L_\infty^{1/2}(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\\
&\quad+ \|D_if\|_{L_\infty^{1/2}(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\|\partial_t((t+2)^{1/2}g)\|_{L_p(\mathbf{R}_+,L_q(\mathbf{R}^3_-))}\Big)\\
&\quad\leq C(p,q)\|\mathbf{z}\|_X^2
\end{aligned}$$

for $i = 1, 2, 3$ with some positive constant $C(p, q)$. Furthermore, in the same manner as we obtain (4.5.24) and (4.5.27), it follows from Lemma 4.5.3 that

$$\|(\nabla f)g\|_{L^1_p(\mathbf{R}_+, W^1_q(\mathbf{R}^3_-)) \cap L^1_p(\mathbf{R}_+, W^1_{q/2}(\mathbf{R}^3_-)) \cap \mathbb{A}_3} \leq C(p, q)\|\mathbf{z}\|^2_X,$$

which, combined with the above two estimates, yields that $\|\mathbf{G}(\mathbf{u}, H)\|_{\widetilde{\mathbb{G}}(3/4,1) \cap \mathbb{A}_3} \leq C(p, q)\|\mathbf{z}\|^2_X$. This completes the required inequality of $\mathbf{G}(\mathbf{u}, H)$ in (4.5.9).

Here we set

$$\mathbf{N}(\mathbf{u}, \theta, h, H) = (\mathbf{F}(\mathbf{u}, H), G_h(\mathbf{u}, H), \mathbf{F}_d(\mathbf{u}, H), F_d(\mathbf{u}, H), \mathbf{G}(\mathbf{u}, H)),$$

and then, as mentioned above, there exists a sufficiently small $\delta_0 > 0$ such that $\mathbf{N} : X_{\delta_0} \to \mathbb{F} \times \mathbb{G}_h \times \mathbb{F}^1_d \times \mathbb{F}^2_d \times \mathbb{G}$. It is possible to consider the following map Φ:

$$\Phi(\mathbf{z}) = L^{-1}(\mathbf{N}(\mathbf{z}), \mathbf{u}_0, h_0) \quad \text{for } \mathbf{z} \in X_{\delta_0},$$

where L is the linear operator of the left hand side on the equations (4.5.1) and (4.5.2). From now on, we show that Φ has a fixed point in X_{δ_0} by taking a smaller $\delta_0 > 0$ if necessary.

By Theorem 4.4.1 and (4.5.9), there exists a positive constant $M_1(p, q)$, depending only on p and q, such that

$$\begin{aligned} \|\Phi(\mathbf{z})\|_X &\leq M_1(p, q)\Big(\|(\mathbf{u}_0, h_0)\|_{\mathbb{I}_1 \times \mathbb{I}_2} + \|\mathbf{z}\|^2_X\Big) \\ &\leq M_1(p, q)(\varepsilon_0 + \delta_0^2) < \delta_0 \end{aligned} \tag{4.5.30}$$

by choosing the positive constants δ_0 and ε_0 in such a way that

$$M_1(p, q)\delta_0 < \frac{1}{2}, \quad M_1(p, q)\varepsilon_0 < \frac{\delta_0}{2}.$$

Analogously, we have, for $\mathbf{z}_1, \mathbf{z}_2 \in X_{\delta_0}$,

$$\|\Phi(\mathbf{z}_1) - \Phi(\mathbf{z}_2)\|_X \leq M_2(p, q)\delta_0\|\mathbf{z}_1 - \mathbf{z}_2\|_X \leq \frac{1}{2}\|\mathbf{z}_1 - \mathbf{z}_2\|_X$$

by choosing $M_2(p, q)\delta_0 < 1/2$ if necessary, which, combined with (4.5.30), furnishes that Φ is a contraction mapping on X_{δ_0}. By Banach's fixed point theorem, we have a unique fixed point $\mathbf{z}^* \in X_{\delta_0}$ of Φ. The $\mathbf{z}^*$ is a solution to the equations (4.5.1) and (4.5.2). □

Proof of Theorem 4.1.1. Note that the compatibility conditions of Theorem 4.1.1 are satisfied if and only if (4.5.4). The mapping Θ_0 given by

$$\Theta_0(\xi', \xi_N) = (\xi', \xi_N + H_0(\xi')) \quad \text{for } (\xi', \xi_N) \in \mathbf{R}^3_-$$

defines, for $h_0 \in B^{3-1/p-1/q}_{q,p}(\mathbf{R}^2)$, a C^2-diffeomorphism from $\dot{\mathbf{R}}^N$ onto Ω_0 (cf. Section 4.2 and Remark 4.2.3 (1)). In addition, by the inequality (4.2.14), the smallness condition in Theorem 4.1.1 implies the smallness condition in Theorem 4.5.1. Theorem 4.5.1 then yields a unique solution $(\mathbf{u}, \theta, h, H) \in X_{\delta_0}$ of the equations (4.5.1) and (4.5.2). Finally, setting

$$(\mathbf{v}, \pi) = (\Theta_*\mathbf{u}, \Theta_*\theta) = (\mathbf{u} \circ \Theta^{-1}, \theta \circ \Theta^{-1}),$$

where Θ_* is defined as (4.2.3), we obtain a unique solution $(\mathbf{v}, \pi, h, H)$ of the original problem (4.1.2) and the auxiliary problem (4.1.3). The proof is complete.

Part 3

Two-phase flows of generalized Newtonian fluids: Whole space type

CHAPTER 5

Strong solutions for two-phase free boundary problems for a class of non-Newtonian fluids

5.1. Main results

As was seen in Chapter 1, the motion of the two immiscible, incompressible, and viscous fluids is governed by the set of equations

$$
(5.1.1)\qquad \begin{cases}
\rho(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) = \operatorname{Div}\mathbf{T} - \rho c_g \mathbf{e}_N, & \text{in } \Omega(t),\ t>0,\\
\operatorname{div}\mathbf{v} = 0 & \text{in } \Omega(t),\ t>0,\\
-[\![\mathbf{T}\mathbf{n}_\Gamma]\!] = c_\sigma \kappa_\Gamma \mathbf{n}_\Gamma & \text{on } \Gamma(t),\ t>0,\\
[\![\mathbf{v}]\!] = 0 & \text{on } \Gamma(t),\ t>0,\\
\partial_t h + \mathbf{v}'\cdot\nabla' h - \mathbf{v}\cdot\mathbf{e}_N = 0 & \text{on } \Gamma(t),\ t>0,\\
\mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega_0,\\
h|_{t=0} = h_0 & \text{on } \mathbf{R}^{N-1}.
\end{cases}
$$

We first remind the notation being in (5.1.1). Let $\mathbf{v}_0 = \mathbf{v}_0(x)$ and $h_0 = h_0(x')$ be given N-component vector and scalar functions for $x \in \Omega_0$ and $x' \in \mathbf{R}^{N-1}$, respectively, and also $\Omega_0 = \Omega_{10} \cup \Omega_{20}$ with

$$\Omega_{i0} = \{(x', x_N) \in \mathbf{R}^N \mid x' \in \mathbf{R}^{N-1},\ (-1)^i (x_N - h_0(x')) > 0\}$$

for $i = 1, 2$. Here, some viscous fluids, *fluid*$_1$ and *fluid*$_2$, occupy Ω_{10} and Ω_{20}, respectively. Then $\Omega_i(t)$ denote the regions fulfilled with *fluid*$_i$ at time t, and besides, $\Omega(t) = \Omega_1(t) \cup \Omega_2(t)$ and

$$\Gamma(t) = \{(x', x_N) \in \mathbf{R}^N \mid x' \in \mathbf{R}^{N-1},\ x_N = h(t, x')\} \quad (t > 0).$$

The normal field on $\Gamma(t)$, pointing from $\Omega_1(t)$ into $\Omega_2(t)$, is denoted by $\mathbf{n}_\Gamma$, and also κ_Γ is the mean curvature of $\Gamma(t)$ with respect to $\mathbf{n}_\Gamma$. Analogously, $\mathbf{n}_0$ is the unit normal field on $\Gamma_0 = \mathbf{R}^N \setminus \Omega_0$.

The parameters $c_g \geq 0$ and $c_\sigma > 0$ denote the gravitational acceleration and the surface tension coefficient, respectively. Let $\rho_i > 0$ be the densities of *fluid*$_i$ for $i = 1, 2$, and then $\rho = \rho_1 \chi_{\Omega_1(t)} + \rho_2 \chi_{\Omega_2(t)}$. In addition, the quantity $[\![f]\!] = [\![f]\!](x,t)$ is the jump of the quantity f, defined on $\Omega(t)$, across the free boundary $\Gamma(t)$ as

$$[\![f]\!](x,t) = \lim_{\varepsilon\to 0+} \{f(x + \varepsilon\mathbf{n}_\Gamma, t) - f(x - \varepsilon\mathbf{n}_\Gamma, t)\} \quad \text{for } x \in \Gamma(t).$$

We here note that the stress tensor $\mathbf{T}$ is given by $\mathbf{T} = -\pi\mathbf{I} + \boldsymbol{\tau}$ for some shear stress $\boldsymbol{\tau}$ in general. In this chapter, we consider generalized Newtonian fluids, as a class of non-Newtonian fluids, such that the shear stress $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 \chi_{\Omega_1(t)} + \boldsymbol{\tau}_2 \chi_{\Omega_2(t)}, \quad \boldsymbol{\tau}_i = 2\mu_i(|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})$$

for $\mathbf{D}(\mathbf{v}) = (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)/2$ and scalar functions $\mu_1, \mu_2 : [0,\infty) \to \mathbf{R}$.

The problem then is to find the velocity field $\mathbf{v} = (v_1(x,t), \ldots, v_N(x,t))^T$ of the fluids, the pressure field $\pi = \pi(x,t)$, and the height function $h = h(x',t)$ satisfying equations (5.1.1).

In our main result, we show that system (5.1.1) admits a unique strong solution on $(0,T)$ for arbitrary $T > 0$, provided that the viscosity functions μ_1, μ_2 satisfy suitable conditions and the initial data are sufficiently small in their natural norms. More precisely, we have the following result.

THEOREM 5.1.1. *Let $N+2 < p < \infty$ and $J = (0,T)$ for some $T > 0$. Suppose that $\rho_1 > 0, \rho_2 > 0, c_g \geq 0, c_\sigma > 0$, and*

$$\mu_i \in C^3([0,\infty)), \quad \mu_i(0) > 0 \quad (i = 1,2).$$

Then there exists $\varepsilon_0 = \varepsilon_0(p,T) > 0$ such that for every

$$(\mathbf{v}_0, h_0) \in W_p^{2-2/p}(\Omega_0)^N \times W_p^{3-2/p}(\mathbf{R}^{N-1})$$

satisfying the smallness condition:

$$\|\mathbf{v}_0\|_{W_p^{2-2/p}(\Omega_0)} + \|h_0\|_{W_p^{3-2/p}(\mathbf{R}^{N-1})} < \varepsilon_0$$

as well as the compatibility conditions:

$$[\![\mu(|\mathbf{D}(\mathbf{v}_0)|^2)\mathbf{D}(\mathbf{v}_0)\mathbf{n}_0 - \big(\mathbf{n}_0 \cdot \mu(|\mathbf{D}(\mathbf{v}_0)|^2)\mathbf{D}(\mathbf{v}_0)\mathbf{n}_0\big)\,\mathbf{n}_0]\!] = 0 \quad \text{on } \Gamma_0,$$
$$\operatorname{div} \mathbf{v}_0 = 0 \quad \text{in } \Omega_0, \quad [\![\mathbf{v}_0]\!] = 0 \quad \text{on } \Gamma_0,$$

the system (5.1.1) admits a unique solution $(\mathbf{v}, \pi, h)$ within the regularity classes:

$$\begin{aligned}
&\mathbf{v} \in H_p^1(J, L_p(\Omega(\cdot)) \cap L_p(J, H_p^2(\Omega(\cdot)))^N,\\
&\pi \in L_p(J, \widehat{W}_p^1(\Omega(\cdot))),\\
&h \in W_p^{2-1/(2p)}(J, L_p(\mathbf{R}^{N-1})) \cap H_p^1(J, W_p^{2-1/p}(\mathbf{R}^{N-1}))\\
&\qquad \cap W_p^{1/2-1/(2p)}(J, H_p^2(\mathbf{R}^{N-1})) \cap L_p(J, W_p^{3-1/p}(\mathbf{R}^{N-1})).
\end{aligned}$$

REMARK 5.1.2. (1) Some remarks on notation are in order at this point. By $\mathbf{v} \in H_p^1(J, L_p(\Omega(\cdot))) \cap L_p(J, H_p^2(\Omega(\cdot)))^N$, we mean that

$$\Theta^*\mathbf{v} = \mathbf{v} \circ \Theta \in H_p^1(J, L_p(\dot{\mathbf{R}}^N)) \cap L_p(J, H_p^2(\dot{\mathbf{R}}^N))^N,$$

where Θ and Θ^* are defined in the following section by (5.2.2) and (5.2.3), respectively. The regularity statement for π is understood in the same way.

(2) The assumption $p > N+2$ implies that

$$h \in BUC(J, BUC^2(\mathbf{R}^{N-1})), \quad \partial_t h \in BUC(J, BUC^1(\mathbf{R}^{N-1})),$$

which means that the condition on the free boundary can be understood in the classical sense.

(3) Typical examples of viscosity functions μ satisfying our conditions are given by

$$\begin{aligned}
\mu(s) &= \nu s^{\frac{d-2}{2}} && \text{with } d = 2,4,6, \text{ or } d \geq 8,\\
\mu(s) &= \nu(1 + s^{\frac{d-2}{2}}) && \text{with } d = 2,4,6, \text{ or } d \geq 8,\\
\mu(s) &= \nu(1+s)^{\frac{d-2}{2}} && \text{with } 1 \leq d < \infty
\end{aligned}$$

for $\nu > 0$. For more information and details, we refer e.g. to the works of [**DR05**], [**MNR01**], and [**PR01**]. Obviously, if $d = 2$, then all viscosity functions above corresponds to the Newtonian situation.

Let us remark at this point that the original paper of this chapter is [**HS15**], and that our proof of Theorem 5.1.1 is inspired by the work due to Prüss and Simonett in [**PS10b**] and [**PS11**]. Its strategy may be described as follows: In Section 5.2 we transform (5.1.1) to a problem on a fixed domain. In Section 5.3, maximal regularity properties of the linearized problem will be investigated, hereby making use of the results [**PS10b**], [**PS11**] due to Prüss and Simonett. In Section 5.4, we introduce some function space $\widetilde{\mathbb{F}}_3(a)$ which plays an important role to control nonlinear terms arising from the boundary condition. Finally, in Section 5.5, we give a proof of our main theorem.

5.2. Reduction to a fixed domain problem

We start this section by calculating the shear stress $\boldsymbol{\tau}$, that is, by calculating explicitly $\mathrm{Div}\{\mu_d(|\mathbf{D}(\mathbf{u})|^2)\mathbf{D}(\mathbf{u})\}$ for $d = 1, 2$. We then obtain

$$\begin{aligned}
&\text{the } i\text{-th component of } \mathrm{Div}\{\mu_d(|\mathbf{D}(\mathbf{u})|^2)\mathbf{D}(\mathbf{u})\}\\
&= \frac{1}{2}\sum_{j,k,l=1}^{N}\{2\dot{\mu}_d(|\mathbf{D}(\mathbf{u})|^2)D_{ij}(\mathbf{u})D_{kl}(\mathbf{u}) + \mu_d(|\mathbf{D}(\mathbf{u})|^2)\delta_{ik}\delta_{jl}\}(\partial_j\partial_k u_l + \partial_j\partial_l u_k),
\end{aligned}$$

where $\partial_j = \partial/\partial x_j$ and $\dot{\varphi}(s) = (\partial\varphi/\partial s)(s)$ for scalar functions $\varphi(s)$ defined on $[0,\infty)$, and besides, $\ddot{\varphi}(s), \dddot{\varphi}(s), \dots$ are defined similarly. In view of this calculation, for vector functions $\mathbf{u}$ and $\mathbf{v}$, setting $\mathbf{A}_d(\mathbf{u})\mathbf{v} = (A_{d,1}(\mathbf{u})\mathbf{v}, \dots, A_{d,N}(\mathbf{u})\mathbf{v})^T$ such that

$$\begin{aligned}
A_{d,i}(\mathbf{u})\mathbf{v} &= -\sum_{j,k,l=1}^{N} A_{d,i}^{j,k,l}(\mathbf{D}(\mathbf{u}))(\partial_j\partial_k v_l + \partial_j\partial_l v_k),\\
A_{d,i}^{j,k,l}(\mathbf{D}(\mathbf{u})) &= \frac{1}{2}\Big(2\dot{\mu}_d(|\mathbf{D}(\mathbf{u})|^2)D_{ij}(\mathbf{u})D_{kl}(\mathbf{u}) + \mu_d(|\mathbf{D}(\mathbf{u})|^2)\delta_{ik}\delta_{jl}\Big)
\end{aligned}$$

with $d = 1, 2$ and $i, j, k, l = 1, \dots, N$, we see that

$$\mathbf{A}_d(\mathbf{u})\mathbf{u} = -\mathrm{Div}\{\mu_d(|\mathbf{D}(\mathbf{u})|^2)\mathbf{D}(\mathbf{u})\}, \quad \mathbf{A}_d(0)\mathbf{u} = -\mu_d(0)(\Delta\mathbf{u} + \nabla\,\mathrm{div}\,\mathbf{u}).$$

In addition, we set

$$\mathbf{A}(\mathbf{u})\mathbf{v} = \chi_{\Omega_1(t)}\mathbf{A}_1(\mathbf{u})\mathbf{v} + \chi_{\Omega_2(t)}\mathbf{A}_2(\mathbf{u})\mathbf{v}, \quad \widetilde{\pi} = \pi + \rho c_g x_N.$$

Thus the equations (5.1.1) may be rewritten as

$$\text{(5.2.1)}\quad \begin{cases}
\rho(\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \mu(0)\Delta\mathbf{v} + \nabla\widetilde{\pi} = -(\mathbf{A}(\mathbf{v}) - \mathbf{A}(0))\mathbf{v} & \text{in } \Omega(t),\ t > 0,\\
\mathrm{div}\,\mathbf{v} = 0 & \text{in } \Omega(t),\ t > 0,\\
-[\![\widetilde{\mathbf{T}}\mathbf{n}_\Gamma]\!] = \sigma\kappa_\Gamma\mathbf{n}_\Gamma + [\![\rho]\!]c_g h & \text{on } \Gamma(t),\ t > 0\\
[\![\mathbf{v}]\!] = 0 & \text{on } \Gamma(t),\ t > 0,\\
\partial_t h + \mathbf{v}'\cdot\nabla' h - \mathbf{v}\cdot\mathbf{e}_N = 0 & \text{on } \Gamma(t),\ t > 0,\\
\mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega_0,\\
h|_{t=0} = h_0 & \text{on } \mathbf{R}^{N-1},
\end{cases}$$

where $\widetilde{\mathbf{T}} = -\widetilde{\pi}\mathbf{I} + \boldsymbol{\tau}$ and $\mu(0) = \chi_{\Omega_1(t)}\mu_1(0) + \chi_{\Omega_2(t)}\mu_2(0)$.

We next transform the problem (5.2.1) to a problem on the fixed domain $\dot{\mathbf{R}}^N$. To this end, we define the transformation Θ on $\dot{\mathbf{R}}^N \times J$ for $J = (0, T)$ with $T > 0$

as follows:

$$\begin{aligned}\Theta : \dot{\mathbf{R}}^N \times J \ni (\xi', \xi_N, \tau) &\mapsto (x', x_N, t) \in \bigcup_{s\in J} \Omega(s) \times \{s\} \\ \text{with } x' = \xi',\ \ x_N &= \xi_N + h(\xi', t),\ \ t = \tau.\end{aligned} \tag{5.2.2}$$

Note that $\det J_\Theta = 1$, where J_Θ denotes the Jacobian matrix of Θ. Set

$$\begin{aligned}\mathbf{u}(\xi, \tau) &= \Theta^* \mathbf{v}(x, t) := \mathbf{v}(\Theta(\xi, \tau)), \\ \theta(\xi, \tau) &= \Theta^* \pi(x, t) := \pi(\Theta(\xi, \tau))\end{aligned} \tag{5.2.3}$$

as well as

$$\Theta_* f(\xi, \tau) := f(\Theta^{-1}(x, t)) \quad \text{for } f \text{ defined on } \dot{\mathbf{R}}^N \times J, \tag{5.2.4}$$

where Θ^{-1} is given by $\Theta^{-1}(x, t) = (x', x_N - h(x', t), t)$. This change of coordinates implies the relations:

$$\begin{aligned}&\partial_t = \partial_\tau - (\partial_\tau h) D_N,\ \ \partial_j = D_j - (D_j h) D_N,\ \ \partial_j \partial_k = D_j D_k - \mathcal{F}_{jk}(h), \\ &\mathcal{F}_{jk}(h) := (D_j D_k h) D_N + (D_j h) D_N D_k + (D_k h) D_j D_N - (D_j h)(D_k h) D_N^2\end{aligned} \tag{5.2.5}$$

for $j, k = 1, \ldots, N$, $\partial_\tau = \partial/\partial\tau$, and $D_j = \partial/\partial\xi_j$, because $D_N h = 0$. In a similar way to (5.2.5), we have

$$\begin{aligned}\mathbf{D}_x(\mathbf{v}) &= \mathbf{E}_\xi(\mathbf{u}, h) := \mathbf{D}_\xi(\mathbf{u}) - \boldsymbol{\mathcal{E}}_\xi(\mathbf{u}, h), \\ \boldsymbol{\mathcal{E}}_\xi(\mathbf{u}, h) &:= (D_N \mathbf{u}) \begin{bmatrix} \nabla'_\xi h \\ 0 \end{bmatrix}^T + \begin{bmatrix} \nabla'_\xi h \\ 0 \end{bmatrix} (D_N \mathbf{u})^T,\end{aligned} \tag{5.2.6}$$

where subscripts x and ξ denote their coordinates. Following [**PS10b**, Section 2] we see that

$$\begin{aligned}\kappa_\Gamma &= \sum_{j=1}^{N-1} D_j \left(\frac{\nabla'_{\xi'} h(t, \xi')}{\sqrt{1 + |\nabla'_{\xi'} h(t, \xi')|^2}} \right) = \Delta'_{\xi'} h - G_\kappa(h), \\ G_\kappa(h) &:= \frac{|\nabla'_{\xi'} h|^2 \Delta'_{\xi'} h}{\left(1 + \sqrt{1 + |\nabla'_{\xi'} h|^2}\right) \sqrt{1 + |\nabla'_{\xi'} h|^2}} + \sum_{j,k=1}^{N-1} \frac{(D_j h)(D_k h)(D_j D_k h)}{\left(1 + |\nabla'_{\xi'} h|^2\right)^{3/2}}.\end{aligned}$$

Hence the equations (5.2.1) are reduced to the following problem in $\dot{\mathbf{R}}^N$:

$$\left\{\begin{aligned}\rho \partial_\tau \mathbf{u} - \mu(0) \Delta \mathbf{u} + \nabla\theta &= \mathbf{F}(\mathbf{u}, \theta, h) && \text{in } \dot{\mathbf{R}}^N,\ t > 0, \\ \operatorname{div} \mathbf{u} &= F_d(\mathbf{u}, h) && \text{in } \dot{\mathbf{R}}^N,\ t > 0, \\ -[\![\mu(0)(D_N u_j + D_j u_N)]\!] &= G_j(\mathbf{u}, [\![\theta]\!], h) && \text{on } \mathbf{R}^N_0,\ t > 0, \\ [\![\theta]\!] - 2[\![\mu(0) D_N u_N]\!] - ([\![\rho]\!] c_g + \sigma\Delta') h &= G_N(\mathbf{u}, h) && \text{on } \mathbf{R}^N_0,\ t > 0, \\ [\![\mathbf{u}]\!] &= 0 && \text{on } \mathbf{R}^N_0,\ t > 0, \\ \partial_\tau h - \mathbf{u} \cdot \mathbf{e}_N &= G_h(\mathbf{u}, h) && \text{on } \mathbf{R}^N_0,\ t > 0, \\ \mathbf{u}|_{\tau=0} &= \mathbf{u}_0 && \text{on } \dot{\mathbf{R}}^N, \\ h|_{\tau=0} &= h_0 && \text{on } \mathbf{R}^{N-1}\end{aligned}\right. \tag{5.2.7}$$

for $j = 1, \dots, N-1$ and $\mathbf{F} = (F_1, \dots, F_N)^T$. Here, for $j = 1, \dots, N-1$ and $i = 1, \dots, N$, the right members of (5.2.7) are given by

$$\begin{aligned}
F_i(\mathbf{u}, \theta, h) =& \rho\{(\partial_\tau h) D_N u_i - (\mathbf{u} \cdot \nabla) u_i + (\mathbf{u}' \cdot \nabla' h) D_N u_i\} \\
& - \mu(0) \sum_{j=1}^N \mathcal{F}_{jj}(h) u_i + (D_i h) D_N \theta + \mathcal{A}_i(\mathbf{u}, h), \\
G_j(\mathbf{u}, [\![\theta]\!], h) =& c_\sigma G_\kappa(h) D_j h - \{([\![\rho]\!] c_g + \sigma \Delta') h\} D_j h + [\![\theta]\!] D_j h + \mathcal{B}_j(\mathbf{u}, h), \\
G_N(\mathbf{u}, h) =& - c_\sigma G_\kappa(h) + \mathcal{B}_N(\mathbf{u}, h), \\
F_d(\mathbf{u}, h) =& (D_N \mathbf{u}') \cdot \nabla' h = D_N(\mathbf{u}' \cdot \nabla' h), \\
G_h(\mathbf{u}, h) =& - \mathbf{u}' \cdot \nabla' h,
\end{aligned}$$

where $\mathcal{A}_i(\mathbf{u}, h)$, $\mathcal{B}_j(\mathbf{u}, h)$ and $\mathcal{B}_N(\mathbf{u}, h)$ are defined as

$$\begin{aligned}
\mathcal{A}_i(\mathbf{u}, h) =& \sum_{j,k,l=1}^N \left(A_i^{j,k,l}(\mathbf{E}(\mathbf{u}, h)) - A_i^{j,k,l}(0) \right) (D_j D_k u_l + D_j D_l u_k) \\
& - \sum_{j,k,l=1}^N \left(A_i^{j,k,l}(\mathbf{E}(\mathbf{u}, h)) - A_i^{j,k,l}(0) \right) (\mathcal{F}_{jk}(h) u_l + \mathcal{F}_{jl}(h) u_k), \\
\mathcal{B}_j(\mathbf{u}, h) =& - [\![\mu(|\mathbf{E}(\mathbf{u}, h)|^2) D_N u_N]\!] D_j h \\
& + [\![\{\mu(|\mathbf{E}(\mathbf{u}, h)|^2) - \mu(0)\}(D_N u_j + D_j u_N)]\!] \\
& - \sum_{k=1}^{N-1} [\![\mu(|\mathbf{E}(\mathbf{u}, h)|^2)(D_j u_k + D_k u_j)]\!] D_k h \\
& + \sum_{k=1}^{N-1} [\![\mu(|\mathbf{E}(\mathbf{u}, h)|^2)(D_N u_j D_k h + D_N u_k D_j h)]\!] D_k h, \\
\mathcal{B}_N(\mathbf{u}, h) =& 2[\![\{\mu(|\mathbf{E}(\mathbf{u}, h)|^2) - \mu(0)\} D_N u_N]\!] + [\![\mu(|\mathbf{E}(\mathbf{u}, h)|^2) D_N u_N]\!] |\nabla' h|^2 \\
& - \sum_{k=1}^{N-1} [\![\mu(|\mathbf{E}(\mathbf{u}, h)|^2)(D_N u_k + D_k u_N)]\!] D_k h
\end{aligned}$$

with

$$A_i^{j,k,l}(\mathbf{E}(\mathbf{u}, h)) = \chi_{\mathbf{R}_-^N} A_{i,1}^{j,k,l}(\mathbf{E}(\mathbf{u}, h)) + \chi_{\mathbf{R}_+^N} A_{i,2}^{j,k,l}(\mathbf{E}(\mathbf{u}, h)).$$

In particular, we note that

$$\begin{aligned}
&\mu(|\mathbf{E}(\mathbf{u}, h)|^2) = \chi_{\mathbf{R}_-^N} \mu_1(|\mathbf{E}(\mathbf{u}, h)|^2) + \chi_{\mathbf{R}_+^N} \mu_2(|\mathbf{E}(\mathbf{u}, h)|^2), \\
&\rho = \chi_{\mathbf{R}_-^N} \rho_1 + \chi_{\mathbf{R}_+^N} \rho_2, \quad \mu(0) = \chi_{\mathbf{R}_-^N} \mu_1(0) + \chi_{\mathbf{R}_+^N} \mu_2(0).
\end{aligned}$$

Finally, in order to simplify our notation, we set

$$\begin{aligned}
\mathbf{G}(\mathbf{u}, [\![\theta]\!], h) &= (G_1(\mathbf{u}, [\![\theta]\!], h), \dots, G_{N-1}(\mathbf{u}, [\![\theta]\!], h), G_N(\mathbf{u}, h))^T \\
\boldsymbol{\mathcal{A}}(\mathbf{u}, h) &= (\mathcal{A}_1(\mathbf{u}, h), \dots, \mathcal{A}_N(\mathbf{u}, h))^T, \\
\boldsymbol{\mathcal{B}}(\mathbf{u}, h) &= (\mathcal{B}_1(\mathbf{u}, h), \dots, \mathcal{B}_N(\mathbf{u}, h))^T.
\end{aligned}$$

5.3. Linearized problem

The set of equations (5.2.7) leads to the following linear problem whose optimal regularity properties will be of central importance below.

$$
(5.3.1)\quad \begin{cases}
\rho\partial_t\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\theta = \mathbf{f} & \text{in } \dot{\mathbf{R}}^N,\\
\operatorname{div}\mathbf{u} = f_d & \text{in } \dot{\mathbf{R}}^N,\\
-[\![\nu(D_N u_j + D_j u_N)]\!] = g_j & \text{on } \mathbf{R}^N_0,\\
[\![\theta]\!] - 2[\![\mu D_N u_N]\!] - ([\![\rho]\!]c_g + c_\sigma\Delta')h = g_N & \text{on } \mathbf{R}^N_0,\\
[\![\mathbf{u}]\!] = 0 & \text{on } \mathbf{R}^N_0,\\
\partial_t h - \mathbf{u}\cdot\mathbf{e}_N = g_h & \text{on } \mathbf{R}^N_0,\\
\mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \dot{\mathbf{R}}^N,\\
h|_{t=0} = h_0 & \text{on } \mathbf{R}^{N-1}
\end{cases}
$$

for $j = 1, \dots, N-1$, and set $\mathbf{g} = (g_1, \dots, g_N)^T$. Here,

$$\rho = \rho_1\chi_{\mathbf{R}^N_-} + \rho_2\chi_{\mathbf{R}^N_+}, \quad \nu = \nu_1\chi_{\mathbf{R}^N_-} + \nu_2\chi_{\mathbf{R}^N_+}$$

with $\rho_i > 0$ and $\nu_i > 0$ for $i = 1, 2$.

The following result due to Prüss and Simonett characterizes the set of data on the right-hand sides of (5.3.1) to obtain solutions of (5.3.1) in the maximal regularity space (cf. [**PS10b**, Theorem 5.1], [**PS11**, Theorem 3.1]).

PROPOSITION 5.3.1. *Let $1 < p < \infty$, $p \neq 3/2, 3$, and $a > 0$, and set $J = (0, a)$. Suppose that for $i = 1, 2$*

$$\rho_i > 0,\ \nu_i > 0,\ c_g \geq 0 \text{ and } c_\sigma > 0.$$

Then the equations (5.3.1) admits a unique solution $(\mathbf{u}, \theta, h)$ with the regularity

$$
\begin{aligned}
&\mathbf{u} \in (H^1_p(J, L_p(\dot{\mathbf{R}}^N)) \cap L_p(J, H^2_p(\dot{\mathbf{R}}^N)))^N,\\
&\theta \in L_p(J, \widehat{W}^1_p(\dot{\mathbf{R}}^N)),\\
&[\![\theta]\!] \in W^{1/2-1/(2p)}_p(J, L_p(\mathbf{R}^{N-1})) \cap L_p(J, W^{1-1/p}_p(\mathbf{R}^{N-1})),\\
&h \in W^{2-1/(2p)}_p(J, L_p(\mathbf{R}^{N-1})) \cap H^1_p(J, W^{2-1/(2p)}_p(\mathbf{R}^{N-1})) \cap L_p(J, W^{3-1/p}_p(\mathbf{R}^{N-1}))
\end{aligned}
$$

if and only if the data $(\mathbf{f}, f_d, \mathbf{g}, g_h, \mathbf{u}_0, h_0)$ satisfy the following regularity and compatibility conditions:

$$
\begin{aligned}
&\mathbf{f} \in L_p(J, L_p(\dot{\mathbf{R}}^N))^N,\\
&f_d \in H^1_p(J, \widehat{W}^{-1}_p(\mathbf{R}^N)) \cap L_p(J, H^1_p(\dot{\mathbf{R}}^N)),\\
&\mathbf{g} \in (W^{1/2-1/(2p)}_p(J, L_p(\mathbf{R}^{N-1})) \cap L_p(J, W^{1-1/p}_p(\mathbf{R}^{N-1})))^N,\\
&g_h \in W^{1-1/(2p)}_p(J, L_p(\mathbf{R}^{N-1})) \cap L_p(J, W^{2-1/p}_p(\mathbf{R}^{N-1})),\\
&\mathbf{u}_0 \in W^{2-2/p}_p(\dot{\mathbf{R}}^N)^N,\ h_0 \in W^{3-2/p}_p(\mathbf{R}^{N-1}),\\
&\operatorname{div}\mathbf{u}_0 = f_d(0) \text{ in } \dot{\mathbf{R}}^N,\ [\![\mathbf{u}_0]\!] = \mathbf{0} \text{ on } \mathbf{R}^{N-1} \text{ if } p > 3/2,\\
&-[\![\nu(D_N u_{0j} + D_j u_{0N})]\!] = g_j(0) \text{ on } \mathbf{R}^{N-1} \text{ if } p > 3
\end{aligned}
$$

for all $j = 1, \dots, N-1$. Moreover, the solution map $[(\mathbf{f}, f_d, \mathbf{g}, g_h, \mathbf{u}_0, h_0) \mapsto (\mathbf{u}, \theta, h)]$ is continuous between the corresponding spaces.

5.4. Properties of function spaces involved

In order to derive estimates for the nonlinear mappings occurring on the right-hand sides of (5.2.7), we first study embedding properties of the functions spaces involved. For $a > 0$, let $J = (0, a)$ and set

$$\begin{aligned}
\mathbb{E}_1(a) =& \{\mathbf{u} \in (H_p^1(J, L_p(\dot{\mathbf{R}}^N)) \cap L_p(J, H_p^2(\dot{\mathbf{R}}^N)))^N \mid [\![\mathbf{u}]\!] = 0\},\\
\mathbb{E}_2(a) =& L_p(J, \widehat{W}_p^1(\dot{\mathbf{R}}^N)),\\
\mathbb{E}_3(a) =& W_p^{1/2-1/(2p)}(J, L_p(\mathbf{R}^{N-1})) \cap L_p(J, W_p^{1-1/p}(\mathbf{R}^{N-1})),\\
\mathbb{E}_4(a) =& W_p^{2-1/(2p)}(J, L_p(\mathbf{R}^{N-1})) \cap H_p^1(J, W_p^{2-1/p}(\mathbf{R}^{N-1}))\\
& \cap W_p^{1/2-1/(2p)}(J, H_p^2(\mathbf{R}^{N-1})) \cap L_p(J, W_p^{3-1/p}(\mathbf{R}^{N-1}))
\end{aligned}$$

as well as

$$\begin{aligned}
\mathbb{F}_1(a) &= L_p(J, L_p(\dot{\mathbf{R}}^N))^N,\\
\mathbb{F}_2(a) &= H_p^1(J, \widehat{W}_p^{-1}(\mathbf{R}^N)) \cap L_p(J, H_p^1(\dot{\mathbf{R}}^N)),\\
\mathbb{F}_3(a) &= (W_p^{1/2-1/(2p)}(J, L_p(\mathbf{R}^{N-1})) \cap L_p(J, W_p^{1-1/p}(\mathbf{R}^{N-1})))^N,\\
\mathbb{F}_4(a) &= W_p^{1-1/(2p)}(J, L_p(\mathbf{R}^{N-1})) \cap L_p(J, W_p^{2-1/p}(\mathbf{R}^{N-1})).
\end{aligned}$$

We then have the following result (cf. [**PS10b**, Lemma 6.1] and [**MS12**, Proposition 3.2]).

Lemma 5.4.1. *Let $N + 2 < p < \infty$, $a > 0$, and $J = (0, a)$. Then the following properties hold.*

(1) $\mathbb{E}_3(a)$ *and* $\mathbb{F}_4(a)$ *are multiplication algebras.*
(2) $\mathbb{E}_1(a) \hookrightarrow (BUC(J, BUC^1(\dot{\mathbf{R}}^N)) \cap BUC(J, BUC(\mathbf{R}^N)))^N$, *and also* $\mathbb{E}_1(a) \hookrightarrow H_p^{1/2}(J, H_p^1(\dot{\mathbf{R}}^N))^N$.
(3) $\mathbb{E}_3(a) \hookrightarrow BUC(J, BUC(\mathbf{R}^{N-1}))$.
(4) $\mathbb{E}_4(a) \hookrightarrow BUC^1(J, BUC^1(\mathbf{R}^{N-1})) \cap BUC(J, BUC^2(\mathbf{R}^{N-1}))$.
(5) $W_p^{2-1/(2p)}(J, L_p(\mathbf{R}^{N-1})) \cap H_p^1(J, W_p^{2-1/p}(\mathbf{R}^{N-1})) \cap L_p(J, W_p^{3-1/p}(\mathbf{R}^{N-1})) \hookrightarrow \mathbb{E}_4(a)$.

A crucial point of our proof is the treatment of viscosity functions μ. To this end, we introduce, for $a > 0$, the function space $\widetilde{\mathbb{F}}_3(a)$ as

$$\widetilde{\mathbb{F}}_3(a) = \{g \in BUC(J, BUC(\mathbf{R}^{N-1})) \mid \|g\|_{\widetilde{\mathbb{F}}_3(a)} = \|g\|_{BUC(J, BUC(\mathbf{R}^{N-1}))} + |g|_{\mathbb{F}_3(a)} < \infty\},$$

where $|g|_{\mathbb{F}_3(a)} = |g|_{\mathbb{F}_3(a),1} + |g|_{\mathbb{F}_3(a),2}$ with

$$|g|_{\mathbb{F}_3(a),1} = \Big(\int_J \int_J \frac{\|g(t) - g(s)\|_{L_p(\mathbf{R}^{N-1})}^p}{|t - s|^{\frac{1}{2} + \frac{p}{2}}} \, dtds\Big)^{1/p} \quad \text{and}$$

$$|g|_{\mathbb{F}_3(a),2} = \Big(\int_J \int_{\mathbf{R}^{N-1}} \int_{\mathbf{R}^{N-1}} \frac{|g(x', t) - g(y', t)|^p}{|x' - y'|^{N-2+p}} \, dx'dy'dt\Big)^{1/p}.$$

We then obtain the following result.

Lemma 5.4.2. *Let $N + 2 < p < \infty$, $a > 0$, and $J = (0, a)$. Then the following properties hold true.*

(1) *$\mathbb{F}_3(a)$ and $\widetilde{\mathbb{F}}_3(a)$ are multiplication algebras. In addition,*

$$\mathbb{F}_3(a) \hookrightarrow BUC(J, BUC(\mathbf{R}^{N-1}))^N, \quad \widetilde{\mathbb{F}}_3(a) \hookrightarrow BUC(J, BUC(\mathbf{R}^{N-1})).$$

(2) *If $\varphi \in BUC^1(\mathbf{R})$ and $g \in \widetilde{\mathbb{F}}_3(a)$, then*

$$\|\varphi(g)\|_{\widetilde{\mathbb{F}}_3(a)} \leq \|\varphi\|_{BUC(\mathbf{R})} + \|\dot{\varphi}\|_{BUC(\mathbf{R})}|g|_{\mathbb{F}_3(a)}.$$

(3) *There exists a positive constant C such that for $f \in \mathbb{F}_3(a)$ and $g \in \widetilde{\mathbb{F}}_3(a)$*

$$\|fg\|_{\mathbb{F}_3(a)} \leq C\|f\|_{\mathbb{F}_3(a)}\|g\|_{\widetilde{\mathbb{F}}_3(a)}.$$

Proof. (1) The properties for $\mathbb{F}_3(a)$ is essentially given in Lemma 5.4.1 (1) and (3). The embedding $\widetilde{\mathbb{F}}_3(a) \hookrightarrow BUC(J, BUC(\mathbf{R}^{N-1}))$ follows from the definition of $\widetilde{\mathbb{F}}_3(a)$. We here only show that $\widetilde{\mathbb{F}}_3(a)$ is a multiplication algebra. For $f, g \in \widetilde{\mathbb{F}}_3(a)$, it follows that

$$\begin{aligned}\|fg\|_{BUC(J,BUC(\mathbf{R}^{N-1}))} &\leq \|f\|_{BUC(J,BUC(\mathbf{R}^{N-1}))}\|g\|_{BUC(J,BUC(\mathbf{R}^{N-1}))} \\ &\leq \|f\|_{\widetilde{\mathbb{F}}_3(a)}\|g\|_{\widetilde{\mathbb{F}}_3(a)}.\end{aligned}$$

Considering $|\cdot|_{\mathbb{F}_3(a),1}$, we see that

$$\begin{aligned}|fg|_{\mathbb{F}_3(a),1} \leq{}& \|f\|_{BUC(J,BUC(\mathbf{R}^{N-1}))}\left(\int_J\int_J \frac{\|g(t)-g(s)\|^p_{L_p(\mathbf{R}^{N-1})}}{|t-s|^{\frac{1}{2}+\frac{p}{2}}}\,dtds\right)^{1/p} \\ &+ \|g\|_{BUC(J,BUC(\mathbf{R}^{N-1}))}\left(\int_J\int_J \frac{\|f(t)-f(s)\|^p_{L_p(\mathbf{R}^{N-1})}}{|t-s|^{\frac{1}{2}+\frac{p}{2}}}\,dtds\right)^{1/p} \\ \leq{}& \|f\|_{\widetilde{\mathbb{F}}_3(a)}|g|_{\mathbb{F}_3(a),1} + \|g\|_{\widetilde{\mathbb{F}}_3(a)}|f|_{\mathbb{F}_3(a),1}.\end{aligned}$$

Similarly, $|fg|_{\mathbb{F}_3(a),2} \leq \|f\|_{\widetilde{\mathbb{F}}_3(a)}|g|_{\mathbb{F}_3(a),2} + \|g\|_{\widetilde{\mathbb{F}}_3(a)}|f|_{\mathbb{F}_3(a),2}$. These yield

$$\|fg\|_{\widetilde{\mathbb{F}}_3(a)} \leq C\|f\|_{\widetilde{\mathbb{F}}_3(a)}\|g\|_{\widetilde{\mathbb{F}}_3(a)}$$

for some positive constant C, which implies that $\widetilde{\mathbb{F}}_3(a)$ is a multiplication algebra.
(2) By the mean value theorem,

$$\begin{aligned}|\varphi(g)|_{\mathbb{F}_3(a)} ={}& \left(\int_J\int_J \frac{\|\varphi(g(t))-\varphi(g(s))\|^p_{L_p(\mathbf{R}^{N-1})}}{|t-s|^{\frac{1}{2}+\frac{p}{2}}}\,dtds\right)^{1/p} \\ &+ \left(\int_J\int_{\mathbf{R}^{N-1}}\int_{\mathbf{R}^{N-1}} \frac{|\varphi(g(x',t))-\varphi(g(y',t))|^p}{|x'-y'|^{N-2+p}}\,dx'dy'dt\right)^{1/p} \\ \leq{}& \|\dot{\varphi}\|_{BUC(\mathbf{R})}\left\{\left(\int_J\int_J \frac{\|g(t)-g(s)\|^p_{L_p(\mathbf{R}^{N-1})}}{|t-s|^{\frac{1}{2}+\frac{p}{2}}}\,dtds\right)^{1/p}\right. \\ &\left.+ \left(\int_J\int_{\mathbf{R}^{N-1}}\int_{\mathbf{R}^{N-1}} \frac{|g(x',t)-g(y',t)|^p}{|x'-y'|^{N-2+p}}\,dx'dy'dt\right)^{1/p}\right\} \\ \leq{}& \|\dot{\varphi}\|_{BUC(\mathbf{R})}|g|_{\mathbb{F}_3(a)},\end{aligned}$$

which furnishes that the required inequality holds.
(3) It obviously holds that

$$\begin{aligned}\|fg\|_{L_p(J,L_p(\mathbf{R}^{N-1}))} &\leq \|f\|_{L_p(J,L_p(\mathbf{R}^{N-1}))}\|g\|_{BUC(J,BUC(\mathbf{R}^{N-1}))} \\ &\leq \|f\|_{\mathbb{F}_3(a)}\|g\|_{\widetilde{\mathbb{F}}_3(a)}.\end{aligned}$$

On the other hand, we see, by calculations similar to (1), that there exists a constant $C > 0$ such that for $i = 1, 2$

$$\begin{aligned}|fg|_{\mathbb{F}_3(a),i} &\le \|f\|_{BUC(J,BUC(\mathbf{R}^{N-1}))}|g|_{\mathbb{F}_3(a),i} + \|g\|_{BUC(J,BUC(\mathbf{R}^{N-1}))}|f|_{\mathbb{F}_3(a),i}\\ &\le C\|f\|_{\mathbb{F}_3(a)}\|g\|_{\widetilde{\mathbb{F}}_3(a)},\end{aligned}$$

which, combined with the inequality above, completes the proof. □

We next recall basic properties of functions which are Fréchet differentiable. Let X and Y be Banach spaces and $U \subset X$ be open. We then denote the Fréchet derivative of a differentiable mapping $\Phi : U \to Y$ by $D\Phi : U \to \mathcal{L}(X, Y)$, and its evaluation of $u \in U$ and $v \in X$ by $[D\Phi(u)]v \in Y$. Moreover, a mapping $\Phi : U \to Y$ is called continuously Fréchet differentiable if and only if Φ is Fréchet differentiable on U and its Fréchet derivative $D\Phi$ is continuous on U. The set of such continuously Fréchet differentiable mappings from U to Y is denoted by $C^1(U, Y)$.

In the sequel, we will make use of the chain and product rule for Fréchet differentiable functions. Let Z be a further Banach space and suppose that the mappings $f : U \to Y$ and $g : Y \to Z$ are continuously Fréchet differentiable. Then the composition $F = g \circ f : U \to Z$ is also continuously Fréchet differentiable, and its evaluation of $x \in U$ and $\bar{x} \in X$ is given by

$$[DF(x)]\bar{x} = [Dg(f(x))][Df(x)]\bar{x}.$$

For the product rule, suppose that there exists a constant $M > 0$ such that for every $y \in Y$ and $z \in Z$

$$\|yz\|_Y \le M\|y\|_Y\|z\|_Z,$$

and also that $f : U \to Y$ and $g : U \to Z$ are continuously Fréchet differentiable. Set $F(x) = f(x)g(x)$ for $x \in U$. Then $F : U \to Y$ is also continuously Fréchet differentiable and its evaluation of $x \in U$ and $\bar{x} \in X$ is given by

$$[DF(x)]\bar{x} = g(x)[Df(x)]\bar{x} + f(x)[Dg(x)]\bar{x}.$$

We now define the solution space $\mathbb{E}(a)$ and the data space $\mathbb{F}(a)$ for $a > 0$ by

$$\begin{aligned}\mathbb{E}(a) &= \{(\mathbf{u}, \theta, \pi, h) \in \mathbb{E}_1(a) \times \mathbb{E}_2(a) \times \mathbb{E}_3(a) \times \mathbb{E}_4(a) \mid [\![\theta]\!] = \pi\},\\ \mathbb{F}(a) &= \mathbb{F}_1(a) \times \mathbb{F}_2(a) \times \mathbb{F}_3(a) \times \mathbb{F}_4(a).\end{aligned}$$

The spaces $\mathbb{E}(a)$ and $\mathbb{F}(a)$ are endowed with their natural norms, that is,

$$\begin{aligned}\|(\mathbf{u}, \theta, \pi, h)\|_{\mathbb{E}(a)} &= \|\mathbf{u}\|_{\mathbb{E}_1(a)} + \|\theta\|_{\mathbb{E}_2(a)} + \|\pi\|_{\mathbb{E}_3(a)} + \|h\|_{\mathbb{E}_4(a)},\\ \|(\mathbf{f}, f_d, \mathbf{g}, g_h)\|_{\mathbb{F}(a)} &= \|\mathbf{f}\|_{\mathbb{F}_1(a)} + \|f_d\|_{\mathbb{F}_2(a)} + \|\mathbf{g}\|_{\mathbb{F}_3(a)} + \|g_h\|_{\mathbb{F}_4(a)}.\end{aligned}$$

Finally, we consider for $(\mathbf{u}, \theta, \pi, h) \in \mathbb{E}(a)$ the nonlinear mapping $\mathbf{N}$ which is defined as

$$\mathbf{N}(\mathbf{u}, \theta, \pi, h) = (\mathbf{F}(\mathbf{u}, \theta, h), F_d(\mathbf{u}, h), \mathbf{G}(\mathbf{u}, \pi, h), G_h(\mathbf{u}, h)), \tag{5.4.1}$$

where the terms on the right-hand side are defined as in Section 5.2. For functions $\mathbf{u} = (u_1, \ldots, u_N)^T$ defined on $\dot{\mathbf{R}}^N$, we set

$$\begin{aligned}\mathbf{u}^1 &= (u_1^1, \ldots, u_N^1), \quad u_j^1 = \chi_{\mathbf{R}_-^N} u_j,\\ \mathbf{u}^2 &= (u_1^2, \ldots, u_N^2), \quad u_j^2 = \chi_{\mathbf{R}_+^N} u_j.\end{aligned} \tag{5.4.2}$$

Recalling the definition of $\mathbf{E}(\mathbf{u}, h)$ and $\boldsymbol{\mathcal{E}}(\mathbf{u}, h)$ in (5.2.6), we give, in the following lemma and its corollary, assertions on the Fréchet dfferentiability of certain functions occurring in the nonlinear mapping $\mathbf{N}$.

LEMMA 5.4.3. *Let $N+2<p<\infty$, $a>0$, and $J=(0,a)$. Then the following assertions hold.*

(1) *For $\varphi \in BUC^1(\mathbf{R})$, the mapping*

$$\varphi : BUC(J, BUC(\dot{\mathbf{R}}^N)) \to BUC(J, BUC(\dot{\mathbf{R}}^N))$$

is continuously Fréchet differentiable.

(2) *For $\psi \in BUC^3(\mathbf{R})$, the mapping*

$$\psi : \widetilde{\mathbb{F}}_3(a) \to \widetilde{\mathbb{F}}_3(a)$$

is continuously Fréchet differentiable.

(3) *Let $\varphi \in BUC^1(\mathbf{R})$ and set, by using (5.4.2),*

$$\Phi^d(\mathbf{u}, h) = \varphi(|\mathbf{E}(\mathbf{u}^d, h)|^2) \quad \text{for } d=1,2.$$

Then $\Phi^d : \mathbb{E}_1(a) \times \mathbb{E}_4(a) \to BUC(J, BUC(\dot{\mathbf{R}}^N))$ is continuously Fréchet differentiable.

(4) *Let $\psi \in BUC^3(\mathbf{R})$ and set, by using (5.4.2),*

$$\Psi^d(\mathbf{u}, h) = \psi(|\gamma_0 \mathbf{E}(\mathbf{u}^d, h)|^2) \quad \text{for } d=1,2,$$

where γ_0 denotes the trace to $\mathbf{R}^N_0$. Then $\Psi^d : \mathbb{E}_1(a) \times \mathbb{E}_4(a) \to \widetilde{\mathbb{F}}_3(a)$ is continuously Fréchet differentiable.

PROOF. (1) We first show that the mapping φ is Fréchet differentiable. To this end, let $f, \bar{f} \in Z = BUC(J, BUC(\dot{\mathbf{R}}^N))$. Then

$$\varphi(f+\bar{f}) - \varphi(f) - \dot{\varphi}(f)\bar{f} = \int_0^1 (\dot{\varphi}(f+\theta\bar{f}) - \dot{\varphi}(f))\, d\theta \bar{f},$$

which implies

$$\|\varphi(f+\bar{f}) - \varphi(f) - \dot{\varphi}(f)\bar{f}\|_Z / \|\bar{f}\|_Z \le \int_0^1 \|\dot{\varphi}(f+\theta\bar{f}) - \dot{\varphi}(f)\|_Z \, d\theta.$$

Since $\dot{\varphi} \in BUC(\mathbf{R})$, the term on the right-hand side above tends to 0 as $\|\bar{f}\|_Z \to 0$. Thus, $[D\varphi(f)]\bar{f} = \dot{\varphi}(f)\bar{f}$.

Next, we shall show the continuity of the Fréchet derivative at $f_0 \in Z$. For $h \in Z$, we have

$$\begin{aligned}
\|D\varphi(f_0+h) - D\varphi(f_0)\|_{\mathcal{L}(Z)} &= \sup_{\|f\|_Z=1} \|[D\varphi(f_0+h)]f - [D\varphi(f_0)]f\|_Z \\
&= \sup_{\|\bar{f}\|_Z=1} \|\dot{\varphi}(f_0+h)f - \dot{\varphi}(f_0)f\|_Z \\
&\le \|\dot{\varphi}(f_0+h) - \dot{\varphi}(f_0)\|_Z,
\end{aligned}$$

which tends to 0 as $\|h\|_Z \to 0$, since $\dot{\varphi} \in BUC(\mathbf{R})$.

(2) For $f, \bar{f} \in \widetilde{\mathbb{F}}_3(a)$, we obtain

$$\psi(f+\bar{f}) - \psi(f) - \dot{\psi}(f)\bar{f} = \int_0^1 (1-\theta)\ddot{\psi}(f+\theta\bar{f})\bar{f}\bar{f}\, d\theta.$$

By Lemma 5.4.2 (1) and (2),

$$\begin{aligned}
\|\ddot{\psi}(f+\theta\bar{f})\bar{f}\|_{\widetilde{\mathbb{F}}_3(a)} \le & C\{\|\ddot{\psi}\|_{BUC(\mathbb{R})} + \|\dddot{\psi}\|_{BUC(\mathbb{R})} |f+\theta\bar{f}|_{\mathbb{F}_3(a)}\} \|\bar{f}\|^2_{\widetilde{\mathbb{F}}_3(a)} \\
\le & C(1 + \|f\|_{\widetilde{\mathbb{F}}_3(a)} + \theta\|\bar{f}\|_{\widetilde{\mathbb{F}}_3(a)}) \|\bar{f}\|^2_{\widetilde{\mathbb{F}}_3(a)},
\end{aligned}$$

which implies that $[D\psi(f)]\bar{f} = \dot{\psi}(f)\bar{f}$.

Next, let $h \in \widetilde{\mathbb{F}}_3(a)$ to show the continuity of the Fréchet derivative at $f_0 \in \widetilde{\mathbb{F}}_3(a)$. Then

$$\begin{aligned}\|D\psi(f_0+h) - D\psi(f_0)\|_{\mathcal{L}(\widetilde{\mathbb{F}}_3(a))} = &\sup_{\|f\|_{\widetilde{\mathbb{F}}_3(a)}=1} \|[D\psi(f_0+h)]f - [D\psi(f_0)]f\|_{\widetilde{\mathbb{F}}_3(a)} \\ \leq &C\|\dot{\psi}(f_0+h) - \dot{\psi}(f_0)\|_{\widetilde{\mathbb{F}}_3(a)}\end{aligned}$$

by Lemma 5.4.2 (1). On the other hand, Taylor's formula and Lemma 5.4.2 (1) yield that

$$\|\dot{\psi}(f_0+h) - \dot{\psi}(f_0)\|_{\widetilde{\mathbb{F}}_3(a)} \leq C \int_0^1 \|\ddot{\psi}(f_0+\theta h)\|_{\widetilde{\mathbb{F}}_3(a)}\, d\theta \|h\|_{\widetilde{\mathbb{F}}_3(a)},$$

which tends to 0 as $\|h\|_{\widetilde{\mathbb{F}}_3(a)} \to 0$. This completes the proof.
(3) By Lemma 5.4.1 (2) and (4), the mappings

$$\begin{aligned}&(\mathbf{u}, h) \mapsto \mathbf{E}(\mathbf{u}^d, h) : \mathbb{E}_1(a) \times \mathbb{E}_4(a) \to Z^{N\times N} \\ &\text{as well as } x \mapsto |x|^2 : Z^{N\times N} \to Z,\end{aligned} \tag{5.4.3}$$

where $Z = BUC(J, BUC(\dot{\mathbf{R}}^N))$, are continuously Fréchet differentiable for $d = 1, 2$. The chain rule thus yields that for $d = 1, 2$

$$\mathbb{E}_1(a) \times \mathbb{E}_4(a) \to Z : (\mathbf{u}, h) \mapsto |\mathbf{E}(\mathbf{u}^d, h)|^2$$

is also continuously Fréchet differentiable. Applying the assertion (1) of this lemma and the chain rule again implies that $\Phi^d : \mathbb{E}_1(a)\times\mathbb{E}_4(a) \to Z$ is continuously Fréchet differentiable for $d = 1, 2$.
(4) Note that, for $(\mathbf{u}, h) \in \mathbb{E}_1(a) \times \mathbb{E}_4(a)$ and $i, j = 1, \ldots, N$, we have

$$\begin{aligned}\|(\gamma_0 D_i u_j^d, D_i h, D_i D_j h)\|_{\widetilde{\mathbb{F}}_3(a)} + \|(\gamma_0 D_i \mathbf{u}^d, \nabla h, D_i \nabla h)\|_{\mathbb{F}_3(a)}& \\ \leq C(a,p)\|(\mathbf{u}, h)\|_{\mathbb{E}_1(a)\times\mathbb{E}_4(a)}&\end{aligned} \tag{5.4.4}$$

for $d = 1, 2$ with some positive constant $C(a, p)$. In fact, Lemma 5.4.1 (2) and [**MS12**, Theorem 4.5] for $s = 1/2$, $m = 1$, and $\mu = 1$ yield that

$$\begin{aligned}&\|\gamma_0 D_i u_j^d\|_{\widetilde{\mathbb{F}}_3(a)} + \|\gamma_0 D_i \mathbf{u}^d\|_{\mathbb{F}_3(a)} \leq \|\gamma_0 D_i u_j^d\|_{BUC(J,BUC(\mathbf{R}^{N-1}))} + \|\gamma_0 D_i \mathbf{u}^d\|_{\mathbb{F}_3(a)} \\ &\leq C(a,p)(\|D_i u_j^d\|_{BUC(J,BUC(\mathbf{R}^N_+))} + \|D_i \mathbf{u}^d\|_{H_p^{1/2}(J,L_p(\mathbf{R}^N_+))\cap L_p(J,H_p^1(\mathbf{R}^N_+))}) \\ &\leq C(a,p)\|\mathbf{u}\|_{\mathbb{E}_1(a)},\end{aligned}$$

which furnishes the required properties of $\mathbf{u}$ in (5.4.4). Concerning h, the desired properties follow from the definition of $\mathbb{E}_4(a)$ and Lemma 5.4.1 (4). By (5.4.4) and Lemma 5.4.2 (1), the mappings

$$\begin{aligned}&(\mathbf{u}, h) \mapsto \gamma_0 \mathbf{E}(\mathbf{u}^d, h) : \mathbb{E}_1(a) \times \mathbb{E}_4(a) \to \widetilde{\mathbb{F}}_3(a)^{N\times N}, \\ &x \mapsto |x|^2 : \widetilde{\mathbb{F}}_3(a)^{N\times N} \to \widetilde{\mathbb{F}}_3(a)\end{aligned}$$

are continuously Fréchet differentiable. Thus, by the chain rule,

$$(\mathbf{u}, h) \mapsto |\gamma_0 \mathbf{E}(\mathbf{u}^d, h)|^2 : \mathbb{E}_1(a) \times \mathbb{E}_4(a) \to \widetilde{\mathbb{F}}_3(a)$$

is also continuously Fréchet differentiable. Together with the assertion (2) of this lemma, the chain rule yields the required property. □

Corollary 5.4.4. *Let $N + 2 < p < \infty$, $a > 0$, and $J = (0, a)$. Then the following assertions hold.*

(1) *Let* $\varphi \in BUC^1(\mathbf{R})$*,* $i,j,k,l,m,q,r = 1,\dots,N$*, and* $d = 1,2$*. By using* (5.2.5) *and* (5.4.2)*, we set*

$$\begin{aligned}\Phi^d_{ijklmqr}(\mathbf{u},h) &= \varphi(|\mathbf{E}(\mathbf{u}^d,h)|^2)E_{ij}(\mathbf{u}^d,h)E_{kl}(\mathbf{u}^d,h)D_mD_qu^d_r,\\ \Lambda^d_{ijklpqr}(\mathbf{u},h) &= \varphi(|\mathbf{E}(\mathbf{u}^d,h)|^2)E_{ij}(\mathbf{u}^d,h)E_{kl}(\mathbf{u}^d,h)\mathcal{F}_{mq}(h)u^d_r,\\ \Phi^d_{ijk}(\mathbf{u},h) &= (\varphi(|\mathbf{E}(\mathbf{u}^d,h)|^2)-\varphi(0))D_iD_ju^d_k,\\ \Lambda^d_{ijk}(\mathbf{u},h) &= (\varphi(|\mathbf{E}(\mathbf{u}^d,h)|^2)-\varphi(0))\mathcal{F}_{ij}(h)u^d_k.\end{aligned}$$

Then the mappings

$$\Phi^d_{ijklmqr}, \Lambda^d_{ijklmqr}, \Phi^d_{ijk}, \Lambda^d_{ijk} : \mathbb{E}_1(a)\times\mathbb{E}_4(a) \to L_p(J, L_p(\dot{\mathbf{R}}^N))$$

are continuously Fréchet differentiable. Moreover, their values and their Fréchet derivatives at $(\mathbf{u},h) = (0,0)$ *vanish.*

(2) *Let* $\psi \in BUC^3(\mathbf{R})$*,* $i,j,k = 1,\dots,N$*, and* $d = 1,2$*. By using* (5.4.2)*, we set*

$$\begin{aligned}\Psi^d_i(\mathbf{u},h) &= \{\psi(|\gamma_0\mathbf{E}(\mathbf{u}^d,h)|^2)-\psi(0)\}\gamma_0D_i\mathbf{u}^d,\\ \Theta^d_{ij}(\mathbf{u},h) &= \psi(|\gamma_0\mathbf{E}(\mathbf{u}^d,h)|^2)(\gamma_0D_i\mathbf{u}^d)D_jh,\\ \Xi^d_{ijk}(\mathbf{u},h) &= \psi(|\gamma_0\mathbf{E}(\mathbf{u}^d,h)|^2)(\gamma_0D_i\mathbf{u}^d)D_jhD_kh.\end{aligned}$$

Then the mappings

$$\Psi^d_i, \Theta^d_{ij}, \Xi^d_{ijk} : \mathbb{E}_1(a)\times\mathbb{E}_4(a)\to\mathbb{F}_3(a)$$

are continuously Frécht differentiable. In addition, their values and Fréchet derivatives at $(\mathbf{u},h) = (0,0)$ *vanish.*

Proof. (1) We only prove the case of $\Phi^d_{ijklmqr}$ here. The other cases may then be proved similarly. By (5.4.3), Lemma 5.4.3 (3), and the product rule,

$$(\mathbf{u},h)\mapsto\varphi(|\mathbf{E}(\mathbf{u}^d,h)|^2)E_{ij}(\mathbf{u}^d,h)E_{kl}(\mathbf{u}^d,h) : \mathbb{E}_1(a)\times\mathbb{E}_4(a)\to BUC(J, BUC(\dot{\mathbf{R}}^N))$$

is continuously Fréchet differentiable. Moreover,

$$\mathbf{u}\mapsto D_mD_qu^d_r : \mathbb{E}_1(a)\to L_p(J, L_p(\dot{\mathbf{R}}^N))$$

is continuously Fréchet differentiable, which, combined with the above assertion and the product rule with $X = \mathbb{E}_1(a)\times\mathbb{E}_4(a)$, $Y = L_p(J, L_p(\dot{\mathbf{R}}^N))$, and $Z = BUC(J, BUC(\dot{\mathbf{R}}^N))$, furnishes that $\Phi^d_{ijklmqr} : \mathbb{E}_1(a)\times\mathbb{E}_1(a)\to L_p(J, L_p(\dot{\mathbf{R}}^N))$ is continuously Fréchet differentiable. In addition, it is clear that $\Phi^d_{ijklmqr}(0,0) = 0$ and $D\Phi^d_{ijklmqr}(0,0) = 0$.

(2) We only consider the case of Ψ^d_i here. Since

$$\mathbf{u}\mapsto\gamma_0D_i\mathbf{u}^d : \mathbb{E}_1(a)\to\mathbb{F}_3(a) \text{ or } \widetilde{\mathbb{F}}_3(a) \tag{5.4.5}$$

is continuously Fréchet differentiable, it follows from (5.4.4) that

$$\mathbf{u}\mapsto\psi(0)\gamma_0D_i\mathbf{u}^d : \mathbb{E}_1(a)\to\mathbb{F}_3(a)$$

is also continuously Fréchet differentiable. On the other hand, by Lemma 5.4.3 (4), (5.4.5), and the product rule applied to $X = \mathbb{E}_1(a)\times\mathbb{E}_4(a)$, $Y = \mathbb{F}_3(a)$, and $Z = \widetilde{\mathbb{F}}_3(a)$, the mapping

$$(\mathbf{u},h)\mapsto\psi(|\gamma_0\mathbf{E}(\mathbf{u}^d,h)|^2)\gamma_0D_i\mathbf{u}^d : \mathbb{E}_1(a)\times\mathbb{E}_4(a)\to\mathbb{F}_3(a)$$

is continuously Fréchet differentiable. Finally, it clearly holds that $\Psi_i^d(0,0) = 0$, and also that, thanks to the product rule,

$$
\begin{aligned}
&D\Psi_i^d(\mathbf{u}, h) \\
&= \gamma_0 D_i \mathbf{u}^d [D(\psi(|\gamma_0 \mathbf{E}(\mathbf{u}^d, h)|^2) - \psi(0))] + (\psi(|\gamma_0 \mathbf{E}(\mathbf{u}^d, h)|^2) - \psi(0))[D(\gamma_0 D_i \mathbf{u}^d)],
\end{aligned}
$$

which furnishes that $D\Psi_i^d(0,0) = 0$. The proof is complete. □

5.5. Nonlinear problem

We start this section by examining properties of the nonlinear mapping $\mathbf{N}$ defined as (5.4.1).

LEMMA 5.5.1. *Let $N + 2 < p < \infty$, $a > 0$, and $r > 0$. Suppose that $\mu_d \in C^3([0,\infty))$ for $d = 1, 2$, and also that $\rho_1 > 0$, $\rho_2 > 0$, $c_g \geq 0$, and $c_\sigma > 0$. Then there hold*

$$
\mathbf{N} \in C^1(B_{\mathbb{E}(a)}(r), \mathbb{F}(a)), \quad \mathbf{N}(0) = 0, \quad \text{and } D\mathbf{N}(0) = 0.
$$

PROOF. We here treat in detail the terms $\boldsymbol{\mathcal{A}}(\mathbf{u}, h)$ and $\boldsymbol{\mathcal{B}}(\mathbf{u}, h)$, which are defined as in Section 5.2. The remaining terms were investigated in [**PS10b**, Proposition 6.2] and [**PS11**, Proposition 4.1].

Case $\boldsymbol{\mathcal{A}}(\mathbf{u}, h)$ Let $(\mathbf{u}, \theta, \pi, h) \in B_{\mathbb{E}(a)}(r)$ and recall, for $i = 1, \ldots, N$, that $\mathcal{A}_i(\mathbf{u}, h)$ is given by

$$
\begin{aligned}
\mathcal{A}_i(\mathbf{u}, h) = &\sum_{j,k,l=1}^{N} (A_i^{j,k,l}(\mathbf{E}(\mathbf{u}, h)) - A_i^{j,k,l}(0))(D_j D_k u_l + D_j D_l u_k) \\
&+ \sum_{j,k,l=1}^{N} (A_i^{j,k,l}(\mathbf{E}(\mathbf{u}, h)) - A_i^{j,k,l}(0))(\mathcal{F}_{jk}(h) u_l + \mathcal{F}_{jl}(h) u_k),
\end{aligned}
$$

where $\mathcal{F}_{jk}(h)$ are defined as in (5.2.5), and also $A_i^{j,k,l}(\mathbf{E}(\mathbf{u}, h))$ as in Section 5.2 by

$$
\begin{aligned}
A_i^{j,k,l}(\mathbf{E}(\mathbf{u}, h)) &= \chi_{\mathbf{R}_-^N} A_{1,i}^{j,k,l}(\mathbf{E}(\mathbf{u}, h)) + \chi_{\mathbf{R}_+^N} A_{2,i}^{j,k,l}(\mathbf{E}(\mathbf{u}, h)), \\
A_{d,i}^{j,k,l}(\mathbf{E}(\mathbf{u}, h)) &= \frac{1}{2}\Big(2\dot{\mu}_d(|\mathbf{E}(\mathbf{u}, h)|^2) E_{ij}(\mathbf{u}, h) E_{kl}(\mathbf{u}, h) + \mu_d(|\mathbf{E}(\mathbf{u}, h)|^2)\delta_{ik}\delta_{jl}\Big)
\end{aligned}
$$

with $d = 1, 2$. Using the notation introduced in (5.4.2), we see that $A_i^{j,k,l}(\mathbf{E}(\mathbf{u}, h))$ may be represented as

$$
\begin{aligned}
&A_i^{j,k,l}(\mathbf{E}(\mathbf{u}, h)) \\
&\quad = \frac{1}{2}\sum_{d=1}^{2} \Big(2\dot{\mu}_d(|\mathbf{E}(\mathbf{u}^d, h)|^2) E_{ij}(\mathbf{u}^d, h) E_{kl}(\mathbf{u}^d, h) + \mu_d(|\mathbf{E}(\mathbf{u}^d, h)|^2)\delta_{ik}\delta_{jl}\Big),
\end{aligned}
$$

and thus

$$\begin{aligned}\mathcal{A}_i(\mathbf{u},h) =&\sum_{d=1}^{2}\sum_{j,k,l=1}^{N}\dot{\mu}_d(|\mathbf{E}(\mathbf{u}^d,h)|^2)E_{ij}(\mathbf{u}^d,h)E_{kl}(\mathbf{u}^d,h)(D_jD_ku_l^d+D_jD_lu_k^d)\\&+\frac{1}{2}\sum_{d=1}^{2}\sum_{j,k,l=1}^{N}(\mu_d(|\mathbf{E}(\mathbf{u}^d,h)|^2)-\mu_d(0))\delta_{ik}\delta_{jl}(D_jD_ku_l^d+D_jD_lu_k^d)\\&+\sum_{d=1}^{2}\sum_{j,k,l=1}^{N}\dot{\mu}_d(|\mathbf{E}(\mathbf{u}^d,h)|^2)E_{ij}(\mathbf{u}^d,h)E_{kl}(\mathbf{u}^d,h)(\mathcal{F}_{jk}(h)u_l^d+\mathcal{F}_{jl}(h)u_k^d)\\&+\frac{1}{2}\sum_{d=1}^{2}\sum_{j,k,l=1}^{N}(\mu_d(|\mathbf{E}(\mathbf{u}^d,h)|^2)-\mu_d(0))\delta_{ik}\delta_{jl}(\mathcal{F}_{jk}(h)u_l^d+\mathcal{F}_{jl}(h)u_k^d)\end{aligned}$$

for $i=1,\dots,N$. By Corollary 5.4.4 (1),

$$\boldsymbol{\mathcal{A}}\in C^1(B_{\mathbb{E}(a)}(r),\mathbb{F}_1(a)),\quad \boldsymbol{\mathcal{A}}(0,0)=0,\quad\text{and } D\boldsymbol{\mathcal{A}}(0,0)=0.$$

Case $\boldsymbol{\mathcal{B}}(\mathbf{u},h)$ Let $(\mathbf{u},\theta,\pi,h)\in B_{\mathbb{E}(a)}(r)$. By Lemma 5.4.1 (2) and (4), each term being in $\boldsymbol{\mathcal{B}}(\mathbf{u},h)$ is continuous with respect to the space variable x. In particular, this furnishes that

$$\gamma_0\dot{\mu}_d(|\mathbf{E}(\mathbf{u}^d,h)|^2))=\dot{\mu}_d(|\gamma_0\mathbf{E}(\mathbf{u}^d,h)|^2)),\quad \gamma_0\{(D_Nu_N^d)(\partial_jh)\}=(\gamma_0D_Nu_N^d)(\gamma_0D_jh).$$

Thus, $\boldsymbol{\mathcal{B}}(\mathbf{u},h)=(\mathcal{B}_1(\mathbf{u},h),\dots,\mathcal{B}_N(\mathbf{u},h))^T$ may be rewritten as

$$\begin{aligned}\mathcal{B}_j(\mathbf{u},h)=&-\sum_{d=1}^{2}(-1)^d\mu_d(|\gamma_0\mathbf{E}(\mathbf{u}^d,h)|^2)(\gamma_0D_Nu_N^d)D_jh\\&+\sum_{d=1}^{2}(-1)^d\left(\mu_d(|\gamma_0\mathbf{E}(\mathbf{u}^d,h)|^2)-\mu_d(0)\right)\left(\gamma_0D_Nu_j^d+\gamma_0D_ju_N^d\right)\\&-\sum_{d=1}^{2}\sum_{k=1}^{N-1}(-1)^d\mu_d(|\gamma_0\mathbf{E}(\mathbf{u}^d,h)|^2)\left(\gamma_0D_ju_k^d+\gamma_0D_ku_j^d\right)\partial_kh\\&+\sum_{d=1}^{2}\sum_{k=1}^{N-1}(-1)^d\mu_d(|\gamma_0\mathbf{E}(\mathbf{u}^d,h)|^2)\left(D_kh\gamma_0D_Nu_j^d+D_jh\gamma_0D_Nu_k^d\right)D_kh,\\ \mathcal{B}_N(\mathbf{u},h)=&2\sum_{d=1}^{2}(-1)^d\left(\mu_d(|\gamma_0\mathbf{E}(\mathbf{u}^d,h)|^2)-\mu_d(0)\right)\gamma_0D_Nu_N^d\\&+\sum_{d=1}^{2}\sum_{k=1}^{N-1}(-1)^d\mu_d(|\gamma_0\mathbf{E}(\mathbf{u}^d,h)|^2)(D_kh)^2\gamma_0D_Nu_N^d\\&-\sum_{d=1}^{2}\sum_{k=1}^{N-1}(-1)^d\mu_d(|\gamma_0\mathbf{E}(\mathbf{u}^d,h)|^2)\left(\gamma_0D_Nu_k^d+\gamma_0D_ku_N^d\right)D_kh.\end{aligned}$$

These representation and Corollary 5.4.4 (2) combined yield that

$$\boldsymbol{\mathcal{B}}\in C^1(B_{\mathbb{E}(a)}(r),\mathbb{F}_3(a)),\quad \boldsymbol{\mathcal{B}}(0,0)=0,\quad\text{and } D\boldsymbol{\mathcal{B}}(0,0)=0,$$

which completes the proof of the lemma. □

Finally, we return to the nonlinear problem (5.2.7). We define the function space $\mathbb{I}$ of initial data by

$$\mathbb{I} = W_p^{2-2/p}(\dot{\mathbf{R}}^N)^N \times W_p^{3-2/p}(\mathbf{R}^{N-1}),$$

and also define, for $\mathbf{z} \in \mathbb{E}(a)$ and $(\mathbf{u}_0, h_0) \in \mathbb{I}$, the mapping Φ by

$$\Phi(z) = L^{-1}(\mathbf{N}(z), \mathbf{u}_0, h_0).$$

Here the linear operator L is given by the left-hand side of the linear problem (5.3.1) with $\nu = \mu(0)$. Observe that the invertiblility of L is guaranteed by Proposition 5.3.1, because $\mu_i(0) > 0$ for $i = 1, 2$ and $\mathbf{N}(z) \in \mathbb{F}(a)$ for $\mathbf{z} \in \mathbb{E}(a)$ by Lemma 5.5.1. The following result shows that the problem (5.2.7) on the fixed domain $\dot{\mathbf{R}}^N$ admits a unique strong solution, provided that the initial data $\mathbf{u}_0, h_0$ are sufficiently small in their corresponding norms.

Proposition 5.5.2. *Let $N + 2 < p < \infty$ and $a > 0$. Suppose that $\rho_1 > 0, \rho_2 > 0, c_g \geq 0$, and $c_\sigma > 0$, and also that*

$$\mu_i(0) > 0, \quad \mu_i \in C^3([0, \infty)) \quad (i = 1, 2).$$

Then there exist positive constants ε_0 and δ_0, which depend on a and p, such that the equations (5.2.7) admits a unique solution $(\mathbf{u}, \theta, h)$ in $B_{\mathbb{E}(a)}(\delta_0)$, provided that the initial data $(\mathbf{u}_0, h_0) \in \mathbb{I}$ satisfies the compatibility conditions:

(5.5.1)

$$\begin{aligned}
&[\![\mu(|\mathbf{E}(\mathbf{u}_0, h_0)|^2)\mathbf{E}(\mathbf{u}_0, h_0)\mathbf{n}_0 - \{\mathbf{n}_0 \cdot \mu(|\mathbf{E}(\mathbf{u}_0, h_0)|^2)\mathbf{E}(\mathbf{u}_0, h_0)\mathbf{n}_0\}\mathbf{n}_0]\!] = 0 \quad \text{on } \mathbf{R}_0^N,\\
&[\![\theta_0]\!] = [\![\mathbf{n}_0 \cdot \mu(|\mathbf{E}(\mathbf{u}_0, h_0)|^2)\mathbf{E}(\mathbf{u}_0, h_0)\mathbf{n}_0]\!] + ([\![\rho]\!]c_g + c_\sigma\Delta')h_0 - c_\sigma G_\kappa(h_0) \quad \text{on } \mathbf{R}_0^N,\\
&\operatorname{div}\mathbf{u}_0 = F_d(\mathbf{u}_0, h_0) \quad \text{in } \dot{\mathbf{R}}^N, \quad [\![\mathbf{u}_0]\!] = 0 \quad \text{on } \mathbf{R}_0^N
\end{aligned}$$

as well as the smallness condition: $\|(\mathbf{u}_0, h_0)\|_{\mathbb{I}} < \varepsilon_0$.

Remark 5.5.3. The compatibility conditions (5.5.1) are equivalent to

$$\begin{aligned}
-[\![\mu(0)(\partial_N u_j + \partial_j u_N)]\!] &= G_j(\mathbf{u}_0, [\![\theta_0]\!], h_0) \quad \text{on } \mathbf{R}_0^N,\\
[\![\theta_0]\!] - 2[\![\mu(0)\partial_N u_{0N}]\!] - ([\![\rho]\!]\gamma_a + \sigma\Delta')h_0 &= G_N(\mathbf{u}_0, h_0) \quad \text{on } \mathbf{R}_0^N,\\
\operatorname{div}\mathbf{u}_0 = F_d(\mathbf{u}_0, h_0) \quad \text{in } \dot{\mathbf{R}}^N, &\quad [\![\mathbf{u}_0]\!] = 0 \quad \text{on } \mathbf{R}_0^N,
\end{aligned}$$

where $j = 1, \ldots, N-1$ and $\mathbf{G} = (G_1, \ldots, G_N)^T$ defined as in Section 5.2.

Proof. Observe that $D\mathbf{N}$ is continuous and $D\mathbf{N}(0) = 0$ by Lemma 5.5.1. Thus we may choose $\delta_0 > 0$ small enough such that

$$\sup_{\mathbf{z} \in B_{\mathbb{E}(a)}(2\delta_0)} \|D\mathbf{N}(\mathbf{z})\|_{\mathcal{L}(B_{\mathbb{E}(a)}(r), \mathbb{F}(a))} \leq \frac{1}{2\|L^{-1}\|_{\mathcal{L}(\mathbb{F}(a) \times \mathbb{I}, \mathbb{E}(a))}},$$

where $r > 0$ is a sufficiently large number.

For $\mathbf{z} \in B_{\mathbb{E}(a)}(\delta_0)$, the mean value theorem implies that

$$\begin{aligned}
&\|\Phi(\mathbf{z})\|_{\mathbb{E}(a)} \\
&\leq \|L^{-1}\|_{\mathcal{L}(\mathbb{F}(a)\times\mathbb{I},\mathbb{E}(a))} \left(\|\mathbf{N}(\mathbf{z})\|_{\mathbb{F}(a)} + \|(\mathbf{u}_0, h_0)\|_{\mathbb{I}}\right) \\
&= \|L^{-1}\|_{\mathcal{L}(\mathbb{F}(a)\times\mathbb{I},\mathbb{E}(a))} \left(\|\mathbf{N}(\mathbf{z}) - \mathbf{N}(0)\|_{\mathbb{F}(a)} + \|(\mathbf{u}_0, h_0)\|_{\mathbb{I}}\right) \\
&\leq \|L^{-1}\|_{\mathcal{L}(\mathbb{F}(a)\times\mathbb{I},\mathbb{E}(a))} \left\{ \left(\sup_{\bar{\mathbf{z}} \in B_{\mathbb{E}(a)}(\delta_0)} \|D\mathbf{N}(\bar{\mathbf{z}})\|_{\mathcal{L}(B_{\mathbb{E}(a)}(r),\mathbb{F}(a))} \right) \|\mathbf{z}\|_{\mathbb{E}(a)} + \varepsilon_0 \right\} \\
&< \frac{\delta_0}{2} + \varepsilon_0 \|L^{-1}\|_{\mathcal{L}(\mathbb{F}(a)\times\mathbb{I},\mathbb{E}(a))}.
\end{aligned}$$

Choosing ε_0 in such a way that $0 < \varepsilon_0 < \delta_0/(2\|L^{-1}\|_{\mathcal{L}(\mathbb{F}(a)\times\mathbb{I},\mathbb{E}(a))})$, we obtain $\|\Phi(\mathbf{z})\|_{\mathbb{E}(a)} < \delta_0$. Hence, Φ is a mapping from $B_{\mathbb{E}(a)}(\delta_0)$ into itself.

Let $\mathbf{z}_1, \mathbf{z}_2 \in B_{\mathbb{E}(a)}(\delta_0)$. Noting that $\Phi(\mathbf{z}_1) - \Phi(\mathbf{z}_2) = L^{-1}(\mathbf{N}(\mathbf{z}_1) - \mathbf{N}(\mathbf{z}_2), 0, 0)$, we obtain, by the mean value theorem,

$$\begin{aligned}
&\|\Phi(z_1) - \Phi(z_2)\|_{\mathbb{E}(a)} \\
&\leq \|L^{-1}\|_{\mathcal{L}(\mathbb{F}(a)\times\mathbb{I},\mathbb{E}(a))} \|\mathbf{N}(\mathbf{z}_1) - \mathbf{N}(\mathbf{z}_2)\|_{\mathbb{F}(a)} \\
&\leq \|L^{-1}\|_{\mathcal{L}(\mathbb{F}(a)\times\mathbb{I},\mathbb{E}(a))} \left(\sup_{\bar{\mathbf{z}} \in B_{\mathbb{E}(a)}(2\delta_0)} \|D\mathbf{N}(\bar{\mathbf{z}})\|_{\mathcal{L}(B_{\mathbb{E}(a)}(r),\mathbb{F}(a))} \right) \|\mathbf{z}_1 - \mathbf{z}_2\|_{\mathbb{E}(a)} \\
&\leq \frac{1}{2} \|\mathbf{z}_1 - \mathbf{z}_2\|_{\mathbb{E}(a)},
\end{aligned}$$

which furnishes that Φ is a contraction mapping on $B_{\mathbb{E}(a)}(\delta_0)$. By the contraction principle, we obtain a unique solution of the equations (5.2.7) in $B_{\mathbb{E}(a)}(\delta_0)$ □

Proof of Theorem 5.1.1. Note that the compatibility conditions of Theorem 5.1.1 are satisfied if and only if (5.5.1) holds. For $h_0 \in W_p^{3-2/p}(\mathbf{R}^{N-1})$, the mapping Θ_{h_0} given by

$$\Theta_{h_0}(\xi', \xi_N) = (\xi', \xi_N + h_0(\xi')) \text{ for } (\xi', \xi_N) \in \dot{\mathbf{R}}^N$$

defines a C^2-diffeomorphism, from $\dot{\mathbf{R}}^N$ onto Ω_0, with the inverse $\Theta_{h_0}^{-1}(x', x_N) = (x', x_N - h_0(x'))$. Thus there exists a positive constant $C(h_0)$ such that

$$C(h_0)^{-1} \|\mathbf{v}_0\|_{W_p^{2-2/p}(\Omega_0)} \leq \|\mathbf{u}_0\|_{W_p^{2-2/p}(\dot{\mathbf{R}}^N)} \leq C(h_0) \|\mathbf{v}_0\|_{W_p^{2-2/p}(\Omega_0)}.$$

Hence the smallness condition in Theorem 5.1.1 implies the smallness condition in Proposition 5.5.2. Proposition 5.5.2 then yields a unique solution $(\mathbf{u}, \theta, h) \in B_{\mathbb{E}(a)}(\delta_0)$ of the equations (5.2.7). Finally, setting

$$(\mathbf{v}, \pi) = (\Theta_* \mathbf{u}, \Theta_* \theta) = (\mathbf{u} \circ \Theta^{-1}, \theta \circ \Theta^{-1}),$$

where Θ_* is defined as (5.2.4), we obtain a unique solution $(\mathbf{v}, \pi, h)$ of the original problem (5.1.1) with the regularities stated in Theorem 5.1.1. The proof is complete.

Appendix

A

Our aim here is to show Lemma 2.6.2 and Lemma 2.6.3. By using (2.4.16), we see that $\widetilde{L}_{k,2\ell}$ $(k = 1, 2, 3, 4, \ell = 1, 2)$, defined as (2.4.14), are given by

$$
\begin{aligned}
\widetilde{L}_{1,2} =& 2B\{-(A + Be^{-2A\delta}) - (B^2 - 2AB - A^2)e^{-A\delta}\mathcal{M}(\delta) \\
&+ A(B^2 - A^2)\mathcal{M}(\delta)^2\}, \\
\widetilde{L}_{1,4} =& 2B\{-(B + Ae^{-2A\delta})e^{-A\delta} - (B^2 + A^2 + 2A(B - A)e^{-2A\delta})\mathcal{M}(\delta) \\
&- A(B - A)^2e^{-A\delta}\mathcal{M}(\delta)^2\}, \\
\widetilde{L}_{2,2} =& -(B^2 + A^2) - (B^2 + 2AB - A^2)e^{-2A\delta} \\
&- 2(B^2(B + A) - A^2(B - A))e^{-A\delta}\mathcal{M}(\delta) \\
&- (B^2 - A^2)(B^2 + A^2)\mathcal{M}(\delta)^2, \\
\widetilde{L}_{2,4} =& (B^2 + 2AB - A^2 + (B^2 + A^2)e^{-2A\delta})e^{-A\delta} \\
&+ 2(-2BA^2 + (B - A)(B^2 + A^2)e^{-2A\delta})\mathcal{M}(\delta) \\
&+ (B - A)^2(B^2 + A^2)e^{-A\delta}\mathcal{M}(\delta)^2, \\
\widetilde{L}_{3,2} =& (D_1(-A, B) - D_2(A, B)e^{-2A\delta})e^{-A\delta} \\
&- 2(2AB(B^2 + A^2) + (B - A)D_2(A, B)e^{-2A\delta})\mathcal{M}(\delta) \\
&- (B - A)^2D_2(A, B)e^{-A\delta}\mathcal{M}(\delta)^2, \\
\widetilde{L}_{3,4} =& D_2(A, B) - D_1(-A, B)e^{-2A\delta} \\
&- 2(B^4 - 2AB^3 + 2A^2B^2 + 2A^3B + A^4)e^{-A\delta}\mathcal{M}(\delta) \\
&- (B^2 - A^2)D_2(-A, B)\mathcal{M}(\delta)^2, \\
\widetilde{L}_{4,2} =& B\{-(D_3(A, B) + D_2(A, B)e^{-2A\delta})e^{-A\delta} \\
&+ 2(2A^2(B^2 + A^2) - (B - A)D_2(A, B)e^{-2A\delta})\mathcal{M}(\delta) \\
&- (B - A)^2D_2(A, B)e^{-A\delta}\mathcal{M}(\delta)^2\}, \\
\widetilde{L}_{4,4} =& B\{-D_2(A, B) - D_3(A, B)e^{-2A\delta} - 2(B^4 + 4A^3B - A^4)e^{-A\delta}\mathcal{M}(\delta) \\
&- (B^2 - A^2)D_2(-A, B)\mathcal{M}(\delta)^2\}.
\end{aligned} \tag{A.1}
$$

Concerning (2.4.16) and (A.1), we have the following lemma by Lemma 1.2.5, Lemma 1.2.6, and the fact that $A\mathcal{M}(\delta)$ and $B\mathcal{M}(\delta)$ belong to $\mathbb{M}_{0,2,\varepsilon,\gamma_0}$.

Lemma A.1. *Let $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Then it holds that*

$$L_{1,2},\ L_{1,4},\ L_{2,2},\ L_{2,4} \in \mathbb{M}_{1,2,\varepsilon,\gamma_0},$$

$$L_{3,2},\ L_{3,4},\ \widetilde{L}_{1,2},\ \widetilde{L}_{1,4},\ \widetilde{L}_{2,2},\ \widetilde{L}_{2,4} \in \mathbb{M}_{2,2,\varepsilon,\gamma_0},$$
$$L_{4,2},\ L_{4,4},\ \widetilde{L}_{3,2},\ \widetilde{L}_{3,4} \in \mathbb{M}_{3,2,\varepsilon,\gamma_0},$$
$$\widetilde{L}_{4,2}, \widetilde{L}_{4,4} \in \mathbb{M}_{4,2,\varepsilon,\gamma_0}.$$

To estimate $\det L$ from below, we introduce the following functions:

$$\begin{aligned}\ell_2(\xi',\lambda) =& (1+2D^2+5D^4)(1+e^{-2A\delta})(1+e^{-2B\delta}) - (16D^2+16D^4)e^{-A\delta}e^{-B\delta}\\ &- (D+6D^3+D^5)(1-e^{-2A\delta})(1-e^{-2B\delta}),\\ \ell_3(\xi',\lambda) =& (1+5D^{-1}+6D^{-2}+2D^{-3}+D^{-4}+D^{-5})A^2\mathcal{M}(\delta)^2\\ &+ 2(1-D^{-1}+3D^{-2}+D^{-3})e^{-A\delta}e^{-B\delta}\\ &- (1-3D^{-1}-D^{-2}-D^{-3})(1+e^{-2A\delta}e^{-2B\delta}),\end{aligned}$$

where we have set $D = AB^{-1}$. Then we have the relation:

$$\det L = (B-A)^{-2}\ell_1(\xi',\lambda) = (B-A)^{-2}B^5\ell_2(\xi',\lambda) = A^3\ell_3(\xi',\lambda),$$

where $\ell_1(\xi',\lambda)$ is defined as (2.4.12). The following lemma was proved in [**Abe04**, Section 3].

Lemma A.2. *Let $0 < \varepsilon < \pi/2$ and $(\xi',\lambda) \in \mathbf{R}^{N-1} \times \Sigma_\varepsilon$.*

(1) *Let $\alpha > 0$ and $|\lambda| \geq \alpha$. Then there exist positive constants $\sigma_1 = \sigma_1(\varepsilon,\mu,\delta,\alpha)$ and $\delta_1 = \delta_1(\varepsilon,\mu,\delta,\alpha)$ such that there holds the estimate:*

$$|\ell_1(\xi',\lambda)| \geq \delta_1(|\lambda|^{1/2}+A)^5$$

provided that $|\xi'| \leq \sigma_1$, where σ_1 is sufficiently small.

(2) *Let $\beta > 0$ and $|\xi'| \geq \beta$. Then there exist positive constants $\sigma_2 = \sigma_2(\varepsilon,\mu,\delta,\beta)$ and $\delta_2 = \delta_2(\varepsilon,\mu)$ such that there holds the estimate:*

$$|\ell_2(\xi',\lambda)| \geq \delta_2$$

provided that (ξ',λ) satisfies the condition: $|\xi'|^2 \leq \sigma_2|\lambda|$. In particular, σ_2 is sufficiently small.

(3) *Let $\gamma > 0$ and $|\xi'|^2 \geq \gamma|\lambda|$. Then there exist positive constants $\sigma_3 = \sigma_3(\varepsilon,\mu,\delta,\gamma)$ and $\delta_3 = \delta_3(\varepsilon,\mu)$ such that there holds the estimate:*

$$|\ell_3(\xi',\lambda)| \geq \delta_3$$

provided that $|\xi'| \geq \sigma_3$, where σ_3 is sufficiently large.

Corresponding to Lemma A.2, we constitute the cut-off functions $\zeta_j(\xi',\lambda)$ ($j = 1,2,3,4$) showed in (2.6.1) as follows: Let $\gamma_0 > 0$. First, in Lemma A.2 (1), we take $\alpha = \gamma_0$, then a positive number $\sigma_1 = \sigma_1(\varepsilon,\mu,\delta,\gamma_0)$ exists. Next, in Lemma A.2 (2), we take $\beta = \sigma_1/4$, then a positive number $\sigma_2 = \sigma_2(\varepsilon,\mu,\delta,\sigma_1)$ exists. Finally, in Lemma A.2 (3), we take $\gamma = \sigma_2/2$, then a positive number $\sigma_3 = \sigma_3(\varepsilon,\mu,\delta,\sigma_2)$ exists. By such positive numbers σ_1, σ_2, and σ_3, we define the cut-off functions $\zeta_j(\xi',\lambda)$ ($j = 1,2,3,4$) as (2.6.1).

Lemma A.3. *Let $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Then the following assertions hold.*

(1) *Let $\zeta_j(\xi',\lambda)$ ($j = 1,2,3,4$) be given by (2.6.1). Then $\zeta_j(\xi',\lambda)$ belong to $\mathbb{M}_{0,2,\varepsilon,\gamma_0}$.*

(2) *Let $\ell = 0,1$, $\alpha' \in \mathbb{N}_0^{N-1}$, and $s \geq 0$. Then we have, on* $\operatorname{supp}\zeta_4$,

$$|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^\ell A^{-s}\}| \leq C(|\lambda|^{1/2}+A)^{-s}A^{-|\alpha'|},\ \ |D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^\ell D^{-1}\}| \leq CA^{-|\alpha'|}$$

with some positive constant $C = C(\alpha', \varepsilon, \mu, \gamma_0, \delta)$ *Moreover,* $\zeta_4(\xi', \lambda)A^{-s} \in \mathbb{M}_{-r,2,\varepsilon,\gamma_0}$ *for any* $0 \le r \le s$.

(3) *It holds that*

$$\zeta_1(\xi', \lambda)\ell_1(\xi', \lambda)^{-1} \in \mathbb{M}_{-5,2,\varepsilon,\gamma_0},$$
$$\zeta_2(\xi', \lambda)\ell_2(\xi', \lambda)^{-1} \in \mathbb{M}_{0,2,\varepsilon,\gamma_0},$$
$$\zeta_4(\xi', \lambda)\ell_3(\xi', \lambda)^{-1} \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}.$$

Remark A.4. We note that $\operatorname{supp}\zeta_3$ is compact in $\mathbf{R}^{N-1} \times \mathbf{C}$, and also that $|\det L| > 0$ on $\operatorname{supp}\zeta_3$ by Lemma 2.4.1. By this fact, we see that, in Lemma 2.6.2 and Lemma 2.6.3, the terms multiplied by $\zeta_3(\xi', \lambda)$ satisfy the required properties.

Proof. (1) We first consider $\zeta_1(\xi', \lambda)$. Noting that

$$\operatorname{supp}\zeta_1 \subset \{\xi' \in \mathbf{R}^{N-1} \mid |\xi'|/\sigma_1 \le 3/4\}, \ \ \zeta_1(\xi', \lambda) = \zeta(\xi'/\sigma_1),$$

we have, for any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$,

$$\begin{aligned}\left|D_{\xi'}^{\alpha'}\zeta_1(\xi', \lambda)\right| &\le \sigma_1^{-|\alpha'|}\left|(D_{\xi'}^{\alpha'}\zeta)\left(\frac{\xi'}{\sigma_1}\right)\right| \\ &\le \left(\frac{|\xi'|}{\sigma_1}\right)^{|\alpha'|}\left\{\sup_{\xi' \in \mathbf{R}^{N-1}}|(D_{\xi'}^{\alpha'}\zeta)(\xi')|\right\}|\xi'|^{-|\alpha'|} \le C(\alpha')|\xi'|^{-|\alpha'|}\end{aligned}$$

with some positive constant $C(\alpha')$. We therefore obtain $|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^\ell\zeta_1(\xi', \lambda)\}| \le C(\alpha')|\xi'|^{-|\alpha'|}$ for $\ell = 0, 1$. In particular, $\zeta_1(\xi', \lambda)$ belongs to $\mathbb{M}_{0,2,\varepsilon,\gamma_0}$.

Next we show that $\zeta_2(\xi', \lambda) \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}$. By Leibniz's rule, we have, for any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$,

(A.2)

$$\begin{aligned}&D_{\xi'}^{\alpha'}\zeta_2(\xi', \lambda) = D_{\xi'}^{\alpha'}\Big[\big\{1 - \zeta\big(\frac{\xi'}{\sigma_1}\big)\big\}\zeta\big(\frac{\xi'}{(\sigma_2\lambda)^{1/2}}\big)\Big] \\ &= \Big\{1 - \zeta\Big(\frac{\xi'}{\sigma_1}\Big)\Big\}(D_{\xi'}^{\alpha'}\zeta)\Big(\frac{\xi'}{(\sigma_2\lambda)^{1/2}}\Big)\Big(\frac{1}{(\sigma_2\lambda)^{1/2}}\Big)^{|\alpha'|} \\ &- \sum_{\substack{\beta' \le \alpha' \\ |\beta'| \ne 0}}\binom{\alpha'}{\beta'}(D_{\xi'}^{\beta'}\zeta)\Big(\frac{\xi'}{\sigma_1}\Big)(D_{\xi'}^{\alpha'-\beta'}\zeta)\Big(\frac{\xi'}{(\sigma_2\lambda)^{1/2}}\Big)\Big(\frac{1}{\sigma_1}\Big)^{|\beta'|}\Big(\frac{1}{(\sigma_2\lambda)^{1/2}}\Big)^{|\alpha'|-|\beta'|}.\end{aligned}$$

Since it holds that

$$\operatorname{supp}\zeta\left(\frac{\xi'}{\sigma_1}\right) \subset \left\{\xi' \in \mathbf{R}^{N-1} \mid \frac{|\xi'|}{\sigma_1} \le \frac{3}{4}\right\},$$
$$\operatorname{supp}\zeta\left(\frac{\xi'}{(\sigma_2\lambda)^{1/2}}\right) \subset \left\{\xi' \in \mathbf{R}^{N-1} \mid \frac{|\xi'|}{(\sigma_2|\lambda|)^{1/2}} \le \frac{3}{4}\right\},$$

we have $|D_{\xi'}^{\alpha'}\zeta_2(\xi', \lambda)| \le C(\alpha')|\xi'|^{-|\alpha'|}$ for any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$ with some positive constant $C(\alpha')$ by (A.2). We also have $|D_{\xi'}^{\alpha'}\{\tau\partial_\tau\zeta_2(\xi', \lambda)\}| \le C(\alpha')|\xi'|^{-|\alpha'|}$ similarly, noting that

$$\tau\partial_\tau\Big\{\zeta\Big(\frac{\xi'}{(\sigma_2\lambda)^{1/2}}\Big)\Big\} = -\sum_{j=1}^{N-1}(D_j\zeta)\Big(\frac{\xi'}{(\sigma_2\lambda)^{1/2}}\Big)\Big(\frac{i\xi_j\tau}{2\sigma_2^{1/2}\lambda^{3/2}}\Big).$$

Summing up the last two inequalities, we see that $\zeta_2(\xi',\lambda)$ belongs to $\mathbb{M}_{0,2,\varepsilon,\gamma_0}$. Analogously it holds that $\zeta_3(\xi',\lambda)$ and $\zeta_4(\xi',\lambda)$ belong to $\mathbb{M}_{0,2,\varepsilon,\gamma_0}$.
(2) We first note that

$$|\lambda|^{1/2}/A \leq (2/\sigma_2)^{1/2}, \quad A \geq \sigma_3 \quad \text{on } \operatorname{supp}\zeta_4. \tag{A.3}$$

By Leibniz's rule and (A.3), we have, on $\operatorname{supp}\zeta_4$

$$|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^\ell A^{-s}\}| \leq CA^{-s-|\alpha'|} \leq C(|\lambda|^{1/2}+A)^{-s}A^{-|\alpha'|} \tag{A.4}$$

for $\ell = 0,1$, any multi-index $\alpha' \in \mathbf{N}_0^N$, and any $s \geq 0$ with some positive constant $C = C(\alpha', s, \sigma_2)$, which implies that the first required inequality holds.

Secondly, since $D^{-1} = A^{-1}B = (\lambda/(\mu AB)) + (A/B)$, we have the second required inequality by (A.4) and Leibniz's rule. The last property: $\zeta_4(\xi',\lambda)A^{-s} \in \mathbb{M}_{-r,2,\varepsilon,\gamma_0}$ $(0 \leq r \leq s)$ follows from (A.3) and (A.4).
(3) We first prove $\zeta_1(\xi',\lambda)\ell_1(\xi',\lambda)^{-1} \in \mathbb{M}_{-5,2,\varepsilon,\gamma_0}$. By Lemma 1.2.5, Lemma 1.2.6, and $D \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}$, it holds that $\ell_2(\xi',\lambda) \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}$. Thus, since $\ell_1(\xi',\lambda) = B^5\ell_2(\xi',\lambda)$, we have

$$\ell_1(\xi',\lambda) \in \mathbb{M}_{5,2,\varepsilon,\gamma_0} \tag{A.5}$$

by using Lemma 1.2.5. The property (A.5), combined with Leibniz's rule, Bell's formula, Lemma A.2 (1), and A.3 (1), yields that

$$|D_{\xi'}^{\alpha'}\{((\tau\partial_\tau)^\ell\zeta_1(\xi',\lambda))\ell_1(\xi',\lambda)^{-s}\}| \leq C(|\lambda|^{1/2}+A)^{-5s}A^{-|\alpha'|} \tag{A.6}$$

for $\ell = 0,1$, $s > 0$, and any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$ with some positive constant $C = C(\alpha', s, \varepsilon, \gamma_0, \mu, \delta)$. In addition, since

$$\begin{aligned}&\tau\partial_\tau\{\zeta_1(\xi',\lambda)\ell_1(\xi',\lambda)^{-1}\}\\ &\quad = \{\tau\partial_\tau\zeta_1(\xi',\lambda)\}\ell_1(\xi',\lambda)^{-1} - \zeta_1(\xi',\lambda)\ell_1(\xi',\lambda)^{-2}\tau\partial_\tau\ell_1(\xi',\lambda),\end{aligned}$$

we obtain, by (A.5), (A.6), and Leibniz's rule,

$$|D_{\xi'}^{\alpha'}\{\tau\partial_\tau(\zeta_1(\xi',\lambda)\ell_1(\xi',\lambda)^{-1})\}| \leq C(|\lambda|^{1/2}+A)^{-5}A^{-|\alpha'|},$$

which, combined with (A.6) for $\ell = 0$ and $s = 1$, furnishes that

$$|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^m(\zeta_1(\xi',\lambda)\ell_1(\xi',\lambda)^{-1})\}| \leq C(|\lambda|^{1/2}+A)^{-5}A^{-|\alpha'|}$$

for $m = 0,1$. This estimate implies that $\zeta_1(\xi',\lambda)\ell_1(\xi',\lambda)^{-1}$ belongs to $\mathbb{M}_{-5,2,\varepsilon,\gamma_0}$. Analogously we have

$$\zeta_2(\xi',\lambda)\ell_2(\xi',\lambda)^{-1} \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}, \quad \zeta_4(\xi',\lambda)\ell_3(\xi',\lambda)^{-1} \in \mathbb{M}_{0,2,\varepsilon,\gamma_0},$$

where we use Lemma A.3 (2) to show the second property. □

We here see that

$$\zeta_j(\xi',\lambda)(\det L)^{-1} \in \mathbb{M}_{-3,2,\varepsilon,\gamma_0} \quad (j = 1,2,4) \tag{A.7}$$

by combining Lemma 1.2.5, Lemma 1.2.6, and Lemma A.3 (2) and (3) with the formulas:

$$\frac{\zeta_1(\xi',\lambda)}{\det L} = \frac{\zeta_1(\xi',\lambda)(B-A)^2}{\ell_1(\xi',\lambda)},$$
$$\frac{\zeta_2(\xi',\lambda)}{\det L} = \frac{\zeta_2(\xi',\lambda)(B-A)^2}{B^5\ell_2(\xi',\lambda)},$$
$$\frac{\zeta_4(\xi',\lambda)}{\det L} = \frac{\zeta_1(\xi',\lambda)}{A^3\ell_3(\xi',\lambda)}.$$

Thus, using Lemma A.1 and Lemma 1.2.5, we have Lemma 2.6.2.

Finally we show Lemma 2.6.3. By Lemma 1.2.5, Lemma A.1, and (A.7), we see that $\zeta_j(\xi',\lambda)\widetilde{L}_{4,2\ell}/\det L \in \mathbb{M}_{1,2,\varepsilon,\gamma_0}$ and $\zeta_j(\xi',\lambda)L_{4,2\ell}/\det L \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}$ for $j=1,2,4$ and $\ell=1,2$, and also that $\zeta_j(\xi',\lambda)A^{-1}L_{4,2\ell-1}/\det L \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}$ by using (2.4.16). Since it holds that

$$\frac{\zeta_j(\xi',\lambda)}{\lambda\det L} = \frac{\zeta_j(\xi',\lambda)(B-A)}{\mu(B+A)\ell_1(\xi',\lambda)} = \frac{\zeta_j(\xi',\lambda)(B-A)}{\mu(B+A)B^5\ell_2(\xi',\lambda)},$$

we have, by Lemma A.3 (3), $\zeta_j(\lambda,\xi')/(\lambda\det L) \in \mathbb{M}_{-5,2,\varepsilon,\gamma_0}$ for $j=1,2$. This fact, combined with Lemma A.1, completes the proof of Lemma 2.6.3.

B

We here introduce some lemmas, used in Chapter 3 and Chapter 4, to show resolvent estimates and maximal regularity properties. To this end, we set

$$\text{(B.1)}\quad [K_0(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[k_0(\xi',\lambda)e^{-B(x_N+y_N)}\widehat{f}(\xi',y_N)\right](x')\,dy_N,$$
$$[K_1(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[k_1(\xi',\lambda)Ae^{-A(x_N+y_N)}\widehat{f}(\xi',y_N)\right](x')\,dy_N,$$
$$[K_2(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[k_2(\xi',\lambda)A^2e^{-Ax_N}\mathcal{M}(y_N)\widehat{f}(\xi',y_N)\right](x')\,dy_N,$$
$$[K_3(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[k_3(\xi',\lambda)Ae^{-B(x_N+y_N)}\widehat{f}(\xi',y_N)\right](x')\,dy_N,$$
$$[K_4(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[k_4(\xi',\lambda)A^2e^{-Bx_N}\mathcal{M}(y_N)\widehat{f}(\xi',y_N)\right](x')\,dy_N,$$
$$[K_5(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[k_5(\xi',\lambda)A\lambda^{1/2}e^{-Bx_N}\mathcal{M}(y_N)\widehat{f}(\xi',y_N)\right](x')\,dy_N,$$
$$[K_6(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[k_6(\xi',\lambda)A^2\mathcal{M}(x_N)e^{-By_N}\widehat{f}(\xi',y_N)\right](x')\,dy_N,$$
$$[K_7(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[k_7(\xi',\lambda)A\lambda^{1/2}\mathcal{M}(x_N)e^{-By_N}\widehat{f}(\xi',y_N)\right](x')\,dy_N,$$
$$[K_8(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[k_8(\xi',\lambda)A^2\mathcal{M}(x_N+y_N)\widehat{f}(\xi',y_N)\right](x')\,dy_N,$$
$$[K_9(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[k_9(\xi',\lambda)A^3\mathcal{M}(x_N)\mathcal{M}(y_N)\widehat{f}(\xi',y_N)\right](x')\,dy_N,$$
$$[K_{10}(\lambda)f](x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[k_{10}(\xi',\lambda)A\lambda\mathcal{M}(x_N)\mathcal{M}(y_N)\widehat{f}(\xi',y_N)\right](x')\,dy_N$$

with multipliers $k_i(\xi', \lambda)$ $(i = 0, 1, \ldots, 10)$, where the symbols: $A, B, \mathcal{M}(a)$ $(a > 0)$, and $\widehat{f}(\xi', y_N)$ are defined as (1.2.1) with $\mu = 1$, (1.2.2), and (1.2.4). Then we have the following lemma.

LEMMA B.1. *Let $0 < \varepsilon < \pi/2$, $1 < q < \infty$, and $\gamma_0 \geq 0$. We here use the operators $K_i(\lambda)$ $(i = 0, 1, \ldots, 10)$, defined as* (B.1)*, for $\lambda \in \Sigma_{\varepsilon,\gamma_0}$. Suppose that*

$$k_0(\xi', \lambda) = \lambda^{1/2}\widetilde{k}_0(\xi', \lambda), \quad \widetilde{k}_0(\xi', \lambda) \in \mathbb{M}_{0,1,\varepsilon,\gamma_0},$$
$$k_i(\xi', \lambda) \in \mathbb{M}_{0,2,\varepsilon,\gamma_0} \quad (i = 1, \ldots, 10).$$

Then, for $l = 0, 1$ and $i = 0, 1, \ldots, 10$, the sets:

$$\{(\tau\partial_\tau)^l K_i(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}$$

are $\mathcal{R}$-bounded families, in $\mathcal{L}(L_q(\mathbf{R}^N_+))$, whose $\mathcal{R}$-bounds do not exceed some positive constant $C(N, q, \varepsilon, \gamma_0)$.

PROOF. The case of $K_i(\lambda)$ for $i = 0, 1, 3, 8$ was already proved by [**SS12**, Lemma 5.4], so that we only show the required property for $K_{10}(\lambda)$ here in a similar way to the proof of [**SS12**, Lemma 5.4].

Setting $k_{10,\lambda}(x', x_N, y_N) = \mathcal{F}^{-1}_{\xi'}[k_{10}(\xi', \lambda)A\lambda\mathcal{M}(x_N)\mathcal{M}(y_N)](x')$, we see that

$$[K_{10}(\lambda)f](x) = \int_{\mathbf{R}^N_+} k_{10,\lambda}(x' - y', x_N, y_N)f(y)\,dy.$$

From now on, we prove that there exists a positive constant $M = M(N, \varepsilon, \gamma_0)$ such that for $\lambda \in \Sigma_{\varepsilon,\gamma_0}$, $x' \in \mathbf{R}^{N-1} \setminus \{0\}$, and $x_N, y_N > 0$

$$\text{(B.2)} \qquad |k_{10,\lambda}(x', x_N, y_N)| \leq \frac{M}{\{|x'|^2 + (x_N + y_N)^2\}^{N/2}},$$
$$|\tau\partial_\tau k_{10,\lambda}(x', x_N, y_N)| \leq \frac{M}{\{|x'|^2 + (x_N + y_N)^2\}^{N/2}}.$$

By the assumption of $k_{10}(\xi', \lambda)$, Leibniz's rule, and Lemma 1.2.6, we have

$$\text{(B.3)} \qquad |D^{\alpha'}_{\xi'}\{k_{10}(\xi', \lambda)A\lambda\mathcal{M}(x_N)\mathcal{M}(y_N)\}| \leq CA^{1-|\alpha'|}e^{-cA(x_N+y_N)}$$

for any multi-index $\alpha' \in \mathbf{N}^{N-1}_0$ with positive constants $c = c(\varepsilon)$, $C = C(\varepsilon, \gamma_0, \alpha')$. In addition, since

$$\begin{aligned}
&\tau\partial_\tau\{A\lambda k_{10}(\xi', \lambda)\mathcal{M}(x_N)\mathcal{M}(y_N)\} \\
&\quad = A(i\tau)k_{10}(\xi', \lambda)\mathcal{M}(x_N)\mathcal{M}(y_N) + A\lambda(\tau\partial_\tau k_{10}(\xi', \lambda))\mathcal{M}(x_N)\mathcal{M}(y_N) \\
&\quad + A\lambda k_{10}(\xi', \lambda)(\tau\partial_\tau\mathcal{M}(x_N))\mathcal{M}(y_N) + A\lambda k_{10}(\xi', \lambda)\mathcal{M}(x_N)(\tau\partial_\tau\mathcal{M}(y_N)),
\end{aligned}$$

we have, in the same manner as (B.3),

$$\text{(B.4)} \qquad |D^{\alpha'}_{\xi'}\tau\partial_\tau\{k_{10}(\xi', \lambda)A\lambda\mathcal{M}(x_N)\mathcal{M}(y_N)\}| \leq CA^{1-|\alpha'|}e^{-cA(x_N+y_N)}.$$

From viewpoint of (B.3) and (B.4), we apply Proposition 1.2.8 with $n = N - 1$, $L = N - 1$, and $\sigma = 1$ to obtain

$$\text{(B.5)} \qquad |k_{10,\lambda}(x', x_N, y_N)| \leq C|x'|^{-N}, \quad |\tau\partial_\tau k_{10,\lambda}(x', x_N, y_N)| \leq C|x'|^{-N}$$

with some positive constant $C = C(N, \varepsilon, \gamma_0)$ independent of x_N, y_N, and λ.

On the other hand, using (B.3) and (B.4) with $\alpha' = 0$, we have, by direct calculations,

$$|k_{10,\lambda}(x', x_N, y_N)| \leq C \int_{\mathbf{R}^{N-1}} |\xi'| e^{-c|\xi'|(x_N+y_N)}\, d\xi' \leq C(x_N + y_N)^{-N},$$
$$|\tau\partial_\tau k_{10,\lambda}(x', x_N, y_N)| \leq C(x_N + y_N)^{-N}$$

for $x_N, y_N > 0$ with some positive constant $C = C(N, \varepsilon, \gamma_0)$, which, combined with (B.5), furnishes that (B.2) holds.

Here we set

$$[L_0 f](x) = \int_{\mathbf{R}^N_+} l_0(x' - y', x_N, y_N) f(y)\, dy,$$
$$l_0(x', x_N, y_N) = \frac{M}{\{|x'|^2 + (x_N + y_N)^2\}^{N/2}},$$

where M is the same constant as in (B.2). We then see that $L_0 \in \mathcal{L}(L_q(\mathbf{R}^N_+))$ by the following steps: First, by Young's inequality, we have

$$\begin{aligned} &\|[L_0 f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\ &\leq \int_0^\infty \|l_0(\cdot, x_N, y_N)\|_{L_1(\mathbf{R}^{N-1})} \|f(\cdot, y_N)\|_{L_q(\mathbf{R}^{N-1})}\, dy_N \\ &\leq C \int_0^\infty \frac{\|f(\cdot, y_N)\|_{L_q(\mathbf{R}^{N-1})}}{x_N + y_N}\, dy_N \end{aligned}$$

with some positive constant $C = C(N, \varepsilon, \gamma_0)$. Second, taking L_q-norm after the the change of variable: $y_N = x_N s$ in the last inequality, we see that for $g(t) = \|f(\cdot, t)\|_{L_q(\mathbf{R}^{N-1})}$

$$\begin{aligned} \|L_0 f\|_{L_q(\mathbf{R}^N_+)} &\leq C \int_0^\infty \frac{\|g(\cdot\, s)\|_{L_q(0,\infty)}}{1+s}\, ds \\ &\leq C\|g\|_{L_q(0,\infty)} \int_0^\infty \frac{ds}{(1+s)s^{1/q}} \leq C\|f\|_{L_q(\mathbf{R}^N_+)} \end{aligned}$$

with a positive constant $C = C(N, q, \varepsilon, \gamma_0)$, which furnishes that $L_0 \in \mathcal{L}(L_q(\mathbf{R}^N_+))$.

Thus, by using Proposition 1.2.4, we obtain

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^l K_{10}(\lambda) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C(N, q, \varepsilon, \gamma_0)$$

for $l = 0, 1$ with some positive constant $C(N, q, \varepsilon, \gamma_0)$. □

As was seen in [**SS12**, Lemma 5.6], we have the following lemma by using Lemma B.1.

Lemma B.2. *Let* $0 < \varepsilon < \pi/2$, $1 < q < \infty$, *and* $\gamma_0 \geq 0$. *We here use the operators* $K_i(\lambda)$ $(i = 0, 1, \ldots, 10)$, *defined as* (B.1), *for* $\lambda \in \Sigma_{\varepsilon,\gamma_0}$. *Suppose that*

$$k_0(\xi', \lambda) \in \mathbb{M}_{-1,1,\varepsilon,\gamma_0}, \quad k_i(\xi', \lambda) \in \mathbb{M}_{-2,2,\varepsilon,\gamma_0} \quad (i = 1, \ldots, 10).$$

Then, for $l = 0, 1$, $i = 0, 1, \ldots, 10$, *and* $j, k = 1, \ldots, N$, *the sets:*

$$\{(\tau\partial_\tau)^l(\lambda K_i(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\},$$
$$\{(\tau\partial_\tau)^l(\gamma K_i(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\},$$
$$\{(\tau\partial_\tau)^l(\lambda^{1/2} D_j K_i(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\},$$

$$\{(\tau\partial_\tau)^l(D_jD_kK_i(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\gamma_0}\}$$

are $\mathcal{R}$-bounded families, in $\mathcal{L}(L_q(\mathbf{R}^N_+))$, whose $\mathcal{R}$-bound do not exceed some positive constant $C(N, q, \varepsilon, \gamma_0)$.

Corollary B.3. *Let $0 < \varepsilon < \pi/2$, $1 < q < \infty$, and $\gamma_0 \geq 0$. We here use the operators $K_i(\lambda)$ $(i = 0, 1, \ldots, 10)$, defined as* (B.1), *for $\lambda \in \Sigma_{\varepsilon,\gamma_0}$.*

(1) *Suppose that for every multi-index $\alpha' \in \mathbf{N}_0^{N-1}$ there exists a positive constant $C = C(\alpha', \gamma_0)$ such that for any $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times \Sigma_{\varepsilon,\gamma_0}$*

$$|D^{\alpha'}_{\xi'} k_i(\xi', \lambda)| \leq CA^{-|\alpha'|} \quad (i = 1, \ldots, 10).$$

Then there exists a positive constant $C = C(\alpha', \gamma_0)$ such that for any $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ and $f \in L_q(\mathbf{R}^N_+)$

$$\|K_i(\lambda)f\|_{L_q(\mathbf{R}^3_-)} \leq C\|f\|_{L_q(\mathbf{R}^3_-)} \quad (i = 1, \ldots, 10).$$

(2) *Suppose that for every multi-index $\alpha' \in \mathbf{N}_0^{N-1}$ there exists a positive constant $C = C(\alpha', \gamma_0)$ such that for any $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times \Sigma_{\varepsilon,\gamma_0}$*

$$|D^{\alpha'}_{\xi'} k_0(\xi', \lambda)| \leq C(|\lambda|^{1/2} + A)^{-1-|\alpha'|},$$
$$|D^{\alpha'}_{\xi'} k_i(\xi', \lambda)| \leq C(|\lambda|^{1/2} + A)^{-2}A^{-|\alpha'|} \quad (i = 1, \ldots, 10).$$

Then there exists a positive constant $C = C(N, q, \varepsilon, \gamma_0)$ such that for any $\lambda \in \Sigma_{\varepsilon,\gamma_0}$, $i = 0, 1 \ldots, 10$, and $f \in L_q(\mathbf{R}^N_+)$

$$\|(\lambda K_i(\lambda)f, \lambda^{1/2}\nabla K_i(\lambda)f, \nabla^2 K_i(\lambda)f)\|_{L_q(\mathbf{R}^N_+)} \leq C\|f\|_{L_q(\mathbf{R}^N_+)}.$$

Proof. It follows from Lemma B.1, B.2, and the definition of $\mathcal{R}$-boundedness (see Definition 1.2.1). □

From now on, we shall show maximal regularity properties of the high frequency parts $\mathbf{v}_\infty$, H_∞ defined as (4.3.21). To this end, for $j = 1, \ldots, N-1$ and $J = 1, \ldots, N$, we here introduce the operators $\mathcal{S}^1_J(\lambda), \mathcal{S}^2_j(\lambda), \mathcal{S}^2_N(\lambda), \mathcal{T}^1(\lambda)$, and $\mathcal{T}^2(\lambda)$ given by

$$\begin{aligned}
&\mathcal{S}^1_J(\lambda)\mathbf{f} \\
&= \sum_{K=1}^N \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty \mathcal{V}^{BB}_{JK}(\xi', \lambda)(c_g + c_\sigma A^2)}{AL(A, B)} Ae^{-B(x_N+y_N)}\widehat{f}_K(y_N)\right] dy_N \\
&+ \sum_{K=1}^N \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty \lambda^{\frac12}\mathcal{V}^{B\mathcal{M}}_{JK}(\xi', \lambda)(c_g + c_\sigma A^2)}{AB^2L(A, B)} A\lambda^{\frac12}e^{-Bx_N}\mathcal{M}(y_N)\widehat{f}_K(y_N)\right] dy_N \\
&+ \sum_{K=1}^N \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty \mathcal{V}^{B\mathcal{M}}_{JK}(\xi', \lambda)(c_g + c_\sigma A^2)}{B^2L(A, B)} A^2e^{-Bx_N}\mathcal{M}(y_N)\widehat{f}_K(y_N)\right] dy_N \\
&+ \sum_{K=1}^N \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty \lambda^{\frac12}\mathcal{V}^{\mathcal{M}B}_{JK}(\xi', \lambda)(c_g + c_\sigma A^2)}{AB^2L(A, B)} A\lambda^{\frac12}\mathcal{M}(x_N)e^{-By_N}\widehat{f}_K(y_N)\right] dy_3 \\
&+ \sum_{K=1}^N \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty \mathcal{V}^{\mathcal{M}B}_{JK}(\xi', \lambda)(c_g + c_\sigma A^2)}{B^2L(A, B)} A^2\mathcal{M}(x_N)e^{-By_N}\widehat{f}_K(y_N)\right] dy_N
\end{aligned}$$

$$+\sum_{K=1}^{N}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty \mathcal{V}_{JK}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)(c_g+c_\sigma A^2)}{AB^2L(A,B)}A\lambda\mathcal{M}(x_N)\mathcal{M}(y_N)\widehat{f}_K(y_N)\right]dy_N$$

$$+\sum_{K=1}^{N}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty \mathcal{V}_{JK}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)(c_g+c_\sigma A^2)}{AB^2L(A,B)}A^3\mathcal{M}(x_N)\mathcal{M}(y_N)\widehat{f}_K(y_N)\right]dy_N,$$

$\mathcal{S}_j^2(\lambda)\nabla^2 g$

$$= -\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty i\xi_j(c_g+c_\sigma A^2)}{A^2L(A,B)}Ae^{-B(x_N+y_N)}\,\widehat{\Delta' g}(y_N)\right]dy_N$$

$$+\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty \xi_j\xi_k(B-A)(c_g+c_\sigma A^2)}{A^3(B+A)L(A,B)}Ae^{-B(x_N+y_N)}\widehat{D_kD_N g}(y_N)\right]dy_N$$

$$-\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty i\xi_j(B^2+A^2)(c_g+c_\sigma A^2)}{A^3(B+A)L(A,B)}A^2\mathcal{M}(x_N+y_N)\,\widehat{\Delta' g}(y_N)\right]dy_N$$

$$-\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty \xi_j\xi_k(B^2+A^2)(c_g+c_\sigma A^2)}{A^4(B+A)L(A,B)}A^2\mathcal{M}(x_N+y_N)\widehat{D_kD_N g}(y_N)\right]dy_N,$$

$\mathcal{S}_N^2(\lambda)\nabla^2 g$

$$= -\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty(B-A)(c_g+c_\sigma A^2)}{A(B+A)L(A,B)}Ae^{-B(x_N+y_N)}\,\widehat{\Delta' g}(y_N)\right]dy_N$$

$$+\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty i\xi_k(c_g+c_\sigma A^2)}{A^2L(A,B)}Ae^{-B(x_N+y_N)}\widehat{D_kD_N g}(y_N)\right]dy_N$$

$$+\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty(B^2+A^2)(c_g+c_\sigma A^2)}{A^2(B+A)L(A,B)}A^2\mathcal{M}(x_N+y_N)\,\widehat{\Delta' g}(y_N)\right](x')\,dy_3$$

$$-\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty i\xi_k(B^2+A^2)(c_g+c_\sigma A^2)}{A^3(B+A)L(A,B)}A^2\mathcal{M}(x_N+y_N)\widehat{D_kD_N g}(y_N)\right]dy_N,$$

$\mathcal{T}^1(\lambda)\mathbf{f}$

$$= -\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty i\xi_k(B-A)}{A(B+A)L(A,B)}Ae^{-A(x_N+y_N)}\widehat{f}_k(y_N)\right]dy_N$$

$$-\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty}{L(A,B)}Ae^{-A(x_N+y_N)}\widehat{f}_N(y_N)\right]dy_N$$

$$+\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty 2i\xi_k B}{A(B+A)L(A,B)}A^2e^{-Ax_N}\mathcal{M}(y_N)\widehat{f}_k(y_N)\right]dy_N$$

$$+\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty 2A}{(B+A)L(A,B)}A^2e^{-Ax_N}\mathcal{M}(y_N)\widehat{f}_N(y_N)\right]dy_N,$$

$\mathcal{T}^2(\lambda)\nabla^2 g$

$$= -\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty D(A,B)}{A^2(B+A)L(A,B)}Ae^{-A(x_N+y_N)}\widehat{\Delta' g}(y_N)\right]dy_N$$

$$+\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty i\xi_k D(A,B)}{A^3(B+A)L(A,B)}Ae^{-A(x_N+y_N)}\widehat{D_kD_N g}(y_N)\right]dy_N$$

where we have set $\varphi_\infty = \varphi_\infty(\xi')$, defined as (3.4.6), and

$$\widehat{f}(y_N) = \widehat{f}(\xi', y_N), \quad \mathcal{F}^{-1}_{\xi'}[g] = \mathcal{F}^{-1}_{\xi'}[g](x').$$

Then, setting

$$\mathcal{S}^1(\lambda)\mathbf{f} = (\mathcal{S}^1_1(\lambda)\mathbf{f}, \dots, \mathcal{S}^1_N(\lambda)\mathbf{f})^T, \;\; \mathcal{S}^2(\lambda)\nabla^2 g = (\mathcal{S}^2_1(\lambda)\nabla^2 g, \dots, \mathcal{S}^2_N(\lambda)\nabla^2 g)^T,$$

we see that

$$\begin{aligned}\mathbf{v}_\infty &= \mathcal{L}^{-1}_\lambda[\mathcal{S}^1(\lambda)\mathcal{L}[\mathbf{f}](\lambda)](t) + \mathcal{L}^{-1}_\lambda[\mathcal{S}^2(\lambda)\mathcal{L}[\nabla^2 g_h](\lambda)](t),\\ H_\infty &= \mathcal{L}^{-1}_\lambda[\mathcal{T}^1(\lambda)\mathcal{L}[\mathbf{f}](\lambda)](t) + \mathcal{L}^{-1}_\lambda[\mathcal{T}^2(\lambda)\mathcal{L}[\nabla^2 g_h](\lambda)](t).\end{aligned}$$

Now we have the following lemma.

Lemma B.4. *Let $\alpha' \in \mathbf{N}_0^{N-1}$ and $l = 0, 1$. Then there exists a positive constant $C(\alpha')$ such that for any $\xi' \in \mathbf{R}^{N-1} \setminus \{0\}$ and $\lambda = i\tau$ for $\tau \in \mathbf{R} \setminus \{0\}$*

$$\begin{aligned}&|(\tau\partial_\tau)^l D^{\alpha'}_{\xi'}\{\varphi_\infty(\xi')L(A,B)^{-1}\}|\\ &\quad \le C(\alpha')\{|\lambda|(|\lambda|^{1/2} + A)^2 + A(c_g + c_\sigma A^2)\}A^{-|\alpha'|},\end{aligned}$$

where $\varphi_\infty(\xi')$ is given by (3.4.6).

Proof. Let $\psi \in C_0^\infty(\mathbf{R})$, $0 \le \psi \le 1$, and

$$\psi(\tau) = \begin{cases} 1 & (|\tau| \le \dfrac{1}{3}),\\ 0 & (|\tau| \ge \dfrac{2}{3}).\end{cases}$$

We only consider the case of $l = 0$ here. By Lemma 1.2.6 (2) and Lemma 3.6.2 (3), there exists a positive number τ_∞ such that by setting

$$\psi_0(\tau) = \psi(\tau/\tau_\infty), \quad \psi_\infty(\tau) = 1 - \psi(\tau/\tau_\infty),$$

we see that

$$\begin{aligned}&|D^{\alpha'}_{\xi'}\{\varphi_\infty(\xi')\psi_0(\tau)L(A,B)^{-1}\}| \le C(\alpha')A^{-3-|\alpha'|},\\ &|D^{\alpha'}_{\xi'}\{\varphi_\infty(\xi')\psi_\infty(\tau)L(A,B)^{-1}\}|\\ &\quad \le C(\alpha')\{|\lambda|(|\lambda|^{1/2} + A)^2 + A(c_g + c_\sigma A^2)\}\end{aligned} \tag{B.6}$$

for any multi-index $\alpha' \in \mathbf{N}_0^{N-1}$ with some positive constant $C(\alpha')$. Moreover, since

$$\begin{aligned}&|\lambda|(|\lambda|^{1/2} + A)^2|D^{\alpha'}_{\xi'}\{\varphi_\infty(\xi')\psi_0(\tau)L(A,B)^{-1}\}|\\ &\quad \le A^3|D^{\alpha'}_{\xi'}\{\varphi_\infty(\xi')\psi_0(\tau)L(A,B)^{-1}\}| \le C(\alpha')A^{-|\alpha'|},\end{aligned}$$

it holds that

$$\begin{aligned}&|D^{\alpha'}_{\xi'}\{\varphi_\infty(\xi')\psi_0(\tau)L(A,B)^{-1}\}|\\ &\quad \le C(\alpha')\{|\lambda|(|\lambda|^{1/2} + A)^2 + A(c_g + c_\sigma A^2)\},\end{aligned}$$

which, combined with the second inequality of (B.6), completes the proof of the lemma. □

By using Lemma 1.2.6, Lemma B.1, and Lemma B.4, we see that the following lemma holds.

Lemma B.5. *Let $1 < q < \infty$. Then there exists a positive constant $M = M(N,q)$ such that for $l = 0, 1$ and $i = 1, 2$*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{Ni}, L_q(\mathbf{R}^N_+)^N)}(\{(\tau\partial_\tau)^l(\lambda\mathcal{S}^i(\lambda)) \mid \lambda = i\tau,\ \tau \in \mathbf{R}\setminus\{0\}\}) \leq M,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{Ni}, W^2_q(\mathbf{R}^N_+)^N)}(\{(\tau\partial_\tau)^l\mathcal{S}^i(\lambda) \mid \lambda = i\tau,\ \tau \in \mathbf{R}\setminus\{0\}\}) \leq M,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{Ni}, W^2_q(\mathbf{R}^N_+))}(\{(\tau\partial_\tau)^l(\lambda\mathcal{T}^i(\lambda)) \mid \lambda = i\tau,\ \tau \in \mathbf{R}\setminus\{0\}\}) \leq M,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{Ni}, W^3_q(\mathbf{R}^N_+))}(\{(\tau\partial_\tau)^l\mathcal{T}^i(\lambda) \mid \lambda = i\tau,\ \tau \in \mathbf{R}\setminus\{0\}\}) \leq M.$$

Let $1 < p, q < \infty$. Then, by combining Lemma B.5 with Weis' operator valued Fourier multiplier theorem, we obtain

$$\begin{aligned}
&\|(\partial_t\mathbf{v}_\infty, \mathbf{v}_\infty, \nabla\mathbf{v}_\infty, \nabla^2\mathbf{v}_\infty)\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_+))} \\
&\quad + \|(\partial_t H_\infty)\|_{L_p((0,\infty), W^2_q(\mathbf{R}^3_+))} + \|H_\infty\|_{L_p(\mathbf{R}_+, W^3_q(\mathbf{R}^3_+))} \\
&\leq C(p,q)\left(\|\mathbf{f}\|_{L_p(\mathbf{R}_+, L_q(\mathbf{R}^3_+))} + \|g_h\|_{L_p(\mathbf{R}_+, W^{2-1/q}_q(\mathbf{R}^3_0))}\right)
\end{aligned} \tag{B.7}$$

for some positive constant $C(p,q)$.

C

We here introduce some properties of the Fourier-Laplace transform. To this end, we first define a class of multipliers. For some $\gamma_0 \geq 0$, set

$$\mathbf{C}_{+,\gamma_0} = \{\lambda \in \mathbf{C}\setminus\{0\} \mid \operatorname{Re}\lambda \geq \gamma_0\}.$$

Let $m(\xi', \lambda)$ be functions defined on $(\mathbf{R}^{N-1}\setminus\{0\}) \times \mathbf{C}_{+,\gamma_0}$, and suppose that $m(\xi', \lambda)$ is C^∞ and holomorphic with respect to ξ' and λ, respectively. In addition, there exist $s \in \mathbf{R}$ and a positive constant $C(s, \gamma_0)$ such that for any $(\xi', \lambda) \in (\mathbf{R}^{N-1}\setminus\{0\}) \times \mathbf{C}_{+,\gamma_0}$

$$|m(\xi', \lambda)| \leq C(s, \gamma_0)(|\lambda|^{1/2} + A)^s.$$

We then denote the set of all such functions defined on $(\mathbf{R}^{N-1}\setminus\{0\}) \times \mathbf{C}_{+,\gamma_0}$ by $\mathbb{L}_{s,\gamma_0}$. Now, we have the following proposition.

Proposition C.1. *Let $f(x', t) \in C^\infty_0(\mathbf{R}^{N-1} \times \mathbf{R}_+)$, and suppose that $m(\xi', \lambda) \in \mathbb{L}_{s,\gamma_0}$ for some $s > -\max\{N-1, 2\}$ and $\gamma_0 \geq 0$. We here use the symbols defined as in* (1.2.1), (1.2.3), (1.2.4), *and* (4.3.5). *In addition, for $x_N > 0$ and $\lambda = \gamma + i\tau$ ($\gamma \geq \gamma_0$), we set*

$$I(x,t) = \mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}\left[m(\xi', \lambda)e^{-Bx_N}\widetilde{f}(\xi', \lambda)\right](x', t),$$
$$J(x,t) = \mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}\left[m(\xi', \lambda)\mathcal{M}(x_N)\widetilde{f}(\xi', \lambda)\right](x', t).$$

It then holds that for $t > 0$ and $\lambda = \gamma + i\tau$ ($\gamma \geq \gamma_0$)

$$I(x,t) = \int_0^t \mathcal{F}^{-1}_{\xi'}\left[\mathcal{L}^{-1}_\lambda\left[m(\xi', \lambda)e^{-Bx_N}\right](t-s)\widehat{f}(\xi', s)\right](x')\,ds,$$

$$\begin{aligned}
&J(x,t) \\
&= -x_N\int_0^t\int_0^1 \mathcal{F}^{-1}_{\xi'}\left[\mathcal{L}^{-1}_\lambda\left[m(\xi', \lambda)e^{-(B\theta + A(1-\theta))x_N}\right](t-s)\widehat{f}(\xi', s)\right](x')\,d\theta ds.
\end{aligned}$$

Proof. Noting that there exists a positive constant C, independent of γ and τ, such that

$$C^{-1}(\gamma^{1/2}+|\tau|^{1/2}) \le |\lambda|^{1/2} \le C(\gamma^{1/2}+|\tau|^{1/2}),$$

we have, by Lemma 1.2.6,

$$|m(\xi',\lambda)e^{-Bx_N}| \le C(\gamma^{1/2}+|\tau|^{1/2}+A)^s e^{-C(\gamma^{1/2}+|\tau|^{1/2}+A)x_N} \tag{C.1}$$

for any $s,\tau \in \mathbf{R}$, $\xi' \in \mathbf{R}^{N-1}$, and $x_N > 0$ with some positive constant C independent of γ, τ, ξ', and x_N. If $s \ge 0$, then it is clear that $m(\xi',\lambda)e^{-Bx_N} \in L_1(\mathbf{R}^{N-1}_{\xi'} \times \mathbf{R}_\tau)$ by (C.1), where subscripts ξ' and τ denote their variables. In the case of $-\max\{(N-1),2\} < s < 0$, we have, by (C.1) and $\gamma \ge \gamma_0 \ge 0$,

$$|m(\xi',\lambda)e^{-Bx_N}| \le \begin{cases} C|\tau|^{-|s|/2}e^{-C(\gamma^{1/2}+|\tau|^{1/2}+A)x_N} & (N=2), \\ CA^{-|s|}e^{-C(\gamma^{1/2}+|\tau|^{1/2}+A)x_N} & (N \ge 3) \end{cases}$$

with some positive constant C independent of γ,τ,ξ', and x_N, so that it holds that $m(\xi',\lambda)e^{-Bx_N} \in L_1(\mathbf{R}^{N-1}_{\xi'} \times \mathbf{R}_\tau)$. The integrability of $m(\xi',\lambda)e^{-Bx_N}$ and Fubini's theorem implies that

$$I(x,t) = \mathcal{F}^{-1}_{\xi'}\left[\left(\mathcal{L}^{-1}_\lambda\left[m(\xi',\lambda)e^{-Bx_N}\right] * \widehat{f}(\xi',\cdot)\right)(t)\right](x').$$

On the other hand, since $f(x',t)=0$ for $t<0$ and $\mathcal{L}^{-1}_\lambda\left[m(\xi',\lambda)e^{-Bx_N}\right](t)=0$ for $t<0$ by Cauchy's integral theorem,

$$\left(\mathcal{L}^{-1}_\lambda\left[m(\xi',\lambda)e^{-Bx_N}\right] * \widehat{f}(\xi',\cdot)\right)(t) = \int_0^t \mathcal{L}^{-1}_\lambda\left[m(\xi',\lambda)e^{-Bx_N}\right](t-s)\widehat{f}(\xi',s)\,ds.$$

We thus obtain the required formula of $I(x,t)$. Concerning $J(x,t)$, we have

$$J(x,t) = -x_N\int_0^1 \mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}\left[m(\xi',\lambda)e^{-(B\theta+A(1-\theta))x_N}\widetilde{f}(\xi',\lambda)\right](x',t)\,d\theta$$

by using (1.2.3), which furnishes the required formal of $J(x,t)$ in the same manner as in $I(x,t)$. □

We next show another convolution formula of Fourier-Laplace transform in the following proposition.

Proposition C.2. *Let $0<\varepsilon<\pi/2$, $\gamma_0>0$, and $s<0$. Suppose that $m(\xi',\lambda) \in \mathbb{M}_{s,2,\varepsilon,\gamma_0}$. We here use the symbols defined as in* (1.2.1), (1.2.3), (1.2.4), *and* (4.3.5). *Then the following assertions hold.*

(1) *Let $f(x,t)$ be a function in $C^\infty_0(\mathbf{R}^N_+ \times \mathbf{R}_+)$, and suppose that for $x_N \ge 0$ and $\lambda = \gamma + i\tau$ $(\gamma \ge \gamma_0)$*

$$K_1(x,t) = \int_0^\infty \mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}\left[m(\xi',\lambda)e^{-A(x_N+y_N)}\widetilde{f}(\xi',y_N,\lambda)\right](x',t)\,dy_N,$$

$$K_2(x,t) = \int_0^\infty \mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}\left[m(\xi',\lambda)e^{-Ax_N}\mathcal{M}(y_N)\widetilde{f}(\xi',y_N,\lambda)\right](x',t)\,dy_N,$$

$$K_3(x,t) = \int_0^\infty \mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}\left[m(\xi',\lambda)e^{-B(x_N+y_N)}\widetilde{f}(\xi',y_N,\lambda)\right](x',t)\,dy_N,$$

$$K_4(x,t) = \int_0^\infty \mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}\left[m(\xi',\lambda)e^{-Bx_N}\mathcal{M}(y_N)\widetilde{f}(\xi',y_N,\lambda)\right](x',t)\,dy_N,$$

$$K_5(x,t) = \int_0^\infty \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[m(\xi',\lambda)\mathcal{M}(x_N)e^{-By_N}\widetilde{f}(\xi',y_N,\lambda)\right](x',t)\,dy_N,$$
$$K_6(x,t) = \int_0^\infty \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[m(\xi',\lambda)\mathcal{M}(x_N)\mathcal{M}(y_N)\widetilde{f}(\xi',y_N,\lambda)\right](x',t)\,dy_N.$$

Then there exists $\delta_0 \geq \gamma_0$ *such that for any* $\varepsilon \leq \varepsilon' < \pi/2$ *and*

$$\Gamma(\delta_0,\varepsilon') = \{\lambda \in \mathbf{C} \mid \lambda = \delta_0 + se^{\pm i(\pi-\varepsilon')},\ \ s : 0 \to \infty\},$$

we have, for $t > 0$,

$$K_1(x,t) = \int_0^t \Big(\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\int_{\Gamma(\delta_0,\varepsilon')} e^{\lambda(t-s)}m(\xi',\lambda)e^{-A(x_N+y_N)}\,d\lambda \times \widehat{f}(\xi',y_N,s)\Big](x')\,dy_N\Big)ds,$$
$$K_2(x,t) = \int_0^t \Big(\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\int_{\Gamma(\delta_0,\varepsilon')} e^{\lambda(t-s)}m(\xi',\lambda)e^{-Ax_N}\mathcal{M}(y_N)\,d\lambda \times \widehat{f}(\xi',y_N,s)\Big](x')\,dy_N\Big)ds,$$
$$K_3(x,t) = \int_0^t \Big(\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\int_{\Gamma(\delta_0,\varepsilon')} e^{\lambda(t-s)}m(\xi',\lambda)e^{-B(x_N+y_N)}\,d\lambda \times \widehat{f}(\xi',y_N,s)\Big](x')\,dy_N\Big)ds,$$
$$K_4(x,t) = \int_0^t \Big(\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\int_{\Gamma(\delta_0,\varepsilon')} e^{\lambda(t-s)}m(\xi',\lambda)e^{-Bx_N}\mathcal{M}(y_N)\,d\lambda \times \widehat{f}(\xi',y_N,s)\Big](x')\,dy_N\Big)ds,$$
$$K_5(x,t) = \int_0^t \Big(\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\int_{\Gamma(\delta_0,\varepsilon')} e^{\lambda(t-s)}m(\xi',\lambda)\mathcal{M}(x_N)e^{-By_N}\,d\lambda \times \widehat{f}(\xi',y_N,s)\Big](x')\,dy_N\Big)ds,$$
$$K_6(x,t) = \int_0^t \Big(\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\int_{\Gamma(\delta_0,\varepsilon')} e^{\lambda(t-s)}m(\xi',\lambda)\mathcal{M}(x_N)\mathcal{M}(y_N)\,d\lambda \times \widehat{f}(\xi',y_N,s)\Big](x')\,dy_N\Big)ds.$$

(2) *Let* $g(x',t)$ *be a function in* $C_0^\infty(\mathbf{R}^{N-1}\times\mathbf{R}_+)$, *and suppose that for* $x_N \geq 0$ *and* $\lambda = \gamma + i\tau$ $(\gamma \geq \gamma_0)$

$$L_1(x,t) = \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[m(\xi',\lambda)e^{-Ax_N}\widetilde{g}(\xi',\lambda)\right](x',t),$$
$$L_2(x,t) = \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[m(\xi',\lambda)e^{-Bx_N}\widetilde{g}(\xi',\lambda)\right](x',t),$$
$$L_3(x,t) = \mathcal{L}_\lambda^{-1}\mathcal{F}_{\xi'}^{-1}\left[m(\xi',\lambda)\mathcal{M}(x_N)\,\widetilde{g}(\xi',\lambda)\right](x',t).$$

Then there exists $\delta_0 \geq \gamma_0$ *such that for any* $\varepsilon \leq \varepsilon' < \pi/2$ *and*

$$\Gamma(\delta_0,\varepsilon') = \{\lambda \in \mathbf{C} \mid \lambda = \delta_0 + se^{\pm i(\pi-\varepsilon')},\ \ s : 0 \to \infty\},$$

we have, for $t > 0$,

$$L_1(x,t) = \int_0^t \mathcal{F}_{\xi'}^{-1}\Big[\int_{\Gamma(\delta_0,\varepsilon')} e^{\lambda(t-s)}m(\xi',\lambda)e^{-Ax_N}\,d\lambda\,\widehat{g}(\xi',s)\Big](x')\,ds,$$

$$L_2(x,t) = \int_0^t \mathcal{F}_{\xi'}^{-1}\Big[\int_{\Gamma(\delta_0,\varepsilon')} e^{\lambda(t-s)} m(\xi',\lambda) e^{-Bx_N}\,d\lambda\,\widehat{g}(\xi',s)\Big](x')\,ds,$$

$$L_3(x,t) = \int_0^t \mathcal{F}_{\xi'}^{-1}\Big[\int_{\Gamma(\delta_0,\varepsilon')} e^{\lambda(t-s)} m(\xi',\lambda) \mathcal{M}(x_N)\,d\lambda\,\widehat{g}(\xi',s)\Big](x')\,ds.$$

Proof. We here consider $K_1(x,t)$ only, since the others can be proved analogously. It holds that for $\lambda = \gamma + i\tau$ $(\gamma \geq \gamma_0)$

(C.2)

$$K_1(x,t) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\left(\mathcal{L}_\lambda^{-1}\left[m(\xi',\lambda)e^{-A(x_N+y_N)}\right] * \widehat{f}(\xi',y_N,\cdot)\right)(t)\right](x')\,dy_N,$$

where we see $\mathcal{L}_\lambda^{-1}\left[m(\xi',\lambda)e^{-A(x_N+y_N)}\right](t)$ as the distribution, and the symbol $*$ denotes the standard convolution. Noting $m(\xi',\gamma+i\tau) \in L_p(\mathbf{R}_\tau)$, where the subscript τ denotes its variable, for a sufficiently large $p \geq 1$ since $m(\xi',\lambda) \in \mathbb{M}_{s,2,\varepsilon,\gamma_0}$ $(s<0)$, we have, by [**KS07**, Proposition 3.40],

$$\mathcal{L}_\lambda^{-1}[m(\xi',\lambda)e^{-A(x_N+y_N)}](t) = \lim_{R\to\infty} \frac{1}{2\pi}\int_{\gamma-Ri}^{\gamma+Ri} e^{\lambda t} m(\xi',\lambda)e^{-A(x_N+y_N)}\,d\lambda.$$

In addition, we have, by Cauchy's integral theorem and choosing $\delta_0 = (2\gamma_0)/\sin\varepsilon$,

$$\begin{aligned}\frac{1}{2\pi}&\int_{\gamma-iR}^{\gamma+iR} e^{\lambda t} m(\xi',\lambda)e^{-A(x_N+y_N)}\,d\lambda \\ &= \frac{1}{2\pi}\int_{\Gamma(\delta_0,\varepsilon';R)} e^{\lambda t} m(\xi',\lambda)e^{-A(x_N+y_N)}\,d\lambda \quad (R>0)\end{aligned}$$

with $\Gamma(\delta_0,\varepsilon';R) = \{\lambda \in \mathbf{C} \mid \lambda = \delta_0 + se^{\pm i(\pi-\varepsilon')},\ s: 0 \to R\}$ $(\varepsilon \leq \varepsilon' < \pi/2)$. By the last two identities combined, we obtain

$$\text{(C.3)}\qquad \mathcal{L}_\lambda^{-1}\left[m(\xi',\lambda)e^{-A(x_N+y_N)}\right](t) = \frac{1}{2\pi}\int_{\Gamma(\delta_0,\varepsilon')} e^{\lambda t} m(\xi',\lambda)e^{-A(x_N+y_N)}\,d\lambda.$$

Since $\widehat{f}(\xi',y_N,t) = 0$ for $t<0$ and

$$\frac{1}{2\pi}\int_{\Gamma(\delta_0,\varepsilon')} e^{\lambda t} m(\xi',\lambda)e^{-A(x_N+y_N)}\,d\lambda = 0 \quad (t<0)$$

by Cauchy's integral theorem, it follows from (C.2) and (C.3) that the required formula holds. □

Bibliography

[Abe04] T. Abe, *On a resolvent estimate of the Stokes equation with Neumann-Dirichlet-type boundary condition on an infinite layer*, Math. Methods Appl. Sci. **27** (2004), no. 9, 1007–1048.

[Abe05a] H. Abels, *The initial-value problem for the Navier-Stokes equations with a free surface in L_q-Sobolev spaces*, Adv. Differential Equations **10** (2005), no. 1, 45–64.

[Abe05b] ———, *Reduced and generalized Stokes resolvent equations in asymptotically flat layers, part I: Unique solvability*, J. Math. Fluid Mech. **7** (2005), no. 2, 201–222.

[Abe06] ———, *Generalized Stokes resolvent equations in an infinite layer with mixed boundary conditions*, Math. Nachr. **279** (2006), no. 4, 351–367.

[Abe07] ———, *On generalized solutions of two-phase flows for viscous incompressible fluids*, Interfaces Free Bound. **9** (2007), no. 1, 31–65.

[AF03] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, 2nd ed., Pure and applied mathematics, vol. 140, Elsevier/Academic Press, Amsterdam, 2003.

[All87] G. Allain, *Small-time existence for the Navier-Stokes equations with a free surface*, Appl. Math. Optim. **16** (1987), no. 1, 37–50.

[Ama94] H. Amann, *Stability of the rest state of a viscous incompressible fluid*, Arch. Rational Mech. Anal. **126** (1994), no. 3, 231–242.

[Ama96] ———, *Stability and bifurcation in viscous incompressible fluids*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **233** (1996), 9–29.

[Bae11] H. Bae, *Solvability of the free boundary value problem of the Navier-Stokes equations*, Discrete Contin. Dyn. Syst. **29** (2011), no. 3, 769–801.

[Bea81] J. T. Beale, *The initial value problem for the Navier-Stokes equations with a free surface*, Comm. Pure Appl. Math. **34** (1981), no. 3, 359–392.

[Bea84] ———, *Large-time regularity of viscous surface waves*, Arch. Rational Mech. Anal. **84** (1983/84), no. 4, 307–352.

[BN85] J. T. Beale and T Nishida, *Large-time behavior of viscous surface waves*, Recent Topics in Nonlinear PDE II, North-Holland Math. Stud. **128**, North-Holland, Amsterdam, 1985, pp. 1–14.

[BP07] D. Bothe and J. Prüss, *L_p-theory for a class of non-Newtonian fluids*, SIAM J. Math. Anal. **39** (2007), no. 2, 379–421.

[Den94] I. V. Denisova, *Problem of the motion of two viscous incompressible fluids separated by a closed free interface*, Acta Appl. Math. **37** (1994), no. 1-2, 31–40.

[DGH$^+$11] R. Denk, M. Geissert, M. Hieber, J. Saal, and O. Sawada, *The spin-coating process: Analysis of the free boundary value problem*, Comm. Partial Differential Equations **36** (2011), no. 7, 1145–1192.

[DHP03] R. Denk, M. Hieber, and J. Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc. **166** (2003), no. 788, viii+114 pp.

[DHP07] ———, *Optimal L_p-L_q-estimates for parabolic boundary value problems with inhomogeneous data*, Math. Z. **257** (2007), no. 1, 193–224.

[DR05] L. Diening and M. Růžička, *Strong solutions for generalized Newtonian fluids*, J. Math. Fluid Mech. **7** (2005), no. 3, 413–450.

[ES13] Y. Enomoto and Y. Shibata, *On the $\mathcal{R}$-sectoriality and the initial boundary value problem for the viscous compressible fluid flow*, Funkcial. Ekvac. **56** (2013), no. 3, 441–505.

[FMS03] J. Frehse, J. Málek, and M. Steinhauer, *On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method*, SIAM J. Math. Anal. **34** (2003), no. 5, 1064–1083.

[FS94] R. Farwig and H. Sohr, *Generalized resolvent estimates for the Stokes system in bounded and unbounded domains*, J. Math. Soc. Japan **46** (1994), no. 4, 607–643.

[Gal11] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, second ed., Springer Monogr. Math., Springer, New York, 2011.

[Göt12] D. Götz, *Three topics in fluid dynamics: Viscoelastic, generalized Newtonian, and compressible fluids*, Ph.D. thesis, Technischen Universität Darmstadt, 2012.

[Hat11] Y. Hataya, *A remark on Beale-Nishida's paper*, Bull. Inst. Math. Acad. Sin. (N. S.) **6** (2011), no. 3, 293–303.

[HK09] Y. Hataya and S. Kawashima, *Decaying solution of the Navier-Stokes flow of infinite volume without surface tension*, Nonlinear Anal. **71** (2009), no. 12, e2535–e2539.

[HS15] M. Hieber and H. Saito, *Strong solutions for two-phase free boundary problems for a class of non-Newtonian fluids*, submitted.

[KS07] T. Kakita and Y. Shibata, *From vector analysis to fluids*, Nippon Hyoron Sha, 2007, in Japanese.

[KS12] T. Kubo and Y. Shibata, *Nonlinear partial differential equations*, Asakura, 2012, in Japanese.

[KW04] P. C. Kunstmann and L. Weis, *Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus*, Functional Analytic Methods for Evolution Equations, Lect. Notes in Math. **1855**, Springer, Berlin, 2004, pp. 65–311.

[Mik65] S. G. Mikhlin, *Multidimensional Singular Integrals and Integral Equations*, Pure and Applied Mathematics Monograph, Pergamon Press, Oxford-New York-Paris, 1965.

[MNR01] J. Málek, J. Nečas, and M. Růžička, *On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains: The case $p \geq 2$*, Adv. Differential Equations **6** (2001), no. 3, 257–302.

[MS12] M. Meyries and R. Schnaubelt, *Interpolation, embeddings and traces of anisotropic fractional Sobolev spaces with temporal weights*, J. Funct. Anal. **262** (2012), no. 3, 1200–1229.

[PR01] A. Prohl and M. Růžička, *On fully implicit space-time discretization for motions of incompressible fluids with shear-dependent viscosities: The case $p \leq 2$*, SIAM J. Numer. Anal. **39** (2001), no. 1, 214–249.

[PS10a] J. Prüss and G. Simonett, *On the Rayleigh-Taylor instability for the two-phase Navier-Stokes equations*, Indiana Univ. Math. J. **59** (2010), no. 6, 1853–1871.

[PS10b] ———, *On the two-phase Navier-Stokes equations with surface tension*, Interfaces Free Bound. **12** (2010), no. 3, 311–345.

[PS11] ———, *Analytic solutions for the two-phase Navier-Stokes equations with surface tension and gravity*, Parabolic Problems, Progr. Nonlinear Differential Equations Appl. **80**, Birkhäuser/Springer Basel AG, Basel, 2011, pp. 507–540.

[Sai15] H. Saito, *On the $\mathcal{R}$-boundedness of solution operator families of the generalized stokes resolvent problem in an infinite layer*, Math. Methods Appl. Sci. (2015), in press.

[Shi13] Y. Shibata, *Generalized resolvent estimate of the Stokes equations with first order boundary condition in a general domain*, J. Math. Fluid Mech. **15** (2013), no. 1, 1–40.

[Shi15] Y. Shibata, *On some free boundary problem of the navier-stokes equaions in the maximal L_p-L_q regularity class*, submitted.

[SS01] Y. Shibata and S. Shimizu, *A decay property of the Fourier transform and its application to the Stokes problem*, J. Math. Fluid Mech. **3** (2001), no. 3, 213–230.

[SS03] ———, *On a resolvent estimate for the Stokes system with Neumann boundary condition*, Differential Integral Equations **16** (2003), no. 4, 385–426.

[SS08] ———, *On the L_p-L_q maximal regularity of the Neumann problem for the Stokes equations in a bounded domain*, J. Reine Angew. Math. **615** (2008), 157–209.

[SS09] ———, *On a resolvent estimate of the Stokes system in a half space arising from a free boundary problem for the Navier-Stokes equations*, Math. Nachr. **282** (2009), no. 3, 482–499.

[SS11] ———, *Maximal L_p-L_q regularity for the two-phase Stokes equations; model problems*, J. Differential Equations **251** (2011), no. 2, 373–419.

[SS12] ———, *On the maximal L_p-L_q regularity of the Stokes problem with first order boundary condition; model problems*, J. Math. Soc. Japan **64** (2012), no. 2, 561–626.

[SS15] H. Saito and Y. Shibata, *On decay properties of solutions to the Stokes equations with surface tension and gravity in the half space*, submitted.

[Tan95] N. Tanaka, *Two-phase free boundary problem for viscous incompressible thermocapillary convection*, Japan. J. Math. (N.S.) **21** (1995), no. 1, 1–42.

[Tan96] A. Tani, *Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface*, Arch. Rational Mech. Anal. **133** (1996), no. 4, 299–331.

[Tri83] H. Triebel, *Theory of function spaces*, 1st ed., Monographs in Mathematics, vol. 78, Birkhüser Verlag, Basel, 1983.

[TT95] A. Tani and N. Tanaka, *Large-time existence of surface waves in incompressible viscous fluids with or without surface tension*, Arch. Rational Mech. Anal. **130** (1995), no. 4, 303–314.

[Wei01] L. Weis, *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann. **319** (2001), no. 4, 735–758.

[Zim89] F. Zimmermann, *On vector-valued Fourier multiplier theorems*, Studia Math. **93** (1989), no. 3, 201–222.

List of original papers

[1] *On the $\mathcal{R}$-boundedness of solution operator families of the generalized Stokes resolvent problem in an infinite layer*, Math. Methods Appl. Sci., in press.