Asymptotic behaviors of solutions for free boundary problems of nonlinear diffusion equations

非線形拡散方程式の自由境界問題に対する解の漸近挙動

December 2014

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Acknowledgements

I would like to express hearty thanks to my supervisor, Professor Yoshio Yamada, who is the chief examiner responsible for my doctoral thesis. He led me to the study of partial differential equations and mathematical modeling, and spent a lot of time having discussions, reading my manuscripts, and instructing and advising me. I was always encouraged and energized by him. He also gave me a lot of opportunities to experience and participate in various conferences and seminars for presentations. I am very happy to get lots of instruction, advice and support from him.

I would like to greatly thank Professor Mitsuharu $\widehat{\mathrm{O}}$tani, Professor Kazunaga Tanaka and Professor Kousuke Kuto, who act as examiners for my doctoral thesis. I learned much about evolution equations from the lecture of Professor $\widehat{\mathrm{O}}$tani. He also gave me a lot of opportunities to talk in various seminars. I also learned a lot from lectures about variational methods by Professor Tanaka and reaction-diffusion systems by Professor Kuto.

I would like to express my gratitude to Professor Yihong Du, who gave a lecture to graduate students, during his stay at Waseda University in 2010, about his joint work [18] on a mathematical model which is mainly discussed in this thesis. I learned much about a free boundary problem from his lecture and started to study the model. I would also like to thank Professor Zhigui Lin for helpful suggestions and comments.

I would like to greatly thank Professor Yoshihiro Shibata, Professor Hideo Kozono, Professor Matthias Hieber, Professor Reinhard Farwig and the secretaries, who gave me opportunities to stay for short periods in Darmstadt, Germany in 2013 and 2014 and to study in Technische Universität Darmstadt (TU Darmstadt) as a member of “the JSPS Japanese-German Graduate Externship”. I would also like to give my gratitude to Professor Dieter Bothe and Professor Herbert Egger, who acted as host professors in TU Darmstadt, and Dr. Jan-Frederik Pietschmann for having many discussions during my stay at TU Darmstadt.

I am very grateful to Professor Hiroshi Matsuzawa and Professor Kazuhiro Oeda for joint work. They spent much time having discussions with me. They always gave me helpful advice and encouragement.

I am very thankful to Professor Shoji Yotsutani, Professor Eiji Yanagida and Professor Toyohiko Aiki for giving me many opportunities to talk in various conferences and seminars and also for helpful advice and encouragement; to Professor Hirokazu Ninomiya and Professor Goro Akagi for many discussions on the free boundary problem and for valuable suggestions and comments; to Professor Daisuke Takahashi for helpful advice on numerical analysis and also for encouragement; to Professor Tohru Ozawa, Professor Kazuhiro Kurata and Professor Satoshi Tanaka for valuable suggestions and comments; to Professor Yuzo Hosono, Professor Tatsuya Watanabe and Professor Yoshitsugu Kabeya for giving me a chance to talk in a seminar and for helpful comments; to Professor Shingo Takeuchi and Professor Tohru Wakasa for a lot of advice and encouragement; to Professor Hirokazu Ohya, Professor Yusuke Yamauchi, Professor Jun-ichi Harada, Professor Norihisa Ikoma, Dr. Yoichi Enatsu, Dr. Atsushi Kosaka and all the students in the room of analysis at Waseda University for encouragement.

Finally I would like to thank my family for their hearty support.

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Chapter 1

Introduction

Problems and settings

We study the following free boundary problems for nonlinear diffusion equations:

$$\begin{cases} u_t - d\Delta u = f(u), & t > 0,\ x \in \Omega(t), \\ Bu = 0, & t > 0,\ x \in \partial\Omega(t), \\ \mathrm{v} = -\mu \nabla u \cdot \mathbf{n}, & t > 0,\ x \in \Gamma(t), \\ \Omega(0) = \Omega_0,\ u(0,x) = u_0(x), & x \in \overline{\Omega}_0, \end{cases} \tag{1.1}$$

where $d,\ \mu,\ h_0$ are positive constants, Ω_0 is a bounded and radially symmetric domain in $\mathbb{R}^N$ and u_0 is a radially symmetric function satisfying $u_0 \in C^2(\overline{\Omega}_0)$, $u_0 > 0$ in Ω_0 and the same boundary condition at $\partial\Omega_0$ as that of u for $t > 0$. Moreover $\Omega(t)$ is also an N-dimensional radially symmetric domain with $N \geq 1$ whose moving boundary $\partial\Omega(t)$ consists of $(N-1)$-dimensional sets $\Gamma(t) = \{x \in \mathbb{R}^N |\ |x| = h(t)\}$ and Γ_{fix} in $\mathbb{R}^N$ ($\partial\Omega(t) := \Gamma(t) \cup \Gamma_{\text{fix}}$, $\Gamma(t) \cap \Gamma_{\text{fix}} = \emptyset$). In the third equation in (1.1), v denotes the outward normal velocity of free boundary $\Gamma(t)$ and $\mathbf{n}$ denotes an outward normal vector against $\Gamma(t)$. In the second condition in (1.1), B represents a boundary operator which, together with $\Omega(t)$, is defined as follows:

(1-a) $N = 1$ and $\Omega(t)$ has a fixed boundary ($\Gamma_{\text{fix}} \neq \emptyset$)   We set

$$\Delta u = u_{xx}, \quad \Omega(t) = (0, h(t)), \quad \Omega_0 = (0, h_0), \quad h_0 > 0.$$

Then we find $\Gamma(t) = \{h(t)\}$, $\Gamma_{\text{fix}} = \{0\}$, and $\mathrm{v} = -\mu\nabla u \cdot \mathbf{n}$ ($t > 0,\ x \in \Gamma(t)$) implies

$$h'(t) = -\mu u_x(t, h(t)) \quad \text{for} \quad t > 0.$$

Moreover we define $Bu = 0$ ($t > 0,\ x \in \partial\Omega(t)$) by one of the following conditions:

$$u(t,0) = 0, \quad u(t,h(t)) = 0 \quad \text{for} \quad t > 0; \tag{1.2}$$

$$u_x(t,0) = 0, \quad u(t,h(t)) = 0 \quad \text{for} \quad t > 0. \tag{1.3}$$

(1-b) $N = 1$ and $\Omega(t)$ has no fixed boundaries ($\Gamma_{\text{fix}} = \emptyset$)   We put

$$\Delta u = u_{xx}, \quad \Omega(t) = (g(t), h(t)), \quad \Omega_0 = (g_0, h_0), \quad g_0 < 0 < h_0,$$

Then we see $\Gamma(t) = \{h(t), g(t)\}$, and $\mathrm{v} = -\mu\nabla u \cdot \mathbf{n}$ $(t > 0,\ x \in \Gamma(t))$ becomes

$$\begin{aligned} h'(t) &= -\mu u_x(t, h(t)) \quad \text{for} \quad t > 0, \\ g'(t) &= -\mu u_x(t, g(t)) \quad \text{for} \quad t > 0. \end{aligned}$$

We denote $Bu = 0$ $(t > 0,\ x \in \partial\Omega(t))$ by

$$u(t, g(t)) = 0, \quad u(t, h(t)) = 0 \quad \text{for} \quad t > 0.$$

In the case of $N \geq 2$, we assume that $\Omega(t)$ is radially symmetric and denote $u = u(t, r)$ with $r = |x|$ $(x \in \mathbb{R}^N)$. We consider the following two cases:

(N-a) $N \geq 2$ and $\Omega(t)$ has a fixed boundary $(\Gamma_{\text{fix}} \neq \emptyset)$ We set

$$\begin{aligned} &\Delta u = u_{rr} + ((N-1)/r)u_r, \quad \Omega(t) = \{x \in \mathbb{R}^N |\ R < |x| < h(t)\}, \quad R \geq 0, \\ &\Omega_0 = \{x \in \mathbb{R}^N |\ R < |x| < h_0\}, \quad h_0 > R. \end{aligned}$$

Here $x \in \Omega(t)$ means $r = |x| \in (R, h(t))$ and $x \in \partial\Omega(t)$ does $r = |x| = R,\ h(t)$. Then we find $\Gamma(t) = \{|x| = h(t)\}$, $\Gamma_{\text{fix}} = \{|x| = R\}$, and $\mathrm{v} = -\mu\nabla u \cdot \mathbf{n}$ $(t > 0,\ x \in \Gamma(t))$ implies

$$h'(t) = -\mu u_r(t, h(t)) \quad \text{for} \quad t > 0.$$

Moreover we define $Bu = 0$ $(t > 0,\ x \in \partial\Omega(t))$ by one of the following conditions

$$\begin{aligned} u(t, R) = 0, \quad u(t, h(t)) = 0 \quad \text{for} \quad t > 0; \\ u_r(t, R) = 0, \quad u(t, h(t)) = 0 \quad \text{for} \quad t > 0 \end{aligned}$$

if $R > 0$. On the other hand if $R = 0$, then we set $Bu = 0$ $(t > 0,\ x \in \partial\Omega(t))$ by

$$u_r(t, 0) = 0, \quad u(t, h(t)) = 0 \quad \text{for} \quad t > 0.$$

(N-b) $N \geq 2$ and $\Omega(t)$ has no fixed boundaries $(\Gamma_{\text{fix}} = \emptyset)$ We put

$$\begin{aligned} &\Delta u = u_{rr} + ((N-1)/r)u_r, \quad \Omega(t) = \{x \in \mathbb{R}^N |\ g(t) < |x| < h(t)\}, \\ &\Omega_0 = \{x \in \mathbb{R}^N |\ g_0 < |x| < h_0\}, \quad 0 < g_0 < h_0, \end{aligned}$$

Here $x \in \Omega(t)$ means $r = |x| \in (g(t), h(t))$ and $x \in \partial\Omega(t)$ does $r = |x| = g(t),\ h(t)$. Then we find $\Gamma(t) = \{|x| = h(t), |x| = g(t)\}$, and $\mathrm{v} = -\mu\nabla u \cdot \mathbf{n}$ $(t > 0,\ x \in \Gamma(t))$ implies

$$\begin{aligned} h'(t) &= -\mu u_r(t, h(t)) \quad \text{for} \quad t > 0, \\ g'(t) &= -\mu u_r(t, g(t)) \quad \text{for} \quad t > 0. \end{aligned}$$

We define $Bu = 0$ $(t > 0,\ x \in \partial\Omega(t))$ by

$$u(t, g(t)) = 0, \quad u(t, h(t)) = 0 \quad \text{for} \quad t > 0.$$

Problem (1.1) is gifted with some backgrounds of both mathematics and natural phenomenon. From a mathematical view-point, (1.1) is called a free boundary problem, where we seek for positive solutions $u(t,x)$ together with the moving interface $\Gamma(t)$ between $\Omega(t) = \{x \in \mathbb{R}^N | \ u(t,x) > 0\}$ and $\{x \in \mathbb{R}^N | \ u(t,x) = 0\}$. The problem consists of four parts: a nonlinear reaction-diffusion equation, boundary conditions, a one-phase Stefan condition and initial conditions. Until now a nonlinear reaction-diffusion equation of the form:

$$u_t - d\Delta u = f(u), \quad t > 0, \ x \in \Omega \tag{1.4}$$

with a fixed domain $\Omega \subset \mathbb{R}^N$, has been studied by a lot of researchers. In (1.4), the unknown function u is determined by the effect of diffusion $d\Delta u$ with a diffusive coefficient $d > 0$ and nonlinear interaction $f(u)$ in fixed domain Ω. The problem has been studied as one of the most important equations to reveal nonlinear phenomena because it is a simple extension of heat equation $u_t = d\Delta u$, and it also creates rich nonlinear phenomena (cf. Smoller [61], Cantrell-Cosner [10] and Henry [35]). Differently from (1.4), the equation in (1.1) is defined in a domain part of whose boundary is moving as time passes, and its behavior is determined by so called Stefan condition

$$\mathrm{v} = -\mu \nabla u \cdot \mathbf{n}.$$

In a modeling of natural phenomena, problem (1.1) has been also studied since J. Stefan started his work concerning (1.1) in 1889. He modeled the melting of ice to water by (1.1) with $f(u) \equiv 0$, where u represents temperature of water and $\Gamma(t)$ is an interface of ice and water at $t > 0$. This model is now called the Stefan problem, and the existence, uniqueness and asymptotic behaviors of solutions for the Stefan problem have been investigated in detail (cf. Rubinstein [57], Friedman [25], Meirmanov [50], Gupta [32] and Nogi-Yamaguchi [54]). In addition, problem (1.1) can be used to model various phenomena. For example, we can observe a blow-up phenomenon for (1.1) in a case where $f(u) = u^p$ for $p > 1$. Indeed it was first observed by Fujita [27] that the solution u of (1.4) with $f(u) = u^p$ $(p > 1)$ with initial data can blow up as $t \to \infty$. By virtue of a comparison principle, we find that the solution of free boundary problem (1.1) can also blow up as $t \to \infty$. Moreover all the time global solution have to decay as $t \to \infty$. Such interesting phenomena were shown in e.g. Aiki [1], Ghidouche-Souplet-Tarzia [28], Souplet [63], Zhang-Lin [68] (see also the references therein). In this thesis, we discuss (1.1) where the nonlinearity basically satisfies

$$f \in C^1[0,\infty), \quad f(0) = f(1) = 0, \quad f(u) < 0 \ \text{ for } \ u > 1$$

which includes monostable/logistic nonlinearity (e.g. $f(u) = u(1-u)$), bistable nonlinearity (e.g. $u(u-c)(1-u)$ for $0 < c < 1$) and polystable nonlinearity (e.g. $f(u) = u(u-\alpha_1)\cdots(u-\alpha_n)(1-u)$ for $0 < \alpha_1 \le \cdots \le \alpha_n \le 1$). These nonlinearities are usually used in study of population dynamics in mathematical ecology and they imply population growth or competition among species. As we have seen above, problem (1.1) has many variations and it can be seen as one of the most important and interesting problem in pure and applied mathematics.

Mathematical models for the spread of species

The spread of species is one of the most important problem in mathematical ecology. The species here include new or native animals or plants; non-native or disease infected ones; and ones which have a bad influence on humans or ecosystems. There are lots of works concerning the invasion ecology (cf. e.g. Lockwood-Hoopes-Marchetti [48] and Shigesada-Kawasaki [59]). For example, Skellam showed in his work [60] constant (linear) spreading speed of muskrat which had been introduced to Europe in 1905 (see Figure 1).

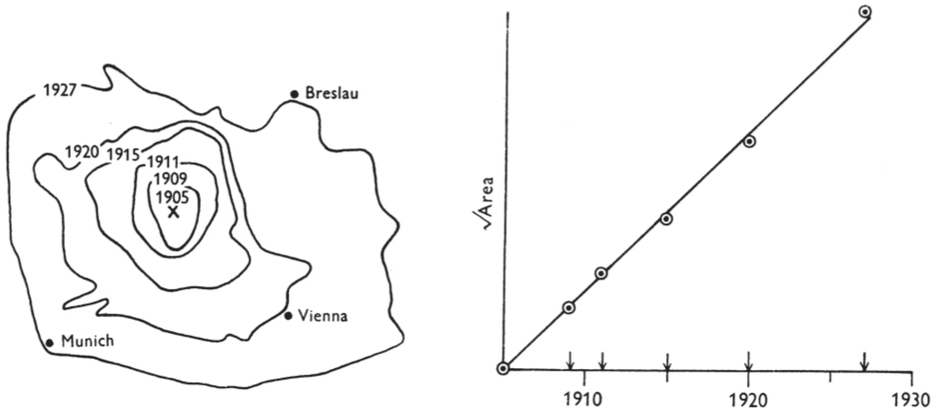

Figure1. Spread of muskrat (reference to Skellam [60, p 200 (Fig.1, Fig.2)])

To model the linear spread speed, traveling wave solutions have been used so far. Consider the Fisher-KPP equation (cf. Fisher [24] and Kolmogorov-Petrovsky-Piskunov [42])

$$u_t - du_{xx} = u(a - bu), \quad t > 0, \ x \in \mathbb{R}, \tag{1.5}$$

where a, b are positive constants and u represents population density. We seek for a solution of (1.5) of the form: $u(t, x) = w(x - ct)$, where c is a constant representing spread speed. Then we find a unique positive solution (**traveling wave solution**) of (1.5) with $\lim_{x\to-\infty} u(t, x) = a/b$, $\lim_{x\to+\infty} u(t, x) = 0$ for all $t > 0$, if $|c| \geq 2\sqrt{ad}$ (see Figure 2). This result implies that the whole region must be occupied at a linear speed by the wave of the population with a front $x = ct$ as $t \to \infty$.

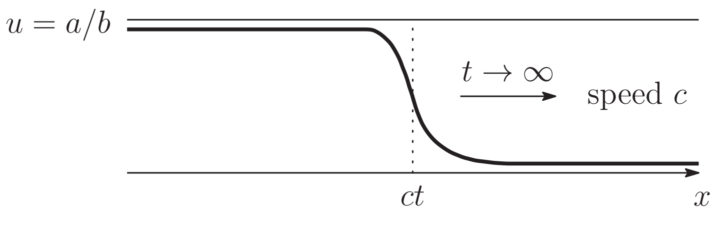

Figure 2. Traveling wave solutions for (1.5)

It is known that there are spreads of nonlinear speed in invasion phenomena (cf. Hastings et al. [34]). For these processes, the traveling wave solutions have also greatly helped us to understand the spatial spread of invasive species.

We consider another model for the spread of species, using a free boundary with a Stefan condition. In population dynamics, a free boundary problem with a kind of

Stefan conditions was studied by Mimura-Yamada-Yotsutani [51, 52, 53] and Lin [45] for two-species models. Also for a single species case, a new model was proposed by Du-Lin [18] in 2010. The model is described as a free boundary problem:

$$\begin{cases} u_t - du_{xx} = u(a - bu), & t > 0,\ 0 < x < h(t), \\ u_x(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \le x \le h_0, \end{cases} \tag{1.6}$$

where d, μ, a, b and h_0 are positive constants, u_0 is a smooth function. In (1.6), $u = u(t,x)$ means a population density in time t and location x, and moving boundary $x = h(t)$ denotes a spreading front of one-dimensional habitat $(0, h(t))$. A characteristic point of solutions for (1.6), compared with (1.5), is that the spreading front is precisely described as a free boundary (see Figure 3).

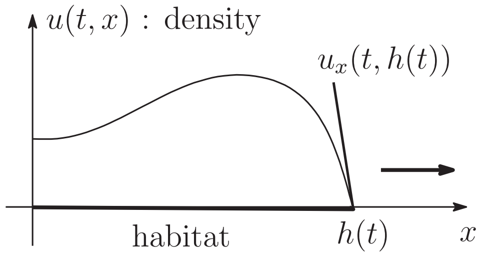

Figure 3. The solution $(u(t,x), h(t))$ of free boundary problem (1.6)

Moreover the behavior of the free boundary is determined by Stefan condition $h'(t) = -\mu u_x(t,h(t))$, which ecologically means that the speed of the propagation front is proportional to the population pressure at the spreading front. To consider the validness of the introduction of the Stefan condition to this ecological problem, it may be better to go back to Skellam's investigation. It has been proved in [18] that, for any solution of (1.6), the free boundary have to satisfy $\lim_{t\to\infty} h(t)/t = k_0$ for some constant $k_0 > 0$. This result is suited to the situation observed by Skellam that the propagation front spreads at a constant speed. We can refer to Bunting-Du-Krakowski [6] and Lin [45] for more numerical and theoretical information. We can also find the development of spreading speed analysis in Chapter 4. In [18], they also showed the existence and uniqueness of solutions for (1.6) and proved a remarkable result on the asymptotic behaviors of solutions as $t \to \infty$. We still have other approaches to model the spread of species, for which we can refer to e.g. [48].

Spreading and vanishing

It is characteristic of free boundary model (1.1) that the asymptotic behaviors of solutions as $t \to \infty$ are divided into two cases called **spreading (propagation)** and **vanishing (extinction)**. This phenomenon was first observed in (1.6) by Du-Lin [18]. They proved that any solution $(u(t,x), h(t))$ of (1.6) satisfies one of the following properties as $t \to \infty$:

(i) Spreading: $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} u(t,x) = a/b$ locally uniformly in $[0,\infty)$;

(ii) Vanishing: $\lim_{t\to\infty} h(t) \le (\pi/2)\sqrt{d/a}$, $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(0,h(t))} = 0$,

where spreading means the species succeed to spread to the whole region and get a constant spatial distribution there, while vanishing implies the habitat of the species must stay in a bounded interval and the population density converges uniformly to 0 in large time. They call this phenomenon **spreading-vanishing dichotomy**. Since the work of Du-Lin [18], such dichotomy results have been studied by many researchers from various aspects. It has turned out that such a dichotomy phenomenon still holds true in any dimension, for various boundary conditions at fixed boundary and various nonlinearities. However the behaviors of solutions as $t \to \infty$ are quite different from each factor. The dichotomy result obtained by [18] was extended to a multi-dimensional radially symmetric problem for a monostable equation in Du-Guo [14], and Kaneko-Yamada [39] have first discussed the problem with different boundary condition and more general nonlinearity including monostable and bistable ones. We now have understood more detailed results focusing on various nonlinear terms by Du-Lou [20], Kaneko-Oeda-Yamada [38] and Liu-Lou [46, 47]; a sharp estimate of spreading speed and an asymptotic profile of solutions by Du-Matsuzawa-Zhou [22, 23]; a spreading speed analysis by Du-Liang [17]; multi-dimensional radially symmetric problems with various nonlinearities and boundary conditions by Kaneko [36]; a multi-dimensional problem in a general domain by Du-Guo [15] and Du-Matano-Wang [21].

In this thesis we will further prove **a general dichotomy theorem** which allows, as in (1.1), more general polystable nonlinearity, various boundary conditions and any spatial dimension. The theorem shows, for any solution of (1.1) where the nonlinear function satisfies

$$f \in C^1[0,\infty), \quad f(0) = f(1) = 0, \quad f(u) < 0 \text{ for } u > 1, \quad f'(0) \neq 0, \tag{1.7}$$

either (i) or (ii) holds true as $t \to \infty$:

(i) Spreading: $\lim_{t\to\infty} \Omega(t) = \mathbb{R}^N \setminus B_R$, $\liminf_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} > 0$;

(ii) Vanishing: $\lim_{t\to\infty} \Omega(t)$ is a bounded set in $\mathbb{R}^N \setminus B_R$, $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} = 0$. Moreover $\|u(t,\cdot)\|_{C(\Omega(t))} = O(e^{-\beta t})$ for some $\beta > 0$ as $t \to \infty$,

where R is a non-negative constant defined in (1.1), constant β depends on $f'(0)$, B_R is a multi-dimensional ball centered in $\mathbb{R}^N$ with radius R and $\lim_{t\to\infty} \Omega(t) = \mathbb{R}^N \setminus B_R$ means $\lim_{t\to\infty} \Omega(t) \supset M$ for any subset M in $\mathbb{R}^N \setminus B_R$. Although spreading leaves a possibility of classifying the behavior of u in more detail, it is easily understood from this dichotomy theorem whether the species can spread and survive or not. For the proof, we need to reveal an underlying principle to determine spreading and vanishing.

Criteria for spreading and vanishing

It is very important to give criteria for spreading and vanishing; that is, for any given initial data (u_0, h_0) and arbitrary given coefficients in the problem, we need to consider weather the species will actually spread or vanish in large time. It also corresponds to giving sufficient conditions of solutions for spreading or vanishing as $t \to \infty$. Generally speaking, we can show that, as $t \to \infty$, spreading occurs if initial data u_0 and the radius of Ω_0 are sufficiently large, while vanishing occurs if both u_0 and the radius of Ω_0 are small enough. One of important ways to give a criterion for spreading or vanishing is to introduce a new parameter $\sigma > 0$ and a sample function ϕ which belongs to the same class of functions as u_0. Then we consider the solution of (1.1) where f satisfies (1.7), and vary the parameter to determine a threshold number $\sigma^* > 0$ depending on ϕ and the radius of Ω_0 such that

- if $u_0 > \sigma^*\phi$ in Ω_0, then spreading occurs;
- if $u_0 < \sigma^*\phi$ in Ω_0, then vanishing occurs;
- if $u_0 = \sigma^*\phi$ in Ω_0, then vanishing occurs for $f'(0) > 0$, while spreading occurs for $f'(0) < 0$.

A similar criterion has been discussed by [20] in one dimension. Moreover, besides this criterion, we can give another criterion and observe different behaviors of solutions from each sign of $f'(0)$. If f is of (1.7) with $f'(0) > 0$, then we can vary the speed parameter μ in the Stefan condition. We find that there exists a threshold radius $|x| = R_N^*$, depending on d, R and $f'(0)$, which separates spreading and vanishing. In other words, when vanishing occurs, outer free boundary $h(t)$ must satisfy $\lim_{t\to\infty} h(t) \leq R_N^*$, and spreading always occur if $h_0 \geq R_N^*$. Hence it is suggested that, even if $h_0 < R_N^*$, the free boundary can go across the radius for sufficiently large μ, and thus the species necessarily spread to the whole region as $t \to \infty$. In fact there exists a threshold number $\mu^* < \infty$ depending on u_0 and the radius of Ω_0 such that

- if $\mu > \mu^*$, then spreading occurs;
- if $\mu \leq \mu^*$, then vanishing occurs.

This result is not true to the case where f satisfies (1.7) with $f'(0) < 0$ because of an Allee effect. However we may find a threshold (transition) phenomenon if $u_0 = \sigma^*\phi$ in $\overline{\Omega}_0$. In fact this phenomenon has been observed in several papers for $N = 1$ and bistable nonlinearity (see Remark 2.6).

Main purpose

The main purpose of this paper is to show the following:

- the existence and uniqueness of solutions for (1.1) (well-posedness of the model)
- spreading and vanishing for the asymptotic behaviors of solutions as $t \to \infty$ (eventual habitat and spatial distribution)

The existence and uniqueness of global solutions and the continuous dependence on parameters are essential to study ecological models. We will reveal an underlying principle to determine spreading or vanishing, and then we can prove the general dichotomy theorem, give criteria for spreading or vanishing, and show a decay rate of solutions when vanishing occurs as $t \to \infty$.

Our main methods of analysis are (strong) maximum principles for parabolic and elliptic equations, comparison principles, monotone methods, construction of upper and lower solutions, an energy method, a zero number argument, and some ODE methods including Sturm's comparison theorem.

Related free boundary models

We will here briefly introduce papers about related free boundary problems; a model in time-periodic environment by Du-Guo-Peng [16] and one in heterogeneous environment by Zhou-Xiao [70]; a free boundary problem for advection-diffusion equations by Gu-Lin-Lou [29, 30] and Kaneko-Matsuzawa [37]; a free boundary problem with another Stefan-like condition by Cai [7] and Cai-Lou-Zhou [8]; a free boundary model for a prey-predator system by Wang [66] and Zhao-Wang [69], and one for a competition system by Guo-Wu [31], Du-Lin [19] and Wang-Zhang [67]; a free boundary model for seasonal succession by Peng-Zhao [55]. There are interesting applications for a SIR model by Kim-Lin-Zhang [40] and Kim-Lin-Zhu [41], and also for information diffusion by Lei-Lin-Wang [44]. Although we could not cover all papers and preprints concerning these models in the thesis, we may find them from the above papers and references therein.

The rest of the thesis is organized as follows; in Chapter 2 we study free boundary problems (1.1) in the case of (1-a); in Chapter 3 we discuss the multi-dimensional problems for (N-a) and (N-b); in Chapter 4 we consider the spreading speed of the propagation front for the problem with (1-b).

Chapter 2

A free boundary problem in one dimension

2.1 Problem

In this chapter we study the following free boundary problem:

$$
\text{(FBP)} \qquad \begin{cases} u_t - du_{xx} = f(u), & t > 0,\ 0 < x < h(t), \\ u(t,0) = 0 \text{ (resp. } u_x(t,0) = 0), & t > 0, \\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \le x \le h_0, \end{cases}
$$

where $\mu,\ d$ and h_0 are positive constants, and initial function u_0 satisfies

$$
u_0 \in C^2[0,h_0],\ u_0 > 0 \text{ in } (0,h_0),\ u_0(0) = u_0(h_0) = 0 \text{ (resp. } u_0'(0) = u_0(h_0) = 0). \quad (2.1)
$$

Throughout sections 2.2 - 2.4, we discuss problem (FBP) for nonlinear reaction-diffusion equations with a nonlinear function $f = f(u)$ belonging to the following space:

$$
S_f := \{f : [0,\infty) \to \mathbb{R} |\ f \text{ is locally Lipschitz continuous},\ f(0) = 0,\ f(u) < 0 \text{ for } u > 1\},
$$

where f is called **locally Lipschitz continuous** if and only if, for any bounded set $K \subset [0,\infty)$, there exists a positive constant L such that

$$
|f(x) - f(y)| \le L|x - y| \quad \text{for} \quad x, y \in K. \quad (2.2)
$$

We again recall that $u = u(t,x)$ represents the population density of a species whose habitat is an interval $(0,h(t))$ (see Figure 4). At the fixed boundary $x = 0$, $u(t,0) = 0$ is called **the Dirichlet boundary condition** which implies that a region $(-\infty,0]$ is a hostile environment for the species, while $u_x(t,0) = 0$ is called **the Neumann boundary condition** which means that the species cannot enter the region.

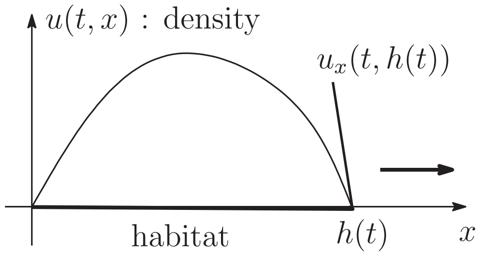

Figure 4. The solution of (FBP) with the Dirichlet boundary condition at $x = 0$.

The main purpose of this chapter is to show

- the existence and uniqueness of solutions for (FBP) (well-posedness of the model)
- the large time behaviors of solutions
- give a criterion for spreading or vanishing

Moreover we will find that the asymptotic behaviors of solutions for (FBP) as $t \to \infty$ are closely related to an elliptic boundary-value problem in the bounded domai

$$\begin{cases} dq_{xx} + f(q) = 0, \quad 0 < x < l, \\ q(x) > 0, \quad 0 < x < l, \\ q(0) = q(l) = 0 \text{ (resp. } q_x(0) = q(l) = 0) \end{cases} \tag{2.3}$$

and an elliptic problem in a half interval:

$$\begin{cases} dv_{xx} + f(v) = 0, \quad x > 0, \\ v(x) > 0, \quad x > 0, \\ v(0) = 0 \text{ (resp. } v_x(0) = 0). \end{cases} \tag{2.4}$$

We denote $\Omega(t) = (0, h(t))$ and $D(t) = \bigcup_{0<s\le t}\{s\} \times \Omega(s)$ (see figure 1). The main results and their proofs in this chapter are based on works [39] and [38].

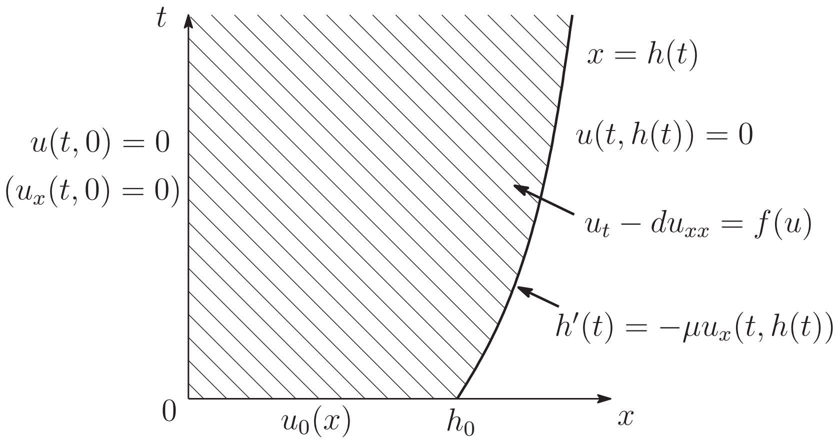

Figure 5. the domain of (FBP)

2.2 Existence and uniqueness of solutions

In this section we prove there exists a unique global solution for (FBP) and the solution depends continuously on initial data and coefficients in the equation. For this purpose we show the local existence and uniqueness of solutions and a priori estimates for $u(t,x)$ and $h'(t)$ that is essential to extend the local solution globally in time.

The main theorems are given as follows.

Theorem 2.1. *Let* (u_0, h_0) *satisfy* (2.1) *and let* $f \in S_f$. *For any given constant* $\alpha \in (0,1)$, *there exists a positive number* T *such that* (FBP) *has a unique solution*

$$(u,h) \in \{C^{\frac{(1+\alpha)}{2},1+\alpha}(\overline{D(T)}) \cap C^{1+\frac{\alpha}{2},2+\alpha}(D(T))\} \times C^{1+\frac{\alpha}{2}}[0,T],$$

where T *is depending on* h_0, α *and* $\|u_0\|_{C^2[0,h_0]}$.

Theorem 2.2. *Problem* (FBP) *has a unique classical solution* (u,h) *and there exist positive constants* C_1, C_2 *depending on* $\|u_0\|_{C(0,\infty)}$ *and* $\|u_0\|_{C^1(0,\infty)}$ *respectively such that*

$$0 < u(t,x) \le C_1 \quad \text{for } (t,x) \in \bigcup_{t\ge 0}(\{t\} \times \Omega(t)), \quad 0 < h'(t) \le \mu C_2 \quad \text{for } t \ge 0.$$

Theorem 2.3. *The solution of* (FBP) *depends continuously on initial data* (u_0, h_0), *coefficients* d, μ *and nonlinearity* f *in the equation.*

We prove Theorems 2.1 –2.3 in the following.

Proof of Theorem 2.1. We prove this theorem in the same way as [11], [18] and [45] and especially follow the argument in [18]. The difference is that we deal with general nonlinearity $f(u)$ in the equation. We divide the proof into three steps.
Step1. Change the variable from free boundary to fixed boundary.

Let $\zeta(y)$ be a $C^\infty[0,\infty)$-function satisfying

$$|\zeta'(y)| < \frac{6}{h_0}, \quad \zeta(y) = \begin{cases} 1, & |y-h_0| < \frac{h_0}{4}, \\ 0, & |y-h_0| > \frac{h_0}{2}. \end{cases}$$

Using this function, we change the variable by

$$(t,x) \longrightarrow (t,y); \quad x = \varphi(y) := y + \zeta(y)(h(t)-h_0).$$

The transformation means, if $|y-h_0| > h_0/2$, then $x = y$, while if $|y-h_0| < h_0/4$, then $x = y + h(t) - h_0$, and it especially changes the free boundary $x = h(t)$ to the fixed line $y = h_0$. Moreover $x = \varphi(y)$ is monotone increasing with respect to $y \in [0,\infty)$ as long as

$$|h(t) - h_0| \le \frac{h_0}{8} \tag{2.5}$$

because $\varphi'(y) = 1 + \zeta'(y)(h(t) - h_0) > 1/4$. Hence the above transformation is a diffeomorphism from $[0, \infty)$ onto $[0, \infty)$. We calculate

$$\begin{aligned} \frac{\partial y}{\partial x} &= \frac{1}{1 + \zeta'(y)(h(t) - h_0)} =: \sqrt{A(t, y)}, \\ \frac{\partial^2 y}{\partial x^2} &= -\frac{\zeta''(y)(h(t) - h_0)}{[1 + \zeta'(y)(h(t) - h_0)]^3} =: B(t, y), \\ -\frac{1}{h'(t)} \frac{\partial y}{\partial t} &= \frac{\zeta(y)}{1 + \zeta'(y)(h(t) - h_0)} =: C(t, y). \end{aligned}$$

Then, setting $w(t, y) = u(t, y + \zeta(y)(h(t) - h_0))$, we get

$$u_t = w_t - h'(t)C(t, y)w_y, \quad u_x = \sqrt{A(t, y)}w_y,$$

$$u_{xx} = A(t, y)w_{yy} + B(t, y)w_y.$$

Therefore (FBP) is equivalent to

$$\begin{cases} w_t - dA(t, y)w_{yy} - (dB(t, y) + h'(t)C(t, y))w_y = f(w), & t > 0, \ 0 < y < h_0, \\ w(t, 0) = 0 \text{ (resp. } w_y(t, 0) = 0), & t > 0, \\ w(t, h_0) = 0, & t > 0, \\ w(0, y) = u_0(y), & 0 \le y \le h_0 \end{cases} \tag{2.6}$$

and

$$h'(t) = -\mu w_y(t, h_0) \ \text{ for } \ t > 0, \quad h(0) = h_0, \tag{2.7}$$

where $A(t, y)$, $B(t, y)$ and $C(t, y)$ are smooth functions for $(t, y) \in [0, \infty) \times [0, h_0]$. For $0 < T < h_0/(8(1 + H_0))$, we define

$$\begin{aligned} D_{1T} &= \{w \in C(Q_T) \mid w(0, y) = u_0(y), \ \|w - u_0\|_{C(Q_T)} \le 1\}, \\ D_{2T} &= \{h \in C^1[0, T] \mid h(0) = h_0, \ h'(0) = H_0, \ \|h' - H_0\|_{C[0,T]} \le 1\}, \end{aligned}$$

where $Q_T = [0, T] \times [0, h_0]$ and $H_0 := -\mu u_0'(h_0)$. Then $D = D_{1T} \times D_{2T}$ is metric space with the metric

$$d((w_1, h_1), (w_2, h_2)) = \|w_1 - w_2\|_{C(Q_T)} + \|h_1' - h_2'\|_{C[0,T]}.$$

Also, by $\|h_1 - h_2\|_{C[0,T]} \le T\|h_1' - h_2'\|_{C[0,T]}$ for all $h_1, h_2 \in D_{2T}$, we have

$$\begin{aligned} |h(t) - h_0| &\le T|h'(t)| \\ &\le T(1 + H_0) \\ &\le h_0/8, \end{aligned}$$

and hence the transformation $(t, x) \to (t, y)$ is valid by (2.5).

Step 2. *Define a mapping $F : \ D \to C(Q_T) \times C^1[0, T]$.*

To construct the solution for the problem (2.6), we will use the contraction mapping theorem. For any $(w, h) \in D$, consider the following linear partial differential equation with time dependent coefficients:

$$\begin{cases} \overline{w}_t - dA(t,y)\overline{w}_{yy} - (dB(t,y) + h'(t)C(t,y))\overline{w}_y = f(w), & 0 < t < T,\ 0 < y < h_0, \\ \overline{w}(t,0) = 0 \text{ (resp. } \overline{w}_y(t,0) = 0), & 0 < t < T, \\ \overline{w}(t,h_0) = 0, & 0 < t < T, \\ \overline{w}(0,y) = u_0(y), & 0 \le y \le h_0, \end{cases}$$

where d, h_0, $A(t,y)$, $B(t,y)$ and $C(t,y)$ are the same as those in (2.6). It is well known that, by the L^p-estimate for parabolic equations, the problem admits a unique solution $\overline{w} \in W_p^{1,2}(Q_T)$ for any $p > 1$ with

$$\|\overline{w}\|_{W_p^{1,2}(Q_T)} \le C_0(\|u_0\|_{W_p^2[0,h_0]} + \|f(w)\|_{L^p(Q_T)}). \tag{2.8}$$

for some constant $C_0 > 0$ (cf. [43]). In addition we can easily choose $C_1 = C_1(h_0) > 0$ to satisfy

$$\|u_0\|_{W_p^2[0,h_0]} \le C_1\|u_0\|_{C^2[0,h_0]}. \tag{2.9}$$

For large p, Sobolev's embedding theorem (cf. [43]) shows $\overline{w} \in C^{\frac{1+\alpha}{2},1+\alpha}(Q_T)$ and there exists $C_2 = C_2(\alpha) > 0$ such that

$$\|\overline{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(Q_T)} \le C_2\|\overline{w}\|_{W_p^{1,2}(Q_T)}. \tag{2.10}$$

Therefore it follows from (2.8) – (2.10) that

$$\|\overline{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(Q_T)} \le C, \tag{2.11}$$

where $C > 0$ is a constant depending on h_0, α and $\|u_0\|_{C^2[0,h_0]}$.

We next define

$$\overline{h}(t) = h_0 - \mu\int_0^t \overline{w}_y(s,h_0)\, ds.$$

Then it follows that

$$\overline{h}'(t) = -\mu\overline{w}_y(t,h_0),\ \overline{h}(0) = h_0,\ \overline{h}'(0) = H_0. \tag{2.12}$$

Hence $\overline{h} \in C^1[0,T]$ and it holds from (2.11) and (2.12) that

$$\|h'\|_{C^{\frac{\alpha}{2}}[0,T]} \le \mu C.$$

We now define the mapping $F:\ D \to C(Q_T) \times C^1[0,T]$ by $F(w,h) = (\overline{w}, \overline{h})$. Then,

F maps D into itself. Indeed

$$\begin{aligned}
\|\overline{w}-u_0\|_{C(Q_T)} &= \max_{t\in[0,T],\ y\in[0,h_0]} |\overline{w}(t,y)-\overline{w}(0,y)| \\
&\le \max_{t\in[0,T],\ y\in[0,h_0]} \left\{ \frac{|\overline{w}(t,y)-\overline{w}(0,y)|}{t^{\frac{1+\alpha}{2}}}\, t^{\frac{1+\alpha}{2}} \right\} \\
&\le \max_{t,s\in[0,T],\ y\in[0,h_0],\ t\neq s} \left\{ \frac{|\overline{w}(t,y)-\overline{w}(s,y)|}{|t-s|^{\frac{1+\alpha}{2}}} \right\} T^{\frac{1+\alpha}{2}} \\
&= \|\overline{w}\|_{C^{\frac{(1+\alpha)}{2},0}(Q_T)} T^{\frac{1+\alpha}{2}} \\
&\le CT^{\frac{1+\alpha}{2}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|\overline{h}' - H_0\|_{C[0,T]} &= \max_{t\in[0,T]} |\overline{h}'(t)-\overline{h}'(0)| \\
&\le \max_{t,s\in[0,T],\ t\neq s} \left\{ \frac{|\overline{h}'(t)-\overline{h}'(s)|}{|t-s|^{\frac{\alpha}{2}}} \right\} T^{\frac{\alpha}{2}} \\
&= \|\overline{h}'\|_{C^{\frac{\alpha}{2}}[0,T]} T^{\frac{\alpha}{2}} \\
&\le \mu C T^{\frac{\alpha}{2}}.
\end{aligned}$$

Setting $T \le \min\{C^{-2/(1+\alpha)},\ (\mu C)^{-2/\alpha}\}$, we see

$$\|\overline{w}-u_0\|_{C(Q_T)} \le 1,\ \ \|\overline{h}'-h_0\|_{C[0,T]} \le 1.$$

Hence, recalling (2.12) and $\overline{w}(0,y)=u_0(y)$, we find $F(w,h)=(\overline{w},\overline{h})\in D$.

Step 3. *F is a contraction mapping and there exists a unique solution for (FBP).*

For any $(w_i,h_i)\in D$ $(i=1,2)$, we set $(\overline{w}_i,\overline{h}_i)=F(w_i,h_i)$ $(i=1,2)$ and $U=\overline{w}_1-\overline{w}_2$. Then $U=U(t,y)$ satisfies

$$\begin{cases}
U_t - dA_2'(t,y)U_{yy} - (dB_2(t,y)+h_2'(t)C_2(t,y))U_y = g(t,y), & 0<t<T,\ 0<y<h_0,\\
U(t,0)=0 \text{ (resp. } U_y(t,0)=0), & 0<t<T,\\
U(t,h_0)=0, & 0<t<T\ ,\\
U(0,y)=0, & 0\le y\le h_0,
\end{cases}$$

where $A_i(t,y),\ B_i(t,y),\ C_i(t,y)$ $(i=1,2)$ is generated by $h_i(t)$, and $g(t,y)=d(A_1(t,y)-A_2(t,y))(\overline{w}_1)_{yy}+d(B_1(t,y)-B_2(t,y))(\overline{w}_1)_y+(h_1'(t)C_1(t,y)-h_2'(t)C_2(t,y))(\overline{w}_1)_y+f(w_1)-f(w_2)$. Using the L^p-estimate for parabolic equations, we find that

$$\|\overline{w}_1-\overline{w}_2\|_{W_p^{1,2}(Q_T)} = \|U\|_{W_p^{2,1}(Q_T)} \le C_0\|g\|_{L^p(Q_T)}. \tag{2.13}$$

Moreover, by the direct calculations, the right hand side of (2.13) is estimated as follows:

$$\begin{aligned}
\|d(A_1-A_2)(\overline{w}_1)_{yy}\|_{L^p(Q_T)} &\le D_1\|h_1-h_2\|_{C[0,T]},\\
\|d(B_1-B_2)(\overline{w}_1)_y\|_{L^p(Q_T)} &\le D_2\|h_1-h_2\|_{C[0,T]},\\
\|(h_1'C_1-h_2'C_2)(\overline{w}_1)_y\|_{L^p(Q_T)} &\le D_3\|h_1-h_2\|_{C[0,T]}+D_4\|h_1'-h_2'\|_{C[0,T]},\\
\|f(w_1)-f(w_2)\|_{L^p(Q_T)} &\le D_5\|w_1-w_2\|_{C(Q_T)},
\end{aligned}$$

where $D_1,\ D_2,\ D_3,\ D_4$ and D_5 are positive constants depending on h_0. Thus, there exists $D_6 = D_6(h_0)$ such that

$$\|g\|_{L^p(Q_T)} \le D_6(\|w_1 - w_2\|_{C(Q_T)} + \|h_1 - h_2\|_{C^1[0,T]}). \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$\|\overline{w}_1 - \overline{w}_2\|_{W_p^{1,2}(Q_T)} \le C_0 D_6(\|w_1 - w_2\|_{C(Q_T)} + \|h_1 - h_2\|_{C^1[0,T]}).$$

Hence, for large p, Sobolev's embedding theorem shows

$$\|\overline{w}_1 - \overline{w}_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}(Q_T)} \le D(\|w_1 - w_2\|_{C(Q_T)} + \|h_1 - h_2\|_{C^1[0,T]}), \tag{2.15}$$

where the constant D depends on h_0 and α. On the other hand

$$\begin{aligned} \|\overline{h}_1' - \overline{h}_2'\|_{C^{\frac{\alpha}{2}}[0,T]} &\le \mu\|\overline{w}_{1,y} - \overline{w}_{2,y}\|_{C^{\frac{\alpha}{2},0}(Q_T)} \\ &\le \mu\|\overline{w}_1 - \overline{w}_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}(Q_T)}. \end{aligned} \tag{2.16}$$

In addition, referring to the estimate in Step 2, we get

$$\begin{aligned} \|\overline{w}_1 - \overline{w}_2\|_{C(Q_T)} &\le \|\overline{w}_1 - \overline{w}_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}(Q_T)} T^{\frac{1+\alpha}{2}}, \\ \|\overline{h}_1' - \overline{h}_2'\|_{C[0,T]} &\le \|\overline{h}_1' - \overline{h}_2'\|_{C^{\frac{\alpha}{2}}[0,T]} T^{\frac{\alpha}{2}}. \end{aligned} \tag{2.17}$$

By using (2.17) and $T^{\frac{1+\alpha}{2}} \le T^{\frac{\alpha}{2}}$ for small T, we obtain

$$\begin{aligned} \|F(w_1,h_1) - F(w_2,h_2)\| &= d((\overline{w}_1, \overline{h}_1), (\overline{w}_2, \overline{h}_2)) \\ &= \|\overline{w}_1 - \overline{w}_2\|_{C(Q_T)} + \|\overline{h}_1' - \overline{h}_2'\|_{C[0,T]} \\ &\le T^{\frac{\alpha}{2}}(\|\overline{w}_1 - \overline{w}_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}(Q_T)} + \|\overline{h}_1' - \overline{h}_2'\|_{C^{\frac{\alpha}{2}}[0,T]}. \end{aligned}$$

It follows from (2.15) and (2.16) that

$$\begin{aligned} \|F(w_1,h_1) - F(w_2,h_2)\| &\le T^{\frac{\alpha}{2}} D(\mu+1)(\|w_1 - w_2\|_{C(Q_T)} + \|h_1 - h_2\|_{C^1[0,T]}) \\ &\le T^{\frac{\alpha}{2}} D(\mu+1)(\|w_1 - w_2\|_{C(Q_T)} + (1+T)\|h_1' - h_2'\|_{C[0,T]}) \\ &\le 2T^{\frac{\alpha}{2}} D(\mu+1)(\|w_1 - w_2\|_{C(Q_T)} + \|h_1' - h_2'\|_{C[0,T]}). \end{aligned}$$

Finally we will choose

$$T \le T^* = \min\Big\{1,\ \Big(\frac{1}{4D(\mu+1)}\Big)^{\frac{2}{\alpha}},\ (\mu C)^{-\frac{2}{\alpha}},\ C^{-\frac{2}{1+\alpha}},\ \frac{h_0}{8(1+H_0)}\Big\},$$

and then

$$\|F(w_1,h_1) - F(w_2,h_2)\| \le \frac{1}{2}(\|w_1 - w_2\|_{C(Q_T)} + \|h_1 - h_2\|_{C^1[0,T]}).$$

Therefore, the contraction mapping theorem shows that a fixed point (w^*, h^*) uniquely exists and (w^*, h^*) is the unique solution of (2.6) and (2.7) which belongs to

$$w^* \in C^{\frac{1+\alpha}{2}, 1+\alpha}(Q_T), \quad h^* \in C^{1+\frac{\alpha}{2}}[0, T].$$

Due to $f(w^*) \in C^{\frac{\alpha}{2}, \alpha}(Q_T)$, the Schauder estimate shows

$$w^* \in C^{1+\frac{\alpha}{2}, 2+\alpha}((0, T] \times [0, h_0]).$$

Since (FBP) is equivalent to problem (2.6) and (2.7), we find that there is a unique classical solution of (FBP) for $0 < t \leq T$. □

Proof of Theorem 2.2. We first show a priori estimates of u and h in $D(T)$ for any given $T > 0$. By the strong maximum principle (cf. Protter and Weinberger [56], Smoller [61]) we find

$$u(t, x) > 0 \ \text{ for } \ (t, x) \in D(T) \ \text{ and } \ u_x(t, h(t)) < 0 \ \text{ for } \ t \in (0, T]. \tag{2.18}$$

Define $C_1 := \max\{\|u_0\|_{C(\overline{\Omega}_0)}, 1\}$ and let $\overline{u} = \overline{u}(t)$ be the solution of an initial value problem:

$$\begin{cases} \dfrac{d\overline{u}}{dt} = f(\overline{u}), \quad t > 0, \\ \overline{u}(0) = C_1. \end{cases}$$

We can easily find $\overline{u}(t) \leq C_1$ for $t \geq 0$ and we get from the comparison theorem (cf. Protter and Weinberger [56], Smoller [61]) that $u(t, x) \leq \overline{u}(t)$ in $\overline{D(T)}$. Hence we have

$$0 < u(t, x) \leq C_1 \ \text{ for } \ (t, x) \in D(T).$$

By (2.18) we can see $h'(t) = -\mu u_x(t, h(t)) > 0$ for $0 < t \leq T$. and it remains to show the boundedness of $h'(t)$. We set

$$\begin{aligned} &w(t, x) = -C_1 M^2 (x - h(t))(x - h(t) + 2/M), \\ &D_M = \{(t, x) \in \mathbb{R}^2 \mid h(t) - 1/M < x < h(t) \ \text{ for } \ 0 \leq t \leq T\} \end{aligned}$$

to compare w with u in D_M, where

$$M = \max\{\sqrt{N/(2dC_1)}, \ \|u_0'\|_{C(\overline{\Omega}_0)}/C_1\}, \quad N = \max_{0 \leq u \leq C_1} f(u). \tag{2.19}$$

Direct calculations show

$$\begin{aligned} w_t &= 2C_1 M h'(t)\{1 - M(h(t) - x)\} \geq 0, \\ w_{xx} &= -2C_1 M^2 \end{aligned}$$

for $(t, x) \in D_M$, and hence

$$w_t - d w_{xx} \geq 2dC_1 M^2 \quad \text{in} \ \ D_M.$$

Setting $U = w - u$ and , we see from (2.19) that

$$\begin{aligned} U_t - dU_{xx} &\geq 2dC_1M^2 - f(u) \\ &\geq u(2dC_1M^2 - N) \\ &\geq 0 \end{aligned}$$

for $(t, x) \in D_M$. Moreover $U(t, h(t)) = w(t, h(t)) - u(t, h(t)) = 0$ and

$$\begin{aligned} U(t, h(t) - M^{-1}) &= w(t, h(t) - M^{-1}) - u(t, h(t) - M^{-1}) \\ &= C_1 - u(t, h(t) - M^{-1}) \\ &\geq 0. \end{aligned}$$

Finally we compare the initial function. Note that in $[h_0 - 1/M, h_0]$

$$\begin{aligned} w(0, x) &= C_1M^2(h_0 - x)(x - h_0 + 2/M) \geq C_1M(h_0 - x) \\ u_0(x) &= \int_{h_0}^{x} u_0'(y)dy \leq \|u_0'\|_{C(\overline{\Omega}_0)}(h_0 - x). \end{aligned}$$

Hence we find from (2.19) that

$$w(0, x) \geq C_1M(h_0 - x) \geq \|u_0'\|_{C(\overline{\Omega}_0)}(h_0 - x) \geq u_0(x)$$

for $h_0 - 1/M \leq x \leq h_0$. Thus, using the maximum principle, we can show

$$w(t, x) - u(t, x) = U(t, x) \geq 0 \quad \text{in} \quad D_M.$$

Noting that $w(t, h(t)) = u(t, h(t)) = 0$ that $u_x(t, h(t)) \geq w_x(t, h(t)) = -2C_1M$, we get

$$h'(t) = -\mu u_x(t, h(t)) \leq -\mu w_x(t, h(t)) \leq \mu(2C_1M) =: \mu C_2$$

for $0 \leq t \leq T$.

By Theorem 2.1, there is a unique solution for $0 < t \leq T$ for some $T < \infty$. We next prove the unique solution is extended globally in time (cf. [18]). Let $[0, T_{max})$ be the maximal existence time in which the unique solution exists. To prove $T_{\max} = \infty$, we assume $T_{max} < \infty$. By the above a priori estimates we find

$$0 \leq u(t, x) \leq C_1, \quad h_0 \leq h(t) \leq h_0 + \mu C_2 T_{max}$$

for $0 \leq t \leq T_{max}$, $0 \leq x \leq h(t)$, where C_1 and C_2 are independent of T_{max}. For any $\delta_0 \in (0, T_{max})$ and any $M > T_{max}$, using the parabolic estimates and Sobolev's embedding theorem, we have

$$\|u(t, \cdot)\|_{C^2(\overline{\Omega(t)})} \leq C_3 \quad \text{for all} \quad t \in [\delta_0, T_{max}),$$

where C_3 only depends on δ_0, M, C_1 and C_2. Hence we can get a time interval $\tau > 0$ which is independent of $t \in [\delta_0, T_{max})$. Then, applying Theorem 2.1, we can extend the solution with initial data at $t = T_{max} - \tau/2$ uniquely to $t = T_{max} - \tau/2 + \tau = T_{max} + \tau/2 > T_{max}$. However this result contradicts the definition of T_{max}, and thus

$T_{max} = \infty$. The proof is complete. □

Proof of Theorem 2.3. Let (u_ϵ, h_ϵ) be the solution of

$$\begin{cases} (u_\epsilon)_t - d_\epsilon (u_\epsilon)_{xx} = f_\epsilon(u_\epsilon), & t > 0,\ 0 < x < h_\epsilon(t), \\ u_\epsilon(t,0) = 0 \text{ (resp. } (u_\epsilon)_x(t,0) = 0), & t > 0, \\ u_\epsilon(t, h_\epsilon(t)) = 0, & t > 0, \\ h'_\epsilon(t) = -\mu_\epsilon (u_\epsilon)_x(t, h(t)), & t > 0, \\ h_\epsilon(0) = (h_0)_\epsilon,\ u_\epsilon(0,x) = (u_0)_\epsilon(x), & 0 \le x \le (h_0)_\epsilon, \end{cases}$$

where d_ϵ, μ_ϵ, $(h_0)_\epsilon$ are positive constants, $f_\epsilon \in S_f$ and $(u_0)_\epsilon$ satisfies (2.1) with h_0 replaced by $(h_0)_\epsilon$, at least one of which is different from d, μ, h_0, f and u_0. Moreover assume that as $\epsilon \to 0$

$$d_\epsilon \to d, \quad \mu_\epsilon \to \mu, \quad (h_0)_\epsilon \to h_0, \quad f_\epsilon(u) \to f(u) \ \text{ for all } \ u \ge 0,$$
$$(u_0)_\epsilon\Big(\frac{(h_0)_\epsilon x}{h_0}\Big) \to u_0(x) \ \text{ in } \ C^2(\Omega_0).$$

By Theorem 2.2, we find for any $\alpha \in (0,1)$

$$u_\epsilon \in C^{1+\alpha, 2+\alpha}(D_\epsilon), \quad h_\epsilon \in C^{1+\alpha}(0,\infty),$$

where $D_\epsilon = \cup_{0<s<\infty}\{s\} \times (0, h_\epsilon(s))$. Using the Ascoli-Arzela's theorem, we can choose a sub-sequence $\{\epsilon'\}$ and some function $u^* \in C^{1,2}(D)$ (where $D = \cup_{0<s<\infty}\{s\} \times (0, h(s))$) and some function $h^* \in C^1(0,\infty)$ such that along the sub-sequence

$$\lim_{\epsilon' \to 0} \|u_{\epsilon'} - u^*\|_{C^{1,2}(D_\epsilon)} = 0, \quad \lim_{\epsilon' \to 0} \|h_{\epsilon'} - h^*\|_{C^1(0,\infty)} = 0.$$

Noting the uniqueness of solutions (u,h) of (FBP), we get $u^* \equiv u$ and $h^* \equiv h$, and thus along any sequence

$$\lim_{\epsilon \to 0} \|u_\epsilon - u\|_{C^{1,2}(D_\epsilon)} = 0, \quad \lim_{\epsilon \to 0} \|h_\epsilon - h\|_{C^1(0,\infty)} = 0.$$

The proof is complete. □

2.3 Energy identity and comparison principle

To discuss the asymptotic behaviors of solutions for (FBP) we prepare, in this section, an energy identity and a comparison principle.

Lemma 2.1. *Let (u,h) be any solution of* (FBP) *and define $F(u) = \int_0^u f(s)\,ds$. Then the following identity holds true*:

$$\frac{d}{2}\|u_x(t,\cdot)\|^2_{L^2(\Omega(t))} + \int_0^t \|u_t(s,\cdot)\|^2_{L^2(\Omega(s))}\,ds + \frac{d}{2\mu^2}\int_0^t h'(s)^3\,ds$$
$$= \frac{d}{2}\|u_0'\|^2_{L^2(\Omega_0)} + \int_{\Omega(t)} F(u(t,x))\,dx - \int_{\Omega_0} F(u_0(x))\,dx.$$

Lemma 2.2. *For any given* $T > 0$*, let* $\overline{h} \in C^1[0,T]$ *and* $\overline{u} \in C(\overline{D_1(T)}) \cap C^{1,2}(D_1(T))$ *satisfy*

$$\begin{cases} \overline{u}_t - d\overline{u}_{xx} \geq f(\overline{u}), & (t,x) \in D_1(T), \\ \overline{u}(t,0) \geq 0 \ (\textit{resp. } \overline{u}_x(t,0) \leq 0), & t \in (0,T], \\ \overline{u}(t,\overline{h}(t)) = 0, & t \in (0,T], \\ \overline{h}'(t) \geq -\mu \overline{u}_x(t,\overline{h}(t)), & t \in (0,T], \end{cases}$$

where d *and* μ *are positive constants and* $D_1(T) = \bigcup_{0\leq s\leq T}(\{s\} \times (0,\overline{h}(s)))$*. Moreover let* (u,h) *be the solution of* (FBP) *with initial data* $(u_0(x), h_0)$*. If* $h_0 \leq \overline{h}(0)$ *and* $u_0(x) \leq \overline{u}(0,x)$ *in* $[0,h_0]$*, then it holds that*

$$h(t) \leq \overline{h}(t) \quad \textit{in} \quad [0,T] \quad \textit{and} \quad u(t,x) \leq \overline{u}(t,x) \quad \textit{in} \ \bigcup_{0\leq s\leq T}(\{s\} \times (0,h(s))).$$

Lemma 2.3. *For any given* $T > 0$*, let* $\underline{h} \in C^1[0,T]$ *and* $\underline{u} \in C(\overline{D_2(T)}) \cap C^{1,2}(D_2(T))$ *satisfy*

$$\begin{cases} \underline{u}_t - d\underline{u}_{xx} \leq f(\underline{u}), & (t,x) \in D_2(T), \\ \underline{u}(t,0) \leq 0 \ (\textit{resp. } \underline{u}_x(t,0) \geq 0), & t \in (0,T], \\ \underline{u}(t,\underline{h}(t)) = 0, & t \in (0,T], \\ \underline{h}'(t) \leq -\mu \underline{u}_x(t,\underline{h}(t)), & t \in (0,T], \end{cases}$$

where d *and* μ *are positive constants and* $D_2(T) = \bigcup_{0\leq s\leq T}(\{s\}\times(0,\underline{h}(s)))$*. Moreover let* (u,h) *be the solution of* (FBP) *with initial data* $(u_0(x), h_0)$*. If* $\underline{h}(0) \leq h_0$ *and* $\underline{u}(0,x) \leq u_0(x)$ *in* $[0,\underline{h}(0)]$*, then it holds that*

$$\underline{h}(t) \leq h(t) \quad \textit{in} \quad [0,T] \quad \textit{and} \quad \underline{u}(t,x) \leq u(t,r) \quad \textit{in} \ \bigcup_{0\leq s\leq T}(\{s\} \times (0,\underline{h}(s))).$$

Definition 2.1. The couple of functions $(\overline{u}, \overline{h})$ in Lemma 2.2 is called **an upper (super-) solution** of (FBP) for $0 \leq t \leq T$. In a similar way we can denote **a lower (sub-) solution** of (FBP) for $0 \leq t \leq T$ by $(\underline{u}, \underline{h})$ as in Lemma 2.3.

Lemma 2.4. *For any given* $T > 0$*, let* (u_{μ_i}, h_{μ_i}) $(i = 1, 2)$ *satisfy*

$$\begin{cases} (u_{\mu_i})_t - d(u_{\mu_i})_{xx} = f(u_{\mu_i}), & 0 < t \leq T, \ 0 < x < h_{\mu_i}(t), \\ u_{\mu_i}(t,0) = 0 \ (\textit{resp. } (u_{\mu_i})_x(t,0) = 0), & 0 < t \leq T, \\ u_{\mu_i}(t, h_{\mu_i}(t)) = 0, & 0 < t \leq T, \\ h'_{\mu_i}(t) = -\mu_i (u_{\mu_i})_x(t, h_{\mu_i}(t)), & 0 < t \leq T, \end{cases}$$

where d *is positive constant. If* $\mu_1 \leq \mu_2$*,* $h_{\mu_1}(0) \leq h_{\mu_2}(0)$*, and* $u_{\mu_1}(0,x) \leq u_{\mu_2}(0,x)$ *in* $[0, h_{\mu_1}(0)]$*, then*

$$h_{\mu_1}(t) \leq h_{\mu_2}(t) \quad \textit{in} \quad [0,T] \quad \textit{and} \quad u_{\mu_1}(t,x) \leq u_{\mu_2}(t,x) \quad \textit{in} \ \bigcup_{0\leq s\leq T}(\{s\} \times (0, h_{\mu_1}(s))).$$

Proof of Lemma 2.1. Consider the following identity.

$$\frac{d}{dt}\left\{\frac{d}{2}\|u_x(t,\cdot)\|^2_{L^2(\Omega(t))}\right\} = \frac{d}{2}u_x(t,h(t))^2h'(t) + d\int_{\Omega(t)} u_x(t,x)u_{xt}(t,x)\ dx. \tag{2.20}$$

Using the Stefan condition we find in (2.20)

$$\frac{d}{2}u_x(t,h(t))^2h'(t) = \frac{d}{2\mu^2}h'(t)^3.$$

We differentiate $u(t,h(t)) = 0$ with respect to t to get

$$u_x(t,h(t))h'(t) + u_t(t,h(t)) = 0.$$

Then, for the second term in the right-hand side of (2.20), integration by parts and the above identity lead to

$$\begin{aligned}
d\int_{\Omega(t)} u_x(t,x)u_{xt}(t,x)\ dx &= d\Big[u_x(t,x)u_t(t,x)\Big]_{x=0}^{x=h(t)} - d\int_{\Omega(t)} u_{xx}(t,x)u_t(t,x)\ dx\\
&= du_x(t,h(t))u_t(t,h(t)) + \int_{\Omega(t)} u_t(t,x)(-du_{xx}(t,x))\ dx\\
&= -du_x(t,h(t))^2h'(t) + \int_{\Omega(t)} u_t(t,x)(f(u(t,x)) - u_t(t,x))\ dx\\
&= -\frac{d}{\mu^2}h'(t)^3 + \int_{\Omega(t)} u_t(t,x)(f(u(t,x)) - u_t(t,x))\ dx.
\end{aligned}$$

where we have used $u_t(t,0) = 0$ (or $u_x(t,0) = 0$ for the Neumann problem). It follows that

$$\begin{aligned}
\frac{d}{dt}\left\{\frac{d}{2}\|u(t,\cdot)\|^2_{L^2(\Omega(t))}\right\} &= -\frac{d}{2\mu^2}h'(t)^3 - \|u_t(t,\cdot)\|^2_{L^2(\Omega(t))} + \int_{\Omega(t)} \frac{\partial}{\partial t}F(u(t,x))\ dx\\
&= -\frac{d}{2\mu^2}h'(t)^3 - \|u_t(t,\cdot)\|^2_{L^2(\Omega(t))} + \frac{d}{dt}\int_{\Omega(t)} F(u(t,x))\ dx.
\end{aligned}$$

Hence integrating this identity over $[0,t]$, we get the conclusion. □

Proof of Lemma 2.2. We basically follow the argument of [18] to prove this lemma. Let $(u_\varepsilon, h_\varepsilon)$ be any solution of

$$\begin{cases}
(u_\varepsilon)_t - d(u_\varepsilon)_{xx} = f(u_\varepsilon), & 0 < t \le T,\ 0 < x < h_\varepsilon(t),\\
u_\varepsilon(t,0) = 0 \text{ (resp. } (u_\varepsilon)_x(t,0) = 0), & 0 < t \le T,\\
u_\varepsilon(t,h_\varepsilon(t)) = 0, & 0 < t \le T,\\
h'_\varepsilon(t) = -\mu(1-\varepsilon)(u_\varepsilon)_x(t,h_\varepsilon(t)), & 0 < t \le T,\\
h_\varepsilon(0) = (1-\varepsilon)h_0,\ u_\varepsilon(0,x) = u_0(h_0x/h_\varepsilon(0)), & 0 \le x \le h_\varepsilon(0),
\end{cases}$$

where ε is so small that $u_\varepsilon(0,x) \le \overline{u}(0,x)$ for $0 \le x \le h_\varepsilon(0)$. We can apply the strong maximum principle to show

$$u_\varepsilon(t,x) > 0 \ \text{ in } \ D_\varepsilon(T), \quad (u_\varepsilon)_x(t,h_\varepsilon(t)) < 0 \ \text{ for } \ 0 \le t \le T, \tag{2.21}$$

where $D_\varepsilon(T) := \bigcup_{0\le s\le T}(\{s\}\times(0,h_\varepsilon(s)))$. Moreover, by the continuous dependence on parameters, we see that

$$\lim_{\varepsilon\to 0} h_\varepsilon(t) = h(t) \ \text{ in } \ C^1[0,T], \quad \lim_{\varepsilon\to 0}\|u_\varepsilon(t,\cdot)-u(t,\cdot)\|_{C(\overline{D_\varepsilon(T)})} = 0.$$

We will prove

$$h_\varepsilon(t) \le \overline{h}(t) \ \text{ in } \ [0,T] \quad \text{and} \quad u_\varepsilon(t,x) \le \overline{u}(t,x) \ \text{ in } \ D_\varepsilon(T) \tag{2.22}$$

because, by taking $\varepsilon \to 0$ in (2.22), we obtain

$$h(t) \le \overline{h}(t) \ \text{ in } \ [0,T] \quad \text{and} \quad u(t,x) \le \overline{u}(t,x) \ \text{ in } \ D(T).$$

Since $h_\varepsilon(0) < \overline{h}(0)$, we have $h_\varepsilon(t) < \overline{h}(t)$ for small $t > 0$, and we may assume

$$h_\varepsilon(t) < \overline{h}(t) \ \text{ in } \ [0,t^*), \quad h_\varepsilon(t^*) = \overline{h}(t^*) \ \text{ and } \ h'_\varepsilon(t^*) \ge \overline{h}'(t^*) \tag{2.23}$$

for some $t^* \in (0,T)$. Let $w(t,x) = e^{kt}(\overline{u}(t,x) - u_\varepsilon(t,x))$. Then, by (2.2), direct calculations show that

$$\begin{aligned} w_t - dw_{xx} &\ge kw + e^{kt}(f(\overline{u}) - f(u_\varepsilon)) \\ &\ge kw - \operatorname{sgn}(\overline{u}-u_\varepsilon)e^{kt}(\overline{u}-u_\varepsilon)L \\ &\ge (k - Le^{kt})w \ \text{ in } D_\varepsilon(T), \end{aligned}$$

where $\operatorname{sgn}(u) = 1$ if $u > 0$, $\operatorname{sgn}(u) = 0$ if $u = 0$ and $\operatorname{sgn}(u) = -1$ if $u < 0$. Taking suitably large k, we see that $w_t - dw_{xx} \ge 0$ in $D_\varepsilon(T)$. Note that $w(t,0) \ge 0$ (resp. $w_x(t,0) \ge 0$), $w(t,h_\varepsilon(t)) \ge 0$ for $0 < t \le t^*$ and $w(0,x) = \overline{u}(0,x) - u_\varepsilon(0,x) \ge 0$ in $[0,h_\varepsilon(0)]$. Therefore, by the maximum principle, we find that $w(t,x) \ge 0$ in $D_\varepsilon(T)$. Since $h_\varepsilon(t^*) = \overline{h}(t^*)$ and $w(t^*,h_\varepsilon(t^*)) = e^{-kt^*}(\overline{u}(t^*,\overline{h}(t^*)) - u_\varepsilon(t^*,h_\varepsilon(t^*))) = 0$, we find $w_x(t^*,h_\varepsilon(t^*)) \le 0$. Hence, taking account of (2.21), we see that

$$\overline{h}'(t^*) - h'_\varepsilon(t^*) \ge -\mu e^{-kt^*}w_x(t^*,h_\varepsilon(t^*)) - \varepsilon\mu(u_\varepsilon)_x(t^*,h_\varepsilon(t^*)) > 0$$

This contradicts (2.23), and consequently $h_\varepsilon(t) \le \overline{h}(t)$ in $[0,T]$. Finally we use the maximum principle again in $D_\varepsilon(T)$ to get $u_\varepsilon(t,x) \le \overline{u}(t,x)$ and (2.22). This completes the proof. □

We omit the proof of Lemma 2.3 because it is almost same as that of Lemma 2.2.

Proof of Lemma 2.4. Since $h'_{\mu_2}(t) = -\mu_2(u_{\mu_2})_x(t,h_{\mu_2}(t)) \ge -\mu_1(u_{\mu_2})_x(t,h_{\mu_2}(t))$, the solution (u_{μ_2},h_{μ_2}) is an upper solution of (FBP) with $\mu = \mu_1$. Hence, using Lemma 2.2, we get the conclusion. □

2.4 Properties of spreading and vanishing

In this section we show the underlying principle to determine the asymptotic behaviors of solutions as $t \to \infty$. Let (u, h) be any solution of (FBP). Since the free boundary is strictly increasing by Theorem 2.2, there exists a limit which satisfies

$$\lim_{t\to\infty} h(t) = \infty \quad \text{or} \quad \lim_{t\to\infty} h(t) < \infty.$$

In each case, the behavior of u as $t \to \infty$ is different which is called **spreading** or **vanishing**.

The following theorem gives a property of vanishing.

Theorem 2.4. *If* $\lim_{t\to\infty} h(t) < \infty$, *then* $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} = 0$.

On the other hand, a property of spreading is given by the following theorem.

Theorem 2.5. *Let* (u, h) *be the solution of* (FBP) *with initial data* (q, l)*, where* $l > 0$ *and* $q(x)$ *satisfies the differential inequity:*

$$\begin{cases} dq_{xx} + f(q) \geq 0, \quad 0 < x < l, \\ q > 0, \quad 0 < x < l, \\ q(0) = 0 \ (resp.\ q_x(0) = 0), \ q(l) = 0. \end{cases}$$

Then the following (i) – (iii) *hold true:*

(i) $\lim_{t\to\infty} h(t) = \infty$; *that is* $\lim_{t\to\infty} \Omega(t) = (0, \infty)$,

(ii) $u(t, x)$ *is non-decreasing with respect to* $t > 0$ *in* $\Omega(t)$,

(iii) $\lim_{t\to\infty} u(t, x) = v^*(x)$ *uniformly in any compact subset of* $[0, \infty)$*, where* v^* *is a minimal positive solution of*

$$\begin{cases} dv_{xx} + f(v) = 0, \quad 0 < x < \infty, \\ v(0) = 0 \ (resp.\ v_x(0) = 0) \end{cases}$$

satisfying $v^*(x) \geq q(x)$ *in* $[0, l]$.

Remark 2.1. *In Theorem 2.5 the boundary condition of the initial function,* $q(0) = 0$ $(resp.\ q_x(0) = 0)$, $q(l) = 0$, *may be replaced by* $q(0) \leq 0$ $(resp.\ q_x(0) \geq 0)$, $q(l) \leq 0$. *In other words* $q(r)$ *is regarded as a lower solution for* (2.3).

We have the following property on spreading from Theorem 2.5.

Corollary 2.1. *Suppose that functions* $q(x)$ *and* $v^*(x)$ *and positive number* l *are defined as in Theorem 2.5. If* $h_0 \geq l$ *and* $u_0(x) \geq q(x)$ *in* $[0, l]$, *then*

$$\lim_{t\to\infty} \Omega(t) = (0, \infty) \quad \textit{and} \quad \liminf_{t\to\infty} u(t, x) \geq v^*(x) \quad \textit{for} \ \ 0 < x < \infty.$$

Proof. Let $(w(t,x), y(t))$ be a solution of (FBP) with initial data (q,l). Then it follows from Lemma 2.2 that

$$h(t) \geq y(t) \quad \text{for } t>0 \quad \text{and} \quad u(t,x) \geq w(t,x) \quad \text{for} \quad t>0,\ 0<x<y(t).$$

Taking $t \to \infty$ in the above inequality, we get the conclusion because Theorem 2.5 shows $\lim_{t\to\infty} y(t) = \infty$ and $\lim_{t\to\infty} w(t,x) = v^*(x)$. □

Before giving the proof of Theorem 2.4, we prepare two important lemmas.

Lemma 2.5. *Assume* $\lim_{t\to\infty} h(t) < \infty$. *If* $v(t,y)$ *is defined by* $v(t,y) = u(t,h(t)y)$, *then* $\{v(t,y) |\ t \geq 1\}$ *is relatively compact in* $C^1[0,1]$.

Proof. We use the energy estimate. Note that

$$\sup_{t\geq 0,\ 0<x<h(t)} F(u(t,x)) \leq \max_{0\leq u\leq C_1} F(u) =: C_3 < \infty.$$

Since $h_\infty := \lim_{t\to\infty} h(t) < \infty$, it follows from Lemma 2.1 that for every $t \geq 0$

$$\begin{aligned}\frac{d}{2}\|u_x(t,\cdot)\|^2_{L^2(\Omega(t))} &+ \int_0^t \|u_t(s,\cdot)\|^2_{L^2(\Omega(s))}\, ds \\ &+ \frac{d}{2\mu^2}\int_0^t h'(s)^3\, ds \leq C_0 + C_3 h_\infty,\end{aligned} \tag{2.24}$$

where $C_0 = (d/2)\|u_0'\|^2_{L^2(\Omega_0)} - \int_0^{h_0} F(u_0(x))\, dx$. The inequality gives

$$\sup_{t\geq 0} \|u_x(t,\cdot)\|_{L^2(\Omega(t))} < \infty. \tag{2.25}$$

By the definition of $v = v(t,y)$, we calculate

$$u_x = \frac{v_y}{h(t)}, \quad u_{xx} = \frac{v_{yy}}{h(t)^2}, \quad u_t = v_t + v_y\Big(-\frac{h'(t)y}{h(t)}\Big),$$

and v satisfies the following problem:

$$\begin{cases} v_t = a(t)v_{yy} + b(t,y)v_y + f(v), & t>0,\ 0<y<1, \\ v(t,0) = 0 \text{ (resp. } v_y(t,0)=0), & t>0, \\ v(t,1) = 0, & t>0, \\ v(0,y) = v_0(y) := u_0(h_0 y), & 0\leq y\leq 1, \end{cases} \tag{2.26}$$

where $a(t) = d/h(t)^2$ and $b(t,y) = h'(t)y/h(t)$. We treat (2.26) as an evolution equation in $L^2(0,1)$. Let A be a closed linear operator in $L^2(0,1)$ with domain $D(A) = H^2(0,1)\cap H_0^1(0,1)$ (resp. $D(A) = H^2(0,1)$) and

$$Av = -v_{yy} \quad \text{for} \quad v \in D(A).$$

For each $t \in [0, \infty)$, define $A(t)$ by

$$A(t)v = a(t)Av \quad \text{for} \quad v \in D(A).$$

By Theorem 2.2, we find $0 < h'(t) \leq \mu C_2$ for all $t \geq 0$, and hence

$$\|(A(t) - A(s))A(0)^{-1}\| = |(a(t) - a(s))a(0)^{-1}| \leq \frac{2\mu C_2 h_\infty}{h_0^2} \, |t - s|.$$

It is well known that $-A(t)$ is a sectorial operator generating an analytic semi-group in $L^2(0, 1)$. Hence, making use of general theory of evolution equations (see Friedman [25], Henry [35] or Tanabe [65]), we can construct evolution operators $\{U(t, s)\}_{0 \leq s \leq t}$ satisfying

$$\|A^\alpha U(t, s)\| = M(\alpha)(t - s)^{-\alpha}, \quad 0 \leq s < t < \infty$$

for each $0 < \alpha < 2$, where $\| \cdot \|$ is the operator norm in $L^2(0, 1)$ and $M(\alpha)$ is a positive number depending only on α (see also Mimura, Yamada and Yotsutani [51, Lemma 4.2]). Using the evolution operators, we can represent the solution $v(t)$ of (2.26) as

$$v(t) = U(t, \tau)v(\tau) + \int_\tau^t U(t, s)g(s) \, ds, \tag{2.27}$$

where $g(t, y) = b(t, y)v_y(t, y) + f(v(t, y))$ and $b(t, y) = h'(t)y/h(t)$ is uniformly bounded in $[0, \infty) \times [0, 1]$. Moreover, by the a priori estimate and (2.25),

$$\begin{aligned} m_1 : &= \sup_{t \geq 0} \|g(t, \cdot)\|_{L^2(0,1)} \\ &\leq \frac{\mu C_2}{h_0} \sup_{t \geq 0} \|v_y(t)\|_{L^2(0,1)} + \max_{0 \leq v \leq C_1} |f(v)| \\ &\leq \frac{\mu C_2 h_\infty}{h_0} \sup_{t \geq 0} \|v_x(t)\|_{L^2(0,1)} + \max_{0 \leq v \leq C_1} |f(v)| < \infty. \end{aligned}$$

We operate v in (2.27) by A^α for $\alpha \in (0, 1)$ to get

$$A^\alpha v(t) = A^\alpha U(t, \tau)v(\tau) + \int_\tau^t A^\alpha U(t, s)g(s) \, ds.$$

Then we can show from the above estimate and $\sup_{t \geq 0} \|v(t)\|_{L^2(0,1)} \leq C_1$ (where C_1 is defined in Theorem 2.2)

$$\begin{aligned} \|A^\alpha v(t)\|_{L^2(0,1)} &\leq \|A^\alpha U(t, \tau)\| \cdot \|v(\tau)\|_{L^2(0,1)} + \int_\tau^t \|A^\alpha U(t, s)\| \cdot \|g(s)\|_{L^2(0,1)} \, ds \\ &\leq M(\alpha)(t - \tau)^{-\alpha} C_1 + M(\alpha) \int_\tau^t (t - s)^{-\alpha} m_1 \, ds \\ &\leq M(\alpha) \Big\{ C_1 (t - \tau)^{-\alpha} + \frac{m_1}{1 - \alpha} (t - \tau)^{1-\alpha} \Big\} \end{aligned}$$

for $t \geq \tau \geq 0$. For any $t \geq 1$, taking $\tau \geq 0$ such that $t - \tau = 1$, we have

$$\|A^\alpha v(t)\|_{L^2(0,1)} \leq M(\alpha) \Big\{ C_1 + \frac{m_1}{1 - \alpha} \Big\}.$$

This estimate implies that $\{v(t)|\ t \geq 1\}$ is uniformly bounded in $D(A^\alpha)$. Then we employ the following embedding result; the inclusion

$$D(A^\alpha) \subset C^1[0,1]$$

is continuous and compact for $3/4 < \alpha < 1$ (see Henry [35, Theorems 1.4.8 and 1.6.1]). Hence the proof is complete. □

Lemma 2.6. *Assume* $\lim_{t\to\infty} h(t) < \infty$. *Then both* $h'(t)$ *and* $\|u_t(t)\|_{L^2(\Omega(t))}$ *are uniformly continuous with respect to* $t \in [1, \infty)$.

Proof. Define $v(t,y) = u(t, h(t)y)$. Since $h_\infty = \lim_{t\to\infty} h(t) < 0$, we can use the same argument in Lemma 2.5 to find that the inclusion $D(A^\alpha) \subset C^1[0,1]$ is continuous for $3/4 < \alpha < 1$. Following the arguments used in the proof of [51, Theorem 4.3], we can derive

$$\|A^\alpha(v(t) - v(\tau))\|_{L^2(0,1)} \leq M_1(\theta,\alpha)(t-\tau)^\theta, \quad t \geq \tau \geq 1$$

for any $0 \leq \alpha < 1$ and any $\theta \in (0, 1-\alpha)$, where $M_1(\theta,\alpha)$ is a positive number depending on θ and α. Hence

$$t \longmapsto v_y(t,y) \text{ is uniformly Hölder continuous in } C[0,1]\text{-norm.} \tag{2.28}$$

This fact, in particular, implies that

$$t \longmapsto h'(t) = -\frac{\mu}{h(t)} v_y(t,1) \text{ is uniformly Hölder continuous.} \tag{2.29}$$

We next prove the uniform continuity of $t \longmapsto \|u_t(t)\|_{L^2(\Omega(t))}$, using the theory of evolution equations (see Tanabe [65]). Recall that v is expressed as (2.27) and that $t \longmapsto g(t,y) = (h'(t)y/h(t))v_y(t,y) + f(v(t,y))$ is uniformly Hölder continuous in $C[0,1]$-norm by (2.28) and (2.29). Hence we can use fundamental estimates for $\{U(t,s)\}$ to derive

$$t \longmapsto v_t(t,\cdot) \text{ is uniformly continuous in } L^2(0,1). \tag{2.30}$$

Noting from the transformation that

$$\|u_t(t)\|_{L^2(\Omega(t))} = h(t)^{1/2}\|v_t(t) - b(t,\cdot)v_y(t)\|_{L^2(0,1)}$$

where $b(t,y) = h'(t)y/h(t)$, we get the conclusion due to (2.28) – (2.30). □

Proof of Theorem 2.4. We make use of the energy identity to show the convergence as $t \to \infty$. Employing (2.24) in the proof of Lemma 2.1 (this is possible because $h_\infty := \lim_{t\to\infty} h(t) < \infty$), we have

$$\int_0^\infty \left\{ \|u_t(t,\cdot)\|^2_{L^2(\Omega(t))} + h'(t)^3 \right\} dt \leq C_5,$$

where C_5 is some positive constant. Since both $\|u_t(t)\|_{L^2(\Omega(t))}$ and $h'(t)$ are uniformly continuous for $t \geq 1$ (Lemma 2.6), the inequality enables us to prove

$$\lim_{t\to\infty} \|u_t(t,\cdot)\|_{L^2(\Omega(t))} = 0 \quad \text{and} \quad \lim_{t\to\infty} h'(t) = 0. \tag{2.31}$$

We introduce a new function $v(t,y)$ by $v(t,y) = u(t,h(t)y)$. As in Lemma 2.5, the function satisfies (2.26), for convenience, which is denoted by

$$\begin{cases} v_t = a(t)v_{yy} + b(t,y)v_y + f(v), & t > 0,\ 0 < y < 1, \\ v(t,0) = 0 \text{ (resp. } v_y(t,0) = 0), & t > 0, \\ v(t,1) = 0, & t > 0, \\ v(0,y) = v_0(y) := u_0(h_0 y), & 0 \leq y \leq 1, \end{cases}$$

where $a(t) = d/h(t)^2$ and $b(t,y) = h'(t)y/h(t)$. Also, since $\{v(t,y)|\ t \geq 1\}$ is relatively compact in $C^1[0,1]$ (Lemma 2.5), we are able to choose a sequence of $\{t_n\} \nearrow \infty$ and a nonnegative function $\hat{v}(y)$ to satisfy

$$\lim_{n\to\infty} v(t_n,y) = \hat{v}(y) \ \text{ in } \ C^1[0,1] \tag{2.32}$$

Observing

$$u_t(t,x) = v_t(t,y) - \frac{h'(t)y}{h(t)} v_y(t,y) = v_t(t,y) - b(t,y)v_y(t,y)$$

and (2.31), we find $\|v_t(t_n)\|_{L^2(0,1)} \to 0$ and $g(t_n,y) \to 0$ as $n \to \infty$. Hence $\hat{v}$ satisfies

$$\frac{d}{h_\infty^2}\hat{v}_{yy} + f(\hat{v}) = 0 \quad \text{for} \ \ 0 < y < 1 \tag{2.33}$$

in the $L^2(0,1)$-sense with $\hat{v}(0) = 0$ (resp. $\hat{v}_y(0) = 0$) and $\hat{v}(1) = 0$, and it also holds in the classical sense by the elliptic regularity. Note that

$$h'(t_n) = -\mu u_x(t_n, h(t_n)) = -\frac{\mu}{h(t_n)} v_y(t_n, 1).$$

Letting $n \to \infty$ in the above relation, we get $0 = -\mu\hat{v}_y(1)/h_\infty$ by (2.31) and (2.32). Hence $\hat{v}_y(1) = 0$. It can be seen that $\hat{v}$ satisfies a second-order differential equation (2.33) with $\hat{v}(1) = \hat{v}_y(1) = 0$, and thus $\hat{v} \equiv 0$ from the uniqueness of solutions. Hence

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} = \lim_{t\to\infty} \|v(t,\cdot)\|_{C[0,1]} = 0.$$

The proof is complete. □

Proof of Theorem 2.5. We will prove (ii) by the comparison principle (Lemmas 2.2 and 2.3) and the uniqueness of solutions. To clarify the dependence of solution (u,h) for (FBP) on initial data (q,l) we write, in this paragraph, $u(t,x;q,l)$ and $h(t;q,l)$ instead of $u(t,x)$ and $h(t)$, respectively. Since $(u(t,x),h(t)) \equiv (q(x),l)$ is a lower solution of (FBP), Lemma 2.3 shows that for any $\tau \geq 0$

$$h(\tau;q,l) \geq l \quad \text{and} \quad u(\tau,x;q,l) \geq q(x) \quad \text{in} \quad [0,l] \tag{2.34}$$

We now compare the solution with initial data $(u(\tau,x;q,l),h(\tau;q,l))$ to the solution with $(q(x),l)$. Then, by virtue of (2.34), the comparison principle gives, for every $t \geq 0$,

$$h(t;u(\tau;q,l),h(\tau;q,l)) \geq h(t;q,l)$$

and

$$u(t,x;u(\tau;q,l),h(\tau;q,l)) \geq u(t,x;q,l) \text{ in } [0,h(t;q,l)]. \tag{2.35}$$

By the uniqueness of solutions for (FBP), we find

$$u(t,x;u(\tau;q,l),h(\tau;q,l)) = u(t+\tau,x;q,l)$$

for any $t,\tau \geq 0$. Hence (2.35) becomes

$$u(t+\tau,x;q,l) \geq u(t,x;q,l) \quad \text{in} \quad [0,h(t;q,l)]$$

for any $t,\tau \geq 0$. Thus $u_t(t,x;q,l) \geq 0$ for $0 \leq x \leq h(t;q,l),\ t \geq 0$.

We can easily show (i) using the property of vanishing and (ii). If we assume $\lim_{t\to\infty} h(t) < \infty$, then Theorem 2.4 implies $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} = 0$. By part (ii), however, the function $u(t,x)$ is non-decreasing in $t \geq 0$ for $0 < x < h(t)$, which allows us to get

$$\liminf_{t\to\infty} u(t,x) \geq q(x) > 0 \quad \text{in} \quad (0,l).$$

This is a contradiction, and hence the free boundary $h(t)$ must satisfy $\lim_{t\to\infty} h(t) = \infty$, that is $\lim_{t\to\infty} \Omega(t) = (0,\infty)$.

Finally we prove the result (iii) on the convergence of solutions as $t \to \infty$. Noting that $u(t,x)$ is non-decreasing with respect to t for each $x \in (0,h(t))$ and uniformly bounded, we find a nonnegative function $\hat{v}(x)$ which satisfies

$$\lim_{t\to\infty} u(t,x) = \hat{v}(x) \quad \text{for every } x \geq 0 \tag{2.36}$$

and moreover $\hat{v}(x) = \lim_{t\to\infty} u(t,x) \geq q(x)$ in $[0,l]$. We will show that $\hat{v}(x)$ is a solution of the stationary problem. We multiply the equation by any function $\phi \in C_0^\infty(0,\infty)$ and integrate it in $(t,t+\delta) \times (0,\infty)$ with any $\delta > 0$ to have

$$\begin{aligned}\int_t^{t+\delta}\int_0^\infty u_t(s,x)\phi(x)\ dxds = d\int_t^{t+\delta}\int_0^\infty u(s,x)\phi_{xx}(x)\ dxds \\ + \int_t^{t+\delta}\int_0^\infty f(u(s,x))\phi(x)\ dxds\end{aligned}$$

(Here the integral makes sense because ϕ has a compact support in $(0,\infty)$). By Lebesgue's dominated convergence theorem and the monotone convergence result (2.36), it holds that

$$\begin{aligned}\lim_{t\to\infty}\int_t^{t+\delta}\int_0^\infty u_t(s,x)\phi(x)\ dxds &= \lim_{t\to\infty}\left\{\int_0^\infty u(t+\delta,x)\phi(x)\ dx - \int_0^\infty u(t,x)\phi(x)\ dx\right\} \\ &= 0.\end{aligned}$$

Similarly we find

$$\lim_{t\to\infty}\int_t^{t+\delta}\int_0^\infty u(s,x)\phi_{xx}(x)\ dxds = \delta\int_0^\infty \hat{v}(x)\phi_{xx}(x)\ dx,$$
$$\lim_{t\to\infty}\int_t^{t+\delta}\int_0^\infty f(u(s,x))\phi(x)\ dxds = \delta\int_0^\infty f(\hat{v}(x))\phi(x)\ dx.$$

Hence $\hat{v}$ satisfies

$$d\hat{v}_{xx}(x) + f(\hat{v}(x)) = 0 \ \text{ for } \ 0 < x < \infty, \ \ \hat{v}(0) = 0 \ (\text{resp. } \hat{v}_x(0) = 0)$$

in the weak sense, and hence it holds true in the classical sense by the elliptic regularity. Here it should be noted that $\hat{v}$ satisfies $\hat{v} \geq q$ in $[0, l]$. We can also show that $\hat{v} \equiv v^*$; $\hat{v}$ is actually a minimal positive solution satisfying $\hat{v} \geq q$ in $[0, l]$. Indeed let v is any positive solution of (2.4) satisfying $v(x) \geq q(x)$ for $0 \leq x \leq l$. Then by the comparison principle we get $u(t, x) \leq v(x)$ for $t > 0$, $0 < x < h(t)$. Letting $t \to \infty$ in this inequality, we deduce

$$\hat{v}(x) = \lim_{t\to\infty} u(t, x) \leq v(x) \ \text{ for } \ 0 < x < \infty.$$

Since v is an arbitrary function satisfying $v(x) \geq q(x)$ for $0 \leq x \leq l$, we obtain $\hat{v} \equiv v^*$. Noting that v^* is smooth (in particular, continuous) in $[0, \infty)$ and u is monotone in t, we find from Dini's theorem that

$$\lim_{t\to\infty} u(t, x) = v^*(x) \ \text{ uniformly in any compact subset of } [0, \infty).$$

Hence we complete the proof. □

2.5 General dichotomy theorem

Using the properties of spreading and vanishing constructed in the preceding section, we can get the dichotomy theorem for a certain class of nonlinear functions.

Theorem 2.6. *Let* (u, h) *be any solution of* (FBP). *Suppose that* f *satisfies*

$$f \in C^1[0, \infty), \ f(0) = f(1) = 0, \ f(u) < 0 \textit{ for } \ u > 1\ , \ f'(0) \neq 0. \tag{2.37}$$

Then either (i) *or* (ii) *holds true as* $t \to \infty$:

(i) *Spreading*: $\lim_{t\to\infty} \Omega(t) = (0, \infty)$, $\liminf_{t\to\infty} \|u(t, \cdot)\|_{C(\Omega(t))} > 0$;

(ii) *Vanishing*: $\lim_{t\to\infty} \Omega(t)$ *is a bounded set in* $(0, \infty)$, $\lim_{t\to\infty} \|u(t, \cdot)\|_{C(\Omega(t))} = 0$.

Remark 2.2. *A general dichotomy theorem in multi-dimensions is shown in Theorem 3.6. In section 3.5, we can also get criteria for spreading and vanishing if* f *satisfies* (2.37).

To prove Theorem 2.6, we need two propositions.

Proposition 2.1. *Let* (u, h) *be any solution of* (FBP)*. Suppose that* f *satisfies* (2.37) *and* $f'(0) > 0$*. If* $\lim_{t\to\infty} h(t) = \infty$*, then* $\liminf_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} > 0$.

Proof. Let $q(x)$ be an eigenfunction for the first eigenvalue $\lambda_1(l)$ of the problem

$$\begin{cases} dq_{xx} + \lambda_1 q = 0, \quad 0 < x < l, \\ q(0) = q(l) = 0 \text{ (resp. } q_x(0) = q(l) = 0), \end{cases}$$

where $l > h_0$ is a large positive number so that $f'(0) > \lambda_1(l)$. It is actually possible for any given $f'(0) > 0$ because $\lambda_1(l)$ is continuous and monotone decreasing with respect to l, $\lambda_1(l) > 0$ for all $l > 0$ and $\lim_{l\to+\infty} \lambda_1(l) = 0$. By the assumption there is some $T > 0$ such that $h(T) = l$, and moreover we can choose small $\varepsilon > 0$ to satisfy $\varepsilon q(x) \le u(T, x)$ in $(0, l)$. Since $f'(0) > \lambda_1(l)$, choosing ε sufficiently small if necessary, we find that $\phi := \varepsilon q$ satisfy

$$\begin{cases} d\phi_{xx} + f(\phi) \ge 0, \quad 0 < x < l, \\ \phi(0) = \phi(l) = 0 \text{ (resp. } \phi_x(0) = \phi(l) = 0). \end{cases}$$

Consider the solution $(w(t, x), s(t))$ with initial data (ϕ, l). Then by Lemma 2.3 we deduce

$$s(t) \le h(t + T) \ \text{ for } \ t > 0 \ \text{ and } \ w(t, x) \le u(t + T, x) \ \text{ for } \ t > 0,\ 0 < x < s(t).$$

Moreover, using Theorem 2.5, we find $\lim_{t\to\infty} s(t) = \infty$ and $\lim_{t\to\infty} w(t, x) = v(x)$ uniformly in any compact set of $[0, \infty)$, where v is a unique solution of (2.4), and consequently it follows that

$$\liminf_{t\to\infty} u(t, x) \ge \lim_{t\to\infty} w(t, x) = v(x) > 0 \ \text{ in } \ (0, \infty).$$

In particular $\liminf_{t\to\infty} \|u(t, \cdot)\|_{C(\Omega(t))} \ge \liminf_{t\to\infty} u(t, x) > 0$. □

Proposition 2.2. *Let* (u, h) *be any solution of* (FBP)*. Suppose that* f *satisfies* (2.37) *and* $f'(0) < 0$*. If* $\liminf_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} = 0$*, then* $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} = 0$ *and* $\lim_{t\to\infty} h(t) < \infty$.

Proof. By the assumption of f there is some $\sigma \in (0, 1]$ such that $f(u) < 0$ for $u \in (0, \sigma)$. Moreover, since $f(0) = 0$, we can set $f(u) = ug(u)$ and g satisfies

$$g(u) < 0 \ \ u \in [0, \sigma).$$

We will construct an upper solution $(v(t, x), s(t))$ for (FBP). Define

$$v(t, x) = \sigma e^{-\alpha t} \cos\left(\frac{\pi x}{2s(t)}\right) \ \text{ and } \ s(t) = s_0(1 + \delta(1 - e^{-\alpha t}))$$

for positive constants α, δ and s_0. Then we find $s'(t) = \alpha s_0 \delta e^{-\alpha t} > 0$. By direct calculations we have

$$\begin{aligned}
v_x &= \sigma e^{-\alpha t}\Big(-\frac{\pi}{2s(t)}\Big)\sin\Big(\frac{\pi x}{2s(t)}\Big),\\
v_{xx} &= \sigma e^{-\alpha t}\Big(-\frac{\pi^2}{4s(t)^2}\Big)\cos\Big(\frac{\pi x}{2s(t)}\Big),\\
v_t &= -\alpha\sigma e^{-\alpha t}\cos\Big(\frac{\pi x}{2s(t)}\Big)+\sigma e^{-\alpha t}\Big(\frac{s'(t)\pi x}{2s(t)^2}\Big)\sin\Big(\frac{\pi x}{2s(t)}\Big)\\
&\geq -\alpha\sigma e^{-\alpha t}\cos\Big(\frac{\pi x}{2s(t)}\Big)
\end{aligned}$$

for $t > 0$ and $0 \leq x \leq s(t)$. Hence it follows that

$$v_t - dv_{xx} - f(v) \geq \sigma e^{-\alpha t}\cos\Big(\frac{\pi x}{2s(t)}\Big)\Big\{-\alpha + \frac{d\pi^2}{4s(t)^2} - g(v)\Big\}.$$

Since $0 \leq v \leq \sigma$, we see that $-g(v) \geq \min\{-g(v);\ 0 \leq v \leq \sigma\} =: m_1 > 0$. Then, taking $\alpha \leq m_1$, we deduce

$$-\alpha + \frac{d\pi^2}{4s(t)^2} - f(v) \geq -\alpha + \frac{d\pi^2}{4s_0^2(1+\delta)^2} + m_1 > 0,$$

and consequently it holds that

$$v_t - dv_{xx} - f(v) \geq 0 \quad \text{for} \quad t > 0,\ 0 \leq x \leq s(t). \tag{2.38}$$

Moreover we can easily find

$$v(t,0) > 0 \ \text{ and } \ v(t,s(t)) = 0 \ \text{ for } \ t \geq 0. \tag{2.39}$$

Next, choosing $\delta \geq \pi\mu\sigma/(2\alpha s_0^2)$, we obtain

$$\begin{aligned}
s'(t) - (-\mu v_x(t,s(t))) &= \alpha s_0\delta e^{-\alpha t} - \frac{\pi\mu\sigma}{2s(t)}e^{-\alpha t}\\
&\geq \alpha s_0\Big(\delta - \frac{\pi\mu\sigma}{2\alpha s_0^2}\Big)e^{-\alpha t}\\
&\geq 0
\end{aligned} \tag{2.40}$$

for $t > 0$. Finally we will check the initial condition. By the assumption that $\liminf_{t\to\infty}\|u(t,\cdot)\|_{C(\Omega(t))} = 0$, there is some $T^* > 0$ such that $u(T^*,x) \leq \sigma/2$ for $0 \leq x \leq h(T^*)$. Choosing large s_0 satisfying $s_0 \geq 3h(T^*)/2$ $(\geq h_0)$, we get

$$u(T^*,x) \leq \frac{\sigma}{2} = \sigma\cos\frac{\pi}{3} \leq \sigma\cos\Big(\frac{\pi h(T^*)}{2s_0}\Big) \quad \text{for} \quad 0 \leq x \leq h(T^*).$$

Hence

$$u(T^*,x) \leq \sigma\cos\Big(\frac{\pi h(T^*)}{2s_0}\Big) \leq \sigma\cos\Big(\frac{\pi x}{2s_0}\Big) = v(0,x) \quad \text{for} \quad 0 \leq x \leq h(T^*). \tag{2.41}$$

It follows from (2.38) – (2.41) and the comparison principle (Lemma 2.2) that

$$u(t+T^*, x) \le v(t,x) \quad \text{and} \quad h(t+T^*) \le s(t) \quad \text{for} \quad t \ge 0,\ 0 \le x \le h(t).$$

Thus

$$\lim_{t\to\infty} h(t) \le \lim_{t\to\infty} s(t) = s_0(1+\delta), \quad \lim_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} \le \lim_{t\to\infty} \|v(t,\cdot)\|_{C(\Omega(t))} = 0,$$

and we complete the proof. □

Proof of Theorem 2.6. Since $h(t)$ is strictly increasing, we find that $\Omega(t) = (0, h(t))$ becomes a bounded set in $(0,\infty)$ or $(0,\infty)$ as $t \to \infty$. In the former case Theorem 2.4 implies $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} = 0$. In the latter case we discuss the problem for each case of $f'(0) > 0$ and $f'(0) < 0$, and we obtain $\liminf_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} > 0$ by Propositions 2.1 and 2.2. □

2.6 Spreading and vanishing for logistic equations

We have already shown the general dichotomy result of spreading and vanishing in Theorem 2.6. In this section, putting a more restrictive condition on f, we will show more detailed asymptotic behaviors of solutions. To be more precise we assume that

$$f \in C^1[0,\infty),\ f(0) = f(1) = 0,\ f'(0) > 0,\ f'(1) < 0,\ f(u) > 0\ (0 < u < 1),$$
$$f(u) < 0\ (u > 1),\ f(u)/u \text{ is decreasing with respect to } u \in [0,1].$$

We call **Case (A) (or a monostable/logistic case)** when the nonlinear function satisfies the above condition. In section 3.6 we will consider more general polystable nonlinearity satisfying (2.37) and $f'(0) > 0$, and generalize main theorems of this section.

2.6.1 Main theorems

The following theorem is a dichotomy result for Case (A).

Theorem 2.7. *Let* (u,h) *be any solution of* (FBP). *Then, either spreading* (i) *or vanishing* (ii) *holds true*:

(i) $\lim_{t\to\infty} \Omega(t) = (0,\infty)$ *and* $\lim_{t\to\infty} u(t,x) = v^*(x)$ *uniformly in any compact subset of* $[0,\infty)$, *where* v^* *is a unique positive solution of* (2.4);

(ii) $\lim_{t\to\infty} \Omega(t) \subset (0, R_1^*)$ *and* $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} = 0$, *where* $R_1^* = \pi\sqrt{d/f'(0)}$ *(resp.* $(\pi/2)\sqrt{d/f'(0)}$*). Moreover* $\|u(t,\cdot)\|_{C(\Omega(t))} = O(e^{-\beta t})$ *for some* $\beta > 0$ *as* $t \to \infty$.

Remark 2.3. *The number R_1^* in Theorem 2.7 is determined by $f'(0) = \lambda_1(R_1^*)$, where $\lambda_1(l)$ is the least eigenvalue of*

$$\begin{cases} -d\phi_{xx} = \lambda_1 \phi, \quad 0 < x < l, \\ \phi > 0, \quad 0 < x < l, \\ \phi(0) = \phi(l) = 0 \quad (\textit{resp. } \phi_r(0) = \phi(l) = 0) \end{cases}$$

for $l > 0$. Indeed it is well known that $\lambda_1(l)$ is continuous and decreasing with respect to l and that it satisfies $\lim_{l\to 0} \lambda_1(l) = +\infty$ and $\lim_{l\to+\infty} \lambda_1(l) = 0$. Thus, there exists a unique positive number R_1^* such that $f'(0) = \lambda_1(R_1^*)$ and $f'(0) > \lambda_1(l)$ for $l > R_1^*$.

Remark 2.4. *When we put the Neumann boundary condition at $x = 0$ ($u_x(t, 0) = 0$) in* (FBP)*, we find $v^*(x) \equiv 1$. Putting the Dirichlet boundary condition, we see that $v_x^*(x) > 0$ in $[0, \infty)$ and $\lim_{x\to\infty} v^*(x) = 1$.*

In the following theorems we use the number R_1^* given in Theorem 2.7.

Theorem 2.8. *Let (u, h) be any solution of* (FBP) *and the following results hold true:*

(i) *Suppose $h_0 \geq R_1^*$. Then spreading occurs.*

(ii) *Suppose $h_0 < R_1^*$.*

  (a) *If initial function u_0 is small enough to satisfy $u_0(x) \leq w(x)$ in Ω_0 for a positive function w defined in Ω_0, then vanishing occurs.*

  (b) *If initial data satisfies*

$$\int_0^{h_0} x u_0(x)\, dx > \frac{d}{2\mu}((R_1^*)^2 - h_0^2), \quad h_0 > \frac{R_1^*}{\sqrt{\mu/d + 1}}$$

$$\left(\textit{resp.} \int_0^{h_0} u_0(x)\, dx > \frac{d}{\mu}(R_1^* - h_0), \quad h_0 > \frac{R_1^*}{\mu/d + 1}\right),$$

  then spreading occurs.

We find from Theorem 2.8 that spreading always occurs when initial habitat is large ($\Omega_0 = (0, h_0) \supset (0, R_1^*]$). On the other hand if the habitat is small enough to satisfy $\Omega_0 \subset (0, R_1^*)$, then there still exist two possibilities. In that case initial population density u_0 or the Stefan coefficient μ determines the asymptotic behaviors of solutions.

Theorem 2.9. *Suppose $h_0 < R_1^*$. Let $\phi \in C^2(\Omega_0) \cap C(\overline{\Omega}_0)$ be any function which satisfies $\phi(0) = \phi(h_0) = 0$ (resp. $\phi_x(0) = \phi(h_0) = 0$). Then there exists a number $\sigma^* = \sigma^*(\phi, h_0) \in (0, \infty]$ such that spreading occurs if $u_0 > \sigma^*\phi$ in Ω_0 and vanishing occurs if $u_0 \leq \sigma^*\phi$ in Ω_0.*

Theorem 2.10. *Suppose $h_0 < R_1^*$. Then there exists some number $\mu^* = \mu^*(u_0, h_0) \in [0, \infty)$ such that spreading occurs for $\mu > \mu^*$, while vanishing occurs for $\mu \leq \mu^*$. Moreover, if $f(u) \leq f'(0)u$ for $u \geq 0$, then $\mu^* \in (0, \infty)$.*

2.6.2 Preliminaries

In this subsection we prepare some results on the fixed boundary problems and some key propositions to prove the main theorems. We first consider the following problem:

$$\begin{cases} u_t = du_{xx} + f(u) & t > 0,\ 0 < x < l, \\ u(t,0) = 0 \text{ (resp. } u_x(t,0) = 0), & t > 0, \\ u(t,l) = 0, & t > 0, \\ u(0,x) = \varphi(x), & 0 \le x \le l, \end{cases} \tag{2.42}$$

where l is a positive number and φ is a nonnegative continuous function and $\varphi \not\equiv 0$. It is well known that any solution $u(t,x)$ of (2.42) converges to a positive solution of the corresponding stationary problem as $t \to \infty$ (see e.g. Brunovsky-Chow [5], Hale-Massatt [33] or Matano [49]):

$$\begin{cases} dq_{xx} + f(q) = 0, \quad 0 < x < l, \\ q(0) = 0 \text{ (resp. } q_x(0) = 0), \\ q(l) = 0. \end{cases} \tag{2.43}$$

To be more precise, we have the following result.

Proposition 2.3. *Let $u = u(t,x)$ be any solution of (2.42) and $R_1^* = \pi\sqrt{d/f'(0)}$ (resp. $(\pi/2)\sqrt{d/f'(0)}$).*

(i) *If $l \le R_1^*$, then $q \equiv 0$ is a unique solution of (2.43) and $\lim_{t\to\infty} u(t,x) = 0$ uniformly in $[0,l]$;*

(ii) *If $l > R_1^*$, then (2.43) has a unique positive solution $q = q_l(x)$ and $\lim_{t\to\infty} u(t,x) = q_l(x)$ uniformly in $[0,l]$.*

For the proof, see Cantrell and Cosner [10, Corollary 3.4]. The number R_1^* is often called "*minimal patch size*" in the sense that the population establishes themselves as time tends to infinity if the length of their habitat is larger than R_1^*. Theorem 2.7 also implies that this patch size has an important role in the free boundary problem.

We next prepare two key propositions for solutions (u,h) of (FBP).

Proposition 2.4. *If $\lim_{t\to\infty} h(t) < \infty$, then $\lim_{t\to\infty} h(t) \le R_1^*$*

Proof. Assume that there exists $T > 0$ such that $l := h(T) > R_1^*$ and consider the solution $w(t,x)$ of problem (2.42) with initial data satisfying $\varphi(x) \equiv u(T,x)$. Then the standard comparison problem shows

$$u(t+T,x) \ge w(t,x) \ \text{ for } \ t > 0,\ 0 < x < l.$$

Note that, by Proposition 2.3, the function $w(t)$ converges to the solution q_l of (2.43) as $t \to \infty$. Then we find

$$\liminf_{t\to\infty} u(t,x) \ge q_l(x) > 0 \quad \text{for} \ \ 0 < x < l,$$

However this fact gives an contradiction because any solution of (FBP) with the property $\lim_{t\to\infty} h(t) < \infty$ must satisfy $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} = 0$ by Theorem 2.4. Hence $\lim_{t\to\infty} h(t) \le R_1^*$ and the proof is complete. □

Proposition 2.5. *If* $\lim_{t\to\infty} h(t) = \infty$*, then* $u(t)$ *converges to* v^* *as* $t \to \infty$ *uniformly in any compact subset of* $[0,\infty)$*, where* $v^*(x)$ *is a unique positive solution of* (2.4).

Proof. Since $\lim_{t\to\infty} h(t) = \infty$, for any given positive number $l > R_1^*$ we can take $T > 0$ such that $h(T) = l$. Then, as in the proof of Proposition 2.1, we obtain

$$\liminf_{t\to\infty} u(t,x) \ge \lim_{t\to\infty} w(t,x) = v^*(x) \quad \text{for} \quad x \ge 0. \tag{2.44}$$

Here we note that $w(t)$ converges to v^* uniformly in any compact subset of $[0,\infty)$ because of Dini's theorem and the non-decreasing of $w(t)$ by Theorem 2.5.

We next consider the solution $\overline{u}(t,x)$ of

$$\begin{cases} \overline{u}_t - d\overline{u}_{xx} = f(\overline{u}), & t>0,\ x>0, \\ \overline{u}(t,0) = 0 \ (\text{resp. } \overline{u}_x(t,0) = 0), & t>0, \\ \overline{u}(0,x) = \max\{1, \|u_0\|_{C(\Omega_0)}\}, & x>0. \end{cases}$$

It follows from $u_0(x) \le \overline{u}(0,x)$ for $x \ge 0$ that the standard comparison principle (see Protter and Weinberger [56] or Smoller [61]) shows

$$u(t,x) \le \overline{u}(t,x) \quad \text{for} \quad t \ge 0,\ 0 < x < h(t). \tag{2.45}$$

Moreover, because $v \equiv M := \max\{1, \|u_0\|_{C(\Omega_0)}\}$ is regarded as an upper solution of (2.4), the function $\overline{u}$ is monotone decreasing with respect to t and $\overline{u}(t)$ converges to $v^*(x)$ as $t \to \infty$ uniformly in any compact subset of $[0,\infty)$ (see Sattinger [58] or Smoller [61]). Hence, letting $t \to \infty$ in (2.45), we obtain

$$\limsup_{t\to\infty} u(t,x) \le \lim_{t\to\infty} \overline{u}(t,x) = v^*(x) \quad \text{for} \quad 0 < x < \infty. \tag{2.46}$$

From (2.44), (2.46) and the uniform convergence of w and $\overline{u}$ to v^*, we find that $u(t,x)$ converges to $v^*(x)$, as $t \to \infty$, uniformly in any compact subset of $[0,\infty)$, and the proof is complete. □

2.6.3 Proofs of main theorems

Proof of Theorem 2.7. We can apply Theorem 2.6 to this case. Moreover Proposition 2.5 shows if $\Omega(t) = (0,\infty)$, then $u(t,x)$ converges to $v^*(x)$ as $t \to \infty$ uniformly in any compact subset of $(0,\infty)$, while Proposition 2.4 implies if $\Omega(t)$ is bounded for all $t > 0$, then it is included by the interval $(0, R_1^*)$. When $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} = 0$, there is some $T > 0$ such that $u(T,x)$ is sufficiently small. Then, as in the proof of part (ii-a) in Theorem 2.8, we can prove

$$u(t,x) \le \varepsilon_0 e^{-\beta t} \sin\left(\frac{\pi-\gamma}{s(t)}x + \gamma\right) \left(\text{resp. } u(t,x) \le \varepsilon_0 e^{-\beta t} \cos\left(\frac{\pi}{2s(t)}x\right)\right)$$

for $t \geq T,\ 0 < x < h(t)$, where $s(t) = s_0(1 + \delta(1 - e^{-\alpha t}))$ and s_0, ε_0, α, β, γ and δ are suitable constants. This implies $\|u(t, \cdot)\|_{C(\Omega(t))} = O(e^{-\beta t})$ as $t \to \infty$, and we complete the proof. $\square$

Proof of Theorem 2.8. (i) We remark that, when vanishing occurs, the free boundary must satisfy $\lim_{t\to\infty} h(t) \leq R_1^*$ by Theorem 2.7. Since $h_0 \geq R_1^*$ and $h(t)$ is strictly increasing with respect to t, we find $\lim_{t\to\infty} h(t) > R_1^*$. Hence Theorem 2.7 implies $\lim_{t\to\infty} h(t) = \infty$ and spreading occurs.

(ii-a) We first discuss the case $R_1^* = \pi\sqrt{d/f'(0)}$ (where we put the Dirichlet boundary condition at $x = 0$ in (FBP)). Define

$$s(t) = s_0(1 + \delta(1 - e^{-\alpha t})) \ \text{ and } \ v(t, x) = \varepsilon_0 e^{-\beta t} \sin\Big(\frac{\pi - \gamma}{s(t)} x + \gamma\Big),$$

where $s_0 \in [h_0, R_1^*)$, $\gamma \in (0, \pi/2)$, and α, β, δ and ε_0 are positive constants. We will prove that $(v(t, x), s(t))$ is an upper solution of (FBP) by choosing suitable positive numbers. Since $s_0 < \pi\sqrt{d/f'(0)}$, we have $d\pi^2/s_0^2 - f'(0) > 0$ and there exist small positive constants γ and δ such that

$$\frac{d(\pi - \gamma)^2}{s_0^2(1 + \delta)^2} - f'(0) \geq 2\delta \tag{2.47}$$

For such $\delta > 0$, we can choose small $\varepsilon_0 > 0$ to satisfy

$$f(v) \leq (f'(0) + \delta)v \ \text{ for } \ 0 < v \leq \varepsilon_0.$$

Then direct calculation gives

$$\begin{aligned}
v_x &= \varepsilon_0 e^{-\beta t}\Big(\frac{\pi - \gamma}{s(t)}\Big) \cos\Big(\frac{\pi - \gamma}{s(t)} x + \gamma\Big),\\
v_{xx} &= -\varepsilon_0 e^{-\beta t}\Big(\frac{\pi - \gamma}{s(t)}\Big)^2 \sin\Big(\frac{\pi - \gamma}{s(t)} x + \gamma\Big),\\
v_t &= -\beta\varepsilon_0 e^{-\beta t} \sin\Big(\frac{\pi - \gamma}{s(t)} x + \gamma\Big) + \varepsilon_0 e^{-\beta t}\Big(\frac{-s'(t)(\pi - \gamma)x}{s(t)^2}\Big) \cos\Big(\frac{\pi - \gamma}{s(t)} x + \gamma\Big).
\end{aligned}$$

Hence we get

$$\begin{aligned}
v_t - dv_{xx} - f(v) &\geq \Big\{ -\beta + \frac{d(\pi - \gamma)^2}{s(t)^2} - f'(0) - \delta\Big\}\varepsilon_0 e^{-\beta t} \sin\Big(\frac{\pi - \gamma}{s(t)} x + \gamma\Big)\\
&\qquad - \frac{s'(t)(\pi - \gamma)x}{s(t)^2}\varepsilon_0 e^{-\beta t} \cos\Big(\frac{\pi - \gamma}{s(t)} x + \gamma\Big)\\
&\geq \Big\{ -\beta + \frac{d(\pi - \gamma)^2}{s_0^2(1 + \delta)^2} - f'(0) - \delta\Big\}\varepsilon_0 e^{-\beta t} \sin\Big(\frac{\pi - \gamma}{s(t)} x + \gamma\Big)\\
&\qquad - \frac{s'(t)(\pi - \gamma)x}{s(t)^2}\varepsilon_0 e^{-\beta t} \cos\Big(\frac{\pi - \gamma}{s(t)} x + \gamma\Big).
\end{aligned} \tag{2.48}$$

By (2.47), we can take $\beta < \delta$ to obtain

$$-\beta + \frac{d(\pi-\gamma)^2}{s_0^2(1+\delta)^2} - f'(0) - \delta \geq -\beta + \delta > 0.$$

Hence

$$\Big\{ -\beta + \frac{d(\pi-\gamma)^2}{s_0^2(1+\delta)^2} - f'(0) - \delta \Big\} \sin\Big(\frac{\pi-\gamma}{s(t)}x + \gamma\Big) > 0 \tag{2.49}$$

for $t > 0,\ 0 \leq x < s(t)$. If the second term in the right-hand side of (2.48) is negative, then we can immediately obtain $v_t - dv_{xx} - f(v) \geq 0$. However we actually find

$$\cos\Big(\frac{\pi-\gamma}{s(t)}x + \gamma\Big) \begin{cases} \geq 0 & \text{for } \ t > 0,\ 0 \leq x \leq X_t := (\pi - 2\gamma)s(t)/\{2(\pi-\gamma)\}, \\ \leq 0 & \text{for } \ t > 0,\ X_t \leq x \leq s(t). \end{cases}$$

For $t > 0$, $0 \leq x \leq X_t$, because of $0 < s'(t) \leq s_0\alpha\delta$, it holds that

$$\begin{aligned} -\frac{s'(t)(\pi-\gamma)x}{s(t)^2}\cos\Big(\frac{\pi-\gamma}{s(t)}x + \gamma\Big) &\geq -\frac{s_0\alpha\delta(\pi-\gamma)X_t}{s(t)^2}\cos\gamma \\ &\geq -\frac{\alpha\delta(\pi-2\gamma)}{2}\cos\gamma \end{aligned} \tag{2.50}$$

and

$$\sin\Big(\frac{\pi-\gamma}{s(t)}x + \gamma\Big) \geq \sin\gamma. \tag{2.51}$$

Choosing the constant α satisfying

$$\alpha \leq \alpha^* := \frac{2\tan\gamma}{(\pi-2\gamma)\delta}\Big\{ -\beta + \frac{d(\pi-\gamma)^2}{s_0^2(1+\delta)^2} - f'(0) - \delta \Big\}.$$

we can deduce from (2.48) – (2.51) that $v_t - dv_{xx} - f(v) \geq 0$ for $t > 0$ and $0 \leq x \leq X_t$. Hence

$$v_t - dv_{xx} - f(v) \geq 0 \quad \text{for} \quad t > 0,\ 0 \leq x \leq s(t). \tag{2.52}$$

We will next check

$$s'(t) - (-\mu v_x(t, s(t))) \geq 0. \tag{2.53}$$

Indeed we observe

$$\begin{aligned} s'(t) - (-\mu v_x(t, s(t))) &= s_0\alpha\delta e^{-\alpha t} - \frac{\mu(\pi-\gamma)}{s(t)}\varepsilon_0 e^{-\beta t} \\ &\geq \Big\{ s_0\alpha\delta - \frac{\mu\varepsilon_0(\pi-\gamma)}{s_0}e^{(\alpha-\beta)t} \Big\} e^{-\alpha t}. \end{aligned}$$

If necessary, we will choose again small α and small ε_0 such that

$$0 < \alpha \leq \min\{\alpha^*,\ \beta\}, \quad \varepsilon_0 \leq \frac{s_0^2\alpha\delta}{\mu(\pi-\gamma)}$$

to get (3.32). Note that for $t > 0$

$$u(t,0) = 0 \leq \varepsilon_0 e^{-\beta t} \sin\gamma = v(t,0) \quad \text{and} \quad v(t,s(t)) = 0 \tag{2.54}$$

and we can choose u_0 small to satisfy

$$u_0(x) \leq w(x) := v(0,x) = \varepsilon_0 \sin\left(\frac{\pi-\gamma}{s_0}x + \gamma\right) \quad \text{in} \quad \Omega_0, \tag{2.55}$$

then it follows from (2.52) – (2.55) that Lemma 2.2 shows

$$h(t) \leq s(t) \quad \text{for} \ \ t > 0 \quad \text{and} \quad u(t,x) \leq v(t,x) \quad \text{for} \ \ t > 0, \ 0 \leq x \leq h(t).$$

Hence $\lim_{t\to\infty} h(t) \leq \lim_{t\to\infty} s(t) = s_0(1+\delta) < \infty$ and vanishing occurs by Theorem 2.7. Moreover $\|u(t,\cdot)\|_{C(\Omega(t))} = O(e^{-\beta t})$ as $t \to \infty$.

We finally remark on the case of the Neumann boundary condition at $x = 0$ (that is $R_1^* = (\pi/2)\sqrt{d/f'(0)}$). Replacing $v(t,x)$ in the Dirichlet problem to

$$v_1(t,x) = \varepsilon_0 e^{-\beta t} \cos\left(\frac{\pi}{2s(t)}x\right),$$

we get the conclusion more easily in an almost same way.

(ii-b) We consider a Stefan problem for the heat equation:

$$\begin{cases} w_t - dw_{xx} = 0, & t > 0, \ 0 < x < y(t), \\ w(t,0) = 0, \ (\text{resp. } w_x(t,0) = 0), & t > 0, \\ w(t,y(t)) = 0, & t > 0, \\ y'(t) = -\mu w_x(t,y(t)), & t > 0, \\ y(0) = h_0, \ w(0,x) = u_0(x), & 0 \leq x \leq y_0, \end{cases} \tag{2.56}$$

where d, μ, h_0 and u_0 are same as those of (FBP). It is well known that problem (2.56) has a unique classical solution globally in time. Define

$$Y^* = \left(h_0^2 + \frac{2\mu}{d}\int_0^{h_0} xu_0(x)\,dx\right)^{1/2} \ \left(\text{resp. } h_0 + \frac{\mu}{d}\int_0^{h_0} u_0(x)\,dx\right).$$

Then we will show $\lim_{t\to\infty} y(t) = Y^*$. Indeed by the Green's theorem

$$\int_{D_y(t)} v(dw_{xx} - w_t) - w(dv_{xx} + v_t)\,dxdt = \int_{\partial D_y(t)} d(vw_x - wv_x)\,dt + wv\,dx$$

for any smooth function $v = v(t,x)$, where $D_y(t) = \bigcup_{0\leq s\leq t}(\{s\} \times (0,y(s)))$. We set $v = -x$ (resp. $v = 1$) in the above identity to get

$$\int_{\partial D_y(t)} d(-xw_x + w)\,dt - xw\,dx = 0 \ \left(\text{resp. } \int_{\partial D_y(t)} dw_x\,dt + w\,dx = 0\right).$$

Using (2.56), we deduce from the direct calculations

$$y(t)^2 = (Y^*)^2 - \frac{2\mu}{d}\int_0^{y(t)} xw(t,x)\,dx \ \left(\text{resp. } y(t) = Y^* - \frac{\mu}{d}\int_0^{y(t)} w(t,x)\,dx\right)$$

(cf. Nogi and Yamaguchi [54], Cannon and Denson [9]). Note that $y(t) \le Y^*$ for $t \ge 0$ and $\|w(t,\cdot)\|_{C(0,y(t))} \to 0$ as $t \to \infty$, and hence $\int_0^{y(t)} xw(t,x)\,dx$ and $\int_0^{y(t)} w(t,x)\,dx$ converge to 0 as $t \to \infty$. Thus $\lim_{t\to\infty} y(t) = Y^*$.

We assume $\|u_0\|_{C(\Omega_0)} < 1$. Then $0 < w(t,x) < 1$ for $t > 0$, $0 < x < y(t)$, and hence $f(w(t,x)) \ge 0$ for $t > 0$, $0 < x < y(t)$. Since $w_t - dw_{xx} = 0 \le f(w)$, the solution $(w(t,x), y(t))$ is regarded as a lower solution of (FBP). Hence Lemma 2.3 shows $h(t) \ge y(t)$ for $t \ge 0$. By the assumption, letting $t \to \infty$ in the above inequality implies

$$\begin{aligned}
&\lim_{t\to\infty} h(t) \ge \lim_{t\to\infty} y(t) = Y^* = \left(h_0^2 + \frac{2\mu}{d}\int_0^{h_0} xu_0(x)\,dx\right)^{1/2} > R_1^* \\
&\left(\text{resp. } \lim_{t\to\infty} h(t) \ge \lim_{t\to\infty} y(t) = Y^* = h_0 + \frac{\mu}{d}\int_0^{h_0} u_0(x)\,dx > R_1^*\right).
\end{aligned} \tag{2.57}$$

Thus we can conclude from Theorem 2.7 that spreading occurs for the solution. We finally consider any solution of (FBP) with an initial function larger than $u_0(x)$. Then Lemma 2.3 shows spreading occurs as $t \to \infty$. Hence the proof is complete. □

Proof of Theorem 2.9. Define

$$\sigma^* := \inf\{\,\rho \mid \text{For any } \sigma > \rho, \text{ spreading occurs for the solution with initial data } (\sigma\phi, h_0)\}.$$

We recall by part (ii) of Theorem 2.8 that vanishing occurs for small u_0. Hence we find $\sigma^* \in (0, \infty]$. By the definition of σ^* and the dichotomy theorem (Theorem 2.7), for any $\sigma < \sigma^*$, we can choose a number $\tau \in [\sigma, \sigma^*]$ such that the solution $(u_\tau(t,x), h_\tau(t))$ of (FBP) with initial data $(\tau\phi, h_0)$ satisfies the property of vanishing as $t \to \infty$. In other words

$$\lim_{t\to\infty} h_\tau(t) = 0 \ \text{ and } \ \lim_{t\to\infty} \|u_\tau(t,\cdot)\|_{C(\Omega(t))} = 0.$$

By the comparison principle, we see that $h_\sigma(t) \le h_\tau(t)$ for $t > 0$ and $u_\sigma(t,x) \le u_\tau(t,x)$ for $t > 0$, $0 < x < h_\sigma(t)$ if $\sigma < \tau$ because of $\sigma\phi \le \tau\phi$. Hence $h_\sigma(t)$ is finite and $u_\sigma(t,x)$ also converges to 0 as $t \to \infty$. Hence vanishing occurs for all $\sigma < \sigma^*$.

We will show that the solution $(u_{\sigma^*}(t,x), h_{\sigma^*}(t))$ satisfies the property of vanishing. Otherwise there is some constant $T > 0$ satisfying $h(T) > R_1^*$. By the continuous dependence of solutions on initial data, we can take small $\delta > 0$ such that $h_\sigma(T) > R_1^*$ for any $\sigma \in [\sigma^* - \delta, \sigma^* + \delta]$. Using this fact and part (i) of Theorem 2.8, we find that spreading occurs for the solution $(u_{\sigma^*-\delta}(t,x), h_{\sigma^*-\delta}(t))$ of (FBP) as $t \to \infty$. This is a contradiction to the definition of σ^*. Hence Theorem 2.7 implies that vanishing occurs for $\sigma = \sigma^*$.

Thus, if initial data satisfies $u_0 \le \sigma^*\phi$, then the comparison theorem shows

$$h(t) \le h_{\sigma^*}(t) \ \text{ for } \ t > 0, \quad u(t,x) \le u_{\sigma^*}(t,x) \ \text{ for } \ t > 0,\ 0 < x < h(t),$$

and vanishing occurs. On the other hand, if $u_0 > \sigma^*\phi$, then spreading occurs because of the definition of σ^* and the comparison principle. We complete the proof. $\square$

Proof of Theorem 2.10. We first show that spreading occurs for large μ. By same way as the proof of part (ii-b) of Theorem 2.8, if μ is large enough to satisfy

$$\mu > \frac{d}{2\int_0^{h_0} xu_0(x)\,dx}((R_1^*)^2 - h_0^2) \quad \left(\text{resp. } \mu > \frac{d}{\int_0^{h_0} u_0(x)\,dx}(R_1^* - h_0)\right)$$

instead of choosing large u_0, then (2.57) also holds true. Hence Theorem 2.7 implies spreading for the solution.

We next show a threshold number on μ. Define

$$\mu^* := \inf\{\rho > 0 \mid \text{vanishing occurs for any } \mu > \rho\}.$$

Since spreading occurs for large μ, we find that μ^* is finite. For any $\mu > \mu^*$, there is some $\mu_1 \in [\mu^*, \mu]$ such that spreading occurs for a solution (u_{μ_1}, h_{μ_1}) of (FBP) with $\mu = \mu_1$. Taking any $\mu_1 < \mu_2$, we can compare (u_{μ_2}, h_{μ_2}) with (u_{μ_1}, h_{μ_1}) by Theorem 2.4 and find that spreading occurs for $\mu = \mu_2$. Thus spreading occurs for all $\mu > \mu^*$. We finally show vanishing occurs for $\mu = \mu^*$. Otherwise there is some $T > 0$ such that $h_{\mu^*}(T) > R_1^*$. By the continuous dependence of solutions on μ, we also find $h_\mu(T) > R_1^*$ for $\mu \in [\mu^* - a, \mu^* + a]$ with some small $a > 0$. This result, together with part (i) of Theorem 2.8, implies that spreading occurs for such μ. This is a contradiction to the definition on μ^*. Hence vanishing occurs for $\mu \leq \mu^*$.

Assuming that $f(u) \leq f'(0)u$ for $u \geq 0$, we find that vanishing occurs for small μ. Indeed define $s(t)$ and $v(t,x)$ (resp. $v_1(t,x)$) as in the the proof of Theorem 2.8, where $s_0 \in [h_0, R_1^*)$, $\gamma \in (0, \pi/2)$, and α, β, δ and M are positive constants. We choose α, β, γ and δ in the same way as the proof of Theorem 2.8. Moreover taking

$$\mu \leq \frac{s_0^2\alpha\delta}{M(\pi - \delta)}, \quad M \geq \frac{\|u_0\|_{C(\Omega(t))}}{S^*}, \quad S^* = \min_{0\leq x\leq h_0} \sin\left(\frac{\pi - \gamma}{s_0}x + \gamma\right),$$

we can show that (v, s) is an upper solution of (FBP) (Here we need the condition $f(u) \leq f'(0)u$ for $u \geq 0$ to get $v_t - dv_{xx} - f(v) \geq 0$). Hence it holds that

$$h(t) \leq s(t) \quad \text{for} \quad t > 0, \quad u(t,x) \leq v(t,x) \quad \text{for} \quad t > 0,\ 0 < x < h(t).$$

Since $\lim_{t\to\infty} h(t) \leq \lim_{t\to\infty} s(t) = s_0(1 + \delta) < \infty$, vanishing occurs by Theorem 2.7. We complete the proof. $\square$

2.7 Spreading and vanishing for bistable equations

We consider in this section the case where the nonlinear function is bistable. Here we call **Case (B) (or a bistable case)** when f satisfies

$$f \in C^1[0,\infty),\ f(0) = f(c^*) = f(1) = 0,\ f'(0) < 0,\ f'(c^*) > 0,\ f'(1) < 0,$$

$$f(u) < 0\ (0 < u < c^*,\ u > 1),\ f(u) > 0\ (c^* < u < 1) \text{ and } \int_0^1 f(u)\,du > 0.$$

In Section 3.7 we will consider more general polystable nonlinearity satisfying (2.37) and $f'(0) < 0$, and generalize some theorems of this section.

2.7.1 Main theorems

Let (u, h) be any solution of (FBP). A dichotomy theorem in Case (B) is given in Theorem 2.6. We recall, as $t \to \infty$, either **spreading**

$$\lim_{t\to\infty} \Omega(t) = (0, \infty) \quad \text{and} \quad \liminf_{t\to\infty} \|u(t, \cdot)\|_{C(\Omega(t))} > 0$$

or **vanishing**

$$\lim_{t\to\infty} \Omega(t) \text{ is a bounded set in } (0, \infty) \quad \text{and} \quad \lim_{t\to\infty} \|u(t, \cdot)\|_{C(\Omega(t))} = 0$$

occurs. Also we find more detailed behaviors of solutions.

Theorem 2.11. *The following results hold true*:

(i) *Let $l > 0$ be a large number such that* (2.3) *has a solution $q(x)$ in $[0, l]$. If $h_0 \geq l$, $u_0(x) \geq q(x)$ in $[0, l]$, then spreading occurs and*

$$\liminf_{t\to\infty} u(t, x) \geq v^*(x) \quad \text{in} \quad [0, \infty),$$

where $v^*(x)$ is a minimal positive solution of (2.4).

(ii) *If $\|u_0\|_{C(\Omega_0)} < c^*$, then vanishing occurs. Moreover, when vanishing occurs, for any $\varepsilon > 0$ there exist positive numbers T_ε and C_ε such that*

$$u(t, x) \leq C_\varepsilon e^{-(k^*-\varepsilon)(t-T_\varepsilon)} \quad \text{for} \quad t \geq T_\varepsilon, \ 0 \leq x \leq h(t),$$

where $k^* = -f'(0) > 0$.

Remark 2.5. *It is well known that* (2.3) *have at least one positive solutions in* $[0, l]$ *if $l > 0$ is sufficiently large. Moreover, when f has a special form: $u(u - c)(1 - u)$ for $0 < c < 1/2$, we have the precise structure of solutions (see Smoller [61, Theorem 24.13] or Smoller and Wasserman [62]); There exists a positive number L such that*

- *If $l < L$, then* (2.3) *has a unique trivial solution $q \equiv 0$;*
- *If $l = L$, then* (2.3) *has a unique positive solution $q(x)$;*
- *If $l > L$, then* (2.3) *has two positive solutions $q_1(x)$ and $q_2(x)$ which satisfy $q_1(x) < q_2(x)$ in $(0, l)$.*

Also, in part (i) *of Theorem* (2.11)*, we get the uniform convergence of $\lim_{t\to\infty} u(t, x) = v^*(x)$ in any compact subset of $[0, \infty)$ where v^* is a unique solution of* (2.4).

We can give a criterion for spreading and vanishing.

Theorem 2.12. *Let* $\phi \in C^2(\Omega_0) \cap C(\overline{\Omega}_0)$ *be any function which satisfies* $\phi(0) = \phi(h_0) = 0$ (*resp.* $\phi_x(0) = \phi(h_0) = 0$). *Then there exists a number* $\sigma^* = \sigma^*(\phi, h_0) \in (0, \infty]$ *with the following properties:*

- *spreading occurs if* $u_0 \geq \sigma^*\phi$ *in* Ω_0. *Moreover there exists a positive number* $\sigma^{**} \geq \sigma^*$ *such that if* $u_0 > \sigma^{**}\phi$ *in* Ω_0, *then* $\liminf_{t\to\infty} u(t,x) \geq v^*(x)$ *in* $[0,\infty)$, *where* $v^*(x)$ *is a minimal positive solution of* (2.4).
- *vanishing occurs if* $u_0 < \sigma^*\phi$ *in* Ω_0.

Moreover if h_0 *is sufficiently large, then* $\sigma^* \leq \sigma^{**} < \infty$.

Remark 2.6. *Concerning a more complete classification of the behavior of* u, *we can refer to recent papers of Du-Lou [20] and Liu-Lou [47]. In [20], they have considered a related free boundary problem whose results are applicable to* (FBP) *with homogeneous Neumann boundary condition at* $x = 0$ ($u_x(t,0) = 0$). *They classify the spreading* $\liminf_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} > 0$ *into two cases*; *if* $u_0 > \sigma^*\phi$, *then the solution converges locally uniformly to a stationary solution of* (2.4), *and if* $u_0 = \sigma^*\phi$, *then* **transition** *occurs in the following sense*:

$$\lim_{t\to\infty} |u(t,x) - V(x)| = 0 \quad \textit{locally uniformly in } [0,\infty), \tag{2.58}$$

where V *is a unique solution of*

$$\begin{cases} dV_{xx} + f(V) = 0, & x \in \mathbb{R}, \\ V_x(0) = 0, \\ \lim_{x\to\pm\infty} V(x) = 0. \end{cases}$$

Moreover, in [47], they have studied (FBP) *with Robin boundary condition at* $x = 0$ ($au_x(t,0) - (1-a)u(t,0) = 0$ *for any* $a \in [0,1]$), *and, furthermore, they have found the transition-phenomenon*: *if* $u_0 = \sigma^*\phi$, *then*

$$\lim_{t\to\infty} \|u(t,\cdot) - V(\cdot + \gamma(t))\|_{L^\infty(0,h(t))} = 0$$

for some continuous function $\gamma(t)$. *They also investigate the behavior of* $\gamma(t)$. *In case of the Dirichlet boundary condition, they obtain* $\lim_{t\to\infty} \gamma(t) = +\infty$.

Remark 2.7. *In Theorem 2.12, it is also important whether* σ^* *is finite or not, if* $h_0 < L$. *It is proved in [20] and [47] that* $\sigma^* < \infty$ *when* f *has other conditions. Moreover, for special kind of* f, *spreading cannot occur* ($\sigma^* = \infty$) *even if* u_0 *is large.*

2.7.2 Proofs of main theorems

We will prove the main theorem in the following.

Proof of Theorem 2.11. (i) Let $(\underline{u}, \underline{h})$ be a solution of (FBP) with initial data $(q(x), l)$. Then Corollary 2.1 shows $\lim_{t\to\infty} \underline{h}(t) = \infty$ and

$$\liminf_{t\to\infty} u(t,x) \geq \lim_{t\to\infty} \underline{u}(t,x) = v^*(x) \quad \text{for} \quad x \geq 0.$$

(ii) Let $w = w(t)$ be the solution of

$$\begin{cases} \dfrac{dw}{dt} = f(w), \quad t > 0, \\ w(0) = \|u_0\|_{C(\Omega_0)} < c^*. \end{cases} \tag{2.59}$$

Then, since $f(w) < 0$ for $0 < w < c^*$ and $f(0) = 0$, the function $w(t)$ is decreasing and $\lim_{t\to\infty} w(t) = 0$. Noting that

$$u(t,0) = 0 < w(t) \quad \text{and} \quad u(t,h(t)) = 0 < w(t) \quad \text{for} \quad t > 0.$$

we find from the standard comparison principle that

$$u(t,x) \leq w(t) \quad \text{for} \quad t \geq 0,\ 0 \leq x \leq h(t). \tag{2.60}$$

Hence $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(\Omega(t))} \leq \lim_{t\to\infty} w(t) = 0$. Thus we conclude from Proposition 2.2 that $\lim_{t\to\infty} h(t) < \infty$ and vanishing occurs.

We will next show some decay properties of solutions. Using the mean-value theorem, one can get $f(w) = f'(\theta w)w$ for some $\theta \in [0,1]$. Since f' is continuous in $[0,\infty)$ and $\lim_{t\to\infty} w(t) = 0$, for any small $\varepsilon > 0$, there is a positive number T_ε such that

$$f(w(t)) \leq (f'(0) + \varepsilon)w(t) \ \text{ for } \ t \geq T_\varepsilon.$$

Hence for $t \geq T_\varepsilon$ the solution of (2.59) has a linear estimate $dw/dt \leq (f'(0) + \varepsilon)w$, which implies

$$w(t) \leq w(T_\varepsilon)e^{-(k^*-\varepsilon)(t-T_\varepsilon)} \quad \text{for} \ \ t \geq T_\varepsilon \tag{2.61}$$

with $k^* = -f'(0) > 0$. Thus it follows from (2.60) and (2.61) that

$$u(t,x) \leq w(t) \leq w(T_\varepsilon)e^{-(k^*-\varepsilon)(t-T_\varepsilon)} =: C_\varepsilon e^{-(k^*-\varepsilon)(t-T_\varepsilon)}$$

for $t \geq T_\varepsilon$ and $0 \leq x \leq h(t)$.

Finally we remark that, when vanishing occurs, there exist some $T > 0$ such that the solution satisfies $\|u(T,\cdot)\|_{C(\Omega(T))} < c^*$. Then we can use the same argument to the case of $\|u_0\|_{C(\Omega_0)} < c^*$, which enables us to obtain the same decay estimate and the proof is complete. □

Proof of Theorem 2.12. We consider the solution of (FBP) with initial data $(\sigma\phi, h_0)$ for a parameter $\sigma > 0$. Define $\sigma^* := \inf\{\ \rho > 0;\ \text{spreading occurs for any } \sigma > \rho\}$. Using Theorem 2.11, we can show $\sigma^* \in (0,\infty]$ and prove that spreading occurs if $u_0 > \sigma^*\phi$ and vanishing occurs if $u_0 < \sigma^*\phi$ as in the proof of Theorem 2.9. Moreover, in the spreading case, we also define $\sigma^{**} := \inf\{\ \rho > 0;\ \liminf_{t\to\infty} u(t,x) \geq v^*(x) \text{ in } [0,\infty)\}$. If u_0 is so large that there exists a number $T \in (0,\infty]$ with $h(T) \geq l$ and $u(T,x) \geq q(x)$

where q is a positive solution of (2.3) for large $l > 0$, then part (i) of Theorem 2.11 implies $\liminf_{t\to\infty} u(t,x) \geq v^*(x)$ in $[0,\infty)$. This result, together with the comparison principle, shows $\sigma^{**} \in [\sigma^*, \infty]$. In particular, if $h_0 \geq l$, then we can choose $\sigma^{**} < \infty$ such that the solution of (FBP) with initial function $u_0 \geq q$ in $(0,l)$ satisfies $\liminf_{t\to\infty} u(t,x) \geq v^*(x)$ in $[0,\infty)$.

We remark that spreading occurs for $u_0 = \sigma^*\phi$. Otherwise, vanishing occurs and we can choose some $T > 0$ such that $u_{\sigma^*}(T,x) < c^*/2$ for the solution of (FBP) with initial data $(\sigma^*\phi, h_0)$. We continue to use such a notation. Then there exists small number $\delta > 0$ such that

$$u_\sigma(T,x) \leq \frac{c^*}{2} \quad \text{for any } \sigma \in [\sigma^* - \delta, \sigma^* + \delta]$$

by the continuous dependence of solutions on initial data. Hence Theorem 2.11 implies the solution $u_{\sigma^*-\delta}(t,x)$ must vanish as $t \to \infty$. This is a contradiction to the definition of σ^*. Hence we get the conclusion by the dichotomy result (Theorem 2.6). □

Chapter 3

A free boundary problem in multi-dimensions

3.1 Problem

In this chapter we study problem (1.1) in multi-dimensions with radially symmetric settings. Through sections 3.1–3.8, we discuss the case where $\Omega(t)$ has a fixed boundary. The problem is given in the following:

$$(\mathrm{P}) \qquad \begin{cases} u_t - d\Delta u = f(u), & t > 0,\ R < r < h(t), \\ Bu = 0, & t > 0,\ r = R,\ h(t), \\ h'(t) = -\mu u_r(t, h(t)), & t > 0, \\ h(0) = h_0,\ u(0, r) = u_0(r), & R \le r \le h(t), \end{cases}$$

where d, μ and h_0 are positive constants, R is a non-negative constant, $r = |x|$ ($x \in \mathbb{R}^N, N \ge 2$) and $\Delta u = u_{rr} + ((N-1)/r)u_r$. Moreover, concerning the boundary condition, $Bu = 0$ for $t > 0$, $r = R$, $h(t)$ means

$$u(t, R) = 0, \quad u(t, h(t)) = 0, \quad t > 0; \tag{3.1}$$

$$\text{or} \quad u_r(t, R) = 0, \quad u(t, h(t)) = 0, \quad t > 0 \tag{3.2}$$

if $R > 0$ in (P), and

$$u_r(t, 0) = 0, \quad u(t, h(t)) = 0, \quad t > 0 \tag{3.3}$$

if $R = 0$ in (P). We basically assume that the nonlinear function f is in

$$S_f := \{f : [0, \infty) \to \mathbb{R} |\ f \text{ is locally Lipschitz continuous},\ f(0) = 0,\ f(u) < 0 \text{ for } u > 1\},$$

Moreover initial function u_0 is assumed to satisfy

$$\begin{aligned} &u_0 \in C^2[R, h_0], \quad u_0 > 0 \ \text{ in } \ (R, h_0),\ u_0(h_0) = 0, \\ &\text{and the same boundary condition as that of } u \text{ for } t > 0 \text{ at } r = R. \end{aligned} \tag{3.4}$$

From the view of mathematical ecology, as stated in Introduction, $u = u(t, r)$ is a population density of non-native species whose habitat is

$$\Omega(t) = \{x \in \mathbb{R}^N |\ R < |x| < h(t)\},$$

where we assume, for simplicity, the distribution of the species and their habitat are radially symmetric. The habitat is a multi-dimensional annulus and the inner boundary $r = R$ is fixed, while the outer boundary $r = h(t)$ is moving depending on time, which implies that the outer free boundary is a propagation front of the species. We put three different types of boundary conditions (3.1) – (3.3) in (P); (3.1) implies that an inner ball $B_R := \{x \in \mathbb{R}^N |\ |x| \le R\}$ is a hostile environment for the species and they cannot survive in this region; (3.2) means B_R is a barrier and they cannot enter the region; (3.3) does no hostile environment and no barrier inside the spreading front.

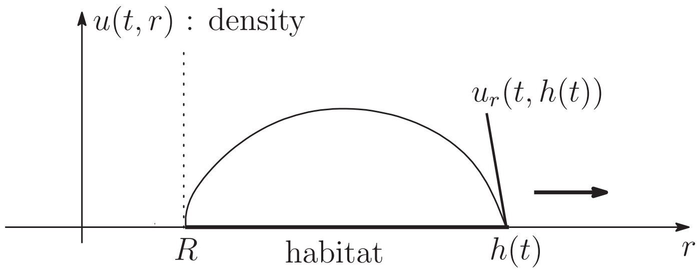

Figure 6. The solution (u, h) for problem (P) with (3.1)

The main purpose in this chapter is to study

- the existence and uniqueness of solutions for (P) (well-posedness for the model)
- spreading and vanishing in multi-dimensions as $t \to \infty$.

It is possible to extend the results in one dimension to this multi-dimensional case. In other words we get a general dichotomy theorem for spreading and vanishing, and show a close relation to an elliptic problem in annulus

$$\begin{cases} d\Delta q + f(q) = 0, \quad R < r < l, \\ q > 0, \quad R < r < l, \\ q(R) = q(l) = 0 \ \ (\text{resp. } q_r(R) = q(l) = 0) \end{cases} \tag{3.5}$$

for some $l > 0$ and an elliptic problem in an exterior domain in $\mathbb{R}^N$

$$\begin{cases} d\Delta v + f(v) = 0, \quad R < r < \infty, \\ v > 0, \quad R < r < \infty, \\ v(R) = 0 \ \ (\text{resp. } v_r(R) = 0). \end{cases} \tag{3.6}$$

Moreover we will present criteria for spreading and vanishing in (P), where the nonlinear function satisfies (1.7) with $f'(0) > 0$ or $f'(0) < 0$. Hence the results which will be proved in this chapter are more general than those in the one-dimensional case.

For the proof, we need lots of techniques to handle the multi-dimensional problem. For example, we have to construct upper and lower solutions more precisely, and also the existence and uniqueness of solutions for (3.5) and (3.6) is not so trivial. For this reason we will show some results and remarks, which partly supports our main results.

We denote, by (P1), (P2) or (P3), problem (P) with conditions (3.1), (3.2), or (3.3) respectively, and we also represent

$$D(t) = \bigcup_{0<s\le t} (\{s\} \times (R, h(s))), \quad D = \bigcup_{t>0}(\{t\} \times (R, h(t))).$$

We remark that the main results and their proofs in this chapter are based on the author's work [36].

3.2 Existence and uniqueness of solutions

In this section we will show the global existence and uniqueness of solutions and continuous dependence of the solutions on initial data, coefficients and the nonlinearity in (P).

The following theorem means a local existence and uniqueness of classical solutions.

Theorem 3.1. *Suppose that initial data satisfies* (3.4) *and* $f \in S_f$. *For any given constant* $\alpha \in (0, 1)$, *there exists a positive number* T *such that* (P) *has a unique solution*

$$(u, h) \in \{C^{\frac{(1+\alpha)}{2},1+\alpha}(\overline{D(T)}) \cap C^{1+\frac{\alpha}{2},2+\alpha}(D(T))\} \times C^{1+\frac{\alpha}{2}}[0, T],$$

where T *is depending on* R, h_0, α *and* $\|u_0\|_{C^2[R,h_0]}$.

We show an a priori estimate of solutions. This estimate also helps us to find that the local solutions are extended uniquely to all $t > 0$.

Theorem 3.2. *Problem* (P) *has a unique classical solution* (u, h) *such that*

$$0 < u(t, r) \le C_1 \ \ \textit{for} \ \ (t, r) \in D, \quad 0 < h'(t) \le \mu C_2 \ \ \textit{for} \ \ t > 0,$$

where positive constants C_1 *and* C_2 *depend on* $\|u_0\|_{C(R,h_0)}$ *and* $\|u_0\|_{C^1(R,h_0)}$, *respectively.*

The following theorem is a property of the continuous dependence of parameters.

Theorem 3.3. *The solution of* (P) *depends continuously on initial data* (u_0, h_0), *coefficients* d, μ *and nonlinearity* f *in* (P).

We show the proofs of these theorems in the following.

Proof of Theorem 3.1. We can prove this theorem in an almost same way as Theorem 2.1 in one dimension or by some modifications of the proof in Du-Guo [14, Theorem 4.1]. □

Proof of Theorem 3.2. We first prove the a priori estimate in $[0,T]$ for some constant $T>0$. Using the strong maximum principle (cf. Protter-Weinberger [56] or Cantrell-Cosner [10]), we can see

$$u(t,r)>0, \quad u_r(t,h(t))<0 \quad \text{for } \ 0\le t\le T,\ R<r<h(t). \tag{3.7}$$

Define $C_1 := \max\{\|u_0\|_{C[R,h_0]},1\}$ and let $\overline{u}=\overline{u}(t)$ be the solution of

$$\begin{cases} \dfrac{d\overline{u}}{dt}=f(\overline{u}), \quad t>0, \\ \overline{u}(0)=C_1. \end{cases}$$

Then, since $u_0(x)\le \overline{u}(0)=C_1$ in $[R,h_0]$, the comparison principle (cf. [56] or Smoller [61]) shows

$$u(t,r)\le \overline{u}(t) \quad \text{for } \ 0\le t\le T,\ R\le r\le h(t).$$

By the condition of $f\in S_f$, $\overline{u}(t)$ is decreasing with respect to t as long as $\overline{u}(t)>1$, and $\overline{u}$ satisfies $\overline{u}(t)\le C_1$ for all $t\ge 0$. Hence we have

$$u(t,r)\le \sup_{0\le t\le T}\overline{u}(t)\le C_1.$$

Combining this result with (3.7), we get the a priori estimate for $u(t,r)$.

We will next show $0<h'(t)\le \mu C_2$ for $0<t\le T$ with some $C_2>0$. By (3.7), one can easily get $h'(t)=-\mu u_x(t,h(t))>0$ for $0\le t\le T$. Set

$$\begin{aligned} &w(t,r)=-C_1M^2(r-h(t))(r-h(t)+2/M), \\ &D_M=\{(t,r)\in\mathbb{R}^2 \mid 0\le t\le T,\ h(t)-1/M<r<h(t)\}, \end{aligned}$$

where

$$M=\max\{1/(h_0-R),\ \|u_0'\|_{C[R,h_0]}/C_1,\ \sqrt{L/(2dC_1)}\}, \quad L=\max_{0\le u\le C_1} f(u).$$

It should be noted that $R\le h_0-1/M\le h(t)-1/M$ for $t\ge 0$ because of $M\ge (h_0-R)^{-1}$. It follows from direct calculations that

$$\begin{aligned} w_t &= 2C_1Mh'(t)\{1-M(h(t)-r)\}\ge 0 \ \text{ in } \ D_M, \\ w_r &= -2C_1M\{1-M(h(t)-r)\}\le 0 \ \text{ in } \ D_M, \\ w_{rr} &= -2C_1M^2. \end{aligned}$$

Hence

$$w_t-d\Delta w-f(w)\ge 2dC_1M^2-L\ge 0 \ \text{ in } \ D_M.$$

We observe

$$w(t,h(t))=0, \quad w(t,h(t)-1/M)=C_1\ge u(t,h(t)-1/M)$$

for $0 \le t \le T$. Noting from the definition of M that

$$w(0,r) = C_1M^2(h_0 - r)(r - h_0 + 2/M) \ge C_1M(h_0 - r) \ge (h_0 - r)\|u_0'\|_{C[R,h_0]}$$
$$u_0(r) = \int_{h_0}^{r} u_0'(y)dy \le (h_0 - r)\|u_0'\|_{C[R,h_0]}$$

for $h_0 - 1/M \le r \le h_0$, we get $w(0,r) \le u_0(r)$ in D_M. Hence the comparison principle presents

$$w(t,r) \ge u(t,r) \quad \text{in} \quad D_M.$$

This inequality together with $w(t,h(t)) = u(t,h(t)) = 0$ enables us to get

$$u_r(t,h(t)) \ge w_r(t,h(t)) = -2C_1M \quad \text{for} \quad 0 \le t \le T,$$

and thus

$$h'(t) = -\mu u_r(t,h(t)) \le \mu(2C_1M) =: \mu C_2 \quad \text{for} \quad 0 \le t \le T.$$

Since we have the local existence and the a priori estimate, we can prove the global existence of solutions for (P) in the standard manner (cf. the proof of Theorem 2.2 or [14]). Let $[0,T_{max})$ be the maximal existence time in which the unique solution exists. We assume $T_{max} < \infty$ to get a contradiction. For any $\delta_0 \in (0,T_{max})$ and any $M > T_{max}$, using the a priori estimate, the parabolic estimates and Sobolev's embedding theorem, we have

$$\|u(t,\cdot)\|_{C^2[R,h(t)]} \le C_3 \quad \text{for all} \quad t \in [\delta_0, T_{max}),$$

where the constant C_3 only depends on δ_0, M, C_1 and C_2 (C_1 and C_2 are also independent of T_{max}). Hence we can get a time interval $\tau > 0$ which is independent of $t \in [\delta_0, T_{max})$. Applying the local existence result of Theorem 3.1, we can extend the solution with initial data at $t = T_{max} - \tau/2$ uniquely to $t = T_{max} - \tau/2 + \tau = T_{max} + \tau/2 > T_{max}$. However this result contradicts the definition of T_{max}, and thus we obtain $T_{max} = \infty$. We complete the proof. □

Proof of Theorem 3.3. Consider the solution (u_ϵ, h_ϵ) of

$$\begin{cases} (u_\epsilon)_t - d_\epsilon \Delta u_\epsilon = f_\epsilon(u_\epsilon), & t > 0,\ R_\epsilon < r < h_\epsilon(t), \\ Bu_\epsilon = 0, & t > 0,\ r = R_\epsilon,\ h_\epsilon(t), \\ h_\epsilon'(t) = -\mu_\epsilon (u_\epsilon)_r(t,h(t)), & t > 0, \\ h_\epsilon(0) = (h_0)_\epsilon,\ u_\epsilon(0,r) = (u_0)_\epsilon(r), & R_\epsilon \le r \le (h_0)_\epsilon, \end{cases}$$

where d_ϵ, R_ϵ, μ_ϵ, $(h_0)_\epsilon$ are positive constants, $f_\epsilon \in S_f$ and $(u_0)_\epsilon$ satisfies (3.4) with h_0 replaced by $(h_0)_\epsilon$. Moreover at least one of them is different from d, R, μ, h_0, f and u_0 in (P), respectively. Assume that as $\epsilon \to 0$

$$d_\epsilon \to d, \quad R_\epsilon \to R, \quad \mu_\epsilon \to \mu, \quad (h_0)_\epsilon \to h_0, \quad f_\epsilon(u) \to f(u) \quad \text{for all} \quad u \ge 0,$$
$$(u_0)_\epsilon\Big(\frac{r-R}{(h_0)_\epsilon - R}(h_0)_\epsilon + \frac{(h_0)_\epsilon - r}{(h_0)_\epsilon - R}R_\epsilon\Big) \to u_0(r) \quad \text{in} \quad C^2(R,h_0).$$

Then, as in the proof of Theorem 2.3, the compactness argument implies

$$\lim_{\epsilon\to 0}\|u_\epsilon - u\|_{C^{1,2}(D_\epsilon)} = 0, \quad \lim_{\epsilon\to 0}\|h_\epsilon - h\|_{C^1(0,\infty)} = 0,$$

where $D_\epsilon = \cup_{0<s<\infty}\{s\} \times (R_\epsilon, h_\epsilon(s))$. Hence we complete the proof. □

3.3 Energy identity and comparison principle

We will prepare an energy identity and comparison results which help us to study the asymptotic behaviors of solutions as $t \to \infty$.

Proposition 3.1. *The following identity holds true for any solution* (u, h) *of* (P):

$$\begin{aligned}\frac{d}{2}\int_R^{h(t)} r^{N-1}u_r(t,r)^2\,dr &+ \int_0^t \Big\{\int_R^{h(s)} r^{N-1}u_t(s,r)^2\,dr\Big\}ds \\ &+\frac{d}{2\mu^2}\int_0^t h(s)^{N-1}h'(s)^3\,ds = \frac{d}{2}\int_R^{h_0} r^{N-1}u_0'(r)^2\,dr \\ &\qquad + \int_R^{h(t)} r^{N-1}F(u(t,r))\,dr - \int_R^{h_0} r^{N-1}F(u_0(r))\,dr,\end{aligned}$$

where $F(u) = \displaystyle\int_0^u f(s)\,ds$.

We define $(\overline{u}, \overline{h})$ in the following comparison principle as **an upper (super) solution of (P)** for $0 < t \le T$.

Lemma 3.1. *For any given* $T > 0$*, let* $\overline{h} \in C^1[0,T]$ *and* $\overline{u} \in C(\overline{D_1(T)}) \cap C^{1,2}(D_1(T))$ *satisfy*

$$\begin{cases} \overline{u}_t - d\Delta\overline{u} \ge f(\overline{u}), & (t,r) \in D_1(T), \\ \overline{u}(t,R) \ge 0 \ (resp.\ \overline{u}_r(t,R) \le 0), & t \in (0,T], \\ \overline{u}(t,\overline{h}(t)) = 0, & t \in (0,T], \\ \overline{h}'(t) \ge -\mu\overline{u}_r(t,\overline{h}(t)), & t \in (0,T], \end{cases}$$

where d*,* μ *and* R *are positive constants (resp.* R *is a non-negative constant) and* $D_1(T) = \bigcup_{0\le s\le T}(\{s\} \times (R, \overline{h}(s)))$*. Moreover let* (u, h) *be the solution of* (P) *with initial data* $(u_0(r), h_0)$*. If* $h_0 \le \overline{h}(0)$ *and* $u_0(r) \le \overline{u}(0,r)$ *in* $[R, h_0]$*, then it holds that*

$$h(t) \le \overline{h}(t) \quad in \quad [0,T] \quad and \quad u(t,r) \le \overline{u}(t,r) \quad in \quad \textstyle\bigcup_{0\le s\le T}(\{s\} \times (R, h(s))).$$

The function $(\underline{u}, \underline{h})$ in the following comparison principle is called **an lower (sub) solution of (P)** for $0 < t \le T$.

Lemma 3.2. *For any given* $T > 0$*, let* $\underline{h} \in C^1[0,T]$ *and* $\underline{u} \in C(\overline{D_2(T)}) \cap C^{1,2}(D_2(T))$ *satisfy*

$$\begin{cases} \underline{u}_t - d\Delta\underline{u} \le f(\underline{u}), & (t,r) \in D_2(T), \\ \underline{u}(t,R) \le 0 \ (resp.\ \underline{u}_r(t,R) \ge 0), & t \in (0,T], \\ \underline{u}(t,\underline{h}(t)) = 0, & t \in (0,T], \\ \underline{h}'(t) \le -\mu\underline{u}_r(t,\underline{h}(t)), & t \in (0,T], \end{cases}$$

where d*,* μ *and* R *are positive constants (resp.* R *is a non-negative constant) and* $D_2(T) = \bigcup_{0\le s\le T}(\{s\} \times (R, \underline{h}(s)))$*. Moreover let* (u, h) *be the solution of* (P) *with initial data* $(u_0(r), h_0)$*. If* $(R \le)\ \underline{h}(0) \le h_0$ *and* $\underline{u}(0,r) \le u_0(r)$ *in* $[R, \underline{h}(0)]$*, then it holds that*

$$\underline{h}(t) \le h(t) \quad in \quad [0,T] \quad and \quad \underline{u}(t,r) \le u(t,r) \quad in \quad \textstyle\bigcup_{0\le s\le T}(\{s\}\times(R,\underline{h}(s))).$$

We will also prepare a variant of the above comparison principles.

Lemma 3.3. *For any given $T>0$, let (u_{μ_i}, h_{μ_i}) $(i=1,2)$ satisfy*

$$\begin{cases} (u_{\mu_i})_t - d\Delta u_{\mu_i} = f(u_{\mu_i}), & 0<t\le T,\ R<r<h_{\mu_i}(t),\\ u_{\mu_i}(t,R)=0 \ (resp.\ (u_{\mu_i})_r(t,R)=0), & 0<t\le T,\\ u_{\mu_i}(t,h_{\mu_i}(t))=0, & 0<t\le T,\\ h'_{\mu_i}(t) = -\mu_i (u_{\mu_i})_r(t,h_{\mu_i}(t)), & 0<t\le T,\\ h_{\mu_i}(0)=h_0,\ u_{\mu_i}(0,r)=u_0(r), & R\le r\le h_0, \end{cases}$$

where d, h_0 and R are positive constants (resp. R is a non-negative constant) and u_0 satisfies (3.4). *If $\mu_1 \le \mu_2$, then*

$$h_{\mu_1}(t) \le h_{\mu_2}(t) \quad in \quad [0,T] \quad and \quad u_{\mu_1}(t,r) \le u_{\mu_2}(t,r) \quad in \quad \textstyle\bigcup_{0\le s\le T}(\{s\}\times(R,h_{\mu_1}(s))).$$

In the following, we will prove Proposition 3.1 and Lemmas 3.1–3.3.

Proof of Proposition 3.1. We start with the following calculation.

$$\begin{aligned} \frac{d}{dt}\Big\{\frac{d}{2}\int_R^{h(t)} r^{N-1}u_r(t,r)^2\,dr\Big\} &= \frac{d}{2}h(t)^{N-1}u_r(t,h(t))^2h'(t)\\ &\quad + d\int_R^{h(t)} r^{N-1}u_r(t,r)u_{rt}(t,r)\,dr. \end{aligned} \tag{3.8}$$

We calculate the right-hand side of (3.8). By the Stefan condition $h'(t) = -\mu u_r(t,h(t))$, it holds for the first term that

$$\frac{d}{2}h(t)^{N-1}u_r(t,h(t))^2h'(t) = \frac{d}{2\mu^2}h(t)^{N-1}h'(t)^3.$$

We next integrate the second term by parts to get

$$\begin{aligned} &d\int_R^{h(t)} r^{N-1}u_r(t,r)u_{rt}(t,r)\,dr\\ &= d\Big[r^{N-1}u_r(t,r)u_t(t,r)\Big]_{r=R}^{r=h(t)} - d\int_R^{h(t)}(r^{N-1}u_r(t,r))_r u_t(t,r)\,dr\\ &= dh(t)^{N-1}u_r(t,h(t))u_t(t,h(t)) - d\int_R^{h(t)}(r^{N-1}u_r(t,r))_r u_t(t,r)\,dr, \end{aligned}$$

where we have used $u_t(t,R)=0$ in (P1) or $u_r(t,R)=0$ in (P2) and (P3). Observe that

$$u_r(t,h(t))h'(t) + u_t(t,h(t)) = 0$$

by differentiation of $u(t,h(t)) = 0$ with respect to t. We also represent the diffusion equation as

$$r^{N-1}u_t(t,r) - d(r^{N-1}u_r(t,r))_r = r^{N-1}f(u).$$

Hence it follows from the above relations that

$$\begin{aligned} &d\int_R^{h(t)} r^{N-1}u_r(t,r)u_{rt}(t,r)\ dr \\ = &-\frac{d}{\mu^2}h(t)^{N-1}h'(t)^3 - \int_R^{h(t)} r^{N-1}u_t(t,r)^2\ dr + \int_R^{h(t)} r^{N-1}u_t(t,r)f(u(t,r))\ dr. \end{aligned}$$

Substituting these to the right-hand side of (3.8), we have

$$\begin{aligned} \frac{d}{dt}\Big\{\frac{d}{2}\int_R^{h(t)} r^{N-1}u_r(t,r)^2\ dr\Big\} = &-\frac{d}{2\mu^2}h(t)^{N-1}h'(t)^3 \\ &- \int_R^{h(t)} r^{N-1}u_t(t,r)^2\ dr + \int_R^{h(t)} r^{N-1}\frac{\partial}{\partial t}F(u(t,r))\ dr, \end{aligned} \tag{3.9}$$

where $F(u) = \int_0^u f(s)\ ds$. Noting from $F(u(t,h(t))) = F(0) = 0$ that

$$\begin{aligned} \int_R^{h(t)} r^{N-1}\frac{\partial}{\partial t}F(u(t,r))\ dr &= \frac{d}{dt}\int_R^{h(t)} r^{N-1}F(u(t,r))\ dr - h(t)^{N-1}F(u(t,h(t)))h'(t) \\ &= \frac{d}{dt}\int_R^{h(t)} r^{N-1}F(u(t,r))\ dr, \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{d}{2}\int_R^{h(t)} r^{N-1}u_r(t,r)^2\ dr - \int_R^{h(t)} r^{N-1}F(u(t,r))\ dr\Big\} \\ &\qquad = -\frac{d}{2\mu^2}h(t)^{N-1}h'(t)^3 - \int_R^{h(t)} r^{N-1}u_t(t,r)^2\ dr. \end{aligned}$$

We finally integrate this identity over $[0,t]$ to get the conclusion. $\square$

Proof of Lemma 3.1. This theorem is proved with a slight modification of the proof of Lemma 2.2 (see also [14, Lemma 3.2]). The argument basically follows [18, Lemma 3.5]. Consider the following problem

$$\begin{cases} (u_\varepsilon)_t - d\Delta u_\varepsilon = f(u_\varepsilon), & 0 < t \le T,\ R < r < h_\varepsilon(t), \\ Bu_\varepsilon = 0, & 0 < t \le T,\ r = R,\ h_\varepsilon(t), \\ h_\varepsilon'(t) = -\mu(1-\varepsilon)(u_\varepsilon)_r(t,h_\varepsilon(t)), & 0 < t \le T, \\ h_\varepsilon(0) = (1-\varepsilon)h_0,\ u_\varepsilon(0,r) = \tilde{u}_0(r), & R \le r \le h_\varepsilon(0), \end{cases}$$

where

$$\tilde{u}_0(r) := u_0\Big(\frac{r-R}{h_\varepsilon(0)-R}h_0 + \frac{h_\varepsilon(0)-r}{h_\varepsilon(0)-R}R\Big),$$

and ε is a sufficiently small positive constant such that $\tilde{u}_0(r) \le \overline{u}(0,r)$ for $R \le r \le h_\varepsilon(0)$. Then we find that the above problem has a unique global solution $(u_\varepsilon, h_\varepsilon)$ by Theorem 3.2. Using the strong maximum principle, we have

$$u_\varepsilon(t,r) > 0 \ \text{ in } \ D_\varepsilon(T), \quad (u_\varepsilon)_r(t,h_\varepsilon(t)) < 0 \ \text{ for } \ 0 \le t \le T, \tag{3.10}$$

where $D_\varepsilon(T) := \bigcup_{0\le s\le T}(\{s\} \times (R, h_\varepsilon(s)))$. Since $h_\varepsilon(0) < \overline{h}(0)$, we have $h_\varepsilon(t) < \overline{h}(t)$ for small $t > 0$, and we may assume that for some $t^* \in (0,T)$

$$h_\varepsilon(t) < \overline{h}(t) \ \text{ in } \ [0,t^*), \quad h_\varepsilon(t^*) = \overline{h}(t^*) \ \text{ and } \ h'_\varepsilon(t^*) \ge \overline{h}'(t^*). \tag{3.11}$$

Then we can show by the maximum principle that $\overline{u}(t,r) \ge u_\varepsilon(t,r)$ in $D_\varepsilon(T)$ (see e.g. the proof of Lemma 2.2). Since $h_\varepsilon(t^*) = \overline{h}(t^*)$ and $\overline{u}(t^*, \overline{h}(t^*)) = u_\varepsilon(t^*, h_\varepsilon(t^*)) = 0$, we deduce $\overline{u}_r(t^*, \overline{h}(t^*)) - (u_\varepsilon)_r(t^*, h_\varepsilon(t^*)) \le 0$. Hence it follows from (3.10) that

$$\begin{aligned}\overline{h}'(t^*) - h'_\varepsilon(t^*) &\ge -\mu\overline{u}_r(t^*, h_\varepsilon(t^*)) + \mu(1-\varepsilon)(u_\varepsilon)_r(t^*, h_\varepsilon(t^*)) \\ &= -\mu\{\overline{u}_r(t^*, h_\varepsilon(t^*)) - (u_\varepsilon)_r(t^*, h_\varepsilon(t^*)\} - \varepsilon\mu(u_\varepsilon)_r(t^*, h_\varepsilon(t^*)) \\ &> 0.\end{aligned}$$

This contradicts the assumption of (3.11), and thus we obtain $h_\varepsilon(t) \le \overline{h}(t)$ in $[0,T]$. Moreover we use the maximum principle to get $u_\varepsilon(t,r) \le \overline{u}(t,r)$ in $D_\varepsilon(T)$. Noting from Theorem 3.3 that

$$\lim_{\varepsilon\to 0} h_\varepsilon(t) = h(t) \ \text{ in } \ C^1[0,T], \quad \lim_{\varepsilon\to 0} \|u_\varepsilon(t,\cdot) - u(t,\cdot)\|_{C(\overline{D_\varepsilon(T)})} = 0,$$

we are able to prove, by taking $\varepsilon \to 0$ in the above inequality,

$$h(t) \le \overline{h}(t) \ \text{ in } \ [0,T] \ \text{ and } \ u(t,r) \le \overline{u}(t,r) \ \text{ in } \ D(T).$$

The proof is complete. □

We can prove Lemma 3.2 in the same way as Lemma 3.1. Hence we omit the proof of Lemma 3.2.

Proof of Lemma 3.3. Since $h'_{\mu_2}(t) = -\mu_2(u_{\mu_2})_r(t, h_{\mu_2}(t)) \ge -\mu_1(u_{\mu_2})_r(t, h_{\mu_2}(t))$ in $[0,T]$ because of $\mu_1 \le \mu_2$, we can regard $(u_{\mu_2}(t,r), h_{\mu_2}(t))$ as an upper solution of (P) with $\mu = \mu_1$. Hence we apply Lemma 3.1 to get the conclusion. □

3.4 Properties of spreading and vanishing

In this section we will discuss the asymptotic behaviors of solutions for (P) as $t \to \infty$. Since the free boundary $h(t)$ is strictly increasing for all $t \ge 0$ (Theorem 3.2), we get its limit which is allowed to be infinity. Hence it holds that

$$\lim_{t\to\infty} h(t) = \infty \ \text{ or } \ \lim_{t\to\infty} h(t) < \infty.$$

The asymptotic behavior of $u(t,r)$ is different from each of the above cases. We will characterize this phenomenon as spreading and vanishing.

3.4.1 Main theorems

We will show main theorems of this section. The following theorem means a property of vanishing.

Theorem 3.4. *If* $\lim_{t\to\infty} h(t) < \infty$*, then* $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} = 0$.

We will show a property of spreading as follows.

Theorem 3.5. *Let* (u,h) *be the solution of* (P1) (*resp.* (P2) *or* (P3)) *with initial data* $(q(r),l)$*, where the function* $q(r)$ *and the number* $l > R$ *satisfy*

$$\begin{cases} d\Delta q + f(q) \geq 0, & R < r < l, \\ q(r) > 0, & R < r < l, \\ q(R) = 0 \;\; (resp.\ q_r(R) = 0), & q(l) = 0. \end{cases} \tag{3.12}$$

Then the following properties hold true:

(i) $\lim_{t\to\infty} h(t) = \infty$*; that is* $\lim_{t\to\infty} \Omega(t) = \mathbb{R}^N \setminus B_R$*,*

(ii) $u_t(t,r)$ *is non-decreasing with respect to* $t \geq 0$ *for* $R \leq r \leq h(t)$*,*

(iii) $\lim_{t\to\infty} u(t,r) = v^*(r)$*: uniformly for* r *in any compact subset of* $[R,\infty)$*, where* $v^*(r)$ *is a minimal positive solution of*

$$\begin{cases} d\Delta v + f(v) = 0, & R < r < \infty, \\ v(R) = 0 \;\; (resp.\ v_r(R) = 0) \end{cases}$$

satisfying $v^*(r) \geq q(r)$ *in* $[R,l]$.

Remark 3.1. *In Theorem 3.5, problem* (3.12) *may be replaced by* (3.5)*. In particular, the solution of* (3.5) *always satisfies* (3.12).

The following property on spreading is an immediate consequence of Theorem 3.5.

Corollary 3.1. *Suppose that* $q(r)$*,* $v^*(r)$ *and* $l > R$ *are defined as in Theorem 3.5. If* $h_0 \geq l$ *and* $u_0(r) \geq q(r)$ *in* $[R,l]$*, then*

$$\lim_{t\to\infty} \Omega(t) = \mathbb{R}^N \setminus B_R \quad and \quad \liminf_{t\to\infty} u(t,r) \geq v^*(r) \quad for \quad R \leq r < \infty.$$

Remark 3.2. *In Corollary 3.1, assumption "*$u_0(r) \geq q(r)$ *in* $[R,l]$*" may be replaced by "* $u_0(r)$ *is an upper solution of* (3.5)*".*

3.4.2 Proofs of main theorems

We prepare two lemmas.

Lemma 3.4. *Let* (u,h) *be any solution of* (P). *Assume* $\lim_{t\to\infty} h(t) < \infty$. *If* $v(t,y)$ *is defined by* $v(t,y) = u(t,(h(t)-R)y+R)$, *then* $\{v(t,\cdot)|\ t \geq 1\}$ *is relatively compact in* $C^1[0,1]$.

Lemma 3.5. *Let* (u,h) *be any solution of* (P). *Assume* $\lim_{t\to\infty} h(t) < \infty$. *Then both* $h'(t)$ *and* $U(t) := \int_R^{h(t)} r^{N-1} u_t(t,r)^2 dr$ *are uniformly continuous with respect to* $t \in [1,\infty)$.

The proofs are basically similar to those in one-dimensional case (see Lemmas 2.5 and 2.6), and we omit details here.

We will prove the main theorems in the following.

Proof of Theorem 3.4. We may assume that (u,h) is a solution of (P1) because, in the other cases, one can also get the conclusion in the same way. Let $U(t) := \int_R^{h(t)} r^{N-1} u_t(t,r)^2 dr$. We will first prove

$$\lim_{t\to\infty} h'(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} U(t) = 0. \tag{3.13}$$

Indeed, since we see from Theorem 3.2

$$\sup_{R\leq r\leq h(t),\ t\geq 0} F(u(t,r)) \leq \max_{0\leq u\leq C_1} F(u) =: C_3 < \infty,$$

it follows from the energy identity (Proposition 3.1) and assumption $h_\infty := \lim_{t\to\infty} h(t) < \infty$ that

$$\begin{aligned} &\frac{d}{2}\int_R^{h(t)} r^{N-1} u_r(t,r)^2\ dr + \int_0^t U(s)ds + \frac{d}{2\mu^2}\int_0^t h(s)^{N-1} h'(s)^3\ ds \\ &\leq C_0 + C_3 \frac{h_\infty^N - R^N}{N} =: C_4 \end{aligned}$$

for all $t \geq 0$, where $C_0 := \frac{d}{2}\int_R^{h_0} r^{N-1} u_0'(r)^2\ dr - \int_R^{h_0} r^{N-1} F(u_0(r))\ dr$. Hence it holds that

$$\sup_{t\geq 0} \int_R^{h(t)} r^{N-1} u_r(t,r)^2\ dr \leq C_4, \quad \int_0^\infty U(s)\ ds \leq C_4$$

and

$$\frac{dh_0^{N-1}}{2\mu^2}\int_0^\infty h'(s)^3\ ds \leq \frac{d}{2\mu^2}\int_0^\infty h(s)^{N-1} h'(s)^3\ ds \leq C_4.$$

Since Lemma 3.5 means that $h'(t)$ and $U(t)$ are uniformly continuous with respect to t, it follows from the above estimates that $h'(t) \to 0$ and $U'(t) \to 0$ as $t \to \infty$. Hence we have (3.13).

We now change variable by $r = (h(t) - R)y + R$ with

$$v(t,y) := u(t,(h(t) - R)y + R).$$

One can check that v satisfies the following problem:

$$\begin{cases} v_t = a(t)v_{yy} + b(t,y)v_y + f(v), & t > 0,\ 0 < y < 1, \\ v(t,0) = 0,\ v(t,1) = 0, & t > 0, \\ v(0,y) = v_0(y) := u_0((h_0 - R)y + R), & 0 \le y \le 1, \end{cases} \tag{3.14}$$

where

$$a(t) = \frac{d}{(h(t) - R)^2}, \quad b(t,y) = \frac{h'(t)y}{h(t) - R} + \frac{(N-1)d}{(h(t) - R)^2 y + R(h(t) - R)}.$$

By Lemma 3.4, there exist a sequence $\{t_n\} \nearrow \infty$ and a non-negative function $\hat{v}(y)$ such that

$$\lim_{n\to\infty} v(t_n,y) = \hat{v}(y) \ \text{ in } \ C^1[0,1]. \tag{3.15}$$

Note that

$$\begin{aligned} u_t(t_n,r) &= v_t(t_n,y) - \frac{h'(t_n)y}{h(t_n) - R} v_y(t_n,y), \\ h'(t_n) &= -\mu u_r(t_n,h(t_n)) = -\frac{\mu}{h(t_n) - R} v_y(t_n,1), \end{aligned} \tag{3.16}$$

and we also recall from (3.13) that

$$\lim_{t\to\infty} h'(t) = 0 \ \text{ and } \ \lim_{t\to\infty} U(t) = \lim_{t\to\infty} \int_R^{h(t)} r^{N-1} u_t(t,r)^2 dr = 0.$$

Then, using (3.16), we have $v_t(t_n,\cdot) \to 0$ in $L^2(0,1)$ as $n \to \infty$. Hence it is possible from (3.13), (3.14) and (3.15) to show in the standard manner that, as $n \to \infty$, $\hat{v}$ satisfies

$$\frac{d}{(h_\infty - R)^2} \hat{v}_{yy} + \frac{(N-1)d}{(h_\infty - R)^2 y + R(h_\infty - R)} \hat{v}_y + f(\hat{v}) = 0 \tag{3.17}$$

for $0 < y < 1$ with $\hat{v}(0) = \hat{v}(1) = 0$. Moreover we also get $0 = -\mu\hat{v}_y(1)/(h_\infty - R)$, and hence $\hat{v}_y(1) = 0$. Thus $\hat{v}$ satisfies second-order differential equation (3.17) with initial condition $\hat{v}(1) = \hat{v}_y(1) = 0$. Using the uniqueness of solutions, we obtain $\hat{v} \equiv 0$. This implies that

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} = \lim_{t\to\infty} \|v(t,\cdot)\|_{C(0,1)} = 0.$$

The proof is complete. □

Proof of Theorem 3.5. The proof is similar to the one dimensional case (see the proof of Theorem 2.5). We apply the comparison principle for the free boundary problem to prove (ii). Let $\underline{u}(t,r) \equiv q(r)$ and $\underline{h}(t) \equiv l$. Then we can regard $(\underline{u}, \underline{h})$ as a lower solution of (P) because $\underline{u}$ satisfies the equation of (P) with the boundary conditions and it holds that

$$\underline{h}'(t) = 0 \leq -\mu q_r(l) = -\mu \underline{u}_r(t, \underline{h}(t)).$$

Hence it follows from Lemma 3.1 that

$$h(\tau) \geq \underline{h}(\tau) \equiv l \quad \text{and} \quad u(\tau, r) \geq \underline{u}(\tau, r) \equiv q(r) \quad \text{in} \ \ [R, l] \tag{3.18}$$

for all $\tau > 0$. In this paragraph, to clarify the dependence of solution (u, h) for (P) on initial data (q, l), we write $u(t, r; q, l)$ and $h(t; q, l)$ instead of $u(t, r)$ and $h(t)$, respectively. We will compare $(u(t, r; u(\tau, r; q, l), h(\tau; q, l)), h(t; u(\tau, r; q, l), h(\tau; q, l)))$ with $(u(t, r; q, l), h(t; q, l))$ by Lemma 3.1. It follows from (3.18) that for every $t \geq 0$

$$h(t; u(\tau; q, l), h(\tau; q, l)) \geq h(t; q, l),$$
$$u(t, r; u(\tau; q, l), h(\tau; q, l)) \geq u(t, r; q, l) \quad \text{in} \quad [R, h(t; q, l)].$$

Here, noting the uniqueness of solutions of (P), we find that

$$h(t; u(\tau; q, l), h(\tau; q, l)) = h(t + \tau; q, l),$$
$$u(t, r; u(\tau; q, l), h(\tau; q, l)) = u(t + \tau, r; q, l).$$

Hence it follows that

$$u(t + \tau, r; q, l) \geq u(t, r; q, l) \quad \text{for any } t, \tau \geq 0 \ \text{ and } \ R < r < h(t; q, l).$$

Thus $u_t(t, r; q, l) \geq 0$ for $t > 0$ and $R < r < h(t; q, l)$.

We will next prove (i). By virtue of (ii), we find

$$\liminf_{t \to \infty} u(t, r) \geq q(r) > 0 \quad \text{for} \ \ R < r < l. \tag{3.19}$$

If we assume $\lim_{t \to \infty} h(t) < \infty$, then we deduce from Theorem 3.4 that

$$\lim_{t \to \infty} \|u(t, \cdot)\|_{C(R, h(t))} = 0.$$

This is a contradiction to (3.19). Hence the free boundary must satisfy $\lim_{t \to \infty} h(t) = \infty$ and $\lim_{t \to \infty} \Omega(t) = \mathbb{R}^N \setminus B_R$.

Finally we will prove (iii). Since $u(t, r)$ is nondecreasing with respect to t for $r \geq R$ and uniformly bounded by Theorem 3.2, there exists a nonnegative function $\hat{v}(r)$ such that

$$\lim_{t \to \infty} u(t, r) = \hat{v}(r) \ \text{ for every } r \geq R \ \text{ with } \ \hat{v}(r) \geq q(r) \ \text{ in } \ [R, l]. \tag{3.20}$$

We will show that $\hat{v} \equiv v^*$ in $[R, \infty)$. Multiplying the equation of (P) by any function $\phi \in C_0^\infty(R, \infty)$ and integrating it over $(t, t + \delta) \times (R, \infty)$ for any positive number δ, we find that

$$\int_t^{t+\delta} \int_R^\infty u_t(s, r)\phi(r) \ drds = d \int_t^{t+\delta} \int_R^\infty u(s, r)\phi_{rr}(r) \ drds$$
$$-(N-1)d \int_t^{t+\delta} \int_R^\infty u(s, r)\Big(\frac{\phi(r)}{r}\Big)_r drds + \int_t^{t+\delta} \int_R^\infty f(u(s, r))\phi(r) \ drds.$$

We calculate the above identity in detail. Using Lebesgue's dominated convergence theorem and (3.20) , we see

$$\begin{aligned}
&\lim_{t\to\infty}\int_t^{t+\delta}\int_R^\infty u_t(s,r)\phi(r)\ drds \\
&=\lim_{t\to\infty}\left\{\int_R^\infty u(t+\delta,r)\phi(r)\ dr-\int_R^\infty u(t,r)\phi(r)\ dr\right\} \\
&=0.
\end{aligned}$$

Similarly it holds that

$$\begin{aligned}
&\lim_{t\to\infty}\int_t^{t+\delta}\int_R^\infty u(s,r)\phi_{rr}(r)\ drds=\delta\int_R^\infty \hat{v}(r)\phi_{rr}(r)\ dr, \\
&\lim_{t\to\infty}\int_t^{t+\delta}\int_R^\infty u(s,r)\Big(\frac{\phi(r)}{r}\Big)_r\ drds=\delta\int_R^\infty \hat{v}(r)\Big(\frac{\phi(r)}{r}\Big)_r\ dr, \\
&\lim_{t\to\infty}\int_t^{t+\delta}\int_R^\infty f(u(s,r))\phi(r)\ drds=\delta\int_R^\infty \hat{f}(\hat{v}(r))\phi(r)\ dr.
\end{aligned}$$

Hence $\hat{v}$ satisfies

$$d\Delta\hat{v}(r)+f(\hat{v}(r))=0$$

in the sense of distribution with $\hat{v}(R)=0$ for (P1) or $\hat{v}_y(R)=0$ for (P2) and (P3). By the standard manner, $\hat{v}$ satisfies (3.6) in a classical sense with $\hat{v}(r)\geq q(r)$ for $r\in[R,l]$.

We will show that $\hat{v}$ is equal to the minimal solution v^* of (3.6) satisfying $v^*(r)\geq q(r)$ in $[R,l]$. Let $v(r)$ be any positive solution of (3.6) satisfying $v(r)\geq q(r)$ in $[R,l]$. We find from the standard comparison principle that $u(t,r)\leq v(r)$ for $t>0$ and $R\leq r\leq h(t)$. Therefore

$$\hat{v}(r)=\lim_{t\to\infty}u(t,r)\leq v(r)\quad\text{for every}\quad r\geq R.$$

Since v is arbitrary solution of (3.6), this inequality implies $\hat{v}\equiv v^*$ in $[R,\infty)$. We conclude from the monotone convergence of (3.20) and Dini's theorem that

$$\lim_{t\to\infty}u(t,r)=v^*(r)\quad\text{uniformly for } r \text{ in any compact subset of } [R,\infty).$$

The proof is complete. □

Proof of Corollary 3.1. Let $(\underline{u},\underline{h})$ be a solution of (P) with initial data (q,l). By the assumption that $h_0\geq l$ and $u_0(r)\geq q(r)$ in $[R,l]$, the comparison principle (Lemma 3.1) shows

$$h(t)\geq\underline{h}(t)\quad\text{and}\quad u(t,r)\geq\underline{u}(t,r)$$

for $t>0$ and $R\leq r\leq\underline{h}(t)$. Note from Proposition 3.5 that

$$\lim_{t\to\infty}\underline{h}(t)=\infty\ \text{ and }\ \lim_{t\to\infty}\underline{u}(t,r)=v^*(r)\ \text{ for }\ R\leq r<\infty.$$

Hence, taking $t\to\infty$ in the above inequalities, we see that

$$\lim_{t\to\infty}h(t)=\infty\ \text{ and }\ \liminf_{t\to\infty}u(t,r)\geq v^*(r)\ \text{ for }\ R\leq r<\infty.$$

This completes the proof. □

3.5 General dichotomy theorem

In this section, we will prove a general dichotomy theorem and a criterion for spreading and vanishing of solutions (u, h) for (P) in multi-dimensions.

The following theorem shows a general dichotomy.

Theorem 3.6. *Suppose that f satisfies*

$$f \in C^1[0,\infty), \quad f(0) = f(1) = 0, \;\; f(u) < 0 \;\; \text{for} \;\; u > 1, \quad f'(0) \neq 0. \tag{3.21}$$

Then any solution (u, h) of (P) satisfies either (i) or (ii) as $t \to \infty$:

(i) *Spreading*: $\lim_{t\to\infty} \Omega(t) = \mathbb{R}^N \setminus B_R, \;\; \liminf_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} > 0$;

(ii) *Vanishing*: $\lim_{t\to\infty} \Omega(t)$ *is a bounded set in* $\mathbb{R}^N \setminus B_R$, $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} = 0$. *Moreover* $\|u(t,\cdot)\|_{C(R,h(t))} = O(e^{-\beta t})$ *as* $t \to \infty$ *for a positive constant* β *depending on* $f'(0)$.

We can also give a criterion for spreading and vanishing.

Theorem 3.7. *Let ϕ be a function in $C^2(R, h_0) \cap C[R, h_0]$ which satisfies the same boundary condition as u. Then there exists a number $\sigma^* > 0$ depending on ϕ, h_0 and R such that the following criterion holds true*:

- *If $u_0 > \sigma^*\phi$ in (R, h_0), then spreading occurs*;
- *If $u_0 < \sigma^*\phi$ in (R, h_0), then vanishing occurs*;
- *If $u_0 = \sigma^*\phi$ in (R, h_0), then vanishing occurs for $f'(0) > 0$, while spreading occurs for $f'(0) < 0$.*

We can prove Theorem 3.7, combining Theorems 3.10 and 3.14. Hence we omit the proof of Theorem 3.7. To prove Theorem 3.6, we need two key propositions.

Proposition 3.2. *Suppose that f satisfies (3.21) and $f'(0) > 0$. If $\lim_{t\to\infty} h(t) = \infty$, then $\liminf_{t\to\infty} u(t, r) > 0$ in (R, ∞).*

Proof. Consider the eigenvalue problem:

$$\begin{cases} d\Delta q + \lambda_1 q = 0, & R < r < l, \\ q(R) = q(l) = 0 \text{ (resp. } q_r(R) = q(l) = 0) \end{cases}$$

for $l > R$. By the Sturm-Liouville theory, there exist the first eigenvalue $\lambda_1(l)$ and the corresponding eigenfunction $q(r)$ of the problem. Since $\lambda_1(l)$ is continuous and monotone decreasing with respect to l, $\lambda_1(l) > 0$ for all $l > 0$ and $\lim_{l\to+\infty} \lambda_1(l) = 0$, we can take $l > h_0$ large enough to satisfy $f'(0) > \lambda_1(l)$ for given $f'(0) > 0$. Choosing ε sufficiently small, we can show that $\phi := \varepsilon q$ satisfies

$$\begin{cases} d\Delta\phi + f(\phi) \geq 0, \quad R < r < l, \\ \phi(R) = \phi(l) = 0 \text{ (resp. } \phi_r(R) = \phi(l) = 0). \end{cases}$$

Let $(w(t,r), s(t))$ be the solution of (P) with initial data (ϕ, l). By $\lim_{t\to\infty} h(t) = \infty$, there exists some $T > 0$ such that $h(T) = l$, and also, if necessary, we can choose small $\varepsilon > 0$ so that $\phi(r) = \varepsilon q(r) \leq u(T, r)$ in (R, l). Hence we apply Lemma 3.2 to get

$$s(t) \leq h(t+T) \text{ for } t > 0 \text{ and } w(t,r) \leq u(t+T, r) \text{ for } t > 0,\ R < r < s(t).$$

Noting from Theorem 3.5 that $\lim_{t\to\infty} s(t) = \infty$ and $\lim_{t\to\infty} w(t,r) = v^*(r)$ uniformly in any compact set of $[R, \infty)$ where v^* is a minimal solution of (3.6), we deduce

$$\liminf_{t\to\infty} u(t,r) \geq \lim_{t\to\infty} w(t,r) = v^*(r) > 0 \text{ in } (R, \infty). \tag{3.22}$$

The proof is complete. □

Proposition 3.3. *Suppose f satisfies* (3.21) *and* $f'(0) < 0$. *If* $\liminf_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} = 0$, *then* $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} = 0$ *and* $\lim_{t\to\infty} h(t) < \infty$.

Proof. Let $f(v) = vg(v)$ and define $c^* = \inf\{c > 0;\ f(c) = 0\}$. Since $f'(0) < 0$, we see $g(v) < 0$ for $0 < v < c^*$. Take $0 < c_1 < c^*$. Then, by the assumption, there exists a positive number T^* such that

$$u(T^*, r) \leq \frac{\sqrt{3}c_1}{2} \quad \text{for} \quad R \leq r \leq h(T^*). \tag{3.23}$$

We fix such $T^* > 0$ and set

$$s(t) = s_0(1 + \delta(1 - e^{-\alpha t})), \quad v(t,r) = c_1 e^{-\alpha t} \cos\left(\frac{\pi(r-R)}{2(s(t)-R)}\right),$$

where $\alpha,\ s_0$ and δ are chosen as follows:

$$\begin{aligned} &0 < \alpha \leq m_1, \quad m_1 := \min\{-g(v);\ 0 \leq v \leq c_1\} > 0, \\ &s_0 \geq \max\{3h(T^*) - 2R,\ h_0\} \text{ and } \delta \geq \frac{\pi\mu c_1}{2\alpha s_0(s_0 - R)}. \end{aligned} \tag{3.24}$$

We will prove that (v, s) is an upper solution of (P). Indeed direct calculations give

$$\begin{aligned} v_r &= c_1 e^{-\alpha t}\left\{-\frac{\pi}{2(s(t)-R)}\right\} \sin\left(\frac{\pi(r-R)}{2(s(t)-R)}\right), \\ v_{rr} &= c_1 e^{-\alpha t}\left\{-\frac{\pi^2}{4(s(t)-R)^2}\right\} \cos\left(\frac{\pi(r-R)}{2(s(t)-R)}\right), \\ v_t &= -\alpha c_1 e^{-\alpha t} \cos\left(\frac{\pi(r-R)}{2(s(t)-R)}\right) + c_1 e^{-\alpha t}\left\{\frac{s'(t)\pi(r-R)}{2(s(t)-R)^2}\right\} \sin\left(\frac{\pi(r-R)}{2(s(t)-R)}\right). \end{aligned}$$

Hence we get

$$v_r \le 0 \quad \text{and} \quad v_t \ge -\alpha c_1 e^{-\alpha t} \cos\left(\frac{\pi(r-R)}{2(s(t)-R)}\right)$$

for $t > 0$, $R \le r \le s(t)$. Note from $0 \le v \le c_1$ for $t > 0$, $R \le r \le s(t)$ that $-g(v) \ge m_1 > 0$. Then it follows that

$$\begin{aligned} & v_t - d\Delta v - f(v) \\ & \ge c_1 e^{-\alpha t} \cos\left(\frac{\pi(r-R)}{2(s(t)-R)}\right)\left\{-\alpha + \frac{\pi^2 d}{4(s(t)-R)^2} - g(v)\right\} \\ & \ge c_1 e^{-\alpha t} \cos\left(\frac{\pi(r-R)}{2(s(t)-R)}\right)\left\{-\alpha + \frac{\pi^2 d}{4(s_0(1+\delta)-R)^2} + m_1\right\}. \end{aligned}$$

Since $\alpha \le m_1$ by (3.24), we find

$$-\alpha + \frac{\pi^2 d}{4(s_0(1+\delta)-R)^2} + m_1 > 0.$$

Thus it holds that

$$v_t - d\Delta v - f(v) \ge 0 \quad \text{for} \quad t > 0,\ R \le r \le s(t). \tag{3.25}$$

Using (3.24), we can also get

$$\begin{aligned} s'(t) - (-\mu v_r(t, s(t))) &= \alpha s_0 \delta e^{-\alpha t} - \frac{\pi \mu c_1}{2(s(t)-R)} e^{-\alpha t} \\ &\ge \alpha s_0 \left\{\delta - \frac{\pi \mu c_1}{2\alpha s_0(s_0-R)}\right\} e^{-\alpha t} \\ &\ge 0 \end{aligned} \tag{3.26}$$

for $t > 0$. We next compare $u(t+T^*)$ with $v(t,r)$ at $t=0$. We recall (3.23) and note from (3.24) that

$$0 < \frac{\pi(h(T^*)-R)}{2(s_0-R)} \le \frac{\pi}{6}.$$

Then it follows that

$$u(T^*, r) \le \frac{\sqrt{3}c_1}{2} = c_1 \cos\frac{\pi}{6} \le c_1 \cos\left(\frac{\pi(h(T^*)-R)}{2(s_0-R)}\right)$$

and

$$c_1 \cos\left(\frac{\pi(h(T^*)-R)}{2(s_0-R)}\right) \le c_1 \cos\left(\frac{\pi(r-R)}{2(s_0-R)}\right) = v(0,r)$$

for $R \le r \le h(T^*)$. Hence we obtain

$$u(T^*, r) \le v(0,r) \quad \text{for} \quad R \le r \le h(T^*). \tag{3.27}$$

We finally check the boundary conditions in the following:

$$\begin{aligned} &u(t+T^*, R) = 0 \le c_1 e^{-\alpha t} = v(t,R) \quad \text{for (P1)},\\ &u_r(t+T^*, R) = v_r(t,R) = 0 \quad \text{for (P2) and (P3)},\\ &u(t+T^*, h(t+T^*)) = v(t, s(t)) = 0 \end{aligned} \tag{3.28}$$

for $t > 0$.

From (3.25) – (3.28), the comparison principle for upper solutions (Lemma 3.1) shows

$$h(t+T^*) \le s(t) \ \text{ for } \ t>0, \quad u(t+T^*, r) \le v(t,r) \ \text{ for } \ t>0, \ R \le r \le h(t+T^*).$$

Thus we obtain

$$\begin{aligned} &\lim_{t\to\infty} h(t) \le \lim_{t\to\infty} s(t) = s_0(1+\delta),\\ &\lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} \le \lim_{t\to\infty} \|v(t,\cdot)\|_{C(R,h(t))} = 0. \end{aligned}$$

We complete the proof. □

Proof of Theorem 3.6. Since the free boundary is strictly increasing, we find that it satisfies $\lim_{t\to\infty} h(t) < \infty$ or $\lim_{t\to\infty} h(t) = \infty$. If $\lim_{t\to\infty} h(t) < \infty$, then $\lim_{t\to\infty} \Omega(t)$ is a bounded set in $\mathbb{R}^N \setminus B_R$ and we deduce $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} = 0$ by Theorem 3.4. On the other hand, if $\lim_{t\to\infty} h(t) = \infty$, then $\lim_{t\to\infty} \Omega(t) = \mathbb{R}^N \setminus B_R$. For the behavior of u as $t \to \infty$, we need to discuss each case of $f'(0) > 0$ and $f'(0) < 0$. In the former case, Propositions 3.2 implies $\liminf_{t\to\infty} u(t,r) > 0$ in (R, ∞), which leads to $\liminf_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} > 0$. In the latter case we get the same inequality by Proposition 3.3. When vanishing occurs, we can prove decay estimate $\|u(t,\cdot)\|_{C(R,h(t))} = O(e^{-\beta t})$ for some $\beta > 0$ as $t \to \infty$. The proof of this estimate is divided into two cases: $f'(0) > 0$ and $f'(0) < 0$. For each case, we can prove the decay rate by Theorems 3.8 and 3.12, respectively. We complete the proof. □

3.6 Spreading and vanishing for Case $f'(0) > 0$

We discuss free boundary problem (P) under the assumption that

$$f \text{ satisfies (3.21) and } f'(0) > 0.$$

We will give a more precise dichotomy theorem in this case and give some sufficient conditions and criteria for spreading and vanishing. As we have seen in Section 3.1, the results in this section are applicable to general polystable (in particular, monostable/logistic) nonlinearities.

3.6.1 Main theorems

A dichotomy theorem is given in the following:

Theorem 3.8. *Let* (u, h) *be any solution of* (P) *and let* B_R *be a multi-dimensional ball with radius* R*. Suppose that* f *satisfies* (3.21) *and* $f'(0) > 0$*. Then either spreading* (i) *or vanishing* (ii) *holds true*:

(i) $\lim_{t\to\infty} \Omega(t) = \mathbb{R}^N \setminus B_R$ *and* $\liminf_{t\to\infty} u(t, r) \geq v^*(r)$ *in* $[R, \infty)$*, where* $v^*(r)$ *is a minimal positive solution of* (3.6)*. Moreover if* $f(u)/u$ *is strictly decreasing with respect to* $u \in [0, 1]$*, then* $\lim_{t\to\infty} u(t, r) = v^*(r)$ *uniformly in any compact subset of* $[R, \infty)$;

(ii) $\lim_{t\to\infty} \Omega(t) \subset B_{R_N^*} \setminus B_R$ *and* $\lim_{t\to\infty} \|u(t, \cdot)\|_{C(R,h(t))} = 0$ *with a positive number* $R_N^* = R_N^*(d, R, f'(0))$*. Moreover* $\|u(t, \cdot)\|_{C(R,h(t))} = O(e^{-\beta t})$ *for some* $\beta > 0$ *as* $t \to \infty$*.*

Remark 3.3. *The number* R_N^* *in Theorem 3.8 is determined by* $f'(0) = \lambda_1(d, R, R_N^*)$*, where, for* (P1) (*resp.* (P2) *or* (P3)), $\lambda_1(d, R, l)$ *is the least eigenvalue of*

$$\begin{cases} -d\Delta\phi = \lambda_1\phi, \quad R < r < l, \\ \phi > 0, \quad R < r < l, \\ \phi(R) = \phi(l) = 0 \quad (\textit{resp. } \phi_r(R) = \phi(l) = 0) \end{cases} \tag{3.29}$$

with $l > R$*. Indeed it is well known that* $\lambda_1(d, R, l)$ *is continuous and decreasing with respect to* l*, and satisfies* $\lim_{l\to R+} \lambda_1(d, R, l) = +\infty$ *and* $\lim_{l\to+\infty} \lambda_1(d, R, l) = 0$*. Thus, for given* d*,* R *and* f*, there exists a unique positive number* $R_N^* = R_N^*(d, R, f'(0))$ *such that*

$$f'(0) = \lambda_1(d, R, R_N^*) \quad \text{and} \quad f'(0) > \lambda_1(d, R, l) \ \text{ for } \ l > R_N^*. \tag{3.30}$$

In the case of $N = 1$*,* R^* *is given explicitly by* $\pi\sqrt{d/f'(0)}+R$ *for* (P1) *or* $(\pi/2)\sqrt{d/f'(0)}$ $+R$ *for* (P2) *and* (P3).

Remark 3.4. *In Theorem 3.8, if the nonlinear term is logistic,* $f(u) = u(a - bu)$ *for* $a, b > 0$*, then* $f(u)/u$ *is strictly decreasing for* $u > 0$ *and we get the convergence of* u *as* $t \to \infty$*. In particular,* $v^*(r) \equiv a/b$ *for* (P2) *and* (P3)*. When* $f(u)$ *is logistic with inhomogeneous coefficients, Du-Guo [14] have obtained a similar dichotomy result for* (P3)*.*

Proposition 3.4. *The function* $R_N^* = R_N^*(d, R, \alpha)$ *with* $\alpha = f'(0)$ *in Theorem 3.8 is monotone increasing with respect to* d *and* R*, and monotone decreasing with respect to* α*. Moreover, for fixed numbers* d *and* R*,* $\lim_{\alpha\to+\infty} R_N^* = R$ *and* $\lim_{\alpha\to 0} R_N^* = \infty$*.*

This proposition implies that, as the hostile environment (barrier) becomes larger, the threshold radius between spreading and vanishing gets larger.

Some sufficient conditions are given in the following theorem.

Theorem 3.9. *Let* (u, h) *be any solution of* (P) *and define* $a^* = \inf\{a > 0;\ f(a) = 0\}$.

(i) *Suppose that* $h_0 \geq R_N^*$; *then spreading occurs. If initial function* u_0 *also satisfies* $\|u_0\|_{C(R,h_0)} \leq a^*$ *and* $f(u)/u$ *is decreasing with respect to* $u \in [0, a^*]$, *then* $\lim_{t\to\infty} u(t, r) = v^*(r)$ *uniformly in any compact subset of* $[R, \infty)$, *where* v^* *is a minimal solution of* (3.6).

(ii) *Suppose that* $h_0 < R_N^*$. *If initial function* u_0 *is small enough to satisfy* $0 \leq u_0(r) \leq w(r)$ *in* $[R, h_0]$ *for a positive function* w *defined in* $[R, h_0]$, *then vanishing occurs. Moreover* $\|u(t, \cdot)\|_{C(R,h(t))} = O(e^{-\beta t})$ *for some* $\beta > 0$ *as* $t \to \infty$.

Consider the solution of (P1) (resp. (P2) or (P3)) with initial data (u_0, h_0). We can find a sharp threshold on initial data which separates spreading and vanishing.

Theorem 3.10. *Suppose that* $h_0 < R_N^*$. *Let* $\phi \in C^2(R, h_0) \cap C[R, h_0]$ *be any function which satisfies* $\phi(R) = \phi(h_0) = 0$ *(resp.* $\phi_r(R) = \phi(h_0) = 0$*). Then there exists a number* $\sigma^* = \sigma^*(\phi, h_0) \in (0, \infty]$ *such that spreading occurs if* $u_0 > \sigma^*\phi$ *in* (R, h_0) *and vanishing occurs if* $u_0 \leq \sigma^*\phi$ *in* (R, h_0).

We can also give another criterion, focusing on the speed parameter.

Theorem 3.11. *Suppose that* $h_0 < R_N^*$. *Then there exists some number* $\mu^* = \mu^*(u_0, h_0)$ $\in [0, \infty)$ *such that spreading occurs for* $\mu > \mu^*$, *while vanishing occurs for* $\mu \leq \mu^*$. *Moreover, if* $f(u) \leq f'(0)u$ *for* $u \geq 0$, *then* $\mu^* \in (0, \infty)$.

3.6.2 Proofs of main theorems

We prepare for the proof of the main theorems.

Proposition 3.5. *If* $\lim_{t\to\infty} h(t) < \infty$, *then* $\lim_{t\to\infty} h(t) \leq R_N^*$.

Proof. We assume $\lim_{t\to\infty} h(t) > R_N^*$ to get a contradiction. By the assumption, there exists a constant $T > 0$ such that $l := h(T) > R_N^*$, and we find that $f'(0) > \lambda_1(l)$, where $\lambda_1(l)$ is the least eigenvalue of (3.29). As in the proof of Proposition 3.2, we deduce

$$\liminf_{t\to\infty} u(t, r) \geq v^*(r) > 0 \quad \text{in} \quad (R, \infty),$$

where v^* is a minimal solution of (3.6). This contradicts the fact of Theorem 3.4 that $\lim_{t\to\infty} \|u(t, \cdot)\|_{C(R,h(t))} = 0$ if $\lim_{t\to\infty} h(t) < \infty$. Thus the free boundary must satisfy $\lim_{t\to\infty} h(t) \leq R_N^*$. □

We now prove the main theorems.

Proof of Theorem 3.8. Since the free boundary is strictly increasing, it must satisfy $\lim_{t\to\infty} h(t) = \infty$ or $\lim_{t\to\infty} h(t) < \infty$. In the former case, we have $\lim_{t\to\infty} \Omega(t) =$

$\mathbb{R} \setminus B_R$ and $\liminf_{t\to\infty} u(t,r) \geq v^*(r)$ for $R \leq r < \infty$ by Proposition 3.2. In the latter case, we find from Theorem 3.4 and Proposition 3.5 that $\lim_{t\to\infty} \Omega(t) \subset B_{R_N^*} \setminus B_R$ and $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} = 0$. When vanishing occurs, there exists some $T > 0$ such that $u(T,r) \leq w(r)$ in $[R, h_0]$ for a positive function w. Then we can prove $\|u(t,\cdot)\|_{C(R,h(t))} = O(e^{-\beta t})$ for some $\beta > 0$ as $t \to \infty$ (see Theorem 3.9).

In the case of spreading, if we further assume $f(u)/u$ is decreasing with respect to $u \in [0,1]$, then (3.6) has a unique solution (see Section 3.8). Then we find the unique convergence of solutions as $t \to \infty$ for any initial data. Indeed one can construct a suitable upper solution for (P1) (resp. (P2) or (P3)). Let $\overline{u}(t,r)$ be the solution of

$$\begin{cases} \overline{u}_t - d\Delta\overline{u} = f(\overline{u}), & t > 0,\ R < r < \infty, \\ \overline{u}(t,R) = 0 \text{ (resp. } \overline{u}_r(t,R) = 0), & t > 0, \\ \overline{u}(0,r) = M := \max\{1, \|u_0\|_{C(R,h_0)}\}, & R \leq r < \infty. \end{cases}$$

Noting that $v \equiv M$ is regarded as an upper solution of (3.6), we find from the monotone method and the uniqueness of solutions for (3.6) that $\overline{u}(t,\cdot)$ is decreasing and converges to $v^*(r)$ uniformly for r in any compact subset of $[R,\infty)$ as $t \to \infty$ (cf. Sattinger [58]). It also follows from the standard comparison principle that $u(t,r) \leq \overline{u}(t,r)$ for $R \leq r < \infty$. Hence letting $t \to \infty$ in this inequality shows

$$\limsup_{t\to\infty} u(t,r) \leq \lim_{t\to\infty} \overline{u}(t,r) = v^*(r) \quad \text{for} \quad R \leq r < \infty.$$

Thus, combining the above inequality and (3.22), we conclude that $\lim_{t\to\infty} u(t,r) = v^*(r)$ uniformly for r in any compact subset of $[R,\infty)$. We complete the proof. □

Proof of Proposition 3.4. Consider the least eigenvalue $\lambda_1(d,R,l)$ of (3.29). One can represent it as

$$\inf\left\{ \frac{d\int_R^l r^{N-1}\phi_r(r)^2\,dr}{\int_R^l r^{N-1}\phi(r)^2\,dr} \,\middle|\, \phi \in H_0^1(R,l) \text{ (resp. } \phi \in H^1(R,l),\ \phi_r(R) = \phi(l) = 0)\right\}.$$

By the representation, $\lambda_1(d,R,l)$ is monotone increasing with respect to d. We recall that $R_N^*(d,R,\alpha)$ is determined by (3.30). Hence we find that $R_N^*(d,R,\alpha)$ is monotone increasing with respect to d.

Next we will show that $R_N^*(d,R,\alpha)$ is monotone increasing with respect to R. We assume $R_N^*(d,R_1,\alpha) > R_N^*(d,R_2,\alpha)$ if $R_1 < R_2$ to get a contradiction. Since $\lambda_1(d,R,l)$ is monotone decreasing with respect to l, we get

$$\alpha = \lambda_1(d,R_2,R_N^*(d,R_2,\alpha)) > \lambda_1(d,R_2,R_N^*(d,R_1,\alpha)).$$

On the other hand, we have

$$\lambda_1(d,R_2,R_N^*(d,R_1,\alpha)) > \lambda_1(d,R_1,R_N^*(d,R_1,\alpha)) = \alpha$$

because $\lambda_1(d,R,l)$ is monotone increasing with respect to R. This result gives us a contradiction. Hence $R_N^*(d,R_1,\alpha) \leq R_N^*(d,R_2,\alpha)$.

Finally, note that $\lambda_1(d,R,l)$ is decreasing with respect to l, $\lim_{l\to R+}\lambda_1(d,R.l) = +\infty$, $\lim_{l\to\infty}\lambda_1(d,R.l)=0$ and $l=R_N^*$ is a point of the intersection between $\lambda_1(d,R,l)$ and α. Then we get $R_N^*(d,R,\alpha_1)\geq R_N^*(d,R,\alpha_2)$ if $\alpha_1<\alpha_2$, and moreover R_N^* satisfies

$$\lim_{\alpha\to+\infty} R_N^* = R \quad \text{and} \quad \lim_{\alpha\to 0} R_N^* = \infty.$$

We complete the proof. □

Proof of Theorem 3.9. (i) Since $h_0\geq R_N^*$ and the free boundary is strictly increasing, it must satisfy $h(t)>R_N^*$ for all $t>0$, and $\lim_{t\to\infty}h(t)>R_N^*$. Hence the dichotomy theorem (Theorem 3.8) implies $\lim_{t\to\infty}h(t)=\infty$ and spreading occurs. Next let $\overline{u}(t,r)$ be the solution of

$$\begin{cases} \overline{u}_t - d\Delta\overline{u} = f(\overline{u}), & t>0,\ R<r<\infty, \\ \overline{u}(t,R)=0 \text{ (resp. } \overline{u}_r(t,R)=0), & t>0, \\ \overline{u}(0,r)=a^*, & R\leq r<\infty \end{cases}$$

for (P1) (resp. (P2) or (P3)). If $\|u_0\|_{C(R,h_0)}\leq a^*$, then the standard comparison principle shows $u(t,r)\leq\overline{u}(t,r)$ for $t>0$, $R\leq r\leq h(t)$. Moreover the assumption of $h_0\geq R_N^*$ implies that the solution satisfies the property of spreading, and in particular $\lim_{t\to\infty}h(t)=\infty$. Note that the solution v of (3.6) satisfying $v(r)\leq a^*$ in $[R,\infty)$ is unique, that is $v=v^*$, under the assumption that $f(u)/u$ is decreasing in $u\in[0,a^*]$ (See Section 3.8). Hence we obtain

$$\limsup_{t\to\infty} u(t,r)\leq \lim_{t\to\infty}\overline{u}(t,r)=v^*(r) \ \text{ in } \ [R,\infty).$$

As in the proof of Proposition 3.2, we get

$$\liminf_{t\to\infty} u(t,r)\geq v^*(r) \ \text{ in } \ [R,\infty).$$

Thus we can prove that $u(t,r)$ converges to $v^*(r)$ uniformly for r in any compact set of $[R,\infty)$ as $t\to\infty$.

(ii) We will construct an upper solution for (P1) (resp. (P2) or (P3)). Let $\lambda_1(\gamma)$ be the least eigenvalue and let $\varphi(y;\gamma)$ be the corresponding eigenfunction for

$$\begin{cases} -d\Delta_y\varphi=\lambda_1\varphi, \quad \varphi>0, \quad \gamma<y<s_0, \\ \varphi(\gamma)=\varphi(s_0)=0 \ \text{ (resp. } \varphi_y(\gamma)=\varphi(s_0)=0), \end{cases}$$

where $\Delta_y\varphi=\varphi_{yy}+(N-1)\varphi_y/y$, $s_0\in[h_0,R_N^*)$, $\gamma\in(0,R)$ for (P1) and (P2), and $\gamma=0$ for (P3). Define

$$s(t)=s_0(1+\delta(1-e^{-\alpha t})) \ \text{ and } \ v(t,r)=\varepsilon_0 e^{-\beta t}\varphi\Big(\frac{s_0}{s(t)}r;\gamma\Big)$$

with positive constants $\alpha,\ \beta,\ \delta$ and ε_0. If $v(t,r)$ and $s(t)$ satisfy

$$\begin{cases} v_t - d\Delta v \geq f(v), & 0 < t < T,\ R < r < s(t), \quad (3.31)\\ v(t,R) \geq 0 \text{ (resp. } v_r(t,R) \leq 0), & 0 < t \leq T, \quad (3.32)\\ v(t,s(t)) = 0, & 0 < t \leq T, \quad (3.33)\\ s'(t) \geq -\mu v_r(t,s(t)), & 0 < t \leq T \quad (3.34)\end{cases}$$

and u_0 is sufficiently small, then we can regard (v,s) as an upper solution of (P). We choose $\alpha,\ \beta,\ \gamma,\ \delta$ and ε_0 in the following way. Since $s_0 < R_N^*$, we get $f'(0) < \lambda_1(R)$. Hence there exists some small $\delta > 0$ satisfying

$$\frac{\lambda_1(R)}{(1+\delta)^2} - f'(0) > 2\delta.$$

By the continuous dependence of $\lambda_1(\gamma)$ on γ, we choose γ sufficiently close to R and, if necessary, take small δ again such that

$$\frac{\lambda_1(\gamma)}{(1+\delta)^2} - f'(0) \geq 2\delta, \quad \gamma < \frac{R}{1+\delta} \tag{3.35}$$

(The second condition on γ is not necessary when we consider $\gamma = 0$ in (P3)). In what follows, we simply denote λ_1 and $\varphi((s_0/s(t))r)$ in place of $\lambda_1(\gamma)$ and $\varphi((s_0/s(t))r;\gamma)$, respectively. Taking $\beta < \delta$, we see from (3.35) that

$$m := -\beta + \frac{\lambda_1}{(1+\delta)^2} - f'(0) - \delta > 0. \tag{3.36}$$

Here we note that δ depends on $f'(0)$ by (3.35); so does β. Remark that $\varphi((s_0/s(t))r) > 0$ for $t > 0,\ R \leq r < s(t)$ because $\gamma < (s_0/s(t))r < s_0$ for $t > 0,\ R \leq r < s(t)$ by the second inequality in (3.35). By using these numbers, we take α satisfying

$$0 < \alpha \leq \min\{\alpha^*,\ \beta\}, \tag{3.37}$$

where

$$\alpha^* := \frac{mL}{\delta s_0 M},\ L = \varphi\Big(\frac{R}{1+\delta}\Big),\ M = \|\varphi_y\|_{C[\gamma,s_0]}.$$

Moreover there exists a constant $\varepsilon_0 > 0$ such that

$$0 < \varepsilon_0 \leq \frac{\alpha\delta s_0}{\mu(-\varphi_y(s_0))}, \quad f(v) \leq (f'(0)+\delta)v \text{ for } 0 < v \leq \varepsilon_0. \tag{3.38}$$

We will first show (3.31). Direct calculation gives

$$\begin{aligned} v_r &= \varepsilon_0 e^{-\beta t}\Big(\frac{s_0}{s(t)}\Big)\varphi_y\Big(\frac{s_0}{s(t)}r\Big),\\ v_{rr} &= \varepsilon_0 e^{-\beta t}\Big(\frac{s_0}{s(t)}\Big)^2\varphi_{yy}\Big(\frac{s_0}{s(t)}r\Big),\\ v_t &= -\beta\varepsilon_0 e^{-\beta t}\varphi\Big(\frac{s_0}{s(t)}r\Big) + \varepsilon_0 e^{-\beta t}\Big(\frac{-s'(t)s_0 r}{s(t)^2}\Big)\varphi_y\Big(\frac{s_0}{s(t)}r\Big).\end{aligned}$$

Then we have

$$\begin{aligned}&v_t - d\Delta v - f(v)\\&= \varepsilon_0 e^{-\beta t}\Big\{ -\beta\varphi\Big(\frac{s_0}{s(t)}r\Big) - \Big(\frac{s'(t)s_0 r}{s(t)^2}\Big)\varphi_y\Big(\frac{s_0}{s(t)}r\Big) - d\Delta\varphi\Big(\frac{s_0}{s(t)}r\Big) - \varphi\Big(\frac{s_0}{s(t)}r\Big)(f'(0)+\delta)\Big\}.\end{aligned}$$

Note that $y = (s_0/s(t))r$ and

$$\begin{aligned}-d\Delta\varphi\Big(\frac{s_0}{s(t)}r\Big) &= -d\Big(\frac{s_0}{s(t)}\Big)^2\varphi_{yy}\Big(\frac{s_0}{s(t)}r\Big) - \frac{(N-1)d}{r}\Big(\frac{s_0}{s(t)}\Big)\varphi_y\Big(\frac{s_0}{s(t)}r\Big)\\&= \Big(\frac{s_0}{s(t)}\Big)^2\Big(-d\Delta_y\varphi\Big(\frac{s_0}{s(t)}r\Big)\Big)\\&\geq \frac{\lambda_1}{(1+\delta)^2}\varphi\Big(\frac{s_0}{s(t)}r\Big).\end{aligned}$$

Hence we obtain

$$v_t - d\Delta v - f(v) \geq \varepsilon_0 e^{-\beta t}\Big[m\varphi\Big(\frac{s_0}{s(t)}r\Big) - \Big(\frac{s'(t)s_0 r}{s(t)^2}\Big)\varphi_y\Big(\frac{s_0}{s(t)}r\Big)\Big], \tag{3.39}$$

where m is given in (3.36). Since $\varphi(y)$ has only one critical point in (γ, s_0) for (P1) (no critical point in (γ, s_0) for (P2) and (P3)), we find that

$$\varphi_y\Big(\frac{s_0}{s(t)}r\Big) \leq 0 \quad \text{for } \ t>0 \ \text{ and } \ R<r<s(t) \tag{3.40}$$

for (P2) and (P3), while

$$\varphi_y\Big(\frac{s_0}{s(t)}r\Big)\begin{cases} >0 & \text{for } \ t>0,\ r\in[R,R_t),\\ =0 & \text{for } \ t>0,\ r=R_t,\\ <0 & \text{for } \ t>0,\ r\in(R_t,s(t)]\end{cases} \tag{3.41}$$

for (P1), where $R_t > 0$ is a positive number depending on t (recalling that γ is sufficiently close to R). When (3.40) holds, the second term in the right-hand side of (3.39) is nonnegative. This result together with $m>0$ enables us to get (3.31). When (3.41) holds, we can similarly show the inequality for $(t,r)\in(0,\infty)\times[R_t,s(t)]$. For $(t,r)\in(0,\infty)\times[R,R_t)$, we see

$$s'(t) \leq \alpha\delta s_0 \quad \text{and} \quad -\frac{s'(t)s_0 r}{s(t)^2} \geq -\frac{\alpha\delta s_0^2}{s(t)} \geq -\alpha\delta s_0.$$

Hence it follows from (3.37) that

$$\begin{aligned}v_t - d\Delta v - f(v) &\geq \varepsilon_0 e^{-\beta t}\Big[m\varphi\Big(\frac{s_0}{s(t)}r\Big) - \alpha\delta s_0\varphi_y\Big(\frac{s_0}{s(t)}r\Big)\Big]\\&\geq \varepsilon_0 e^{-\beta t}(mL - \alpha\delta s_0 M)\\&\geq 0\end{aligned}$$

for $t > 0$, $R \le r < R_t$. We thus obtain (3.31).

We can easily check boundary conditions (3.32) and (3.33) as follows:

$$\begin{aligned} &u(t,R) = 0 < v(t,R) \quad \text{for (P1)}, \\ &u_r(t,R) = 0 \ge v_r(t,R) \quad \text{for (P2) and (P3)}, \\ &v(t,s(t)) = 0 \end{aligned}$$

for $t > 0$. We will next show (3.34). By calculation, it holds that

$$\begin{aligned} s'(t) - (-\mu v_r(t,s(t))) &= \alpha\delta s_0 e^{-\alpha t} + \frac{\mu s_0 \varphi_y(s_0)}{s(t)} \varepsilon_0 e^{-\beta t} \\ &\ge \Big\{ \alpha\delta s_0 - \varepsilon_0 \mu(-\varphi_y(s_0)) e^{(\alpha-\beta)t} \Big\} e^{-\alpha t}. \end{aligned}$$

Using (3.38), we deduce

$$s'(t) - (-\mu v_r(t,s(t))) \ge \mu(-\varphi_y(s_0)) \Big\{ \frac{\alpha\delta s_0}{\mu(-\varphi_y(s_0))} - \varepsilon_0 \Big\} e^{-\alpha t} \ge 0$$

for $t > 0$. Thus (3.31) – (3.34) holds true.

Finally, taking initial function u_0 so small that

$$u_0(r) \le v(0,r) = \varepsilon_0 \varphi(r) =: w(r) \quad \text{in} \quad [R, h_0],$$

we can apply Lemma 3.1 to show

$$u(t,r) \le v(t,r) \quad \text{and} \quad h(t) \le s(t) \quad \text{for} \quad t > 0,\ R \le r \le h(t).$$

Therefore $u(t,r) \le \varepsilon_0 \|\varphi\|_{C(\gamma,s_0)} e^{-\beta t}$ for $t > 0$, $R \le r \le h(t)$ and $\lim_{t\to\infty} h(t) \le \lim_{t\to\infty} s(t) = s_0(1+\delta)$. Hence we conclude that vanishing occurs as $t \to \infty$. □

Proof of Theorem 3.10. Theorem 3.9 has an important role in the proof. Let (u_σ, h_σ) be the solution of (P) with initial data $(\sigma\phi, h_0)$ and define

$$\sigma^* := \inf\{\, \rho \ge 0 \mid \text{spreading occurs for any } \sigma > \rho \}.$$

By the definition, spreading occurs for all solutions with $\sigma > \sigma^*$ as $t \to \infty$. Since part (ii) of Theorem 3.9 shows that vanishing occurs for small initial data, we find $\sigma^* > 0$. By the definition of σ^* and Theorem 3.8, for any $\sigma < \sigma^*$, there exists a number $\tau \in [\sigma, \sigma^*)$ such that the solution $(u_\tau(t,r), h_\tau(t))$ of (P) satisfies the property of vanishing as $t \to \infty$. Noting that $\sigma\phi \le \tau\phi$ in (R, h_0), we deduce from Lemma 3.1 that

$$h_\sigma(t) \le h_\tau(t) \quad \text{and} \quad u_\sigma(t,r) \le u_\tau(t,r) \quad \text{for} \quad t \ge 0,\ R < r < h_\sigma(t).$$

Hence, by letting $t \to \infty$ in the above inequality, we get

$$\lim_{t\to\infty} h_\sigma(t) < \infty \quad \text{and} \quad \lim_{t\to\infty} \|u_\sigma(t,\cdot)\|_{(R,h_\sigma(t))} = 0.$$

This result implies that vanishing occurs for all solutions with $\sigma < \sigma^*$ as $t \to \infty$. Moreover, using Lemma 3.1, we can easily show that, as $t \to \infty$, spreading occurs if $u_0 > \sigma^*\phi$ in (R, h_0), while vanishing occurs if $u_0 < \sigma^*\phi$ in (R, h_0).

We will show that vanishing occurs when $u_0 = \sigma^*\phi$. If $\sigma^* = \infty$, then there is nothing to prove. Hence we may assume that $\sigma^* < \infty$ in the rest of the proof. Assume that spreading occurs for the solution as $t \to \infty$, and then we can take a constant $T > 0$ such that $h(T) > R_N^*$. By the continuous dependence of solutions on initial data (Theorem 3.3), there exists so small $\delta > 0$ that

$$h_\sigma(T) > R_N^* \quad \text{for any } \sigma \in [\sigma^* - \delta, \sigma^* + \delta].$$

In particular, by virtue of part (i) of Theorem 3.9, the solution $(u_{\sigma^*-\delta}(t,r), h_{\sigma^*-\delta}(t))$ of (P) satisfies the property of spreading as $t \to \infty$. This result contradicts the definition of σ^*. Hence, by Theorem 3.8, vanishing occurs when $u_0 = \sigma^*\phi$ as $t \to \infty$. We complete the proof. □

Proof of Theorem 3.11. We can prove this theorem with suitable modification of the proof of Theorem 2.10 in one dimension. Hence we omit details here. □

3.7 Spreading and vanishing for Case $f'(0) < 0$

In this section we discuss multi-dimensional free boundary problem (P) where

the nonlinear function satisfies (3.21) and $f'(0) < 0$.

We show a dichotomy theorem and a criterion for spreading and vanishing.

3.7.1 Main theorems

A dichotomy theorem is given in the following:

Theorem 3.12. *Let* (u, h) *be any solution of* (P). *Then any solution of* (P) *satisfies either spreading* (i) *or vanishing* (ii) *as* $t \to \infty$:

(i) $\lim_{t\to\infty} \Omega(t) = \mathbb{R}^N \setminus B_R$ *and* $\liminf_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} > 0$;

(ii) $\lim_{t\to\infty} \Omega(t)$ *is a bounded set in* $\mathbb{R}^N \setminus B_R$ *and* $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} = 0$. *In addition,* $\|u(t,\cdot)\|_{C(R,h(t))} = O(e^{-kt})$ *for any* $k \in (0, -f'(0))$ *as* $t \to \infty$.

Define $c^* := \inf\{c > 0;\ f(c) = 0\} \in (0, 1]$ and suppose that the nonlinear function also satisfies

$$\int_0^1 f(u)\,du > 0.$$

Then we will also show some sufficient conditions for spreading or vanishing.

Theorem 3.13. *Let* (u, h) *be any solution of* (P). *The following results hold true:*

(i) *If* $h_0 \geq l$ *and* $u_0(r) \geq q(r)$ *in* $[R, l]$ *for a positive solution* $q(r)$ *of* (3.5) *with a sufficiently large constant* $l > R$*, then spreading occurs and*

$$\liminf_{t\to\infty} u(t, r) \geq v^*(r) \quad \textit{for} \quad R \leq r < \infty,$$

where $v^*(r)$ *is a minimal positive solution of* (3.6) *satisfying* $v^*(r) \geq q(r)$ *in* $[R, l]$.

(ii) *If* $\|u_0\|_{C(R,h_0)} < c^*$*, then vanishing occurs. Moreover* $\|u(t,\cdot)\|_{C(R,h(t))} = O(e^{-kt})$ *for any* $k \in (0, -f'(0))$ *as* $t \to \infty$.

Remark 3.5. *In part (i) of Theorem 3.13, assumption "*$u_0(r) \geq q(r)$ *in* $[R, l]$ *for a positive solution* $q(r)$ *of* (3.5) *" may be replaced by "* $u_0(r)$ *is an upper solution of* (3.5)*".*

Consider the solution of (P1) (resp. (P2) or (P3)) with initial data (u_0, h_0). We can give a criterion for spreading and vanishing, focusing on initial data.

Theorem 3.14. *Let* $\phi \in C^2(R, h_0) \cap C[R, h_0]$ *be any function which satisfies* $\phi(R) = \phi(h_0) = 0$ *(resp.* $\phi_r(R) = \phi(h_0) = 0$*). Then there exists a number* $\sigma^* = \sigma^*(\phi, h_0) \in (0, \infty]$ *with the following properties:*

- *spreading occurs if* $u_0 \geq \sigma^*\phi$ *in* (R, h_0)*. Moreover there exists a positive number* $\sigma^{**} \geq \sigma^*$ *such that if* $u_0 > \sigma^{**}\phi$ *in* (R, h_0)*, then* $\liminf_{t\to\infty} u(t, r) \geq v^*(r)$ *in* $[R, \infty)$*, where* $v^*(x)$ *is a minimal positive solution of* (3.6).
- *vanishing occurs if* $u_0 < \sigma^*\phi$ *in* (R, h_0).

Moreover if h_0 *is sufficiently large, then* $\sigma^* \leq \sigma^{**} < \infty$.

In Theorem 3.14, we remark that, differently from the case of $f'(0) > 0$, spreading occurs if $u_0 = \sigma^*\phi$.

Remark 3.6. *In Theorems 3.12 and 3.14, when spreading occurs, the large time behaviors of* $u(t, r)$ *might be divided into some cases. For example, we consider problem* (P) *with a bistable nonlinearity:*

$$f \in C^1[0, \infty), \quad f(u) < 0 \;\; \textit{for} \;\; 0 < u < c^*, \; u > 1, \quad f(u) > 0 \;\; \textit{for} \;\; c^* < u < 1,$$

$$f(0) = f(c^*) = f(1) = 0, \quad f'(0) < 0, \quad f'(c^*) > 0, \quad f'(1) < 0 \;\; \textit{and} \;\; \int_0^1 f(u)\,du > 0.$$

If $u_0 > \sigma^{**}\phi$*, then for* (P1)

$$\lim_{t\to\infty} u(t, r) = v^*(r) \;\; \textit{uniformly for } r \textit{ in any compact subset of } [R, \infty),$$

where v^* *is a positive solution of* (3.6)*, or*

$$\lim_{t\to\infty} u(t, r) = 1 \;\; \textit{uniformly for } r \textit{ in any compact subset of } [R, \infty)$$

for (P2) *or* (P3)*. On the other hand, if* $u_0 = \sigma^*\phi$*, then transition phenomena as in one-dimensional case might occur. For the proof, we need more precise information on solutions for* (3.5) *and* (3.6)*. We may also require additional assumptions on* f.

3.7.2 Proofs of main theorems

We give the proofs of the main theorems.

Proof of Theorem 3.12. We can apply Theorem 3.6 to this case, and we get this dichotomy theorem. When vanishing occurs, there exists a constant $T > 0$ such that $u(T, r) \le c^*$ in $[R, h(T)]$. As in the proof of Theorem 3.13, we can show $\|u(t,\cdot)\|_{C(R,h(t))} = O(e^{-kt})$ for any $k \in (0, -f'(0))$ as $t \to \infty$. □

Proof of Theorem 3.13. (i) Note that elliptic problem (3.5) has at least two positive solutions q_i $(i = 1, 2)$ for sufficiently large $l > R$ (see Section 3.8). Let $(\underline{u}(t, r), \underline{h}(t))$ be a solution of (P) with initial data $(q_i(r), l)$. Since $h_0 \ge l$ and $u_0(r) \ge q_i(r)$ in $[R, l]$, it follows from Corollary 3.1 that

$$\liminf_{t\to\infty} u(t,r) \ge \lim_{t\to\infty} \underline{u}(t,r) = v_i^*(r) \quad \text{for} \quad R \le r < \infty,$$

where v_i^* $(i = 1, 2)$ is a minimal positive solutions of (3.6) satisfying $v_i^*(r) \ge q_i(r)$ in (R, l).

(ii) We can prove this property in the same way as the one-dimensional case. Let $w = w(t)$ be the solution of

$$\begin{cases} \dfrac{dw}{dt} = f(w), \quad t > 0, \\ w(0) = c_1 \in [\|u_0\|_{C(R,h_0)}, c^*). \end{cases}$$

Then $w(t)$ is regarded as an upper solution for (P), and

$$u(t,r) \le w(t) \ \text{ for } \ t > 0, \ R < r < h(t). \tag{3.42}$$

Since $f(w) < 0$ for $0 < w < c^*$ and $f(0) = 0$, the function $w(t)$ is decreasing and satisfies $\lim_{t\to\infty} w(t) = 0$. Hence $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} \le \lim_{t\to\infty} w(t) = 0$. Thus it follows from Proposition 3.3 that $\lim_{t\to\infty} h(t) < \infty$ and vanishing occurs as $t \to \infty$.

We will next show a decay estimate of vanishing. Noting that $f'(w)$ is continuous in $[0, \infty)$ and $\lim_{t\to\infty} w(t) = 0$, for any small $\varepsilon > 0$, one can choose a positive number T_ε such that

$$f(w(t)) \le (f'(0) + \varepsilon) w(t) \ \text{ for } \ t \ge T_\varepsilon.$$

Then $w(t)$ satisfies

$$\frac{dw}{dt} \le (f'(0) + \varepsilon) w \ \text{ for } \ t \ge T_\varepsilon.$$

Hence we have

$$w(t) \le w(T_\varepsilon) e^{-(k^*-\varepsilon)(t-T_\varepsilon)} \quad \text{for} \ t \ge T_\varepsilon$$

with $k^* = -f'(0) > 0$. Using (3.42) and the above estimate, we get

$$u(t,r) \le w(t) \le w(T_\varepsilon) e^{-(k^*-\varepsilon)(t-T_\varepsilon)} =: C_\varepsilon e^{-(k^*-\varepsilon)(t-T_\varepsilon)}$$

for $t \geq T_\varepsilon$, $R \leq r \leq h(t)$. Hence $\|u(t,\cdot)\|_{C(R,h(t))} = O(e^{-kt})$ for any $k \in (0, -f'(0))$ as $t \to \infty$. We complete the proof. □

Proof of Theorem 3.14. For solutions of (P) with initial data $(\sigma\phi, h_0)$, define

$$\sigma^* := \inf\{\ \rho \geq 0 \mid \text{spreading occurs for any } \sigma > \rho\}.$$

In the same way as the proof of Theorem 3.10, we can prove by Theorem 3.13 that $\sigma^* \in (0, \infty]$ and, as $t \to \infty$, spreading occurs if $u_0 > \sigma^*\phi$ in (R, h_0), while vanishing occurs if $u_0 < \sigma^*\phi$ in (R, h_0). As in the proof of Theorem 2.12, we can show another threshold number σ^{**} for the case of spreading.

We assume that vanishing occurs for $u_0 = \sigma^*\phi$ as $t \to \infty$. Then there exists a constant $T > 0$ such that $u(T, r) < c^*/2$. By the continuous dependence of solutions on initial data (Theorem 3.3), we can choose a number $\delta > 0$ such that, for $\sigma \in [\sigma^* - \delta, \sigma^* + \delta]$, u satisfies

$$u(T, r) < \frac{c^*}{2} \quad \text{for} \quad R \leq r \leq h(t).$$

By part (ii) of Theorem 3.13, such a solution satisfies the property of vanishing as $t \to \infty$. This result contradicts the definition of σ^*. Hence the dichotomy theorem (Theorem 3.12) implies that spreading occurs when $u_0 = \sigma^*\phi$ as $t \to \infty$. We complete the proof. □

3.8 Semilinear elliptic equations

In this section we will show some results and remarks on semilinear elliptic equations in an annulus or an exterior domain in $\mathbb{R}^N$, where the nonlinear function satisfies one of the following conditions:

$$(3.21) \quad \text{with} \quad f'(0) > 0 \tag{3.43}$$

and

$$(3.21) \quad \text{with} \quad f'(0) < 0 \quad \text{and} \quad \int_0^1 f(s)\, ds > 0. \tag{3.44}$$

The results in this section partly support main theorems.

3.8.1 Semilinear elliptic equations in annulus

We consider semilinear elliptic equations in an annulus given by

$$(3.5) \quad \begin{cases} d\Delta q + f(q) = 0, \quad R < r < l, \\ q > 0, \quad R < r < l, \\ q(R) = q(l) = 0 \ \ (\text{resp. } q_r(R) = q(l) = 0) \end{cases}$$

for some $l > R$. We have the following results.

Theorem 3.15. *Suppose that f satisfies* (3.43). *Then problem* (3.5) *have at least one positive solutions if l is sufficiently large.*

Proof. We will first construct an lower solution for (3.5). Let φ be an eigenfunction corresponding to the first eigenvalue λ_1 for the problem:

$$\begin{cases} d\Delta\varphi + \lambda_1\varphi = 0, \quad R < r < l, \\ \varphi(R) = \varphi(l) = 0 \text{ (resp. } \varphi_r(R) = \varphi(l) = 0) \end{cases}$$

for $l > R$. Consider a function $\phi := \varepsilon\varphi$ for a constant $\varepsilon > 0$. As in the proof of Proposition 3.2, choosing sufficiently large $l > R$ and sufficiently small ε, we find that ϕ satisfies

$$\begin{cases} d\Delta\phi + f(\phi) \geq 0, \quad R < r < l, \\ \phi(R) = \phi(l) = 0 \text{ (resp. } \phi_r(R) = \phi(l) = 0), \end{cases}$$

and ϕ is a lower solution of (3.5). Next, we can easily see that $\Phi = 1$ is an upper solution for (3.5). Hence, by the monotone method (see Sattinger [58]), we can show that there exists at least one positive solution q satisfying

$$\phi(r) \leq q(r) \leq \Phi(r) \quad \text{in} \quad R < r < l.$$

We complete the proof. □

Theorem 3.16. *Suppose that f satisfies* (3.44). *Then there exists a sufficiently large number $l^* > 0$ such that, for every $l > l^*$,* (3.5) *has at least two positive and radially symmetric solutions q_1 and q_2 satisfying*

$$I(q_1) < 0 < I(q_2),$$

where

$$I(q) = \frac{d}{2}\int_R^l r^{N-1} q_r(r)^2 \, dr + \int_R^l r^{N-1} F(q(r)) \, dr$$

with $F(q) = -\int_0^q f(s)\, ds$.

We omit the proof of Theorem 3.16 here.

Remark 3.7. *The existence of positive solutions has been also shown by Stakgold-Payne [64] for a monostable nonlinearity and by Clément-Sweers [12] for a bistable nonlinearity. In the above theorem we have given the results on existence of solutions of* (3.5)*, where f satisfies more general nonlinearity.*

3.8.2 Semilinear elliptic equations in exterior domain

We consider semilinear elliptic equations in an exterior domain in $\mathbb{R}^N$ given by

$$(3.6)\quad \begin{cases} d\Delta v + f(v) = 0, \quad R < r < \infty, \\ v > 0, \quad R < r < \infty, \\ v(R) = 0 \ \ (\text{resp. } v_r(R) = 0). \end{cases}$$

We have the following theorem on the existence of solutions.

Theorem 3.17. *Suppose that f satisifies* (3.43) *or* (3.44)*. Then there exist at least one positive solutions for* (3.6)*.*

Proof. The proof is derived immediately from the existence of positive solution of (3.5) by using Theorem 3.5. Hence the proof is complete from Theorems 3.15 and 3.16. □

Let $a^* := \inf\{a > 0 \mid f(a) = 0\}$. Then we can show that a solution of (3.6) satisfying $0 < v \le a^*$ is unique if f satisfies (3.43) and another condition.

Theorem 3.18. *Suppose that f satisfies* (3.43) *and that $f(u)/u$ is decreasing with respect to $u \in [0, a^*]$. Then a solution $v(r)$ of* (3.6) *satisfying $0 < v \le a^*$ in (R, ∞) is unique. Moreover the solution satisfies $v_r(r) > 0$ for all $r \ge R$ and $\lim_{r\to\infty} v(r) = a^*$ with $v_r(r) = o(1/r^{N-1})$ as $r \to \infty$ under the Dirichlet boundary condition at $r = R$, while $v(r) \equiv a^*$ under the Neumann boundary condition at $r = R$.*

We prepare the following propositions.

Proposition 3.6. *Suppose that f satisfies* (3.43) *and that $f(u)/u$ is decreasing with respect to $u \in [0, a^*]$. Let $v \in C^2(R, \infty)$ be any positive solution of* (3.6) *under the Dirichlet boundary condition at $r = R$ and let v satisfy $0 < v \le a^*$ in (R, ∞). Then $v_r(r) > 0$ for all $r \ge R$ and $\lim_{r\to\infty} v(r) = a^*$ with $v_r(r) = o(1/r^{N-1})$ as $r \to \infty$.*

Proof. We basically follow the proof of [36, Proposition 10]. We will first prove $\lim_{r\to\infty} v(r) = a^*$. Assume $\lim_{r\to\infty} v(r) \ne a^*$ to get a contradiction. Let $P(r) := dr^{N-1}v_r(r)$. Since $0 < v(r) \le a^*$ for $R < r < \infty$, we see $f(v(r)) \ge 0$ for $R < r < \infty$. It follows from (3.6) that

$$P_r(r) = -r^{N-1} f(v(r)) \le 0 \quad \text{for} \quad R < r < \infty.$$

By $P(R) = R^{N-1}v_r(R) > 0$, v_r changes its sign at most only once in (R, ∞). Hence we find that there exists a limit number $\nu := \lim_{r\to\infty} v(r)$ with $\nu \in [0, a^*)$. Since $f(0) = f(a^*) = 0$, we denote $f(v) = vg(v)$ for some positive function g. Then, for sufficiently small $\varepsilon > 0$, there exists large $R_{1,\varepsilon} > 0$ such that $g(v(r)) \ge g(\nu) - \varepsilon > 0$ in $[R_{1,\varepsilon}, \infty)$. Consider (3.6) in $[R_{1,\varepsilon}, R_2]$ for $R_2 > R_{1,\varepsilon}$. Then v satisfies

$$(p(r)v_r)_r + q(r)v = 0, \quad v > 0 \quad \text{for} \quad r \in (R_{1,\varepsilon}, R_2),$$

where $p(r) = dr^{N-1}$ and $q(r) = r^{N-1}g(v(r))$. Note that for $r \in (R_{1,\varepsilon}, R_2)$

$$p(r) = dr^{N-1} \le dR_2^{N-1} =: C_1^2,$$
$$q(r) = r^{N-1}g(v(r)) \ge R_{1,\varepsilon}^{N-1}(g(\nu) - \varepsilon) =: C_{2,\varepsilon}^2.$$

We now compare $v(r)$ with

$$w(r) = \sin\Big(\frac{C_{2,\varepsilon}}{C_1}(r - R_{1,\varepsilon})\Big),$$

which is the positive solution of

$$C_1^2 w_{rr} + C_{2,\varepsilon}^2 w = 0 \quad \text{for} \ \ r \in (R_{1,\varepsilon}, R_2), \quad w(R_{1,\varepsilon}) = 0.$$

Choosing $R_2 \le (C_1/C_{2,\varepsilon})\pi + R_{1,\varepsilon}$, if necessary, we see from Sturm's comparison theorem (see e.g. [13]) that v have at least one zero points in $[R_{1,\varepsilon}, R_2]$. This contradicts $0 < v \le a^*$. Thus $\lim_{r\to\infty} v(r) = a^*$.

We next prove $v_r(r) > 0$ for all $r \ge R$. Assume that there exists $r^* > R$ satisfying $v_r(r^*) = 0$. Since $P(r) = r^{N-1}v_r(r)$ is strictly decreasing with respect to r, we find $v_r(r) < 0$ for all $r > r^*$. This implies a contradiction, $\lim_{r\to\infty} v(r) \ne a^*$. Hence the derivative of v with respect to r is positive for all $r \ge R$.

We finally prove the rate of convergence. Let $P(r) = dr^{N-1}v_r(r) \to \beta \in [0, P(R))$ as $r \to \infty$. We will prove that β must be 0. Suppose that $\beta > 0$ to get a contradiction. For any $\eta > 0$, we find

$$\frac{\beta}{r^{N-1}} \le v_r(r) < \frac{\beta + \eta}{r^{N-1}} \quad \text{if } r \text{ is sufficiently large.} \tag{3.45}$$

Integrating this inequality in (ρ, M) for large M and $\rho > R$ gives

$$\beta \int_\rho^M \frac{1}{r^{N-1}}\, dr \le \int_\rho^M v_r(r)\, dr = v(M) - v(\rho) < (\beta + \eta) \int_\rho^M \frac{1}{r^{N-1}}\, dr. \tag{3.46}$$

When $N = 2$, the left-hand side of the inequality implies

$$a^* > v(M) - v(\rho) \ge \beta(\log M - \log \rho) \to +\infty \quad \text{as} \quad M \to +\infty.$$

This gives us a contradiction, and $\beta = 0$. When $N \ge 3$, it holds from (3.46) that

$$\frac{\beta}{N-2}(\rho^{2-N} - M^{2-N}) \le v(M) - v(\rho) < \frac{\beta + \eta}{N-2}(\rho^{2-N} - M^{2-N}).$$

Letting $M \to +\infty$ and replacing ρ with r, we have

$$\frac{\beta}{N-2} r^{2-N} \le a^* - v(r) \le \frac{\beta + \eta}{N-2} r^{2-N} \tag{3.47}$$

for large r. Since $f(a^*) = 0$, we represent $f(v) = v(a^* - v)g_1(v)$ for some positive function g_1. Recall (3.45), (3.47) and

$$v_{rr}(r) = -\frac{(N-1)}{r}v_r(r) - \frac{v(r)(a^* - v(r))g_1(v(r))}{d}.$$

Then it follows that

$$-\frac{(N-1)}{r}\cdot\frac{\beta+\eta}{r^{N-1}}-\frac{\beta+\eta}{N-2}\cdot\frac{v(r)g_1(v(r))}{dr^{N-2}}\le v_{rr}(r)\le-\frac{\beta}{N-2}\cdot\frac{v(r)g_1(v(r))}{dr^{N-2}} \quad (3.48)$$

when r is large. Since $a^*-\eta<v(r)<a^*$ and $C<g_1(v(r))<C+\eta$ for large r and some positive constant C, we see from (3.48) that

$$-\frac{(N-1)(\beta+\eta)}{r^N}-\frac{\beta+\eta}{d(N-2)}\cdot\frac{C+\eta}{r^{N-2}}\le v_{rr}(r)\le-\frac{\beta(1-\eta)}{d(N-2)}\cdot\frac{C}{r^{N-2}} \quad (3.49)$$

for large r. We integrate the right-hand side of (3.49) over (r,M) to see

$$v_r(M)-v_r(r)\le-\frac{\beta C(1-\eta)}{d(N-2)}(\log M-\log r)\quad (N=3),$$
$$v_r(M)-v_r(r)\le-\frac{\beta C(1-\eta)}{d(N-2)(N-3)}\Big(\frac{1}{r^{N-3}}-\frac{1}{M^{N-3}}\Big)\quad (N\ge 4).$$

Letting $M\to+\infty$, we obtain for $N=3$

$$v_r(r)\ge\frac{\beta C(1-\eta)}{d(N-2)}(\log M-\log r)\to+\infty.$$

This result enables us to get $\beta=0$. Similarly for $N\ge 4$, we deduce from (3.45) that

$$\beta+\eta\ge\frac{\beta C(1-\eta)}{d(N-2)(N-3)}r^2$$

for large r. This inequality implies that β must be zero. Hence $r^{N-1}v_r(r)\to 0$ as $r\to+\infty$, and $v_r(r)=o(1/r^{N-1})$ as $r\to+\infty$. We complete the proof. □

Proposition 3.7. *Suppose that f satisfies* (3.43) *and that $f(u)/u$ is decreasing with respect to $u\in[0,a^*]$. Then $v\equiv a^*$ is a unique positive solution of* (3.6) *under the Neumann boundary condition at $r=R$.*

Proof. We can easily check that $v\equiv a^*$ is a solution of (3.6). If there exists another solution for (3.6) with $v_r(R)=0$ and $v(R)\in(0,a^*)$, then we find that v is non-increasing because $P(R)=0$ and $P_r(r)\le 0$ for $R<r<\infty$, following the proof of Proposition 3.6. Hence we obtain the same contradiction in Proposition 3.6 by employing Sturm's comparison principle. □

Proof of Theorem 3.18. We basically follow the proof of [36, Theorem 10]. It has been already shown in Proposition 3.7 that there exists a unique solution for (3.6) with the Neumann boundary condition at $r=R$. For convenience, we prove the existence of solutions for the Dirichlet problem by a different way from the proof of Theorem 3.17. Let $w(r)$ be a function represented as

$$w(r) = \begin{cases} \phi(r), & r \in [R, l], \\ 0, & r \in (l, \infty), \end{cases}$$

where l is a positive number and ϕ is a solution of (3.29). Then we find that, for any small $\delta > 0$, δw is a lower solution of (3.6) in the distribution sense. On the other hand, $v \equiv a^*$ is an upper solution of (3.6). Hence the standard monotone method (see Sattinger [58]) shows that there exists a solution v satisfying $\delta w(r) \leq v(r) \leq a^*$ for $r \in [R, \infty)$. By the elliptic regularity theory, v satisfies (3.6) in the classical sense. We recall from Proposition 3.6 that

$$v_r(r) > 0 \quad \text{for} \quad r \geq R, \quad \lim_{r\to\infty} v(r) = a^* \quad \text{and} \quad \lim_{r\to\infty} r^{N-1} v_r(r) = 0. \tag{3.50}$$

We next prove the uniqueness of solutions for (3.6). Since δ is any small positive number, the uniqueness of solutions v for (3.6) satisfying $\delta w(r) \leq v(r) \leq a^*$ for $r \in [R, \infty)$ enables us to get the conclusion. Suppose that w_* (resp. w^*) is a minimal (resp. maximal) positive solution of (3.6), which is generated from $\delta w(r)$ (resp. 1) by the monotone method. Then

$$\begin{aligned} d(r^{N-1} w_{*,r}(r))_r + r^{N-1} f(w_*(r)) = 0, \quad R < r < \infty, \quad w_*(R) = 0 \\ (\text{resp. } d(r^{N-1} w^*_r(r))_r + r^{N-1} f(w^*(r)) = 0, \quad R < r < \infty, \quad w^*(R) = 0) \end{aligned} \tag{3.51}$$

with

$$w_*(r) \leq w^*(r) \quad \text{for} \quad R < r < \infty.$$

Multiplying (3.51) by w^* (resp. w_*) and subtracting the both sides of (3.51), we obtain

$$\begin{aligned} d\{(r^{N-1} w_{*,r}(r))_r w^*(r) - (r^{N-1} w^*_r(r))_r w_*(r)\} &= r^{N-1}\{w_*(r) f(w^*(r)) - w^*(r) f(w_*(r))\} \\ &= r^{N-1} w^*(r) w_*(r) \Big(\frac{f(w^*(r))}{w^*(r)} - \frac{f(w_*(r))}{w_*(r)} \Big). \end{aligned}$$

Moreover integrating the identity in $[R, \rho]$ for $\rho > R$, we have

$$\begin{aligned} &\int_R^\rho (r^{N-1} w_{*,r}(r))_r w^*(r) - (r^{N-1} w^*_r(r))_r w_*(r) \, dr \\ &\quad = \frac{1}{d} \int_R^\rho r^{N-1} w^*(r) w_*(r) \Big(\frac{f(w^*(r))}{w^*(r)} - \frac{f(w_*(r))}{w_*(r)} \Big) \, dr. \end{aligned}$$

Integrating by parts the left-hand side of the identity implies

$$\begin{aligned} &\rho^{N-1} w_{*,r}(\rho) w^*(\rho) - \rho^{N-1} w^*_r(\rho) w_*(\rho) \\ &\quad = \frac{1}{d} \int_R^\rho r^{N-1} w^*(r) w_*(r) \Big(\frac{f(w^*(r))}{w^*(r)} - \frac{f(w_*(r))}{w_*(r)} \Big) \, dr. \end{aligned} \tag{3.52}$$

We apply (3.50) to have

$$\lim_{\rho\to\infty} \rho^{N-1} w_{*,r}(\rho) = \lim_{\rho\to\infty} \rho^{N-1} w_r^*(\rho) = 0,$$
$$\lim_{\rho\to\infty} w^*(\rho) = \lim_{\rho\to\infty} w_*(\rho) = a^*.$$

Hence, taking $\rho \to \infty$ in (3.52), we get

$$\int_R^\infty r^{N-1} w^*(r) w_*(r) \Big(\frac{f(w^*(r))}{w^*(r)} - \frac{f(w_*(r))}{w_*(r)} \Big)\, dr = 0.$$

Since $f(u)/u$ is decreasing with respect to $u \in [0, a^*]$ and $w^* \geq w_* > 0$ in $[R, \infty)$, the above identity implies $w^* \equiv w_*$ in $[R, \infty)$. The proof is complete. □

3.9 Problem

Through sections 3.9 - 3.11, we consider free boundary problem (1.1) where $\Omega(t)$ has no fixed boundary. The problem is given by

$$\text{(DP)} \quad \begin{cases} u_t - d\Delta u = f(u), & t > 0,\ g(t) < r < h(t), \\ u(t, g(t)) = 0,\ u(t, h(t)) = 0, & t > 0, \\ g'(t) = -\mu u_r(t, g(t)), & t > 0, \\ h'(t) = -\mu u_r(t, h(t)), & t > 0, \\ g(0) = g_0,\ h(0) = h_0,\ u(0, r) = u_0(r), & g_0 \leq r \leq h_0, \end{cases}$$

where $d,\ \mu,\ g_0$ and h_0 are positive constants with $g_0 < h_0$, $r = |x|$ $(x \in \mathbb{R}^N, N \geq 2)$, $\Delta u := u_{rr} + ((N-1)/r)u_r$, and the initial function is assumed to satisfy

$$u_0 \in C^2[g_0, h_0], \quad u_0 > 0 \ \text{ in } \ (g_0, h_0), \quad u_0(g_0) = u_0(h_0) = 0.$$

In (DP), we basically allow nonlinear function to be

$$f \in C^1[0, \infty), \quad f(0) = f(1) = 0, \quad f(u) < 0 \ (u > 1), \tag{3.53}$$

and also, when we discuss the asymptotic behaviors of solutions, we further assume

$$\begin{aligned} &f \in C^1[0, \infty),\ f(0) = f(1) = 0,\ f'(0) > 0,\ f'(1) < 0,\ f(u) > 0\ (0 < u < 1), \\ &f(u) < 0\ (u > 1),\ \text{and } f(u)/u \text{ is decreasing with respect to } u \in [0, 1]. \end{aligned} \tag{3.54}$$

Problem (DP) corresponds to (P) with inner fixed boundary $r = |x| = R$ replaced by free boundary $r = g(t)$ (see Figure 7). It will be proved that $g'(t) < 0$ and $h'(t) > 0$. Hence the domain

$$\Omega(t) = \{x \in \mathbb{R}^N |\ g(t) < |x| < h(t)\}$$

is expanding as time passes, and hence outer radius $r = h(t)$ can go to infinity as $t \to \infty$ and also inner radius $r = g(t)$ can reach the origin at some time.

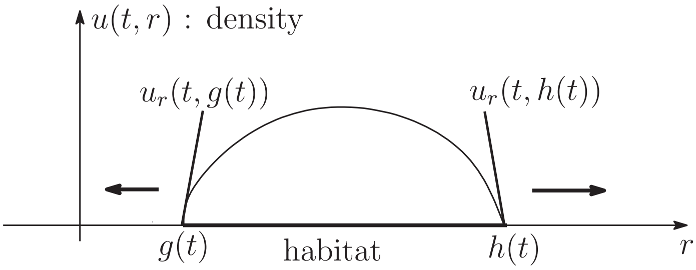

Figure 7. The solution $(u(t,r), g(t), h(t))$ for problem (DP)

Such a phenomenon makes a big difference between (P) and other free boundary problems. Hence we call this phenomenon **singularity**.

Definition 3.1. *Singularity: there exists some* $T > 0$ *such that* $g(T) = 0$.

After singularity appears, we cannot extend the solution of (DP) in a classical sense. However we can extend the solution weakly to all $t > 0$. For this purpose we need to define a weak solution, which will be given in next section.

The main purpose through sections 3.9 - 3.11 is to study

- the existence and uniqueness of classical and weak solutions;
- the asymptotic behaviors of solutions;
- a sufficient condition for singularity.

We finally mention that the situation described by (DP) arises from multi-dimensional free boundary problems; for example the case where some parts of the boundaries happen to connect each other at some time (see Figure 8).

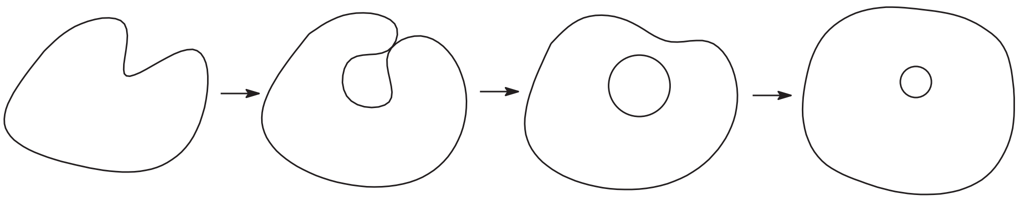

Figure 8. Development of the free boundaries ($N = 2$)

3.10 Classical and weak solutions

In this section we will first show a result on the existence and uniqueness of classical. Next we define a weak solution for (DP) and give a result on the unique existence of weak solutions for (DP). Finally we will show a relation between classical and weak solutions.

The following theorem means a unique existence of classical solutions and some estimates of the density function and the free boundaries.

Theorem 3.19. *For any given $\alpha \in (0,1)$, there exists a number $T > 0$ depending on g_0, h_0, α and $\|u_0\|_{C^2(g_0,h_0)}$ such that* (DP) *has a unique classical solution*

$$(u,g,h) \in \{C^{\frac{(1+\alpha)}{2},1+\alpha}(\overline{D(T)}) \cap C^{1+\frac{\alpha}{2},2+\alpha}(D(T))\} \times C^{1+\frac{\alpha}{2}}[0,T] \times C^{1+\frac{\alpha}{2}}[0,T],$$

where $D(T) = \bigcup_{0<t\leq T}(\{t\} \times (g(t),h(t)))$. Moreover let T_{max} be a maximal existence time of classical solutions for some $T_{max} \in (0,\infty]$, and then it holds that

$$\begin{aligned} 0 < u(t,r) \leq C_1 \quad &in \quad D(T_{max}), \\ -\mu C_2 \leq g'(t) < 0 < h'(t) \leq \mu C_2 \quad &for \quad 0 < t < T_{max}, \\ g(t) > 0 \quad &for \quad 0 \leq t < T_{max}, \end{aligned}$$

where C_1 and C_2 are positive constants depending on $\|u_0\|_{C(g_0,h_0)}$ and $\|u_0\|_{C^1(g_0,h_0)}$, respectively.

Proof. We prove this theorem with slight modification of Theorem 3.2. Hence we omit the details here. We can show $g(t) > 0$ for $0 \leq t < T_{\max}$ because, if $g(T_1) = 0$ for some $T_1 < T_{\max}$, then $g'(T_1) < 0$ gives a contradiction to $g(t) = r = |x| \geq 0$ for all $t \in (0, T_{\max})$. □

We next prepare for weak solutions for (DP).

Definition 3.2. *Let $G_T := (0,T) \times G$ for some $T > 0$ and a large domain $G \supset (g_0,h_0)$. A non-negative function $u \in L^\infty(G_T) \cap H^1(G_T)$ is called* **a weak solution** *of* (DP) *over G_T when $u(t,r)$ satisfies*

$$\iint_{G_T} dr^{N-1}u_r\phi_r - r^{N-1}\alpha(u)\phi_t \; drdt - \int_G r^{N-1}\alpha(\tilde{u}_0)\phi(0,r) \; dr = \iint_{G_T} r^{N-1}f(u)\phi \; drdt$$

for any $\phi \in C^1(G_T)$ which satisfies $\phi = 0$ on $(\{T\} \times G) \cup ([0,T] \times \partial G)$, where

$$\alpha(u) = \begin{cases} u & if \;\; u > 0, \\ u - d/\mu & if \;\; u \leq 0, \end{cases} \qquad \tilde{u}_0 = \begin{cases} u_0, & r \in [g_0,h_0], \\ 0 & r \in G \setminus (g_0,h_0). \end{cases}$$

We can prove the global existence and uniqueness of weak solutions for (DP).

Theorem 3.20. *For any $T > 0$ and any domain $G \supset (g_0,h_0)$, there exists a unique weak solution for* (DP) *over $[0,T] \times G$.*

Proof. We can apply a result on the unique existence of weak solutions by Du-Guo [15, Theorems 3.1 and 3.5]. Hence we get the conclusions. □

By this theorem, we are able to consider the problem after inner boundary $r = g(t)$ reaches the origin.

We finish this section by showing the relation between classical solutions and weak solutions.

Theorem 3.21. *The following results hold true.*

(i) *Let $u = u(t, r)$ be a classical solution of* (DP). *Then the function*

$$v(t,r) = \begin{cases} u(t,r), & (t,r) \in (0,T) \times (g(t), h(t)), \\ 0, & (t,r) \in (0,T) \times (G \setminus (g(t), h(t))) \end{cases}$$

is a weak solution of (DP) *over $G_T = (0, T) \times G$.*

(ii) *Let v be a weak solution of* (DP) *over $(0, T) \times G$ and assume the functions $h, g \in C^1(0, T)$ $(g(t) < h(t)$ for $t \geq 0)$ satisfy*

$$\{r \in G|\ g(t) < r < h(t)\} = \{r \in G|\ v(t,r) > 0\},$$
$$\{r \in G|\ r \leq g(t), h(t) \leq r\} = \{r \in G|\ v(t,r) = 0\}$$

for $0 \leq t \leq T$. If

- *$u = v$ for $(t, r) \in (0, T) \times [g(t), h(t)]$,*
- *u and u_r is continuous in $[0, T) \times [g(t), h(t)]$,*
- *u_{rr} and u_t is continuous in $(0, T) \times (g(t), h(t))$,*

then (u, g, h) is a classical solution of (DP).

Proof. (i) Note that the equation is also written as $r^{N-1}u_t - d(r^{N-1}u_r)_r = r^{N-1}f(u)$. Multiplying the equation by $\phi \in C^1(G_T)$ $(\phi_0(r) := \phi(0, r))$ which must vanish on $(\{T\} \times G) \cup ([0, T] \times \partial G)$ and integrating the equation over $(0, T) \times (g(t), h(t))$, we obtain

$$\int_0^T \int_{g(t)}^{h(t)} r^{N-1} u_t \phi \ drdt - d \int_0^T \int_{g(t)}^{h(t)} (r^{N-1}u_r)_r \phi \ drdt = \int_0^T \int_{g(t)}^{h(t)} r^{N-1} f(u) \phi \ drdt. \tag{3.55}$$

Since $u(t, h(t)) = u(t, g(t)) = 0$ for $t \geq 0$ and the choice of ϕ, integration by parts gives

$$\int_0^T \int_{g(t)}^{h(t)} r^{N-1} u_t \phi \ drdt = -\int_{g_0}^{h_0} r^{N-1} u_0 \phi_0 \ dr - \int_0^T \int_{g(t)}^{h(t)} r^{N-1} u \phi_t \ drdt \tag{3.56}$$

and

$$\int_0^T \int_{g(t)}^{h(t)} (r^{N-1}u_r)_r \phi \ drdt = \int_0^T h(t)^{N-1} u_r(t, h(t)) \phi(t, h(t)) \ dt - \int_0^T g(t)^{N-1} u_r(t, g(t)) \phi(t, g(t)) \ dt - \int_0^T \int_{g(t)}^{h(t)} r^{N-1} u_r \phi_r \ drdt. \tag{3.57}$$

On the other hand, it follows from the divergence theorem

$$\int_0^T \int_{G\setminus(g(t),h(t))} r^{N-1}\phi_t \, drdt = -\int_{G\setminus(g_0,h_0)} r^{N-1}\phi_0 \, dr + \int_0^T h(t)^{N-1}h'(t)\phi(t,h(t)) \, dt \\ - \int_0^T g(t)^{N-1}g'(t)\phi(t,g(t)) \, dt. \tag{3.58}$$

Using the Stefan conditions, we find that

$$\int_0^T \int_{G\setminus(g(t),h(t))} r^{N-1}\phi_t \, drdt = -\int_{G\setminus(g_0,h_0)} r^{N-1}\phi_0 \, dr \\ -\mu\Big\{ \int_0^T h(t)^{N-1}u_r(t,h(t))\phi(t,h(t)) \, dt - \int_0^T g(t)^{N-1}u_r(t,g(t))\phi(t,g(t)) \, dt\Big\}. \tag{3.59}$$

Substituting (3.56) and (3.57) into (3.55) and using (3.59), we get

$$- \int_{g_0}^{h_0} r^{N-1}u_0\phi_0 \, dr - \int_0^T \int_{g(t)}^{h(t)} r^{N-1}u\phi_t \, drdt + d\int_0^T \int_{g(t)}^{h(t)} r^{N-1}u_r\phi_r \, drdt \\ + \frac{d}{\mu}\int_{G\setminus(g_0,h_0)} r^{N-1}\phi_0 \, dr + \frac{d}{\mu}\int_0^T \int_{G\setminus(g(t),h(t))} r^{N-1}\phi_t \, drdt = \int_0^T \int_{g(t)}^{h(t)} r^{N-1}f(u)\phi \, drdt.$$

We finally replace $u(t,r)$ and u_0 with $v(t,r)$ and v_0 to see $v = v_r = 0$ on $(0,T)\times(G\setminus(g(t),h(t)))$, and hence $f(v)=0$ on that region. Thus we can observe that v satisfies the weak form.

(ii) By the assumption, we find that u satisfies the boundary condition and the equation in (DP). By the definition of weak solutions, u satisfies

$$\int_0^T \int_{g(t)}^{h(t)} (dr^{N-1}u_r\phi_r - r^{N-1}u\phi_t) \, drdt + \frac{d}{\mu}\int_0^T \int_{G\setminus(g(t),h(t))} r^{N-1}\phi_t \, drdt \\ - \int_{\Omega_0} r^{N-1}u_0\phi_0 \, dr + \frac{d}{\mu}\int_{G\setminus(g_0,h_0)} r^{N-1}\phi_0 \, dr = \int_0^T r^{N-1}f(u)\phi \, drdt. \tag{3.60}$$

Let ϕ get its support on $[0,T)\times(g(t),h(t))$. Then it follows that

$$\int_0^T \int_{g(t)}^{h(t)} (dr^{N-1}u_r\phi_r - r^{N-1}u\phi_t) \, drdt - \int_{g_0}^{h_0} r^{N-1}u_0\phi_0 \, dr = \int_0^T r^{N-1}f(u)\phi \, drdt.$$

This identity together with (3.55) and (3.56) gives

$$\int_0^T \int_{g(t)}^{h(t)} (r^{N-1}u_t - d(r^{N-1}u_r)_r - r^{N-1}f(u))\phi \, drdt = 0$$

for any $\phi \in C^1(G_T)$, which implies that u is the solution of $u_t - d\Delta u = f(u)$ in $(0,T)\times(g(t),h(t))$.

We next check the initial condition and the Stefan condition. We must recall (3.58). Substituting this identity to (3.60), we get

$$\begin{aligned}
&\int_0^T \int_{g(t)}^{h(t)} (dr^{N-1}u_r\phi_r - r^{N-1}u\phi_t)\ drdt \\
&= \int_{g_0}^{h_0} r^{N-1}u_0\phi_0\ dr + \int_0^T \int_{g(t)}^{h(t)} r^{N-1}f(u)\phi\ drdt \\
&\quad - \frac{d}{\mu}\Big\{ \int_0^T h(t)^{N-1}h'(t)\phi(t,h(t))\ dt - \int_0^T g(t)^{N-1}g'(t)\phi(t,g(t))\ dt \Big\}.
\end{aligned} \tag{3.61}$$

On the other hand, integrating by parts the following quantity as in (3.55) and (3.56), we can show for any $\delta > 0$

$$\begin{aligned}
&\int_\delta^T \int_{g(t)}^{h(t)} (dr^{N-1}u_r\phi_r - r^{N-1}u\phi_t)\ drdt \\
&= \int_\delta^T \int_{g(t)}^{h(t)} \{r^{N-1}u_t - (dr^{N-1}u_r)_r\}\phi\ drdt + \int_{g_0}^{h_0} r^{N-1}u(\delta,r)\phi(\delta,r)\ dr \\
&+ d\Big\{ \int_\delta^T h(t)^{N-1}u_r(t,h(t))\phi(t,h(t))\ dt - \int_\delta^T g(t)^{N-1}u_r(t,g(t))\phi(t,g(t))\ dt \Big\}.
\end{aligned}$$

Letting $\delta \to 0$, we get from the continuity of u and u_r at $t = 0$

$$\begin{aligned}
&\int_0^T \int_{g(t)}^{h(t)} (dr^{N-1}u_r\phi_r - r^{N-1}u\phi_t)\ drdt \\
&= \int_0^T \int_{g(t)}^{h(t)} \{r^{N-1}u_t - (dr^{N-1}u_r)_r\}\phi\ drdt + \int_{g_0}^{h_0} r^{N-1}u(0,r)\phi_0\ dr \\
&+ d\Big\{ \int_0^T h(t)^{N-1}u_r(t,h(t))\phi(t,h(t))\ dt - \int_0^T g(t)^{N-1}u_r(t,g(t))\phi(t,g(t))\ dt \Big\}.
\end{aligned} \tag{3.62}$$

Comparing (3.61) and (3.62), we obtain

$$\begin{aligned}
&\int_{g_0}^{h_0} r^{N-1}(u(0,r) - u_0)\phi_0\ dr + \frac{d}{\mu}\int_0^T h(t)^{N-1}\phi(t,h(t))(\mu u_r(t,h(t)) + h'(t))\ dt \\
&\qquad - \frac{d}{\mu}\int_0^T g(t)^{N-1}\phi(t,g(t))(\mu u_r(t,g(t)) + g'(t))\ dt = 0.
\end{aligned} \tag{3.63}$$

Taking the support of ϕ in $[0,T) \times (g(t),h(t))$, we find that

$$\int_{g_0}^{h_0} r^{N-1}(u(0,r) - u_0)\phi_0\ dr = 0.$$

This means that $u(0,r) = u_0(r)$ in (g_0, h_0), and

$$\begin{aligned}
&\frac{d}{\mu}\int_0^T h(t)^{N-1}\phi(t,h(t))(\mu u_r(t,h(t)) + h'(t))\ dt \\
&\quad - \frac{d}{\mu}\int_0^T g(t)^{N-1}\phi(t,g(t))(\mu u_r(t,g(t)) + g'(t))\ dt = 0.
\end{aligned}$$

By taking the support of ϕ in $[0,T) \times \{r \in \mathbb{R} |\ 0 \le r < h(t)\}$, it follows from the above identity that

$$g'(t) = -\mu u_r(t, g(t)) \ \text{ for } \ 0 \le t \le T,$$

and hence $h'(t) = -\mu u_r(t, h(t))$ for $0 \le t \le T$. We complete the proof. □

3.11 Generation of singularity

Since $h(t)$ is strictly increasing and $g(t)$ is strictly decreasing, there exist their limits which satisfy

$$\lim_{t\to\infty} g(t) \in [0, g_0), \quad \lim_{t\to\infty} h(t) \in (h_0, \infty].$$

The main purpose of this section is to prove the following dichotomy theorem in (DP).

Theorem 3.22. *Let* (u, g, h) *be any solution of* (DP). *Then either* (i) *or* (ii) *holds true as* $t \to \infty$

(i) $\lim_{t\to\infty} \Omega(t) \subset B_{R^*(t)}$ *for all* $t \ge 0$, $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(g(t),h(t))} = 0$, *where* $R^*(t)$ *is a bounded and continuous and decreasing function with respect to* t.

(ii) *Singularity appears and* $\lim_{t\to\infty} \Omega(t) = \mathbb{R}^N$.

In particular, if $h_0 \ge R^*(0)$, *then part* (ii) *occurs.*

We define $R^*(t)$ in the following. Let λ_1 be the least eigenvalue and let $\phi_1 = \phi_1(r)$ be the corresponding eigenfunction for the problem

$$\begin{cases} -d\Delta\phi_1 = \lambda_1\phi_1, & r \in \Omega, \\ \phi_1 = 0, & r \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}$. It is well known that $\lambda_1 = \lambda_1(d;\Omega)$ is continuous with respect to d and Ω, and $\lambda_1(d;\Omega_1) > \lambda_1(d;\Omega_2)$ if $\Omega_1 \subset \Omega_2$ ($\Omega_1 \ne \Omega_2$). We now replace Ω to $I(t) = (g(t), l)$ with a positive number l. Then we find that $I(t_1) \subset I(t_2)$ for $t_1 < t_2$ because $g(t)$ is decreasing. For each $t \ge 0$, we can uniquely determine some number $l = R^*(d, g(t), f'(0)) =: R^*(t) > 0$ such that

$$f'(0) = \lambda_1(d; (g(t), R^*(t))) \ \text{ and } \ f'(0) > \lambda_1(d; (g(t), l)) \ (l > R^*(t)).$$

Moreover we find that $R^*(t)$ is bounded, continuous and monotone decreasing with respect to t.

To prove Theorem 3.22, we need some propositions and lemmas. We establish an energy identity.

Proposition 3.8. *Let* (u,g,h) *be solutions of* (DP). *Then the following identity holds true*:

$$\begin{aligned}
&\frac{d}{2}\int_{\Omega(t)} r^{N-1}u_r(t,r)^2\ dr + \iint_{\cup_{s=0}^{s=t}\Omega(s)} r^{N-1}u_t(s,r)^2\ drds \\
&\quad + \frac{d}{2\mu^2}\int_0^t h(s)^{N-1}h'(s)^3\ ds - \frac{d}{2\mu^2}\int_0^t g(s)^{N-1}g'(s)^3\ ds \\
&\quad = \int_{\Omega(t)} r^{N-1}F(u(t,r))\ dr - \int_{\Omega_0} r^{N-1}F(u_0(r))\ dr + \frac{d}{2}\int_{\Omega_0} r^{N-1}u_0'(r)^2\ dr,
\end{aligned}$$

where $F(u) = \int_0^u f(s)\ ds$.

Proof. We can show this theorem almost in the same way as Proposition 3.1. Hence we omit details here. □

The following result is a property of vanishing.

Proposition 3.9. *If* $\lim_{t\to\infty} h(t) < \infty$, *then* $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(g(t),h(t))} = 0$.

Proof. The proof is similar to that of Theorem 3.4. We introduce a new function by

$$v(t,y) = u(t,(h(t)-g(t))y + g(t)).$$

Then direct calculation gives

$$\begin{aligned}
u_t &= v_t - \frac{g'(t) + (h'(t)-g'(t))y}{h(t)-g(t)} v_y, \\
u_r &= \frac{1}{h(t)-g(t)} v_y,\ \ u_{rr} = \frac{1}{(h(t)-g(t))^2} v_{yy}.
\end{aligned}$$

Hence v satisfies

$$\begin{cases}
v_t = a(t)v_{yy} + (b(t,y)+c(t,y))v_y + f(v), & t>0,\ 0<y<1, \\
v(t,0) = 0,\ v(t,1) = 0, & t>0, \\
v(0,y) = v_0(y) := u_0((h_0-g_0)y + g_0), & 0 \le y \le 1,
\end{cases} \tag{3.64}$$

where

$$\begin{aligned}
a(t) &= \frac{d}{(h(t)-g(t))^2},\ \ b(t,y) = \frac{g'(t) + (h'(t)-g'(t))y}{h(t)-g(t)}, \\
c(t,y) &= \frac{(N-1)d}{(h(t)-g(t))^2 y + g(t)(h(t)-g(t))}.
\end{aligned}$$

We will prove that $v(t,y)$ converges to a stationary solution as $t\to\infty$. By the assumption that $\lim_{t\to\infty} h(t) < \infty$, Theorem 3.8 implies

$$\int_{g(t)}^{h(t)} r^{N-1}u_r(t,r)^2\ dr + \int_0^\infty U(t)\ dt \le C$$

with some positive constant C independent of t and $U(t) = \int_{g(t)}^{h(t)} r^{N-1}u_t(t,r)^2\ dr$. Hence there exists a sequence $\{t_n\} \nearrow \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} h(t_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} U(t_n) = 0.$$

Since $U(t)$ is uniformly continuous with respect to t (cf. Lemma 3.5), we see $\lim_{t\to\infty} U(t) = 0$. Moreover, noting that $\{v(t,\cdot)|\ t \geq 1\}$ is relatively compact in $C^1[0,1]$ (cf. Lemma 3.4), we find that

$$\lim_{n\to\infty} v(t_n, y) = \hat{v}(y) \quad \text{in} \quad C^1[0,1]$$

for some function $\hat{v}$. In addition, we can prove that $\hat{v}$ satisfies

$$\begin{cases} a_\infty \hat{v}_{yy} + c_\infty(y)\hat{v}_y + \hat{f}(\hat{v}) = 0, & 0 < y < 1, \\ \hat{v}(0) = \hat{v}(1) = 0, \end{cases} \tag{3.65}$$

where $a_\infty = d/(h_\infty - g_\infty)$ and $c_\infty(y) = (N-1)d/\{(h_\infty - g_\infty)^2 y + g_\infty(h_\infty - g_\infty)\}$. Taking account of the Stefan condition, we see from $h'(t_n) = -\mu v_y(t_n, 1)/(h_\infty - g_\infty)$ that $\hat{v}_y(1) = 0$. Hence the strong maximum principle shows $\hat{v} \equiv 0$. Thus

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C(g(t),h(t))} = \lim_{n\to\infty} \|v(t_n,\cdot)\|_{C[0,1]} = 0.$$

The proof is complete. □

We prepare a comparison principle for (DP).

Lemma 3.6. *Let $T_{\max} > 0$ be a maximal existence time of classical solutions for* (DP) *For any $T \in (0, T_{\max})$, suppose that $\underline{g}$, $\underline{h} \in C^1[0,T]$ and $\underline{u} \in C(\overline{D_1(T)}) \cap C^{1,2}(D_1(T))$ with $D_2(T) = \{(t,r) \in \mathbb{R}^2|\ 0 < t \leq T,\ \underline{g}(t) < r < \underline{h}(t)\}$ satisfy*

$$\begin{cases} \underline{u}_t - d\Delta \underline{u} \leq f(\underline{u}), & 0 < t \leq T,\ \underline{g}(t) < r < \underline{h}(t), \\ \underline{u}(t, \underline{g}(t)) = 0,\ \underline{u}(t, \underline{h}(t)) = 0, & 0 < t \leq T, \\ \underline{g}'(t) \geq -\mu \underline{u}_r(t, \underline{g}(t)),\ \underline{h}'(t) \leq -\mu \underline{u}_r(t, \underline{h}(t)), & 0 < t \leq T. \end{cases}$$

Let (u, g, h) be a solution of (DP) *with initial data (u_0, g_0, h_0). If*

$$g_0 \leq \underline{g}(0) \leq \underline{h}(0) \leq h_0, \quad \underline{u}(0,r) \leq u_0(r) \quad \text{for} \quad \underline{g}(0) \leq r \leq \underline{h}(0),$$

then

$$\begin{aligned} &g(t) \leq \underline{g}(t) \leq \underline{h}(t) \leq h(t) \quad \text{for} \quad 0 < t \leq T, \\ &\underline{u}(t,r) \leq u(t,r) \quad \text{for} \quad 0 < t \leq T,\ \underline{g}(t) < r < \underline{h}(t). \end{aligned}$$

We have the following result.

Proposition 3.10. *If $\lim_{t\to\infty} h(t) = \infty$, then there exists some $T \in (0, \infty)$ such that $g(T) = 0$.*

To prove this proposition, we need the following lemma.

Lemma 3.7. *Let a function* $\varphi \in C^2(0,\infty)$ *satisfy*

$$\begin{cases} d\Delta\varphi + f(\varphi) = 0, & 0 < r < \infty, \\ 0 \le \varphi \le 1, & 0 \le r < \infty. \end{cases}$$

Then φ *is represented as*

$$\varphi(r) = \varphi(0) - \frac{1}{d}\int_0^r \frac{1}{z^{N-1}} \int_0^z y^{N-1} f(\varphi(y))\, dydz,$$

where $\varphi(0) := \lim_{r\to 0}\varphi(r)$*. Moreover the function satisfies*

$$\varphi \in C^2[0,\infty) \quad and \quad |\varphi_r(r)| \le \frac{\max_\varphi |f(\varphi)|}{dN} r \quad for \quad r \ge 0.$$

Proof. The function φ also satisfies $d(r^{N-1}\varphi_r)_r + r^{N-1}f(\varphi) = 0$. Hence integrating this equation in (ε, r) for $\varepsilon < r$ implies

$$dr^{N-1}\varphi_r(r) - d\varepsilon^{N-1}\varphi_r(\varepsilon) = -\int_\varepsilon^r s^{N-1} f(\varphi(s))\, ds. \tag{3.66}$$

Since φ is bounded in $[0,\infty)$ and satisfies $\varphi \in C^2(0,\infty)$ and (3.66), the second term in the left-hand side of (3.66) converges to a finite value as $\varepsilon \to 0$. Hence we denote $a := \lim_{\varepsilon\to 0} \varepsilon^{N-1}\varphi_r(\varepsilon) < \infty$. We can show $a = 0$. Indeed, by (3.66), we calculate

$$\begin{aligned} \varphi_r(r) &= \frac{\varepsilon^{N-1}\varphi_r(\varepsilon)}{r^{N-1}} - \frac{1}{dr^{N-1}} \int_\varepsilon^r s^{N-1} f(\varphi(s))\, ds \\ &\to \frac{a}{r^{N-1}} - \frac{1}{dr^{N-1}} \int_0^r s^{N-1} f(\varphi(s))\, ds \end{aligned} \tag{3.67}$$

as $\varepsilon \to 0$. We again integrate the above equation over (ε, r) to obtain

$$\varphi(r) - \varphi(\varepsilon) = a\int_\varepsilon^r \frac{1}{s^{N-1}}\, ds - \frac{1}{d}\int_\varepsilon^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} f(\varphi(s))\, dsdt. \tag{3.68}$$

Note that $\varphi(\varepsilon)$ and the second term of the right-hand side of (3.68) is bounded as $\varepsilon \to 0$, and that

$$\int_\varepsilon^r \frac{1}{s^{N-1}}\, ds = \begin{cases} \log r - \log\varepsilon & (N = 2), \\ \dfrac{1}{2-N}\left(\dfrac{1}{r^{N-2}} - \dfrac{1}{\varepsilon^{N-2}}\right) & (N \ge 3) \end{cases}$$

is not bounded as $\varepsilon \to 0$. Hence, if $a > 0$, then letting $\varepsilon \to 0$ in (3.68) gives a contradiction. We thus obtain $a = 0$, and it follows from (3.67) and (3.68) that

$$\varphi_r(r) = -\frac{1}{dr^{N-1}} \int_0^r s^{N-1} f(\varphi(s))\, ds, \tag{3.69}$$

$$\varphi(r) = \varphi(\varepsilon) - \frac{1}{d}\int_\varepsilon^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} f(\varphi(s))\, dsdt. \tag{3.70}$$

In particular we get by (3.69)

$$|\varphi_r(r)| \le \frac{\max_\varphi |f(\varphi)|}{dr^{N-1}} \int_0^r s^{N-1}\, ds \le \frac{\max_\varphi |f(\varphi)|}{dN} r$$

for $r > 0$. Moreover we can find from (3.70) that

$$\limsup_{\varepsilon\to 0} \varphi(\varepsilon) = \liminf_{\varepsilon\to 0} \varphi(\varepsilon) = \varphi(r) + \frac{1}{d}\int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} f(\varphi(s))\, dsdt.$$

Hence we have

$$\varphi(0) := \lim_{\varepsilon\to 0} \varphi(\varepsilon) = \varphi(r) + \frac{1}{d}\int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} f(\varphi(s))\, dsdt,$$

and it holds that

$$\varphi(r) = \varphi(0) - \frac{1}{d}\int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} f(\varphi(s))\, dsdt.$$

Finally, by calculations, we find $\varphi_r(0) := \lim_{\varepsilon\to 0}\varphi_r(\varepsilon) = 0$ and $\varphi_{rr}(0) := \lim_{\varepsilon\to 0}\varphi_{rr}(\varepsilon) = 0$, and hence $\varphi \in C^2[0,\infty)$. □

Proof of Proposition 3.10. We first assume $g_\infty := \lim_{t\to\infty} g(t) > 0$ to get a contradiction. Since $g(t)$ is decreasing, $\lim_{t\to\infty} g(t) = g_\infty$ and $g'(t)$ is uniformly continuous with respect to t, we have $\lim_{t\to\infty} g'(t) = 0$. On the other hand, since $\lim_{t\to\infty} h(t) = \infty$, there exists some $T > 0$ such that $h(T) > R^*(T)$. Define $l_1 = g(T)$ and $l_2 = h(T)$, and in the same way as the proof of Proposition 3.2, we get

$$\liminf_{t\to\infty} u(t,r) \ge v(r) \quad \text{for} \quad r \in [g_\infty, \infty),$$

where v is a unique positive solution of

$$\begin{cases} d\Delta v + f(v) = 0, & g_\infty < r < \infty, \\ v(g_\infty) = 0. \end{cases} \tag{3.71}$$

Moreover the positive solution $w = w(t,r)$ of

$$\begin{cases} w_t - d\Delta w = f(w), & t > 0,\ g_\infty < r < \infty, \\ w(t, g_\infty) = 0, & t > 0, \\ w(0,r) = \max\{1, \|u_0\|_{C(g_0,h_0)}\} \end{cases}$$

is an upper solution for (DP). Hence

$$\limsup_{t\to\infty} u(t,r) \le \lim_{t\to\infty} w(t,r) = v(r) \quad \text{for} \quad r \in [g_\infty, \infty)$$

by the unique existence of solutions for (3.71) (see Theorem 3.18). Thus we find that $\lim_{t\to\infty} u(t,r) = v(r)$ uniformly in any compact subset of (g_∞, ∞). Note that, by the maximum principle, $v_r(g_\infty) > 0$. Moreover there exists a sequence $\{t_n\} \nearrow \infty$ as $n \to \infty$ to satisfy

$$\lim_{n\to\infty} u_r(t_n, r) = v_r(r) \quad \text{for} \quad g_\infty \le r < \infty.$$

Hence we obtain

$$\lim_{n\to\infty} g'(t_n) = \lim_{n\to\infty} -\mu u_r(t_n, g(t_n)) = -\mu v_r(g_\infty) < 0.$$

This contradicts $\lim_{t\to\infty} g'(t) = 0$.

We next prove that $g(t)$ does not satisfy $g_\infty = 0$. To show this result, we prepare a solution $(w(t,r), s(t), \rho(t))$ of (DP) with initial data $(\varepsilon\varphi(r), g_0, l)$, where ε and l are suitable constants determined later, and φ is an eigenfunction corresponding to the first eigenvalue λ_1 for the problem:

$$\begin{cases} d\Delta\varphi + \lambda_1\varphi = 0, & g_0 < r < l, \\ \varphi(g_0) = \varphi(l) = 0. \end{cases}$$

We can choose sufficiently small ε and large l such that $\phi := \varepsilon\varphi$ satisfies

$$\begin{cases} d\Delta\phi + f(\phi) \ge 0, & g_0 < r < l, \\ \phi(g_0) = \phi(l) = 0. \end{cases}$$

It is possible to choose a constant $T_1 > 0$ such that $\rho(0) = l = h(T_1)$ because we assume $\lim_{t\to\infty} h(t) = \infty$. If we prove $s(T) = 0$ for some $T < \infty$, then Theorem 3.6 shows $g(T + T_1) \le s(T) = 0$. Hence $g(T + T_1) = 0$. Therefore we will prove that free boundary $s(t)$ reaches the origin at finite time. Since we have already found, by the same way as above, that $\lim_{t\to\infty} s(t) > 0$ does not occur, we assume $\lim_{t\to\infty} s(t) = 0$ to get a contradiction. As in Theorem 3.5 for (P), we also obtain the following results:

- $\lim_{t\to\infty} \rho(t) = \infty$,
- $w(t,r)$ is non-decreasing with respect to t for $0 < r < \rho(t)$, (3.72)

where we allow w to satisfy $w(t,r) = 0$ in $[0, g(t))$ for $t \ge 0$. Since w is bounded by Theorem 3.19 and (3.72), there exists a function $\hat{v}$ such that

$$\lim_{t\to\infty} w(t,r) = \hat{v}(r) \text{ in } (0,\infty).$$

Then, as in similar way to the proof of Theorem 3.5, we can show that $\hat{v}$ satisfies

$$d\Delta\hat{v} + f(\hat{v}) = 0, \quad 0 < r < \infty$$

in the sense of distribution. Hence we find that $\hat{v} \in C^2(0,\infty)$ and $\hat{v}$ satisfies

$$\begin{cases} d(r^{N-1}\hat{v}_r)_r + r^{N-1}f(\hat{v}) = 0, & 0 < r < \infty, \\ 0 \le \hat{v} \le 1, & 0 \le r < \infty. \end{cases}$$

Hence we can apply Lemma 3.7 to see that $\hat{v}$ satisfies $\hat{v} \in C^2[0,\infty)$ and

$$\hat{v}(r) = \hat{v}(0) - \frac{1}{d}\int_0^r \frac{1}{z^{N-1}}\int_0^z y^{N-1} f(\hat{v}(y))\, dydz,$$

where $\hat{v}(0) := \lim_{r\to 0}\hat{v}(r)$. Then, by the monotone convergence of $w(t,r)$ to $\hat{v}(r)$ as $t\to\infty$ and $\hat{v}\in C^2[0,\infty)$, we can show that $w(t,r)$ converges to $\hat{v}(r)$ uniformly in any compact subset of $[0,\infty)$ as $t\to\infty$. Hence it follows from $w(t,0)=0$ that $\hat{v}(0)=0$ with $\hat{v}(r)\geq u_0(r)$ in (g_0,h_0), and

$$\hat{v}(r) = -\frac{1}{d}\int_0^r \frac{1}{z^{N-1}}\int_0^z y^{N-1} f(\hat{v}(y))\, dydz.$$

Noting that $f(u)\geq 0$ for $u\in[0,1]$, we deduce from the above identity that $\hat{v}(r)\leq 0$ in $[0,\infty)$. This is a contradiction to $\hat{v}(r)\geq u_0(r)$ in (g_0,h_0). Thus there exists some finite $T>0$ such that $s(T)=0$. Hence we complete the proof. □

To prove Theorem 3.22, we also need the following property.

Lemma 3.8. *If $\lim_{t\to\infty} h(t) < \infty$, then $h(t)\leq R^*(t)$ for all $t\geq 0$.*

Proof. Assume that there exists some $T>0$ satisfying $h(T)>R^*(T)$. Then we consider a solution $w(t,r)$ of the following problem:

$$\begin{cases} w_t - d\Delta w = f(w), & t>T,\ g(T)<r<h(T),\\ w(t,g(t))=0,\ w(t,h(t))=0, & t>T,\\ w(T,r)=u(T,r), & g(T)\leq r\leq h(T). \end{cases}$$

By the standard comparison principle, we see that

$$u(t,r)\geq w(t,r) \ \text{ for } \ t\geq T,\ g(T)\leq r\leq h(T).$$

Since $h(T)>R^*(T)$, letting $t\to\infty$ in the above inequality, we have

$$\liminf_{t\to\infty} u(t,r)\geq q^*(r)>0 \ \text{ for } \ g(T)<r<h(T), \tag{3.73}$$

where $q^*(r)$ is a unique positive solution of

$$\begin{cases} \Delta q^* + f(q^*)=0, & g(T)<r<h(T),\\ q^*(g(T))=q^*(h(T))=0. \end{cases}$$

However this leads to a contradiction. Indeed, by the assumption of $\lim_{t\to\infty} h(t)<\infty$ and Proposition 3.9, the function $u(t,r)$ must satisfy

$$\lim_{t\to\infty}\|u(t,\cdot)\|_{C(g(t),h(t))}=0,$$

which contradicts (3.73). Hence we can conclude $h(t) \le R^*(t)$ for all $t \ge 0$. □

Proof of Theorem 3.22. Since $h(t)$ is strictly increasing, we find that $\lim_{t\to\infty} h(t) < \infty$ or $\lim_{t\to\infty} h(t) = \infty$. In the case that $\lim_{t\to\infty} h(t) < \infty$, it follows from Proposition 3.9 and Lemma 3.8 that

$$\Omega(t) \subset B_{R^*(t)} \text{ for all } t > 0, \quad \lim_{t\to\infty} \|u(t,\cdot)\|_{C(g(t),h(t))} = 0.$$

If $\lim_{t\to\infty} h(t) = \infty$, then, by Proposition 3.10, we can find some finite $T > 0$ such that $g(T) = 0$, and singularity appears. Hence $\lim_{t\to\infty} \Omega(t) = \mathbb{R}^N$. In particular, if $h_0 \ge R^*(0)$, then $h(t) > R^*(t)$ for all $t \ge 0$. Thus Lemma 3.8 implies that singularity (ii) occurs as $t \to \infty$. □

3.12 Remarks on problem in general domain

In this section we will briefly introduce some recent results in multi-dimensions. We consider the following free boundary problem:

$$\begin{cases} u_t - d\Delta u = f(u), & t > 0,\ x \in \Omega(t), \\ u = 0, \quad u_t = \mu|\nabla_x u|^2 & t > 0,\ x \in \Gamma(t) \\ u(0,x) = u_0(x), & x \in \overline{\Omega}_0, \end{cases} \tag{3.74}$$

where d and μ are positive constants, $\Omega(t)$ is an N-dimensional domain in $\mathbb{R}^N$, free boundary $\Gamma(t) := \partial\Omega(t)$ has no fixed boundary, and Ω_0 is a bounded domain in $\mathbb{R}^N$. The initial function satisfies

$$u_0 \in C(\overline{\Omega}_0) \cap H^1(\Omega_0), \quad u_0 > 0 \text{ in } \Omega_0.$$

Moreover f is assumed to be monostable, bistable or combustion type of nonlinearity. In Du-Guo [15] and Du-Matano-Wang [21], they introduced a weak form of (3.74), and get a global existence and uniqueness of weak solutions for (3.74).

For the asymptotic behavior of solutions of (3.74), we have remarkable results. Let $\overline{\text{co}}(\Omega_0)$ denote a closed convex hull of Ω_0, and d_0 is a diameter of Ω_0.

Theorem 3.23 (Du-Matano-Wang [21] Theorem 1.1)**.** *For any fixed $t > 0$, $\tilde\Gamma(t) := \Gamma(t) \setminus \overline{\text{co}}(\Omega_0)$ is a $C^{2+\alpha}$ hypersurface in $\mathbb{R}^N$, and $\tilde\Gamma := \{(t,x) : x \in \tilde\Gamma(t),\ t > 0\}$ is a $C^{2+\alpha}$ hypersurface in $\mathbb{R}^{N+1}$. In particular, the free boundary is always $C^{2+\alpha}$ smooth if Ω_0 is convex.*

Theorem 3.24 ([21] Theorem 1.2)**.** *$\Omega(t)$ is expanding in the sense that $\overline{\Omega}_0 \subset \Omega(t) \subset \Omega(s)$ if $0 < t < s$. Moreover, $\Omega_\infty := \cup_{t>0}\Omega(t)$ is either the entire space $\mathbb{R}^N$, or it is a bounded set. Furthermore, when $\Omega_\infty = \mathbb{R}^N$, for large t, $\Gamma(t)$ is a smooth closed hypersurface in $\mathbb{R}^N$, and there exists a continuous function $M(t)$ such that*

$$\Gamma(t) \subset \left\{x \ :\ M(t) - \frac{d_0}{2}\pi \le |x| \le M(t)\right\};$$

and when Ω_∞ is bounded, $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty}(\Omega(t)) = 0$.

If we assume a logistic nonlinearity, we get more precise information.

Theorem 3.25 ([21] Theorem 1.3). *If $f(u) = au - bu^2$ with a, b positive constants, then there exists $\mu^* \geq 0$ such that $\Omega_\infty = \mathbb{R}^N$ if $\mu > \mu^*$, and Ω_∞ is bounded if $\mu \in (0, \mu^*]$. Moreover, when $\Omega_\infty = \mathbb{R}^N$, the following holds:*

$$\lim_{t\to\infty} \frac{M(t)}{t} = k_0(\mu), \quad \lim_{t\to\infty} \max_{|x|\leq ct} \left| u(t,x) - \frac{a}{b} \right| = 0 \ \ \textit{for all } c \in (0, k_0(\mu)),$$

where $k_0(\mu)$ is a positive increasing function of μ satisfying $\lim_{\mu\to\infty} k_0(\mu) = 2\sqrt{ad}$.

Chapter 4

Spreading speed analysis for a free boundary problem

4.1 Problem

In this chapter we will study propagation speed of the free boundary and sharp asymptotic profiles of solutions for a free boundary problem in one dimension given by

$$\text{(DFBP)} \qquad \begin{cases} u_t - u_{xx} + \beta u_x = f(u), & t > 0,\ g(t) < x < h(t), \\ u(t, g(t)) = 0,\ u(t, h(t)) = 0, & t > 0, \\ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ g(0) = -h_0,\ h(0) = h_0,\ u(0, x) = u_0(x), & -h_0 \le x \le h_0. \end{cases}$$

where μ and h_0 are positive constants, and $\beta \in [0, c_0)$ with some constant $c_0 > 0$ defined later. The first equation in (DFBP) is called a reaction-advection-diffusion equation and β is an advection-term which implies advective environments such as water flow and wind.

In the case of $\beta = 0$, problem (DFBP) becomes

$$\begin{cases} u_t - u_{xx} = f(u), & t > 0,\ g(t) < x < h(t), \\ u(t, g(t)) = 0,\ u(t, h(t)) = 0, & t > 0, \\ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ g(0) = -h_0,\ h(0) = h_0,\ u(0, x) = u_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{4.1}$$

and this problem corresponds to (1.1) with $d = 1$ for Case (1-b). Moreover, as one handles (4.1), we can deal with problem (1.1) for Case (1-a) with Neumann boundary condition (1.3), for which it is possible to get the corresponding results to (4.1). In Du and Lou [20], they have obtained the global existence and uniqueness of solutions for (4.1) and the asymptotic behaviors as $t \to \infty$ when f is a nonlinearity of logistic, bistable or combustion type. They have shown spreading, vanishing and transition phenomenon for large time behaviors, and give sufficient conditions for these phenomenon. Moreover they give propagation speed of the free boundaries; when spreading occurs, there exists a constant $c^* > 0$ such that

$$\lim_{t\to\infty} \frac{g(t)}{t} = -c^*, \quad \lim_{t\to\infty} \frac{h(t)}{t} = c^*.$$

Here **spreading** in (4.1) and (DFBP) means

$$\begin{aligned} &\lim_{t\to\infty} g(t) = -\infty, \quad \lim_{t\to\infty} h(t) = +\infty, \\ &\lim_{t\to\infty} u(t,x) = 1 \ \text{ uniformly in any compact subset of } (-\infty, \infty), \end{aligned} \tag{4.2}$$

which is a special version of the general definition of spreading in Theorem 2.6. For (4.1), Du, Matsuzawa and Zhou [22] have further studied the sharp asymptotic profiles of the free boundaries and the density function, and proved that there exist some constants $\tilde{H}, \tilde{G} \in \mathbb{R}$ such that

$$\begin{aligned} &\lim_{t\to\infty} \sup_{x\in[g(t),0]} |u(t,x) - q^*(x - g(t))| = 0, \\ &\lim_{t\to\infty} \sup_{x\in[0,h(t)]} |u(t,x) - q^*(h(t) - x)| = 0, \\ &\lim_{t\to\infty} (h(t) - c^*t - \tilde{H}) = 0, \ \lim_{t\to\infty} h'(t) = c^*, \\ &\lim_{t\to\infty} (g(t) + c^*t - \tilde{G}) = 0, \ \lim_{t\to\infty} g'(t) = -c^*, \end{aligned} \tag{4.3}$$

where $q^*(x)$ is called *a semi-wave with speed* c^*, which is uniquely determined by the following problem:

$$\begin{cases} q_{xx} - cq_x + f(q) = 0, \quad q > 0 \ \text{ in } \ (0,\infty), \\ q(0) = 0, \ \lim_{x\to\infty} q(x) = 1 \end{cases}$$

with $q_x(0) = c/\mu$ (see [20] for more details).

We will briefly introduce the results on the spreading speed in other cases of (1.1). There are few results on the spreading speed for problem (1.1) with Dirichlet boundary condition (1.2). In Liu and Lou [47], however, they considered (1.1) with a bistable nonlinearity and a Robin boundary condition, and obtained a similar result to (4.3). For multi-dimensional radially symmetric cases, it was shown in Du, Matsuzawa and Zhou [23] that, if the domain is a ball, then there exist constants $c_* > 0$ and $\hat{h} \in \mathbb{R}$ which satisfy

$$\lim_{t\to\infty} (h(t) - c^*t + (N-1)c_* \log t - \hat{h}) = 0, \ \lim_{t\to\infty} h'(t) = c^*.$$

If the domain is an annulus as in Chapter 3, there is no result on the spreading speed. On the other hand, for a general multi-dimensional case, it was shown in Du, Matano and Wang [21] that the propagating speed approaches a constant in some sense.

We intend to consider the case where the diffusion equation is replaced by more general diffusion equations. For example, advection-diffusion equations in (DFBP) are one of interesting extensions; it will be conjectured for (DFBP) that left and right spreading speeds, $\lim_{t\to\infty} g(t)/t$ and $\lim_{t\to\infty} h(t)/t$, are different by an effect of the advective environment. In fact Gu, Lin and Lou [29] have shown that, if $f(u) = u(1-u)$, then there exist constants $c_l^* > 0$ and $c_r^* > 0$ with $c_l^* < c^* < c_r^*$ (c^* is defined below (4.3)) such that

$$\lim_{t\to\infty} \frac{g(t)}{t} = -c_l^*, \quad \lim_{t\to\infty} \frac{h(t)}{t} = c_r^*.$$

We will further investigate the different spreading speeds as $t \to \infty$ and prove much sharper estimate of speeds of the propagation fronts, and show asymptotic profiles of solutions as $t \to \infty$, assuming that f is in Case (A) or Case (B) which we again recall as follows:

$$\text{Case (A)} \begin{cases} f \in C^1[0,\infty),\ f(0) = f(1) = 0,\ f'(0) > 0,\ f'(1) < 0, \\ f(u) > 0\ (0 < u < 1),\ f(u) < 0\ (u > 1); \end{cases}$$

$$\text{Case (B)} \begin{cases} f \in C^1[0,\infty),\ f(0) = f(c^*) = f(1) = 0,\ f'(0) < 0,\ f'(c^*) > 0,\ f'(1) < 0, \\ f(u) < 0\ (0 \le u < c^*,\ u > 1),\ f(u) > 0\ (c^* < u < 1) \text{ and } \int_0^1 f(u)\,du > 0. \end{cases}$$

The main purpose in the present chapter is

- to study the spreading speeds of the free boundaries for this model;
- to investigate more precise asymptotic profiles of solutions (u, g, h);
- to learn more about the effects of the advection term.

To state main results in the next section, we should prepare the definition of speeds c_0, c_l^* and c_r^*. Consider the following problem:

$$\begin{cases} Q_{xx} - cQ_x + f(Q) = 0, \quad Q > 0, \quad \text{in } \mathbb{R}, \\ Q(0) = 1/2, \\ \lim_{x\to-\infty} Q(x) = 0, \quad \lim_{x\to+\infty} Q(x) = 1. \end{cases} \tag{4.4}$$

It is well known (cf. Aronson and Weinberger [3, 4]) that there exists a number $c_0 > 0$ (called **minimal speed**) such that problem (4.4) has a unique **traveling wave solution** Q_c for $c \ge c_0$ when f is a function of Case (A), while (4.4) has a unique solution Q_c only for $c = c_0$ when f is a function of Case (B). Next consider

$$\begin{cases} q_{xx} - (c-\beta)q_x + f(q) = 0, \quad q > 0 \text{ in } (0,\infty), \\ q(0) = 0,\ \lim_{x\to\infty} q(x) = 1. \end{cases} \tag{4.5}$$

$$\begin{cases} q_{xx} - (c+\beta)q_x + f(q) = 0, \quad q > 0 \text{ in } (0,\infty), \\ q(0) = 0,\ \lim_{x\to\infty} q(x) = 1. \end{cases} \tag{4.6}$$

Then we have the following propositions whose proofs are completed with an obvious modifications of [20, Proposition 1.8].

Proposition 4.1. *Suppose that f satisfies Case* (A) *or* (B). *Then the following results hold true*:

(i) *For any $\mu > 0$ there exist a unique constant $c_r^* \in (0, c_0 + \beta)$ and a unique solution q_r^* of* (4.5) *with $c = c_r^*$ such that $(q_r^*)_x(0) = c_r^*/\mu$.*

(ii) *For any $\mu > 0$ there exist a unique constant $c_l^* \in (0, c_0 - \beta)$ and a unique solution q_l^* of* (4.6) *with $c = c_l^*$ such that $(q_l^*)_x(0) = c_l^*/\mu$.*

The solutions q_r^* and q_l^* are called *semi-waves with speeds c_r^* and c_l^** respectively because $v(t,x) := q_r^*(c_r^* t - x)$ and $w(t,x) := q_l^*(x + c_l^* t)$ satisfy

$$\begin{cases} v_t - v_{xx} + \beta v_x = f(v), \quad t \in \mathbb{R},\ x < c_r^* t, \\ v(t, c_r^* t) = 0,\ v_x(t, c_r^* t) = -c_r^*/\mu, \quad t \in \mathbb{R}, \\ \lim_{x \to -\infty} v(t,x) = 1, \quad t \in \mathbb{R}; \end{cases}$$

$$\begin{cases} w_t - w_{xx} + \beta w_x = f(w), \quad t \in \mathbb{R},\ -c_l^* t < x, \\ w(t, -c_l^* t) = 0,\ -\mu w_x(t, -c_l^* t) = -c_l^*, \quad t \in \mathbb{R}, \\ \lim_{x \to +\infty} w(t,x) = 1, \quad t \in \mathbb{R}. \end{cases}$$

In this chapter we always assume that spreading of (4.2) occurs for solutions as $t \to \infty$ because it can actually occur if the solutions has large initial data (u_0, h_0) (this will be proved by the same way as in Chapter 2). In addition, some sufficient conditions have been given in Gu, Lin and Lou [30] if f satisfies Case (A).

We remark that main results and their proofs in this chapter are found in the work of Kaneko and Matsuzawa [37].

4.2 Main theorems

Let (u, g, h) be the solution of (DFBP) for which spreading of (4.2) occurs, and suppose that the nonlinear function f satisfies the conditions of Case (A) or Case (B).

We show, in the following, a result on the spreading speed of the free boundaries and give some rough estimates of them.

Theorem 4.1. *Let c_r^* and c_l^* be positive numbers given by Proposition 4.1. There exists a constant $C > 0$ such that*

$$|g(t) + c_l^* t|,\ |h(t) - c_r^* t| \le C \quad \textit{for all} \quad t > 0,$$

and then it holds that

$$\lim_{t \to \infty} \frac{g(t)}{t} = -c_l^*, \quad \lim_{t \to \infty} \frac{h(t)}{t} = c_r^*.$$

We also give a convergence estimate on the density function u.

Theorem 4.2. *Let c_r^* and c_l^* be positive numbers defined in Proposition 4.1. Then, for any $\varepsilon > 0$, there exist constants $T > 0$, $M > 0$ and $\delta^* \in (0, -f'(1))$ such that the following estimate holds true*:

$$\sup_{x \in [-(c_l^* - \varepsilon)t, (c_r^* - \varepsilon)t]} |u(t,x) - 1| \le M e^{-\delta^* t} \quad \text{for} \quad t \ge T.$$

We will finally show the following theorem which means the sharp asymptotic profiles of $(u(t,x), g(t), h(t))$ as $t \to \infty$.

Theorem 4.3. *Let (c_r^*, q_r^*) and (c_l^*, q_l^*) be given by Proposition 4.1. Then there exist $\hat{H}$, $\hat{G} \in \mathbb{R}$ such that*

$$\begin{aligned}
&\lim_{t\to\infty} \|u(t,\cdot) - q_r^*(h(t) - \cdot)\|_{C^2[0,h(t)]} = 0, \\
&\lim_{t\to\infty} \|u(t,\cdot) - q_l^*(\cdot - g(t))\|_{C^2[g(t),0]} = 0, \\
&\lim_{t\to\infty} (h(t) - c_r^* t - \hat{H}) = 0, \ \lim_{t\to\infty} h'(t) = c_r^*, \\
&\lim_{t\to\infty} (g(t) + c_l^* t - \hat{G}) = 0, \ \lim_{t\to\infty} g'(t) = -c_l^*.
\end{aligned}$$

Remark 4.1. *Theorems 4.1 – 4.3 also hold true when the nonlinear function f is replaced by a combustion type nonlinearity: there exist some $\theta \in (0,1)$ and $\delta_0 > 0$ such that $f \in C^1[0,\infty)$ is non-decreasing in $(\theta, \theta + \delta_0)$ and*

$$f(u) = 0 \quad in \quad [0,\theta], \quad f(u) > 0 \quad in \quad (\theta, 1), \quad f'(1) < 0, \quad f(u) < 0 \quad in \quad [1,\infty).$$

We can refer to [37] for these results.

4.3 Preliminaries

We consider Case (A) or (B) for nonlinear term $f(u)$. A result for the existence and uniqueness of solutions is given as follows.

Proposition 4.2. *For any $\alpha \in (0,1)$ and any constant $T > 0$, problem* (DFBP) *has a unique global solution*

$$(u,g,h) \in \{C^{\frac{(1+\alpha)}{2}, 1+\alpha}(\overline{D(T)}) \cap C^{1+\frac{\alpha}{2}, 2+\alpha}(D(T))\} \times C^{1+\frac{\alpha}{2}}[0,T] \times C^{1+\frac{\alpha}{2}}[0,T],$$

where $D(T) = \bigcup_{0<t\le T}(\{t\} \times (g(t), h(t)))$. Moreover the solution satisfies

$$\begin{aligned}
0 < u(t,x) \le C_1 \quad &in \quad D(T), \\
-\mu C_2 \le g'(t) < 0 < h'(t) \le \mu C_2 \quad &for \quad 0 < t \le T,
\end{aligned}$$

where C_1 and C_2 are positive constants independent of T.

Proof. Although problem (DFBP) has free boundaries on both ends of the one-dimensional interval, we can prove this theorem essentially in the same way as the proofs of Theorems 2.1 and 2.2. We can also refer to [20] (if $\beta = 0$) for more detail. □

We prepare various comparison principles. Their proofs are almost similar to that of Lemma 2.2, and hence we omit the details here.

Lemma 4.1. *For any $T \in (0, \infty)$, suppose that $\overline{g}$, $\overline{h} \in C^1[0, T]$ and $\overline{u} \in C(\overline{D_1(T)}) \cap C^{1,2}(D_1(T))$ with $D_1(T) = \{(t, x) \in \mathbb{R}^2 |\ 0 < t \leq T,\ \overline{g}(t) < x < \overline{h}(t)\}$ satisfy*

$$\begin{cases} \overline{u}_t - \overline{u}_{xx} + \beta\overline{u}_x \geq f(\overline{u}), & 0 < t \leq T,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u}(t, \overline{g}(t)) = 0,\ \overline{u}(t, \overline{h}(t)) = 0, & 0 < t \leq T, \\ \overline{g}'(t) \leq -\mu\overline{u}_x(t, \overline{g}(t)),\ \overline{h}'(t) \geq -\mu\overline{u}_x(t, \overline{h}(t)), & 0 < t \leq T. \end{cases}$$

Let (u, g, h) be a solution of (DFBP) *with initial data $(u_0, -h_0, h_0)$. If*

$$\overline{g}(0) \leq -h_0, \quad h_0 \leq \overline{h}(0), \quad u_0(x) \leq \overline{u}(0, x) \quad \text{for} \quad -h_0 \leq x \leq h_0,$$

then

$$\begin{aligned} &\overline{g}(t) \leq g(t), \quad h(t) \leq \overline{h}(t) \quad \text{for}\ \ 0 < t \leq T, \\ &u(t, x) \leq \overline{u}(t, x) \quad \text{for}\ \ 0 < t \leq T,\ g(t) < x < h(t). \end{aligned}$$

Lemma 4.2. *Suppose that T, $\overline{g}$, $\overline{h}$ and $\overline{u}$ are the same as in Lemma 4.1 and satisfy*

$$\begin{cases} \overline{u}_t - \overline{u}_{xx} + \beta\overline{u}_x \geq f(\overline{u}), & 0 < t \leq T,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u}(t, \overline{h}(t)) = 0, & 0 < t \leq T, \\ \overline{h}'(t) \geq -\mu\overline{u}_x(t, \overline{h}(t)), & 0 < t \leq T. \end{cases}$$

Let (u, g, h) be a solution of (DFBP) *with initial data $(u_0, -h_0, h_0)$. If*

$$\begin{aligned} &g(t) \leq \overline{g}(t),\ u(t, \overline{g}(t)) \leq \overline{u}(t, \overline{g}(t)) \quad \text{for}\ \ 0 \leq t \leq T, \\ &h_0 \leq \overline{h}(0),\ u_0(x) \leq \overline{u}(0, x) \quad \text{for}\ \ \overline{g}(0) \leq x \leq h_0, \end{aligned}$$

then

$$\begin{aligned} &h(t) \leq \overline{h}(t) \quad \text{for}\ \ 0 < t \leq T, \\ &u(t, x) \leq \overline{u}(t, x) \quad \text{for}\ \ 0 < t \leq T,\ \overline{g}(t) < x < h(t). \end{aligned}$$

Lemma 4.3. *For any $T \in (0, \infty)$, suppose that $\underline{g}$, $\underline{h} \in C^1[0, T]$ and $\underline{u} \in C(\overline{D_1(T)}) \cap C^{1,2}(D_1(T))$ with $D_2(T) = \{(t, x) \in \mathbb{R}^2 |\ 0 < t \leq T,\ \underline{g}(t) < x < \underline{h}(t)\}$ satisfy*

$$\begin{cases} \underline{u}_t - \underline{u}_{xx} + \beta\underline{u}_x \leq f(\underline{u}), & 0 < t \leq T,\ \underline{g}(t) < x < \underline{h}(t), \\ \underline{u}(t, \underline{g}(t)) = 0,\ \underline{u}(t, \underline{h}(t)) = 0, & 0 < t \leq T, \\ \underline{g}'(t) \geq -\mu\underline{u}_x(t, \underline{g}(t)),\ \underline{h}'(t) \leq -\mu\underline{u}_x(t, \underline{h}(t)), & 0 < t \leq T. \end{cases}$$

Let (u, g, h) *be a solution of* (DFBP) *with initial data* $(u_0, -h_0, h_0)$*. If*

$$-h_0 \le \underline{g}(0) \le \underline{h}(0) \le h_0, \quad \underline{u}(0, x) \le u_0(x) \quad \text{for} \quad \underline{g}(0) \le x \le \underline{h}(0),$$

then

$$\begin{aligned} &g(t) \le \underline{g}(t) \le \underline{h}(t) \le h(t) \quad \text{for} \quad 0 < t \le T, \\ &\underline{u}(t, x) \le u(t, x) \quad \text{for} \quad 0 < t \le T, \ \underline{g}(t) < x < \underline{h}(t). \end{aligned}$$

Lemma 4.4. *Suppose that* T*,* $\underline{g}$*,* $\underline{h}$ *and* $\underline{u}$ *are the same as in Lemma 4.3 and satisfy*

$$\begin{cases} \underline{u}_t - \underline{u}_{xx} + \beta \underline{u}_x \le f(\underline{u}), & 0 < t \le T, \ \underline{g}(t) < x < \underline{h}(t), \\ \underline{u}(t, \underline{h}(t)) = 0, & 0 < t \le T, \\ \underline{h}'(t) \le -\mu \underline{u}_x(t, \underline{h}(t)), & 0 < t \le T. \end{cases}$$

Let (u, g, h) *be a solution of* (DFBP) *with initial data* $(u_0, -h_0, h_0)$*. If*

$$\begin{aligned} &g(t) \le \underline{g}(t), \ \underline{u}(t, \underline{g}(t)) \le u(t, \underline{g}(t)) \quad \text{for} \quad 0 \le t \le T, \\ &\underline{h}(0) \le h_0, \ \underline{u}(0, x) \le u_0(x) \quad \text{for} \quad \underline{g}(0) \le x \le h_0, \end{aligned}$$

then

$$\begin{aligned} &\underline{h}(t) \le h(t) \quad \text{for} \quad 0 < t \le T, \\ &\underline{u}(t, x) \le u(t, x) \quad \text{for} \quad 0 < t \le T, \ \underline{g}(t) < x < \underline{h}(t). \end{aligned}$$

Remark 4.2. *We can show similar results to Lemmas 4.2 and 4.4, focusing on the other free boundary* $g(t)$*.*

Definition 4.1. *The triple of functions* $(\overline{u}, \overline{g}, \overline{h})$ *given in Lemmas 4.1 and 4.2 is called* **an upper (super-) solution for (DFBP)***. Similarly, the triple of functions* $(\underline{u}, \underline{g}, \underline{h})$ *denoted in Lemmas 4.3 and 4.4 is said to be* **a lower (sub-) solution for (DFBP)***.*

We are now ready to show a key proposition to prove the main theorems. For the proof, we can basically follow an argument in [20] ($\beta = 0$). However, because of presence of the advection term, we need some modification.

Proposition 4.3. *Let* (u, g, h) *be the solution of* (DFBP) *and suppose that spreading occurs for the solution. For any* $c_1 \in (0, c_l^*)$*,* $c_2 \in (0, c_r^*)$ *and* $\delta \in (0, -f'(1))$ *where* c_r^* *and* c_l^* *are positive constants given by Proposition 4.1, there exist positive numbers* T*,* M_1*,* M_2 *and* $\tilde{\delta} = \tilde{\delta}(c_1) \in (0, \delta)$ *such that the following properties hold true for* $t \ge T$ *:*

(i) $[g(t), h(t)] \supset [-c_1 t, c_2 t]$;

(ii) $u(t,x) \geq 1 - M_1 e^{-\tilde{\delta} t}$ *in* $[-c_1 t, c_1 t]$;

(iii) $u(t,x) \leq 1 + M_2 e^{-\delta t}$ *in* $[g(t), h(t)]$.

Proof. (i) We prepare a unique solution $q_l(z;c)$ of

$$\begin{cases} q_{xx} - (c+\beta) q_x + f(q) = 0, \\ q(0) = 0, \ q_x(0) = c_l^*/\mu \end{cases}$$

and a unique solution $q_r(z;c)$ of

$$\begin{cases} q_{xx} - (c-\beta) q_x + f(q) = 0, \\ q(0) = 0, \ q_x(0) = c_r^*/\mu, \end{cases}$$

where μ is the same number as that in (DFBP). It is shown in [37, Proposition 2.4] that, for any $c \in (0, c_l^*)$ there is a unique constant $z_l(c) > 0$ such that $(q_l)_x(z_l(c); c) = 0$ and $(q_l)_x(z;c) > 0$ for $0 \leq z < z_l(c)$. Similarly for any $c \in (0, c_r^*)$, there is a unique constant $z_r(c) > 0$ such that $(q_r)_x(z_r(c); c) = 0$ and $(q_r)_x(z;c) > 0$ for $0 \leq z < z_r(c)$. We set

$$k_l(t;c) := ct + z_l(c), \quad k_r(t;c) := ct + z_r(c)$$

and

$$w(t,x;c_1,c_2) = \begin{cases} q_r(k_r(t;c_2) - x; c_2), & x \in (c_2 t, k_r(t;c_2)], \\ q_r(z_r(c_2); c_2), & x \in (0, c_2 t], \\ q_l(z_l(c_1); c_1), & x \in (-c_1 t, 0], \\ q_l(k_l(t;c_1) + x; c_1), & x \in [-k_l(t;c_1), -c_1 t]. \end{cases}$$

Also we can choose $c_1' \in (c_1, c_l^*)$ and $c_2' \in (c_2, c_r^*)$ close to c_l^* and c_r^* respectively such that

$$q_l(z_l(c_1'); c_1') = q_r(z_r(c_2'); c_2') =: Q, \tag{4.7}$$

which in particular means that $w(t,x;c_1',c_2')$ is continuous at $x = 0$. Choosing a suitably large constant $T_1 > 0$, we can prove that $(w(t,x;c_1',c_2'), -k_l(t;c_1'), k_r(t;c_2'))$ is a lower solution of (DFBP) for $t \geq T_1$ (see the proof of [37, Proposition 3.2] in detail). Hence it follows from Lemma 4.3 that

$$[g(t), h(t)] \supset [-k_l(t-T_1; c_1'), k_r(t-T_1; c_2')] \quad \text{for} \quad t \geq T_1,$$
$$u(t,x) \geq w(t-T_1, x; c_1', c_2') \quad \text{for} \quad t \geq T_1, \ -k_l(t-T_1; c_1') \leq x \leq k_r(t-T_1; c_2').$$

Taking large constant $T_2 > \max\{c_1' T_1/(c_1' - c_1), c_2' T_1/(c_2' - c_2)\}$, we have

$$[-k_l(t-T_1; c_1'), k_r(t-T_1; c_2')] \supset [-c_1 t, c_2 t] \quad \text{for} \quad t \geq T_2.$$

Thus we obtain

$$\begin{aligned}&[g(t), h(t)] \supset [-c_1 t, c_2 t] \quad \text{for} \ \ t \geq T_2,\\ &u(t,x) \geq w(t - T_1, x; c_1', c_2') \quad \text{for} \ \ t \geq T_2, \ -c_1 t \leq x \leq c_2 t.\end{aligned}$$

(ii) For any $\delta \in (0, -f'(1))$, noting the condition of f, we can take a constant $\rho = \rho(\delta) > 0$ which satisfies

$$f(u) \geq \delta(1-u) \ \ (u \in [1-\rho, 1]) \ \ \text{and} \ \ f(u) \leq \delta(1-u) \ \ (u \in [1, 1+\rho]). \tag{4.8}$$

By the proof of (i) and the choice of T_2, it holds that

$$u(t,x) \geq w(t - T_1, x; c_1', c_2') = Q \ \ \text{for} \ \ t \geq T_2, \ -c_1 t \leq x \leq c_2 t,$$

where Q is defined in (4.7). Taking c_1' (resp. c_2') in (4.7) sufficiently close to c_l^* (resp. c_r^*) such that $Q \geq 1 - \rho$, we get

$$1 - \rho < Q \leq u(t,x) \quad \text{for} \quad -c_1 t \leq x \leq c_2 t, \ t \geq T_2.$$

Without loss of generality, we assume $c_1 < c_2$. Fix any $T \geq T_2$ and let $\psi = \psi(t,x)$ be the solution of

$$\begin{cases}\psi_t = \psi_{xx} - \beta\psi_x + \delta(1-\psi), & t > 0, \ -c_1 T < x < c_1 T,\\ \psi(t, -c_1 T) = \psi(t, c_1 T) = Q, & t > 0,\\ \psi(0,x) = Q, & -c_1 T \leq x \leq c_1 T.\end{cases}$$

Then, since $\psi_1 = 1$ is a lower solution and $\psi_2 = Q$ is an upper solution for the above problem, we find $Q \leq \psi(t,x) \leq 1$ for $t > 0$, $-c_1 T < x < c_1 T$. Hence we get $f(\psi) \geq \delta(1-\psi)$ for $t > 0$, $-c_1 T < x < c_1 T$, and the standard comparison principle implies

$$u(t+T, x) \geq \psi(t,x) \quad \text{for} \ \ t > 0, \ -c_1 T \leq x \leq c_1 T.$$

Set $\Psi = (\psi - Q)e^{\delta t}$; and then

$$u(t+T, x) \geq e^{-\delta t}\Psi + Q \quad \text{for} \ \ t > 0, \ -c_1 T \leq x \leq c_1 T \tag{4.9}$$

and Ψ satisfies

$$\begin{cases}\Psi_t = \Psi_{xx} - \beta\Psi_x + \delta(1-Q)e^{\delta t}, & t > 0, \ -c_1 T < x < c_1 T,\\ \Psi(t, -c_1 T) = \Psi(t, c_1 T) = 0, & t > 0,\\ \Psi(0,x) = 0, & -c_1 T \leq x \leq c_1 T.\end{cases}$$

Moreover, using the standard comparison principle, we can check

$$\Psi(t,x) \geq \underline{\Psi}(t,x) := \delta(1-Q)\int_0^t e^{\delta s}\Big(\int_{-c_1 T}^{c_1 T} G_1(t-s, x, z)\ dz\Big)ds \tag{4.10}$$

for $t > 0$, $-c_1T \le x \le c_1T$, where

$$G_1(t-s,x,z) := G(t-s,x-z) - e^{\beta(x-c_1T)}G(t-s,x+z-2c_1T+2\beta(t-s)) - e^{\beta(x+c_1T)}G(t-s,x+z+2c_1T+2\beta(t-s)),$$

$$G(t,x) := \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-\beta t)^2}{4t}}$$

(Note that $\Psi_t \le \Psi_{xx} - \beta\Psi_x + \delta(1-Q)e^{\delta t}$ for $t > 0$, $-c_1T < x < c_1T$, $\underline{\Psi}(t, \pm c_1T) \le 0$ for $t > 0$ and $\underline{\Psi}(0,x) = 0$ for $-c_1T \le x \le c_1T$). We will further estimate $\underline{\Psi}(t,x)$ in $\Omega(\varepsilon)$, where

$$\Omega(\varepsilon) := \left\{(t,x) \in \mathbb{R}^2 |\ |x| \le (1-\varepsilon)c_1T,\ \ 0 < t \le \varepsilon^2 c_1 T\right\}$$

for large $T \ge 4/(c_1(1-\beta\varepsilon)^2)$ and small $\varepsilon < 1/(6\beta)$. It is possible to obtain

$$\underline{\Psi}(t,x) \ge (1-Q)(e^{\delta t}-1)\left(1 - \frac{4}{\sqrt{\pi}}e^{-\frac{1-6\beta\varepsilon}{8}c_1T}\right) \ \text{ in }\ \Omega(\varepsilon) \tag{4.11}$$

(see the proof of [37, Proposition 3.2] in detail). Thus it follows from (4.9), (4.10) and (4.11) that

$$u(t+T,x) \ge 1 - e^{-\delta t} - \frac{4}{\sqrt{\pi}}e^{-\frac{1-6\beta\varepsilon}{8}c_1T} \ \text{ in }\ \Omega(\varepsilon).$$

Taking $t = \varepsilon^2 c_1 T$ and small $\varepsilon > 0$ such that $\delta\varepsilon^2 < (1-6\beta\varepsilon)/8$, we obtain

$$\begin{aligned} u(\varepsilon^2c_1T + T, x) &\ge 1 - e^{-\delta\varepsilon^2 c_1 T} - \frac{4}{\sqrt{\pi}}e^{-\frac{1-6\beta\varepsilon}{8}c_1T} \\ &\ge 1 - M_1 e^{-\delta\varepsilon^2 c_1 T} \end{aligned}$$

for $|x| \le (1-\varepsilon)c_1T$ with $T \ge T_3 := \max\{T_2, 4/(c_1(1-\beta\varepsilon)^2)\}$ and $M_1 := 4/\sqrt{\pi}+1$. We recall that T is an arbitrary large positive number. Then setting

$$t = \varepsilon^2 c_1 T + T \ \ (T = (1+\varepsilon^2 c_1)^{-1}t),$$

we have

$$u(t,x) \ge 1 - M_1 e^{-\tilde{\delta}t} \ \text{ for }\ |x| \le (1-\varepsilon)(1+\varepsilon^2c_1)^{-1}c_1t,$$

where $\tilde{\delta} := \frac{\varepsilon^2 c_1}{1+\varepsilon^2 c_1}\delta$. Letting $\varepsilon \to 0$ in the above inequality, we get

$$u(t,x) \ge 1 - M_1 e^{-\tilde{\delta}t} \ \text{ for }\ t \ge T_3,\ |x| \le c_1 t.$$

(iii) Consider the solution $\overline{u} = \overline{u}(t)$ of

$$\begin{cases} \dfrac{d\overline{u}}{dt} = f(\overline{u}), \quad t > 0, \\ \overline{u}(0) = C_0 > \max\{\|u_0\|_{C[-h_0,h_0]}, 1\}. \end{cases}$$

Then the standard comparison principle shows

$$u(t,x) \le \overline{u}(t) \quad \text{for} \;\; t > 0, \;\; g(t) \le x \le h(t). \tag{4.12}$$

Since $f(u) < 0$ for $u > 1$ and $f(1) = 0$, we can check that $\overline{u}(t)$ is monotone decreasing and converges to 1 as $t \to \infty$. Hence, for the given constant $\rho > 0$ in (4.8), there exists $T_4 > 0$ such that $\overline{u}(t) \le 1 + \rho$ for $t \ge T_4$. Hence we find from (4.8) that $\overline{u}(t)$ satisfies

$$\begin{cases} \dfrac{d\overline{u}}{dt} = f(\overline{u}) \le \delta(1 - \overline{u}), & t > T_4, \\ \overline{u}(T_4) \le 1 + \rho. \end{cases}$$

By direct calculations, we get $\overline{u}(t) \le 1 + M_2 e^{-\delta t}$ for $t \ge T_4$, $g(t) \le x \le h(t)$, where $M_2 = \rho e^{\delta T_4}$. Thus it follows from (4.12) that

$$u(t,x) \le 1 + M_2 e^{-\delta t} \quad \text{for} \;\; t \ge T_4, \;\; g(t) \le x \le h(t).$$

If we define $T = \max\{T_2, T_3, T_4\}$, then we see that the assertions of (i)-(iii) hold for all $t \ge T$. Hence the proof is complete. □

4.4 Proofs of main results

In this section we will prove the main theorems of this chapter.

Proof of Theorem 4.1. We construct sharp upper and lower solutions for (DFBP). In the construction of suitable functions, one can also refer to [22]. By Proposition 4.3, for any $c_1 \in (0, c_l^*)$ and any $\delta \in (0, -f'(1))$, we can choose positive constants T, M_1, M_2 and $\tilde{\delta} \in (0, \delta)$ such that for $t \ge T$

$$\begin{aligned} & g(t) \le -c_1 t, \;\; c_1 t \le h(t), \\ & u(t,x) \ge 1 - M_1 e^{-\tilde{\delta} t} \;\; \text{in} \;\; [-c_1 t, c_1 t], \\ & u(t,x) \le 1 + M_2 e^{-\delta t} \;\; \text{in} \;\; [g(t), h(t)]. \end{aligned}$$

Moreover, since $\delta < -f'(1)$, there exists some $\eta \in (0,1)$ such that

$$f(u) \ge 0 \;\; \text{in} \; [1-\eta, 1], \quad \delta \le -f'(u) \;\; \text{for} \;\; [1-\eta, 1+\eta].$$

We will again take a large number T satisfying

$$\max\{M_1 e^{-\tilde{\delta} T}, M_2 e^{-\delta T}\} < \eta/2. \tag{4.13}$$

For $\sigma > 0$, define a lower solution by

$$\begin{aligned} & \underline{u}(t,x) := (1 - M_1 e^{-\tilde{\delta} t}) q_r^*(\underline{h}(t) - x), \\ & \underline{g}(t) := -c_1 t, \quad \underline{h}(t) := c_r^*(t - T) + c_1 T - \sigma M_1 (e^{-\tilde{\delta} T} - e^{-\tilde{\delta} t}). \end{aligned}$$

Then, choosing sufficiently large $\sigma > 0$, we can check from Lemma 4.4 that

$$\begin{aligned} &\underline{h}(t) \le h(t) \quad \text{for} \quad t \ge T, \\ &\underline{u}(t,x) \le u(t,x) \quad \text{for} \quad t \ge T, \ -c_1 t \le x \le \underline{h}(t) \end{aligned}$$

(see [37, Lemma 3.3] in detail). On the other hand, we take $M_2' > M_2$ satisfying $M_2' e^{-\delta T} < \eta$. Since $q_r^*(x) \to 1$ as $x \to \infty$, we can find $X_0 > 0$ such that $(1 + M_2' e^{-\delta T}) q_r^*(X_0) \ge 1 + M_2 e^{-\delta T}$. We now define an upper solution for (DFBP) by

$$\begin{aligned} &\overline{u}(t,x) := (1 + M_2' e^{-\delta t}) q_r^*(\overline{h}(t) - x), \\ &\overline{g}(t) := g(t), \quad \overline{h}(t) := c_r^*(t - T) + \sigma M_2'(e^{-\delta T} - e^{-\delta t}) + h(T) + X_0, \end{aligned}$$

for some constant $\sigma > 0$. For suitably large $\sigma > 0$, we can get by Lemma 4.2

$$\begin{aligned} &h(t) \le \overline{h}(t) \quad \text{for} \quad t \ge T, \\ &u(t,x) \le \overline{u}(t,x) \quad \text{for} \quad t \ge T, \ g(t) \le x \le h(t) \end{aligned}$$

(cf. [37, Lemma 3.5]). We combine the above estimates to have

$$(c_1 - c_r^*)T - \sigma M_1(e^{-\tilde{\delta} T} - e^{-\tilde{\delta} t}) \le h(t) - c_r^* t \le h(T) - c_r^* T + \sigma M_2'(e^{-\delta T} - e^{-\delta t}) + X_0$$

for all $t \ge T$. Setting

$$\begin{aligned} &C_0 := \max_{0 < t \le T} |h(t) - c_r^* t|, \\ &C_1 := \max\{C_0, \ (c_r^* - c_1)T + \sigma M_1 e^{-\tilde{\delta} T}, \ h(T) - c_r^* T + \sigma M_2' e^{-\delta T} + X_0\}, \end{aligned}$$

we deduce

$$|h(t) - c_r^* t| \le C_1 \quad \text{for} \quad t > 0.$$

In a similar way it follows that

$$|g(t) + c_l^* t| \le C_2 \quad \text{for} \quad t > 0$$

with some $C_2 > 0$. Let $C := \max\{C_1, C_2\}$. Thus we finally obtain

$$|g(t) + c_l^* t|, \ |h(t) - c_r^* t| \le C \quad \text{for} \quad t > 0.$$

The spreading speeds are easily deduced by the above estimates, and the proof is complete. □

Proof of Theorem 4.2. From the proof of Theorem 4.1 we have

$$u(t,x) \ge q_r^*(\underline{h}(t) - x) - M_1 e^{-\tilde{\delta} t} \quad \text{for} \quad t \ge T, \ -c_1 t \le x \le \underline{h}(t),$$

where $\underline{h}(t) = c_r^*(t - T) + c_1 T - \sigma M_1(e^{-\tilde{\delta} T} - e^{-\tilde{\delta} t})$ (T, c_1, M_1 and $\tilde{\delta}$ are the same numbers as in the proof of Theorem 4.1). Fix any constant $\kappa \in (0, c_r^* - c_2)$, and then there exists a constant $T^* \ge T$ such that

$$\underline{h}(t) - x \ge (c_r^* - c_2)t + (c_1 - c_r^*)T - \sigma M_1 \ge \kappa t$$

for $t \geq T^*$, $-c_1 t \leq x \leq c_2 t$, which in particular implies $\underline{h}(t) \geq c_2 t$ for $t \geq T^*$. Noting that $q_r^*(x) \geq 1 - ae^{-bx}$ for $x \geq 0$ for some constants $a, b > 0$, we find that

$$\begin{aligned} u(t,x) &\geq 1 - ae^{-b(\underline{h}(t)-x)t} - M_1 e^{-\tilde{\delta}t} \\ &\geq 1 - ae^{-b\kappa t} - M_1 e^{-\tilde{\delta}t} \\ &\geq 1 - M_1' e^{-\tilde{\delta}' t} \end{aligned}$$

for $t \geq T^*$, $-c_1 t \leq x \leq c_2 t$, where $M_1' = a + M_1$ and $\tilde{\delta}' = \min\{b\kappa, \tilde{\delta}\}$. For any $\varepsilon > 0$, we may take $c_1 = c_l^* - \varepsilon$ and $c_2 = c_r^* - \varepsilon$. Hence the above inequality implies

$$u(t,x) \geq 1 - M_1' e^{-\tilde{\delta}' t} \quad \text{for} \quad t \geq T^*, \ -(c_l^* - \varepsilon)t \leq x \leq (c_r^* - \varepsilon)t. \tag{4.14}$$

On the other hand we find from Proposition 4.3 that

$$u(t,x) \leq 1 + M_2 e^{-\delta t} \quad \text{in} \quad [g(t), h(t)]$$

with some $\delta, M_2 > 0$. By Theorem 4.1, there is some $T^{**} > 0$ such that for given $\varepsilon > 0$

$$g(t) \leq -(c_l^* - \varepsilon)t, \quad (c_r^* - \varepsilon)t \leq h(t) \quad \text{for} \quad t \geq T^{**}.$$

This result gives

$$u(t,x) \leq 1 + M_2 e^{-\delta t} \quad \text{for} \quad t \geq T^{**}, \ -(c_l^* - \varepsilon)t \leq x \leq (c_r^* - \varepsilon)t. \tag{4.15}$$

Hence (4.14) and (4.15) implies the conclusion by letting $T = \max\{T^*, T^{**}\}$, $M = \max\{M_1', M_2\}$ and $\delta^* = \min\{\tilde{\delta}', \delta\}$. The proof is complete. □

We will prepare for the proof of Theorem 4.3. Define $v(t,z) := u(t, z + c_r^* t)$ and $H(t) := h(t) - c_r^* t$. Then (v, H) satisfies

$$\begin{cases} v_t - v_{zz} - (c_r^* - \beta)v_z = f(v), & t > 0, \ g(t) - c_r^* t < z < H(t), \\ v(t, g(t) - c_r^* t) = 0, \ v(t, H(t)) = 0, & t > 0, \\ H'(t) = -\mu v_z(t, H(t)) - c_r^*, & t > 0, \\ H(0) = h_0, \ v(0,z) = u_0(z), & -h_0 \leq z \leq h_0. \end{cases}$$

We also denote $w(t,y) := v(t, y + H(t))$. Then (w, H) satisfies

$$\begin{cases} w_t - w_{yy} - (c_r^* - \beta + H'(t))w_y = f(w), & t > 0, \ g(t) - c_r^* t - H(t) < y < 0, \\ w(t, g(t) - c_r^* t - H(t)) = 0, \ w(t, 0) = 0, & t > 0, \\ H'(t) = -\mu w_y(t, 0) - c_r^*, & t > 0, \\ H(0) = h_0, \ w(0,y) = u_0(y), & -h_0 \leq y \leq h_0. \end{cases}$$

Observe that $H(t)$ and $H'(t) = h'(t) - c_r^*$ is bounded by Proposition 4.2 and Theorem 4.1. Moreover we get the following results.

Lemma 4.5. *For any $b \in \mathbb{R}$, $H(t) - b$ changes its sign at most finitely many times.*

For the proof, see [37, Lemma 3.7]. This lemma is proved by using a zero number argument developed by Angenent [2]. See also Cai, Lou and Zhou [8] for an application of this theory to a free boundary problem.

Proof of Theorem 4.3. By Theorem 4.1, we find that the function $H(t)$ is bounded for all $t \geq 0$. Hence there exist a sequence $\{t_n\} \subset \mathbb{R}$ with $\lim_{n\to\infty} t_n = \infty$ and a constant $\hat{H} \in \mathbb{R}$ such that $H(t_n) \to \hat{H}$ as $n \to \infty$. Assume that there exist another sequence $\{\tilde{t}_n\} \subset \mathbb{R}$ with $\lim_{n\to\infty} \tilde{t}_n = \infty$ and a constant $\hat{H}_1 \neq \hat{H}$ such that $H(\tilde{t}_n) \to \hat{H}_1$ as $n \to \infty$. Then it follows from part (i) of Lemma 4.5 that $H(t) - b$ changes its sign at most finite times for $\min\{\hat{H}_1, \hat{H}\} < b < \max\{\hat{H}_1, \hat{H}\}$. This contradicts $H(\tilde{t}_n) \to \hat{H}$ and $H(\tilde{t}_n) \to H_1$ as $n \to \infty$. Hence $H(t)$ converges to $\hat{H}$ as $t \to \infty$, that is,

$$\lim_{t\to\infty}(h(t) - c_r^* t - \hat{H}) = 0.$$

Moreover suppose that $\lim_{n\to\infty} h'(t_n) \neq c_r^*$ for some sequence $\{t_n\} \subset \mathbb{R}$ with $\lim_{n\to\infty} t_n = \infty$. Then there is a subsequence $\{\tilde{t}_n\}$ with $\lim_{n\to\infty} H(\tilde{t}_n + \cdot\,) = \hat{H}$ in $C^1_{loc}(\mathbb{R})$ (cf. [37, Proposition 3.9]). Hence we see that $\lim_{n\to\infty} H'(\tilde{t}_n) = 0$ and $\lim_{n\to\infty} h'(\tilde{t}_n) = c_r^*$. This is a contradiction, and consequently $\lim_{t\to\infty} h'(t) = c_r^*$.

It remains to prove the convergence of u to the semi-waves as $t \to \infty$. First we get the following results by [37, Proposition 3.8]; for any constants $K, L > 0$

$$\begin{aligned} &\lim_{t\to\infty} \|w(t,\,\cdot\,) - q_r^*(-\,\cdot\,)\|_{C^2[-K,0]} = 0, \\ &\lim_{t\to\infty} \|v(t,\,\cdot\,) - q_r^*(\hat{H} - \,\cdot\,)\|_{C^2[-c_r^* t, \hat{H}-L]} = 0 \end{aligned} \tag{4.16}$$

(These properties are proved by the parabolic estimates and some important arguments used in Du, Matsuzawa and Zhou [23]). We now fix some K and some $L > 0$ to satisfy

$$[0, \hat{H} + c_r^* t - L] \cup [h(t) - K, h(t)] = [0, h(t)] \tag{4.17}$$

for sufficiently large $t > 0$. It is actually possible to take such constants; since $H(t) = h(t) - c_r^* t$ is bounded, there are constants K, $L > 0$ such that $\hat{H} + K > L + \sup_{t\geq 0} H(t)$, and hence we have $\hat{H} + c_r^* t - L > h(t) - K$. Using (4.16), for any $\varepsilon > 0$ we find a constant $T_0 > 0$ such that

$$\begin{aligned} &\hat{H} - \varepsilon \leq H(t) \leq \hat{H} + \varepsilon, \\ &\|u(t,\,\cdot\,) - q_r^*(h(t) - \,\cdot\,)\|_{C^2[h(t)-K,h(t)]} \leq \varepsilon, \\ &\|u(t,\,\cdot\,) - q_r^*(\hat{H} + c_r^* t - \,\cdot\,)\|_{C^2[0,\hat{H}+c_r^* t-L]} \leq \varepsilon \end{aligned}$$

for $t \geq T_0$. Then it follows from the above inequality that

$$\begin{aligned} \|u(t,\,\cdot) - q_r^*(h(t) - \,\cdot\,)\|_{C^2[0,\hat{H}+c_r^* t-L]} &\leq \|u(t,\,\cdot\,) - q_r^*(\hat{H} + c_r^* t - \,\cdot\,)\|_{C^2[0,\hat{H}+c_r^* t-L]} \\ &\quad + \|q_r^*(\hat{H} + c_r^* t - \,\cdot\,) - q_r^*(h(t) - \,\cdot\,)\|_{C^2[0,\hat{H}+c_r^* t-L]} \\ &\leq (1 + C\|q_r^*\|_{C^2[0,\infty)})\varepsilon \end{aligned}$$

for $t \geq T_0$ and some constant $C > 0$ independent of ε. Hence it holds for $t \geq T_0$ that

$$\begin{aligned}\|u(t,\,\cdot\,) - q_r^*(h(t) -\,\cdot\,)\|_{C^2[0,h(t)]} &\leq \|u(t,\,\cdot) - q_r^*(h(t) -\,\cdot\,)\|_{C^2[0,\hat{H}+c_r^*t-L]} \\ &\quad + \|u(t,\,\cdot\,) - q_r^*(h(t) -\,\cdot\,)\|_{C^2[h(t)-K,h(t)]} \\ &\leq (2 + C\|q_r^*\|_{C^2[0,\infty)})\varepsilon.\end{aligned}$$

Thus we obtain

$$\limsup_{t\to\infty} \|u(t,\,\cdot\,) - q_r^*(h(t) -\,\cdot\,)\|_{C^2[0,h(t)]} \leq (2 + C\|q_r^*\|_{C^2[0,\infty)})\varepsilon.$$

Since ε is an arbitrary small positive number, we conclude

$$\lim_{t\to\infty} \|u(t,\,\cdot\,) - q_r^*(h(t) -\,\cdot\,)\|_{C^2[0,h(t)]} = 0.$$

We can prove in a similar way above

$$\begin{aligned}&\lim_{t\to\infty} \|u(t,\cdot) - q_l^*(\cdot - g(t))\|_{C^2[g(t),0]} = 0,\\ &\lim_{t\to\infty} (g(t) + c_l^* t - \hat{G}) = 0,\ \lim_{t\to\infty} g'(t) = -c_l^*.\end{aligned}$$

We complete the proof. □

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