Applications of the non-commutative Specker phenomenon
スペッカー現象の応用

February 2015

Jun NAKAMURA
中村　順

Applications of the non-commutative Specker phenomenon
スペッカー現象の応用

February 2015

Waseda University
Graduate School of Fundamental Science and Engineering
Major in Mathematics, Research on Set Theory

Jun NAKAMURA
中村　順

Contents

1. Preface

This thesis demonstrates applications of non-commutative Specker phenomenon to set theory and group theory. Non-commutative Specker phenomenon plays an important role in wild topology. More precisely, the notion "n-slender group" is a central notion in non-commutative Specker phenomenon and it plays central roles in studies of the fundamental groups of wild spaces, detecting points in groups. Historically n-slender groups were introduced as the non-commutative version of slender groups and it is known that an abelian group A is n-slender if and only if A is slender ([**8**] Theorem 3.3). There is a nice characterization of slender groups by J. Nunke (see [**15**],[**17**]), which implies that any countable torsion-free abelian group containing no rational group $\mathbb{Q}$ is slender. The main result in this thesis is that certain HNN extensions and amalgamated free product preserve the atomic property of the Hawaiian earring group. Roughly speaking, the atomic property is the property that the group is hard to be decomposed to non-trivial free products. As corollaries to the main results we prove that any surface group except real projective plane and Baumslag-Solitar group are n-slender. Free groups are basic examples of n-slender groups and in this thesis we shows other examples of n-slender groups, which are known in combinatorial group theory.

Apart from the main results we demonstrate results of the non-commutative Specker phenomenon respect to cardinal invariants and also in uncountable cases. The commutative case of cardinal invariants has already been studied by K. Eda, S. Kamo, A. Blass and J. Brendle, S. Shelah. Specker-Eda number $\mathfrak{se}$ is introduced by A. Blass [**2**] and is finally shown that $\mathfrak{se} = \mathfrak{e}_{\mathfrak{l}}$ =min$\{\mathfrak{e}, \mathfrak{b}\}$ by J. Brendle and S. Shelah [**3**], where $\mathfrak{e}$ and $\mathfrak{b}$ are known cardinal invariants defined combinatorially. We introduce non-commutative Specker-Eda number $\mathfrak{se}_{\mathfrak{nc}}$ in the same way as the commutative case and prove $\mathfrak{se}_{\mathfrak{nc}} = \mathfrak{se}$. Actually the proof of this equality is done by reducing a problem to the commutative case. Uncountable Specker phenomenon also have been studied around 1955. The well-known result is that, if κ is less than the least measurable cardinal, then $\mathbb{Z}^\kappa$ exhibits Specker phenomenon by J. Łoś and E. C. Zeeman [**25**] independently. After then it was generalized to arbitrary cardinality by K. Eda [**7, 15**] in 1983.

On the other hand, the non-commutative version to this direction is two-fold. Specker phenomenon does not occur in the first uncountable cardinal by S. Shelah and L. Strüngmann [**23**] for complete free products. But, the unrestricted free products satisfies the non-commutative version of the Łoś-Eda theorem [**15**], which was proved by K. Eda and S. Shelah [**14**]. When we study the case over measurable cardinals, in a non-commutative version of ultraproducts, homogeneous elements and the subgroup consisting of them play an important role. We show some basic results of it. One of them is that, $H^* = \{W \in \times_{n<\omega} \mathbb{Z}_n | W \text{ is homogeneous}\}$ is n-slender.

In this thesis we demonstrate results which show relationships between set theory and combinatorial group theory. Definitions and basics come from concepts of set theory, but essential ideas in proofs lie in finite combinatorics of group theory. We also use basic techniques in set theory for cardinal invariants or uncountable Specker phenomenon.

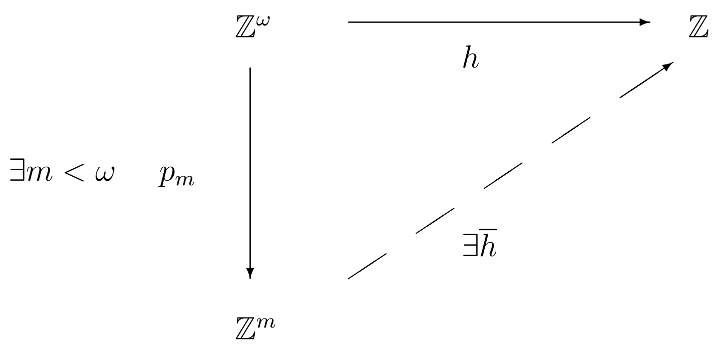
In the introduction we summarize basic techniques of the non-commutative Specker phenomenon and in the following sections we state our results.

2. Introductions of Specker phenomenon

In this section, we introduce Specker phenomenon and its applications to wild spaces.

2.1. The commutative case. E. Specker showed $\mathbb{Z}$ satisfies the following diagram, which is the best example of Specker phenomenon.

$h : \mathbb{Z}^\omega \to \mathbb{Z}$ a homomorphism.



$h = \overline{h} \circ p_m \quad p_m$:projection. $\quad \overline{h}(x) = \sum_{i=0}^{m-1} x(i)h(e_i)$

Specker's theorem says that for any homomorphism h, its value $h(x)$ is determined by only finite components of x. Let e_i is the element of $\mathbb{Z}^\omega$ which i-th component is 1, other components are all zero. We can represent an elements x of $\mathbb{Z}^\omega$ for $m < \omega$ as,

$$x = \sum_{i<\omega} x(i)e_i = \sum_{i=0}^{m-1} x(i)e_i + \sum_{m \le i < \omega} x(i)e_i.$$

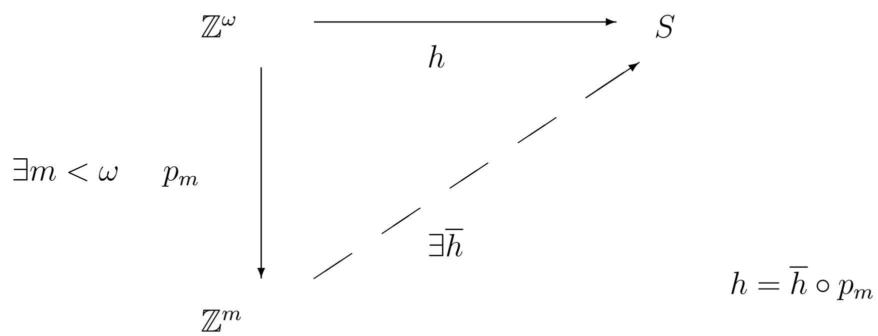
Then, $h(x)$ is represented,

$$h(x) = h(\sum_{i=0}^{m-1} x(i)e_i) + h(\sum_{m \le i < \omega} x(i)e_i) = \sum_{i=0}^{m-1} x(i)h(e_i) + h(\sum_{m \le i < \omega} x(i)e_i).$$

Specker's theorem says that $\sum_{m \le i < \omega} x(i)e_i$ is in $\mathrm{Ker}(h)$, we have $h(x) = \sum_{i=0}^{m-1} x(i)h(e_i)$.

We introduce the slenderness, which is based on Specker phenomenon. The slenderness was introduced by J. Łoś. An abelian group S is slender, if S satisfies the following diagram.

$h : \mathbb{Z}^\omega \to S$ a homomorphism.

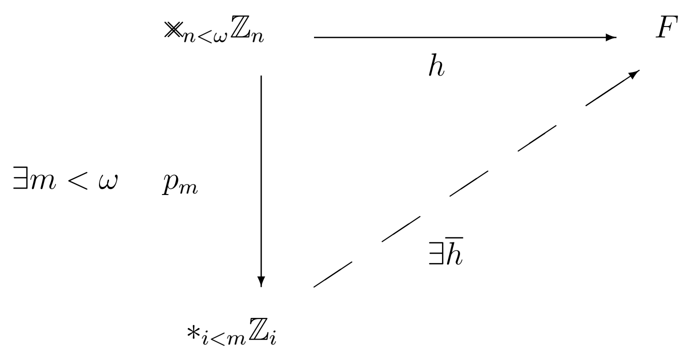


We can say that, a slender group S satisfies Specker's theorem. $\mathbb{Z}$ is a typical example of slender groups. Slender groups have the good characterization proved by R.J. Nunke ([**15**]).

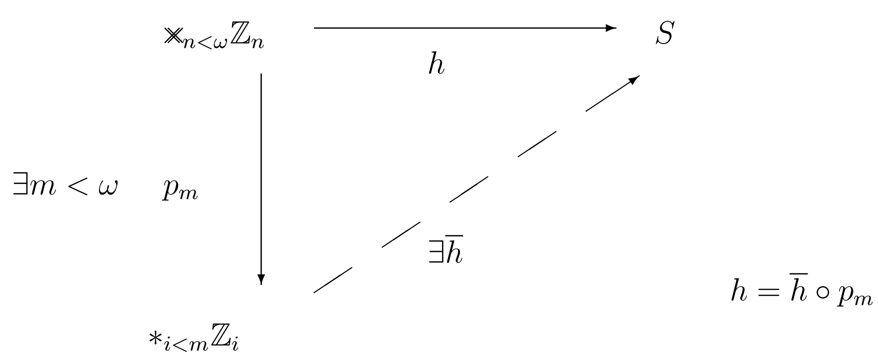
THEOREM 2.1. *An abelian group is slender if and only if, it is torsion-free and contains no copy of* $\mathbb{Q}, \mathbb{Z}^\omega$*, or p-adic integer group* $\mathbb{J}_p$ *for any prime p.*

2.2. The non-commutative case. Now, we introduce the non-commutative Specker phenomenon. G. Higman showed the next diagram, which is called the non-commutative Specker's theorem.

Let F be a free group and $h : \circledast_{n<\omega}\mathbb{Z}_n \to F$ a homomorphism.



N-slenderness was introduced by K. Eda in 1992. A group S is n-slender if, G satisfies the following diagram.



An n-slender group satisfies non-commutative Specker's theorem. $\mathbb{Z}$ is also a good example of n-slender groups. The next theorem means that the n-slenderness is a generalization about the slenderness to non-commutative groups [**8**] .

Theorem 2.2. *Let A be an abelian group. A is slender if and only if, A is n-slender.*

The n-slenderness also is preserved by the restricted direct products and free products, which is a generalization that the slenderness is preserved by direct sums [**17**], [**8**] .

Theorem 2.3. *Let $G_i (i \in I)$ be n-slender. Then, the free product $*_{i\in I} G_i$ and the restricted direct product $\prod^r_{i\in I} G_i = \{x \in \prod_{i\in I} G_i | \{i \in I | x(i) \neq e\}$ is finite $\}$ are n-slender.*

There is a characterization of n-slender groups using fundamental groups [**8**].

Theorem 2.4. *$\pi_1(X, x)$ is n-slender if and only if, for any homomorphism $h : \pi_1(\mathbb{H}, o) \to \pi_1(X, x)$, there exists a continuous map $f : (\mathbb{H}, o) \to (X, x)$ such that $h = f_*$ where f_* is the induced homomorphism.*

We can rephrase Higman's theorem in topological terms as follows: Let h be a homomorphism from $\pi_1(\mathbb{H}, o)$ to $\pi_1(\mathbb{S}^1)$. Then, there exists a continuous map $f : \mathbb{H} \to \mathbb{S}^1$ such that $h = f_*$.

Many things about wild algebraic topology can be reduced to the Hawaiian earring and how the homomorphic image of the fundamental group of the Hawaiian earring can detect a point in the space in question. It is due to the non-commutative Specker phenomenon. the following two theorems are examples [**10**], [**11**].

Theorem 2.5. *Let X and Y be a one-dimensional Peano continua which are not semi-locally simply connected at any point. Then, X and Y are homeomorphic if and only if, the fundamental groups of X and Y are isomorphic.*

Theorem 2.6. *Let X and Y be one-dimensional Peano continua. If the fundamental groups of X and Y are isomorphic, then X and Y are homotopy equivalent.*

3. Atomic properties of the Hawaiian earring group

The atomic property of the Hawaiian earring group and n-slenderness of free groups play central roles in the study of the fundamental groups of wild spaces and according to them certain spaces are recovered from their fundamental groups [**9, 11, 6**]. In addition, the fundamental groups of wild Peano continua also have the atomic property for free products with injective homomorphisms [**11**].

In the present thesis we will show the the atomic property for certain HNN extensions and will show that such HNN extensions preserve n-slenderness using it. We will also show the atomic property and the preservation of n-slenderness for certain amalgamated free products.

The atomic property for HNN extensions means that for any homomorphism h from $\times_{n<\omega}\mathbb{Z}_n$ to an HNN extension of a group G, $G^* = \langle G, t | tAt^{-1} = B\rangle$, there exists a natural number N such that $h[\times_{n\geq N}\mathbb{Z}_n]$ is contained in a conjugate subgroup to G. In other words, "almost all" of a homomorphic image is contained in a subgroup conjugate to the base group. For amalgamated free products we also introduce the atomic property as follows: for any homomorphism h from the the Hawaiian earring group $\times_{n<\omega}\mathbb{Z}_n$ to an amalgamated free product $A \underset{U}{*} B$, there exists a natural number N such that $h[\times_{n\geq N}\mathbb{Z}_n]$ is contained in a conjugate subgroup to A or B. In this thesis, we prove that any HNN extension with a certain condition has the atomic property and apply it to prove n-slenderness of the Baumslag-Solitar groups.

The property of n-slenderness which is a non-commutative version of slenderness of abelian groups was introduced by K. Eda in 1992 [**8**]. The class of slender groups is known as a remarkable class of torsion-free abelian groups and a nice characterization of it due to J. Nunke is known (see [**15**],[**17**]). The property of n-slenderness is related to the fundamental group of the Hawaiian earring which is the plane continuum $\mathbb{H} = \bigcup_{n<\omega}\{(x, y) \mid (x - \frac{1}{n+1})^2 + y^2 = \frac{1}{(n+1)^2}\}$, the union of countably many circles in the Euclidean plane. It is known that the Hawaiian earring group is isomorphic to $\times_{n<\omega}\mathbb{Z}_n$ where $\mathbb{Z}_n$ is a copy of the integer group $\mathbb{Z}$. A group S is n-slender[1] if, for any homomorphism h from $\times_{n<\omega}\mathbb{Z}_n$ to S, $h(\delta_n)$ is identity for all but finitely many $n < \omega$ where δ_n is the generator of $\mathbb{Z}_n$. It is proved that an abelian group A is n-slender, if and only if A is slender [**8**]. There

[1] There is a straightforward generalization of slenderness to non-commutative groups G [**18**]. But it depends on only abelian subgroups of G. The property of n-slenderness is essentially different from it.

is a topological characterization of n-slender groups, that is, G is n-slender if and only if, for any pointed space (X, x) with $\pi_1(X, x) \cong G$ and for any homomorphism $h : \pi_1(\mathbb{H}, b^*) \to \pi_1(X, x)$, there exists a continuous map $f : (\mathbb{H}, b^*) \to (X, x)$ such that $h = f_*$ where f_* is a naturally induced homomorphism. There are some theorems about wild algebraic topology of one-dimensional spaces which are reduced to the Hawaiian earring group. The property of n-slenderness is a supporting concept in these theorems (see [**8, 9, 10**]).

There are many questions left about n-slenderness. Since every torsion-free abelian group of finite rank, in particular finitely generated, is slender, it is natural to question whether every finitely generated group is n-slender. G. R. Conner advised us on this problem and we find that there exits a torsion-free finitely presented group which contains a subgroup isomorphic to $\mathbb{Q}$. It is due to the two famous theorems. One of them is that every countable group C can be embedded in a group G generated by two elements of infinite order and the other is Higman's embedding theorem, that is, a finitely generated group G can be embedded in some finitely presented group if and only if G can be recursively presented (see [**22**, Chap. IV, Theorems 3.1 and 7.1]). By the first theorem, we obtain a two-generated recursively presented group G_0 which contains $\mathbb{Q}$ because $\mathbb{Q}$ is recursively presented and the construction of this theorem preserves it. Then we obtain a finitely presented group containing G_0 by the Higman's embedding theorem. In proof of above two theorems, the desired groups are constructed by amalgamated free products or HNN extensions. Since these constructions do not add a new torsion element, we obtain the counterexample as above.

Now, we conjecture that any one-relator torsion-free group is n-slender. It is related to whether HNN extensions preserve n-slenderness because any one-relator group is embedded in a HNN extension of the one-relator group which has a shorter length than the given one [**22**]. This thesis is a first step to investigate the atomic property for HNN extensions and to this conjecture.

3.1. Definitions and Basics.

This section is devoted to introduce the basics of infinitary words and HNN extensions. We follow the notation in [**8**] and [**22**], but state definitions and propositions for the reader's convenience. Sometimes, we shall abuse the notation of infinitary words and HNN extensions without confusion.

Definition 3.1. Let G_i $(i \in I)$ be groups such that $G_i \cap G_j = \{e\}$ for any $i \neq j \in I$. We call elements of $\bigcup_{i\in I} G_i \setminus \{e\}$ letters. A word W is a function $W : \overline{W} \to \bigcup_{i\in I} G_i \setminus \{e\}$ where $\overline{W}$ is a linearly ordered set and $\{\alpha \in \overline{W} \mid W(\alpha) \in G_i\}$ is finite for any $i \in I$. The class of all words is denoted by $\mathcal{W}(G_i : i \in I)$.(abbreviated by $\mathcal{W}$)

Definition 3.2. Let U,V be words. U and V are isomorphic, which is denoted by $U \equiv V$, if there exists an order isomorphism $\varphi : \overline{U} \to \overline{V}$ such that $\forall \alpha \in \overline{U}$ $(U(\alpha) = V(\varphi(\alpha))$. It is easily seen that $\mathcal{W}$ becomes a set under this identification. Consider the cardinality of a domain of a word W, it is greater than of equal to $\mathrm{Max}\{|I|, \omega\} = \kappa$ because elements of G_i appear only finitely in range of W for any $i \in I$. Then we restrict a domain of a word function to subsets of κ. Thus, $\mathcal{W}$ becomes a set under this identification.

Definition 3.3. For a subset $X \subseteq I$, the restricted word W_X of W is given by the function

$$W_X : \overline{W_X} \ \to \ \bigcup_{i\in I} G_i \ \text{ where } \overline{W_X} = \{\alpha \in \overline{W} | W(\alpha) \in \bigcup_{i\in X} G_i\} \ \text{ and}$$

$W_X(\alpha) = W(\alpha)$ for all $\alpha \in \overline{W_X}$. Hence $W_X \in \mathcal{W}$. If X is finite, then we can regard W_X as an element of the free product $*_{i\in X} G_i$.

Definition 3.4. Let U,V be words. U and V are equivalent, which is denoted by $U \sim V$, if $U_F = V_F$ for all $F \subset\subset I$ where we regard U_F and V_F as elements of the free product $*_{i\in F} G_i$.
So, "$U_F = V_F$" means that they are equal in the sense of the free product $*_{i\in F} G_i$.

Let $[W]$ be the equivalence class of a word W. The composition of two words and the inverse of a word are defined naturally. Thus $\mathcal{W}/\sim = \{[W] \mid W \in \mathcal{W}\}$ becomes a group.

Definition 3.5. $⋇_{i\in I} G_i$ is the group $\mathcal{W}(G_i : i \in I)/\sim$. Clearly, if I is finite, then $⋇_{i\in I} G_i$ is isomorphic to the free product $*_{i\in I} G_i$.

Definition 3.6. W is reduced if $W \equiv UXV$ implies $[X] \neq e$ for any non-empty word X where e is the identity, and for any contiguous elements α and β of $\overline{W}$, it never occurs that $W(\alpha)$ and $W(\beta)$ belong to the same G_i.

Definition 3.7. $l_i(W)$ is the cardinality of $\{\alpha \in X \mid X(\alpha) \in G_i\}$ where X is the reduced word of W.

Lemma 3.8. ([**8**, Theorem 1.4])*For any word W, there exists a reduced word V such that $[W] = [V]$ and V is unique up to isomorphism.*

Proposition 3.9. ([**8**, Proposition 1.9]) *If $g_\lambda (\lambda \in \Lambda)$ are elements of $⋇_{i\in I} G_i$ such that $\{\lambda \in \Lambda \mid l_i(g_\lambda) \neq 0\}$ are finite for all $i \in I$, then there exists a natural homomorphism $\varphi : ⋇_{\lambda\in\Lambda} \mathbb{Z}_\lambda \to ⋇_{i\in I} G_i$ via $\delta_\lambda \mapsto g_\lambda$ $(\lambda \in \Lambda)$ where $\mathbb{Z}_\lambda (\lambda \in \Lambda)$ are copies of the integer group and δ_λ is the generator of $\mathbb{Z}_\lambda$.*

Definition 3.10. Let G be a group, A and B be subgroups of G with $\varphi : A \to B$ an isomorphism. The HNN extension of G relative to A,B and φ is the group $G^* =< G, t | tat^{-1} = \varphi(a), a \in A >$.

We call the group G is the base group of G^* and denote $G^* =< G, t | tAt^{-1} = B >$ without confusion. In this thesis, G^* always denotes an HNN extension of G.

Definition 3.11. [**22**, p.181] A sequence alternate with t,t^{-1} and elements of G, $g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n$ $(n \geq 0,\ g_i \in G,\ \epsilon_i = \pm 1)$ is reduced if there is no subsequence $t g_i t^{-1}$ with $g_i \in A$ or $t^{-1} g_i t$ with $g_i \in B$.

Clearly, any element of G^* is represented by some reduced sequence. Since this notion is very different from the reducedness of words for free products, we explain it by an example. Suppose that $tat^{-1} = b$. We remark that the both sequences $1, t, b^{-1}, t, a, 1$ and $1, t, 1, t, 1$ are reduced and represent the same element of G^*. The following lemma is basic for reducedness of HNN extension.

Proposition 3.12. (*Britton's Lemma* [**22**, p.181]) *If the sequence* $g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n$ *and* $n \geq 1$, *then* $g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n$ *is not the identity of* G^*

Definition 3.13. For $w \in G^*$, Let $l(w)$ be the length of a reduced sequence which represents w.

The well-definedness of the length is due [**22**, Chap. IV, Lemma 2.3]. We remark the difference between free product and HNN extension about the lengths of words. In a free product, $l(w) = 0$ means w is the empty word. On the other hand, in an HNN extension, $l(w) = 0$ means w is an element of G, so w could be non-trivial.

Definition 3.14. An element $w = g_0 t^{\epsilon_1} \cdots t^{\epsilon_n}$ is cyclically reduced if all cyclic permutations of the sequence are reduced.

If $n \geq 1$, it is equivalent to $w^2 = g_0 t^{\epsilon_1} \cdots t^{\epsilon_n} g_0 t^{\epsilon_1} \cdots t^{\epsilon_n}$ is reduced. We note that any element of G is cyclically reduced for the case $n = 0$.

Proposition 3.15. *Any element of* G^* *is conjugate to some* $g \in G$ *or cyclically reduced element whose length is greater than* 0.

We omit proof because it is due to the fact, every element of G^* is conjugate to a cyclically reduced element, which is used to prove the Torsion Theorem for HNN Extensions [**22**, Chap. IV,Theorem 2.4].

Definition 3.16. Let C_G be the set of all elements of G^* which is conjugate to an element of G.

We remark that $C_G = \bigcup_{w \in G^*} wGw^{-1}$ is closed under conjugacy and $l(x^n) \leq l(x)$ for any $n \geq 2$ and $x \in C_G$. In addition, if $x \notin C_G$, then x is conjugate to some cyclically reduced element with non-zero length and $l(x), n < l(x^n)$ for any $n \geq 2$.

Proposition 3.17. *Let* $w = g_0 t^{\epsilon_1} \cdots t^{\epsilon_n} g_n t^{\epsilon_{n+1}} \cdots t^{\epsilon_{2n}} g_{2n}$ *be reduced. Then* $w \in C_G$ *if and only if there exists* $x \in A \cup B$ *such that* $g_0 t^{\epsilon_1} \cdots t^{\epsilon_n} = (t^{\epsilon_{n+1}} \cdots t^{\epsilon_{2n}} g_{2n})^{-1} x$

PROOF. We prove it by the induction of n. The case $n = 0$ is clear. Let consider the case of $n+1$. The necessity is easy, we only argue the sufficiency. Let $w = g_0 t^{\epsilon_1}(g_1 t^{\epsilon_2} \cdots t^{\epsilon_n} g_n t^{\epsilon_{n+1}} \cdots t^{\epsilon_{2n+1}} g_{2n+1}) t^{\epsilon_{2n+2}} g_{2n+2} \in C_G$. We consider the sub word $t^{\epsilon_{2n+2}}(g_{2n+2}g_0)t^{\epsilon_1}$ of the second term. If $\epsilon_1 = \epsilon_{2n+2}$, we can easily conclude that $l(w^2) > l(w)$ which contradicts to $w \in C_G$. We obtain $\epsilon_1 = -\epsilon_{2n+2}$ and can assume $\epsilon_1 = -1$. If $g_{2n+2}g_0 \notin A$, then w is conjugate to a cyclically reduced element, which is a contradiction. We find out there exists $b \in B$ such that $t(g_{2n+2}g_0)t^{-1} = b$. It implies $g_0 t^{-1} = g_{2n+2}^{-1} t^{-1} \cdot b$ and $w = g_{2n+2}^{-1} t^{-1}(b g_1 t^{\epsilon_2} \cdots t^{\epsilon_n} g_n t^{\epsilon_{n+1}} \cdots t^{\epsilon_{2n+1}} g_{2n+1}) t g_{2n+2}$. Since $w \in C_G$ and C_G is closed under conjugacy, we can apply the inductive assumption to $b g_1 t^{\epsilon_2} \cdots t^{\epsilon_n} g_n t^{\epsilon_{n+1}} \cdots t^{\epsilon_{2n+1}} g_{2n+1}$. It deduce that $b g_1 t^{\epsilon_2} \cdots t^{\epsilon_n} = (t^{\epsilon_{n+1}} \cdots t^{\epsilon_{2n+1}} g_{2n+1})^{-1} x$ for some $x \in A \cup B$. Multiply $g_{2n+2}^{-1} t^{-1}$ from the left side to the both sides of this equation, then we have the conclusion. □

3.2. Atomic properties for HNN extensions. In this section, we prove HNN extensions with a certain condition have the atomic property. The first lemma is essentially the same as Lemmas 11.5, 11.6 in Chapter I of [**22**], which are results for amalgamated free products. Here, we arrange them to HNN extensions.

LEMMA 3.18. *Let WxW^{-1}, VyV^{-1} be reduced where $W, V \in G^*$ and $x, y \in G$. If $WxW^{-1}VyV^{-1} \in C_G$, then exactly one of the following holds:*

(1) *$W^{-1}V \in G$;*
(2) *there exists a non-empty word W such that XW is reduced, $V = XW$ and $X^{-1}xX \in A \cup B$; or*
(3) *there exists a non-empty word W such that XV is reduced, $W = XV$ and $X^{-1}xX \in A \cup B$.*

PROOF. It is sufficient to prove that the negation of (1) implies (2) or (3). Since C_G is closed under conjugacy, we consider $V^{-1}WxW^{-1}Vy$ which is conjugate to $WxW^{-1}VyV^{-1}$ by V^{-1}. That is, we can assume $V = e$ and $l(W) \neq 0$ in the statement. We have $WxW^{-1}y \in C_G$ and apply Proposition 3.17 to it. We conclude that $W^{-1}yW \in A \cup B$, which is the desired case (2). If you consider $xW^{-1}VyV^{-1}W^{-1}$, then we obtain the case (3). □

It says that if $l(W) > l(V)$, then $VyV^{-1} = WuW^{-1}$ for some $u \in A \cup B$ and $WxW^{-1}VyV^{-1} = W(xu)W^{-1}$. In other word, If $x, y \in C_G$, then $l(xy) = Max\{l(x), l(y)\}$.

The next lemma is an analogy of [**20**, Lemma 2].

LEMMA 3.19. *Let $x_n, y_n \in G^*$ $(n < \omega)$, $f \in {}^{\omega}\omega$ and $y_n = x_n y_{n+1}^{f(n)}$ for any natural number n.*

(*a*) *If* $\sum_{i\leq n} l(x_i) + n \leq f(n)$ *and* $l(y_{n+1}) \leq l(y_{n+1}^{f(n)})$ *for any natural number* n*, then there exists a natural number* m *such that* $y_n \in C_G$ *for any* $n \geq m$*.*

(*b*) *If* $\sum_{i\leq n} l(x_i) + n \leq f(n)$ *for any natural number* n*, then the set* $\{n < \omega | y_n \in C_G\}$ *is infinite.*

Proof. We prove (a) and (b) by contradiction. For (a), suppose its negation. Then, we can take a natural number N satisfying $l(y_0) < N$ and $y_{N+1} \notin C_G$. By the remark of Definition 2.16, $\sum_{i\leq N} l(x_i) + N \leq l(y_{N+1}^{f(N)})$. We show that $\sum_{i\leq N-k-1} l(x_i) + N \leq l(y_{N-k})$ for any $0 \leq k \leq N-1$ by the induction of k. Consider the case $k = 0$. By $y_{N+1} \notin C_G$ and the assumption of f, $y_{N+1}^{f(N)} \geq f(N) \geq \sum_{i\leq N} l(x_i) + N$. Then, we have $l(y_{N+1}) \geq l(x_N)$ and $l(y_N) \geq l(y_{N+1}^{f(N)}) - l(x_N)$ by $y_N = x_n y_{N+1}^{f(N)}$. We conclude $l(y_N) \geq l(y_{N+1}^{f(N)}) - l(x_N) \geq \sum_{i\leq N-1} l(x_i) + N$. Consider the case $k+1$. By the inductive hypothesis and $l(y_{N-k}) \leq l(y_{N-k}^{f(N-k-1)})$, $l(y_{N-k}^{f(N-k-1)}) \geq l(y_{N-k}) \geq \sum_{i\leq N-k-1} l(x_i) + N$. By the same argument of the case $k = 0$, we have the desired. Then, we have $\sum_{i\leq N-k-1} l(x_i) + N \leq l(y_{N-k})$ for any $0 \leq k \leq N-1$, it concludes $N \leq l(y_0)$, which contradicts to $N > l(y_0)$.

Next for (b), suppose its negation. Then, we can take a natural number n_0 such that $y_n \notin C_G$ for any $n \geq n_0$. Take $N > l(y_{n_0}), n_0$. We can apply (a) to x_n, y_n ($n_0 \leq n$), which deduces a contradiction. □

The following two lemmas are related to infinitary words and HNN extensions.

Lemma 3.20. *Let* $h : \divideontimes_{n<\omega}\mathbb{Z}_n \to G^*$ *be a homomorphism and* $Im(h) \subseteq C_G$*. Then, there exists a natural number* N *such that* $h[\divideontimes_{n\geq N}\mathbb{Z}_n]$ *is contained in a subgroup conjugate to* G*.*

Proof. Firstly, we claim $\exists N < \omega \exists m < \omega \forall x \in \divideontimes_{n\geq N}\mathbb{Z}_n (l(h(x)) \leq m)$. To show this by contradiction, suppose its negation, then we obtain inductively $X_n \in \divideontimes_{i\geq n}\mathbb{Z}_i$ such that $h(X_n) \in C_G$ and $n < l(h(X_n)) < l(h(X_{n+1}))$ for any $n < \omega$. By Proposition 3.9, we can define the following infinitary words.

$$V_0 = X_0 X_1 \cdots X_{n-1} X_n \cdots \tag{1}$$

$$= X_0 X_1 \cdots X_{n-1} V_n \tag{2}$$

To comprehend to infinitary words, we explain its constructions. Firstly we define words of $\divideontimes_{n<\omega}\mathbb{Z}_n$, W_n for any $n < \omega$ such that $W_n(i) = \delta_i$ for $n \geq i$ and $W_n(i) = e$ for others. These words satisfy the followings.

$$W_n = \delta_n \delta_{n+1} \delta_{n+2} \cdots \tag{3}$$
$$= \delta_n W_{n+1} \tag{4}$$
$$W_0 = \delta_0 \delta_1 \cdots \delta_{n-1} \delta_n \cdots \tag{5}$$
$$= \delta_0 \delta_1 \cdots \delta_{n-1} W_n \tag{6}$$

By Proposition 3.9, there exists a homomorphism $\varphi : \times_{n<\omega} \mathbb{Z}_n \to \times_{n<\omega} \mathbb{Z}_n$ via $\delta_n \mapsto X_n$ $(n < \omega)$, then $V_n = \varphi(W_n)$.

This definition is not exact since X_n may not be a letter. But, we can regard them as the composition of infinitary many words because X_n does not contain δ_i for any $i < n$. We find out $V_n = X_n V_{n+1}$ and $h(V_n) \in C_G$ for any $n < \omega$. To deduce a contradiction, we show $h(V_m) = e$ for any $m > l(h(V_0))$, which contradicts to that $h(X_n)$ is non-trivial for any n. If $h(V_m) \neq e$, then $l(h(V_m)) \geq l(h(X_m))$ by $V_m = X_m V_{m+1}$ and Lemma 3.18, which says that $l(h(V_m)) = Max\{l(X_m), l(V_{m+1})\}$. Because of $l(h(X_n)) < l(h(X_{n+1}))$, we inductively conclude that $l(h(V_{m-k})) \geq l(h(X_m))$ for any $k < m$. It implies that $l(h(V_0)) \geq m$, which is a contradiction.

Now, there exists a natural number N such that the image of $\times_{n \geq N} \mathbb{Z}_n$ has the maximum length. We can take $w \in G^*$ satisfying $wgw^{-1} \in h[\times_{n \geq N} \mathbb{Z}_n]$ (reduced) and $l(wgw^{-1})$ is equal to the maximum. By Lemma 3.18 and $\mathrm{Im}(h) \subseteq C_G$, $h[\times_{n \geq N} \mathbb{Z}_n]$ is contained in wGw^{-1}.

□

Lemma 3.21. *Let $h : \times_{n<\omega} \mathbb{Z}_n \to G^*$ be a homomorphism and $h[\times_{n \geq N} \mathbb{Z}_n] \nsubseteq C_G$ for any $N < \omega$. Then, for any $m, n < \omega$, there exists an element $w \in \times_{i \geq n} \mathbb{Z}_i$ such that $h(w) \in C_G$ and $l(h(w)) \geq m$.*

Proof. To show this by contradiction, suppose the negation of the conclusion. We may assume $\forall x \in \times_{n<\omega} \mathbb{Z}_n (h(x) \in C_G \to l(h(x)) < m)$ for some m. By the assumption, there exists $X_n \in \times_{i \geq n} \mathbb{Z}_i$ such that $h(X_n) \notin C_G$ and $l(h(X_n)) \geq 2m$ for any n. Let $f(n) = \sum_{i \leq n} l(h(X_i)) + n + 2m$. Now, we define $V_n \in \times_{i<\omega} \mathbb{Z}_i$ for any $n < \omega$ from X_n and f. This construction appeared in [**8**] and [**20**]. Let T be the tree $\langle \bigcup_{n<\omega} \prod_{m<n} f(m), \subseteq \rangle$ and t_n be an arbitrary element of $\mathrm{Lev}_n(T) = \prod_{m<n} f(m)$. Clearly, T is linearly ordered by the lexicographical order. Let $\overline{V_n} = \{t \in T \mid t_n \subseteq t\}$ and $V_n(x) = X_k$ iff $x \in \mathrm{Lev}_k(T)$ for any n. Then, $V_n = X_n V_{n+1}^{f(n)}$ for any n. Applying Lemma 3.19 (b), there exists $M < \omega$ such that $l(h(V_0)) + 1 < M$ and $h(V_{M+1}) \in C_G$. By a similar way as the previous lemma, we inductively conclude that $h(V_{M-k}) \notin C_G$ and $l(h(V_{M-k})) \geq \sum_{i \leq M-k-1} l(h(X_i)) + M - 1 + 2m$ for $1 \leq k < M$ which implies $l(h(V_0)) > M - 1$, it is a contradiction. To prove it, consider the case $k = 1$. Since $h(V_{M+1}) \in C_G$, $h(V_{M+1})^{f(M)} \in C_G$ and $l(h(V_{M+1})^{f(M)}) < m$. We have $h(V_M) \notin C_G$ because $2m \leq l(h(X_M))$,

$V_M = X_M V_{M+1}^{f(M)}$ and $\forall x \in \ast_{n<\omega}\mathbb{Z}_n(h(x) \in C_G \to l(h(x)) < m)$. Then $l(h(V_M)^{f(M-1)}) \geq f(M-1) = \sum_{i\leq M-1} l(h(X_i)) + M - 1 + 2m$ since $h(V_M) \notin C_G$. We deduce $l(h(V_{M-1})) \geq \sum_{i\leq M-2} l(h(X_i)) + M - 1 + 2m$ by $h(V_{M-1}) = h(X_{M-1})h(V_M)^{f(M)}$. Next, consider the case $k+1$. By the inductive hypothesis, we have $l(h(V_{M-k})^{f(M-k-1)}) \geq l(h(V_{M-k})) \geq \sum_{i\leq M-k-1} l(h(X_i)) + M - 1 + 2m$. Then, by a similar way, we conclude the desired. □

Now, we show the atomic property for HNN extensions.

Theorem 3.22. *Let $G^* = \langle G, t | tAt^{-1} = B\rangle$ be an HNN extension of G and satisfying the following condition.*
$(*)$ $2 \leq \exists p < \omega \ \forall g \in G \setminus A \ (g^p \notin A) \wedge \forall g \in G \setminus B \ (g^p \notin B)$
Then, for any homomorphism $h : \ast_{n<\omega}\mathbb{Z}_n \to G^*$, there exists a natural number N such that $h[\ast_{n\geq N}\mathbb{Z}_n]$ is contained in a subgroup conjugate to G. That is, G^* has the atomic property.

Proof. By lemma 3.20, it is sufficient to show that for any homomorphism h, there exists N such that $h[\ast_{n\geq N}\mathbb{Z}_n] \subseteq C_G$. Assume not. By Lemma 3.21, we obtain inductively $X_n \in \ast_{i\geq n}\mathbb{Z}_i$ such that $h(X_n) \in C_G$ and $n < l(h(X_n)) < l(h(X_{n+1}))$ for any $n < \omega$. Take p as in $(*)$ and $f : \omega \to \omega$ such that $\sum_{i\leq n} l(h(X_i)) + n < p^{f(n)}$. We remark that $l(x^p) = l(x)$ for any $x \in C_G$. As above, we obtain $V_n = X_n V_{n+1}^{p^{f(n)}}$ for any $n < \omega$. Applying lemma 3.19 (a) to $h(X_n), h(V_n)$, there exists $M < \omega$ such that $h(V_n) \in C_G$ for any $n \geq M$. By the same argument of lemma 3.20, we can deduce a contradiction. □

Clearly, adding the assumption that G is n-slender to the assumptions of Theorem 3.22 we have the fact that G^* is also n-slender. The next theorem is about ascending HNN extensions (or mapping tori).

Theorem 3.23. *Any ascending HNN extension $G^* = \langle G, t | tGt^{-1} = B\rangle$ has the atomic property.*

Proof. Like the previous theorem, it is sufficient to show that for any homomorphism h, there exists N such that $h[\ast_{n\geq N}\mathbb{Z}_n] \subseteq C_G$. Suppose the negation. We obtain $X_n \in \ast_{i\geq n}\mathbb{Z}_i$ such that $h(X_n) \notin C_G$. Let $\rho : G^* \to < t >$ be a natural homomorphism such that $\rho \restriction G$ is trivial. Considering the reduced forms of ascending HNN extensions which is known $t^{-q}gt^p$ where p, q are non-negative integer and $g \in G$, it is easy to find that $\forall x \notin C_G(\rho(x) \neq 0)$. Since the integer group is n-slender, there exists m such that $\rho \circ h[\ast_{n\geq m}\mathbb{Z}_n] = \{0\}$. It contradicts to $\rho \circ h(X_m) \neq 0$. □

The proof of Theorem 3.22 can be translated to amalgamated free products. So, we obtain the same result for amalgamated free products.

Corollary 3.24. *(Corollary to the proof of Theorem 3.22)*

Let $A \underset{U}{*} B$ be an amalgamated free product of A and B. If it satisfies the following condition. $(*)' 2 \leq \exists p < \omega \; \forall g \in A \setminus U \; (g^p \notin U) \wedge \forall g \in B \setminus U \; (g^p \notin U)$

Then, $A \underset{U}{*} B$ has the atomic property.

We show some one-relator groups are n-slender using the atomic property for HNN extensions.

COROLLARY 3.25. *The Baumslag-Solitar group is n-slender*

PROOF. Since $BS(m,n) = \langle a, b | ab^m a^{-1} = b^n \rangle = \langle \langle b \rangle, t | tb^m t^{-1} = b^n \rangle$, $BS(m,n)$ is the HNN extension of the integer group. Because we can take a prime number p which is greater than $|m|$ and $|n|$, it satisfies $(*)$. We can apply Theorem 3.22. □

We mention n-slenderness of surface groups, which are the fundamental groups of closed surfaces. Since any subgroup of a surface group with infinite index is free and the class of n-slender groups is closed under extensions as like of slender groups, any torsion-free surface group is n-slender. More precisely, such a group has a homomorphism to $\mathbb{Z}$ the kernel of which is free. But, this is also an example of applications of the atomic property.

COROLLARY 3.26. *Any surface group except real projective plane is n-slender.*

PROOF. We only explain how to regard surface groups as amalgamated free products. Firstly, the case of closed orientable surface M_g with genus of g. It is known $\pi_1(M_g) = \langle x_1, \cdots, x_{2g} | [x_1, x_2] \cdots [x_{2g-1}, x_{2g}] \rangle$.

Let $A = \langle x_1 \rangle * \langle x_2 \rangle$, U_0 be the subgroup generated by $[x_1, x_2]$, $B = \langle x_3 \rangle * \cdots * \langle x_{2g} \rangle$ and U_1 be the subgroup generated by $[x_3, x_4] \cdots [x_{2g-1}, x_{2g}]$. We amalgamate U_0 and U_1 according to the isomorphism which maps $[x_1, x_2]$ to $([x_3, x_4] \cdots [x_{2g-1}, x_{2g}])^{-1}$.

The non-orientable case, $\pi_1(N_g) = \langle x_1, \cdots, x_g | x_1 x_1 \cdots x_g x_g \rangle$ is also so by the similar way. Such amalgamated free products satisfy $(*)'$. □

3.3. Problems. Finally, we introduce two open problems, the one is about the atomic property and the other is about n-slender groups .

QUESTION 3.27. Does every HNN extension have the atomic property?

In other words, can we drop the assumptions of Theorems 3.22 or not. The affirmative answer implies that HNN extensions preserve n-slenderness and consequently does that any torsion-free one-relator group is n-slender and that an amalgamated free product of two n-slender groups is also n-slender.

QUESTION 3.28. Does there exist a good characterization of n-slender groups ?

This problem would be important to solve other problems about n-slender groups. Especially, we wonder whether a torsion-free countable group which does not contain $\mathbb{Q}$ and is n-slender or not.

4. The non-commutative Specker-Eda number

4.1. Introduction. We show that the non-commutative Specker-Eda number $\mathfrak{se}_{\mathfrak{nc}}$ is equal to the Specker-Eda number $\mathfrak{se}$ and the subgroup of $\circledast_{\alpha<\kappa}\mathbb{Z}_\alpha$ consisting of all words which have no subword with uncountable cofinality or coinitiality exhibits the non-commutative Specker phenomenon. $\mathfrak{se}_{\mathfrak{nc}}$ is the smallest cardinality of subgroups of $\circledast_{n<\omega}\mathbb{Z}_n$ which exhibit the non-commutative Specker phenomenon. And $\mathfrak{se}$ is the smallest cardinality of subgroups of $\mathbb{Z}^\omega$ which exhibit the Specker phenomenon.

E. Specker also established subgroups of $\mathbb{Z}^\omega$ which exhibit the Specker phenomenon. But these subgroup have the cardinality of the continuum $2^{\aleph_0}$. So, the next question naturally arises whether the smallest cardinality of subgroups which exhibit the Specker phenomenon is $2^{\aleph_0}$. It turned out that this question is undecidable on ZFC by K. Eda [**7**] in 1983. S. Kamo [**21**] also considered a related question in Cohen extensions in 1986. A. Blass [**2**] studied the cardinal invariant and named it the Specker-Eda number, $\mathfrak{se}$. He pointed out K. Eda's proof established that $\mathfrak{p} \leq \mathfrak{se} \leq \mathfrak{d}$ and proved that $\mathfrak{e}_{\mathfrak{l}} \leq \mathfrak{se} \leq \mathfrak{b}$ in 1994. Finally, in 1996, J. Brendle and S. Shelah [**3**] proved that $\mathfrak{se} = \mathfrak{e}_{\mathfrak{l}}$ =min$\{\mathfrak{e}, \mathfrak{b}\}$. Now, we consider the non-commutative case.

4.2. $\mathfrak{se}_{\mathfrak{nc}} = \mathfrak{se}$**.** We have described Specker' theorem and Higman's theorem as the non-commutative version of Specker's theorem at Section 2. Then, we can naturally introduce a new cardinal invariant, the non-commutative version of the Specker-Eda number.

Definition 4.1. Let G be a subgroup of $\circledast_{i\in I}\mathbb{Z}_i$ containing all δ_i. G exhibits the non-commutative Specker phenomenon if, there is a finite subset $F \subseteq I$ such that h factors through $*_{i\in F}\mathbb{Z}_i$ for any homomorphism h from G to Z.

Definition 4.2. The non-commutative Specker-Eda number is denoted by $\mathfrak{se}_{\mathfrak{nc}}$. It is the least cardinal of G which is a subgroup of $\circledast_{n<\omega}\mathbb{Z}_n$ and exhibits the non-commutative Specker phenomenon.

Definition 4.3. A subgroup G of $\circledast_{n<\omega}\mathbb{Z}_n$ is weakly fine if, for any reduced word W in G and for any $m < \omega$,
G contains $X_0, X_1, \cdots, X_{i_n} \in *_{n<m}\mathbb{Z}_n$ and $Y_0, \cdots, Y_{i_n+1} \in \circledast_{m\leq n<\omega}\mathbb{Z}_n$ where $W \equiv Y_0X_0Y_1X_1\cdots X_{i_n}Y_{i_n+1}$.

The next lemma says that it suffices to show that there are only finitely n satisfying $h(\delta_n) \neq 0$ for any homomorphism h from G to $\mathbb{Z}$ when we prove G exhibits the non-commutative Specker phenomenon in this section.

LEMMA 4.4. *Let G be a subgroup of $\times_{n<\omega}\mathbb{Z}_n$ containing all δ_n. If there are only finitely n satisfying $h(\delta_n) \neq 0$ for any homomorphism h from G to $\mathbb{Z}$, then there exists a subgroup H such that $|G| = |H|$ and H exhibits the non-commutative Specker phenomenon.*

PROOF. For any reduced word W, let φ_W be a homomorphism : $\times_{n<\omega}\mathbb{Z}_n \to \times_{n<\omega}\mathbb{Z}_n$ via $\delta_m \mapsto W_{\omega\setminus m}$ by Proposition 3.9. Then, we can take a subgroup H satisfying followings;

(1)$G \subseteq H$ and $|G| = |H|$

(2)$\forall W \in H(\varphi_W[H] \subseteq H)$ (where W is reduced.)

(3) G is weakly fine.

Now, we show that H is the desired subgroup. Suppose the negation. Then, for every n, there exists $W \in H$ such that $h \circ p_n(W) \neq h(W)$ for some homomorphism h. Since $G \subseteq H$, there exists n_0 such that $\forall n \geq n_0(h(\delta_n) = 0)$. Take W such that $h \circ p_{n_0}(W) \neq h(W)$. Since $h(W) = h(Y_0)h(X_0)\cdots h(X_{i_{n_0}})h(Y_{i_{n_0}+1})$, $h(Y_i) \neq 0$ for some i. $h \circ \varphi_{Y_i}$ is a homomorphism from H to $\mathbb{Z}$ and $h \circ \varphi_{Y_i}(\delta_n) = h(Y_i)$ for all n. Because, if $n \geq n_0$,then $(Y_i)_{\omega\setminus n} \in *_{n_0\leq k<n}\mathbb{Z}_k * \times_{n\leq k<\omega}\mathbb{Z}_k$, otherwise $(Y_i)_{\omega\setminus n} = Y_i$. It contradicts. □

THEOREM 4.5. $\mathfrak{se}_{\mathfrak{nc}} = \mathfrak{se}$.

PROOF. Firstly, we show that $\mathfrak{se} \leq \mathfrak{se}_{\mathfrak{nc}}$. Let $\sigma : \times_{n<\omega}\mathbb{Z}_n \to \mathbb{Z}^\omega$ be the canonical homomorphism such that $\sigma(W)(n) = W_{\{n\}}$ $(n < \omega)$ and G be a subgroup of $\times_{n<\omega}\mathbb{Z}_n$ which exhibits non-commutative Specker phenomenon and whose cardinality is $\mathfrak{se}_{\mathfrak{nc}}$. Then $\sigma[G]$ also exhibits Specker phenomenon. Because, let $h : \sigma[G] \to \mathbb{Z}$ be a homomorphism. The composition of h and σ is a homomorphism from G to $\mathbb{Z}$. Therefore, $h(e_n) = h \circ \sigma(\delta_n) = 0$ for all but finitely many n. Then, we get $\mathfrak{se} \leq |\sigma[G]| \leq |G| = \mathfrak{se}_{\mathfrak{nc}}$.

Next, we show that $\mathfrak{se}_{\mathfrak{nc}} \leq \mathfrak{se}$. To show this, two lemmas are necessary.

LEMMA 4.6. $x, a \in \mathbb{Z}$

$$\forall n < \omega\left(n! \mid x - \sum_{i=1}^{n-1} i!a\right) \Rightarrow x = 0 \textit{ and } a = 0$$

PROOF. we can prove by induction that $2 \leq n$ implies $\sum_{i=1}^{n-1} i! \leq 2(n-1)!$. Therefore, we can easily find a natural number n such that $|x - \sum_{i=1}^{n-1} i!a| < n!$ and $|x - \sum_{i=1}^{n} i!a| < (n+1)!$. It means that $x - \sum_{i=1}^{n-1} i!a = 0 = x - \sum_{i=1}^{n} i!a$. □

To show the next lemma, we consider the following words:U_∞, U_n. For $W \in \mathsf{X}_{n<\omega}\mathbb{Z}_n$, let $V_n = W_{\omega\setminus n}$. To define U_∞, U_n , we consider such a tree $T = \langle \bigcup_{n<\omega} (\prod_{1\leq m\leq n+1} m), \subseteq \rangle$ like the binary tree $< 2^{<\omega}, \subseteq >$. Then we order T lexicographically, i.e; If $x, y \in T$, define $x \lhd y$ iff $x(n) < y(n)$ where $n \in \mathrm{dom}(x) \cap \mathrm{dom}(y)$ is the least natural number such that $x(n) \neq y(n)$, or $\mathrm{dom}(x) < \mathrm{dom}(y)$. Consequently, T is linearly ordered by this lexicographical order. Now, we define U_∞, U_n as follows.

$$\overline{U_\infty} = T,\ U_\infty(x) = V_n \quad (x \in \mathrm{Lev}_n(T) = \prod_{1\leq m\leq n+1} m),$$

$$\overline{U_n} = \{t \in T \mid\ y_n \subseteq t\} \text{ where } y_n \text{ is an arbitrary element of } \mathrm{Lev}_n(T),$$

$$U_n(x) = V_k \quad (x \in \mathrm{Lev}_k(T)).$$

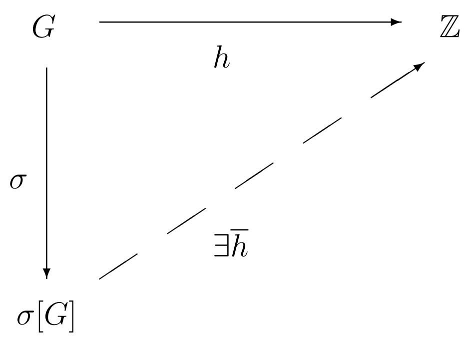
This definition is not exact because V_n may not be a letter. But we can naturally regard them as the composition of infinitary many words, since V_n does not contain letters δ_i for $i < n$. And we find that they really become a word.

$$\begin{array}{cccccc}
 & & & V_3 & \cdots & \\
 & & & V_3 & \cdots & \\
 & & & V_3 & \cdots & \\
 & & V_2 & V_3 & \cdots & \\
 & V_1 & V_2 & \cdots & & \\
 & & V_2 & \ddots & & \\
U_\infty = V_0 & & & & & \\
 & & V_2 & \cdots & & \\
 & V_1 & V_2 & \cdots & & \\
 & & V_2 & V_3 & \cdots & \\
 & & & V_3 & \cdots & \\
 & & & V_3 & \cdots & \\
 & & & V_3 & \cdots & \cdots \\
 & & & & \ddots &
\end{array}$$

$$
\begin{array}{llll}
 & & V_{n+2} & \cdots \\
 & & & \ddots \\
 & V_{n+1} & \vdots & \\
 & & V_{n+2} & \cdots \\
 & & & \ddots \\
U_n = V_n & \vdots & & \\
 & & V_{n+2} & \cdots \\
 & & & \ddots \\
 & V_{n+1} & \vdots & \\
 & & V_{n+2} & \cdots \\
 & & & \ddots
\end{array}
$$

Now, we state the second lemma.

LEMMA 4.7. *Let G be a subgroup of $\circledast_{n<\omega}\mathbb{Z}_n$ containing all δ_n. If, for every $W \in ker(\sigma) \cap G$, G contains U_∞ and all U_n coressponding to W and G is weakly fine, then any homomorphism h from G to $\mathbb{Z}$ factors thorough $\sigma[G]$.*



$h = \overline{h} \circ \sigma$ σ:canonical homomorphism

PROOF. It is sufficient to show that $\ker(\sigma) \cap G \subseteq \ker(h)$. Let G' be a commutator subgroup of G and $[W]$ be a element of G/G'. Let $W \in \ker(\sigma) \cap G$. Then $[W] = [V_n]$ for all n because G/G' is abelian and weakly fine. By the figure of U_∞, U_n, we have

$$
\begin{aligned}
[U_\infty] &= \sum_{i=1}^{n-1} i![V_{i-1}] + n![U_{n-1}] \\
&= \sum_{i=1}^{n-1} i![W] + n![U_{n-1}]
\end{aligned}
$$

And there exists a homomorphism $h_0 : G' \to \mathbb{Z}$ s.t $h(x) = h_0([x])$ for any $x \in G$ by the homomorphism theorem because $G' \subseteq \ker(h)$. Therefore, we get

$$n! \mid h_0([U_\infty]) - \sum_{i=1}^{n-1} i! h_0([W]) \quad \text{for all } n$$

So, we have $h(W) = h_0([W]) = 0$ by Lemma 4.6 □

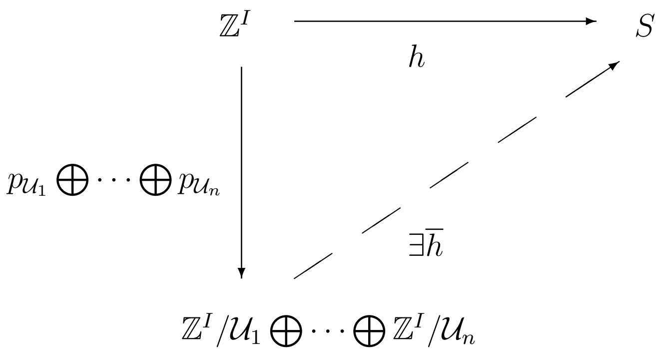
Now, we return to the proof of $\mathfrak{se}_{\mathfrak{nc}} \leq \mathfrak{se}$. Our goal is getting a subgroup of $\divideontimes_{n<\omega}\mathbb{Z}_n$ whose cardinality is $\mathfrak{se}$ and which exhibits the non-commutative Specker phenomenon. In the diagram of Lemma 4.7, if $\sigma[G]$ exhibits the Specker phenomenon, then G also exhibits the non-commutative Specker phenomenon because $h(\delta_n) = \overline{h}(e_n)$. So, we take a subgroup H of $\mathbb{Z}^\omega$ whose cardinality is $\mathfrak{se}$ and which exhibits the Specker phenomenon. $\sigma^{-1}[H]$ also exhibits the non-commutative Specker phenomenon, but, unfortunately, the cardinality of $\ker(\sigma)$ is $2^{\aleph_0}$. Let X be a set such that $\sigma[X] = H$ and $|X| = \mathfrak{se}$. Then let G be the smallest subgroup which contains X and satisfies the clause of Lemma 4.7. Obviously, the size of G is $\mathfrak{se}$. And $\sigma[G]$ contains H, so $\sigma[G]$ also exhibits the Specker phenomenon. Therefore, G is the desired subgroup. □

5. The Specker phenomenon in the uncountable case

The uncountable Specker phenomenon in the commutative case was studied around 1955. J. Łoś and E. C. Zeeman [**25**] independently showed that $\mathbb{Z}^\kappa$ exhibits the Specker phenomenon iff κ is less than the least measurable cardinal. There exists similar results of non-commutative version. These results deduce that, If κ is uncountable, $\divideontimes_{\alpha<\kappa}\mathbb{Z}_\alpha$ fails the non-commutative Specker phenomenon. But, there exist subgroups of $\divideontimes_{\alpha<\kappa}\mathbb{Z}_\alpha$ which exhibit the Specker phenomenon. We introduce and show these results.

Theorem 5.1. (*Łoś-Eda theorem* [**15**]) *Let S be a slender group. For any homomorphism $h : \mathbb{Z}^I \to S$, there exist ω_1-complete ultrafilters $\mathcal{U}_1, \cdots, \mathcal{U}_n$ on I such that $h = \overline{h} \circ p_{\mathcal{U}_1} \bigoplus \cdots \bigoplus p_{\mathcal{U}_n}$.*

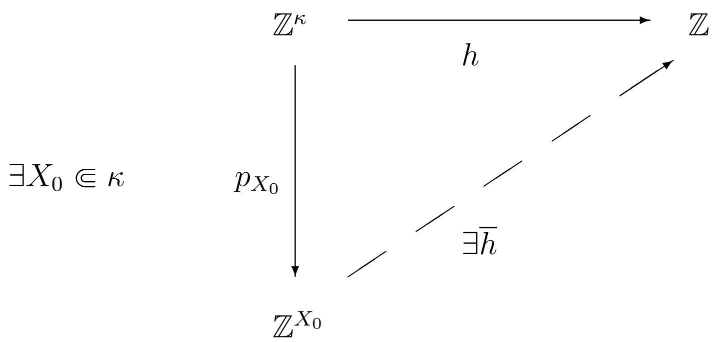
This theorem says that the following diagram satisfies.



It deduces the next result of uncountable Specker phenomenon.

Corollary 5.2. *If κ is less than the least measurable cardinal, then $\mathbb{Z}^\kappa$ satisfies Specker's theorem.*

It means that the following diagram.



We remark that, let κ be the least cardinal which has a non-principal ω_1-complete ultrafilters on κ, then, κ is measurable.

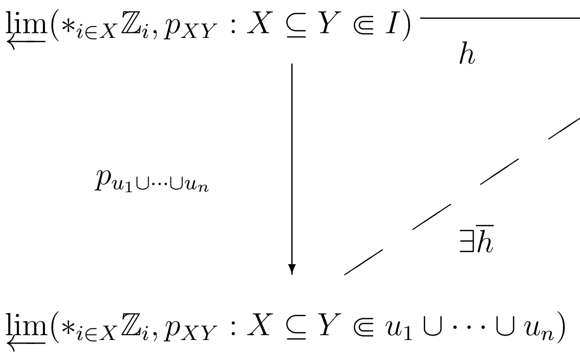
The next theorem says that the non-commutative Specker phenomenon can not exhibit in uncountable cardinals.

Theorem 5.3. *(S. Shelah and L. Strüngmann* [**23**]*)*
$\circledast_{\alpha<\omega_1}\mathbb{Z}_\alpha$ fails the non-commutative Specker phenomenon, i.e; there exists a homomorphism $h : \circledast_{\alpha<\omega_1}\mathbb{Z}_\alpha \to \mathbb{Z}$ such that $h(\delta_\alpha) = 0$ for any $\alpha < \omega_1$ but also h is non-trivial.

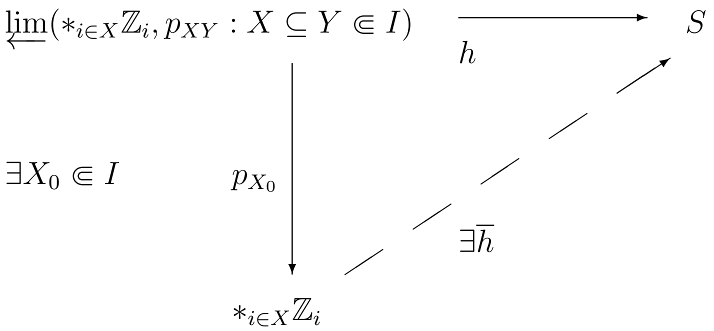
The following theorem is the non-commutative version of the above result of J. Łoś and E. C. Zeeman.

Theorem 5.4. *(S. Shelah and K. Eda* [**14**]*) Let S be a n-slender group. For any homomorphism $h : \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I) \to S$, there exist ω_1-complete ultrafilters $\mathcal{U}_1, \cdots, \mathcal{U}_n$ on I such that $h = \overline{h} \circ p_{u_1\cup\cdots\cup u_n}$ for any $u_1 \in \mathcal{U}_1, \cdots, u_n \in \mathcal{U}_n$.*

It means that the following diagram satisfies.



If $|I|$ is less than the least measurable cardinal, then the above diagram is changed to the following.



Proposition 5.5. *For any homomorphism h from $\divideontimes_{\alpha<\kappa}\mathbb{Z}_\alpha$ to $\mathbb{Z}$, there are only finitely α such that $h(\delta_\alpha) \neq 0$.*

Proof. Assume not. Let h be a homomorphism from $\divideontimes_{\alpha<\kappa}\mathbb{Z}_\alpha$ to $\mathbb{Z}$ and $\alpha_n (n < \omega)$ be elements of κ such that $h(\delta_{\alpha_n}) \neq 0$ for all n. Then, we can take the homomorphism $\varphi : \divideontimes_{n<\omega}\mathbb{Z}_n \to \divideontimes_{\alpha<\kappa}\mathbb{Z}_\alpha$ via $\delta_n \mapsto \delta_{\alpha_n}$. The composition of h and φ is a homomorphism from $\divideontimes_{n<\omega}\mathbb{Z}_n$ to $\mathbb{Z}$ and $h \circ \varphi(\delta_n) \neq 0$ for all n. It contradicts. $\square$

We remark If κ is uncountable, $\divideontimes_{\alpha<\kappa}\mathbb{Z}_\alpha$ fails the non-commutative Specker phenomenon. Our interest is whether any homomorphism from $\divideontimes_{\alpha<\kappa}\mathbb{Z}_\alpha$ to $\mathbb{Z}$ factors through $*_{\alpha\in X}\mathbb{Z}_\alpha$ for some finite subset X of κ.

5.1. Subgroups exhibit the uncountable Specker phenomenon. We have introduced non-commutative uncountable Specker phenomenon and related results, such that the unrestricted free product $\varprojlim(*_{i\in X}G_i, p_{XY} : X \subseteq Y \Subset I)$ exhibits the Specker phenomenon iff the cardinality of the index set I is less than the least measurable cardinal [**14**]. We consider another non-commutative case, that is, $\divideontimes_{\alpha<\kappa}\mathbb{Z}_\alpha$ and its subgroups where κ is an uncountable cardinal. We have mentioned, S. Shelah and L. Strüngmann [8] showed a counter example, i.e: there exists a homomorphism h from $\times_{\alpha<\omega_1}\mathbb{Z}_\alpha$ to $\mathbb{Z}$ such that $h(\delta_\alpha)=0$ for all $\alpha<\omega_1$ but h is not trivial. On the other hand, there exists a subgroup of $\times_{\alpha<\kappa}\mathbb{Z}_\alpha$ which exhibits the non-commutative Specker phenomenon.

PROPOSITION 5.6. *([**8**] Proposition 3.5.) Let $h:\times^\sigma_{\alpha<\kappa}\mathbb{Z}_\alpha\to\mathbb{Z}$ be a homomorphism. Then there exists a finite subset F and a homomorphism $\overline{h}:*_{\alpha\in F}\mathbb{Z}_\alpha\to\mathbb{Z}$ such that $h=\overline{h}\circ p_F$.*

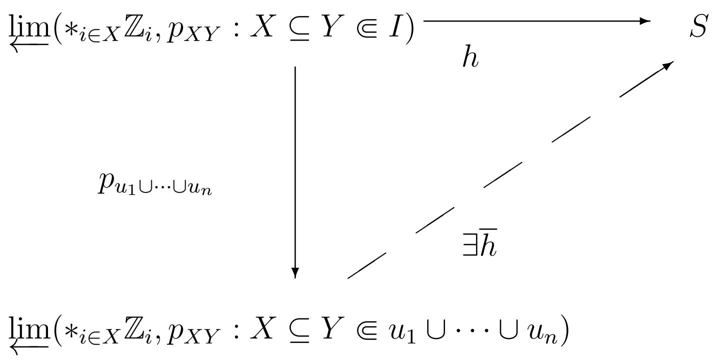
It means that $\times^\sigma_{\alpha<\kappa}\mathbb{Z}_\alpha$ exhibits the uncountable Specker phenomenon. We show that there is another subgroup which exhibits the uncountable Specker phenomenon.

THEOREM 5.7. *Let G be the subgroup of $\times_{\alpha<\kappa}\mathbb{Z}_\alpha$ consisting of all words which have no subword with uncountable cofinality or coinitiality. G is a maximal subgroup which exhibits the non-commutative Specker phenomenon.*

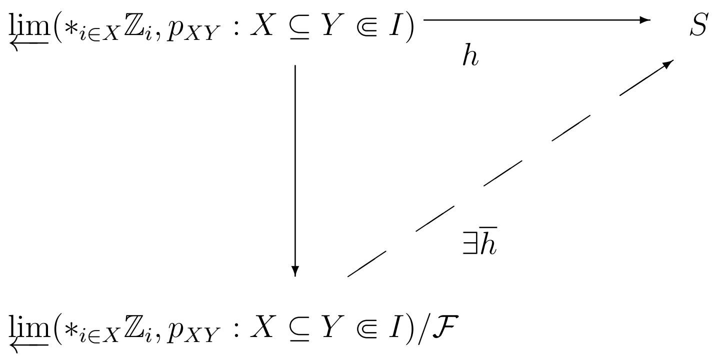
PROOF. If a subgroup of $\times_{\alpha<\kappa}\mathbb{Z}_\alpha$ has a word with uncountable cofinality or coinitiality, then it does not exhibits the non-commutative Specker phenomenon by [**23**]. Therefore, it suffices to show that G exhibits the non-commutative Specker phenomenon. Assume not. Let F be a set of all α which satisfies $h(\delta_\alpha)\neq 0$ for some homomorphism h from G to $\mathbb{Z}$. Like Proposition 5.5, F is finite. Then, we can find a reduced word W such that $h(W)\neq 0$ and $W\in\times_{\alpha\in\kappa\setminus F}\mathbb{Z}_\alpha$. Let T_0 be a maximal tail subword of W such that $h(T)=0$ for any proper tail subword of T_0. T_0 could be empty, but it is easy case. So, we consider the case T_0 is non-empty. Then, let W_0 be the subword such that $W\equiv W_0T_0$ and d be the left end point of $\overline{T_0}$. We claim that $h(T_0)=0$. Let $+\infty$ be the right end point of $\overline{T_0}$. Take a descending sequence $\{b_n\}_{n\in A}$ such that $\mathrm{Inf}\{b_n|\ n\in A\}=d$. Since $W\in G$, A is countable. It is sufficient to consider the case that $A=\omega$. Let $V_0=T_0\restriction_{(b_0,+\infty)}$ and $V_{n+1}=T_0\restriction_{(b_{n+1},b_n)}$ for $n<\omega$. Clearly, $h(V_n)=0$ for all n. We can take a homomorphism $\varphi:\times_{n<\omega}\mathbb{Z}_n\to G$ via $\delta_n\mapsto V_n$ by Proposition 3.9. $h(T_0)=h(\cdots V_1V_0)=h(\varphi(\cdots\delta_1\delta_0))=0$ because $h\circ\varphi$ is trivial. Since any tail subword of W_0 is uncountable and the cofinality of W_0 is countable, we can take an ascending sequence $\{a_n\}_{n<\omega}$ such that $\sup\{a_n|\ n<\omega\}=d$ and $h(W_0\restriction_{(a_n,d)})\neq 0$ for all n. It is easily seen that $\{W_0\restriction_{(a_n,d)}\mid n<\omega\}$ satisfies the clause of Proposition 3.9. It contradicts. □

5.2. Homogeneous elements and ultrafilters. We are motivated from [**13**]. The author of [**13**] mentioned personally that a proof about Question 5.16 does not work and hence Question 5.16 is still open. The uncountable Specker phenomenon in the commutative case was studied around 1955. J. Łoś and E. C. Zeeman [**25**] independently showed that $\mathbb{Z}^\kappa$ exhibits the Specker phenomenon if κ is less than the least measurable cardinal. There is a similar result in the non-commutative case. S. Shelah and K. Eda [**14**] showed that the unrestricted free product $\varprojlim(*_{i\in X}G_i, p_{XY} : X \subseteq Y \Subset I)$ exhibits the Specker phenomenon if the cardinality of the index set I is less than the least measurable cardinal. Some problems occur from the non-commutative case and we investigate them.

THEOREM 5.8. *(S. Shelah and K. Eda* [**14**]*) Let S be a n-slender group. For any homomorphism $h : \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I) \to S$, there exist ω_1-complete ultrafilters $\mathcal{U}_1, \cdots, \mathcal{U}_n$ on I such that $h = \overline{h} \circ p_{u_1\cup\cdots\cup u_n}$ for any $u_1 \in \mathcal{U}_1, \cdots, u_n \in \mathcal{U}_n$. Moreover, if the cardinality of I is less than the least measurable cardinal, then h factors through some finitely generated free group.*



Let $\mathcal{F} = \{X | \exists u_1 \in \mathcal{U}_1 \cdots \exists u_n \in \mathcal{U}_n(\bigcup_{i\leq n} u_i \subseteq X)\}$. It becomes an ultrafilter on I. We introduce an equivalence relation $\sim_{\mathcal{F}}$ on $\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I)$. $x \sim_{\mathcal{F}} y$ if and only if there exists $u \in \mathcal{F}$ such that $p_u(x) = p_u(y)$. Then, we get the following diagram.



It is a problem that what kind of group is $\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I)/\mathcal{F}$. We remark that $\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I)/\mathcal{F}$ could not be equal to $\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I)/\mathcal{U}_1 * \cdots * \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I)/\mathcal{U}_n$. For the first step, we consider the case $n = 1$ and we investigate its cardinality.

DEFINITION 5.9. Let $F, G \Subset \omega$ with $|F| = |G|$ and e_{FG} be the order isomorphism from F to G. Then, we naturally regard e_{FG} as an isomorphism from $*_{i\in F}\mathbb{Z}_i$ to $*_{i\in G}\mathbb{Z}_i$. An element $x \in \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \omega)$ is homogeneous if and only if for any $F, G \Subset \omega$ with $|F| = |G|$, $e_{FG}(p_F(x)) = p_G(x)$.

Let H be the subgroup consisting of all homogeneous elements.

THEOREM 5.10. *Let κ be a measurable cardinal and $\mathcal{U}$ be a κ-complete normal ultrafilter on κ. Then, $\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \kappa)/\mathcal{U} \simeq H$.*

PROOF. Let $\mathcal{U}^n = \{X \in [\kappa]^n | \exists u \in \mathcal{U}([u]^n \subseteq X)\}$ for $n \geq 2$ and $x \in \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \kappa)$. By the assumption, $\mathcal{U}^n$ is a κ-complete ultrafilter for any n. Since $[\kappa]^n = \bigcup_{W\in *_{i<n}\mathbb{Z}_i}\{F | e_{Fn}(p_F(x)) = W\}$, there exist $W_{n,x} \in *_{i<n}\mathbb{Z}_i$ such that $\{F | e_{Fn}(p_F(x)) = W_{n,x}\} \in \mathcal{U}$. We define a homomorphism $h : \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \kappa)/\mathcal{U} \to H$ as $h([x])(n) = W_{n,x}$. It is easily seen that h is an isomorphism.

□

PROPOSITION 5.11. *the cardinality of H is 2^ω. In addition, H is not n-slender.*

To prove it, we prepare the following lemma.

DEFINITION 5.12. Let $x_n \in \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \omega)$ for any $n < \omega$ and $[x_0, x_1] = x_0 x_1 x_0^{-1} x_1^{-1}$. We define inductively $[x_0, \cdots, x_n]$ as the following.

$$[x_0, \cdots, x_{n+1}] := [x_0, \cdots, x_n]x_{n+1}[x_0, \cdots, x_n]^{-1}x_{n+1}^{-1}$$

LEMMA 5.13. *There exists $y_n \in H(n < \omega)$ such that $\forall i < n(y_n(i) = e)$ for any $n < \omega$.*

PROOF. Let δ_i be the 1 of $\mathbb{Z}_i$. We define y_n as the following.

$$\begin{aligned} y_n(n) &= [\delta_0, \cdots, \delta_{n-1}] \\ y_n(n+1) &= [\delta_0, \cdots, \delta_n][\delta_0, \cdots, \delta_{n-1}][\delta_0, \cdots, \delta_{n-2}, \delta_n]\cdots[\delta_1, \cdots, \delta_n] \\ \vdots &= \vdots \end{aligned}$$

More precisely, let $A_{n+k,l} = \{f \in^l (n+k) | f \text{ is order preserving }\}$ with $n \leq l \leq n+k$. $A_{n+k,l} = \{f_i | i < l\}(i < j \to f_i < f_j)$ is linear ordered by the lexicographical order.

$$\prod_{f\in A_{n+k,l}} [\delta_{f(0)}, \cdots \delta_{f(l-1)}] := [\delta_{f_0(0)}, \cdots \delta_{f_0(l-1)}] \cdots [\delta_{f_{l-1}(0)}, \cdots \delta_{f_{l-1}(l-1)}]$$

$$y_n(n+k) = \prod\nolimits_{f\in A_{n+k,n+k}}[\delta_{f(0)}, \cdots \delta_{f(l-1)}] \cdots \prod\nolimits_{f\in A_{n+k,n}}[\delta_{f(0)}, \cdots \delta_{f(l-1)}]$$

Clearly, these are desired elements.

□

Proof of Proposition 5.11. Let $y_n (n < \omega)$ be as Lemma 4.4. There exists an homomorphism $h : \varprojlim(*_{i \in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \omega) \to \varprojlim(*_{i \in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \omega)$ which maps δ_n to y_n for any $n < \omega$. Clearly, the image of h is contained by H. Therefore, H is not n-slender. By Lemma 2.6 in [**6**], we can conclude $|H| = 2^\omega$. □

Proposition 5.14. *$H^* = \{W \in \times_{n<\omega}\mathbb{Z}_n | W$ is homogeneous$\}$ is n-slender.*

Proof. Firstly, we claim that $W \in H^* \setminus \{e\}$ implies $l_i(W) \neq 0$ for any $i < \omega$. Suppose the negation. Let n be the least natural number such that $W_{\{0,\cdots,n-1\}} \neq e$ and take $i < \omega$ with $l_i(W) = 0$. If $i < n$, then $W_{\{0,\cdots,n-1\}} = W_{n\setminus\{i\}}$. Since W is homogeneous, $e_{(n\setminus\{i\})(n-1)}(W_{n\setminus\{i\}}) = W_{\{0,\cdots,n-2\}} \neq e$. It is a contradiction to the minimality of n. If $n \leq i$, we can deduce a contradiction as well. Now, we show the n-slenderness of H^*. Assume not, then there exists a homomorphism $h : \times_{n<\omega}\mathbb{Z}_n \to H^*$ such that $h(\delta_n) \neq e$ for all $n < \omega$. By theorem 2.3 in [**12**], there exists a standard homomorphism $\overline{h}$ and $u \in \times_{n<\omega}\mathbb{Z}_n$ such that $h = u\overline{h}u^{-1}$. Because $\{n | l_0(\overline{h}(\delta_n)) \neq 0\}$ is finite, we can take N with $l_0(\overline{h}(\delta_N)) = 0$. On the other hand, $\overline{h}(\delta_N)$ is a non-trivial homogeneous word which is a contradiction. □

5.3. Problems.

Question 5.15. What is the cardinality of $\varprojlim(*_{i \in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \kappa)/\mathcal{U}$ when there are only finitely n such that $\mathcal{U}^n$ is an ultrafilter.

In the proof of theorem 5.10, the fact $\mathcal{U}^n$ is a σ-complete ultrafilter for all n is essential. It is clear that $\mathcal{U}^{n+1}$ is an ultrafilter implies $\mathcal{U}^n$ is so. Therefore, the case there are only finitely n such that $\mathcal{U}^n$ is an ultrafilter is left. We conjecture $|\varprojlim(*_{i \in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \kappa)/\mathcal{U}| \geq \kappa$ in the case.

Question 5.16. Is the cardinality of H^* countable or uncountable ?

6. Acknowledgment

I would like to thank to Prof. Katsuya Eda for giving helpful advice. I also thank the members of the seminar within the Eda laboratory.

Bibliography

[1] P. Ackermann, B. Fine, and G. Rosenberger, *On surface groups: motivating examples in combinatorial group theory*, Groups St. Andrews 2005 vol. 1, 96-129, London Math. Soc. Lecture Notes Ser., **339**, Cambridge Univ. Press, Cambridge, 2007.

[2] A. Blass, *Cardinal characteristic and the product of countably many infinite cyclic groups*, J. Algebra, **169** (1994), 512-540.

[3] J. Brendle and S. Shelah, *Evasion and Prediction II*, J. London Math. Soc., **53** (1996), 19-27.

[4] J. W. Cannon and G. R. Conner, *The combinatorial structure of the Hawaiian earring group*, Top. Appl. **106** (2000), 225-271.

[5] D. J. Collins, R. I. Grigorchuk, P. F. Kurchanov and H. Zieschang, *Combinatorial Group Theory and Applications to Geometry*, Encyclopaedia of math. sci., Vol. 58, Springer, 1998.

[6] G. Conner and K. Eda, *Fundamental groups having the whole information of spaces*, Top. Appl., **146-147** (2005), 317-328.

[7] K. Eda, *On a Boolean power of a torsion-free abelian group*, J. Algebra, **82** (1983), 84-93.

[8] ______, *Free σ-products and noncommutatively slender groups*, J. Algebra, **148** (1992), 243-263.

[9] ______, *The fundamental groups of one-dimensional spaces and spatial homomorphisms*, Top. Appl., **123** (2002), 479-505.

[10] ______, *Homotopy types of one-dimensional Peano continua*, Fund. Math., **209** (2010), 27-42.

[11] ______, *Atomic property of the fundamental groups of the Hawaiian earring and wild locally path-connected spaces*, Jour. Math. Soc. Japan, **63** (2011), 769-787.

[12] ______, *Free σ-product and fundamental groups of subspaces of the plane*, Top. Appl., **84** (1998), 283-306.

[13] ______, , RIMS Kokyuroku, **1530** (2007), 1-4.

[14] K. Eda and S. Shelah, *The non-commutative Specker phenomenon in the uncountable case*, J. Algebra, **252** (2002), 22-26.

[15] P. C. Eklof and A. H. Mekler, *Almost free modules*, North-Holland, 1990.

[16] H. Fischer and A. Zastrow, *The fundamental groups of the subsets of closed surfaces inject into their first shape groups*, Alg. & Geom. Topology, **5** (2005), 1655-1676.

[17] L. Fuchs, *Infinite Abelian Groups*, Vol.2, Academic Press, New York, 1970.

[18] R .Göbel, *Stout and slender groups*, J. Algebra, **35** (1975), 39-55.

[19] M. Hall, Jr, *The theory of groups*, Chelsea Pub. Co., 1959

[20] G. Higman, *Unrestricted free products and varieties of topological groups*, J. London Math. Soc., **27** (1952), 73-81.

[21] S. Kamo, *On the slender property of certain Boolean algebras*, J. Math. Soc. Japan, **38** (1986), 493-500.

[22] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Ergebn. Math. Grenzgeb, Springer, 1977.

[23] S. Shelah and L. Strüngmann, *The failure of the uncountable non- commutative Specker Phenomenon*, J. Group Theory, **4** (2001), 417-426.

[24] E. Specker, *Additive Grupen von Folgen ganzer Zahlen*, Portugal. Math., **9** (1950), 131-140.

[25] E. C. Zeeman, *On the direct sum of free cycles*, J. London Math. Soc., **30** (1955), 195-212.

早稲田大学　博士（理学）　学位申請　研究業績書

（2014　年　12　月　現在）

種 類 別	題名、　発表・発行掲載誌名、　発表・発行年月、　連名者（申請者含む）
口頭発表	「Specker phenomenon in the countable case」 Workshop on Combinatorial and Descriptive Set Theory　2008 年 8 月
口頭発表	「無限語と Specker-Eda number」数学基礎論若手の会 2009　2009 年 11 月
口頭発表	「Amalgamated free products of n-slender groups」 RIMS Set Theory Workshop 2010　2010 年 10 月
口頭発表	「Uncountable Specker phenomenon and n-slender groups」 数学基礎論若手の会 2010　2010 年 11 月
口頭発表	「finitely generated torsion free groups and non-commutatively slender groups」 数学基礎論若手の会 2011　2011 年 12 月
研究会誌（査読なし）	「Specker phenomenon in the countable case」 数理解析研究所講究録　(1619) 76-82　2008 年 12 月
共著論文（査読あり）	「The classification of the inverse limits of sequences of free groups of finite rank」Bull. London Math. Soc. (2013) 45(4): 671-676.　2013 年 2 月 K. Eda, J. Nakamura
単著論文（査読あり）	「Atomic properties of the Hawaiian earring group for HNN extensions」 Communications in Algebra　2015 年 1 月出版予定