Mathematical studies for a system describing the double-diffusive convection

二重拡散対流を記述する方程式系の数学的解析

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Contents

Chapter 1

Introduction

1.1 Introduction

In this dissertation, we study the following system of equations (DCBF), which describes double-diffusive convection phenomena of incompressible fluid contained in a porous medium.

$$
\text{(DCBF)}\begin{cases}
\partial_t \boldsymbol{u} = \nu\Delta\boldsymbol{u} - a\boldsymbol{u} - \nabla p + \boldsymbol{g}T + \boldsymbol{h}C + \boldsymbol{f}_1 & (x,t) \in \Omega\times[0,S], \\
\partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + f_2 & (x,t) \in \Omega\times[0,S], \\
\partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta C + \rho\Delta T + f_3 & (x,t) \in \Omega\times[0,S], \\
\nabla\cdot\boldsymbol{u} = 0 & (x,t) \in \Omega\times[0,S],
\end{cases}
$$

where Ω is a domain (open connected subset) in N-dimensional Euclidean space $\mathbb{R}^N$ and S denotes a length of time interval. Unknown functions of (DCBF) are

$$
\begin{aligned}
&\boldsymbol{u} = \boldsymbol{u}(x,t) = (u^1(x,t), u^2(x,t), \cdots, u^N(x,t)) &&: \text{ Fluid velocity,} \\
&T = T(x,t) &&: \text{ Temperature of fluid,} \\
&C = C(x,t) &&: \text{ Concentration of solute,} \\
&p = p(x,t) &&: \text{ Pressure of fluid.}
\end{aligned}
$$

As for given data in (DCBF), ν, a and ρ are positive constants and $\boldsymbol{g} = (g^1, g^2, \cdots, g^N)$, $\boldsymbol{h} = (h^1, h^2, \cdots, h^N)$ are constant vectors. The given external forces are denoted by $\boldsymbol{f}_1 = \boldsymbol{f}_1(x,t) = (f_1^1(x,t), f_1^2(x,t), \cdots, f_1^N(x,t))$, $f_2 = f_2(x,t)$ and $f_3 = f_3(x,t)$.

The time partial differential operator is designated by ∂_t and the gradient operator and the Laplace operator are written by $\nabla_x := (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_N})$ and $\Delta_x := \sum_{\mu=1}^N \partial_{x_\mu}^2$ respectively, where ∂_{x_μ} describes the x_μ-directional spatial partial differential operator ($\mu = 1, 2, \cdots, N$ and $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$). We simply write ∇ and Δ if there is no ambiguity of the variable. We here note that

$$
\boldsymbol{u}\cdot\nabla T = \sum_{\mu=1}^N u^\mu \partial_{x_\mu} T, \quad \boldsymbol{u}\cdot\nabla C = \sum_{\mu=1}^N u^\mu \partial_{x_\mu} C,
$$

that is, the inner product in $\mathbb{R}^N$ is written by $x \cdot y := \sum_{\mu=1}^N x_\mu y_\mu$ for each $x, y \in \mathbb{R}^N$ in this thesis.

As the boundary condition, we impose either

$$\boldsymbol{u} = 0, \quad T = 0, \quad C = 0 \qquad (x,t) \in \partial\Omega \times [0,S] \tag{1.1}$$

or

$$\boldsymbol{u} = 0, \quad \frac{\partial T}{\partial n} = 0, \quad \frac{\partial C}{\partial n} = 0 \qquad (x,t) \in \partial\Omega \times [0,S] \tag{1.2}$$

on (DCBF) if the boundary of the domain Ω, designated by $\partial\Omega$, is not an empty set. Here $\partial T/\partial n := \boldsymbol{n} \cdot \nabla T$ and $\boldsymbol{n}$ denotes the unit outward normal vector on $\partial\Omega$. Throughout this thesis, problems with the boundary condition (1.1) is said to be "(homogeneous) Dirichlet (boundary condition) case" and problems with (1.2) is said to be "(homogeneous) Neumann (boundary condition) case".

In this chapter, we introduce a physical background and some previous mathematical studies for (DCBF). We also state our aims and plans of this dissertation in the end of this chapter.

1.1.1 Physical Background

In this subsection, we give a brief review of physical background of the system (DCBF).

When the hot fluid saturated by some solute exists over the cold and fresh fluid, the sedimentation of solute with finger-shaped-like distribution occurs in the fluid. This phenomenon, called salt fingering, have been observed by the experiment in the field of oceanography for more than one hundred years. In 1960, the mechanism of this phenomenon was explained by Melvin Stern. According to his paper [54], the salt fingering is mainly characterized by the buoyancy of the fluid (e.g., heat expansion) and the difference of two diffusion speeds between the heat and the solute. Subsequently, it was revealed that his theory also can be applied to some unusual diffusion processes which arise in the fluid possessing two physical quantities with distinct diffusion speeds and heterogeneous distributions. Such complex phenomena are generally called "Double-diffusive convection phenomena" and have been investigated since the pioneer result by Stern.

Double-diffusive convection phenomenon appears in various situations, not only in oceanography. In astrophysics, the semiconvection process of massive stars can be explained within the framework of double-diffusive convection. In geology, the layers of volcanic rocks can be regarded as a result of double-diffusive convection of magma. In material engineering, double-diffusive convection of melting stuff causes the freckling of products (we can find more details or examples of double-diffusive convection in, e.g., Radko [51] and Brandt-Fernando [6]). It is well known that double-diffusive convection also occurs in the fluid contained in a porous medium. For example, we have to consider the effects of double-diffusive convection in models of the soil pollution, the storage of heat-generating materials such as grain and coal, the reservoir of radioactive substances and the chemical reaction in catalysts (see, e.g., Nield–Bejan [40]). Due to these important applications, double-diffusive convection in porous medium is one of the significant subjects in engineering.

The first equation of (DCBF) originates from the following equation, the so-called Brinkman-Forchheimer equations, which describes the relationship between the fluid velocity $\boldsymbol{u}$ and the pressure p in some porous medium (see, e.g., Vafai–Tien [61] and Chapter 1 of Nield–Bejan [40]).

$$\varrho \left(\frac{1}{\varphi} \partial_t \boldsymbol{u} + \frac{1}{\varphi^2} \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = -\nabla p + \frac{\mu}{\varphi \varrho} \Delta \boldsymbol{u} - \frac{\mu}{K} \boldsymbol{u} - \frac{c\varrho}{K^{1/2}} |\boldsymbol{u}| \boldsymbol{u}, \tag{1.3}$$

where c, μ and K are some physical constants and ϱ denotes the density of fluid (if we deal with incompressible fluid, ϱ is also a constant). Moreover, φ is a function of space variable $x \in \Omega$ which designate the porosity, the rate of void space in the medium. The right-hand side of (1.3) is a modified Darcy's law and the left-hand side is added on the analogy of the Navier-Stokes equations. However, some researchers, e.g., Beck [4] and Nield [39] pointed out that the effect from the convection term $\varphi^{-2}\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ is much less than those from other terms. Based on this background, we neglect $\varphi^{-2}\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ in our system (DCBF) (we note that dropping $\varphi^{-2}\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ dose not conflict the momentum conservation principle, since the momentum of fluid is lost by the collision with the porous medium). We also omit the quadratic term $c\varrho/K^{1/2}|\boldsymbol{u}|\boldsymbol{u}$ under the assumption that the fluid velocity $\boldsymbol{u}$ is sufficiently small due to the disturbance by the porous medium (strictly speaking, this omission of the quadratic term is not necessarily valid from physical viewpoint. The study for (DCBF) with the term $c\varrho/K^{1/2}|\boldsymbol{u}|\boldsymbol{u}$ should be an important future problem). Moreover, we assume that the medium is homogeneous, i.e., the porosity φ is a constant. In order to describe the effects of buoyancy, we add the terms $\boldsymbol{g}T$ and $\boldsymbol{h}C$ based on the Oberbeck-Boussinesq approximation (see, e.g., Joseph [31]). Then, by the normalization of constants, we obtain the first equation of (DCBF).

According to results of irreversible thermodynamics (see, e.g., Pottier [50], Førland–Førland–Ratkje [23]), the behavior of temperature T and concentration of solute C can be described by the following equations:

$$\begin{aligned} \partial_t T + \boldsymbol{u} \cdot \nabla T &= \nabla \cdot (D_T \nabla T + \rho_D \nabla C), \\ \partial_t C + \boldsymbol{u} \cdot \nabla C &= \nabla \cdot (D_C \nabla C + \rho_S \nabla T), \end{aligned}$$

where D_T and D_C are diffusion coefficients. The nonlinear terms $\boldsymbol{u} \cdot \nabla T$ and $\boldsymbol{u} \cdot \nabla C$ represent the advection of heat and solute (throughout this thesis, these terms $\boldsymbol{u} \cdot \nabla T$ and $\boldsymbol{u} \cdot \nabla C$ are called nonlinear diffusion terms, advection terms or convection terms) and the terms $\nabla \cdot (\rho_D \nabla C)$ and $\nabla \cdot (\rho_S \nabla T)$ describe interactions between the temperature and the concentration of solute, which are called Dufour's effect and Soret's effect respectively (ρ_D and ρ_S are called Dufour's coefficient and Soret's coefficient). In (DCBF), we only consider the contribution from Soret's effect, since Dufore's effect is much smaller than Soret's effect, particularly when we deal with liquid (see, e.g., Platten–Legros [49], Mojtabi–Charrier-Mojtabi [37] and Chapter 3 of Nield–Bejan [40]). To be precise, the coefficients D_T, D_C and ρ_S depend on T and C. However, we assume that ρ_S is constant and we set $D_T = D_C = 1$ in (DCBF) for simplicity (our arguments in this thesis can be applied to the case where diffusion coefficients are arbitrary positive constants).

1.1.2 Previous Results

Here, we exhibit some previous results from mathematical viewpoint.

In Piniewski [48], the initial boundary value problem is considered for the system where the first equation of (DCBF) is replaced by the Navier-Stokes equations and Soret's effect is neglected in 2-dimensional rectangle domains. By the application of Galerkin's method, the existence of a unique weak solution is assured in this paper. Establishing some a priori estimates, Piniewski [48] also showed the existence of global attractor.

As for the coupling of the Navier-Stokes equations with the second equation of (DCBF), which is called Boussinesq system or heat convection system, there exist earlier studies. In Inoue–Ôtani [29] and [30], for instance, the initial boundary value problem and the time periodic problem for the heat convection system in bounded domains with moving boundary are considered respectively. By reducing the system to an abstract equation in some Hilbert space and using the result given in Ôtani [41], where the solvability of Cauchy problem for abstract equations governed by subdifferential operators with non-monotone perturbations is discussed, they showed the global solvability of the initial boundary value problem with arbitrarily large initial data and external forces for $N = 2$ and with sufficiently small data for $N = 3$ respectively in Inoue–Ôtani [29]. Similarly, applying the abstract non-monotone perturbation theory by Ôtani [42], they assured the solvability of the periodic problem with large external forces for $N = 2$ and with small data for $N = 3$ in Inoue–Ôtani [30]. In Hishida [28], it is shown that the initial boundary value problem of Boussinesq system in bounded domains with $N \geqslant 2$ possesses a global solution in suitable L^q-space for sufficiently small initial data via the L^q-theory of semigroups generated by the Stokes operator and the Laplace operator. In Taniuchi [57], the solvability of Boussinesq system for non-decaying initial data is assured by the semigroup approach in some suitable Bezov spaces. By the application of semigroup theory, the time periodic problem of heat convection system in unbounded domains with the dimension $N \geqslant 3$ was also showed by Villamizar-Roa–Rodríguez-Bellido–Rojas-Medar [62].

The system where the first equation of (DCBF) is replaced by the steady linear Brinkman-Forchheimer equations (i.e., (DCBF) without the term $\partial_t \boldsymbol{u}$) is considered in, e.g., Straughan–Hutter [55], Payne–Song [44] and Lin–Payne [34]. In Payne–Song [44], some a priori estimates are established and a spatial decay estimate of solutions is obtained for cylindrical domains in $\mathbb{R}^3$. By Straughan–Hutter [55] and Lin–Payne [34], the continuous dependences of solutions on Soret's coefficient ρ and constant vectors $\boldsymbol{g}$, $\boldsymbol{h}$ were studied respectively in 3-dimensional bounded domains (see also Payne–Song [46], the case where $\boldsymbol{g}$ and $\boldsymbol{h}$ depend on the space variable $x \in \Omega$).

To the best our knowledge, it seems that the first result for the solvability of (DCBF) itself is given in Terasawa–Ôtani [60]. In Terasawa–Ôtani [60], the initial boundary value problem in bounded domains with homogeneous Dirichlet boundary condition is considered. Their main strategy follows that in Inoue–Ôtani [29]. That is to say, reducing (DCBF) to some abstract equation governed by subdifferential operators with non-monotone perturbations and applying the result of Ôtani [41], they assured the existence of a unique global solution provided that the space dimension is up to 3.

1.1.3 Main Purpose and Plan

Other than the above, a great number of mathematical studies have been devoted to investigations of Boussinesq system and the system (DCBF) where the first equation is replaced by the Navier-Stokes equations. In almost all such researches, the main strategies and concerns for the problem are to apply mathematical tools developed in the studies of the Navier-Stokes equations. Since the nonlinear diffusion terms $\boldsymbol{u} \cdot \nabla T$ and $\boldsymbol{u} \cdot \nabla C$ quite resemble to the convection term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ of the Navier-Stokes equations, applications of techniques in the Navier-Stokes equations were successful in the previous works. However, under such strategies, the peculiarities of the nonlinear diffusion terms $\boldsymbol{u} \cdot \nabla T$ and $\boldsymbol{u} \cdot \nabla C$ are concealed behind the difficulty of the convection term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$. That is to say, the terms $\boldsymbol{u} \cdot \nabla T$ and $\boldsymbol{u} \cdot \nabla C$ are handled with the same argument as that for $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and the difference between them is ignored. In fact, as is stated in the previous subsection, we already know that more advanced analysis can be accomplished and more precise results can be obtained for the double-diffusive convection system with the linear Brinkman-Forchheimer equations than known results for the Navier-Stokes equations. In Terasawa–Ôtani [60], for instance, the global solvability of (DCBF) for large data with the dimension $N = 3$ can be obtained, which is not achieved for the Navier-Stokes equation yet.

Main purpose of this thesis is to reveal the structures and the difficulties of the system (DCBF), which arise from the the nonlinear diffusion terms $\boldsymbol{u} \cdot \nabla T$, $\boldsymbol{u} \cdot \nabla C$, the buoyancy terms $\boldsymbol{g}T$, $\boldsymbol{h}C$ and the term of Soret's effect $\rho \Delta T$ under the simplification of equations associated with the fluid.

This dissertation consists of six chapters.

The next chapter is devoted to preparation of notations and mathematical tools. For example, we introduce the uniform C^k-domain, one of the typical examples of unbounded domains which allow Sobolev's embedding theorem and elliptic estimates of Laplace operator, to be used in Chapter 4. We also state the definition and some properties about the Helmholtz decomposition of Lebesgue spaces. Moreover, we give the definition and some examples of subdifferential operator and some known results for evolution equations governed by subdifferential operators for later use in Chapter 3. In Chapter 2, we define the dynamical system and its attractors and we prepare some abstract results for the construction of attractors in Chapter 6.

Chapter 3 deals with the system (DCBF) in bounded domains. Basic strategy in the third chapter relies on those in Inoue–Ôtani [29], [30] and Terasawa–Ôtani [60], i.e., we reduce (DCBF) to an abstract equation in some suitable Hilbert space, which is governed by the subdifferential operators, and we apply solvability results given by Ôtani [41] and [42] (see Section 2.3). To begin with, we check that the global solvability of the initial boundary value problem also holds for the homogeneous Neumann boundary condition case (1.2). By almost the same procedure as that in Terasawa–Ôtani [60], it is shown that the initial boundary value problem of (DCBF) with Neumann condition possesses a unique global solution in Section 3.2. We also consider the solvability of time periodic problem for (DCBF) in Section 3.3 and 3.4. We here note that the required conditions in the abstract result Ôtani [42] (periodic problem) are stricter than those in Ôtani [41] (Cauchy problem) and the direct application of [42] to the time periodic problem of (DCBF) seems to be difficult (required conditions in [41], [42] will be stated in Section 2.3). To cope with this difficulty, we introduce some approximate equations of (DCBF) with dissipation terms and cut-off approximations. By using the abstract result in [42], we first assure the existence of solutions for the approximate problem. Discussing the convergences of approximate solutions and equations, we shall show the solvability of the time periodic problem of (DCBF) with Dirichlet condition (1.1) in Section 3.3. As for Neumann condition case, we need another step of relaxation approximation due to the lack of coercivity of the Laplace operator. In order to assure the convergence of these relaxation terms, we assume the following additional condition for the external forces:

$$\int_\Omega f_2 dx = \int_\Omega f_3 dx = 0,$$

which is also one of the necessary conditions for the existence of periodic solution of (DCBF) with Neumann condition. Under the above condition, the solvability of time periodic problem can be derived for Neumann case in Section 3.4.

In Chapter 4, we consider the initial boundary value problem of (DCBF) in unbounded domains. We here remark that it is impossible to follow the same procedure as that in the Chapter 3, since the φ-level set compactness is imposed among other required conditions in [41] (see Section 2.3) and this condition is usually satisfied by Rellich-Kondrachov's theorem, which requires the boundedness of Ω. In Chapter 4, we introduce another strategy which relies on Banach's contraction mapping principle and we shall assure the existence of a unique global solution in uniform C^2-domains (see Section 2.1) with the dimension $N \leqslant 4$ for large initial data and external forces. We note that our result in fourth chapter completely cover those in Terasawa–Ôtani [60] and Chapter 3.

Chapter 5 is concerned with the time periodic problem of (DCBF) in the whole domain $\mathbb{R}^N$ with the dimension $N = 3, 4$. Since the solvability can be assured for large data, i.e., arbitrary large initial data and external forces for the initial boundary value problem (Chapter 4), we can expect the solvability of time periodic problem without the smallness condition of external forces. However, there are very few results for the solvability of time periodic problem in unbounded domains with large data, in particular, for parabolic equations with non-monotone perturbations, where the uniqueness of solutions is not obtained. As for the studies for the parabolic equations with non-monotone perturbations, the time periodic problems in unbounded domains have been investigated by, e.g., Maremonti [35], Kozono–Nakao [33] for the Navier-Stokes equations and Villamizar-Roa–Rodríguez-Bellido–Rojas-Medar [62] for Boussinesq system. However, their procedures do not fit our aim, since the smallness of given data seems to be essential in order to assure the convergence of iterations in their argument. On the other hand, as for the solvability of periodic problem with large data, abstract evolution equations associated with subdifferential operators in Hilbert space have been studied by, for instance, in Bénilan–Brézis [5], Nagai [38], Yamada [63] and Ôtani [42]. In these abstract results, some required conditions are guaranteed by the boundedness of space domains (e.g., the coercivity of subdifferential operators, φ-level set compactness) and the direct application of their argument to the problem in unbounded domains seems to be impossible (for this reason, we can not use our strategy in Chapter 3). We also note that the strategy for semi-linear parabolic equations given in, e.g., Pao [43] is not available, since the term of Soret's effect $\rho\Delta T$ makes difficult to assure the comparison theorem. In spite of these difficulties, the existence of time periodic solutions of (DCBF) in $\mathbb{R}^N$ with $N = 3, 4$ will be shown for large data in fifth chapter, via the convergence of solutions of some approximate equations in bounded domains.

Chapter 6 deals with the study for the large time behavior of solutions whose existence is assured in Chapter 4 in terms of global attractor and exponential attractor (definition of attractors will be stated in Section 2.4). We consider the case where Ω is bounded for both cases with homogeneous Dirichlet and Neumann boundary condition and external forces do not depend on the time variable t. The construction of global and exponential attractor of the dynamical system relies on the abstract results, e.g., Babin–Vishik [3], Chepyzhov–Vishik [16], Robinson [52], Temam [59] for global attractor and Eden–Foais–Nicolaenko–Temam [18], Efendiev [19], Efendiev–Miranville–Zelik [20] for exponential attractor (see Section 2.4). In Piniewski [48], the existence of global attractor is already shown for $N = 2$. However, we need to establish more precise a priori estimates for the case $N \geqslant 3$ than those in [48] and previous chapters so that abstract results stated above can be applied. In order to derive such minute estimates of solutions for higher regularity, we introduce some abstract result given in Brézis [11] and its modification. As mentioned in Chapter 6, when the homogeneous Neumann condition is imposed to (DCBF), there is no global and exponential attractors in the standard sense due to the so-called mass conservation property. Hence we introduce the restricted dynamical system by the same way as that in Brochet-Hilhorst [13] and we shall show the existence of attractors for this dynamical system.

Chapter 2

Preliminary

In this chapter, we define some notations and prepare basic mathematical tools which will be used in the following chapters of this thesis. Almost all of propositions and corollaries in this chapter are exhibited without any proof. However, we can find their demonstrations and more details in the references listed near the statements.

2.1 Lebesgue Space and Sobolev Space

We first recall fundamental facts about the Lebesgue space, the Sobolev space and the Bochner space in this section.

2.1.1 Notations and Basic Properties

Let Ω be a domain of N-dimensional Euclidean space $\mathbb{R}^N$. In this dissertation, $L^q(\Omega)$ and $W^{k,q}(\Omega)$ stand for the standard Lebesgue spaces and Sobolev spaces ($1 \leqslant q \leqslant \infty$, $k \in \mathbb{N}$) and $H^k(\Omega) := W^{k,2}(\Omega)$. The usual norm in $L^q(\Omega)$, $W^{k,q}(\Omega)$ and $H^k(\Omega)$ are designated by $|\cdot|_{L^q(\Omega)}$, $|\cdot|_{W^{k,q}(\Omega)}$ and $|\cdot|_{H^k(\Omega)}$ (definitions and details can be found in, e.g., Adams [1], Brézis [12] and Folland [22]). To be precise, for each $V \in W^{1,q}(\Omega)$, ∇V is defined as vector valued functions. However, in this thesis, we simply write

$$|\nabla V|^q_{L^q(\Omega)} := \sum_{\mu=1}^{N} |\partial_{x_\mu} V|^q_{L^q(\Omega)}.$$

Let $C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact supports in Ω. Then we define $W_0^{k,q}(\Omega)$ and $H_0^k(\Omega)$ by the closure of $C_0^\infty(\Omega)$ under the norm in $W^{k,q}(\Omega)$ and $H^k(\Omega)$ respectively.

According to, e.g., Adams [1], Brézis [12] and Folland [22], Lebesgue and Sobolev spaces hold the following basic properties.

Proposition 2.1.1 (Hölder's inequality). *Let $q \in [1, \infty]$ and define q' by*

$$q' := \begin{cases} q/(q-1) & (\text{ if } \quad 1 < q < \infty), \\ \infty & (\text{ if } \quad q = 1), \\ 1 & (\text{ if } \quad q = \infty) \end{cases}$$

(henceforth, called the conjugate Hölder exponent of q). Then for any $V_1 \in L^q(\Omega)$ and $V_2 \in L^{q'}(\Omega)$, $V_1V_2 \in L^1(\Omega)$ holds and

$$|V_1V_2|_{L^1(\Omega)} \leqslant |V_1|_{L^q(\Omega)}|V_2|_{L^{q'}(\Omega)}.$$

Proposition 2.1.2 (Duality). *Let $q \in [1, \infty)$ and let q' be the conjugate Hölder exponent, i.e., $q' := q/(q-1)$ for $1 < q < \infty$ and $q' := \infty$ for $q = 1$. Then $L^{q'}(\Omega)$ is the dual space of $L^q(\Omega)$.*

Proposition 2.1.3 (Density). *For any $q \in [1, \infty)$, $C_0^\infty(\Omega)$ is dense in $L^q(\Omega)$.*

Proposition 2.1.4 (Reflexivity). *Let $q \in (1, \infty)$, then $L^q(\Omega)$, $W^{k,q}(\Omega)$ and $W_0^{k,q}(\Omega)$ are reflexive for any $k \in \mathbb{N}$.*

From Hölder's inequality, we can derive the following.

Corollary 2.1.1 (Logarithmic convexity of L^q-norms). *Let $1 \leqslant q_1 < q_2 < q_3 \leqslant \infty$. Then $L^{q_1}(\Omega) \cap L^{q_3}(\Omega) \subset L^{q_2}(\Omega)$ is valid. Moreover, for any $V \in L^{q_1}(\Omega) \cap L^{q_3}(\Omega)$,*

$$|V|_{L^{q_2}(\Omega)} \leqslant |V|_{L^{q_1}(\Omega)}^{1-\alpha}|V|_{L^{q_3}(\Omega)}^{\alpha}$$

holds, where $\alpha := (1/q_1 - 1/q_2)/(1/q_1 - 1/q_3)$ (if $q_3 = \infty$, $\alpha := 1 - q_1/q_2$).

2.1.2 Embedding Inequalities

In this subsection, we recall the following important inequality, the so-called Sobolev's embedding inequality (see Adams [1], Brézis [12] and Evans [21]).

Proposition 2.1.5 (Sobolev's embedding theorem). *Let $1 \leqslant q < N$ and $q^* := qN/(N-q)$. Then the embedding $W^{1,q}(\mathbb{R}^N) \subset L^{q^*}(\mathbb{R}^N)$ holds. Moreover, there exist a constant γ which depends only on q and N such that*

$$|V|_{L^{q^*}(\mathbb{R}^N)} \leqslant \gamma|\nabla V|_{L^q(\mathbb{R}^N)} \qquad \forall V \in W^{1,q}(\mathbb{R}^N). \tag{2.1}$$

If $q = N$, then the embedding $W^{1,q}(\mathbb{R}^N) \subset L^r(\mathbb{R}^N)$ holds for any $r \in [q, \infty)$. Moreover, if $q > N$, then the embedding $W^{1,q}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$ holds.

By zero-extension of functions, Proposition 2.1.5 immediately yields the following.

Corollary 2.1.2 (Sobolev's embedding theorem)**.** *Let* $1 \leqslant q < N$ *and* $q^* := qN/(N-q)$. *Then the embedding* $W_0^{1,q}(\Omega) \subset L^{q^*}(\Omega)$ *holds. Moreover, there exist a constant* γ *which depends only on* q *and* N *such that*

$$|V|_{L^{q^*}(\Omega)} \leqslant \gamma |\nabla V|_{L^q(\Omega)} \qquad \forall V \in W_0^{1,q}(\Omega). \tag{2.2}$$

If $q = N$, *then the embedding* $W_0^{1,q}(\Omega) \subset L^r(\Omega)$ *holds for any* $r \in [q, \infty)$. *Moreover, if* $q > N$, *then the embedding* $W_0^{1,q}(\Omega) \subset L^\infty(\Omega)$ *holds.*

Here we introduce the definition of C^k-class domain.

Definition 2.1.1 (C^k-class domain)**.** *The domain* $\Omega \subset \mathbb{R}^N$ *with the boundary* $\partial\Omega$ *is said to be* C^k*-class domain, if for any* $x \in \partial\Omega$, *there exist a neighborhood* $\mathcal{O}_x$ *of* x *in* $\mathbb{R}^N$ *and a* C^k*-diffeomorphism* $\psi_x : \mathcal{O}_x \to \mathbb{B}^N := \{x \in \mathbb{R}^N; |x| < 1\}$ *such that*

$$\psi_x(\mathcal{O}_x \cap \Omega) = \{x = (x_1, x_2, \cdots, x_N) \in \mathbb{B}^N; x_N > 0\},$$
$$\psi_x(\mathcal{O}_x \cap \partial\Omega) = \{x = (x_1, x_2, \cdots, x_N) \in \mathbb{B}^N; x_N = 0\}.$$

According to, e.g., Adams [1], Brézis [12] and Evans [21], Sobolev's embedding theorem also holds for C^1-class domain with bounded boundary.

Proposition 2.1.6. *Let* Ω *be* C^1*-class domain with bounded boundary* $\partial\Omega$ *or the half space* $\mathbb{R}^N_+ := \{x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N;\ x_N > 0\}$. *Moreover, let* $1 \leqslant q < N$ *and* $q^* := qN/(N-q)$. *Then the embedding* $W^{1,q}(\Omega) \subset L^{q^*}(\Omega)$ *holds and there exist a constant* γ *which depends on* Ω, q *and* N *such that*

$$|V|_{L^{q^*}(\Omega)} \leqslant \gamma |V|_{W^{1,q}(\Omega)} \qquad \forall V \in W^{1,q}(\Omega). \tag{2.3}$$

If $q = N$, *then the embedding* $W^{1,q}(\Omega) \subset L^r(\Omega)$ *holds for any* $r \in [q, \infty)$. *Moreover, if* $q > N$, *then the embedding* $W^{1,q}(\Omega) \subset L^\infty(\Omega)$ *holds.*

Here we remark that the coefficient γ appearing in (2.3) depends on the shape or the radius of Ω, although the coefficient in (2.1) and (2.2) depends only on N and q.

We also recall the following compact embedding theorem which holds for bounded domain case.

Proposition 2.1.7 (Rellich-Kondrachov's theorem)**.** *Let* Ω *be a bounded* C^1*-class domain and let* $1 \leqslant q < N$ *and* $r \in [1, q^*)$, *where* $q^* := qN/(N-q)$. *Then* $W^{1,q}(\Omega)$ *is compactly embedded in* $L^r(\Omega)$.

From now on, the exponent $q^* := qN/(N-q)$ is called the critical Sobolev exponent associated with q.

2.1.3 Elliptic Estimates

In this subsection, we mention the elliptic estimates of the Laplace operator $-\Delta$. Proofs of propositions below can be found in, e.g., Brézis [12].

From now on, we write $-\Delta_D$ and $-\Delta_N$ in order to represent the Laplace operator $-\Delta$ with the homogeneous Dirichlet or Neumann boundary condition respectively. That is to say, $-\Delta_D$ and $-\Delta_N$ describe the Laplace operator $-\Delta$ defined on

$$D(-\Delta_D) := H^2(\Omega) \cap H_0^1(\Omega), \qquad D(-\Delta_N) := \left\{ V \in H^2(\Omega);\ \frac{\partial V}{\partial n} = 0 \quad \text{on } \partial\Omega \right\}$$

respectively ($D(-\Delta_D)$ and $D(-\Delta_N)$ are called the domain of $-\Delta_D$ and $-\Delta_N$). We here remark that $-\Delta_D = -\Delta_N$ is valid if $\Omega = \mathbb{R}^N$.

Then $-\Delta_D$ and $-\Delta_N$ possess the following properties.

Proposition 2.1.8 (Elliptic estimate for $-\Delta_D$)**.** *Let Ω be the whole space $\mathbb{R}^N$, the half space $\mathbb{R}^N_+$ or C^2-domain with bounded boundary $\partial\Omega$ and let $F \in L^2(\Omega)$. Moreover, assume that $V \in H_0^1(\Omega)$ is a weak solution of $-\Delta_D V + V = F$, i.e.,*

$$\begin{cases} -\Delta V + V = F & \text{in } \Omega, \\ V = 0 & \text{on } \partial\Omega. \end{cases}$$

That is to say, assume that $V \in H_0^1(\Omega)$ satisfies

$$\int_\Omega \nabla V \cdot \nabla W + \int_\Omega VW = \int_\Omega FW \qquad \forall W \in H_0^1(\Omega).$$

Then, V belongs to $D(-\Delta_D)$ and becomes a strong solution of $-\Delta_D V + V = F$. Moreover, there exist a constant γ_D which depends only on Ω and N such that $|V|_{H^2(\Omega)} \leqslant \gamma_D |F|_{L^2(\Omega)}$, i.e.

$$|V|_{H^2(\Omega)} \leqslant \gamma_D(|\Delta V|_{L^2(\Omega)} + |V|_{L^2(\Omega)}) \tag{2.4}$$

holds.

Proposition 2.1.9 (Elliptic estimate for $-\Delta_N$)**.** *Let Ω be the whole space $\mathbb{R}^N$, the half space $\mathbb{R}^N_+$ or C^2-domain with bounded boundary $\partial\Omega$ and let $F \in L^2(\Omega)$. Moreover, assume that $V \in H^1(\Omega)$ is a weak solution of $-\Delta_N V + V = F$, i.e.,*

$$\begin{cases} -\Delta V + V = F & \text{in } \Omega, \\ \frac{\partial V}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

That is to say, assume that $V \in H^1(\Omega)$ satisfies

$$\int_\Omega \nabla V \cdot \nabla W + \int_\Omega VW = \int_\Omega FW \qquad \forall W \in H^1(\Omega).$$

Then, V belongs to $D(-\Delta_N)$ and becomes a strong solution of $-\Delta_N V + V = F$. Moreover, there exist a constant γ_N which depends only on Ω and N such that $|V|_{H^2(\Omega)} \leqslant \gamma_N |F|_{L^2(\Omega)}$, i.e.

$$|V|_{H^2(\Omega)} \leqslant \gamma_N(|\Delta V|_{L^2(\Omega)} + |V|_{L^2(\Omega)}) \tag{2.5}$$

holds.

Propositions 2.1.8 and 2.1.9 imply that if V belongs to $D(-\Delta_D)$ or $D(-\Delta_N)$, then H^2-norm of V is bounded only by L^2-norm of $-\Delta V$ and V. The inequalities (2.4) and (2.5) are called elliptic estimate of $-\Delta_D$ or $-\Delta_N$ respectively.

2.1.4 Uniform C^k-Domain

Here we introduce the concept of uniform C^k-domain.

Definition 2.1.2 (Uniform C^k-Regular Class). *The domain $\Omega \subset \mathbb{R}^N$ with the boundary $\partial\Omega$ is said to be uniformly regular of class C^k (or said to be a uniform C^k-domain), if there exist a family of coordinate chart $(\mathcal{O}_j, \psi_j)$ $(j \in \mathbb{N})$ of $\overline{\Omega}$ (the closure of Ω) which satisfies the following conditions (called the uniform C^k-regularity condition):*

1. *Each ψ_j is a C^k-diffeomorphism from $\mathcal{O}_j$ onto $\mathbb{B}^N := \{x \in \mathbb{R}^N; |x| < 1\}$ such that*
$$\psi_j(\mathcal{O}_j \cap \Omega) = \{x = (x_1, x_2, \cdots, x_N) \in \mathbb{B}^N;\ x_N > 0\},$$
$$\psi_j(\mathcal{O}_j \cap \partial\Omega) = \{x = (x_1, x_2, \cdots, x_N) \in \mathbb{B}^N;\ x_N = 0\}.$$
2. *There exist some constant M independent of $j \in \mathbb{N}$ such that*
$$\sum_{|\alpha| \leqslant k} \left\{ \sup_{x \in \mathcal{O}_j} |D^\alpha \psi_j(x)| + \sup_{y \in \mathbb{B}^N} |D^\alpha \psi_j^{-1}(y)| \right\} \leqslant M,$$
where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N) \in \mathbb{N}^N$, $D^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_N}^{\alpha_N}$ and $|\alpha| := \sum_{i=1,2,\cdots,N} \alpha_i$.
3. *There exist some positive constant ε such that ε-neighbourhood of $\partial\Omega$ in $\overline{\Omega}$ is contained by $\bigcup_{j \in \mathbb{N}} \psi_j^{-1}(\frac{1}{2}\mathbb{B}^N)$, where $\frac{1}{2}\mathbb{B}^N := \{x/2;\ x \in \mathbb{B}^N\}$.*
4. *There exists some natural number n_0 such that any (n_0+1)-distinct $\mathcal{O}_j$ does not possess intersection.*

The simplest examples of uniform C^k-domain are the half space $\mathbb{R}^N_+$ and C^k-class domains with bounded boundary. Moreover, if the boundary of Ω can be represented by C^k-class bounded function from $\mathbb{R}^{N-1}$ to $\mathbb{R}$, then the domain Ω is uniformly regular of class C^k.

As is mentioned in Amann [2] and Browder [14], if Ω is a uniform C^k-domain and satisfies the uniform C^k-regularity condition in Definition 2.1.2, then there exist a partition of unity on Ω which possesses good properties. Then, we can apply almost the same argument to the uniform C^k-domain case as those for C^k-class domains with bounded boundary (see Proposition 2.1.6, 2.1.8 and 2.1.9) and we can assure the following facts.

Proposition 2.1.10 (Sovolev's embedding inequality for unbounded domains)**.** *Let* Ω *be uniformly regular of class* C^1*. Moreover, let* $1 \leqslant q < N$ *and* $q^* := qN/(N-q)$*. Then the embedding* $W^{1,q}(\Omega) \subset L^{q^*}(\Omega)$ *holds and there exist a constant* γ *which depends on* Ω*,* q *and* N *such that*

$$|V|_{L^{q^*}(\Omega)} \leqslant \gamma |V|_{W^{1,q}(\Omega)} \qquad \forall V \in W^{1,q}(\Omega).$$

If $q = N$*, then the embedding* $W^{1,q}(\Omega) \subset L^r(\Omega)$ *holds for any* $r \in [q, \infty)$*. Moreover, if* $q > N$*, then the embedding* $W^{1,q}(\Omega) \subset L^\infty(\Omega)$ *holds.*

Proposition 2.1.11 (Elliptic estimate for $-\Delta_D$ in unbounded domains)**.** *Let* Ω *be a uniform* C^2*-domain and let* $F \in L^2(\Omega)$*. Moreover, assume that* $V \in H_0^1(\Omega)$ *is a weak solution of* $-\Delta_D V + V = F$*. That is to say, assume that* $V \in H_0^1(\Omega)$ *satisfies*

$$\int_\Omega \nabla V \cdot \nabla W + \int_\Omega VW = \int_\Omega FW \qquad \forall W \in H_0^1(\Omega).$$

Then, V *belongs to* $D(-\Delta_D)$ *and becomes a strong solution of* $-\Delta_D V + V = F$*. Moreover, there exist a constant* γ_D *which depends only on* Ω *and* N *such that* $|V|_{H^2(\Omega)} \leqslant \gamma_D |F|_{L^2(\Omega)}$*, i.e.*

$$|V|_{H^2(\Omega)} \leqslant \gamma_D(|\Delta V|_{L^2(\Omega)} + |V|_{L^2(\Omega)}) \tag{2.6}$$

holds.

Proposition 2.1.12 (Elliptic estimate for $-\Delta_N$ in unbounded domains)**.** *Let* Ω *be a uniform* C^2*-domain and let* $F \in L^2(\Omega)$*. Moreover, assume that* $V \in H^1(\Omega)$ *is a weak solution of* $-\Delta_N V + V = F$*. That is to say, assume that* $V \in H^1(\Omega)$ *satisfies*

$$\int_\Omega \nabla V \cdot \nabla W + \int_\Omega VW = \int_\Omega FW \qquad \forall W \in H^1(\Omega).$$

Then, V *belongs to* $D(-\Delta_N)$ *and becomes a strong solution of* $-\Delta_N V + V = F$*. Moreover, there exist a constant* γ_N *which depends only on* Ω *such that* $|V|_{H^2(\Omega)} \leqslant \gamma_N |F|_{L^2(\Omega)}$*, i.e.*

$$|V|_{H^2(\Omega)} \leqslant \gamma_N(|\Delta V|_{L^2(\Omega)} + |V|_{L^2(\Omega)}) \tag{2.7}$$

holds.

2.1.5 Bochner Space

Let X denote a Banach space with the norm $\|\cdot\|_X$. We define the space $L^q(0,S;X)$ by the set of X-valued functions $U : [0,S] \to X$ such that $|U|_{L^q(0,S;X)} < \infty$, where

$$|U|_{L^q(0,S;X)} := \begin{cases} \left(\displaystyle\int_0^S \|U(t)\|_X^q dt \right)^{1/q} & \text{if } 1 \leqslant q < \infty, \\ \operatorname{ess\,sup}_{t \in [0,S]} \|U(t)\|_X & \text{if } q = \infty. \end{cases}$$

In this subsection, we state some properties of functions belonging to $L^q(0,S;X)$, the so-called Bochner space (see, e.g., Adams [1] and Yosida [64]).

First, by almost the same argument as that for scalar valued functions, the following density property of Bochner space $L^q(0,S;X)$ can be guaranteed.

Proposition 2.1.13 (Density)**.** *Let $1 \leqslant q < \infty$. Then $C_0^\infty((0,S);X)$ is dense in $L^q(0,S;X)$.*

Here we consider the special case where $X = L^{q_1}(\Omega)$ with $q_1 \in [1,\infty)$. Due to the facts that $C_0^\infty(\Omega)$ is dense in $L^{q_1}(\Omega)$ (Proposition 2.1.3) and arbitrary function of $L^{q_2}(0,S;X)$ can be approximated by some step functions with value in X for $q_2 \in [1,\infty)$, we can show that any function belonging to $L^{q_2}(0,S;L^{q_1}(\Omega))$ ($q_1, q_2 \in [1,\infty)$) is approximated by some step functions with value in $C_0^\infty(\Omega)$ (this argument is also valid for the case where $X = W_0^{k,q_1}(\Omega)$). This argument yields the following density property.

Proposition 2.1.14 (Density)**.** *Let $q_1, q_2 \in [1,\infty)$. Then $C_0^\infty((0,S);C_0^\infty(\Omega))$ is dense in $L^{q_2}(0,S;L^{q_1}(\Omega))$. Moreover, $C_0^\infty((0,S);C_0^\infty(\Omega))$ is dense in $L^{q_2}(0,S;W_0^{k,q_1}(\Omega))$ for any $k \in \mathbb{N}$.*

In general, we can not characterize the dual of Bochner space $L^q(0,S;X)$. However, when X is reflexive, we can obtain almost the same result as those for scalar valued functions (see, e.g., Phillips [47]).

Proposition 2.1.15 (Duality)**.** *Let $1 \leqslant q < \infty$ and let q' be the conjugate Hölder exponent of q. Moreover, assume that X is reflexive. Then the dual space of $L^q(0,S;X)$ coincides with $L^{q'}(0,S;X^*)$, where X^* is the dual of X.*

Corollary 2.1.3 (Reflexivity)**.** *Let $1 < q < \infty$ and assume that X is reflexive. Then $L^q(0,S;X)$ is reflexive.*

Next we consider the space $W^{1,q}(0,S;X)$ ($q \in [1,\infty]$) defined by

$$W^{1,q}(0,S;X) := \left\{ U \in L^q(0,S;X);\ \begin{array}{c} \exists G \in L^q(0,S;X) \text{ such that} \\ U(t) = U(0) + \displaystyle\int_0^t G(s)ds \quad \forall t \in [0,S] \end{array} \right\}.$$

Generally speaking, the absolutely continuity does not necessarily lead to the existence of primitive function for X-valued functions. However, when X is reflexive, the following properties are equivalent:

1. $U \in W^{1,q}(0,S;X)$.
2. U is absolutely continuous on $[0,S]$ and differentiable at a.e. $t \in [0,S]$. Moreover, the time derivative of U belongs to $L^q(0,S;X)$.

Moreover, we can find the following fact in, e.g., appendix of Brézis [11].

Proposition 2.1.16. *Let X be a reflexive Banach space and let $U \in L^q(0,S;X)$ with $1 < q < \infty$. Moreover, assume that there exist a constant γ such that*

$$\int_0^{T-h} \|U(t+h) - U(t)\|_X^q dt \leqslant \gamma h^q \qquad \forall h \in (0,T)$$

holds. Then U belongs to $W^{1,q}(0,S;X)$.

2.2 Helmholtz Decomposition

We here deal with N-component vector valued functions in order to describes the fluid velocity. In particular, this section is mainly devoted to the definition and the fundamental facts of Helmholtz decomposition.

We define the spaces $\mathbb{L}^q(\Omega) := (L^q(\Omega))^N$, $\mathbb{W}^{k,q}(\Omega) := (W^{k,q}(\Omega))^N$ and $\mathbb{H}^k(\Omega) := \mathbb{W}^{k,q}(\Omega)$ with the norm

$$|\boldsymbol{w}|_{\mathbb{L}^q(\Omega)} := \sum_{\mu=1}^N |w^\mu|_{L^q(\Omega)}, \quad |\boldsymbol{w}|_{\mathbb{W}^{k,q}(\Omega)} := \sum_{\mu=1}^N |w^\mu|_{W^{k,q}(\Omega)},$$

where $\boldsymbol{w} = (w^1, w^2, \cdots, w^N)$. Moreover, we define $\mathbb{W}_0^{k,q}(\Omega)$ and $\mathbb{H}_0^k(\Omega)$ by the closure of $\mathbb{C}_0^\infty(\Omega)$ in $\mathbb{W}^{k,q}(\Omega)$ and $\mathbb{H}^k(\Omega)$, where $\mathbb{C}_0^\infty(\Omega) := (C_0^\infty(\Omega))^N$. Strictly speaking, for each $\boldsymbol{w} \in \mathbb{W}^{1,q}(\Omega)$, $\nabla \boldsymbol{w}$ is defined as tensor valued function. However, we use the following notation:

$$|\nabla \boldsymbol{w}|_{\mathbb{L}^q(\Omega)}^q := \sum_{\mu=1}^N |\partial_{x_\mu} \boldsymbol{w}|_{\mathbb{L}^q(\Omega)}^q.$$

Moreover, for any $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathbb{W}^{1,q}(\Omega)$, we write

$$\int_\Omega \nabla \boldsymbol{w}_1 \cdot \nabla \boldsymbol{w}_2 := \sum_{\mu=1}^N \int_\Omega \nabla w_1^\mu \cdot \nabla w_2^\mu$$

throughout this thesis, where $\boldsymbol{w}_i = (w_i^1, w_i^2, \cdots, w_i^N)$ $(i = 1, 2)$.

We remark that the propositions stated in Section 2.1.1, 2.1.2 and 2.1.4 (Hölder's inequality, duality, density, reflexivity and Sobolev's embedding) also holds for $\mathbb{L}^q(\Omega)$, $\mathbb{W}^{k,q}(\Omega)$ and $\mathbb{H}^k(\Omega)$.

2.2.1 Helmholtz Decomposition

We first define the following spaces:

$$\begin{aligned}
&\mathbb{C}_\sigma^\infty(\Omega) := \{\boldsymbol{w} \in \mathbb{C}_0^\infty(\Omega);\ \nabla \cdot \boldsymbol{w}(x) = 0 \ \ \forall x \in \Omega\},\\
&\mathbb{L}_\sigma^q(\Omega) : \text{ the closure of } \mathbb{C}_\sigma^\infty(\Omega) \text{ under the norm of } \mathbb{L}^q(\Omega),\\
&G_q(\Omega) := \{\boldsymbol{w} \in \mathbb{L}^q(\Omega);\ \exists p \in W_{\mathrm{loc}}^{1,q}(\overline{\Omega}), \ \ \text{s.t., } \boldsymbol{w} = \nabla p\},
\end{aligned}$$

where $\overline{\Omega}$ is the closure of Ω in $\mathbb{R}^N$.

It is well known that the following result can be assured for $q = 2$ (see, e.g., Chapter III of Galdi [27]).

Proposition 2.2.1 (Helmholtz Decomposition with $q = 2$)**.** *Let Ω be any domain in $\mathbb{R}^N$ with $N \geqslant 2$. Then $\mathbb{L}^2_\sigma(\Omega)$ and $G_2(\Omega)$ become orthogonal subspaces in $\mathbb{L}^2(\Omega)$. Moreover, $\mathbb{L}^2(\Omega) = \mathbb{L}^2_\sigma(\Omega) \oplus G_2(\Omega)$ holds, i.e., any $\boldsymbol{v} \in \mathbb{L}^2(\Omega)$ is uniquely decomposed by $\boldsymbol{w}_1 \in \mathbb{L}^2_\sigma(\Omega)$ and $\boldsymbol{w}_2 \in G_2(\Omega)$:*

$$\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2 \qquad \boldsymbol{w}_1 \in \mathbb{L}^2_\sigma(\Omega),\ \ \boldsymbol{w}_2 \in G_2(\Omega).$$

We here remark that

$$|\boldsymbol{w}_1|_{\mathbb{L}^2(\Omega)} + |\boldsymbol{w}_2|_{\mathbb{L}^2(\Omega)} \leqslant 2|\boldsymbol{v}|_{\mathbb{L}^2(\Omega)}$$

also holds for any $\boldsymbol{v} \in \mathbb{L}^2(\Omega)$ and its decomposition $\boldsymbol{w}_1 \in \mathbb{L}^2_\sigma(\Omega)$ and $\boldsymbol{w}_2 \in G_2(\Omega)$ due to the properties of orthogonal projections in Hilbert space.

For $q \neq 2$, such decomposition does not necessarily hold for arbitrary domain Ω. However, in special cases, we can obtain the same result as that for $q = 2$ (see, e.g., Chapter III of Galdi [27]).

Proposition 2.2.2 (Helmholtz Decomposition with $q \neq 2$)**.** *Let $1 < q < \infty$ and let Ω be either the whole space $\mathbb{R}^N$, the half space $\mathbb{R}^N_+$ or C^2-class domain with bounded boundary with $N \geqslant 2$. Then $\mathbb{L}^q(\Omega) = \mathbb{L}^q_\sigma(\Omega) \oplus G_q(\Omega)$ holds, i.e., any $\boldsymbol{v} \in \mathbb{L}^q(\Omega)$ is uniquely decomposed by*

$$\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2 \qquad \boldsymbol{w}_1 \in \mathbb{L}^q_\sigma(\Omega),\ \ \boldsymbol{w}_2 \in G_q(\Omega).$$

Moreover, there exist some constant γ which depends only on Ω, q and N such that

$$|\boldsymbol{w}_1|_{\mathbb{L}^q(\Omega)} + |\boldsymbol{w}_2|_{\mathbb{L}^q(\Omega)} \leqslant \gamma|\boldsymbol{v}|_{\mathbb{L}^q(\Omega)}$$

holds for any $\boldsymbol{v} \in \mathbb{L}^q(\Omega)$ and its decomposition $\boldsymbol{w}_1 \in \mathbb{L}^q_\sigma(\Omega)$, $\boldsymbol{w}_2 \in G_q(\Omega)$.

According to Fujiwara–Morimoto [26] and Miyakawa [36], we can obtain the following.

Proposition 2.2.3 (Duality of $\mathbb{L}^q_\sigma(\Omega)$)**.** *Let $1 < q < \infty$ and let Ω be either the whole space $\mathbb{R}^N$, the half space $\mathbb{R}^N_+$ or C^2-class domain with bounded boundary with $N \geqslant 2$. Then the dual space of $\mathbb{L}^q_\sigma(\Omega)$ coincides with $\mathbb{L}^{q'}_\sigma(\Omega)$, where $q' := q/(q-1)$.*

Moreover, recalling Proposition 2.1.14 and the fact that $\mathbb{C}^\infty_\sigma(\Omega)$ is dense in $\mathbb{L}^{q_1}_\sigma(\Omega)$, we can see the following.

Proposition 2.2.4 (Density)**.** *For each $q_1 \in [1, \infty)$ and $q_2 \in (1, \infty)$, $C^\infty_0((0, S); \mathbb{C}^\infty_\sigma(\Omega))$ is dense in $L^{q_2}(0, S; \mathbb{L}^{q_1}_\sigma(\Omega))$.*

2.2.2 Stokes Operator

In this subsection, we mention the Stokes operator and its basic properties. Since our arguments are carried out in L^2-framework throughout this thesis, we only deal with the case where the exponent $q = 2$, i.e., $\mathbb{L}^2(\Omega)$-case for simplicity.

Definition 2.2.1 (Stokes Operator)**.** *Let $\mathcal{P}_\Omega$ denote the orthogonal projection from $\mathbb{L}^2(\Omega)$ onto $\mathbb{L}^2_\sigma(\Omega)$. Then we define the Stokes operator $\mathcal{A}_\Omega$ by*

$$\mathcal{A}_\Omega := -\mathcal{P}_\Omega\Delta \qquad \textit{with domain } D(\mathcal{A}_\Omega) = \mathbb{H}^2(\Omega) \cap \mathbb{H}^1_\sigma(\Omega),$$

where $\mathbb{H}^1_\sigma(\Omega)$ is the closure of $\mathbb{C}^\infty_\sigma(\Omega)$ under the norm of $\mathbb{H}^1(\Omega)$.

If there is no confusion, the orthogonal projection and the Stokes operator are simply designated by $\mathcal{P}$ and $\mathcal{A}$ respectively.

Here, operating the orthogonal projection $\mathcal{P}_\Omega$ to the first equation of (DCBF), we obtain the following equations:

$$\text{(2.8)} \qquad \begin{cases} \partial_t \boldsymbol{u} + \nu\mathcal{A}_\Omega\boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_\Omega\boldsymbol{g}T + \mathcal{P}_\Omega\boldsymbol{h}C + \mathcal{P}_\Omega\boldsymbol{f}_1 & (x,t) \in \Omega\times[0,S], \\ \partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + f_2 & (x,t) \in \Omega\times[0,S], \\ \partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta C + \rho\Delta T + f_3 & (x,t) \in \Omega\times[0,S]. \end{cases}$$

We remark that the system (2.8) is equivalent to the original system (DCBF). Indeed, if we can find a solution $(\boldsymbol{u}, T, C)$ of the system (2.8) in $\mathbb{L}^2_\sigma(\Omega) \times L^2(\Omega) \times L^2(\Omega)$-framework, the solution $\boldsymbol{u}$ satisfies the solenoidal condition $\nabla \cdot \boldsymbol{u} = 0$ and the existence of pressure p which satisfies the first equation of (DCBF) can be deduced automatically from the definition and properties of Helmholtz decomposition of $\mathbb{L}^2(\Omega)$ (more detail, see, Sohr [53] and Temam [58]). Therefore, throughout this dissertation, we treat the system (2.8) instead of the original system (DCBF) and we consider the solution $(\boldsymbol{u}, T, C)$ of (2.8) instead of $(\boldsymbol{u}, T, C, p)$ of (DCBF). For this reason, the system (2.8) is also called (DCBF) henceforth.

As for the fundamental fact of the Stoke operator, the following elliptic estimate holds for $\mathcal{A}_\Omega$ defined on uniform C^2-domain Ω (see Sohr [53]).

Proposition 2.2.5 (Elliptic estimate for $\mathcal{A}_\Omega$)**.** *Let Ω be either the whole space $\mathbb{R}^N$ or uniform C^2-domain and let $\boldsymbol{F} \in \mathbb{L}^2_\sigma(\Omega)$. Moreover, assume that $\boldsymbol{w} \in \mathbb{H}^1_\sigma(\Omega)$ is a weak solution of $\mathcal{A}_\Omega\boldsymbol{w} + \boldsymbol{w} = \boldsymbol{F}$. That is to say, assume that $\boldsymbol{w} \in \mathbb{H}^1_\sigma(\Omega)$ satisfies*

$$\int_\Omega \nabla\boldsymbol{w}\cdot\nabla\boldsymbol{v} + \int_\Omega \boldsymbol{w}\cdot\boldsymbol{v} = \int_\Omega \boldsymbol{F}\cdot\boldsymbol{v} \qquad \forall\boldsymbol{v} \in \mathbb{H}^1_\sigma(\Omega).$$

Then, $\boldsymbol{w}$ belongs to $D(\mathcal{A}_\Omega)$ and becomes a strong solution of $\mathcal{A}_\Omega\boldsymbol{w} + \boldsymbol{w} = \boldsymbol{F}$. Moreover, there exist a constant γ_S which depends only on Ω and N such that $|\boldsymbol{w}|_{\mathbb{H}^2(\Omega)} \leqslant \gamma_S|\boldsymbol{F}|_{\mathbb{L}^2(\Omega)}$, i.e.

$$\text{(2.9)} \qquad |\boldsymbol{w}|_{\mathbb{H}^2(\Omega)} \leqslant \gamma_S(|\mathcal{A}_\Omega\boldsymbol{w}|_{\mathbb{L}^2(\Omega)} + |\boldsymbol{w}|_{\mathbb{L}^2(\Omega)})$$

holds.

This Proposition implies that the Stoke operator $\mathcal{A}_\Omega$ becomes a maximal monotone operator in $\mathbb{L}^2_\sigma(\Omega)$ (see example of subdifferential operators in Section 2.3).

In the particular case where $\Omega = \mathbb{R}^N$, we obtain the following (see Constantin-Foais [17], Sohr [53] and Temam [58]).

Proposition 2.2.6 (Stokes operator in $\mathbb{R}^N$)**.** *If $\boldsymbol{w}$ belongs to $D(\mathcal{A}_{\mathbb{R}^N}) = \mathbb{H}^2(\mathbb{R}^N) \cap \mathbb{H}^1_\sigma(\mathbb{R}^N)$, then $\Delta\boldsymbol{w}$ belongs to $\mathbb{L}^2_\sigma(\mathbb{R}^N)$, i.e., $\mathcal{A}_{\mathbb{R}^N}\boldsymbol{w} = -\Delta\boldsymbol{w}$ holds.*

2.3 Maximal Monotone Operator and Subdifferential Operator

In this section, we introduce the definition of subdifferential operator and some known results for the abstract equation governed by subdifferential operators defined in the Hilbert space. Throughout this section, H stands for a real Hilbert space with the inner product $(\cdot,\cdot)_H$ and the norm $\|\cdot\|_H$. Moreover, $\overline{\ \cdot\ }^H$ designates the closure in H.

2.3.1 Definition

Let φ be a lower semi-continuous convex function from H onto $(-\infty, +\infty]$. The function $\varphi : H \to (-\infty, +\infty]$ is said to be "proper" if the set defined by

$$D(\varphi) := \{U \in H; \varphi(U) < +\infty\} \qquad : \text{ Effective domain of } \varphi$$

is not empty set. Then we define the subdifferential of a proper lower semi-continuous convex function φ by

$$\partial\varphi(U_0) := \{h \in H;\ (h, U - U_0)_H \leqslant \varphi(U) - \varphi(U_0) \quad \forall U \in H\}.$$

The set $D(\partial\varphi) := \{U \in H; \partial\varphi(U) \neq \varnothing\}$ is called the domain of subdifferential operator $\partial\varphi$.

It is well known that the subdifferential operator becomes a maximal monotone operator (see Brézis [10], [11]). Here, the operator $A : H \to 2^H$ is said to be a maximal monotone operator if the following conditions are satisfied:

1. $(U_1 - U_2, W_1 - W_2)_H \geqslant 0$ holds for any $U_1, U_2 \in D(A)$ (domain of A) and $W_1 \in AU_1, W_2 \in AU_2$.

2. If $(U, W) \in H \times H$ satisfies $(U - U_1, W - W_1)_H \geqslant 0$ for any $U_1 \in D(A)$ and any $W_1 \in AU_1$, then $U \in D(A)$ and $W \in AU$ hold.

The condition 2, the so-called maximality of A, and the following condition $2'$ and condition $2''$ are equivalent.

$2'$. There exist a positive constant λ_0 such that for any $F \in H$, the equation $U + \lambda_0 AU \ni F$ possesses a unique solution $U \in D(A)$.

2″. For any $\lambda > 0$ and for any $F \in H$, the equation $U + \lambda AU \ni F$ possesses a unique solution $U \in D(A)$.

According to the condition 2″, for any maximal monotone operator A and positive parameter λ, we can define the resolvent $J_\lambda : H \to D(A)$ by $J_\lambda := (I + \lambda A)^{-1}$, where I is the identity mapping on H. It is well known that for any $U \in \overline{D(A)}^H$, $J_\lambda U$ strongly converges to U as $\lambda \to 0$ in H.

Based on this fact, by measuring how fast the resolvent $J_\lambda U$ converges to U, we can define a nonlinear interpolation class associated with A between $D(A)$ and $\overline{D(A)}^H$ (see Brézis [7], [8] and [9]). Let $0 < \alpha < 1,\ 1 \leqslant q \leqslant \infty$. Then we define the set $\mathcal{B}_{\alpha,q}(A)$, called Brézis class, by

$$\mathcal{B}_{\alpha,q}(A) := \left\{ U \in \overline{D(A)}^H;\ t^{-\alpha}\|U - J_t U\|_H \in L^q_*(0,1) \right\},$$

where

$$L^q_*(0,\tau) := \left\{ g : [0,\tau] \to \mathbb{R};\ |g|_{L^q_*(0,\tau)} := \left(\int_0^\tau |g(t)|^q \frac{1}{t} dt \right)^{1/q} < \infty \right\} \quad (\text{ if } q \in [1,\infty)),$$

$$L^\infty_*(0,\tau) := L^\infty(0,\tau).$$

We also write

$$|U|_{\mathcal{B}_{\alpha,q}(A)} := |t^{-\alpha}\|U - J_t U\|_H|_{L^q_*(0,1)} = \left(\int_0^1 \|U - J_t U\|_H^q \frac{1}{t^{1+\alpha q}} dt \right)^{1/q}.$$

The nonlinear interpolation class $\mathcal{B}_{\alpha,q}(A)$ covers several known interpolation spaces such as Lorentz space, Marcinkiewicz space and Besov spaces. We note that Brézis class $\mathcal{B}_{\alpha,q}(A)$ generally dose not become a linear space, even, a convex set. However, the following fundamental facts are valid (see Brézis [7], [8] and [9]):

$$\begin{aligned} &\mathcal{B}_{\alpha,q_1}(A) \subset \mathcal{B}_{\alpha,q_1}(A) && \forall \alpha \in (0,1),\ 1 \leqslant q_1 < q_2 \leqslant \infty, \\ &\mathcal{B}_{\alpha_1,q_1}(A) \subset \mathcal{B}_{\alpha_2,q_2}(A) && \forall q_1, \forall q_2 \in [1,\infty],\ 0 < \alpha_2 < \alpha_1 < 1. \end{aligned}$$

Moreover, if $A = \partial\varphi$, we have

$$\mathcal{B}_{\frac{1}{2},2}(\partial\varphi) = D(\varphi).$$

Here we give some examples of subdifferential operators, which will be used in this thesis.

Example 1. Let $H := L^2(\Omega)$. We define the function $\varphi_q : L^2(\Omega) \to \mathbb{R}$ by

$$\varphi_q(V) := \begin{cases} \frac{1}{q}|V|^q_{L^q(\Omega)} & \text{if } U \in L^q(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $q \geqslant 2$. It is easy to see that φ_q is a proper lower semi-continuous convex function in H. Here we define a function $\Theta : \mathbb{R} \to \mathbb{R}$ by $\Theta(s) := |s|^{q-2}s\ \ (s \in \mathbb{R})$ and we define a operator $\Theta' : L^{2(q-1)}(\Omega) \to H$ by $\Theta'(V)(x) := \Theta(V(x))$ $(V \in L^{2(q-1)}(\Omega),\ x \in \Omega)$. Then, we can see that

$$\varphi_q(W) - \varphi_q(V) \geqslant (\Theta'(V), W - V)_H$$

holds for any $W \in H$ and $V \in L^{2(q-1)}(\Omega)$. Therefore, the monotone operator Θ' satisfies $\Theta' \subset \partial\varphi_q$. Moreover, Θ' satisfies the maximal condition $2'$ with $\lambda_0 = 1$. Indeed, for any $W \in H$, we can define the function V_W by the solution of the following algebraic equation:

$$\Theta(V_W(x)) + V_W(x) = W(x) \qquad (\forall x \in \Omega)$$

(since $s \mapsto \Theta(s) + s$ is bijective function, the equation above possesses a unique solution $V_W(x)$ for each given number $W(x)$). Then, since

$$|W(x)|^2 = |V_W(x)|^2 + 2|V_W(x)|^q + |V_W(x)|^{2(q-1)}$$

and $W \in H = L^2(\Omega)$, we have $V_W \in L^2(\Omega) \cap L^{2(q-1)}(\Omega)$. This implies that V_W becomes a unique solution of $\Theta'(V_W) + V_W = W$ in H. Hence, from the maximality of Θ', we have $\Theta' = \partial\varphi_q$, i.e. $\partial\varphi_q(V(\cdot)) = |V(\cdot)|^{q-2}V(\cdot)$ with domain $D(\partial\varphi_q) = L^2(\Omega) \cap L^{2(q-1)}(\Omega)$.

Example 2. Let $H := L^2(\Omega)$, where Ω is either the whole space or uniform C^2-domain. Define $\varphi_D : L^2(\Omega) \to \mathbb{R}$ by

$$\varphi_D(U) := \begin{cases} \frac{1}{2}|\nabla V|^2_{L^2(\Omega)} & \text{if } V \in H^1_0(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We can easily show that φ_D is a lower semi-continuous function on $L^2(\Omega)$. Then the subdifferential operator $\partial\varphi_D$ coincides with $-\Delta_D$. Similarly, define the lower semi-continuous convex function $\varphi_N : L^2(\Omega) \to \mathbb{R}$ by

$$\varphi_N(U) := \begin{cases} \frac{1}{2}|\nabla V|^2_{L^2(\Omega)} & \text{if } V \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then the subdifferential operator $\partial\varphi_N$ coincides with $-\Delta_N$.

Example 3. Let $H := \mathbb{L}^2_\sigma(\Omega)$, where Ω is either the whole space or uniform C^2-domain. Define the lower semi-continuous convex function $\varphi_S : \mathbb{L}^2_\sigma(\Omega) \to \mathbb{R}$ by

$$\varphi_S(\boldsymbol{w}) := \begin{cases} \frac{1}{2}|\nabla \boldsymbol{w}|^2_{\mathbb{L}^2(\Omega)} & \text{if } \boldsymbol{w} \in \mathbb{H}^1_\sigma(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then the subdifferential operator $\partial\varphi_S$ coincides with the Stokes operator $\mathcal{A}_\Omega$.

These facts can be guaranteed by the elliptic estimates of $-\Delta_D$, $-\Delta_N$ and $\mathcal{A}_\Omega$ (recall Section 2.1 and 2.2).

Example 4. Let $H := L^2(\Omega)$, where Ω is either the whole space or uniform C^2-domain. Then, $\varphi'_D := \varphi_D + \varphi_q$ and $\varphi'_N := \varphi_N + \varphi_q$ become proper lower semi-continuous convex functions. It is easy to see that $\partial\varphi_D + \partial\varphi_q \subset \partial\varphi'_D$ and $\partial\varphi_N + \partial\varphi_q \subset \partial\varphi'_N$. In general, the sum of maximal monotone operators does not necessarily possess maximality and it is not obvious that $\partial\varphi_D + \partial\varphi_q$ and $\partial\varphi_N + \partial\varphi_q$ coincide with the subdifferential of $\varphi_D + \varphi_q$ and $\varphi_N + \varphi_q$ respectively. However, by virtue of the following useful lemma (see Proposition 2.17 in Brézis [11]), we can assure the maximality of $\partial\varphi_D + \partial\varphi_q$ and $\partial\varphi_N + \partial\varphi_q$.

Lemma 2.3.1. *Let A be a maximal monotone operator in the real Hilbert space H and $\varphi : H \to (-\infty, +\infty]$ be a proper lower semi-continuous convex function. Moreover, assume that there exist some constant γ such that*

$$\varphi(J_\lambda U) \leqslant \varphi(U) + \gamma\lambda \qquad \forall U \in H, \quad \lambda > 0$$

holds, where $J_\lambda := (I + \lambda A)^{-1}$. Then $A + \partial\varphi$ is a maximal monotone operator on H and

$$\overline{D(A + \partial\varphi)}^H = \overline{D(A) \cap D(\partial\varphi)}^H = \overline{D(A)}^H \cap \overline{D(\partial\varphi)}^H$$

holds.

Recalling mappings $\Theta : \mathbb{R}^N \to \mathbb{R}^N$ and $\Theta' : L^{2(q-1)}(\Omega) \to H$ in Example 1, we can assure the existence of $V_W^\lambda \in L^{2(q-1)}(\Omega)$ such that

$$\lambda\Theta'(V_W^\lambda) + V_W^\lambda = W \qquad \text{in } H$$

for any $\lambda > 0$, $W \in H$. Moreover, if $W \in H_0^1(\Omega)$ or $W \in H^1(\Omega)$, then V_W^λ also becomes $H_0^1(\Omega)$ or $H^1(\Omega)$-function respectively, due to the following identity:

$$\nabla W(x) = \nabla(\lambda\Theta'(V_W^\lambda(x)) + V_W^\lambda(x)) = \lambda(q-1)|V_W^\lambda(x)|^{q-2}\nabla V_W^\lambda(x) + \nabla V_W^\lambda.$$

This identity also yields $|\nabla W|_{L^2(\Omega)} \geqslant |\nabla V_W^\lambda|_{L^2(\Omega)}$. Using this inequality and recalling that $V_W^\lambda = (I + \lambda\partial\varphi_q)^{-1}W$, we obtain $\varphi_D((I + \lambda\partial\varphi_q)^{-1}W) \leqslant \varphi_D(W)$ and $\varphi_N((I + \lambda\partial\varphi_q)^{-1}W) \leqslant \varphi_N(W)$ for any $W \in H$ and any $\lambda > 0$. By virtue of Lemma 2.3.1, we can assure the maximality of $\partial\varphi_D+\partial\varphi_q$ and $\partial\varphi_N+\partial\varphi_q$, i.e., we obtain $\partial\varphi_D+\partial\varphi_q = \partial(\varphi_D+\varphi_q)$ and $\partial\varphi_N + \partial\varphi_q = \partial(\varphi_N + \varphi_q)$.

2.3.2 Known Result

We exhibit known results for the abstract evolution equations governed by the subdifferential operators in this subsection. Let $\varphi : H \to (-\infty, +\infty]$ be a proper lower semi-continuous convex function.

According to Theorem 3.6 in Brézis [11], the following solvability result for the Cauchy problem is assured.

Proposition 2.3.1 (Solvability of Cauchy problem)**.** *Let $U_0 \in \overline{D(\varphi)}^H$ and let $f \in L^1(0, S; H)$, $\sqrt{t}f \in L^2(0, S; H)$. Then the problem*

$$\begin{cases} \dfrac{dU}{dt}(t) + \partial\varphi(U(t)) \ni f(t) & a.e.\ t \in [0, S], \\ U(0) = U_0 \end{cases}$$

possesses a unique solution $U \in C([0,S];H)$ *satisfying*

$$\varphi(U) \in L^1(0,S), \qquad \sqrt{t}\frac{dU}{dt},\ \sqrt{t}g \in L^2(0,S;H),$$

where g *is the function such that* $g(t) \in \partial\varphi(U(t))$ *and* $\frac{dU}{dt}(t) + g(t) = f(t)$ *for a.e.* $t \in [0,S]$.

Moreover, if $f \in L^2(0,S;H)$ *and* $U_0 \in D(\varphi)$*, then the solution* U *satisfies*

$$\varphi(U) \in W^{1,1}(0,S), \qquad \frac{dU}{dt},\ g \in L^2(0,S;H).$$

In the proof of Proposition 2.3.1 given in Brézis [11], the following lemma is effectively applied. This lemma is also useful for assuring the continuity of solution in H^1-space.

Lemma 2.3.2. *Let* $U \in W^{1,2}(0,S;H)$ *and* $U(t) \in D(\partial\varphi)$ *for a.e.* $t \in [0,S]$*. Moreover, assume that there exist some* $g \in L^2(0,S;H)$ *such that* $g(t) \in \partial\varphi(U(t))$ *for a.e.* $t \in [0,S]$*. Then* $\varphi(U(\cdot)) \in W^{1,1}(0,S)$ *is valid.*

As for the periodic problem, Theorem 3.15 and Corollary 3.4 in Brézis [11] assure the following. Henceforth, the space of periodic continuous X-valued functions is designated by $C_\pi([0,S];X)$, i.e.,

$$C_\pi([0,S];X) := \{U \in C([0,S];X);\ U(0) = U(S) \quad \text{in } X\}.$$

Proposition 2.3.2 (Solvability of periodic problem)**.** *Assume that the subdifferential operator* $\partial\varphi$ *is coercive, i.e.,*

$$\lim_{\|U\|_H \to \infty,\ U \in D(\varphi)} \frac{\varphi(U)}{\|U\|_H} = +\infty.$$

Then for any $f \in L^2(0,S;H)$*, the problem*

$$\begin{cases} \dfrac{dU}{dt}(t) + \partial\varphi(U(t)) \ni f(t) & a.e.\ t \in [0,S], \\ U(0) = U(S) \end{cases}$$

possesses a solution $U \in C_\pi([0,S];H)$ *satisfying*

$$\varphi(U) \in W^{1,1}(0,S), \qquad \frac{dU}{dt},\ g \in L^2(0,S;H).$$

We next introduce the abstract results given in Ôtani [41] and [42] for some abstract equations associated with the subdifferential operators $\partial\varphi$ with non-monotone perturbations B in the Hilbert space H. Since we only deal with the single-valued subdifferential operator $\partial\varphi$ and non-monotone perturbation B in Chapter 3, we restrict ourselves to consider the following simplified problem (CP) and (AP) (we note that more general case where, e.g., $\partial\varphi$ and B are multivalued mappings and φ has a t-dependence is investigated in Ôtani [41] and [42]).

$$\text{(CP)} \begin{cases} \dfrac{dU}{dt}(t) + \partial\varphi(U(t)) + B(U(t)) = F(t) \qquad t \in [0, S], \\ U(0) = U_0, \end{cases}$$

$$\text{(AP)} \begin{cases} \dfrac{dU}{dt}(t) + \partial\varphi(U(t)) + B(U(t)) = F(t) \qquad t \in [0, S], \\ U(0) = U(S). \end{cases}$$

To formulate solvability results, we introduce the following conditions:

(A1) For each $L \in (0, +\infty)$, the set $\{U \in H;\, \varphi(U) + \|U\|_H^2 \leqslant L\}$ is compact in H.

(A2) $B(\cdot)$ is φ-demiclosed in the following sense: if $\{U_n\}_{n\in\mathbb{N}}$ strongly converges U in $C([0, S]; H)$, $\{\partial\varphi(U_n)\}_{n\in\mathbb{N}}$ weakly converges $\partial\varphi(U)$ in $L^2(0, S; H)$ and $\{B(U_n)\}_{n\in\mathbb{N}}$ weakly converges to b in $L^2(0, S; H)$ as $n \to \infty$, then $b(t) = B(U(t))$ holds for a.e. $t \in [0, S]$.

(A3)$_\alpha$ For a given exponent $\alpha \in (0, 1/2)$, there exists a monotone increasing function $\ell(\cdot)$ such that

$$\|B(U)\|_H \leqslant \ell(\|U\|_H) \left\{ \varepsilon\|\partial\varphi(U)\|_H + \frac{1}{\varepsilon}|\varphi(U)|^{\frac{1-\alpha}{1-2\alpha}} + 1 \right\} \quad \forall U \in D(\partial\varphi),$$

where ε is some positive constant determined by the initial data U_0 and the external force F, more precisely, ε is some monotone decreasing function of $|U_0|_H + |U_0|_{\mathcal{B}_{\alpha,p}(\partial\varphi)} + |F|_{L^2(0,S;H)}$.

(A4) There exists a monotone increasing function $\ell(\cdot)$ and $k \in (0, 1)$ such that

$$\|B(U)\|_H^2 \leqslant k\|\partial\varphi(U)\|_H^2 + \ell(\varphi(U) + \|U\|_H^2) \qquad \forall U \in D(\partial\varphi).$$

(A5) There exists a monotone increasing function $\ell(\cdot)$ and a constant $k \in [0, 1)$ such that

$$\|B(U)\|_H^2 \leqslant k\|\partial\varphi(U)\|_H^2 + \ell(\|U\|_H)(\varphi(U) + 1)^2 \qquad \forall U \in D(\partial\varphi).$$

(A6) There exist positive constants $\alpha,\ K$ such that

$$(-\partial\varphi(U) - B(U), U)_H + \alpha\varphi(U) \leqslant K \qquad \forall U \in D(\partial\varphi).$$

Then the following facts can be found in Ôtani [41] and [42].

Proposition 2.3.3 (Cauchy problem (CP) for $\mathcal{B}_{\alpha,q}(\partial\varphi)$ with $\alpha \in (0,1/2)$). *Let $U_0 \in \mathcal{B}_{\alpha,q}(\partial\varphi)$ with $q \in [1,2]$ and $F \in L^2(0,S;H)$. Moreover, let the conditions* (A1), (A2) *and* (A3)$_\alpha$ *be satisfied. Then there exists $S_0 \in (0,S]$ depending on $\|U_0\|_H$ and $|U_0|_{\mathcal{B}_{\alpha,q}(\partial\varphi)}$ such that* (CP) *has a solution $U \in C([0,S_0];H)$ satisfying*

$$
\begin{aligned}
&t^{1/2-\alpha}\frac{dU}{dt},\ t^{1/2-\alpha}\partial\varphi(U),\ t^{1/2-\alpha}B(U) \in L^2(0,S_0;H),\\
&t^{-\alpha}\|U(\cdot)-U_0\|_H,\ t^{1/2-\alpha}|\varphi(U(\cdot))|^{1/2} \in L^q_*(0,S_0) \qquad \forall q \in [2,\infty].
\end{aligned}
$$

Proposition 2.3.4 (Cauchy problem (CP) for $D(\varphi)$). *Let $F \in L^2(0,S;H)$ and $U_0 \in D(\varphi)$. Moreover, let the conditions* (A1), (A2) *and* (A4) *be satisfied. Then there exists $S_0 \in (0,S]$ depending on $\|U_0\|_H$ and $\varphi(U_0)$ such that* (CP) *has a solution $U \in C([0,S_0];H)$ satisfying*

$$
\begin{aligned}
&\frac{dU}{dt},\ \partial\varphi(U),\ B(U) \in L^2(0,S_0;H),\\
&\varphi(U(\cdot)) \in W^{1,1}(0,S_0).
\end{aligned}
$$

Proposition 2.3.5 (Periodic problem (AP)). *Let the conditions* (A1), (A2), (A5) *and* (A6) *be satisfied. Moreover, assume that*

- *There exist some constant γ_0 and $q \in (1,\infty)$ such that*

$$
\|U\|_H^q \leqslant \gamma\varphi(U) \qquad \forall U \in D(\varphi).
$$

- *The operator $\partial\varphi$ is strictly monotone, i.e., if $(U_1-U_2, W_1-W_2)_H = 0$ with $U_i \in D(\partial\varphi)$ and $W_i = \partial\varphi(U_i)$ $(i=1,2)$, then $U_1 = U_2$.*

Then for every $F \in L^2(0,S;H)$, (AP) *has a solution $U \in C_\pi([0,S];H)$ such that*

$$
\begin{aligned}
&\frac{dU}{dt},\ \partial\varphi(U),\ B(U) \in L^2(0,S;H),\\
&\varphi(U(\cdot)) \in W^{1,1}(0,S),\quad \varphi(U(0)) = \varphi(U(S)).
\end{aligned}
$$

2.4 Dynamical System and Attractor

As the preparation for Chapter 6, we here define the dynamical system and its attractors. We also mention some abstract result for the construction of attractors in this section.

2.4.1 Definition

Let E be a closed subset of Banach space X with the norm $\|\cdot\|_X$. A family of mappings $\mathscr{S}(t) : E \to E$ defined with respect to the variables $t \geqslant 0$, which is denoted by $\{\mathscr{S}(t)\}_{t\geqslant 0}$, is said to be semigroup acting on E, if the followings are satisfied:

1. $\mathscr{S}(0)$ is the identity operator on E.

2. $\mathscr{S}(t_1)\mathscr{S}(t_2) = \mathscr{S}(t_1 + t_2)$ holds for any $t_1, t_2 \geqslant 0$.

The pair $(\{\mathscr{S}(t)\}_{t\geqslant 0}, E)$ is called dynamical system in this thesis. Then, attractors of dynamical system $(\{\mathscr{S}(t)\}_{t\geqslant 0}, E)$ are defined as follows.

Definition 2.4.1 (Global Attractor)**.** *A nonempty subset $\mathscr{A} \subset E$ is said to be global attractor of the dynamical system $(\{\mathscr{S}(t)\}_{t\geq 0}, E)$, if $\mathscr{A}$ satisfies the following properties:*

1. *$\mathscr{A}$ is compact in E.*

2. *$\mathscr{A}$ is strictly invariant under $\{\mathscr{S}(t)\}_{t\geq 0}$, i.e., $\mathscr{A}$ satisfies $\mathscr{S}(t)\mathscr{A} = \mathscr{A}$ for each $t \geqslant 0$.*

3. *$\mathscr{A}$ satisfies the following "attracting property"; for any bounded subset $B \subset E$, $\mathscr{A}$ satisfies*
$$\lim_{t\to+\infty} dist_X(\mathscr{S}(t)B, \mathscr{A}) = 0,$$
where $dist_X(X_1, X_2) := \sup_{x_1\in X_1} \inf_{x_2\in X_2} \|x_1 - x_2\|_X \quad (X_1, X_2 \subset X)$.

Definition 2.4.2 (Exponential Attractor)**.** *A nonempty subset $\mathscr{M} \subset E$ is said to be exponential attractor of the dynamical system $(\{\mathscr{S}(t)\}_{t\geq 0}, E)$, if $\mathscr{M}$ satisfies the following properties:*

1. *$\mathscr{M}$ is compact in E and $\mathscr{M}$ has a finite fractal dimension in X. Here the fractal dimension of a compact set $\mathscr{K} \subset X$ is defined by*
$$dim_F(\mathscr{K}, X) := \limsup_{\varepsilon\to 0,\ \varepsilon>0} \frac{\mathbb{H}_\varepsilon(\mathscr{K}, X)}{\log_2 1/\varepsilon},$$
where $\mathbb{H}_\varepsilon(\mathscr{K}, X) := \log_2 N_\varepsilon(\mathscr{K}, X)$ and $N_\varepsilon(\mathscr{K}, X)$ is the minimal number of ε-open balls in X which cover $\mathscr{K}$.

2. *$\mathscr{M}$ is positively invariant under $\{\mathscr{S}(t)\}_{t\geq 0}$, i.e., $\mathscr{M}$ satisfies $\mathscr{S}(t)\mathscr{M} \subset \mathscr{M}$ for each $t \geqslant 0$.*

3. *$\mathscr{M}$ satisfies the following "exponential attracting property"; there exist a monotone function $Q(\cdot)$ and a positive constant α such that $\mathscr{M}$ satisfies*
$$dist_X(\mathscr{S}(t)B, \mathscr{M}) \leqslant Q(\|B\|_X)e^{-\alpha t}$$
for any bounded subset $B \subset E$, where $\|B\|_X := \sup_{y\in B} \|y\|_X$.

We here note that the global attractor of dynamical system is uniquely provided. Indeed, by Definition 2.4.1, we can see that the global attractor is characterized by the smallest closed attracting set (a set satisfying attracting property). On the other hand, the exponential attractor of dynamical system is not necessarily determined uniquely.

2.4.2 Existence of Attractors

In this thesis, the construction of attractors relies on the following abstract results.

Proposition 2.4.1 (Existence of global attractor)**.** *Let* $(\{\mathscr{S}(t)\}_{t\geq 0}, E)$ *be a dynamical system. Assume that* $\mathscr{S}(t)$ *is continuous on* E *for each* $t \geqslant 0$ *and* $(\{\mathscr{S}(t)\}_{t\geq 0}, E)$ *has a compact attracting set* K*. That is to say, we assume that there exist some compact set* $K \subset E$ *satisfying*

$$\lim_{t\to+\infty} dist_X(\mathscr{S}(t)B, K) = 0,$$

for any bounded subset $B \subset E$*. Then the dynamical system* $(\{\mathscr{S}(t)\}_{t\geq 0}, E)$ *possesses a global attractor.*

The demonstrations for Proposition 2.4.1 can be found in, e.g., Babin–Vishik [3], Chepyzhov–Vishik [16], Robinson [52] and Temam [59]. This proposition immediately leads to the following corollary. Here, the set $\mathscr{B} \subset E$ is said to be an absorbing set of $(\{\mathscr{S}(t)\}_{t\geq 0}, E)$, if for any bounded subset $B \subset E$, there exist $t_B \geqslant 0$ such that $\mathscr{S}(t)B \subset \mathscr{B}$ holds for any $t \geqslant t_B$ (obviously, the absorbing set satisfies the definition of attracting set).

Corollary 2.4.1 (Existence of global attractor)**.** *Let* $(\{\mathscr{S}(t)\}_{t\geq 0}, E)$ *be a dynamical system. Assume that* $\mathscr{S}(t)$ *is continuous on* E *for each* $t \geqslant 0$ *and* $(\{\mathscr{S}(t)\}_{t\geq 0}, E)$ *has a compact absorbing set* $\mathscr{B}$*. Then the dynamical system* $(\{\mathscr{S}(t)\}_{t\geq 0}, E)$ *possesses a global attractor.*

Corollary 2.4.1 is sometimes more convenient than Proposition 2.4.1, since the existence of absorbing set can be deduced by the standard a priori estimates.

Next we mention the abstract theory for exponential attractor.

Proposition 2.4.2 (Existence of exponential attractor)**.** *Let* Y *be a normed subspace with the norm* $\|\cdot\|_Y$ *which is compactly embedded in* X*. Assume that there exist a compact absorbing set* $\mathscr{B}_0 \subset E$ *of the dynamical system* $(\{\mathscr{S}(t)\}_{t\geq 0}, E)$ *and* $\mathscr{B}_0$ *is positively invariant under* $\{\mathscr{S}(t)\}_{t\geq 0}$*, i.e.,* $\mathscr{S}(t)\mathscr{B}_0 \subset \mathscr{B}_0$ *is valid for any* $t \geqslant 0$*. Moreover, we assume that there exist* $t_* > 0$ *and positive constants* $\alpha_1, \alpha_2, \alpha_3$ *and* $\beta \in (0, 1]$ *satisfying the followings.*

1. $\|\mathscr{S}(t_*)U_1 - \mathscr{S}(t_*)U_2\|_Y \leqslant \alpha_1\|U_1 - U_2\|_X$ *holds for any* $U_1, U_2 \in \mathscr{B}_0$*.*
2. $\|\mathscr{S}(t)U_1 - \mathscr{S}(t)U_2\|_X \leqslant \alpha_2\|U_1 - U_2\|_X$ *holds for any* $U_1, U_2 \in \mathscr{B}_0$*,* $t \in [0, t_*]$*.*
3. $\|\mathscr{S}(t)U_1 - \mathscr{S}(s)U_1\|_X \leqslant \alpha_3|t - s|^\beta$ *holds for any* $U_1 \in \mathscr{B}_0$*,* $t, s \in [0, t_*]$*.*

Then $(\{\mathscr{S}(t)\}_{t\geq 0}, E)$ *possesses an exponential attractor.*

By virtue of abstract theory of Efendiev [19] and Efendiev–Miranville–Zelik [20], the condition *1* yields the existence of exponential attractor $\mathscr{M}_*$ of the discrete dynamical system $(\{\mathscr{S}_*^n\}_{n\in\mathbb{N}}, \mathscr{B}_0)$, where $\mathscr{S}_* := \mathscr{S}(t_*)$. Moreover, applying the standard argument in Eden–Foais–Nicolaenko–Temam [18], and using the condition *2* and *3*, we can assure that the set

$$\mathscr{M} := \bigcup_{0\leqslant t\leqslant t_*} \mathscr{S}(t)\mathscr{M}_*$$

becomes an exponential attractor of the dynamical system $(\{\mathscr{S}(t)\}_{t\geq 0}, \mathscr{B}_0)$. Since $\mathscr{B}_0$ is absorbing set, $\mathscr{M}$ also satisfies all the required conditions of exponential attractor of the original dynamical system $(\{\mathscr{S}(t)\}_{t\geq 0}, E)$.

2.5 Other Basic Tools

The reminder of this chapter is devoted to list up some fundamental facts other than above.

We first recall Banach's fixed point principle (see e.g., Evans [21]).

Proposition 2.5.1 (Banach's contraction mapping principle)**.** *Let X be a complete metric space with the metric $d(\cdot,\cdot)$. Moreover, we assume that the mapping $\Phi : X \to X$ is a contraction mapping, i.e., there exist some $k \in [0,1)$ satisfying*

$$d(\Phi(z_1), \Phi(z_2)) \leqslant kd(z_1, z_2), \qquad \forall z_1, \forall z_2 \in X.$$

Then Φ possesses a unique fixed point $z_0 \in X$. That is to say, there exist a unique $z_0 \in X$ such that $\Phi(z_0) = z_0$.

We also recall Shauder–Tychonoff's fixed point theorem. The original statement and its proof can be found in, e.g., Browder [15]. For simplicity, we here restrict ourselves to the particular case.

Proposition 2.5.2 (Schauder–Tychonoff's fixed point theorem)**.** *Let X be a reflexive Banach space endowed with the weak topology and let $\mathscr{C} \subset X$ be convex and compact in the weak topology of X. Moreover, we assume that $\Phi : \mathscr{C} \to \mathscr{C}$ is weakly continuous in X. Then Φ possesses at least one fixed point in $\mathscr{C}$.*

Next we introduce the following fact, the so-called Ascoli's theorem.

Proposition 2.5.3 (Ascoli's theorem)**.** *Let X be a Banach space and $\mathscr{G} \subset C([0,S];X)$. Then $\mathscr{G}$ is relatively compact in $C([0,S];X)$ if and only if the followings are satisfied.*

- *For any $t \in [0,S]$, the set $\{g(t); g \in \mathscr{G}\}$ is relatively compact in X.*
- *$\mathscr{G}$ is equi-continuous, i.e., for any $s \in [0,S]$ and any $\varepsilon > 0$, there exist some $\delta = \delta(s,\varepsilon)$ such that $\|g(t) - g(s)\|_X < \varepsilon$ holds for any $t \in (s-\delta, s+\delta)$ and any $g \in \mathscr{G}$.*

The demonstration can be carried out by exactly the same procedure as those for Ascoli-Arzela's theorem (see Brézis [12] and Yosida [64])

If the space domain Ω is bounded, we obtain the following inequality (see Brézis [12] and Evans [21]).

Proposition 2.5.4 (Poincaré's inequality)**.** *Let* Ω *be a bounded domain and let* $q \in [1, \infty]$*. Then there exist a constant* κ *depending on* q *and* Ω *such that*

$$|V|_{L^q(\Omega)} \leqslant \kappa |\nabla V|_{W^{1,q}(\Omega)} \qquad \forall V \in W_0^{1,q}(\Omega).$$

Obviously, we can obtain

$$|\boldsymbol{w}|_{\mathbb{L}^q(\Omega)} = \sum_{\mu=1}^{N} |w^\mu|_{L^q(\Omega)} \leqslant \kappa \sum_{\mu=1}^{N} |\nabla w^\mu|_{L^q(\Omega)} = \kappa |\nabla \boldsymbol{w}|_{\mathbb{W}^{1,q}(\Omega)}$$

for any $\boldsymbol{w} \in \mathbb{W}_0^{1,q}(\Omega)$. Moreover, since

$$\begin{aligned} |\nabla \boldsymbol{w}|^2_{\mathbb{L}^2(\Omega)} &= \int_\Omega \mathcal{A}_\Omega \boldsymbol{w} \cdot \boldsymbol{w} dx \\ &\leqslant |\mathcal{A}_\Omega \boldsymbol{w}|_{\mathbb{L}^2(\Omega)} |\boldsymbol{w}|_{\mathbb{L}^2(\Omega)} \leqslant \kappa |\mathcal{A}_\Omega \boldsymbol{w}|_{\mathbb{L}^2(\Omega)} |\nabla \boldsymbol{w}|_{\mathbb{L}^2(\Omega)}, \\ |\nabla V|^2_{L^2(\Omega)} &\leqslant \kappa |\Delta V|_{L^2(\Omega)} |\nabla V|_{L^2(\Omega)} \end{aligned}$$

hold for any $\boldsymbol{w} \in D(\mathcal{A}_\Omega)$ and any $V \in D(-\Delta_D)$, we have

$$\begin{aligned} |\nabla \boldsymbol{w}|_{\mathbb{L}^2(\Omega)} &\leqslant \kappa |\mathcal{A}_\Omega \boldsymbol{w}|_{\mathbb{L}^2(\Omega)} \qquad \forall \boldsymbol{w} \in D(\mathcal{A}_\Omega), \\ |\nabla V|^2_{L^2(\Omega)} &\leqslant \kappa |\Delta V|_{L^2(\Omega)} \qquad \forall V \in D(-\Delta_D). \end{aligned}$$

Moreover, using Poincaré's inequality, we can obtain the following elliptic estimates:

$$\begin{aligned} |\boldsymbol{w}|_{\mathbb{H}^2(\Omega)} &\leqslant \gamma_S (|\mathcal{A}_\Omega \boldsymbol{w}|_{\mathbb{L}^2(\Omega)} + |\boldsymbol{w}|_{\mathbb{L}^2(\Omega)}) \leqslant \gamma_S (|\mathcal{A}_\Omega \boldsymbol{w}|_{\mathbb{L}^2(\Omega)} + \kappa |\nabla \boldsymbol{w}|_{\mathbb{L}^2(\Omega)}) \\ &\leqslant \gamma'_S |\mathcal{A}_\Omega \boldsymbol{w}|_{\mathbb{L}^2(\Omega)} \end{aligned}$$

for any $\boldsymbol{w} \in D(\mathcal{A}_\Omega)$ and

$$\begin{aligned} |V|_{H^2(\Omega)} &\leqslant \gamma_D (|\Delta V|_{L^2(\Omega)} + |V|_{L^2(\Omega)}) \leqslant \gamma_D (|\Delta V|_{L^2(\Omega)} + \kappa |\nabla V|_{L^2(\Omega)}) \\ &\leqslant \gamma'_D |\Delta V|_{L^2(\Omega)} \end{aligned}$$

for any $V \in D(-\Delta_D)$, namely, H^2-norm of $\boldsymbol{w}$ and V are bounded only by the L^2-norm of $\mathcal{A}_\Omega \boldsymbol{w}$ and $\Delta_D V$. We here note that Poincaré's inequality holds only for functions belonging to $W_0^{1,q}(\Omega)$. More generally, we have the following inequality (see Evans [21]).

Proposition 2.5.5 (Poincaré-Wirtinger's inequality)**.** *Let* Ω *be a bounded domain and let* $q \in [1, \infty]$*. Then there exist a constant* κ *depending on* q *and* Ω *such that*

$$\left| V - \frac{1}{|\Omega|} \int_\Omega V dx \right|_{L^q(\Omega)} \leqslant \kappa |\nabla V|_{W^{1,q}(\Omega)} \qquad \forall V \in W^{1,q}(\Omega),$$

where $|\Omega|$ *stands for the Lebesgue measure of* Ω*.*

Poincaré-Wirtinger's inequality also yields

$$|\nabla V|^2_{L^2(\Omega)} \leqslant \kappa |\Delta V|_{L^2(\Omega)} \qquad \forall V \in D(-\Delta_N),$$

since

$$\begin{aligned}|\nabla V|^2_{L^2(\Omega)} = |\nabla V'|^2_{L^2(\Omega)} &= \int_\Omega V'(-\Delta_N)V' dx \\ &\leqslant |V'|_{L^2(\Omega)}|\Delta V'|_{L^2(\Omega)} \leqslant \kappa |\nabla V'|_{L^2(\Omega)}|\Delta V'|_{L^2(\Omega)} \\ &= \kappa |\nabla V|_{L^2(\Omega)}|\Delta V|_{L^2(\Omega)},\end{aligned}$$

where $V' := V - \frac{1}{|\Omega|}\int_\Omega V dx$ (remark that $\frac{1}{|\Omega|}\int_\Omega V dx$ is a constant).

We here state some Gronwall's type inequalities (proofs can be found in Evans [21] and Brézis [11]).

Lemma 2.5.1. *Let $\eta \in W^{1,1}(0,S)$ and $g, \phi, \psi \in L^1(0,S)$. Moreover, we assume that η, g, ϕ, and ψ satisfy the following inequality:*

$$\frac{d}{dt}\eta(t) + g(t) \leqslant \phi(t)\eta(t) + \psi(t) \qquad a.e.\ t \in [0,S].$$

Then

$$\begin{aligned}&\eta(t) + \int_{t_0}^t g(s) \exp\left(\int_s^t \phi(\tau)d\tau\right) ds \\ &\quad \leqslant \eta(t_0)\exp\left(\int_{t_0}^t \phi(s)ds\right) + \int_{t_0}^t \psi(s)\exp\left(\int_s^t \phi(\tau)d\tau\right) ds\end{aligned}$$

holds for any $t_0 \in [0,S]$ and any $t \in [t_0,S]$.

Lemma 2.5.2. *Let $\eta \in C([0,S])$, let k be a non-negative constant and let $\phi \in L^1(0,S)$ be a non-negative function. Moreover, we assume that η, ϕ, and k satisfy the following inequality:*

$$\frac{1}{2}\eta^2(t) \leqslant \frac{1}{2}k^2 + \int_0^t \phi(s)\eta(s)ds \qquad a.e.\ t \in [0,S].$$

Then

$$|\eta(t)| \leqslant k + \int_0^t \phi(s)ds$$

holds for any $t \in [0,S]$.

Finally, we state several comments for the fractional power of $\mathcal{A}_\Omega$, $-\Delta_D$ and $-\Delta_N$. It is well known that $\mathcal{A}_\Omega$, $-\Delta_D$ and $-\Delta_N$ become non-negative self-adjoint linear operators and the fractional power of these operators denoted by $\mathcal{A}_\Omega^\alpha$, $(-\Delta_D)^\alpha$ and $(-\Delta_N)^\alpha$ with $\alpha \in (0,1)$ can be defined. The characterization for the domains of the fractional power of operators can be found in Fujita–Morimoto [24] and Fujiwara [25]. For example, we have

$$D(\mathcal{A}_\Omega^{1/2}) = \mathbb{H}^1_\sigma(\Omega), \quad D((-\Delta_D)^{1/2}) = H^1_0(\Omega), \quad D((-\Delta_N)^{1/2}) = H^1(\Omega)$$

and

$$\begin{aligned} &|\mathcal{A}_\Omega^{1/2}\boldsymbol{w}|_{\mathbb{L}^2(\Omega)} = |\nabla \boldsymbol{w}|_{\mathbb{L}^2(\Omega)}, \\ &|(-\Delta_D)^{1/2}V|_{L^2(\Omega)} = |(-\Delta_N)^{1/2}V|_{L^2(\Omega)} = |\nabla V|_{L^2(\Omega)}. \end{aligned}$$

As for the relationship between the fractional power and the Brézis class, we have $D(A^\alpha) = \mathcal{B}_{\alpha,2}(A)$, where A is either $\mathcal{A}_\Omega$, $-\Delta_D$ or $-\Delta_N$ (see Brézis [7], [8] and [9]). Since $A = \mathcal{A}_\Omega, -\Delta_D, -\Delta_N$ are linear maximal monotone operators, the resolvent J_λ and the fractional power A^α are commutative. Then we can show that

$$U \in D(A^\alpha) \ \Rightarrow \ A^\alpha J_\lambda U = J_\lambda A^\alpha U \to A^\alpha U \quad \lambda \to 0 \quad \text{in } H$$

(see, e.g., Tanabe [56]). From these facts, we can derive the following smoothing approximation, to be used in Chapter 4.

Proposition 2.5.6. *Let $\boldsymbol{w}$ belong to $C([0,S];\mathbb{H}^1_\sigma(\Omega)) \cap L^2(0,S;\mathbb{H}^2(\Omega))$. Then there exist a sequence $\{\boldsymbol{w}_n\}_{n\in\mathbb{N}}$ such that*

- *$\boldsymbol{w}_n \in C([0,S];D(\mathcal{A}^2))$ for any $n \in \mathbb{N}$,*
- *$\{\boldsymbol{w}_n\}_{n\in\mathbb{N}}$ strongly converges to $\boldsymbol{w}$ in $C([0,S];\mathbb{H}^1_\sigma(\Omega)) \cap L^2(0,S;\mathbb{H}^2(\Omega))$.*

Indeed, for example, the sequence $\{\boldsymbol{w}_n\}_{n\in\mathbb{N}}$ defined by $\boldsymbol{w}_n := \left(\rho_{1/n} * \widetilde{J_n\boldsymbol{w}}\right)\Big|_{[0,S]}$ satisfies required properties. Here, $\rho_{1/n}$ denotes the Friedrichs mollifier with parameter $1/n$ and $J_n := (I + \frac{1}{n}\mathcal{A})^{-1}$. The operator $*$ designates the convolution and $\widetilde{\cdot}$ stands for the extension of functions belonging to $C([0,S];\mathbb{L}^2(\Omega))$ defined by

$$\widetilde{\boldsymbol{v}}(t) := \begin{cases} \boldsymbol{v}(t) & (t \in [0,S]), \\ \boldsymbol{v}(-t) & (t \in [-S,0]), \\ \boldsymbol{v}(2S-t) & (t \in [S,2S]), \\ 0 & (t \in \mathbb{R} \setminus [-S,2S]). \end{cases}$$

Moreover, $\cdot|_{[0,S]}$ is the restriction of functions onto the interval $[0,S]$.

Since $\boldsymbol{w}(t) \in D(\mathcal{A})$ (a.e. $t \in [0,S]$) is assumed, $J_n\boldsymbol{w}(t) \in D(\mathcal{A}^2)$ (a.e. $t \in [0,S]$) is valid, which implies that $\boldsymbol{w}_n \in C([0,S];D(\mathcal{A}^2))$.

Then, by the definition of $\boldsymbol{w}_n$, we have

$$\begin{aligned} \boldsymbol{w}(t) - \boldsymbol{w}_n(t) &= \boldsymbol{w}(t) - J_n\boldsymbol{w}(t) + J_n\boldsymbol{w}(t) - \boldsymbol{w}_n(t) \\ &= \boldsymbol{w}(t) - J_n\boldsymbol{w}(t) + \int_{-1/n}^{1/n} \rho_{1/n}(s)\left(J_n\boldsymbol{w}(t) - \widetilde{J_n\boldsymbol{w}}(t-s)\right) ds \\ &= \boldsymbol{w}(t) - J_n\boldsymbol{w}(t) + \int_{-1/n}^{1/n} \rho_{1/n}(s)\left(J_n\boldsymbol{w}(t) - J_n\boldsymbol{w}(t-s)\right) ds \end{aligned}$$

for $t \in [1/n, S-1/n]$,

$$\begin{aligned}\boldsymbol{w}(t) - \boldsymbol{w}_n(t) &= \boldsymbol{w}(t) - J_n\boldsymbol{w}(t) + J_n\boldsymbol{w}(t) - \boldsymbol{w}_n(t) \\ &= \boldsymbol{w}(t) - J_n\boldsymbol{w}(t) + \int_{-1/n}^{1/n} \rho_{1/n}(s) \left(J_n\boldsymbol{w}(t) - \widetilde{J_n\boldsymbol{w}}(t-s)\right) ds \\ &= \boldsymbol{w}(t) - J_n\boldsymbol{w}(t) + \int_{-1/n}^{t} \rho_{1/n}(s) \left(J_n\boldsymbol{w}(t) - J_n\boldsymbol{w}(t-s)\right) ds \\ &\qquad + \int_{t}^{1/n} \rho_{1/n}(s) \left(J_n\boldsymbol{w}(t) - J_n\boldsymbol{w}(s-t)\right) ds\end{aligned}$$

for $t \in [0, 1/n)$ and

$$\begin{aligned}\boldsymbol{w}(t) - \boldsymbol{w}_n(t) &= \boldsymbol{w}(t) - J_n\boldsymbol{w}(t) + J_n\boldsymbol{w}(t) - \boldsymbol{w}_n(t) \\ &= \boldsymbol{w}(t) - J_n\boldsymbol{w}(t) + \int_{-1/n}^{1/n} \rho_{1/n}(s) \left(J_n\boldsymbol{w}(t) - \widetilde{J_n\boldsymbol{w}}(t-s)\right) ds \\ &= \boldsymbol{w}(t) - J_n\boldsymbol{w}(t) + \int_{t-S}^{1/n} \rho_{1/n}(s) \left(J_n\boldsymbol{w}(t) - J_n\boldsymbol{w}(t-s)\right) ds \\ &\qquad + \int_{-1/n}^{t-S} \rho_{1/n}(s) \left(J_n\boldsymbol{w}(t) - J_n\boldsymbol{w}(2S+s-t)\right) ds\end{aligned}$$

for $t \in (S-1/n, S]$ (remark $2S+s-t \in [-1/n, 1/n]$).

Since J_n is a contraction mapping on $\mathbb{L}^2_\sigma(\Omega)$ and $\boldsymbol{w}$ belongs to $C([0,S]; \mathbb{L}^2_\sigma(\Omega))$, i.e., $\boldsymbol{w}$ is uniformly continuous on $[0,S]$, we can see that $\boldsymbol{w}_n(t) \to \boldsymbol{w}(t)$ in $\mathbb{L}^2_\sigma(\Omega)$ as $n \to \infty$ for any $t \in [0,S]$ and this convergence is uniform over $[0,S]$, namely, we can assure that $\boldsymbol{w}_n \to \boldsymbol{w}$ strongly in $C([0,S]; \mathbb{L}^2_\sigma(\Omega))$.

Similarly, we get

$$\begin{aligned}&\mathcal{A}^{1/2}\boldsymbol{w}(t) - \mathcal{A}^{1/2}\boldsymbol{w}_n(t) \\ =&\mathcal{A}^{1/2}\boldsymbol{w}(t) - J_n\mathcal{A}^{1/2}\boldsymbol{w}(t) + \int_{-1/n}^{1/n} \rho_{1/n}(s) \left(J_n\boldsymbol{w}(t) - J_n\mathcal{A}^{1/2}\boldsymbol{w}(t-s)\right) ds\end{aligned}$$

for $t \in [1/n, S-1/n]$,

$$\begin{aligned}&\mathcal{A}^{1/2}\boldsymbol{w}(t) - \mathcal{A}^{1/2}\boldsymbol{w}_n(t) \\ =&\mathcal{A}^{1/2}\boldsymbol{w}(t) - J_n\mathcal{A}^{1/2}\boldsymbol{w}(t) + \int_{-1/n}^{t} \rho_{1/n}(s) \left(J_n\mathcal{A}^{1/2}\boldsymbol{w}(t) - J_n\mathcal{A}^{1/2}\boldsymbol{w}(t-s)\right) ds \\ &\qquad + \int_{t}^{1/n} \rho_{1/n}(s) \left(J_n\mathcal{A}^{1/2}\boldsymbol{w}(t) - J_n\mathcal{A}^{1/2}\boldsymbol{w}(s-t)\right) ds\end{aligned}$$

for $t \in [0, 1/n)$ and

$$\begin{aligned}
&\mathcal{A}^{1/2}\boldsymbol{w}(t) - \mathcal{A}^{1/2}\boldsymbol{w}_n(t) \\
=&\mathcal{A}^{1/2}\boldsymbol{w}(t) - J_n\mathcal{A}^{1/2}\boldsymbol{w}(t) + \int_{t-S}^{1/n} \rho_{1/n}(s) \left(J_n\mathcal{A}^{1/2}\boldsymbol{w}(t) - J_n\mathcal{A}^{1/2}\boldsymbol{w}(t-s)\right) ds \\
&\qquad + \int_{-1/n}^{t-S} \rho_{1/n}(s) \left(J_n\mathcal{A}^{1/2}\boldsymbol{w}(t) - J_n\mathcal{A}^{1/2}\boldsymbol{w}(2S+s-t)\right) ds
\end{aligned}$$

for $t \in (S-1/n, S]$. By the same argument as above, we can assure that $\mathcal{A}^{1/2}\boldsymbol{w}_n \to \mathcal{A}^{1/2}\boldsymbol{w}$ in $C([0,S];\mathbb{L}^2_\sigma(\Omega))$, that is, $\boldsymbol{w}_n$ converges to $\boldsymbol{w}$ in $C([0,S];\mathbb{H}^1_\sigma(\Omega))$.

Since $\widetilde{\mathcal{A}\boldsymbol{w}} \in L^2(\mathbb{R}^1;\mathbb{L}^2_\sigma(\Omega))$, we can see that $\rho_{1/n} * \widetilde{\mathcal{A}\boldsymbol{w}}|_{[0,S]} \to \mathcal{A}\boldsymbol{w}$ in $L^2(0,S;\mathbb{L}^2_\sigma(\Omega))$ as $n \to \infty$. We can also derive that

$$\left|\rho_{1/n} * \widetilde{\mathcal{A}\boldsymbol{w}}(t) - \rho_{1/n} * \widetilde{J_n\mathcal{A}\boldsymbol{w}}(t)\right|_{\mathbb{L}^2_\sigma(\Omega)} \leqslant \rho_{1/n} * \left|\widetilde{\mathcal{A}\boldsymbol{w}} - \widetilde{J_n\mathcal{A}\boldsymbol{w}}\right|_{\mathbb{L}^2_\sigma(\Omega)}(t).$$

Therefore, by using Young's inequality, we have

$$\begin{aligned}
\left|\rho_{1/n} * \widetilde{\mathcal{A}\boldsymbol{w}} - \rho_{1/n} * \widetilde{J_n\mathcal{A}\boldsymbol{w}}\right|_{L^2(\mathbb{R}^1;\mathbb{L}^2_\sigma(\Omega))} &\leqslant |\rho_{1/n}|_{L^1(\mathbb{R}^1)} \left|\widetilde{\mathcal{A}\boldsymbol{w}} - \widetilde{J_n\mathcal{A}\boldsymbol{w}}\right|_{L^2(\mathbb{R}^1;\mathbb{L}^2_\sigma(\Omega))} \\
&\leqslant 3\left|\mathcal{A}\boldsymbol{w} - J_n\mathcal{A}\boldsymbol{w}\right|_{L^2(0,S;\mathbb{L}^2_\sigma(\Omega))}.
\end{aligned}$$

Since the right hand side converges to zero by virtue of Lebesgue's dominated convergence theorem, we can assure that $\rho_{1/n} * \widetilde{J_n\mathcal{A}\boldsymbol{w}}$ strongly converges to $\mathcal{A}\boldsymbol{w}$ in $L^2(0,S;\mathbb{L}^2_\sigma(\Omega))$.

Chapter 3

Global Solvability in Bounded Domains

3.1 Problems and Main Theorems

In this chapter, we consider the system (DCBF) in bounded domain Ω with sufficiently smooth boundary $\partial\Omega$.

$$
\text{(DCBF)}\begin{cases}
\partial_t \boldsymbol{u} + \nu\mathcal{A}_\Omega \boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_\Omega \boldsymbol{g}T + \mathcal{P}_\Omega \boldsymbol{h}C + \mathcal{P}_\Omega \boldsymbol{f}_1 & (x,t)\in\Omega\times[0,S],\\
\partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + f_2 & (x,t)\in\Omega\times[0,S],\\
\partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta C + \rho\Delta T + f_3 & (x,t)\in\Omega\times[0,S],\\
\boldsymbol{u}(\cdot,0)=\boldsymbol{u}_0,\ T(\cdot,0)=T_0,\ C(\cdot,0)=C_0 & \text{(Initial condition)},\\
\qquad\qquad\text{or} & \\
\boldsymbol{u}(\cdot,0)=\boldsymbol{u}(\cdot,S),\ T(\cdot,0)=T(\cdot,S),\ C(\cdot,0)=C(\cdot,S) & \text{(Periodic condition)}.
\end{cases}
$$

We deal with both Dirichlet boundary condition case:

$$\boldsymbol{u}=0,\quad T=0,\quad C=0 \qquad (x,t)\in\partial\Omega\times[0,S]$$

and Neumann boundary condition case:

$$\boldsymbol{u}=0,\quad \frac{\partial T}{\partial n}=0,\quad \frac{\partial C}{\partial n}=0 \qquad (x,t)\in\partial\Omega\times[0,S].$$

We simply write $\mathcal{A}$ and $\mathcal{P}$ in order to represent the Stokes operator $\mathcal{A}_\Omega$ and the orthogonal projection $\mathcal{P}_\Omega$. Throughout this chapter, the norms of $L^q(\Omega)$, $W^{k,q}(\Omega)$, $H^k(\Omega)$, $\mathbb{L}^q(\Omega)$, $\mathbb{W}^{k,q}(\Omega)$ and $\mathbb{H}^k(\Omega)$ will be simply denoted by $|\cdot|_{L^q}$, $|\cdot|_{W^{k,q}}$, $|\cdot|_{H^k}$, $|\cdot|_{\mathbb{L}^q}$, $|\cdot|_{\mathbb{W}^{k,q}}$, and $|\cdot|_{\mathbb{H}^k}$ respectively (in subsequent chapters, we also use these notation if no vagueness arises)

In Terasawa–Ôtani [60], the following solvability of the initial boundary value problem with Dirichlet condition is given.

Proposition 3.1.1. *Let $N = 2, 3$ and let $\boldsymbol{f}_1 = \boldsymbol{0}$, $f_2, f_3 = 0$. Then for each initial data $\boldsymbol{u}_0 \in \mathbb{H}^1_\sigma(\Omega)$, $T_0, C_0 \in H^1_0(\Omega)$, the initial boundary value problem of* (DCBF) *with the homogeneous Dirichlet boundary condition admits a unique solution* $(\boldsymbol{u}, T, C)$ *satisfying*

$$\begin{aligned}
&\boldsymbol{u} \in C([0, S]; \mathbb{H}^1_\sigma(\Omega)) \cap L^2(0, S; \mathbb{H}^2(\Omega)) \cap W^{1,2}(0, S; \mathbb{L}^2_\sigma(\Omega)),\\
&T, C \in C([0, S]; H^1_0(\Omega)) \cap L^2(0, S; H^2(\Omega)) \cap W^{1,2}(0, S; L^2(\Omega))
\end{aligned}$$

for any $S > 0$.

Motivated by this result, we aim to solve other types of problems in a bounded domain, i.e., the solvability of the initial boundary value problem with homogeneous Neumann boundary condition and the time periodic problem with Dirichlet and Neumann boundary conditions in this chapter.

In the next section, we first show the existence of a unique global solution for the initial boundary value problem with Neumann boundary condition. To this end, we follow the strategy of Terasawa–Ôtani [60], i.e., we reduce (DCBF) to an abstract problem in some Hilbert space and we apply Proposition 2.3.3 and Proposition 2.3.4, the abstract result given in Ôtani [41] to this problem. In this way, the following result will be demonstrated in Section 3.2 (see Section 2.5 about the fractional power of operators $\mathcal{A}^\alpha$ and $(-\Delta_N)^\alpha$).

Theorem 3.1.1 (Initial boundary value problem with Neumann boundary condition)**.** *Let $N = 2, 3$ and let $\boldsymbol{f}_1 \in L^2(0, S; \mathbb{L}^2(\Omega))$, $f_2, f_3 \in L^2(0, S; L^2(\Omega))$. Then for each initial data $\boldsymbol{u}_0 \in D(\mathcal{A}^\alpha)$, $T_0, C_0 \in D((-\Delta_N)^\alpha)$ with $\alpha \in [1/4, 1/2]$, the initial boundary value problem of* (DCBF) *with the homogeneous Neumann boundary condition admits a unique solution* $(\boldsymbol{u}, T, C)$ *satisfying*

$$(\#)_\alpha \quad \begin{cases}
\boldsymbol{u} \in C([0, S]; \mathbb{L}^2_\sigma(\Omega)), \quad T, C \in C([0, S]; L^2(\Omega)),\\
t^{1/2-\alpha}\partial_t \boldsymbol{u},\ t^{1/2-\alpha}\mathcal{A}\boldsymbol{u} \in L^2(0, S; \mathbb{L}^2_\sigma(\Omega)),\\
t^{1/2-\alpha}|\nabla \boldsymbol{u}|_{\mathbb{L}^2(\Omega)} \in L^q_*(0, S) \quad \forall q \in [2, \infty],\\
t^{1/2-\alpha}\partial_t T,\ t^{1/2-\alpha}\partial_t C,\ t^{1/2-\alpha}\Delta T,\ t^{1/2-\alpha}\Delta C \in L^2(0, S; L^2(\Omega)),\\
t^{1/2-\alpha}|\nabla T|_{L^2(\Omega)},\ t^{1/2-\alpha}|\nabla C|_{L^2(\Omega)} \in L^q_*(0, S) \qquad \forall q \in [2, \infty].
\end{cases}$$

Here we note $L^q_* = L^q(dt/t)$, i.e., $|g|_{L^q_*(0,S)} := \left(\int_0^1 |g(t)|^q t^{-1} dt\right)^{1/q}$ for $q \in [1, \infty)$ and $L^\infty_*(0, S) = L^\infty(0, S)$.

Remarks

(1) If $\alpha = 1/2$, i.e., if $\boldsymbol{u}_0 \in D(\mathcal{A}^{1/2}) = \mathbb{H}^1_\sigma(\Omega)$ and $T_0, C_0 \in D(A_N^{1/2}) = H^1(\Omega)$, then $(\#)_{1/2}$ implies that the solution satisfies

$$\begin{aligned}
&\boldsymbol{u} \in C([0, S]; \mathbb{H}^1_\sigma(\Omega)) \cap L^2(0, S; \mathbb{H}^2(\Omega)) \cap W^{1,2}(0, S; \mathbb{L}^2_\sigma(\Omega)),\\
&T, C \in C([0, S]; H^1(\Omega)) \cap L^2(0, S; H^2(\Omega)) \cap W^{1,2}(0, S; L^2(\Omega)).
\end{aligned}$$

(2) Even if Neumann boundary condition is replaced by Dirichlet boundary condition, our argument which will be employed in Section 3.2 can be carried out with obvious modifications and the following result, which includes that of Terasawa–Ôtani [60], can be assured.

Corollary 3.1.1 (Initial boundary value problem with Dirichlet boundary condition)**.** *Let* $N = 2, 3$ *and let* $\boldsymbol{f}_1 \in L^2(0, S; \mathbb{L}^2(\Omega))$, $f_2, f_3 \in L^2(0, S; L^2(\Omega))$. *Then for each initial data* $\boldsymbol{u}_0 \in D(\mathcal{A}^\alpha)$, $T_0, C_0 \in D((-\Delta_D)^\alpha)$ *with* $\alpha \in [1/4, 1/2]$, *the initial boundary value problem of* (DCBF) *with the homogeneous Dirichlet boundary condition admits a unique solution* $(\boldsymbol{u}, T, C)$ *satisfying*

$$\begin{cases} \boldsymbol{u} \in C([0, S]; \mathbb{L}^2_\sigma(\Omega)), \quad T, C \in C([0, S]; L^2(\Omega)), \\ t^{1/2-\alpha}\partial_t \boldsymbol{u},\ t^{1/2-\alpha}\mathcal{A}\boldsymbol{u} \in L^2(0, S; \mathbb{L}^2_\sigma(\Omega)), \\ t^{1/2-\alpha}|\nabla \boldsymbol{u}|_{\mathbb{L}^2(\Omega)} \in L^q_*(0, S) \quad \forall q \in [2, \infty], \\ t^{1/2-\alpha}\partial_t T,\ t^{1/2-\alpha}\partial_t C,\ t^{1/2-\alpha}\Delta T,\ t^{1/2-\alpha}\Delta C \in L^2(0, S; L^2(\Omega)), \\ t^{1/2-\alpha}|\nabla T|_{L^2(\Omega)},\ t^{1/2-\alpha}|\nabla C|_{L^2(\Omega)} \in L^q_*(0, S) \qquad \forall q \in [2, \infty]. \end{cases}$$

Next we consider the periodic problem of (DCBF) with Dirichlet and Neumann boundary conditions. Basic strategies is the application of Proposition 2.3.5, i.e., the abstract result given by Ôtani [42]. Here we remark that the required conditions in Proposition 2.3.5 are stricter than those in Proposition 2.3.3 and Proposition 2.3.4. Then, due to the presence of convection terms $\boldsymbol{u} \cdot \nabla T$, $\boldsymbol{u} \cdot \nabla C$ and the buoyancy terms $\boldsymbol{g}T$, $\boldsymbol{h}C$, it is difficult to apply Proposition 2.3.5 directly to (DCBF). In order to cope with this difficulty, we introduce some approximation system involving some dissipation terms and cut-off functions and we show the solvability of these systems by using Proposition 2.3.5. In particular, for the Neumann boundary condition case, we need to introduce another step of approximation and impose some additional condition on the external forces f_2 and f_3, since the operator $-\Delta_N$ does not possess the coercivity. Finally, discussing the convergence of solutions for approximate equations, we shall prove the following results in Section 3.3 and 3.4 respectively (recall $C_\pi([0, S]; X) := \{U \in C([0, S]; X); U(0) = U(S)\}$).

Theorem 3.1.2 (Periodic problem with Dirichlet boundary condition)**.** *Let* $N = 2, 3$ *and let* $\boldsymbol{f}_1 \in L^2(0, S; \mathbb{L}^2(\Omega))$, $f_2, f_3 \in L^2(0, S; L^2(\Omega))$. *Then the time periodic problem of* (DCBF) *with the homogeneous Dirichlet boundary condition possesses at least one periodic solution* $(\boldsymbol{u}, T, C)$ *satisfying*

$$\begin{aligned} &\boldsymbol{u} \in C_\pi([0, S]; \mathbb{H}^1_\sigma(\Omega)) \cap L^2(0, S; \mathbb{H}^2(\Omega)) \cap W^{1,2}(0, S; \mathbb{L}^2_\sigma(\Omega)), \\ &T,\ C \in C_\pi([0, S]; H^1_0(\Omega)) \cap L^2(0, S; H^2(\Omega)) \cap W^{1,2}(0, S; L^2(\Omega)). \end{aligned}$$

Theorem 3.1.3 (Periodic problem with Neumann boundary condition)**.** *Let* $N = 2, 3$ *and let* $\boldsymbol{f}_1 \in L^2(0, S; \mathbb{L}^2(\Omega))$, $f_2, f_3 \in L^2(0, S; L^2(\Omega))$. *Moreover, we assume that*

$$\int_0^S \int_\Omega f_2(x, t) dx dt = \int_0^S \int_\Omega f_3(x, t) dx dt = 0. \tag{3.1}$$

Then the time periodic problem of (DCBF) *with the homogeneous Neumann boundary condition possesses at least one periodic solution* $(\boldsymbol{u}, T, C)$ *satisfying*

$$\boldsymbol{u} \in C_\pi([0, S]; \mathbb{H}^1_\sigma(\Omega)) \cap L^2(0, S; \mathbb{H}^2(\Omega)) \cap W^{1,2}(0, S; \mathbb{L}^2_\sigma(\Omega)),$$
$$T,\ C \in C_\pi([0, S]; H^1(\Omega)) \cap L^2(0, S; H^2(\Omega)) \cap W^{1,2}(0, S; L^2(\Omega)).$$

Remark We can easily show that (3.1) assumed in Theorem 3.1.3 is also a necessary condition under the homogeneous Neumann boundary condition. Indeed, let $(\boldsymbol{u}, T, C)$ be a periodic solution derived from Theorem 3.1.3. From the homogeneous Neumann boundary condition and the solenoidal condition of $\boldsymbol{u}$, we have

$$\int_\Omega \Delta T dx = 0, \quad \int_\Omega \Delta C dx = 0,$$
$$\int_\Omega \boldsymbol{u} \cdot \nabla T dx = -\int_\Omega T \nabla \cdot \boldsymbol{u} dx = 0, \quad \int_\Omega \boldsymbol{u} \cdot \nabla C dx = 0.$$

Therefore, integrating the second and the third equation over Ω, we get

$$\frac{d}{dt}\int_\Omega T dx = \int_\Omega f_2 dx, \quad \frac{d}{dt}\int_\Omega C dx = \int_\Omega f_3 dx.$$

Since T and C are time-periodic with period S, integration over $[0, S]$ yields

$$\int_0^S \int_\Omega f_2(x, t) dx dt = \int_0^S \int_\Omega f_3(x, t) dx dt = 0.$$

Furthermore, for Dirichlet boundary condition case, we can obtain the following uniqueness result.

Theorem 3.1.4 (Uniqueness of periodic solution)**.** *There exists a constant* θ *depending on* $|\boldsymbol{g}|, |\boldsymbol{h}|, \nu, \rho$ *and* $|\boldsymbol{f}_1|_{L^2(0,S;\mathbb{L}^2_\sigma(\Omega))}$ *satisfying the following: if* $|f_2|_{L^2(0,S;L^2(\Omega))} \leqslant \theta$ *and* $|f_3|_{L^2(0,S;L^2(\Omega))} \leqslant \theta$*, then the periodic solution of* (DCBF) *given in* Theorem 3.1.2 *is unique.*

Remark Under the homogeneous Neumann boundary condition, the uniqueness of periodic solution (given in Theorem 3.1.3) generally does not hold, even if the external forces are vary small. Indeed, we can see that $\boldsymbol{g} = (g^1, \cdots, g^N)$ and $\boldsymbol{h} = (h^1, \cdots, h^N)$ possess potential functions, i.e., $\boldsymbol{g} = \nabla G$ and $\boldsymbol{h} = \nabla H$ are valid, where

$$G(x) := g^1(x^1 - x^1_0) + \cdots + g^N(x^N - x^N_0),$$
$$H(x) := h^1(x^1 - x^1_0) + \cdots + h^N(x^N - x^N_0)$$

$(x = (x^1, \cdots, x^N) \in \mathbb{R}^N$ is the variable and $x^1_0, \cdots, x^N_0$ are arbitrary fixed numbers). Since Ω is assume to be bounded, $G, H \in H^1(\Omega)$ holds. This fact implies that $\boldsymbol{g}$ and $\boldsymbol{h}$ belong to $G_2(\Omega)$ (the orthogonal complement of $\mathbb{L}^2_\sigma(\Omega)$), that is to say, we obtain $\mathcal{P}\boldsymbol{g} = \mathcal{P}\boldsymbol{h} = 0$. Therefore, if we can deduce a time-periodic solution $(\boldsymbol{u}, T, C)$ of (DCBF) with the homogeneous Neumann boundary condition, we can assure that $(\boldsymbol{u}, T + M_T, C + M_C)$ also becomes periodic solution under the Neumann boundary condition for any real number M_T and M_C.

3.2 Initial Boundary Value Problem with Neumann Boundary Condition

3.2.1 Reduction to an Abstract Problem

To begin with, we prove Theorem 3.1.1. To this end, we first reduce the system

$$\text{(DCBF)}\begin{cases} \partial_t \boldsymbol{u} + \nu \mathcal{A}\boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}\boldsymbol{g}T + \mathcal{P}\boldsymbol{h}C + \mathcal{P}\boldsymbol{f}_1, \\ \partial_t T - \Delta_N T + \boldsymbol{u}\cdot\nabla T = f_2, \\ \partial_t C - \Delta_N C + \boldsymbol{u}\cdot\nabla C = \rho\Delta_N T + f_3 \end{cases}$$

to an abstract equation.

We define the Hilbert space H by

$$H := \mathbb{L}^2_\sigma(\Omega)\times L^2(\Omega)\times L^2(\Omega).$$

Moreover, for each parameter $\eta \in (0,1]$, H_η designates the Hilbert space H endowed with the following inner product:

$$(U_1, U_2)_H = (\boldsymbol{u}_1, \boldsymbol{u}_2)_{\mathbb{L}^2} + (T_1, T_2)_{L^2} + \frac{\eta^2}{9\rho^2}(C_1, C_2)_{L^2}, \tag{3.2}$$

where $U_i = (\boldsymbol{u}_i, T_i, C_i)^t$ $(i = 1, 2)$ and $(\cdot,\cdot)_{\mathbb{L}^2}$, $(\cdot,\cdot)_{L^2}$ describe the inner product in $\mathbb{L}^2(\Omega)$, $L^2(\Omega)$ respectively, i.e.,

$$(\boldsymbol{w}_1, \boldsymbol{w}_2)_{\mathbb{L}^2} := \int_\Omega \boldsymbol{u}_1\cdot\boldsymbol{u}_2 dx, \qquad (V_1, V_2)_{L^2} := \int_\Omega V_1 V_2 dx$$

for $\boldsymbol{w}_i \in \mathbb{L}^2(\Omega)$ and $V_i \in L^2(\Omega)$ $(i = 1, 2)$. The wight $\eta^2/9\rho^2$ in the third component of the inner product (3.2) is added so that we can deal with the term $\rho\Delta T$ as a sufficiently small perturbation (see next subsection, Check of $(\mathrm{A3})_\alpha$). For elements belonging to H_η, we here put

$$U = \begin{pmatrix} \boldsymbol{u} \\ T \\ C \end{pmatrix}, \quad \frac{dU}{dt} = \begin{pmatrix} \partial_t\boldsymbol{u} \\ \partial_t T \\ \partial_t C \end{pmatrix}, \quad F = \begin{pmatrix} \mathcal{P}\boldsymbol{f}_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

Next we define $\varphi : H_\eta \to (-\infty, +\infty]$ by

$$\varphi(U) = \begin{cases} \frac{\nu}{2}|\nabla\boldsymbol{u}|^2_{L^2} + \frac{1}{2}|\nabla T|^2_{L^2} + \frac{\eta^2}{18\rho^2}|\nabla C|^2_{L^2} & \text{if } U \in D(\varphi), \\ +\infty & \text{if } U \in H_\eta \setminus D(\varphi) \end{cases} \tag{3.3}$$

with the effective domain $D(\varphi) := \mathbb{H}^1_\sigma(\Omega)\times H^1(\Omega)\times H^1(\Omega)$. Then, recalling φ_S and φ_N defined in Section 2.3, we can show that φ is a proper lower semi-continuous convex function on H_η and its subdifferential $\partial\varphi$ coincides with

$$\partial\varphi(U) = \begin{pmatrix} \nu\mathcal{A}\Delta\boldsymbol{u} \\ -\Delta_N T \\ -\Delta_N C \end{pmatrix}$$

with domain $D(\partial\varphi) = D(\mathcal{A}) \times D(-\Delta_N) \times D(-\Delta_N)$. We note that $\partial\varphi$ is single-valued operator, although the subdifferential operators could be generally multi-valued operators. Collecting the other remainder terms, we define the single-valued non-monotone perturbation B by

$$B(U) = \begin{pmatrix} a\boldsymbol{u} - \mathcal{P}\boldsymbol{g}T - \mathcal{P}\boldsymbol{h}C \\ \boldsymbol{u}\cdot\nabla T \\ \boldsymbol{u}\cdot\nabla C - \rho\Delta_N T \end{pmatrix}.$$

In this way, we can reduce our problem to the following abstract Cauchy problem (CP) in the Hilbert space H_η:

$$\text{(CP)} \begin{cases} \dfrac{dU}{dt} + \partial\varphi(U) + B(U) = F, \\ U(0) = U_0. \end{cases}$$

3.2.2 Existence of Local Solution

According to Ôtani [41], i.e., Proposition 2.3.3 and Proposition 2.3.4, the existence of time local solution of (CP) satisfying $(\#)_\alpha$ is assured, provided that our system (DCBF) satisfies the conditions (A1), (A2), $(\mathrm{A3})_\alpha$ and (A4) (recall and see Section 2.3). In this subsection, we check these conditions and show the local existence.

We substitute ε, the exponent appearing in the condition $(\mathrm{A3})_\alpha$ (see Section 2.3.2), by η, the parameter of the Hilbert space H_η. If there is no confusion, the Hilbert space $H_\eta = H_\varepsilon$ is simply designated by H henceforth.

• **Check of (A1) :** For any $L \in (0, +\infty)$, the set $\{U \in H; \varphi(U) + \|U\|_H^2 \leqslant L\}$ is compact in H.

Since $\{U \in H; \varphi(U) + \|U\|_H^2 \leqslant L\}$ is a closed and bounded subset in $\mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times H^1(\Omega)$, Rellich-Kondrachov's theorem (Proposition 2.1.7) immediately leads to the compactness of this set in $\mathbb{L}^2_\sigma(\Omega) \times L^2(\Omega) \times L^2(\Omega)$.

• **Check of (A2) :** $B(\cdot)$ is φ-demiclosed.

Assume that the sequence $\{U_k\}_{k\in\mathbb{N}} = \{(\boldsymbol{u}_k, T_k, C_k)^t\}_{k\in\mathbb{N}}$ and its limit $U = (\boldsymbol{u}, T, C)^t$ satisfy

$$\begin{cases} \boldsymbol{u}_k \to \boldsymbol{u} & \text{strongly in } C([0,S];\mathbb{L}^2_\sigma(\Omega)), \\ T_k \to T & \text{strongly in } C([0,S];L^2(\Omega)), \\ C_k \to C & \text{strongly in } C([0,S];L^2(\Omega)), \end{cases}$$

$$\begin{cases} \nu\mathcal{A}\boldsymbol{u}_k \rightharpoonup \nu\mathcal{A}\boldsymbol{u} & \text{weakly in } L^2(0,S;\mathbb{L}^2_\sigma(\Omega)), \\ -\Delta_N T_k \rightharpoonup -\Delta_N T & \text{weakly in } L^2(0,S;L^2(\Omega)), \\ -\Delta_N C_k \rightharpoonup -\Delta_N C & \text{weakly in } L^2(0,S;L^2(\Omega)), \end{cases}$$

$$\begin{cases} a\boldsymbol{u}_k - \mathcal{P}\boldsymbol{g}T_k - \mathcal{P}\boldsymbol{h}C_k \rightharpoonup h_1 & \text{weakly in } L^2(0,S;\mathbb{L}^2_\sigma(\Omega)), \\ \boldsymbol{u}_k \cdot \nabla T_k \rightharpoonup h_2 & \text{weakly in } L^2(0,S;L^2(\Omega)), \\ \boldsymbol{u}_k \cdot \nabla C_k - \rho\Delta_N T_k \rightharpoonup h_3 & \text{weakly in } L^2(0,S;L^2(\Omega)). \end{cases}$$

We can derive $\boldsymbol{u}(t) \in D(\mathcal{A})$ and $T(t), C(t) \in D(-\Delta_N)$ for a.e. $t \in [0,S]$ from the fact that $\partial\varphi(U)$ belongs to $L^2(0,S;H)$. By the strong convergences of $\{U_k\}_{k\in\mathbb{N}}$, we can easily get $h_1 = a\boldsymbol{u} - \mathcal{P}\boldsymbol{g}T - \mathcal{P}\boldsymbol{h}C$. Fix $\phi \in C_0^\infty((0,S);C_0^\infty(\Omega))$. Using the solenoidal condition and boundary condition of $\boldsymbol{u}_k$ and applying the integration by parts, we have

$$\begin{aligned} &\int_0^S \int_\Omega \phi \boldsymbol{u}_k \cdot \nabla T_k dxdt \\ &= \int_0^S \int_{\partial\Omega} \phi \boldsymbol{u}_k T_k \cdot \boldsymbol{n} dSdt - \int_0^S \int_\Omega \boldsymbol{u}_k T_k \cdot \nabla\phi dxdt - \int_0^S \int_\Omega \phi T_k \nabla \cdot \boldsymbol{u}_k dxdt \\ &= -\int_0^S \int_\Omega \boldsymbol{u}_k T_k \cdot \nabla\phi dxdt. \end{aligned}$$

Then taking the limit as $k \to \infty$ and using the integration by parts again (recalling $\boldsymbol{u} \in D(\mathcal{A})$, $T \in D(-\Delta_N)$, which implies that $\nabla \cdot (\boldsymbol{u}T) = \boldsymbol{u} \cdot \nabla T$ is well defined in $L^2(\Omega)$), we obtain

$$\begin{aligned} \int_0^S \int_\Omega h_2 \phi dxdt &= -\int_0^S \int_\Omega \boldsymbol{u}T \cdot \nabla\phi dxdt \\ &= \int_0^S \int_\Omega \boldsymbol{u} \cdot \nabla T \phi dxdt \end{aligned}$$

for any $\phi \in C_0^\infty((0,S);C_0^\infty(\Omega))$. Since $C_0^\infty((0,S);C_0^\infty(\Omega))$ is dense in $L^2(0,S;L^2(\Omega))$, we can assure that $h_2 = \boldsymbol{u} \cdot \nabla T$. By exactly the same reasoning, we can get $h_3 = \boldsymbol{u} \cdot \nabla C - \rho\Delta_N T$.

- **Check of $\textbf{(A3)}_\alpha$:** For each given $\varepsilon > 0$, $\exists \ell(\cdot)$ such that

$$\|B(U)\|_H \leqslant \ell(\|U\|_H) \left\{ \varepsilon \|\partial\varphi(U)\|_H + \frac{1}{\varepsilon} |\varphi(U)|^{\frac{1-\alpha}{1-2\alpha}} + 1 \right\} \quad \forall U \in D(\partial\varphi).$$

By the definition of B and the inner product of $H = H_\varepsilon$, we get

$$\begin{aligned} \|B(U)\|_H \leqslant\ & a|\boldsymbol{u}|_{\mathbb{L}^2} + |\boldsymbol{g}||T|_{L^2} + |\boldsymbol{h}||C|_{L^2} \\ &+ |\boldsymbol{u} \cdot \nabla T|_{L^2} + \frac{\varepsilon}{3\rho}(|\boldsymbol{u} \cdot \nabla C|_{L^2} + \rho|\Delta T|_{L^2}). \end{aligned}$$

Here applying Hölder's inequality and using the fact $|U|^2_{L^3} \leqslant |U|_{L^2}|U|_{L^6}$ (recall Corollary 2.1.1 in Section 2.1), we have

$$\begin{aligned} |\boldsymbol{u} \cdot \nabla T|_{L^2} = \left(\int_\Omega |\boldsymbol{u}|^2 |\nabla T|^2 dx \right)^{1/2} &\leqslant ||\boldsymbol{u}|^2|^{1/2}_{\mathbb{L}^3} ||\nabla T|^2|^{1/2}_{L^{3/2}} \\ &\leqslant |\boldsymbol{u}|_{\mathbb{L}^6} |\nabla T|_{L^3} \leqslant |\boldsymbol{u}|_{\mathbb{L}^6} |\nabla T|^{1/2}_{L^2} |\nabla T|^{1/2}_{L^6}. \end{aligned}$$

Moreover, from Sobolev's inequality and elliptic estimate,

$$\begin{aligned}|\boldsymbol{u}|_{\mathbb{L}^6}|\nabla T|_{L^2}^{1/2}|\nabla T|_{L^6}^{1/2} &\leqslant \gamma_0|\boldsymbol{u}|_{\mathbb{H}^1}|\nabla T|_{L^2}^{1/2}|\nabla T|_{H^1}^{1/2} \\ &\leqslant \gamma_0|\nabla \boldsymbol{u}|_{\mathbb{L}^2}|\nabla T|_{L^2}^{1/2}|\Delta T|_{L^2}^{1/2}\end{aligned}$$

can be derived for $N = 2, 3$, where γ_0 is some general constant. Here, we use the following fact:

$$\begin{aligned}|\nabla T|_{H^1} = |\nabla T'|_{H^1} &\leqslant |T'|_{H^2} \leqslant \gamma_0(|\Delta T'|_{L^2} + |T'|_{L^2}) \\ &\leqslant \gamma_0(|\Delta T'|_{L^2} + \gamma_0|\nabla T'|_{L^2}) \leqslant \gamma_0|\Delta T'|_{L^2} = \gamma_0|\Delta T|_{L^2},\end{aligned}$$

where $T' := T - \frac{1}{|\Omega|}\int_\Omega T dx$ and we apply Poincaré-Wirtinger's inequality (Proposition 2.5.5 in Section 2.5). Then we can obtain

$$\begin{aligned}|\boldsymbol{u} \cdot \nabla T|_{L^2} &\leqslant \gamma_0|\nabla \boldsymbol{u}|_{\mathbb{L}^2}|\nabla T|_{L^2}^{1/2}|\Delta T|_{L^2}^{1/2} \\ &\leqslant \frac{\varepsilon}{4}|\Delta T|_{L^2} + \frac{\gamma_0}{\varepsilon}|\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2|\nabla T|_{L^2}.\end{aligned}$$

Similarly,

$$|\boldsymbol{u} \cdot \nabla C|_{L^2} \leqslant \frac{\varepsilon}{4}|\Delta C|_{L^2} + \frac{\gamma_0}{\varepsilon}|\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2|\nabla C|_{L^2}$$

holds for $N = 2, 3$. From these inequalities above, we can derive

$$\begin{aligned}\|B(U)\|_H &\leqslant \gamma_0\|U\|_H + \frac{\varepsilon}{4}|\Delta T|_{L^2} + \frac{\gamma_0}{\varepsilon}|\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2|\nabla T|_{L^2} \\ &\qquad + \frac{\varepsilon}{3}|\Delta T|_{L^2} + \frac{\varepsilon}{4}\frac{\varepsilon}{3\rho}|\Delta C|_{L^2} + \frac{\gamma_0}{\varepsilon}\frac{\varepsilon}{3\rho}|\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2|\nabla C|_{L^2} \\ &\leqslant \gamma_0\|U\|_H + \frac{7\varepsilon}{12}\|\partial\varphi(U)\|_H + \frac{\gamma_0}{\varepsilon}\varphi^{3/2}(U),\end{aligned}$$

which guarantees the condition $(\mathrm{A3})_\alpha$ with $\alpha \in [1/4, 1/2)$.

- **Check of (A4) :** $\exists \ell(\cdot)$ and $\exists k \in [0, 1)$ such that

$$\|B(U)\|_H^2 \leqslant k\|\partial\varphi(U)\|_H^2 + \ell(\varphi(U) + \|U\|_H^2) \qquad \forall U \in D(\partial\varphi).$$

Let $\varepsilon = 1$ in the procedure above, Check of $(\mathrm{A3})_\alpha$. Then we can get

$$\|B(U)\|_H \leqslant \frac{7}{12}\|\partial\varphi(U)\|_H + \gamma_0\left(\|U\|_H + \varphi^{3/2}(U)\right) \quad (N = 2, 3),$$

which obviously assures the condition (A4).

Therefore, if $N = 2, 3$, (DCBF) satisfies all the required conditions in Proposition 2.3.3 with $\alpha \in [1/4, 1/2)$ and in Proposition 2.3.4. That is to say, we can assure the existence of time local solution $(\boldsymbol{u}, T, C)^t$ which satisfies $(\#)_\alpha$ $(1/4 \leqslant \alpha < 1/2)$ for initial data $(\boldsymbol{u}_0, T_0, C_0)^t \in \mathcal{B}_{\alpha,2} = D(\mathcal{A}^\alpha) \times D((-\Delta_N)^\alpha) \times D((-\Delta_N)^\alpha)$ and satisfies $(\#)_{1/2}$ for initial data $(\boldsymbol{u}_0, T_0, C_0)^t \in D(\varphi) = \mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times H^1(\Omega)$.

3.2.3 Global Existence and Uniqueness

In this subsection, we show that the time local solutions derived in the previous subsection can be extended to the whole interval $[0, S]$ by establishing some a priori estimates. We also discuss the uniqueness of solution in the end of this section.

Let $S_0 \in (0, S]$ and let $Q(z_1, z_2, \cdots)$ denote a general constant depending on the variables $z_1, z_2, \cdots$. Multiplying the second equation of (DCBF) by T and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}|T|_{L^2}^2 + |\nabla T|_{L^2}^2 = \int_\Omega f_2 T dx \leqslant |f_2|_{L^2}|T|_{L^2}. \tag{3.4}$$

In (3.4), we use the fact that

$$\int_\Omega (\boldsymbol{u}\cdot\nabla T)T dx = \frac{1}{2}\int_\Omega \boldsymbol{u}\cdot\nabla\left(T^2\right)dx = -\frac{1}{2}\int_\Omega T^2\nabla\cdot\boldsymbol{u}dx = 0. \tag{3.5}$$

Applying Proposition 2.5.2 (see Section 2.5) to (3.4) with $\eta = |T|_{L^2}$, $k = |T_0|_{L^2}$ and $\phi = |f_2|_{L^2}$, we have

$$|T(t)|_{L^2} \leqslant |T_0|_{L^2} + \int_0^t |f_2(s)|_{L^2}ds \leqslant |T_0|_{L^2} + \int_0^S |f_2(s)|_{L^2}ds$$

which yields

$$\sup_{0\le t\le S_0}|T(t)|_{L^2} \leqslant |T_0|_{L^2} + |f_2|_{L^1(0,S;L^2(\Omega))}.$$

Integrating (3.4) again, we get

$$\begin{aligned}\int_0^{S_0}|\nabla T(t)|_{L^2}^2dt &\leqslant |T_0|_{L^2}^2 + \int_0^{S_0}|f_2(s)|_{L^2}|T(s)|_{L^2}ds\\ &\leqslant |T_0|_{L^2}^2 + (|T_0|_{L^2} + |f_2|_{L^1(0,S;L^2(\Omega))})\int_0^S|f_2(s)|_{L^2}ds\\ &\leqslant Q(|T_0|_{L^2}, |f_2|_{L^1(0,S;L^2(\Omega))}).\end{aligned}$$

Next, multiplying the third equation of (DCBF) by C and using (3.5) with T replaced by C, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|C|_{L^2}^2 + |\nabla C|_{L^2}^2 &= -\int_\Omega \rho\nabla T\cdot\nabla C dx + \int_\Omega f_3 C dx\\ &\leqslant \rho|\nabla T|_{L^2}|\nabla C|_{L^2} + |f_3|_{L^2}|C|_{L^2}\\ &\leqslant \frac{1}{2}|\nabla C|_{L^2}^2 + \frac{\rho^2}{2}|\nabla T|_{L^2}^2 + |f_3|_{L^2}|C|_{L^2},\end{aligned}$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}|C|_{L^2}^2 + \frac{1}{2}|\nabla C|_{L^2}^2 \leqslant \frac{\rho^2}{2}|\nabla T|_{L^2}^2 + |f_3|_{L^2}|C|_{L^2}. \tag{3.6}$$

Applying Proposition 2.5.2 to (3.6) with $\eta = |C|_{L^2}$, $k = |C_0|_{L^2} + \frac{\rho}{\sqrt{2}}|\nabla T|_{L^2(0,S_0;L^2(\Omega))}$ and $\phi = |f_2|_{L^2}$, we have

$$\sup_{0\le t\le S_0} |C(t)|_{L^2} \leqslant |C_0|_{L^2} + Q(|T_0|_{L^2}, |f_2|_{L^1(0,S;L^2(\Omega))}) + \int_0^S |f_3(s)|_{L^2} ds.$$

Moreover, integration of (3.6) also yields

$$\begin{aligned}\int_0^{S_0} |\nabla C(t)|^2_{L^2} dt &\leqslant |C_0|^2_{L^2} + \int_0^{S_0} |\nabla T(s)|^2_{L^2} ds + \int_0^{S_0} |f_2(s)|_{L^2}|C(s)|_{L^2} ds \\ &\leqslant Q(|T_0|_{L^2}, |C_0|_{L^2}, |f_2|_{L^1(0,S;L^2(\Omega))}, |f_3|_{L^1(0,S;L^2(\Omega))}).\end{aligned}$$

Multiplying the first equation of (DCBF) by $\partial_t \boldsymbol{u}$ and using Hölder's inequality, we get

$$\begin{aligned}&|\partial_t \boldsymbol{u}|^2_{\mathbb{L}^2} + \frac{\nu}{2}\frac{d}{dt}|\nabla \boldsymbol{u}|^2_{\mathbb{L}^2} + \frac{a}{2}\frac{d}{dt}|\boldsymbol{u}|^2_{\mathbb{L}^2} \\ &\qquad \leqslant |\partial_t \boldsymbol{u}|_{\mathbb{L}^2}(|\boldsymbol{g}||T|_{L^2} + |\boldsymbol{h}||C|_{L^2} + |\boldsymbol{f}_1|_{\mathbb{L}^2}) \\ &\qquad \leqslant \frac{1}{2}|\partial_t \boldsymbol{u}|^2_{\mathbb{L}^2} + \frac{1}{2}(|\boldsymbol{g}||T|_{L^2} + |\boldsymbol{h}||C|_{L^2} + |\boldsymbol{f}_1|_{\mathbb{L}^2})^2.\end{aligned} \tag{3.7}$$

Integrating (3.7) over $[0, t]$, we have

$$\begin{aligned}&\sup_{0\le t\le S_0} \nu|\nabla \boldsymbol{u}(t)|^2_{\mathbb{L}^2} + \sup_{0\le t\le S_0} a|\boldsymbol{u}(t)|^2_{\mathbb{L}^2} + \int_0^{S_0} |\partial_t \boldsymbol{u}(s)|^2_{\mathbb{L}^2} ds \\ &\qquad \leqslant Q(S, |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}, \|U_0\|_H, |F|_{L^2(0,S;H)}),\end{aligned} \tag{3.8}$$

From (3.8) and the first equation of (DCBF), we can derive

$$\int_0^{S_0} |\mathcal{A}\boldsymbol{u}(s)|^2_{\mathbb{L}^2} ds \leqslant Q(S, |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}, \|U_0\|_H, |F|_{L^2(0,S;H)}). \tag{3.9}$$

Multiplying the second equation of (DCBF) by $-\Delta T$ and recalling the estimates of convection term $\boldsymbol{u}\cdot\nabla T$ in the previous section (Check of $(\mathrm{A3})_\alpha$), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla T|^2_{L^2} + |\Delta T|^2_{L^2} &\leqslant |\Delta T|_{L^2}|\boldsymbol{u}\cdot\nabla T|_{L^2} + |\Delta T|_{L^2}|f_2|_{L^2} \\ &\leqslant \frac{1}{2}|\Delta T|^2_{L^2} + |\boldsymbol{u}\cdot\nabla T|^2_{L^2} + |f_2|^2_{L^2} \\ &\leqslant \frac{1}{2}|\Delta T|^2_{L^2} + \gamma_0|\nabla \boldsymbol{u}|^2_{\mathbb{L}^2}|\nabla T|_{L^2}|\Delta T|_{L^2} + |f_2|^2_{L^2},\end{aligned}$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}|\nabla T|^2_{L^2} + \frac{1}{4}|\Delta T|^2_{L^2} \leqslant \gamma_0|\nabla \boldsymbol{u}|^4_{\mathbb{L}^2}|\nabla T|^2_{L^2} + |f_2|^2_{L^2}. \tag{3.10}$$

By applying Gronwall's inequality to (3.10) (Proposition 2.5.1 with $\eta = |\nabla T|^2_{L^2}$) we can derive

$$\sup_{0\le t\le S_0} |\nabla T(t)|^2_{L^2} \leqslant Q(S, |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}, |\nabla T_0|_{L^2}, \|U_0\|_H, |F|_{L^2(0,S;H)}).$$

From (3.10), we also get

$$\int_0^{S_0} |\Delta T(t)|^2_{L^2} dt \leqslant Q(S, |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}, |\nabla T_0|_{L^2}, \|U_0\|_H, |F|_{L^2(0,S;H)}).$$

Multiplying the third equation of (DCBF) by $-\Delta C$ and using the same argument as for (3.10), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla C|^2_{L^2} + |\Delta C|^2_{L^2} \leqslant |\Delta C|_{L^2} \left(|\boldsymbol{u}\cdot\nabla C|_{L^2} + \rho|\Delta T|_{L^2} + |f_3|_{L^2}\right)\\
\Rightarrow &\frac{1}{2}\frac{d}{dt}|\nabla C|^2_{L^2} + \frac{1}{4}|\Delta C|^2_{L^2} \leqslant \gamma_0|\nabla \boldsymbol{u}|^4_{\mathbb{L}^2}|\nabla C|^2_{L^2} + 3\rho^2|\Delta T|^2_{L^2} + 3|f_3|^2_{L^2},
\end{aligned}$$

which leads to

$$\begin{aligned}
\sup_{0\le t\le S_0} |\nabla C(t)|^2_{L^2} &+ \int_0^{S_0} |\Delta C(t)|^2_{L^2} dt \\
&\leqslant Q(S, |\nabla \boldsymbol{u}_0|_{L^2}, |T_0|_{H^1}, |C_0|_{H^1}, |F|_{L^2(0,S;H)}).
\end{aligned} \tag{3.11}$$

Collecting the estimates above, we can assure the following boundedness of solution $(\boldsymbol{u}, T, C)^t$:

$$\begin{aligned}
\sup_{0\le t\le S_0} |\boldsymbol{u}(t)|_{\mathbb{H}^1} &+ \sup_{0\le t\le S_0} |T(t)|_{H^1} + \sup_{0\le t\le S_0} |C(t)|_{H^1} \\
&\leqslant Q(S, |\nabla \boldsymbol{u}_0|_{L^2}, |T_0|_{H^1}, |C_0|_{H^1}, |F|_{L^2(0,S;H)}).
\end{aligned} \tag{3.12}$$

We remark that this a priori bounds (the right hand side of 3.12) is independent of $S_0 \in (0, S]$. Therefore, time local solutions with the initial data $U_0 \in D(\varphi)$ can be continued globally up to $[0, S]$. Moreover, recalling the regularity $(\#)_\alpha$, we can assure that every local solutions with the initial data $U_0 \in D(\mathcal{A}^\alpha) \times D((-\Delta_N)^\alpha) \times D((-\Delta_N)^\alpha)$ possesses $t_0 \in (0, S_0)$ such that $U(t_0) \in D(\varphi)$. Then, regarding $U(t_0)$ as an initial data and applying the global existence result for the case where $U_0 \in D(\varphi)$, we can also extend time local solutions globally for the general case where $\alpha \in [1/4, 1/2]$.

We next show the uniqueness of the solution. Let $U_i = (\boldsymbol{u}_i, T_i, C_i)^t$ $(i = 1, 2)$ be solutions with the same initial data and let $\delta U = (\delta \boldsymbol{u}, \delta T, \delta C)^t$ be the difference of these two solutions, i.e.,

$$\delta \boldsymbol{u} := \boldsymbol{u}_1 - \boldsymbol{u}_2, \quad \delta T := T_1 - T_2, \quad \delta C := C_1 - C_2.$$

From (DCBF), δU satisfies the following equations:

$$\text{(D)} \begin{cases} \partial_t \delta \boldsymbol{u} + \nu \mathcal{A} \delta \boldsymbol{u} + a\delta \boldsymbol{u} = \mathcal{P}\boldsymbol{g}\delta T + \mathcal{P}\boldsymbol{h}\delta C, \\ \partial_t \delta T - \Delta \delta T = -\boldsymbol{u}_1 \cdot \nabla \delta T + \delta \boldsymbol{u} \cdot \nabla T_2, \\ \partial_t \delta C - \Delta \delta C = \rho \Delta \delta T - \boldsymbol{u}_1 \cdot \nabla \delta C + \delta \boldsymbol{u} \cdot \nabla C_2. \end{cases}$$

Multiplying the first equation of (D) by $\delta\boldsymbol{u}$, we get

$$
\begin{aligned}
(3.13)\qquad \frac{1}{2}\frac{d}{dt}|\delta\boldsymbol{u}|_{\mathbb{L}^2}^2 + \nu|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}^2 &\leqslant |\boldsymbol{g}||\delta T|_{L^2}|\delta\boldsymbol{u}|_{\mathbb{L}^2} + |\boldsymbol{h}||\delta C|_{L^2}|\delta\boldsymbol{u}|_{\mathbb{L}^2} \\
&\leqslant \frac{1}{2}(|\boldsymbol{g}|^2 + |\boldsymbol{h}|^2)|\delta\boldsymbol{u}|_{\mathbb{L}^2}^2 + \frac{|\boldsymbol{g}|^2}{2}|\delta T|_{L^2}^2 + \frac{|\boldsymbol{h}|^2}{2}|\delta C|_{L^2}^2.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
&\int_\Omega (\boldsymbol{u}_1\cdot\nabla\delta T)\delta T dx = -\frac{1}{2}\int_\Omega \delta T^2\nabla\cdot\boldsymbol{u}_1 dx = 0,\\
&\int_\Omega (\delta\boldsymbol{u}\cdot\nabla T_2)\delta T dx = -\int_\Omega T_2(\delta\boldsymbol{u}\cdot\nabla\delta T + \delta T\nabla\cdot\delta\boldsymbol{u})dx = -\int_\Omega (\delta\boldsymbol{u}\cdot\nabla\delta T)T_2 dx
\end{aligned}
$$

and multiplying the second equation of (D) by δT, we have

$$
\begin{aligned}
(3.14)\qquad \frac{1}{2}\frac{d}{dt}|\delta T|_{L^2}^2 + |\nabla\delta T|_{L^2}^2 &\leqslant |\delta\boldsymbol{u}T_2|_{\mathbb{L}^2}|\nabla\delta T|_{L^2} \leqslant |\delta\boldsymbol{u}|_{\mathbb{L}^3}|T_2|_{L^6}|\nabla\delta T|_{L^2} \\
&\leqslant \frac{1}{2}|\nabla\delta T|_{L^2}^2 + \gamma_1|\delta\boldsymbol{u}|_{\mathbb{L}^2}|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}|T_2|_{H^1}^2 \\
&\leqslant \frac{1}{2}|\nabla\delta T|_{L^2}^2 + \frac{\nu}{4}|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}^2 + \gamma_1|\delta\boldsymbol{u}|_{\mathbb{L}^2}^2|T_2|_{H^1}^4,
\end{aligned}
$$

where γ_1 is some suitable general constant. Similarly, multiplying the third equation of (D) by δC, we obtain

$$
\begin{aligned}
(3.15)\qquad \frac{1}{2}\frac{d}{dt}|\delta C|_{L^2}^2 + |\nabla\delta C|_{L^2}^2 &\leqslant \gamma_1|\delta\boldsymbol{u}|_{\mathbb{L}^2}^{1/2}|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}^{1/2}|C_2|_{H^1}|\nabla\delta C|_{L^2} \\
&\qquad + \rho|\nabla\delta T|_{L^2}|\nabla\delta C|_{L^2} \\
&\leqslant \frac{1}{2}|\nabla\delta C|_{L^2}^2 + \rho^2|\nabla\delta T|_{L^2} \\
&\qquad + \frac{\nu\rho^2}{4}|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}^2 + \gamma_1|\delta\boldsymbol{u}|_{\mathbb{L}^2}^2|C_2|_{H^1}^4.
\end{aligned}
$$

Let $y(t) = |\delta\boldsymbol{u}(t)|_{\mathbb{L}^2}^2 + |\delta T(t)|_{L^2}^2 + \frac{1}{2\rho^2}|\delta C(t)|_{L^2}^2$. Then, summing up (3.13), (3.14) and $\frac{1}{2\rho^2}\times$ (3.15), we obtain

$$
\frac{d}{dt}y(t) \leqslant \gamma_1\left(1 + |T_2(t)|_{H^1}^4 + |C_2(t)|_{H^1}^4\right)y(t).
$$

Here we note that $(\#)_\alpha$ with $\alpha\in[1/4,1/2]$ implies that

$$
t^{1/2-\alpha}|\nabla T_2|_{L^2}, t^{1/2-\alpha}|\nabla C_2|_{L^2} \in L_*^4(0,S) \Rightarrow |\nabla T_2|_{L^2}, |\nabla C_2|_{L^2} \in L^4(0,S).
$$

Hence, the uniqueness follows from Gronwall's inequality. □

3.3 Time Periodic Problem with Dirichlet Boundary Condition

In this section, we consider the solvability of time periodic problem with homogeneous Dirichlet boundary condition (Theorem 3.1.2). Comparing the required condition (A5) in Proposition 2.3.5 (see section 2.3)

$$\|B(U)\|_H^2 \leqslant k\|\partial\varphi(U)\|_H^2 + \ell(\|U\|_H)(\varphi(U)+1)^2,$$

with the estimate of non-monotone perturbation term $B(U)$ derived in Check of $(\mathrm{A3})_\alpha$

$$\|B(U)\|_H^2 \leqslant \varepsilon^2\|\partial\varphi(U)\|_H^2 + \frac{\gamma}{\varepsilon^2}(\varphi(U)^3 + \|U\|_H^2),$$

we realize that it is difficult to apply Proposition 2.3.5 directly to (DCBF). We also face some difficulties in checking the condition (A6).

3.3.1 Approximate Equations

To cope with the difficulties above, we first introduce the following approximate equations with parameter $\epsilon > 0$.

$$(\mathrm{DCBF})_\epsilon \begin{cases} \partial_t \boldsymbol{u} + \nu\mathcal{A}\boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}\boldsymbol{g}[T]_\epsilon + \mathcal{P}\boldsymbol{h}[C]_\epsilon + \mathcal{P}\boldsymbol{f}_1, \\ \partial_t T - \Delta_D T + \epsilon|T|^{q-2}T + \boldsymbol{u}\cdot\nabla T = f_2, \\ \partial_t C - \Delta_D C + \epsilon|C|^{q-2}C + \boldsymbol{u}\cdot\nabla C = \rho\Delta_D T + f_3, \end{cases}$$

where the cut-off function $[\cdot]_\epsilon$ is defined by

$$[T]_\epsilon(x,t) := \begin{cases} T(x,t) & \text{if } |T(x,t)| \leqslant 1/\epsilon, \\ (\mathrm{Sgn}\ T(x,t)) \times 1/\epsilon & \text{if } |T(x,t)| \geqslant 1/\epsilon \end{cases}$$

($\mathrm{Sgn}T := T/|T|$: the sign of T) and q is a large exponent to be fixed later on.

Next we reduce $(\mathrm{DCBF})_\epsilon$ to an abstract equation. Here, we choose $\eta = 1$ in (3.2), definition of the inner product of $H = \mathbb{L}^2_\sigma(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. We define proper lower semi-continuous convex functions φ' and ψ_ϵ by

$$\varphi'(U) = \begin{cases} \frac{\nu}{2}|\nabla\boldsymbol{u}|_{L^2}^2 + \frac{1}{2}|\nabla T|_{L^2}^2 + \frac{1}{18\rho^2}|\nabla C|_{L^2}^2 & \text{if } U \in D(\varphi'), \\ +\infty & \text{if } U \in H_\eta \setminus D(\varphi'), \end{cases}$$

$$\psi_\epsilon(U) = \begin{cases} \frac{\epsilon}{q}|T|_{L^q}^q + \frac{\epsilon}{9\rho^2 q}|C|_{L^q}^q & \text{if } U \in D(\psi_\epsilon), \\ +\infty & \text{if } U \in H \setminus D(\psi_\epsilon), \end{cases}$$

where $D(\varphi') := \mathbb{H}^1_\sigma(\Omega) \times H^1_0(\Omega) \times H^1_0(\Omega)$ and $D(\psi_\epsilon) := \mathbb{L}^2_\sigma(\Omega) \times L^q(\Omega) \times L^q(\Omega)$. Then, recalling φ_S, φ_D and φ_q defined in Example 1, 2 and 3 (Section 2.3), we have

$$\partial\varphi'(U) = \begin{pmatrix} \nu\mathcal{A}\boldsymbol{u} \\ -\Delta_D T \\ -\Delta_D C \end{pmatrix}, \quad \partial\psi_\epsilon(U) = \begin{pmatrix} \mathbf{0} \\ \epsilon|T|^{q-2}T \\ \epsilon|C|^{q-2}C \end{pmatrix}$$

with domain $D(\partial\varphi') = D(\mathcal{A}) \times D(-\Delta_D) \times D(-\Delta_D)$ and $D(\partial\psi_\epsilon) = \mathbb{L}^2_\sigma(\Omega) \times L^{2(q-1)}(\Omega) \times L^{2(q-1)}(\Omega)$. Moreover, by the same reasoning as that explained in Example 4 of Section 2.3, we can assure that $\partial(\varphi' + \psi_\epsilon) = \partial\varphi' + \partial\psi_\epsilon$ and $D(\partial(\varphi' + \psi_\epsilon)) = D(\partial\varphi') \cap D(\partial\psi_\epsilon)$, since we get

$$\begin{aligned} (\partial\varphi'(U), \partial\psi_\epsilon(U))_H &= (-\Delta T, \epsilon|T|^{q-2}T)_{L^2} + (-\Delta C, \epsilon|C|^{q-2}C)_{L^2} \\ &= \epsilon(p-1)\int_\Omega |T|^{q-2}|\nabla T|^2 dx + \epsilon(q-1)\int_\Omega |C|^{q-2}|\nabla C|^2 dx \geqslant 0. \end{aligned} \tag{3.16}$$

Then putting

$$B_\epsilon(U) = \begin{pmatrix} a\boldsymbol{u} - \mathcal{P}\boldsymbol{g}[T]_\epsilon - \mathcal{P}\boldsymbol{h}[C]_\epsilon \\ \boldsymbol{u}\cdot\nabla T \\ \boldsymbol{u}\cdot\nabla C - \rho\Delta_D T \end{pmatrix},$$

we can reduce approximate problems (DCBF)$_\epsilon$ to the following abstract problem.

$$(\mathrm{AP})_\epsilon \begin{cases} \dfrac{dU(t)}{dt} + \partial\varphi_\epsilon(U(t)) + B_\epsilon(U(t)) = F(t) \quad t \in [0, S], \\ U(0) = U(S), \end{cases}$$

where $\varphi_\epsilon := \varphi' + \psi_\epsilon$.

3.3.2 Existence of Approximate Solutions

In this subsection, we check that the conditions (A1), (A2), (A5) and (A6) in Proposition 2.3.5 (section 2.3) are satisfied for (DCBF)$_\epsilon$ (we note that Poincaré's inequality guarantees the condition

- $\|U\|_H^2 \leqslant \gamma\varphi_\epsilon(U)$ holds for any $U \in D(\varphi_\epsilon)$ and $\partial\varphi_\epsilon$ is strictly monotone,

which is also required in Proposition 2.3.5).

• **Check of (A1) :** For any $L \in (0, +\infty)$, the set $\{U \in H; \varphi_\epsilon(U) + \|U\|_H^2 \leqslant L\}$ is compact in H.

Since $\{U \in H; \varphi_\epsilon(U) + \|U\|_H^2 \leqslant L\}$ is closed and bounded in $\mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times H^1(\Omega)$, this set becomes compact by virtue of Rellich-Kondrachov's theorem (Proposition 2.1.7) in H.

• **Check of (A2) :** $B_\epsilon(\cdot)$ is φ_ϵ-demiclosed.
Assume that

$$\begin{cases} \boldsymbol{u}_k \to \boldsymbol{u} & \text{strongly in } C([0,S];\mathbb{L}^2_\sigma(\Omega)), \\ T_k \to T & \text{strongly in } C([0,S];L^2(\Omega)), \\ C_k \to C & \text{strongly in } C([0,S];L^2(\Omega)), \end{cases}$$

$$\begin{cases} \nu\mathcal{A}\boldsymbol{u}_k \rightharpoonup \nu\mathcal{A}\boldsymbol{u} & \text{weakly in } L^2(0,S;\mathbb{L}^2_\sigma(\Omega)), \\ -\Delta_D T_k + \epsilon|T_k|^{q-2}T_k \rightharpoonup -\Delta_D T + \epsilon|T|^{q-2}T & \text{weakly in } L^2(0,S;L^2(\Omega)), \\ -\Delta_D C_k + \epsilon|C_k|^{q-2}C_k \rightharpoonup -\Delta_D C + \epsilon|C|^{q-2}C & \text{weakly in } L^2(0,S;L^2(\Omega)), \end{cases}$$

$$\begin{cases} a\boldsymbol{u}_k - \mathcal{P}\boldsymbol{g}[T_k]_\epsilon - \mathcal{P}\boldsymbol{h}[C_k]_\epsilon \rightharpoonup h'_1 & \text{weakly in } L^2(0,S;\mathbb{L}^2_\sigma(\Omega)), \\ \boldsymbol{u}_k\cdot\nabla T_k \rightharpoonup h'_2 & \text{weakly in } L^2(0,S;L^2(\Omega)), \\ \boldsymbol{u}_k\cdot\nabla C_k - \rho\Delta_D T_k \rightharpoonup h'_3 & \text{weakly in } L^2(0,S;L^2(\Omega)). \end{cases}$$

From the strong convergences, we can easily derive $h'_1 = a\boldsymbol{u} - \mathcal{P}\boldsymbol{g}[T]_\epsilon - \mathcal{P}\boldsymbol{h}[C]_\epsilon$. Using the angular condition (3.16), the strong convergence of $\{U_k\}_{k\in\mathbb{N}}$ and the weak convergences of $\{\partial\varphi_\epsilon(U_k)\}_{k\in\mathbb{N}}$, we obtain

$$\begin{cases} -\Delta_D T_k \rightharpoonup -\Delta_D T & \text{weakly in } L^2(0,S;L^2(\Omega)), \\ \epsilon|T_k|^{q-2}T_k \rightharpoonup \epsilon|T|^{q-2}T & \text{weakly in } L^2(0,S;L^2(\Omega)), \\ -\Delta_D C_k \rightharpoonup -\Delta_D C & \text{weakly in } L^2(0,S;L^2(\Omega)), \\ \epsilon|C_k|^{q-2}C_k \rightharpoonup \epsilon|T|^{q-2}T & \text{weakly in } L^2(0,S;L^2(\Omega)). \end{cases}$$

Therefore, we can repeat exactly the same argument that in Check of (A2), Section 3.2 and we can assure that $h'_2 = \boldsymbol{u}\cdot\nabla T$ and $h'_3 = \boldsymbol{u}\cdot\nabla C - \rho\Delta_D T$.

• **Check of (A5) :** There exists a monotone increasing function $\ell(\cdot)$ and a constant $k \in [0,1)$ such that

$$\|B_\epsilon(U)\|_H^2 \leqslant k\|\partial\varphi_\epsilon(U)\|_H^2 + \ell(\|U\|_H)(\varphi_\epsilon(U)+1)^2 \qquad \forall U \in D(\partial\varphi_\epsilon).$$

By the definition of the inner product in H and $B_\epsilon(U)$, we get

$$\|B_\epsilon(U)\|_H^2 \leqslant \gamma_2\|U\|_H^2 + \frac{2}{9}|\Delta T|_{L^2}^2 + |\boldsymbol{u}\cdot\nabla T|_{L^2}^2 + \frac{2}{9\rho^2}|\boldsymbol{u}\cdot\nabla C|_{L^2}^2,$$

where γ_2 is a suitable general constant and we use the fact that

$$|[T]_\epsilon|_{L^2} \leqslant |T|_{L^2}, \quad |[C]_\epsilon|_{L^2} \leqslant |C|_{L^2}.$$

Using the integration by parts, the condition $\nabla\cdot\boldsymbol{u}=0$ and applying Hölder's inequality, we obtain

$$\begin{aligned}
|\boldsymbol{u}\cdot\nabla T|_{L^2}^2 &= \int_\Omega \nabla T\cdot\boldsymbol{u}(\boldsymbol{u}\cdot\nabla T)dx\\
&= -\int_\Omega T\boldsymbol{u}\nabla(\boldsymbol{u}\cdot\nabla T)dx - \int_\Omega T(\boldsymbol{u}\cdot\nabla T)\nabla\cdot\boldsymbol{u}dx\\
&\leqslant \int_\Omega |T||\boldsymbol{u}||\nabla(\boldsymbol{u}\cdot\nabla T)|dx\\
&\leqslant \int_\Omega |T||\boldsymbol{u}|\sum_{\mu=1}^N|\nabla u^\mu|\sum_{\mu=1}^N|\partial_{x_\mu}T|dx + \int_\Omega |T||\boldsymbol{u}|\sum_{\mu=1}^N|u^\mu|\sum_{\mu=1}^N|\nabla\partial_{x_\mu}T|dx\\
&\leqslant |T|_{L^{12}}|\boldsymbol{u}|_{\mathbb{L}^6}|\nabla\boldsymbol{u}|_{\mathbb{L}^4}|\nabla T|_{L^2} + |T|_{L^{12}}|\boldsymbol{u}|_{\mathbb{L}^6}|\boldsymbol{u}|_{\mathbb{L}^4}\sum_{\mu=1}^N|\partial_{x_\mu}\nabla T|_{L^2}.
\end{aligned}$$

By Sobolev's inequality, elliptic estimates and the fact that $|U|_{L^4}^4\leqslant|U|_{L^2}|U|_{L^6}^3$ (see Corollary 2.1.1 in Section 2.1), we have

$$\begin{aligned}
|T|_{L^{12}}|\boldsymbol{u}|_{\mathbb{L}^6}|\nabla\boldsymbol{u}|_{\mathbb{L}^4}|\nabla T|_{L^2} &\leqslant \gamma_2|T|_{L^{12}}|\nabla\boldsymbol{u}|_{\mathbb{L}^2}|\nabla\boldsymbol{u}|_{\mathbb{L}^2}^{1/4}|\nabla\boldsymbol{u}|_{\mathbb{L}^6}^{3/4}|\nabla T|_{L^2}\\
&\leqslant \gamma_2|T|_{L^{12}}|\nabla\boldsymbol{u}|_{\mathbb{L}^2}^{5/4}|\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^{3/4}|\nabla T|_{L^2}\\
&\leqslant \frac{\nu}{12}|\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 + \gamma_2|T|_{L^{12}}^{8/5}|\nabla\boldsymbol{u}|_{\mathbb{L}^2}^2|\nabla T|_{L^2}^{8/5}\\
&\leqslant \frac{\nu}{12}|\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 + \gamma_2|\nabla\boldsymbol{u}|_{\mathbb{L}^2}^4 + \gamma_2|T|_{L^{12}}^{16/5}|\nabla T|_{L^2}^{16/5}\\
&\leqslant \frac{\nu}{12}|\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 + \gamma_2(|\nabla\boldsymbol{u}|_{\mathbb{L}^2}^4 + |\nabla T|_{L^2}^4 + |T|_{L^{12}}^{16}),
\end{aligned}$$

$$\begin{aligned}
|T|_{L^{12}}|\boldsymbol{u}|_{\mathbb{L}^6}|\boldsymbol{u}|_{\mathbb{L}^4}\sum_{\mu=1}^N|\partial_{x_\mu}\nabla T|_{L^2} &\leqslant |T|_{L^{12}}|\boldsymbol{u}|_{\mathbb{L}^6}^{7/4}|\boldsymbol{u}|_{\mathbb{L}^2}^{1/4}|T|_{H^2}\\
&\leqslant \gamma_2|T|_{L^{12}}|\nabla\boldsymbol{u}|_{\mathbb{L}^2}^{7/4}|\boldsymbol{u}|_{\mathbb{L}^2}^{1/4}|\Delta T|_{L^2}\\
&\leqslant \frac{1}{9}|\Delta T|_{L^2}^2 + \gamma_2|T|_{L^{12}}^2|\nabla\boldsymbol{u}|_{\mathbb{L}^2}^{7/2}|\boldsymbol{u}|_{\mathbb{L}^2}^{1/2}\\
&\leqslant \frac{1}{9}|\Delta T|_{L^2}^2 + \gamma_2(|\nabla\boldsymbol{u}|_{\mathbb{L}^2}^4 + |T|_{L^{12}}^{16}|\boldsymbol{u}|_{\mathbb{L}^2}^4).
\end{aligned}$$

Therefore, we can deduce

$$\begin{aligned}
|\boldsymbol{u}\cdot\nabla T|_{L^2}^2 \leqslant{}& \frac{\nu}{12}|\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 + \frac{1}{9}|\Delta T|_{L^2}^2\\
&+ \gamma_2(|\nabla\boldsymbol{u}|_{\mathbb{L}^2}^4 + |\nabla T|_{L^2}^4 + |T|_{L^{12}}^{16}(1+|\boldsymbol{u}|_{\mathbb{L}^2}^4)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
|\boldsymbol{u}\cdot\nabla C|_{L^2}^2 \leqslant{}& \frac{\nu}{12}\frac{9\rho^2}{2}|\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 + \frac{1}{9}|\Delta C|_{L^2}^2\\
&+ \gamma_2(|\nabla\boldsymbol{u}|_{\mathbb{L}^2}^4 + |\nabla C|_{L^2}^4 + |C|_{L^{12}}^{16}(1+\|U\|_H^4)).
\end{aligned}$$

Therefore, by taking $q \geqslant 12$ in ψ_ϵ, we can derive

$$\begin{aligned}\|B_\epsilon(U)\|_H^2 &\leqslant \gamma_2\|U\|_H^2 + \gamma_2(\varphi_\epsilon^2(U)+1)(1+\|U\|_H^4)\\ &\qquad + \frac{\nu}{6}|\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 + \frac{1}{3}|\Delta T|_{L^2}^2 + \frac{1}{9}\frac{2}{9\rho^2}|\Delta C|_{L^2}^2\\ &\leqslant \gamma_2\|U\|_H^2 + \gamma_2(\varphi_\epsilon^2(U)+1)(1+\|U\|_H^4) + \frac{1}{3}\|\partial\varphi_\epsilon(U)\|_H^2,\end{aligned}$$

whence follows (A5) with $k = 1/3$, provided that $q \geqslant 12$ in ψ_ϵ.

• **Check of (A6) :** There exist positive constants $\alpha,\ K$ such that

$$(-\partial\varphi_\epsilon(U) - B_\epsilon(U), U)_H + \alpha\varphi_\epsilon(U) \leqslant K \qquad \forall U \in D(\partial\varphi_\epsilon).$$

The definition of the inner product of H yields

$$\begin{aligned}(\partial\varphi_\epsilon(U), U)_H &= \nu|\nabla\boldsymbol{u}|_{\mathbb{L}^2}^2 + |\nabla T|_{L^2}^2 + \frac{1}{9\rho^2}|\nabla C|_{L^2}^2 + \epsilon|T|_{L^q}^q + \frac{\epsilon}{9\rho^2}|C|_{L^q}^q\\ &\geqslant 2\varphi_\epsilon(U).\end{aligned}$$

Noting that $(\boldsymbol{u}\cdot\nabla T, T)_{L^2} = (\boldsymbol{u}\cdot\nabla C, C)_{L^2} = 0$ (see (3.5)) and $|[T]_\epsilon(x,t)| \leqslant 1/\epsilon$, $|[C]_\epsilon(x,t)| \leqslant 1/\epsilon$, we have

$$\begin{aligned}&(B_\epsilon(U), U)_H\\ \geqslant& a|\boldsymbol{u}|_{\mathbb{L}^2}^2 - |\boldsymbol{g}||\boldsymbol{u}|_{\mathbb{L}^2}|[T]_\epsilon|_{L^2} - |\boldsymbol{h}||\boldsymbol{u}|_{\mathbb{L}^2}|[C]_\epsilon|_{L^2} - \frac{1}{9\rho}|\nabla T|_{L^2}|\nabla C|_{L^2}\\ \geqslant& a|\boldsymbol{u}|_{\mathbb{L}^2}^2 - \frac{|\Omega|^{1/2}}{\epsilon}|\boldsymbol{u}|_{\mathbb{L}^2}(|\boldsymbol{g}|+|\boldsymbol{h}|) - \frac{1}{2}|\nabla T|_{L^2}^2 - \frac{1}{18\rho^2}|\nabla C|_{L^2}^2\\ \geqslant& \frac{a}{2}|\boldsymbol{u}|_{\mathbb{L}^2}^2 - \frac{|\Omega|}{2a\epsilon^2}(|\boldsymbol{g}|+|\boldsymbol{h}|)^2 - \varphi_\epsilon(U).\end{aligned}$$

Hence we obtain

$$(-\partial\varphi_\epsilon(U) - B_\epsilon(U), U)_H \leqslant -\varphi_\epsilon(U) + \frac{|\Omega|}{2a\epsilon^2}(|\boldsymbol{g}|+|\boldsymbol{h}|)^2,$$

i.e., (A6) is satisfied with $\alpha = 1$ and $K = \frac{|\Omega|}{2a\epsilon^2}(|\boldsymbol{g}|+|\boldsymbol{h}|)^2$.

Thus, for any parameter ϵ, the existence of a periodic solution $(\boldsymbol{u}_\epsilon, T_\epsilon, C_\epsilon)^t$ of approximate equations (DCBF)$_\epsilon$ can be assured by Proposition 2.3.5.

3.3.3 Convergence

In this subsection, discussing the convergence of $\{(\boldsymbol{u}_\epsilon, T_\epsilon, C_\epsilon)^t\}_{\epsilon>0}$ as $\epsilon \to 0$, we conclude the solvability of the original system (DCBF).

To this end, we establish some a priori estimates of $(\boldsymbol{u}_\epsilon, T_\epsilon, C_\epsilon)^t$. Throughout this subsection, γ_3 denotes the general constant independent of ϵ. Multiplying the second equation of $(\mathrm{DCBF})_\epsilon$ by T_ϵ, integrating over Ω and using the fact that $(\boldsymbol{u}_\epsilon \cdot \nabla T_\epsilon, T_\epsilon)_{L^2} = 0$ (recall (3.5)), we get

$$\frac{1}{2}\frac{d}{dt}|T_\epsilon|_{L^2}^2 + |\nabla T_\epsilon|_{L^2}^2 + \epsilon |T_\epsilon|_{L^q}^q = \int_\Omega f_2 T_\epsilon dx.$$

Applying Poincaré's inequality, we have

$$\int_\Omega f_2 T_\epsilon dx \leqslant \kappa^{1/2} |f_2|_{L^2} |\nabla T_\epsilon|_{L^2} \leqslant \frac{1}{2}|\nabla T_\epsilon|_{L^2}^2 + \frac{\kappa}{2}|f_2|_{L^2}^2$$

(here, κ is a constant satisfying $|V|_{L^2}^2 \leqslant \kappa |\nabla V|_{L^2}$ for any $V \in H_0^1(\Omega)$). Then we obtain

$$\frac{d}{dt}|T_\epsilon|_{L^2}^2 + |\nabla T_\epsilon|_{L^2}^2 + 2\epsilon |T_\epsilon|_{L^q}^q \leqslant \kappa |f_2|_{L^2}^2. \tag{3.17}$$

We here note that

$$\int_0^S \frac{d}{ds}|T_\epsilon(s)|_{L^2}^2 ds = |T_\epsilon(S)|_{L^2}^2 - |T_\epsilon(0)|_{L^2}^2 = 0,$$

since T_ϵ satisfies the periodic condition. Therefore, integrating (3.17) over $[0, S]$, we have

$$\int_0^S |\nabla T_\epsilon(s)|_{L^2}^2 ds + \epsilon \int_0^S |T_\epsilon(s)|_{L^q}^q ds \leqslant \kappa |f_2|_{L^2(0,S;L^2(\Omega))}^2.$$

Moreover, Poincaré's inequality yields

$$\int_0^S |T_\epsilon(s)|_{L^2}^2 ds \leqslant \kappa^2 |f_2|_{L^2(0,S;L^2(\Omega))}^2.$$

Here, since T_ϵ belongs to $C([0, S]; L^2(\Omega))$, there exist $t_1^\epsilon \in [0, S]$ such that $|T_\epsilon(\cdot)|_{L^2}$ attains its minimum at t_1^ϵ, i.e.,

$$|T_\epsilon(t_1^\epsilon)|_{L^2} = \min_{0 \leqslant t \leqslant S} |T_\epsilon(t)|_{L^2}.$$

Then, we get

$$|T_\epsilon(t_1^\epsilon)|_{L^2} \leqslant \frac{1}{S}\int_0^S |T_\epsilon(s)|_{L^2}^2 ds \leqslant \frac{\kappa^2}{S}|f_2|_{L^2(0,S;L^2(\Omega))}^2,$$

namely, $|T_\epsilon(t_1^\epsilon)|_{L^2}$ is bounded independently of ϵ. Then integrating (3.17) over $[t_1^\epsilon, t]$ with $t \in [t_1^\epsilon, t_1^\epsilon + S]$ and recalling the time periodicity of T_ϵ, we can obtain

$$\sup_{0 \leq t \leq S} |T_\epsilon(t)|_{L^2}^2 \leqslant \left(\frac{\kappa^2}{S} + \kappa\right) |f_2|_{L^2(0,S;L^2(\Omega))}^2.$$

Hence multiplication of the second equation of $(\mathrm{DCBF})_\epsilon$ by T_ϵ yields

$$\sup_{0 \leq t \leq S} |T_\epsilon(t)|_{L^2}^2 + \int_0^S |\nabla T_\epsilon(s)|_{L^2}^2 ds + \epsilon \int_0^S |T_\epsilon(s)|_{L^q}^q ds \leqslant \gamma_3. \tag{3.18}$$

Next, multiplying the third equation of $(\mathrm{DCBF})_\epsilon$ by C_ϵ, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|C_\epsilon|_{L^2}^2 + |\nabla C_\epsilon|_{L^2}^2 + \epsilon|C_\epsilon|_{L^q}^q &\leqslant \rho|\nabla T_\epsilon|_{L^2}|\nabla C_\epsilon|_{L^2} + \kappa^{1/2}|f_3|_{L^2}|\nabla C_\epsilon|_{L^2}\\ &\leqslant \frac{1}{2}|\nabla C_\epsilon|_{L^2}^2 + \rho^2|\nabla T_\epsilon|_{L^2}^2 + \kappa|f_3|_{L^2}^2,\end{aligned}$$

i.e.,

$$\frac{d}{dt}|C_\epsilon|_{L^2}^2 + |\nabla C_\epsilon|_{L^2}^2 + 2\epsilon|C_\epsilon|_{L^q}^q \leqslant 2\rho^2|\nabla T_\epsilon|_{L^2}^2 + 2\kappa|f_3|_{L^2}^2. \tag{3.19}$$

Using $|\nabla T_\epsilon|_{L^2(0,S;L^2(\Omega))}^2 \leqslant \gamma_3$ and repeating the same procedure as above, we can obtain

$$\sup_{0\le t\le S}|C_\epsilon(t)|_{L^2}^2 + \int_0^S |\nabla C_\epsilon(s)|_{L^2}^2 ds + \epsilon\int_0^S |C_\epsilon(s)|_{L^q}^q ds \leqslant \gamma_3. \tag{3.20}$$

Multiplying the first equation of $(\mathrm{DCBF})_\epsilon$ by $\boldsymbol{u}_\epsilon$ and noting that $|[T_\epsilon]_\epsilon(x,t)| \leqslant |T_\epsilon(x,t)|$ and $|[C_\epsilon]_\epsilon(x,t)| \leqslant |C_\epsilon(x,t)|$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 + \nu|\nabla\boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 + a|\boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 &\leqslant |\boldsymbol{u}_\epsilon|_{\mathbb{L}^2}(|\boldsymbol{g}||T_\epsilon|_{L^2} + |\boldsymbol{h}||C_\epsilon|_{L^2} + |\boldsymbol{f}_1|_{L^2})\\ &\leqslant \frac{\nu}{2}|\nabla\boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 + \frac{\kappa}{2\nu}(|\boldsymbol{g}||T_\epsilon|_{L^2} + |\boldsymbol{h}||C_\epsilon|_{L^2} + |\boldsymbol{f}_1|_{L^2})^2.\end{aligned}$$

Recalling $\sup_{0\le t\le S}|T_\epsilon(t)|_{L^2}^2 + \sup_{0\le t\le S}|C_\epsilon(t)|_{L^2}^2 \leqslant \gamma_3$, we obtain

$$\frac{d}{dt}|\boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 + \nu|\nabla\boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 + 2a|\boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 \leqslant \gamma_3(1 + |\boldsymbol{f}_1|_{L^2}^2),$$

which yields

$$\sup_{0\le t\le S}|\boldsymbol{u}_\epsilon(t)|_{\mathbb{L}^2}^2 + \int_0^S |\nabla\boldsymbol{u}_\epsilon(s)|_{\mathbb{L}^2}^2 ds \leqslant \gamma_3. \tag{3.21}$$

Multiplying the first equation of $(\mathrm{DCBF})_\epsilon$ by $\partial_t\boldsymbol{u}_\epsilon$, we get

$$\frac{1}{2}|\partial_t\boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 + \frac{\nu}{2}\frac{d}{dt}|\nabla\boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 + \frac{a}{2}\frac{d}{dt}|\boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 \leqslant \gamma_3(1 + |\boldsymbol{f}_1|_{\mathbb{L}^2}^2). \tag{3.22}$$

In view of (3.18), (3.20) and (3.21), we obtain

$$\int_0^S \varphi_\epsilon(U_\epsilon(s))ds \leqslant \gamma_3.$$

Recalling the regularities in Proposition 2.3.5, i.e., $\varphi_\epsilon(U_\epsilon(\cdot)) \in C([0,S])$, we can assure the existence of $t_2^\epsilon \in [0,S]$ where $\varphi_\epsilon(U_\epsilon(\cdot))$ attains its minimum. From these facts, we can derive $\varphi_\epsilon(U_\epsilon(t_2^\epsilon)) \leqslant \gamma_3$, i.e.,

$$|\nabla\boldsymbol{u}_\epsilon(t_2^\epsilon)|_{\mathbb{L}^2}^2 + |\nabla T_\epsilon(t_2^\epsilon)|_{L^2}^2 + |\nabla C_\epsilon(t_2^\epsilon)|_{L^2}^2 + \epsilon|T_\epsilon(t_2^\epsilon)|_{L^q}^q + \epsilon|C_\epsilon(t_2^\epsilon)|_{L^q}^q \leqslant \gamma_3. \tag{3.23}$$

Using (3.23) and integrating (3.22) over $[t_2^\epsilon, t]$ $(t \in [t_2^\epsilon, t_2^\epsilon + S])$. we have

$$\sup_{0\le t\le S} |\nabla \boldsymbol{u}_\epsilon(t)|_{\mathbb{L}^2} + \int_0^S |\partial_t \boldsymbol{u}_\epsilon(s)|_{\mathbb{L}^2}^2 ds \leqslant \gamma_3. \tag{3.24}$$

From the first equation of $(\mathrm{DCBF})_\epsilon$, we can also obtain

$$\int_0^S |\mathcal{A}\boldsymbol{u}_\epsilon(s)|_{\mathbb{L}^2}^2 ds \leqslant \gamma_3. \tag{3.25}$$

Multiplying the second equation of $(\mathrm{DCBF})_\epsilon$ by $-\Delta T_\epsilon$ and using

$$\int_\Omega -\Delta T_\epsilon \epsilon |T_\epsilon|^{q-2} T_\epsilon dx = \epsilon(q-1)\int_\Omega |\nabla T_\epsilon|^2 |T_\epsilon|^{q-2} dx,$$
$$|\boldsymbol{u}_\epsilon \cdot \nabla T_\epsilon|_{L^2}^2 \leqslant |\boldsymbol{u}_\epsilon|_{\mathbb{L}^6} |\nabla T_\epsilon|_{L^3}^2 \leqslant \gamma_3 |\nabla \boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 |\nabla T_\epsilon|_{L^2} |\Delta T_\epsilon|_{L^2}$$

(see (3.16) and the estimate for the convection terms given in Check of $(\mathrm{A3})_\alpha$ in Section 3.2), we obtain

$$\frac{1}{2}\frac{d}{dt}|\nabla T_\epsilon|_{L^2}^2 + \frac{1}{2}|\Delta T_\epsilon|_{L^2}^2 \leqslant \gamma_3 |\nabla \boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^4 |\nabla T_\epsilon|_{L^2}^2 + 2|f_2|_{L^2}^2.$$

By applying Gronwall's inequality and integrating over $[t_2^\epsilon, t]$ with $t \in [t_2^\epsilon, t_2^\epsilon + S]$ (see (3.23)), we get

$$\sup_{0\le t\le S} |\nabla T_\epsilon(t)|_{L^2}^2 + \int_0^S |\Delta T_\epsilon(s)|_{L^2}^2 ds \leqslant \gamma_3. \tag{3.26}$$

Moreover, multiplying the second equation of $(\mathrm{DCBF})_\epsilon$ by $\partial_t T_\epsilon$, we have

$$\begin{aligned}\frac{1}{2}|\partial_t T_\epsilon|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}|\nabla T_\epsilon|_{L^2}^2 + \frac{\epsilon}{q}\frac{d}{dt}|T_\epsilon|_{L^q}^q &\leqslant |\boldsymbol{u}_\epsilon \cdot \nabla T_\epsilon|_{L^2}^2 + |f_2|_{L^2}^2 \\ &\leqslant \gamma_3 |\nabla \boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 |\nabla T_\epsilon|_{L^2} |\Delta T_\epsilon|_{L^2} + |f_2|_{L^2}^2.\end{aligned}$$

In view of (3.24), integration over $[t_2, t]$ with $t \in [t_2, t_2 + S]$ gives

$$\sup_{0\le t\le S} \epsilon |T_\epsilon(t)|_{L^q}^q + \int_0^S |\partial_t T_\epsilon(s)|_{L^2}^2 ds \leqslant \gamma_3. \tag{3.27}$$

Similarly, multiplying the third equation of $(\mathrm{DCBF})_\epsilon$ by $-\Delta C_\epsilon$ and $\partial_t C_\epsilon$, we obtain

$$\frac{1}{2}\frac{d}{dt}|\nabla C_\epsilon|_{L^2}^2 + \frac{1}{4}|\Delta C_\epsilon|_{L^2}^2 \leqslant \gamma_3 |\nabla \boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^4 |\nabla C_\epsilon|_{L^2}^2 + \rho^2 |\nabla T_\epsilon|_{L^2}^2 + |f_3|_{L^2}^2$$

and

$$\begin{aligned}&\frac{1}{4}|\partial_t C_\epsilon|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}|\nabla C_\epsilon|_{L^2}^2 + \frac{\epsilon}{q}\frac{d}{dt}|C_\epsilon|_{L^q}^q \\ &\qquad \leqslant \gamma_3 |\nabla \boldsymbol{u}_\epsilon|_{\mathbb{L}^2}^2 |\nabla C_\epsilon|_{L^2} |\Delta C_\epsilon|_{L^2} + \rho^2 |\Delta T_\epsilon|_{L^2}^2 + |f_3|_{L^2}^2,\end{aligned}$$

which yield

$$
\begin{aligned}
&\sup_{0\le t\le S}|\nabla C_\epsilon(t)|_{L^2}^2+\sup_{0\le t\le S}\epsilon|C_\epsilon(t)|_{L^q}^q\leqslant\gamma_3,\\
&\int_0^S|\Delta C_\epsilon(s)|_{L^2}^2ds+\int_0^S|\partial_tC_\epsilon(s)|_{L^2}^2ds\leqslant\gamma_3.
\end{aligned}
\tag{3.28}
$$

Furthermore, from the second and the third equation, we can derive

$$
\int_0^S|\epsilon|T_\epsilon|^{q-2}T_\epsilon(s)|_{L^2}^2ds+\int_0^S|\epsilon|C_\epsilon|^{q-2}C_\epsilon(s)|_{L^2}^2ds\leqslant\gamma_3.
\tag{3.29}
$$

Making use of a priori estimates (3.18), (3.20), (3.21), (3.24), (3.25), (3.26), (3.27), (3.28) and (3.29), we can discuss the convergence of solutions $\{U_\epsilon\}_{\epsilon>0}=\{(\boldsymbol{u}_\epsilon,T_\epsilon,C_\epsilon)^t\}_{\epsilon>0}$.

We obtain

$$
\sup_{0\leqslant t\leqslant S}(\|U_\epsilon(t)\|_H+\varphi_\epsilon(U_\epsilon(t)))\leqslant\gamma_3,
\tag{3.30}
$$

which implies that the sequence $\{U_\epsilon(t)\}_{\epsilon>0}$ is pre-compact in H for arbitrary $t\in[0,S]$, by virtue of Rellich-Kondrachov's compactness theorem. Moreover, by using the estimates for $\partial_t\boldsymbol{u}_\epsilon$, ∂_tT_ϵ and ∂_tC_ϵ, we get

$$
\begin{aligned}
&|\boldsymbol{u}_\epsilon(t)-\boldsymbol{u}_\epsilon(s)|_{\mathbb{L}^2}\leqslant\int_s^t|\partial_t\boldsymbol{u}_\epsilon(\tau)|_{\mathbb{L}^2}d\tau\leqslant\gamma_3|t-s|^{1/2},\\
&|T_\epsilon(t)-T_\epsilon(s)|_{L^2}\leqslant\gamma_3|t-s|^{1/2},\qquad|C_\epsilon(t)-C_\epsilon(s)|_{L^2}\leqslant\gamma_3|t-s|^{1/2}
\end{aligned}
$$

for any $t,s\in[0,S]$, which imply the equi-continuity of $\{U_\epsilon(t)\}_{\epsilon>0}$ in H. Hence, by virtue of Ascoli's theorem (Proposition 2.5.3), there exists a subsequence $\{U_{\epsilon_n}\}_{n\in\mathbb{N}}$, simply denoted by $\{U_n\}_{n\in\mathbb{N}}$, with $\epsilon_n\to0$ as $n\to\infty$ such that

$$
U_n\to U\quad\text{strongly in }C_\pi([0,S];H)\text{ as }n\to\infty.
\tag{3.31}
$$

From (3.29), there exist subsequences of $\{\epsilon_n|T_{\epsilon_n}|^{q-2}T_{\epsilon_n}\}_{n\in\mathbb{N}}$ and $\{\epsilon_n|C_{\epsilon_n}|^{q-2}C_{\epsilon_n}\}_{n\in\mathbb{N}}$ which weakly converge in $L^2(0,S;L^2(\Omega))$. Moreover, since

$$
|\epsilon|T_\epsilon|^{q-2}T_\epsilon(t)|_{L^{q'}}^{q'}=\epsilon^{p'}|T_\epsilon(t)|_{L^q}^q=\epsilon^{p'-1}(\epsilon|T_\epsilon(t)|_{L^q}^q)\leqslant\epsilon^{q'-1}\gamma_3\to0
$$

as $\epsilon\to0$ for any $t\in[0,S]$ ($q':=q/(q-1)$), we can see that the weak limits of $\{\epsilon_n|T_{\epsilon_n}|^{q-2}T_{\epsilon_n}\}_{n\in\mathbb{N}}$ and $\{\epsilon_n|C_{\epsilon_n}|^{q-2}C_{\epsilon_n}\}_{n\in\mathbb{N}}$ coincide with 0 (recall that $C_0^\infty((0,S);C_0^\infty(\Omega))$ is dense in $L^2(0,S;L^2(\Omega))$). We can also show that $[T_{\epsilon_n}]_{\epsilon_n}\to T$, $[C_{\epsilon_n}]_{\epsilon_n}\to C$ strongly in $L^2(0,S;L^2(\Omega))$, since $|[T_{\epsilon_n}]_{\epsilon_n}(x,t)|\leqslant|T_{\epsilon_n}(x,t)|$, $|[C_{\epsilon_n}]_{\epsilon_n}(x,t)|\leqslant|C_{\epsilon_n}(x,t)|$ and strong convergence (3.31) holds.

From the strong convergence (3.31) and uniform a priori bounds for $L^2(0,S;H)$-norm, we can assure that

$$
\frac{dU_n}{dt}\rightharpoonup\frac{dU}{dt}=(\partial_t\boldsymbol{u},\partial_tT,\partial_tC)^t,
\tag{3.32}
$$

$$
\partial\varphi_{\epsilon_n}(U_n)\rightharpoonup\partial\varphi'(U)=(\mathcal{A}\boldsymbol{u},-\Delta_DT,-\Delta_DC)^t
\tag{3.33}
$$

weakly in $L^2(0,S;H)$ as $n \to \infty$. Then, we obtain the limit satisfies $U \in W^{1,2}(0,S;H)$ and $\partial\varphi'(U) \in L^2(0,S;H)$. By virtue of Lemma 2.3.2, we can deduce $\varphi'(U(\cdot)) \in W^{1,1}(0,S)$, namely, the absolute continuity of $\varphi'(U(\cdot))$. Moreover, $\sup_{0\leqslant t\leqslant S}\varphi_\epsilon(U_\epsilon(t)) \leqslant \gamma_3$ and strong convergence (3.31) yield

$$\begin{aligned}
&\partial_{x_\mu}\boldsymbol{u}_n \rightharpoonup \partial_{x_\mu}\boldsymbol{u} && *\text{-weakly in } L^\infty(0,S;\mathbb{L}^2(\Omega)),\\
&\partial_{x_\mu}T_n \rightharpoonup \partial_{x_\mu}T && *\text{-weakly in } L^\infty(0,S;L^2(\Omega)),\\
&\partial_{x_\mu}C_n \rightharpoonup \partial_{x_\mu}C && *\text{-weakly in } L^\infty(0,S;L^2(\Omega))
\end{aligned}$$

for all $\mu = 1,2,\cdots,N$, which imply the weak continuity of $\partial_{x_\mu}\boldsymbol{u}(\cdot)$ in $\mathbb{L}^2(\Omega)$ and $\partial_{x_\mu}T(\cdot)$, $\partial_{x_\mu}C(\cdot)$ in $L^2(\Omega)$ on $[0,S]$. Since we have the norm-continuity and weak continuity of $\partial_{x_\mu}\boldsymbol{u}(\cdot)$, $\partial_{x_\mu}T(\cdot)$ and $\partial_{x_\mu}C(\cdot)$, we can deduce $\partial_{x_\mu}\boldsymbol{u} \in C([0,S];\mathbb{L}^2(\Omega))$ and $\partial_{x_\mu}T, \partial_{x_\mu}C \in C([0,S];L^2(\Omega))$ for each $\mu = 1,2,\cdots,N$. Furthermore, the periodicity of U in H immediately leads to

$$\boldsymbol{u} \in C_\pi([0,S];\mathbb{H}^1_\sigma(\Omega)), \quad T, C \in C_\pi([0,S];H^1_0(\Omega)).$$

Hence the limit $(\boldsymbol{u},T,C)^t$ satisfies all the regularities required in Theorem 3.1.2.

Finally, since the weak convergences of convection terms $\{\boldsymbol{u}_{\epsilon_n}\cdot\nabla T_{\epsilon_n}\}_{n\in\mathbb{N}}$ and $\{\boldsymbol{u}_{\epsilon_n}\cdot\nabla C_{\epsilon_n}\}_{n\in\mathbb{N}}$ can be assured by exactly the same argument as given above (see Check of (A2) in Section 3.2),

$$B_{\epsilon_n}(U_{\epsilon_n}) \rightharpoonup B(U) \quad \text{weakly in } L^2(0,S;L^2(\Omega)) \quad \text{as } n\to\infty \tag{3.34}$$

is valid.

Thus, it follows from (3.31), (3.33), (3.32) and (3.34) that solutions of approximate system (DCBF)$_\epsilon$ weakly converge to the solution of the original system (DCBF) $(\boldsymbol{u},T,C)^t$ in $L^2(0,S;H)$. □

3.3.4 Uniqueness

In this subsection, we show the uniqueness of the periodic solution (Theorem 3.1.4).

Let $U_i = (\boldsymbol{u}_i, T_i, C_i)^t$ $(i=1,2)$ be periodic solutions and let

$$\delta\boldsymbol{u} := \boldsymbol{u}_1 - \boldsymbol{u}_2, \quad \delta T := T_1 - T_2, \quad \delta C := C_1 - C_2.$$

Recall that δU satisfies the following (Section 3.2):

$$\text{(D)}\begin{cases}
\partial_t\delta\boldsymbol{u} + \nu\mathcal{A}\delta\boldsymbol{u} + a\delta\boldsymbol{u} = \mathcal{P}\boldsymbol{g}\delta T + \mathcal{P}\boldsymbol{h}\delta C,\\
\partial_t\delta T - \Delta\delta T = -\boldsymbol{u}_1\cdot\nabla\delta T + \delta\boldsymbol{u}\cdot\nabla T_2,\\
\partial_t\delta C - \Delta\delta C = \rho\Delta\delta T - \boldsymbol{u}_1\cdot\nabla\delta C + \delta\boldsymbol{u}\cdot\nabla C_2.
\end{cases}$$

Multiplying each equation of (D) by $\mathcal{A}\delta\boldsymbol{u}$, δT and δC respectively, we get

$$\begin{aligned}
&\frac{d}{dt}|\delta T|^2_{L^2} + |\nabla\delta T|^2_{L^2} \leqslant \gamma_4|\nabla\delta\boldsymbol{u}|^2_{\mathbb{L}^2}|\nabla T_2|^2_{L^2},\\
&\frac{d}{dt}|\delta C|^2_{L^2} + |\nabla\delta C|^2_{L^2} \leqslant 2\gamma_4|\nabla\delta\boldsymbol{u}|^2_{\mathbb{L}^2}|\nabla C_2|^2_{L^2} + 2\rho^2|\nabla\delta T|_{L^2},\\
&\frac{d}{dt}|\nabla\delta\boldsymbol{u}|^2_{\mathbb{L}^2} + \nu|\mathcal{A}\delta\boldsymbol{u}|^2_{\mathbb{L}^2} \leqslant \frac{2\kappa|\boldsymbol{g}|^2}{\nu}|\nabla\delta T|^2_{L^2} + \frac{2\kappa|\boldsymbol{h}|^2}{\nu}|\nabla\delta C|^2_{L^2}
\end{aligned}$$

(recall (3.13), (3.14) and (3.15) in Section 3.2 and use Poncaré's inequality $|V|_{L^2}^2 \leqslant \kappa|\nabla V|_{L^2}^2$). Here, γ_4 is the constant appearing in

$$\begin{aligned}|V\boldsymbol{w}|_{\mathbb{L}^2} &\leqslant |V|_{L^4}|\boldsymbol{w}|_{\mathbb{L}^4} \leqslant \gamma_4|V|_{H^1}|\boldsymbol{w}|_{\mathbb{H}^1}\\ &\leqslant \gamma_4|\nabla V|_{L^2}|\nabla \boldsymbol{w}|_{\mathbb{L}^2} \qquad \forall V \in H_0^1(\Omega),\ \forall \boldsymbol{w} \in \mathbb{H}_0^1(\Omega).\end{aligned}$$

Then, putting $\eta(t) := |\delta T|_{L^2}^2 + \frac{1}{4\rho^2}|\delta C|_{L^2}^2 + \frac{\nu\beta}{8\chi\kappa}|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}^2$, where $\chi := \max\{|\boldsymbol{g}|, |\boldsymbol{h}|\}$ and $\beta := \min\{1, 1/2\rho^2\}$, we can see that $\eta(t)$ satisfies

$$\frac{d}{dt}\eta(t) + \frac{\sigma}{\kappa}\eta(t) \leqslant \gamma_4'\left(|\nabla T_1|_{L^2}^2 + \frac{1}{2\rho^2}|\nabla C_1|_{L^2}^2\right)\eta(t), \tag{3.35}$$

where $\sigma := \min\{1/4, \nu\}$, $\gamma_4' := \max\{\gamma_4, \frac{8\chi\kappa}{\nu\beta}\gamma_4\}$. Therefore, if

$$\sup_{0\le t\le S}|\nabla T_1(t)|_{L^2}^2 + \sup_{0\le t\le S}\frac{1}{2\rho^2}|\nabla C_1(t)|_{L^2}^2 < \frac{\sigma}{\kappa\gamma_4'}, \tag{3.36}$$

then we have $\int_0^S y(t)dt \leqslant 0$, which implies the uniqueness.

In order to show (3.36), we establish a priori estimates. Multiplying each equation (DCBF) by $\mathcal{A}\boldsymbol{u}$, T, C and repeating almost the same calculations as those in the previous section (see (3.17), (3.19) and (3.22)), we have

$$\begin{aligned}&\frac{d}{dt}|T|_{L^2}^2 + |\nabla T|_{L^2}^2 \leqslant \kappa|f_2|_{L^2},\\ &\frac{d}{dt}|C|_{L^2}^2 + |\nabla C|_{L^2}^2 \leqslant 2\rho^2|\nabla T|_{L^2}^2 + 2\kappa|f_3|_{L^2}^2,\\ &\frac{d}{dt}|\nabla\boldsymbol{u}|_{\mathbb{L}^2}^2 + \nu|\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 \leqslant \frac{3\kappa|\boldsymbol{g}|^2}{\nu}|\nabla T|_{L^2}^2 + \frac{3\kappa|\boldsymbol{h}|^2}{\nu}|\nabla C|_{L^2}^2 + \frac{3}{\nu}|\boldsymbol{f}_1|_{L^2}^2.\end{aligned} \tag{3.37}$$

Integrating each inequality of (3.37) and using the periodicity of solution $T, C, \nabla\boldsymbol{u}$, we have

$$\begin{aligned}&\int_0^S |\nabla T(s)|_{L^2}^2 ds \leqslant \kappa|f_2|_{L^2(0,S;L^2(\Omega))}^2 =: Q_1,\\ &\int_0^S |\nabla C(s)|_{L^2}^2 ds \leqslant 2\rho^2\int_0^S |\nabla T(s)|_{L^2}^2 ds + 2\kappa|f_3|_{L^2(0,S;L^2(\Omega))}^2\\ &\qquad\qquad \leqslant 2\rho^2\kappa|f_2|_{L^2(0,S;L^2(\Omega))}^2 + 2\kappa|f_3|_{L^2(0,S;L^2(\Omega))}^2 =: Q_2\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}&\int_0^S |\mathcal{A}\boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds\\ \leqslant&\frac{3\kappa|\boldsymbol{g}|^2}{\nu^2}\int_0^S |\nabla T(s)|_{L^2}^2 ds + \frac{3\kappa|\boldsymbol{h}|^2}{\nu^2}\int_0^S |\nabla C(s)|_{L^2}^2 ds + \frac{3}{\nu^2}|\boldsymbol{f}_1|_{L^2(0,S;\mathbb{L}^2(\Omega))}^2\\ \leqslant&\frac{3\kappa^2}{\nu^2}\left(|\boldsymbol{g}|^2 + 2\rho^2|\boldsymbol{h}|^2\right)|f_2|_{L^2(0,S;L^2(\Omega))}^2\\ &\qquad + \frac{6\kappa^2|\boldsymbol{h}|^2}{\nu^2}|f_3|_{L^2(0,S;L^2(\Omega))}^2 + \frac{3}{\nu^2}|\boldsymbol{f}_1|_{L^2(0,S;\mathbb{L}^2(\Omega))}^2 =: Q_3.\end{aligned} \tag{3.39}$$

By the continuity of solution and Poincaré's inequality, (3.38) and (3.39) imply that there exist some $t_3, t_4, t_5 \in [0, S]$ such that

$$|\nabla T(t_3)|_{L^2}^2 \leqslant \frac{Q_1}{S}, \quad |\nabla C(t_4)|_{L^2}^2 \leqslant \frac{Q_2}{S}, \quad \frac{1}{\kappa}|\nabla \boldsymbol{u}(t_5)|_{\mathbb{L}^2}^2 \leqslant \frac{Q_3}{S}. \tag{3.40}$$

Then integrating the third inequality of (3.37) over $[t_5, t]$, we have

$$\sup_{0 \le t \le S} |\nabla \boldsymbol{u}(t)|_{\mathbb{L}^2}^2 \leqslant \frac{\kappa Q_3}{S} + Q_3 =: Q_4.$$

Multiplying the second and the third equations by $-\Delta T$ and $-\Delta C$ respectively, we get

$$\begin{aligned}
\frac{1}{4}|\Delta T|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}|\nabla T|_{L^2}^2 &\leqslant \gamma_5^2 |\nabla \boldsymbol{w}|_{\mathbb{L}^2}^4 |\nabla T|_{L^2}^2 + |f_2|_{L^2}^2 \\
&\leqslant \gamma_5^2 Q_4^2 |\nabla T|_{L^2}^2 + |f_2|_{L^2}^2, \\
\frac{1}{4}|\Delta C|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}|\nabla C|_{L^2}^2 &\leqslant \frac{9}{4}\gamma_5^2 |\nabla \boldsymbol{u}|_{\mathbb{L}^2}^4 |\nabla C|_{L^2}^2 + \frac{3\rho^2}{2}|\Delta T|_{L^2}^2 + \frac{3}{2}|f_3|_{L^2}^2 \\
&\leqslant \frac{9}{4}\gamma_5^2 Q_4^2 |\nabla C|_{L^2}^2 + \frac{3\rho^2}{2}|\Delta T|_{L^2}^2 + \frac{3}{2}|f_3|_{L^2}^2,
\end{aligned} \tag{3.41}$$

where γ_5 is a constant appearing in

$$|\boldsymbol{w} \cdot \nabla V|_{L^2}^2 \leqslant \gamma_5 |\nabla \boldsymbol{w}|_{\mathbb{L}^2}^2 |\nabla V|_{L^2} |\Delta V|_{L^2}$$

for any $\boldsymbol{w} \in \mathbb{H}_0^1(\Omega)$ and $V \in D(-\Delta_D)$ (see Check of $(\mathrm{A3})_\alpha$ in Section 3.2). Integrating (3.41) over $[0, S]$, we have

$$\begin{aligned}
&\int_0^S |\Delta T(s)|_{L^2}^2 ds \leqslant 4\gamma_5^2 Q_4^2 Q_1 + 4|f_2|_{L^2(0,S;L^2(\Omega))}^2 =: Q_5, \\
&\int_0^S |\Delta C(s)|_{L^2}^2 ds \leqslant 9\gamma_5^2 Q_4^2 Q_2 + 6\rho^2 Q_5 + 6|f_3|_{L^2(0,S;L^2(\Omega))}^2 =: Q_6.
\end{aligned}$$

Then applying Gronwall's inequality to (3.41) and recalling (3.40), we have

$$\begin{aligned}
|\nabla T(t)|_{L^2}^2 \leqslant{}& \exp(2\gamma_5^2 Q_4^2 (t - t_3)) |\nabla T(t_3)|_{L^2}^2 \\
&+ 2\int_{t_3}^t |f_2(s)|_{L^2}^2 \exp(2\gamma_5^2 Q_4^2 (t - s)) ds
\end{aligned}$$

for any $t \in [t_3, t_3 + S]$ and

$$\begin{aligned}
|\nabla C(t)|_{L^2}^2 \leqslant{}& \exp\left(\frac{9}{2}\gamma_5^2 Q_4^2 (t - t_4)\right) |\nabla C(t_4)|_{L^2}^2 \\
&+ 3\int_{t_4}^t \left(\rho^2 |\Delta T|_{L^2}^2 + |f_3|_{L^2}^2\right) \exp\left(\frac{9}{2}\gamma_5^2 Q_4^2 (t - s)\right) ds
\end{aligned}$$

for any $t \in [t_4, t_4 + S]$. Therefore, we can obtain

$$
(3.42) \qquad
\begin{aligned}
&\sup_{0\le t\le S} |\nabla T(t)|_{L^2}^2 \leqslant \left(\frac{Q_1}{S} + 2|f_2|^2_{L^2(0,S;L^2(\Omega))} \right) \exp(2\gamma_5^2 Q_4^2 S), \\
&\sup_{0\le t\le S} |\nabla C(t)|_{L^2}^2 \leqslant 3 \left(\frac{Q_2}{3S} + \rho^2 Q_5 + |f_3|^2_{L^2(0,S;L^2(\Omega))} \right) \exp\left(\frac{9}{2}\gamma_5^2 Q_4^2 S \right).
\end{aligned}
$$

Since Q_1, Q_2, Q_5 is monotone decreasing to 0 and Q_4 does not increase as $|f_2|_{L^2(0,S;L^2(\Omega))} \to 0$ and $|f_3|_{L^2(0,S;L^2(\Omega))} \to 0$, inequalities (3.42) imply that for each fixed $\nu, \rho, \boldsymbol{g}, \boldsymbol{h}, \rho$ and $\boldsymbol{f}_1$, there exist some sufficiently small f_2 and f_3 satisfying (3.36), whence follows the uniqueness of time periodic solution. □

3.4 Time Periodic Problem with Neumann Boundary Condition

In this section, we consider the solvability of time periodic problem of (DCBF) with the homogeneous Neumann boundary condition.

The replacement of the boundary condition does not make it difficult to construct approximate solutions. That is to say, we can guarantee the existence of a time periodic solution of the following equations by exactly the same argument as those in the previous section, Dirichlet boundary condition case.

$$
\begin{cases}
\partial_t \boldsymbol{u} + \nu \mathcal{A}\boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}\boldsymbol{g}[T]_\epsilon + \mathcal{P}\boldsymbol{h}[C]_\epsilon + \mathcal{P}\boldsymbol{f}_1, \\
\partial_t T - \Delta_N T + \epsilon |T|^{q-2} T + \boldsymbol{u} \cdot \nabla T = f_2, \\
\partial_t C - \Delta_N C + \epsilon |C|^{q-2} C + \boldsymbol{u} \cdot \nabla C = \rho \Delta_N T + f_3,
\end{cases}
$$

where the cut-off function $[\cdot]_\epsilon$ is defined by

$$
[T]_\epsilon(x,t) := \begin{cases}
T(x,t) & \text{if } |T(x,t)| \leqslant 1/\epsilon, \\
\operatorname{Sgn} T(x,t)/\epsilon = T(x,t)/\epsilon |T(x,t)| & \text{if } |T(x,t)| \geqslant 1/\epsilon.
\end{cases}
$$

and q is a sufficiently large exponent (see Check of (A5) in section 2.3.2). However, since Poincaré's inequality plays essential role in a priori estimates for Dirichlet boundary condition case, it is difficult to deduce the uniform boundedness of approximate solutions and discuss the convergence for Neumann boundary condition case.

In order to manage this difficulty, we introduce another approximation step.

3.4.1 Approximate Equations

We consider the following system with two approximation parameters $\epsilon, \lambda > 0$.

$$(\text{DCBF})_{\epsilon,\lambda} \begin{cases} \partial_t \boldsymbol{u} + \nu \mathcal{A}\boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}\boldsymbol{g}[T]_\epsilon + \mathcal{P}\boldsymbol{h}[C]_\epsilon + \mathcal{P}\boldsymbol{f}_1, \\ \partial_t T - \Delta_N T + \lambda T + \epsilon |T|^{q-2} T + \boldsymbol{u} \cdot \nabla T = f_2, \\ \partial_t C - \Delta_N C + \lambda C + \epsilon |C|^{q-2} C + \boldsymbol{u} \cdot \nabla C = \rho \Delta_N T + f_3. \end{cases}$$

By the same way as that in the previous section, we can reduce $(\text{DCBF})_{\epsilon,\lambda}$ to the following abstract problem $(\text{AP})_{\epsilon,\lambda}$:

$$(\text{AP})_{\epsilon,\lambda} \begin{cases} \dfrac{dU(t)}{dt} + \partial\varphi_{\epsilon,\lambda}(U(t)) + B_\epsilon(U(t)) = F(t) \quad t \in [0, S], \\ U(0) = U(S) \end{cases}$$

in the Hilbert space $H = \mathbb{L}^2_\sigma(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ with the inner product

$$(U_1, U_2)_H := (\boldsymbol{u}_1, \boldsymbol{u}_2)_{\mathbb{L}^2} + (T_1, T_2)_{L^2} + \frac{1}{9\rho^2}(C_1, C_2)_{L^2}.$$

Here,

$$U = \begin{pmatrix} \boldsymbol{u} \\ T \\ C \end{pmatrix}, \quad \frac{dU}{dt} = \begin{pmatrix} \partial_t \boldsymbol{u} \\ \partial_t T \\ \partial_t C \end{pmatrix}, \quad F = \begin{pmatrix} \mathcal{P}\boldsymbol{f}_1 \\ f_2 \\ f_3 \end{pmatrix}$$

and

$$\partial\varphi_{\epsilon,\lambda}(U) = \begin{pmatrix} \nu \mathcal{A}\boldsymbol{u} \\ -\Delta_N T + \epsilon |T|^{q-2} T + \lambda T \\ -\Delta_N C + \epsilon |C|^{q-2} C + \lambda C \end{pmatrix},$$

$$B_\epsilon(U) = \begin{pmatrix} a\boldsymbol{u} - \mathcal{P}\boldsymbol{g}[T]_\epsilon - \mathcal{P}\boldsymbol{h}[C]_\epsilon \\ \boldsymbol{u} \cdot \nabla T \\ \boldsymbol{u} \cdot \nabla C - \rho \Delta T \end{pmatrix},$$

where

$$\varphi(U) = \begin{cases} \frac{\nu}{2} |\nabla \boldsymbol{u}|^2_{\mathbb{L}^2} + \frac{1}{2} |\nabla T|^2_{L^2} + \frac{1}{18\rho^2} |\nabla C|^2_{L^2} & \text{if } U \in D(\varphi), \\ +\infty & \text{if } U \in H \backslash D(\varphi) \end{cases}$$

with domain $D(\varphi) := \mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times H^1(\Omega)$,

$$\psi_{\epsilon,\lambda}(U) = \begin{cases} \frac{\epsilon}{q} |T|^q_{L^q} + \frac{\epsilon}{9\rho^2 q} |C|^q_{L^q} + \frac{\lambda}{2} |T|^2_{L^2} + \frac{\lambda}{18\rho^2} |C|^2_{L^2} & \text{if } U \in D(\psi_{\epsilon,\lambda}), \\ +\infty & \text{if } U \in H \backslash D(\psi_{\epsilon,\lambda}) \end{cases}$$

with domain $D(\psi_{\epsilon,\lambda}) := \mathbb{L}^2_\sigma(\Omega) \times L^q(\Omega) \times L^q(\Omega)$,

$\varphi_{\epsilon,\lambda} = \varphi + \psi_{\epsilon,\lambda}$.

Since the presence of the relaxation terms λT and λC does not prevent us from repeating arguments in Section 3.3, we can assure the existence of a periodic solution $(\boldsymbol{u}_{\epsilon,\lambda}, T_{\epsilon,\lambda}, C_{\epsilon,\lambda})^t$ of $(\mathrm{DCBF})_{\epsilon,\lambda}$ by applying Proposition 2.3.5. Moreover, in a priori estimates, the coercive approximation terms λT and λC can shoulder the role of Poincaré's inequality. Therefore, due to the terms λT and λC, we can deduce the following estimates by almost the same procedure in the previous section.

$$\sup_{0\le t\le S} \|U_{\epsilon,\lambda}(t)\|_H + \sup_{0\le t\le S} \varphi_{\epsilon,\lambda}(U_{\epsilon,\lambda}(t)) \leqslant \gamma_6,$$
$$\left|\frac{dU_{\epsilon,\lambda}}{dt}\right|_{L^2(0,S;H)} + |\partial\varphi_{\epsilon,\lambda}(U_{\epsilon,\lambda})|_{L^2(0,S;H)} \leqslant \gamma_6,$$

where γ_6 is a constant independent of the parameter ϵ (which may depend on λ). These uniform boundedness guarantee the convergence of solutions of $(\mathrm{DCBF})_{\epsilon,\lambda}$ as the parameter ϵ tends to 0. Therefore, by repeating the same argument as those carried out in Dirichlet boundary condition case (Section 3.3), we can assure that the following system $(\mathrm{DCBF})_\lambda$ possesses a periodic solution $(\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)^t$.

$$(\mathrm{DCBF})_\lambda \begin{cases} \partial_t \boldsymbol{u}_\lambda + \nu\mathcal{A}\boldsymbol{u}_\lambda + a\boldsymbol{u}_\lambda = \mathcal{P}\boldsymbol{g}T_\lambda + \mathcal{P}\boldsymbol{h}C_\lambda + \mathcal{P}\boldsymbol{f}_1, \\ \partial_t T_\lambda - \Delta_N T_\lambda + \lambda T_\lambda + \boldsymbol{u}_\lambda\cdot\nabla T_\lambda = f_2, \\ \partial_t C_\lambda - \Delta_N C_\lambda + \lambda C_\lambda + \boldsymbol{u}_\lambda\cdot\nabla C_\lambda = \rho\Delta_N T_\lambda + f_3, \end{cases}$$

where $(\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)^t$ satisfies the following regularities:

$$\boldsymbol{u}_\lambda \in C_\pi([0,S];\mathbb{H}^1_\sigma(\Omega)) \cap L^2(0,S;\mathbb{H}^2(\Omega)) \cap W^{1,2}(0,S;\mathbb{L}^2_\sigma(\Omega)),$$
$$T_\lambda, C_\lambda \in C_\pi([0,S];H^1(\Omega)) \cap L^2(0,S;H^2(\Omega)) \cap W^{1,2}(0,S;L^2(\Omega)).$$

3.4.2 Convergence

To complete our proof, we establish some appropriate a priori estimates independent of λ and we discuss the convergence of $U_\lambda = (\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)^t$. In this subsection, γ_7 denotes a general constant independent of the approximation parameter λ.

Integrating the second and the third equations of $(\mathrm{DCBF})_\lambda$ over Ω, we get

$$\frac{d}{dt}\int_\Omega T_\lambda(x,t)dx + \lambda\int_\Omega T_\lambda(x,t)dx = \int_\Omega f_2(x,t)dx,$$
$$\frac{d}{dt}\int_\Omega C_\lambda(x,t)dx + \lambda\int_\Omega C_\lambda(x,t)dx = \int_\Omega f_3(x,t)dx. \tag{3.43}$$

Here we used the following facts:

$$\int_\Omega \Delta_N T_\lambda dx = \int_{\partial\Omega} \frac{\partial T_\lambda}{\partial n} dS = 0,$$
$$\int_\Omega \boldsymbol{u}_\lambda\cdot\nabla T_\lambda dx = \int_\Omega \nabla\cdot(\boldsymbol{u}_\lambda T_\lambda)dx - \int_\Omega T_\lambda\nabla\cdot\boldsymbol{u}_\lambda dx = \int_{\partial\Omega} \boldsymbol{u}_\lambda T_\lambda dS = 0.$$

Integrating (3.43) with respect to t over $[0, S]$ and using the periodic condition and (3.1), we find that

$$\lambda \int_0^S \int_\Omega T_\lambda(x,t)dxdt = 0, \quad \lambda \int_0^S \int_\Omega C_\lambda(x,t)dxdt = 0.$$

Then, by the continuity of T_λ, C_λ and the intermediate value theorem, there exist some $t_6^\lambda, t_7^\lambda \in [0, S]$ such that

$$\int_\Omega T_\lambda(x, t_6^\lambda)dx = 0, \quad \int_\Omega C_\lambda(x, t_7^\lambda)dx = 0.$$

Applying Gronwall's inequality to (3.43), we have

$$\int_\Omega T_\lambda(x,t)dx = \int_{t_6^\lambda}^t e^{-\lambda(t-s)} \int_\Omega f_2(x,s)dxds \quad \forall t \in [t_6^\lambda, t_6^\lambda + S]. \tag{3.44}$$

From Poincaré-Wirtinger's inequality (Proposition 2.5.5):

$$\left| V - \frac{1}{|\Omega|}\int_\Omega V dx \right|_{L^2} \leqslant \kappa_W |\nabla V|_{L^2} \qquad \forall V \in H^1(\Omega),$$

we obtain

$$\begin{aligned} |T_\lambda(t)|_{L^2} &\leqslant \kappa_W |\nabla T_\lambda(t)|_{L^2} + \left| \frac{1}{|\Omega|} \int_\Omega T_\lambda(t)dx \right|_{L^2} \\ &= \kappa_W |\nabla T_\lambda(t)|_{L^2} + \frac{1}{|\Omega|^{1/2}} \left| \int_\Omega T_\lambda(t)dx \right| \\ &= \kappa_W |\nabla T_\lambda(t)|_{L^2} + \frac{1}{|\Omega|^{1/2}} \left| \int_{t_6^\lambda}^t e^{-\lambda(t-s)} \int_\Omega f_2(s)dxds \right| \\ &\leqslant \kappa_W |\nabla T_\lambda(t)|_{L^2} + S^{1/2}|f_2|_{L^2(0,S;L^2(\Omega))} \end{aligned} \tag{3.45}$$

for any $t \in [t_6^\lambda, t_6^\lambda + S]$. Similarly, we get

$$|C_\lambda(t)|_{L^2} \leqslant \kappa_W |\nabla C_\lambda(t)|_{L^2} + S^{1/2}|f_3|_{L^2(0,S;L^2(\Omega))} \tag{3.46}$$

for any $t \in [t_7^\lambda, t_7^\lambda + S]$.

Replacing Poincaré's inequality by the inequalities (3.45) and (3.46) in our argument of Section 3.3, we can derive uniform a priori bounds. Multiplying the second equation of $(\mathrm{DCBF})_\lambda$ by T_λ, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|T_\lambda|_{L^2}^2 + |\nabla T_\lambda|_{L^2}^2 + \lambda |T_\lambda|_{L^2}^2 \\ =& \int_\Omega f_2 T_\lambda dx \leqslant |f_2|_{L^2}|T_\lambda|_{L^2} \\ \leqslant& \kappa_W |\nabla T_\lambda|_{L^2}|f_2|_{L^2} + S^{1/2}|f_2|_{L^2(0,S;L^2(\Omega))}|f_2|_{L^2}, \end{aligned}$$

i.e.,

$$\frac{d}{dt}|T_\lambda|_{L^2}^2 + |\nabla T_\lambda|_{L^2}^2 + \lambda|T_\lambda|_{L^2}^2 \leqslant \kappa_W^2|f_2|_{L^2}^2 + 2S^{1/2}|f_2|_{L^2(0,S;L^2(\Omega))}|f_2|_{L^2}. \tag{3.47}$$

Integrating (3.47) over $[0, S]$, we have

$$\int_0^S |\nabla T_\lambda(s)|_{L^2}^2 ds + \lambda \int_0^S |T_\lambda(s)|_{L^2}^2 ds \leqslant (\kappa_W^2 + 2S)|f_2|_{L^2(0,S;L^2(\Omega))}^2, \tag{3.48}$$

which immediately yields, together with (3.45),

$$\begin{aligned}\int_0^S |T_\lambda(s)|_{L^2}^2 ds &\leqslant \int_0^S 2\kappa_W^2|\nabla T_\lambda(s)|_{L^2}^2 + 2S|f_2(s)|_{L^2}^2 ds \\ &\leqslant 2\left(\kappa_W^2(\kappa_W^2 + 2S) + S\right)|f_2|_{L^2(0,S;L^2(\Omega))}^2.\end{aligned} \tag{3.49}$$

Combining the continuity of T_λ with (3.49), we can assure that there exist a $t_8^\lambda \in [0, S]$ such that

$$|T_\lambda(t_8^\lambda)|_{L^2}^2 \leqslant \frac{2}{S}\left(\kappa_W^2(\kappa_W^2 + 2S) + S\right)|f_2|_{L^2(0,S;L^2(\Omega))}^2.$$

Then integrating (3.47) over $[t_8^\lambda, t]$ with $t \in [t_8^\lambda, t_8^\lambda + S]$, we obtain

$$\sup_{0\leq t\leq S} |T_\lambda(t)|_{L^2}^2 \leqslant \left(\frac{2\kappa_W^4}{S} + 5\kappa_W^2 + 2S + 2\right)|f_2|_{L^2(0,S;L^2(\Omega))}^2. \tag{3.50}$$

Similarly, multiplication of the third equation of $(\mathrm{DCBF})_\lambda$ by C_λ gives

$$\begin{aligned}&\frac{d}{dt}|C_\lambda|_{L^2}^2 + |\nabla C_\lambda|_{L^2}^2 + 2\lambda|C|_{L^2}^2 \\ &\qquad \leqslant 2\rho^2|\nabla T_\lambda|_{L^2}^2 + 2\kappa_W^2|f_3|_{L^2} + 2S^{1/2}|f_2|_{L^2(0,S;L^2(\Omega))}|f_3|_{L^2},\end{aligned}$$

which implies

$$\sup_{0\leq t\leq S} |C_\lambda(t)|_{L^2} + \int_0^S |\nabla C_\lambda(s)|_{L^2}^2 ds + \lambda \int_0^S |C_\lambda(s)|_{L^2}^2 ds \leqslant \gamma_7. \tag{3.51}$$

Moreover, multiplying the first equation of $(\mathrm{DCBF})_\lambda$ by $\mathcal{A}\boldsymbol{u}_\lambda$ and $\partial_t \boldsymbol{u}_\lambda$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla \boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2 + \frac{\nu}{2}|\mathcal{A}\boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2 + a|\nabla \boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2 &\leqslant \frac{1}{2\nu}\left(|\boldsymbol{g}||T_\lambda|_{L^2} + |\boldsymbol{h}||C_\lambda|_{L^2} + |\boldsymbol{f}_1|_{\mathbb{L}^2}\right)^2 \\ &\leqslant \frac{1}{2\nu}\left(\gamma_7 + |\boldsymbol{f}_1|_{\mathbb{L}^2}\right)^2, \\ \frac{1}{2}|\partial_t \boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2 + \frac{\nu}{2}\frac{d}{dt}|\nabla \boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2 + \frac{a}{2}\frac{d}{dt}|\boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2 &\leqslant \frac{1}{2}\left(|\boldsymbol{g}||T_\lambda|_{L^2} + |\boldsymbol{h}||C_\lambda|_{L^2} + |\boldsymbol{f}_1|_{\mathbb{L}^2}\right)^2 \\ &\leqslant \frac{1}{2}\left(\gamma_7 + |\boldsymbol{f}_1|_{\mathbb{L}^2}\right)^2.\end{aligned}$$

From these inequality, we obtain

$$\sup_{0\le t\le S}|\boldsymbol{u}_\lambda(t)|_{\mathbb{H}^1}+\int_0^S|\mathcal{A}\boldsymbol{u}_\lambda(s)|_{\mathbb{L}^2}^2ds+\int_0^S|\partial_t\boldsymbol{u}_\lambda(s)|_{\mathbb{L}^2}^2ds\leqslant\gamma_7. \tag{3.52}$$

Recalling our calculations in Section 3.2 and 3.3, (see (3.10) or procedures for (3.27), (3.28)), we can derive the followings from the multiplication of the second and the third equation of (DCBF)$_\lambda$ by $-\Delta T_\lambda$, $\partial_t T_\lambda$ and $-\Delta C_\lambda$, $\partial_t C_\lambda$.

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla T_\lambda|_{L^2}^2+\frac{1}{4}|\Delta T_\lambda|_{L^2}^2+\lambda|\nabla T_\lambda|_{L^2}^2\leqslant\gamma_7|\nabla\boldsymbol{u}_\lambda|_{\mathbb{L}^2}^4|\nabla T_\lambda|_{L^2}^2+|f_2|_{L^2}^2,\\
&\frac{1}{2}|\partial_t T_\lambda|_{L^2}^2+\frac{1}{2}\frac{d}{dt}|\nabla T_\lambda|_{L^2}^2+\lambda\frac{d}{dt}|T_\lambda|_{L^2}^2\leqslant\gamma_7|\nabla\boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2|\nabla T_\lambda|_{L^2}|\Delta T_\lambda|_{L^2}+|f_2|_{L^2}^2,\\
&\frac{1}{2}\frac{d}{dt}|\nabla C_\lambda|_{L^2}^2+\frac{1}{4}|\Delta C_\lambda|_{L^2}^2+\lambda|\nabla C_\lambda|_{L^2}^2\leqslant\gamma_7|\nabla\boldsymbol{u}_\lambda|_{\mathbb{L}^2}^4|\nabla C_\lambda|_{L^2}^2+3\rho^2|\Delta T_\lambda|_{L^2}^2+3|f_3|_{L^2}^2,\\
&\frac{1}{4}|\partial_t C_\lambda|_{L^2}^2+\frac{1}{2}\frac{d}{dt}|\nabla C_\lambda|_{L^2}^2+\frac{\lambda}{2}\frac{d}{dt}|C_\lambda|_{L^2}^2\leqslant\gamma_7|\nabla\boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2|\nabla C_\lambda|_{L^2}|\Delta C_\lambda|_{L^2}+\rho^2|\Delta T_\lambda|_{L^2}^2+|f_3|_{L^2}^2,
\end{aligned}$$

which yield

$$\begin{aligned}
&\sup_{0\le t\le S}|\nabla T_\lambda(t)|_{L^2}+\sup_{0\le t\le S}|\nabla C_\lambda(t)|_{L^2}\leqslant\gamma_7,\\
&\int_0^S|\Delta T_\lambda(s)|_{L^2}^2ds+\int_0^S|\Delta C_\lambda(s)|_{L^2}^2ds\leqslant\gamma_7,\\
&\int_0^S|\partial_t T_\lambda(s)|_{L^2}^2ds+\int_0^S|\partial_t C_\lambda(s)|_{L^2}^2ds\leqslant\gamma_7.
\end{aligned}\tag{3.53}$$

From the uniform bounds (3.48), (3.50), (3.51), (3.52) and (3.53), we can accomplish the convergence argument and we can assure the existence of the original system (DCBF). Indeed, by (3.48) and (3.51), we can show that $\lambda T_\lambda\to 0$, $\lambda C_\lambda\to 0$ strongly in $L^2(0,S;L^2(\Omega))$ as $\lambda\to 0$, since

$$|\lambda T_\lambda|_{L^2(0,S;L^2(\Omega))}^2=\lambda\int_0^S\lambda|T_\lambda(s)|_{L^2}^2ds\leqslant\lambda\gamma_7.$$

Moreover, (3.48), (3.50), (3.51), (3.52) and (3.53) imply that $U_\lambda=(\boldsymbol{u}_\lambda,T_\lambda,C_\lambda)^t$ satisfies

$$\begin{aligned}
&\sup_{0\le t\le S}\|U_\lambda(t)\|_H+\sup_{0\le t\le S}\varphi(U_\lambda(t))\leqslant\gamma_7,\\
&\left|\frac{dU_\lambda}{dt}\right|_{L^2(0,S;H)}+|\partial\varphi(U_\lambda)|_{L^2(0,S;H)}\leqslant\gamma_7.
\end{aligned}$$

Thus, we can employ the same convergence argument as that in Section 3.3.3 and we can assure the existence of a time periodic solution for Neumann boundary condition case. □

Chapter 4

Initial Boundary Value Problem in Unbounded Domains

4.1 Problems and Main Theorems

In this chapter, we deal with the initial boundary value problem of (DCBF) in general domains.

$$
\text{(DCBF)}\begin{cases}
\partial_t \boldsymbol{u} + \nu\mathcal{A}\boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}\boldsymbol{g}T + \mathcal{P}\boldsymbol{h}C + \mathcal{P}\boldsymbol{f}_1 & (x,t) \in \Omega\times[0,S],\\
\partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + f_2 & (x,t) \in \Omega\times[0,S],\\
\partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta C + \rho\Delta T + f_3 & (x,t) \in \Omega\times[0,S],\\
\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0,\ T(\cdot,0) = T_0,\ C(\cdot,0) = C_0.
\end{cases}
$$

We mainly deal with the Neumann boundary condition case in this chapter (in the end of this chapter, we shall show that the same solvability result holds for Dirichlet condition case as that for Neumann case).

In addition to the notations fixed in Chapter 2, we use the followings in order to state our result.

$$W_S := C([0,S];\mathbb{H}^1_\sigma(\Omega)) \cap L^2(0,S;\mathbb{H}^2(\Omega)),$$

$$X_S := \{f \in L^1(0,S;L^2(\Omega));\sqrt{t}f \in L^2(0,S;L^2(\Omega))\},$$

$$Y_S := \left\{U \in C([0,S];L^2(\Omega)) \cap L^2(0,S;H^1(\Omega));\sqrt{t}\Delta U, \sqrt{t}\partial_t U \in L^2(0,S;L^2(\Omega))\right\},$$

$$Z_S := \left\{\begin{pmatrix}\boldsymbol{u}\\T\\C\end{pmatrix} \in C([0,S];\mathbb{L}^2_\sigma(\Omega)\times L^2(\Omega)\times L^2(\Omega));\ \begin{matrix}\boldsymbol{u} \in W_S,\ T, C \in Y_S,\\ \partial_t\boldsymbol{u} \in L^2(0,S;\mathbb{L}^2_\sigma(\Omega))\end{matrix}\right\}.$$

The norm of W_S, X_S and Y_S are defined by

$$
\begin{aligned}
\|\boldsymbol{u}\|_{W_S} &:= \sup_{0\leqslant t\leqslant S} |\boldsymbol{u}(t)|_{\mathbb{H}^1_\sigma(\Omega)} + \left(\int_0^S |\mathcal{A}\boldsymbol{u}(t)|^2_{\mathbb{L}^2(\Omega)} dt\right)^{1/2}, \\
\|f\|_{X_S} &:= |f|_{L^1(0,S;L^2(\Omega))} + \left(\int_0^S t|f(t)|^2_{L^2(\Omega)} dt\right)^{1/2}, \\
\|U\|_{Y_S} &:= \sup_{0\leqslant t\leqslant S} |U(t)|_{L^2(\Omega)} + |\nabla U|_{L^2(0,S;L^2(\Omega))} \\
&\qquad + \left(\int_0^S t|\Delta U(t)|^2_{L^2(\Omega)} dt\right)^{1/2} + \left(\int_0^S t|\partial_t U(t)|^2_{L^2(\Omega)} dt\right)^{1/2}.
\end{aligned}
$$

The main purpose of this chapter is to show the following solvability result for (DCBF).

Theorem 4.1.1. *Let $N \leqslant 4$ and let the space domain Ω be either the whole space $\mathbb{R}^N$ or uniform C^2-domain. Moreover, assume that the initial data satisfy $\boldsymbol{u}_0 \in \mathbb{H}^1_\sigma(\Omega)$, $T_0, C_0 \in L^2(\Omega)$ and the external forces satisfy $\boldsymbol{f}_1 \in L^2(0,S;\mathbb{L}^2(\Omega))$, $f_2, f_3 \in X_S$. Then the initial boundary value problem of* (DCBF) *with the homogeneous Neumann boundary condition admits a unique solution $(\boldsymbol{u}, T, C)^t \in Z_S$.*

Our proof in this chapter consists of the following four steps.

Step 1: Fix $\underline{\boldsymbol{u}} \in W_S$. Then we find a unique solution $(\underline{T}, \underline{C})$ of the following problem in $Y_S \times Y_S$:

$$
\begin{cases}
\partial_t \underline{T} - \Delta \underline{T} + \underline{\boldsymbol{u}} \cdot \nabla \underline{T} = f_2, \\
\partial_t \underline{C} - \Delta \underline{C} + \underline{\boldsymbol{u}} \cdot \nabla \underline{C} = \rho \Delta \underline{T} + f_3, \\
\frac{\partial \underline{T}}{\partial n}|_{\partial\Omega} = 0, \quad \frac{\partial \underline{C}}{\partial n}|_{\partial\Omega} = 0, \\
\underline{T}(\cdot, 0) = T_0, \quad \underline{C}(\cdot, 0) = C_0.
\end{cases} \tag{4.1}
$$

We define the mapping $\Phi_{T_0,C_0} : W_S \to Y_S \times Y_S$ by the relationship $\Phi_{T_0,C_0}(\underline{\boldsymbol{u}}) := (\underline{T}, \underline{C})$ based on the solvability of (4.1).

Step 2: Replacing T, C in the first equation of (DCBF) by the unique solution $\underline{T}$, $\underline{C}$ derived in Step 1, we consider the following problem:

$$
\begin{cases}
\partial_t \overline{\boldsymbol{u}} + \nu \mathcal{A} \overline{\boldsymbol{u}} + a \overline{\boldsymbol{u}} = \mathcal{P}\boldsymbol{g}\underline{T} + \mathcal{P}\boldsymbol{h}\underline{C} + \mathcal{P}\boldsymbol{f}_1, \\
\overline{\boldsymbol{u}}|_{\partial\Omega} = 0, \qquad \overline{\boldsymbol{u}}(\cdot, 0) = \boldsymbol{u}_0,
\end{cases} \tag{4.2}
$$

and we show that (4.2) possesses a unique global solution $\overline{\boldsymbol{u}}$ in W_S. Then we define $\Psi_{\boldsymbol{u}_0} : Y_S \times Y_S \to W_S$ by the correspondence $\Psi_{\boldsymbol{u}_0}((\underline{T}, \underline{C})) := \overline{\boldsymbol{u}}$.

Step 3: We check that the mapping $\Psi_{\boldsymbol{u}_0} \circ \Phi_{T_0,C_0}$ becomes a contraction mapping in W_{S_0} for a sufficiently small $S_0 \in (0, S]$. Then we can show the existence of time-local solution for (DCBF) belonging to Z_{S_0}.

Step 4: Establishing some a priori estimates, we assure that local solutions can be extended up to the prescribed interval $[0, S]$.

4.2 Construction of Solutions for Steps 1 and 2

In this section, we check Steps 1 and 2. That is to say, we assure the solvability of (4.1) and (4.2).

4.2.1 Well-Definedness of Φ_{T_0,C_0}

We first consider (4.1) in this subsection.

Lemma 4.2.1. *Let $N \leqslant 4$ and let the space domain Ω be either the whole space $\mathbb{R}^N$ or uniform C^2-domain. Then for any $T_0 \in L^2(\Omega)$, $\boldsymbol{u} \in W_S$ and $f_2 \in X_S$, the following initial boundary value problem* (4.3) *possesses a unique global solution T belonging to Y_S.*

$$\begin{cases} \partial_t T - \Delta T + \boldsymbol{u} \cdot \nabla T = f_2 & \text{in} \ \ \Omega \times [0, S], \\ \frac{\partial T}{\partial n}|_{\partial\Omega} = 0, \quad T(\cdot, 0) = T_0. \end{cases} \tag{4.3}$$

Proof. To begin with, we consider the case where $T_0 \in H^1(\Omega)$ and $f_2 \in L^2(0, S; L^2(\Omega))$. According to Proposition 2.3.1 (solvability of evolution equation governed by subdifferential operators), Proposition 2.1.12 and Example 2 in Section 2.3.1 (maximal monotonicity of $-\Delta_N$), there exists a unique global solution T of the following problem (4.4) with $T_0 \in H^1(\Omega)$ and $f_2 \in L^2(0, S; L^2(\Omega))$:

$$\begin{cases} \partial_t T - \Delta_N T = f_2 & \text{in} \ \ \Omega \times [0, S], \\ T(\cdot, 0) = T_0, \end{cases} \tag{4.4}$$

where T satisfies

$$T \in C([0, S]; H^1(\Omega)) \cap L^2(0, S; H^2(\Omega)) \cap W^{1,2}(0, S; L^2(\Omega)).$$

Here we define a Banach space Y'_S by

$$Y'_S := C([0, S]; H^1(\Omega)) \cap L^2(0, S; H^2(\Omega))$$

with the norm

$$\|U\|^2_{Y'_S} := \sup_{0 \le t \le S} |U(t)|^2_{H^1} + \int_0^S |\Delta U(s)|^2_{L^2} ds.$$

Let $\boldsymbol{w} \in C([0, S]; D(\mathcal{A}))$ and $U \in Y'_S$. Then $\boldsymbol{w} \cdot \nabla U$ belongs to $L^2(0, S; L^2(\Omega))$, since

$$\begin{aligned} |\boldsymbol{w} \cdot \nabla U|^2_{L^2} &\leqslant |\boldsymbol{w}|^2_{\mathbb{L}^8} |\nabla U|^2_{L^{8/3}} \leqslant \gamma |\boldsymbol{w}|^2_{\mathbb{W}^{1,8/3}} |\nabla U|^2_{L^{8/3}} \\ &\leqslant \gamma |\boldsymbol{w}|_{\mathbb{H}^1} |\boldsymbol{w}|_{\mathbb{W}^{1,4}} |\nabla U|_{L^2} |\nabla U|_{L^4} \\ &\leqslant \gamma |\boldsymbol{w}|_{\mathbb{H}^1} |\boldsymbol{w}|_{\mathbb{H}^2} |\nabla U|_{L^2} |U|_{H^2} \\ &\leqslant \gamma |\boldsymbol{w}|_{\mathbb{H}^1} |\boldsymbol{w}|_{\mathbb{H}^2} |\nabla U|_{L^2} \left(|\Delta U|_{L^2} + |U|_{L^2}\right) \end{aligned} \tag{4.5}$$

holds with $N \leqslant 4$, where γ is some suitable general constant and we use Hölder's inequality, Sobolev's inequality and the elliptic estimate (recall properties in Section

2.1) in (4.5). Therefore, for any $U \in Y'_S$ and $\boldsymbol{w} \in C([0,S];D(\mathcal{A}))$, the following problem (4.6) also admits a unique global solution:

$$\begin{cases} \partial_t T - \Delta_N T + \boldsymbol{w}\cdot\nabla U = f_2 \quad \text{in} \;\; \Omega\times[0,S], \\ T(\cdot,0) = T_0, \end{cases} \tag{4.6}$$

where T belongs to $C([0,S];H^1(\Omega))\cap L^2(0,S;H^2(\Omega))\cap W^{1,2}(0,S;L^2(\Omega))$. Based on this fact, we define a mapping $\Sigma^{\boldsymbol{w}}_{S,T_0} : Y'_S \to Y'_S$ by the relationship $\Sigma^{\boldsymbol{w}}_{S,T_0}(U) := T$.

Let $\Sigma^{\boldsymbol{w}}_{S,T_0}(U_i) := T_i$ $(i = 1,2)$ and $\delta U := U_1 - U_2$, $\delta T := T_1 - T_2$. Obviously, δU and δT satisfy the following problem:

$$\begin{cases} \partial_t \delta T - \Delta_N \delta T + \boldsymbol{w}\cdot\nabla \delta U = 0 \quad \text{in} \;\; \Omega\times[0,S], \\ \delta T(\cdot,0) = 0. \end{cases} \tag{4.7}$$

Multiplying (4.7) by δT, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\delta T|^2_{L^2} + |\nabla\delta T|^2_{L^2} &= -\int \delta T \boldsymbol{w} \nabla \delta U dx = -\int \delta U \boldsymbol{w}\nabla\delta T dx \\ &\leqslant |\boldsymbol{w}\delta U|_{\mathbb{L}^2}|\nabla\delta T|_{L^2} \leqslant \frac{1}{2}|\nabla\delta T|^2_{L^2} + \frac{1}{2}|\boldsymbol{w}|^2_{\mathbb{L}^4}|\delta U|^2_{L^4} \\ &\leqslant \frac{1}{2}|\nabla\delta T|^2_{L^2} + \gamma|\boldsymbol{w}|^2_{\mathbb{H}^1}|\delta U|^2_{H^1}, \end{aligned}$$

i.e.,

$$\frac{d}{dt}|\delta T|^2_{L^2} + |\nabla\delta T|^2_{L^2} \leqslant \gamma|\delta U|^2_{H^1} \tag{4.8}$$

(recall $\boldsymbol{w} \in C([0,S];D(\mathcal{A})) \subset L^\infty(0,S;\mathbb{H}^2(\Omega))$). Hence, integrating (4.8) over $[0,t]$ for each $t \in [0,S]$, we obtain

$$\sup_{0\leqslant t\leqslant S}|\delta T(t)|^2_{L^2} \leqslant \gamma S \sup_{0\leqslant t\leqslant S}|\delta U(t)|^2_{H^1}. \tag{4.9}$$

Next, multiplying (4.7) by $-\Delta\delta T$ and using (4.5), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\nabla\delta T|^2_{L^2} + |\Delta\delta T|^2_{L^2} &\leqslant |\boldsymbol{w}\cdot\nabla\delta U|_{L^2}|\Delta\delta T|_{L^2} \\ &\leqslant \frac{1}{2}|\Delta\delta T|^2_{L^2} + \gamma|\boldsymbol{w}|^2_{\mathbb{H}^2}|\nabla\delta U|_{L^2}\left(|\Delta\delta U|_{L^2} + |\delta U|_{L^2}\right), \end{aligned}$$

i.e.,

$$\frac{d}{dt}|\nabla\delta T|^2_{L^2} + |\Delta\delta T|^2_{L^2} \leqslant \gamma|\nabla\delta U|_{L^2}\left(|\Delta\delta U|_{L^2} + |\delta U|_{L^2}\right),$$

which yields

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant S}|\nabla\delta T(t)|_{L^2}^2+|\Delta\delta T|^2_{L^2(0,S;L^2(\Omega))}\\
&\leqslant \gamma S^{1/2}\sup_{0\leqslant t\leqslant S}|\nabla\delta U(t)|_{L^2}\left(|\Delta\delta U|_{L^2(0,S;L^2(\Omega))}+S^{1/2}\sup_{0\leqslant t\leqslant S}|\delta U(t)|_{L^2}\right)\\
&\leqslant \gamma(S^{1/2}+S)\left(|\Delta\delta U|^2_{L^2(0,S;L^2(\Omega))}+\sup_{0\leqslant t\leqslant S}|\delta U(t)|^2_{H^1}\right).
\end{aligned}\tag{4.10}$$

From (4.9) and (4.10), we can derive

$$\|\delta T\|^2_{Y'_S}\leqslant \gamma(S^{1/2}+S)\|\delta U\|^2_{Y'_S},$$

which implies that $\Sigma^{\boldsymbol{w}}_{S_0,T_0}$ becomes a contraction mapping in Y'_{S_0} for sufficiently small $S_0\in(0,S]$, namely, we can assure that the following problem has a unique local solution $T\in Y'_{S_0}$ for any $\boldsymbol{w}\in C([0,S];D(\mathcal{A}))$.

$$\begin{cases}\partial_t T-\Delta_N T+\boldsymbol{w}\cdot\nabla T=f_2 \quad \text{in}\ \ \Omega\times[0,S_0],\\ T(\cdot,0)=T_0.\end{cases}\tag{4.11}$$

Furthermore, multiplications of (4.11) by T and $-\Delta T$ yield

$$\frac{1}{2}\frac{d}{dt}|T|^2_{L^2}+|\nabla T|^2_{L^2}\leqslant |f_2|_{L^2}|T|_{L^2}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\nabla T|^2_{L^2}+|\Delta T|^2_{L^2}&\leqslant |\boldsymbol{w}\cdot\nabla T|_{L^2}|\Delta T|_{L^2}+|f_2|_{L^2}|\Delta T|_{L^2}\\
&\leqslant \frac{1}{2}|\Delta T|^2_{L^2}+\gamma|\nabla T|_{L^2}(|\Delta T|_{L^2}+|T|_{L^2})+|f_2|^2_{L^2}\\
&\leqslant \frac{3}{4}|\Delta T|^2_{L^2}+\gamma|\nabla T|^2_{L^2}+|T|^2_{L^2}+|f_2|^2_{L^2}.
\end{aligned}$$

Integrating them over $[0,t]$ ($t\in[0,S_0]$) and $[0,S_0]$, we obtain

$$\sup_{0\leqslant t\leqslant S_0}|T(t)|^2_{H^1}+\int_0^{S_0}|\Delta T(s)|^2_{L^2}ds\leqslant(1+S)Q(\gamma,|T_0|_{H^1},|f_2|_{L^2(0,S;L^2(\Omega))})$$

($Q(z_1,z_2,\cdots)$ denotes a monotone increasing function of $z_1,z_2,\cdots$) for any $S_0\in[0,S]$, which implies that the local solution of (4.11) can be extended globally up to S.

Let $\boldsymbol{u}\in W_S$. Then, recalling Proposition 2.5.6 in Chapter 2, we can assure the existence of a sequence $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ satisfying $\boldsymbol{u}_n\in C([0,S];D(\mathcal{A}))$ and $\boldsymbol{u}_n\to\boldsymbol{u}$ strongly in $C([0,S];\mathbb{H}^1_\sigma(\Omega))\cap L^2(0,S;\mathbb{H}^2(\Omega))$. Therefore, for each $n\in\mathbb{N}$ and $f_2\in L^2(0,S;L^2(\Omega))$, the following problem possesses a unique global solution T_n.

$$\begin{cases}\partial_t T_n-\Delta T_n+\boldsymbol{u}_n\cdot\nabla T_n=f_2 \quad \text{in}\ \ \Omega\times[0,S],\\ \frac{\partial T_n}{\partial n}|_{\partial\Omega}=0,\quad T_n(\cdot,0)=T_0\in H^1(\Omega),\end{cases}\tag{4.12}$$

where

$$T_n \in C([0,S];H^1(\Omega)) \cap L^2(0,S;H^2(\Omega)) \cap W^{1,2}(0,S;L^2(\Omega)).$$

Multiplying (4.12) by T_n, we have

$$\frac{1}{2}\frac{d}{dt}|T_n|_{L^2}^2 + |\nabla T_n|_{L^2}^2 \leqslant |f_2|_{L^2}|T_n|_{L^2}, \tag{4.13}$$

since $\int_\Omega T_n \boldsymbol{u}_n \cdot \nabla T_n dx = 0$. Hence, we obtain

$$\sup_{0\leqslant t\leqslant S} |T_n(t)|_{L^2} + |\nabla T_n|_{L^2(0,S;L^2(\Omega))} \leqslant \gamma_1, \tag{4.14}$$

where γ_1 denotes a general constant independent of n. Next, multiplying (4.12) by $-\Delta T_n$, we get

$$\frac{1}{2}\frac{d}{dt}|\nabla T_n|_{L^2}^2 + \frac{1}{4}|\Delta T_n|_{L^2}^2 \leqslant |\boldsymbol{u}_n \cdot \nabla T_n|_{L^2}^2 + \frac{1}{2}|f_2|_{L^2}^2,$$

which, together with (4.5) and the fact that $\sup_{0\leq t\leq S} |\boldsymbol{u}_n(t)|_{\mathbb{H}^1} \leqslant \gamma_1$, leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\nabla T_n|_{L^2}^2 + \frac{1}{8}|\Delta T_n|_{L^2}^2 \\ &\qquad \leqslant \gamma_1 |\boldsymbol{u}_n|_{\mathbb{H}^2}^2 |\nabla T_n|_{L^2}^2 + \frac{1}{2}|f_2|_{L^2}^2 + \frac{1}{2}|T_n|_{L^2}^2. \end{aligned} \tag{4.15}$$

Applying Gronwall's inequality to (4.15), we have

$$\begin{aligned} &|\nabla T_n(t)|_{L^2}^2 \\ &\leqslant \left(|\nabla T_0|_{L^2}^2 + \int_0^S \left(|f_2(s)|_{L^2}^2 + |T_n(s)|_{L^2}^2\right) ds \right) \exp\left(\gamma_1 \int_0^S |\boldsymbol{u}_n(s)|_{\mathbb{H}^2}^2 ds \right). \end{aligned}$$

From (4.14) and the fact that $\int_0^S |\boldsymbol{u}_n(s)|_{\mathbb{H}^2}^2 ds \leqslant \gamma_1$, we can derive

$$\sup_{0\leqslant t\leqslant S} |\nabla T_n(t)|_{L^2}^2 \leqslant \gamma_1. \tag{4.16}$$

Integrating (4.15) over $[0,S]$ and using (4.16), we get

$$\int_0^S |\Delta T_n(s)|_{L^2}^2 ds \leqslant \gamma_1. \tag{4.17}$$

Similarly, multiplication of (4.12) by $\partial_t T_n$ gives

$$\frac{1}{2}|\partial_t T_n|_{L^2}^2 + \frac{d}{dt}\frac{1}{2}|\nabla T_n|_{L^2}^2 \leqslant \gamma_1 |\boldsymbol{u}_n|_{\mathbb{H}^2} |\nabla T_n|_{L^2} \left(|\Delta T_n|_{L^2} + |T_n|_{L^2}\right) + |f_2|_{L^2}^2, \tag{4.18}$$

which yields

$$\int_0^S |\partial_t T_n(s)|_{L^2}^2 ds \leqslant \gamma_1. \tag{4.19}$$

By using these estimates, we are going to show that $\{T_n\}_{n\in\mathbb{N}}$ becomes a Cauchy sequence in Banach space

$$Y''_S := C([0,S];H^1(\Omega)) \cap L^2(0,S;H^2(\Omega)) \cap W^{1,2}(0,S;L^2(\Omega))$$

with the norm

$$\|U\|^2_{Y''_S} := \sup_{0\le t\le S} |U(t)|^2_{H^1} + \int_0^S |\Delta U(s)|^2_{L^2} ds + \int_0^S |\partial_t U(s)|^2_{L^2} ds.$$

Let $\delta\boldsymbol{u} := \boldsymbol{u}_m - \boldsymbol{u}_n,\ \delta T := T_m - T_n$. Then $\delta\boldsymbol{u}$ and δT satisfy the following equation:

$$\text{(4.20)} \qquad \begin{cases} \partial_t \delta T - \Delta \delta T + \delta\boldsymbol{u}\cdot\nabla T_m + \boldsymbol{u}_n\cdot\nabla\delta T = 0 \quad \text{in} \ \ \Omega\times[0,S], \\ \frac{\partial \delta T}{\partial n}|_{\partial\Omega} = 0, \quad \delta T(\cdot,0) = 0. \end{cases}$$

Multiplication of (4.20) by δT yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\delta T|^2_{L^2} + |\nabla\delta T|^2_{L^2} \\ =& -\int_\Omega \delta T \delta\boldsymbol{u}\cdot\nabla T_m dx = \int_\Omega T_m \delta\boldsymbol{u}\cdot\nabla\delta T dx \\ \leqslant& \frac{1}{2}|\nabla\delta T|^2_{L^2} + \frac{1}{2}|T_m\delta\boldsymbol{u}|^2_{\mathbb{L}^2} \leqslant \frac{1}{2}|\nabla\delta T|^2_{L^2} + \gamma_1 |T_m|^2_{H^1}|\delta\boldsymbol{u}|^2_{\mathbb{H}^1}, \end{aligned}$$

i.e.,

$$\text{(4.21)} \qquad \frac{d}{dt}|\delta T|^2_{L^2} + |\nabla\delta T|^2_{L^2} \leqslant \gamma_1 |T_m|^2_{H^1}|\delta\boldsymbol{u}|^2_{\mathbb{H}^1}.$$

Integrating (4.21) over $[0,t]$, we have

$$\text{(4.22)} \qquad \sup_{0\leqslant t\leqslant S}|\delta T(t)|^2_{L^2} + \int_0^S |\nabla\delta T(s)|^2_{L^2} ds \leqslant \gamma_1 \sup_{0\leqslant t\leqslant S}|\delta\boldsymbol{u}(t)|^2_{\mathbb{H}^1}.$$

Next, multiplying (4.20) by $-\Delta\delta T$, we get

$$\text{(4.23)} \qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}|\nabla\delta T|^2_{L^2} + \frac{1}{8}|\Delta\delta T|^2_{L^2} \\ \leqslant& \gamma_1|\boldsymbol{u}_n|^2_{\mathbb{H}^2}|\nabla\delta T|^2_{L^2} + \frac{1}{2}|\delta T|^2_{L^2} \\ &\qquad + \gamma_1|\delta\boldsymbol{u}|_{\mathbb{H}^1}|\delta\boldsymbol{u}|_{\mathbb{H}^2}|\nabla T_m|_{L^2}\left(|\Delta T_m|_{L^2} + |T_m|_{L^2}\right), \\ \leqslant& \gamma_1|\boldsymbol{u}_n|^2_{\mathbb{H}^2}|\nabla\delta T|^2_{L^2} + \frac{1}{2}|\delta T|^2_{L^2} \\ &\qquad\qquad + \gamma_1|\delta\boldsymbol{u}|_{\mathbb{H}^1}|\delta\boldsymbol{u}|_{\mathbb{H}^2}\left(|\Delta T_m|_{L^2} + 1\right), \end{aligned}$$

where we used (4.14), (4.16) and the following estimates of convection terms which can be given by almost the same procedures as that for (4.5):

$$\begin{aligned}
&|\delta \boldsymbol{u}\cdot\nabla T_m|_{L^2}^2\\
&\qquad\leqslant \gamma_1|\delta\boldsymbol{u}|_{\mathbb{H}^1}|\delta\boldsymbol{u}|_{\mathbb{H}^2}|\nabla T_m|_{L^2}\left(|\Delta T_m|_{L^2}+|T_m|_{L^2}\right),\\
&|\boldsymbol{u}_n\cdot\nabla\delta T|_{L^2}^2\\
&\qquad\leqslant \gamma_1|\boldsymbol{u}_n|_{\mathbb{H}^1}|\boldsymbol{u}_n|_{\mathbb{H}^2}|\nabla\delta T|_{L^2}\left(|\Delta\delta T|_{L^2}+|\delta T|_{L^2}\right).
\end{aligned}\tag{4.24}$$

Then using uniform bounds

$$\sup_{0\leqslant t\leqslant S}|\boldsymbol{u}_n(t)|_{\mathbb{H}^1}+\int_0^S|\boldsymbol{u}_n(s)|_{\mathbb{H}^2}^2ds\leqslant\gamma_1$$

and applying Gronwall's inequality, we have

$$\begin{aligned}
&|\nabla\delta T(t)|_{L^2}^2\\
\leqslant&\gamma_1\int_0^S\left\{|\delta\boldsymbol{u}(s)|_{\mathbb{H}^1}|\delta\boldsymbol{u}(s)|_{\mathbb{H}^2}\left(|\Delta T_m(s)|_{L^2}+1\right)+|\delta T(s)|_{L^2}^2\right\}ds\\
&\qquad\qquad\qquad\qquad\times\exp\left(\gamma_1\int_0^S|\boldsymbol{u}_n(s)|_{\mathbb{H}^2}^2ds\right)\\
\leqslant&\gamma_1\int_0^S\left\{|\delta\boldsymbol{u}(s)|_{\mathbb{H}^1}|\delta\boldsymbol{u}(s)|_{\mathbb{H}^2}\left(|\Delta T_m(s)|_{L^2}+1\right)+|\delta T(s)|_{L^2}^2\right\}ds.
\end{aligned}$$

Moreover, by using the uniform boundedness $\int_0^S|\Delta T_m(t)|_{L^2}^2dt\leqslant\gamma_1$ and (4.22), we obtain

$$\sup_{0\leqslant t\leqslant S}|\nabla\delta T(t)|_{L^2}^2\leqslant\gamma_1\left(\sup_{0\leqslant t\leqslant S}|\delta\boldsymbol{u}(t)|_{\mathbb{H}^1}^2+|\delta\boldsymbol{u}|_{L^2(0,S;\mathbb{H}^2(\Omega))}^2\right).\tag{4.25}$$

Similarly, multiplying (4.20) by $\partial_t\delta T$, we get

$$\begin{aligned}
&\frac12|\partial_t\delta T|_{L^2}^2+\frac12\frac{d}{dt}|\nabla\delta T|_{L^2}^2\\
\leqslant&\frac12|\delta\boldsymbol{u}\cdot\nabla T_m|_{L^2}^2+|\boldsymbol{u}_n\cdot\nabla\delta T|_{L^2}^2\\
\leqslant&\gamma_1|\delta\boldsymbol{u}|_{\mathbb{H}^1}|\delta\boldsymbol{u}|_{\mathbb{H}^2}|\nabla T_m|_{L^2}\\
&\qquad+\gamma_1|\boldsymbol{u}_n|_{\mathbb{H}^1}|\boldsymbol{u}_n|_{\mathbb{H}^2}|\nabla\delta T|_{L^2}\left(|\Delta\delta T|_{L^2}+|\delta T|_{L^2}\right),
\end{aligned}$$

which implies

$$\int_0^S|\partial_t\delta T(t)|_{L^2}^2dt\leqslant\gamma_1\left(\sup_{0\leqslant t\leqslant S}|\delta\boldsymbol{u}(t)|_{\mathbb{H}^1}^2+|\delta\boldsymbol{u}|_{L^2(0,S;\mathbb{H}^2(\Omega))}^2\right).\tag{4.26}$$

Therefore, from (4.22), (4.25) and (4.26), we can assure that

$$\|\delta T\|_{Y_S''}^2\leqslant\gamma_1\left(\sup_{0\leqslant t\leqslant S}|\delta\boldsymbol{u}(t)|_{\mathbb{H}^1}^2+|\delta\boldsymbol{u}|_{L^2(0,S;\mathbb{H}^2(\Omega))}^2\right),$$

i.e., $\{T_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y_S''. Hence (4.3) has a unique global solution in Y_S'' for any $T_0 \in H^1(\Omega)$ and $f_2 \in L^2(0,S;L^2(\Omega))$.

Next we consider the solvability of (4.3) with L^2-initial data. Let $T_0 \in L^2(\Omega)$ and let $\{T_{0m}\}_{m\in\mathbb{N}} \subset H^1(\Omega)$ satisfy $T_{0m} \to T_0$ in $L^2(\Omega)$ as $m \to \infty$. Then for each $m \in \mathbb{N}$ and for any $f_2 \in L^2(0,S;L^2(\Omega))$, the following problem possesses a unique solution $T_m \in Y_S''$:

$$\begin{cases} \partial_t T_m - \Delta_N T_m + \boldsymbol{u}\cdot\nabla T_m = f_2 \quad \text{in } \ \Omega\times[0,S], \\ T(\cdot,0) = T_{0m}. \end{cases} \tag{4.27}$$

Then $\{T_m\}_{m\in\mathbb{N}}$ becomes a Cauchy sequence in Y_S. Indeed, for any $m_1, m_2 \in \mathbb{N}$, $\delta T := T_{m_1} - T_{m_2}$ satisfies

$$\begin{cases} \partial_t \delta T - \Delta_N \delta T + \boldsymbol{u}\cdot\nabla \delta T = 0 \quad \text{in } \ \Omega\times[0,S], \\ \delta T(\cdot,0) = T_{0m_1} - T_{0m_2}. \end{cases} \tag{4.28}$$

Multiplying (4.28) by δT, $-t\Delta\delta T$ and $t\partial_t\delta T$, we get (see (4.23) and use (4.5))

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\delta T|_{L^2}^2 + |\nabla\delta T|_{L^2}^2 = 0, \\ &\frac{1}{2}\frac{d}{dt}t|\nabla\delta T|_{L^2}^2 - \frac{1}{2}|\nabla\delta T|_{L^2}^2 + t|\Delta\delta T|_{L^2}^2 \\ &\qquad \leqslant \gamma|\boldsymbol{u}|_{\mathbb{H}^1}^2|\boldsymbol{u}|_{\mathbb{H}^2}^2 t|\nabla\delta T|_{L^2}^2 + \frac{t}{2}|\Delta\delta T|_{L^2}^2 + t|\delta T|_{L^2}^2, \\ &t|\partial_t\delta T|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}t|\nabla\delta T|_{L^2}^2 - \frac{1}{2}|\nabla\delta T|_{L^2}^2 \\ &\qquad \leqslant \frac{t}{2}|\partial_t\delta T|_{L^2}^2 + t\gamma|\boldsymbol{u}|_{\mathbb{H}^1}|\boldsymbol{u}|_{\mathbb{H}^2}|\nabla\delta T|_{L^2}(|\Delta\delta T|_{L^2} + |\delta T|_{L^2}), \end{aligned} \tag{4.29}$$

which yield

$$\|\delta T\|_{Y_S}^2 \leqslant \gamma|T_{0m_1} - T_{0m_2}|_{L^2}^2 \tag{4.30}$$

with some suitable constant γ. Therefore, we can assure that (4.3) possesses a unique global solution $T \in Y_S$ for any initial data T_0 belonging to $L^2(\Omega)$ and $f_2 \in L^2(0,S;L^2(\Omega))$.

Finally, we consider the case where $f_2 \in X_S$. Here, we define

$$\chi_\varepsilon(t) := \begin{cases} 0 & (\text{ if } 0 \leqslant t < \varepsilon \ ), \\ 1 & (\text{ if } \varepsilon \leqslant t \leqslant S \ ). \end{cases} \tag{4.31}$$

Since it is easy to see that $f_2^\varepsilon := \chi_\varepsilon f_2$ belongs to $L^2(0,S;L^2(\Omega))$, the following problem (4.32) possesses a unique global solution $T_\varepsilon \in Y_S$ for each $\varepsilon > 0$ and $T_0 \in L^2(\Omega)$.

$$\begin{cases} \partial_t T_\varepsilon - \Delta_N T_\varepsilon + \boldsymbol{u}\cdot\nabla T_\varepsilon = f_2^\varepsilon \quad \text{in } \ \Omega\times[0,S], \\ T(\cdot,0) = T_0. \end{cases} \tag{4.32}$$

Then, for each $\varepsilon_1, \varepsilon_2 > 0$, $\delta T := T_{\varepsilon_1} - T_{\varepsilon_2}$ and $\delta f_2 := f_2^{\varepsilon_1} - f_2^{\varepsilon_2}$ satisfy

$$\begin{cases} \partial_t \delta T - \Delta_N \delta T + \boldsymbol{u} \cdot \nabla \delta T = \delta f_2 \quad \text{in} \ \ \Omega \times [0, S], \\ \delta T(\cdot, 0) = 0. \end{cases} \tag{4.33}$$

Multiplying (4.33) by δT, we get

$$\frac{1}{2}\frac{d}{dt}|\delta T|_{L^2}^2 + |\nabla \delta T|_{L^2}^2 \leqslant |\delta T|_{L^2} |\delta f_2|_{L^2}.$$

By Gronwall's inequality, we have

$$\sup_{0 \leqslant t \leqslant S} |\delta T(t)|_{L^2} \leqslant |\delta f_2|_{L^1(0,S;L^2(\Omega))},$$

which also yields

$$\int_0^S |\nabla \delta T(s)|_{L^2}^2 ds \leqslant |\delta f_2|_{L^1(0,S;L^2(\Omega))}^2.$$

By almost the same argument as those for (4.29) and (4.30), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} t|\nabla \delta T|_{L^2}^2 - |\nabla \delta T|_{L^2}^2 + t|\Delta \delta T|_{L^2}^2 \\ &\qquad \leqslant \gamma |\boldsymbol{u}|_{\mathbb{H}^1}^2 |\boldsymbol{u}|_{\mathbb{H}^2}^2 t|\nabla \delta T|_{L^2}^2 + \frac{t}{2}|\Delta \delta T|_{L^2}^2 + t|\delta T|_{L^2}^2 + t|\delta f_2|_{L^2}^2, \\ &t|\partial_t \delta T|_{L^2}^2 + \frac{1}{2}\frac{d}{dt} t|\nabla \delta T|_{L^2}^2 - \frac{1}{2}|\nabla \delta T|_{L^2}^2 \\ &\qquad \leqslant \frac{t}{2}|\partial_t \delta T|_{L^2}^2 + t\gamma |\boldsymbol{u}|_{\mathbb{H}^1} |\boldsymbol{u}|_{\mathbb{H}^2} |\nabla \delta T|_{L^2} (|\Delta \delta T|_{L^2} + |\delta T|_{L^2}) + t|\delta f_2|_{L^2}^2. \end{aligned}$$

From these inequalities, we can derive

$$\sup_{0 \leqslant t \leqslant S} t|\nabla \delta T(t)|_{L^2}^2 + \int_0^S s|\Delta \delta T(s)|_{L^2}^2 ds + \int_0^S s|\partial_t \delta T(s)|_{L^2}^2 ds \leqslant \gamma' \int_0^S t|\delta f_2(s)|_{L^2}^2 ds$$

with some general constant γ' independent of ε_1, ε_2. Hence,

$$\|\delta T\|_{Y_S}^2 \leqslant \gamma' \|\delta f_2\|_{X_S}^2$$

holds. Since $f_2^{\varepsilon} \to f_2$ in X_S as $\varepsilon \to 0$, we can assure that $\{T_\varepsilon\}_{\varepsilon > 0}$ becomes a Cauchy sequence in Y_S.

Consequently, we can assure that (4.3) possesses a unique global solution $T \in Y_S$ for any initial data T_0 belonging to $L^2(\Omega)$ and $f_2 \in X_S$. □

Next, we consider the third equation.

Lemma 4.2.2. *Let $N \leqslant 4$ and let the space domain Ω be either the whole space $\mathbb{R}^N$ or uniform C^2-domain. Moreover, assume that $C_0 \in L^2(\Omega)$, $\boldsymbol{u} \in W_S$, $T \in Y_S$ and $f_3 \in X_S$. Then the following problem (4.34) has a unique global solution $C \in Y_S$.*

$$\begin{cases} \partial_t C - \Delta C + \boldsymbol{u} \cdot \nabla C = \rho \Delta T + f_3 \quad \text{in} \ \ \Omega \times [0, S], \\ \frac{\partial C}{\partial n}|_{\partial\Omega} = 0, \quad C(\cdot, 0) = C_0. \end{cases} \tag{4.34}$$

This problem is quite similar to the previous problem (4.3). However, we can not apply Lemma 4.2.1 directly, since it is not known whether $\Delta T \in X_S$.

Proof of Lemma 4.2.2. Let $\chi_\varepsilon : [0, S] \to \mathbb{R}$ be the cut-off function defined by

$$\chi_\varepsilon(t) := \begin{cases} 0 & (\text{ if } 0 \leqslant t < \varepsilon\ ), \\ 1 & (\text{ if } \varepsilon \leqslant t \leqslant S\ ). \end{cases}$$

Since $T \in Y_S$ implies that $\rho \chi_\varepsilon \Delta T \in X_S$, we can show that the following problem admits a unique global solution $C_\varepsilon \in Y_S$ for each parameter $\varepsilon > 0$ by applying Lemma 4.2.1.

$$\begin{cases} \partial_t C_\varepsilon - \Delta C_\varepsilon + \boldsymbol{u} \cdot \nabla C_\varepsilon = \rho \chi_\varepsilon \Delta T + f_3 \quad \text{in} \ \ \Omega \times [0, S], \\ \frac{\partial C_\varepsilon}{\partial n}|_{\partial\Omega} = 0, \quad C_\varepsilon(\cdot, 0) = C_0. \end{cases} \tag{4.35}$$

Then, by assuring that the sequence $\{C_\varepsilon\}_{\varepsilon > 0}$ is a Cauchy sequence in Y_S, we conclude the existence of solution for (4.34). Let $\chi_{\tilde{\varepsilon}} := \chi_{\varepsilon_1} - \chi_{\varepsilon_2}$, i.e.,

$$\chi_{\tilde{\varepsilon}}(t) := \begin{cases} 1 & (\varepsilon_1 \leqslant t < \varepsilon_2), \\ 0 & (\text{ otherwise }). \end{cases}$$

Then, $\delta C := C_{\varepsilon_1} - C_{\varepsilon_2}$ satisfies the following problem in Y_S:

$$\begin{cases} \partial_t \delta C - \Delta \delta C + \boldsymbol{u} \cdot \nabla \delta C = \rho \chi_{\tilde{\varepsilon}} \Delta T \quad \text{in} \ \ \Omega \times [0, S], \\ \frac{\partial \delta C}{\partial n}|_{\partial\Omega} = 0, \quad \delta C(\cdot, 0) = 0. \end{cases} \tag{4.36}$$

Multiplying (4.36) by δC, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\delta C|_{L^2}^2 + |\nabla \delta C|_{L^2}^2 &\leqslant \rho \chi_{\tilde{\varepsilon}} |\nabla \delta C|_{L^2} |\nabla T|_{L^2} \\ &\leqslant \frac{1}{2}|\nabla \delta C|_{L^2}^2 + \frac{\rho^2}{2}\chi_{\tilde{\varepsilon}}^2 |\nabla T|_{L^2}^2. \end{aligned} \tag{4.37}$$

Integrating (4.37) over $[0, t]$ and $[0, S]$, we obtain

$$\sup_{0 \leqslant t \leqslant S} |\delta C(t)|_{L^2}^2 + |\nabla \delta C|_{L^2(0,S;L^2(\Omega))}^2 \leqslant \rho^2 \int_{\varepsilon_1}^{\varepsilon_2} |\nabla T|_{L^2}^2 ds. \tag{4.38}$$

Moreover multiplication of (4.36) by $-t\Delta\delta C$ and (4.24), estimates for the convection terms, yield

$$\begin{aligned}(4.39)\qquad &\frac{1}{2}\frac{d}{dt}t|\nabla\delta C|_{L^2}^2+\frac{t}{8}|\Delta\delta C|_{L^2}^2\\ &\quad\leqslant \gamma_2|\boldsymbol{u}|_{\mathbb{H}^2}^2 t|\nabla\delta C|_{L^2}^2+\frac{t\rho^2\chi_{\tilde{\varepsilon}}^2}{2}|\Delta T|_{L^2}^2+\frac{t}{2}|\delta C|_{L^2}^2+\frac{1}{2}|\nabla\delta C|_{L^2}^2,\end{aligned}$$

where the coefficient γ_2 is a general constant independent of $\varepsilon_1, \varepsilon_2$. Applying Gronwall's inequality to (4.39), we obtain

$$\begin{aligned}(4.40)\qquad t|\nabla\delta C(t)|_{L^2}^2 \leqslant &\int_0^S\left\{s\rho^2\chi_{\tilde{\varepsilon}}^2|\Delta T(s)|_{L^2}^2+s|\delta C(s)|_{L^2}^2+|\nabla\delta C(s)|_{L^2}^2\right\}ds\\ &\times\exp\left(2\gamma_2\int_0^S|\boldsymbol{u}(s)|_{\mathbb{H}^2}^2ds\right).\end{aligned}$$

Hence from (4.38), we can derive

$$\sup_{0\leqslant t\leqslant S} t|\nabla\delta C(t)|_{L^2}^2 \leqslant \gamma_2\int_{\varepsilon_1}^{\varepsilon_2}\left\{s|\Delta T(s)|_{L^2}^2+|\nabla T(s)|_{L^2}^2\right\}ds.$$

Moreover, integrating (4.39) over $[0, S]$, we have

$$(4.41)\qquad \int_0^S t|\Delta\delta C(t)|_{L^2}^2dt \leqslant \gamma_2\int_{\varepsilon_1}^{\varepsilon_2}\left\{s|\Delta T(s)|_{L^2}^2+|\nabla T(s)|_{L^2}^2\right\}ds.$$

Multiplying (4.36) by $t\partial_t\delta C$, we can obtain (see our arguments for (4.26))

$$(4.42)\qquad \int_0^S t|\partial_t\delta C(t)|_{L^2}^2dt \leqslant \gamma_2\int_{\varepsilon_1}^{\varepsilon_2}\left\{s|\Delta T(s)|_{L^2}^2+|\nabla T(s)|_{L^2}^2\right\}ds.$$

Thus, we can assure that $\{C_\varepsilon\}_{\varepsilon>0}$ forms a Cauchy sequence in Y_S since $T \in Y_S$. Hence, the problem (4.34) possesses a unique global solution. □

Hence it follows that we can obtain a unique global solution $\underline{T}, \underline{C}$ of (4.1) and we can guarantee the well-definedness of Φ_{T_0,C_0}.

4.2.2 Well-Definedness of $\Psi_{\boldsymbol{u}_0}$

The Step 2 can be immediately accomplished by virtue of Proposition 2.5.6 in Chapter 2. In fact, applying Proposition 2.5.6 with

$$\varphi_S'(\boldsymbol{w}) := \begin{cases}\frac{\nu}{2}|\nabla\boldsymbol{w}|_{\mathbb{L}^2(\Omega)}^2+\frac{a}{2}|\boldsymbol{w}|_{\mathbb{L}^2(\Omega)}^2 & \text{if } \boldsymbol{w}\in\mathbb{H}_\sigma^1(\Omega),\\ +\infty & \text{otherwise,}\end{cases}$$

we can assure that the following.

Lemma 4.2.3. *Let $N \in \mathbb{N}$ and assume that $\boldsymbol{u}_0 \in \mathbb{H}^1_\sigma(\Omega)$ and $\boldsymbol{F} \in L^2(0,S;\mathbb{L}^2_\sigma(\Omega))$. Then the following problem* (4.43) *admits a unique global solution $\boldsymbol{u} \in W_S$ satisfying $\partial_t \boldsymbol{u} \in L^2(0,S;\mathbb{L}^2_\sigma(\Omega))$.*

$$\begin{cases} \partial_t \boldsymbol{u} + \nu \mathcal{A}\boldsymbol{u} + a\boldsymbol{u} = \boldsymbol{F} \quad \text{in} \ \ \Omega\times[0,S], \\ \boldsymbol{u}|_{\partial\Omega} = 0, \quad \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0. \end{cases} \tag{4.43}$$

Then, taking $\boldsymbol{F} := \mathcal{P}\boldsymbol{g}\underline{T} + \mathcal{P}\boldsymbol{h}\underline{C} + \mathcal{P}\boldsymbol{f}_1$ in (4.43), we can show the existence of a unique global solution $\overline{\boldsymbol{u}}$ of (4.2) and the well-definedness of $\Psi_{\boldsymbol{u}_0}$.

4.3 Application of Contraction Mapping Principle

In this section, we assure the local existence of a unique solution by using Banach's contraction mapping principle.

Let $\underline{\boldsymbol{u}}_i \in W_S$ $(i = 1,2)$, $(\underline{T}_i, \underline{C}_i) := \Phi_{T_0,C_0}(\underline{\boldsymbol{u}}_i)$ and $\overline{\boldsymbol{u}}_i := \Psi_{\boldsymbol{u}_0}((\underline{T}_i, \underline{C}_i))$, namely,

$$\begin{cases} \partial_t \overline{\boldsymbol{u}}_i + \nu\mathcal{A}\overline{\boldsymbol{u}}_i + a\overline{\boldsymbol{u}}_i = \mathcal{P}\boldsymbol{g}\underline{T}_i + \mathcal{P}\boldsymbol{h}\underline{C}_i + \mathcal{P}\boldsymbol{f}_1, \\ \partial_t \underline{T}_i - \Delta \underline{T}_i + \underline{\boldsymbol{u}}_i \cdot \nabla \underline{T}_i = f_2, \\ \partial_t \underline{C}_i - \Delta \underline{C}_i + \underline{\boldsymbol{u}}_i \cdot \nabla \underline{C}_i = \rho\Delta \underline{T}_i + f_3, \\ \delta\overline{\boldsymbol{u}}_i|_{\partial\Omega} = 0, \quad \left.\dfrac{\partial \underline{T}_i}{\partial n}\right|_{\partial\Omega} = 0, \quad \left.\dfrac{\partial \underline{C}_i}{\partial n}\right|_{\partial\Omega} = 0, \\ \overline{\boldsymbol{u}}_i(\cdot,0) = \boldsymbol{u}_0, \ \underline{T}_i(\cdot,0) = T_0, \ \underline{C}_i(\cdot,0) = C_0. \end{cases} \tag{4.44}$$

Moreover, let $\delta\underline{\boldsymbol{u}} := \underline{\boldsymbol{u}}_1 - \underline{\boldsymbol{u}}_2$, $\delta T := \underline{T}_1 - \underline{T}_2$, $\delta C := \underline{C}_1 - \underline{C}_2$ and $\delta\overline{\boldsymbol{u}} := \overline{\boldsymbol{u}}_1 - \overline{\boldsymbol{u}}_2$. Then $\delta\underline{\boldsymbol{u}}$, δT, δC and $\delta\overline{\boldsymbol{u}}$ satisfy the following equations:

$$\begin{cases} \partial_t \delta\overline{\boldsymbol{u}} + \nu\mathcal{A}\delta\overline{\boldsymbol{u}} + a\delta\overline{\boldsymbol{u}} = \mathcal{P}_\Omega \boldsymbol{g}\delta T + \mathcal{P}_\Omega \boldsymbol{h}\delta C, \\ \partial_t \delta T - \Delta\delta T + \underline{\boldsymbol{u}}_1 \cdot \nabla\delta T + \delta\underline{\boldsymbol{u}} \cdot \nabla \underline{T}_2 = 0, \\ \partial_t \delta C - \Delta\delta C + \underline{\boldsymbol{u}}_1 \cdot \nabla\delta C + \delta\underline{\boldsymbol{u}} \cdot \nabla \underline{C}_2 = \rho\Delta\delta T, \\ \delta\overline{\boldsymbol{u}}|_{\partial\Omega} = 0, \quad \left.\dfrac{\partial \delta T}{\partial n}\right|_{\partial\Omega} = 0, \quad \left.\dfrac{\partial \delta C}{\partial n}\right|_{\partial\Omega} = 0, \\ \delta\overline{\boldsymbol{u}}(\cdot,0) = 0, \ \delta T(\cdot,0) = 0, \ \delta C(\cdot,0) = 0. \end{cases} \tag{4.45}$$

Multiplying the first equation of (4.45) by $\delta\overline{\boldsymbol{u}}$ and $\mathcal{A}\delta\overline{\boldsymbol{u}}$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\delta\overline{\boldsymbol{u}}|^2_{\mathbb{L}^2} + \nu|\nabla\delta\overline{\boldsymbol{u}}|^2_{\mathbb{L}^2} + a|\delta\overline{\boldsymbol{u}}|^2_{\mathbb{L}^2} \\ &\qquad\qquad \leqslant (|\boldsymbol{g}||\delta T|_{L^2} + |\boldsymbol{h}||\delta C|_{L^2})\,|\delta\overline{\boldsymbol{u}}|_{\mathbb{L}^2}, \\ &\frac{1}{2}\frac{d}{dt}|\nabla\delta\overline{\boldsymbol{u}}|^2_{\mathbb{L}^2} + \frac{\nu}{4}|\mathcal{A}\delta\overline{\boldsymbol{u}}|^2_{\mathbb{L}^2} + a|\nabla\delta\overline{\boldsymbol{u}}|^2_{\mathbb{L}^2} \\ &\qquad\qquad \leqslant \frac{|\boldsymbol{g}|^2}{2\nu}|\delta T|^2_{L^2} + \frac{|\boldsymbol{h}|^2}{\nu}|\delta C|^2_{L^2}. \end{aligned} \tag{4.46}$$

Applying Gronwall's inequality (Proposition 2.5.2) to (4.46), we have

$$\begin{aligned}\sup_{0\leqslant t\leqslant S}|\delta\overline{\boldsymbol{u}}(t)|_{\mathbb{L}^2} &\leqslant |\boldsymbol{g}|\int_0^S |\delta T(s)|_{L^2}ds + |\boldsymbol{h}|\int_0^S |\delta C(s)|_{L^2}ds \\ &\leqslant S\left(|\boldsymbol{g}|\sup_{0\leqslant t\leqslant S}|\delta T(t)|_{L^2} + |\boldsymbol{h}|\sup_{0\leqslant t\leqslant S}|\delta C(t)|_{L^2}\right).\end{aligned} \tag{4.47}$$

Moreover, integrating (4.46) over $[0, S]$, we obtain

$$\begin{aligned}&\sup_{0\leqslant t\leqslant S}|\nabla\delta\overline{\boldsymbol{u}}(t)|^2_{\mathbb{L}^2} + \frac{\nu}{2}\int_0^S |\mathcal{A}\delta\overline{\boldsymbol{u}}(s)|^2_{\mathbb{L}^2}ds \\ &\qquad\leqslant \frac{S}{\nu}\left(|\boldsymbol{g}|^2\sup_{0\leqslant t\leqslant S}|\delta T(t)|^2_{L^2} + 2|\boldsymbol{h}|^2\sup_{0\leqslant t\leqslant S}|\delta C(t)|^2_{L^2}\right).\end{aligned} \tag{4.48}$$

Next, from the facts that

$$\begin{aligned}&\int_\Omega \delta T\delta\underline{\boldsymbol{u}}\cdot\nabla\underline{T_2}dx = -\int_\Omega \underline{T_2}\delta\underline{\boldsymbol{u}}\cdot\nabla\delta T dx, \\ &\int_\Omega \delta T\underline{\boldsymbol{u}}_1\cdot\nabla\delta T dx = 0\end{aligned}$$

and from the multiplication of the second equation by δT, we can derive

$$\begin{aligned}\frac{d}{dt}|\delta T|^2_{L^2} + |\nabla\delta T|^2_{L^2} &\leqslant |\underline{T_2}\delta\underline{\boldsymbol{u}}|^2_{\mathbb{L}^2} \leqslant |\underline{T_2}|^2_{L^4}|\delta\underline{\boldsymbol{u}}|^2_{\mathbb{L}^4} \\ &\leqslant \gamma_3|\underline{T_2}|^2_{H^1}|\delta\underline{\boldsymbol{u}}|^2_{\mathbb{H}^1},\end{aligned} \tag{4.49}$$

where γ_3 is a constant depending only on Sobolev's embedding constant. Similarly, multiplying the third equation of (4.45) by δC, we have

$$\frac{d}{dt}|\delta C|^2_{L^2} + \frac{1}{2}|\nabla\delta C|^2_{L^2} \leqslant \rho^2|\nabla\delta T|^2_{L^2} + 2\gamma_3|\underline{C_2}|^2_{H^1}|\delta\underline{\boldsymbol{u}}|^2_{\mathbb{H}^1}. \tag{4.50}$$

Then (4.49) and (4.50) yield

$$\begin{aligned}&\sup_{0\leqslant t\leqslant S}|\delta T(t)|^2_{L^2} + \int_0^S |\nabla\delta T(s)|^2_{L^2}ds \\ &\qquad\qquad\leqslant \gamma_3\sup_{0\leqslant t\leqslant S}|\delta\underline{\boldsymbol{u}}(t)|^2_{\mathbb{H}^1}\int_0^S |\underline{T_2}(s)|^2_{H^1}ds, \\ &\sup_{0\leqslant t\leqslant S}|\delta C(t)|^2_{L^2} \\ &\qquad\leqslant \gamma_3\sup_{0\leqslant t\leqslant S}|\delta\underline{\boldsymbol{u}}(t)|^2_{\mathbb{H}^1}\left\{\rho^2\int_0^S |\underline{T_2}(s)|^2_{H^1}ds + 2\int_0^S |\underline{C_2}(s)|^2_{H^1}ds\right\}.\end{aligned} \tag{4.51}$$

Hence, combining (4.47) and (4.48) with (4.51), we can derive

$$(4.52)\qquad \begin{aligned} &\sup_{0\leqslant t\leqslant S} |\delta\overline{\boldsymbol{u}}(t)|^2_{\mathbb{H}^1} + \int_0^S |\mathcal{A}\delta\overline{\boldsymbol{u}}(s)|^2_{\mathbb{L}^2} ds \\ &\qquad \leqslant \gamma_4 S(1+S) \sup_{0\leqslant t\leqslant S} |\delta\underline{\boldsymbol{u}}(t)|^2_{\mathbb{H}^1} \left\{ \int_0^S |\underline{T_2}(s)|^2_{H^1} ds + \int_0^S |\underline{C_2}(s)|^2_{H^1} ds \right\}, \end{aligned}$$

where γ_4 is a constant depending only on $\nu, |\boldsymbol{g}|, |\boldsymbol{h}|, \rho$ and γ_3. Here, multiplying the second and the third equation of (4.44) with $i = 2$ by $\underline{T_2}$ and $\underline{C_2}$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\underline{T_2}|^2_{L^2} + |\nabla \underline{T_2}|^2_{L^2} \leqslant |\underline{T_2}|_{L^2} |f_2|_{L^2}, \\ &\frac{1}{2}\frac{d}{dt}|\underline{C_2}|^2_{L^2} + |\nabla \underline{C_2}|^2_{L^2} \leqslant \rho |\nabla \underline{T_2}|_{L^2} |\nabla \underline{C_2}|_{L^2} + |\underline{C_2}|_{L^2} |f_3|_{L^2}, \end{aligned}$$

which implies

$$\begin{aligned} &\sup_{0\leq t\leq S} |\underline{T_2}(t)|_{L^2} \leqslant |T_0|_{L^2} + |f_2|_{L^1(0,S;L^2(\Omega))}, \\ &\int_0^S |\nabla \underline{T_2}(s)|^2_{L^2} ds \leqslant |T_0|^2_{L^2} + \left(|T_0|_{L^2} + |f_2|_{L^1(0,S;L^2(\Omega))}\right) |f_2|_{L^1(0,S;L^2(\Omega))}, \\ &\sup_{0\leq t\leq S} |\underline{C_2}(t)|_{L^2} \leqslant |C_0|_{L^2} + \rho |\nabla \underline{T_2}|_{L^2(0,S;L^2(\Omega))} + |f_3|_{L^1(0,S;L^2(\Omega))}, \\ &\int_0^S |\nabla \underline{C_2}(s)|^2_{L^2} ds \leqslant |C_0|^2_{L^2} + \rho^2 \int_0^S |\nabla \underline{T_2}(s)|^2_{L^2} ds + \sup_{0\leq t\leq S} |\underline{C_2}(t)|_{L^2} |f_3|_{L^1(0,S;L^2(\Omega))}. \end{aligned}$$

Hence $\int_0^S |\underline{T_2}(s)|^2_{H^1} ds$ and $\int_0^S |\underline{C_2}(s)|^2_{H^1} ds$ are bounded only by L^2-norm of the initial data and $L^1(0,S;L^2(\Omega))$-norm of the external forces. Thus, we can assure that $\Psi_{\boldsymbol{u}_0} \circ \Phi_{T_0,C_0}$ becomes a contraction mapping on W_{S_0} for a sufficiently small $S_0 \in (0, S]$, whence follows the existence of a unique local solution of (DCBF).

4.4 Global Existence

In this section, we shall show that the unique time-local solution constructed in the previous section can be extended up to S by establishing some a priori estimates.

Multiplying the second and the third equations of (DCBF) by T and C (and repeating exactly the same calculations as above), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|T|^2_{L^2} + |\nabla T|^2_{L^2} \leqslant |T|_{L^2} |f_2|_{L^2}, \\ &\frac{1}{2}\frac{d}{dt}|C|^2_{L^2} + |\nabla C|^2_{L^2} \leqslant \rho |\nabla T|_{L^2} |\nabla C|_{L^2} + |C|_{L^2} |f_3|_{L^2}, \end{aligned}$$

which yield

$$\sup_{0\le t\le S}|T(t)|_{L^2}+\int_0^S|\nabla T(s)|^2_{L^2}ds\leqslant Q(|T_0|_{L^2},|f_2|_{L^1(0,S;L^2(\Omega))}),$$

$$\begin{aligned}&\sup_{0\le t\le S}|C(t)|_{L^2}+\int_0^S|\nabla C(s)|^2_{L^2}ds\\&\qquad\leqslant Q(|T_0|_{L^2},|C_0|_{L^2},|f_2|_{L^1(0,S;L^2(\Omega))},|f_3|_{L^1(0,S;L^2(\Omega))}),\end{aligned}$$

where $Q(z_1,z_2,\cdots)$ is some monotone increasing function of $z_1,z_2,\cdots$. Multiplying the first equation of (DCBF) by $\boldsymbol{u}$ and $\mathcal{A}\boldsymbol{u}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|^2_{\mathbb{L}^2}+\nu|\nabla\boldsymbol{u}|^2_{\mathbb{L}^2}+a|\boldsymbol{u}|^2_{\mathbb{L}^2}\\&\qquad\leqslant(|\boldsymbol{g}||T|_{L^2}+|\boldsymbol{h}||C|_{L^2}+|\boldsymbol{f}_1|_{\mathbb{L}^2})\,|\boldsymbol{u}|_{\mathbb{L}^2},\\&\frac{1}{2}\frac{d}{dt}|\nabla\boldsymbol{u}|^2_{\mathbb{L}^2}+\frac{\nu}{2}|\mathcal{A}\boldsymbol{u}|^2_{\mathbb{L}^2}+a|\nabla\boldsymbol{u}|^2_{\mathbb{L}^2}\\&\qquad\leqslant\frac{3|\boldsymbol{g}|^2}{2\nu}|T|^2_{L^2}+\frac{3|\boldsymbol{h}|^2}{2\nu}|C|^2_{L^2}+\frac{3}{2\nu}|\boldsymbol{f}_1|_{\mathbb{L}^2},\end{aligned}$$

i.e.,

$$\sup_{0\leqslant t\leqslant S}|\boldsymbol{u}(t)|_{\mathbb{H}^1}+\int_0^S|\mathcal{A}\boldsymbol{u}(s)|^2_{\mathbb{L}^2}ds\leqslant\gamma_5(1+S),$$

where γ_5 is a general constant which depends on ν, $|\boldsymbol{g}|$, $|\boldsymbol{h}|$, $|T_0|_{L^2}$, $|C_0|_{L^2}$, $|f_2|_{L^1(0,S;L^2(\Omega))}$, $|f_3|_{L^1(0,S;L^2(\Omega))}$, $|\boldsymbol{f}_1|_{L^2(0,S;\mathbb{L}^2(\Omega))}$, $|\boldsymbol{u}_0|_{\mathbb{H}^1}$ and independent of S. Therefore, for any $S_0\in(0,S)$, we can assure that

$$\sup_{0\leqslant t\leqslant S_0}\{|T(t)|_{L^2}+|C(t)|_{L^2}+|\boldsymbol{u}(t)|_{\mathbb{H}^1}\}\leqslant\gamma_5(1+S).$$

This uniform bound independent of S_0 implies that the local solution constructed in the previous section can be extended onto the whole of the prescribed interval $[0,S]$, whence follows our result. □

4.5 Remarks

4.5.1 For H^1-Initial Data

If the initial data belong to $\mathbb{H}^1_\sigma(\Omega)\times H^1(\Omega)\times H^1(\Omega)$, we can derive the following result.

Corollary 4.5.1. *Let $N\leqslant 4$ and let the space domain Ω be either the whole space $\mathbb{R}^N$ or uniform C^2-domain. Then for any $\boldsymbol{u}_0\in\mathbb{H}^1_\sigma(\Omega)$, $T_0,C_0\in H^1(\Omega)$ and for any $\boldsymbol{f}_1\in L^2(0,S;\mathbb{L}^2(\Omega))$, $f_2,f_3\in L^2(0,S;L^2(\Omega))$,* (DCBF) *with the homogeneous Neumann boundary condition admits a unique solution $(\boldsymbol{u},T,C)$ satisfying the following regularities:*

$$\begin{aligned}&\boldsymbol{u}\in C([0,S];\mathbb{H}^1_\sigma(\Omega))\cap L^2(0,S;\mathbb{H}^2(\Omega))\cap W^{1,2}(0,S;\mathbb{L}^2_\sigma(\Omega)),\\&T,\ C\in C([0,S];H^1(\Omega))\cap L^2(0,S;H^2(\Omega))\cap W^{1,2}(0,S;L^2(\Omega)).\end{aligned}$$

In our proof for Lemma 4.2.1, we already saw that the following lemma is valid.

Lemma 4.5.1. *Let $N \leqslant 4$ and let the space domain Ω be either the whole space $\mathbb{R}^N$ or uniform C^2-domain. Assume that $T_0 \in H^1(\Omega)$, $\boldsymbol{u} \in W_S$ and $f_2 \in L^2(0,S;L^2(\Omega))$. Then the following problem* (4.53) *has a unique global solution T in Y_S''.*

$$\begin{cases} \partial_t T - \Delta T + \boldsymbol{u}\cdot\nabla T = f_2 \quad \text{in } \ \Omega\times[0,S], \\ \frac{\partial T}{\partial n}|_{\partial\Omega} = 0, \quad T(\cdot,0) = T_0. \end{cases} \tag{4.53}$$

Here we recall that

$$Y_S'' := C([0,S];H^1(\Omega)) \cap L^2(0,S;H^2(\Omega)) \cap W^{1,2}(0,S;L^2(\Omega)).$$

Lemma 4.5.1 immediately assure the solvability of the third equation:

$$\begin{cases} \partial_t C - \Delta C + \boldsymbol{u}\cdot\nabla C = \rho\Delta T + f_3 \quad \text{in } \ \Omega\times[0,S], \\ \frac{\partial C}{\partial n}|_{\partial\Omega} = 0, \quad C(\cdot,0) = C_0, \end{cases}$$

since $T \in Y_S''$ implies $\Delta T \in L^2(0,S;L^2(\Omega))$.

Therefore, we can show the existence of local solution of (DCBF) for H^1-initial data by almost the same argument as above.

Finally, we check the uniform boundedness of solution. Recall that

$$\sup_{0\leqslant t\leqslant S_0} \{|T(t)|_{L^2} + |C(t)|_{L^2} + |\boldsymbol{u}(t)|_{\mathbb{H}^1}\} \leqslant \gamma_6,$$
$$\int_0^{S_0} |\nabla T(s)|_{L^2}^2 ds + \int_0^{S_0} |\nabla C(s)|_{L^2}^2 ds + \int_0^{S_0} |\mathcal{A}\boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds \leqslant \gamma_6$$

hold for any $S_0 \in [0,S]$, where γ_6 is a general constant independent of S_0. Moreover, multiplying the second and the third equation of (DCBF) by $-\Delta T$, $-\Delta C$ respectively and repeating our procedures for (4.15), we obtain

$$\frac{1}{2}\frac{d}{dt}|\nabla T|_{L^2}^2 + \frac{1}{8}|\Delta T|_{L^2}^2 \leqslant \gamma_6|\boldsymbol{u}|_{\mathbb{H}^2}^2|\nabla T|_{L^2}^2 + \frac{1}{2}|f_2|_{L^2}^2 + \frac{1}{2}|T|_{L^2}^2,$$
$$\frac{1}{2}\frac{d}{dt}|\nabla C|_{L^2}^2 + \frac{1}{16}|\Delta C|_{L^2}^2 \leqslant \gamma_6|\boldsymbol{u}|_{\mathbb{H}^2}^2|\nabla C|_{L^2}^2 + \frac{1}{2}|f_3|_{L^2}^2 + \frac{1}{2}|C|_{L^2}^2 + 4\rho^2|\Delta T|_{L^2}^2.$$

By applying Granwall's inequality to these inequalities, we have

$$\sup_{0\leqslant t\leqslant S_0} \{|\nabla T(t)|_{L^2} + |\nabla C(t)|_{L^2}\} \leqslant \gamma_6,$$

which implies the time-local solution can be extended up to S.

4.5.2 Dirichlet Boundary Condition Case

Evidently, our argument in this chapter can be carried out for the initial boundary value problem with Dirichlet boundary condition without any changing and modification. Namely, we can assure the following.

Corollary 4.5.2. *Let $N \leqslant 4$ and let the space domain Ω be either the whole space $\mathbb{R}^N$ or uniform C^2-domain. Then for any $\boldsymbol{u}_0 \in \mathbb{H}^1_\sigma(\Omega)$, $T_0, C_0 \in L^2(\Omega)$ and any $\boldsymbol{f}_1 \in L^2(0,S;\mathbb{L}^2(\Omega))$, $f_2, f_3 \in X_S$,* (DCBF) *with the homogeneous Dirichlet boundary condition admits a unique solution $(\boldsymbol{u},\ T,\ C)^t \in Z'_S$, where*

$$Y'''_S := \left\{ U \in C([0,S];L^2(\Omega)) \cap L^2(0,S;H^1_0(\Omega)); \sqrt{t}\Delta U, \sqrt{t}\partial_t U \in L^2(0,S;L^2(\Omega)) \right\},$$

$$Z'_S := \left\{ \begin{pmatrix} \boldsymbol{u} \\ T \\ C \end{pmatrix} \in C([0,S];\mathbb{L}^2_\sigma(\Omega) \times L^2(\Omega) \times L^2(\Omega)); \begin{array}{l} \boldsymbol{u} \in W_S,\ T, C \in Y'''_S, \\ \partial_t \boldsymbol{u} \in L^2(0,S;\mathbb{L}^2_\sigma(\Omega)) \end{array} \right\}$$

with the norm $\|U\|_{Y'''_S} := \|U\|_{Y_S}$.

Moreover, if $\boldsymbol{f}_1 \in L^2(0,S;\mathbb{L}^2(\Omega))$, $f_2, f_3 \in L^2(0,S;L^2(\Omega))$ and $\boldsymbol{u}_0 \in \mathbb{H}^1_\sigma(\Omega)$, $T_0, C_0 \in H^1_0(\Omega)$, Then the solution satisfies the following regularities:

$$\begin{aligned} &\boldsymbol{u} \in C([0,S];\mathbb{H}^1_\sigma(\Omega)) \cap L^2(0,S;\mathbb{H}^2(\Omega)) \cap W^{1,2}(0,S;\mathbb{L}^2_\sigma(\Omega)), \\ &T,\ C \in C([0,S];H^1_0(\Omega)) \cap L^2(0,S;H^2(\Omega)) \cap W^{1,2}(0,S;L^2(\Omega)). \end{aligned}$$

Chapter 5

Time Periodic Problem in the Whole Space

5.1 Problems and Main Theorems

We here consider the following time periodic problem of (DCBF) in the whole space $\mathbb{R}^N$.

$$\text{(DCBF)}\begin{cases} \partial_t \boldsymbol{u} = \nu\Delta\boldsymbol{u} - a\boldsymbol{u} - \nabla p + \boldsymbol{g}T + \boldsymbol{h}C + \boldsymbol{f}_1 & (x,t) \in \mathbb{R}^N \times [0,S], \\ \partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + f_2 & (x,t) \in \mathbb{R}^N \times [0,S], \\ \partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta C + \rho\Delta T + f_3 & (x,t) \in \mathbb{R}^N \times [0,S], \\ \nabla\cdot\boldsymbol{u} = 0 & (x,t) \in \mathbb{R}^N \times [0,S], \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}(\cdot,S),\ T(\cdot,0) = T(\cdot,S),\ C(\cdot,0) = C(\cdot,S). \end{cases}$$

The main purpose of this chapter is to show the existence of solution in the following sense (recall that the conjugate Hölder exponent and the critical Sobolev exponent associated with q are designated by $q' := q/(q-1)$ and $q^* := qN/(N-q)$ respectively).

Definition 5.1.1 (Periodic solution of (DCBF))**.** *Let $N = 3$ or 4. Then $(\boldsymbol{u}, T, C)$ is said to be a periodic solution of (DCBF), if*

1. *$(\boldsymbol{u}, T, C)$ satisfies the following regularities:*

$$\begin{aligned} &\boldsymbol{u} \in C_\pi([0,S];\mathbb{L}^{2^*}_\sigma(\mathbb{R}^N)), && T, C \in C_\pi([0,S];L^{2^*}(\mathbb{R}^N)), \\ &\partial_{x_\mu}\boldsymbol{u} \in C_\pi([0,S];\mathbb{L}^2(\mathbb{R}^N)), && \partial_{x_\mu}T,\ \partial_{x_\mu}C \in C_\pi([0,S];L^2(\mathbb{R}^N)), \\ &\partial_t\boldsymbol{u} \in L^2(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N)), && \partial_t T,\ \partial_t C \in L^2(0,S;L^2(\mathbb{R}^N)), \\ &\partial_{x_\iota}\partial_{x_\mu}\boldsymbol{u} \in L^2(0,S;\mathbb{L}^2(\mathbb{R}^N)), && \partial_{x_\iota}\partial_{x_\mu}T,\ \partial_{x_\iota}\partial_{x_\mu}C \in L^2(0,S;L^2(\mathbb{R}^N)), \\ &\Delta\boldsymbol{u} \in L^2(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N)) \end{aligned}$$

   for all $\iota, \mu = 1, 2, \cdots, N$.

2. *$(\boldsymbol{u}, T, C)$ satisfies the second and the third equation of* (DCBF) *in $L^2(0,S;L^2(\mathbb{R}^N))$.*

3. For all $\boldsymbol{\phi} \in L^2(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N)) \cap L^2(0,S;\mathbb{L}^{(2^*)'}_\sigma(\mathbb{R}^N))$, $(\boldsymbol{u},T,C)$ *satisfies*

$$\int_0^S \int_{\mathbb{R}^N} (\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + a\boldsymbol{u} - \boldsymbol{g}T - \boldsymbol{h}C - \boldsymbol{f}_1) \cdot \boldsymbol{\phi} \, dxdt = 0. \tag{5.1}$$

Our argument in this chapter is divided into the following three steps:

Step 1: We show the solvability of the following problem with two approximation parameters $n \in \mathbb{N}$ and $\lambda > 0$.

$$\begin{cases} \partial_t \boldsymbol{u} + \nu \mathcal{A}_{\Omega_n} \boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_{\Omega_n} \boldsymbol{g}T + \mathcal{P}_{\Omega_n} \boldsymbol{h}C + \mathcal{P}_{\Omega_n} \boldsymbol{f}_1|_{\Omega_n} & (x,t) \in \Omega_n \times [0,S], \\ \partial_t T + \boldsymbol{u} \cdot \nabla T + \lambda T = \Delta T + f_2|_{\Omega_n} & (x,t) \in \Omega_n \times [0,S], \\ \partial_t C + \boldsymbol{u} \cdot \nabla C + \lambda C = \Delta C + \rho \Delta T + f_3|_{\Omega_n} & (x,t) \in \Omega_n \times [0,S], \\ \boldsymbol{u} = 0, \quad T = 0, \quad C = 0 & (x,t) \in \partial\Omega_n \times [0,S], \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}(\cdot,S), \ T(\cdot,0) = T(\cdot,S), \ C(\cdot,0) = C(\cdot,S). \end{cases}$$

Throughout this chapter, Ω_R denotes the open ball centered at the origin with radius $R > 0$, i.e., $\Omega_R := \{x \in \mathbb{R}^N; |x| < R\}$ and $\cdot|_\omega$ designates the restriction of function onto $\omega \subset \mathbb{R}^N$.

Step 2: We discuss the convergence of solutions given in Step 1 as $n \to \infty$ and we assure that the following problem possesses a solution for each parameter $\lambda > 0$.

$$\begin{cases} \partial_t \boldsymbol{u} + \nu \mathcal{A}_{\mathbb{R}^N} \boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_{\mathbb{R}^N} \boldsymbol{g}T + \mathcal{P}_{\mathbb{R}^N} \boldsymbol{h}C + \mathcal{P}_{\mathbb{R}^N} \boldsymbol{f}_1 & (x,t) \in \mathbb{R}^N \times [0,S], \\ \partial_t T + \boldsymbol{u} \cdot \nabla T + \lambda T = \Delta T + f_2 & (x,t) \in \mathbb{R}^N \times [0,S], \\ \partial_t C + \boldsymbol{u} \cdot \nabla C + \lambda C = \Delta C + \rho \Delta T + f_3 & (x,t) \in \mathbb{R}^N \times [0,S], \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}(\cdot,S), \ T(\cdot,0) = T(\cdot,S), \ C(\cdot,0) = C(\cdot,S). \end{cases}$$

Step 3: We show that the solutions of Step 2 converge to a periodic solution of the original problem (DCBF) by letting the relaxation parameter λ tend to 0.

In this way, we demonstrate the following main theorem in this chapter.

Theorem 5.1.1. *Let* $N = 3$ *or* 4 *and* $a > 0$. *Moreover, assume that*

$$\boldsymbol{f}_1 \in W^{1,2}(0,S;\mathbb{L}^2(\mathbb{R}^N)), \qquad \boldsymbol{f}_1(0) = \boldsymbol{f}_1(S),$$
$$f_2, f_3 \in L^2(0,S;L^2(\mathbb{R}^N)) \cap L^2(0,S;L^{(2^*)'}(\mathbb{R}^N)).$$

Then (DCBF) *possesses at least one periodic solution* $(\boldsymbol{u},T,C)$.

Remark. We can show that the identity (5.1) is equivalent to the first equation of (DCBF). Since $T, C \in C_\pi([0,S];L^{2^*}(\mathbb{R}^N))$ and $\boldsymbol{f}_1 \in C_\pi([0,S];\mathbb{L}^2(\mathbb{R}^N))$, then $\boldsymbol{g}T(t), \boldsymbol{h}C(t)$ and $\boldsymbol{f}_1(t)$ can be decomposed as follows:

$$\boldsymbol{g}T(t) = \boldsymbol{v}_T(t) + \boldsymbol{w}_T(t), \qquad \boldsymbol{h}C(t) = \boldsymbol{v}_C(t) + \boldsymbol{w}_C(t), \qquad \boldsymbol{f}_1(t) = \boldsymbol{v}_{\boldsymbol{f}_1}(t) + \boldsymbol{w}_{\boldsymbol{f}_1}(t)$$

for any $t \in [0, S]$, where

$$\begin{aligned} &\boldsymbol{v}_T(t), \boldsymbol{v}_C(t) \in \mathbb{L}^{2^*}_\sigma(\mathbb{R}^N), \quad &&\boldsymbol{w}_T(t), \boldsymbol{w}_C(t) \in G_{2^*}(\mathbb{R}^N), \\ &\boldsymbol{v}_{\boldsymbol{f}_1}(t) \in \mathbb{L}^2_\sigma(\mathbb{R}^N), &&\boldsymbol{w}_{\boldsymbol{f}_1}(t) \in G_2(\mathbb{R}^N). \end{aligned}$$

Here we recall that

$$|\boldsymbol{v}_T(t) - \boldsymbol{v}_T(s)|_{\mathbb{L}^{2^*}(\mathbb{R}^N)} + |\boldsymbol{w}_T(t) - \boldsymbol{w}_T(s)|_{\mathbb{L}^{2^*}(\mathbb{R}^N)} \leqslant \alpha\, |\boldsymbol{g}T(t) - \boldsymbol{g}T(s)|_{\mathbb{L}^{2^*}(\mathbb{R}^N)}$$

holds for any $t, s \in [0, S]$, where α is some suitable constant independent of t and s. This inequality implies that $\boldsymbol{v}_T, \boldsymbol{w}_T \in C_\pi([0, S]; \mathbb{L}^{2^*}(\mathbb{R}^N))$. By exactly the same reasoning, $\boldsymbol{v}_C, \boldsymbol{w}_C \in C_\pi([0, S]; \mathbb{L}^{2^*}(\mathbb{R}^N))$ and $\boldsymbol{v}_{\boldsymbol{f}_1}, \boldsymbol{w}_{\boldsymbol{f}_1} \in C_\pi([0, S]; \mathbb{L}^2(\mathbb{R}^N))$ are also valid. Then, since $L^2(0, S; \mathbb{L}^2_\sigma(\mathbb{R}^N)) \cap L^2(0, S; \mathbb{L}^{(2^*)'}_\sigma(\mathbb{R}^N))$ is the dual space of $L^2(0, S; \mathbb{L}^2_\sigma(\mathbb{R}^N)) + L^2(0, S; \mathbb{L}^{2^*}_\sigma(\mathbb{R}^N))$, (5.1) yields

$$\text{(5.2)} \qquad \begin{aligned} &\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + a\boldsymbol{u} - \boldsymbol{v}_T - \boldsymbol{v}_C - \boldsymbol{v}_{\boldsymbol{f}_1} = 0 \\ \Leftrightarrow \quad &\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + a\boldsymbol{u} + \boldsymbol{w}_T + \boldsymbol{w}_C + \boldsymbol{w}_{\boldsymbol{f}_1} = \boldsymbol{g}T + \boldsymbol{h}C + \boldsymbol{f}_1 \end{aligned}$$

in $L^2(0, S; \mathbb{L}^2(\mathbb{R}^N)) + L^2(0, S; \mathbb{L}^{2^*}(\mathbb{R}^N))$. By the definition of $G_q(\mathbb{R}^N)$, there exist p_1 and p_2 such that

$$\begin{aligned} &p_1(\cdot, t) \in W^{1,2^*}_{\mathrm{loc}}(\mathbb{R}^N), \quad p_2(\cdot, t) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N) \qquad \forall t \in [0, S], \\ &\nabla p_1 = \boldsymbol{w}_T + \boldsymbol{w}_C \in C_\pi([0, S]; \mathbb{L}^{2^*}(\mathbb{R}^N)), \\ &\nabla p_2 = \boldsymbol{w}_{\boldsymbol{f}_1} \in C_\pi([0, S]; \mathbb{L}^2(\mathbb{R}^N)). \end{aligned}$$

Therefore (5.2) is equivalent to the first equation of (DCBF) with $p = p_1 + p_2$.

5.2 Bounded Domain Case

We first consider Step 1, i.e., the solvability of the following equations in bounded domains with large data.

Lemma 5.2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$ with $N \leqslant 4$. Moreover, assume that $\boldsymbol{F}_1 \in L^2(0, S; \mathbb{L}^2(\Omega))$ and $F_2, F_3 \in L^2(0, S; L^2(\Omega))$. Then for any non-negative constants a and λ, the following system* (5.3) *possesses at least one periodic solution $(\boldsymbol{u}, T, C)$.*

$$\text{(5.3)} \qquad \begin{cases} \partial_t \boldsymbol{u} + \nu \mathcal{A}_\Omega \boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_\Omega \boldsymbol{g}T + \mathcal{P}_\Omega \boldsymbol{h}C + \mathcal{P}_\Omega \boldsymbol{F}_1 & (x,t) \in \Omega \times [0, S], \\ \partial_t T + \boldsymbol{u} \cdot \nabla T + \lambda T = \Delta T + F_2 & (x,t) \in \Omega \times [0, S], \\ \partial_t C + \boldsymbol{u} \cdot \nabla C + \lambda C = \Delta C + \rho \Delta T + F_3 & (x,t) \in \Omega \times [0, S], \\ \boldsymbol{u} = 0, \quad T = 0, \quad C = 0 & (x,t) \in \partial\Omega \times [0, S]. \end{cases}$$

Here $(\boldsymbol{u}, T, C)$ is said to be a periodic solution of (5.3)*, if*

1. $(\boldsymbol{u}, T, C)$ *satisfies*

$$\boldsymbol{u} \in C_\pi([0,S]; \mathbb{H}^1_\sigma(\Omega)) \cap L^2(0,S;\mathbb{H}^2(\Omega)) \cap W^{1,2}(0,S;\mathbb{L}^2_\sigma(\Omega)),$$
$$T, C \in C_\pi([0,S]; H^1_0(\Omega)) \cap L^2(0,S;H^2(\Omega)) \cap W^{1,2}(0,S;L^2(\Omega)). \tag{5.4}$$

2. $(\boldsymbol{u}, T, C)$ *satisfies the second and the third equation of* (5.3) *in* $L^2(0,S;L^2(\Omega))$.

3. $(\boldsymbol{u}, T, C)$ *satisfies the first equation of* (5.3) *in* $L^2(0,S;\mathbb{L}^2_\sigma(\Omega))$.

Solvability of (5.3) has been already proved in Chapter 3 for $N \leqslant 3$. By using almost the same procedures, we can also guarantee the solvability of (5.3) with $N = 4$ (for instance, we first consider the following approximate system:

$$\begin{cases} \partial_t \boldsymbol{u} + \nu \mathcal{A}_\Omega \boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_\Omega \boldsymbol{g} T + \mathcal{P}_\Omega \boldsymbol{h} C + \mathcal{P}_\Omega \boldsymbol{F}_1 \\ \partial_t T + \epsilon \Delta^2 T - \Delta_D T + \boldsymbol{u} \cdot \nabla T = F_2 \\ \partial_t C + \epsilon \Delta^2 C - \Delta_D C + \boldsymbol{u} \cdot \nabla C = \rho \Delta_D T + F_3, \end{cases}$$

where $\Delta^2 := (-\Delta_D)^2$. After we show the solvability of these approximate equations by Proposition 2.3.5, we discuss the convergence of approximate solutions as the parameter ϵ tends to 0).

However, we here give another way to prove this fact.

Proof. Fix $\underline{T}, \underline{C} \in L^2(0,S;L^2(\Omega))$ arbitrarily. Then by virtue of Proposition 2.3.2, the following problem (5.5) possesses a periodic solution $\boldsymbol{u}$ belonging to $C_\pi([0,S];\mathbb{H}^1_\sigma(\Omega)) \cap L^2(0,S;\mathbb{H}^2(\Omega)) \cap W^{1,2}(0,S;\mathbb{L}^2_\sigma(\Omega))$.

$$\begin{cases} \partial_t \boldsymbol{u} + \nu \mathcal{A}_\Omega \boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_\Omega \boldsymbol{g} \underline{T} + \mathcal{P}_\Omega \boldsymbol{h} \underline{C} + \mathcal{P}_\Omega \boldsymbol{F}_1 & (x,t) \in \Omega \times [0,S], \\ \boldsymbol{u} = 0 & (x,t) \in \partial\Omega \times [0,S], \\ \boldsymbol{u}(0) = \boldsymbol{u}(S). \end{cases} \tag{5.5}$$

It is easy to see that the solution of (5.5) is unique. Indeed, let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be solutions of (5.5). Then $\boldsymbol{u}_1 - \boldsymbol{u}_2$ satisfies

$$\begin{aligned} &\partial_t(\boldsymbol{u}_1 - \boldsymbol{u}_2) + \nu \mathcal{A}_\Omega(\boldsymbol{u}_1 - \boldsymbol{u}_2) + a(\boldsymbol{u}_1 - \boldsymbol{u}_2) = 0 \\ \Rightarrow &\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2_{\mathbb{L}^2(\Omega)} + \left(\frac{\nu}{\kappa} + a\right)|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2_{\mathbb{L}^2(\Omega)} \leqslant 0, \end{aligned} \tag{5.6}$$

where κ is a constant appearing in Poincaré's inequality:

$$\begin{aligned} &|U|^2_{L^2(\Omega)} \leqslant \kappa |\nabla U|^2_{L^2(\Omega)}, \quad |\boldsymbol{v}|^2_{L^2(\Omega)} \leqslant \kappa |\nabla \boldsymbol{v}|^2_{L^2(\Omega)}, \\ &|\nabla U|^2_{L^2(\Omega)} \leqslant \kappa |\Delta U|^2_{L^2(\Omega)}, \quad |\nabla \boldsymbol{v}|^2_{L^2(\Omega)} \leqslant \kappa |\mathcal{A}_\Omega \boldsymbol{v}|^2_{L^2(\Omega)} \end{aligned}$$

for any $\boldsymbol{v} \in D(\mathcal{A}_\Omega)$ and $U \in D(-\Delta_D)$. Integrating (5.6) over $[0,S]$ and using the periodicity of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we obtain

$$\int_0^S |\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)|^2_{\mathbb{L}^2(\Omega)} ds \leqslant 0,$$

which implies $\boldsymbol{u}_1 = \boldsymbol{u}_2$. Let $\boldsymbol{u}$ be the unique periodic solution of (5.5). Then we can define $\Phi : X_S \to Y_S$ by the relationship $\Phi((\underline{T}, \underline{C})) := \boldsymbol{u}$, where

$$X_S := L^2(0, S; L^2(\Omega)) \times L^2(0, S; L^2(\Omega)),$$
$$Y_S := C([0, S]; \mathbb{H}^1_\sigma(\Omega)) \cap L^2(0, S; \mathbb{H}^2(\Omega)).$$

Fix $\underline{\boldsymbol{u}} \in Y_S$ arbitrarily. Then we next consider the following system:

$$\text{(5.7)} \qquad \begin{cases} \partial_t T + \underline{\boldsymbol{u}} \cdot \nabla T + \lambda T = \Delta T + F_2 & (x,t) \in \Omega \times [0,S], \\ \partial_t C + \underline{\boldsymbol{u}} \cdot \nabla C + \lambda C = \Delta C + \rho \Delta T + F_3 & (x,t) \in \Omega \times [0,S], \\ T = 0, \quad C = 0 & (x,t) \in \partial\Omega \times [0,S]. \end{cases}$$

According to Chapter 4, the initial boundary value problem of (5.7) has a unique solution for any initial data $T(0) \in L^2(\Omega)$ and $C(0) \in L^2(\Omega)$. Moreover, this solution satisfies the following regularities:

$$\text{(5.8)} \qquad \begin{aligned} &T, C \in C([0,S]; L^2(\Omega)) \cap L^2(0, S; H^1_0(\Omega)), \\ &\sqrt{t}\Delta T,\ \sqrt{t}\partial_t T,\ \sqrt{t}\Delta C,\ \sqrt{t}\partial_t C \in L^2(0, S; L^2(\Omega)). \end{aligned}$$

Let (T_1, C_1) and (T_2, C_2) be two solutions of the initial boundary value problem for (5.7). Then $\delta T := T_1 - T_2$ and $\delta C := C_1 - C_2$ satisfy

$$\begin{cases} \partial_t \delta T + \underline{\boldsymbol{u}} \cdot \nabla \delta T + \lambda \delta T = \Delta \delta T, \\ \partial_t \delta C + \underline{\boldsymbol{u}} \cdot \nabla \delta C + \lambda \delta C = \Delta \delta C + \rho \Delta \delta T. \end{cases}$$

Multiplying each equation by δT and δC respectively, we get

$$\text{(5.9)} \qquad \frac{1}{2}\frac{d}{dt}|\delta T|^2_{L^2(\Omega)} + |\nabla \delta T|^2_{L^2(\Omega)} + \lambda |\delta T|^2_{L^2(\Omega)} = 0$$

and

$$\text{(5.10)} \qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}|\delta C|^2_{L^2(\Omega)} + |\nabla \delta C|^2_{L^2(\Omega)} + \lambda |\delta C|^2_{L^2(\Omega)} \\ =& -\rho \int_\Omega \nabla \delta T \cdot \nabla \delta C \\ \leqslant& \rho |\nabla \delta T|_{L^2(\Omega)} |\nabla \delta C|_{L^2(\Omega)} \leqslant \frac{1}{2}|\nabla \delta C|^2_{L^2(\Omega)} + \frac{\rho^2}{2}|\nabla \delta T|^2_{L^2(\Omega)}. \end{aligned}$$

Then (5.9) and (5.10) yield

$$\frac{d}{dt}\left(|\delta T|^2_{L^2(\Omega)} + \frac{1}{\rho^2}|\delta C|^2_{L^2(\Omega)}\right) + \left(|\nabla \delta T|^2_{L^2(\Omega)} + \frac{1}{\rho^2}|\nabla \delta C|^2_{L^2(\Omega)}\right) \leqslant 0.$$

Therefore, by applying Poincaré's inequality and Gronwall's inequality, we have

$$\left(|\delta T(S)|^2_{L^2(\Omega)} + \frac{1}{\rho^2}|\delta C(S)|^2_{L^2(\Omega)}\right) \leqslant \left(|\delta T(0)|^2_{L^2(\Omega)} + \frac{1}{\rho^2}|\delta C(0)|^2_{L^2(\Omega)}\right) e^{-\frac{S}{\kappa}}.$$

Therefore, we can derive the existence of a unique periodic solution of (5.7) from the application of Banach's contraction mapping principle in $L^2(\Omega) \times L^2(\Omega)$.

Let $(\overline{T}, \overline{C})$ be the periodic solution of (5.7) and define $\Psi : Y_S \to X_S$ by the correspondence $\Psi(\underline{\boldsymbol{u}}) := (\overline{T}, \overline{C})$. Since $\overline{T}(0), \overline{C}(0) \in H_0^1(\Omega)$ hold by the periodicity and (5.8), the following regularities of $\Psi(\underline{\boldsymbol{u}}) = (\overline{T}, \overline{C})$ can be obtained for any given $\underline{\boldsymbol{u}} \in Y_S$ (recall Lemma 4.5.1 and Corollary 4.5.2 in Chapter 4).

$$\overline{T}, \overline{C} \in C_\pi([0,S]; H_0^1(\Omega)) \cap L^2(0,S; H^2(\Omega)) \cap W^{1,2}(0,S; L^2(\Omega)).$$

Moreover, multiplying each equation of (5.7) by $\overline{T}$ and $\overline{C}$ respectively, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\overline{T}|^2_{L^2(\Omega)} + |\nabla \overline{T}|^2_{L^2(\Omega)} \leqslant |F_2|_{L^2(\Omega)} |\overline{T}|_{L^2(\Omega)} \leqslant \sqrt{\kappa} |F_2|_{L^2(\Omega)} |\nabla \overline{T}|_{L^2(\Omega)} \\
&\Rightarrow \frac{d}{dt}|\overline{T}|^2_{L^2(\Omega)} + |\nabla T|^2_{L^2(\Omega)} \leqslant \kappa |F_2|^2_{L^2(\Omega)} \\
&\Rightarrow \int_0^S |\nabla \overline{T}(s)|^2_{L^2(\Omega)} ds \leqslant \kappa \int_0^S |F_2(s)|^2_{L^2(\Omega)} ds
\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\overline{C}|^2_{L^2(\Omega)} + |\nabla \overline{C}|^2_{L^2(\Omega)} \leqslant \rho |\nabla \overline{T}|_{L^2(\Omega)} |\nabla \overline{C}|_{L^2(\Omega)} + |F_3|_{L^2(\Omega)} |\overline{C}|_{L^2(\Omega)} \\
&\Rightarrow \frac{d}{dt}|\overline{C}|^2_{L^2(\Omega)} + |\nabla \overline{C}|^2_{L^2(\Omega)} \leqslant 2\rho^2 |\nabla \overline{T}|^2_{L^2(\Omega)} + 2\kappa |F_3|^2_{L^2(\Omega)} \\
&\Rightarrow \int_0^S |\nabla \overline{C}(s)|^2_{L^2(\Omega)} ds \leqslant 2\rho^2 \int_0^S |\nabla \overline{T}(s)|^2_{L^2(\Omega)} ds + 2\kappa \int_0^S |F_3(s)|^2_{L^2(\Omega)} ds
\end{aligned} \tag{5.12}$$

for arbitrary $\underline{\boldsymbol{u}} \in Y_S$.

Define the set $K \subset X_S$ by

$$K := \left\{ (U_1, U_2) \in X_S; \begin{array}{ll} |U_1|^2_{L^2(0,S;L^2(\Omega))} & \leqslant \kappa^2 |F_2|^2_{L^2(0,S;L^2(\Omega))} \\ |U_2|^2_{L^2(0,S;L^2(\Omega))} & \leqslant 2\rho^2\kappa^2 |F_2|^2_{L^2(0,S;L^2(\Omega))} + 2\kappa^2 |F_3|^2_{L^2(0,S;L^2(\Omega))} \end{array} \right\}.$$

Then (5.11) and (5.12) imply that K satisfies $\Psi \circ \Phi(K) \subset K$. Obviously, K is convex and compact with respect to the weak topology of X_S.

Let $\{\underline{T}_k\}_{k\in\mathbb{N}}$ and $\{\underline{C}_k\}_{k\in\mathbb{N}}$ be sequences which weakly converge in $L^2(0,S;L^2(\Omega))$ and let $\underline{T}$ and $\underline{C}$ denote their limits respectively. Furthermore, we define $\underline{\boldsymbol{u}}_k := \Phi((\underline{T}_k, \underline{C}_k))$, $\underline{\boldsymbol{u}} := \Phi((\underline{T}, \underline{C}))$, $(\overline{T}_k, \overline{C}_k) := \Psi(\underline{\boldsymbol{u}}_k)$ and $(\overline{T}, \overline{C}) := \Psi(\underline{\boldsymbol{u}})$. Multiplying

$$\partial_t \underline{\boldsymbol{u}}_k + \nu \mathcal{A}_\Omega \underline{\boldsymbol{u}}_k + a \underline{\boldsymbol{u}}_k = \mathcal{P}_\Omega \boldsymbol{g} \underline{T}_k + \mathcal{P}_\Omega \boldsymbol{h} \underline{C}_k + \mathcal{P}_\Omega \boldsymbol{F}_1$$

by $\underline{\boldsymbol{u}}_k$, $\mathcal{A}_\Omega \underline{\boldsymbol{u}}_k$ and $\partial_t \underline{\boldsymbol{u}}_k$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\underline{\boldsymbol{u}}_k|^2_{\mathbb{L}^2(\Omega)} + \frac{\nu}{2}|\nabla \underline{\boldsymbol{u}}_k|^2_{\mathbb{L}^2(\Omega)} + a|\underline{\boldsymbol{u}}_k|^2_{\mathbb{L}^2(\Omega)} \\
&\qquad \leqslant \frac{3|\boldsymbol{g}|^2\kappa}{2\nu} |\underline{T}_k|^2_{L^2(\Omega)} + \frac{3|\boldsymbol{h}|^2\kappa}{2\nu} |\underline{C}_k|^2_{L^2(\Omega)} + \frac{3\kappa}{2\nu} |\boldsymbol{F}_1|^2_{\mathbb{L}^2(\Omega)},
\end{aligned} \tag{5.13}$$

$$\text{(5.14)}\qquad \begin{aligned}&\frac{1}{2}\frac{d}{dt}|\nabla \underline{\boldsymbol{u}}_k|^2_{\mathbb{L}^2(\Omega)} + \frac{\nu}{2}|\mathcal{A}_\Omega \underline{\boldsymbol{u}}_k|^2_{\mathbb{L}^2(\Omega)}\\ &\qquad \leqslant \frac{3|\boldsymbol{g}|^2}{2\nu}|\underline{T}_k|^2_{L^2(\Omega)} + \frac{3|\boldsymbol{h}|^2}{2\nu}|\underline{C}_k|^2_{L^2(\Omega)} + \frac{3}{2\nu}|\boldsymbol{F}_1|^2_{\mathbb{L}^2(\Omega)},\end{aligned}$$

$$\text{(5.15)}\qquad \begin{aligned}&\frac{1}{2}|\partial_t \underline{\boldsymbol{u}}_k|^2_{\mathbb{L}^2(\Omega)} + \frac{\nu}{2}\frac{d}{dt}|\nabla \underline{\boldsymbol{u}}_k|^2_{\mathbb{L}^2(\Omega)} + \frac{a}{2}\frac{d}{dt}|\underline{\boldsymbol{u}}_k|^2_{\mathbb{L}^2(\Omega)}\\ &\qquad \leqslant \frac{3|\boldsymbol{g}|^2}{2}|\underline{T}_k|^2_{L^2(\Omega)} + \frac{3|\boldsymbol{h}|^2}{2}|\underline{C}_k|^2_{L^2(\Omega)} + \frac{3}{2}|\boldsymbol{F}_1|^2_{\mathbb{L}^2(\Omega)}.\end{aligned}$$

Here we remark that $|\underline{T}_k|^2_{L^2(0,S;L^2(\Omega))}$ and $|\underline{C}_k|^2_{L^2(0,S;L^2(\Omega))}$ possess uniform bounds independent of $k \in \mathbb{N}$. Then, by integrating inequalities (5.13), (5.14) and (5.15) over $[0, S]$, we obtain the following estimates for $\underline{\boldsymbol{u}}_k$:

$$\text{(5.16)}\qquad \int_0^S |\underline{\boldsymbol{u}}_k(s)|^2_{\mathbb{H}^1(\Omega)}\,ds + \int_0^S |\mathcal{A}_\Omega \underline{\boldsymbol{u}}_k(s)|^2_{\mathbb{L}^2(\Omega)}\,ds + \int_0^S |\partial_t \underline{\boldsymbol{u}}_k(s)|^2_{\mathbb{L}^2(\Omega)} ds \leqslant \gamma_1.$$

Here and henceforth, γ_1 designates some general constant independent of $k \in \mathbb{N}$. Since $\underline{\boldsymbol{u}}_k \in C([0, S]; \mathbb{H}^1_\sigma(\Omega))$, there exists $t_0^k \in [0, S]$ where $|\underline{\boldsymbol{u}}_k(\cdot)|_{\mathbb{H}^1(\Omega)}$ attains its minimum. From (5.16), we can immediately derive

$$\left|\underline{\boldsymbol{u}}_k(t_0^k)\right|^2_{\mathbb{H}^1(\Omega)} \leqslant \frac{1}{S}\int_0^S |\underline{\boldsymbol{u}}_k(s)|^2_{\mathbb{H}^1(\Omega)}\,ds \leqslant \gamma_1.$$

Integrating (5.13) and (5.14) over $[t_0^k, t]$ with $t \in [t_0^k, t_0^k + S]$, we obtain

$$\text{(5.17)}\qquad \sup_{0\leqslant t\leqslant S} |\underline{\boldsymbol{u}}_k(t)|^2_{\mathbb{H}^1(\Omega)} \leqslant \gamma_1.$$

By virtue of (5.16), (5.17) and Ascoli's theorem (Proposition 2.5.3), we can extract a subsequence $\{\underline{\boldsymbol{u}}_{k_j}\}_{j\in\mathbb{N}}$ which converges strongly in $C_\pi([0, S]; \mathbb{L}^2_\sigma(\Omega))$. Let $\boldsymbol{u}'$ designate its limit. From (5.16) again, we can derive

$$\begin{aligned}\mathcal{A}_\Omega \underline{\boldsymbol{u}}_{k_j} &\rightharpoonup \mathcal{A}_\Omega \boldsymbol{u}' &&\text{weakly in } L^2(0, S; \mathbb{L}^2_\sigma(\Omega)),\\ \partial_t \underline{\boldsymbol{u}}_{k_j} &\rightharpoonup \partial_t \boldsymbol{u}' &&\text{weakly in } L^2(0, S; \mathbb{L}^2_\sigma(\Omega)).\end{aligned}$$

Recalling that $\underline{\boldsymbol{u}}_{k_j}$ is a solution of

$$\partial_t \underline{\boldsymbol{u}}_{k_j} = -\nu \mathcal{A}_\Omega \underline{\boldsymbol{u}}_{k_j} - a\underline{\boldsymbol{u}}_{k_j} + \mathcal{P}_\Omega \boldsymbol{g}\underline{T}_{k_j} + \mathcal{P}_\Omega \boldsymbol{h}\underline{C}_{k_j} + \mathcal{P}_\Omega \boldsymbol{F}_1$$

and taking the limit as $j \to \infty$, we can show that $\boldsymbol{u}'$ becomes a periodic solution of the following equation:

$$\partial_t \boldsymbol{u}' = -\nu \mathcal{A}_\Omega \boldsymbol{u}' - a\boldsymbol{u}' + \mathcal{P}_\Omega \boldsymbol{g}\underline{T} + \mathcal{P}_\Omega \boldsymbol{h}\underline{C} + \mathcal{P}_\Omega \boldsymbol{F}_1.$$

Due to the uniqueness of periodic solution of (5.5), $\boldsymbol{u}'$ coincides with $\underline{\boldsymbol{u}}$. Since arguments above do not depend on the choice of subsequences, we can assure that $\underline{\boldsymbol{u}}_k \to \underline{\boldsymbol{u}}$ strongly in $C_\pi([0, S]; \mathbb{L}^2_\sigma(\Omega))$.

Multiplying

$$\partial_t \overline{T}_k - \Delta \overline{T}_k + \underline{\boldsymbol{u}}_k \cdot \nabla \overline{T}_k + \lambda \overline{T}_k = F_2 \tag{5.18}$$

by $\overline{T}_k$, we get

$$\begin{aligned} &\frac{d}{dt}|\overline{T}_k|^2_{L^2(\Omega)} + |\nabla \overline{T}_k|^2_{L^2(\Omega)} \leqslant \kappa |F_2|^2_{L^2(\Omega)} \\ \Rightarrow &\int_0^S |\overline{T}_k(s)|^2_{H^1(\Omega)} ds \leqslant \gamma_1. \end{aligned} \tag{5.19}$$

Since $\overline{T}_k \in C([0,S]; H^1_0(\Omega))$, there exists $t^k_1 \in [0,S]$ such that $|\overline{T}_k(t^k_1)|_{H^1(\Omega)} \leqslant \gamma_1$. Then, from (5.19), we obtain

$$\sup_{0 \leqslant t \leqslant S} \left|\overline{T}_k(t)\right|^2_{L^2(\Omega)} \leqslant \gamma_1.$$

Next multiplying (5.18) by $-\Delta \overline{T}_k$ and $\partial_t \overline{T}_k$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\nabla \overline{T}_k|^2_{L^2(\Omega)} + \frac{1}{2}|\Delta \overline{T}_k|^2_{L^2(\Omega)} \leqslant |\underline{\boldsymbol{u}}_k \cdot \nabla \overline{T}_k|^2_{L^2(\Omega)} + |F_2|^2_{L^2(\Omega)}, \\ &\frac{1}{2}|\partial_t \overline{T}_k|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}|\nabla \overline{T}_k|^2_{L^2(\Omega)} + \frac{\lambda}{2}\frac{d}{dt}|\overline{T}_k|^2_{L^2(\Omega)} \leqslant |\underline{\boldsymbol{u}}_k \cdot \nabla \overline{T}_k|^2_{L^2(\Omega)} + |F_2|^2_{L^2(\Omega)}. \end{aligned}$$

We here recall the following estimates (see (4.5) in Section 4.2 and use Poincaré's inequality).

$$\begin{aligned} |\boldsymbol{w} \cdot \nabla U|^2_{L^2(\Omega)} &\leqslant |\boldsymbol{w}|^2_{\mathbb{L}^8(\Omega)} |\nabla U|^2_{L^{8/3}(\Omega)} \leqslant \eta |\boldsymbol{w}|^2_{\mathbb{W}^{1,8/3}(\Omega)} |\nabla U|^2_{L^{8/3}(\Omega)} \\ &\leqslant \eta |\boldsymbol{w}|_{\mathbb{W}^{1,2}(\Omega)} |\boldsymbol{w}|_{\mathbb{W}^{1,4}(\Omega)} |\nabla U|_{L^2(\Omega)} |\nabla U|_{L^4(\Omega)} \\ &\leqslant \eta |\boldsymbol{w}|_{\mathbb{H}^1(\Omega)} |\boldsymbol{w}|_{\mathbb{H}^2(\Omega)} |\nabla U|_{L^2(\Omega)} |U|_{H^2(\Omega)} \\ &\leqslant \eta |\nabla \boldsymbol{w}|_{\mathbb{L}^2(\Omega)} |\mathcal{A}_\Omega \boldsymbol{w}|_{\mathbb{L}^2(\Omega)} |\nabla U|_{L^2(\Omega)} |\Delta U|_{L^2(\Omega)}, \end{aligned} \tag{5.20}$$

where $\boldsymbol{w} \in D(\mathcal{A}_\Omega)$, $U \in D(-\Delta_D)$ and η is some suitable constant. From this inequality, we can derive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\nabla \overline{T}_k|^2_{L^2(\Omega)} + \frac{1}{4}|\Delta \overline{T}_k|^2_{L^2(\Omega)} \\ &\qquad \leqslant \gamma_1 |\nabla \underline{\boldsymbol{u}}_k|^2_{\mathbb{L}^2(\Omega)} |\mathcal{A}_\Omega \underline{\boldsymbol{u}}_k|^2_{\mathbb{L}^2(\Omega)} |\nabla \overline{T}_k|^2_{L^2(\Omega)} + |F_2|^2_{L^2(\Omega)}, \\ &\frac{1}{4}|\partial_t \overline{T}_k|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}|\nabla \overline{T}_k|^2_{L^2(\Omega)} + \frac{\lambda}{2}\frac{d}{dt}|\overline{T}_k|^2_{L^2(\Omega)} \\ &\qquad \leqslant \gamma_1 |\nabla \underline{\boldsymbol{u}}_k|_{\mathbb{L}^2(\Omega)} |\mathcal{A}_\Omega \underline{\boldsymbol{u}}_k|_{\mathbb{L}^2(\Omega)} |\nabla \overline{T}_k|_{L^2(\Omega)} |\Delta \overline{T}_k|_{L^2(\Omega)} + |F_2|^2_{L^2(\Omega)}, \end{aligned}$$

which, together with (5.16) and (5.17), yield

$$\sup_{0 \leqslant t \leqslant S} \left|\overline{T}_k(t)\right|^2_{H^1(\Omega)} + \int_0^S \left|\Delta \overline{T}_k(s)\right|^2_{L^2(\Omega)} ds + \int_0^S |\partial_t \overline{T}_k(s)|^2_{L^2(\Omega)} ds \leqslant \gamma_1. \tag{5.21}$$

Similarly, multiplications of

$$\partial_t\overline{C}_k - \Delta\overline{C}_k + \underline{\boldsymbol{u}}_k\cdot\nabla\overline{C}_k + \lambda\overline{C}_k = \rho\Delta\overline{T}_k + F_3$$

by $\overline{C}_k$, $-\Delta\overline{C}_k$ and $\partial_t\overline{C}_k$ give

$$\begin{aligned}
&\frac{d}{dt}|\overline{C}_k|^2_{L^2(\Omega)} + |\nabla\overline{C}_k|^2_{L^2(\Omega)} \leqslant 2\kappa|F_3|^2_{L^2(\Omega)} + 2\rho^2|\nabla\overline{T}_k|^2_{L^2(\Omega)},\\
&\frac{1}{2}\frac{d}{dt}|\nabla\overline{C}_k|^2_{L^2(\Omega)} + \frac{1}{4}|\Delta\overline{C}_k|^2_{L^2(\Omega)}\\
&\qquad\leqslant \gamma_1|\nabla\underline{\boldsymbol{u}}_k|^2_{\mathbb{L}^2(\Omega)}|\mathcal{A}_\Omega\underline{\boldsymbol{u}}_k|^2_{\mathbb{L}^2(\Omega)}|\nabla\overline{C}_k|^2_{L^2(\Omega)} + \frac{3\rho^2}{2}|\Delta\overline{T}_k|^2_{L^2(\Omega)} + \frac{3}{2}|F_3|^2_{L^2(\Omega)},\\
&\frac{1}{4}|\partial_t\overline{C}_k|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}|\nabla\overline{C}_k|^2_{L^2(\Omega)} + \frac{\lambda}{2}\frac{d}{dt}|\overline{C}_k|^2_{L^2(\Omega)}\\
&\qquad\leqslant \gamma_1|\nabla\underline{\boldsymbol{u}}_k|_{\mathbb{L}^2(\Omega)}|\mathcal{A}_\Omega\underline{\boldsymbol{u}}_k|_{\mathbb{L}^2(\Omega)}|\nabla\overline{C}_k|_{L^2(\Omega)}|\Delta\overline{C}_k|_{L^2(\Omega)} + \frac{3\rho^2}{2}|\Delta\overline{T}_k|_{L^2(\Omega)} + \frac{3}{2}|F_3|^2_{L^2(\Omega)},
\end{aligned}$$

which yield

$$\sup_{0\leqslant t\leqslant S}\left|\overline{C}_k(t)\right|^2_{H^1(\Omega)} + \int_0^S \left|\Delta\overline{C}_k(s)\right|^2_{L^2(\Omega)} ds + \int_0^S |\partial_t\overline{C}_k(s)|^2_{L^2(\Omega)} ds \leqslant \gamma_1. \tag{5.22}$$

From (5.21) and (5.22), there exist subsequences $\{\overline{T}_{k_j}\}_{j\in\mathbb{N}}$ and $\{\overline{C}_{k_j}\}_{j\in\mathbb{N}}$ which converge strongly in $C_\pi([0,S];L^2(\Omega))$. Let T' and C' designate their limits as $j\to\infty$ respectively. Furthermore (5.21) and (5.22) yield

$$\begin{aligned}
&\Delta\overline{T}_{k_j} \rightharpoonup \Delta T', \qquad \partial_t\overline{T}_{k_j} \rightharpoonup \partial_t T' \qquad \text{weakly in } L^2(0,S;L^2(\Omega)),\\
&\Delta\overline{C}_{k_j} \rightharpoonup \Delta C', \qquad \partial_t\overline{C}_{k_j} \rightharpoonup \partial_t C' \qquad \text{weakly in } L^2(0,S;L^2(\Omega)).
\end{aligned}$$

We can also assure that $\{\underline{\boldsymbol{u}}_{k_j}\cdot\nabla\overline{T}_{k_j}\}_{j\in\mathbb{N}}$ and $\{\underline{\boldsymbol{u}}_{k_j}\cdot\nabla\overline{C}_{k_j}\}_{j\in\mathbb{N}}$ weakly converge to $\underline{\boldsymbol{u}}\cdot\nabla T'$ and $\underline{\boldsymbol{u}}\cdot\nabla C'$ in $L^2(0,S;L^2(\Omega))$ respectively. Indeed, for any $\phi\in C_0^\infty(\Omega\times(0,S))$,

$$\int_\Omega \phi\underline{\boldsymbol{u}}_{k_j}\cdot\nabla\overline{T}_{k_j}dx = -\int_\Omega \overline{T}_{k_j}\underline{\boldsymbol{u}}_{k_j}\cdot\nabla\phi dx \xrightarrow[j\to\infty]{} -\int_\Omega T'\underline{\boldsymbol{u}}\cdot\nabla\phi dx = -\int_\Omega \phi\underline{\boldsymbol{u}}\cdot\nabla T' dx$$

holds by virtue of the strong convergences of $\{\underline{\boldsymbol{u}}_{k_j}\}_{j\in\mathbb{N}}$ and $\{\overline{T}_{k_j}\}_{j\in\mathbb{N}}$ (recall our argument for Check of (A2), in Chapter 3). Hence, taking the limit as $j\to\infty$ in

$$\begin{aligned}
&\partial_t\overline{T}_{k_j} - \Delta\overline{T}_{k_j} + \underline{\boldsymbol{u}}_{k_j}\cdot\nabla\overline{T}_{k_j} + \lambda\overline{T}_{k_j} = F_2,\\
&\partial_t\overline{C}_{k_j} - \Delta\overline{C}_{k_j} + \underline{\boldsymbol{u}}_{k_j}\cdot\nabla\overline{C}_{k_j} + \lambda\overline{C}_{k_j} = \rho\Delta\overline{T}_{k_j} + F_3,
\end{aligned}$$

we can see that (T',C') becomes a periodic solution of the following equations:

$$\begin{aligned}
&\partial_t T' - \Delta T' + \underline{\boldsymbol{u}}\cdot\nabla T' + \lambda T' = F_2,\\
&\partial_t C' - \Delta C' + \underline{\boldsymbol{u}}\cdot\nabla C' + \lambda C' = \rho\Delta T' + F_3.
\end{aligned}$$

Since the periodic solution of (5.7) is unique, we can show that $T' = \overline{T}$ and $C' = \overline{C}$, i.e., $\{\overline{T}_k\}_{k\in\mathbb{N}}$ and $\{\overline{C}_k\}_{k\in\mathbb{N}}$ strongly converge to $\overline{T}$ and $\overline{C}$ in $C_\pi([0,S]; L^2(\Omega))$ respectively. This implies the continuity of the mapping $\Psi\circ\Phi$ under the weak topology of X_S.

Thus, applying Schauder–Tychonoff's fixed point theorem (Proposition 2.5.2) to the mapping $\Psi\circ\Phi$ on K endowed with the weak topology of $L^2(0,S;L^2(\Omega))\times L^2(0,S;L^2(\Omega))$, we can guarantee the existence of a periodic solution of (5.3). □

5.3 Relaxation Problem in $\mathbb{R}^N$

Next we consider the following periodic problems in $\mathbb{R}^N$ with relaxation terms λT and λC.

Lemma 5.3.1. *Let $N = 3$ or 4 and assume that $\boldsymbol{f}_1 \in L^2(0,S;\mathbb{L}^2(\mathbb{R}^N))$ and $f_2, f_3 \in L^2(0,S;L^2(\mathbb{R}^N))$. Then for any positive constants a and λ, the following system* (5.23) *possesses at least one periodic solution $(\boldsymbol{u}, T, C)$.*

$$(5.23)\qquad \begin{cases} \partial_t \boldsymbol{u} + \nu\mathcal{A}_{\mathbb{R}^N}\boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_{\mathbb{R}^N}\boldsymbol{g}T + \mathcal{P}_{\mathbb{R}^N}\boldsymbol{h}C + \mathcal{P}_{\mathbb{R}^N}\boldsymbol{f}_1 & (x,t)\in\mathbb{R}^N\times[0,S],\\ \partial_t T + \boldsymbol{u}\cdot\nabla T + \lambda T = \Delta T + f_2 & (x,t)\in\mathbb{R}^N\times[0,S],\\ \partial_t C + \boldsymbol{u}\cdot\nabla C + \lambda C = \Delta C + \rho\Delta T + f_3 & (x,t)\in\mathbb{R}^N\times[0,S]. \end{cases}$$

Here $(\boldsymbol{u}, T, C)$ is called periodic solution of (5.23), *if*

1. *$(\boldsymbol{u}, T, C)$ satisfies*

$$(5.24)\qquad \begin{aligned} &\boldsymbol{u} \in C_\pi([0,S];\mathbb{H}^1_\sigma(\mathbb{R}^N)) \cap L^2(0,S;\mathbb{H}^2(\mathbb{R}^N)) \cap W^{1,2}(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N)),\\ &T, C \in C_\pi([0,S];H^1(\mathbb{R}^N)) \cap L^2(0,S;H^2(\mathbb{R}^N)) \cap W^{1,2}(0,S;L^2(\mathbb{R}^N)). \end{aligned}$$

2. *$(\boldsymbol{u}, T, C)$ satisfies the second and the third equation of* (5.23) *in $L^2(0,S;L^2(\mathbb{R}^N))$.*

3. *$(\boldsymbol{u}, T, C)$ satisfies the first equation of* (5.23) *in $L^2(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N))$.*

Proof. According to Lemma 5.2.1, for each natural number $n\in\mathbb{N}$ and positive number λ, the following equations $(\mathrm{DCBF})_{n,\lambda}$ possess a periodic solution $(\boldsymbol{u}_n, T_n, C_n)$:

$$(\mathrm{DCBF})_{n,\lambda}\begin{cases} \partial_t \boldsymbol{u}_n + \nu\mathcal{A}_{\Omega_n}\boldsymbol{u}_n + a\boldsymbol{u}_n = \mathcal{P}_{\Omega_n}\boldsymbol{g}T_n + \mathcal{P}_{\Omega_n}\boldsymbol{h}C_n + \mathcal{P}_{\Omega_n}\boldsymbol{f}_1|_{\Omega_n} & (x,t)\in\Omega_n\times[0,S],\\ \partial_t T_n + \boldsymbol{u}_n\cdot\nabla T_n + \lambda T_n = \Delta T_n + f_2|_{\Omega_n} & (x,t)\in\Omega_n\times[0,S],\\ \partial_t C_n + \boldsymbol{u}_n\cdot\nabla C_n + \lambda C_n = \Delta C_n + \rho\Delta T_n + f_3|_{\Omega_n} & (x,t)\in\Omega_n\times[0,S],\\ \boldsymbol{u}_n = 0,\ T_n = 0,\ C_n = 0 & (x,t)\in\partial\Omega_n\times[0,S], \end{cases}$$

where $(\boldsymbol{u}_n, T_n, C_n)$ satisfies

$$(5.25)\qquad \begin{aligned} &\boldsymbol{u}_n \in C_\pi([0,S];\mathbb{H}^1_\sigma(\Omega_n)) \cap L^2(0,S;\mathbb{H}^2(\Omega_n)) \cap W^{1,2}(0,S;\mathbb{L}^2_\sigma(\Omega_n)),\\ &T_n, C_n \in C_\pi([0,S];H^1_0(\Omega_n)) \cap L^2(0,S;H^2(\Omega_n)) \cap W^{1,2}(0,S;L^2(\Omega_n)). \end{aligned}$$

To begin with, we establish some a priori estimates for $(\boldsymbol{u}_n, T_n, C_n)$ independent of n. Throughout this section, γ_2 stands for a general constant independent of n. Multiplying the second equation of $(\mathrm{DCBF})_{n,\lambda}$ by T_n, we have

$$\frac{1}{2}\frac{d}{dt}|T_n|^2_{L^2(\Omega_n)} + |\nabla T_n|^2_{L^2(\Omega_n)} + \lambda |T_n|^2_{L^2(\Omega_n)} \leqslant \frac{1}{2\lambda}|f_2|_{\Omega_n}|^2_{L^2(\Omega_n)} + \frac{\lambda}{2}|T_n|^2_{L^2(\Omega_n)},$$

namely,

$$\frac{d}{dt}|T_n|^2_{L^2(\Omega_n)} + 2|\nabla T_n|^2_{L^2(\Omega_n)} + \lambda |T_n|^2_{L^2(\Omega_n)} \leqslant \frac{1}{\lambda}|f_2|_{\Omega_n}|^2_{L^2(\Omega_n)} \leqslant \frac{1}{\lambda}|f_2|^2_{L^2(\mathbb{R}^N)}. \tag{5.26}$$

Integrating (5.26) over $[0, S]$ and recalling $T(\cdot, 0) = T(\cdot, S)$, we get

$$\int_0^S |T_n(s)|^2_{H^1(\Omega_n)} ds \leqslant \gamma_2. \tag{5.27}$$

Then, by the continuity of T_n, there exists $t_2^n \in [0, S]$ satisfying

$$|T_n(t_2^n)|^2_{H^1(\Omega_n)} = \min_{0 \leqslant t \leqslant S} |T_n(t)|^2_{H^1(\Omega_n)}.$$

From (5.27) again, we can derive

$$|T_n(t_2^n)|^2_{H^1(\Omega_n)} \leqslant \frac{1}{S}\int_0^S |T_n(s)|^2_{H^1(\Omega_n)} ds \leqslant \gamma_2. \tag{5.28}$$

Then integrating (5.26) over $[t_2^n, t]$ ($t \in [t_2^n, t_2^n + S]$) and using the boundedness (5.28), we obtain

$$\sup_{0 \leqslant t \leqslant S} |T_n(t)|^2_{L^2(\Omega_n)} \leqslant \gamma_2. \tag{5.29}$$

Multiplying the third equation of $(\mathrm{DCBF})_{n,\lambda}$ by C_n and repeating almost the same procedure as above, we have

$$\frac{d}{dt}|C_n|^2_{L^2(\Omega_n)} + |\nabla C_n|^2_{L^2(\Omega_n)} + \lambda |C_n|^2_{L^2(\Omega_n)} \leqslant \rho^2 |\nabla T_n|^2_{L^2(\Omega_n)} + \frac{1}{\lambda}|f_3|^2_{L^2(\mathbb{R}^N)},$$

which yields

$$\int_0^S |C_n(s)|^2_{H^1(\Omega_n)} ds \leqslant \gamma_2$$

and

$$\sup_{0 \leqslant t \leqslant S} |C_n(t)|^2_{L^2(\Omega_n)} \leqslant \gamma_2. \tag{5.30}$$

Moreover, multiplying the first equation of $(\mathrm{DCBF})_{n,\lambda}$ by $\boldsymbol{u}_n$, $\mathcal{A}_{\Omega_n}\boldsymbol{u}_n$ and $\partial_t\boldsymbol{u}_n$ (see (5.13), (5.14) and (5.15)), we have

$$\begin{aligned}
&\frac{d}{dt}|\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} + 2\nu|\nabla\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} + a|\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)}\\
&\qquad\leqslant \frac{3|\boldsymbol{g}|^2}{a}|T_n|^2_{L^2(\Omega_n)} + \frac{3|\boldsymbol{h}|^2}{a}|C_n|^2_{L^2(\Omega_n)} + \frac{3}{a}|\boldsymbol{f}_1|^2_{\mathbb{L}^2(\mathbb{R}^N)},\\
&\frac{d}{dt}|\nabla\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} + \nu|\mathcal{A}_{\Omega_n}\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)}\\
&\qquad\leqslant \frac{3|\boldsymbol{g}|^2}{\nu}|T_n|^2_{L^2(\Omega_n)} + \frac{3|\boldsymbol{h}|^2}{\nu}|C_n|^2_{L^2(\Omega_n)} + \frac{3}{\nu}|\boldsymbol{f}_1|^2_{\mathbb{L}^2(\mathbb{R}^N)},\\
&|\partial_t\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} + \nu\frac{d}{dt}|\nabla\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} + a\frac{d}{dt}|\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)}\\
&\qquad\leqslant 3|\boldsymbol{g}|^2|T_n|^2_{L^2(\Omega_n)} + 3|\boldsymbol{h}|^2|C_n|^2_{L^2(\Omega_n)} + 3|\boldsymbol{f}_1|^2_{\mathbb{L}^2(\mathbb{R}^N)},
\end{aligned}$$

which immediately lead to

$$\int_0^S |\boldsymbol{u}_n(s)|^2_{\mathbb{H}^1(\Omega_n)}ds + \int_0^S |\mathcal{A}_{\Omega_n}\boldsymbol{u}_n(s)|^2_{\mathbb{L}^2(\Omega_n)}ds + \int_0^S |\partial_t\boldsymbol{u}_n(s)|^2_{\mathbb{L}^2(\Omega_n)}ds \leqslant \gamma_2 \tag{5.31}$$

and

$$\sup_{0\leqslant t\leqslant S}|\boldsymbol{u}_n(t)|^2_{\mathbb{H}^1(\Omega_n)} \leqslant \gamma_2. \tag{5.32}$$

We here prepare the following lemma concerning the elliptic estimate and the estimate for convection terms so that we can accomplish second energy estimates of T_n and C_n.

Lemma 5.3.2. *Let $R > 0$ and let $\boldsymbol{w} \in \mathbb{H}^2(\Omega_R) \cap \mathbb{H}^1_\sigma(\Omega_R)$ and $U \in H^2(\Omega_R) \cap H^1_0(\Omega_R)$. Then there exist some constant β which is independent of R such that the following inequalities hold:*

$$|\boldsymbol{w}\cdot\nabla U|^2_{L^2(\Omega_R)} \leqslant \beta|\nabla\boldsymbol{w}|^2_{\mathbb{L}^2(\Omega_R)}|\nabla U|_{L^2(\Omega_R)}|\Delta U|_{L^2(\Omega_R)}, \tag{5.33}$$

if $N = 3$.

$$|\boldsymbol{w}\cdot\nabla U|^2_{L^2(\Omega_R)} \leqslant \beta|\nabla\boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)}|\mathcal{A}_{\Omega_R}\boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)}|\nabla U|_{L^2(\Omega_R)}|\Delta U|_{L^2(\Omega_R)}, \tag{5.34}$$

if $N = 4$.

$$\left|\partial_{x_\iota}\partial_{x_\mu}U\right|_{L^2(\Omega_R)} \leqslant \beta\,|\Delta U|_{L^2(\Omega_R)}\,,\quad \left|\partial_{x_\iota}\partial_{x_\mu}\boldsymbol{w}\right|_{\mathbb{L}^2(\Omega_R)} \leqslant \beta\,|\mathcal{A}_{\Omega_R}\boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)} \tag{5.35}$$

for $N = 3, 4$ and for all $\iota, \mu = 1, 2, \cdots, N$.

Proof of Lemma 5.3.2. Let $N = 3$, then Hölder's inequality yields

$$|\boldsymbol{w}\cdot\nabla U|^2_{L^2(\Omega_R)} \leqslant |\boldsymbol{w}|^2_{\mathbb{L}^6(\Omega_R)}|\nabla U|_{L^2(\Omega_R)}|\nabla U|_{L^6(\Omega_R)}.$$

By Sobolev's inequality, $\boldsymbol{w} \in \mathbb{H}^1_\sigma(\Omega_R)$ satisfies $|\boldsymbol{w}|^2_{\mathbb{L}^6(\Omega_R)} \leqslant \gamma |\nabla \boldsymbol{w}|^2_{\mathbb{L}^2(\Omega_R)}$ with some constant γ independent of R. Moreover, by using Sobolev's inequality, elliptic estimate and Poincaré's inequality, we have

$$|\nabla U|_{L^6(\Omega_R)} \leqslant \beta_{\Omega_R} |U|_{H^2(\Omega_R)} \leqslant \beta_{\Omega_R} |\Delta U|_{L^2(\Omega_R)} \tag{5.36}$$

for $U \in H^2(\Omega_R) \cap H^1_0(\Omega_R)$, where β_{Ω_R} is a suitable constant which may depend on the radius R. Therefore, we can assure that (5.33) holds with $\beta = \gamma \beta_{\Omega_R}$ and we only have to show that the coefficient β can be taken independently of R.

For any $U \in H^2(\Omega_R) \cap H^1_0(\Omega_R)$, we define $U_R \in H^2(\Omega_1) \cap H^1_0(\Omega_1)$ by $U_R(y) := U(Ry)$, where $y \in \Omega_1$. We here remark that,

$$\partial_{x_\mu} U(x) = \frac{1}{R} \partial_{y_\mu} U_R(y) \quad (\mu = 1, 2, 3)$$

holds under the change of variable $y = x/R$ $(x \in \Omega_R,\ y \in \Omega_1)$. Therefore, we get

$$|\nabla_x U|^6_{L^6(\Omega_R)} = \int_{\Omega_1} \sum_{\mu=1}^{3} \left| \frac{1}{R} \partial_{y_\mu} U_R(y) \right|^6 R^3 dy = R^{-3} |\nabla_y U_R|^6_{L^6(\Omega_1)},$$

$$|\Delta_x U|^2_{L^2(\Omega_R)} = \int_{\Omega_1} \left| \sum_{\mu=1}^{3} \frac{1}{R^2} \partial^2_{y_\mu} U(x) \right|^2 R^3 dy = R^{-1} |\Delta_y U_R|^2_{L^2(\Omega_1)}.$$

Then using (5.36) again with $R = 1$, we obtain

$$\begin{aligned} |\nabla_x U|_{L^6(\Omega_R)} &= R^{-1/2} |\nabla_y U_R|_{L^6(\Omega_1)} \\ &\leqslant R^{-1/2} \beta_{\Omega_1} |\Delta_y U_R|_{L^2(\Omega_1)} = R^{-1/2} \beta_{\Omega_1} R^{1/2} |\Delta_x U|_{L^2(\Omega_R)}, \end{aligned} \tag{5.37}$$

which implies that (5.33) is valid for any $R > 0$ with the coefficient $\beta = \gamma \beta_{\Omega_1}$, which is independent of R.

Let $N = 4$ and $U \in H^2(\Omega_R) \cap H^1_0(\Omega_R)$, $\boldsymbol{w} \in \mathbb{H}^2(\Omega_R) \cap \mathbb{H}^1_\sigma(\Omega_R)$. Then we have

$$\begin{aligned} |\boldsymbol{w} \cdot \nabla U|^2_{L^2(\Omega_R)} &\leqslant |\boldsymbol{w}|^2_{\mathbb{L}^8(\Omega_R)} |\nabla U|^2_{L^{8/3}(\Omega_R)} \\ &\leqslant |\boldsymbol{w}|^2_{\mathbb{L}^8(\Omega_R)} |\nabla U|_{L^2(\Omega_R)} |\nabla U|_{L^4(\Omega_R)}. \end{aligned} \tag{5.38}$$

Moreover, since

$$|\nabla U|_{L^4(\Omega_R)} \leqslant \beta_{\Omega_R} |U|_{H^2(\Omega_R)} \leqslant \beta_{\Omega_R} |\Delta U|_{L^2(\Omega_R)} \tag{5.39}$$

and

$$\begin{aligned} |\boldsymbol{w}|^2_{\mathbb{L}^8(\Omega_R)} &\leqslant \beta'_{\Omega_R} |\boldsymbol{w}|^2_{\mathbb{W}^{1,8/3}(\Omega_R)} \leqslant \beta'_{\Omega_R} |\boldsymbol{w}|_{\mathbb{W}^{1,2}(\Omega_R)} |\boldsymbol{w}|_{\mathbb{W}^{1,4}(\Omega_R)} \\ &\leqslant \beta'_{\Omega_R} |\boldsymbol{w}|_{\mathbb{H}^1(\Omega_R)} |\boldsymbol{w}|_{\mathbb{H}^2(\Omega_R)} \leqslant \beta'_{\Omega_R} |\nabla \boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)} |\mathcal{A}_{\Omega_R} \boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)} \end{aligned} \tag{5.40}$$

hold with some general constant β_{Ω_R} and β'_{Ω_R}, then (5.34) is valid for each $R > 0$ with $\beta = \beta_{\Omega_R} \beta'_{\Omega_R}$ (see (5.20)).

Here we define $U_R \in H^2(\Omega_1)\cap H_0^1(\Omega_1)$ and $\boldsymbol{w}_R \in \mathbb{H}^2(\Omega_1)\cap\mathbb{H}_\sigma^1(\Omega_1)$ by $U_R(y) := U(Ry)$ and $\boldsymbol{w}_R(y) := \boldsymbol{w}(Ry)$ $(y \in \Omega_1)$ respectively. Then, by the same reasoning as that for (5.37), i.e., from (5.39) and identities

$$|\nabla_x U|^4_{L^4(\Omega_R)} = \int_{\Omega_1} \sum_{\mu=1}^{4} \left| \frac{1}{R} \partial_{y_\mu} U_R(y) \right|^4 R^4 dy = |\nabla_y U_R|^4_{L^4(\Omega_1)},$$
$$|\Delta_x U|^2_{L^2(\Omega_R)} = \int_{\Omega_1} \left| \sum_{\mu=1}^{4} \frac{1}{R^2} \partial^2_{y_\mu} U(x) \right|^2 R^4 dy = |\Delta_y U_R|^2_{L^2(\Omega_1)},$$

we can derive

$$\begin{aligned} |\nabla_x U|_{L^4(\Omega_R)} &= |\nabla_y U_R|_{L^4(\Omega_1)} \\ &\leqslant \beta_{\Omega_1} |\Delta_y U_R|_{L^2(\Omega_1)} = \beta_{\Omega_1} |\Delta_x U|_{L^2(\Omega_R)}, \end{aligned} \tag{5.41}$$

where $x \in \Omega_R$ and $y \in \Omega_1$. Under the change of variable $y = x/R$, the following identities also hold:

$$\begin{aligned} |\boldsymbol{w}|^8_{\mathbb{L}^8(\Omega_R)} &= \int_{\Omega_1} |\boldsymbol{w}_R(y)|^8 R^4 dy = R^4 |\boldsymbol{w}_R|^8_{\mathbb{L}^8(\Omega_1)}, \\ |\nabla_x \boldsymbol{w}|^2_{\mathbb{L}^2(\Omega_R)} &= \int_{\Omega_1} \sum_{\mu=1}^{4} \left| \frac{1}{R} \partial_{y_\mu} \boldsymbol{w}_R(x) \right|^2 R^4 dy = R^2 |\nabla_y \boldsymbol{w}_R|^2_{\mathbb{L}^2(\Omega_1)}. \end{aligned} \tag{5.42}$$

We here remark that the following identity also can be verified:

$$\mathcal{P}_{\Omega_R} \Delta_x \boldsymbol{w}(x) = \frac{1}{R^2} \mathcal{P}_{\Omega_1} \Delta_y \boldsymbol{w}_R(y). \tag{5.43}$$

Indeed, since $\boldsymbol{w} \in \mathbb{H}^2(\Omega_R)$, the Helmholtz decomposition $\Delta_x \boldsymbol{w} = \boldsymbol{v}^1 + \boldsymbol{v}^2$ holds, where $\boldsymbol{v}^1 \in \mathbb{L}^2_\sigma(\Omega_R)$ and $\boldsymbol{v}^2 \in G_2(\Omega_R)$. We recall that the definition of $G_2(\Omega_R)$ implies that there exists $P \in W^{1,2}(\Omega_R)$ such that $\boldsymbol{v}^2 = \nabla_x P$. Let $\boldsymbol{v}^1_R(y) := \boldsymbol{v}^1(Ry)$, $\boldsymbol{v}^2_R(y) := \boldsymbol{v}^2(Ry)$ and $P_R(y) := P(Ry)$ with $y \in \Omega_1$. Then, $\boldsymbol{v}^1_R \in \mathbb{L}^2_\sigma(\Omega_1)$ is clear by the definition of $\mathbb{L}^2_\sigma(\Omega_R)$ and $\boldsymbol{v}^2_R \in G_2(\Omega_1)$ is also evident by $P_R \in W^{1,2}(\Omega_1)$. These facts yield the decomposition $\frac{1}{R^2}\Delta_y \boldsymbol{w}_R = \boldsymbol{v}^1_R + \boldsymbol{v}^2_R$ under the change of variable $y = x/R$. Therefore, since the decomposition is unique, we can assure the identity (5.43).

Then, (5.43) gives us

$$\begin{aligned} |\mathcal{A}_{\Omega_R} \boldsymbol{w}|^2_{\mathbb{L}^2(\Omega_R)} &= \int_{\Omega_R} |\mathcal{P}_{\Omega_R} \Delta_x \boldsymbol{w}(x)|^2 dx \\ &= \int_{\Omega_1} \left| \frac{1}{R^2} \mathcal{P}_{\Omega_1} \Delta_y \boldsymbol{w}_R(y) \right|^2 R^4 dy = |\mathcal{A}_{\Omega_1} \boldsymbol{w}_R|^2_{\mathbb{L}^2(\Omega_1)}. \end{aligned} \tag{5.44}$$

Recalling (5.40), together with (5.42) and (5.44), we obtain

$$\begin{aligned} |\boldsymbol{w}|^2_{\mathbb{L}^8(\Omega_R)} = R|\boldsymbol{w}_R|^2_{\mathbb{L}^8(\Omega_1)} &\leqslant R\beta'_{\Omega_1} |\nabla_y \boldsymbol{w}_R|_{\mathbb{L}^2(\Omega_1)} |\mathcal{A}_{\Omega_1} \boldsymbol{w}_R|_{\mathbb{L}^2(\Omega_1)} \\ &= R\beta'_{\Omega_1} R^{-1} |\nabla_x \boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)} |\mathcal{A}_{\Omega_R} \boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)}. \end{aligned} \tag{5.45}$$

Hence, by (5.38), (5.41) and (5.45), we can assure that the coefficient in (5.34) can be taken as $\beta = \beta_{\Omega_1}\beta'_{\Omega_1}$ for any R.

Let $\boldsymbol{w} \in \mathbb{H}^2(\Omega_R) \cap \mathbb{H}^1_\sigma(\Omega_R)$ and $U \in H^2(\Omega_R) \cap H^1_0(\Omega_R)$. Combining the elliptic estimates with Poincaré's inequality, we get

$$\left|\partial_{x_\iota}\partial_{x_\mu} U\right|_{L^2(\Omega_R)} \leqslant \beta_{\Omega_R} \left|\Delta U\right|_{L^2(\Omega_R)}, \qquad \left|\partial_{x_\iota}\partial_{x_\mu} \boldsymbol{w}\right|_{\mathbb{L}^2(\Omega_R)} \leqslant \beta_{\Omega_R} \left|\mathcal{A}_{\Omega_R}\boldsymbol{w}\right|_{\mathbb{L}^2(\Omega_R)}$$

for any $\iota, \mu = 1, 2, \cdots, N$. Then we can immediately assure that (5.35) holds with $\beta = \beta_{\Omega_1}$ for arbitrary R, since the identities

$$\begin{aligned}
&\left|\partial_{x_\iota}\partial_{x_\mu} U\right|_{L^2(\Omega_R)} = R^{\frac{N-4}{2}} \left|\partial_{y_\iota}\partial_{y_\mu} U_R\right|_{L^2(\Omega_1)}, \qquad \left|\Delta_x U\right|_{L^2(\Omega_R)} = R^{\frac{N-4}{2}} \left|\Delta_y U_R\right|_{L^2(\Omega_1)}, \\
&\left|\partial_{x_\iota}\partial_{x_\mu} \boldsymbol{w}\right|_{\mathbb{L}^2(\Omega_R)} = R^{\frac{N-4}{2}} \left|\partial_{y_\iota}\partial_{y_\mu} \boldsymbol{w}_R\right|_{\mathbb{L}^2(\Omega_1)}, \qquad \left|\mathcal{A}_{\Omega_R}\boldsymbol{w}\right|_{\mathbb{L}^2(\Omega_R)} = R^{\frac{N-4}{2}} \left|\mathcal{A}_{\Omega_1}\boldsymbol{w}_R\right|_{\mathbb{L}^2(\Omega_1)}
\end{aligned}$$

are valid (use (5.43) again). □

Proof of Lemma 5.3.1 (continued). Multiplying the second equation of $(\mathrm{DCBF})_{n,\lambda}$ by $-\Delta T_n$ and using (5.33) and (5.34), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + |\Delta T_n|^2_{L^2(\Omega_n)} \\
&\qquad \leqslant |\boldsymbol{u}_n \cdot \nabla T_n|_{L^2(\Omega_n)}|\Delta T_n|_{L^2(\Omega_n)} + |f_2|_{\Omega_n}|_{L^2(\Omega_n)}|\Delta T_n|_{L^2(\Omega_n)} \\
&\qquad \leqslant \gamma_2|\nabla \boldsymbol{u}_n|_{\mathbb{L}^2(\Omega_n)}|\nabla T_n|^{1/2}_{L^2(\Omega_n)}|\Delta T_n|^{3/2}_{L^2(\Omega_n)} + |f_2|_{\Omega_n}|_{L^2(\Omega_n)}|\Delta T_n|_{L^2(\Omega_n)} \\
&\qquad \leqslant \frac{1}{2}|\Delta T_n|^2_{L^2(\Omega_n)} + \gamma_2|\nabla \boldsymbol{u}_n|^4_{\mathbb{L}^2(\Omega_n)}|\nabla T_n|^2_{L^2(\Omega_n)} + |f_2|^2_{L^2(\mathbb{R}^N)} \\
&\Rightarrow \frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + |\Delta T_n|^2_{L^2(\Omega_n)} \leqslant \gamma_2|\nabla \boldsymbol{u}_n|^4_{\mathbb{L}^2(\Omega_n)}|\nabla T_n|^2_{L^2(\Omega_n)} + 2|f_2|^2_{L^2(\mathbb{R}^N)}
\end{aligned} \tag{5.46}$$

for $N = 3$ and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + |\Delta T_n|^2_{L^2(\Omega_n)} \\
&\quad \leqslant \gamma_2|\nabla \boldsymbol{u}_n|^{1/2}_{\mathbb{L}^2(\Omega_n)}|\mathcal{A}_{\Omega_n}\boldsymbol{u}_n|^{1/2}_{\mathbb{L}^2(\Omega_n)}|\nabla T_n|^{1/2}_{L^2(\Omega_n)}|\Delta T_n|^{3/2}_{L^2(\Omega_n)} + |f_2|_{L^2(\mathbb{R}^N)}|\Delta T_n|_{L^2(\Omega_n)} \\
&\quad \leqslant \frac{1}{2}|\Delta T_n|^2_{L^2(\Omega_n)} + \gamma_2|\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)}|\mathcal{A}_{\Omega_n}\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)}|\nabla T_n|^2_{L^2(\Omega_n)} + |f_2|^2_{L^2(\mathbb{R}^N)} \\
&\Rightarrow \frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + |\Delta T_n|^2_{L^2(\Omega_n)} \\
&\quad \leqslant \gamma_2|\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)}|\mathcal{A}_{\Omega_n}\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)}|\nabla T_n|^2_{L^2(\Omega_n)} + 2|f_2|^2_{L^2(\mathbb{R}^N)}
\end{aligned} \tag{5.47}$$

for $N = 4$. We here recall that there exist some $t_2^n \in [0, S]$ such that $|T_n(t_2^n)|^2_{H^1(\Omega_n)} \leqslant \gamma_2$ (see (5.28)). Applying Gronwall's inequality to (5.46) and (5.47) over $[t_2^n, t]$ with $t \in [t_2^n, t_2^n + S]$ and using the uniform boundedness of $\boldsymbol{u}_n$ ((5.31) and (5.32)), we obtain

$$\sup_{0\leqslant t\leqslant S} |\nabla T_n(t)|^2_{L^2(\Omega_n)} \leqslant \gamma_2. \tag{5.48}$$

Integrating (5.46) and (5.47) over $[0, S]$ and using (5.48), we have

$$\int_0^S |\Delta T_n(s)|^2_{L^2(\Omega_n)} ds \leqslant \gamma_2. \tag{5.49}$$

From (5.35) and (5.49), we can derive

$$\int_0^S |\partial_{x_\iota} \partial_{x_\mu} T_n(s)|^2_{L^2(\Omega_n)} ds \leqslant \gamma_2 \qquad \forall \iota, \forall \mu = 1, 2, \cdots, N. \tag{5.50}$$

Multiplying the second equation of $(\mathrm{DCBF})_{n,\lambda}$ by $\partial_t T_n$, we get

$$\begin{aligned}
&|\partial_t T_n|^2_{L^2(\Omega_n)} + \frac{1}{2}\frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + \frac{\lambda}{2}\frac{d}{dt}|T_n|^2_{L^2(\Omega_n)} \\
&\quad \leqslant \frac{1}{2}|\partial_t T_n|^2_{L^2(\Omega_n)} + |\boldsymbol{u}_n \cdot \nabla T_n|^2_{L^2(\Omega_n)} + |f_2|^2_{L^2(\mathbb{R}^N)} \\
&\quad \leqslant \frac{1}{2}|\partial_t T_n|^2_{L^2(\Omega_n)} + \gamma_2 |\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} |\nabla T_n|_{L^2(\Omega_n)} |\Delta T_n|_{L^2(\Omega_n)} + |f_2|^2_{L^2(\mathbb{R}^N)} \\
\Rightarrow &|\partial_t T_n|^2_{L^2(\Omega_n)} + \frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + \lambda \frac{d}{dt}|T_n|^2_{L^2(\Omega_n)} \\
&\quad \leqslant \gamma_2 |\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} |\nabla T_n|_{L^2(\Omega_n)} |\Delta T_n|_{L^2(\Omega_n)} + 2|f_2|^2_{L^2(\mathbb{R}^N)}
\end{aligned} \tag{5.51}$$

for $N = 3$ and

$$\begin{aligned}
&|\partial_t T_n|^2_{L^2(\Omega_n)} + \frac{1}{2}\frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + \frac{\lambda}{2}\frac{d}{dt}|T_n|^2_{L^2(\Omega_n)} \\
&\quad \leqslant \frac{1}{2}|\partial_t T_n|^2_{L^2(\Omega_n)} + |f_2|^2_{L^2(\mathbb{R}^N)} \\
&\qquad + \gamma_2 |\nabla \boldsymbol{u}_n|_{\mathbb{L}^2(\Omega_n)} |\mathcal{A}_{\Omega_n} \boldsymbol{u}_n|_{\mathbb{L}^2(\Omega_n)} |\nabla T_n|_{L^2(\Omega_n)} |\Delta T_n|_{L^2(\Omega_n)} \\
\Rightarrow &|\partial_t T_n|^2_{L^2(\Omega_n)} + \frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + \lambda \frac{d}{dt}|T_n|^2_{L^2(\Omega_n)} \\
&\quad \leqslant \gamma_2 |\nabla \boldsymbol{u}_n|_{\mathbb{L}^2(\Omega_n)} |\mathcal{A}_{\Omega_n} \boldsymbol{u}_n|_{\mathbb{L}^2(\Omega_n)} |\nabla T_n|_{L^2(\Omega_n)} |\Delta T_n|_{L^2(\Omega_n)} + 2|f_2|^2_{L^2(\mathbb{R}^N)}
\end{aligned} \tag{5.52}$$

for $N = 4$, which yield

$$\int_0^S |\partial_t T_n(s)|^2_{L^2(\Omega_n)} ds \leqslant \gamma_2. \tag{5.53}$$

By almost the same calculations as above, multiplication of the third equation by $-\Delta C_n$ and $\partial_t C_n$ give

$$\begin{aligned}
&\frac{d}{dt}|\nabla C_n|^2_{L^2(\Omega_n)} + |\Delta C_n|^2_{L^2(\Omega_n)} \\
&\qquad \leqslant \gamma_2 |\nabla \boldsymbol{u}_n|^4_{\mathbb{L}^2(\Omega_n)} |\nabla C_n|^2_{L^2(\Omega_n)} + 3\rho^2 |\Delta T_n|^2_{L^2(\Omega_n)} + 3|f_3|^2_{L^2(\mathbb{R}^N)}, \\
&|\partial_t C_n|^2_{L^2(\Omega_n)} + \frac{d}{dt}|\nabla C_n|^2_{L^2(\Omega_n)} + \lambda \frac{d}{dt}|C_n|^2_{L^2(\Omega_n)} \\
&\qquad \leqslant \gamma_2 |\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} |\nabla C_n|_{L^2(\Omega_n)} |\Delta C_n|_{L^2(\Omega_n)} + 3\rho^2 |\Delta T_n|^2_{L^2(\Omega_n)} + 3|f_3|^2_{L^2(\mathbb{R}^N)}
\end{aligned}$$

for $N=3$ and

$$\frac{d}{dt}|\nabla C_n|^2_{L^2(\Omega_n)} + |\Delta C_n|^2_{L^2(\Omega_n)}$$
$$\leqslant \gamma_2 |\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} |\mathcal{A}_{\Omega_n}\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} |\nabla C_n|^2_{L^2(\Omega_n)} + 3\rho^2 |\Delta T_n|^2_{L^2(\Omega_n)} + 3|f_3|^2_{L^2(\mathbb{R}^N)},$$
$$|\partial_t C_n|^2_{L^2(\Omega_n)} + \frac{d}{dt}|\nabla C_n|^2_{L^2(\Omega_n)} + \lambda\frac{d}{dt}|C_n|^2_{L^2(\Omega_n)}$$
$$\leqslant \gamma_2 |\nabla \boldsymbol{u}_n|_{\mathbb{L}^2(\Omega_n)} |\mathcal{A}_{\Omega_n}\boldsymbol{u}_n|_{\mathbb{L}^2(\Omega_n)} |\nabla C_n|_{L^2(\Omega_n)} |\Delta C_n|_{L^2(\Omega_n)} + 3\rho^2 |\Delta T_n|^2_{L^2(\Omega_n)} + 3|f_3|^2_{L^2(\mathbb{R}^N)}$$

for $N=4$. From (5.31), (5.32), (5.49) and (5.35), we can derive

$$(5.54)\qquad \begin{aligned} &\sup_{0\leqslant t\leqslant S} |\nabla C_n|^2_{L^2(\Omega_n)} + \int_0^S |\Delta C_n(s)|^2_{L^2(\Omega_n)} ds + \int_0^S |\partial_{x_\iota}\partial_{x_\mu} C_n(s)|^2_{L^2(\Omega_n)} ds \leqslant \gamma_2, \\ &\int_0^S |\partial_t C_n(s)|^2_{L^2(\Omega_n)} ds \leqslant \gamma_2 \end{aligned}$$

for any $\iota, \mu = 1, 2, \cdots, N$.

Let $\hat{\cdot}$ and $[\cdot]^\wedge$ designate the zero-extension of function to $\mathbb{R}^N$, i.e.,

$$\widehat{T_n}(x,t) = [T_n]^\wedge(x,t) := \begin{cases} T_n(x,t) & (\text{ if } x\in\Omega_n), \\ 0 & (\text{ otherwise }). \end{cases}$$

We remark that

$$\partial_t[\boldsymbol{u}_n]^\wedge = [\partial_t \boldsymbol{u}_n]^\wedge, \qquad \partial_t[T_n]^\wedge = [\partial_t T_n]^\wedge, \qquad \partial_t[C_n]^\wedge = [\partial_t C_n]^\wedge.$$

We also have

$$\nabla[\boldsymbol{u}_n]^\wedge = [\nabla \boldsymbol{u}_n]^\wedge, \qquad \nabla[T_n]^\wedge = [\nabla T_n]^\wedge, \qquad \nabla[C_n]^\wedge = [\nabla C_n]^\wedge,$$

since $\boldsymbol{u}_n(t) \in \mathbb{H}^1_\sigma(\Omega_n)$ and $T_n(t), C_n(t) \in H^1_0(\Omega_n)$ for any $t\in[0,S]$. Then, from (5.29), (5.30), (5.31), (5.32), (5.48), (5.49), (5.50), (5.53) and (5.54), we can derive

$$(5.55)\qquad \begin{aligned} &\sup_{0\leqslant t\leqslant S} \left|\widehat{T_n}(t)\right|^2_{H^1(\mathbb{R}^N)} + \int_0^S |[\Delta T_n]^\wedge(s)|^2_{L^2(\mathbb{R}^N)}\, ds + \int_0^S \left|\partial_t\widehat{T_n}(s)\right|^2_{L^2(\mathbb{R}^N)} ds \leqslant \gamma_2, \\ &\sup_{0\leqslant t\leqslant S} \left|\widehat{C_n}(t)\right|^2_{H^1(\mathbb{R}^N)} + \int_0^S |[\Delta C_n]^\wedge(s)|^2_{L^2(\mathbb{R}^N)}\, ds + \int_0^S \left|\partial_t\widehat{C_n}(s)\right|^2_{L^2(\mathbb{R}^N)} ds \leqslant \gamma_2, \\ &\sup_{0\leqslant t\leqslant S} |\widehat{\boldsymbol{u}_n}(t)|^2_{\mathbb{H}^1(\mathbb{R}^N)} + \int_0^S |[\mathcal{A}_{\Omega_n}\boldsymbol{u}_n]^\wedge(s)|^2_{\mathbb{L}^2(\mathbb{R}^N)}\, ds + \int_0^S |\partial_t\widehat{\boldsymbol{u}_n}(s)|^2_{\mathbb{L}^2(\mathbb{R}^N)} ds \leqslant \gamma_2 \end{aligned}$$

and

$$(5.56)\qquad \begin{aligned} &\int_0^S \left|[\partial_{x_\iota}\partial_{x_\mu} T_n]^\wedge(s)\right|^2_{L^2(\mathbb{R}^N)} ds \leqslant \gamma_2, \qquad \int_0^S \left|[\partial_{x_\iota}\partial_{x_\mu} C_n]^\wedge(s)\right|^2_{L^2(\mathbb{R}^N)} ds \leqslant \gamma_2, \\ &\int_0^S \left|[\partial_{x_\iota}\partial_{x_\mu} \boldsymbol{u}_n]^\wedge(s)\right|^2_{\mathbb{L}^2(\mathbb{R}^N)} ds \leqslant \gamma_2 \end{aligned}$$

for any $\iota, \mu = 1, 2, \cdots, N$. By (5.33) and (5.34), we also have

$$\int_0^S |[\boldsymbol{u}_n \cdot \nabla T_n]^\wedge(s)|^2_{L^2(\mathbb{R}^N)}\, ds + \int_0^S |[\boldsymbol{u}_n \cdot \nabla C_n]^\wedge(s)|^2_{L^2(\mathbb{R}^N)}\, ds \leqslant \gamma_2. \tag{5.57}$$

We here remark that

$$\boldsymbol{v} \in \mathbb{L}^2_\sigma(\Omega) \Rightarrow \widehat{\boldsymbol{v}} \in \mathbb{L}^2_\sigma(\mathbb{R}^N), \qquad \boldsymbol{v} \in \mathbb{H}^1_\sigma(\Omega) \Rightarrow \widehat{\boldsymbol{v}} \in \mathbb{H}^1_\sigma(\mathbb{R}^N)$$

hold for any $\Omega \subset \mathbb{R}^N$, due to the definition of $\mathbb{L}^2_\sigma(\Omega)$ and $\mathbb{H}^1_\sigma(\Omega)$. Therefore we obtain $\widehat{\boldsymbol{u}_n} \in C_\pi([0, S]; \mathbb{H}^1_\sigma(\mathbb{R}^N))$ and $\partial_t \widehat{\boldsymbol{u}_n}$, $[\mathcal{A}_{\Omega_n} \boldsymbol{u}_n]^\wedge \in L^2(0, S; \mathbb{L}^2_\sigma(\mathbb{R}^N))$ for each $n \in \mathbb{N}$.

By using (5.55), (5.56) and (5.57), we can assure that there exists a subsequence $\{(\widehat{\boldsymbol{u}_{n_i}},\ \widehat{T_{n_i}},\ \widehat{C_{n_i}})\}_{i \in \mathbb{N}}$ of $\{(\widehat{\boldsymbol{u}_n},\ \widehat{T_n},\ \widehat{C_n})\}_{n \in \mathbb{N}}$, which is simply denoted by $\{U_i\}_{i \in \mathbb{N}} := \{(\widehat{\boldsymbol{u}}_i,\ \widehat{T}_i,\ \widehat{C}_i)\}_{i \in \mathbb{N}}$, such that

$$\begin{aligned}
&\widehat{T}_i \rightharpoonup T_* && *\text{-weakly in } L^\infty(0, S; H^1(\mathbb{R}^N)),\\
&\partial_t \widehat{T}_i \rightharpoonup T_{**} && \text{weakly in } L^2(0, S; L^2(\mathbb{R}^N)),\\
&[\partial_{x_\iota} \partial_{x_\mu} T_i]^\wedge \rightharpoonup T^{\iota,\mu}_{***} && \text{weakly in } L^2(0, S; L^2(\mathbb{R}^N)) \qquad (\forall \iota, \forall \mu = 1, 2, \cdots N),\\
&[\Delta T_i]^\wedge \rightharpoonup T_{****} && \text{weakly in } L^2(0, S; L^2(\mathbb{R}^N)),
\end{aligned} \tag{5.58}$$

$$\begin{aligned}
&\widehat{C}_i \rightharpoonup C_* && *\text{-weakly in } L^\infty(0, S; H^1(\mathbb{R}^N)),\\
&\partial_t \widehat{C}_i \rightharpoonup C_{**} && \text{weakly in } L^2(0, S; L^2(\mathbb{R}^N)),\\
&[\partial_{x_\iota} \partial_{x_\mu} C_i]^\wedge \rightharpoonup C^{\iota,\mu}_{***} && \text{weakly in } L^2(0, S; L^2(\mathbb{R}^N)) \qquad (\forall \iota, \forall \mu = 1, 2, \cdots N),\\
&[\Delta C_i]^\wedge \rightharpoonup C_{****} && \text{weakly in } L^2(0, S; L^2(\mathbb{R}^N)),
\end{aligned} \tag{5.59}$$

$$\begin{aligned}
&\widehat{\boldsymbol{u}}_i \rightharpoonup \boldsymbol{u}_* && *\text{-weakly in } L^\infty(0, S; \mathbb{H}^1_\sigma(\mathbb{R}^N)),\\
&\partial_t \widehat{\boldsymbol{u}}_i \rightharpoonup \boldsymbol{u}_{**} && \text{weakly in } L^2(0, S; \mathbb{L}^2_\sigma(\mathbb{R}^N)),\\
&[\partial_{x_\iota} \partial_{x_\mu} \boldsymbol{u}_i]^\wedge \rightharpoonup \boldsymbol{u}^{\iota,\mu}_{***} && \text{weakly in } L^2(0, S; \mathbb{L}^2(\mathbb{R}^N)) \qquad (\forall \iota, \forall \mu = 1, 2, \cdots N),\\
&[\mathcal{A}_{\Omega_{n_i}} \boldsymbol{u}_i]^\wedge \rightharpoonup \boldsymbol{u}_{****} && \text{weakly in } L^2(0, S; \mathbb{L}^2_\sigma(\mathbb{R}^N)),
\end{aligned} \tag{5.60}$$

$$\begin{aligned}
&[\boldsymbol{u}_i \cdot \nabla T_i]^\wedge \rightharpoonup \chi_1 && \text{weakly in } L^2(0, S; L^2(\mathbb{R}^N)),\\
&[\boldsymbol{u}_i \cdot \nabla C_i]^\wedge \rightharpoonup \chi_2 && \text{weakly in } L^2(0, S; L^2(\mathbb{R}^N)).
\end{aligned} \tag{5.61}$$

Let $\phi_1 \in C^1_0((0, S); L^2(\mathbb{R}^N))$. Then

$$\int_0^S \int_{\mathbb{R}^N} \partial_t \widehat{T}_i \phi_1 dx dt = - \int_0^S \int_{\mathbb{R}^N} \widehat{T}_i \partial_t \phi_1 dx dt.$$

Since $\widehat{T}_i$ weakly converges to T_* in $L^2(0, S; L^2(\mathbb{R}^N))$, taking the limit as $i \to \infty$, we get

$$\int_0^S \int_{\mathbb{R}^N} T_{**} \phi_1 dx dt = - \int_0^S \int_{\mathbb{R}^N} T_* \partial_t \phi_1 dx dt$$

for any $\phi_1 \in C^1_0((0, S); L^2(\mathbb{R}^N))$, which implies that $T_{**} = \partial_t T_*$ holds in $L^2(0, S; L^2(\mathbb{R}^N))$.

Next we assume that $\phi_2 \in C_0^\infty(\mathbb{R}^N \times (0,S))$. Since ϕ_2 possesses a compact support in $\mathbb{R}^N \times (0,S)$, there exist some natural number $M \in \mathbb{N}$ such that $supp\phi_2 \subset \Omega_M \times (0,S)$ holds, where $supp\phi_2$ denotes the support of ϕ_2. Then we have

$$\begin{aligned}\int_0^S \int_{\mathbb{R}^N} \phi_2 [\partial_{x_\iota}\partial_{x_\mu} T_i]^\wedge &= \int_0^S \int_{\Omega_{n_i}} \phi_2|_{\Omega_{n_i}} \partial_{x_\iota}\partial_{x_\mu} T_i dxdt = -\int_0^S \int_{\Omega_{n_i}} \partial_{x_\iota}\left(\phi_2|_{\Omega_{n_i}}\right)\partial_{x_\mu} T_i dxdt \\ &= -\int_0^S \int_{\Omega_{n_i}} (\partial_{x_\iota}\phi_2)|_{\Omega_{n_i}} \partial_{x_\mu} T_i dxdt = -\int_0^S \int_{\mathbb{R}^N} \partial_{x_\iota}\phi_2 \partial_{x_\mu}\widehat{T}_i dxdt\end{aligned}$$

for any i such that $n_i \geqslant M$. Taking the limit as $i \to \infty$, we obtain

$$\int_0^S \int_{\mathbb{R}^N} T_{***}^{\iota,\mu}\phi_2 dxdt = -\int_0^S \int_{\mathbb{R}^N} \partial_{x_\mu} T_* \partial_{x_\iota}\phi_2 dxdt.$$

Therefore $T_{***}^{\iota,\mu} = \partial_{x_\iota}\partial_{x_\mu} T_*$ and $\partial_{x_\iota}\partial_{x_\mu} T_* \in L^2(0,S;L^2(\mathbb{R}^N))$ are valid for any $\iota,\mu = 1,2,\cdots,N$. Moreover, for $n_i \geqslant M$, we obtain

$$\int_0^S \int_{\mathbb{R}^N} \phi_2 [\Delta T_i]^\wedge dxdt = \int_0^S \int_{\Omega_{n_i}} \phi_2|_{\Omega_{n_i}} \Delta T_i dxdt = -\int_0^S \int_{\mathbb{R}^N} \nabla\phi_2 \cdot \nabla\widehat{T}_i dxdt,$$

which yields

$$\int_0^S \int_{\mathbb{R}^N} \phi_2 T_{****} dxdt = -\int_0^S \int_{\mathbb{R}^N} \nabla\phi_2 \cdot \nabla T_* dxdt = \int_0^S \int_{\mathbb{R}^N} \phi_2 \Delta T_* dxdt.$$

Hence $T_{****} = \Delta T_*$ in $L^2(0,S;L^2(\mathbb{R}^N))$.

By exactly the same argument as above, we have

$$C_{**} = \partial_t C_*, \quad C_{***}^{\iota,\mu} = \partial_\iota\partial_\mu C_*, \quad C_{****} = \Delta C_*, \quad \boldsymbol{u}_{**} = \partial_t \boldsymbol{u}_*, \quad \boldsymbol{u}_{***}^{\iota,\mu} = \partial_\iota\partial_\mu \boldsymbol{u}_*.$$

Fix $\boldsymbol{\phi}_3 \in C_0^\infty((0,S);\mathbb{C}_\sigma^\infty(\mathbb{R}^N))$ arbitrarily and let a natural number $M \in \mathbb{N}$ satisfy $supp\boldsymbol{\phi}_3 \subset \Omega_M \times (0,S)$. Since $\boldsymbol{\phi}_3|_{\Omega_{n_i}} \in C_0^\infty((0,S);\mathbb{C}_\sigma^\infty(\Omega_{n_i}))$ for $n_i \geqslant M$, we get

$$\begin{aligned}\int_0^S \int_{\mathbb{R}^N} \boldsymbol{\phi}_3 \cdot [\mathcal{A}_{\Omega_{n_i}}\boldsymbol{u}_i]^\wedge dxdt &= \int_0^S \int_{\Omega_{n_i}} \boldsymbol{\phi}_3|_{\Omega_{n_i}} \cdot \mathcal{A}_{\Omega_i}\boldsymbol{u}_i dxdt = -\int_0^S \int_{\Omega_{n_i}} \boldsymbol{\phi}_3|_{\Omega_{n_i}} \cdot \Delta\boldsymbol{u}_i dxdt \\ &= \int_0^S \int_{\Omega_{n_i}} (\nabla\boldsymbol{\phi}_3)|_{\Omega_{n_i}} \cdot \nabla\boldsymbol{u}_i dxdt = \int_0^S \int_{\mathbb{R}^N} \nabla\boldsymbol{\phi}_3 \cdot \nabla\widehat{\boldsymbol{u}}_i dxdt.\end{aligned}$$

Taking the limit as $i \to \infty$ and using the fact that $\boldsymbol{u}_*(t) \in D(\mathcal{A}_{\mathbb{R}^N})$ for a.e. $t \in [0,S]$, we obtain

$$\int_0^S \int_{\mathbb{R}^N} \boldsymbol{\phi}_3 \cdot \boldsymbol{u}_{****} dxdt = \int_0^S \int_{\mathbb{R}^N} \boldsymbol{\phi}_3 \cdot \mathcal{A}_{\mathbb{R}^N}\boldsymbol{u}_* dxdt.$$

Since $C_0^\infty((0,S);\mathbb{C}_\sigma^\infty(\mathbb{R}^N))$ is dense in $L^2(0,S;\mathbb{L}_\sigma^2(\mathbb{R}^N))$, we can show that $\boldsymbol{u}_{****} = \mathcal{A}_{\mathbb{R}^N}\boldsymbol{u}_*$ holds.

Moreover, using the fact that $\boldsymbol{u}_* \in W^{1,2}(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N))$ and $\mathcal{A}_{\mathbb{R}^N}\boldsymbol{u}_* \in L^2(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N))$ and applying Lemma 2.3.2, we can assure the absolute continuity of $|\nabla \boldsymbol{u}_*(\cdot)|^2_{\mathbb{L}^2(\mathbb{R}^N)}$. Combining the continuity of $|\nabla \boldsymbol{u}_*(\cdot)|^2_{\mathbb{L}^2(\mathbb{R}^N)}$ with the fact that $\boldsymbol{u}_* \in L^\infty(0,S;\mathbb{H}^1_\sigma(\mathbb{R}^N))$, we obtain $\boldsymbol{u}_* \in C([0,S];\mathbb{H}^1_\sigma(\mathbb{R}^N))$. Likewise, we have $T_*, C_* \in C([0,S];H^1(\mathbb{R}^N))$. Hence $\boldsymbol{u}_*, T_*$ and C_* satisfy required regularities (5.24) except their periodicity.

From (5.55), it is obvious that

$$\sup_{0\leqslant t\leqslant S}\left|\widehat{T}_i\Big|_{\Omega_n}(t)\right|^2_{H^1(\Omega_n)} + \int_0^S \left|\partial_t\widehat{T}_i\Big|_{\Omega_n}(s)\right|^2_{L^2(\Omega_n)} ds \leqslant \gamma_2,$$

$$\sup_{0\leqslant t\leqslant S}\left|\widehat{C}_i\Big|_{\Omega_n}(t)\right|^2_{H^1(\Omega_n)} + \int_0^S \left|\partial_t\widehat{C}_i\Big|_{\Omega_n}(s)\right|^2_{L^2(\Omega_n)} ds \leqslant \gamma_2,$$

$$\sup_{0\leqslant t\leqslant S}\left|\widehat{\boldsymbol{u}}_i|_{\Omega_n}(t)\right|^2_{\mathbb{H}^1(\Omega_n)} + \int_0^S \left|\partial_t\widehat{\boldsymbol{u}}_i|_{\Omega_n}(s)\right|^2_{\mathbb{L}^2(\Omega_n)} ds \leqslant \gamma_2$$

for any $i \in \mathbb{N}$ and $n \in \mathbb{N}$. These inequalities imply that we can apply Ascoli's theorem to the $\{U_i\}_{i\in\mathbb{N}}$ and its subsequence on Ω_n for any $n \in \mathbb{N}$.

Therefore, by applying Ascoli's theorem on Ω_1, we can extract a subsequence of $\{U_i\}_{i\in\mathbb{N}} = \{(\widehat{\boldsymbol{u}}_i,\ \widehat{T}_i,\ \widehat{C}_i)\}_{i\in\mathbb{N}}$, which is denoted by $\{U_{i^1_j}\}_{j\in\mathbb{N}} := \{(\widehat{\boldsymbol{u}_{i^1_j}},\ \widehat{T_{i^1_j}},\ \widehat{C_{i^1_j}})\}_{j\in\mathbb{N}}$, satisfying the following convergences:

$$\begin{aligned}
\widehat{T_{i^1_j}}|_{\Omega_1} &\to T^1 && \text{strongly in } C([0,S];L^2(\Omega_1)),\\
\widehat{C_{i^1_j}}|_{\Omega_1} &\to C^1 && \text{strongly in } C([0,S];L^2(\Omega_1)),\\
\widehat{\boldsymbol{u}_{i^1_j}}|_{\Omega_1} &\to \boldsymbol{u}^1 && \text{strongly in } C([0,S];\mathbb{L}^2(\Omega_1)).
\end{aligned}$$

Let $U^1 := (\boldsymbol{u}^1, T^1, C^1)$. Here we remark that since U_i possesses the time-periodicity for each $i \in \mathbb{N}$, U^1 is also time-periodic function, i.e.,

$$U^1 \in C_\pi([0,S];\mathbb{L}^2(\Omega_1)\times L^2(\Omega_1)\times L^2(\Omega_1)).$$

Next, applying Ascoli's theorem again with $n = 2$, we obtain the existence of a subsequence $\{U_{i^2_j}\}_{j\in\mathbb{N}}$ of $\{U_{i^1_j}\}_{j\in\mathbb{N}}$ which satisfies

$$U_{i^2_j} \to U^2 \quad \text{strongly in } C_\pi([0,S];\mathbb{L}^2(\Omega_2)\times L^2(\Omega_2)\times L^2(\Omega_2)).$$

As for the relationship between U^1 and U^2, we can easily show that

$$U^1(x,t) = U^2(x,t) \quad \forall t \in [0,S], \quad \text{for a.e. } x \in \Omega_1.$$

Repeating the same procedure as above inductively and applying the diagonal argument, i.e., along the diagonal subsequence $\{U_{i^l_l}\}_{l\in\mathbb{N}}$, simply denoted by $\{U_l\}_{l\in\mathbb{N}}$ (i.e., $\{i^l_l\}$ is denoted by $\{l\}$ for simplicity), we obtain the following convergences:

$$\begin{aligned}
\widehat{T}_l|_{\Omega_n} &\to T^n && \text{strongly in } C([0,S];L^2(\Omega_n)),\\
\widehat{C}_l|_{\Omega_n} &\to C^n && \text{strongly in } C([0,S];L^2(\Omega_n)),\\
\widehat{\boldsymbol{u}}_l|_{\Omega_n} &\to \boldsymbol{u}^n && \text{strongly in } C([0,S];\mathbb{L}^2(\Omega_n))
\end{aligned}$$

for any $n \in \mathbb{N}$, where $U^n := (\boldsymbol{u}^n, T^n, C^n)$ belongs to $C_\pi([0,S]; \mathbb{L}^2(\Omega_n) \times L^2(\Omega_n) \times L^2(\Omega_n))$. Moreover,

$$U^{n_1}(x,t) = U^{n_2}(x,t) \quad \forall t \in [0,S], \quad \text{for a.e. } x \in \Omega_{n_1}$$

holds for $n_2 \geqslant n_1$. We note that $\{U_l\}_{l\in\mathbb{N}}$ still satisfies (5.58), (5.59), (5.60) and (5.61).

Define $U := (\boldsymbol{u}, T, C)$ by

$$U(x,t) := U^n(x,t) \qquad\qquad \text{if } x \in \Omega_n.$$

Let $\phi_4 \in C_0^\infty(\mathbb{R}^N \times (0,S))$ and assume that $supp\phi_4 \subset \Omega_M \times (0,S)$ for some $M \in \mathbb{N}$. From (5.58), we have

$$\begin{aligned}
&\int_0^S \int_{\mathbb{R}^N} \widehat{T}_l \phi_4 dxdt = \int_0^S \int_{\Omega_M} \widehat{T}_l|_{\Omega_M} \phi_4|_{\Omega_M} dxdt \\
\xrightarrow[l\to\infty]{} &\int_0^S \int_{\mathbb{R}^N} T_* \phi_4 dxdt = \int_0^S \int_{\Omega_M} T^M \phi_4|_{\Omega_M} dxdt,
\end{aligned}$$

which implies that T coincides with T_*. Similarly, we obtain $C = C_*$ and $\boldsymbol{u} = \boldsymbol{u}_*$. From the periodicity of T, C and $\boldsymbol{u}$, we can derive $T_*, C_* \in C_\pi([0,S]; H^1(\mathbb{R}^N))$ and $\boldsymbol{u}_* \in C_\pi([0,S]; \mathbb{H}^1_\sigma(\mathbb{R}^N))$. Moreover,

$$\begin{aligned}
&\int_0^S \int_{\mathbb{R}^N} [\boldsymbol{u}_l \cdot \nabla T_l]^\wedge \phi_4 dxdt \\
&= \int_0^S \int_{\Omega_{n_l}} \boldsymbol{u}_l \cdot \nabla T_l \phi_4|_{\Omega_{n_l}} dxdt = -\int_0^S \int_{\Omega_{n_l}} \boldsymbol{u}_l T_l \cdot \nabla \phi_4|_{\Omega_{n_l}} dxdt \\
&= -\int_0^S \int_{\Omega_M} \boldsymbol{u}_l|_{\Omega_M} T_l|_{\Omega_M} \cdot \nabla \phi_4|_{\Omega_M} dxdt \qquad (\forall l \text{ s.t. } n_l \geqslant M) \\
\xrightarrow[l\to\infty]{} &\int_0^S \int_{\mathbb{R}^N} \chi_1 \phi_4 dxdt = -\int_0^S \int_{\Omega_M} \boldsymbol{u}^M T^M \cdot \nabla \phi_4|_{\Omega_M} dxdt \\
&= -\int_0^S \int_{\mathbb{R}^N} \boldsymbol{u} T \cdot \nabla \phi_4 dxdt.
\end{aligned}$$

Since $T = T_*$, $\boldsymbol{u} = \boldsymbol{u}_*$ and $\boldsymbol{u}_* \cdot \nabla T_*$ belongs to $L^2(0,S; L^2(\mathbb{R}^N))$, $\chi_1 = \boldsymbol{u} \cdot \nabla T$ is valid in $L^2(0,S; L^2(\mathbb{R}^N))$. By exactly the same procedure as above, we can show that $\chi_2 = \boldsymbol{u} \cdot \nabla C$ in $L^2(0,S; L^2(\mathbb{R}^N))$.

Thus, we can assure that $(\boldsymbol{u}_*, T_*, C_*)$ becomes a periodic solution of (5.23). □

5.4 Convergence as $\lambda \to 0$

Let $(\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)$ designate the periodic solution of (5.23) with parameter $\lambda > 0$. In this section, we discuss the convergences of solutions $\{(\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)\}_{\lambda>0}$ as $\lambda \to 0$ and we complete our proof.

We first introduce the uniform boundedness of $\{(\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)\}_{\lambda>0}$ independent of λ by establishing some a priori estimates. To this end, we here prepare the following lemma for convection terms and elliptic estimates.

Lemma 5.4.1. *Let $\boldsymbol{w} \in \mathbb{H}^2(\mathbb{R}^N)$ and $U \in H^2(\mathbb{R}^N)$. Then there exists a constant β satisfying the following inequalities:*

$$|\boldsymbol{w}\cdot\nabla U|^2_{L^2(\mathbb{R}^N)} \leqslant \beta|\nabla\boldsymbol{w}|^2_{\mathbb{L}^2(\mathbb{R}^N)}|\nabla U|_{L^2(\mathbb{R}^N)}|\Delta U|_{L^2(\mathbb{R}^N)}, \tag{5.62}$$

if $N = 3$.

$$|\boldsymbol{w}\cdot\nabla U|^2_{L^2(\mathbb{R}^N)} \leqslant \beta|\nabla\boldsymbol{w}|_{\mathbb{L}^2(\mathbb{R}^N)}|\Delta\boldsymbol{w}|_{\mathbb{L}^2(\mathbb{R}^N)}|\nabla U|_{L^2(\mathbb{R}^N)}|\Delta U|_{L^2(\mathbb{R}^N)}, \tag{5.63}$$

if $N = 4$.

$$\left|\partial_{x_\iota}\partial_{x_\mu}U\right|_{L^2(\mathbb{R}^N)} \leqslant \beta\,|\Delta U|_{L^2(\mathbb{R}^N)}, \qquad \left|\partial_{x_\iota}\partial_{x_\mu}\boldsymbol{w}\right|_{\mathbb{L}^2(\mathbb{R}^N)} \leqslant \beta\,|\Delta\boldsymbol{w}|_{\mathbb{L}^2(\mathbb{R}^N)} \tag{5.64}$$

for $N = 3, 4$ and for all $\iota, \mu = 1, 2, \cdots, N$.

Proof. Let $\boldsymbol{w} \in \mathbb{H}^2(\mathbb{R}^N)$, $U \in H^2(\mathbb{R}^N)$ and let $\{\boldsymbol{w}_k\}_{k\in\mathbb{N}} \subset \mathbb{C}_0^\infty(\mathbb{R}^N)$ and $\{U_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^N)$ be sequences satisfying $\boldsymbol{w}_k \to \boldsymbol{w}$ in $\mathbb{H}^2(\mathbb{R}^N)$ and $U_k \to U$ in $H^2(\mathbb{R}^N)$ as $k \to \infty$ respectively. Since $\boldsymbol{w}_k$ and U_k possess compact supports, we can apply almost the same procedures as those for Lemma 5.3.2 and we can assure that (5.62), (5.63) and (5.64) are valid with $\boldsymbol{w}_k$, U_k and suitable coefficient β independent of k. Then immediately, (5.62), (5.63) and (5.64) can be verified for all $\boldsymbol{w} \in \mathbb{H}^2(\mathbb{R}^N)$ and $U \in H^2(\mathbb{R}^N)$ by letting $k \to \infty$. □

From now on, we write simply $|\cdot|_{L^p}$ and $|\cdot|_{H^k}$ in order to designate the norm of $L^p(\mathbb{R}^N)$ and $H^k(\mathbb{R}^N)$ respectively, if no confusion arises. Multiplying the second equation of (5.23) by T_λ, we get

$$\frac{1}{2}\frac{d}{dt}|T_\lambda|^2_{L^2} + |\nabla T_\lambda|^2_{L^2} + \lambda|T_\lambda|^2_{L^2} = \int_{\mathbb{R}^N} f_2 T_\lambda dx \leqslant \gamma_3|f_2|_{L^{(2^*)'}}|\nabla T_\lambda|_{L^2},$$

where we use Hölder's inequality and Sobolev's inequality. Here and henceforth, γ_3 stands for a general constant independent of λ. Then we obtain

$$\frac{1}{2}\frac{d}{dt}|T_\lambda|^2_{L^2} + \frac{1}{2}|\nabla T_\lambda|^2_{L^2} + \lambda|T_\lambda|^2_{L^2} \leqslant \gamma_3|f_2|^2_{L^{(2^*)'}},$$

which yields

$$\int_0^S |\nabla T_\lambda(s)|^2_{L^2}ds + \lambda\int_0^S |T_\lambda(s)|^2_{L^2}ds \leqslant \gamma_3, \tag{5.65}$$

since f_2 belongs to $L^2(0, S; L^{(2^*)'}(\mathbb{R}^N))$. Multiplying the third equation of (5.23) by C_λ, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|C_\lambda|^2_{L^2} + |\nabla C_\lambda|^2_{L^2} + \lambda|C_\lambda|^2_{L^2} &= \rho\int_{\mathbb{R}^N} C_\lambda\Delta T_\lambda dx + \int_{\mathbb{R}^N} f_3 C_\lambda dx\\ &\leqslant \rho|\nabla C_\lambda|_{L^2}|\nabla T_\lambda|_{L^2} + \gamma_3|f_3|_{L^{(2^*)'}}|\nabla C_\lambda|_{L^2},\end{aligned}$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}|C_\lambda|^2_{L^2} + \frac{1}{2}|\nabla C_\lambda|^2_{L^2} + \lambda|C_\lambda|^2_{L^2} \leqslant \rho^2|\nabla T_\lambda|^2_{L^2} + \gamma_3|f_3|^2_{L^{(2^*)'}}.$$

Integrating this over $[0, S]$ and using (5.65), we obtain

$$\int_0^S |\nabla C_\lambda(s)|^2_{L^2} ds + \lambda \int_0^S |C_\lambda(s)|^2_{L^2} ds \leqslant \gamma_3, \tag{5.66}$$

since $f_3 \in L^2(0, S; L^{(2^*)'}(\mathbb{R}^N))$. Multiplying the first equation of (5.23) by $-\Delta \boldsymbol{u}_\lambda$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&|\nabla \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + \nu|\Delta \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + a|\nabla \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} \\ &= -\int_{\mathbb{R}^N} \Delta \boldsymbol{u}_\lambda \cdot \mathcal{P}_{\mathbb{R}^N} \boldsymbol{g} T_\lambda dx - \int_{\mathbb{R}^N} \Delta \boldsymbol{u}_\lambda \cdot \mathcal{P}_{\mathbb{R}^N} \boldsymbol{h} C_\lambda dx - \int_{\mathbb{R}^N} \mathcal{P}_{\mathbb{R}^N} \boldsymbol{f}_1 \cdot \Delta \boldsymbol{u}_\lambda dx.\end{aligned}$$

Here we recall (5.24), i.e., $\Delta \boldsymbol{u}_\lambda(t) \in \mathbb{L}^2_\sigma(\mathbb{R}^N)$ for a.e. $t \in [0, S]$. Then by integration by parts, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} \Delta \boldsymbol{u}_\lambda \cdot \mathcal{P}_{\mathbb{R}^N} \boldsymbol{g} T_\lambda dx \\ &= \int_{\mathbb{R}^N} \Delta \boldsymbol{u}_\lambda \cdot \boldsymbol{g} T_\lambda dx = \sum_{\mu=1}^N \int_{\mathbb{R}^N} \Delta u^\mu_\lambda g^\mu T_\lambda dx = -\sum_{\mu=1}^N \int_{\mathbb{R}^N} \nabla u^\mu_\lambda \cdot g^\mu \nabla T_\lambda dx \\ &\leqslant \sum_{\mu=1}^N |\nabla u^\mu_\lambda|_{L^2} |g^\mu| |\nabla T_\lambda|_{L^2} \leqslant \sum_{\mu=1}^N \left(\frac{a}{4}|\nabla u^\mu_\lambda|^2_{L^2} + \frac{|g^\mu|^2}{a}|\nabla T_\lambda|^2_{L^2} \right) \\ &= \frac{a}{4}|\nabla \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + \frac{|\boldsymbol{g}|^2}{a}|\nabla T_\lambda|^2_{L^2}.\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^N} \Delta \boldsymbol{u}_\lambda \cdot \mathcal{P}_{\mathbb{R}^N} \boldsymbol{g} T_\lambda dx \leqslant \frac{a}{4}|\nabla \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + \frac{|\boldsymbol{h}|^2}{a}|\nabla C_\lambda|^2_{L^2}$$

holds. Therefore, multiplication of the first equation by $-\Delta \boldsymbol{u}_\lambda$ yields

$$\frac{1}{2}\frac{d}{dt}|\nabla \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + \frac{\nu}{2}|\Delta \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + \frac{a}{2}|\nabla \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} \leqslant \frac{|\boldsymbol{g}|^2}{a}|\nabla T_\lambda|^2_{L^2} + \frac{|\boldsymbol{h}|^2}{a}|\nabla C_\lambda|^2_{L^2} + \frac{1}{2\nu}|\boldsymbol{f}_1|^2_{\mathbb{L}^2}. \tag{5.67}$$

Integrating (5.67) over $[0, S]$ and using the elliptic estimate (5.64), we have

$$\int_0^S |\Delta \boldsymbol{u}_\lambda(s)|^2_{\mathbb{L}^2} ds + \int_0^S |\nabla \boldsymbol{u}_\lambda(s)|^2_{\mathbb{L}^2} ds + \int_0^S |\partial_{x_\iota}\partial_{x_\mu} \boldsymbol{u}_\lambda(s)|^2_{\mathbb{L}^2} ds \leqslant \gamma_3 \tag{5.68}$$

for $\iota, \mu = 1, 2, \cdots, N$. From the fact that $\boldsymbol{u}_\lambda \in C_\pi([0, S]; \mathbb{H}^1_\sigma(\mathbb{R}^N))$ and (5.68), there exists $t^\lambda_3 \in [0, S]$ where $|\nabla \boldsymbol{u}(t^\lambda_3)|_{\mathbb{L}^2} \leqslant \gamma_3$ holds. Therefore integrating (5.67) over $[t^\lambda_3, t]$ with $t \in [t^\lambda_3, t^\lambda_3 + S]$, we obtain

$$\sup_{0 \leqslant t \leqslant S} |\nabla \boldsymbol{u}_\lambda(t)|_{\mathbb{L}^2} ds \leqslant \gamma_3. \tag{5.69}$$

Moreover, Sobolev's inequality and (5.69) lead to the fact that $\boldsymbol{u}_\lambda \in C([0,S];\mathbb{L}^{2^*}_\sigma(\mathbb{R}^N))$ and

$$\sup_{0\leqslant t\leqslant S} |\boldsymbol{u}_\lambda(t)|_{\mathbb{L}^{2^*}} ds \leqslant \gamma_3. \tag{5.70}$$

Multiplying the second equation of (5.23) by $-\Delta T_\lambda$ and $\partial_t T_\lambda$ (by exactly the same procedures as those for (5.46), (5.47), (5.51), (5.52) and by using (5.62), (5.63)), we get

$$\begin{aligned} &\frac{d}{dt}|\nabla T_\lambda|^2_{L^2} + |\Delta T_\lambda|^2_{L^2} \leqslant \gamma_3 |\nabla \boldsymbol{u}_\lambda|^4_{\mathbb{L}^2} |\nabla T_\lambda|^2_{L^2} + 2|f_2|^2_{L^2}, \\ &|\partial_t T_\lambda|^2_{L^2} + \frac{d}{dt}|\nabla T_\lambda|^2_{L^2} + \lambda \frac{d}{dt}|T_\lambda|^2_{L^2} \leqslant \gamma_3 |\nabla \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} |\nabla T_\lambda|_{L^2} |\Delta T_\lambda|_{L^2} + 2|f_2|^2_{L^2} \end{aligned} \tag{5.71}$$

for $N=3$ and

$$\begin{aligned} &\frac{d}{dt}|\nabla T_\lambda|^2_{L^2} + |\Delta T_\lambda|^2_{L^2} \leqslant \gamma_3 |\nabla \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} |\Delta \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} |\nabla T_\lambda|^2_{L^2} + 2|f_2|^2_{L^2}, \\ &|\partial_t T_\lambda|^2_{L^2} + \frac{d}{dt}|\nabla T_\lambda|^2_{L^2} + \lambda \frac{d}{dt}|T_\lambda|^2_{L^2} \leqslant \gamma_3 |\nabla \boldsymbol{u}_\lambda|_{\mathbb{L}^2} |\Delta \boldsymbol{u}_\lambda|_{\mathbb{L}^2} |\nabla T_n|_{L^2} |\Delta T_\lambda|_{L^2} + 2|f_2|^2_{L^2} \end{aligned} \tag{5.72}$$

for $N=4$. From the fact that $T_\lambda \in C([0,S];H^1(\mathbb{R}^N))$ and (5.65) holds, there exists $t_4^\lambda \in [0,S]$ such that

$$|\nabla T_\lambda(t_4^\lambda)|^2_{L^2} + \lambda |T_\lambda(t_4^\lambda)|^2_{L^2} = \min_{0\leqslant t\leqslant S} \left(|\nabla T_\lambda(t)|^2_{L^2} + \lambda |T_\lambda(t)|^2_{L^2} \right) \leqslant \gamma_3.$$

Therefore, integrating (5.71) and (5.72) over $[t_4^\lambda, t]$ and applying Gronwall's inequality, we obtain

$$\sup_{0\leqslant t\leqslant S} |T_\lambda(t)|^2_{L^{2^*}} + \sup_{0\leqslant t\leqslant S} |\nabla T_\lambda(t)|^2_{L^2} + \int_0^S |\Delta T_\lambda(s)|^2_{L^2} ds + \int_0^S |\partial_t T_\lambda(s)|^2_{L^2} ds \leqslant \gamma_3. \tag{5.73}$$

Similarly, we get the followings from the third equation of (5.23).

$$\begin{aligned} &\frac{d}{dt}|\nabla C_\lambda|^2_{L^2} + |\Delta C_\lambda|^2_{L^2} \leqslant \gamma_3 |\nabla \boldsymbol{u}_\lambda|^4_{\mathbb{L}^2} |\nabla C_\lambda|^2_{L^2} + 3\rho^2 |\Delta T_\lambda|^2_{L^2} + 3|f_3|^2_{L^2}, \\ &|\partial_t C_\lambda|^2_{L^2} + \frac{d}{dt}|\nabla C_\lambda|^2_{L^2} + \lambda \frac{d}{dt}|C_\lambda|^2_{L^2} \\ &\qquad\qquad \leqslant \gamma_3 |\nabla \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} |\nabla C_\lambda|_{L^2} |\Delta C_\lambda|_{L^2} + 3\rho^2 |\Delta T_\lambda|^2_{L^2} + 3|f_3|^2_{L^2} \end{aligned}$$

for $N=3$ and

$$\begin{aligned} &\frac{d}{dt}|\nabla C_\lambda|^2_{L^2} + |\Delta C_\lambda|^2_{L^2} \leqslant \gamma_3 |\nabla \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} |\Delta \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} |\nabla C_\lambda|^2_{L^2} + 3\rho^2 |\Delta T_\lambda|^2_{L^2} + 3|f_3|^2_{L^2}, \\ &|\partial_t C_\lambda|^2_{L^2} + \frac{d}{dt}|\nabla C_\lambda|^2_{L^2} + \lambda \frac{d}{dt}|C_\lambda|^2_{L^2} \\ &\qquad \leqslant \gamma_3 |\nabla \boldsymbol{u}_\lambda|_{\mathbb{L}^2} |\Delta \boldsymbol{u}_\lambda|_{\mathbb{L}^2} |\nabla C_\lambda|_{L^2} |\Delta C_\lambda|_{L^2} + 3\rho^2 |\Delta T_\lambda|^2_{L^2} + 3|f_2|^2_{L^2} \end{aligned}$$

for $N = 4$. Then we can obtain

$$(5.74)\quad \sup_{0\leqslant t\leqslant S}|C_\lambda(t)|^2_{L^{2^*}} + \sup_{0\leqslant t\leqslant S}|\nabla C_\lambda(t)|^2_{L^2} + \int_0^S |\Delta C_\lambda(s)|^2_{L^2}ds + \int_0^S |\partial_t C_\lambda(s)|^2_{L^2}ds \leqslant \gamma_3.$$

By virtue of the elliptic estimates (5.64), (5.73) and (5.74) yield

$$(5.75)\qquad \int_0^S |\partial_{x_\iota}\partial_{x_\mu}T_\lambda(s)|^2_{L^2}ds + \int_0^S |\partial_{x_\iota}\partial_{x_\mu}C_\lambda(s)|^2_{L^2}ds \leqslant \gamma_3$$

for any $\iota, \mu = 1, 2, \cdots, N$.

Let $D_h\boldsymbol{u}_\lambda(t) := \boldsymbol{u}_\lambda(t+h) - \boldsymbol{u}_\lambda(t)$, $D_hT_\lambda(t) := T_\lambda(t+h) - T_\lambda(t)$, $D_hC_\lambda(t) := C_\lambda(t+h) - C_\lambda(t)$ and $D_h\boldsymbol{f}_1(t) := \boldsymbol{f}_1(t+h) - \boldsymbol{f}_1(t)$ $(h > 0)$. Then from the first equation of (5.23), we get

$$(5.76)\qquad \partial_t D_h\boldsymbol{u}_\lambda - \nu\Delta D_h\boldsymbol{u}_\lambda + aD_h\boldsymbol{u}_\lambda = \mathcal{P}_{\mathbb{R}^N}\boldsymbol{g}D_hT_\lambda + \mathcal{P}_{\mathbb{R}^N}\boldsymbol{h}D_hC_\lambda + \mathcal{P}_{\mathbb{R}^N}D_h\boldsymbol{f}_1.$$

Multiplying (5.76) by $D_h\boldsymbol{u}_\lambda$, we have

$$\frac{d}{dt}|D_h\boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + 2\nu|\nabla D_h\boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + a|D_h\boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} \leqslant \frac{3|\boldsymbol{g}|^2}{a}|D_hT_\lambda|^2_{L^2} + \frac{3|\boldsymbol{h}|^2}{a}|D_hC_\lambda|^2_{L^2} + \frac{3}{a}|D_h\boldsymbol{f}_1|^2_{\mathbb{L}^2}.$$

Since $D_h\boldsymbol{u}_\lambda \in C_\pi([0,S];\mathbb{L}^2_\sigma(\mathbb{R}^N))$, $\boldsymbol{f}_1 \in W^{1,2}(0,S;\mathbb{L}^2(\mathbb{R}^N))$ and we already have estimates for ∂_tT_λ and ∂_tC_λ in (5.73) and (5.74), integration of this inequality over $[0,S]$ gives us

$$\int_0^S |D_h\boldsymbol{u}_\lambda(s)|^2_{\mathbb{L}^2}ds \leqslant \gamma_3h^2$$

for any $h > 0$. Hence we get (by virtue of Proposition 2.1.16)

$$(5.77)\qquad \int_0^S |\partial_t\boldsymbol{u}_\lambda(s)|^2_{\mathbb{L}^2}ds \leqslant \gamma_3.$$

Moreover, from (5.62) and (5.63), we can also derive

$$(5.78)\qquad \int_0^S |\boldsymbol{u}_\lambda\cdot\nabla T_\lambda(s)|^2_{L^2}\,ds + \int_0^S |\boldsymbol{u}_\lambda\cdot\nabla C_\lambda(s)|^2_{L^2}\,ds \leqslant \gamma_3.$$

By using the uniform boundedness (5.68), (5.69), (5.70), (5.73), (5.74), (5.75), (5.77), (5.78) and the standard arguments of convex analysis, we can extract a subsequence of $\{(\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)\}_{\lambda>0}$, denoted by $\{V_i\}_{i\in\mathbb{N}} := \{(\boldsymbol{u}_i, T_i, C_i)\}_{i\in\mathbb{N}}$, which satisfies the following convergences:

$$(5.79)\quad \begin{aligned}
&T_i \rightharpoonup T_\star && *\text{-weakly in } L^\infty(0,S;L^{2^*}(\mathbb{R}^N)),\\
&\partial_{x_\mu}T_i \rightharpoonup T^\mu_{\star\star} && *\text{-weakly in } L^\infty(0,S;L^2(\mathbb{R}^N)) \qquad (\forall\mu = 1,2,\cdots N),\\
&\partial_tT_i \rightharpoonup T_{\star\star\star} && \text{weakly in } L^2(0,S;L^2(\mathbb{R}^N)),\\
&\partial_{x_\iota}\partial_{x_\mu}T_i \rightharpoonup T^{\iota,\mu}_{\star\star\star\star} && \text{weakly in } L^2(0,S;L^2(\mathbb{R}^N)) \qquad (\forall\iota,\forall\mu = 1,2,\cdots N),\\
&\Delta T_i \rightharpoonup T_{\star\star\star\star\star} && \text{weakly in } L^2(0,S;L^2(\mathbb{R}^N)),
\end{aligned}$$

$$
(5.80)\quad
\begin{aligned}
&C_i \rightharpoonup C_\star && *\text{-weakly in } L^\infty(0,S;L^{2^*}(\mathbb{R}^N)),\\
&\partial_{x_\mu} C_i \rightharpoonup C^\mu_{\star\star} && *\text{-weakly in } L^\infty(0,S;L^2(\mathbb{R}^N)) \qquad (\forall \mu = 1,2,\cdots N),\\
&\partial_t C_i \rightharpoonup C_{\star\star\star} && \text{weakly in } L^2(0,S;L^2(\mathbb{R}^N)),\\
&\partial_{x_\iota}\partial_{x_\mu} C_i \rightharpoonup C^{\iota,\mu}_{\star\star\star\star} && \text{weakly in } L^2(0,S;L^2(\mathbb{R}^N)) \qquad (\forall \iota, \forall \mu = 1,2,\cdots N),\\
&\Delta C_i \rightharpoonup C_{\star\star\star\star\star} && \text{weakly in } L^2(0,S;L^2(\mathbb{R}^N)),
\end{aligned}
$$

$$
(5.81)\quad
\begin{aligned}
&\boldsymbol{u}_i \rightharpoonup \boldsymbol{u}_\star && *\text{-weakly in } L^\infty(0,S;\mathbb{L}^{2^*}_\sigma(\mathbb{R}^N)),\\
&\partial_{x_\mu} \boldsymbol{u}_i \rightharpoonup \boldsymbol{u}^\mu_{\star\star} && *\text{-weakly in } L^\infty(0,S;\mathbb{L}^2(\mathbb{R}^N)) \qquad (\forall \mu = 1,2,\cdots N),\\
&\partial_t \boldsymbol{u}_i \rightharpoonup \boldsymbol{u}_{\star\star\star} && \text{weakly in } L^2(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N)),\\
&\partial_{x_\iota}\partial_{x_\mu} \boldsymbol{u}_i \rightharpoonup \boldsymbol{u}^{\iota,\mu}_{\star\star\star\star} && \text{weakly in } L^2(0,S;\mathbb{L}^2(\mathbb{R}^N)) \qquad (\forall \iota, \forall \mu = 1,2,\cdots N),\\
&\Delta \boldsymbol{u}_i \rightharpoonup \boldsymbol{u}_{\star\star\star\star\star} && \text{weakly in } L^2(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N)),
\end{aligned}
$$

$$
(5.82)\quad
\begin{aligned}
&\boldsymbol{u}_i \cdot \nabla T_i \rightharpoonup \chi_3 && \text{weakly in } L^2(0,S;L^2(\mathbb{R}^N)),\\
&\boldsymbol{u}_i \cdot \nabla C_i \rightharpoonup \chi_4 && \text{weakly in } L^2(0,S;L^2(\mathbb{R}^N)).
\end{aligned}
$$

Moreover, λT_λ and λC_λ strongly converge to zero in $L^2(0,S;L^2(\mathbb{R}^N))$ as $\lambda \to 0$, since, from (5.65) and (5.66),

$$
\int_0^S |\lambda T_\lambda|^2_{L^2} dt = \lambda \int_0^S \lambda |T_\lambda|^2_{L^2} dt \leqslant \lambda\gamma_3.
$$

For each $n \in \mathbb{N}$, V_i satisfies

$$
(5.83)\quad
\begin{aligned}
&\sup_{0\leqslant t\leqslant S} \left|\nabla T_i|_{\Omega_n}(t)\right|^2_{L^2(\Omega_n)} + \int_0^S \left|\partial_t T_i|_{\Omega_n}(t)\right|^2_{L^2(\Omega_n)} ds \leqslant \gamma_3,\\
&\sup_{0\leqslant t\leqslant S} \left|\nabla C_i|_{\Omega_n}(t)\right|^2_{L^2(\Omega_n)} + \int_0^S \left|\partial_t C_i|_{\Omega_n}(t)\right|^2_{L^2(\Omega_n)} ds \leqslant \gamma_3,\\
&\sup_{0\leqslant t\leqslant S} \left|\nabla \boldsymbol{u}_i|_{\Omega_n}(t)\right|^2_{\mathbb{L}^2(\Omega_n)} + \int_0^S \left|\partial_t \boldsymbol{u}_i|_{\Omega_n}(t)\right|^2_{\mathbb{L}^2(\Omega_n)} ds \leqslant \gamma_3.
\end{aligned}
$$

Moreover, using (5.70), (5.73), (5.74) and Hölder's inequality, we get

$$
\sup_{0\leqslant t\leqslant S} \left|T_i|_{\Omega_n}(t)\right|^2_{L^2(\Omega_n)} + \sup_{0\leqslant t\leqslant S} \left|C_i|_{\Omega_n}(t)\right|^2_{L^2(\Omega_n)} + \sup_{0\leqslant t\leqslant S} \left|\boldsymbol{u}_i|_{\Omega_n}(t)\right|^2_{\mathbb{L}^2(\Omega_n)} \leqslant \gamma_3 \left|\Omega_n\right|^{\frac{2^*-2}{2^*}\cdot\frac{1}{2}}.
$$

Therefore, we can repeat exactly the same argument as that in Section 5.3 and we can assure that there exists a subsequence of $\{V_i\}_{i\in\mathbb{N}}$, which is simply denoted by $\{V_l\}_{l\in\mathbb{N}}$, and there exists $\{V^n\}_{n\in\mathbb{N}} := \{(\boldsymbol{u}^n, T^n, C^n)\}_{n\in\mathbb{N}}$ such that

$$
\begin{aligned}
&T_l|_{\Omega_n} \to T^n && \text{strongly in } C_\pi([0,S];L^2(\Omega_n)),\\
&C_l|_{\Omega_n} \to C^n && \text{strongly in } C_\pi([0,S];L^2(\Omega_n)),\\
&\boldsymbol{u}_l|_{\Omega_n} \to \boldsymbol{u}^n && \text{strongly in } C_\pi([0,S];\mathbb{L}^2(\Omega_n))
\end{aligned}
$$

hold for each $n \in \mathbb{N}$ and $V^{n_1}(x,t) = V^{n_2}(x,t)$ holds for any $t \in [0,S]$ and for a.e. $x \in \Omega_{n_1}$ with $n_2 \geqslant n_1$.

As in Section 5.3, we here define $V := (\boldsymbol{u}, T, C)$ by

$$V(x,t) := V^n(x,t) \qquad\qquad \text{if } x \in \Omega_n.$$

Then we can show that $\nabla T, \partial_t T \in L^2(0,S;L^2_{\mathrm{loc}}(\mathbb{R}^N))$. Indeed, the uniform boundedness (5.83) implies that, for each fixed $n \in \mathbb{N}$, there exists a subsequence of $\{T_l\}_{l\in\mathbb{N}}$, which is denoted by $\{T_{l^n_m}\}_{m\in\mathbb{N}}$ satisfying the following convergences:

$$\begin{aligned}
\partial_{x_\mu} T_{l^n_m}\big|_{\Omega_n} &\rightharpoonup T^{\mu,n}_{\#} && \text{weakly in } L^2(0,S;L^2(\Omega_n)),\\
\partial_t T_{l^n_m}\big|_{\Omega_n} &\rightharpoonup T^n_{\#\#} && \text{weakly in } L^2(0,S;L^2(\Omega_n))
\end{aligned}$$

for $\mu = 1, 2, \cdots, N$. Since $T|_{\Omega_n} = T^n$ belongs to $L^2(0,S;L^2(\Omega_n))$, we can assure that $T^{\mu,n}_{\#} = \partial_{x_\mu} T\big|_{\Omega_n}$ and $T^n_{\#\#} = \partial_t T|_{\Omega_n}$ in $L^2(0,S;L^2(\Omega_n))$. This guarantees the well-definedness of ∇T and $\partial_t T$ as functions belonging to $L^2(0,S;L^2_{\mathrm{loc}}(\mathbb{R}^N))$. By the same reasoning, we can show that $\nabla C, \partial_t C \in L^2(0,S;L^2_{\mathrm{loc}}(\mathbb{R}^N))$ and $\nabla \boldsymbol{u}, \partial_t \boldsymbol{u} \in L^2(0,S;\mathbb{L}^2_{\mathrm{loc}}(\mathbb{R}^N))$.

Let $\phi_5 \in C^\infty_0(\mathbb{R}^N \times (0,S))$ and let $M \in \mathbb{N}$ satisfy $supp\phi_5 \subset \Omega_M \times [0,S]$. Then,

$$\begin{aligned}
&\int_0^S \int_{\mathbb{R}^N} T_l \phi_5 dxdt = \int_0^S \int_{\Omega_M} T_l|_{\Omega_M} \phi_5|_{\Omega_M} dxdt \\
\xrightarrow[l\to\infty]{} &\int_0^S \int_{\mathbb{R}^N} T_\star \phi_5 dxdt = \int_0^S \int_{\Omega_M} T^M \phi_5|_{\Omega_M} dxdt,
\end{aligned}$$

which yields $T_\star = T$ and $T \in L^\infty(0,S;L^{2^*}(\mathbb{R}^N))$. Likewise, $C_\star = C$ in $L^\infty(0,S;L^{2^*}(\mathbb{R}^N))$ and $\boldsymbol{u}_\star = \boldsymbol{u}$ in $L^\infty(0,S;\mathbb{L}^{2^*}_\sigma(\mathbb{R}^N))$ are verified. Moreover, we can see that

$$\begin{aligned}
&\int_0^S \int_{\mathbb{R}^N} \partial_{x_\mu} T_i \phi_5 dxdt = -\int_0^S \int_{\mathbb{R}^N} T_i \partial_{x_\mu} \phi_5 dxdt = -\int_0^S \int_{\Omega_M} T_i|_{\Omega_M} \partial_{x_\mu} \phi_5|_{\Omega_M} dxdt \\
\xrightarrow[l\to\infty]{} &\int_0^S \int_{\mathbb{R}^N} T^\mu_{\star\star} \phi_5 dxdt = -\int_0^S \int_{\mathbb{R}^N} T \partial_{x_\mu} \phi_5 dxdt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^S \int_{\mathbb{R}^N} \partial_t T_l \phi_5 dxdt = -\int_0^S \int_{\mathbb{R}^N} T_l \partial t \phi_5 dxdt = -\int_0^S \int_{\Omega_M} T_l|_{\Omega_M} \partial_t \phi_5|_{\Omega_M} dxdt \\
\xrightarrow[l\to\infty]{} &\int_0^S \int_{\mathbb{R}^N} T_{\star\star\star} \phi_5 dxdt = -\int_0^S \int_{\mathbb{R}^N} T \partial_t \phi_5 dxdt,
\end{aligned}$$

which imply that $T_{\star\star\star} = \partial_t T$ and $T^\mu_{\star\star} = \partial_{x_\mu} T$ hold in the distribution sense. Since ∇T and $\partial_t T$ are well defined in $L^2(0,S;L^2_{\mathrm{loc}}(\mathbb{R}^N))$, we can assure that $\partial_t T \in L^2(0,S;L^2(\mathbb{R}^N))$ and $\partial_{x_\mu} T \in L^\infty(0,S;L^2(\mathbb{R}^N))$ for $\mu = 1, 2, \cdots, N$. Moreover, for any $\phi_6 \in L^2(0,S;H^1(\mathbb{R}^N))$, we have

$$\begin{aligned}
&\int_0^S \int_{\mathbb{R}^N} \partial_{x_\iota} \partial_{x_\mu} T_l \phi_6 dxdt = -\int_0^S \int_{\mathbb{R}^N} \partial_{x_\mu} T_l \partial_{x_\iota} \phi_6 dxdt \\
\xrightarrow[l\to\infty]{} &\int_0^S \int_{\mathbb{R}^N} T^{\iota,\mu}_{\star\star\star\star} \phi_6 dxdt = -\int_0^S \int_{\mathbb{R}^N} \partial_{x_\mu} T \partial_{x_\iota} \phi_6 dxdt.
\end{aligned}$$

Therefore $T^{\iota,\mu}_{\star\star\star\star} = \partial_{x_\iota}\partial_{x_\mu} T$ holds in $L^2(0,S;H^{-1}(\mathbb{R}^N))$. Hence $\partial_{x_\iota}\partial_{x_\mu} T \in L^2(0,S;L^2(\mathbb{R}^N))$ for each $\iota,\mu = 1,2,\cdots,N$, which immediately leads to $T_{\star\star\star\star\star} = \Delta T$ in $L^2(0,S;L^2(\mathbb{R}^N))$. By exactly the same argument, we can derive the followings:

$$
\begin{aligned}
&C_\star = C, \quad C^{\mu}_{\star\star} = \partial_{x_\mu} C, \quad C_{\star\star\star} = \partial_t C, \quad C^{\iota,\mu}_{\star\star\star\star} = \partial_{x_\iota}\partial_{x_\mu} C, \quad C_{\star\star\star\star\star} = \Delta C,\\
&\boldsymbol{u}_\star = \boldsymbol{u}, \quad \boldsymbol{u}^{\mu}_{\star\star} = \partial_{x_\mu}\boldsymbol{u}, \quad \boldsymbol{u}_{\star\star\star} = \partial_t \boldsymbol{u}, \quad \boldsymbol{u}^{\iota,\mu}_{\star\star\star\star} = \partial_{x_\iota}\partial_{x_\mu}\boldsymbol{u}, \quad \boldsymbol{u}_{\star\star\star\star\star} = \Delta\boldsymbol{u}
\end{aligned}
$$

for all $\iota,\mu = 1,2,\cdots N$. Moreover, we can also show that $\chi_3 = \boldsymbol{u}\cdot\nabla T$ and $\chi_4 = \boldsymbol{u}\cdot\nabla C$ in $L^2(0,S;L^2(\mathbb{R}^N))$ by exactly the same argument as that in previous section.

Finally, we check the continuity of $\nabla\boldsymbol{u}$, ∇T and ∇C. We here remark that the standard argument via the abstract result Lemma 2.3.2 can not be applied, since it is difficult to check whether $\boldsymbol{u} \in L^2(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N))$ and $T, C \in L^2(0,S;L^2(\mathbb{R}^N))$.

Recalling (5.69) and (5.77), the uniform boundedness of $\boldsymbol{u}_\lambda$, we obtain

$$
|\boldsymbol{u}_l(t) - \boldsymbol{u}_l(s)|_{\mathbb{L}^2} \leqslant \int_s^t |\partial_t \boldsymbol{u}_l(\tau)|_{\mathbb{L}^2}\,d\tau \leqslant \gamma_3|t-s|^{1/2} \tag{5.84}
$$

and

$$
|\nabla\boldsymbol{u}_l(t) - \nabla\boldsymbol{u}_l(s)|_{\mathbb{L}^2} \leqslant \gamma_3,
$$

which implies that $\{\boldsymbol{u}_l(t) - \boldsymbol{u}_l(s)\}_{l\in\mathbb{N}}$ has a subsequence which weakly converges in $\mathbb{H}^1_\sigma(\mathbb{R}^N)$ for each fixed $s,t \in [0,S]$. Moreover, from the space-local strong convergences of $\{\boldsymbol{u}_l\}_{l\in\mathbb{N}}$, i.e., from the fact that

$$
\boldsymbol{u}_l|_{\Omega_n} \to \boldsymbol{u}^n \qquad \text{strongly in } C_\pi([0,S];\mathbb{L}^2(\Omega_n))
$$

for any $n \in \mathbb{N}$, it is easy to see that the weak limit of $\{\boldsymbol{u}_l(t) - \boldsymbol{u}_l(s)\}_{l\in\mathbb{N}}$ coincides with $\boldsymbol{u}(t) - \boldsymbol{u}(s)$ (use the density of $C_0^\infty((0,S);\mathbb{C}_0^\infty(\mathbb{R}^N))$ in $L^2(0,S;\mathbb{L}^2(\mathbb{R}^N))$), in particular, it can be shown that

$$
\boldsymbol{u}(t) - \boldsymbol{u}(s) \in \mathbb{H}^1_\sigma(\mathbb{R}^N) \qquad \forall t, \forall s \in [0,S].
$$

Moreover, (5.84) yields

$$
|\boldsymbol{u}(t) - \boldsymbol{u}(s)|_{\mathbb{L}^2} \leqslant \liminf_{l\to\infty} |\boldsymbol{u}_l(t) - \boldsymbol{u}_l(s)|_{\mathbb{L}^2} \leqslant \gamma_3|t-s|^{1/2}, \tag{5.85}
$$

which implies that $\boldsymbol{u}(\cdot) - \boldsymbol{u}(s)$ belongs to $C([0,S];\mathbb{L}^2_\sigma(\mathbb{R}^N))$ for arbitrary fixed $s \in [0,S]$. Recalling the regularities of $\boldsymbol{u}$ derived above, we can fix $t_5 \in [0,S]$ such that

$$
\partial_t\boldsymbol{u}(t_5) \in \mathbb{L}^2_\sigma(\mathbb{R}^N), \quad \nabla\boldsymbol{u}(t_5), \partial_{x_\iota}\partial_{x_\mu}\boldsymbol{u}(t_5) \in \mathbb{L}^2(\mathbb{R}^N) \qquad (\forall\iota, \forall\mu = 1,2,\cdots,N).
$$

Therefore, the time $t_5 \in [0,S]$ also satisfies

$$
\begin{aligned}
&\boldsymbol{u}(\cdot) - \boldsymbol{u}(t_5) \in W^{1,2}(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N)),\\
&\boldsymbol{u}(t) - \boldsymbol{u}(t_5) \in D(\mathcal{A}_{\mathbb{R}^N}) \text{ for a.e. } t \in [0,S],\\
&\mathcal{A}_{\mathbb{R}^N}(\boldsymbol{u}(\cdot) - \boldsymbol{u}(t_5)) = \Delta(\boldsymbol{u}(\cdot) - \boldsymbol{u}(t_5)) \in L^2(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N)).
\end{aligned}
$$

These imply that we can apply the standard argument via Lemma 2.3.2 to $\boldsymbol{u}(\cdot) - \boldsymbol{u}(t_5)$, i.e., we can assure $\nabla \boldsymbol{u}(\cdot) - \nabla \boldsymbol{u}(t_5) \in C([0,S];\mathbb{L}^2(\mathbb{R}^N))$. This fact immediately leads to $\nabla \boldsymbol{u} \in C([0,S];\mathbb{L}^2(\mathbb{R}^N))$ since $\nabla \boldsymbol{u}(t_5) \in \mathbb{L}^2(\mathbb{R}^N)$.

Moreover, since $\boldsymbol{u}(t) - \boldsymbol{u}(s) \in \mathbb{H}^1_\sigma(\mathbb{R}^N)$, we can apply the Sobolev's inequality and we have

$$|\boldsymbol{u}(t) - \boldsymbol{u}(s)|_{\mathbb{L}^{2^*}} \leqslant \gamma_3 \, |\nabla \boldsymbol{u}(t) - \nabla \boldsymbol{u}(s)|_{\mathbb{L}^2} \, .$$

Together with the fact that $\nabla \boldsymbol{u} \in C([0,S];\mathbb{L}^2(\mathbb{R}^N))$, we obtain $\boldsymbol{u} \in C([0,S];\mathbb{L}^{2^*}_\sigma(\mathbb{R}^N))$. By almost the same arguments above, we can show that $T, C \in C([0,S];L^{2^*}(\mathbb{R}^N))$ and $\nabla T, \nabla C \in C([0,S];L^2(\mathbb{R}^N))$.

Thus, we can assure that $(\boldsymbol{u}, T, C)$, constructed above, becomes a periodic solution of the original system (DCBF), whence follows our result. □

Chapter 6

Existence of Attractors

6.1 Problems and Main Theorems

We consider the existence of global and exponential attractor for solutions of (DCBF).

$$
\text{(DCBF)}\begin{cases}
\partial_t \boldsymbol{u} + \mathcal{A}\boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}\boldsymbol{g}T + \mathcal{P}\boldsymbol{h}C + \mathcal{P}\boldsymbol{f}_1 & (x,t) \in \Omega\times[0,S],\\
\partial_t T - \Delta T + \boldsymbol{u}\cdot\nabla T = f_2 & (x,t) \in \Omega\times[0,S],\\
\partial_t C - \Delta C + \boldsymbol{u}\cdot\nabla C = \rho\Delta T + f_3 & (x,t) \in \Omega\times[0,S],\\
\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0,\ T(\cdot,0) = T_0,\ C(\cdot,0) = C_0.
\end{cases}
$$

In this chapter, we deal with the case where Ω is bounded domain with sufficiently smooth boundary $\partial\Omega$ and the autonomous case, i.e., the external forces $\boldsymbol{f}_1$, f_2 and f_3 depend only on the space variable x. As for the boundary condition for (DCBF), we impose either Dirichlet boundary condition:

$$
\boldsymbol{u} = 0,\ T = 0,\ C = 0, \qquad (x,t) \in \partial\Omega\times[0,S],
$$

or Neumann boundary condition:

$$
\boldsymbol{u} = 0,\ \frac{\partial T}{\partial n} = 0,\ \frac{\partial C}{\partial n} = 0, \qquad (x,t) \in \partial\Omega\times[0,S].
$$

Throughout this chapter, We use the following notations:

$$
\begin{aligned}
&\mathcal{H} := \mathbb{H}^1_\sigma(\Omega) \times L^2(\Omega) \times L^2(\Omega),\\
&\mathcal{H}^1_D := \mathbb{H}^1_\sigma(\Omega) \times H^1_0(\Omega) \times H^1_0(\Omega),\\
&\mathcal{H}^1_N := \mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times H^1(\Omega),\\
&\mathcal{H}^{2,1} := (\mathbb{H}^2(\Omega) \cap \mathbb{H}^1_\sigma(\Omega)) \times H^1(\Omega) \times H^1(\Omega),\\
&\mathcal{H}^2 = \mathcal{H}^{2,2} := (\mathbb{H}^2(\Omega) \cap \mathbb{H}^1_\sigma(\Omega)) \times H^2(\Omega) \times H^2(\Omega),
\end{aligned}
$$

where each spaces are endowed with the norm defined by

$$\begin{aligned}
&\|U\|_{\mathcal{H}}^2 := |\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2 + |T|_{L^2}^2 + |C|_{L^2}^2,\\
&\|U\|_{\mathcal{H}_D^1}^2 = \|U\|_{\mathcal{H}_N^1}^2 := |\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2 + |T|_{H^1}^2 + |C|_{H^1}^2,\\
&\|U\|_{\mathcal{H}^{2,1}}^2 := |\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 + |T|_{H^1}^2 + |C|_{H^1}^2,\\
&\|U\|_{\mathcal{H}^2}^2 := |\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 + |T|_{H^2}^2 + |C|_{H^2}^2.
\end{aligned}$$

According to our results in Chapter 4, we can assure the following solvability.

Proposition 6.1.1 (Dirichlet case). *Let $N \leqslant 4$ and let Dirichlet boundary condition be imposed. Moreover, we assume that $\boldsymbol{f}_1 \in \mathbb{L}^2(\Omega)$ and $f_2, f_3 \in L^2(\Omega)$. Then, for any initial data $(\boldsymbol{u}_0, T_0, C_0)$ belonging to $\mathcal{H}$,* (DCBF) *possesses a unique solution $(\boldsymbol{u}, T, C)$ which satisfies the following regularities:*

$$\begin{aligned}
&\boldsymbol{u} \in C([0,S]; \mathbb{H}_\sigma^1(\Omega)) \cap L^2(0,S;\mathbb{H}^2(\Omega)),\\
&\partial_t \boldsymbol{u} \in L^2(0,S;\mathbb{L}_\sigma^2(\Omega)),\\
&T,\ C \in C([0,S]; L^2(\Omega)) \cap L^2(0,S;H_0^1(\Omega)),\\
&\sqrt{t}\Delta T,\ \sqrt{t}\partial_t T,\ \sqrt{t}\Delta C,\ \sqrt{t}\partial_t C \in L^2(0,S;L^2(\Omega))
\end{aligned}$$

for any time interval $S > 0$. Furthermore, if the initial data $(\boldsymbol{u}_0, T_0, C_0)$ belongs to $\mathcal{H}_D^1$, (DCBF) *possesses a unique solution $(\boldsymbol{u}, T, C)$ which satisfies the following regularities:*

$$\begin{aligned}
&\boldsymbol{u} \in C([0,S]; \mathbb{H}_\sigma^1(\Omega)) \cap L^2(0,S;\mathbb{H}^2(\Omega)) \cap W^{1,2}(0,S;\mathbb{L}_\sigma^2(\Omega)),\\
&T,\ C \in C([0,S]; H_0^1(\Omega)) \cap L^2(0,S;H^2(\Omega)) \cap W^{1,2}(0,S;L^2(\Omega))
\end{aligned}$$

for any $S > 0$.

Proposition 6.1.2 (Neumann case). *Let $N \leqslant 4$ and let Neumann boundary condition be imposed. Moreover, we assume that $\boldsymbol{f}_1 \in \mathbb{L}^2(\Omega)$ and $f_2, f_3 \in L^2(\Omega)$. Then, for any initial data $(\boldsymbol{u}_0, T_0, C_0)$ belonging to $\mathcal{H}$,* (DCBF) *possesses a unique solution $(\boldsymbol{u}, T, C)$ which satisfies the following regularities:*

$$\begin{aligned}
&\boldsymbol{u} \in C([0,S]; \mathbb{H}_\sigma^1(\Omega)) \cap L^2(0,S;\mathbb{H}^2(\Omega)),\\
&\partial_t \boldsymbol{u} \in L^2(0,S;\mathbb{L}_\sigma^2(\Omega)),\\
&T,\ C \in C([0,S]; L^2(\Omega)) \cap L^2(0,S;H^1(\Omega)),\\
&\sqrt{t}\Delta T,\ \sqrt{t}\partial_t T,\ \sqrt{t}\Delta C,\ \sqrt{t}\partial_t C \in L^2(0,S;L^2(\Omega))
\end{aligned}$$

for any time interval $S > 0$. Furthermore, if the initial data $(\boldsymbol{u}_0, T_0, C_0)$ belongs to $\mathcal{H}_N^1$, (DCBF) *possesses a unique solution $(\boldsymbol{u}, T, C)$ which satisfies the following regularities:*

$$\begin{aligned}
&\boldsymbol{u} \in C([0,S]; \mathbb{H}_\sigma^1(\Omega)) \cap L^2(0,S;\mathbb{H}^2(\Omega)) \cap W^{1,2}(0,S;\mathbb{L}_\sigma^2(\Omega)),\\
&T,\ C \in C([0,S]; H^1(\Omega)) \cap L^2(0,S;H^2(\Omega)) \cap W^{1,2}(0,S;L^2(\Omega))
\end{aligned}$$

for any $S > 0$.

On the basis of Proposition 6.1.1, we can define the semigroup $\{\mathscr{S}_D(t)\}_{t\geq 0}$ acting on $\mathcal{H}$ and $\mathcal{H}^1_D$ by the correspondence $\mathscr{S}_D(t)(\boldsymbol{u}_0, T_0, C_0) := (\boldsymbol{u}(t), T(t), C(t))$, where $(\boldsymbol{u}, T, C)$ is the unique solution of (DCBF) given in Proposition 6.1.1 with the initial data $(\boldsymbol{u}_0, T_0, C_0)$. In the same manner, we define the semigroup $\{\mathscr{S}_N(t)\}_{t\geq 0}$ acting on $\mathcal{H}$ and $\mathcal{H}^1_N$, based on Proposition 6.1.2.

We first show the following results for the Dirichlet boundary condition case.

Theorem 6.1.1. *The dynamical system* $(\{\mathscr{S}_D(t)\}_{t\geq 0}, \mathcal{H})$ *possesses a global attractor* $\mathscr{A}_D$.

Theorem 6.1.2. *The dynamical system* $(\{\mathscr{S}_D(t)\}_{t\geq 0}, \mathcal{H})$ *possesses an exponential attractor* $\mathscr{M}_D$.

Theorem 6.1.3. *The dynamical system* $(\{\mathscr{S}_D(t)\}_{t\geq 0}, \mathcal{H}^1_D)$ *possesses a global attractor* $\mathscr{A}^1_D$ *and an exponential attractor* $\mathscr{M}^1_D$.

In order to apply the abstract results stated in Section 2 (Proposition 2.4.1, Corollary 2.4.1 and Proposition 2.4.2), we establish some minute a priori estimates in the next section. From almost the same procedures as those in the previous chapters, we can derive the second energy estimates for solutions of (DCBF). However, in order to assure the existence of compact absorbing set for the dynamical system $(\{\mathscr{S}_D(t)\}_{t\geq 0}, \mathcal{H}^1_D)$, we have to establish the pointwise estimates for $|\boldsymbol{u}(t)|_{\mathbb{H}^2}$, $|T(t)|_{H^2}$ and $|C(t)|_{H^2}$. In order to cope with this difficulty, we introduce the abstract result given in Brézis [11] (Proposition 6.2.1) and we prepare Lemma 6.2.1 and Corollary 6.2.1, which can be proved by some argument similar to that for Proposition 6.2.1. In section 6.3, we discuss the estimate for differences of two distinct solutions so that we can guarantee some continuity required in Proposition 2.4.1, Corollary 2.4.1 and Proposition 2.4.2 and we demonstrate the existence of global and exponential attractors for Dirichlet boundary condition case. In this argument, Proposition 6.2.1, Lemma 6.2.1 and Corollary 6.2.1 play an essential role again.

When we consider the Neumann boundary condition case, it is easy to see that attractors can not exist in the usual sense. Indeed, integrating the second equation of (DCBF) over Ω and $[0, t]$, we have

$$\int_\Omega T(t)dx = \int_\Omega T_0 dx + t\int_\Omega f_2 dx,$$

which yields the following inequality:

$$\left|\int_\Omega T_0 dx + t\int_\Omega f_2 dx\right| \leqslant \int_\Omega |T(t)|dx \leqslant |\Omega|^{1/2}|T(t)|_{L^2},$$

where $|\Omega|$ is the measure of Ω. Therefore, if $\int_\Omega f_2 dx \neq 0$, then $|T(t)|_{L^2}$ strictly increases as $t \to \infty$. Moreover, even if $\int_\Omega f_2 dx = 0$, $|T(t)|_{L^2}$ is always bounded from below by the mean value of initial data T_0. This implies that there is no bounded subset in $\mathcal{H}$ which attracts all orbit of solution.

Based on this fact, we assume

$$\int_\Omega f_2 dx = \int_\Omega f_3 dx = 0$$

for the Neumann boundary condition case. Then we can define the dynamical systems $(\{\mathscr{S}_N(t)\}_{t\geq 0}, \mathcal{H}_{m_T,m_C})$ and $(\{\mathscr{S}_N(t)\}_{t\geq 0}, \mathcal{H}^1_{N,m_T,m_C})$, where

$$\begin{aligned} &L^2_m(\Omega) := \left\{ U \in L^2(\Omega);\ \left| \frac{1}{|\Omega|} \int_\Omega U dx \right| \leqslant m \right\}, \\ &\mathcal{H}_{m_T,m_C} := \mathbb{H}^1_\sigma(\Omega) \times L^2_{m_T}(\Omega) \times L^2_{m_C}(\Omega), \\ &\mathcal{H}^1_{N,m_T,m_C} := \mathcal{H}_{m_T,m_C} \cap \mathcal{H}^1_N. \end{aligned}$$

We can show the existence of attractors for these dynamical systems in the last section of this chapter.

Theorem 6.1.4. *Assume that*

$$\int_\Omega f_2 dx = \int_\Omega f_3 dx = 0.$$

Then, for any positive numbers m_T *and* m_C*, the dynamical system* $(\{\mathscr{S}_N(t)\}_{t\geq 0}, \mathcal{H}_{m_T,m_C})$ *possesses a global attractor* $\mathscr{A}_{N,m_T,m_C}$ *and an exponential attractor* $\mathscr{M}_{N,m_T,m_C}$*. Furthermore, for arbitrary* $m_T, m_C > 0$*, the dynamical system* $(\{\mathscr{S}_N(t)\}_{t\geq 0}, \mathcal{H}^1_{N,m_T,m_C})$ *admits a global attractor* $\mathscr{A}^1_{N,m_T,m_C}$ *and an exponential attractor* $\mathscr{M}^1_{N,m_T,m_C}$*.*

Remarks.
(1) To be precise, $\mathscr{A}_D = \mathscr{A}^1_D$ holds true. In fact, since $\mathscr{A}^1_D$ is bounded in $\mathcal{H}$, $\mathscr{A}^1_D$ is attracted to $\mathscr{A}_D$ by the semigroup $\{\mathscr{S}_D(t)\}_{t\geq 0}$. Moreover, from the strict invariance, i.e., $\mathscr{S}(t)\mathscr{A}^1_D = \mathscr{A}^1_D$, we can derive

$$dist_{\mathcal{H}}(\mathscr{A}^1_D, \mathscr{A}_D) = \lim_{t\to\infty} dist_{\mathcal{H}}(\mathscr{S}(t)\mathscr{A}^1_D, \mathscr{A}_D) = 0.$$

This identity and the compactness of $\mathscr{A}_D$ in $\mathcal{H}$ implies that $\mathscr{A}^1_D \subset \mathscr{A}_D$. Conversely, we can obtain the fact that $\mathscr{A}_D$ is bounded in $\mathcal{H}^1_D$, since $\mathscr{A}_D \subset \mathscr{B}_0$, where $\mathscr{B}_0$ will be defined the end of Section 6.2 (recall that $\mathscr{A}_D$ is the smallest compact absorbing set of $(\{\mathscr{S}_D(t)\}_{t\geq 0}, \mathcal{H})$). Then, together with the strict invariance of $\mathscr{A}_D$ and the compactness of $\mathscr{A}^1_D$, we can assure that $dist_{\mathcal{H}^1_D}(\mathscr{A}_D, \mathscr{A}^1_D) = 0$, which yields $\mathscr{A}_D \subset \mathscr{A}^1_D$. By the same reasoning, $\mathscr{A}_{N,m_T,m_C} = \mathscr{A}^1_{N,m_T,m_C}$ holds for arbitrary $m_T, m_C > 0$.

(2) If $|f_2|_{L^2}$ and $|f_3|_{L^2}$ are sufficiently small (the smallness will be given concretely by (6.61) in Section 6.3), then we can show that the global attractor $\mathscr{A}_D$ consists only one element and $\mathscr{A}_D$ satisfies all the definitions of exponential attractor (see estimates (6.60) in Section 6.3).

(3) Let

$$\mathscr{A}_N := \bigcup_{m_T, m_C \geqslant 0} \mathscr{A}_{N,m_T,m_C}.$$

Then we can assure that $\mathscr{A}_N$ is strictly invariant and satisfies attracting property for the dynamical system $(\{\mathscr{S}_N(t)\}_{t\geq 0}, \mathcal{H})$. Similarly,

$$\mathscr{A}_N^1 := \bigcup_{m_T, m_C \geqslant 0} \mathscr{A}_{N,m_T,m_C}^1$$

becomes a strictly invariant attracting set of $(\{\mathscr{S}_N(t)\}_{t\geq 0}, \mathcal{H}_N^1)$. Immediately, $\mathscr{A}_N = \mathscr{A}_N^1$ is valid (see remark (1)).

6.2 A priori Estimates

We first establish some a priori estimates in this section so that we can construct a compact absorbing set. Throughout this chapter, κ is the coefficient appearing in Poincaré's inequality:

$$\begin{aligned} &|U|_{L^2}^2 \leqslant \kappa |\nabla U|_{L^2}^2, \quad |\nabla U|_{L^2}^2 \leqslant \kappa |\Delta U|_{L^2}^2, \quad \forall U \in H^2(\Omega) \cap H_0^1(\Omega), \\ &|\boldsymbol{u}|_{\mathbb{L}^2}^2 \leqslant \kappa |\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2, \quad |\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2 \leqslant \kappa |\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 \quad \forall \boldsymbol{u} \in \mathbb{H}^2(\Omega) \cap \mathbb{H}_\sigma^1(\Omega) \end{aligned}$$

and the constant b is defined by

$$b := \frac{1}{2} \min \left\{ \frac{1}{\kappa}, \frac{\nu}{\kappa} \right\}.$$

Fix a positive number $\mu > 0$ arbitrary. Let the initial data $(\boldsymbol{u}_0, T_0, C_0)$ belong to

$$B_\mu := \{(\boldsymbol{u}_0, T_0, C_0) \in \mathcal{H};\ |\boldsymbol{u}_0|_{\mathbb{H}^1}^2 + |T_0|_{L^2}^2 + |C_0|_{L^2}^2 \leqslant \mu\}. \tag{6.1}$$

Multiplying the second equation of (DCBF) by T, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|T|_{L^2}^2 + |\nabla T|_{L^2}^2 \leqslant |f_2|_{L^2}|T|_{L^2} \leqslant \sqrt{\kappa}|f_2|_{L^2}|\nabla T|_{L^2} \\ \Rightarrow\ &\frac{d}{dt}|T|_{L^2}^2 + |\nabla T|_{L^2}^2 \leqslant \kappa |f_2|_{L^2}^2 \\ \Rightarrow\ &\frac{d}{dt}|T|_{L^2}^2 + b|T|_{L^2}^2 + (1 - b\kappa)|\nabla T|_{L^2}^2 \leqslant \kappa |f_2|_{L^2}^2. \end{aligned} \tag{6.2}$$

Using Gronwall's inequality, we have

$$|T(t)|_{L^2}^2 + (1 - b\kappa) \int_0^t e^{-b(t-s)} |\nabla T(s)|_{L^2}^2 ds \leqslant |T_0|_{L^2}^2 e^{-bt} + \kappa |f_2|_{L^2}^2 \int_0^t e^{-b(t-s)} ds. \tag{6.3}$$

Here define

$$t_\mu^1 := \max \left\{ 0, -\frac{1}{b} \log \left(\frac{\kappa}{b\mu} |f_2|_{L^2}^2 \right) \right\}$$

and let $t \geqslant t_\mu^1$. Then, since

$$\mu e^{-bt} \leqslant \mu e^{-bt_\mu^1} \leqslant \frac{\kappa |f_2|_{L^2}^2}{b}$$

holds for $t \geqslant t_\mu^1$, we obtain

$$|T(t)|_{L^2}^2 + (1-b\kappa)\int_0^t e^{-b(t-s)}|\nabla T(s)|_{L^2}^2 ds \leqslant \frac{2\kappa}{b}|f_2|_{L^2}^2 =: M_1. \tag{6.4}$$

Integration of (6.2) over $[t, t+1]$ yields

$$\int_t^{t+1} |\nabla T(s)|_{L^2}^2 ds \leqslant |T(t)|_{L^2}^2 + \kappa |f_2|_{L^2}^2 \leqslant M_1 + \kappa|f_2|_{L^2}^2 =: M_2. \tag{6.5}$$

Similarly, multiplication of the third equation by C and Gronwall's inequality give

$$\begin{aligned}
&\frac{d}{dt}|C|_{L^2}^2 + |\nabla C|_{L^2}^2 \leqslant 2\kappa|f_3|_{L^2}^2 + 2\rho^2|\nabla T|_{L^2}^2 \\
\Rightarrow\ &\frac{d}{dt}|C|_{L^2}^2 + b|C|_{L^2}^2 + (1-b\kappa)|\nabla C|_{L^2}^2 \leqslant 2\kappa|f_3|_{L^2}^2 + 2\rho^2|\nabla T|_{L^2}^2 \\
\Rightarrow\ &|C(t)|_{L^2}^2 + (1-b\kappa)\int_0^t e^{-b(t-s)}|\nabla C(s)|_{L^2}^2 ds \\
&\quad \leqslant |C_0|_{L^2}^2 e^{-bt} + 2\kappa|f_3|_{L^2}^2 \int_0^t e^{-b(t-s)} ds + 2\rho^2 \int_0^t e^{-b(t-s)}|\nabla T(s)|_{L^2}^2 ds.
\end{aligned} \tag{6.6}$$

Here we define $t_\mu^2 := \max\left\{t_\mu^1, -\frac{1}{b}\log\left(\frac{\kappa}{b\mu}|f_3|_{L^2}^2\right)\right\}$. Then, the estimates (6.1), (6.4), (6.5) and (6.6) yield

$$|C(t)|_{L^2}^2 + (1-b\kappa)\int_0^t e^{-b(t-s)}|\nabla C(s)|_{L^2}^2 ds \leqslant \frac{3\kappa}{b}|f_3|_{L^2}^2 + \frac{2\rho^2}{1-b\kappa}M_1 =: M_3 \tag{6.7}$$

and

$$\begin{aligned}
\int_t^{t+1} |\nabla C(s)|_{L^2}^2 ds &\leqslant |C(t)|_{L^2}^2 + 2\kappa|f_3|_{L^2}^2 + 2\rho^2 \int_t^{t+1} |\nabla T(s)|_{L^2}^2 ds \\
&\leqslant M_3 + 2\kappa|f_3|_{L^2}^2 + 2\rho^2 M_2 =: M_4
\end{aligned} \tag{6.8}$$

for any $t \leqslant t_\mu^2$. Multiplying the first equation of (DCBF) by $\mathcal{A}\boldsymbol{u}$, we get

$$\begin{aligned}
&\frac{d}{dt}|\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2 + \nu|\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 \leqslant \frac{3|\boldsymbol{g}|^2}{\nu}|T|_{L^2}^2 + \frac{3|\boldsymbol{h}|^2}{\nu}|C|_{L^2}^2 + \frac{3}{\nu}|\boldsymbol{f}_1|_{\mathbb{L}^2}^2 \\
\Rightarrow\ &\frac{d}{dt}|\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2 + b|\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2 \leqslant \frac{3\kappa|\boldsymbol{g}|^2}{\nu}|\nabla T|_{L^2}^2 + \frac{3\kappa|\boldsymbol{h}|^2}{\nu}|\nabla C|_{L^2}^2 + \frac{3}{\nu}|\boldsymbol{f}_1|_{\mathbb{L}^2}^2 \\
\Rightarrow\ &|\nabla \boldsymbol{u}(t)|_{\mathbb{L}^2}^2 \leqslant |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2 e^{-bt} + \frac{3}{b\nu}|\boldsymbol{f}_1|_{\mathbb{L}^2}^2 \\
&\quad + \frac{3\kappa|\boldsymbol{g}|^2}{\nu}\int_0^t e^{-b(t-s)}|\nabla T(s)|_{L^2}^2 ds + \frac{3\kappa|\boldsymbol{h}|^2}{\nu}\int_0^t e^{-b(t-s)}|\nabla C(s)|_{L^2}^2 ds.
\end{aligned} \tag{6.9}$$

Here we define

$$t_\mu^0 := \max\left\{t_\mu^2, -\frac{1}{b}\log\left(\frac{|\boldsymbol{f}_1|_{\mathbb{L}^2}^2}{b\nu\mu}\right)\right\}.$$

Then, by virtue of (6.1), (6.4), (6.7) and (6.9), we obtain

$$|\nabla \boldsymbol{u}(t)|_{\mathbb{L}^2}^2 \leqslant \frac{4}{b\nu}|\boldsymbol{f}_1|_{\mathbb{L}^2}^2 + \frac{3\kappa|\boldsymbol{g}|^2}{\nu(1-b\kappa)}M_1 + \frac{3\kappa|\boldsymbol{h}|^2}{\nu(1-b\kappa)}M_3 =: M_5 \tag{6.10}$$

and in view of (6.5), (6.8), (6.9) and (6.10), we have

$$\int_t^{t+1} |\mathcal{A}\boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds \leqslant \frac{1}{\nu}\left(M_5 + \frac{3\kappa|\boldsymbol{g}|^2}{\nu}M_2 + \frac{3\kappa|\boldsymbol{h}|^2}{\nu}M_4 + \frac{3}{\nu}|\boldsymbol{f}_1|_{\mathbb{L}^2}^2\right) =: M_6 \tag{6.11}$$

for $t \geqslant t_\mu^0$. Multiplying the first equation of (DCBF) by $\partial_t \boldsymbol{u}$, we get

$$|\partial_t \boldsymbol{u}|_{\mathbb{L}^2}^2 + \nu\frac{d}{dt}|\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2 + a\frac{d}{dt}|\boldsymbol{u}|_{\mathbb{L}^2}^2 \leqslant 3\kappa|\boldsymbol{g}|^2|\nabla T|_{L^2}^2 + 3\kappa|\boldsymbol{h}|^2|\nabla C|_{L^2}^2 + 3|\boldsymbol{f}_1|_{\mathbb{L}^2}^2. \tag{6.12}$$

From (6.5), (6.8) and (6.10), we can derive

$$\int_t^{t+1} |\partial_t \boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds \leqslant \nu M_5 + a\kappa M_5 + 3\kappa|\boldsymbol{g}|^2 M_2 + 3\kappa|\boldsymbol{h}|^2 M_4 + 3|\boldsymbol{f}_1|_{\mathbb{L}^2}^2 =: M_7 \tag{6.13}$$

for any $t \geqslant t_\mu^0$. Next multiplying the second equation of (DCBF) by $-t\Delta T$, we have

$$\frac{d}{dt}t|\nabla T|_{L^2}^2 + t|\Delta T|_{L^2}^2 \leqslant |\nabla T|_{L^2}^2 + \frac{27}{2}t\gamma_0^2|\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2|\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2|\nabla T|_{L^2}^2 + 2t|f_2|_{L^2}^2. \tag{6.14}$$

Here γ_0 is a constant satisfying the following inequality (see estimates for the convection terms in Chapter 4 and Chapter 5):

$$|\boldsymbol{w}\cdot\nabla V|_{L^2}^2 \leqslant \gamma_0|\nabla \boldsymbol{w}|_{\mathbb{L}^2}|\mathcal{A}\boldsymbol{w}|_{\mathbb{L}^2}|\nabla V|_{L^2}|\Delta V|_{L^2} \qquad \forall \boldsymbol{w}\in D(\mathcal{A}),\ \forall V\in D(-\Delta_D). \tag{6.15}$$

Integrate (6.14) over $[s, t+1]$ with $s\in[t,t+1]$ and $t\geqslant t_\mu^0$. Then applying Gronwall's inequality, we obtain

$$\begin{aligned}
&(t+1)|\nabla T(t+1)|_{L^2}^2 \\
&\leqslant \left(s|\nabla T(s)|_{L^2}^2 + \int_s^{t+1}|\nabla T|_{L^2}^2 d\tau + 2|f_2|_{L^2}^2\int_s^{t+1}\tau d\tau\right) \\
&\qquad\qquad \times \exp\left(\int_s^{t+1}\frac{27}{2}\gamma_0^2|\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2|\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 d\tau\right) \\
&\leqslant \big((t+1)|\nabla T(s)|_{L^2}^2 + M_2 + 2|f_2|_{L^2}^2(t+1)\big)\exp\left(\frac{27}{2}\gamma_0^2 M_5 M_6\right).
\end{aligned} \tag{6.16}$$

Integrating (6.16) over $[t, t+1]$ with respect to the s-variable and using (6.5), we can deduce

$$(t+1)|\nabla T(t+1)|_{L^2}^2 \leqslant \big((t+1)M_2 + M_2 + 2|f_2|_{L^2}^2(t+1)\big)\exp\left(\frac{27}{2}\gamma_0^2 M_5 M_6\right).$$

Hence,

$$|\nabla T(t)|_{L^2}^2 \leqslant \left(2M_2 + 2|f_2|_{L^2}^2\right) \exp\left(\frac{27}{2}\gamma_0^2 M_5 M_6\right) =: M_8 \tag{6.17}$$

holds for $t \geqslant t_\mu^0 + 1$. Moreover, integration (6.14) over $[t, t+1]$ ($t \geqslant t_\mu^0 + 1$) gives

$$\begin{aligned}
&t \int_t^{t+1} |\Delta T(s)|_{L^2}^2 ds \\
\leqslant &\int_t^{t+1} s|\Delta T(s)|_{L^2}^2 ds \\
\leqslant &t|\nabla T(t)|_{L^2}^2 + \int_t^{t+1} |\nabla T(s)|_{L^2}^2 ds \\
&\quad + \frac{27}{2}\gamma_0^2 \int_t^{t+1} s|\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2 |\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 |\nabla T|_{L^2}^2 ds + 2|f_2|_{L^2}^2 \int_t^{t+1} sds,
\end{aligned}$$

which implies that

$$\int_t^{t+1} |\Delta T(s)|_{L^2}^2 ds \leqslant M_8 + M_2 + 27\gamma_0^2 M_5 M_6 M_8 + 4|f_2|_{L^2}^2 =: M_9 \tag{6.18}$$

is valid for any $t \geqslant t_\mu^0 + 1$. Multiplying the second equation of (DCBF) by $t\partial_t T$ and using (6.15), we get

$$\begin{aligned}
&t|\partial_t T|_{L^2}^2 + \frac{d}{dt} t|\nabla T|_{L^2}^2 \\
&\qquad \leqslant |\nabla T|_{L^2}^2 + 2t\gamma_0 |\nabla \boldsymbol{u}|_{\mathbb{L}^2} |\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2} |\nabla T|_{L^2} |\Delta T|_{L^2} + 2t|f_2|_{L^2}^2.
\end{aligned} \tag{6.19}$$

Integration of (6.19) over $[t, t+1]$ ($t \geqslant t_\mu^0 + 1$) with respect to the variable s yields

$$\begin{aligned}
&t \int_t^{t+1} |\partial_t T(s)|_{L^2}^2 ds \\
&\qquad \leqslant tM_8 + M_2 + 2(t+1)\gamma_0 M_5^{1/2} M_6^{1/2} M_8^{1/2} M_9^{1/2} + 2|f_2|_{L^2}^2 (t+1),
\end{aligned}$$

namely,

$$\int_t^{t+1} |\partial_t T(s)|_{L^2}^2 ds \leqslant M_8 + M_2 + 4\gamma_0 M_5^{1/2} M_6^{1/2} M_8^{1/2} M_9^{1/2} + 4|f_2|_{L^2}^2 =: M_{10}. \tag{6.20}$$

By almost the same procedures as above, the following inequalities can be derived from the third equation of (DCBF):

$$\begin{aligned}
&\frac{d}{dt} t|\nabla C|_{L^2}^2 + t|\Delta C|_{L^2}^2 \\
&\qquad \leqslant |\nabla C|_{L^2}^2 + \frac{27}{2} t\gamma_0^2 |\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2 |\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 |\nabla C|_{L^2}^2 + 4\rho^2 t|\Delta T|_{L^2}^2 + 4t|f_3|_{L^2}^2, \\
&t|\partial_t C|_{L^2}^2 + \frac{d}{dt} t|\nabla C|_{L^2}^2 \\
&\qquad \leqslant |\nabla C|_{L^2}^2 + 3t\gamma_0 |\nabla \boldsymbol{u}|_{\mathbb{L}^2} |\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2} |\nabla C|_{L^2} |\Delta C|_{L^2} + 3\rho^2 t|\Delta T|_{L^2}^2 + 3t|f_3|_{L^2}^2.
\end{aligned} \tag{6.21}$$

Then we can assure that

$$
\begin{aligned}
|\nabla C(t+1)|_{L^2}^2 &\leqslant \left(2\int_t^{t+1}|\nabla C(s)|_{L^2}^2 ds + 4|f_3|_{L^2}^2 + 4\rho^2\int_t^{t+1}|\Delta T(s)|_{L^2}^2 ds\right)\\
&\qquad\times \exp\left(\int_t^{t+1}\frac{27}{2}\gamma_0^2|\nabla \boldsymbol{u}(s)|_{\mathbb{L}^2}^2|\mathcal{A}\boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds\right)\\
&\leqslant \left(2M_4 + 4|f_3|_{L^2}^2 + 4\rho^2 M_9\right)\exp\left(\frac{27}{2}\gamma_0^2 M_5 M_6\right) =: M_{11}
\end{aligned}
\tag{6.22}
$$

for any $t \geqslant t_\mu^0 + 1$ and

$$
\begin{aligned}
&\int_t^{t+1}|\Delta C(s)|_{L^2}^2 ds\\
&\quad\leqslant M_{11} + M_4 + 27\gamma_0^2 M_5 M_6 M_{11} + 8\rho^2 M_9 + 8|f_3|_{L^2}^2 =: M_{12},
\end{aligned}
\tag{6.23}
$$

$$
\begin{aligned}
&\int_t^{t+1}|\partial_t C(s)|_{L^2}^2 ds\\
&\quad\leqslant M_{11} + M_4 + 6\gamma_0 M_5^{1/2}M_6^{1/2}M_{11}^{1/2}M_{12}^{1/2} + 6\rho^2 M_9 + 6|f_3|_{L^2}^2 =: M_{13}
\end{aligned}
\tag{6.24}
$$

for any $t \geqslant t_\mu^0 + 2$.

Here we define

$$
D_h\boldsymbol{u}(t) := \boldsymbol{u}(t+h)-\boldsymbol{u}(t),\ D_hT(t) := T(t+h)-T(t),\ D_hC(t) := C(t+h)-C(t)
\tag{6.25}
$$

for each fixed $h \in (0,1)$. Then, we get

$$
\partial_t D_h\boldsymbol{u}(t) + \nu\mathcal{A}D_h\boldsymbol{u}(t) + aD_h\boldsymbol{u}(t) = \mathcal{P}_\Omega\boldsymbol{g}D_hT(t) + \mathcal{P}_\Omega\boldsymbol{h}D_hC(t)
\tag{6.26}
$$

from the first equation of (DCBF). Multiplying (6.26) by $D_h\boldsymbol{u}$, we have

$$
\frac{d}{dt}|D_h\boldsymbol{u}(t)|_{\mathbb{L}^2}^2 + \nu|\nabla D_h\boldsymbol{u}(t)|_{\mathbb{L}^2}^2 \leqslant \frac{2\kappa|\boldsymbol{g}|^2}{\nu}|D_hT(t)|_{L^2}^2 + \frac{2\kappa|\boldsymbol{h}|^2}{\nu}|D_hC(t)|_{L^2}^2.
\tag{6.27}
$$

Let $t \geqslant t_\mu^0 + 2$ and $s \in [t, t+1]$. Integrating (6.27) over $[s, t+1]$, we obtain

$$
\begin{aligned}
&|D_h\boldsymbol{u}(t+1)|_{\mathbb{L}^2}^2\\
&\quad\leqslant |D_h\boldsymbol{u}(s)|_{\mathbb{L}^2}^2 + \frac{2\kappa|\boldsymbol{g}|^2}{\nu}\int_s^{t+1}|D_hT(\tau)|_{L^2}^2 d\tau + \frac{2\kappa|\boldsymbol{h}|^2}{\nu}\int_s^{t+1}|D_hC(\tau)|_{L^2}^2 d\tau.
\end{aligned}
\tag{6.28}
$$

We here note that

$$
\begin{aligned}
&\int_t^{t+1}|D_h\boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds \leqslant 2h^2 M_7,\\
&\int_t^{t+1}|D_hT(s)|_{L^2}^2 ds \leqslant 2h^2 M_{10},\\
&\int_t^{t+1}|D_hC(s)|_{L^2}^2 ds \leqslant 2h^2 M_{13}
\end{aligned}
$$

are valid for $t \geqslant t_\mu^0 + 2$, since estimates (6.13), (6.20) and (6.24) hold. Using these inequalities, we have

$$\left|\frac{\boldsymbol{u}(t+h)-\boldsymbol{u}(t)}{h}\right|_{\mathbb{L}^2}^2 \leqslant \left(2M_7 + \frac{4\kappa|\boldsymbol{g}|^2 M_{10}}{\nu} + \frac{4\kappa|\boldsymbol{h}|^2 M_{13}}{\nu}\right) =: M_{14} \tag{6.29}$$

for any $t \geqslant t_\mu^0 + 3$. Here we introduce the following abstract result given in Brézis [11] so that the estimate of $|\boldsymbol{u}(t)|_{\mathbb{H}^2}$ can be derived form (6.29).

Proposition 6.2.1. *Let H be a Hilbert space and A be a (single-valued) maximal monotone operator in H. Moreover, assume that $U \in C([0,S];H)$ is a solution of the equation*

$$\frac{dU}{dt} + AU = F \qquad \text{in } H,$$

where $F \in C([0,S];H)$. Then for each fixed $t_0 \in [0,S)$, the following conditions are equivalent:

1. *$U(t_0) \in D(A)$ (where $D(A)$ is the domain of A).*
2. $\liminf_{h\to 0,\ h>0} \left|\frac{U(t_0+h)-U(t_0)}{h}\right|_H < \infty$.
3. *U is right differentiable at t_0 and the right derivative of U at t_0, denoted by $\frac{d^+U}{dt}(t_0)$, satisfies*
$$\frac{d^+U}{dt}(t_0) = -AU(t_0) + F(t_0).$$

Proof. See Theorem 3.5 of Brézis [11] or see our proof of Lemma 6.2.1 and Corollary 6.2.1 given later on. □

From (6.29), we can assure that the first equation of (DCBF) satisfies the condition *2* of Proposition 6.2.1 for $t \geqslant t_\mu^0 + 3$ with $H = \mathbb{L}^2_\sigma(\Omega)$, $A = \nu\mathcal{A} + aI$ and $F = \mathcal{P}\boldsymbol{g}T + \mathcal{P}\boldsymbol{h}C + \mathcal{P}\boldsymbol{f}_1$. Therefore, the condition *3* of Proposition 6.2.1 yields

$$\left|\frac{d^+\boldsymbol{u}}{dt}(t)\right|_{\mathbb{L}^2} \leqslant M_{14}^{1/2} \tag{6.30}$$

for $t \geqslant t_\mu^0 + 3$. Moreover, the condition *3* of Proposition 6.2.1 also give us

$$\begin{aligned}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbb{L}^2}^2 &\leqslant \frac{1}{\nu^2}\left(a\kappa^{1/2}M_5^{1/2} + |\boldsymbol{g}|M_1^{1/2} + |\boldsymbol{h}|M_3^{1/2} + |\boldsymbol{f}_1|_{\mathbb{L}^2} + M_{14}^{1/2}\right)^2 \\ &=: M_{15},\end{aligned} \tag{6.31}$$

where $t \geqslant t_\mu^0 + 3$.

Integrating (6.27) over $[t, t+1]$ with $t \geqslant t_\mu^0 + 3$ again and using (6.29), we get

$$\int_t^{t+1} |\nabla D_h \boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds \leqslant \frac{1}{\nu}\left(M_{14} + \frac{4\kappa|\boldsymbol{g}|^2 M_{10}}{\nu} + \frac{4\kappa|\boldsymbol{h}|^2 M_{13}}{\nu}\right) h^2.$$

According to Proposition 2.1.16 in Section 2.1, this inequality implies that

$$\nabla \boldsymbol{u} \in W^{1,2}(t, t+1; \mathbb{L}^2(\Omega))$$

and

$$\int_t^{t+1} |\partial_t \nabla \boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds \leqslant \frac{1}{\nu} \left(M_{14} + \frac{4\kappa |\boldsymbol{g}|^2 M_{10}}{\nu} + \frac{4\kappa |\boldsymbol{h}|^2 M_{13}}{\nu} \right) =: M_{16} \tag{6.32}$$

for any $t \geqslant t_\mu^0 + 3$.

Here we introduce the following lemma in order to establish estimates for $|T(t)|_{H^2}$ and $|C(t)|_{H^2}$.

Lemma 6.2.1. *Let $N \leqslant 4$ and $\boldsymbol{w} \in C([0,S]; \mathbb{H}_\sigma^1(\Omega))$. Moreover, assume that $V \in C([0,S]; L^2(\Omega))$ is a solution of the following equation:*

$$\partial_t V - \Delta_D V + \boldsymbol{w} \cdot \nabla V = F \qquad (x,t) \in \Omega \times (0,S),$$

where $F \in C([0,S]; L_w^2(\Omega))$ and $L_w^2(\Omega)$ designates the space $L^2(\Omega)$ endowed with the weak topology. Then, for any $t_0 \in [0,S)$, the following conditions are equivalent:

1. *$V(t_0) \in D(-\Delta_D) = H^2(\Omega) \cap H_0^1(\Omega)$.*
2. *$\liminf_{h \to 0,\ h>0} \left| \frac{V(t_0+h) - V(t_0)}{h} \right|_{L^2} < \infty$.*
3. *V is weakly right differentiable at t_0 and the weak right derivative at t_0, denoted by $\text{w-}\frac{d^+V}{dt}(t_0)$, satisfies*

$$\text{w-}\frac{d^+V}{dt}(t_0) = -\boldsymbol{w}(t_0) \cdot \nabla V(t_0) - (-\Delta_D V(t_0)) + F(t_0).$$

Proof. It is obvious that the condition *3* implies the condition *2*.

Assume the condition *2*. Then we can extract a subsequence of $\left\{ \frac{V(t_0+h) - V(t_0)}{h} \right\}_{h>0}$ which weakly converges in $L^2(\Omega)$. Let this subsequence be denoted by $\left\{ \frac{V(t_0+h_n) - V(t_0)}{h_n} \right\}_{n \in \mathbb{N}}$ and its weak limit be designated by χ. Fix an arbitrary $W \in D(-\Delta_D)$. Then

$$\partial_t (V - W) - \Delta_D (V - W) + \boldsymbol{w} \cdot \nabla V = f - (-\Delta_D) W$$

holds. Multiplying this equation by $V(t) - W$ and using the monotonicity of $-\Delta_D$, we get

$$\frac{1}{2} \frac{d}{dt} |V(t) - W|_{L^2}^2 \leqslant (-\boldsymbol{w}(t) \cdot \nabla V(t) + F(t) - (-\Delta_D) W,\ V(t) - W)_{L^2}, \tag{6.33}$$

where $(\cdot, \cdot)_{L^2}$ stands for the usual inner product of $L^2(\Omega)$. Here we note that

$$(\boldsymbol{w}(t) \cdot \nabla V(t), V(t) - W)_{L^2} = (\boldsymbol{w}(t) \cdot \nabla W,\ V(t) - W)_{L^2}$$

holds by the solenoidal condition of $\boldsymbol{w}$. Then integration of (6.33) over $[t_0, t_0 + h_n]$, we have

$$\begin{aligned}(6.34)\qquad &(V(t_0+h_n) - V(t_0), V(t_0) - W)_{L^2} \\ &\leqslant \int_{t_0}^{t_0+h_n} (-\boldsymbol{w}(t') \cdot \nabla W + F(t') - (-\Delta_D)W,\ V(t') - W)_{L^2}\, dt',\end{aligned}$$

where we use the fact that

$$\frac{1}{2}|V(t) - W|_{L^2}^2 - \frac{1}{2}|V(s) - W|_{L^2}^2 \geqslant (V(t) - V(s), V(s) - W)_{L^2}$$

for any $t, s \in [0, S]$. Here, we remark that the integrand on the right hand side $(-\boldsymbol{w}(\cdot) \cdot \nabla W + F(\cdot) - (-\Delta_D)W,\ V(\cdot) - W)_{L^2}$ is continuous on $[0, S]$ for $N \leqslant 4$, since we assume that $\boldsymbol{w} \in C([0, S]; \mathbb{H}^1_\sigma(\Omega))$, $V \in C([0, S]; L^2(\Omega))$ and $F \in C([0, S]; L^2_w(\Omega))$. Hence, dividing (6.34) by h_n and taking the limit as $n \to \infty$, we can obtain

$$\begin{aligned}&(\chi, V(t_0) - W)_{L^2} \\ &\leqslant (-\boldsymbol{w}(t_0) \cdot \nabla W + F(t_0) - (-\Delta_D)W,\ V(t_0) - W)_{L^2} \\ &= (-\boldsymbol{w}(t_0) \cdot \nabla V(t_0) + F(t_0) - (-\Delta_D)W,\ V(t_0) - W)_{L^2},\end{aligned}$$

that is to say,

$$0 \leqslant (-\chi - \boldsymbol{w}(t_0) \cdot \nabla V(t_0) + F(t_0) - (-\Delta_D)W,\ V(t_0) - W)_{L^2}$$

for any $W \in D(-\Delta_D)$. By the maximal monotonicity of $-\Delta_D$ in $L^2(\Omega)$ (recall the definition in Section 2.3.1), we can assure that $V(t_0) \in D(-\Delta_D)$ and

$$\text{w-}\lim_{h_n \to +0} \frac{V(t_0 + h_n) - V(t_0)}{h_n} - \Delta_D V(t_0) + \boldsymbol{w}(t_0) \cdot \nabla V(t_0) = F(t_0).$$

Since this argument does not depend on the choice of subsequence $\{h_n\}$, we conclude that the original sequence $\left\{\frac{V(t_0+h) - V(t_0)}{h}\right\}_{h>0}$ also weakly converges to χ as $h \to +0$.

Finally, assume that $V(t_0) \in D(-\Delta_D)$. Repeating almost the same procedures as those for (6.33) and integrating over $[t_0, t_0 + h]$, we get

$$|V(t_0 + h) - V(t_0)|_{L^2} \leqslant \int_{t_0}^{t_0+h} | - \boldsymbol{w}(t') \cdot \nabla V(t_0) + F(t') + \Delta V(t_0)|_{L^2} dt'$$

for any $h > 0$. Since $\boldsymbol{w} \in C([0, S]; \mathbb{H}^1_\sigma(\Omega))$ and $F \in C([0, S]; L^2_w(\Omega)) \subset L^\infty(0, S; L^2(\Omega))$, there exist some suitable constant M^0 independent of $t \in [0, S]$ such that

$$| - \boldsymbol{w}(t) \cdot \nabla V(t_0) + F(t) + \Delta V(t_0)|_{L^2} \leqslant M^0$$

for a.e. $t \in [0, S]$ with $N \leqslant 4$, which yields

$$\left|\frac{V(t_0 + h) - V(t_0)}{h}\right|_{L^2} \leqslant M^0.$$

Hence, we can repeat our argument in the previous step and assure the existence of a weak limit of $\left\{\frac{V(t_0+h) - V(t_0)}{h}\right\}_{h>0}$ which coincides with $-\boldsymbol{w}(t_0) \cdot \nabla V(t_0) + F(t_0) + \Delta V(t_0)$. □

If F is strongly continuous, we can assure the strong right-differentiability of V:

Corollary 6.2.1. *In addition to assumptions in* Lemma 6.2.1, *we assume that F belongs to $C([0,S];L^2(\Omega))$. Then, for any $t_0 \in [0,S)$, the following conditions are equivalent:*

1. $V(t_0) \in D(-\Delta_D) = H^2(\Omega) \cap H_0^1(\Omega)$.

2. $\liminf_{h\to 0,\ h>0} \left| \frac{V(t_0+h)-V(t_0)}{h} \right|_{L^2} < \infty$.

3'. V is strongly right differentiable at t_0 and the right derivative at t_0, denoted by $\frac{d^+V}{dt}(t_0)$, satisfies

$$\frac{d^+V}{dt}(t_0) = -\boldsymbol{w}(t_0)\cdot\nabla V(t_0) - (-\Delta_D V(t_0)) + F(t_0).$$

Proof. According to our argument in the proof of Lemma 6.2.1, we only have to prove that the condition *1* leads to the condition *3'*, in particular, to the fact that the sequence $\left\{\frac{V(t_0+h)-V(t_0)}{h}\right\}_{h>0}$ strongly converges in $L^2(\Omega)$.

Let $V(t_0) \in D(-\Delta_D)$ (condition *1*). Recalling our proof of Lemma 6.2.1, we get

$$\begin{aligned} &|V(t_0+h) - V(t_0)|_{L^2} \\ &\quad \leqslant \int_{t_0}^{t_0+h} |-\boldsymbol{w}(t')\cdot\nabla V(t_0) + F(t') - (-\Delta_D V(t_0))|_{L^2} dt'. \end{aligned}$$

If F belongs to $C([0,S];L^2(\Omega))$, then the integrand on the right hand side becomes continuous due to $\boldsymbol{w} \in C([0,S];\mathbb{H}^1_\sigma(\Omega))$. Therefore, dividing this inequality by $h > 0$ and taking the limit as $h \to 0$, we obtain

$$\begin{aligned} &\limsup_{h\to+0} \left| \frac{V(t_0+h)-V(t_0)}{h} \right|_{L^2} \\ &\quad \leqslant |-\boldsymbol{w}(t_0)\cdot\nabla V(t_0) + F(t_0) - (-\Delta_D V(t_0))|_{L^2}. \end{aligned}$$

We here recall that $\left\{\frac{V(t_0+h)-V(t_0)}{h}\right\}_{h>0}$ converges weakly to $\chi := -\boldsymbol{w}(t_0)\cdot\nabla V(t_0) + F(t_0) + \Delta V(t_0)$ (arguments for Lemma 6.2.1). Hence we obtain

$$\begin{aligned} |\chi|_{L^2} &\leqslant \liminf_{h\to+0} \left| \frac{V(t_0+h)-V(t_0)}{h} \right|_{L^2} \\ &\leqslant \limsup_{h\to+0} \left| \frac{V(t_0+h)-V(t_0)}{h} \right|_{L^2} \leqslant |\chi|_{L^2}, \end{aligned}$$

which implies that $\left\{\frac{V(t_0+h)-V(t_0)}{h}\right\}_{h>0}$ strongly converges to χ as $h \to +0$. Thus, we assure the strong right-differentiability of V at t_0. $\square$

Remark.
In Lemma 6.2.1 and Corollary 6.2.1, the homogeneous Dirichlet boundary condition can be replaced by any boundary condition which guarantees the maximal monotonicity of the operator $-\Delta$, e.g., the homogeneous Neumann boundary condition.

From the second equation of (DCBF),

$$\partial_t D_h T(t) - \Delta D_h T(t) + \boldsymbol{u}(t+h) \cdot \nabla D_h T(t) + D_h \boldsymbol{u}(t) \cdot \nabla T(t) = 0$$

holds, where $D_h T(t) := T(t+h) - T(t)$ and $D_h \boldsymbol{u}(t) := \boldsymbol{u}(t+h) - \boldsymbol{u}(t)$ $(0 < h < 1)$. Multiplication of this equation by $D_h T$ gives us

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|D_h T(t)|_{L^2}^2 + |\nabla D_h T(t)|_{L^2}^2 \\
=&-\int_\Omega D_h T(t)\boldsymbol{u}(t+h) \cdot \nabla D_h T(t)dx - \int_\Omega D_h T(t) D_h \boldsymbol{u}(t) \cdot \nabla T(t)dx \\
=&0 + \int_\Omega T(t) D_h \boldsymbol{u}(t) \cdot \nabla D_h T(t)dx \\
\leqslant&\frac{1}{2}|\nabla D_h T(t)|_{L^2}^2 + \frac{1}{2}|T(t) D_h \boldsymbol{u}(t)|_{\mathbb{L}^2}^2.
\end{aligned}$$

Therefore

$$\frac{d}{dt}|D_h T(t)|_{L^2}^2 + |\nabla D_h T(t)|_{L^2}^2 \leqslant \gamma_1 |\nabla T(t)|_{L^2}^2 |\nabla D_h \boldsymbol{u}(t)|_{\mathbb{L}^2}^2. \tag{6.35}$$

Here and henceforth, γ_1 denotes the coefficient of the following inequality:

$$|U\boldsymbol{w}|_{\mathbb{L}^2}^2 \leqslant |U|_{L^4}^2 |\boldsymbol{w}|_{\mathbb{L}^4}^2 \leqslant \gamma_1 |\nabla U|_{L^2}^2 |\nabla \boldsymbol{w}|_{\mathbb{L}^2}^2 \quad \forall \boldsymbol{w} \in \mathbb{H}_\sigma^1(\Omega), \forall U \in H_0^1(\Omega)$$

for $N \leqslant 4$. Integration of (6.35) over $[s, t+1]$ and again over $[t, t+1]$ with respect to the variable s yields

$$\begin{aligned}
|D_h T(t+1)|_{L^2}^2 &\leqslant \int_t^{t+1} |D_h T(s)|_{L^2}^2 ds + 2\gamma_1 M_8 M_{16} h^2 \\
&\leqslant 2(M_{10} + \gamma_1 M_8 M_{16}) h^2
\end{aligned} \tag{6.36}$$

for $t \geqslant t_\mu^0 + 3$. Then, the second equation of (DCBF) satisfies all requirements in Corollary 6.2.1 with $V = T$, $\boldsymbol{w} = \boldsymbol{u}$ and $F \equiv f_2$ and condition *2* for $t \geqslant t_\mu^0 + 4$. Hence condition *3'* yields

$$\left|\frac{d^+ T}{dt}(t)\right|_{L^2}^2 \leqslant 2(M_{10} + \gamma_1 M_8 M_{16}) =: M_{17}.$$

for $t \geqslant t_\mu^0 + 4$. Since

$$
\begin{aligned}
&|\Delta T(t)|_{L^2} \\
&\leqslant \left|\frac{d^+T}{dt}(t)\right|_{L^2} + |\boldsymbol{u}(t)\cdot\nabla T(t)|_{L^2} + |f_2|_{L^2} \\
&\leqslant \left|\frac{d^+T}{dt}(t)\right|_{L^2} + \gamma_0^{1/2}|\nabla\boldsymbol{u}(t)|_{\mathbb{L}^2}^{1/2}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbb{L}^2}^{1/2}|\nabla T(t)|_{L^2}^{1/2}|\Delta T(t)|_{L^2}^{1/2} + |f_2|_{L^2} \\
&\leqslant \left|\frac{d^+T}{dt}(t)\right|_{L^2} + \frac{1}{2}|\Delta T(t)|_{L^2} + \frac{\gamma_0}{2}|\nabla\boldsymbol{u}(t)|_{\mathbb{L}^2}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbb{L}^2}|\nabla T(t)|_{L^2} + |f_2|_{L^2},
\end{aligned}
$$

condition *3'* also guarantees the following estimates:

$$
\begin{aligned}
|\Delta T(t)|_{L^2} &\leqslant 2M_{17}^{1/2} + 2|f_2|_{L^2} + \gamma_0|\nabla\boldsymbol{u}(t)|_{\mathbb{L}^2}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbb{L}^2}|\nabla T(t)|_{L^2} \\
&\leqslant 2M_{17}^{1/2} + 2|f_2|_{L^2} + \gamma_0 M_5^{1/2}M_{15}^{1/2}M_8^{1/2} =: M_{18}^{1/2}
\end{aligned}
\tag{6.37}
$$

for $t \geqslant t_\mu^0 + 4$. Integrating (6.35) over $[t, t+1]$ with $t \geqslant t_\mu^0 + 4$ again, using estimates (6.36) and applying Proposition 2.1.16, we can deduce

$$\partial_t \nabla T \in L^2(t, t+1; L^2(\Omega))$$

and

$$\int_t^{t+1} |\partial_t \nabla T(s)|_{L^2}^2 ds \leqslant (M_{17} + 2\gamma_1 M_8 M_{16}) =: M_{19} \tag{6.38}$$

for $t \geqslant t_\mu^0 + 4$.

Similarly, from the third equation,

$$\partial_t D_h C(t) - \Delta D_h C(t) + \boldsymbol{u}(t+h)\cdot\nabla D_h C(t) + D_h\boldsymbol{u}(t)\cdot\nabla C(t) = \rho\Delta D_h T(t)$$

is valid, where

$$D_h T(t) := T(t+h) - T(t), \quad D_h C(t) := C(t+h) - C(t), \quad D_h\boldsymbol{u}(t) := \boldsymbol{u}(t+h) - \boldsymbol{u}(t).$$

Multiplying this equation by $D_h C(t)$, we get

$$
\begin{aligned}
&\frac{d}{dt}|D_h C(t)|_{L^2}^2 + |\nabla D_h C(t)|_{L^2}^2 \\
&\qquad \leqslant 2\gamma_1|\nabla C(t)|_{L^2}^2|\nabla D_h\boldsymbol{u}(t)|_{\mathbb{L}^2}^2 + 2\rho^2|\nabla D_h T(t)|_{L^2}^2
\end{aligned}
\tag{6.39}
$$

By almost the same argument as that for (6.36), we have

$$|D_h C(t+1)|_{L^2}^2 \leqslant (2M_{13} + 4\gamma_1 M_{11}M_{16} + 2\rho^2 M_{19})h^2 \tag{6.40}$$

for $t \geqslant t_\mu^0 + 4$. Here we note that the third equation of (DCBF) satisfies all hypotheses in Lemma 6.2.1 with $V = C$, $\boldsymbol{w} = \boldsymbol{u}$ and $F = f_3 + \rho\Delta T$, since $T \in C([0, +\infty); H^1(\Omega))$

and (6.37) yield the weak continuity of $\Delta T(\cdot)$ in $L^2(\Omega)$ on $[t_\mu^0+4,\infty)$. Moreover, (6.40) implies that condition *2* of Lemma 6.2.1 is satisfied for any $t \in [t_\mu^0+5,\infty)$. Therefore, for any $t \geqslant t_\mu^0+5$, we can assure that $C(t) \in D(-\Delta_D)$ and

$$\begin{aligned}
&\liminf_{h\to+0}\left|\frac{C(t+h)-C(t)}{h}\right|_{L^2}^2 \leqslant 2M_{13}+4\gamma_1 M_{11}M_{16}+2\rho^2 M_{19} =: M_{20},\\
&\text{w-}\lim_{h\to+0}\frac{C(t+h)-C(t)}{h} - \Delta C(t) + \boldsymbol{u}(t)\cdot\nabla C(t) = \rho\Delta T(t) + f_3.
\end{aligned}$$

These immediately lead to

$$\begin{aligned}
|\Delta C(t)|_{L^2} &\leqslant 2M_{20}^{1/2} + 2\rho|\Delta T(t)|_{L^2} + 2|f_3|_{L^2}\\
&\qquad\qquad + \gamma_0|\nabla\boldsymbol{u}(t)|_{\mathbb{L}^2}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbb{L}^2}|\nabla C(t)|_{L^2}\\
&\leqslant 2M_{20}^{1/2} + 2\rho M_{18}^{1/2} + 2|f_3|_{L^2} + \gamma_0 M_5^{1/2}M_{15}^{1/2}M_{11}^{1/2}\\
&=: M_{21}^{1/2}
\end{aligned} \tag{6.41}$$

for $t \geqslant t_\mu^0+5$. Moreover, the third equation of (DCBF) and the estimate (6.41) give us

$$\begin{aligned}
&|\partial_t C(t)|_{L^2}\\
\leqslant&|\Delta C(t)|_{L^2} + \gamma_0^{1/2}|\nabla\boldsymbol{u}(t)|_{\mathbb{L}^2}^{1/2}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbb{L}^2}^{1/2}|\nabla C(t)|_{L^2}^{1/2}|\Delta C(t)|_{L^2}^{1/2}\\
&\qquad\qquad + \rho|\Delta T(t)|_{L^2} + |f_3|_{L^2}\\
\leqslant&M_{21}^{1/2} + \gamma_0^{1/2}M_5^{1/4}M_{15}^{1/4}M_{11}^{1/4}M_{21}^{1/4} + \rho M_{18}^{1/2} + |f_3|_{L^2}\\
=:&M_{22}^{1/2}
\end{aligned}$$

for a.e. $t \geqslant t_\mu^0+5$. Integrating (6.39) over $[t,t+1]$ again, we can obtain

$$\int_t^{t+1}|\nabla D_h C(s)|_{L^2}^2 ds \leqslant (M_{20}+4\gamma_1 M_{11}M_{16}+4\rho^2 M_{19})h^2$$

and by the same reasoning as for (6.32), we obtain

$$\int_t^{t+1}|\partial_t\nabla C(s)|_{L^2}^2 ds \leqslant M_{20}+4\gamma_1 M_{11}M_{16}+4\rho^2 M_{19} =: M_{23}. \tag{6.42}$$

Thus, estimates (6.3), (6.7), (6.10), (6.17), (6.22) and (6.31) imply that the set $\mathscr{B}_0$ defined by

$$\mathscr{B}_0 := \left\{ (\boldsymbol{u}_0, T_0, C_0) \in \mathcal{H};\ \begin{array}{ll} |\nabla\boldsymbol{u}_0|_{\mathbb{L}^2}^2 \leqslant M_5, & |\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2 \leqslant M_{15}\\ |T_0|_{L^2}^2 \leqslant M_1, & |\nabla T_0|_{L^2}^2 \leqslant M_8\\ |C_0|_{L^2}^2 \leqslant M_3, & |\nabla C_0|_{L^2}^2 \leqslant M_{11} \end{array}\right\}$$

becomes an absorbing set of $(\{\mathscr{S}_D(t)\}_{t\geq 0}, \mathcal{H})$. Obviously, $\mathscr{B}_0$ is compact in $\mathcal{H}$ by virtue of Rellich-Kondrachov's compactness theorem. Moreover, from (6.37) and (6.41), we can assure the set $\mathscr{B}_0^1$ defined by

$$\mathscr{B}_0^1 := \left\{ (\boldsymbol{u}_0, T_0, C_0) \in \mathcal{H}_D^1; \begin{array}{rrr} & |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2 \leqslant M_5, & |\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2 \leqslant M_{15} \\ |T_0|_{L^2}^2 \leqslant M_1, & |\nabla T_0|_{L^2}^2 \leqslant M_8, & |\Delta T_0|_{L^2}^2 \leqslant M_{18} \\ |C_0|_{L^2}^2 \leqslant M_3, & |\nabla C_0|_{L^2}^2 \leqslant M_{11}, & |\Delta C_0|_{L^2}^2 \leqslant M_{21} \end{array} \right\}$$

becomes an compact absorbing set of $(\{\mathscr{S}_D(t)\}_{t\geq 0}, \mathcal{H}_D^1)$.

Estimates derived above are not sufficient to show the continuity of $\mathscr{S}_D(t)$ on $\mathcal{H}_D^1$ and to construct an exponential attractor, since our estimates are established for adequately large t. Therefore we have to prepare some additional estimates of solutions on time interval $[0, t]$, namely, we show the dependence on initial data and t by applying techniques employed above. Let $U = (\boldsymbol{u}, T, C)$ be a solution of (DCBF) with the initial data $U_0 = (\boldsymbol{u}_0, T_0, C_0)$. Integrating or applying Gronwall's inequality to (6.2), (6.6), (6.9), (6.12) over $[0, t]$, we can assure

$$\begin{aligned}
&\sup_{0\leqslant s\leqslant t} |T(s)|_{L^2}^2 + \int_0^t |\nabla T(s)|_{L^2}^2 ds \leqslant Q(t, |T_0|_{L^2}^2), \\
&\sup_{0\leqslant s\leqslant t} |C(s)|_{L^2}^2 + \int_0^t |\nabla C(s)|_{L^2}^2 ds \leqslant Q(t, |T_0|_{L^2}^2, |C_0|_{L^2}^2), \\
&\sup_{0\leqslant s\leqslant t} |\nabla \boldsymbol{u}(s)|_{\mathbb{L}^2}^2 + \int_0^t |\mathcal{A}\boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds \leqslant Q(t, |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2, |T_0|_{L^2}^2, |C_0|_{L^2}^2), \\
&\int_0^t |\partial_t \boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds \leqslant Q(t, |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2, |T_0|_{L^2}^2, |C_0|_{L^2}^2).
\end{aligned} \tag{6.43}$$

From now on, let $Q(z_1, z_2, \cdots)$ stand for some general monotone increasing function of variables $z_1, z_2, \cdots$. Moreover, by almost the same ways as those for (6.14), (6.19) and (6.21) (repeating the same calculations without the weight t), we obtain

$$\begin{aligned}
&\frac{d}{dt}|\nabla T|_{L^2}^2 + |\Delta T|_{L^2}^2 \leqslant \frac{27}{2}\gamma_0^2 |\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2 |\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 |\nabla T|_{L^2}^2 + 2|f_2|_{L^2}^2, \\
&|\partial_t T|_{L^2}^2 + \frac{d}{dt}|\nabla T|_{L^2}^2 \leqslant 2\gamma_0 |\nabla \boldsymbol{u}|_{\mathbb{L}^2} |\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2} |\nabla T|_{L^2} |\Delta T|_{L^2} + 2|f_2|_{L^2}^2,
\end{aligned} \tag{6.44}$$

$$\begin{aligned}
&\frac{d}{dt}|\nabla C|_{L^2}^2 + |\Delta C|_{L^2}^2 \\
&\qquad \leqslant \frac{27}{2}\gamma_0^2 |\nabla \boldsymbol{u}|_{\mathbb{L}^2}^2 |\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2}^2 |\nabla C|_{L^2}^2 + 4\rho^2 |\Delta T|_{L^2}^2 + 4|f_3|_{L^2}^2, \\
&|\partial_t C|_{L^2}^2 + \frac{d}{dt}|\nabla C|_{L^2}^2 \\
&\qquad \leqslant 3\gamma_0 |\nabla \boldsymbol{u}|_{\mathbb{L}^2} |\mathcal{A}\boldsymbol{u}|_{\mathbb{L}^2} |\nabla C|_{L^2} |\Delta C|_{L^2} + 3\rho^2 |\Delta T|_{L^2}^2 + 3|f_3|_{L^2}^2.
\end{aligned} \tag{6.45}$$

Integrating each inequality of (6.44) and (6.45) over $[0,t]$, we have

$$\begin{aligned}
\sup_{0\leqslant s\leqslant t}|\nabla T(s)|_{L^2}^2 &\leqslant Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^1}^2,|C_0|_{L^2}^2),\\
\int_0^t|\Delta T(s)|_{L^2}^2ds+\int_0^t|\partial_tT(s)|_{L^2}^2ds &\leqslant Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^1}^2,|C_0|_{L^2}^2),\\
\sup_{0\leqslant s\leqslant t}|\nabla C(s)|_{L^2}^2 &\leqslant Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^1}^2,|C_0|_{H^1}^2),\\
\int_0^t|\Delta C(s)|_{L^2}^2ds+\int_0^t|\partial_tC(s)|_{L^2}^2ds &\leqslant Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^1}^2,|C_0|_{H^1}^2).
\end{aligned}\tag{6.46}$$

Repeating the same procedures as those for (6.28), we obtain

$$\begin{aligned}
&|D_h\boldsymbol{u}(t)|_{\mathbb{L}^2}^2+\nu\int_0^t|\nabla D_h\boldsymbol{u}(s)|_{\mathbb{L}^2}^2ds\\
&\qquad\leqslant|D_h\boldsymbol{u}(0)|_{\mathbb{L}^2}^2+h^2Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^1}^2,|C_0|_{H^1}^2),
\end{aligned}\tag{6.47}$$

where $D_h\boldsymbol{u}(t):=\boldsymbol{u}(t+h)-\boldsymbol{u}(t)$ with $h>0$. Therefore, if $\boldsymbol{u}_0$ belongs to $D(\mathcal{A})$, then $\boldsymbol{u}(t)\in D(\mathcal{A})$ also holds for any $t>0$ and

$$\begin{aligned}
|\mathcal{A}\boldsymbol{u}(t)|_{\mathbb{L}^2}^2&\leqslant\left|\frac{d^+\boldsymbol{u}}{dt}(0)\right|_{\mathbb{L}^2}^2+Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^1}^2,|C_0|_{H^1}^2)\\
&\leqslant Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^1}^2,|C_0|_{H^1}^2)
\end{aligned}\tag{6.48}$$

is valid by virtue of Proposition 6.2.1. Immediately, we get

$$|\partial_t\boldsymbol{u}(t)|_{\mathbb{L}^2}^2\leqslant Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^1}^2,|C_0|_{H^1}^2)\tag{6.49}$$

for a.e. $t\geqslant 0$. Moreover, (6.47) also gives us $\nabla\boldsymbol{u}\in W^{1,2}(0,t;\mathbb{L}^2(\Omega))$ and

$$\int_0^t|\partial_t\nabla\boldsymbol{u}(s)|_{\mathbb{L}^2}^2ds\leqslant Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^1}^2,|C_0|_{H^1}^2)\tag{6.50}$$

for any $t\geqslant 0$. Integrating (6.35) over $[0,t]$ and using (6.43), (6.46) and (6.50), we get

$$\begin{aligned}
&|D_hT(t)|_{L^2}^2+\int_0^t|\nabla D_hT(s)|_{L^2}^2ds\\
&\qquad\leqslant|D_hT(0)|_{L^2}^2+h^2Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^1}^2,|C_0|_{H^1}^2).
\end{aligned}$$

Therefore, under the assumption of $T_0\in D(-\Delta_D)$, taking the limit as $h\to 0$, applying Corollary 6.2.1 and Proposition 2.1.16, we can deduce

$$\begin{aligned}
|\Delta T(t)|_{L^2}^2&\leqslant\left|\frac{d^+T}{dt}(0)\right|_{L^2}^2+Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^1}^2,|C_0|_{H^1}^2)\\
&\leqslant Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^2}^2,|C_0|_{H^1}^2),\\
\int_0^t|\partial_t\nabla T(s)|_{L^2}^2ds&\leqslant Q(t,|\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2,|\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2,|T_0|_{H^2}^2,|C_0|_{H^1}^2)
\end{aligned}\tag{6.51}$$

for $t \geqslant 0$. Obviously,

$$|\partial_t T(t)|_{L^2}^2 \leqslant Q(t, |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2, |\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2, |T_0|_{H^2}^2, |C_0|_{H^1}^2) \tag{6.52}$$

holds for a.e. $t \geqslant 0$. Integrating (6.39) over $[0,t]$ and using (6.43), (6.46), (6.50) and (6.51), we get

$$\begin{aligned} &|D_h C(t)|_{L^2}^2 + \int_0^t |\nabla D_h C(s)|_{L^2}^2 ds \\ &\qquad \leqslant |D_h C(0)|_{L^2}^2 + h^2 Q(t, |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2, |\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2, |T_0|_{H^2}^2, |C_0|_{H^1}^2). \end{aligned} \tag{6.53}$$

Let $C_0 \in D(-\Delta_D)$. Since the equation

$$\partial_t (C - C_0) - \Delta(C - C_0) + \boldsymbol{u} \cdot \nabla (C - C_0) = \rho \Delta T + \Delta C_0 + \boldsymbol{u} \cdot \nabla C_0 + f_3$$

is valid, multiplication of the above equation by $C - C_0$ yields

$$\begin{aligned} |C(h) - C(0)|_{L^2} &\leqslant \int_0^h |\rho \Delta T(s) + \Delta C_0 + \boldsymbol{u}(s) \cdot \nabla C_0 + f_3|_{L^2} ds \\ &\leqslant h Q(h, |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2, |\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2, |T_0|_{H^2}^2, |C_0|_{H^2}^2). \end{aligned} \tag{6.54}$$

Therefore applying Lemma 6.2.1 to the third equation of (DCBF), together with (6.53) and (6.54), we obtain

$$|\Delta C(t)|_{L^2}^2 \leqslant Q(t, |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2, |\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2, |T_0|_{H^2}^2, |C_0|_{H^2}^2). \tag{6.55}$$

Moreover, we have (see (6.51) and (6.52))

$$\begin{aligned} &\int_0^t |\partial_t \nabla C(s)|_{L^2}^2 ds \leqslant Q(t, |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2, |\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2, |T_0|_{H^2}^2, |C_0|_{H^2}^2), \\ &\qquad |\partial_t C(t)|_{L^2}^2 \leqslant Q(t, |\nabla \boldsymbol{u}_0|_{\mathbb{L}^2}^2, |\mathcal{A}\boldsymbol{u}_0|_{\mathbb{L}^2}^2, |T_0|_{H^2}^2, |C_0|_{H^2}^2). \end{aligned} \tag{6.56}$$

Let τ_0 be a time such that the absorbing set $\mathscr{B}_0$ is absorbed in $\mathscr{B}_0$ itself by the semigroup $\{\mathscr{S}_D(t)\}_{t\geq 0}$, i.e., $\mathscr{S}_D(t)\mathscr{B}_0 \subset \mathscr{B}_0$ for any $t \geqslant \tau_0$. Then we define $\mathscr{B}_{00}$ by

$$\mathscr{B}_{00} := \bigcup_{0 \leqslant t \leqslant \tau_0} \mathscr{S}_D(t)\mathscr{B}_0.$$

By using (6.43), (6.46) and (6.48), we can assure that there exist a constant M_{24} depending only on M_1, M_3, M_5, M_8, M_{11}, M_{15} and τ_0 such that

$$\|\mathscr{S}_D(t)U_0\|_{\mathcal{H}^{2,1}} \leqslant M_{24} \tag{6.57}$$

for any $U_0 \in \mathscr{B}_0$ and any $t \in [0, \tau_0]$. This fact guarantees the compactness of the set $\mathscr{B}_{00}$ in $\mathcal{H}$. It easy to see that $\mathscr{B}_{00}$ is positively invariant under the semigroup $\{\mathscr{S}_D(t)\}_{t\geq 0}$ and $\mathscr{B}_{00}$ is also an absorbing set.

Similarly, we define

$$\mathscr{B}_{00}^1 := \bigcup_{0\leqslant t\leqslant \tau_0^1} \mathscr{S}_D(t)\mathscr{B}_0^1,$$

where τ_0^1 is a time such that $\mathscr{S}_D(t)\mathscr{B}_0 \subset \mathscr{B}_0$ for any $t \geqslant \tau_0^1$. Owing to (6.43), (6.46), (6.48), (6.51) and (6.55), there exist some constant M_{25} which depends only on M_1, M_3, M_5, M_8, M_{11}, M_{15}, M_{18}, M_{21} and τ_0 such that

$$\|\mathscr{S}_D(t)U_0\|_{\mathcal{H}^2} \leqslant M_{25} \tag{6.58}$$

for any $U_0 \in \mathscr{B}_0^1$ and any $t \in [0, \tau_0^1]$. Therefore, the set $\mathscr{B}_{00}^1$ is compact and positively invariant absorbing set in $\mathcal{H}_D^1$.

6.3 Continuity of Semigroup

Since we can assure the existence of compact absorbing sets $\mathscr{B}_0$, $\mathscr{B}_0^1$, $\mathscr{B}_{00}$ and $\mathscr{B}_{00}^1$, we only have to show some continuity of solutions in order to apply Corollary 2.4.1 and Proposition 2.4.2 so that we complete our proof of Theorem 6.1.1, 6.1.2 and 6.1.3, existence of attractors for Dirichlet boundary condition case. Throughout this section, $U_i = (\boldsymbol{u}_i, T_i, C_i)$ denote the unique solutions of (DCBF) (with Dirichlet boundary condition) with the initial data $U_{i0} = (\boldsymbol{u}_{i0}, T_{i0}, C_{i0})$ $(i = 1, 2)$. Then $\delta\boldsymbol{u} := \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\delta T := T_1 - T_2$ and $\delta C := C_1 - C_2$ satisfy the following equations:

$$\text{(D)} \begin{cases} \partial_t \delta\boldsymbol{u} + \nu\mathcal{A}\delta\boldsymbol{u} + a\delta\boldsymbol{u} = \mathcal{P}\boldsymbol{g}\delta T + \mathcal{P}\boldsymbol{h}\delta C, \\ \partial_t\delta T - \Delta\delta T + \delta\boldsymbol{u}\cdot\nabla T_1 + \boldsymbol{u}_2\cdot\nabla\delta T = 0, \\ \partial_t\delta C - \Delta\delta C + \delta\boldsymbol{u}\cdot\nabla C_1 + \boldsymbol{u}_2\cdot\nabla\delta C = \rho\Delta\delta T. \end{cases}$$

Multiplying each equation of (D) by $\mathcal{A}\delta\boldsymbol{u}$, δT and δC respectively, we get

$$\begin{aligned} &\frac{d}{dt}|\delta T|_{L^2}^2 + |\nabla\delta T|_{L^2}^2 \leqslant \gamma_1|\nabla T_1|_{L^2}^2|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}^2, \\ &\frac{d}{dt}|\delta C|_{L^2}^2 + |\nabla\delta C|_{L^2}^2 \leqslant 2\gamma_1|\nabla C_1|_{L^2}^2|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}^2 + 2\rho^2|\nabla\delta T|_{L^2}^2, \\ &\frac{d}{dt}|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}^2 + \nu|\mathcal{A}\delta\boldsymbol{u}|_{\mathbb{L}^2}^2 \leqslant \frac{2|\boldsymbol{g}|^2\kappa}{\nu}|\nabla\delta T|_{L^2}^2 + \frac{2|\boldsymbol{h}|^2\kappa}{\nu}|\nabla\delta C|_{L^2}^2 \end{aligned} \tag{6.59}$$

(recall that κ denotes the coefficient of Poincaré's inequality and γ_1 is a coefficient appearing in

$$|U\boldsymbol{w}|_{\mathbb{L}^2}^2 \leqslant \gamma_1|\nabla U|_{L^2}^2|\nabla\boldsymbol{w}|_{\mathbb{L}^2}^2,$$

where $\boldsymbol{w} \in \mathbb{H}_\sigma^1(\Omega)$ and $U \in H_0^1(\Omega)$). We here define

$$\eta(t) := |\delta T(t)|_{L^2}^2 + \frac{1}{4\rho^2}|\delta C(t)|_{L^2}^2 + \frac{\nu\beta}{8\chi\kappa}|\nabla\delta\boldsymbol{u}(t)|_{\mathbb{L}^2}^2,$$

where $\chi := \max\{|\boldsymbol{g}|^2, |\boldsymbol{h}|^2\}$ and $\beta := \min\{1, \frac{1}{2\rho^2}\}$. From (6.59), we have

$$\frac{d}{dt}\eta(t) + \frac{\vartheta}{\kappa}y(t) \leqslant \gamma_1' \left(|\nabla T_1|_{L^2}^2 + \frac{1}{2\rho^2}|\nabla C_1|_{L^2}^2 \right) \eta(t),$$

where $\vartheta := \min\{\frac{1}{4}, \nu\}$ and $\gamma_1' := \max\{\gamma_1, \frac{8\chi\kappa}{\nu\beta}\gamma_1\}$. Hence applying Gronwall's inequality, we can obtain

$$\begin{aligned}\eta(t) &\leqslant \eta(0)\exp\left(\gamma_1'\int_0^t |\nabla T_1(s)|_{L^2}^2 ds + \frac{\gamma_1'}{2\rho^2}\int_0^t |\nabla C_1(s)|_{L^2}^2 ds - \frac{\vartheta}{\kappa}t\right) \\ &\leqslant \eta(0)\exp\left(2\gamma_1'|T_{10}|_{L^2}^2 + \frac{\gamma_1'}{2\rho^2}|C_{10}|_{L^2}^2 + 2\gamma_1'\kappa t|f_2|_{L^2}^2 + \frac{\gamma_1'\kappa t}{\rho^2}|f_3|_{L^2}^2 - \frac{\vartheta}{\kappa}t\right),\end{aligned} \tag{6.60}$$

where we use the following a priori estimates derived from the second and third equations of (DCBF) (integrate (6.2) and (6.6) over $[0,t]$):

$$\begin{aligned}&\int_0^t |\nabla T_1(s)|_{L^2}^2 ds \leqslant |T_1(0)|_{L^2}^2 + \kappa t|f_2|_{L^2}^2, \\ &\int_0^t |\nabla C_1(s)|_{L^2}^2 ds \leqslant |C_1(0)|_{L^2}^2 + 2\rho^2\int_0^t |\nabla T_1(s)|_{L^2}^2 ds + 2\kappa t|f_2|_{L^2}^2.\end{aligned}$$

Then (6.60) guarantees the continuity of mappings $\mathscr{S}_D(t)$ on $\mathcal{H}$ for each fixed $t \geqslant 0$.

Hence we can assure Theorem 6.1.1, i.e., the existence of global attractor $\mathscr{A}_D$ of the dynamical system $(\{\mathscr{S}_D(t)\}_{t\geq 0}, \mathcal{H})$ by the existence of compact absorbing set $\mathscr{B}_0$, the continuity of $\mathscr{S}_D(t)$ and the application of Corollary 2.4.1. In addition, if

$$2\gamma_1'\kappa|f_2|_{L^2}^2 + \frac{\gamma_1'\kappa}{\rho^2}|f_3|_{L^2}^2 < \frac{\vartheta}{\kappa}, \tag{6.61}$$

then (6.60) implies that $\eta(t) \to$ as $t \to +\infty$. Hence, when (6.61) is satisfied, we can assure that $\mathscr{A}_D$ consists only one element and $\mathscr{A}_D$ satisfies the definition of exponential attractor (recall remark (2) in Section 6.1).

According to (6.60),

$$\|\delta U(t)\|_{\mathcal{H}} \leqslant Q(t, |T_{10}|_{L^2}^2, |C_{10}|_{L^2}^2)\|\delta U(0)\|_{\mathcal{H}} \tag{6.62}$$

holds (Q is some suitable monotone increasing function). Integrating inequalities of (6.59), we have

$$\begin{aligned}&\int_0^t |\mathcal{A}\delta\boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds \leqslant Q(t, |T_{10}|_{L^2}^2, |C_{10}|_{L^2}^2)\|\delta U(0)\|_{\mathcal{H}}^2, \\ &\int_0^t |\nabla\delta T(s)|_{L^2}^2 ds \leqslant Q(t, |T_{10}|_{L^2}^2, |C_{10}|_{L^2}^2)\|\delta U(0)\|_{\mathcal{H}}^2, \\ &\int_0^t |\nabla\delta C(s)|_{L^2}^2 ds \leqslant Q(t, |T_{10}|_{L^2}^2, |C_{10}|_{L^2}^2)\|\delta U(0)\|_{\mathcal{H}}^2.\end{aligned} \tag{6.63}$$

Multiplying the second and third equation of (D) by $-\Delta\delta T$ and $-\Delta\delta C$ respectively, we have (see (6.14), (6.19), (6.21), (6.44) and (6.45))

$$
\begin{aligned}
&\frac{d}{dt}|\nabla\delta T|_{L^2}^2 + |\Delta\delta T|_{L^2}^2 \\
&\qquad \leqslant \frac{27}{2}\gamma_0^2|\nabla \boldsymbol{u}_2|_{\mathbb{L}^2}^2|\mathcal{A}\boldsymbol{u}_2|_{\mathbb{L}^2}^2|\nabla\delta T|_{L^2}^2 + 2\gamma_0|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}|\mathcal{A}\delta\boldsymbol{u}|_{\mathbb{L}^2}|\nabla T_1|_{L^2}|\Delta T_1|_{L^2}, \\
&\frac{d}{dt}|\nabla\delta C|_{L^2}^2 + |\Delta\delta C|_{L^2}^2 \\
&\qquad \leqslant \frac{27}{2}\gamma_0^2|\nabla \boldsymbol{u}_2|_{\mathbb{L}^2}^2|\mathcal{A}\boldsymbol{u}_2|_{\mathbb{L}^2}^2|\nabla\delta C|_{L^2}^2 \\
&\qquad\qquad + 4\gamma_0|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}|\mathcal{A}\delta\boldsymbol{u}|_{\mathbb{L}^2}|\nabla C_1|_{L^2}|\Delta C_1|_{L^2} + 4\rho^2|\Delta\delta T|_{L^2}^2.
\end{aligned}
\tag{6.64}
$$

Applying Gronwall's inequality to (6.64), we obtain

$$
\begin{aligned}
&|\nabla\delta T(t)|_{L^2}^2 + |\nabla\delta C(t)|_{L^2}^2 \\
&\quad \leqslant Q(t, |T_{10}|_{H^1}^2, |C_{10}|_{H^1}^2|\nabla\boldsymbol{u}_{10}|_{\mathbb{L}^2}^2, |\nabla\boldsymbol{u}_{20}|_{\mathbb{L}^2}^2, |T_{20}|_{L^2}^2, |C_{20}|_{L^2}^2)\|\delta U(0)\|_{\mathcal{H}_D^1}^2.
\end{aligned}
\tag{6.65}
$$

From (6.62) and (6.65), we can derive the following estimate which implies the continuity of mappings $\mathscr{S}_D(t)$ on $\mathcal{H}_D^1$ for each fixed $t \geqslant 0$.

$$
\begin{aligned}
&\|\delta U(t)\|_{\mathcal{H}_D^1}^2 \\
&\quad \leqslant Q(t, |T_{10}|_{H^1}^2, |C_{10}|_{H^1}^2|\nabla\boldsymbol{u}_{10}|_{\mathbb{L}^2}^2, |\nabla\boldsymbol{u}_{20}|_{\mathbb{L}^2}^2, |T_{20}|_{L^2}^2, |C_{20}|_{L^2}^2)\|\delta U(0)\|_{\mathcal{H}_D^1}^2.
\end{aligned}
\tag{6.66}
$$

Hence the existence of global attractor $\mathscr{A}_D^1$ of the dynamical system $(\{\mathscr{S}_D(t)\}_{t\geq 0}, \mathcal{H}_D^1)$ (Theorem 6.1.3) can be assured by the existence of $\mathscr{B}_0^1$ and (6.66).

Next we consider the existence of exponential attractor. Recall the compact positively invariant absorbing set $\mathscr{B}_{00}$ defined by

$$
\mathscr{B}_{00} := \bigcup_{0\leqslant t\leqslant \tau_0} \mathscr{S}_D(t)\mathscr{B}_0.
$$

From (6.57) and positive invariance of $\mathscr{B}_{00}$, we can derive the following boundedness for any $t \geqslant 0$ and for any U_0 belonging to $\mathscr{B}_{00}$.

$$
\|\mathscr{S}_D(t)U_0\|_{\mathcal{H}^{2,1}} \leqslant M_{24}. \tag{6.67}
$$

According to Proposition 2.4.2, it is sufficient that we show the following lemma to assure the existence of exponential attractor of $(\{\mathscr{S}_D(t)\}_{t\geq 0}, \mathcal{H})$.

Lemma 6.3.1. *There exist constants* M_{26} *and* M_{27} *satisfying*

$$
\|\mathscr{S}_D(1)U_{10} - \mathscr{S}_D(1)U_{20}\|_{\mathcal{H}^{2,1}} \leqslant M_{26}\|U_{10} - U_{20}\|_{\mathcal{H}}, \tag{6.68}
$$

$$
\|\mathscr{S}_D(t)U_{10} - \mathscr{S}_D(s)U_{10}\|_{\mathcal{H}} \leqslant M_{27}|t-s|^{1/2} \tag{6.69}
$$

for any $U_{i0} \in \mathscr{B}_{00}$ $(i = 1, 2)$ *and any* $t, s \in [0, 1]$.

Proof. By (6.46), we have

$$\begin{aligned}\int_0^t |\Delta T_i(s)|_{L^2}^2 ds + \int_0^t |\partial_t T_i(s)|_{L^2}^2 ds \leqslant Q(t, \|U_{i0}\|_{\mathcal{H}_D^1}),\\ \int_0^t |\Delta C_i(s)|_{L^2}^2 ds + \int_0^t |\partial_t C_i(s)|_{L^2}^2 ds \leqslant Q(t, \|U_{i0}\|_{\mathcal{H}_D^1}).\end{aligned} \tag{6.70}$$

for any $t \geqslant 0$. Moreover, from (6.49) and (6.50), we can derive

$$|\partial_t \boldsymbol{u}_i(t)|_{\mathbb{L}^2}^2 \leqslant Q(t, \|U_{i0}\|_{\mathcal{H}^{2,1}}) \tag{6.71}$$

for a.e. $t \geqslant 0$ and

$$\int_0^t |\partial_t \nabla \boldsymbol{u}_i(s)|_{\mathbb{L}^2}^2 ds \leqslant Q(t, \|U_{i0}\|_{\mathcal{H}^{2,1}}) \tag{6.72}$$

for any $t \geqslant 0$ respectively. Therefore, we can obtain

$$\begin{aligned}&\|\mathscr{S}_D(t)U_{10} - \mathscr{S}_D(s)U_{10}\|_{\mathcal{H}}\\ \leqslant& \int_s^t |\partial_t T_1(\tau)|_{L^2} d\tau + \int_s^t |\partial_t C_1(\tau)|_{L^2} d\tau + \int_s^t |\partial_t \nabla \boldsymbol{u}_1(\tau)|_{\mathbb{L}^2} d\tau\\ \leqslant& Q(\|U_{10}\|_{\mathcal{H}^{2,1}})|t-s|^{1/2} \leqslant Q(M_{24})|t-s|^{1/2}\end{aligned}$$

for any $t, s \in [0,1]$, which implies (6.69).

We recall (6.62) and (6.63), i.e.,

$$\|\delta U(t)\|_{\mathcal{H}} \leqslant Q(t, \|U_{10}\|_{\mathcal{H}})\|\delta U(0)\|_{\mathcal{H}}$$

and

$$\begin{aligned}\int_0^t |\mathcal{A}\delta\boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds \leqslant Q(t, \|U_{10}\|_{\mathcal{H}})\|\delta U(0)\|_{\mathcal{H}}^2,\\ \int_0^t |\nabla \delta T(s)|_{L^2}^2 ds \leqslant Q(t, \|U_{10}\|_{\mathcal{H}})\|\delta U(0)\|_{\mathcal{H}}^2,\\ \int_0^t |\nabla \delta C(s)|_{L^2}^2 ds \leqslant Q(t, \|U_{10}\|_{\mathcal{H}})\|\delta U(0)\|_{\mathcal{H}}^2.\end{aligned}$$

Moreover, multiplying the first equation of (D) by $\partial_t \delta \boldsymbol{u}$ and integrating over $[0,t]$, we get

$$\int_0^t |\partial_t \delta \boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds \leqslant Q(t, \|U_{10}\|_{\mathcal{H}})\|\delta U(0)\|_{\mathcal{H}}^2. \tag{6.73}$$

Multiplying the second equation of (D) by $-t\Delta\delta T$ and $t\partial_t\delta T$, we have (see (6.14) and (6.19))

$$\begin{aligned}
&\frac{d}{dt}t|\nabla\delta T|_{L^2}^2 + t|\Delta\delta T|_{L^2}^2 \leqslant |\nabla\delta T|_{L^2}^2 + 2t\gamma_0|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}|\mathcal{A}\delta\boldsymbol{u}|_{\mathbb{L}^2}|\nabla T_1|_{L^2}|\Delta T_1|_{L^2}\\
&\qquad\qquad + \frac{27}{2}t\gamma_0^2|\nabla\boldsymbol{u}_2|_{\mathbb{L}^2}^2|\mathcal{A}\boldsymbol{u}_2|_{\mathbb{L}^2}^2|\nabla\delta T|_{L^2}^2,\\
&t|\partial_t\delta T|_{L^2}^2 + \frac{d}{dt}t|\nabla\delta T|_{L^2}^2 \leqslant |\nabla\delta T|_{L^2}^2 + 2t\gamma_0|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}|\mathcal{A}\delta\boldsymbol{u}|_{\mathbb{L}^2}|\nabla T_1|_{L^2}|\Delta T_1|_{L^2}\\
&\qquad\qquad + 2t\gamma_0|\nabla\boldsymbol{u}_2|_{\mathbb{L}^2}|\mathcal{A}\boldsymbol{u}_2|_{\mathbb{L}^2}|\nabla\delta T|_{L^2}|\Delta\delta T|_{L^2}.
\end{aligned}\tag{6.74}$$

Therefore, together with (6.43), (6.46), (6.62) and (6.63), we obtain

$$\begin{aligned}
&t|\nabla\delta T(t)|_{L^2}^2\\
&\leqslant \left(\int_0^t |\nabla\delta T(s)|_{L^2}^2 ds + 2\gamma_0\int_0^t s|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}|\mathcal{A}\delta\boldsymbol{u}|_{\mathbb{L}^2}|\nabla T_1|_{L^2}|\Delta T_1|_{L^2}ds\right)\\
&\qquad\qquad \times\exp\left(\int_0^t \frac{27}{2}\gamma_0^2|\nabla\boldsymbol{u}_2|_{\mathbb{L}^2}^2|\mathcal{A}\boldsymbol{u}_2|_{\mathbb{L}^2}^2 ds\right)\\
&\leqslant Q(t, \|U_{10}\|_{\mathcal{H}_D^1}, \|U_{20}\|_{\mathcal{H}})\|\delta U(0)\|_{\mathcal{H}}^2
\end{aligned}\tag{6.75}$$

and

$$\begin{aligned}
&\int_0^t s|\Delta\delta T(s)|_{L^2}^2 ds \leqslant Q(t, \|U_{10}\|_{\mathcal{H}_D^1}, \|U_{20}\|_{\mathcal{H}})\|\delta U(0)\|_{\mathcal{H}}^2,\\
&\int_0^t s|\partial_t\delta T(s)|_{L^2}^2 ds \leqslant Q(t, \|U_{10}\|_{\mathcal{H}_D^1}, \|U_{20}\|_{\mathcal{H}})\|\delta U(0)\|_{\mathcal{H}}^2.
\end{aligned}\tag{6.76}$$

Similarly, from the third equation of (D), we have (see (6.21))

$$\begin{aligned}
&\frac{d}{dt}t|\nabla\delta C|_{L^2}^2 + t|\Delta\delta C|_{L^2}^2\\
&\quad\leqslant |\nabla\delta C|_{L^2}^2 + 4t\gamma_0|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}|\mathcal{A}\delta\boldsymbol{u}|_{\mathbb{L}^2}|\nabla C_1|_{L^2}|\Delta C_1|_{L^2}\\
&\qquad + \frac{27}{2}t\gamma_0^2|\nabla\boldsymbol{u}_2|_{\mathbb{L}^2}^2|\mathcal{A}\boldsymbol{u}_2|_{\mathbb{L}^2}^2|\nabla\delta C|_{L^2}^2 + 4\rho^2 t|\Delta\delta T|_{L^2}^2,\\
&\frac{d}{dt}t|\nabla\delta C|_{L^2}^2 + t|\partial_t\delta C|_{L^2}^2\\
&\quad\leqslant |\nabla\delta C|_{L^2}^2 + 3t\gamma_0|\nabla\delta\boldsymbol{u}|_{\mathbb{L}^2}|\mathcal{A}\delta\boldsymbol{u}|_{\mathbb{L}^2}|\nabla C_1|_{L^2}|\Delta C_1|_{L^2}\\
&\qquad + 3t\gamma_0|\nabla\boldsymbol{u}_2|_{\mathbb{L}^2}|\mathcal{A}\boldsymbol{u}_2|_{\mathbb{L}^2}|\nabla\delta C|_{L^2}|\Delta\delta C|_{L^2} + 3\rho^2 t|\Delta\delta T|_{L^2}^2,
\end{aligned}\tag{6.77}$$

which yield

$$\begin{aligned}
&t|\nabla\delta C(t)|_{L^2}^2 + \int_0^t s|\Delta\delta C(s)|_{L^2}^2 ds + \int_0^t s|\partial_t\delta C(s)|_{L^2}^2 ds\\
&\qquad\qquad \leqslant Q(t, \|U_{10}\|_{\mathcal{H}_D^1}, \|U_{20}\|_{\mathcal{H}})\|\delta U(0)\|_{\mathcal{H}}^2
\end{aligned}\tag{6.78}$$

for any $t \geqslant 0$.

Let $h > 0$. Then $D_h\delta\boldsymbol{u}(t) := \delta\boldsymbol{u}(t+h) - \delta\boldsymbol{u}(t)$, $D_h\delta T(t) := \delta T(t+h) - \delta T(t)$ and $D_h\delta C(t) := \delta C(t+h) - \delta C(t)$ satisfy

$$\partial_t D_h\delta\boldsymbol{u}(t) + \nu\mathcal{A}D_h\delta\boldsymbol{u}(t) + aD_h\delta\boldsymbol{u}(t) = \mathcal{P}_\Omega\boldsymbol{g}D_h\delta T(t) + \mathcal{P}_\Omega\boldsymbol{h}D_h\delta C(t),$$

which yields (see (6.27))

$$\frac{d}{dt}|D_h\delta\boldsymbol{u}(t)|^2_{\mathbb{L}^2} + \nu|\nabla D_h\delta\boldsymbol{u}(t)|^2_{\mathbb{L}^2} \leqslant \frac{2\kappa|\boldsymbol{g}|^2}{\nu}|D_h\delta T(t)|^2_{L^2} + \frac{2\kappa|\boldsymbol{h}|^2}{\nu}|D_h\delta C(t)|^2_{L^2}. \tag{6.79}$$

Integrating (6.79) over $[s,t]$ with $t > 0$ and using estimates for $\partial_t\delta T$ and $\partial_t\delta C$ ((6.76) and (6.78)) we have (see (6.28))

$$\begin{aligned} &|D_h\delta\boldsymbol{u}(t)|^2_{\mathbb{L}^2} \\ &\leqslant |D_h\delta\boldsymbol{u}(s)|^2_{\mathbb{L}^2} + \frac{2\kappa|\boldsymbol{g}|^2}{\nu}\int_s^t |D_h\delta T(\tau)|^2_{L^2}d\tau + \frac{2\kappa|\boldsymbol{h}|^2}{\nu}\int_s^t |D_h\delta C(\tau)|^2_{L^2}d\tau \\ &\leqslant |D_h\delta\boldsymbol{u}(s)|^2_{\mathbb{L}^2} + \frac{h^2}{s}Q(t, \|U_{10}\|_{\mathcal{H}^1_D}, \|U_{20}\|_{\mathcal{H}})\|\delta U(0)\|^2_{\mathcal{H}}. \end{aligned} \tag{6.80}$$

Integrating (6.80) again over $[1/2, t]$ with s-variable and using (6.73), we obtain,

$$\left(t - \frac{1}{2}\right)\left|\frac{\delta\boldsymbol{u}(t+h) - \delta\boldsymbol{u}(t)}{h}\right|^2_{\mathbb{L}^2} \leqslant Q(t, \|U_{10}\|_{\mathcal{H}^1_D}, \|U_{20}\|_{\mathcal{H}})\left(\log t - \log\frac{1}{2}\right)\|\delta U(0)\|^2_{\mathcal{H}}$$

i.e.,

$$\left|\frac{\delta\boldsymbol{u}(t+h) - \delta\boldsymbol{u}(t)}{h}\right|^2_{\mathbb{L}^2} \leqslant Q(t, \|U_{10}\|_{\mathcal{H}^1_D}, \|U_{20}\|_{\mathcal{H}})\frac{\log t + \log 2}{t - \frac{1}{2}}\|\delta U(0)\|^2_{\mathcal{H}}. \tag{6.81}$$

for $t > 1/2$. Especially, at $t = 1$,

$$\left|\frac{\delta\boldsymbol{u}(1+h) - \delta\boldsymbol{u}(1)}{h}\right|^2_{\mathbb{L}^2} \leqslant (2\log 2)Q(\|U_{10}\|_{\mathcal{H}^1_D}, \|U_{20}\|_{\mathcal{H}})\|\delta U(0)\|^2_{\mathcal{H}} \tag{6.82}$$

can be acquired. Hence applying Proposition 6.2.1 to the first equation of (D), we can assure

$$\begin{aligned} &|\mathcal{A}\delta\boldsymbol{u}(1)|_{\mathbb{L}^2} \\ \leqslant&\frac{1}{\nu}\left(a|\delta\boldsymbol{u}(1)|_{\mathbb{L}^2} + |\boldsymbol{g}||\delta T(1)|_{L^2} + |\boldsymbol{h}||\delta C(1)|_{L^2} + \left|\frac{d^+\delta\boldsymbol{u}}{dt}(1)\right|_{\mathbb{L}^2}\right) \\ \leqslant&Q(\|U_{10}\|_{\mathcal{H}^1_D}, \|U_{20}\|_{\mathcal{H}})\|\delta U(0)\|^2_{\mathcal{H}}. \end{aligned} \tag{6.83}$$

Consequently, from (6.62), (6.75), (6.78) and (6.83), we can derive

$$\|\delta U(1)\|_{\mathcal{H}^{2,1}} \leqslant Q(\|U_{10}\|_{\mathcal{H}^1_D}, \|U_{20}\|_{\mathcal{H}})\|\delta U(0)\|_{\mathcal{H}} \leqslant Q(M_{24})\|\delta U(0)\|_{\mathcal{H}}, \tag{6.84}$$

(recall (6.57)), which implies (6.68). □

Thus, by applying Proposition 2.4.2, we can assure Theorem 6.1.2, namely, the existence of exponential attractor $\mathscr{M}_D$ of $(\{\mathscr{S}_D(t)\}_{t\geq 0}, \mathcal{H})$.

Finally, we demonstrate the reminder of Theorem 6.1.3, i.e., the existence of exponential attractor $\mathscr{M}_D^1$ of $(\{\mathscr{S}_D(t)\}_{t\geq 0}, \mathcal{H}_D^1)$ via the abstract result Proposition 2.4.2.

We here recall that the existence of compact positively invariant absorbing set $\mathscr{B}_{00}^1$ in $\mathcal{H}_D^1$. It is easy to see that

$$\|\mathscr{S}_D(t)U_0\|_{\mathcal{H}^{2,2}} \leqslant M_{25} \tag{6.85}$$

holds for any $U_0 \in \mathscr{B}_{00}^1$ and any $t \geqslant 0$ (see (6.58)). In order to apply Proposition 2.4.2, we need to check the following estimates.

Lemma 6.3.2. *There exist constants M_{28}, M_{29} and M_{30} satisfying*

$$\|\mathscr{S}_D(t)U_{10} - \mathscr{S}_D(t)U_{20}\|_{\mathcal{H}_D^1} \leqslant M_{28}\|U_{10} - U_{20}\|_{\mathcal{H}_D^1}, \tag{6.86}$$

$$\|\mathscr{S}_D(1)U_{10} - \mathscr{S}_D(1)U_{20}\|_{\mathcal{H}^{2,2}} \leqslant M_{29}\|U_{10} - U_{20}\|_{\mathcal{H}_D^1}, \tag{6.87}$$

$$\|\mathscr{S}_D(t)U_{10} - \mathscr{S}_D(s)U_{10}\|_{\mathcal{H}_D^1} \leqslant M_{30}|t-s|^{1/2} \tag{6.88}$$

for any $U_{i0} \in \mathscr{B}_{00}^1$ $(i = 1, 2)$ and any $t, s \in [0, 1]$.

Proof. The inequality (6.86) is obvious from (6.66).

From (6.85) and each equation of (DCBF), we have

$$|\partial_t \boldsymbol{u}_i(t)|_{\mathbb{L}^2}^2 + |\partial_t T_i(t)|_{L^2}^2 + |\partial_t C_i(t)|_{L^2}^2 \leqslant Q(\|U_{i0}\|_{\mathcal{H}^{2,2}}) \leqslant Q(M_{25}) \tag{6.89}$$

for a.e. $t \geqslant 0$. Moreover, from (6.50), (6.51) and (6.56) we have

$$\int_0^t |\partial_t \nabla \boldsymbol{u}_i(s)|_{\mathbb{L}^2}^2 ds + \int_0^t |\partial_t \nabla T_i(s)|_{L^2}^2 ds + \int_0^t |\partial_t \nabla C_i(s)|_{L^2}^2 ds \leqslant Q(t), \tag{6.90}$$

which immediately yields (6.88). Here and henceforth, $Q(t, \|U_{10}\|_{\mathcal{H}^{2,2}}, \|U_{20}\|_{\mathcal{H}^{2,2}})$ and $Q(t, M_{25})$ will be simply denoted by $Q(t)$.

Here, we recall (6.63) and (6.66), i.e.,

$$\begin{aligned} &\|\delta U(t)\|_{\mathcal{H}_D^1}^2 \leqslant Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1}^2, \\ &\int_0^t |\mathcal{A}\delta \boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds + \int_0^t |\nabla \delta T(s)|_{L^2}^2 ds + \int_0^t |\nabla \delta C(s)|_{L^2}^2 ds \leqslant Q(t)\|\delta U(0)\|_{\mathcal{H}}^2. \end{aligned} \tag{6.91}$$

Moreover, integrating (6.64) over $[0, t]$, we have

$$\int_0^t |\Delta \delta T(s)|_{L^2}^2 ds + \int_0^t |\Delta \delta C(s)|_{L^2}^2 ds \leqslant Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1}^2. \tag{6.92}$$

Multiplying each equation of (D) by $\partial_t \delta \boldsymbol{u}$, $\partial_t \delta T$, $\partial_t \delta C$ respectively and integrating over $[0, t]$, we obtain (see (6.74) and (6.77))

$$\int_0^t |\partial_t \delta \boldsymbol{u}(s)|_{\mathbb{L}^2}^2 ds + \int_0^t |\partial_t \delta T(s)|_{L^2}^2 + \int_0^t |\partial_t \delta C(s)|_{L^2}^2 \leqslant Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1}^2. \tag{6.93}$$

Then integration of (6.79) over $[s,t]$ and over $[0,t]$ with respect to s-variable again, together with (6.93), yields

$$t\,|\delta\boldsymbol{u}(t+h)-\delta\boldsymbol{u}(t)|_{\mathbb{L}^2}^2 \leqslant h^2Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1}^2 \tag{6.94}$$

for any $t\geqslant 0$ and $h>0$. Applying Proposition 6.2.1 and taking the limit as $h\to +0$, we obtain

$$|\mathcal{A}\delta\boldsymbol{u}(t)|_{\mathbb{L}^2}^2 \leqslant \frac{1}{t}Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1}^2 \tag{6.95}$$

for any $t>0$. Integrating (6.79) again over $[1/4,t]$, we have

$$\int_{1/4}^{t}|\partial_t\nabla\delta\boldsymbol{u}(s)|_{\mathbb{L}^2}^2\,ds \leqslant Q(t)\|\delta U(0)\|_{\mathcal{H}}^2 \tag{6.96}$$

for any $t\geqslant 1/4$.

From the second equation of (D),

$$\begin{aligned}\partial_t D_h\delta T(t)-\Delta\delta D_hT(t)+D_h\delta\boldsymbol{u}(t)\cdot\nabla T_1(t+h)+\delta\boldsymbol{u}(t)\cdot\nabla D_hT_1(t)&\\ +\boldsymbol{u}_2(t+h)\cdot\nabla D_h\delta T(t)+D_h\boldsymbol{u}_2(t)\cdot\nabla\delta T(t)&=0\end{aligned} \tag{6.97}$$

is satisfied by $D_h\delta T(t):=\delta T(t+h)-\delta T(t)$ and $D_h\delta\boldsymbol{u}(t):=\delta\boldsymbol{u}(t+h)-\delta\boldsymbol{u}(t)$ with $h>0$. Multiplying (6.97) by $D_h\delta T(t)$, we obtain

$$\begin{aligned}&\frac{d}{dt}|D_h\delta T(t)|_{L^2}^2+|\nabla\delta D_hT(t)|_{L^2}^2\\ \leqslant\ &3|D_h\delta\boldsymbol{u}(t)T_1(t+h)|_{\mathbb{L}^2}^2+3|\delta\boldsymbol{u}(t)D_hT_1(t)|_{\mathbb{L}^2}^2+3|D_h\boldsymbol{u}_2(t)\delta T(t)|_{\mathbb{L}^2}^2\\ \leqslant\ &3\gamma_1|\nabla D_h\delta\boldsymbol{u}(t)|_{\mathbb{L}^2}^2|\nabla T_1(t+h)|_{L^2}^2\\ &\quad+3\gamma_1|\nabla\delta\boldsymbol{u}(t)|_{\mathbb{L}^2}^2|\nabla D_hT_1(t)|_{L^2}^2+3\gamma_1|\nabla D_h\boldsymbol{u}_2(t)|_{\mathbb{L}^2}^2|\nabla\delta T(t)|_{\mathbb{L}^2}^2.\end{aligned} \tag{6.98}$$

Let $s\in[1/4,t]$. Integrating (6.98) over $[s,t]$ and using (6.66), (6.85), (6.90) and (6.96), we have

$$|D_h\delta T(t)|_{L^2}^2 \leqslant |D_h\delta T(s)|_{L^2}^2+h^2Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1}^2 \tag{6.99}$$

for any $t\geqslant s\geqslant 1/4$. Integrating (6.99) over $[1/4,t]$ with respect to the variable s, we obtain, by (6.93),

$$\left(t-\frac{1}{4}\right)|D_h\delta T(t)|_{L^2}^2 \leqslant h^2Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1}^2. \tag{6.100}$$

We here mention that the second equation of (D) satisfies all requirements in Lemma 6.2.1 with $U=\delta T$, $\boldsymbol{w}=\boldsymbol{u}_2$ and $F=-\delta\boldsymbol{u}\cdot\nabla T_1$, since $\delta\boldsymbol{u}\in C([0,\infty);\mathbb{H}_\sigma^1(\Omega))$, $T_1\in C([0,\infty);H^1(\Omega))$ and $|\delta\boldsymbol{u}\cdot\nabla T_1|_{L^2}\in L^\infty(0,t)$ are varied. Moreover, (6.100) implies that condition *2* of Lemma 6.2.1 is satisfied by the second equation of (D) for $t > 1/4$. Hence condition *3* of Lemma 6.2.1 and (6.100) yield

$$\begin{aligned}
&|\Delta\delta T(t)|_{L^2} \\
&\leqslant \left|\text{w-}\lim_{h\to 0}\frac{\delta T(t+h)-\delta T(t)}{h}\right|_{L^2} + |\delta\boldsymbol{u}(t)\cdot\nabla T_1(t)|_{L^2} + |\boldsymbol{u}_2(t)\cdot\nabla\delta T(t)|_{L^2} \\
&\leqslant \liminf_{h\to+0}\left|\frac{\delta T(t+h)-\delta T(t)}{h}\right|_{L^2} + \gamma_1|\nabla\delta\boldsymbol{u}(t)|_{\mathbb{L}^2}|T_1(t)|_{H^2} \\
&\qquad\qquad + \gamma_0^{1/2}|\nabla\boldsymbol{u}_2(t)|_{\mathbb{L}^2}^{1/2}|\mathcal{A}\boldsymbol{u}_2(t)|_{\mathbb{L}^2}^{1/2}|\nabla\delta T(t)|_{L^2}^{1/2}|\Delta\delta T(t)|_{L^2}^{1/2} \\
&\leqslant \frac{1}{\sqrt{t-\frac{1}{4}}}Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1} + Q(t)\|\delta U(0)\|_{\mathcal{H}} + Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1}^{1/2}|\Delta\delta T(t)|_{L^2}^{1/2}.
\end{aligned}$$

Hence

$$|\Delta\delta T(t)|_{L^2}^2 \leqslant \frac{1}{t-\frac{1}{4}}Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1}^2. \tag{6.101}$$

holds for any $t > 1/4$ Moreover, from (6.98) and (6.100), we can derive the following for any $t \geqslant 1/2$:

$$\int_{1/2}^{t}|D_h\nabla\delta T(s)|_{L^2}^2 ds \leqslant h^2 Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1}^2,$$

which implies

$$\int_{1/2}^{t}|\partial_t\nabla\delta T(s)|_{L^2}^2 ds \leqslant Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1}^2 \tag{6.102}$$

for $t \geqslant 1/2$. Similarly, multiplying

$$\begin{aligned}
\partial_t D_h\delta C(t) - \Delta\delta D_h C(t) + D_h\delta\boldsymbol{u}(t)\cdot\nabla C_1(t+h) + \delta\boldsymbol{u}(t)\cdot\nabla D_h C_1(t) \\
+ \boldsymbol{u}_2(t+h)\cdot\nabla D_h\delta C(t) + D_h\boldsymbol{u}_2(t)\cdot\nabla\delta C(t) = \rho\Delta D_h\delta T(t)
\end{aligned}$$

by $D_h\delta C(t)$, we obtain

$$\begin{aligned}
\frac{d}{dt}|D_h\delta C(t)|_{L^2}^2 &\leqslant 4\rho^2|\nabla D_h\delta T(t)|_{L^2}^2 + 4\gamma_1|\nabla D_h\delta\boldsymbol{u}(t)|_{\mathbb{L}^2}^2|\nabla C_1(t+h)|_{L^2}^2 \\
&\quad + 4\gamma_1|\nabla\delta\boldsymbol{u}(t)|_{\mathbb{L}^2}^2|\nabla D_h C_1(t)|_{L^2}^2 + 4\gamma_1|\nabla D_h\boldsymbol{u}_2(t)|_{\mathbb{L}^2}^2|\nabla\delta C(t)|_{\mathbb{L}^2}^2.
\end{aligned} \tag{6.103}$$

From (6.66), (6.85), (6.90), (6.93), (6.96) and (6.102), integration of (6.103) over $[s,t]$ and over $[1/2,t]$ with respect to s-variable again gives us

$$\left(t-\frac{1}{2}\right)|D_h\delta T(t)|_{L^2}^2 \leqslant h^2 Q(t)\|\delta U(0)\|_{\mathcal{H}_D^1}^2. \tag{6.104}$$

We can easily check that the third equation of (D) satisfies requirements in Lemma 6.2.1 with $U = \delta C$, $\boldsymbol{w} = \boldsymbol{u}_2$ and $F = -\delta \boldsymbol{u} \cdot \nabla C_1 + \rho \Delta \delta T$ and we can obtain

$$\begin{aligned}
&|\Delta \delta C(t)|_{L^2} \\
&\leqslant \left| \text{w-}\lim_{h\to 0} \frac{\delta C(t+h) - \delta C(t)}{h} \right|_{L^2} + |\delta \boldsymbol{u}(t) \cdot \nabla C_1(t)|_{L^2} \\
&\qquad\qquad\qquad\qquad + |\boldsymbol{u}_2(t) \cdot \nabla \delta C(t)|_{L^2} + |\rho \Delta \delta T(t)|_{L^2} \\
&\leqslant \frac{1}{\sqrt{t - \frac{1}{2}}} Q(t) \|\delta U(0)\|_{\mathcal{H}^1_D} + Q(t)\|\delta U(0)\|_{\mathcal{H}^1_D} + Q(t)\|\delta U(0)\|^{1/2}_{\mathcal{H}^1_D} |\Delta \delta C(t)|^{1/2}_{L^2},
\end{aligned}$$

that is to say,

$$|\Delta \delta C(t)|^2_{L^2} \leqslant \frac{1}{t - \frac{1}{2}} Q(t) \|\delta U(0)\|^2_{\mathcal{H}^1_D} \tag{6.105}$$

for any $t > 1/2$. Hence we can derive (6.87) from (6.95), (6.101) and (6.105). □

6.4 Neumann Boundary Condition Case

In this section, we consider Neumann boundary condition case (Theorem 6.1.4). Let $(\boldsymbol{u}, T, C)$ be a solution of (DCBF) (with homogeneous Neumann boundary condition) with the initial data $(\boldsymbol{u}_0, T_0, C_0)$. We here recall that

$$\int_\Omega f_2 dx = \int_\Omega f_3 dx = 0.$$

is assumed in Theorem 6.1.4. Under this assumption, we obtain the following mass conservation properties:

$$\int_\Omega T(t) dx = \int_\Omega T_0 dx, \qquad \int_\Omega C(t) dx = \int_\Omega C_0 dx \tag{6.106}$$

for any $t \geqslant 0$, by integrating the second and the third equation over Ω, $[0, t]$ and using the following facts:

$$\begin{aligned}
&\int_\Omega \Delta_N T dx = 0, \qquad \int_\Omega \Delta_N C dx = 0, \\
&\int_\Omega \boldsymbol{u} \cdot \nabla T dx = -\int_\Omega T \nabla \cdot \boldsymbol{u} dx = 0, \qquad \int_\Omega \boldsymbol{u} \cdot \nabla C dx = 0,
\end{aligned}$$

which can be assured by homogeneous Neumann boundary condition and solenoidal condition. Therefore the semigroup $\{\mathscr{S}_N(t)\}_{t\geqslant 0}$ acts on the restricted space $\mathcal{H}_{m_T, m_C}$ and $\mathcal{H}^1_{N, m_T, m_C}$ for arbitrary positive number m_T and m_C, namely, the dynamical systems $(\{\mathscr{S}_N(t)\}_{t\geqslant 0}, \mathcal{H}_{m_T, m_C})$ and $(\{\mathscr{S}_N(t)\}_{t\geqslant 0}, \mathcal{H}^1_{N, m_T, m_C})$ are well defined for any $m_T, m_C > 0$.

Here we define

$$T'(t) := T(t) - \frac{1}{|\Omega|}\int_\Omega T_0 dx, \quad C'(t) := C(t) - \frac{1}{|\Omega|}\int_\Omega C_0 dx.$$

Then combining the mass conservation law (6.106) with Poincaré-Wirtinger's inequality:

$$\left| U - \frac{1}{|\Omega|}\int_\Omega U dx \right|_{L^2}^2 \leqslant \kappa' |\nabla U|_{L^2}^2 \qquad \forall U \in H^1(\Omega)$$

(κ' is a suitable constant), we can assure that T' and C' satisfy

$$|T'|_{L^2}^2 \leqslant \lambda' |\nabla T'|_{L^2}^2, \quad |C'|_{L^2}^2 \leqslant \lambda' |\nabla C'|_{L^2}^2. \tag{6.107}$$

Obviously, T', C' and $\boldsymbol{u}$ satisfies the following equations:

$$\begin{aligned} &\partial_t \boldsymbol{u} + \nu \mathcal{A}\boldsymbol{u} - a\boldsymbol{u} = \mathcal{P}\boldsymbol{g}T' + \mathcal{P}\boldsymbol{g}\frac{1}{|\Omega|}\int_\Omega T_0 dx + \mathcal{P}\boldsymbol{h}C' + \mathcal{P}\boldsymbol{h}\frac{1}{|\Omega|}\int_\Omega C_0 dx + \mathcal{P}\boldsymbol{f}_1, \\ &\partial_t T' - \Delta T' + \boldsymbol{u}\cdot\nabla T' = f_2, \\ &\partial_t C' - \Delta C' + \boldsymbol{u}\cdot\nabla C' = \rho\Delta T' + f_3. \end{aligned} \tag{6.108}$$

Then, under assumptions $\int_\Omega f_2 dx = \int_\Omega f_3 dx = 0$, we can assure the existence of global attractor and exponential attractor by repeating the same arguments as those for Dirichlet boundary condition case with substitution of (6.107) for Poincaré's inequality. Actually, calculations for the second and the third equation can be established by exactly the same procedures as those in previous sections, with T and C replaced by T' and C' respectively (we here note that the replacement of boundary conditions dose not affect our argument for Lemma 6.2.1 and Corollary 6.2.1). As for the first equation of (6.108), we remark

$$\begin{aligned} &\left|\mathcal{P}_\Omega \boldsymbol{g}\frac{1}{|\Omega|}\int_\Omega T_0 dx\right|_{L^2}^2 = |\boldsymbol{g}|^2 \int_\Omega \left(\frac{1}{|\Omega|}\int_\Omega T_0 dx\right)^2 dx \leqslant |\boldsymbol{g}|^2 |\Omega| m_T^2, \\ &\left|\mathcal{P}_\Omega \boldsymbol{h}\frac{1}{|\Omega|}\int_\Omega C_0 dx\right|_{L^2}^2 \leqslant |\boldsymbol{h}|^2 |\Omega| m_C^2, \\ &\left|\mathcal{P}_\Omega \boldsymbol{g}\frac{1}{|\Omega|}\int_\Omega T_{10} dx - \mathcal{P}_\Omega \boldsymbol{g}\frac{1}{|\Omega|}\int_\Omega T_{20} dx\right|_{L^2}^2 \\ &\quad = \left|\mathcal{P}_\Omega \boldsymbol{g}\frac{1}{|\Omega|}\int_\Omega T_{10} - T_{20} dx\right|_{L^2}^2 \leqslant |\boldsymbol{g}|^2 \frac{1}{|\Omega|}\left(\int_\Omega \delta T_{10} - T_{20} dx\right)^2 \leqslant |\boldsymbol{g}|^2 |T_{10} - T_{20}|_{L^2}^2, \\ &\left|\mathcal{P}_\Omega \boldsymbol{h}\frac{1}{|\Omega|}\int_\Omega C_{10} dx - \mathcal{P}_\Omega \boldsymbol{h}\frac{1}{|\Omega|}\int_\Omega C_{20} dx\right|_{L^2}^2 \leqslant |\boldsymbol{h}|^2 |C_{10} - C_{20}|_{L^2}^2, \end{aligned}$$

which allow us to accomplish almost the same procedure for the first equation as those above (obviously, the terms $\mathcal{P}_\Omega \boldsymbol{g}\frac{1}{|\Omega|}\int_\Omega T_0 dx$ and $\mathcal{P}_\Omega \boldsymbol{h}\frac{1}{|\Omega|}\int_\Omega C_0 dx$ do not hinder us applying

Proposition 6.2.1). Moreover, since

$$\begin{aligned}
&|T(t)|_{L^2} \leqslant |T'(t)|_{L^2} + \left| \frac{1}{|\Omega|} \int_\Omega T_0 dx \right|_{L^2} \leqslant |T'(t)|_{L^2} + |T_0|_{L^2} \leqslant |T'(t)|_{L^2} + m_T, \\
&|C(t)|_{L^2} \leqslant |C'(t)|_{L^2} + \left| \frac{1}{|\Omega|} \int_\Omega C_0 dx \right|_{L^2} \leqslant |C'(t)|_{L^2} + |C_0|_{L^2} \leqslant |C'(t)|_{L^2} + m_C, \\
&|T_1'(t) - T_2'(t)|_{L^2} \leqslant |T_1(t) - T_2(t)|_{L^2} + |T_{10} - T_{20}|_{L^2}, \\
&|C_1'(t) - C_2'(t)|_{L^2} \leqslant |C_1(t) - C_2(t)|_{L^2} + |C_{10} - C_{20}|_{L^2}, \\
&\nabla T' = \nabla T, \quad \Delta T' = \Delta T, \quad \partial_t T' = \partial_t T, \\
&\nabla C' = \nabla C, \quad \Delta C' = \Delta C, \quad \partial_t C' = \partial_t C,
\end{aligned}$$

estimates for T' and C' immediately lead to those for T and C.

Hence, for each m_T and m_C, we can assure the existence of a global attractor and an exponential attractor by almost the same reasoning stated in Sections 6.2 and 6.3, whence it follows our results. □

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List of Original Papers

- (with Mitsuharu Ôtani) The existence of periodic solutions of some double-diffusive convection system based on Brinkman-Forchheimer equations, *Adv. Math. Sci. Appl.* **vol. 23 No. 1** (2013), 77–92.
- (with Mitsuharu Ôtani) Global solvability of some double-diffusive convection system coupled with Brinkman-Forchheimer equations, *Lib. Math. (N.S.)* **Vol. 33 No. 1** (2013), 79–107.
- (with Mitsuharu Ôtani) Attractors for autonomous double-diffusive convection systems based on Brinkman-Forchheimer equations, to appear in *Math. Meth. Appl. Sci.*.
- (with Mitsuharu Ôtani) Global solvability for double-diffusive convection system based on Brinkman-Forchheimer equation in general domains, to appear in *Osaka J. Math.*.