Reformulation of Einstein equations with constraint damping and numerical simulation

Constraint damping を用いた Einstein 方程式の再定式化と数値計算

July 2012

Research on Relativity

Major in Pure and Applied Mathematics

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Acknowledgements

I would like to express my sincere gratitude to my supervisor, Professor Gen Yoneda for providing me this precious study opportunity as a Ph.D student in his laboratory. I gratefully acknowledge helpful discussions with Professor Hisa-aki Shinkai of Osaka Institute of Technology on several points in this research. I also thanks to Professors Kimio Ueno, Daisuke Takahashi and Kei-ichi Maeda for helpful suggestions.

Contents

Notation

- signature of the metric : $(-,+,+,\cdots)$.
- $\mathcal{M}^m$: m dimensional Riemannian manifold.
- $\mathcal{T}^a{}_b(\mathcal{M}^m)$: tensor field as $p \in \mathcal{M}^m$. $\mathcal{T}^1_0(\mathcal{M}^m)$ is tangent space, $\mathcal{T}^0_1(\mathcal{M}^m)$ is cotangent space and $\mathcal{T}^0_0(\mathcal{M}^m)$ is scalar field.
- n_μ : normal vector on $\mathcal{T}^1{}_0(\mathcal{M}^{m+1})$
- $g_{\mu\nu}, \gamma_{\mu\nu}$: metric on $\mathcal{M}^{m+1}$ and $\mathcal{M}^m$, respectively.
- g, γ : determinant of $g_{\mu\nu}$ and $\gamma_{\mu\nu}$, respectively.
- $P^\mu{}_\nu$: projection operator from $\mathcal{T}^a_b(\mathcal{M}^{m+1})$ to $\mathcal{T}^a_b(\mathcal{M}^m)$ which is explicitly written as

$$P^\mu{}_\nu \equiv \delta^\mu{}_\nu - \frac{1}{\epsilon} n^\mu n_\nu, \tag{1}$$

  where ϵ express the direction of n_μ.
- ∇_μ : covariant derivative operator associated with $g_{\mu\nu}$.
- D_μ : covariant derivative operator associated with $\gamma_{\mu\nu}$. The relation between ∇_μ and D_μ is $\forall T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} \in \mathcal{T}^a_b(\mathcal{M}^m)$,

$$D_\lambda T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} \equiv P^\omega{}_\lambda P^{\mu_1}{}_{\alpha_1} P^{\mu_2}{}_{\alpha_2} \cdots P^{\beta_1}{}_{\nu_1} P^{\beta_2}{}_{\nu_2} \cdots \nabla_\omega T^{\alpha_1\alpha_2\cdots}{}_{\beta_1\beta_2\cdots}. \tag{2}$$

- ${}^{(m)}R_{\mu\nu\lambda\omega}$: m dimensional Riemann tensor. The definition is given as $\forall T_\omega \in \mathcal{T}^0_1(\mathcal{M}^m)$,

$$\begin{aligned} {}^{(m)}R^\omega{}_{\lambda\mu\nu} T_\omega &\equiv 2D_{[\nu} D_{\mu]} T_\lambda & (3)\\ &= (\partial_\mu {}^{(m)}\Gamma^\omega{}_{\lambda\nu} - \partial_\nu {}^{(m)}\Gamma^\omega{}_{\lambda\mu} + {}^{(m)}\Gamma^\omega{}_{\mu\rho} {}^{(m)}\Gamma^\rho{}_{\lambda\nu} - {}^{(m)}\Gamma^\omega{}_{\nu\rho} {}^{(m)}\Gamma^\rho{}_{\lambda\mu}) T_\omega, & (4) \end{aligned}$$

  where ${}^{(m)}\Gamma^\lambda{}_{\mu\nu}$ is the Levi-Civita connection on $\mathcal{M}^{m+1}$.

$$ {}^{(m)}R^\lambda{}_{\omega\mu\nu} T^\omega = 2D_{[\mu} D_{\nu]} T^\lambda \tag{5}$$

- $\mathcal{L}_\xi(V^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots})$: Lie derivative of $V^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} \in \mathcal{T}^a{}_b$ associated with $\xi^\lambda \in \mathcal{T}^1_0(\mathcal{M}^{m+1})$. The expression with ∇_μ is

$$\begin{aligned} \mathcal{L}_\xi(V^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots}) = \xi^\lambda \nabla_\lambda (V^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots}) &+ V^{\lambda\mu_2\cdots}{}_{\nu_1\nu_2\cdots} (\nabla_\lambda \xi^{\mu_1})\\ &+ V^{\mu_1\lambda\cdots}{}_{\nu_1\nu_2\cdots} (\nabla_\lambda \xi^{\mu_2}) + \cdots - V^{\mu_1\mu_2\cdots}{}_{\lambda\nu_2\cdots} (\nabla_{\nu_1} \xi^\lambda)\\ &- V^{\mu_1\mu_2\cdots}{}_{\nu_1\lambda\cdots} (\nabla_{\nu_2} \xi^\lambda) - \cdots . \end{aligned} \tag{6}$$

- $K_{\mu\nu}$: extrinsic curvature of $\mathcal{M}^m$. The definition is given by

$$K_{\mu\nu} \equiv -\frac{1}{2} \mathcal{L}_n(\gamma_{\mu\nu}) \tag{7}$$

  where n is the normal vector on $\mathcal{T}_p(\mathcal{M}^m)$ and $\gamma_{\mu\nu}$ is the metric of $\mathcal{M}^m$.
- α : lapse function in $\mathcal{T}^0_0(\mathcal{M}^m)$.
- β^μ : shift vector in $\mathcal{T}^1_0(\mathcal{M}^m)$.
- Λ : cosmological constant.

Part I

Introduction of Numerical Relativity

Chapter 1

Introduction and Definition

In this chapter, we state the abstract of the Numerical Relativity and the definitions in this paper. Currently, there are many references about General Relativity and the Numerical Relativity, we mainly refer to [1–3] and [4–7], respectively.

1.1 Introduction

When solving the Einstein equations numerically, the standard way is to split spacetime into space and time. Arnowitt, Deser and Misner (ADM) were first formulated the decomposition of Einstein equations (we call this formulation the *original ADM formulation*) [8]. Smarr and York [9, 10] were re-formulated the original ADM formulation (we call this the *standard ADM formulation*). However, it is well known that in long-term evolutions in strong gravitational fields such as the coalescences of binary neutron stars and/or black holes, simulations with the ADM formulation are unstable and are often interrupted before producing physically interesting results. Finding more robust and stable formulations is known to the *formulation problem* in numerical relativity [4, 11, 12].

Many formulations have been proposed in the last two decades. The most commonly used sets of evolution equations among numerical relativists are the so-called *Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation* [13, 14], the *generalized harmonic (GH) formulation* [15, 16], the *Kidder-Scheel-Teukolsky (KST) formulation* [17], and the *Z4 formulation* [18, 19] (as references of their numerical application, we here cite only well-known articles; [20, 21] for the BSSN formulation, [22] for the GH formulation, [23] for the KST formulation, and [24] for the Z4 formulation).

All of the above modern formulations include the technique of *constraint damping*, which attempts to control the violations of constraints by adding the constraint terms to their evolution equations. Using this technique, more stable and accurate systems are obtained (see e.g. [25,26]). This technique can be described as *adjustment* of the original system.

In [27–29], Yoneda and Shinkai systematically investigated how the adjusted terms change the original systems by calculating the *constraint propagation equations* (dynamical equations of constraints). They suggested some effective adjustments for the ADM and BSSN formulations under the name *adjusted ADM formulation* and *adjusted BSSN formulation*, respectively [27,28]. The actual constraint-damping effect was confirmed by numerical tests [30–33].

Fiske proposed a method of adjusting the original evolution system using the norm of the constraints, C^2, [34], which we call a *C^2-adjusted system.* The new evolution equations force the

constraints to evolve toward their decay if the coefficient parameters of the adjusted terms are set as appropriate positive values. Fiske reported the damping effect of the constraint violations for the Maxwell system [34] and for the linearized ADM and BSSN formulations [35]. He also reported the limitation of the magnitude of the coefficient parameters of the adjusted terms.

In [31], we applied the C^2-adjusted system to the (full) ADM formulation and calculated the *Constraint Amplification Factors (CAFs)*, we confirmed that this system has numerical stability. Besides, we presented some numerical tests, the violations of the constraints are less than those in the original system. In addition, for comparing with the numerical stability of the C^2-adjusted ADM formulation, we performed some simulations with the Detweiler system [40], and confirmed the constraint violations with the C^2-adjusted ADM formulation are damped down in comparison with the Detweiler system. We also reported the differences of the effective range of the coefficient of the adjusted terms.

In [32], we applied the C^2-adjusted system to the (full) BSSN formulation and derived the constraint propagation equations in the flat space. We performed some numerical tests and compare them with three other types of BSSN formulations: the standard BSSN formulation, the $\widetilde{A}$-adjusted BSSN formulation [27, 30], and the C^2-adjusted BSSN formulation. We used the gauge-wave and polarized Gowdy wave testbeds, which are the test problems as is known to apples-with-apples testbeds for comparing evolution systems [41]. Since the models are precisely fixed up to the gauge conditions, boundary conditions, and technical parameters, the testbeds are widely used for comparisons [30, 42, 43].

The structure of this article is as follows. In part I, we derive $3+1$ formulations of Einstein equations in m dimension not only four dimension. We introduce the standard ADM formulation in chapter 2. In chapter 3, we introduce the BSSN formulation. In Part II, we review some tools for analyzing the numerical stability and introduce the C^2-adjusted ADM and BSSN formulations. In chapter 4, we introduce the constraint propagation equations. We review the ideas of the adjusted systems, the CAFs and the C^2-adjusted systems in chapter 5. In chapter 6, we introduce the C^2-adjusted system to the ADM formulation, and the C^2-adjusted BSSN formulation is introduced in chapter 7. In part III, we perform the simulations and summarize this article. In chapter 8, we set the numerical conditions and parameters of the simulations. We perform the simulations with the C^2-adjusted ADM and BSSN formulations in chapter 9 and 10, respectively, In chapter 11, we summarize this paper.

1.2 Definition of Geometrical Values

1.2.1 Projection

For a $\mathcal{M}^{m+1}$, we set the hypersurface $\mathcal{M}^m$ which is satisfied that a global ($m+1$ dimensional) curve ξ is constant. Hence the unit normal vector n_μ on the tangent space $\mathcal{T}_p(\mathcal{M}^m)$ can be expressed as

$$n_\mu = \epsilon N \nabla_\nu \xi, \tag{1.1}$$

where ϵ express the direction of n_μ and N is the positive function such that the norm of n_μ make a unit. For $g^{\mu\nu}$, the inner product of n_μ is satisfied that

$$n_\mu n_\nu g^{\mu\nu} = \epsilon, \tag{1.2}$$

where ϵ is

$$\epsilon = \begin{cases} 1 & : \text{if } n_\mu \text{ is spacelike,} \\ -1 & : \text{if } n_\mu \text{ is timelike.} \end{cases} \tag{1.3}$$

The positive function N is expressed as

$$N = \{\epsilon g^{\mu\nu}(\nabla_\mu \xi)(\nabla_\nu \xi)\}^{-1/2}. \tag{1.4}$$

In addition, n_μ is satisfied $n_\mu(\nabla_\nu n^\mu) = 0$ because

$$n_\mu(\nabla_\nu n^\mu) = \nabla_\nu(n^\mu n_\nu) - (\nabla_\nu n_\mu)n^\mu \tag{1.5}$$
$$= -(\nabla_\nu n^\mu)n_\mu \tag{1.6}$$
$$\therefore \quad n_\mu(\nabla_\nu n^\mu) = 0. \tag{1.7}$$

On the other hand, $n^\lambda \nabla_\lambda n_\mu$ is

$$n^\lambda \nabla_\lambda n_\mu = n^\lambda \nabla_\lambda(\epsilon N \nabla_\mu \xi) \tag{1.8}$$
$$= \epsilon n^\lambda (\nabla_\lambda N)(\nabla_\mu \xi) + \epsilon n^\lambda N(\nabla_\lambda \nabla_\mu \xi) \tag{1.9}$$
$$= \frac{1}{N} n^\lambda (\nabla_\lambda N) n_\mu + \epsilon n^\lambda N(\nabla_\mu \nabla_\lambda \xi) \tag{1.10}$$
$$= n^\lambda n_\mu (\nabla_\lambda \log N) + \epsilon \nabla_\mu(n^\lambda N \nabla_\lambda \xi) - \epsilon \nabla_\mu(n^\lambda N)(\nabla_\lambda \xi) \tag{1.11}$$
$$= n^\lambda n_\mu (\nabla_\lambda \log N) + (\nabla_\mu \epsilon) - \frac{\epsilon}{N}(\nabla_\mu N) \tag{1.12}$$
$$= (\nabla_\lambda \log N)(n^\lambda n_\mu - \epsilon \delta^\lambda{}_\mu) \tag{1.13}$$
$$= -\epsilon(\nabla_\lambda \log N)\left(\delta^\lambda{}_\mu - \frac{1}{\epsilon} n^\lambda n_\mu\right) \tag{1.14}$$
$$= -\epsilon D_\mu \log N. \tag{1.15}$$

Now we define the *projection operator*. For a $P^\mu{}_\nu \in \mathcal{T}^1_1(\mathcal{M}^{m+1})$, if $P^\mu{}_\nu$ satisfies the two conditions:

- $P^\mu{}_\nu n^\nu = P^\mu{}_\nu n_\mu = 0$, and
- $\forall V^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} \in \mathcal{T}^a_b(\mathcal{M}^m)$,
$$V^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} P^{\lambda_1}{}_{\mu_1} P^{\lambda_2}{}_{\mu_2} \cdots P^{\nu_1}{}_{\omega_1} P^{\nu_2}{}_{\omega_2} \cdots = V^{\lambda_1\lambda_2\cdots}{}_{\omega_1\omega_2\cdots} \in \mathcal{T}^a_b(\mathcal{M}^m).$$

then $P^\mu{}_\nu$ is projection operator from $\mathcal{T}^a_b(\mathcal{M}^{m+1})$ to $\mathcal{T}^a_b(\mathcal{M}^m)$. For the metric $g_{\mu\nu}$ on $\mathcal{M}^{m+1}$ and normal vector n_μ on $\mathcal{T}_p(\mathcal{M}^m)$, $P^\mu{}_\nu$ is explicitly written as

$$P^\mu{}_\nu = \delta^\mu{}_\nu - \frac{1}{\epsilon} n^\mu n_\nu, \tag{1.16}$$

because,

- $P^\mu{}_\nu n^\nu = P^\mu{}_\nu n_\mu = 0$, and
- $\forall V^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} \in \mathcal{T}^a_b(\mathcal{M}^m)$,
$$\begin{aligned} & V^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} P^{\lambda_1}{}_{\mu_1} P^{\lambda_2}{}_{\mu_2} \cdots P^{\nu_1}{}_{\omega_1} P^{\nu_2}{}_{\omega_2} \cdots \\ =& V^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} \delta^{\lambda_1}{}_{\mu_1} \delta^{\lambda_2}{}_{\mu_2} \cdots P^{\nu_1}{}_{\omega_1} \delta^{\nu_2}{}_{\omega_2} \cdots \\ =& V^{\lambda_1\lambda_2\cdots}{}_{\omega_1\omega_2\cdots} \in \mathcal{T}^a_b(\mathcal{M}^m). \end{aligned}$$

Now we express the basis in $\mathcal{T}_p^*(\mathcal{M}^{m+1})$ as $(dx^1, \cdots, dx^m, dx^{m+1})$, and that in $\mathcal{T}_p^*(\mathcal{M}^m)$ as $(dX^1, \cdots, dX^m)$. The line element ds^2 can be expressed as

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu && (1.17)\\ &= (g_{\mu\nu} P^\mu{}_\alpha P^\nu{}_\beta) dX^\alpha dX^\beta && (1.18)\\ &\equiv \gamma_{\alpha\beta} dX^\alpha dX^\beta, && (1.19) \end{aligned}$$

therefore $\gamma_{\mu\nu}$ is the metric on $\mathcal{M}^m$. In addition, $\gamma_{\mu\nu}$ is consistent with $P_{\mu\nu}$ because

$$\begin{aligned} P_{\mu\nu} &= g_{\mu\nu} P^\lambda{}_\nu && (1.20)\\ &= g_{\mu\nu} \left(\delta^\lambda{}_\nu - \frac{1}{\epsilon} n^\lambda n_\nu \right) && (1.21)\\ &= g_{\mu\nu} - \frac{1}{\epsilon} n_\mu n_\nu, && (1.22)\\ \gamma_{\mu\nu} &= g_{\alpha\beta} P^\alpha{}_\mu P^\beta{}_\nu && (1.23)\\ &= \left(P_{\alpha\beta} + \frac{1}{\epsilon} n_\alpha n_\beta \right) P^\alpha{}_\mu P^\beta{}_\nu && (1.24)\\ &= P_{\mu\nu}. && (1.25) \end{aligned}$$

1.2.2 Lie Derivative

The *Lie derivative* associated with $v^\mu \in \mathcal{T}_p(\mathcal{M}^{m+1})$ for $T_{\mu_1\mu_2\cdots}{}^{\nu_1\nu_2\cdots} \in \mathcal{T}_b^a$ can be expressed as

$$\begin{aligned} \mathcal{L}_v(T_{\mu_1\mu_2\cdots}{}^{\nu_1\nu_2\cdots}) &= v^\lambda \nabla_\lambda (T_{\mu_1\mu_2\cdots}{}^{\nu_1\nu_2\cdots}) + T_{\lambda\mu_2\cdots}{}^{\nu_1\nu_2\cdots} (\nabla_{\mu_1} v^\lambda) + T_{\mu_1\lambda\cdots}{}^{\nu_1\nu_2\cdots} (\nabla_{\mu_2} v^\lambda)\\ &\quad - T_{\mu_1\mu_2\cdots}{}^{\lambda\nu_2\cdots} (\nabla_\lambda v^{\nu_1}) - T_{\mu_1\mu_2\cdots}{}^{\nu_1\lambda\cdots} (\nabla_\lambda v^{\nu_2}) \cdots, \end{aligned} \tag{1.26}$$

and $\forall S \in \mathcal{R}$, the Lie derivative is $\mathcal{L}_v(S) = v^\lambda (\nabla_v S)$.

$\forall v^\mu \in \mathcal{T}_p(\mathcal{M}^{m+1})$, $\mathcal{L}_v(\delta^\mu{}_\nu) = 0$ because

$$\begin{aligned} \mathcal{L}_v(\delta^\mu{}_\nu) &= v^\lambda \nabla_\lambda \delta^\mu{}_\nu - \delta^\lambda{}_\nu \nabla_\lambda v^\mu + \delta^\mu{}_\lambda \nabla_\nu v^\lambda && (1.27)\\ &= 0. && (1.28) \end{aligned}$$

Hence, we can get the relation of

$$\begin{aligned} 0 &= \mathcal{L}_v(\gamma^{\mu\lambda}\gamma_{\lambda\nu}) && (1.29)\\ &= \mathcal{L}_v(\gamma^{\mu\lambda})\gamma_{\lambda\nu} + \gamma^{\mu\lambda}\mathcal{L}_v(\gamma_{\lambda\nu}), && (1.30)\\ \therefore \quad \mathcal{L}_v(\gamma^{\mu\nu}) &= -\gamma^{\mu\lambda}\gamma^{\nu\omega}\mathcal{L}_v(\gamma_{\lambda\omega}). && (1.31) \end{aligned}$$

For the Lie derivative of $\forall T_{\mu_1\mu_2\cdots\mu_m} \in \mathcal{T}_a^0(\mathcal{M}^m)$ associated with n^μ on $\mathcal{T}_p(\mathcal{M}^m)$ is in $\mathcal{T}_a^0(\mathcal{M}^m)$ because

$$n^{\mu_i}\mathcal{L}_n(T_{\mu_1\mu_2\cdots\mu_m}) = n^{\mu_i}\left\{ n^\lambda(\nabla_\lambda T_{\mu_1\mu_2\cdots\mu_m}) + T_{\lambda\mu_2\cdots\mu_m}(\nabla_{\mu_1} n^\lambda) + \cdots + T_{\mu_1\mu_2\cdots\lambda}(\nabla_{\mu_m} n^\lambda) \right\} \tag{1.32}$$

$$= n^{\mu_i} n^\lambda (\nabla_\lambda T_{\mu_1\mu_2\cdots\mu_m}) + n^{\mu_i} T_{\mu_1\mu_2\cdots \underbrace{\lambda}_{i} \cdots\mu_m} (\nabla_{\mu_i} n^\lambda) \tag{1.33}$$

$$= -(\nabla_\lambda n^{\mu_i}) n^\lambda T_{\mu_1\mu_2\cdots\mu_m} + n^{\mu_i} T_{\mu_1\mu_2\cdots \underbrace{\lambda}_{i} \cdots\mu_m} (\nabla_{\mu_i} n^\lambda) \tag{1.34}$$

$$= 0. \tag{1.35}$$

$\forall N \in \mathcal{R}$, $\forall T_{\mu\nu} \in \mathcal{T}_2^0(\mathcal{M}^m)$ and normal vector n^μ on $\mathcal{T}_p(\mathcal{M}^m)$, the Lie derivative of $T_{\mu\nu}$ associated with Nn^μ is

$$\begin{aligned}
\mathcal{L}_{Nn}(T_{\mu\nu}) &= Nn^\lambda \nabla_\lambda(T_{\mu\nu}) + T_{\lambda\nu}(\nabla_\mu N n^\lambda) + T_{\mu\lambda}(\nabla_\nu N n^\lambda) && (1.36)\\
&= Nn^\lambda \nabla_\lambda(T_{\mu\nu}) + T_{\lambda\nu} N(\nabla_\mu n^\lambda) + T_{\mu\lambda} N(\nabla_\nu n^\lambda) \quad (\because n^\lambda T_{\lambda\mu} = 0) && (1.37)\\
&= N\left\{ n^\lambda \nabla_\lambda(T_{\mu\nu}) + T_{\lambda\nu}(\nabla_\mu n^\lambda) + T_{\mu\lambda}(\nabla_\nu n^\lambda) \right\} && (1.38)\\
&= N\mathcal{L}_n(T_{\mu\nu}). && (1.39)
\end{aligned}$$

1.2.3 Extrinsic Curvature

The *extrinsic curvature* of $\mathcal{M}^m$ is defined as

$$K_{\mu\nu} \equiv -\frac{1}{2}\mathcal{L}_n(\gamma_{\mu\nu}), \tag{1.40}$$

where $\gamma_{\mu\nu}$ is the metric on the $\mathcal{M}^m$ and n^λ is the unit normal on $\mathcal{T}_p(\mathcal{M}^m)$. The right-hand-side of (1.40) is

$$\begin{aligned}
-\frac{1}{2}\mathcal{L}_n(\gamma_{\mu\nu}) &= -\frac{1}{2}n^\lambda(\nabla_\lambda\gamma_{\mu\nu}) - \frac{1}{2}\gamma_{\mu\lambda}(\nabla_\nu n^\lambda) - \frac{1}{2}\gamma_{\nu\lambda}(\nabla_\mu n^\lambda) && (1.41)\\
&= \frac{1}{2\epsilon}n^\lambda \nabla_\lambda(n_\mu n_\nu) - \frac{1}{2}(\nabla_\nu n_\mu) - \frac{1}{2}(\nabla_\mu n_\nu) && (1.42)\\
&= \frac{1}{2\epsilon}n^\lambda n_\nu(\nabla_\lambda n_\mu) + \frac{1}{2\epsilon}n^\lambda n_\mu(\nabla_\lambda n_\nu) - \frac{1}{2}(\nabla_\nu n_\mu) - \frac{1}{2}(\nabla_\mu n_\nu) && (1.43)\\
&= -\frac{1}{2}(P^\lambda{}_\nu - \delta^\lambda{}_\nu)(\nabla_\lambda n_\mu) - \frac{1}{2}(P^\lambda{}_\mu - \delta^\lambda{}_\mu)(\nabla_\lambda n_\nu) - \frac{1}{2}(\nabla_\nu n_\mu) - \frac{1}{2}(\nabla_\mu n_\nu) && (1.44)\\
&= -\frac{1}{2}P^\lambda{}_\nu(\nabla_\lambda n_\mu) - \frac{1}{2}P^\lambda{}_\mu(\nabla_\lambda n_\nu), && (1.45)
\end{aligned}$$

then

$$\begin{aligned}
P^\lambda{}_\mu(\nabla_\lambda n_\nu) &= P^\lambda{}_\mu P^\omega{}_\nu(\nabla_\lambda n_\omega) && (1.46)\\
&= P^\lambda{}_\mu P^\omega{}_\nu \nabla_\lambda(\epsilon N \nabla_\omega \xi) && (1.47)\\
&= P^\lambda{}_\mu P^\omega{}_\nu \left\{ \epsilon(\nabla_\lambda N)(\nabla_\omega \xi) + \epsilon N(\nabla_\lambda \nabla_\omega \xi) \right\} && (1.48)\\
&= P^\lambda{}_\mu P^\omega{}_\nu \left\{ \frac{1}{N}(\nabla_\lambda N) n_\omega + \epsilon N(\nabla_\lambda \nabla_\omega \xi) \right\} && (1.49)\\
&= P^\lambda{}_\mu P^\omega{}_\nu \epsilon N(\nabla_\omega \nabla_\lambda \xi), && (1.50)
\end{aligned}$$

therefore $P^\lambda{}_{[\mu}(\nabla_{|\lambda|} n_{\nu]}) = 0$. The extrinsic curvature can be expressed as

$$K_{\mu\nu} = -P^\lambda{}_\mu(\nabla_\lambda n_\nu). \tag{1.51}$$

Chapter 2

Geometrical Decomposition of Einstein Equations

2.1 Riemann Tensor Decomposition

In this section, we introduce the component of the decomposition of the $m+1$ dimensional Riemann tensor onto $\mathcal{T}^a_b(\mathcal{M}^m)$. The two of them are known as the *Gauss-Codazzi equation* and the *Codazzi-Mainardi equation*. Since these calculations are complicated, we write them in details at Appendix A.

We split ${}^{(m+1)}R_{\mu\alpha\nu\beta}$ to the components onto $\mathcal{T}^a_b(\mathcal{M}^m)$. With $P^\mu{}_\nu$ and n^λ, ${}^{(m+1)}R_{\mu\alpha\nu\beta}$ can be decomposed to the three parts:

- $P^\mu{}_\alpha P^\lambda{}_\rho P^\nu{}_\beta P^\omega{}_\gamma \, {}^{(m+1)}R_{\mu\lambda\nu\omega}$,
- $P^\lambda{}_\rho P^\nu{}_\beta P^\omega{}_\gamma n^\mu \, {}^{(m+1)}R_{\mu\lambda\nu\omega}$, and
- $P^\lambda{}_\alpha P^\omega{}_\beta n^\nu n^\mu {}^{(m+1)}R_{\mu\lambda\nu\omega}$.

The other parts such as $P^\mu{}_\alpha n^\lambda n^\nu n^\omega \, {}^{(m+1)}R_{\mu\lambda\nu\omega}$ are identically zero because of the characters of the Riemann tensor.

2.1.1 Gauss-Codazzi Equation

The projection of ${}^{(m+1)}R_{\mu\lambda\nu\omega}$ with four $P^\mu{}_\nu$s to the components on $\mathcal{M}^m$ is known as the *Gauss-Codazzi equation*:

The relation between ${}^{(m+1)}R_{\mu\nu\lambda\omega}$ and ${}^{(m)}R_{\mu\nu\lambda\omega}$ is given by

$$P^\mu{}_\alpha P^\lambda{}_\rho P^\nu{}_\beta P^\omega{}_\gamma \, {}^{(m+1)}R_{\mu\lambda\nu\omega} = {}^{(m)}R_{\alpha\rho\beta\gamma} - \frac{1}{\epsilon}(K_{\beta\alpha}K_{\gamma\rho} - K_{\gamma\alpha}K_{\beta\rho}). \tag{2.1}$$

where ϵ is the direction of n^μ.

2.1.2 Codazzi-Mainardi Equation

The projection of ${}^{(m+1)}R_{\mu\lambda\nu\omega}$ with three $P^\mu{}_\nu$s and one n^μ is known as the *Codazzi-Mainardi equation*:

The projection of ${}^{(m+1)}R_{\mu\nu\lambda\omega}$ with three $P^{\mu}{}_{\nu}$s and one n^{μ} to the components onto $\mathcal{T}^a_b(\mathcal{M}^m)$ is given by

$$P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}P^{\lambda}{}_{\omega}n^{\rho}\,{}^{(m+1)}R_{\rho\mu\nu\lambda} = -D_{\omega}K_{\beta\alpha} + D_{\beta}K_{\omega\alpha}. \tag{2.2}$$

2.1.3 The Last Part of the Decompositions

The projection with two $P^{\mu}{}_{\nu}$s and two n^{μ}s is below:

The projection of ${}^{(m+1)}R_{\mu\nu\lambda\omega}$ with two $P^{\mu}{}_{\nu}$s and two n^{μ}s to the components onto $\mathcal{T}^a_b(\mathcal{M}^m)$ is given by

$$P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}n^{\omega}\,{}^{(m+1)}R_{\omega\mu\lambda\nu} = K_{\beta}{}^{\lambda}K_{\lambda\alpha} - \frac{\epsilon}{N}D_{\beta}D_{\alpha}N + \frac{1}{N}\mathcal{L}_{Nn}(K_{\alpha\beta}). \tag{2.3}$$

2.2 Ricci Tensor Decomposition

Next, we split ${}^{(m+1)}R_{\mu\nu}$ to the components onto $\mathcal{T}^a_b(\mathcal{M}^m)$. In general, the second-order tensor $V_{\mu\nu} \in \mathcal{T}^0_2(\mathcal{M}^{m+1})$ can be decomposed to the components $u \in \mathcal{T}^0_0(\mathcal{M}^m), v_{\mu} \in \mathcal{T}^0_1(\mathcal{M}^m), w_{\mu\nu} \in \mathcal{T}^0_2(\mathcal{M}^m)$ such that

$$V_{\mu\nu} = un_{\mu}n_{\nu} + 2n_{(\mu}v_{\mu)} + w_{\mu\nu}. \tag{2.4}$$

Then, ${}^{(m+1)}R_{\mu\nu}$ is decoupled to the components which are

- parallel to $n^{\mu}n^{\nu}$,
- parallel to $P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}$, and
- parallel to $n^{\mu}P^{\nu}{}_{\alpha}$.

2.2.1 The Component paralleled with Normal Vectors

The component of the projection of (2.3) with $P^{\alpha\beta}$ is

$$n^{\mu}n^{\nu}\,{}^{(m+1)}R_{\mu\nu} = K_{\mu\nu}K^{\mu\nu} - \frac{\epsilon}{N}D^{\mu}D_{\mu}N + \frac{1}{N}\gamma^{\mu\nu}\mathcal{L}_{Nn}(K_{\mu\nu}). \tag{2.5}$$

Now we calculate the last term of the right-hand-side of the above equation:

$$\begin{aligned}
\frac{1}{N}\gamma^{\mu\nu}\mathcal{L}_{Nn}(K_{\mu\nu}) &= \gamma^{\mu\nu}\mathcal{L}_{n}(K_{\mu\nu}) && (2.6)\\
&= \mathcal{L}_{n}(K) - \mathcal{L}_{n}(\gamma^{\mu\nu})K_{\mu\nu} && (2.7)\\
&= \mathcal{L}_{n}(K) + \mathcal{L}_{n}(\gamma_{\mu\nu})K^{\mu\nu} && (2.8)\\
&= \frac{1}{N}\mathcal{L}_{Nn}(K) - 2K_{\mu\nu}K^{\mu\nu}. && (2.9)
\end{aligned}$$

Therefore,

The component of ${}^{(m+1)}R_{\mu\nu}$ paralleled with $n^{\mu}n^{\nu}$ is

$$n^{\mu}n^{\nu}\,{}^{(m+1)}R_{\mu\nu} = -K_{\mu\nu}K^{\mu\nu} - \frac{\epsilon}{N}D^{\mu}D_{\mu}N + \frac{1}{N}\mathcal{L}_{Nn}(K). \tag{2.10}$$

2.2.2 The Component paralleled with Projection Operators

We project (2.1) with $P^{\rho\gamma}$;

$$P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\ {}^{(m+1)}R_{\mu\nu} - \frac{1}{\epsilon}P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}n^{\omega}\ {}^{(m+1)}R_{\mu\lambda\nu\omega} = {}^{(m)}R_{\alpha\beta} - \frac{1}{\epsilon}(K_{\beta\alpha}K - K_{\gamma\alpha}K_{\beta}{}^{\gamma}). \quad (2.11)$$

With (2.3), (2.11) is

$$\begin{aligned} P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\ {}^{(m+1)}R_{\mu\nu} &= \frac{1}{\epsilon}K_{\beta}{}^{\lambda}K_{\lambda\alpha} - \frac{1}{N}D_{\beta}D_{\alpha}N + \frac{1}{\epsilon N}\mathcal{L}_{Nn}(K_{\alpha\beta}) + {}^{(m)}R_{\alpha\beta} \\ &\quad - \frac{1}{\epsilon}(K_{\beta\alpha}K - K_{\gamma\alpha}K_{\beta}{}^{\gamma}) \quad (2.12) \\ &= {}^{(m)}R_{\alpha\beta} - \frac{1}{\epsilon}(K_{\beta\alpha}K - 2K_{\gamma\alpha}K_{\beta}{}^{\gamma}) - \frac{1}{N}D_{\beta}D_{\alpha}N + \frac{1}{\epsilon N}\mathcal{L}_{Nn}(K_{\alpha\beta}). \quad (2.13) \end{aligned}$$

The component of ${}^{(m+1)}R_{\mu\nu}$ paralleled with $P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}$ is

$$P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\ {}^{(m+1)}R_{\mu\nu} = {}^{(m)}R_{\alpha\beta} - \frac{1}{\epsilon}KK_{\alpha\beta} + \frac{2}{\epsilon}K_{\lambda\beta}K_{\alpha}{}^{\lambda} - \frac{1}{N}D_{\beta}D_{\alpha}N + \frac{1}{N\epsilon}\mathcal{L}_{Nn}(K_{\alpha\beta}). \quad (2.14)$$

2.2.3 The Last Component of the Decomposition of Ricci Tensor

We project (2.2) with $P^{\alpha\omega}$,

$$P^{\nu}{}_{\beta}n^{\rho}\ {}^{(m+1)}R_{\rho\nu} = -D_{\omega}K_{\beta}{}^{\omega} + D_{\beta}K. \quad (2.15)$$

The component of ${}^{(m+1)}R_{\mu\nu}$ paralleled with $P^{\mu}{}_{\alpha}n^{\nu}$ is

$$P^{\mu}{}_{\alpha}n^{\nu}\ {}^{(m+1)}R_{\mu\nu} = -D_{\mu}K_{\alpha}{}^{\mu} + D_{\alpha}K. \quad (2.16)$$

2.3 Scalar Curvature

The projection of the left-hand-side of (2.13) with $P^{\alpha\beta}$ is

$$\begin{aligned} P^{\alpha\beta}\left(P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\ {}^{(m+1)}R_{\mu\nu}\right) &= P^{\mu\nu}\ {}^{(m+1)}R_{\mu\nu} \quad (2.17) \\ &= {}^{(m+1)}R - \frac{1}{\epsilon}n^{\mu}n^{\nu}\ {}^{(m+1)}R_{\mu\nu} \quad (2.18) \\ &= {}^{(m+1)}R + \frac{1}{\epsilon}K_{\mu\nu}K^{\mu\nu} + \frac{1}{N}D^{\mu}D_{\mu}N - \frac{1}{\epsilon N}\mathcal{L}_{Nn}(K). \quad (2.19) \end{aligned}$$

and the right-hand-side is

$$\begin{aligned} & P^{\alpha\beta}\left\{{}^{(m)}R_{\alpha\beta} - \frac{1}{\epsilon}(K_{\beta\alpha}K - 2K_{\gamma\alpha}K_{\beta}{}^{\gamma}) - \frac{1}{N}D_{\beta}D_{\alpha}N + \frac{1}{\epsilon N}\mathcal{L}_{Nn}(K_{\alpha\beta})\right\} \quad (2.20) \\ =& {}^{(m)}R - \frac{1}{\epsilon}(K^2 - 2K_{\mu\nu}K^{\mu\nu}) - \frac{1}{N}D^{\mu}D_{\mu}N + \frac{1}{\epsilon N}\gamma^{\alpha\beta}\mathcal{L}_{Nn}(K_{\alpha\beta}) \quad (2.21) \\ =& {}^{(m)}R - \frac{1}{\epsilon}K^2 - \frac{1}{N}D^{\mu}D_{\mu}N + \frac{1}{\epsilon N}\mathcal{L}_{Nn}(K), \quad (2.22) \end{aligned}$$

therefore, the relation between ${}^{(m+1)}R$ and ${}^{(m)}R$ is

$$ {}^{(m+1)}R = {}^{(m)}R - \frac{1}{\epsilon}K^2 - \frac{1}{\epsilon}K_{\mu\nu}K^{\mu\nu} - \frac{2}{N}D^\mu D_\mu N + \frac{2}{\epsilon N}\mathcal{L}_{Nn}(K). \tag{2.23} $$

The relation between ${}^{(m+1)}R$ and ${}^{(m)}R$ is

$$ {}^{(m+1)}R = {}^{(m)}R - \frac{1}{\epsilon}K^2 - \frac{1}{\epsilon}K_{\mu\nu}K^{\mu\nu} - \frac{2}{N}D^\mu D_\mu N + \frac{2}{\epsilon N}\mathcal{L}_{Nn}(K). \tag{2.24} $$

2.4 Einstein Tensor Decomposition

The $m+1$ dimensional Einstein tensor, ${}^{(m+1)}G_{\mu\nu}$, is defined as

$$ {}^{(m+1)}G_{\mu\nu} \equiv {}^{(m+1)}R_{\mu\nu} - \frac{1}{2}{}^{(m+1)}Rg_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{2.25} $$

The Einstein tensor can be decomposed to the three parts which parallel to $n^\mu n^\nu$, $P^\mu{}_\alpha P^\nu{}_\beta$, and $P^\mu{}_\alpha n^\nu$.

2.4.1 The Component paralleled with Normal Vectors

The component of ${}^{(m+1)}G_{\mu\nu}$ paralleled with $n^\mu n^\nu$ is

$$ \begin{aligned} n^\mu n^{\nu\,(m+1)}G_{\mu\nu} &= n^\mu n^{\nu\,(m+1)}R_{\mu\nu} - \frac{1}{2}\epsilon\ {}^{(m+1)}R + \Lambda\epsilon & (2.26)\\ &= -K_{\mu\nu}K^{\mu\nu} - \frac{\epsilon}{N}D^\mu D_\mu N + \frac{1}{N}\mathcal{L}_{Nn}(K) \\ &\quad - \frac{1}{2}\epsilon\left\{{}^{(m)}R - \frac{1}{\epsilon}K^2 - \frac{1}{\epsilon}K_{\mu\nu}K^{\mu\nu} - \frac{2}{N}D^\mu D_\mu N + \frac{2}{\epsilon N}\mathcal{L}_{Nn}(K)\right\} + \Lambda\epsilon & (2.27)\\ &= \frac{1}{2}\left(-\epsilon\ {}^{(m)}R + K^2 - K_{\mu\nu}K^{\mu\nu}\right) + \Lambda\epsilon. & (2.28) \end{aligned} $$

The component of ${}^{(m+1)}G_{\mu\nu}$ perpendicular to $T(\mathcal{M}^m)$ is

$$ n^\mu n^{\nu\,(m+1)}G_{\mu\nu} = \frac{1}{2}\left(-\epsilon\ {}^{(m)}R + K^2 - K_{\mu\nu}K^{\mu\nu}\right) + \Lambda\epsilon. \tag{2.29} $$

2.4.2 The Component paralleled with Projection Operators

The component of ${}^{(m+1)}G_{\mu\nu}$ paralleled with $P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}$ is

$$P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\,{}^{(m+1)}G_{\mu\nu} = P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\left({}^{(m+1)}R_{\mu\nu} - \frac{1}{2}\,{}^{(m+1)}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right) \tag{2.30}$$

$$= P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\,{}^{(m+1)}R_{\mu\nu} - \frac{1}{2}\,{}^{(m+1)}R\gamma_{\mu\nu} + \Lambda\gamma_{\mu\nu} \tag{2.31}$$

$$\begin{aligned} &= {}^{(m)}R_{\alpha\beta} - \frac{1}{\epsilon}KK_{\alpha\beta} + \frac{2}{\epsilon}K_{\lambda\beta}K^{\lambda}{}_{\alpha} - \frac{1}{N}D_{\beta}D_{\alpha}N + \frac{1}{N\epsilon}\mathcal{L}_{Nn}(K_{\alpha\beta}) \\ &\quad - \frac{1}{2}\gamma_{\alpha\beta}\left\{{}^{(m)}R - \frac{1}{\epsilon}K^2 - \frac{1}{\epsilon}K_{\mu\nu}K^{\mu\nu} - \frac{2}{N}D^{\mu}D_{\mu}N + \frac{2}{\epsilon N}\mathcal{L}_{Nn}(K)\right\} \\ &\quad + \Lambda\gamma_{\alpha\beta} \end{aligned} \tag{2.32}$$

$$\begin{aligned} &= {}^{(m)}R_{\alpha\beta} - \frac{1}{\epsilon}KK_{\alpha\beta} + \frac{2}{\epsilon}K_{\lambda\beta}K^{\lambda}{}_{\alpha} - \frac{1}{N}D_{\beta}D_{\alpha}N + \frac{1}{N\epsilon}\mathcal{L}_{Nn}(K_{\alpha\beta}) \\ &\quad + \frac{1}{\epsilon}\gamma_{\alpha\beta}\left\{\frac{1}{2}\left(-\epsilon\,{}^{(m)}R + K^2 - K_{\mu\nu}K^{\mu\nu}\right) + \Lambda\epsilon\right\} \\ &\quad + \gamma_{\alpha\beta}\left(\frac{1}{N}D^{\mu}D_{\mu}N - \frac{1}{N\epsilon}\mathcal{L}_{Nn}(K) + \frac{1}{\epsilon}K_{\mu\nu}K^{\mu\nu}\right). \end{aligned} \tag{2.33}$$

The component of ${}^{(m+1)}G_{\mu\nu}$ paralleled with $P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}$ is

$$\begin{aligned} P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}{}^{(m+1)}G_{\mu\nu} &= {}^{(m)}R_{\alpha\beta} - \frac{1}{\epsilon}KK_{\alpha\beta} + \frac{2}{\epsilon}K_{\lambda\beta}K^{\lambda}{}_{\alpha} - \frac{1}{N}D_{\beta}D_{\alpha}N + \frac{1}{N\epsilon}\mathcal{L}_{Nn}(K_{\alpha\beta}) \\ &\quad + \frac{1}{\epsilon}\gamma_{\alpha\beta}\left\{\frac{1}{2}\left(-\epsilon\,{}^{(m)}R + K^2 - K_{\mu\nu}K^{\mu\nu}\right) + \Lambda\epsilon\right\} \\ &\quad + \gamma_{\alpha\beta}\left(\frac{1}{N}D^{\mu}D_{\mu}N - \frac{1}{N\epsilon}\mathcal{L}_{Nn}(K) + \frac{1}{\epsilon}K_{\mu\nu}K^{\mu\nu}\right). \end{aligned} \tag{2.34}$$

2.4.3 The Last Component of Einstein Tensor

The component of ${}^{(m+1)}G_{\mu\nu}$ paralleled with $P^{\mu}{}_{\alpha}n^{\nu}$ is

$$P^{\mu}{}_{\alpha}n^{\nu}\,{}^{(m+1)}G_{\mu\nu} = P^{\mu}{}_{\alpha}n^{\nu}\,{}^{(m+1)}R_{\mu\nu} \tag{2.35}$$

$$= -D_{\omega}K^{\omega}{}_{\alpha} + D_{\alpha}K. \tag{2.36}$$

The component of ${}^{(m+1)}G_{\mu\nu}$ paralleled with $P^{\mu}{}_{\alpha}n^{\nu}$ is

$$P^{\mu}{}_{\alpha}n^{\nu}\,{}^{(m+1)}G_{\mu\nu} = -D_{\mu}K^{\mu}{}_{\alpha} + D_{\alpha}K. \tag{2.37}$$

2.5 Energy Momentum Tensor Decomposition

Before decoupled the Einstein equations, we split the energy momentum tensor. The energy-momentum tensor $T_{\mu\nu} \in \mathcal{T}^0_2(\mathcal{M}^{m+1})$ can be expressed as

$$T_{\mu\nu} = \rho_H n_{\mu}n_{\nu} + 2J_{(\mu}n_{\nu)} + S_{\mu\nu}, \tag{2.38}$$

where $\rho_H \in \mathcal{T}^0_0(\mathcal{M}^m), J_\mu \in \mathcal{T}^0_1(\mathcal{M}^m), S_{\mu\nu} \in \mathcal{T}^0_2(\mathcal{M}^m)$.

The energy momentum tensor $T_{\mu\nu}$ is expressed with the components on $\mathcal{M}^m$ such as

$$\rho_H \equiv n^\mu n^\nu T_{\mu\nu}, \tag{2.39}$$

$$J_\alpha \equiv \frac{1}{\epsilon} P^\mu{}_\alpha n^\nu T_{\mu\nu}, \tag{2.40}$$

$$S_{\alpha\beta} \equiv P^\mu{}_\alpha P^\nu{}_\beta T_{\mu\nu}. \tag{2.41}$$

The trace part of the energy momentum tensor can be expressed with the components on $\mathcal{M}^m$,

$$T \equiv g^{\mu\nu} T_{\mu\nu} \tag{2.42}$$

$$= \left(P^{\mu\nu} + \frac{1}{\epsilon} n^\mu n^\nu \right) T_{\mu\nu} \tag{2.43}$$

$$= P^{\alpha\beta} P^\mu{}_\alpha P^\nu{}_\beta T_{\mu\nu} + \frac{1}{\epsilon} n^\mu n^\nu T_{\mu\nu} \tag{2.44}$$

$$= S + \frac{1}{\epsilon} \rho_H, \tag{2.45}$$

where $S \equiv \gamma^{\mu\nu} S_{\mu\nu}$.

2.6 Einstein Equations Decomposition

The $m+1$ dimensional Einstein equations are

$$^{(m+1)}G_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.46}$$

The relation between of the $^{(m+1)}R$ and $^{(m+1)}T$ is

$$\kappa T = \kappa g^{\mu\nu} T_{\mu\nu} \tag{2.47}$$

$$= g^{\mu\nu}\ {}^{(m+1)}G_{\mu\nu} \tag{2.48}$$

$$= g^{\mu\nu} \left({}^{(m+1)}R_{\mu\nu} - \frac{1}{2} {}^{(m+1)}R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) \tag{2.49}$$

$$= {}^{(m+1)}R - \frac{m+1}{2}\ {}^{(m+1)}R + (m+1)\Lambda \tag{2.50}$$

$$= -\frac{m-1}{2}\ {}^{(m+1)}R + (m+1)\Lambda, \tag{2.51}$$

$$\therefore \quad {}^{(m+1)}R = 2\left(1 + \frac{2}{m-1}\right)\Lambda - \frac{2\kappa}{m-1}\left(S + \frac{1}{\epsilon}\rho_H\right). \tag{2.52}$$

The $m+1$ dimensional Einstein equations can be expressed as

$$^{(m+1)}R_{\mu\nu} - \frac{2}{m-1}\Lambda g_{\mu\nu} = \kappa \left\{ T_{\mu\nu} - \frac{1}{m-1} g_{\mu\nu} \left(S + \frac{1}{\epsilon}\rho_H \right) \right\}. \tag{2.53}$$

Now we set the second-order (covariant) tensor $^{(m+1)}E_{\mu\nu} \in \mathcal{T}^0_2(\mathcal{M}^{m+1})$ such that

$$^{(m+1)}E_{\mu\nu} \equiv {}^{(m+1)}R_{\mu\nu} - \frac{2}{m-1}\Lambda g_{\mu\nu} - \kappa \left\{ T_{\mu\nu} - \frac{1}{m-1} g_{\mu\nu} \left(S + \frac{1}{\epsilon}\rho_H \right) \right\}, \tag{2.54}$$

and with ${}^{(m+1)}G_{\mu\nu}$, (2.54) can be expressed as

$$ {}^{(m+1)}E_{\mu\nu} = {}^{(m+1)}G_{\mu\nu} - \kappa T_{\mu\nu} - \frac{1}{m-1} g_{\mu\nu} g^{\lambda\omega} ({}^{(m+1)}G_{\lambda\omega} - \kappa T_{\lambda\omega}). \tag{2.55} $$

Since ${}^{(m+1)}E_{\mu\nu}$ is the second-order tensor, we split this tensor with $H \in \mathcal{T}_0^0(\mathcal{M}^m)$, $M_\mu \in \mathcal{T}_1^0(\mathcal{M}^m)$ and ${}^{(m)}E_{\mu\nu} \in \mathcal{T}_2^0(\mathcal{M}^m)$ such that

$$ {}^{(m+1)}E_{\mu\nu} = H n_\mu n_\nu + 2M_{(\mu} n_{\nu)} + {}^{(m)}E_{\mu\nu}. \tag{2.56} $$

2.6.1 The Component paralleled with Normal Vectors

The component of the Einstein equations paralleled with $n^\mu n^\nu$ is

$$ \begin{aligned} H &= n^\mu n^\nu \, {}^{(m+1)}E_{\mu\nu} && (2.57)\\ &= n^\mu n^\nu \left({}^{(m+1)}G_{\mu\nu} - \kappa T_{\mu\nu} \right) - \frac{1}{m-1} \epsilon g^{\lambda\omega} \left({}^{(m+1)}G_{\lambda\omega} - \kappa T_{\lambda\omega} \right) && (2.58)\\ &= \frac{m-2}{2(m-1)} \left(-\epsilon \, {}^{(m)}R + K^2 - K_{\mu\nu}K^{\mu\nu} + 2\Lambda\epsilon - 2\kappa\rho_H \right) \\ &\quad - \frac{1}{m-1} \epsilon P^{\lambda\omega} \left({}^{(m+1)}G_{\lambda\omega} - \kappa T_{\lambda\omega} \right). && (2.59) \end{aligned} $$

The component of the Einstein equations paralleled with normal vectors is

$$ \begin{aligned} H \equiv &\frac{m-2}{2(m-1)} \left(-\epsilon \, {}^{(m)}R + K^2 - K_{\mu\nu}K^{\mu\nu} + 2\Lambda\epsilon - 2\kappa\rho_H \right) \\ &- \frac{1}{m-1} \epsilon P^{\lambda\omega} \left({}^{(m+1)}G_{\lambda\omega} - \kappa T_{\lambda\omega} \right). \end{aligned} \tag{2.60} $$

2.6.2 The Component paralleled with Projection Operators

The component of the Einstein equations paralleled with ${P^\mu}_\alpha {P^\nu}_\beta$ is

$$ \begin{aligned} {P^\mu}_\alpha {P^\nu}_\beta \, {}^{(m+1)}E_{\mu\nu} &= {P^\mu}_\alpha {P^\nu}_\beta \Bigg[{}^{(m+1)}R_{\mu\nu} - \frac{2}{m-1}\Lambda g_{\mu\nu} \\ &\qquad - \kappa \left\{ T_{\mu\nu} - \frac{1}{m-1} g_{\mu\nu} \left(S + \frac{1}{\epsilon}\rho_H \right) \right\} \Bigg] && (2.61)\\ &= {}^{(m)}R_{\alpha\beta} - \frac{1}{\epsilon}(K_{\beta\alpha}K - 2K_{\gamma\alpha}K_\beta{}^\gamma) - \frac{1}{N} D_\beta D_\alpha N + \frac{1}{\epsilon N} \mathcal{L}_{Nn}(K_{\alpha\beta}) \\ &\quad - \frac{2}{m-1}\Lambda\gamma_{\alpha\beta} - \kappa \left\{ S_{\alpha\beta} - \frac{1}{m-1} \left(S + \frac{1}{\epsilon}\rho_H \right) \gamma_{\alpha\beta} \right\}. && (2.62) \end{aligned} $$

The component of the Einstein equations paralleled with ${P^\mu}_\alpha {P^\nu}_\beta$ is expressed as

$$ \begin{aligned} {}^{(m)}E_{\alpha\beta} = {}^{(m)}R_{\alpha\beta} &- \frac{1}{\epsilon}(K_{\beta\alpha}K - 2K_{\gamma\alpha}K_\beta{}^\gamma) - \frac{1}{N} D_\beta D_\alpha N + \frac{1}{\epsilon N} \mathcal{L}_{Nn}(K_{\alpha\beta}) \\ &- \frac{2}{m-1}\Lambda\gamma_{\alpha\beta} - \kappa \left\{ S_{\alpha\beta} - \frac{1}{m-1} \left(S + \frac{1}{\epsilon}\rho_H \right) \gamma_{\alpha\beta} \right\}. \end{aligned} \tag{2.63} $$

2.6.3 Last Part of the Decomposition

The component of the Einstein equations paralleled with $P^{\mu}{}_{\alpha}n^{\nu}$ is

$$M_\alpha = \frac{1}{\epsilon} P^{\mu}{}_{\alpha} n^{\nu} {}^{(m+1)}\mathcal{E}_{\mu\nu} \tag{2.64}$$

$$= \frac{1}{\epsilon} P^{\mu}{}_{\alpha} n^{\nu} \left({}^{(m+1)}G_{\mu\nu} - \kappa T_{\mu\nu} \right) \tag{2.65}$$

$$= \frac{1}{\epsilon} \left(-D_\mu K^{\mu}{}_{\alpha} + D_\alpha K \right) - \kappa J_\alpha. \tag{2.66}$$

The component of the Einstein equations paralleled with $P^{\mu}{}_{\alpha}n^{\nu}$ is

$$M_\alpha \equiv \frac{1}{\epsilon} \left(-D_\mu K^{\mu}{}_{\alpha} + D_\alpha K \right) - \kappa J_\alpha. \tag{2.67}$$

2.7 Standard ADM Formulation

In this section, we introduce the m dimensional standard ADM formulation. We adopt the normal vector as timelike, that is $\epsilon = -1$, and we set ξ as time line t. In addition, we express N as α.

2.7.1 Lie Derivative along with αn^{μ}

In general, αn^{λ} and ∂_t are not parallel. α mean the distance between $\mathcal{M}(t)$ and $\mathcal{M}(t+dt)$, it is called the *lapse function*. Hence, the difference between αn^{λ} and ∂_t is expressed as

$$\beta^{\mu} \equiv \partial_t - \alpha n^{\mu}, \tag{2.68}$$

where β^{μ} is called the *shift vector*. Since the Lie derivative operator associated with ∂_t is consistent with ∂_t, $\forall T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} \in \mathcal{T}^a_b$, the Lie derivative along with αn^{μ} can be expressed as

$$\mathcal{L}_{\alpha n}(T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots}) = \partial_t T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} - \mathcal{L}_{\beta}(T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots}). \tag{2.69}$$

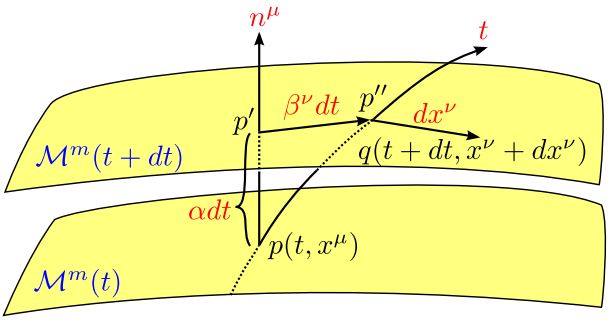

Figure 2.1: Decomposition of spacetime.

2.7.2 Metric

Let be $\mathcal{M}^m$ which is satisfied that the time line is constant. For $\forall p \in \mathcal{M}^m(t)$, the line element between p and $\forall q \in \mathcal{M}^m(t+dt)$ is

$$ds^2 = -(\alpha dt)^2 + \gamma_{\mu\nu}(\beta^\mu dt + dx^\mu)(\beta^\nu dt + dx^\nu). \tag{2.70}$$

(The concept of (2.70) is drawn as Fig. 2.1) Hence the metric $g_{\mu\nu} \in \mathcal{M}^{m+1}$ can be written as

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^\mu\beta_\mu & \beta_\mu \\ \beta_\nu & \gamma_{\mu\nu} \end{pmatrix}. \tag{2.71}$$

2.7.3 Constraint Equations

First, we introduce the constraint equations of the ADM formulation. For (2.60), the last term is corresponding to zero, we adopt the constraint equations in the ADM formulation as below;

The m dimensional Hamiltonian constraint equation of the ADM formulation is

$$\mathcal{H} \equiv {}^{(m)}R + K^2 - K_{\mu\nu}K^{\mu\nu} - 2\Lambda - 2\kappa\rho_H, \tag{2.72}$$

and the m dimensional momentum constraint equations of the ADM formulation are

$$\mathcal{M}_\mu \equiv D_\nu K^\nu{}_\mu - D_\mu K - \kappa J_\mu. \tag{2.73}$$

2.7.4 Dynamical Equations

If we set ξ as time-line, the equations of (1.40) and (2.63) denote the dynamics of the spacetime. Therefore these equations express dynamical equations in the standard ADM formulation.

The dynamical equations of the standard ADM formulation are

$$\partial_t\gamma_{\mu\nu} = -2\alpha K_{\mu\nu} + \mathcal{L}_\beta(\gamma_{\mu\nu}), \tag{2.74}$$

$$\begin{aligned}\partial_t K_{\mu\nu} = {} & \alpha\,{}^{(m)}R_{\mu\nu} + \alpha(K_{\mu\nu}K - 2K_{\lambda\mu}K_\nu{}^\lambda) - D_\mu D_\nu\alpha + \mathcal{L}_\beta(K_{\mu\nu}) \\ & - \frac{2\Lambda}{m-1}\alpha\gamma_{\mu\nu} - \kappa\alpha\left\{S_{\mu\nu} - \frac{1}{m-1}(S-\rho_H)\,\gamma_{\mu\nu}\right\}.\end{aligned} \tag{2.75}$$

Chapter 3

Baumgarte-Shapiro-Shibata-Nakamura Formulation

In current numerical simulations such as the binary neutron mergers and/or black hole mergers, the ADM formulation is not used, the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation is widely used. The BSSN formulation was suggested by Shibata and Nakamura [13]. After that, this formulation was re-formulated by Baumgarte and Shapiro, they showed that this formulation is better than the ADM formulation with some simulations [14]. In this chapter, we derive the conformal and traceless transformation of the ADM formulation. Next, the BSSN formulation is derived.

3.1 Connection

First, we calculate the relation between ${}^{(m)}\Gamma^{\lambda}{}_{\mu\nu}$ and ${}^{(m)}\bar{\Gamma}^{\lambda}{}_{\mu\nu}$. The conformal metric is defined by

$$\bar{\gamma}_{\mu\nu} \equiv \phi^{-2}\gamma_{\mu\nu}, \tag{3.1}$$

where ϕ is an arbitrary function. The contravariant expression of the conformal metric is

$$\bar{\gamma}^{\mu\nu} = \phi^{2}\gamma^{\mu\nu}, \tag{3.2}$$

and it is satisfied the condition that $\bar{\gamma}_{\mu\nu}\bar{\gamma}^{\nu\lambda} = \delta^{\lambda}{}_{\mu}$.

The connection in the conformal space is

$${}^{(m)}\bar{\Gamma}^{\omega}{}_{\mu\nu} \equiv \frac{1}{2}\bar{\gamma}^{\omega\lambda}(\partial_{\nu}\bar{\gamma}_{\lambda\mu} + \partial_{\mu}\bar{\gamma}_{\lambda\nu} - \partial_{\lambda}\bar{\gamma}_{\mu\nu}). \tag{3.3}$$

Then the relation between ${}^{(m)}\bar{\Gamma}^{\lambda}{}_{\mu\nu}$ and ${}^{(m)}\Gamma^{\lambda}{}_{\mu\nu}$ is

$$\begin{aligned}
{}^{(m)}\bar{\Gamma}^{\omega}{}_{\mu\nu} &= \frac{1}{2}\phi^2\gamma^{\omega\lambda}\Big\{-2\phi^{-3}(\partial_\nu\phi)\gamma_{\lambda\mu} + \phi^{-2}(\partial_\nu\gamma_{\lambda\mu}) - 2\phi^{-3}(\partial_\mu\phi)\gamma_{\lambda\nu} + \phi^{-2}(\partial_\mu\gamma_{\lambda\nu}) \\
&\qquad + 2\phi^{-3}(\partial_\lambda\phi)\gamma_{\mu\nu} - \phi^{-2}(\partial_\lambda\gamma_{\mu\nu})\Big\} \qquad (3.4)\\
&= -\phi^{-1}\Big\{(\partial_\nu\phi)\delta^{\omega}{}_{\mu} + (\partial_\mu\phi)\delta^{\omega}{}_{\nu} - (\partial_\lambda\phi)\gamma^{\omega\lambda}\gamma_{\mu\nu}\Big\} + {}^{(m)}\Gamma^{\omega}{}_{\mu\nu} \qquad (3.5)\\
&= -\Big\{\delta^{\omega}{}_{\mu}(\partial_\nu\log\phi) + \delta^{\omega}{}_{\nu}(\partial_\mu\log\phi) - \gamma^{\omega\lambda}\gamma_{\mu\nu}(\partial_\lambda\log\phi)\Big\} + {}^{(m)}\Gamma^{\omega}{}_{\mu\nu}. \qquad (3.6)
\end{aligned}$$

The relation between the connection, ${}^{(m)}\Gamma^{\lambda}{}_{\mu\nu}$, and the conformal connection, ${}^{(m)}\bar{\Gamma}^{\lambda}{}_{\mu\nu}$, is

$${}^{(m)}\Gamma^{\omega}{}_{\mu\nu} = {}^{(m)}\bar{\Gamma}^{\omega}{}_{\mu\nu} + \delta^{\omega}{}_{\mu}(\bar{D}_\nu\log\phi) + \delta^{\omega}{}_{\nu}(\bar{D}_\mu\log\phi) - \bar{\gamma}^{\omega\lambda}\bar{\gamma}_{\mu\nu}(\bar{D}_\lambda\log\phi), \qquad (3.7)$$

where $\bar{D}$ is the covariant derivative operator associated with $\bar{\gamma}_{\mu\nu}$.

3.2 Riemann Tensor

Now we calculate the relation between the conformal and standard Riemann tensor.

The relation between the Riemann tensor, ${}^{(m)}R^{\lambda}{}_{\mu\omega\nu}$, and the conformal Riemann tensor, ${}^{(m)}\bar{R}^{\lambda}{}_{\mu\omega\nu}$, can be expressed as

$$\begin{aligned}
{}^{(m)}R^{\lambda}{}_{\mu\omega\nu} &= {}^{(m)}\bar{R}^{\lambda}{}_{\mu\omega\nu} + \delta^{\lambda}{}_{\nu}(\bar{D}_\omega\bar{D}_\mu\log\phi) - \bar{\gamma}_{\mu\nu}(\bar{D}_\omega\bar{D}^{\lambda}\log\phi) - \delta^{\lambda}{}_{\omega}(\bar{D}_\nu\bar{D}_\mu\log\phi) \\
&\quad + \bar{\gamma}_{\mu\omega}(\bar{D}_\nu\bar{D}^{\lambda}\log\phi) + \delta^{\lambda}{}_{\omega}(\bar{D}_\mu\log\phi)(\bar{D}_\nu\log\phi) \\
&\quad - \delta^{\lambda}{}_{\omega}\bar{\gamma}_{\mu\nu}(\bar{D}^{\rho}\log\phi)(\bar{D}_\rho\log\phi) - \bar{\gamma}_{\omega\mu}(\bar{D}_\nu\log\phi)(\bar{D}^{\lambda}\log\phi) \\
&\quad - \delta^{\lambda}{}_{\nu}(\bar{D}_\omega\log\phi)(\bar{D}_\mu\log\phi) + \delta^{\lambda}{}_{\nu}\bar{\gamma}_{\mu\omega}(\bar{D}^{\zeta}\log\phi)(\bar{D}_\zeta\log\phi) \\
&\quad + \bar{\gamma}_{\nu\mu}(\bar{D}^{\lambda}\log\phi)(\bar{D}_\omega\log\phi). \qquad (3.8)
\end{aligned}$$

The derivation of the above equation is expressed in Appendix B.3.

3.3 Ricci Tensor

Next, we compute the relation of the conformal and standard Ricci tensor. For (3.8), we set $\omega = \lambda$, then

$$\begin{aligned}
{}^{(m)}R_{\mu\nu} &= {}^{(m)}\bar{R}_{\mu\nu} + \bar{D}_\nu\bar{D}_\mu\log\phi - \bar{\gamma}_{\mu\nu}(\bar{D}_\lambda\bar{D}^{\lambda}\log\phi) - m(\bar{D}_\nu\bar{D}_\mu\log\phi) \\
&\quad + \bar{D}_\nu\bar{D}_\mu\log\phi + m(\bar{D}_\mu\log\phi)(\bar{D}_\nu\log\phi) - m\bar{\gamma}_{\mu\nu}(\bar{D}^{\rho}\log\phi)(\bar{D}_\rho\log\phi) \\
&\quad - (\bar{D}_\nu\log\phi)(\bar{D}_\mu\log\phi) - (\bar{D}_\nu\log\phi)(\bar{D}_\mu\log\phi) \\
&\quad + \bar{\gamma}_{\mu\nu}(\bar{D}^{\zeta}\log\phi)(\bar{D}_\zeta\log\phi) + \bar{\gamma}_{\nu\mu}(\bar{D}^{\lambda}\log\phi)(\bar{D}_\lambda\log\phi) \qquad (3.9)\\
&= {}^{(m)}\bar{R}_{\mu\nu} + (2-m)\bar{D}_\nu\bar{D}_\mu\log\phi - \bar{\gamma}_{\mu\nu}(\bar{D}_\lambda\bar{D}^{\lambda}\log\phi) \\
&\quad + (m-2)(\bar{D}_\mu\log\phi)(\bar{D}_\nu\log\phi) + (2-m)\bar{\gamma}_{\mu\nu}(\bar{D}^{\rho}\log\phi)(\bar{D}_\rho\log\phi). \qquad (3.10)
\end{aligned}$$

The relation between the Ricci tensor, ${}^{(m)}R_{\mu\nu} \equiv {}^{(m)}R^{\lambda}{}_{\mu\lambda\nu}$, and the conformal Ricci tensor, ${}^{(m)}\bar{R}_{\mu\nu} \equiv {}^{(m)}\bar{R}^{\lambda}{}_{\mu\lambda\nu}$, can be expressed as

$$ {}^{(m)}R_{\mu\nu} = {}^{(m)}\bar{R}_{\mu\nu} + {}^{(m)}R^{\phi}_{\mu\nu}, \tag{3.11} $$

$$ \begin{aligned} \text{where} \quad {}^{(m)}R^{\phi}_{\mu\nu} \equiv{}& (2-m)\bar{D}_\nu\bar{D}_\mu \log\phi - \bar{\gamma}_{\mu\nu}(\bar{D}_\lambda\bar{D}^\lambda \log\phi) \\ &+ (m-2)(\bar{D}_\mu \log\phi)(\bar{D}_\nu \log\phi) + (2-m)\bar{\gamma}_{\mu\nu}(\bar{D}^\rho \log\phi)(\bar{D}_\rho \log\phi). \end{aligned} \tag{3.12} $$

3.4 Scalar Curvature

Next, we compute the relation of the conformal and standard scalar curvature. For (3.12), the trace part of the ${}^{(m)}R_{\mu\nu}$ is

$$ {}^{(m)}R = \gamma^{\mu\nu}({}^{(m)}\bar{R}_{\mu\nu} + {}^{(m)}R^{\phi}_{\mu\nu}) \tag{3.13} $$

$$ \begin{aligned} &= \gamma^{\mu\nu}\Big\{ {}^{(m)}\bar{R}_{\mu\nu} + (2-m)\bar{D}_\nu\bar{D}_\mu \log\phi - \bar{\gamma}_{\mu\nu}(\bar{D}_\lambda\bar{D}^\lambda \log\phi) \\ &\quad + (m-2)(\bar{D}_\mu \log\phi)(\bar{D}_\nu \log\phi) + (2-m)\bar{\gamma}_{\mu\nu}(\bar{D}^\rho \log\phi)(\bar{D}_\rho \log\phi)\Big\} \end{aligned} \tag{3.14} $$

$$ = \phi^{-2}\Big\{ {}^{(m)}\bar{R} - 2(m-1)(\bar{D}_\lambda\bar{D}^\lambda \log\phi) - (m-2)(m-1)(\bar{D}^\rho \log\phi)(\bar{D}_\rho \log\phi)\Big\}. \tag{3.15} $$

The relation between the scalar curvature, ${}^{(m)}R \equiv \gamma^{\mu\nu}\,{}^{(m)}R_{\mu\nu}$, and conformal scalar curvature, ${}^{(m)}\bar{R} \equiv \bar{\gamma}^{\mu\nu}\,{}^{(m)}\bar{R}_{\mu\nu}$, can be expressed as

$$ {}^{(m)}R = \phi^{-2}({}^{(m)}\bar{R} + {}^{(m)}R^{\phi}), \tag{3.16} $$

where ${}^{(m)}R^{\phi} \equiv \bar{\gamma}^{\mu\nu}R^{\phi}_{\mu\nu}$.

3.5 Trace-Free Part of Conformal Value

We first define the traceless decomposition. $\forall V_{\mu\nu} \in \mathcal{T}^0_2(\mathcal{M}^m)$, the trace-free part of $V_{\mu\nu}$ is defined as

$$ V^{\mathrm{TF}}_{\mu\nu} \equiv V_{\mu\nu} - \frac{1}{m}V\gamma_{\mu\nu}, \tag{3.17} $$

where $V \equiv \gamma^{\mu\nu}V_{\mu\nu}$ and $V^{\mathrm{TF}}_{\mu\nu} \in \mathcal{T}^0_2(\mathcal{M}^m)$ because of $V^{\mathrm{TF}}_{\mu\nu}n^\mu = 0$.

The trace-free part of the conformal value $\bar{V}_{\mu\nu} \equiv \phi^{-2}V_{\mu\nu}$ is defined as

$$ \bar{V}^{\mathrm{TF}}_{\mu\nu} \equiv \bar{V}_{\mu\nu} - \frac{1}{m}\bar{V}\bar{\gamma}_{\mu\nu}, \tag{3.18} $$

where $\bar{V} \equiv \bar{V}_{\mu\nu}\bar{\gamma}^{\mu\nu}$. The relation between $V^{\mathrm{TF}}_{\mu\nu}$ and $\bar{V}^{\mathrm{TF}}_{\mu\nu}$ is

$$ \bar{V}^{\mathrm{TF}}_{\mu\nu} = \bar{V}_{\mu\nu} - \frac{1}{m}\bar{V}\bar{\gamma}_{\mu\nu} \tag{3.19} $$

$$ = \phi^{-2}V_{\mu\nu} - \frac{1}{m}(\phi^{-2}V_{\omega\lambda})(\phi^2\gamma^{\omega\lambda})\phi^{-2}\gamma_{\mu\nu} \tag{3.20} $$

$$ = \phi^{-2}V^{\mathrm{TF}}_{\mu\nu}. \tag{3.21} $$

We now calculate the relation between ${}^{(m)}R^{\rm TF}_{\mu\nu}$ and ${}^{(m)}\bar{R}^{\rm TF}_{\mu\nu}$;

$$
\begin{aligned}
{}^{(m)}R^{\rm TF}_{\mu\nu} &= {}^{(m)}R_{\mu\nu} - \frac{1}{m}\,{}^{(m)}R\gamma_{\mu\nu} && (3.22)\\
&= ({}^{(m)}\bar{R}_{\mu\nu} + {}^{(m)}R^{\phi}_{\mu\nu}) - \frac{1}{m}\phi^{-2}({}^{(m)}\bar{R} + {}^{(m)}R^{\phi})\gamma_{\mu\nu} && (3.23)\\
&= ({}^{(m)}\bar{R}_{\mu\nu} + {}^{(m)}R^{\phi}_{\mu\nu})^{\rm TF}. && (3.24)
\end{aligned}
$$

Next, we calculate the relation between $(D_\mu D_\nu\alpha)^{\rm TF}$ and $(\bar{D}_\mu\bar{D}_\nu\alpha)^{\rm TF}$. First, we compute $D_\mu D_\nu\alpha$;

$$
\begin{aligned}
D_\mu D_\nu &= \bar{D}_\mu D_\nu\alpha - (\bar{D}_\nu\log\phi)(D_\mu\alpha) - (\bar{D}_\mu\log\phi)(D_\nu\alpha) + \bar{\gamma}_{\mu\nu}(\bar{D}^\lambda\log\phi)(D_\lambda\alpha) && (3.25)\\
&= \bar{D}_\mu\bar{D}_\nu\alpha - (\bar{D}_\nu\log\phi)(\bar{D}_\mu\alpha) - (\bar{D}_\mu\log\phi)(\bar{D}_\nu\alpha) + \bar{\gamma}_{\mu\nu}(\bar{D}^\lambda\log\phi)(\bar{D}_\lambda\alpha), && (3.26)
\end{aligned}
$$

then,

$$
\begin{aligned}
D_\mu D^\mu\alpha &= \gamma^{\mu\nu}D_\mu D_\nu\alpha && (3.27)\\
&= \phi^{-2}\bar{\gamma}^{\mu\nu}\Big\{\bar{D}_\mu\bar{D}_\nu\alpha - (\bar{D}_\nu\log\phi)(\bar{D}_\mu\alpha) - (\bar{D}_\mu\log\phi)(\bar{D}_\nu\alpha) + \bar{\gamma}_{\mu\nu}(\bar{D}^\lambda\log\phi)(\bar{D}_\lambda\alpha)\Big\} && (3.28)\\
&= \phi^{-2}\Big\{\bar{D}_\mu\bar{D}^\mu\alpha + (m-2)(\bar{D}^\mu\log\phi)(\bar{D}_\mu\alpha)\Big\}, && (3.29)
\end{aligned}
$$

therefore

$$
\begin{aligned}
(D_\mu D_\nu\alpha)^{\rm TF} &= D_\mu D_\nu\alpha - \frac{1}{m}(D_\lambda D^\lambda\alpha)\gamma_{\mu\nu} && (3.30)\\
&= \bar{D}_\mu\bar{D}_\nu\alpha - (\bar{D}_\nu\log\phi)(\bar{D}_\mu\alpha) - (\bar{D}_\mu\log\phi)(\bar{D}_\nu\alpha) + \bar{\gamma}_{\mu\nu}(\bar{D}^\lambda\log\phi)(\bar{D}_\lambda\alpha)\\
&\quad - \frac{1}{m}\bar{\gamma}_{\mu\nu}\Big\{\bar{D}_\lambda\bar{D}^\lambda\alpha + (m-2)(\bar{D}^\lambda\log\phi)(\bar{D}_\lambda\alpha)\Big\} && (3.31)\\
&= (\bar{D}_\mu\bar{D}_\nu\alpha)^{\rm TF} - \big\{(\bar{D}_\nu\log\phi)(\bar{D}_\mu\alpha)\big\}^{\rm TF} - \big\{(\bar{D}_\mu\log\phi)(\bar{D}_\nu\alpha)\big\}^{\rm TF}. && (3.32)
\end{aligned}
$$

3.6 Conformal Traceless Formulation

We now introduce the conformal traceless formulation for the standard ADM formulation. We adopt the dynamical variables $(\bar{\gamma}_{\mu\nu}, \phi, K, \bar{A}_{\mu\nu})$ instead of $(\gamma_{\mu\nu}, K_{\mu\nu})$. Since the conformal factor ϕ is arbitrary, we must set the relation between ϕ and the ADM dynamical variables $(\gamma_{\mu\nu}, K_{\mu\nu})$. If $\phi = \phi(K_{\mu\nu})$, the conformal metric $\bar{\gamma}_{\mu\nu}$ include the character of the normal vector. Therefore, the conformal factor ϕ is constructed with the function of $\gamma_{\mu\nu}$. Since there is a degree of the freedom of ϕ, we adopt the determinant of the conformal metric, $\bar{\gamma}$, as a positive constant value.

3.6.1 Dynamical Variables

The new dynamical variables are defined as

$$
\begin{aligned}
\phi &\equiv p\gamma^{\frac{1}{2m}}, && (3.33)\\
\bar{\gamma}_{\mu\nu} &\equiv \phi^{-2}\gamma_{\mu\nu}, && (3.34)\\
K &\equiv \gamma^{\mu\nu}K_{\mu\nu}, && (3.35)\\
\bar{A}_{\mu\nu} &\equiv \phi^{-2}\left(K_{\mu\nu} - \frac{1}{m}K\gamma_{\mu\nu}\right), && (3.36)
\end{aligned}
$$

where $p = \bar{\gamma}^{-\frac{1}{2m}}$. Note that (3.36) is given by multiplying ϕ^{-2}. This is because $\widetilde{A}_{\mu\nu}$ is raised and lowered indexes with $\bar{\gamma}_{\mu\nu}$.

3.6.2 Constraint Equations

With new variables (3.33)-(3.36), the Hamiltonian constraint equation (2.72) can be expressed as

$$\mathcal{H} = {}^{(m)}R + K^2 - K_{\mu\nu}K^{\mu\nu} - 2\Lambda - 2\kappa\rho_H \tag{3.37}$$

$$= \phi^{-2}\left\{ {}^{(m)}\bar{R} - 2(m-1)(\bar{D}_\lambda \bar{D}^\lambda \log\phi) - (m-2)(m-1)(\bar{D}^\rho \log\phi)(\bar{D}_\rho \log\phi)\right\} + K^2 - \left(\bar{A}_{\mu\nu} + \frac{1}{m}K\bar{\gamma}_{\mu\nu}\right)\left(\bar{A}^{\mu\nu} + \frac{1}{m}K\bar{\gamma}^{\mu\nu}\right) - 2\Lambda - 2\kappa\rho_H \tag{3.38}$$

$$= \phi^{-2}\left\{ {}^{(m)}\bar{R} - 2(m-1)(\bar{D}_\lambda \bar{D}^\lambda \log\phi) - (m-2)(m-1)(\bar{D}^\rho \log\phi)(\bar{D}_\rho \log\phi)\right\} + \frac{m-1}{m}K^2 - \bar{A}_{\mu\nu}\bar{A}^{\mu\nu} - \frac{2}{m}K\bar{A}_{\mu\nu}\bar{\gamma}^{\mu\nu} - 2\Lambda - 2\kappa\rho_H, \tag{3.39}$$

and the momentum constraint equations (2.73) can be expressed as

$$\mathcal{M}_\mu = D_\nu K^\nu{}_\mu - D_\mu K - \kappa J_\mu \tag{3.40}$$

$$= \gamma^{\nu\lambda}\Big\{ \bar{D}_\nu K_{\lambda\mu} - (\bar{D}_\lambda \log\phi)K_{\nu\mu} - (\bar{D}_\nu \log\phi)K_{\lambda\mu} + \bar{\gamma}_{\nu\lambda}(\bar{D}^\omega \log\phi)K_{\omega\mu} - (\bar{D}_\mu \log\phi)K_{\nu\lambda} - (\bar{D}_\nu \log\phi)K_{\mu\lambda} + \bar{\gamma}_{\mu\lambda}(\bar{D}^\omega \log\phi)K_{\omega\nu}\Big\} - \bar{D}_\mu K - \kappa J_\mu \tag{3.41}$$

$$= \phi^{-2}\bar{\gamma}^{\nu\lambda}\Big[\bar{D}_\nu\left\{\phi^2\left(\bar{A}_{\lambda\mu} + \frac{1}{m}K\bar{\gamma}_{\lambda\mu}\right)\right\} -\phi^2(\bar{D}_\lambda\log\phi)\left(\bar{A}_{\nu\mu} + \frac{1}{m}K\bar{\gamma}_{\nu\mu}\right) -\phi^2(\bar{D}_\nu\log\phi)\left(\bar{A}_{\lambda\mu} + \frac{1}{m}K\bar{\gamma}_{\lambda\mu}\right) + \phi^2\bar{\gamma}_{\nu\lambda}(\bar{D}^\omega\log\phi)\left(\bar{A}_{\omega\mu} + \frac{1}{m}K\bar{\gamma}_{\omega\mu}\right) - \phi^2(\bar{D}_\mu\log\phi)\left(\bar{A}_{\nu\lambda} + \frac{1}{m}K\bar{\gamma}_{\nu\lambda}\right) -\phi^2(\bar{D}_\nu\log\phi)\left(\bar{A}_{\mu\lambda} + \frac{1}{m}K\bar{\gamma}_{\mu\lambda}\right) +\phi^2\bar{\gamma}_{\mu\lambda}(\bar{D}^\omega\log\phi)\left(\bar{A}_{\omega\nu} + \frac{1}{m}K\bar{\gamma}_{\omega\nu}\right)\Big\} - \bar{D}_\mu K - \kappa J_\mu \tag{3.42}$$

$$= \phi^{-2}\bar{D}_\nu\left\{\phi^2\left(\bar{A}^\nu{}_\mu + \frac{1}{m}K\delta^\nu{}_\mu\right)\right\} +(m-2)(\bar{D}_\lambda\log\phi)\left(\bar{A}^\lambda{}_\mu + \frac{1}{m}K\delta^\lambda{}_\mu\right) - (\bar{D}_\mu\log\phi)\left(\bar{A}_{\mu\nu}\bar{\gamma}^{\mu\nu} + K\right) - \bar{D}_\mu K - \kappa J_\mu \tag{3.43}$$

$$= m(\bar{D}_\lambda\log\phi)\left(\bar{A}^\lambda{}_\mu + \frac{1}{m}K\delta^\lambda{}_\mu\right) + (\bar{D}_\nu\bar{A}^\nu{}_\mu) + \frac{1}{m}(\bar{D}_\mu K) - (\bar{D}_\mu\log\phi)\left(\bar{A}_{\mu\nu}\bar{\gamma}^{\mu\nu} + K\right) - \bar{D}_\mu K - \kappa J_\mu \tag{3.44}$$

$$= \bar{D}_\nu\bar{A}^\nu{}_\mu + m(\bar{D}_\lambda\log\phi)\bar{A}^\lambda{}_\mu - \frac{m-1}{m}\bar{D}_\mu K - (\bar{D}_\mu\log\phi)\bar{A}_{\mu\nu}\bar{\gamma}^{\mu\nu} - \kappa J_\mu. \tag{3.45}$$

In this conformal transformation, the number of the dynamical variables is two more than the ADM formulation. We should add two constraints to this formulation because of the consistent of the degree of the dynamical variables of the ADM formulation. We define two constraint such as

$$\bar{\mathcal{S}} \equiv \bar{\gamma}^{-\frac{1}{2m}} - p, \tag{3.46}$$
$$\bar{\mathcal{A}} \equiv \bar{\gamma}^{\mu\nu}\bar{A}_{\mu\nu}. \tag{3.47}$$

The constraint equations of the conformal traceless formulation are

$$\begin{aligned}
\bar{\mathcal{H}} &\equiv \phi^{-2}\left\{ {}^{(m)}\bar{R} - 2(m-1)(\bar{D}_\lambda\bar{D}^\lambda\log\phi) - (m-2)(m-1)(\bar{D}^\rho\log\phi)(\bar{D}_\rho\log\phi)\right\} \\
&\quad + \frac{m-1}{m}K^2 - \bar{A}_{\mu\nu}\bar{A}^{\mu\nu} - \frac{2}{m}K\bar{A}_{\mu\nu}\bar{\gamma}^{\mu\nu} - 2\Lambda - 2\kappa\rho_H, && (3.48)\\
\bar{\mathcal{M}}_\mu &\equiv \bar{D}_\nu\bar{A}^\nu{}_\mu + m(\bar{D}_\lambda\log\phi)\bar{A}^\lambda{}_\mu - \frac{m-1}{m}\bar{D}_\mu K - (\bar{D}_\mu\log\phi)\bar{A}_{\mu\nu}\bar{\gamma}^{\mu\nu} - \kappa J_\mu, && (3.49)\\
\bar{\mathcal{S}} &\equiv \bar{\gamma}^{-\frac{1}{2m}} - p, && (3.50)\\
\bar{\mathcal{A}} &\equiv \bar{\gamma}^{\mu\nu}\bar{A}_{\mu\nu}. && (3.51)
\end{aligned}$$

3.6.3 Dynamical Equations

Next, we calculate the dynamical equations of the conformal traceless formulation. The dynamical equations of the standard ADM formulation with the conformal traceless values can be written as

$$\partial_t\gamma_{\mu\nu} = -2\alpha\phi^2\bar{A}_{\mu\nu} - \frac{2}{m}\alpha K\phi^2\bar{\gamma}_{\mu\nu} + \phi^2\bar{\mathcal{L}}_\beta(\bar{\gamma}_{\mu\nu}) + 2\phi^2\bar{\gamma}_{\mu\nu}\bar{\mathcal{L}}_\beta(\log\phi), \tag{3.52}$$

$$\begin{aligned}
\partial_t K_{\mu\nu} &= \alpha\,{}^{(m)}\bar{R}_{\mu\nu} + \alpha\,{}^{(m)}R^\phi_{\mu\nu} + \alpha\phi^2\left(-2\bar{A}_{\mu\lambda}\bar{A}^\lambda{}_\nu + \frac{m-4}{m}K\bar{A}_{\mu\nu} + \frac{m-2}{m^2}K^2\bar{\gamma}_{\mu\nu}\right) \\
&\quad - D_\mu D_\nu\alpha + \phi^2\bar{\mathcal{L}}_\beta(\bar{A}_{\mu\nu}) + \frac{1}{m}\phi^2\bar{\gamma}_{\mu\nu}\bar{\mathcal{L}}_\beta(K) + \frac{1}{m}\phi^2K\bar{\mathcal{L}}_\beta(\bar{\gamma}_{\mu\nu}) + 2\phi^2\bar{A}_{\mu\nu}\bar{\mathcal{L}}_\beta(\log\phi) \\
&\quad + \frac{2}{m}\phi^2K\bar{\gamma}_{\mu\nu}\bar{\mathcal{L}}_\beta(\log\phi) - \frac{2\Lambda}{m-1}\alpha\phi^2\bar{\gamma}_{\mu\nu} - \kappa\alpha\left\{S_{\mu\nu} - \frac{1}{m-1}(S-\rho_H)\phi^2\bar{\gamma}_{\mu\nu}\right\}. && (3.53)
\end{aligned}$$

With these equations (3.52)-(3.53), we compute the dynamical equations.

The time derivative of ϕ is

$$\begin{aligned}
\partial_t\phi &= \frac{1}{2m}\phi\gamma^{\mu\nu}(\partial_t\gamma_{\mu\nu}) && (3.54)\\
&= \frac{1}{2m}\phi^{-1}\bar{\gamma}^{\mu\nu}\left\{-2\alpha\phi^2\left(\bar{A}_{\mu\nu} + \frac{1}{m}K\bar{\gamma}_{\mu\nu}\right) + \phi^2\bar{\mathcal{L}}_\beta(\bar{\gamma}_{\mu\nu}) + 2\phi\bar{\mathcal{L}}_\beta(\phi)\bar{\gamma}_{\mu\nu}\right\} && (3.55)\\
&= -\frac{1}{m}\alpha\phi\bar{\mathcal{A}} - \frac{1}{m}\alpha\phi K + \frac{1}{m}\phi(\bar{D}_\lambda\beta^\lambda) + \bar{\mathcal{L}}_\beta(\phi). && (3.56)
\end{aligned}$$

The time derivative of $\bar{\gamma}_{\mu\nu}$ is

$$\partial_t \bar{\gamma}_{\mu\nu} = -2\phi^{-1}\bar{\gamma}_{\mu\nu}(\partial_t \phi) + \phi^{-2}(\partial_t \gamma_{\mu\nu}) \tag{3.57}$$

$$= -2\phi^{-1}\bar{\gamma}_{\mu\nu}\left\{-\frac{1}{m}\alpha\phi\bar{\mathcal{A}} - \frac{1}{m}\alpha\phi K + \frac{1}{m}\phi(\bar{D}_\lambda\beta^\lambda) + \bar{\mathcal{L}}_\beta(\phi)\right\}$$
$$+ \phi^{-2}\left\{-2\alpha\phi^2\left(\bar{A}_{\mu\nu} + \frac{1}{m}K\bar{\gamma}_{\mu\nu}\right) + \phi^2\bar{\mathcal{L}}_\beta(\bar{\gamma}_{\mu\nu}) + 2\phi\bar{\mathcal{L}}_\beta(\phi)\bar{\gamma}_{\mu\nu}\right\} \tag{3.58}$$

$$= \frac{2}{m}\alpha\bar{\mathcal{A}}\bar{\gamma}_{\mu\nu} - 2\alpha\bar{A}_{\mu\nu} - \frac{2}{m}(\bar{D}_\lambda\beta^\lambda)\bar{\gamma}_{\mu\nu} + \bar{\mathcal{L}}_\beta(\bar{\gamma}_{\mu\nu}). \tag{3.59}$$

The time derivative of K is

$$\partial_t K = -(\partial_t\gamma_{\mu\nu})K^{\mu\nu} + \gamma^{\mu\nu}(\partial_t K_{\mu\nu}) \tag{3.60}$$

$$= -\left(\bar{A}^{\mu\nu} + \frac{1}{m}K\bar{\gamma}^{\mu\nu}\right)\left\{-2\alpha\bar{A}_{\mu\nu} - \frac{2}{m}\alpha K\bar{\gamma}_{\mu\nu} + \bar{\mathcal{L}}_\beta(\bar{\gamma}_{\mu\nu}) + 2\bar{\gamma}_{\mu\nu}\bar{\mathcal{L}}_\beta(\log\phi)\right\}$$
$$+ \phi^{-2}\bar{\gamma}^{\mu\nu}\Bigg[\alpha\,{}^{(m)}\bar{R}_{\mu\nu} + \alpha\,{}^{(m)}R^\phi_{\mu\nu} + \alpha\phi^2\left(-2\bar{A}_{\mu\lambda}\bar{A}^\lambda{}_\nu + \frac{m-4}{m}K\bar{A}_{\mu\nu} + \frac{m-2}{m^2}K^2\bar{\gamma}_{\mu\nu}\right)$$
$$- D_\mu D_\nu\alpha + \phi^2\bar{\mathcal{L}}_\beta(\bar{A}_{\mu\nu}) + \frac{1}{m}\phi^2\bar{\gamma}_{\mu\nu}\bar{\mathcal{L}}_\beta(K) + \frac{1}{m}\phi^2 K\bar{\mathcal{L}}_\beta(\bar{\gamma}_{\mu\nu}) + 2\phi^2\bar{A}_{\mu\nu}\bar{\mathcal{L}}_\beta(\log\phi)$$
$$+ \frac{2}{m}\phi^2 K\bar{\gamma}_{\mu\nu}\bar{\mathcal{L}}_\beta(\log\phi) - \frac{2\Lambda}{m-1}\alpha\phi^2\bar{\gamma}_{\mu\nu} - \kappa\alpha\left\{S_{\mu\nu} - \frac{1}{m-1}(S-\rho_H)\phi^2\bar{\gamma}_{\mu\nu}\right\}\Bigg] \tag{3.61}$$

$$= \alpha\phi^{-2}\,{}^{(m)}\bar{R} + \alpha\phi^{-2}\,{}^{(m)}R^\phi + \alpha K^2 - D_\lambda D^\lambda\alpha + \bar{\mathcal{L}}_\beta(K) - \frac{2m\Lambda}{m-1}\alpha$$
$$+ \frac{\kappa}{m-1}\alpha(S - m\rho_H) + \alpha K\bar{\mathcal{A}} + \bar{\mathcal{L}}_\beta(\bar{\mathcal{A}}). \tag{3.62}$$

The time derivative of $\bar{A}_{\mu\nu}$ is

$$\partial_t \bar{A}_{\mu\nu} = -2\phi^{-1}\bar{A}_{\mu\nu}(\partial_t\phi) + \phi^{-2}(\partial_t K_{\mu\nu}) - \frac{1}{m}\phi^{-2}\gamma_{\mu\nu}(\partial_t K) - \frac{1}{m}\phi^{-2}K(\partial_t\gamma_{\mu\nu}) \tag{3.63}$$

$$\begin{aligned}
&= -2\phi^{-1}\bar{A}_{\mu\nu}\Big\{ -\frac{1}{m}\alpha\phi\bar{\mathcal{A}} - \frac{1}{m}\alpha\phi K + \frac{1}{m}\phi(\bar{D}_\lambda\beta^\lambda) + \bar{\mathcal{L}}_\beta(\phi)\Big\} \\
&\quad + \phi^{-2}\Big[\alpha\,{}^{(m)}\bar{R}_{\mu\nu} + \alpha\,{}^{(m)}R^{\phi}_{\mu\nu} + \alpha\phi^2\left(-2\bar{A}_{\mu\lambda}\bar{A}^\lambda{}_\nu + \frac{m-4}{m}K\bar{A}_{\mu\nu} + \frac{m-2}{m^2}K^2\bar{\gamma}_{\mu\nu}\right) \\
&\quad - D_\mu D_\nu\alpha + \phi^2\bar{\mathcal{L}}_\beta(\bar{A}_{\mu\nu}) + \frac{1}{m}\phi^2\bar{\gamma}_{\mu\nu}\bar{\mathcal{L}}_\beta(K) + \frac{1}{m}\phi^2 K\bar{\mathcal{L}}_\beta(\bar{\gamma}_{\mu\nu}) + 2\phi^2\bar{A}_{\mu\nu}\bar{\mathcal{L}}_\beta(\log\phi) \\
&\quad + \frac{2}{m}\phi^2 K\bar{\gamma}_{\mu\nu}\bar{\mathcal{L}}_\beta(\log\phi) - \frac{2\Lambda}{m-1}\alpha\phi^2\bar{\gamma}_{\mu\nu} - \kappa\alpha\left\{S_{\mu\nu} - \frac{1}{m-1}(S-\rho_H)\phi^2\bar{\gamma}_{\mu\nu}\right\}\Big] \\
&\quad - \frac{1}{m}\bar{\gamma}_{\mu\nu}\Big\{ \alpha\phi^{-2}\,{}^{(m)}\bar{R} + \alpha\phi^{-2}\,{}^{(m)}R^{\phi} + \alpha K^2 - D_\lambda D^\lambda\alpha + \bar{\mathcal{L}}_\beta(K) - \frac{2m\Lambda}{m-1}\alpha \\
&\quad + \frac{\kappa}{m-1}\alpha(S-m\rho_H) + \alpha K\bar{\mathcal{A}} + \bar{\mathcal{L}}_\beta(\bar{\mathcal{A}})\Big\} \\
&\quad - \frac{1}{m}\phi^{-2}K\Big\{ -2\alpha\phi^2\bar{A}_{\mu\nu} - \frac{2}{m}\alpha K\phi^2\bar{\gamma}_{\mu\nu} + \phi^2\bar{\mathcal{L}}_\beta(\bar{\gamma}_{\mu\nu}) + 2\phi^2\bar{\gamma}_{\mu\nu}\bar{\mathcal{L}}_\beta(\log\phi)\Big\}
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
&= \alpha\phi^{-2}({}^{(m)}\bar{R}_{\mu\nu} + {}^{(m)}R^{\phi}_{\mu\nu})^{\mathrm{TF}} - 2\alpha\bar{A}_{\mu\lambda}A^\lambda{}_\nu + \alpha K\bar{A}_{\mu\nu} + \frac{2}{m}\alpha\bar{A}_{\mu\nu}\bar{\mathcal{A}} \\
&\quad - \phi^{-2}(D_\mu D_\nu\alpha)^{\mathrm{TF}} - \frac{2}{m}(\bar{D}_\lambda\beta^\lambda)\bar{A}_{\mu\nu} + \bar{\mathcal{L}}_\beta(\bar{A}_{\mu\nu}) - \kappa\alpha(\phi^{-2}S_{\mu\nu})^{\mathrm{TF}} - \frac{1}{m}\alpha K\bar{\mathcal{A}}\bar{\gamma}_{\mu\nu} \\
&\quad - \frac{1}{m}\bar{\mathcal{L}}_\beta(\bar{\mathcal{A}})\bar{\gamma}_{\mu\nu}.
\end{aligned} \tag{3.65}$$

The dynamical equations of the conformal traceless formulation are

$$\partial_t\phi = -\frac{1}{m}\alpha\phi K + \frac{1}{m}\phi(\bar{D}_\lambda\beta^\lambda) + \bar{\mathcal{L}}_\beta(\phi) - \frac{1}{m}\alpha\phi\bar{\mathcal{A}}, \tag{3.66}$$

$$\partial_t\bar{\gamma}_{\mu\nu} = -2\alpha\bar{A}_{\mu\nu} - \frac{2}{m}(\bar{D}_\lambda\beta^\lambda)\bar{\gamma}_{\mu\nu} + \bar{\mathcal{L}}_\beta(\bar{\gamma}_{\mu\nu}) + \frac{2}{m}\alpha\bar{\mathcal{A}}\bar{\gamma}_{\mu\nu}, \tag{3.67}$$

$$\begin{aligned}
\partial_t K &= \alpha\phi^{-2}({}^{(m)}\bar{R} + {}^{(m)}R^{\phi}) + \alpha K^2 - D_\lambda D^\lambda\alpha + \bar{\mathcal{L}}_\beta(K) - \frac{2m\Lambda}{m-1}\alpha \\
&\quad + \frac{\kappa}{m-1}\alpha(S-m\rho_H) + \alpha K\bar{\mathcal{A}} + \bar{\mathcal{L}}_\beta(\bar{\mathcal{A}}),
\end{aligned} \tag{3.68}$$

$$\begin{aligned}
\partial_t\bar{A}_{\mu\nu} &= \alpha\phi^{-2}({}^{(m)}\bar{R}_{\mu\nu} + {}^{(m)}R^{\phi}_{\mu\nu})^{\mathrm{TF}} - 2\alpha\bar{A}_{\mu\lambda}A^\lambda{}_\nu + \alpha K\bar{A}_{\mu\nu} - \phi^{-2}(D_\mu D_\nu\alpha)^{\mathrm{TF}} \\
&\quad - \frac{2}{m}(\bar{D}_\lambda\beta^\lambda)\bar{A}_{\mu\nu} + \bar{\mathcal{L}}_\beta(\bar{A}_{\mu\nu}) - \kappa\alpha(\phi^{-2}S_{\mu\nu})^{\mathrm{TF}} - \frac{1}{m}\alpha K\bar{\gamma}_{\mu\nu}\bar{\mathcal{A}} + \frac{2}{m}\alpha\bar{A}_{\mu\nu}\bar{\mathcal{A}} \\
&\quad - \frac{1}{m}\bar{\gamma}_{\mu\nu}\bar{\mathcal{L}}_\beta(\bar{\mathcal{A}}).
\end{aligned} \tag{3.69}$$

3.7 Baumgarte-Shapiro-Shibata-Nakamura Formulation

3.7.1 Dynamical Variables

In the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation, the determinant of $\widetilde{\gamma}_{\mu\nu}$ is set as unity and the conformal factor ϕ is adopted as $e^{2\varphi}$ [13, 14]. Therefore, we use φ instead of ϕ as the dynamical variable.

The divergence term such as $\partial_\mu \gamma^{\mu\nu}$ would make increasing the numerical error. Thus, in the BSSN formulation, a new dynamical variable, $\widetilde{\Gamma}^\mu$, is added to that of the conformal traceless formulation, (3.33)-(3.36). In original paper [13], this variable was adopted as $\partial_\nu \widetilde{\gamma}^{\mu\nu}$. In this article, however, we follow [14] and use $\widetilde{\Gamma}^\mu = -\partial_\nu \gamma^{\mu\nu}$ as the new variable.

The dynamical variables of the BSSN formulation are

$$\varphi = \frac{1}{4m}\log(\gamma), \tag{3.70}$$

$$\widetilde{\gamma}_{\mu\nu} = e^{-4\varphi}\gamma_{\mu\nu}, \tag{3.71}$$

$$K = \gamma^{\mu\nu}K_{\mu\nu}, \tag{3.72}$$

$$\widetilde{A}_{\mu\nu} = e^{-4\varphi}\left(K_{\mu\nu} - \frac{1}{m}K\gamma_{\mu\nu}\right), \tag{3.73}$$

$$\widetilde{\Gamma}^\lambda = \widetilde{\Gamma}^\lambda{}_{\mu\nu}\widetilde{\gamma}^{\mu\nu}. \tag{3.74}$$

3.7.2 Geometrical Values

For (3.7), (3.12) and (3.16), we set ϕ as $e^{2\varphi}$, then the connection, the Ricci tensor and the scalar curvature of BSSN formulation become below:

The relation between the connection of the ADM and that of the BSSN formulation can be written as

$$ {}^{(m)}\Gamma^{\omega}{}_{\mu\nu} = {}^{(m)}\widetilde{\Gamma}^{\omega}{}_{\mu\nu} + 2\delta^{\omega}{}_{\mu}(\widetilde{D}_{\nu}\varphi) + 2\delta^{\omega}{}_{\nu}(\widetilde{D}_{\mu}\varphi) - 2\widetilde{\gamma}^{\omega\lambda}\widetilde{\gamma}_{\mu\nu}(\widetilde{D}_{\lambda}\varphi). \tag{3.75} $$

The Ricci tensor is

$$ {}^{(m)}R_{\mu\nu} = {}^{(m)}\widetilde{R}_{\mu\nu} + {}^{(m)}R^{\varphi}_{\mu\nu}, \tag{3.76} $$

where,

$$ \begin{aligned} {}^{(m)}\widetilde{R}_{\mu\nu} &= \widetilde{\gamma}_{\omega(\mu}\partial_{\nu)}\widetilde{\Gamma}^{\omega} + {}^{(m)}\widetilde{\Gamma}_{(\mu\nu)\omega}\widetilde{\Gamma}^{\omega} - \frac{1}{2}\widetilde{\gamma}^{\lambda\omega}\partial_{\lambda}\partial_{\omega}\widetilde{\gamma}_{\mu\nu} + {}^{(m)}\widetilde{\Gamma}^{\lambda\omega}{}_{\nu}\,{}^{(m)}\widetilde{\Gamma}_{\lambda\omega\mu} \\ &\quad + 2\,{}^{(m)}\widetilde{\Gamma}^{\lambda\omega}{}_{(\mu}\,{}^{(m)}\widetilde{\Gamma}_{\nu)\lambda\omega}, \qquad (3.77) \\ {}^{(m)}R^{\varphi}_{\mu\nu} &= -2(m-2)(\widetilde{D}_{\nu}\widetilde{D}_{\mu}\varphi) - 2(\widetilde{D}_{\lambda}\widetilde{D}^{\lambda}\varphi)\widetilde{\gamma}_{\mu\nu} + 4(m-2)(\widetilde{D}_{\mu}\varphi)(\widetilde{D}_{\nu}\varphi) \\ &\quad - 4(m-2)(\widetilde{D}^{\lambda}\varphi)(\widetilde{D}_{\lambda}\varphi)\widetilde{\gamma}_{\mu\nu}, \qquad (3.78) \end{aligned} $$

and $\widetilde{D}_{\mu}$ is the covariant derivative associated with $\widetilde{\gamma}_{\mu\nu}$.
The scalar curvature is

$$ {}^{(m)}R = e^{-4\varphi}\big({}^{(m)}\widetilde{R} + {}^{(m)}R^{\varphi}\big) \tag{3.79} $$

where ${}^{(m)}\widetilde{R} \equiv \widetilde{\gamma}^{\mu\nu}\,{}^{(m)}\widetilde{R}_{\mu\nu}$ and ${}^{(m)}R^{\varphi} \equiv \widetilde{\gamma}^{\mu\nu}\,{}^{(m)}R^{\varphi}_{\mu\nu}$.
$\forall V_{\mu\nu} \in \mathcal{T}^0_2(\mathcal{M}^m)$ and $\widetilde{V}_{\mu\nu} \equiv e^{-4\varphi}V_{\mu\nu}$, the Lie derivative associated with β^{λ} is

$$ \begin{aligned} \widetilde{\mathcal{L}}_{\beta}(\widetilde{V}_{\mu\nu}) &\equiv \beta^{\lambda}(\widetilde{D}_{\lambda}\widetilde{V}_{\mu\nu}) + \widetilde{V}_{\lambda\nu}(\widetilde{D}_{\mu}\beta^{\lambda}) + \widetilde{V}_{\mu\lambda}(\widetilde{D}_{\nu}\beta^{\lambda}) \qquad (3.80) \\ &= -4(\widetilde{D}_{\lambda}\varphi)\beta^{\lambda}\widetilde{V}_{\mu\nu} + e^{-4\varphi}\mathcal{L}_{\beta}(V_{\mu\nu}). \qquad (3.81) \end{aligned} $$

3.7.3 Constraint Equations

The constraint equations of the BSSN formulation are consistent with (3.48)-(3.51) replaced ϕ with $e^{2\varphi}$. In addition, the constraint, $\widetilde{\mathcal{G}}^{\lambda}$, is added to the formulation because of the consistent of the degree of the freedom of the ADM formulation.

The constraint equations of the conformal traceless formulation are

$$ \widetilde{\mathcal{H}} \equiv e^{-4\varphi}\big({}^{(m)}\widetilde{R} + {}^{(m)}R^{\varphi}\big) + \frac{m-1}{m}K^2 - \widetilde{A}_{\mu\nu}\widetilde{A}^{\mu\nu} - \frac{2}{m}K\widetilde{\mathcal{A}} - 2\Lambda - 2\kappa\rho_H, \tag{3.82} $$

$$ \widetilde{\mathcal{M}}_{\mu} \equiv \widetilde{D}_{\nu}\widetilde{A}^{\nu}{}_{\mu} + 2m(\widetilde{D}_{\lambda}\varphi)\widetilde{A}^{\lambda}{}_{\mu} - \frac{m-1}{m}\widetilde{D}_{\mu}K - 2(\widetilde{D}_{\mu}\varphi)\widetilde{\mathcal{A}} - \kappa J_{\mu}, \tag{3.83} $$

$$ \widetilde{\mathcal{S}} \equiv \widetilde{\gamma} - 1, \tag{3.84} $$

$$ \widetilde{\mathcal{A}} \equiv \widetilde{\gamma}^{\mu\nu}\widetilde{A}_{\mu\nu}, \tag{3.85} $$

$$ \widetilde{\mathcal{G}}^{\lambda} \equiv \widetilde{\Gamma}^{\lambda} - \widetilde{\Gamma}^{\lambda}{}_{\mu\nu}\widetilde{\gamma}^{\mu\nu}. \tag{3.86} $$

3.7.4 Dynamical Equations

The time derivatives of φ, $\widetilde{\gamma}_{\mu\nu}$, K and $\widetilde{A}_{\mu\nu}$ can get from (3.66)-(3.69). More precisely, the time derivative of φ can get from the relation of $\partial_t\varphi = \dfrac{\delta\varphi}{\delta\phi}\partial_t\phi$. Besides, the Hamiltonian constraint equations are added to (3.68) for elimination of ${}^{(m)}\widetilde{R}$ since there is the divergence term of $\widetilde{\gamma}_{\mu\nu}$ in the scalar curvature [14].

$$\begin{aligned}\partial_t K &= \alpha e^{-4\varphi}({}^{(m)}\widetilde{R} + {}^{(m)}R^{\varphi}) + \alpha K^2 - D_\lambda D^\lambda\alpha + \widetilde{\mathcal{L}}_\beta(K) - \frac{2m\Lambda}{m-1}\alpha + \frac{\kappa}{m-1}\alpha(S - m\rho_H) \\ &\quad + \alpha K\widetilde{\mathcal{A}} + \widetilde{\mathcal{L}}_\beta(\widetilde{\mathcal{A}}) - \alpha\widetilde{\mathcal{H}} \qquad (3.87)\\ &= \alpha e^{-4\varphi}({}^{(m)}\widetilde{R} + {}^{(m)}R^{\varphi}) + \alpha K^2 - D_\lambda D^\lambda\alpha + \widetilde{\mathcal{L}}_\beta(K) - \frac{2m\Lambda}{m-1}\alpha + \frac{\kappa}{m-1}\alpha(S - m\rho_H) \\ &\quad + \alpha K\widetilde{\mathcal{A}} + \widetilde{\mathcal{L}}_\beta(\widetilde{\mathcal{A}}) - \alpha e^{-4\varphi}({}^{(m)}\widetilde{R} + {}^{(m)}R^{\varphi}) - \frac{m-1}{m}\alpha K^2 + \alpha\widetilde{A}_{\mu\nu}\widetilde{A}^{\mu\nu} + \frac{2}{m}\alpha K\widetilde{\mathcal{A}} \\ &\quad + 2\alpha\Lambda + 2\kappa\alpha\rho_H \qquad (3.88)\\ &= \alpha\widetilde{A}_{\mu\nu}\widetilde{A}^{\mu\nu} + \frac{m+2}{m}\alpha K\widetilde{\mathcal{A}} + \frac{1}{m}\alpha K^2 - D^\lambda D_\lambda\alpha + \widetilde{\mathcal{L}}_\beta(K) + \widetilde{\mathcal{L}}_\beta(\widetilde{\mathcal{A}}) - \frac{2}{m-1}\alpha\Lambda \\ &\quad + \frac{\kappa}{m-1}\alpha\{S + (m-2)\rho_H\}. \qquad (3.89)\end{aligned}$$

Next, we compute the time derivative of the new variable, $\widetilde{\Gamma}^\lambda$. We first calculate the time derivative of ${}^{(m)}\widetilde{\Gamma}_{\lambda\mu\nu}$ is

$$\begin{aligned}\partial_t\,{}^{(m)}\widetilde{\Gamma}_{\lambda\mu\nu} &= \frac{1}{2}\{\partial_\nu(\partial_t\widetilde{\gamma}_{\lambda\mu}) + \partial_\mu(\partial_t\widetilde{\gamma}_{\lambda\nu}) - \partial_\lambda(\partial_t\widetilde{\gamma}_{\mu\nu})\} \qquad (3.90)\\ &= \frac{1}{2}\left\{\widetilde{D}_\nu(\partial_t\widetilde{\gamma}_{\lambda\mu}) + \widetilde{D}_\mu(\partial_t\widetilde{\gamma}_{\lambda\nu}) - \widetilde{D}_\lambda(\partial_t\widetilde{\gamma}_{\mu\nu})\right\} + {}^{(m)}\widetilde{\Gamma}^{\omega}{}_{\mu\nu}(\partial_t\widetilde{\gamma}_{\lambda\omega}). \qquad (3.91)\end{aligned}$$

For the time derivative of $\widetilde{\gamma}_{\mu\nu}$, we adopt

$$\partial_t\widetilde{\gamma}_{\mu\nu} = -2\alpha\widetilde{A}_{\mu\nu} - \frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\widetilde{\gamma}_{\mu\nu} + \widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\mu\nu}) + \frac{1}{m\widetilde{\gamma}}\beta^\rho(\widetilde{D}_\rho\widetilde{\mathcal{S}})\widetilde{\gamma}_{\mu\nu}, \qquad (3.92)$$

so as not to include the constraint terms if this equation is expressed with partial derivative operator. Then, the dynamical equation of $\widetilde{\Gamma}^\lambda$ is

$$\partial_t\widetilde{\Gamma}^\lambda = -\widetilde{\gamma}^{\lambda\mu}\,{}^{(m)}\widetilde{\Gamma}^{\nu\omega}{}_{\omega}(\partial_t\widetilde{\gamma}_{\mu\nu}) - {}^{(m)}\widetilde{\Gamma}^{\lambda\mu\nu}(\partial_t\widetilde{\gamma}_{\mu\nu}) + \widetilde{\gamma}^{\lambda\omega}\widetilde{\gamma}^{\mu\nu}(\partial_t\,{}^{(m)}\widetilde{\Gamma}_{\omega\mu\nu}) \tag{3.93}$$

$$= -\widetilde{\gamma}^{\lambda\mu}\,{}^{(m)}\widetilde{\Gamma}^{\nu\omega}{}_{\omega}(\partial_t\widetilde{\gamma}_{\mu\nu}) - {}^{(m)}\widetilde{\Gamma}^{\lambda\mu\nu}(\partial_t\widetilde{\gamma}_{\mu\nu}) + \frac{1}{2}\widetilde{\gamma}^{\lambda\omega}\widetilde{\gamma}^{\mu\nu}\left\{\widetilde{D}_\nu(\partial_t\widetilde{\gamma}_{\omega\mu}) + \widetilde{D}_\mu(\partial_t\widetilde{\gamma}_{\omega\nu})\right\} - \widetilde{D}_\omega(\partial_t\widetilde{\gamma}_{\mu\nu})\Big\} + {}^{(m)}\widetilde{\Gamma}^{\rho\mu}{}_{\mu}\widetilde{\gamma}^{\lambda\omega}(\partial_t\widetilde{\gamma}_{\omega\rho}) \tag{3.94}$$

$$= -{}^{(m)}\widetilde{\Gamma}^{\lambda\mu\nu}\left\{-2\alpha\widetilde{A}_{\mu\nu} - \frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\widetilde{\gamma}_{\mu\nu} + \widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\mu\nu}) + \frac{1}{m\widetilde{\gamma}}\beta^\rho(\widetilde{D}_\rho\widetilde{\mathcal{S}})\widetilde{\gamma}_{\mu\nu}\right\}$$
$$+\widetilde{\gamma}^{\lambda\omega}\widetilde{\gamma}^{\mu\nu}\widetilde{D}_\nu\left\{-2\alpha\widetilde{A}_{\omega\mu} - \frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\widetilde{\gamma}_{\omega\mu} + \widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\omega\mu}) + \frac{1}{m\widetilde{\gamma}}\beta^\rho(\widetilde{D}_\rho\widetilde{\mathcal{S}})\widetilde{\gamma}_{\omega\mu}\right\}$$
$$- \frac{1}{2}\widetilde{\gamma}^{\lambda\omega}\widetilde{\gamma}^{\mu\nu}\widetilde{D}_\omega\left\{-2\alpha\widetilde{A}_{\mu\nu} - \frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\widetilde{\gamma}_{\mu\nu} + \widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\mu\nu}) + \frac{1}{m\widetilde{\gamma}}\beta^\rho(\widetilde{D}_\rho\widetilde{\mathcal{S}})\widetilde{\gamma}_{\mu\nu}\right\} \tag{3.95}$$

$$= -2(\widetilde{D}_\mu\alpha)\widetilde{A}^{\lambda\mu} - 2\alpha(\widetilde{D}_\mu\widetilde{A}^{\lambda\mu}) + 2\alpha\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu\nu}\widetilde{A}_{\mu\nu} + \widetilde{D}^\lambda(\alpha\widetilde{\mathcal{A}})$$
$$- \frac{2}{m}\widetilde{D}^\lambda\widetilde{D}_\rho\beta^\rho + \widetilde{D}_\rho\widetilde{D}^\lambda\beta^\rho + \widetilde{D}_\mu\widetilde{D}^\mu\beta^\lambda + \frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_{\mu} - 2(\widetilde{D}_\mu\beta^\rho)\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_{\rho}$$
$$- \frac{1}{m\widetilde{\gamma}}\beta^\rho\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_{\mu}(\widetilde{D}_\rho\widetilde{\mathcal{S}}) - \frac{1}{m\widetilde{\gamma}^2}\beta^\rho(\widetilde{D}^\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\rho\widetilde{\mathcal{S}}) + \frac{2-m}{2m\widetilde{\gamma}}(\widetilde{D}^\lambda\beta^\rho)(\widetilde{D}_\rho\widetilde{\mathcal{S}})$$
$$+ \frac{2-m}{2m\widetilde{\gamma}}\beta^\rho(\widetilde{D}^\lambda\widetilde{D}_\rho\widetilde{\mathcal{S}}). \tag{3.96}$$

In the BSSN formulation, the momentum constraint equations add to the dynamical equation of $\widetilde{\Gamma}^\lambda$ so that the divergence term, $\widetilde{D}_\nu\widetilde{A}^{\lambda\nu}$, is eliminated [13]:

$$\partial_t\widetilde{\Gamma}^\lambda = (3.96) + 2\alpha\widetilde{\mathcal{M}}^\lambda \tag{3.97}$$

$$= -2(\widetilde{D}_\mu\alpha)\widetilde{A}^{\lambda\mu} + 2\alpha\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu\nu}\widetilde{A}_{\mu\nu} + \widetilde{D}^\lambda(\alpha\widetilde{\mathcal{A}}) + 4m\alpha(\widetilde{D}_\mu\varphi)\widetilde{A}^{\mu\lambda} - \frac{2(m-1)}{m}\alpha\widetilde{D}^\lambda K$$
$$- 4\alpha(\widetilde{D}^\lambda\varphi)\widetilde{\mathcal{A}} - 2\kappa\alpha\widetilde{\gamma}^{\lambda\mu}J_\mu - \frac{2}{m}\widetilde{D}^\lambda\widetilde{D}_\rho\beta^\rho + \widetilde{D}_\rho\widetilde{D}^\lambda\beta^\rho + \widetilde{D}_\mu\widetilde{D}^\mu\beta^\lambda$$
$$+ \frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_{\mu} - 2\,{}^{(m)}\widetilde{\Gamma}^{\lambda\nu}{}_{\omega}(\widetilde{D}_\nu\beta^\omega) - \frac{1}{m\widetilde{\gamma}}\beta^\rho\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_{\mu}(\widetilde{D}_\rho\widetilde{\mathcal{S}})$$
$$- \frac{1}{m\widetilde{\gamma}^2}\beta^\rho(\widetilde{D}^\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\rho\widetilde{\mathcal{S}}) + \frac{2-m}{2m\widetilde{\gamma}}(\widetilde{D}^\lambda\beta^\rho)(\widetilde{D}_\rho\widetilde{\mathcal{S}}) + \frac{2-m}{2m\widetilde{\gamma}}\beta^\rho(\widetilde{D}^\lambda\widetilde{D}_\rho\widetilde{\mathcal{S}}). \tag{3.98}$$

The calculation of the shift terms of (3.98) are

$$
\begin{aligned}
&-\frac{2}{m}(\widetilde{D}_\nu\widetilde{D}_\rho\beta^\rho)\widetilde{\gamma}^{\lambda\nu}+\widetilde{\gamma}^{\lambda\omega}\widetilde{D}_\nu\widetilde{D}_\omega\beta^\nu+\widetilde{\gamma}^{\mu\nu}\widetilde{D}_\mu\widetilde{D}_\nu\beta^\lambda+\frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_\mu\\
&-2\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_\omega\widetilde{D}_\mu\beta^\omega-\frac{1}{m\widetilde{\gamma}}\beta^\rho\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_\mu(\widetilde{D}_\rho\widetilde{\mathcal{S}})-\frac{1}{m\widetilde{\gamma}^2}\beta^\rho(\widetilde{D}^\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\rho\widetilde{\mathcal{S}})\\
&+\frac{2-m}{2m\widetilde{\gamma}}(\widetilde{D}^\lambda\beta^\rho)(\widetilde{D}_\rho\widetilde{\mathcal{S}})+\frac{2-m}{2m\widetilde{\gamma}}\beta^\rho(\widetilde{D}^\lambda\widetilde{D}_\rho\widetilde{\mathcal{S}}) \qquad (3.99)\\
=&-\frac{2}{m}\widetilde{\gamma}^{\lambda\nu}(\partial_\nu\partial_\rho\beta^\rho)+\frac{1}{m\widetilde{\gamma}^2}\beta^\omega(\widetilde{D}^\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\omega\widetilde{\mathcal{S}})-\frac{1}{m\widetilde{\gamma}}\beta^\omega(\widetilde{D}^\lambda\widetilde{D}_\omega\widetilde{\mathcal{S}})-\frac{1}{m\widetilde{\gamma}}(\widetilde{D}^\lambda\beta^\omega)(\widetilde{D}_\omega\widetilde{\mathcal{S}})\\
&+\widetilde{\gamma}^{\lambda\omega}(\partial_\nu\partial_\omega\beta^\nu)+\widetilde{\gamma}^{\lambda\omega}(\partial_\nu\,{}^{(m)}\widetilde{\Gamma}^\nu{}_{\omega\rho})\beta^\rho-{}^{(m)}\widetilde{\Gamma}^{\rho\lambda}{}_\nu\,{}^{(m)}\widetilde{\Gamma}^\nu{}_{\rho\sigma}\beta^\sigma+\frac{1}{2\widetilde{\gamma}}(\widetilde{D}^\lambda\beta^\rho)(\widetilde{D}_\rho\widetilde{\mathcal{S}})\\
&+\widetilde{\gamma}^{\mu\nu}(\partial_\mu\partial_\nu\beta^\lambda)+\widetilde{\gamma}^{\mu\nu}(\partial_\mu\,{}^{(m)}\widetilde{\Gamma}^\lambda{}_{\nu\omega})\beta^\omega+2\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_\omega(\partial_\mu\beta^\omega)-{}^{(m)}\widetilde{\Gamma}^{\omega\mu}{}_\mu(\partial_\omega\beta^\lambda)\\
&-{}^{(m)}\widetilde{\Gamma}^{\omega\mu}{}_\mu\,{}^{(m)}\widetilde{\Gamma}^\lambda{}_{\omega\rho}\beta^\rho+{}^{(m)}\widetilde{\Gamma}^{\lambda\nu}{}_\omega\,{}^{(m)}\widetilde{\Gamma}^\omega{}_{\nu\rho}\beta^\rho\\
&+\frac{2}{m}(\partial_\rho\beta^\rho)\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_\mu+\frac{1}{m\widetilde{\gamma}}\beta^\omega\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_\mu(\widetilde{D}_\omega\widetilde{\mathcal{S}})-2(\partial_\mu\beta^\omega)\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_\omega-2\beta^\rho\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_\omega\,{}^{(m)}\widetilde{\Gamma}^\omega{}_{\mu\rho}\\
&-\frac{1}{m\widetilde{\gamma}}\beta^\rho\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu}{}_\mu(\widetilde{D}_\rho\widetilde{\mathcal{S}})-\frac{1}{m\widetilde{\gamma}^2}\beta^\rho(\widetilde{D}^\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\rho\widetilde{\mathcal{S}})+\frac{2-m}{2m\widetilde{\gamma}}(\widetilde{D}^\lambda\beta^\rho)(\widetilde{D}_\rho\widetilde{\mathcal{S}})\\
&+\frac{2-m}{2m\widetilde{\gamma}}\beta^\rho(\widetilde{D}^\lambda\widetilde{D}_\rho\widetilde{\mathcal{S}}) \qquad (3.100)\\
=&\left(1-\frac{2}{m}\right)\widetilde{\gamma}^{\lambda\nu}(\partial_\nu\partial_\rho\beta^\rho)+\beta^\rho(\partial_\rho\widetilde{\Gamma}^\lambda)-\beta^\rho(\partial_\rho\widetilde{\mathcal{G}}^\lambda)+\frac{1}{\widetilde{\gamma}^2}\beta^\rho(\widetilde{D}_\rho\widetilde{\mathcal{S}})(\widetilde{D}^\lambda\widetilde{\mathcal{S}})+\frac{1}{\widetilde{\gamma}}\beta^\rho(\widetilde{D}_\rho\widetilde{D}^\lambda\widetilde{\mathcal{S}})\\
&-\frac{1}{2\widetilde{\gamma}}\beta^\omega\,{}^{(m)}\widetilde{\Gamma}^{\lambda\nu}{}_\omega(\widetilde{D}_\nu\widetilde{\mathcal{S}})+\widetilde{\gamma}^{\mu\nu}(\partial_\mu\partial_\nu\beta^\lambda)-(\partial_\omega\beta^\lambda)\widetilde{\Gamma}^\omega+(\partial_\omega\beta^\lambda)\widetilde{\mathcal{G}}^\omega+\frac{2}{m}(\partial_\rho\beta^\rho)\widetilde{\Gamma}^\lambda\\
&-\frac{2}{m}(\partial_\rho\beta^\rho)\widetilde{\mathcal{G}}^\lambda-\frac{1}{2\widetilde{\gamma}}\beta^\rho(\widetilde{D}^\lambda\widetilde{D}_\rho\widetilde{\mathcal{S}}) \qquad (3.101)\\
=&\left(1-\frac{2}{m}\right)\widetilde{\gamma}^{\lambda\nu}(\partial_\nu\partial_\rho\beta^\rho)+\beta^\rho(\partial_\rho\widetilde{\Gamma}^\lambda)+\widetilde{\gamma}^{\mu\nu}(\partial_\mu\partial_\nu\beta^\lambda)-(\partial_\omega\beta^\lambda)\widetilde{\Gamma}^\omega+\frac{2}{m}(\partial_\rho\beta^\rho)\widetilde{\Gamma}^\lambda\\
&+\frac{1}{\widetilde{\gamma}^2}\beta^\omega(\widetilde{D}^\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\omega\widetilde{\mathcal{S}})-\beta^\rho(\widetilde{D}_\rho\widetilde{\mathcal{G}}^\lambda)+\frac{1}{2\widetilde{\gamma}}\beta^\rho(\widetilde{D}_\rho\widetilde{D}^\lambda\widetilde{\mathcal{S}})-\frac{1}{2\widetilde{\gamma}}\beta^\omega\,{}^{(m)}\widetilde{\Gamma}^{\lambda\nu}{}_\omega(\widetilde{D}_\nu\widetilde{\mathcal{S}})+(\widetilde{D}_\omega\beta^\lambda)\widetilde{\mathcal{G}}^\omega\\
&-\frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\widetilde{\mathcal{G}}^\lambda+\frac{1}{m\widetilde{\gamma}}\beta^\omega\widetilde{\mathcal{G}}^\lambda(\widetilde{D}_\omega\widetilde{\mathcal{S}}). \qquad (3.102)
\end{aligned}
$$

We add or eliminate the constraint terms so as not to include the constraint terms in the dynamical equations which are expressed with the partial derivative operators. We call this formulation as the *standard BSSN formulation*:

The dynamical equations of the standard BSSN formulation are

$$\partial_t\varphi = -\frac{1}{2m}\alpha K + \frac{1}{2m}(\widetilde{D}_\lambda\beta^\lambda) + \widetilde{\mathcal{L}}_\beta(\varphi) - \frac{1}{m\widetilde{\gamma}}\beta^\lambda(\widetilde{D}_\lambda\widetilde{\mathcal{S}}) \tag{3.103}$$

$$= -\frac{1}{2m}\alpha K + \frac{1}{2m}(\partial_\lambda\beta^\lambda) + \beta^\lambda(\partial_\lambda\varphi), \tag{3.104}$$

$$\partial_t\widetilde{\gamma}_{\mu\nu} = -2\alpha\widetilde{A}_{\mu\nu} - \frac{2}{m}(\widetilde{D}_\lambda\beta^\lambda)\widetilde{\gamma}_{\mu\nu} + \widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\mu\nu}) + \frac{1}{m\widetilde{\gamma}}\beta^\lambda(\widetilde{D}_\lambda\widetilde{\mathcal{S}})\widetilde{\gamma}_{\mu\nu} \tag{3.105}$$

$$= -2\alpha\widetilde{A}_{\mu\nu} - \frac{2}{m}(\partial_\lambda\beta^\lambda)\widetilde{\gamma}_{\mu\nu} + \beta^\lambda(\partial_\lambda\widetilde{\gamma}_{\mu\nu}) + (\partial_\mu\beta^\lambda)\widetilde{\gamma}_{\lambda\nu} + (\partial_\nu\beta^\lambda)\widetilde{\gamma}_{\lambda\mu}, \tag{3.106}$$

$$\partial_t K = \alpha\widetilde{A}_{\mu\nu}\widetilde{A}^{\mu\nu} + \frac{1}{m}\alpha K^2 - D^\lambda D_\lambda\alpha + \widetilde{\mathcal{L}}_\beta(K) - \frac{2}{m-1}\alpha\Lambda + \frac{\kappa}{m-1}\alpha\{S + (m-2)\rho_H\} \tag{3.107}$$

$$= \alpha\widetilde{A}_{\mu\nu}\widetilde{A}^{\mu\nu} + \frac{1}{m}\alpha K^2 - D^\lambda D_\lambda\alpha + \beta^\lambda(\partial_\lambda K) - \frac{2}{m-1}\alpha\Lambda + \frac{\kappa}{m-1}\alpha\{S + (m-2)\rho_H\}, \tag{3.108}$$

$$\partial_t\widetilde{A}_{\mu\nu} = \alpha e^{-4\varphi}({}^{(m)}\widetilde{R}_{\mu\nu} + {}^{(m)}R^{\varphi}_{\mu\nu})^{\mathrm{TF}} + \alpha K\widetilde{A}_{\mu\nu} - 2\alpha\widetilde{A}_{\mu\lambda}\widetilde{A}^\lambda{}_\nu - e^{-4\varphi}(D_\mu D_\nu\alpha)^{\mathrm{TF}} - \frac{2}{m}(\widetilde{D}_\lambda\beta^\lambda)\widetilde{A}_{\mu\nu} + \widetilde{\mathcal{L}}_\beta(\widetilde{A}_{\mu\nu}) - \kappa\alpha(e^{-4\varphi}S_{\mu\nu})^{\mathrm{TF}} + \frac{1}{m\widetilde{\gamma}}\beta^\lambda(\widetilde{D}_\lambda\widetilde{\mathcal{S}})\widetilde{A}_{\mu\nu} \tag{3.109}$$

$$= \alpha e^{-4\varphi}({}^{(m)}\widetilde{R}_{\mu\nu} + {}^{(m)}R^{\varphi}_{\mu\nu})^{\mathrm{TF}} + \alpha K\widetilde{A}_{\mu\nu} - 2\alpha\widetilde{A}_{\mu\lambda}\widetilde{A}^\lambda{}_\nu - e^{-4\varphi}(D_\mu D_\nu\alpha)^{\mathrm{TF}} - \frac{2}{m}(\partial_\lambda\beta^\lambda)\widetilde{A}_{\mu\nu} + \beta^\lambda(\partial_\lambda\widetilde{A}_{\mu\nu}) + (\partial_\mu\beta^\lambda)\widetilde{A}_{\lambda\nu} + (\partial_\nu\beta^\lambda)\widetilde{A}_{\lambda\mu} - \kappa\alpha(e^{-4\varphi}S_{\mu\nu})^{\mathrm{TF}}, \tag{3.110}$$

$$\partial_t\widetilde{\Gamma}^\lambda = -2(\widetilde{D}_\nu\alpha)\widetilde{A}^{\lambda\nu} + 2\alpha\,{}^{(m)}\Gamma^{\lambda\mu\nu}\widetilde{A}_{\mu\nu} + 4m\alpha(\widetilde{D}_\mu\varphi)\widetilde{A}^{\mu\lambda} - \frac{2(m-1)}{m}\alpha\widetilde{D}^\lambda K - \frac{2}{m}\widetilde{D}^\lambda\widetilde{D}_\rho\beta^\rho + \widetilde{D}_\rho\widetilde{D}^\lambda\beta^\rho + \widetilde{D}_\mu\widetilde{D}^\mu\beta^\lambda + \frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\widetilde{\Gamma}^\lambda - 2\,{}^{(m)}\widetilde{\Gamma}^{\lambda\nu}{}_\omega(\widetilde{D}_\nu\beta^\omega) - 2\kappa\alpha J_\mu\widetilde{\gamma}^{\lambda\mu} - \frac{1}{\widetilde{\gamma}^2}\beta^\omega(\widetilde{D}^\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\omega\widetilde{\mathcal{S}}) + \beta^\rho(\widetilde{D}_\rho\widetilde{\mathcal{G}}^\lambda) - \frac{1}{2\widetilde{\gamma}}\beta^\rho(\widetilde{D}_\rho\widetilde{D}^\lambda\widetilde{\mathcal{S}}) + \frac{1}{2\widetilde{\gamma}}\beta^\omega\,{}^{(m)}\widetilde{\Gamma}^{\lambda\nu}{}_\omega(\widetilde{D}_\nu\widetilde{\mathcal{S}}) - (\widetilde{D}_\omega\beta^\lambda)\widetilde{\mathcal{G}}^\omega + \frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\widetilde{\mathcal{G}}^\lambda - \frac{1}{m\widetilde{\gamma}}\beta^\omega\widetilde{\mathcal{G}}^\lambda(\widetilde{D}_\omega\widetilde{\mathcal{S}}) \tag{3.111}$$

$$= -2(\partial_\nu\alpha)\widetilde{A}^{\lambda\nu} + 2\alpha\,{}^{(m)}\Gamma^{\lambda\mu\nu}\widetilde{A}_{\mu\nu} + 4m\alpha(\partial_\mu\varphi)\widetilde{A}^{\mu\lambda} - \frac{2(m-1)}{m}\alpha\widetilde{\gamma}^{\lambda\omega}\partial_\omega K + \left(1 - \frac{2}{m}\right)\widetilde{\gamma}^{\lambda\nu}(\partial_\nu\partial_\rho\beta^\rho) + \widetilde{\gamma}^{\mu\nu}(\partial_\mu\partial_\nu\beta^\lambda) - (\partial_\rho\beta^\lambda)\widetilde{\Gamma}^\rho + \frac{2}{m}(\partial_\rho\beta^\rho)\widetilde{\Gamma}^\lambda + \beta^\rho(\partial_\rho\widetilde{\Gamma}^\lambda) - 2\kappa\alpha J_\mu\widetilde{\gamma}^{\lambda\mu}. \tag{3.112}$$

3.8 Why BSSN formulation is better than ADM formulation?

Now we discuss the reason of the BSSN formulation is better than the ADM formulation. From the view point of the derivation of the BSSN formulation, some ideas are used for making the formulation. We show the modifications from the standard ADM formulation;

- the metric is decomposed to conformal factor and conformal metric,

- the extrinsic curvature is decomposed to trace part and trace-free part,
- new variable is added, and
- constraint equations are added to the dynamical equations.

The decompositions of the metric and extrinsic curvature would be suitable to the simulations of the coalescences of the black hole and the gravitational waves. However, these technique would be not always suitable for simulations. We can see the influence of adding the new variables by the changing of the expression of the Ricci tensor;

$$
\begin{aligned}
{}^{(m)}\widetilde{R}_{ij} &= \frac{1}{2}\widetilde{\gamma}^{mn}\{(\partial_n\partial_j\widetilde{\gamma}_{mi}) + (\partial_m\partial_i\widetilde{\gamma}_{nj}) - (\partial_m\partial_n\widetilde{\gamma}_{ij}) - (\partial_i\partial_j\widetilde{\gamma}_{mn})\} + {}^{(m)}\widetilde{\Gamma}^{mn}{}_{i}\,{}^{(m)}\widetilde{\Gamma}_{mnj} \\
&\quad - {}^{(m)}\widetilde{\Gamma}_{mij}\,{}^{(m)}\widetilde{\Gamma}^{m}{}_{ab}\widetilde{\gamma}^{ab} \qquad (3.113) \\
&= \gamma_{\ell(i}\partial_{j)}\widetilde{\Gamma}^{\ell} + {}^{(m)}\widetilde{\Gamma}_{(ij)\ell}\widetilde{\Gamma}^{\ell} - \frac{1}{2}\widetilde{\gamma}^{mn}(\partial_m\partial_n\widetilde{\gamma}_{ij}) + {}^{(m)}\widetilde{\Gamma}^{\ell m}{}_{i}\,{}^{(m)}\widetilde{\Gamma}_{\ell mj} + 2\,{}^{(m)}\Gamma^{\ell m}{}_{(i}\,{}^{(m)}\Gamma_{j)\ell m}. \qquad (3.114)
\end{aligned}
$$

We can see the highest order derivative terms (second order derivative terms) are clearly expressed with the wave operator, this would be the reason of the stability of the simulations. However, this is effective in the flat space. Therefore, the reason of the stability is the adding the constraint equations to the dynamical equations. After this chapter, we construct the formulations by adding the constraint equations.

Part II

Numerical Stability and C^2-adjusted Formulations

Chapter 4

Constraint Propagation Equations

To investigate the numerical stability of the numerical relativity, the one of the most important tools is the *constraint propagation equation* which are the dynamical equation of constraint. With the constraint propagation equation, we can predict that the simulations with the formulation are running well or not. After this chapter, we use only three dimension and the vacuum case. The indexes are adopted the Latin $(i, j, k, \cdots)$ instead of the Greek $(\mu, \nu, \lambda, \cdots)$.

4.1 Idea of Constraint Propagation

We review the general procedure of rewriting the evolution equations which we call adjusted systems [27–29, 36]. Suppose we have dynamical variables u^i which evolve along with the evolution equations,

$$\partial_t u^i = f(u^i, \partial_j u^i, \dots), \tag{4.1}$$

and suppose also that the system has the (first class) constraint equations,

$$C^a(u^a, \partial_j u^a, \dots) \approx 0. \tag{4.2}$$

We propose to study the properties of the evolution equation of C^a (which we call the constraint propagation),

$$\partial_t C^a = g(C^a, \partial_i C^a, \dots), \tag{4.3}$$

for predicting the violation behavior of constraints, C^a, in time evolution. Equation (4.3) is theoretically weakly zero, i.e. $\partial_t C^a \approx 0$, since the system is supposed to be the first class. However, the free numerical evolution with the discretized grids introduces constraint violation at least the level of truncation error, which sometimes grows to stop the simulations. The set of the ADM formulation has such a disastrous feature even in the Schwarzschild spacetime, as was shown in [29].

4.2 Constraint Propagation of Standard ADM formulation

The divergence of Einstein equations are given by $\nabla^{\mu\,(3)}E_{\mu\nu} = 0$ where ${}^{(3)}E_{ij}$ is explicitly in (2.54). The constraint propagation equations can get by decomposition of $\nabla^{\mu\,(3)}E_{\mu\nu} = 0$. From Appendix C, we can get the relations with ${}^{(3)}E_{ij} = 0$;

The constraint propagation equations of the standard ADM formulation are

$$\partial_t \mathcal{H} = \mathcal{L}_\beta(\mathcal{H}) + 2\alpha K\mathcal{H} - 2\alpha(D_i\mathcal{M}^i) - 4(D_i\alpha)\mathcal{M}^i, \tag{4.4}$$

$$\partial_t \mathcal{M}_i = \mathcal{L}_\beta(\mathcal{M}_i) - (D_i\alpha)\mathcal{H} + \alpha K\mathcal{M}_i - \frac{1}{2}\alpha(D_i\mathcal{H}). \tag{4.5}$$

4.3 Constraint Propagation of BSSN Formulation

Since the introduction of the constraint propagation equations of the BSSN formulation are complex, the calculations are expressed in Appendix C.4.

The constraint propagation equations of the standard BSSN formulation are

$$\begin{aligned}
\partial_t\widetilde{\mathcal{H}} = {} & \frac{2}{3}\alpha\widetilde{\mathcal{H}} + \widetilde{\mathcal{L}}_\beta(\widetilde{\mathcal{H}}) - 2(\widetilde{D}^\mu\alpha)e^{-4\varphi}\widetilde{\mathcal{M}}_\mu - 4\alpha e^{-4\varphi}(\widetilde{D}_\lambda\varphi)\widetilde{\mathcal{M}}^\lambda \\
& + \frac{4}{9}\alpha K^2\widetilde{\mathcal{A}} + 5\alpha e^{-4\varphi}(\widetilde{D}^\lambda\widetilde{D}_\lambda\widetilde{\mathcal{A}}) - \frac{2}{3}\alpha e^{-4\varphi}({}^{(3)}\widetilde{R} + {}^{(3)}R^\varphi)\widetilde{\mathcal{A}} \\
& + (\widetilde{D}_\mu\alpha)e^{-4\varphi}\widetilde{\mathcal{G}}^\mu\widetilde{\mathcal{A}} - \alpha e^{-4\varphi}\widetilde{\mathcal{G}}^\rho(\widetilde{D}_\rho\widetilde{\mathcal{A}}) + (\widetilde{D}_\lambda\widetilde{D}^\lambda\alpha)e^{-4\varphi}\widetilde{\mathcal{A}} + 2(\widetilde{D}_\lambda\alpha)e^{-4\varphi}(\widetilde{D}^\lambda\widetilde{\mathcal{A}}) \\
& + 4(\widetilde{D}^\lambda\alpha)e^{-4\varphi}(\widetilde{D}_\lambda\varphi)\widetilde{\mathcal{A}} + 8\alpha e^{-4\varphi}(\widetilde{D}_\lambda\varphi)(\widetilde{D}^\lambda\widetilde{\mathcal{A}}) - \frac{2}{3}\alpha\widetilde{A}_{\mu\nu}\widetilde{A}^{\mu\nu}\widetilde{\mathcal{A}} - \frac{2}{3}\widetilde{\mathcal{L}}_\beta(K)\widetilde{\mathcal{A}} \\
& - \frac{2}{m}K\widetilde{\mathcal{L}}_\beta(\widetilde{\mathcal{A}}) - 2\alpha e^{-4\varphi}\widetilde{A}^{\mu\nu}(\widetilde{D}_\nu\widetilde{\mathcal{G}}_\mu) - 2e^{-4\varphi}(\widetilde{D}^\nu\widetilde{D}_{(\lambda}\widetilde{\mathcal{G}}_{\nu)})\beta^\lambda \\
& + \frac{2}{3}(\widetilde{D}_\mu\widetilde{D}_\sigma\beta^\sigma)e^{-4\varphi}\widetilde{\mathcal{G}}^\mu + \frac{16}{3}(\widetilde{D}_\mu\beta^\mu)e^{-4\varphi}(\widetilde{D}_\rho\varphi)\widetilde{\mathcal{G}}^\rho - (\widetilde{D}_\lambda\widetilde{D}_\omega\beta^\lambda)e^{-4\varphi}\widetilde{\mathcal{G}}^\omega \\
& + \frac{2}{3}(\widetilde{D}_\lambda\widetilde{D}_\rho\beta^\rho)e^{-4\varphi}\widetilde{\mathcal{G}}^\lambda + \frac{2}{3}(\widetilde{D}_\rho\beta^\rho)e^{-4\varphi}(\widetilde{D}_\lambda\widetilde{\mathcal{G}}^\lambda) + \beta^\rho e^{-4\varphi}(\widetilde{D}_\lambda\widetilde{D}_\rho\widetilde{\mathcal{G}}^\lambda) \\
& - \frac{1}{3\widetilde{\gamma}}\beta^\omega e^{-4\varphi}(\widetilde{D}_\omega\mathcal{S})(\widetilde{D}_\lambda\widetilde{\mathcal{G}}^\lambda) + \frac{4}{3\widetilde{\gamma}}e^{-4\varphi}({}^{(3)}\widetilde{R} + {}^{(3)}R^\varphi)\beta^\lambda(\widetilde{D}_\lambda\mathcal{S}) \\
& - \frac{1}{2\widetilde{\gamma}}(\widetilde{D}_\lambda\beta^\rho)e^{-4\varphi}(\widetilde{D}_\rho\widetilde{D}^\lambda\widetilde{\mathcal{S}}) + \frac{1}{2\widetilde{\gamma}}(\widetilde{D}_\lambda\beta^\omega)\,{}^{(3)}\widetilde{\Gamma}^{\lambda\nu}{}_\omega e^{-4\varphi}(\widetilde{D}_\nu\widetilde{\mathcal{S}}) \\
& - \frac{1}{3\widetilde{\gamma}}(\widetilde{D}_\lambda\beta^\omega)e^{-4\varphi}\widetilde{\mathcal{G}}^\lambda(\widetilde{D}_\omega\widetilde{\mathcal{S}}) + \frac{37}{6\widetilde{\gamma}^3}\beta^\omega e^{-4\varphi}(\widetilde{D}^\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\omega\widetilde{\mathcal{S}}) \\
& + \frac{-17}{6\widetilde{\gamma}^2}\beta^\omega e^{-4\varphi}(\widetilde{D}_\lambda\widetilde{D}^\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\omega\widetilde{\mathcal{S}}) + \frac{61}{6\widetilde{\gamma}^2}\beta^\rho e^{-4\varphi}(\widetilde{D}_\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\rho\widetilde{D}^\lambda\widetilde{\mathcal{S}}) \\
& + \frac{5}{3\widetilde{\gamma}}\beta^\rho e^{-4\varphi}(\widetilde{D}_\lambda\widetilde{D}_\rho\widetilde{D}^\lambda\widetilde{\mathcal{S}}) - \frac{1}{2\widetilde{\gamma}^2}\beta^\omega e^{-4\varphi}\,{}^{(3)}\widetilde{\Gamma}^{\lambda\nu}{}_\omega(\widetilde{D}_\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\nu\widetilde{\mathcal{S}}) \\
& + \frac{1}{2\widetilde{\gamma}}\beta^\omega e^{-4\varphi}(\widetilde{D}_\lambda\,{}^{(3)}\widetilde{\Gamma}^{\lambda\nu}{}_\omega)(\widetilde{D}_\nu\widetilde{\mathcal{S}}) + \frac{1}{2\widetilde{\gamma}}\beta^\omega e^{-4\varphi}\,{}^{(3)}\widetilde{\Gamma}^{\lambda\nu}{}_\omega(\widetilde{D}_\lambda\widetilde{D}_\nu\widetilde{\mathcal{S}}) \\
& + \frac{1}{3\widetilde{\gamma}^2}\beta^\omega e^{-4\varphi}\widetilde{\mathcal{G}}^\lambda(\widetilde{D}_\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\omega\widetilde{\mathcal{S}}) - \frac{1}{3\widetilde{\gamma}}\beta^\omega e^{-4\varphi}\widetilde{\mathcal{G}}^\lambda(\widetilde{D}_\lambda\widetilde{D}_\omega\widetilde{\mathcal{S}}) \\
& + \frac{1}{2\widetilde{\gamma}}(\widetilde{D}_\nu\beta^\rho)e^{-4\varphi}\widetilde{\Gamma}^\nu(\widetilde{D}_\rho\widetilde{\mathcal{S}}) - \frac{1}{2\widetilde{\gamma}^2}\beta^\sigma e^{-4\varphi}\widetilde{\mathcal{G}}^\omega(\widetilde{D}_\omega\widetilde{\mathcal{S}})(\widetilde{D}_\sigma\widetilde{\mathcal{S}}) \\
& + \frac{1}{2\widetilde{\gamma}}\beta^\rho e^{-4\varphi}\widetilde{\Gamma}^\nu(\widetilde{D}_\nu\widetilde{D}_\rho\widetilde{\mathcal{S}}) + \frac{1}{\widetilde{\gamma}^2}\alpha e^{-4\varphi}\widetilde{A}^{\lambda\omega}(\widetilde{D}_\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\omega\widetilde{\mathcal{S}}) - \frac{1}{\widetilde{\gamma}}\alpha e^{-4\varphi}\widetilde{A}^{\lambda\omega}(\widetilde{D}_\lambda\widetilde{D}_\omega\widetilde{\mathcal{S}})
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\widetilde{\gamma}}\alpha e^{-4\varphi}\,{}^{(3)}\widetilde{\Gamma}^{\mu}{}_{\lambda\omega}\widetilde{A}^{\lambda\omega}(\widetilde{D}_{\mu}\widetilde{\mathcal{S}}) - \frac{1}{6\widetilde{\gamma}^2}\beta^{\rho}e^{-4\varphi}\,{}^{(3)}\widetilde{\Gamma}^{\mu\lambda}{}_{\lambda}(\widetilde{D}_{\mu}\widetilde{\mathcal{S}})(\widetilde{D}_{\rho}\widetilde{\mathcal{S}})\\
&- \frac{1}{3\widetilde{\gamma}}\beta^{\sigma}e^{-4\varphi}\,{}^{(3)}\widetilde{\Gamma}^{\lambda\rho\nu}\,{}^{(3)}\widetilde{\Gamma}_{\nu\lambda\rho}(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}}) - \frac{1}{2\widetilde{\gamma}}(\widetilde{D}^{\omega}\widetilde{D}_{\omega}\beta^{\sigma})e^{-4\varphi}(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})\\
&- \frac{1}{\widetilde{\gamma}}(\widetilde{D}_{\omega}\beta^{\sigma})e^{-4\varphi}(\widetilde{D}^{\omega}\widetilde{D}_{\sigma}\widetilde{\mathcal{S}}) - \frac{1}{2\widetilde{\gamma}}(\widetilde{D}_{\rho}\beta^{\sigma})e^{-4\varphi}\,{}^{(3)}\widetilde{\Gamma}^{\rho\lambda}{}_{\lambda}(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})\\
&- \frac{1}{6\widetilde{\gamma}}\beta^{\sigma}e^{-4\varphi}\,{}^{(3)}\widetilde{\Gamma}^{\rho\lambda}{}_{\lambda}(\widetilde{D}_{\rho}\widetilde{D}_{\sigma}\widetilde{\mathcal{S}}) + \frac{1}{3\widetilde{\gamma}}(\widetilde{D}_{\mu}\beta^{\sigma})e^{-4\varphi}\,{}^{(3)}\widetilde{\Gamma}^{\mu\nu}{}_{\nu}(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})\\
&+ \frac{8}{3\widetilde{\gamma}}\beta^{\sigma}e^{-4\varphi}(\widetilde{D}^{\lambda}\widetilde{D}_{\lambda}\varphi)(\widetilde{D}_{\sigma}\mathcal{S}) + \frac{8}{3\widetilde{\gamma}}\beta^{\sigma}e^{-4\varphi}\,{}^{(3)}\widetilde{\Gamma}^{\rho\lambda}{}_{\lambda}(\widetilde{D}_{\rho}\varphi)(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})\\
&+ \frac{8}{3\widetilde{\gamma}}e^{-4\varphi}(\widetilde{D}^{\nu}\widetilde{D}_{\nu}\beta^{\lambda})(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}}) - \frac{16}{3\widetilde{\gamma}^2}(\widetilde{D}^{\nu}\beta^{\lambda})e^{-4\varphi}(\widetilde{D}_{\nu}\widetilde{\mathcal{S}})(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})\\
&+ \frac{16}{3\widetilde{\gamma}}(\widetilde{D}^{\nu}\beta^{\lambda})e^{-4\varphi}(\widetilde{D}_{\nu}\widetilde{D}_{\lambda}\widetilde{\mathcal{S}}) - 8e^{-4\varphi}(\widetilde{D}_{\lambda}\varphi)\Big\{\frac{1}{3\widetilde{\gamma}}\beta^{\rho}\widetilde{\Gamma}^{\lambda}(\widetilde{D}_{\rho}\widetilde{\mathcal{S}}) - \frac{1}{3\widetilde{\gamma}}\beta^{\rho}\widetilde{\mathcal{G}}^{\lambda}(\widetilde{D}_{\rho}\widetilde{\mathcal{S}})\\
&\quad + \frac{1}{\widetilde{\gamma}^2}\beta^{\rho}(\widetilde{D}^{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\rho}\widetilde{\mathcal{S}}) - \frac{1}{2\widetilde{\gamma}}\beta^{\rho}(\widetilde{D}^{\lambda}\widetilde{D}_{\rho}\widetilde{\mathcal{S}})\Big\} + \frac{8}{3\widetilde{\gamma}}\beta^{\sigma}e^{-4\varphi}(\widetilde{D}^{\omega}\varphi)(\widetilde{D}_{\omega}\varphi)(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})\\
&+ \frac{4}{\widetilde{\gamma}}(\widetilde{D}_{\mu}\beta^{\lambda})e^{-4\varphi}(\widetilde{D}^{\mu}\varphi)(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}}) - \frac{1}{\widetilde{\gamma}^2}\alpha e^{-4\varphi}\widetilde{A}^{\mu\nu}(\widetilde{D}_{\mu}\widetilde{\mathcal{S}})(\widetilde{D}_{\nu}\widetilde{\mathcal{S}})\\
&+ \frac{1}{\widetilde{\gamma}}\alpha e^{-4\varphi}\widetilde{A}^{\mu\nu}(\widetilde{D}_{\mu}\widetilde{D}_{\nu}\widetilde{\mathcal{S}}) + \frac{1}{\widetilde{\gamma}}\alpha e^{-4\varphi}\,{}^{(3)}\widetilde{\Gamma}^{\rho}{}_{\nu\mu}\widetilde{A}^{\mu\nu}(\widetilde{D}_{\rho}\widetilde{\mathcal{S}}), \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
\partial_t\widetilde{\mathcal{M}}_i = {}&(\widetilde{D}^{\nu}\alpha)e^{-4\varphi}(\widetilde{D}_{(\mu}\widetilde{\mathcal{G}}_{\nu)}) - \frac{1}{3}(\widetilde{D}_{\mu}\alpha)\widetilde{\mathcal{H}} - \frac{1}{3}(\widetilde{D}_{\mu}\alpha)K\widetilde{\mathcal{A}} + \alpha K\widetilde{\mathcal{M}}_{\mu} + \frac{1}{6}\alpha(\widetilde{D}_{\mu}\widetilde{\mathcal{H}})\\
&+ 2(m-2)\alpha e^{-4\varphi}(\widetilde{D}^{\lambda}\varphi)(\widetilde{D}_{(\mu}\widetilde{\mathcal{G}}_{\lambda)}) + \frac{1}{2}(\widetilde{D}_{\mu}K)\widetilde{\mathcal{A}} + \frac{1}{9}\alpha K(\widetilde{D}_{\mu}\widetilde{\mathcal{A}}) + \frac{1}{9}\alpha(\widetilde{D}_{\mu}K)\widetilde{\mathcal{A}}\\
&- (\widetilde{D}_{\omega}\alpha)\widetilde{A}^{\omega}{}_{\mu}\widetilde{\mathcal{A}} - \alpha\widetilde{A}^{\omega}{}_{\mu}(\widetilde{D}_{\omega}\widetilde{\mathcal{A}}) + \frac{3}{2\widetilde{\gamma}^2}\beta^{\omega}\widetilde{A}^{\lambda}{}_{\mu}(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\omega}\widetilde{\mathcal{S}})\\
&- \frac{3}{2\widetilde{\gamma}}(\widetilde{D}_{\lambda}\beta^{\omega})\widetilde{A}^{\lambda}{}_{\mu}(\widetilde{D}_{\omega}\widetilde{\mathcal{S}}) - \frac{3}{2\widetilde{\gamma}}\beta^{\omega}\widetilde{A}^{\lambda}{}_{\mu}(\widetilde{D}_{\lambda}\widetilde{D}_{\omega}\widetilde{\mathcal{S}}) - \frac{1}{2\widetilde{\gamma}^2}\beta^{\lambda}(\widetilde{D}_{\mu}\widetilde{\mathcal{S}})(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})\widetilde{\mathcal{A}}\\
&+ \frac{1}{2\widetilde{\gamma}}(\widetilde{D}_{\mu}\beta^{\lambda})(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})\widetilde{\mathcal{A}} + \frac{3}{2\widetilde{\gamma}}\beta^{\lambda}(\widetilde{D}_{\mu}\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})\widetilde{\mathcal{A}} + \widetilde{\mathcal{L}}_{\beta}(\widetilde{\mathcal{M}}_{\mu}), \qquad (4.7)
\end{aligned}
$$

$$
\begin{aligned}
\partial_t\widetilde{\mathcal{G}}^i = {}&2\alpha\widetilde{\mathcal{M}}^{\lambda} - (\widetilde{D}^{\lambda}\alpha)\widetilde{\mathcal{A}} - \alpha(\widetilde{D}^{\lambda}\widetilde{\mathcal{A}}) + 4\alpha(\widetilde{D}^{\lambda}\varphi)\widetilde{\mathcal{A}} + \frac{1}{3\widetilde{\gamma}}\beta^{\rho}\widetilde{\Gamma}^{\lambda}(\widetilde{D}_{\rho}\widetilde{\mathcal{S}}) - \frac{2}{3\widetilde{\gamma}}\beta^{\rho}\widetilde{\mathcal{G}}^{\lambda}(\widetilde{D}_{\rho}\widetilde{\mathcal{S}})\\
&- \frac{2}{3\widetilde{\gamma}^2}\beta^{\rho}(\widetilde{D}^{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\rho}\widetilde{\mathcal{S}}) - \frac{1}{6\widetilde{\gamma}}(\widetilde{D}^{\lambda}\beta^{\rho})(\widetilde{D}_{\rho}\widetilde{\mathcal{S}}) - \frac{1}{3\widetilde{\gamma}}\beta^{\rho}(\widetilde{D}^{\lambda}\widetilde{D}_{\rho}\widetilde{\mathcal{S}}) + \beta^{\rho}(\widetilde{D}_{\rho}\widetilde{\mathcal{G}}^{\lambda})\\
&+ \frac{1}{2\widetilde{\gamma}}\beta^{\omega}\,{}^{(3)}\widetilde{\Gamma}^{\lambda\nu}{}_{\omega}(\widetilde{D}_{\nu}\widetilde{\mathcal{S}}) - (\widetilde{D}_{\omega}\beta^{\lambda})\widetilde{\mathcal{G}}^{\omega} + \frac{2}{3}(\widetilde{D}_{\rho}\beta^{\rho})\widetilde{\mathcal{G}}^{\lambda}, \qquad (4.8)
\end{aligned}
$$

$$
\partial_t\widetilde{\mathcal{A}} = \alpha K\widetilde{\mathcal{A}} + \widetilde{\mathcal{A}}, \qquad (4.9)
$$

$$
\partial_t\widetilde{\mathcal{S}} = -2\alpha\widetilde{\gamma}\widetilde{\mathcal{A}} + \widetilde{\mathcal{L}}_{\beta}(\mathcal{S}). \qquad (4.10)
$$

Chapter 5

Tools for Investigation of Numerical Stability

5.1 Adjusted Systems

Such features of the constraint propagation equations, (4.3), will be changed when we modify the original evolution equations. Suppose we add the constraint terms to the right-hand side of (4.1) as

$$\partial_t u^i = f(u^i, \partial_j u^i, \dots) + F(C^a, \partial_j C^a, \dots), \tag{5.1}$$

where $F(C^a, \dots) \approx 0$ in principle but not exactly zero in numerical evolutions, then (4.3) will also be modified as

$$\partial_t C^a = g(C^a, \partial_i C^a, \dots) + G(C^a, \partial_i C^a, \dots). \tag{5.2}$$

Therefore we are able to control $\partial_t C^a$ by an appropriate adjustment $F(C^a, \partial_j C^a, \dots)$ in (5.1). There exist various combinations of $F(C^a, \partial_j C^a, \dots)$ in (5.1), and all the alternative formulations are using this technique. Therefore, our goal is to find out a better way of adjusting the evolution equations which realizes $\partial_t C^a \leq 0$.

5.2 Constraint Amplification Factors

There are many efforts of re-formulation of the Einstein equations which make the evolution equations in an explicit first-order hyperbolic form (e.g. [17, 18, 44, 45]). This is motivated by the expectations that the symmetric hyperbolic system has well-posed properties in its Cauchy treatment in many systems and that the boundary treatment can be improved if we know the characteristic speed of the system. The advantage of the standard ADM system [9,10] (compared with the original ADM system [8]) is reported by Frittelli [46] from the point of the hyperbolicity of the constraint propagation equations. However, the classification of hyperbolicity(weakly, strongly or symmetric hyperbolic) only uses the characteristic part of evolution equations and ignore the rest. Several numerical experiments [11, 47] reported that such a classification is not enough to predict the stability of the evolution system, especially for highly non-linear system like the Einstein equations.

In order to investigate the stability structure of (5.2), the authors [28] proposed the constraint amplification factors(CAFs). The CAFs are the eigenvalues of the coefficient matrix,

$M^a{}_b$ (below), which is the Fourier-transformed components of the constraint propagation equations, $\partial_t \hat{C}^a$. That is,

$$\begin{aligned} &\partial_t \hat{C}^a = g(\hat{C}^a) = M^a{}_b \hat{C}^b, \\ &\text{where } C^a(x,t) = \int \hat{C}(k,t)^a \exp(\mathrm{i}k \cdot x) d^3k. \end{aligned} \tag{5.3}$$

CAFs include all the contributions of the terms, and enable us to check the eigenvalues. If CAFs have negative real-part, the constraints are forced to be diminished. Therefore, we expect more stable evolution than a system which has CAFs with positive real-part. If CAFs have non-zero imaginary-part, the constraints are supposed to propagate away. Therefore, we expect more stable evolution than a system which has CAFs with zero imaginary-part. The discussion and examples are shown in [11, 36], where several adjusted-ADM systems [11] and adjusted-BSSN systems [27] are proposed.

5.3 C^2-adjusted System

Fiske [34] proposed an adjustment of the evolution equations in the way of

$$\partial_t u^i = f(u^i, \partial_j u^i, \dots) - \kappa^{ij} \left(\frac{\delta C^2}{\delta u^j} \right), \tag{5.4}$$

where κ^{ij} is positive-definite constant coefficient, and C^2 is the norm of constraints which is defined as $C^2 \equiv \int C_a C^a d^3x$. The term $(\delta C^2/\delta u^j)$ is the functional derivative of C^2 with u^j. We call the set of (5.4) with (4.2) as "C^2-adjusted formulation". The associated constraint propagation equation becomes

$$\partial_t C^2 = h(C^a, \partial_i C^a, \dots) - \int d^3x \left(\frac{\delta C^2}{\delta u^i} \right) \kappa^{ij} \left(\frac{\delta C^2}{\delta u^j} \right). \tag{5.5}$$

If we set κ^{ij} so as the second term in the RHS of (5.5) becomes dominant than the first term, then $\partial_t C^2$ becomes negative, which indicates that constraint violations are expected to decay to zero. Fiske presented some numerical examples in the Maxwell system, and concluded that this method actually reduces the constraint violations. He also reported that the coefficient κ^{ij} has a practical upper limit in order not to crash simulations.

Chapter 6

C^2-adjusted ADM Formulation

6.1 Formulation

6.1.1 Standard ADM Formulation

We start by presenting the standard ADM formulation of the Einstein equations. The standard ADM evolution equations in three dimension are written as

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \tag{6.1}$$

$$\partial_t K_{ij} = \alpha({}^{(3)}R_{ij} + KK_{ij} - 2K_{i\ell}K^{\ell}{}_{j}) - D_iD_j\alpha + K_{\ell i}D_j\beta^{\ell} + K_{\ell j}D_i\beta^{\ell} + \beta^{\ell}D_{\ell}K_{ij}, \tag{6.2}$$

The constraint equations are

$$\mathcal{H} \equiv {}^{(3)}R + K^2 - K_{ij}K^{ij} \approx 0, \tag{6.3}$$

$$\mathcal{M}_i \equiv D_jK^j{}_i - D_iK \approx 0. \tag{6.4}$$

6.1.2 C^2-adjusted ADM Formulation

Now we apply C^2-adjustment to the ADM formulation, which can be written as

$$\partial_t \gamma_{ij} = (6.1) - \kappa_{\gamma ijmn}\left(\frac{\delta C^2}{\delta \gamma_{mn}}\right), \tag{6.5}$$

$$\partial_t K_{ij} = (6.2) - \kappa_{Kijmn}\left(\frac{\delta C^2}{\delta K_{mn}}\right), \tag{6.6}$$

where C^2 is the norm of the constraints, which we set

$$C^2 \equiv \int (\mathcal{H}^2 + \gamma^{ij}\mathcal{M}_i\mathcal{M}_j)d^3x, \tag{6.7}$$

and both coefficients of $\kappa_{\gamma ijmn}, \kappa_{Kijmn}$ are supposed to be positive definite. The additional terms in (6.5) and (6.6) are

$$\begin{aligned}\frac{\delta C^2}{\delta \gamma_{mn}} &= 2H_1{}^{mn}\mathcal{H} - 2(\partial_\ell H_2{}^{mn\ell})\mathcal{H} - 2H_2{}^{mn\ell}(\partial_\ell\mathcal{H}) + 2(\partial_k\partial_\ell H_3{}^{mnk\ell})\mathcal{H} \\ &\quad + 4(\partial_\ell H_3{}^{mnk\ell})(\partial_k\mathcal{H}) + 2H_3{}^{mnk\ell}(\partial_k\partial_\ell\mathcal{H}) + 2M_{1i}{}^{mn}\mathcal{M}^i - 2(\partial_\ell M_{2i}{}^{mn\ell})\mathcal{M}^i \\ &\quad - 2M_{2i}{}^{mn\ell}(\partial_\ell\mathcal{M}^i) - \mathcal{M}^m\mathcal{M}^n,\end{aligned} \tag{6.8}$$

$$\frac{\delta C^2}{\delta K_{mn}} = 2H_4{}^{mn}\mathcal{H} + 2M_{3i}{}^{mn}\mathcal{M}^i - 2(\partial_\ell M_{4i}{}^{mn\ell})\mathcal{M}^i - 2M_{4i}{}^{mn\ell}(\partial_\ell\mathcal{M}^i), \tag{6.9}$$

where

$$H_1^{mn} = -2R^{mn} + {}^{(3)}\Gamma^{m}{}^{(3)}\Gamma^{n} - {}^{(3)}\Gamma^{meb}{}^{(3)}\Gamma^{n}{}_{eb} - 2KK^{mn} + 2K^{m}{}_{j}K^{nj}, \tag{6.10}$$

$$H_2^{\ell mn} = -\gamma^{\ell m}{}^{(3)}\Gamma^{n} - \gamma^{\ell n}{}^{(3)}\Gamma^{m} + \gamma^{mn}{}^{(3)}\Gamma^{\ell} + {}^{(3)}\Gamma^{nm\ell} + {}^{(3)}\Gamma^{mn\ell} - {}^{(3)}\Gamma^{\ell nm}, \tag{6.11}$$

$$H_3^{k\ell mn} = \frac{1}{2}\gamma^{m\ell}\gamma^{nk} + \frac{1}{2}\gamma^{km}\gamma^{n\ell} - \gamma^{k\ell}\gamma^{mn}, \tag{6.12}$$

$$H_4^{mn} = 2\gamma^{mn}K - 2K^{mn}, \tag{6.13}$$

$$\begin{aligned} M_1^{mn}{}_i = &-\frac{1}{2}K_{\ell i,j}\gamma^{jm}\gamma^{\ell n} - \frac{1}{2}K_{\ell i,j}\gamma^{jn}\gamma^{\ell m} + \frac{1}{2}{}^{(3)}\Gamma^{n}K^{m}{}_{i} + \frac{1}{2}{}^{(3)}\Gamma^{m}K^{n}{}_{i} + {}^{(3)}\Gamma^{amn}K_{ai} \\ &- \frac{1}{2}K^{mc}\gamma^{nb}\gamma_{bc,i} - \frac{1}{2}K^{nc}\gamma^{mb}\gamma_{bc,i} + K_{ab,i}\gamma^{am}\gamma^{bn}, \end{aligned} \tag{6.14}$$

$$M_2^{\ell mn}{}_i = -\frac{1}{2}\gamma^{n\ell}K^{m}{}_{i} - \frac{1}{2}\gamma^{m\ell}K^{n}{}_{i} + \frac{1}{2}\gamma^{mn}K^{\ell}{}_{i} + \frac{1}{2}K^{nm}\delta^{\ell}{}_{i}, \tag{6.15}$$

$$M_3^{mn}{}_i = -\frac{1}{2}{}^{(3)}\Gamma^{m}\delta^{n}{}_{i} - \frac{1}{2}{}^{(3)}\Gamma^{n}\delta^{m}{}_{i} + \frac{1}{2}\gamma^{na}\gamma^{mb}\gamma_{ab,i}, \tag{6.16}$$

$$M_4^{\ell mn}{}_i = \frac{1}{2}\gamma^{\ell m}\delta^{n}{}_{i} + \frac{1}{2}\gamma^{\ell n}\delta^{m}{}_{i} - \gamma^{mn}\delta^{\ell}{}_{i}, \tag{6.17}$$

H_1^{mn}, H_2^{imn}, H_3^{ijmn}, H_4^{mn}, $M_1^{mn}{}_i$, $M_2^{jmn}{}_i$, $M_3^{mn}{}_i$, $M_4^{jmn}{}_i$ are the same with the appendix of [29] if $(m,n) = (n,m)$.

6.2 Constraint Propagation with C^2-adjusted ADM formulation

In this subsection, we discuss the constraint propagation of the C^2-adjusted ADM formulation, by giving the CAFs on flat background metric. We show CAFs are negative real numbers or complex numbers with negative real-part.

The constraint propagation equations, (4.4) and (4.5), are changed due to C^2-adjusted terms.

$$\begin{aligned} \partial_t\mathcal{H} = &\mathbb{H}_1\mathcal{H} + \mathbb{H}_2{}^{a}(\partial_a\mathcal{H}) + \mathbb{H}_3{}^{ab}(\partial_a\partial_b\mathcal{H}) + \mathbb{H}_4{}^{abc}(\partial_a\partial_b\partial_c\mathcal{H}) + \mathbb{H}_5{}^{abcd}(\partial_a\partial_b\partial_c\partial_d\mathcal{H}) + \mathbb{H}_{6a}\mathcal{M}^{a} \\ &+ \mathbb{H}_{7a}{}^{b}(\partial_b\mathcal{M}^{a}) + \mathbb{H}_{8a}{}^{bc}(\partial_b\partial_c\mathcal{M}^{a}) + \mathbb{H}_{9a}{}^{bcd}(\partial_b\partial_c\partial_d\mathcal{M}^{a}), \end{aligned} \tag{6.18}$$

where

$$\begin{aligned} \mathbb{H}_1 = &2\alpha K - 2\kappa_{\gamma mnij}\Big\{H_1{}^{mn}H_1^{ij} - H_1{}^{mn}(\partial_c H_2{}^{ijc}) + H_1{}^{mn}(\partial_d\partial_c H_3{}^{ijdc}) + H_2{}^{mn\ell}(\partial_\ell H_1{}^{ij}) \\ &- H_2{}^{mn\ell}(\partial_\ell\partial_c H_2{}^{ijc}) + H_2{}^{mn\ell}(\partial_\ell\partial_d\partial_c H_3^{ijdc}) + H_3{}^{mnk\ell}(\partial_k\partial_\ell H_1{}^{ij}) \\ &- H_3{}^{mnk\ell}(\partial_k\partial_\ell\partial_c H_2{}^{ijc}) + H_3{}^{mnk\ell}(\partial_k\partial_\ell\partial_d\partial_c H_3{}^{ijdc})\Big\} \\ &- 2(\partial_\ell\kappa_{\gamma mnij})\Big\{H_2{}^{mn\ell}H_1^{ij} - H_2{}^{mn\ell}(\partial_c H_2{}^{ijc}) + H_2{}^{mn\ell}(\partial_d\partial_c H_3{}^{ijdc}) \\ &+ 2H_3{}^{mn\ell k}(\partial_k H_1{}^{ij}) - 2H_3{}^{mn\ell k}(\partial_k\partial_c H_2{}^{ijc}) + 2H_3{}^{mn\ell k}(\partial_k\partial_d\partial_c H_3^{ijdc})\Big\} \\ &- 2(\partial_k\partial_\ell\kappa_{\gamma mnij})\Big\{H_3{}^{mnk\ell}H_1^{ij} - H_3{}^{mnk\ell}(\partial_c H_2{}^{ijc}) + H_3{}^{mnk\ell}(\partial_d\partial_c H_3{}^{ijdc})\Big\} \\ &- 2\kappa_{Kmnij}H_4{}^{mn}H_4{}^{ij}, \end{aligned} \tag{6.19}$$

$$
\begin{aligned}
\mathbb{H}_2{}^a = \beta^a - 2\kappa_{\gamma mnij}\Big\{ & -H_1{}^{mn}H_2{}^{ija} + 2H_1{}^{mn}(\partial_c H_3{}^{ijac}) + H_2{}^{mna}H_1{}^{ij} - H_2{}^{mna}(\partial_c H_2{}^{ijc}) \\
& - H_2{}^{mn\ell}(\partial_\ell H_2{}^{ija}) + H_2{}^{mna}(\partial_d\partial_c H_3{}^{ijdc}) + 2H_2{}^{mn\ell}(\partial_\ell\partial_c H_3{}^{ijac}) \\
& + H_3{}^{mna\ell}(\partial_\ell H_1{}^{ij}) + H_3{}^{mnka}(\partial_k H_1{}^{ij}) - H_3{}^{mna\ell}(\partial_\ell\partial_c H_2{}^{ijc}) \\
& - H_3{}^{mnka}(\partial_k\partial_c H_2{}^{ijc}) - H_3{}^{mnk\ell}(\partial_k\partial_\ell H_2{}^{ija}) + H_3{}^{mna\ell}(\partial_\ell\partial_d\partial_c H_3^{ijdc}) \\
& + H_3{}^{mnka}(\partial_k\partial_d\partial_c H_3{}^{ijdc}) + 2H_3{}^{mnk\ell}(\partial_k\partial_\ell\partial_c H_3{}^{ijac})\Big\} \\
& - 2(\partial_\ell\kappa_{\gamma mnij})\Big\{ -H_2{}^{mn\ell}H_2{}^{ija} + 2H_2{}^{mn\ell}(\partial_c H_3{}^{ijac}) + 2H_3{}^{mn\ell a}H_1{}^{ij} \\
& - 2H_3{}^{mn\ell a}(\partial_c H_2{}^{ijc}) - 2H_3{}^{mn\ell k}(\partial_k H_2{}^{ija}) + 2H_3{}^{mn\ell a}(\partial_d\partial_c H_3{}^{ijdc}) \\
& + 4H_3{}^{mn\ell k}(\partial_k\partial_c H_3{}^{ijac})\Big\} \\
& - 2(\partial_k\partial_\ell\kappa_{\gamma mnij})\Big\{ -H_3{}^{mnk\ell}H_2{}^{ija} + 2H_3{}^{mnk\ell}(\partial_c H_3{}^{ijac})\Big\}, \qquad (6.20)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{H}_3{}^{ab} = -2\kappa_{\gamma mnij}\Big\{ & H_1{}^{mn}H_3{}^{ijab} - H_2{}^{mna}H_2{}^{ijb} + 2H_2{}^{mna}(\partial_c H_3{}^{ijbc}) + H_2{}^{mn\ell}(\partial_\ell H_3{}^{ijab}) \\
& + H_3{}^{mnab}H_1{}^{ij} - H_3{}^{mnab}(\partial_c H_2{}^{ijc}) - H_3{}^{mna\ell}(\partial_\ell H_2{}^{ijb}) - H_3{}^{mnka}(\partial_k H_2{}^{ijb}) \\
& + H_3{}^{mnab}(\partial_d\partial_c H_3{}^{ijdc}) + 2H_3{}^{mna\ell}(\partial_\ell\partial_c H_3{}^{ijbc}) + 2H_3{}^{mnka}(\partial_k\partial_c H_3{}^{ijbc}) \\
& + H_3{}^{mnk\ell}(\partial_k\partial_\ell H_3{}^{ijab})\Big\} \\
& - 2(\partial_\ell\kappa_{\gamma mnij})\Big\{ H_2{}^{mn\ell}H_3{}^{ijab} - 2H_3^{mn\ell a}H_2{}^{ijb} + 4H_3^{mn\ell a}(\partial_c H_3{}^{ijbc}) \\
& + 2H_3^{mn\ell k}(\partial_k H_3{}^{ijab})\Big\} - 2(\partial_k\partial_\ell\kappa_{\gamma mnij})H_3{}^{mnk\ell}H_3{}^{ijab}, \qquad (6.21)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{H}_4{}^{abc} = -2\kappa_{\gamma mnij}\Big\{ & H_2{}^{mna}H_3{}^{ijbc} - H_3{}^{mnab}H_2{}^{ijc} + 2H_3{}^{mnab}(\partial_e H_3{}^{ijce}) + H_3{}^{mna\ell}(\partial_\ell H_3{}^{ijbc}) \\
& + H_3{}^{mnka}(\partial_k H_3{}^{ijbc})\Big\} - 4(\partial_k\kappa_{\gamma mnij})H_3{}^{mnka}H_3{}^{ijbc}, \qquad (6.22)
\end{aligned}
$$

$$
\mathbb{H}_5{}^{abcd} = -2\kappa_{\gamma mnij}H_3{}^{mnab}H_3{}^{ijcd}, \qquad (6.23)
$$

$$
\begin{aligned}
\mathbb{H}_{6a} = -2\alpha^{(3)}\Gamma^b{}_{ba} - 4\alpha_{,a} - \kappa_{\gamma mnij}\Big\{ & 2H_1{}^{mn}M_{1a}{}^{ij} - 2H_1{}^{mn}(\partial_d M_{2a}{}^{ijd}) - H_1{}^{mn}\mathcal{M}^{(i}\delta^{j)}{}_a \\
& + 2H_2{}^{mn\ell}(\partial_\ell M_{1a}{}^{ij}) - 2H_2{}^{mn\ell}(\partial_\ell\partial_d M_{2a}{}^{ijd}) + 2H_3{}^{mnk\ell}(\partial_k\partial_\ell M_{1a}{}^{ij}) \\
& - 2H_3{}^{mnk\ell}(\partial_k\partial_\ell\partial_d M_{2a}{}^{ijd})\Big\} \\
& - (\partial_\ell\kappa_{\gamma mnij})\Big\{ 2H_2{}^{mn\ell}M_{1a}{}^{ij} - 2H_2{}^{mn\ell}(\partial_d M_{2a}{}^{ijd}) - H_2{}^{mn\ell}\mathcal{M}^{(i}\delta^{j)}{}_a \\
& + 4H_3{}^{mn\ell k}(\partial_k M_{1a}{}^{ij}) - 4H_3{}^{mn\ell k}(\partial_k\partial_d M_{2a}{}^{ijd})\Big\} \\
& - (\partial_k\partial_\ell\kappa_{\gamma mnij})\Big\{ 2H_3{}^{mnk\ell}M_{1a}{}^{ij} - 2H_3{}^{mnk\ell}(\partial_d M_{2a}{}^{ijd}) - H_3{}^{mnk\ell}\mathcal{M}^{(i}\delta^{j)}{}_a\Big\} \\
& - \kappa_{Kmnij}\Big\{ 2H_4{}^{mn}M_{3a}{}^{ij} - 2H_4{}^{mn}(\partial_\ell M_{4a}{}^{ij\ell})\Big\}, \qquad (6.24)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{H}_{7a}{}^{b} = & -2\alpha\delta^{b}{}_{a} - \kappa_{\gamma mnij}\Big\{-2H_1{}^{mn}M_{2a}{}^{ijb} + 2H_2{}^{mnb}M_{1a}{}^{ij} - 2H_2{}^{mnb}(\partial_d M_{2a}{}^{ijd}) \\
& - 2H_2{}^{mn\ell}(\partial_\ell M_{2a}{}^{ijb}) - H_2{}^{mnb}\mathcal{M}^j\delta^i{}_a - H_2{}^{mnb}\mathcal{M}^i\delta^j{}_a + 2H_3{}^{mnb\ell}(\partial_\ell M_{1a}{}^{ij}) \\
& + 2H_3{}^{mnkb}(\partial_k M_{1a}{}^{ij}) - 2H_3{}^{mnb\ell}(\partial_\ell\partial_d M_{2a}{}^{ijd}) - 2H_3{}^{mnkb}(\partial_k\partial_d M_{2a}{}^{ijd}) \\
& - 2H_3{}^{mnk\ell}(\partial_k\partial_\ell M_{2a}{}^{ijb}) - H_3{}^{mnb\ell}(\partial_\ell\mathcal{M}^{(i})\delta^{j)}{}_a - H_3{}^{mnb\ell}(\partial_\ell\mathcal{M}^{(j})\delta^{i)}{}_a\Big\} \\
& - (\partial_\ell\kappa_{\gamma mnij})\Big\{-2H_2{}^{mn\ell}M_{2a}{}^{ijb} + 4H_3{}^{mn\ell b}M_{1a}{}^{ij} - 4H_3{}^{mn\ell b}(\partial_d M_{2a}{}^{ijd}) \\
& - 4H_3{}^{mn\ell k}(\partial_k M_{2a}{}^{ijb}) - 2H_3{}^{mn\ell b}\mathcal{M}^j\delta^i{}_a - 2H_3{}^{mn\ell b}\mathcal{M}^i\delta^j{}_a\Big\} \\
& + 2(\partial_k\partial_\ell\kappa_{\gamma mnij})H_3{}^{mnk\ell}M_{2a}{}^{ijb} + 2\kappa_{Kmnij}H_4{}^{mn}M_{4a}{}^{ijb}, \qquad (6.25)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{H}_{8a}{}^{bc} = & -\kappa_{\gamma mnij}\Big\{-2H_2{}^{mnb}M_{2a}{}^{ijc} + 2H_3{}^{mnbc}M_{1a}{}^{ij} - 2H_3{}^{mnbc}(\partial_d M_{2a}{}^{ijd}) \\
& - 2H_3{}^{mnb\ell}(\partial_\ell M_{2a}{}^{ijc}) - 2H_3{}^{mnkb}(\partial_k M_{2a}{}^{ijc}) - H_3{}^{mnbc}\mathcal{M}^j\delta^i{}_a - H_3{}^{mnbc}\mathcal{M}^i\delta^j{}_a\Big\} \\
& + 4(\partial_k\kappa_{\gamma mnij})H_3{}^{mnkb}M_{2a}{}^{ijc}, \qquad (6.26)
\end{aligned}
$$

$$
\mathbb{H}_{9a}{}^{bcd} = 2\kappa_{\gamma mnij}H_3{}^{mnbc}M_{2a}{}^{ijd}. \qquad (6.27)
$$

The propagation equation of the momentum constraint with C^2-adjusted ADM formulation can be written as

$$
\begin{aligned}
\partial_t\mathcal{M}_a = & \mathbb{M}_{1a}\mathcal{H} + \mathbb{M}_{2a}{}^{b}(\partial_b\mathcal{H}) + \mathbb{M}_{3a}{}^{bc}(\partial_b\partial_c\mathcal{H}) + \mathbb{M}_{4a}{}^{bcd}(\partial_b\partial_c\partial_d\mathcal{H}) + \mathbb{M}_{5ab}\mathcal{M}^b \\
& + \mathbb{M}_{6ab}{}^{c}(\partial_c\mathcal{M}^b) + \mathbb{M}_{7ab}{}^{cd}(\partial_c\partial_d\mathcal{M}^b), \qquad (6.28)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{M}_{1a} = & -\partial_a\alpha - 2\kappa_{\gamma mnij}\Big\{M_{1a}{}^{mn}H_1{}^{ij} - M_{1a}{}^{mn}(\partial_c H_2{}^{ijc}) + M_{1a}{}^{mn}(\partial_d\partial_c H_3{}^{ijdc}) \\
& + M_{2a}{}^{mn\ell}(\partial_\ell H_1{}^{ij}) - M_{2a}{}^{mn\ell}(\partial_\ell\partial_c H_2{}^{ijc}) + M_{2a}{}^{mn\ell}(\partial_\ell\partial_d\partial_c H_3{}^{ijdc})\Big\} \\
& - 2(\partial_\ell\kappa_{\gamma mnij})\Big\{M_{2a}{}^{mn\ell}H_1{}^{ij} - M_{2a}{}^{mn\ell}(\partial_c H_2{}^{ijc}) + M_{2a}{}^{mn\ell}(\partial_d\partial_c H_3{}^{ijdc})\Big\} \\
& - 2\kappa_{Kmnij}\Big\{M_{3a}{}^{mn}H_4{}^{ij} + M_{4a}{}^{mn\ell}(\partial_\ell H_4{}^{ij})\Big\} - 2(\partial_\ell\kappa_{Kmnij})M_{4a}{}^{mn\ell}H_4{}^{ij}, \qquad (6.29)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{M}_{2a}{}^{b} = &-\frac{1}{2}\alpha\delta_a{}^b - 2\kappa_{\gamma mnij}\Big\{-M_{1a}{}^{mn}H_2{}^{ijb} + 2M_{1a}{}^{mn}(\partial_c H_3{}^{ijbc}) + M_{2a}{}^{mnb}H_1{}^{ij} \\
&- M_{2a}{}^{mnb}(\partial_c H_2{}^{ijc}) - M_{2a}{}^{mn\ell}(\partial_\ell H_2{}^{ijb}) + M_{2a}{}^{mnb}(\partial_d\partial_c H_3{}^{ijdc}) \\
&+ 2M_{2a}{}^{mn\ell}(\partial_\ell\partial_c H_3{}^{ijbc})\Big\} \\
&- 2(\partial_\ell\kappa_{\gamma mnij})\Big\{-M_{2a}{}^{mn\ell}H_2{}^{ijb} + 2M_{2a}{}^{mn\ell}(\partial_c H_3{}^{ijbc})\Big\} - 2\kappa_{Kmnij}M_{4a}{}^{mnb}H_4{}^{ij}, \qquad (6.30)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{M}_{3a}{}^{bc} = &-2\kappa_{\gamma mnij}\Big\{M_{1a}{}^{mn}H_3{}^{ijbc} - M_{2a}{}^{mnb}H_2{}^{ijc} + 2M_{2a}{}^{mnb}(\partial_d H_3{}^{ijcd}) \\
&+ M_{2a}{}^{mn\ell}(\partial_\ell H_3{}^{ijbc})\Big\} - 2(\partial_\ell\kappa_{\gamma mnij})M_{2a}{}^{mn\ell}H_3{}^{ijbc}, \qquad (6.31)
\end{aligned}
$$

$$
\mathbb{M}_{4a}{}^{bcd} = -2\kappa_{\gamma mnij}M_{2a}{}^{mnb}H_3{}^{ijcd}, \qquad (6.32)
$$

$$
\begin{aligned}
\mathbb{M}_{5ab} = &\gamma_{mb}\beta^m{}_{,a} + \beta^\ell\gamma_{ab,\ell} + \alpha K\gamma_{ab} - \kappa_{\gamma mnij}\Big\{2M_{1a}{}^{mn}M_{1b}{}^{ij} - 2M_{1a}{}^{mn}(\partial_d M_{2b}{}^{ijd}) \\
&- M_{1a}{}^{mn}\mathcal{M}^{(j}\delta^{i)}{}_b + 2M_{2a}{}^{mn\ell}(\partial_\ell M_{1b}{}^{ij}) - 2M_{2a}{}^{mn\ell}(\partial_\ell\partial_d M_{2b}{}^{ijd})\Big\} \\
&- (\partial_\ell\kappa_{\gamma mnij})\Big\{2M_{2a}{}^{mn\ell}M_{1b}{}^{ij} - 2M_{2a}{}^{mn\ell}(\partial_d M_{2b}{}^{ijd}) - M_{2a}{}^{mn\ell}\mathcal{M}^{(j}\delta^{i)}{}_b\Big\} \\
&- 2\kappa_{Kmnij}\Big\{M_{3a}{}^{mn}M_{3b}{}^{ij} - M_{3a}{}^{mn}(\partial_\ell M_{4b}{}^{ij\ell}) + M_{4a}{}^{mn\ell}(\partial_\ell M_{3b}{}^{ij}) \\
&- M_{4a}{}^{mn\ell}(\partial_\ell\partial_d M_{4b}{}^{ijd})\Big\} \\
&- 2(\partial_\ell\kappa_{Kmnij})\Big\{M_{4a}{}^{mn\ell}M_{3b}{}^{ij} - M_{4a}{}^{mn\ell}(\partial_d M_{4b}{}^{ijd})\Big\}, \qquad (6.33)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{M}_{6ab}{}^{c} = &\beta^c\gamma_{ab} - \kappa_{\gamma mnij}\Big\{-2M_{1a}{}^{mn}M_{2b}{}^{ijc} + 2M_{2a}{}^{mnc}M_{1b}{}^{ij} - 2M_{2a}{}^{mnc}(\partial_d M_{2b}{}^{ijd}) \\
&- 2M_{2a}{}^{mn\ell}(\partial_\ell M_{2b}{}^{ijc}) - M_{2a}{}^{mnc}\mathcal{M}^j\delta^i{}_b - M_{2a}{}^{mnc}\mathcal{M}^i\delta^j{}_b\Big\} \\
&+ 2(\partial_\ell\kappa_{\gamma mnij})M_{2a}{}^{mn\ell}M_{2b}{}^{ijc} \\
&- 2\kappa_{Kmnij}\Big\{-M_{3a}{}^{mn}M_{4b}{}^{ijc} + M_{4a}{}^{mnc}M_{3b}{}^{ij} - M_{4a}{}^{mnc}(\partial_d M_{4b}{}^{ijd}) \\
&- M_{4a}{}^{mn\ell}(\partial_\ell M_{4b}{}^{ijc})\Big\} + 2(\partial_\ell\kappa_{Kmnij})M_{4a}{}^{mn\ell}M_{4b}{}^{ijc}, \qquad (6.34)
\end{aligned}
$$

$$
\mathbb{M}_{7ab}{}^{cd} = 2\kappa_{\gamma mnij}M_{2a}{}^{mnc}M_{2b}{}^{ijd} + 2\kappa_{Kmnij}M_{4a}{}^{mnc}M_{4b}{}^{ijd}. \qquad (6.35)
$$

If we fix the background is flat spacetime, ($\alpha = 1, \beta^i = 0, \gamma_{ij} = \delta_{ij}, K_{ij} = 0$), then CAFs are easily derived. For simplicity, we also set $\kappa_{\gamma ijmn} = \kappa_{Kijmn} = \kappa\delta_{im}\delta_{jn}$, where κ is positive. The Fourier-transformed equations of the constraint propagation equations are

$$
\partial_t \begin{pmatrix} \hat{\mathcal{H}} \\ \hat{\mathcal{M}}_i \end{pmatrix} = \begin{pmatrix} -4\kappa|\vec{k}|^4 & -2\mathrm{i}k_j \\ -(1/2)\mathrm{i}k_i & \kappa(-|\vec{k}|^2\delta_{ij} - 3k_ik_j) \end{pmatrix} \begin{pmatrix} \hat{\mathcal{H}} \\ \hat{\mathcal{M}}_j \end{pmatrix}. \qquad (6.36)
$$

The eigenvalues, λ, of the coefficient matrix of (6.36) are given by solving

$$(\lambda + \kappa|k|^2)^2(\lambda^2 + A\lambda + B) = 0,$$

where $A \equiv 4\kappa|k|^2(|k|^2+1)$ and $B \equiv |k|^2 + 16\kappa^2|k|^6$ (The Mathematica code for solving (6.36) is Appendix E). Therefore, the four eigenvalues are

$$(-\kappa|k|^2, -\kappa|k|^2, \lambda_+, \lambda_-), \tag{6.37}$$

where

$$\lambda_\pm = -2\kappa|k|^2(|k|^2+1) \pm |k|\sqrt{-1 + 4\kappa^2|k|^2(|k|^2-1)^2}. \tag{6.38}$$

From the relation of the coefficients with solutions,

$$\lambda_+ + \lambda_- = -A < 0, \quad \text{and} \quad \lambda_+\lambda_- = B > 0, \tag{6.39}$$

we find both the real parts of λ_+ and λ_- are negative. Therefore, we see all four eigenvalues are complex numbers with negative real-part or negative real numbers.

On the other hand, the CAFs of the standard ADM formulation on flat background [$\kappa = 0$ in (6.37)] are reduced to

$$(0, 0, \pm \mathrm{i}|\vec{k}|), \tag{6.40}$$

where the real-part of all of the CAFs are zero. Therefore the introduction of the C^2-adjusted terms to the evolution equations changes the constraint propagation equations to a self-decay system.

More precisely, CAFs depend to $|k|^2$ if $\kappa \neq 0$. This indicates that adjusted terms affect to reduce high frequency error-growing modes. Since we intend not to change the original evolution equations drastically by adding adjusted terms, we consider only small κ. This limits the robustness of the system to the low frequency error-growing modes. Therefore the system may stop due to the low frequency modes, but the longer evolutions are expected to be obtained.

6.3 Detweiler's ADM Formulation

We review Detweiler's ADM formulation [40] for a comparison with the C^2-adjusted ADM formulation and the standard ADM formulation. Detweiler proposed an evolution system in order to ensure the decay of the norm of constraints, $\partial_t C^2 < 0$. His system can be treated as one of the adjusted ADM systems and the set of evolution equations can be written as

$$\partial_t \gamma_{ij} = (6.1) + LD_{\gamma ij}, \tag{6.41}$$

$$\partial_t K_{ij} = (6.2) + LD_{Kij}, \tag{6.42}$$

where

$$D_{\gamma ij} \equiv -\alpha^3 \gamma_{ij}\mathcal{H}, \tag{6.43}$$

$$\begin{aligned} D_{Kij} \equiv\ & \alpha^3(K_{ij} - (1/3)K\gamma_{ij})\mathcal{H} \\ & + \alpha^2[3(\partial_{(i}\alpha)\delta^k{}_{j)} - (\partial_\ell \alpha)\gamma_{ij}\gamma^{k\ell}]\mathcal{M}_k + \alpha^3[\delta^k{}_{(i}\delta^\ell{}_{j)} - (1/3)\gamma_{ij}\gamma^{k\ell}]D_k\mathcal{M}_\ell, \end{aligned} \tag{6.44}$$

where L is a constant. He found that with this particular combination of adjustments, the evolution of the norm constraints, C^2, can be negative definite when we apply the maximal slicing condition, $K = 0$, for fixing the lapse function, α. Note that the effectiveness with other gauge conditions is remain unknown. The numerical demonstrations with Detweiler's ADM formulation are presented in [12, 28], and there we can see the drastic improvements for stability.

Chapter 7

C^2-adjusted BSSN Formulation

7.1 Formulation

7.1.1 Standard BSSN Formulation

We work with the widely used notation of the BSSN system. That is, the dynamical variables $(\varphi, K, \widetilde{\gamma}_{ij}, \widetilde{A}_{ij}, \widetilde{\Gamma}^i)$ as the replacement of the variables of the ADM formulation, (γ_{ij}, K_{ij}), where

$$\varphi \equiv \frac{1}{12}\log(\gamma), \tag{7.1}$$

$$K \equiv \gamma^{ij}K_{ij}, \tag{7.2}$$

$$\widetilde{\gamma}_{ij} \equiv e^{-4\varphi}\gamma_{ij}, \tag{7.3}$$

$$\widetilde{A}_{ij} \equiv e^{-4\varphi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right), \text{ and} \tag{7.4}$$

$$\widetilde{\Gamma}^i \equiv \widetilde{\gamma}^{mn}\widetilde{\Gamma}^i{}_{mn}. \tag{7.5}$$

The BSSN evolution equations are, then,

$$\partial_t\varphi = -\frac{1}{6}\alpha K + \frac{1}{6}(\partial_i\beta^i) + \beta^i(\partial_i\varphi), \tag{7.6}$$

$$\partial_t K = \alpha\widetilde{A}_{ij}\widetilde{A}^{ij} + \frac{1}{3}\alpha K^2 - D_iD^i\alpha + \beta^i(\partial_i K), \tag{7.7}$$

$$\partial_t\widetilde{\gamma}_{ij} = -2\alpha\widetilde{A}_{ij} - \frac{2}{3}\widetilde{\gamma}_{ij}(\partial_\ell\beta^\ell) + \widetilde{\gamma}_{j\ell}(\partial_i\beta^\ell) + \widetilde{\gamma}_{i\ell}(\partial_j\beta^\ell) + \beta^\ell(\partial_\ell\widetilde{\gamma}_{ij}), \tag{7.8}$$

$$\begin{aligned}\partial_t\widetilde{A}_{ij} = {} & \alpha K\widetilde{A}_{ij} - 2\alpha\widetilde{A}_{i\ell}\widetilde{A}^\ell{}_j + \alpha e^{-4\varphi}R_{ij}{}^{\mathrm{TF}} - e^{-4\varphi}(D_iD_j\alpha)^{\mathrm{TF}} - \frac{2}{3}\widetilde{A}_{ij}(\partial_\ell\beta^\ell) \\ & + (\partial_i\beta^\ell)\widetilde{A}_{j\ell} + (\partial_j\beta^\ell)\widetilde{A}_{i\ell} + \beta^\ell(\partial_\ell\widetilde{A}_{ij}),\end{aligned} \tag{7.9}$$

$$\begin{aligned}\partial_t\widetilde{\Gamma}^i = {} & 2\alpha\left\{6(\partial_j\varphi)\widetilde{A}^{ij} + \widetilde{\Gamma}^i{}_{j\ell}\widetilde{A}^{j\ell} - \frac{2}{3}\widetilde{\gamma}^{ij}(\partial_j K)\right\} - 2(\partial_j\alpha)\widetilde{A}^{ij} + \frac{2}{3}\widetilde{\Gamma}^i(\partial_j\beta^j) + \frac{1}{3}\widetilde{\gamma}^{ij}(\partial_\ell\partial_j\beta^\ell) \\ & + \beta^\ell(\partial_\ell\widetilde{\Gamma}^i) - \widetilde{\Gamma}^j(\partial_j\beta^i) + \widetilde{\gamma}^{j\ell}(\partial_j\partial_\ell\beta^i),\end{aligned} \tag{7.10}$$

where $^{\mathrm{TF}}$ denotes the trace-free part. The Ricci tensor in the BSSN system is normally calculated as

$$R_{ij} \equiv \widetilde{R}_{ij} + R^\varphi_{ij}, \tag{7.11}$$

where

$$\widetilde{R}_{ij} \equiv \widetilde{\gamma}_{n(i}\partial_{j)}\widetilde{\Gamma}^n + \widetilde{\gamma}^{\ell m}(2\,{}^{(3)}\widetilde{\Gamma}^k{}_{\ell(i}\,{}^{(3)}\widetilde{\Gamma}_{j)km} + {}^{(3)}\widetilde{\Gamma}_{n\ell j}\,{}^{(3)}\widetilde{\Gamma}^n{}_{im}) - \frac{1}{2}\widetilde{\gamma}^{m\ell}(\partial_\ell\partial_m\widetilde{\gamma}_{ij}) + \widetilde{\Gamma}^n\,{}^{(3)}\widetilde{\Gamma}_{(ij)n}, \tag{7.12}$$

$$R^{\varphi}_{ij} \equiv -2\widetilde{D}_i\widetilde{D}_j\varphi + 4(\widetilde{D}_i\varphi)(\widetilde{D}_j\varphi) - 2\widetilde{\gamma}_{ij}\widetilde{D}_m\widetilde{D}^m\varphi - 4\widetilde{\gamma}_{ij}(\widetilde{D}^m\varphi)(\widetilde{D}_m\varphi). \tag{7.13}$$

The BSSN system has five constraint equations. The *kinematic* constraint equations, which are the Hamiltonian constraint equation and the momentum constraint equations ($\widetilde{\mathcal{H}}$-constraint and $\widetilde{\mathcal{M}}$-constraint, hereafter), are expressed in terms of the BSSN basic variables as

$$\widetilde{\mathcal{H}} \equiv e^{-4\varphi}\widetilde{R} - 8e^{-4\varphi}(\widetilde{D}_i\widetilde{D}^i\varphi + (\widetilde{D}^m\varphi)(\widetilde{D}_m\varphi)) + \frac{2}{3}K^2 - \widetilde{A}_{ij}\widetilde{A}^{ij} - \frac{2}{3}\widetilde{\mathcal{A}}K \approx 0, \tag{7.14}$$

$$\widetilde{\mathcal{M}}_i \equiv -\frac{2}{3}\widetilde{D}_iK + 6(\widetilde{D}_j\varphi)\widetilde{A}^j{}_i + \widetilde{D}_j\widetilde{A}^j{}_i - 2(\widetilde{D}_i\varphi)\widetilde{\mathcal{A}} \approx 0, \tag{7.15}$$

respectively, where $\widetilde{D}_i$ is the covariant derivative associated with $\widetilde{\gamma}_{ij}$ and $\widetilde{R} = \widetilde{\gamma}^{ij}\widetilde{R}_{ij}$. Because of the introduction of new variables, there are additional *algebraic* constraint equations:

$$\widetilde{\mathcal{G}}^i \equiv \widetilde{\Gamma}^i - \widetilde{\gamma}^{j\ell}\,{}^{(3)}\widetilde{\Gamma}^i{}_{j\ell} \approx 0, \tag{7.16}$$

$$\widetilde{\mathcal{A}} \equiv \widetilde{A}^{ij}\widetilde{\gamma}_{ij} \approx 0, \tag{7.17}$$

$$\widetilde{\mathcal{S}} \equiv \widetilde{\gamma} - 1 \approx 0, \tag{7.18}$$

which we call the $\widetilde{\mathcal{G}}$-, $\widetilde{\mathcal{A}}$-, and $\widetilde{\mathcal{S}}$-constraints, respectively, hereafter. If the algebraic constraint equations, (7.16)-(7.18), are not satisfied, the BSSN formulation and ADM formulation are not equivalent mathematically.

7.1.2 C^2-adjusted BSSN Formulation

The C^2-adjusted BSSN evolution equations are formally written as

$$\partial_t\varphi = (7.6) - \lambda_\varphi\left(\frac{\delta\widetilde{C}^2}{\delta\varphi}\right), \tag{7.19}$$

$$\partial_tK = (7.7) - \lambda_K\left(\frac{\delta\widetilde{C}^2}{\delta K}\right), \tag{7.20}$$

$$\partial_t\widetilde{\gamma}_{ij} = (7.8) - \lambda_{\widetilde{\gamma}ijmn}\left(\frac{\delta\widetilde{C}^2}{\delta\widetilde{\gamma}_{mn}}\right), \tag{7.21}$$

$$\partial_t\widetilde{A}_{ij} = (7.9) - \lambda_{\widetilde{A}ijmn}\left(\frac{\delta\widetilde{C}^2}{\delta\widetilde{A}_{mn}}\right), \tag{7.22}$$

$$\partial_t\widetilde{\Gamma}^i = (7.10) - \lambda^{ij}_{\widetilde{\Gamma}}\left(\frac{\delta\widetilde{C}^2}{\delta\widetilde{\Gamma}^j}\right), \tag{7.23}$$

where all the coefficients λ_φ, λ_K, $\lambda_{\widetilde{\gamma}ijmn}$, $\lambda_{\widetilde{A}ijmn}$, and $\lambda^{ij}_{\widetilde{\Gamma}}$ are positive definite. $\widetilde{C}^2$ is a function of the constraints $\widetilde{\mathcal{H}}$, $\widetilde{\mathcal{M}}_i$, $\widetilde{\mathcal{G}}^i$, $\widetilde{\mathcal{A}}$, and $\widetilde{\mathcal{S}}$, which we set as

$$\widetilde{C}^2 = \int(\widetilde{\mathcal{H}}^2 + \gamma^{ij}\widetilde{\mathcal{M}}_i\widetilde{\mathcal{M}}_j + c_G\gamma_{ij}\widetilde{\mathcal{G}}^i\widetilde{\mathcal{G}}^j + c_A\widetilde{\mathcal{A}}^2 + c_S\widetilde{\mathcal{S}}^2)d^3x, \tag{7.24}$$

where, c_G, c_A, and c_S are Boolean parameters (0 or 1). These three parameters are introduced to prove the necessity of the algebraic constraint terms in (7.24).

The adjusted terms are

$$\begin{aligned}\frac{\delta\widetilde{C}^2}{\delta\varphi} &= 2\bar{H}_1\widetilde{\mathcal{H}} - 2(\partial_a\bar{H}_2^a)\widetilde{\mathcal{H}} - 2\bar{H}_2^a\partial_a\widetilde{\mathcal{H}} + 2(\partial_a\partial_b\bar{H}_3^{ab})\widetilde{\mathcal{H}} + 2(\partial_a\bar{H}_3^{ab})\partial_b\widetilde{\mathcal{H}} + 2(\partial_b\bar{H}_3^{ab})\partial_a\widetilde{\mathcal{H}} \\ &\quad + 2\bar{H}_3^{ab}\partial_a\partial_b\widetilde{\mathcal{H}} - 2(\partial_a\bar{M}_{1i}{}^a)e^{-4\varphi}\widetilde{\gamma}^{ij}\widetilde{\mathcal{M}}_j + 8\bar{M}_{1i}{}^a e^{-4\varphi}(\partial_a\varphi)\widetilde{\gamma}^{ij}\widetilde{\mathcal{M}}_j \\ &\quad - 2\bar{M}_{1i}{}^a e^{-4\varphi}(\partial_a\widetilde{\gamma}^{ij})\widetilde{\mathcal{M}}_j - 2\bar{M}_{1i}{}^a e^{-4\varphi}\widetilde{\gamma}^{ij}\partial_a\widetilde{\mathcal{M}}_j - 4\widetilde{\gamma}^{ij}e^{-4\varphi}\widetilde{\mathcal{M}}_i\widetilde{\mathcal{M}}_j \\ &\quad + 4c_G e^{4\varphi}\widetilde{\gamma}_{ij}\widetilde{\mathcal{G}}^i\widetilde{\mathcal{G}}^j, \end{aligned} \tag{7.25}$$

$$\begin{aligned}\frac{\delta\widetilde{C}^2}{\delta K} &= 2\bar{H}_4\widetilde{\mathcal{H}} - 2(\partial_\ell\bar{M}_{2i}{}^\ell)e^{-4\varphi}\widetilde{\gamma}^{ij}\widetilde{\mathcal{M}}_j + 8\bar{M}_{2i}{}^\ell e^{-4\varphi}(\partial_\ell\varphi)\widetilde{\gamma}^{ij}\widetilde{\mathcal{M}}_j - 2\bar{M}_{2i}{}^\ell e^{-4\varphi}(\partial_\ell\widetilde{\gamma}^{ij})\widetilde{\mathcal{M}}_j \\ &\quad - 2\bar{M}_{2i}{}^\ell e^{-4\varphi}\widetilde{\gamma}^{ij}\partial_\ell\widetilde{\mathcal{M}}_j, \end{aligned} \tag{7.26}$$

$$\begin{aligned}\frac{\delta\widetilde{C}^2}{\delta\widetilde{\gamma}_{mn}} &= 2\bar{H}_5^{mn}\widetilde{\mathcal{H}} - 2(\partial_i\bar{H}_6^{imn})\widetilde{\mathcal{H}} - 2\bar{H}_6^{imn}\partial_i\widetilde{\mathcal{H}} + 2(\partial_i\partial_j\bar{H}_7^{ijmn})\widetilde{\mathcal{H}} + 2(\partial_i\bar{H}_7^{ijmn})\partial_j\widetilde{\mathcal{H}} \\ &\quad + 2(\partial_j\bar{H}_7^{ijmn})\partial_i\widetilde{\mathcal{H}} + 2\bar{H}_7^{ijmn}\partial_i\partial_j\widetilde{\mathcal{H}} + 2\bar{M}_{3i}{}^{mn}e^{-4\varphi}\widetilde{\gamma}^{ij}\widetilde{\mathcal{M}}_j - 2(\partial_c\bar{M}_{4i}{}^{cmn})e^{-4\varphi}\widetilde{\gamma}^{ij}\widetilde{\mathcal{M}}_j \\ &\quad + 8\bar{M}_{4i}{}^{cmn}e^{-4\varphi}(\partial_c\varphi)\widetilde{\gamma}^{ij}\widetilde{\mathcal{M}}_j - 2\bar{M}_{4i}{}^{cmn}e^{-4\varphi}(\partial_c\widetilde{\gamma}^{ij})\widetilde{\mathcal{M}}_j - 2\bar{M}_{4i}{}^{cmn}e^{-4\varphi}\widetilde{\gamma}^{ij}\partial_c\widetilde{\mathcal{M}}_j \\ &\quad - e^{-4\varphi}\widetilde{\gamma}^{im}\widetilde{\gamma}^{jn}\widetilde{\mathcal{M}}_i\widetilde{\mathcal{M}}_j + 2c_G G_1^{imn}e^{4\varphi}\widetilde{\gamma}_{ij}\widetilde{\mathcal{G}}^j - 2c_G(\partial_\ell G_2^{imn\ell})e^{4\varphi}\widetilde{\gamma}_{ij}\widetilde{\mathcal{G}}^j \\ &\quad - 8c_G G_2^{imn\ell}e^{4\varphi}(\partial_\ell\varphi)\widetilde{\gamma}_{ij}\widetilde{\mathcal{G}}^j - 2c_G G_2^{imn\ell}e^{4\varphi}(\partial_\ell\widetilde{\gamma}_{ij})\widetilde{\mathcal{G}}^j - 2c_G G_2^{imn\ell}e^{4\varphi}\widetilde{\gamma}_{ij}\partial_\ell\widetilde{\mathcal{G}}^j \\ &\quad + c_G e^{4\varphi}\widetilde{\mathcal{G}}^m\widetilde{\mathcal{G}}^n + 2c_A A_1^{mn}\widetilde{\mathcal{A}} + 2c_S S_1^{mn}\widetilde{\mathcal{S}}, \end{aligned} \tag{7.27}$$

$$\begin{aligned}\frac{\delta\widetilde{C}^2}{\delta\widetilde{A}_{mn}} &= 2\bar{H}_8^{mn}\widetilde{\mathcal{H}} + 2e^{-4\varphi}\widetilde{\gamma}^{ij}\bar{M}_{5i}{}^{mn}\widetilde{\mathcal{M}}_j - 2(\partial_c\bar{M}_{6i}{}^{cmn})e^{-4\varphi}\widetilde{\gamma}^{ij}\widetilde{\mathcal{M}}_j \\ &\quad + 8\bar{M}_{6i}{}^{cmn}e^{-4\varphi}(\partial_c\varphi)\widetilde{\gamma}^{ij}\widetilde{\mathcal{M}}_j - 2\bar{M}_{6i}{}^{cmn}e^{-4\varphi}(\partial_c\widetilde{\gamma}^{ij})\widetilde{\mathcal{M}}_j - 2\bar{M}_{6i}{}^{cmn}e^{-4\varphi}\widetilde{\gamma}^{ij}\partial_c\widetilde{\mathcal{M}}_j \\ &\quad + 2c_A A_2^{mn}\widetilde{\mathcal{A}}, \end{aligned} \tag{7.28}$$

$$\frac{\delta\widetilde{C}^2}{\delta\widetilde{\Gamma}^a} = 2\bar{H}_{9a}\widetilde{\mathcal{H}} - 2(\partial_b\bar{H}_{10a}^b)\widetilde{\mathcal{H}} - 2\bar{H}_{10a}^b\partial_b\widetilde{\mathcal{H}} + 2c_G G_{3a}^i e^{4\varphi}\widetilde{\gamma}_{ij}\widetilde{\mathcal{G}}^j, \tag{7.29}$$

where

$$\bar{H}_1 = -4e^{-4\varphi}\widetilde{R} + 32e^{-4\varphi}\{\widetilde{D}^i\widetilde{D}_i\varphi + (\widetilde{D}_i\varphi)(\widetilde{D}^i\varphi)\}, \tag{7.30}$$

$$\bar{H}_2^a = 8e^{-4\varphi}(\widetilde{\gamma}^{ij}\widetilde{\Gamma}^a{}_{ij} - 2\widetilde{D}^a\varphi), \tag{7.31}$$

$$\bar{H}_3^{ab} = -8e^{-4\varphi}\widetilde{\gamma}^{ab}, \tag{7.32}$$

$$\bar{H}_4 = (4/3)K - (2/3)\widetilde{\gamma}^{ij}\widetilde{A}_{ij}, \tag{7.33}$$

$$\begin{aligned}\bar{H}_5^{mn} &= -e^{-4\varphi}\widetilde{R}^{mn} + e^{-4\varphi}(\partial_j\widetilde{\Gamma}^{(m})\widetilde{\gamma}^{n)j} - 2e^{-4\varphi}\widetilde{\Gamma}^{km}{}_j\widetilde{\Gamma}^{jn}{}_k - 2e^{-4\varphi}\widetilde{\Gamma}^{i\ell(m}\widetilde{\Gamma}^{n)}{}_{\ell i} \\ &\quad - e^{-4\varphi}\widetilde{\Gamma}^{ami}\widetilde{\Gamma}_{ai}{}^n - e^{-4\varphi}\widetilde{\Gamma}^{mi\ell}\widetilde{\Gamma}^n{}_{\ell i} + \frac{1}{2}e^{-4\varphi}\widetilde{\gamma}_{ij,a\ell}\widetilde{\gamma}^{ij}\widetilde{\gamma}^{am}\widetilde{\gamma}^{\ell n} + 8e^{-4\varphi}\widetilde{D}^m\widetilde{D}^n\varphi \\ &\quad - 8e^{-4\varphi}(\widetilde{D}^{(m}\varphi)\widetilde{\Gamma}^{n)}{}_{ij}\widetilde{\gamma}^{ij} + 8e^{-4\varphi}(\widetilde{D}^m\varphi)(\widetilde{D}^n\varphi) \\ &\quad + 2\widetilde{A}^{mb}\widetilde{A}^n{}_b + (2/3)\widetilde{A}^{mn}K, \end{aligned} \tag{7.34}$$

$$\bar{H}_6^{\ell mn} = e^{-4\varphi}\{\widetilde{\Gamma}^{\ell mn} + 2\widetilde{\Gamma}^{(nm)\ell} + (1/2)\Gamma^{\ell}\widetilde{\gamma}^{mn} + 8\widetilde{\gamma}^{\ell(m}(\widetilde{D}^{n)}\varphi) - 4\widetilde{\gamma}^{mn}\widetilde{D}^{\ell}\varphi\}, \tag{7.35}$$

$$\bar{H}_7^{ijmn} = -(1/2)e^{-4\varphi}\widetilde{\gamma}^{mn}\widetilde{\gamma}^{ij}, \tag{7.36}$$

$$\bar{H}_8^{mn} = -2\widetilde{A}^{mn} - (2/3)\widetilde{\gamma}^{mn}K, \tag{7.37}$$

$$\bar{H}_{9a} = (1/2)e^{-4\varphi}\widetilde{\gamma}^{ij}\widetilde{\gamma}_{ij,a}, \tag{7.38}$$

$$\bar{H}_{10a}^{b} = e^{-4\varphi}\delta^{b}{}_{a}, \tag{7.39}$$

$$\bar{M}_{1i}{}^{a} = 6\widetilde{A}^{a}{}_{i} - 2\widetilde{A}_{mn}\widetilde{\gamma}^{mn}\delta^{a}{}_{i}, \tag{7.40}$$

$$\bar{M}_{2i}{}^{j} = -(2/3)\delta^{j}{}_{i}, \tag{7.41}$$

$$\bar{M}_{3i}{}^{mn} = -6(\widetilde{D}^{(m}\varphi)\widetilde{A}^{n)}{}_{i} + 2(\widetilde{D}_{i}\varphi)\widetilde{A}^{mn} - \widetilde{D}^{(m}\widetilde{A}^{n)}{}_{i} + \widetilde{A}^{a(n}\widetilde{\Gamma}^{m)}{}_{ai} + \widetilde{A}_{i}{}^{(m}\widetilde{\Gamma}^{n)}{}_{j\ell}\widetilde{\gamma}^{j\ell}, \tag{7.42}$$

$$\bar{M}_{4i}{}^{cmn} = -\widetilde{\gamma}^{c(n}\widetilde{A}^{m)}{}_{i} + (1/2)\widetilde{\gamma}^{mn}\widetilde{A}^{c}{}_{i} - (1/2)\widetilde{A}^{nm}\delta^{c}{}_{i}, \tag{7.43}$$

$$\bar{M}_{5i}{}^{mn} = 6(\widetilde{D}^{(m}\varphi)\delta^{n)}{}_{i} - 2(\widetilde{D}_{i}\varphi)\widetilde{\gamma}^{mn} - \delta_{i}{}^{(m}\widetilde{\Gamma}^{n)}{}_{j\ell}\widetilde{\gamma}^{j\ell} + (1/2)\widetilde{\gamma}^{mn}{}_{,i}, \tag{7.44}$$

$$\bar{M}_{6i}{}^{cmn} = \widetilde{\gamma}^{c(m}\delta^{n)}{}_{i}, \tag{7.45}$$

$$G_1^{iab} = \widetilde{\Gamma}^{iab} + \widetilde{\gamma}^{i(b}\widetilde{\Gamma}^{a)}{}_{mn}\widetilde{\gamma}^{mn}, \tag{7.46}$$

$$G_2^{iab\ell} = -\widetilde{\gamma}^{\ell(b}\widetilde{\gamma}^{a)i} + (1/2)\widetilde{\gamma}^{ab}\widetilde{\gamma}^{i\ell}, \tag{7.47}$$

$$G_{3j}^{i} = \delta^{i}{}_{j}, \tag{7.48}$$

$$A_1^{ab} = -\widetilde{A}^{ab}, \tag{7.49}$$

$$A_2^{ab} = \widetilde{\gamma}^{ab}, \tag{7.50}$$

$$S_1^{ab} = (1/2)\varepsilon^{ajk}\varepsilon^{bn\ell}\widetilde{\gamma}_{jn}\widetilde{\gamma}_{k\ell}. \tag{7.51}$$

7.2 Constraint Propagation Equations

Now we discuss the effect of the algebraic constraints. For simplicity, we set $\lambda_{\widetilde{\gamma}ijmn} = \lambda_{\widetilde{\gamma}}\delta_{im}\delta_{jn}$, $\lambda_{\widetilde{A}ijmn} = \lambda_{\widetilde{A}}\delta_{im}\delta_{jn}$, and $\lambda^{ij}_{\widetilde{\Gamma}} = \lambda_{\widetilde{\Gamma}}\delta^{ij}$. The constraint propagation equations of the C^2-adjusted BSSN formulation in flat spacetime are

$$\partial_t\widetilde{\mathcal{H}} = [\text{Original Terms}] + \left(-128\lambda_\varphi\Delta^2 - \frac{3}{2}\lambda_{\widetilde{\gamma}}\Delta^2 + 2\lambda_{\widetilde{\Gamma}}\Delta\right)\widetilde{\mathcal{H}} + c_G\left(-\frac{1}{2}\lambda_{\widetilde{\gamma}}\Delta\partial_m - 2\lambda_{\widetilde{\Gamma}}\partial_m\right)\widetilde{\mathcal{G}}^m + 3c_S\lambda_{\widetilde{\gamma}}\Delta\widetilde{\mathcal{S}}, \tag{7.52}$$

$$\partial_t\widetilde{\mathcal{M}}_a = [\text{Original Terms}] + \left\{\frac{8}{9}\lambda_K\delta^{bc}\partial_a\partial_b + \lambda_{\widetilde{A}}\Delta\delta_a{}^c + \lambda_{\widetilde{A}}\delta^{bc}\partial_a\partial_b\right\}\widetilde{\mathcal{M}}_c - 2c_A\lambda_{\widetilde{A}}\partial_a\widetilde{\mathcal{A}}, \tag{7.53}$$

$$\partial_t\widetilde{\mathcal{G}}^a = [\text{Original Terms}] + \delta^{ab}\left(\frac{1}{2}\lambda_{\widetilde{\gamma}}\partial_b\Delta + 2\lambda_{\widetilde{\Gamma}}\partial_b\right)\widetilde{\mathcal{H}} + c_G\left(\lambda_{\widetilde{\gamma}}\Delta\delta^a{}_b + \frac{1}{2}\lambda_{\widetilde{\gamma}}\delta^{ac}\partial_c\partial_b - 2\lambda_{\widetilde{\Gamma}}\delta^a{}_b\right)\widetilde{\mathcal{G}}^b - c_S\lambda_{\widetilde{\gamma}}\delta^{ab}\partial_b\widetilde{\mathcal{S}}, \tag{7.54}$$

$$\partial_t\widetilde{\mathcal{A}} = [\text{Original Terms}] + 2\lambda_{\widetilde{A}}\delta^{ij}(\partial_i\widetilde{\mathcal{M}}_j) - 6c_A\lambda_{\widetilde{A}}\widetilde{\mathcal{A}}, \tag{7.55}$$

$$\partial_t\widetilde{\mathcal{S}} = [\text{Original Terms}] + 3\lambda_{\widetilde{\gamma}}\Delta\widetilde{\mathcal{H}} + c_G\lambda_{\widetilde{\gamma}}\partial_\ell\widetilde{\mathcal{G}}^\ell - 6c_S\lambda_{\widetilde{\gamma}}\widetilde{\mathcal{S}}, \tag{7.56}$$

From (7.52)-(7.56), we see that the constraints affect each others. The constraint propagation equations of the algebraic constraints, (7.54)-(7.56), include $c_G(\lambda_{\widetilde{\gamma}}\Delta\delta^a{}_b - 2\lambda_{\widetilde{\Gamma}}\delta^a{}_b)\widetilde{\mathcal{G}}^b$, $-6c_A\lambda_{\widetilde{A}}\widetilde{\mathcal{A}}$, and $-6c_S\lambda_{\widetilde{\gamma}}\widetilde{\mathcal{S}}$, respectively. These terms contribute to reduce the violations of each constraint if c_G, c_A, and c_S are non-zero. Therefore, we adopt $c_G = c_A = c_S = 1$ in (7.24);

$$C^2 = \int\left(\widetilde{\mathcal{H}}^2 + \gamma^{ij}\widetilde{\mathcal{M}}_i\widetilde{\mathcal{M}}_j + \gamma_{ij}\widetilde{\mathcal{G}}^i\widetilde{\mathcal{G}}^j + \widetilde{\mathcal{A}}^2 + \widetilde{\mathcal{S}}^2\right)d^3x. \tag{7.57}$$

This discussion is considered only from the viewpoint of the inclusion of the diffusion terms. In order to validate this decision, we perform some numerical examples in Sec.10.

7.3 $\widetilde{A}$-adjusted BSSN Formulation

In [27], two of the authors reported some examples of adjusted systems for the BSSN formulation. The authors investigated the signatures of eigenvalues of the coefficient matrix of the constraint propagation equations, and concluded three of the examples to be the best candidates for the adjustment. The actual numerical tests were performed later [30] using the gauge-wave, linear-wave, and polarized Gowdy wave testbeds. The most robust system among the three examples for these three testbeds was the $\widetilde{A}$-adjusted BSSN formulation, which replaces (7.9) in the standard BSSN system with

$$\partial_t\widetilde{A}_{ij} = (7.9) + \kappa_A\alpha\widetilde{D}_{(i}\widetilde{\mathcal{M}}_{j)}, \tag{7.58}$$

where κ_A is a constant. If κ_A is set as positive, the violations of the constraints are expected to be damped in flat spacetime [27]. We also use the $\widetilde{A}$-adjusted BSSN system for comparison in the following numerical tests.

The constraint propagation equations of this system are

$$\partial_t \widetilde{\mathcal{H}} = [\text{Original Terms}], \tag{7.59}$$

$$\partial_t \widetilde{\mathcal{M}}_i = [\text{Original Terms}] + (1/2)\kappa_A \Delta \widetilde{\mathcal{M}}_i, \tag{7.60}$$

$$\partial_t \widetilde{\mathcal{G}}^i = [\text{Original Terms}], \tag{7.61}$$

$$\partial_t \widetilde{\mathcal{A}} = [\text{Original Terms}] + \kappa_A \delta^{ij} \partial_i \widetilde{\mathcal{M}}_j, \tag{7.62}$$

$$\partial_t \widetilde{\mathcal{S}} = [\text{Original Terms}], \tag{7.63}$$

where Δ is the Laplacian operator in flat space. *Original Terms* refers to the right-hand side of the constraint propagation equations for the standard BSSN formulation. Full expressions for the terms are given in the appendix of [27].

Part III

Numerical Simulations

Chapter 8

Settings

8.1 Gauge-wave Testbed

The metric of the gauge-wave test is

$$ds^2 = -Hdt^2 + Hdx^2 + dy^2 + dz^2, \tag{8.1}$$

where

$$H = 1 - A\sin(2\pi(x-t)/d), \tag{8.2}$$

which describes a sinusoidal gauge wave of amplitude A propagating along the x-axis. The nontrivial extrinsic curvature is

$$K_{xx} = -\frac{\pi A}{d}\frac{\cos(\frac{2\pi(x-t)}{d})}{\sqrt{1 - A\sin\frac{2\pi(x-t)}{d}}}. \tag{8.3}$$

Following [41], we chose the numerical domain and parameters as follows:

- Gauge-wave parameters: $d = 1$ and $A = 10^{-2}$.
- Simulation domain: $x \in [-0.5, 0.5]$, $y = z = 0$.
- Grid: $x^n = -0.5 + (n - 1/2)dx$ with $n = 1, \cdots, 100$, where $dx = 1/100$.
- Time step: $dt = 0.25dx$.
- Boundary conditions: Periodic boundary condition in x-direction and planar symmetry in y- and z-directions.
- Gauge conditions:

$$\partial_t\alpha = -\alpha^2 K, \quad \beta^i = 0. \tag{8.4}$$

- Scheme: second-order iterative Crank-Nicolson.

8.2 Gowdy-wave Testbed

Metric and Parameters

The metric of the polarized Gowdy wave is given by

$$ds^2 = t^{-1/2}e^{\lambda/2}(-dt^2 + dx^2) + t(e^P dy^2 + e^{-P}dz^2), \tag{8.5}$$

where P and λ are functions of x and t. The forward direction of the time coordinate t corresponds to the expanding universe, and $t = 0$ corresponds to the cosmological singularity.

For simple forms of the solutions, P and λ are given by

$$P = J_0(2\pi t)\cos(2\pi x), \tag{8.6}$$

$$\begin{aligned}\lambda = &-2\pi t J_0(2\pi t)J_1(2\pi t)\cos^2(2\pi x) + 2\pi^2 t^2[J_0^2(2\pi t) \\ &+ J_1^2(2\pi t)] - (1/2)\{(2\pi)^2[J_0^2(2\pi) + J_1^2(2\pi)] \\ &- 2\pi J_0(2\pi)J_1(2\pi)\},\end{aligned} \tag{8.7}$$

where J_n is the Bessel function.

Following [41], a new time coordinate τ, which satisfies harmonic slicing, is obtained by the coordinate transformation

$$t(\tau) = ke^{c\tau}, \tag{8.8}$$

where k and c are arbitrary constants. We also follow [41] by setting k, c, and the initial time t_0 as

$$k \sim 9.67076981276405, \quad c \sim 0.002119511921460, \tag{8.9}$$

$$t_0 = 9.87532058290982, \tag{8.10}$$

so that the lapse function in the new time coordinate is unity and $t = \tau$ at the initial time.

We also use the following parameters specified in [41].

- Simulation domain: $x \in [-0.5, 0.5], y = z = 0$.
- Grid: $x_n = -0.5 + (n - (1/2))dx$, $n = 1, \cdots, 100$, where $dx = 1/100$.
- Time step: $dt = 0.25dx$.
- Boundary conditions: Periodic boundary condition in x-direction and planar symmetry in y- and z-directions.
- Gauge conditions: $\partial_t\alpha = -\alpha^2 K$, $\beta^i = 0$.
- Scheme: second-order iterative Crank-Nicolson.

Chapter 9

Simulations with C^2-adjusted ADM Formulation

9.1 Constraint violations and the damping of the violations

Figure 9.1 shows the L2 norm of the Hamiltonian constraint and momentum constraints with a function of backward time $(-t)$ in the case of the standard ADM formulation, (6.1)-(6.2). We see the violations of the momentum constraints are larger than that of the Hamiltonian constraint at the initial stage, and both grow larger with time. The behavior is well-known, and the starting point of the formulation problem.

We, then, compare the evolutions with three formulations: (a) the standard ADM formulation (6.1)-(6.2), (b) Detweiler's formulation (6.41)-(6.42), and (c) the C^2-adjusted ADM formulation (6.5)-(6.6). We tuned the parameters L in (a), and $\kappa_{\gamma ijmn}$ and κ_{Kijmn} in (c) within the expected ranges from the eigenvalue analyses. In the formulation (c), we set $\kappa_{\gamma ijmn} = \kappa_\gamma \delta_{im}\delta_{jn}$ and $\kappa_{Kijmn} = \kappa_K \delta_{im}\delta_{jn}$ for simplicity, and optimized κ_γ and κ_K in their positive ranges. We use $L = -10^{+1.9}$ and $(\kappa_\gamma, \kappa_k) = (-10^{-9.0}, -10^{-3.5})$ for the plots, since the violation of constraints are minimized at $t = -1000$ for those evolutions. Note that the signatures of $(\kappa_\gamma, \kappa_K)$ and L are reversed from the expected one in Sec. 5.3 and Sec. 6.3, respectively, since we integrate time backward.

We plot the L2 norms of C^2 of these three formulations in Figure 9.2. We see the constraint violations of (a)(the standard ADM formulation) and (b)(Detweiler's formulation) grow larger with time, while that of (c)(C^2-adjusted ADM formulation) almost coincide with (a) until $t = -500$, then the violation of (c) begins smaller than (a). The L2 violation level of (c), then, keeps its magnitude at most $O(10^{-3})$, while those of (a) and (b) monotonically grow larger with oscillations. Figure 9.2 shows up to $t = -1000$, but we confirmed this behavior up to $t = -1700$.

Figure 9.2 tells us that the effects of Detweiler's adjustment appear at the initial stage, while C^2-adjustment contributes at the later stage. The time difference can be seen also from the magnitudes of adjustment terms in each evolution equations, which we show in Figure 9.3. The lines (b1), (b2), (c1), and (c2) are the norms of $D_{\gamma ij}$ in (6.43), D_{Kij} in (6.44), $\delta C^2/\delta\gamma_{ij}$ in (6.8), and $\delta C^2/\delta K_{ij}$ in (6.9), respectively.

We see that the L2 norms of the adjusted terms of Detweiler's ADM formulation, $D_{\gamma ij}$ and D_{Kij}, decrease, while that of the C^2-adjusted ADM formulation increase. If the magnitudes of the adjusted terms are smaller, the effects of the constraint damping become small. Therefore, the L2 norm of C^2 of Detweiler's ADM formulation are not damped down in the later stage in

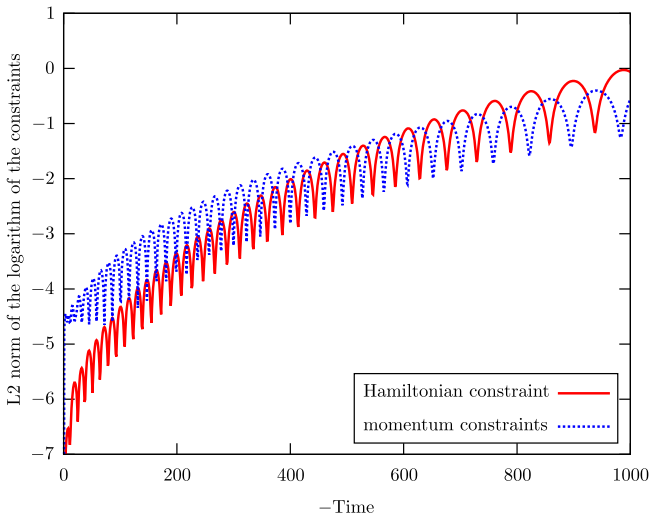

Figure 9.1: The L2 norm of the Hamiltonian and momentum constraints of the Gowdy-wave evolution using the standard ADM formulation. We see that the violation of the momentum constraints is larger initially, and both violations are growing with time.

Figure 9.2.

One possible explanation for the weak effect of Detweiler's adjustment in the later stage is the existence of the lapse function, α (and α^2, α^3), in the adjusted terms in (6.43)-(6.44). The Gowdy-wave testbed is the evolution to the initial singularity of the space-time, and the lapse function becomes smaller with evolution. Note that in previous works [12, 28], we see that the constraint violations are damped down in the simulation with Detweiler's ADM formulation, where the lapse function, α, is adopted by the geodesic condition.

In Figure 9.4, we plotted the magnitude of the original terms and the adjusted terms of C^2-adjusted ADM formulation; the first and second terms in (6.5) and (6.6). We find that there is $O(10^2)$–$O(10^5)$ of differences between them. Therefore, we conclude that the adjustments do not disturb the original ADM formulation, but control the violation of the constraints. We may understand that higher derivative terms in (6.8) and (6.9) work as artificial viscosity terms in numerics.

9.2 Parameter dependence of the C^2-adjusted ADM formulation

There are two parameters, κ_γ and κ_K, in the C^2-adjusted ADM formulation and we next study the sensitivity of these two on the damping effect to the constraint violation.

Figure 9.5 shows the dependences on κ_γ and κ_K. In Figure 9.5 (A), we fix $\kappa_K = 0$ and change κ_γ. In Figure 9.5 (B), we fix $\kappa_\gamma = 0$ and change κ_K. In Figure 9.5 (A), we see that all the simulations stop soon after the damping effect appears. On the other hand, in Figure 9.5 (B), we see that the simulations continue with constraint-damping effects. These results suggest $\kappa_K \neq 0$ or $\kappa_\gamma = 0$ is essential to keep the constraint-damping effects.

We think the trigger for stopping evolutions in the cases of Figure 9.5 (A) (when $\kappa_K = 0$) is the term $\mathbb{H}_5{}^{abcd}(\partial_a\partial_b\partial_c\partial_d\mathcal{H})$ which appears in the constraint propagation equation of the

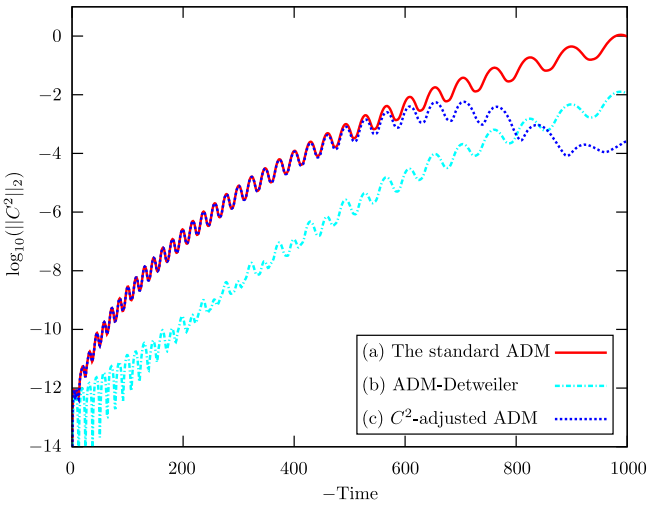

Figure 9.2: The L2 norm of the constraints, C^2, of the polarized Gowdy-wave tests with ADM and two types of adjusted formulations. The vertical axis is the logarithm of the C^2 and the horizontal axis is backward time. The solid line (a) is of the standard ADM formulation. The dot-dashed line (b) is the evolution with Detweiler's ADM with $L = -10^{+1.9}$. The dotted line (c) is the C^2-adjusted ADM with $\kappa_\gamma = -10^{-9.0}$ and $\kappa_K = -10^{-3.5}$. We see the lines (a) and (c) almost overlap until $t = -500$, then the case (c) keeps the L2 norm at the level $\leq 10^{-3}$, while the lines of (a) and (b) monotonically grow larger with oscillations. We confirmed this behavior up to $t \simeq -1700$.

Hamiltonian constraint, (6.18). We evaluated and checked each terms and found that ${\mathbb{H}_5}^{abcd}$ exponentially grows in time and dominates the other terms in (6.18) before the simulation stops. Since ${\mathbb{H}_5}^{abcd}$ is consists of $\gamma^{ij}\gamma^{mn}$ [see (6.12) and (6.23)], the time backward integration of Gowdy spacetime makes this term disastrous. So that, in this Gowdy testbed, the cases $\kappa_\gamma = 0$ reduce this trouble and keep the evolution with constraint-damping effects.

The sudden stops of evolutions in Figure 9.5 (A) can be interpreted due to a non-linear growth of "constraint shocks", since the adjusted terms are highly non-linear[1]. The robustness against a constraint-shock is hard to be proved, but the continuous evolution cases in Figure 9.5 (B) may show that a remedial example is available by tuning parameters.

[1] We appreciate the anonymous referee for pointing out this issue.

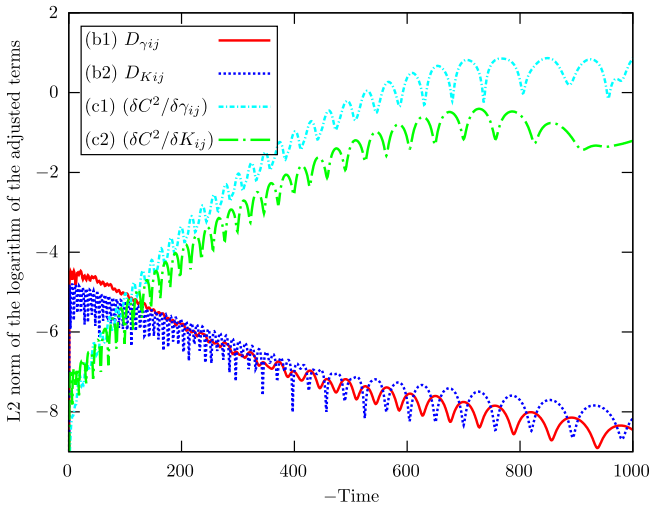

Figure 9.3: The magnitudes of the adjusted terms in each equations for the evolutions shown in Figure 9.2. The vertical axis is the logarithm of the adjusted terms. The horizontal axis is backward time. The lines (b1) and (b2) are the adjusted terms (6.43) and (6.44) respectively. The lines (c1) and (c2) are the adjusted terms (6.8) and (6.9) respectively. We see the adjustments in Detweiler-ADM [the lines (b1) and (b2)] decrease with time, which indicates that these contributions become less effective.

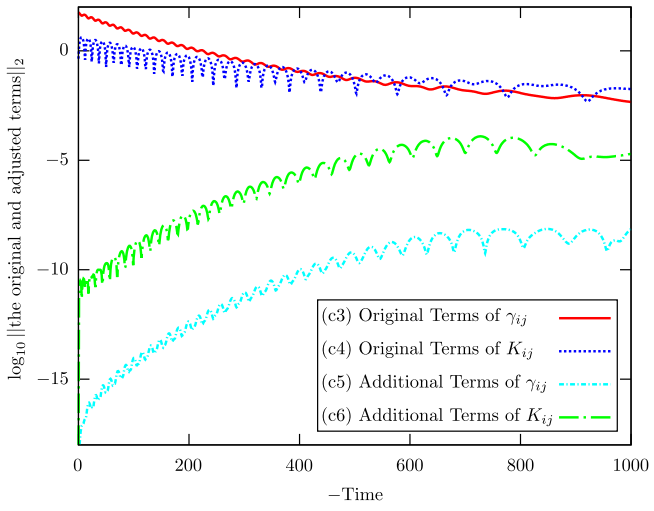

Figure 9.4: Comparison of the magnitude of the original terms and the adjusted terms of the C^2-adjusted ADM formulation, (6.5)-(6.6). The lines (c3) and (c4) are the L2 norm of the original terms [the evolution equations of g_{ij} and K_{ij}, (6.1) and (6.2)], respectively. The lines (c5) and (c6) are the L2 norm of the adjusted terms, which is the second terms of the right-hand side of (6.5) and (6.6), respectively. We see the adjusted terms are "tiny", compared with the original terms.

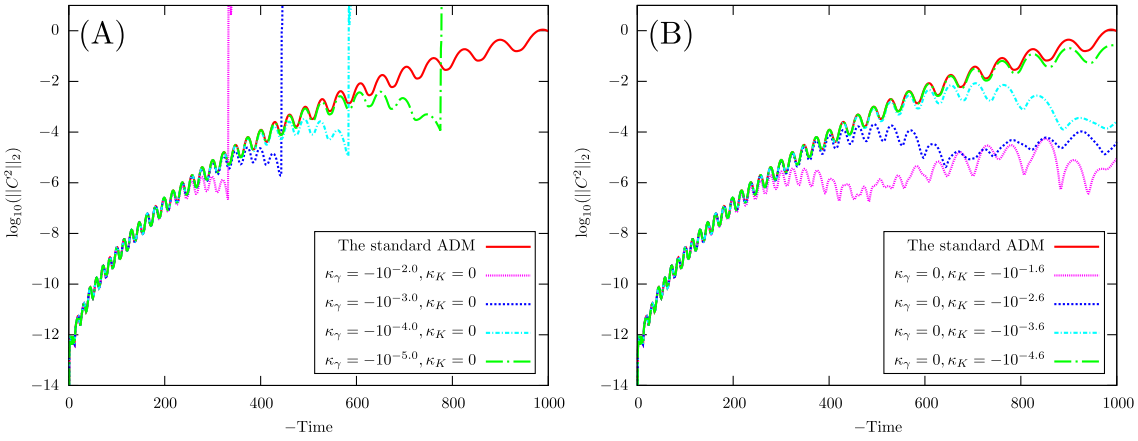

Figure 9.5: Parameter dependence of the C^2-adjusted ADM formulation. The vertical axis is the logarithm of the C^2 and the horizontal axis is backward time. The left panel (A) is the evolutions with $\kappa_K = 0$ and $\kappa_\gamma = -10^{-2.0}, -10^{-3.0}, -10^{-4.0}, -10^{-5.0}$. The right panel (B) is the cases with $\kappa_\gamma = 0$ and $\kappa_K = -10^{-1.6}, -10^{-2.6}, -10^{-3.6}, -10^{-4.6}$. In (A), we see that the simulations stop soon after the constraint dumping effect appears. In (B), we see that the simulations continue with constraint-damping effects.

Chapter 10

Simulations with C^2-adjusted BSSN Formulation

10.1 Gauge wave case

10.1.1 Constraint Violations and their Dampings

Figure 10.1 shows the violations of five constraint equations $\widetilde{\mathcal{H}}$, $\widetilde{\mathcal{M}}_i$, $\widetilde{\mathcal{G}}^i$, $\widetilde{\mathcal{A}}$, and $\widetilde{\mathcal{S}}$ for the gauge-wave evolution using the standard BSSN formulation. The violation of the $\widetilde{\mathcal{M}}$-constraint, line (A-2), is the largest during the evolution, while the violations of both the $\widetilde{\mathcal{A}}$-constraint and $\widetilde{\mathcal{S}}$-constraint are negligible. This is the starting point for improving the BSSN formulation.

Applying the adjustment procedure, the lifetime of the standard BSSN evolution is increased at least 10-fold. In Fig.10.2, we plot the L2 norm of the constraints, (7.57), of three BSSN evolutions: (A) the standard BSSN formulation (7.6)-(7.10), (B) the $\widetilde{A}$-adjusted BSSN formulation (7.6)-(7.8), (7.10), and (7.58), and (C) the C^2-adjusted BSSN formulation (7.19)-(7.23). For the standard BSSN case, we see the violation of constraint monotonically increases in the earlier stage, while other two adjusted cases keep it smaller. We can say that the C^2-adjusted formulation is the most robust one against the violation of constraints between three.

We plot the norm of each constraint equation in Fig.10.3. First, we see that the violation of the $\widetilde{\mathcal{M}}$-constraint for the two adjusted BSSN formulations [the lines (B-2) and (C-2) in Fig.10.3] are less than that of the standard BSSN formulation in Fig.10.1. This behavior would be explained from the constraint propagation equations, where we see the terms $\lambda_{\widetilde{A}}\Delta\widetilde{\mathcal{M}}_a$ and $(1/2)\kappa_A\Delta\widetilde{\mathcal{M}}_i$ in (7.53) and (7.60), respectively. These terms contribute to reduce the violations of the $\widetilde{\mathcal{M}}$-constraint. This is the main consequence of the two adjusted BSSN formulations.

Second, we also find that the violations of the $\widetilde{\mathcal{A}}$-constraint and $\widetilde{\mathcal{S}}$-constraint are larger than those in Fig.10.1. From constraint propagation equations (7.55) and (4.9), the violation of the $\widetilde{\mathcal{A}}$-constraint is triggered by the $\widetilde{\mathcal{M}}$- and $\widetilde{\mathcal{A}}$-constraints. The increase in the violations of the $\widetilde{\mathcal{A}}$-constraint is caused by the term $2\lambda_{\widetilde{A}}\delta^{ij}(\partial_i\widetilde{\mathcal{M}}_j)$. Similarly, in (7.56) and (4.10), the violation of the $\widetilde{\mathcal{S}}$-constraint is triggered by only the $\widetilde{\mathcal{A}}$-constraint since the magnitude of $\lambda_{\widetilde{\gamma}}$ is negligible. Therefore, the increase in the violation of the $\widetilde{\mathcal{S}}$-constraint is due to the violation of the $\widetilde{\mathcal{A}}$-constraint.

From (7.25) and (7.27), it can be seen that the adjusted terms of the evolution equations of φ and $\widetilde{\gamma}_{ij}$ include second-order derivative terms of the $\widetilde{\mathcal{H}}$-constraint. This means that these evolution equations include fourth-order derivative terms of the dynamical variables. In order to

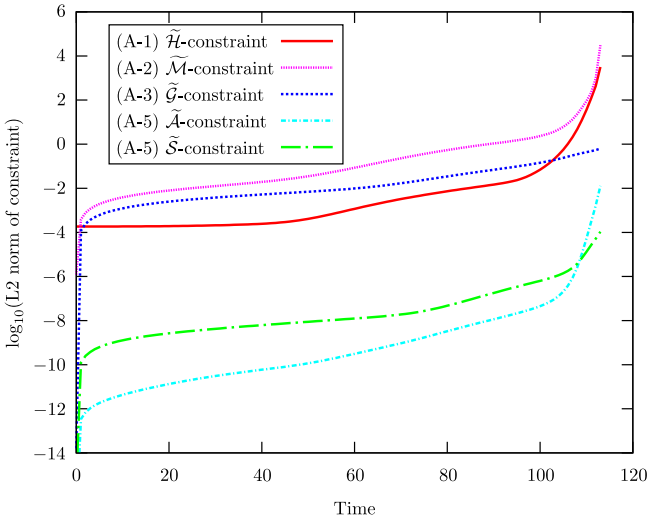

Figure 10.1: L2 norm of each constraint violation in the gauge-wave evolution using the standard BSSN formulation. The vertical axis is the logarithm of the L2 norm of the constraints and the horizontal axis is time. We see the evolution stops at $t = 110$ due to the growth of $\widetilde{\mathcal{H}}$-constraint violation.

investigate the magnitudes of the adjusted terms, we show in Fig.10.4 the ratio of the adjusted terms to that of the original terms in each evolution equation. We see that the magnitudes of the adjusted terms of φ and $\widetilde{\gamma}_{ij}$ are reasonably small.

In the simulations with the C^2-adjusted BSSN formulation, the largest violation is the $\widetilde{\mathcal{S}}$-constraint. The $\widetilde{\mathcal{S}}$-constraint depends only on the dynamical variables $\widetilde{\gamma}_{ij}$, so that there is no other choice than setting $\lambda_{\widetilde{\gamma}}$ for controlling $\widetilde{\mathcal{S}}$-constraint, as can be seen from (7.56). However, we must set $\lambda_{\widetilde{\gamma}}$ to a value as small as possible since the adjusted term of $\widetilde{\gamma}_{ij}$ includes higher derivatives of $\widetilde{\gamma}_{ij}$. Therefore, it is hard to control the $\widetilde{\mathcal{S}}$-constraint, and we have not yet found an appropriate set of parameters. This will remain as a future problem of this C^2-adjusted BSSN system.

We also investigated the sensitivity of the parameters in the C^2-adjusted BSSN evolutions. We compared evolutions with setting only one of the parameters, $(\lambda_\varphi, \lambda_K, \lambda_{\widetilde{\gamma}}, \lambda_{\widetilde{A}}, \lambda_{\widetilde{\Gamma}})$, nonzero. Since the key of the damping of the violation of constraints is the $\widetilde{\mathcal{M}}$-constraint, and $(\lambda_K, \lambda_{\widetilde{A}})$ controls the violation of $\widetilde{\mathcal{M}}$-constraint directly by (7.53), we mention here only the dependence on λ_K and $\lambda_{\widetilde{A}}$. We found that constraint-damping feature changes sensitively by both λ_K and $\lambda_{\widetilde{A}}$, among them setting $\lambda_{\widetilde{A}}$ is important to control the $\widetilde{\mathcal{M}}$-constraint violation. We see the best controlled evolution with $\lambda_{\widetilde{A}} = 10^{-3}$ than 10^{-2} and 10^{-4}.

10.1.2 Contribution of Algebraic Constraints in Definition of C^2

In Sec.7.2, we defined C^2, (7.57), including the algebraic constraints. We check this validity by turning off the algebraic constraints in (7.57) and tested. The result is shown in Fig.10.5, where we see the simulation stops at $t = 800$ due to a sudden increase in the violation of the constraints. This confirms that the algebraic constraints play an important role of damping of the violations of constraints.

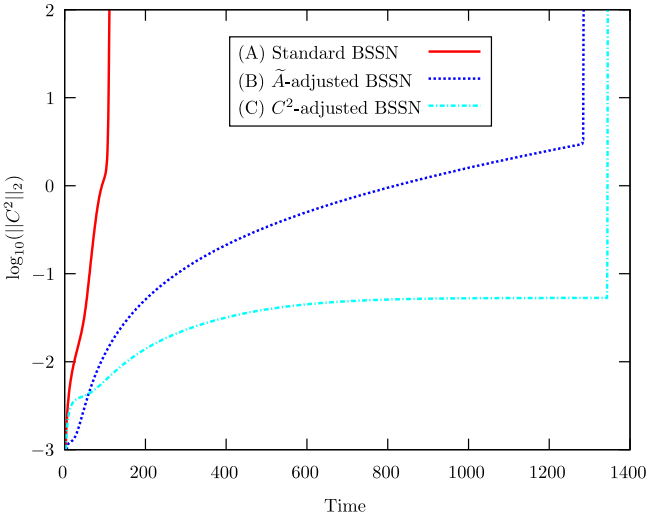

Figure 10.2: L2 norm of all the constraints in gauge-wave evolution comparing three BSSN formulations: (A) standard BSSN formulation (solid line), (B) $\widetilde{A}$-adjusted BSSN formulation (dotted line), and (C) C^2-adjusted BSSN formulation (dot-dashed line). The adopted parameters are $\kappa_A = 10^{-1.6}$ for (B), and $\lambda_\varphi = 10^{-8.5}$, $\lambda_K = 10^{-8.4}$, $\lambda_{\widetilde{\gamma}} = 10^{-7.3}$, $\lambda_{\widetilde{A}} = 10^{-2.5}$, and $\lambda_{\widetilde{\Gamma}} = 10^{-1.8}$ for (C) to minimize C^2 at $t = 1000$. The constraint violations of the $\widetilde{A}$-adjusted BSSN formulation, (B), increase with time and the simulation stops before $t = 1300$, while those of the C^2-adjusted BSSN formulation, (C), remain at $O(10^{-1})$ until $t = 1300$ and the simulation stops at $t = 1350$.

10.2 Gowdy wave case

10.2.1 Constraint Violations and Their Dampings

We begin showing the case of the standard BSSN formulation, (7.6)-(7.10). Figure 10.6 shows the L2 norm of the violations of the constraints as a function of backward time ($-t$). We see that the violation of the $\widetilde{\mathcal{M}}$-constraint is the largest at all times and that all the violations of constraints increase monotonically with time. [Comparing with the result in [30], our code shows that the $\widetilde{\mathcal{H}}$-constraint (A-1) remains at the same level but the $\widetilde{\mathcal{M}}$-constraint (A-2) is smaller.]

Similar to the gauge-wave test, we compare the violations of C^2 for three types of BSSNs in Fig.10.7. In the case of the $\widetilde{A}$-adjusted BSSN formulation, the violation of the constraints increases if we set $|\kappa_A|$ larger than $10^{-0.2}$. In the case of the C^2-adjusted BSSN formulation, it increases if we set $|\lambda_{\widetilde{A}}|$ larger than $10^{-1.2}$. Note that the signatures of the above κ_A and λs are negative, contrary to the predictions in [27] and Sec.7.1, respectively. This is because these simulations are performed with backward time.

As shown in Fig.10.7, the violations of C^2 for the standard BSSN formulation and the $\widetilde{A}$-adjusted BSSN formulation increase monotonically with time, while that for the C^2-adjusted BSSN formulation decreases after $t = -200$. To investigate the reason of this rapid decay after $t = -200$, we plot each constraint violation in Fig.10.8. We see that the violations of the $\widetilde{\mathcal{A}}$-constraint and $\widetilde{\mathcal{S}}$-constraint increase with negative time, in contrast to the standard BSSN formulation, and those of the $\widetilde{\mathcal{M}}$-constraint and $\widetilde{\mathcal{G}}$-constraint decrease after $t = -200$.

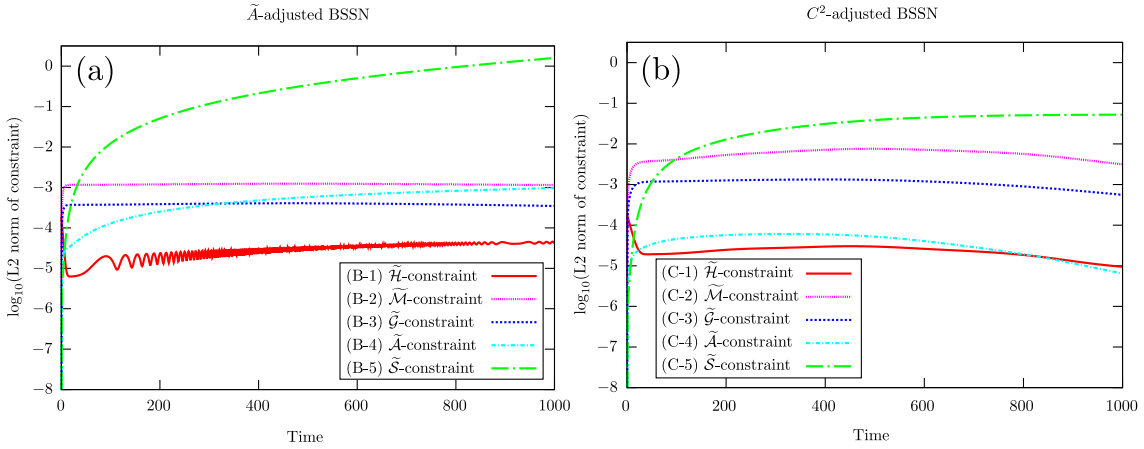

Figure 10.3: L2 norm of each constraint in the gauge-wave evolution using the $\widetilde{A}$-adjusted BSSN formulation [panel (a)] and C^2-adjusted BSSN formulation [panel (b)]. The parameters κ_A, λ_φ, λ_K, $\lambda_{\widetilde{\gamma}}$, $\lambda_{\widetilde{A}}$, and $\lambda_{\widetilde{\Gamma}}$ are the same as those in Fig.10.2. In both panels, we see that the violations of the $\widetilde{\mathcal{H}}$-constraint [the lines (B-1) and (C-1)], the $\widetilde{\mathcal{M}}$-constraint [(B-2) and (C-2)], and the $\widetilde{\mathcal{G}}$-constraint [(B-3) and (C-3)] are less than those for the standard BSSN formulation in Fig.10.1. However, the violations of the $\widetilde{\mathcal{A}}$-constraint [(B-4) and (C-4)] and the $\widetilde{\mathcal{S}}$-constraint [(B-5) and (C-5)] are larger. Line (B-5) overlaps with line (B) in Fig.10.2 after $t = 100$, and line (C-5) overlaps with line (C) in Fig.10.2 after $t = 500$.

The propagation equation of the $\widetilde{\mathcal{M}}$-constraint, (7.53), includes the term $-2c_A\lambda_{\widetilde{A}}\partial_a\widetilde{\mathcal{A}}$, which contributes to constraint damping. Similarly, the propagation equation of the $\widetilde{\mathcal{G}}$-constraint, (7.54), includes $\delta^{ab}\{(1/2)\lambda_{\widetilde{\gamma}}\partial_b\Delta + 2\lambda_{\widetilde{\Gamma}}\partial_b\}\widetilde{\mathcal{H}} - c_S\lambda_{\widetilde{\gamma}}\delta^{ab}\partial_b\widetilde{\mathcal{S}}$; the decay of the violations of the $\widetilde{\mathcal{G}}$-constraint is caused by these terms. Therefore, these terms are considered to become significant of approximately $t = -200$ when the violations of the $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{H}}$, and $\widetilde{\mathcal{S}}$-constraints become a certain order of magnitude.

In contrast to the gauge-wave testbed (Fig.10.4), we prepared Fig.10.9, which shows the magnitudes of the ratio of the adjusted terms to the original terms. Since the magnitudes of the adjusted terms of φ and $\widetilde{\gamma}_{ij}$ can be disregarded, the effect of the reduction of the adjusted terms of φ and $\widetilde{\gamma}_{ij}$ is negligible. Therefore, the C^2-adjusted BSSN evolution in the Gowdy wave can be regarded as maintaining its original hyperbolicity.

10.2.2 Contribution of Algebraic Constraints in Definition of C^2

In Sec.7.2, we investigated the effect of the definition of C^2. Similar to the gauge-wave tests in the previous subsection, we show the effect of constraint damping caused by the algebraic constraints. In Fig.10.10, we plot the violations of all the constraint with $c_G = c_A = c_S = 0$. We see that all the violations of the constraints are larger than those in Fig.10.8. This result is consistent with the discussion in Sec.7.2.

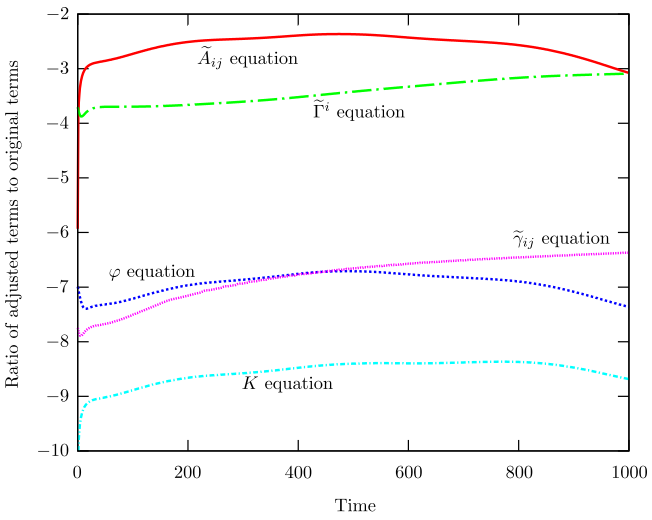

Figure 10.4: L2 norm of the ratio (adjusted terms)/(original terms) of each evolution equation of the C^2-adjusted BSSN formulation, (7.19)-(7.23), in the gauge-wave test. We see that the largest ratio is the evolution equation of $\widetilde{A}_{ij}$. The corrections to φ, K, and $\widetilde{\gamma}_{ij}$ evolution equations are reasonably small.

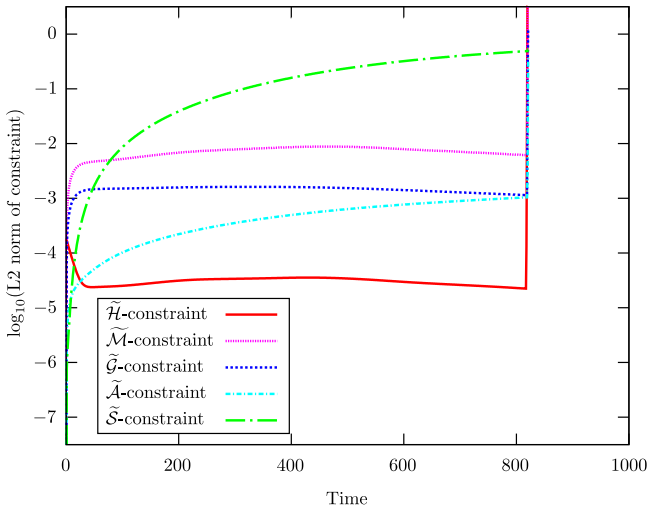

Figure 10.5: Difference with the definition of C^2, (7.57), in the damping of each constraint violation with $c_G = c_A = c_S = 0$. The parameters λ_φ, λ_K, $\lambda_{\widetilde{\gamma}}$, $\lambda_{\widetilde{A}}$, and $\lambda_{\widetilde{\Gamma}}$ are the same as those in Fig.10.2. The simulation stops since the violations of the constraints sudden increase at $t = 800$.

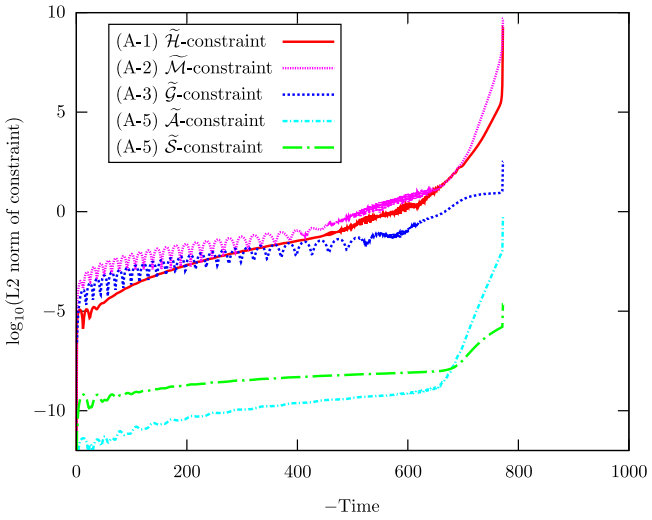

Figure 10.6: L2 norm of each constraint equation in the polarized Gowdy wave evolution using the standard BSSN formulation. The vertical axis is the logarithm of the L2 norm of the constraint and the horizontal axis is backward time.

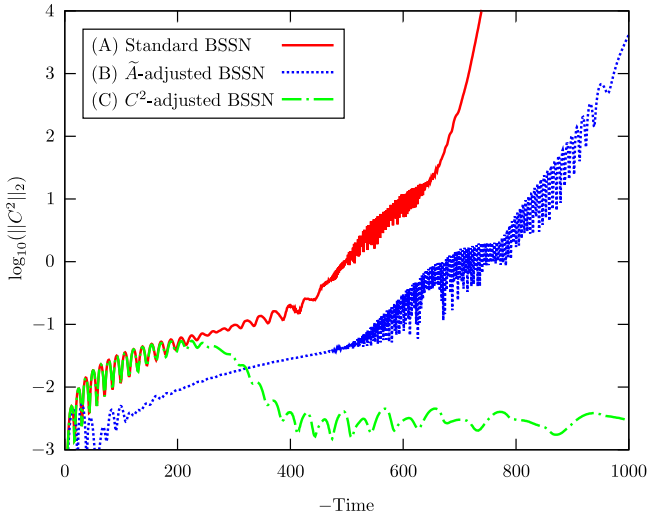

Figure 10.7: L2 norm of the constraints, C^2, of the polarized Gowdy wave tests for the standard BSSN and two adjusted formulations. The vertical axis is the logarithm of the L2 norm of C^2 and the horizontal axis is backward time. The solid line (A) is the standard BSSN formulation, the dotted line (B) is the $\widetilde{A}$-adjusted BSSN formulation with $\kappa_A = -10^{-0.2}$, and the dot-dashed line (C) is the C^2-adjusted BSSN formulation with $\lambda_\varphi = -10^{-10}$, $\lambda_K = -10^{-4.6}$, $\lambda_{\widetilde{\gamma}} = -10^{-11}$, $\lambda_{\widetilde{A}} = -10^{-1.2}$, and $\lambda_{\widetilde{\Gamma}} = -10^{-14.3}$. Note that the signatures of κ_A and λs are negative since the simulations evolve backward. We see that lines (A) and (C) are identical until $t = -200$. Line (C) then decreases and maintains its magnitude under $O(10^{-2})$ after $t = -400$. We confirm this behavior until $t = -1500$.

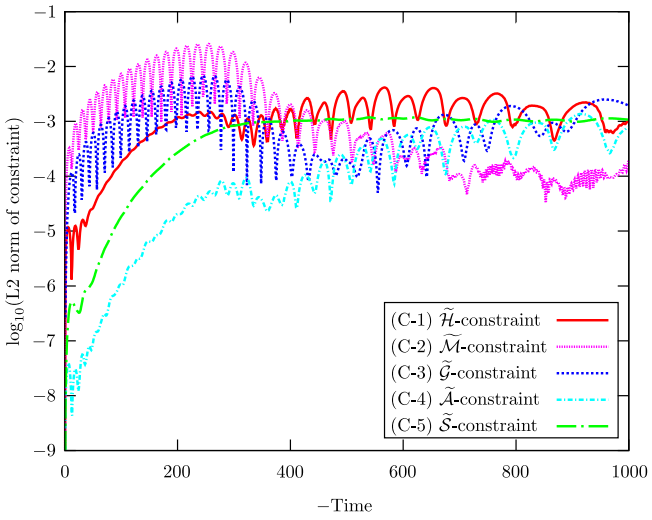

Figure 10.8: The same with Fig.10.6 but for the C^2-adjusted BSSN formulation. The parameters, $(\lambda_\varphi, \lambda_K, \lambda_{\widetilde{\gamma}}, \lambda_{\widetilde{A}}, \lambda_{\widetilde{\Gamma}})$, are the same with those for (C) in Fig.10.7. We see that the violation of the $\widetilde{\mathcal{M}}$-constraint decreases and becomes the lowest after $t = -700$.

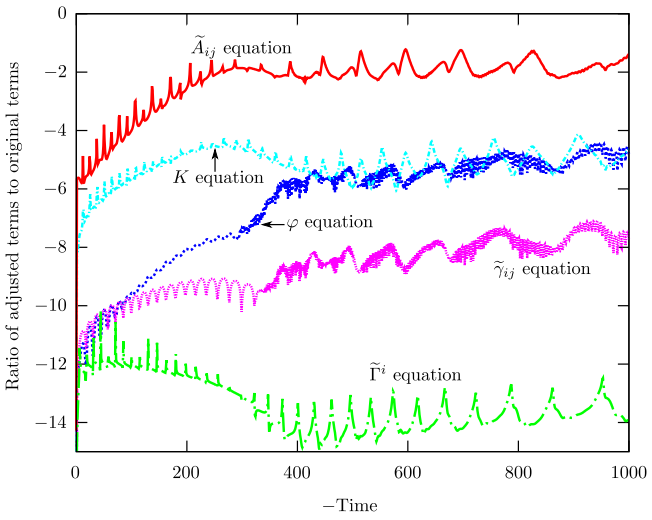

Figure 10.9: L2 norm of the ratio (adjusted terms)/(original terms) of each evolution equation for the C^2-adjusted BSSN formulation, (7.19)-(7.23). We see that the largest ratio is that for the evolution of $\widetilde{A}_{ij}$. The corrections to the $\widetilde{\gamma}_{ij}$ and $\widetilde{\Gamma}^i$ evolution equations are reasonably small.

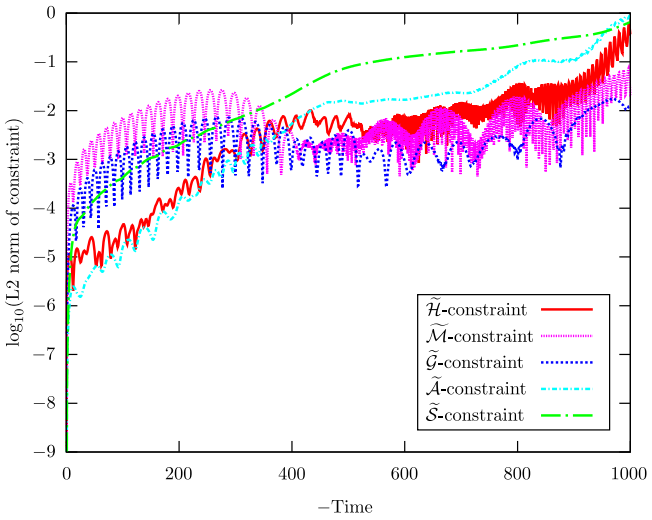

Figure 10.10: Difference with the definition of C^2 with $c_G = c_A = c_S = 0$. The coefficient parameters, λ_φ, λ_K, $\lambda_{\widetilde{\gamma}}$, $\lambda_{\widetilde{A}}$ and $\lambda_{\widetilde{\Gamma}}$, are all the same as those for (C) in Fig.10.7. In comparison with Fig.10.8, all the violations of the constraints are larger.

Chapter 11

Summary

To construct $3+1$ splitting formulation of the Einstein equations, we introduced the standard ADM formulation and the BSSN formulation, and derived both of the constraint propagation equations of these formulations. Next, we proposed new sets of evolution equations, which we call the C^2-adjusted ADM formulation and C^2-adjusted BSSN formulation. We applied the adjusting method suggested by Fiske [34] to the standard ADM and BSSN formulations.

For the C^2-adjusted ADM formulation, we obtained the evolution equations as (6.8)-(6.9) and the constraint propagation equations, (6.18) and (6.28), and also discussed the constraint propagation of this system. We analyzed the constraint amplification factors (CAFs) on the flat background, and confirmed that all of the CAFs have negative real-part which indicate the damping of the constraint violations. We, then, performed numerical tests with the polarized Gowdy-wave and showed the damping of the constraint violations as expected.

On the other hand, for the C^2-adjusted BSSN formulation, we derived evolution equations as (7.19)-(7.23) and the constraint propagation equations, (7.52)-(7.56), in flat spacetime. We performed numerical tests in the gauge-wave and Gowdy wave spacetimes and confirmed that the violations of constraints decrease as expected, and that longer and accurate simulation than that of the standard BSSN evolution is available.

There are two advantages of the C^2-adjusted system. One is that we can uniquely determine the form of the adjustments. The other is that we can specify the effective signature of the coefficiencies (Lagrange multipliers) independent on the background. (The term *effective* means that the system has the property of the damping constraint violations). In [28], Yoneda and Shinkai systematically examined several combinations of adjustments to the ADM evolution equations, and discuss the effective signature of those Lagrange multipliers using CAFs as the guiding principle. However, the C^2-adjusted idea, (5.4), automatically includes this guiding principle. We confirm this fact using CAF-analysis on the flat background.

Although, in BSSN, there are two kinetic constraints and three additional algebraic constraints compared to the ADM system; thus, the definition of C^2 is a matter of concern. By analyzing constraint propagation equations, we concluded that C^2 should include all the constraints. This was also confirmed by numerical tests. The importance of such algebraic constraints suggests the similar treatment when we apply this idea to other formulations of the Einstein equation.

We performed the simulation with the C^2-adjusted ADM formulation on the Gowdy-wave spacetime and confirmed the effect of the constraint dumping. We investigated the parameter dependencies and found that the constraint-damping effect does not continue due to one of the adjusted terms. We also found that the Detweiler's adjustment [40] is not so effective

against constraint violations on this spacetime. Up to this moment, we do not yet know how to choose the ranges of parameters which are suitable to damp the constraint violations unless the simulations are actually performed.

To evaluate the reduction of the violations of the constraints of the C^2-adjusted BSSN formulation, we also compared evolutions with the $\widetilde{A}$-adjusted BSSN formulation proposed in [27]. We concluded that the C^2-adjusted BSSN formulation exhibits superior constraint damping to both the standard and $\widetilde{A}$-adjusted BSSN formulations. In particular, the lifetimes of the simulations of the C^2-adjusted BSSN formulation in the gauge-wave and Gowdy wave testbeds are as ten-times and twice as longer than those of the standard BSSN formulation, respectively.

So far, many trials have been reported to improve BSSN formulation (e.g. [27,48]). Recently, for example, a conformal-traceless Z4 formulation was proposed with its test demonstrations [24]. Among them, Fig.1 of [24] can be compared with our Fig.10.3 [(B-1) and (C-1)] as the same gauge-wave test. The violation of $\widetilde{\mathcal{H}}$-constraint in C^2-adjusted BSSN evolution looks smaller than that of new Z4 evolution, but regarding the blow-up time of simulations, new Z4 system has advantage.

Fiske reported the applications of the idea of C^2-adjustment to *linearized* ADM and BSSN formulations in his dissertation [35]. (As he mentioned, his BSSN is not derived from the standard BSSN equations but from a linearized ADM using a new variable, Γ. His set of BSSN equations also does not include the $\widetilde{\mathcal{A}}$- and $\widetilde{\mathcal{S}}$-constraints in our notation.). He observed damping of the constraint violation of five orders of magnitude and the equivalent solution errors in his numerical evolution tests. Our studies show that the full BSSN set of equations with fully adjusted terms also produces the desired constraint-damping results (Fig.10.2 and Fig.10.7), although apparent improvements are at fewer orders of magnitude.

In the C^2-adjusted ADM and BSSN cases, the associated adjustment parameters (Lagrangian multipliers) are sensitive and require fine-tuning. In future, automatic controlling system such that monitoring the order of constraint violations and maintaining them by tuning the parameters automatically would be helpful. Applications of control theory in this direction are being investigated.

The correction terms of the C^2-adjusted system include higher-order derivatives and are not quasi-linear; thus, little is known mathematically about such systems. These additional terms might effectively act as artificial viscosity terms in fluid simulations, but might also enhance the violation of errors. To investigate this direction further, the next step is to apply the idea to a system in which constraints do not include second-order derivatives of dynamical variables. We are working on the Kidder-Scheel-Teukolsky formulation [17] as an example of such a system, which we will report in the near future.

Appendix A

Riemann Tensor Decomposition

A.1 Gauss-Codazzi Equation

First, we define the relation of the covariant derivative operator between $m+1$ and m dimension. The covariant derivative operator of $\mathcal{M}^m$ is defined as

$$D_\lambda T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} \equiv P^\omega{}_\lambda P^{\mu_1}{}_{\alpha_1} P^{\mu_2}{}_{\alpha_2} \cdots P^{\beta_1}{}_{\nu_1} P^{\beta_2}{}_{\nu_2} \cdots \nabla_\omega T^{\alpha_1\alpha_2\cdots}{}_{\beta_1\beta_2\cdots}. \tag{A.1}$$

where $T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} \in T(\mathcal{M}^m)$. The reason that D_μ is the covariant derivative operator is because of

$$\begin{aligned}
D_\lambda \gamma^{\alpha\beta} &= P^\omega{}_\lambda P^\alpha{}_\rho P^\beta{}_\sigma \nabla_\omega \gamma^{\rho\sigma} && \text{(A.2)}\\
&= P^\omega{}_\lambda P^\alpha{}_\rho P^\beta{}_\sigma \nabla_\omega \left(g^{\rho\sigma} - \frac{1}{\epsilon} n^\rho n^\sigma \right) && \text{(A.3)}\\
&= -\frac{1}{\epsilon} P^\omega{}_\lambda P^\alpha{}_\rho P^\beta{}_\sigma \nabla_\omega (n^\rho n^\sigma) && \text{(A.4)}\\
&= 0. && \text{(A.5)}
\end{aligned}$$

The calculation of the second-order covariant derivative of $\forall T_\lambda \in T(\mathcal{M}^m)$ is

$$\begin{aligned}
D_\mu D_\nu T_\lambda &= P^\alpha{}_\mu P^\beta{}_\nu P^\omega{}_\lambda \nabla_\alpha D_\beta T_\omega && \text{(A.6)}\\
&= P^\alpha{}_\mu P^\beta{}_\nu P^\omega{}_\lambda \nabla_\alpha \left(P^\gamma{}_\beta P^\delta{}_\omega \nabla_\gamma T_\delta \right) && \text{(A.7)}\\
&= P^\alpha{}_\mu P^\beta{}_\nu P^\delta{}_\lambda (\nabla_\alpha P^\gamma{}_\beta)(\nabla_\gamma T_\delta) + P^\alpha{}_\mu P^\gamma{}_\nu P^\omega{}_\lambda (\nabla_\alpha P^\delta{}_\omega)(\nabla_\gamma T_\delta)\\
&\quad + P^\alpha{}_\mu P^\gamma{}_\nu P^\delta{}_\lambda (\nabla_\alpha \nabla_\gamma T_\delta) && \text{(A.8)}\\
&= -\frac{1}{\epsilon} P^\alpha{}_\mu P^\beta{}_\nu P^\delta{}_\lambda n^\gamma (\nabla_\alpha n_\beta)(\nabla_\gamma T_\delta) - \frac{1}{\epsilon} P^\alpha{}_\mu P^\gamma{}_\nu P^\omega{}_\lambda n^\delta (\nabla_\alpha n_\omega)(\nabla_\gamma T_\delta)\\
&\quad + P^\alpha{}_\mu P^\gamma{}_\nu P^\delta{}_\lambda (\nabla_\alpha \nabla_\gamma T_\delta) && \text{(A.9)}\\
&= \frac{1}{\epsilon} P^\delta{}_\lambda n^\gamma K_{\mu\nu} (\nabla_\gamma T_\delta) + \frac{1}{\epsilon} P^\gamma{}_\nu n^\delta K_{\mu\lambda} (\nabla_\gamma T_\delta) + P^\alpha{}_\mu P^\gamma{}_\nu P^\delta{}_\lambda (\nabla_\alpha \nabla_\gamma T_\delta) && \text{(A.10)}\\
&= \frac{1}{\epsilon} P^\delta{}_\lambda n^\gamma K_{\mu\nu} (\nabla_\gamma T_\delta) - \frac{1}{\epsilon} P^\gamma{}_\nu (\nabla_\gamma n^\delta) K_{\mu\lambda} T_\delta + P^\alpha{}_\mu P^\gamma{}_\nu P^\delta{}_\lambda (\nabla_\alpha \nabla_\gamma T_\delta) && \text{(A.11)}\\
&= \frac{1}{\epsilon} P^\delta{}_\lambda n^\gamma K_{\mu\nu} (\nabla_\gamma T_\delta) + \frac{1}{\epsilon} K_\nu{}^\delta K_{\mu\lambda} T_\delta + P^\alpha{}_\mu P^\gamma{}_\nu P^\delta{}_\lambda (\nabla_\alpha \nabla_\gamma T_\delta), && \text{(A.12)}
\end{aligned}$$

the relation between ${}^{(m+1)}R_{\mu\nu\lambda\omega}$ and ${}^{(m)}R_{\mu\nu\lambda\omega}$ is

$$
\begin{aligned}
{}^{(m)}R^{\omega}{}_{\lambda\nu\mu}T_{\omega} &= 2D_{[\mu}D_{\nu]}T_{\lambda} && \text{(A.13)}\\
&= \frac{2}{\epsilon}P^{\delta}{}_{\lambda}n^{\gamma}K_{[\mu\nu]}(\nabla_{\gamma}T_{\delta}) + \frac{2}{\epsilon}K_{[\nu}{}^{\delta}K_{\mu]\lambda}T_{\delta} + 2P^{\alpha}{}_{[\mu}P^{\gamma}{}_{\nu]}P^{\delta}{}_{\lambda}(\nabla_{\alpha}\nabla_{\gamma}T_{\delta}) && \text{(A.14)}\\
&= \frac{2}{\epsilon}(K_{\nu}{}^{\omega}K_{\mu\lambda} - K_{\mu}{}^{\omega}K_{\nu\lambda})T_{\omega} + 2P^{\alpha}{}_{\mu}P^{\gamma}{}_{\nu}P^{\delta}{}_{\lambda}(\nabla_{[\alpha}\nabla_{\gamma]}T_{\delta}) && \text{(A.15)}\\
&= \frac{1}{\epsilon}(K_{\nu}{}^{\omega}K_{\mu\lambda} - K_{\mu}{}^{\omega}K_{\nu\lambda})T_{\omega} + P^{\alpha}{}_{\mu}P^{\gamma}{}_{\nu}P^{\delta}{}_{\lambda}\ {}^{(m+1)}R^{\omega}{}_{\delta\gamma\alpha}T_{\omega}. && \text{(A.16)}
\end{aligned}
$$

Therefore, the **Gauss-Codazzi equation** is

$$
P^{\mu}{}_{\alpha}P^{\lambda}{}_{\rho}P^{\nu}{}_{\beta}P^{\omega}{}_{\gamma}\ {}^{(m+1)}R_{\mu\lambda\nu\omega} = {}^{(m)}R_{\alpha\rho\beta\gamma} - \frac{1}{\epsilon}(K_{\beta\alpha}K_{\gamma\rho} - K_{\gamma\alpha}K_{\beta\rho}). \tag{A.17}
$$

A.2 Codazzi-Mainardi Equation

First, we set a value for convenient,

$$
a_{\mu} \equiv n^{\nu}\nabla_{\nu}n_{\mu}. \tag{A.18}
$$

a_{μ} is in $T(\mathcal{M}^{m})$ because

$$
\begin{aligned}
n^{\mu}a_{\mu} &= n^{\mu}n^{\nu}\nabla_{\nu}n_{\mu} && \text{(A.19)}\\
&= n^{\nu}(n^{\mu}\nabla_{\nu}n_{\mu}) && \text{(A.20)}\\
&= 0. && \text{(A.21)}
\end{aligned}
$$

The relation between $K_{\mu\nu}$ and a_{μ} is

$$
\begin{aligned}
K_{\mu\nu} &= -P^{\lambda}{}_{\mu}\nabla_{\lambda}n_{\nu} && \text{(A.22)}\\
&= -\left(\delta^{\lambda}{}_{\mu} - \frac{1}{\epsilon}n^{\lambda}n_{\mu}\right)\nabla_{\lambda}n_{\nu} && \text{(A.23)}\\
&= -\nabla_{\mu}n_{\nu} + \frac{1}{\epsilon}n_{\mu}a_{\nu}. && \text{(A.24)}
\end{aligned}
$$

The calculation of the second-order covariant derivative of n_{μ} is

$$
\begin{aligned}
P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}P^{\lambda}{}_{\omega}\nabla_{\lambda}\nabla_{\nu}n_{\mu} &= P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}P^{\lambda}{}_{\omega}\nabla_{\lambda}\left(-K_{\nu\mu} + \frac{1}{\epsilon}n_{\nu}a_{\mu}\right) && \text{(A.25)}\\
&= -P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}P^{\lambda}{}_{\omega}(\nabla_{\lambda}K_{\nu\mu}) + \frac{1}{\epsilon}P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}P^{\lambda}{}_{\omega}\nabla_{\lambda}(n_{\nu}a_{\mu}) && \text{(A.26)}\\
&= -D_{\omega}K_{\beta\alpha} + \frac{1}{\epsilon}P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}P^{\lambda}{}_{\omega}(\nabla_{\lambda}n_{\nu})a_{\mu} && \text{(A.27)}\\
&= -D_{\omega}K_{\beta\alpha} - \frac{1}{\epsilon}P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}K_{\omega\nu}a_{\mu} && \text{(A.28)}\\
&= -D_{\omega}K_{\beta\alpha} - \frac{1}{\epsilon}K_{\omega\beta}a_{\alpha}, && \text{(A.29)}
\end{aligned}
$$

the projection of ${}^{(m+1)}R_{\mu\nu\lambda\omega}$ by $P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}P^{\lambda}{}_{\omega}n^{\rho}$ is

$$
\begin{aligned}
P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}P^{\lambda}{}_{\omega}n^{\rho}\ {}^{(m+1)}R_{\rho\mu\nu\lambda} &= 2P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}P^{\lambda}{}_{\omega}n^{\rho}(\nabla_{[\lambda}\nabla_{\nu]}n_{\mu}) && \text{(A.30)}\\
&= -2D_{[\omega}K_{\beta]\alpha} - \frac{2}{\epsilon}K_{[\omega\beta]}a_{\alpha} && \text{(A.31)}\\
&= -2D_{[\omega}K_{\beta]\alpha}, && \text{(A.32)}
\end{aligned}
$$

therefore, the **Codazzi-Mainardi equation** is

$$P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}P^{\lambda}{}_{\omega}n^{\rho}\,{}^{(m+1)}R_{\rho\mu\nu\lambda} = -D_{\omega}K_{\beta\alpha} + D_{\beta}K_{\omega\alpha}. \tag{A.33}$$

A.3 Component paralleled with Normal Vectors

First, we compute the value a_{μ},

$$\begin{aligned}
a_{\mu} &= n^{\lambda}\nabla_{\lambda}n_{\mu} && \text{(A.34)}\\
&= n^{\lambda}\nabla_{\lambda}(\epsilon N\nabla_{\mu}\xi) && \text{(A.35)}\\
&= \epsilon n^{\lambda}(\nabla_{\lambda}N)(\nabla_{\mu}\xi) + \epsilon n^{\lambda}N(\nabla_{\lambda}\nabla_{\mu}\xi) && \text{(A.36)}\\
&= \frac{1}{N}n^{\lambda}(\nabla_{\lambda}N)n_{\mu} + \epsilon n^{\lambda}N(\nabla_{\mu}\nabla_{\lambda}\xi) && \text{(A.37)}\\
&= n^{\lambda}n_{\mu}(\nabla_{\lambda}\log N) + \epsilon\nabla_{\mu}(n^{\lambda}N\nabla_{\lambda}\xi) - \epsilon\nabla_{\mu}(n^{\lambda}N)(\nabla_{\lambda}\xi) && \text{(A.38)}\\
&= n^{\lambda}n_{\mu}(\nabla_{\lambda}\log N) + (\nabla_{\mu}\epsilon) - \frac{\epsilon}{N}(\nabla_{\mu}N) && \text{(A.39)}\\
&= (\nabla_{\lambda}\log N)(n^{\lambda}n_{\mu} - \epsilon\delta^{\lambda}{}_{\mu}) && \text{(A.40)}\\
&= -\epsilon(\nabla_{\lambda}\log N)\left(\delta^{\lambda}{}_{\mu} - \frac{1}{\epsilon}n^{\lambda}n_{\mu}\right) && \text{(A.41)}\\
&= -\epsilon D_{\mu}\log N. && \text{(A.42)}
\end{aligned}$$

Then, the extrinsic curvature is expressed with (A.42),

$$\begin{aligned}
K_{\mu\nu} &= -\nabla_{\mu}n_{\nu} + \frac{1}{\epsilon}n_{\mu}a_{\nu} && \text{(A.43)}\\
&= -\nabla_{\mu}n_{\nu} - n_{\mu}D_{\nu}\log N. && \text{(A.44)}
\end{aligned}$$

Next, we calculate the the projection of the component of ${}^{(m)}R_{\mu\nu\lambda\omega}$ with two $P^{\mu}{}_{\nu}$ and two n^{λ},

$$\begin{aligned}
&P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}n^{\omega}\,{}^{(m+1)}R_{\omega\mu\lambda\nu} && \text{(A.45)}\\
=&P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}(\nabla_{\nu}\nabla_{\lambda}n_{\mu}) - P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}(\nabla_{\lambda}\nabla_{\nu}n_{\mu}) && \text{(A.46)}\\
=&P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}\nabla_{\nu}\left(-K_{\lambda\mu} - n_{\lambda}D_{\mu}\log N\right) - P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}\nabla_{\lambda}\left(-K_{\nu\mu} - n_{\nu}D_{\mu}\log N\right) && \text{(A.47)}\\
=&-P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}(\nabla_{\nu}K_{\lambda\mu}) - P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}\nabla_{\nu}(n_{\lambda}D_{\mu}\log N) + P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}(\nabla_{\lambda}K_{\nu\mu})\\
&+P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}\nabla_{\lambda}(n_{\nu}D_{\mu}\log N) && \text{(A.48)}\\
=&P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}(\nabla_{\nu}n^{\lambda})K_{\lambda\mu} - \epsilon P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\nabla_{\nu}(D_{\mu}\log N) + P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}(\nabla_{\lambda}K_{\nu\mu})\\
&+P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}(\nabla_{\lambda}n_{\nu})(D_{\mu}\log N) && \text{(A.49)}\\
=&-P^{\mu}{}_{\alpha}K_{\beta}{}^{\lambda}K_{\lambda\mu}\boxed{-\epsilon D_{\beta}D_{\alpha}\log N} + P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}(\nabla_{\lambda}K_{\nu\mu})\boxed{-\epsilon(D_{\alpha}\log N)(D_{\beta}\log N)} && \text{(A.50)}\\
=&-K_{\beta}{}^{\lambda}K_{\lambda\alpha}\boxed{-\frac{\epsilon}{N}D_{\beta}D_{\alpha}N} + P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}(\nabla_{\lambda}K_{\nu\mu}). && \text{(A.51)}
\end{aligned}$$

Then Lie derivative of the $K_{\mu\nu}$ associated with Nn^{μ} is

$$\begin{aligned}
\mathcal{L}_{Nn}(K_{\mu\nu}) &= Nn^{\lambda}(\nabla_{\lambda}K_{\mu\nu}) + K_{\lambda\nu}\nabla_{\mu}(Nn^{\lambda}) + K_{\mu\lambda}\nabla_{\nu}(Nn^{\lambda}) && \text{(A.52)}\\
&= Nn^{\lambda}(\nabla_{\lambda}K_{\mu\nu}) + K_{\lambda\nu}N(\nabla_{\mu}n^{\lambda}) + K_{\mu\lambda}N(\nabla_{\nu}n^{\lambda}). && \text{(A.53)}
\end{aligned}$$

For the projection operator $P^{\mu}{}_{\nu}$, the Lie derivative associated with Nn^{λ} is

$$\begin{aligned}
\mathcal{L}_{Nn}(P^{\mu}{}_{\nu}) &= Nn^{\lambda}(\nabla_{\lambda}P^{\mu}{}_{\nu}) - P^{\lambda}{}_{\nu}\nabla_{\lambda}(Nn^{\mu}) + P^{\mu}{}_{\lambda}\nabla_{\nu}(Nn^{\lambda}) && \text{(A.54)}\\
&= -\frac{1}{\epsilon}Nn^{\lambda}(\nabla_{\lambda}n^{\mu})n_{\nu} - \frac{1}{\epsilon}Nn^{\lambda}n^{\mu}(\nabla_{\lambda}n_{\nu}) - P^{\lambda}{}_{\nu}(\nabla_{\lambda}N)n^{\mu} - P^{\lambda}{}_{\nu}N(\nabla_{\lambda}n^{\mu})\\
&\quad + P^{\mu}{}_{\lambda}(\nabla_{\nu}N)n^{\lambda} + P^{\mu}{}_{\lambda}N(\nabla_{\nu}n^{\lambda}) && \text{(A.55)}\\
&= -\frac{1}{\epsilon}Nn^{\lambda}n_{\nu}g^{\mu\omega}(-K_{\lambda\omega} - n_{\lambda}D_{\omega}\log N) - \frac{1}{\epsilon}Nn^{\lambda}n^{\mu}(-K_{\lambda\nu} - n_{\lambda}D_{\nu}\log N)\\
&\quad - (D_{\nu}N)n^{\mu} - P^{\lambda}{}_{\nu}Ng^{\mu\omega}(-K_{\lambda\omega} - n_{\lambda}D_{\omega}\log N)\\
&\quad + P^{\mu}{}_{\lambda}Ng^{\lambda\omega}(-K_{\nu\omega} - n_{\nu}D_{\omega}\log N) && \text{(A.56)}\\
&= Nn_{\nu}(D^{\mu}\log N) + Nn^{\mu}(D_{\nu}\log N) - (D_{\nu}N)n^{\mu} + NK_{\nu}{}^{\mu} - NK_{\nu}{}^{\mu}\\
&\quad - Nn_{\nu}(D^{\mu}\log N) && \text{(A.57)}\\
&= 0. && \text{(A.58)}
\end{aligned}$$

Therefore, the projection of (A.53) with $P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}$ is

$$\begin{aligned}
P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\mathcal{L}_{Nn}(K_{\mu\nu}) &= P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}Nn^{\lambda}(\nabla_{\lambda}K_{\mu\nu}) + P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}K_{\lambda\nu}N(\nabla_{\mu}n^{\lambda})\\
&\quad + P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}K_{\mu\lambda}N(\nabla_{\nu}n^{\lambda}). && \text{(A.59)}\\
\Leftrightarrow \quad P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}(\nabla_{\lambda}K_{\mu\nu}) &= \frac{1}{N}P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}\mathcal{L}_{Nn}(K_{\mu\nu}) - P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}K_{\lambda\nu}(\nabla_{\mu}n^{\lambda})\\
&\quad - P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}K_{\mu\lambda}(\nabla_{\nu}n^{\lambda}) && \text{(A.60)}\\
&= \frac{1}{N}\mathcal{L}_{Nn}(K_{\alpha\beta}) + 2K_{\lambda\beta}K_{\alpha}{}^{\lambda}. \quad (\because \mathcal{L}_{Nn}(P^{\mu}{}_{\nu}) = 0) && \text{(A.61)}
\end{aligned}$$

Therefore, the component is expressed as

$$P^{\mu}{}_{\alpha}P^{\nu}{}_{\beta}n^{\lambda}n^{\omega}\ {}^{(m+1)}R_{\omega\mu\lambda\nu} = K_{\beta}{}^{\lambda}K_{\lambda\alpha} - \frac{\epsilon}{N}D_{\beta}D_{\alpha}N + \frac{1}{N}\mathcal{L}_{Nn}(K_{\alpha\beta}). \qquad \text{(A.62)}$$

Appendix B

Conformal Riemann Tensor

B.1 Conformal Covariant Derivative

We calculate the relation between the normal covariant derivative operator D_μ and the conformal covariant derivative operator $\bar{D}_\mu$.

$\bar{D}_\mu$ operate $T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots}$ as

$$\begin{aligned}\bar{D}_\lambda T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} = \partial_\lambda T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} + {}^{(m)}\bar{\Gamma}^{\mu_1}{}_{\lambda\omega} T^{\omega\mu_2\cdots}{}_{\nu_1\nu_2\cdots} + \bar{\Gamma}^{\mu_2}{}_{\lambda\omega} T^{\mu_1\omega\cdots}{}_{\nu_1\nu_2\cdots} + \cdots \\ - \bar{\Gamma}^{\omega}{}_{\lambda\nu_1} T^{\mu_1\mu_2\cdots}{}_{\omega\nu_2\cdots} - \bar{\Gamma}^{\omega}{}_{\lambda\nu_2} T^{\mu_1\mu_2\cdots}{}_{\nu_1\omega\cdots} - \cdots . \end{aligned} \tag{B.1}$$

$\forall T \in \mathcal{T}_0^0(\mathcal{M}^m), \forall T^\mu \in \mathcal{T}_0^1(\mathcal{M}^m), \forall T_\mu \in \mathcal{T}_1^0(\mathcal{M}^m), \forall T_{\mu\nu} \in \mathcal{T}_2^0(\mathcal{M}^m)$,

$$D_\mu T = \bar{D}_\mu T, \tag{B.2}$$

$$D_\mu T^\nu = \bar{D}_\mu T^\nu + \delta^\nu{}_\mu (\bar{D}_\omega \log\phi) T^\omega + (\bar{D}_\mu \log\phi) T^\nu - \bar{\gamma}_{\mu\omega} (\bar{D}^\nu \log\phi) T^\omega, \tag{B.3}$$

$$D_\mu T_\nu = \bar{D}_\mu T_\nu - (\bar{D}_\nu \log\phi) T_\mu - (\bar{D}_\mu \log\phi) T_\nu + \bar{\gamma}_{\mu\nu} (\bar{D}^\lambda \log\phi) T_\lambda, \tag{B.4}$$

$$\begin{aligned} D_\lambda T_{\mu\nu} = \bar{D}_\lambda T_{\mu\nu} - (\bar{D}_\mu \log\phi) T_{\lambda\nu} - (\bar{D}_\nu \log\phi) T_{\lambda\mu} - 2(\bar{D}_\lambda \log\phi) T_{\mu\nu} + \bar{\gamma}_{\lambda\mu} (\bar{D}^\omega \log\phi) T_{\omega\nu} \\ + \bar{\gamma}_{\lambda\nu} (\bar{D}^\omega \log\phi) T_{\omega\mu}. \end{aligned} \tag{B.5}$$

B.2 Conformal Lie Derivative

We express the Lie derivative operator in conformal manifolds as $\bar{\mathcal{L}}$ in this report. Then, $\forall\phi \in \mathcal{T}_0^0(\mathcal{M}^m), \forall v^\lambda \in \mathcal{T}_0^1(\mathcal{M}^m), \forall V_{\mu\nu} \in \mathcal{T}_2^0(\mathcal{M}^m)$ and the conformal value $\bar{V}_{\mu\nu} \equiv \phi^{-2} V_{\mu\nu}$, we calculate that the relation between $\mathcal{L}_v(V_{\mu\nu})$ and $\bar{\mathcal{L}}_v(\bar{V}_{\mu\nu})$ is

$$\bar{\mathcal{L}}_v(\bar{V}_{\mu\nu}) \equiv v^\lambda (\bar{D}_\lambda \bar{V}_{\mu\nu}) + \bar{V}_{\lambda\nu} (\bar{D}_\mu v^\lambda) + \bar{V}_{\mu\lambda} (\bar{D}_\nu v^\lambda) \tag{B.6}$$

$$= v^\lambda (\partial_\lambda \bar{V}_{\mu\nu}) + \bar{V}_{\lambda\nu} (\partial_\mu v^\lambda) + \bar{V}_{\mu\lambda} (\partial_\nu v^\lambda) \tag{B.7}$$

$$= -2\phi v^\lambda (\partial_\lambda \phi) V_{\mu\nu} + \phi^{-2} v^\lambda (\partial_\lambda V_{\mu\nu}) + \phi^{-2} V_{\lambda\nu} (\partial_\mu v^\lambda) + \phi^{-2} V_{\mu\lambda} (\partial_\nu v^\lambda) \tag{B.8}$$

$$= -2 v^\lambda \bar{V}_{\mu\nu} (\bar{D}_\lambda \log\phi) + \phi^{-2} \mathcal{L}_v(T_{\mu\nu}). \tag{B.9}$$

$$\therefore \quad \mathcal{L}_v(V_{\mu\nu}) = \phi^2 \bar{\mathcal{L}}_v(\bar{V}_{\mu\nu}) + 2\phi^2 \bar{V}_{\mu\nu} \bar{\mathcal{L}}_v(\log\phi). \tag{B.10}$$

B.3 Conformal Riemann Tensor

We calculate the relation between ${}^{(m)}R^{\lambda}{}_{\mu\omega\nu}$ and ${}^{(m)}\bar{R}^{\lambda}{}_{\mu\omega\nu}$;

$$
\begin{aligned}
{}^{(m)}R^{\lambda}{}_{\mu\omega\nu} &= \partial_\omega\, {}^{(m)}\Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu\, {}^{(m)}\Gamma^{\lambda}{}_{\mu\omega} + {}^{(m)}\Gamma^{\lambda}{}_{\omega\rho}\, {}^{(m)}\Gamma^{\rho}{}_{\mu\nu} - {}^{(m)}\Gamma^{\lambda}{}_{\nu\rho}\, {}^{(m)}\Gamma^{\rho}{}_{\mu\omega} \qquad \text{(B.11)}\\
&= \partial_\omega\, {}^{(m)}\bar{\Gamma}^{\lambda}{}_{\mu\nu} + \delta^{\lambda}{}_{\mu}\partial_\omega(\bar{D}_\nu \log\phi) + \delta^{\lambda}{}_{\nu}\partial_\omega(\bar{D}_\mu\log\phi) - (\partial_\omega\bar{\gamma}^{\lambda\rho})\bar{\gamma}_{\mu\nu}(\bar{D}_\rho\log\phi)\\
&\quad -(\partial_\omega\bar{\gamma}_{\mu\nu})(\bar{D}^{\lambda}\log\phi) - \bar{\gamma}^{\lambda\rho}\bar{\gamma}_{\mu\nu}\partial_\omega(\bar{D}_\rho\log\phi)\\
&\quad -\partial_\nu\, {}^{(m)}\bar{\Gamma}^{\lambda}{}_{\mu\omega} - \delta^{\lambda}{}_{\mu}\partial_\nu(\bar{D}_\omega\log\phi) - \delta^{\lambda}{}_{\omega}\partial_\nu(\bar{D}_\mu\log\phi) + (\partial_\nu\bar{\gamma}^{\lambda\rho})\bar{\gamma}_{\mu\omega}(\bar{D}_\rho\log\phi)\\
&\quad +(\partial_\nu\bar{\gamma}_{\mu\omega})(\bar{D}^{\lambda}\log\phi) + \bar{\gamma}^{\lambda\rho}\bar{\gamma}_{\mu\omega}\partial_\nu(\bar{D}_\rho\log\phi)\\
&\quad +{}^{(m)}\bar{\Gamma}^{\lambda}{}_{\omega\rho}\, {}^{(m)}\bar{\Gamma}^{\rho}{}_{\mu\nu} + {}^{(m)}\bar{\Gamma}^{\lambda}{}_{\omega\mu}(\bar{D}_\nu\log\phi) + {}^{(m)}\bar{\Gamma}^{\lambda}{}_{\omega\nu}(\bar{D}_\mu\log\phi)\\
&\quad -\bar{\gamma}_{\mu\nu}\, {}^{(m)}\bar{\Gamma}^{\lambda\zeta}{}_{\omega}(\bar{D}_\zeta\log\phi) + \delta^{\lambda}{}_{\omega}\, {}^{(m)}\bar{\Gamma}^{\rho}{}_{\mu\nu}(\bar{D}_\rho\log\phi) + \delta^{\lambda}{}_{\omega}(\bar{D}_\mu\log\phi)(\bar{D}_\nu\log\phi)\\
&\quad +\delta^{\lambda}{}_{\omega}(\bar{D}_\nu\log\phi)(\bar{D}_\mu\log\phi) - \delta^{\lambda}{}_{\omega}\bar{\gamma}_{\mu\nu}(\bar{D}^{\rho}\log\phi)(\bar{D}_\rho\log\phi)\\
&\quad +{}^{(m)}\bar{\Gamma}^{\lambda}{}_{\mu\nu}(\bar{D}_\omega\log\phi) + \delta^{\lambda}{}_{\mu}(\bar{D}_\nu\log\phi)(\bar{D}_\omega\log\phi) + \delta^{\lambda}{}_{\nu}(\bar{D}_\omega\log\phi)(\bar{D}_\mu\log\phi)\\
&\quad -\bar{\gamma}_{\mu\nu}(\bar{D}^{\lambda}\log\phi)(\bar{D}_\omega\log\phi)\\
&\quad -{}^{(m)}\bar{\Gamma}_{\omega\mu\nu}(\bar{D}^{\lambda}\log\phi) - \bar{\gamma}_{\omega\mu}(\bar{D}_\nu\log\phi)(\bar{D}^{\lambda}\log\phi) - \bar{\gamma}_{\omega\nu}(\bar{D}_\mu\log\phi)(\bar{D}^{\lambda}\log\phi)\\
&\quad +\bar{\gamma}_{\mu\nu}(\bar{D}^{\lambda}\log\phi)(\bar{D}_\omega\log\phi)\\
&\quad -{}^{(m)}\bar{\Gamma}^{\lambda}{}_{\nu\rho}\, {}^{(m)}\bar{\Gamma}^{\rho}{}_{\mu\omega} - {}^{(m)}\bar{\Gamma}^{\lambda}{}_{\nu\mu}(\bar{D}_\omega\log\phi) - {}^{(m)}\bar{\Gamma}^{\lambda}{}_{\nu\omega}(\bar{D}_\mu\log\phi)\\
&\quad +{}^{(m)}\bar{\Gamma}^{\lambda}{}_{\nu\rho}\bar{\gamma}_{\mu\omega}(\bar{D}^{\rho}\log\phi) - \delta^{\lambda}{}_{\nu}\bar{\Gamma}^{\rho}{}_{\mu\omega}(\bar{D}_\rho\log\phi) - \delta^{\lambda}{}_{\nu}(\bar{D}_\mu\log\phi)(\bar{D}_\omega\log\phi)\\
&\quad -\delta^{\lambda}{}_{\nu}(\bar{D}_\omega\log\phi)(\bar{D}_\mu\log\phi) + \delta^{\lambda}{}_{\nu}\bar{\gamma}_{\mu\omega}(\bar{D}^{\zeta}\log\phi)(\bar{D}_\zeta\log\phi)\\
&\quad -{}^{(m)}\bar{\Gamma}^{\lambda}{}_{\mu\omega}(\bar{D}_\nu\log\phi) - \delta^{\lambda}{}_{\mu}(\bar{D}_\omega\log\phi)(\bar{D}_\nu\log\phi) - \delta^{\lambda}{}_{\omega}(\bar{D}_\mu\log\phi)(\bar{D}_\nu\log\phi)\\
&\quad +\bar{\gamma}_{\mu\omega}(\bar{D}_\nu\log\phi)(\bar{D}^{\lambda}\log\phi)\\
&\quad +{}^{(m)}\bar{\Gamma}_{\nu\mu\omega}(\bar{D}^{\lambda}\log\phi) + \bar{\gamma}_{\nu\mu}(\bar{D}^{\lambda}\log\phi)(\bar{D}_\omega\log\phi) + \bar{\gamma}_{\nu\omega}(\bar{D}^{\lambda}\log\phi)(\bar{D}_\mu\log\phi)\\
&\quad -\bar{\gamma}_{\mu\omega}(\bar{D}_\nu\log\phi)(\bar{D}^{\lambda}\log\phi) \qquad \text{(B.12)}\\
&= {}^{(m)}\bar{R}^{\lambda}{}_{\mu\omega\nu} + \delta^{\lambda}{}_{\nu}(\bar{D}_\omega\bar{D}_\mu\log\phi) - \bar{\gamma}_{\mu\nu}(\bar{D}_\omega\bar{D}^{\lambda}\log\phi) - \delta^{\lambda}{}_{\omega}(\bar{D}_\nu\bar{D}_\mu\log\phi)\\
&\quad +\bar{\gamma}_{\mu\omega}(\bar{D}_\nu\bar{D}^{\lambda}\log\phi) + \delta^{\lambda}{}_{\omega}(\bar{D}_\mu\log\phi)(\bar{D}_\nu\log\phi) - \delta^{\lambda}{}_{\omega}\bar{\gamma}_{\mu\nu}(\bar{D}^{\rho}\log\phi)(\bar{D}_\rho\log\phi)\\
&\quad -\bar{\gamma}_{\omega\mu}(\bar{D}_\nu\log\phi)(\bar{D}^{\lambda}\log\phi) - \delta^{\lambda}{}_{\nu}(\bar{D}_\omega\log\phi)(\bar{D}_\mu\log\phi)\\
&\quad +\delta^{\lambda}{}_{\nu}\bar{\gamma}_{\mu\omega}(\bar{D}^{\zeta}\log\phi)(\bar{D}_\zeta\log\phi) + \bar{\gamma}_{\nu\mu}(\bar{D}^{\lambda}\log\phi)(\bar{D}_\omega\log\phi). \qquad \text{(B.13)}
\end{aligned}
$$

Appendix C

Derivation of Constraint Propagation

C.1 Decomposition of Divergence of Second-Order Tensor

$\forall\, {}^{(m+1)}V_{\mu\nu} \in \mathcal{T}_2^0(\mathcal{M}^{m+1})$ such that

$$ {}^{(m+1)}V_{\mu\nu} = {}^{(m)}V n_\mu n_\nu + 2\,{}^{(m)}V_{(\mu} n_{\nu)} + {}^{(m)}V_{\mu\nu}, \tag{C.1}$$

where ${}^{(m)}V \in \mathcal{T}_0^0(\mathcal{M}^m), {}^{(m)}V_\mu \in \mathcal{T}_1^0(\mathcal{M}^m), {}^{(m)}V_{\mu\nu} \in \mathcal{T}_2^0(\mathcal{M}^m)$ and n_μ is the unit normal on $\mathcal{T}_0^1(\mathcal{M}^m)$, we decompose the equations;

$$ I_\mu = \nabla^\nu\, {}^{(m+1)}V_{\mu\nu}, \tag{C.2}$$

into the components paralleled with n^μ and with $P^\mu{}_\nu$.

C.1.1 The Component paralleled with Normal Vector

First we compute the component of (C.2) paralleled with n^μ;

$$\begin{aligned}
n^\nu I_\nu &= g^{\mu\lambda} n^\nu \nabla_\lambda ({}^{(m)}V n_\mu n_\nu + {}^{(m)}V_\mu n_\nu + {}^{(m)}V_\nu n_\mu + {}^{(m)}V_{\mu\nu}) && \text{(C.3)}\\
&= \epsilon n^\lambda (\nabla_\lambda\, {}^{(m)}V) + \epsilon g^{\mu\nu} (\nabla_\mu n_\nu)\, {}^{(m)}V + \epsilon g^{\mu\nu} (\nabla_\mu\, {}^{(m)}V_\nu) + n^\lambda n^\nu (\nabla_\lambda\, {}^{(m)}V_\nu) \\
&\quad + g^{\mu\lambda} n^\nu (\nabla_\lambda\, {}^{(m)}V_{\mu\nu}) && \text{(C.4)}\\
&= \epsilon n^\lambda (\nabla_\lambda\, {}^{(m)}V) + \epsilon \left(P^{\mu\nu} + \frac{1}{\epsilon} n^\mu n^\nu \right) (\nabla_\mu n_\nu)\, {}^{(m)}V + \epsilon \left(P^{\mu\nu} + \frac{1}{\epsilon} n^\mu n^\nu \right) (\nabla_\mu\, {}^{(m)}V_\nu) \\
&\quad + n^\lambda n^\nu (\nabla_\lambda\, {}^{(m)}V_\nu) + g^{\mu\lambda} n^\nu (\nabla_\lambda\, {}^{(m)}V_{\mu\nu}) && \text{(C.5)}\\
&= \epsilon n^\lambda (\nabla_\lambda\, {}^{(m)}V) + \epsilon P^\mu{}_\nu (\nabla_\mu n^\nu)\, {}^{(m)}V + \epsilon P^\mu{}_\lambda P^\nu{}_\omega P^{\lambda\omega} (\nabla_\mu\, {}^{(m)}V_\nu) - 2(\nabla_\lambda n^\nu) n^\lambda\, {}^{(m)}V_\nu \\
&\quad - P^\lambda{}_\mu (\nabla_\lambda n^\nu)\, {}^{(m)}V^\mu{}_\nu && \text{(C.6)}\\
&= \frac{\epsilon}{N} \mathcal{L}_{Nn} ({}^{(m)}V) - \epsilon K\, {}^{(m)}V + \epsilon (D_\mu\, {}^{(m)}V^\mu) + \frac{2\epsilon}{N} (D_\mu N)\, {}^{(m)}V^\mu + K_{\mu\nu}\, {}^{(m)}V^{\mu\nu}. && \text{(C.7)}
\end{aligned}$$

C.1.2 The Component paralleled with Projection Operator

Next, we calculate the component of (C.2) paralleled with $P^{\mu}{}_{\nu}$;

$$P^{\nu}{}_{\omega} I_{\nu} = P^{\nu}{}_{\omega} g^{\mu\lambda} \nabla_{\lambda} \left({}^{(m)}V n_{\mu} n_{\nu} + 2\, {}^{(m)}V_{(\mu} n_{\nu)} + {}^{(m)}V_{\mu\nu} \right) \tag{C.8}$$

$$\begin{aligned} &= P^{\nu}{}_{\omega} n^{\lambda} (\nabla_{\lambda} n_{\nu})\, {}^{(m)}V + P^{\nu}{}_{\omega} P^{\lambda}{}_{\mu} (\nabla_{\lambda} n_{\nu})\, {}^{(m)}V^{\mu} + P^{\nu}{}_{\omega} n^{\lambda} (\nabla_{\lambda}\, {}^{(m)}V_{\nu}) \\ &\quad + P^{\nu}{}_{\omega} P^{\lambda}{}_{\mu} (\nabla_{\lambda} n^{\mu})\, {}^{(m)}V_{\nu} + P^{\nu}{}_{\omega} P^{\mu}{}_{\rho} P^{\lambda}{}_{\sigma} P^{\rho\sigma} (\nabla_{\lambda}\, {}^{(m)}V_{\mu\nu}) - \frac{1}{\epsilon} P^{\nu}{}_{\omega} n^{\lambda} (\nabla_{\lambda} n^{\mu})\, {}^{(m)}V_{\mu\nu} \end{aligned} \tag{C.9}$$

$$\begin{aligned} &= -\epsilon P^{\nu}{}_{\omega} (D_{\nu} \log N)\, {}^{(m)}V - P^{\nu}{}_{\omega} K_{\mu\nu}\, {}^{(m)}V^{\mu} + \frac{1}{N} P^{\nu}{}_{\omega} \mathcal{L}_{Nn} ({}^{(m)}V_{\nu}) + K_{\omega}{}^{\lambda}\, {}^{(m)}V_{\lambda} \\ &\quad - P^{\nu}{}_{\omega} K\, {}^{(m)}V_{\nu} + P^{\sigma\rho} (D_{\rho}\, {}^{(m)}V_{\sigma\omega}) + P^{\nu}{}_{\omega} (D^{\mu} \log N)\, {}^{(m)}V_{\mu\nu} \end{aligned} \tag{C.10}$$

$$= -\frac{\epsilon}{N} (D_{\omega} N)\, {}^{(m)}V + \frac{1}{N} \mathcal{L}_{Nn} ({}^{(m)}V_{\omega}) - K\, {}^{(m)}V_{\omega} + D_{\lambda}\, {}^{(m)}V^{\lambda}{}_{\omega} + \frac{1}{N} (D_{\mu} N)\, {}^{(m)}V^{\mu}{}_{\omega}. \tag{C.11}$$

C.2 Decomposition of Energy Momentum Conservation

Now we calculate the decomposition of the energy momentum conservation equation;

$$\nabla^{\mu}\, {}^{(m+1)}T_{\mu\nu} = 0, \tag{C.12}$$

$${}^{(m+1)}T_{\mu\nu} \equiv \rho_H n_{\mu} n_{\nu} + 2 J_{(\mu} n_{\nu)} + S_{\mu\nu}, \tag{C.13}$$

where $\rho_H \in \mathcal{T}^0_0(\mathcal{M}^m)$, $J_{\nu} \in \mathcal{T}^0_1(\mathcal{M}^m)$ and $S_{\mu\nu} \in \mathcal{T}^0_2(\mathcal{M}^m)$. For (C.7) and (C.11), we adopt n_{μ} as time like ($\epsilon = -1$) and N and N^{μ} are expressed as α and β^{μ}, respectively. The energy conservation equation and the momentum conservation equations are

$$\partial_t \rho_H = \alpha K \rho_H - \alpha (D_{\mu} J^{\mu}) - 2 (D_{\mu} \alpha) J^{\mu} + \alpha K_{\mu\nu} S^{\mu\nu} + \mathcal{L}_{\beta} (\rho_H), \tag{C.14}$$

$$\partial_t J_{\mu} = -(D_{\mu} \alpha) \rho_H + \alpha K J_{\mu} - \alpha (D_{\nu} S^{\nu}{}_{\mu}) - (D_{\nu} \alpha) S^{\nu}{}_{\mu} + \mathcal{L}_{\beta} (J_{\mu}), \tag{C.15}$$

respectively.

C.3 Constraint Propagation of Standard ADM

Now we split the divergence of the Einstein equations into the components paralleled with n_{μ} and with $P^{\mu}{}_{\nu}$. The divergence of the Einstein equation is

$$\nabla^{\mu} ({}^{(m+1)}G_{\mu\nu} - \kappa T_{\mu\nu}) = 0. \tag{C.16}$$

To split the divergence of the Einstein equation in the ADM formulation, we first calculate the trace part of (2.55);

$$g^{\mu\nu}\, {}^{(m+1)}\mathcal{E}_{\mu\nu} = -\frac{2}{m-1} g^{\mu\nu} ({}^{(m+1)}G_{\mu\nu} - \kappa T_{\mu\nu}), \tag{C.17}$$

then we can get the relation;

$${}^{(m+1)}G_{\mu\nu} - \kappa T_{\mu\nu} = {}^{(m+1)}E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\lambda\omega}\, {}^{(m+1)}E_{\lambda\omega}. \tag{C.18}$$

The component of the divergence of (C.18) paralleled with n^μ is

$$n^\mu \nabla^\nu \left({}^{(m+1)}G_{\mu\nu} - \kappa T_{\mu\nu} \right) \tag{C.19}$$

$$= n^\mu \nabla^\nu \, {}^{(m+1)}E_{\mu\nu} - \frac{1}{2} n^\nu \nabla_\nu \left(g^{\lambda\omega} \, {}^{(m+1)}E_{\lambda\omega} \right) \tag{C.20}$$

$$\begin{aligned} =& \frac{\epsilon}{N} \mathcal{L}_{Nn}(\mathcal{H}) - \epsilon K \mathcal{H} + \epsilon (D_\mu \mathcal{M}^\mu) + \frac{2\epsilon}{N} (D_\mu N) \mathcal{M}^\mu + K_{\mu\nu} \, {}^{(m)}E^{\mu\nu} \\ & - \frac{1}{2N} \mathcal{L}_{Nn}(\epsilon \mathcal{H} + \gamma^{\mu\nu} \, {}^{(m)}E_{\mu\nu}) \\ =& \frac{\epsilon}{2N} \mathcal{L}_{Nn}(\mathcal{H}) - \epsilon K \mathcal{H} + \epsilon (D_\mu \mathcal{M}^\mu) + \frac{2\epsilon}{N} (D_\mu N) \mathcal{M}^\mu + K_{\mu\nu} \, {}^{(m)}E^{\mu\nu} - \frac{\epsilon}{2N} \mathcal{L}_{Nn}(\gamma^{\mu\nu} \, {}^{(m)}E_{\mu\nu}). \end{aligned} \tag{C.21}$$

The component of the divergence of (C.18) paralleled with $P^\mu{}_\nu$ is

$$P^\mu{}_\omega \nabla^\nu \left({}^{(m+1)}G_{\mu\nu} - \kappa T_{\mu\nu} \right) \tag{C.22}$$

$$= P^\mu{}_\omega \nabla^\nu \, {}^{(m+1)}E_{\mu\nu} - \frac{1}{2} P^\mu{}_\omega \nabla_\mu \left(g^{\lambda\omega} \, {}^{(m+1)}E_{\lambda\omega} \right) \tag{C.23}$$

$$\begin{aligned} =& - \frac{\epsilon}{N} (D_\omega N) \mathcal{H} + \frac{1}{N} \mathcal{L}_{Nn}(\mathcal{M}_\omega) - K \mathcal{M}_\omega + D_\lambda \, {}^{(m)}E^\lambda{}_\omega + \frac{1}{N} (D_\mu N) \, {}^{(m)}E^\mu{}_\omega \\ & - \frac{1}{2} D_\omega \left(\epsilon \mathcal{H} + P^{\lambda\omega} \, {}^{(m+1)}E_{\lambda\omega} \right). \end{aligned} \tag{C.24}$$

In the standard ADM formulation, n^μ is timelike and ${}^{(m)}E_{\mu\nu} = 0$. The constraint propagation equations of the m dimensional standard ADM formulation are

$$\partial_t \mathcal{H} = \mathcal{L}_\beta(\mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha (D_\mu \mathcal{M}^\mu) - 4 (D_\mu \alpha) \mathcal{M}^\mu, \tag{C.25}$$

$$\partial_t \mathcal{M}_\mu = \mathcal{L}_\beta(\mathcal{M}_\mu) - (D_\mu \alpha) \mathcal{H} + \alpha K \mathcal{M}_\mu - \frac{1}{2} \alpha (D_\mu \mathcal{H}). \tag{C.26}$$

C.4 Constraint Propagation of BSSN Formulation

With $\widetilde{D}_\mu$, the energy conservation equation (C.14) and the momentum conservation equations (C.15) can be written as

$$\begin{aligned} \partial_t \rho_H =& \alpha K \rho_H - \alpha e^{-4\varphi} (\widetilde{D}^\mu J_\mu) - 2(m-2) \alpha e^{-4\varphi} (\widetilde{D}^\mu \varphi) J_\mu - 2 e^{-4\varphi} (\widetilde{D}^\mu \alpha) J_\mu \\ & + \alpha e^{-4\varphi} \widetilde{A}^{\mu\nu} S_{\mu\nu} + \frac{1}{m} \alpha K S + \widetilde{\mathcal{L}}_\beta(\rho_H), \end{aligned} \tag{C.27}$$

$$\begin{aligned} \partial_t J_\mu =& - (\widetilde{D}_\mu \alpha) \rho_H + \alpha K J_\mu - \alpha e^{-4\varphi} (\widetilde{D}^\omega S_{\omega\mu}) + 2\alpha (\widetilde{D}_\mu \varphi) S - 2(m-2) \alpha (\widetilde{D}_\nu \varphi) S^\nu{}_\mu \\ & - (\widetilde{D}_\nu \alpha) S^\nu{}_\mu + \widetilde{\mathcal{L}}_\beta(J_\mu), \end{aligned} \tag{C.28}$$

respectively.

The propagation equation of $\widetilde{\mathcal{S}}$ is

$$\partial_t \widetilde{\mathcal{S}} = \partial_t \widetilde{\gamma} \tag{C.29}$$

$$= \widetilde{\gamma} \widetilde{\gamma}^{\mu\nu} (\partial_t \widetilde{\gamma}_{\mu\nu}) \tag{C.30}$$

$$= \widetilde{\gamma} \widetilde{\gamma}^{\mu\nu} \left(-2\alpha \widetilde{A}_{\mu\nu} - \frac{2}{m} (\widetilde{D}_\lambda \beta^\lambda) \widetilde{\gamma}_{\mu\nu} + \widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\mu\nu}) + \frac{1}{m\widetilde{\gamma}} \beta^\lambda (\widetilde{D}_\lambda \widetilde{\mathcal{S}}) \widetilde{\gamma}_{\mu\nu} \right) \tag{C.31}$$

$$= -2\alpha \widetilde{\gamma} \widetilde{\mathcal{A}} + \widetilde{\mathcal{L}}_\beta(\widetilde{\mathcal{S}}). \tag{C.32}$$

The propagation equation of $\widetilde{\mathcal{A}}$ is

$$\partial_t\widetilde{\mathcal{A}} = -\widetilde{A}^{\mu\nu}(\partial_t\widetilde{\gamma}_{\mu\nu}) + \widetilde{\gamma}^{\mu\nu}(\partial_t\widetilde{A}_{\mu\nu}) \tag{C.33}$$

$$\begin{aligned}
&= -\widetilde{A}^{\mu\nu}\Big\{-2\alpha\widetilde{A}_{\mu\nu} - \frac{2}{m}(\widetilde{D}_\lambda\beta^\lambda)\widetilde{\gamma}_{\mu\nu} + \widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\mu\nu}) + \frac{1}{m\widetilde{\gamma}}\beta^\lambda(\widetilde{D}_\lambda\widetilde{\mathcal{S}})\widetilde{\gamma}_{\mu\nu}\Big\} \\
&\quad + \widetilde{\gamma}^{\mu\nu}\Big\{\alpha K\widetilde{A}_{\mu\nu} - 2\alpha\widetilde{A}_{\mu\lambda}\widetilde{A}^\lambda{}_\nu + \alpha e^{-4\varphi}({}^{(m)}\widetilde{R}_{\mu\nu} + {}^{(m)}R^{\varphi}_{\mu\nu})^{\mathrm{TF}} - e^{-4\varphi}(D_\mu D_\nu\alpha)^{\mathrm{TF}} \\
&\qquad - \frac{2}{m}(\widetilde{D}_\lambda\beta^\lambda)\widetilde{A}_{\mu\nu} + \widetilde{\mathcal{L}}_\beta(\widetilde{A}_{\mu\nu}) - \kappa\alpha(e^{-4\varphi}S_{\mu\nu})^{\mathrm{TF}} + \frac{1}{m\widetilde{\gamma}}\beta^\lambda(\widetilde{D}_\omega\widetilde{\mathcal{S}})\widetilde{A}_{\mu\nu}\Big\}
\end{aligned} \tag{C.34}$$

$$= -\widetilde{A}^{\mu\nu}\widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\mu\nu}) + \alpha K\widetilde{\mathcal{A}} + \widetilde{\gamma}^{\mu\nu}\widetilde{\mathcal{L}}_\beta(\widetilde{A}_{\mu\nu}) \tag{C.35}$$

$$= \alpha K\widetilde{\mathcal{A}} + \widetilde{\mathcal{L}}_\beta(\widetilde{\mathcal{A}}). \tag{C.36}$$

For the right-hand-side of the propagation equations of the $\mathcal{G}^\lambda$, this is just the constraint term which might be zero in the evolution of the Γ^λ, (3.112),

$$\begin{aligned}
\partial_t\widetilde{\mathcal{G}}^\lambda &= 2\alpha\widetilde{\mathcal{M}}^\lambda - (\widetilde{D}^\lambda\alpha)\widetilde{\mathcal{A}} - \alpha(\widetilde{D}^\lambda\widetilde{\mathcal{A}}) + 4\alpha(\widetilde{D}^\lambda\varphi)\widetilde{\mathcal{A}} + \frac{1}{m\widetilde{\gamma}}\beta^\rho\widetilde{\Gamma}^\lambda(\widetilde{D}_\rho\widetilde{\mathcal{S}}) - \frac{2}{m\widetilde{\gamma}}\beta^\rho\widetilde{\mathcal{G}}^\lambda(\widetilde{D}_\rho\widetilde{\mathcal{S}}) \\
&\quad - \frac{m-1}{m\widetilde{\gamma}^2}\beta^\rho(\widetilde{D}^\lambda\widetilde{\mathcal{S}})(\widetilde{D}_\rho\widetilde{\mathcal{S}}) - \frac{2-m}{2m\widetilde{\gamma}}(\widetilde{D}^\lambda\beta^\rho)(\widetilde{D}_\rho\widetilde{\mathcal{S}}) - \frac{1}{m\widetilde{\gamma}}\beta^\rho(\widetilde{D}^\lambda\widetilde{D}_\rho\widetilde{\mathcal{S}}) + \beta^\rho(\widetilde{D}_\rho\widetilde{\mathcal{G}}^\lambda) \\
&\quad + \frac{1}{2\widetilde{\gamma}}\beta^\omega\,{}^{(m)}\widetilde{\Gamma}^{\lambda\nu}{}_\omega(\widetilde{D}_\nu\widetilde{\mathcal{S}}) - (\widetilde{D}_\omega\beta^\lambda)\widetilde{\mathcal{G}}^\omega + \frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\widetilde{\mathcal{G}}^\lambda.
\end{aligned} \tag{C.37}$$

The propagation equation of $\widetilde{\mathcal{M}}_\mu$ is

$$\begin{aligned}
\partial_t\widetilde{\mathcal{M}}_\mu &= (\partial_t\widetilde{\gamma}^{\nu\lambda})(\widetilde{D}_\nu\widetilde{A}_{\lambda\mu}) + \widetilde{\gamma}^{\nu\lambda}\Big\{\partial_\nu(\partial_t\widetilde{A}_{\lambda\mu}) - (\partial_t\widetilde{\gamma}^{\omega\rho})\,{}^{(m)}\widetilde{\Gamma}_{\rho\nu\lambda}\widetilde{A}_{\omega\mu} - (\partial_t\,{}^{(m)}\widetilde{\Gamma}_{\rho\nu\lambda})\widetilde{A}^\rho{}_\mu \\
&\quad - {}^{(m)}\widetilde{\Gamma}^\omega{}_{\nu\lambda}(\partial_t\widetilde{A}_{\omega\mu}) - (\partial_t\widetilde{\gamma}^{\omega\rho})\,{}^{(m)}\widetilde{\Gamma}_{\rho\nu\mu}\widetilde{A}_{\lambda\omega} - (\partial_t\,{}^{(m)}\widetilde{\Gamma}_{\rho\nu\mu})\widetilde{A}_\lambda{}^\rho - {}^{(m)}\widetilde{\Gamma}^\omega{}_{\nu\mu}(\partial_t\widetilde{A}_{\lambda\omega})\Big\} \\
&\quad + 2m\partial_\lambda(\partial_t\varphi)\widetilde{A}^\lambda{}_\mu + 2m(\widetilde{D}_\lambda\varphi)(\partial_t\widetilde{\gamma}^{\lambda\omega})\widetilde{A}_{\omega\mu} + 2m(\widetilde{D}^\omega\varphi)(\partial_t\widetilde{A}_{\omega\mu}) \\
&\quad - \frac{m-1}{m}\partial_\mu(\partial_t K) - 2\partial_\mu(\partial_t\varphi)\widetilde{\mathcal{A}} - 2(\widetilde{D}_\mu\varphi)(\partial_t\widetilde{\mathcal{A}}) - \kappa(\partial_t J_\mu)
\end{aligned} \tag{C.38}$$

$$\begin{aligned}
&= -(\partial_t\widetilde{\gamma}_{\nu\lambda})(\widetilde{D}^\nu\widetilde{A}^\lambda{}_\mu) + \widetilde{\gamma}^{\nu\lambda}\Big[\partial_\nu(\partial_t\widetilde{A}_{\lambda\mu}) + (\partial_t\widetilde{\gamma}_{\omega\rho})\,{}^{(m)}\widetilde{\Gamma}^\rho{}_{\nu\lambda}\widetilde{A}^\omega{}_\mu - \frac{1}{2}\Big\{\partial_\lambda(\partial_t\widetilde{\gamma}_{\rho\nu}) \\
&\quad + \partial_\nu(\partial_t\widetilde{\gamma}_{\rho\lambda}) - \partial_\rho(\partial_t\widetilde{\gamma}_{\nu\lambda})\Big\}\widetilde{A}^\rho{}_\mu - {}^{(m)}\widetilde{\Gamma}^\omega{}_{\nu\lambda}(\partial_t\widetilde{A}_{\omega\mu}) + (\partial_t\widetilde{\gamma}_{\omega\rho})\,{}^{(m)}\widetilde{\Gamma}^\rho{}_{\nu\mu}\widetilde{A}_\lambda{}^\omega \\
&\quad - \frac{1}{2}\Big\{\partial_\mu(\partial_t\widetilde{\gamma}_{\rho\nu}) + \partial_\nu(\partial_t\widetilde{\gamma}_{\rho\mu}) - \partial_\rho(\partial_t\widetilde{\gamma}_{\nu\mu})\Big\}\widetilde{A}_\lambda{}^\rho - {}^{(m)}\widetilde{\Gamma}^\omega{}_{\nu\mu}(\partial_t\widetilde{A}_{\lambda\omega})\Big]
\end{aligned} \tag{C.39}$$

$$
\begin{aligned}
&+2m\widetilde{D}_\lambda(\partial_t\varphi)\widetilde{A}^\lambda{}_\mu-2m(\widetilde{D}^\lambda\varphi)(\partial_t\widetilde{\gamma}_{\lambda\omega})\widetilde{A}^\omega{}_\mu+2m(\widetilde{D}^\omega\varphi)(\partial_t\widetilde{A}_{\omega\mu})\\
&-\frac{m-1}{m}\widetilde{D}_\mu(\partial_tK)-2\widetilde{D}_\mu(\partial_t\varphi)\widetilde{\mathcal{A}}-2(\widetilde{D}_\mu\varphi)(\partial_t\widetilde{\mathcal{A}})-\kappa(\partial_tJ_\mu) \qquad \text{(C.40)}\\
=&-(\widetilde{D}^\nu\widetilde{A}^\lambda{}_\mu)(\partial_t\widetilde{\gamma}_{\nu\lambda})+\widetilde{D}^\lambda(\partial_t\widetilde{A}_{\lambda\mu})-\widetilde{D}^\nu(\partial_t\widetilde{\gamma}_{\omega\nu})\widetilde{A}^\omega{}_\mu+\frac{1}{2}\widetilde{\gamma}^{\nu\lambda}\widetilde{D}_\omega(\partial_t\widetilde{\gamma}_{\nu\lambda})\widetilde{A}^\omega{}_\mu\\
&-\frac{1}{2}\widetilde{D}_\mu(\partial_t\widetilde{\gamma}_{\rho\nu})\widetilde{A}^{\nu\rho}+2m\widetilde{D}_\lambda(\partial_t\varphi)\widetilde{A}^\lambda{}_\mu-2m(\widetilde{D}^\lambda\varphi)(\partial_t\widetilde{\gamma}_{\lambda\omega})\widetilde{A}^\omega{}_\mu+2m(\widetilde{D}^\omega\varphi)(\partial_t\widetilde{A}_{\omega\mu})\\
&-\frac{m-1}{m}\widetilde{D}_\mu(\partial_tK)-2\widetilde{D}_\mu(\partial_t\varphi)\widetilde{\mathcal{A}}-2(\widetilde{D}_\mu\varphi)(\partial_t\widetilde{\mathcal{A}})-\kappa(\partial_tJ_\mu) \qquad \text{(C.41)}\\
=&-(\widetilde{D}^\nu\widetilde{A}^\lambda{}_\mu)\Big\{-2\alpha\widetilde{A}_{\nu\lambda}-\frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\widetilde{\gamma}_{\nu\lambda}+\widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\nu\lambda})+\frac{1}{m\widetilde{\gamma}}\beta^\rho(\widetilde{D}_\rho\widetilde{\mathcal{S}})\widetilde{\gamma}_{\nu\lambda}\Big\}\\
&+\widetilde{D}^\lambda\Big\{\alpha K\widetilde{A}_{\lambda\mu}-2\alpha\widetilde{A}_{\lambda\rho}\widetilde{A}^\rho{}_\mu+\alpha e^{-4\varphi}({}^{(m)}\widetilde{R}_{\lambda\mu}+{}^{(m)}R^{\varphi}_{\lambda\mu})^{\mathrm{TF}}-e^{-4\varphi}(D_\lambda D_\mu\alpha)^{\mathrm{TF}}\\
&-\frac{2}{m}(\widetilde{D}_\omega\beta^\omega)\widetilde{A}_{\lambda\mu}+\widetilde{\mathcal{L}}_\beta(\widetilde{A}_{\lambda\mu})-\kappa\alpha(e^{-4\varphi}S_{\lambda\mu})^{\mathrm{TF}}+\frac{1}{m\widetilde{\gamma}}\beta^\omega(\widetilde{D}_\omega\widetilde{\mathcal{S}})\widetilde{A}_{\lambda\mu}\Big\}\\
&-\widetilde{A}^\omega{}_\mu\widetilde{D}^\nu\Big\{-2\alpha\widetilde{A}_{\omega\nu}-\frac{2}{m}(\widetilde{D}_\lambda\beta^\lambda)\widetilde{\gamma}_{\omega\nu}+\widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\omega\nu})+\frac{1}{m\widetilde{\gamma}}\beta^\lambda(\widetilde{D}_\lambda\widetilde{\mathcal{S}})\widetilde{\gamma}_{\omega\nu}\Big\}\\
&+\frac{1}{2}\widetilde{\gamma}^{\nu\lambda}\widetilde{A}^\omega{}_\mu\widetilde{D}_\omega\Big\{-2\alpha\widetilde{A}_{\nu\lambda}-\frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\widetilde{\gamma}_{\nu\lambda}+\widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\nu\lambda})+\frac{1}{m\widetilde{\gamma}}\beta^\rho(\widetilde{D}_\rho\widetilde{\mathcal{S}})\widetilde{\gamma}_{\nu\lambda}\Big\}\\
&-\frac{1}{2}\widetilde{A}^{\nu\rho}\widetilde{D}_\mu\Big\{-2\alpha\widetilde{A}_{\rho\nu}-\frac{2}{m}(\widetilde{D}_\lambda\beta^\lambda)\widetilde{\gamma}_{\rho\nu}+\widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\rho\nu})+\frac{1}{m\widetilde{\gamma}}\beta^\lambda(\widetilde{D}_\lambda\widetilde{\mathcal{S}})\widetilde{\gamma}_{\rho\nu}\Big\}\\
&+2m\widetilde{A}^\lambda{}_\mu\widetilde{D}_\lambda\Big\{-\frac{1}{2m}\alpha K+\frac{1}{2m}(\widetilde{D}_\omega\beta^\omega)+\widetilde{\mathcal{L}}_\beta(\varphi)-\frac{1}{m\widetilde{\gamma}}\beta^\omega(\widetilde{D}_\omega\widetilde{\mathcal{S}})\Big\}\\
&-2m(\widetilde{D}^\lambda\varphi)\widetilde{A}^\omega{}_\mu\Big\{-2\alpha\widetilde{A}_{\lambda\omega}-\frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\widetilde{\gamma}_{\lambda\omega}+\widetilde{\mathcal{L}}_\beta(\widetilde{\gamma}_{\lambda\omega})+\frac{1}{m\widetilde{\gamma}}\beta^\rho(\widetilde{D}_\rho\widetilde{\mathcal{S}})\widetilde{\gamma}_{\lambda\omega}\Big\}\\
&+2m(\widetilde{D}^\omega\varphi)\Big\{\alpha K\widetilde{A}_{\omega\mu}-2\alpha\widetilde{A}_{\omega\rho}\widetilde{A}^\rho{}_\mu+\alpha e^{-4\varphi}({}^{(m)}\widetilde{R}_{\omega\mu}+{}^{(m)}R^{\varphi}_{\omega\mu})^{\mathrm{TF}}-e^{-4\varphi}(D_\omega D_\mu\alpha)^{\mathrm{TF}}\\
&-\frac{2}{m}(\widetilde{D}_\rho\beta^\rho)\widetilde{A}_{\omega\mu}+\widetilde{\mathcal{L}}_\beta(\widetilde{A}_{\omega\mu})-\kappa\alpha(e^{-4\varphi}S_{\omega\mu})^{\mathrm{TF}}+\frac{1}{m\widetilde{\gamma}}\beta^\rho(\widetilde{D}_\rho\widetilde{\mathcal{S}})\widetilde{A}_{\omega\mu}\Big\}\\
&-\frac{m-1}{m}\widetilde{D}_\mu\Big\{\alpha\widetilde{A}_{\lambda\omega}\widetilde{A}^{\lambda\omega}+\frac{1}{m}\alpha K^2-D^\lambda D_\lambda\alpha+\widetilde{\mathcal{L}}_\beta(K)-\frac{2}{m-1}\alpha\Lambda\\
&+\frac{\kappa}{m-1}\alpha\{S+(m-2)\rho_H\}\Big\}-2(\widetilde{D}_\mu\varphi)\{\alpha K\widetilde{\mathcal{A}}+\widetilde{\mathcal{L}}_\beta(\widetilde{\mathcal{A}})\}\\
&-2\widetilde{D}_\mu\Big\{-\frac{1}{2m}\alpha K+\frac{1}{2m}(\widetilde{D}_\lambda\beta^\lambda)+\widetilde{\mathcal{L}}_\beta(\varphi)-\frac{1}{m\widetilde{\gamma}}\beta^\lambda(\widetilde{D}_\lambda\widetilde{\mathcal{S}})\Big\}\widetilde{\mathcal{A}}\\
&-\kappa\Big\{-(\widetilde{D}_\mu\alpha)\rho_H+\alpha KJ_\mu-\alpha e^{-4\varphi}(\widetilde{D}^\omega S_{\omega\mu})+2\alpha(\widetilde{D}_\mu\varphi)S-2(m-2)\alpha(\widetilde{D}_\nu\varphi)S^\nu{}_\mu\\
&-(\widetilde{D}_\nu\alpha)S^\nu{}_\mu+\widetilde{\mathcal{L}}_\beta(J_\mu)\Big\} \qquad \text{(C.42)}
\end{aligned}
$$

$$
\begin{aligned}
=& \boxed{-{}^{(m)}R_{\rho\mu}D^{\rho}\alpha} \boxed{+\alpha K\widetilde{\mathcal{M}}_{\mu}+\frac{(m-1)(m-2)}{m^2}\alpha K(\widetilde{D}_{\mu}K)-\frac{m-1}{m^2}(\widetilde{D}_{\mu}\alpha)K^2} \\
&\boxed{+\beta^{\lambda}(\widetilde{D}^{\omega}\widetilde{D}_{\lambda}\widetilde{A}_{\omega\mu})-\beta^{\lambda}(\widetilde{D}_{\lambda}\widetilde{D}^{\omega}\widetilde{A}_{\omega\mu})+\widetilde{A}^{\nu}{}_{\lambda}(\widetilde{D}_{\nu}\widetilde{D}_{\mu}\beta^{\lambda})-\widetilde{A}^{\omega}{}_{\mu}(\widetilde{D}_{\rho}\widetilde{D}_{\omega}\beta^{\rho})+\widetilde{\mathcal{L}}_{\beta}(\widetilde{\mathcal{M}}_{\mu})} \\
&\boxed{+(\widetilde{D}^{\lambda}\alpha)e^{-4\varphi}({}^{(m)}\widetilde{R}_{\lambda\mu}+{}^{(m)}R^{\varphi}_{\lambda\mu})}-\frac{1}{m}(\widetilde{D}_{\mu}\alpha)\left(\widetilde{\mathcal{H}}-\frac{m-1}{m}K^2+\frac{3}{m}K\widetilde{\mathcal{A}}\right) \\
&\boxed{+2(m-2)\alpha(\widetilde{D}^{\lambda}\varphi)e^{-4\varphi}({}^{(m)}\widetilde{R}_{\lambda\mu}+{}^{(m)}R^{\varphi}_{\lambda\mu})+\alpha e^{-4\varphi}\widetilde{D}^{\lambda}({}^{(m)}\widetilde{R}_{\lambda\mu}+{}^{(m)}R^{\varphi}_{\lambda\mu})} \\
&+\frac{m-2}{2m}\alpha\Big\{\widetilde{D}_{\mu}\widetilde{\mathcal{H}}\boxed{-\frac{m}{m-2}e^{-4\varphi}\widetilde{D}_{\mu}({}^{(m)}\widetilde{R}+{}^{(m)}R^{\varphi})}-\frac{m-1}{m}(\widetilde{D}_{\mu}K^2)+\frac{m}{m-2}(\widetilde{D}_{\mu}K)\widetilde{\mathcal{A}} \\
&\quad+\frac{2}{m}\widetilde{D}_{\mu}(K\widetilde{\mathcal{A}})\Big\}\boxed{+\widetilde{A}^{\omega}{}_{\mu}(\widetilde{D}_{\omega}\widetilde{D}_{\nu}\beta^{\nu})-\widetilde{A}_{\omega}{}^{\rho}(\widetilde{D}_{\mu}\widetilde{D}_{\rho}\beta^{\omega})}-(\widetilde{D}_{\omega}\alpha)\widetilde{A}^{\omega}{}_{\mu}\widetilde{\mathcal{A}}-\alpha\widetilde{A}^{\omega}{}_{\mu}(\widetilde{D}_{\omega}\widetilde{\mathcal{A}}) \\
&+\frac{3}{2\widetilde{\gamma}^2}\beta^{\omega}\widetilde{A}^{\lambda}{}_{\mu}(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\omega}\widetilde{\mathcal{S}})-\frac{3}{2\widetilde{\gamma}}(\widetilde{D}_{\lambda}\beta^{\omega})\widetilde{A}^{\lambda}{}_{\mu}(\widetilde{D}_{\omega}\widetilde{\mathcal{S}})-\frac{3}{2\widetilde{\gamma}}\beta^{\omega}\widetilde{A}^{\lambda}{}_{\mu}(\widetilde{D}_{\lambda}\widetilde{D}_{\omega}\widetilde{\mathcal{S}}) \\
&-\frac{3}{2m\widetilde{\gamma}^2}\beta^{\lambda}(\widetilde{D}_{\mu}\widetilde{\mathcal{S}})(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})\widetilde{\mathcal{A}}+\frac{3}{2m\widetilde{\gamma}}(\widetilde{D}_{\mu}\beta^{\lambda})(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})\widetilde{\mathcal{A}}+\frac{3}{2\widetilde{\gamma}}\beta^{\lambda}(\widetilde{D}_{\mu}\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})\widetilde{\mathcal{A}} \qquad \text{(C.43)} \\
=& \boxed{(\widetilde{D}^{\nu}\alpha)e^{-4\varphi}(\widetilde{D}_{(\mu}\widetilde{\mathcal{G}}_{\nu)})}-\frac{1}{m}(\widetilde{D}_{\mu}\alpha)\widetilde{\mathcal{H}}-\frac{3}{m^2}(\widetilde{D}_{\mu}\alpha)K\widetilde{\mathcal{A}}+\alpha K\widetilde{\mathcal{M}}_{\mu}+\frac{m-2}{2m}\alpha(\widetilde{D}_{\mu}\widetilde{\mathcal{H}}) \\
&\boxed{+2(m-2)\alpha e^{-4\varphi}(\widetilde{D}^{\lambda}\varphi)(\widetilde{D}_{(\mu}\widetilde{\mathcal{G}}_{\lambda)})}+\frac{1}{2}(\widetilde{D}_{\mu}K)\widetilde{\mathcal{A}}+\frac{m-2}{m^2}\alpha K(\widetilde{D}_{\mu}\widetilde{\mathcal{A}})+\frac{m-2}{m^2}\alpha(\widetilde{D}_{\mu}K)\widetilde{\mathcal{A}} \\
&-(\widetilde{D}_{\omega}\alpha)\widetilde{A}^{\omega}{}_{\mu}\widetilde{\mathcal{A}}-\alpha\widetilde{A}^{\omega}{}_{\mu}(\widetilde{D}_{\omega}\widetilde{\mathcal{A}})+\frac{3}{2\widetilde{\gamma}^2}\beta^{\omega}\widetilde{A}^{\lambda}{}_{\mu}(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\omega}\widetilde{\mathcal{S}})-\frac{3}{2\widetilde{\gamma}}(\widetilde{D}_{\lambda}\beta^{\omega})\widetilde{A}^{\lambda}{}_{\mu}(\widetilde{D}_{\omega}\widetilde{\mathcal{S}}) \\
&-\frac{3}{2\widetilde{\gamma}}\beta^{\omega}\widetilde{A}^{\lambda}{}_{\mu}(\widetilde{D}_{\lambda}\widetilde{D}_{\omega}\widetilde{\mathcal{S}})-\frac{3}{2m\widetilde{\gamma}^2}\beta^{\lambda}(\widetilde{D}_{\mu}\widetilde{\mathcal{S}})(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})\widetilde{\mathcal{A}}+\frac{3}{2m\widetilde{\gamma}}(\widetilde{D}_{\mu}\beta^{\lambda})(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})\widetilde{\mathcal{A}} \\
&+\frac{3}{2\widetilde{\gamma}}\beta^{\lambda}(\widetilde{D}_{\mu}\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})\widetilde{\mathcal{A}}+\widetilde{\mathcal{L}}_{\beta}(\widetilde{\mathcal{M}}_{\mu}). \qquad \text{(C.44)}
\end{aligned}
$$

To calculate the propagation equations of $\widetilde{\mathcal{H}}$, we first ${}^{(m)}\widetilde{R}$ and ${}^{(m)}R^{\varphi}$. The conformal scalar curvature is

$$
{}^{(m)}\widetilde{R}=\partial_{\nu}\widetilde{\Gamma}^{\nu}+{}^{(m)}\widetilde{\Gamma}^{\mu}{}_{\mu\omega}\,{}^{(m)}\widetilde{\Gamma}^{\omega}-\frac{1}{2}\widetilde{\gamma}^{\lambda\omega}\widetilde{\gamma}^{\mu\nu}(\partial_{\lambda}\partial_{\omega}\widetilde{\gamma}_{\mu\nu})+{}^{(m)}\widetilde{\Gamma}^{\lambda\mu\nu}\,{}^{(m)}\widetilde{\Gamma}_{\lambda\mu\nu}+2\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu\nu}\,{}^{(m)}\Gamma_{\nu\lambda\mu}, \qquad \text{(C.45)}
$$

then, the dynamical equation is

$$
\begin{aligned}
\partial_t\,{}^{(m)}\widetilde{R}=&\partial_t\Big\{\partial_{\nu}\widetilde{\Gamma}^{\nu}+{}^{(m)}\widetilde{\Gamma}^{\mu}{}_{\mu\omega}\,{}^{(m)}\widetilde{\Gamma}^{\omega}-\frac{1}{2}\widetilde{\gamma}^{\lambda\omega}\widetilde{\gamma}^{\mu\nu}(\partial_{\lambda}\partial_{\omega}\widetilde{\gamma}_{\mu\nu})+{}^{(m)}\widetilde{\Gamma}^{\lambda\mu\nu}\,{}^{(m)}\widetilde{\Gamma}_{\lambda\mu\nu} \\
&\quad+2\,{}^{(m)}\widetilde{\Gamma}^{\lambda\mu\nu}\,{}^{(m)}\widetilde{\Gamma}_{\nu\lambda\mu}\Big\} \qquad \text{(C.46)} \\
=&\widetilde{D}_{\mu}(\partial_t\widetilde{\Gamma}^{\mu})+\partial_t({}^{(m)}\widetilde{\Gamma}^{\mu}{}_{\mu\nu})\widetilde{\Gamma}^{\nu}+(\partial_{\lambda}\,{}^{(m)}\widetilde{\Gamma}^{\mu}{}_{\omega\mu})\widetilde{\gamma}^{\lambda\rho}\widetilde{\gamma}^{\omega\sigma}(\partial_t\widetilde{\gamma}_{\rho\sigma}) \\
&-{}^{(m)}\widetilde{\Gamma}^{\lambda\rho\nu}\,{}^{(m)}\widetilde{\Gamma}_{\nu\lambda}{}^{\mu}(\partial_t\widetilde{\gamma}_{\mu\rho})-\frac{1}{2}\widetilde{\gamma}^{\mu\nu}\widetilde{\gamma}^{\lambda\omega}\widetilde{D}_{\lambda}\widetilde{D}_{\omega}(\partial_t\widetilde{\gamma}_{\mu\nu})-\frac{1}{2}\,{}^{(m)}\widetilde{\Gamma}^{\rho\lambda}{}_{\lambda}\widetilde{\gamma}^{\mu\nu}\widetilde{D}_{\rho}(\partial_t\widetilde{\gamma}_{\mu\nu}) \\
&+{}^{(m)}\widetilde{\Gamma}^{\mu\nu\lambda}\widetilde{D}_{\mu}(\partial_t\widetilde{\gamma}_{\lambda\nu}). \qquad \text{(C.47)}
\end{aligned}
$$

The ${}^{(m)}R^{\varphi}$ can be expressed as

$$ {}^{(m)}R^{\varphi} = -4(m-1)(\widetilde{D}_\mu\widetilde{D}^\mu\varphi) - 4(m-1)(m-2)(\widetilde{D}^\lambda\varphi)(\widetilde{D}_\lambda\varphi) \tag{C.48} $$

and the dynamical equation of ${}^{(m)}R^{\varphi}$ is

$$ \partial_t\, {}^{(m)}R^{\varphi} = \partial_t\Big\{-4(m-1)(\widetilde{D}_\lambda\widetilde{D}^\lambda\varphi) - 4(m-2)(m-1)(\widetilde{D}^\mu\varphi)(\widetilde{D}_\mu\varphi)\Big\} \tag{C.49} $$

$$ \begin{aligned} &= 4(m-1)(\widetilde{D}^\lambda\widetilde{D}^\omega\varphi)(\partial_t\widetilde{\gamma}_{\lambda\omega}) + 4(m-1)\,{}^{(m)}\widetilde{\Gamma}^{\rho\lambda\omega}(\widetilde{D}_\rho\varphi)(\partial_t\widetilde{\gamma}_{\lambda\omega}) \\ &\quad - 4(m-1)\widetilde{\gamma}^{\mu\nu}\widetilde{D}_\mu\widetilde{D}_\nu(\partial_t\varphi) - 4(m-1)(\widetilde{D}_\omega\varphi)(\partial_t\widetilde{\mathcal{G}}^\lambda) + 4(m-1)(\widetilde{D}_\omega\varphi)(\partial_t\widetilde{\Gamma}^\omega) \\ &\quad + 4(m-2)(m-1)(\widetilde{D}^\omega\varphi)(\widetilde{D}^\lambda\varphi)(\partial_t\widetilde{\gamma}_{\omega\lambda}) - 8(m-2)(m-1)(\widetilde{D}^\mu\varphi)\widetilde{D}_\mu(\partial_t\varphi). \end{aligned} \tag{C.50} $$

The propagation equation of the Hamiltonian constraint equation is

$$ \begin{aligned} \partial_t\widetilde{\mathcal{H}} &= -4e^{-4\varphi}({}^{(m)}\widetilde{R} + {}^{(m)}R^{\varphi})(\partial_t\varphi) + e^{-4\varphi}\Big\{\widetilde{D}_\mu(\partial_t\widetilde{\Gamma}^\mu) + \partial_t({}^{(m)}\widetilde{\Gamma}^\mu{}_{\mu\nu})\widetilde{\Gamma}^\nu \\ &\quad + (\partial_\lambda\,{}^{(m)}\widetilde{\Gamma}^\mu{}_{\omega\mu})\widetilde{\gamma}^{\lambda\rho}\widetilde{\gamma}^{\omega\sigma}(\partial_t\widetilde{\gamma}_{\rho\sigma}) - {}^{(m)}\widetilde{\Gamma}^{\lambda\rho\nu}\,{}^{(m)}\widetilde{\Gamma}_{\nu\lambda}{}^{\mu}(\partial_t\widetilde{\gamma}_{\mu\rho}) - \frac{1}{2}\widetilde{\gamma}^{\mu\nu}\widetilde{\gamma}^{\lambda\omega}\widetilde{D}_\lambda\widetilde{D}_\omega(\partial_t\widetilde{\gamma}_{\mu\nu}) \\ &\quad - \frac{1}{2}\,{}^{(m)}\widetilde{\Gamma}^{\rho\lambda}{}_{\lambda}\widetilde{\gamma}^{\mu\nu}\widetilde{D}_\rho(\partial_t\widetilde{\gamma}_{\mu\nu}) + {}^{(m)}\widetilde{\Gamma}^{\mu\nu\lambda}\widetilde{D}_\mu(\partial_t\widetilde{\gamma}_{\lambda\nu})\Big\} \\ &\quad + e^{-4\varphi}\Big\{4(m-1)(\widetilde{D}^\lambda\widetilde{D}^\omega\varphi)(\partial_t\widetilde{\gamma}_{\lambda\omega}) + 4(m-1)\,{}^{(m)}\widetilde{\Gamma}^{\rho\lambda\omega}(\widetilde{D}_\rho\varphi)(\partial_t\widetilde{\gamma}_{\lambda\omega}) \\ &\quad - 4(m-1)\widetilde{\gamma}^{\mu\nu}\widetilde{D}_\mu\widetilde{D}_\nu(\partial_t\varphi) - 4(m-1)(\widetilde{D}_\lambda\varphi)(\partial_t\widetilde{\mathcal{G}}^\lambda) + 4(m-1)(\widetilde{D}_\omega\varphi)(\partial_t\widetilde{\Gamma}^\omega) \\ &\quad + 4(m-2)(m-1)(\widetilde{D}^\omega\varphi)(\widetilde{D}^\lambda\varphi)(\partial_t\widetilde{\gamma}_{\omega\lambda}) - 8(m-2)(m-1)(\widetilde{D}^\mu\varphi)\widetilde{D}_\mu(\partial_t\varphi)\Big\} \\ &\quad + \frac{2(m-1)}{m}K(\partial_tK) - 2\widetilde{A}^{\mu\nu}(\partial_t\widetilde{A}_{\mu\nu}) + 2\widetilde{A}^\mu{}_\nu\widetilde{A}^{\lambda\nu}(\partial_t\widetilde{\gamma}_{\mu\lambda}) - \frac{2}{m}\partial_t(K\widetilde{\mathcal{A}}) - 2\kappa(\partial_t\rho_H) \end{aligned} \tag{C.51} $$

$$ \begin{aligned} &= \frac{2}{m}\alpha\widetilde{\mathcal{H}} + \widetilde{\mathcal{L}}_\beta(\widetilde{\mathcal{H}}) - 2(\widetilde{D}^\mu\alpha)e^{-4\varphi}\widetilde{\mathcal{M}}_\mu - 4(m-2)\alpha e^{-4\varphi}(\widetilde{D}_\lambda\varphi)\widetilde{\mathcal{M}}^\lambda \\ &\quad + \frac{2(m-1)}{m^2}\alpha K^2\widetilde{\mathcal{A}} + 5\alpha e^{-4\varphi}(\widetilde{D}^\lambda\widetilde{D}_\lambda\widetilde{\mathcal{A}}) - \frac{2}{m}\alpha e^{-4\varphi}({}^{(m)}\widetilde{R} + {}^{(m)}R^{\varphi})\widetilde{\mathcal{A}} \\ &\quad + (\widetilde{D}_\mu\alpha)e^{-4\varphi}\widetilde{\mathcal{G}}^\mu\widetilde{\mathcal{A}} - \alpha e^{-4\varphi}\widetilde{\mathcal{G}}^\rho(\widetilde{D}_\rho\widetilde{\mathcal{A}}) + (\widetilde{D}_\lambda\widetilde{D}^\lambda\alpha)e^{-4\varphi}\widetilde{\mathcal{A}} + 2(\widetilde{D}_\lambda\alpha)e^{-4\varphi}(\widetilde{D}^\lambda\widetilde{\mathcal{A}}) \\ &\quad - \frac{2\kappa}{m}\alpha S\widetilde{\mathcal{A}} + 4(m-2)(\widetilde{D}^\lambda\alpha)e^{-4\varphi}(\widetilde{D}_\lambda\varphi)\widetilde{\mathcal{A}} + 4(m-1)\alpha e^{-4\varphi}(\widetilde{D}_\lambda\varphi)(\widetilde{D}^\lambda\widetilde{\mathcal{A}}) \\ &\quad - \frac{2}{m}\widetilde{\mathcal{A}}\Big\{\alpha\widetilde{A}_{\mu\nu}\widetilde{A}^{\mu\nu} + \widetilde{\mathcal{L}}_\beta(K) - \frac{2}{m-1}\alpha\Lambda + \frac{\kappa}{m-1}\alpha\{S + (m-2)\rho_H\}\Big\} \\ &\quad - \frac{2}{m}K\widetilde{\mathcal{L}}_\beta(\widetilde{\mathcal{A}}) - 2\alpha e^{-4\varphi}\widetilde{A}^{\mu\nu}(\widetilde{D}_\nu\widetilde{\mathcal{G}}_\mu) - 2e^{-4\varphi}(\widetilde{D}^\nu\widetilde{D}_{(\lambda}\widetilde{\mathcal{G}}_{\nu)})\beta^\lambda \\ &\quad + \frac{2}{m}(\widetilde{D}_\mu\widetilde{D}_\sigma\beta^\sigma)e^{-4\varphi}\widetilde{\mathcal{G}}^\mu + \frac{8(m-1)}{m}(\widetilde{D}_\mu\beta^\mu)e^{-4\varphi}(\widetilde{D}_\rho\varphi)\widetilde{\mathcal{G}}^\rho - (\widetilde{D}_\lambda\widetilde{D}_\omega\beta^\lambda)e^{-4\varphi}\widetilde{\mathcal{G}}^\omega \\ &\quad + \frac{2}{m}(\widetilde{D}_\lambda\widetilde{D}_\rho\beta^\rho)e^{-4\varphi}\widetilde{\mathcal{G}}^\lambda + \frac{2}{m}(\widetilde{D}_\rho\beta^\rho)e^{-4\varphi}(\widetilde{D}_\lambda\widetilde{\mathcal{G}}^\lambda) + \beta^\rho e^{-4\varphi}(\widetilde{D}_\lambda\widetilde{D}_\rho\widetilde{\mathcal{G}}^\lambda) \\ &\quad - \frac{1}{m\widetilde{\gamma}}\beta^\omega e^{-4\varphi}(\widetilde{D}_\omega\mathcal{S})(\widetilde{D}_\lambda\widetilde{\mathcal{G}}^\lambda) \end{aligned} \tag{C.52} $$

$$
\begin{aligned}
&+\frac{4}{m\widetilde{\gamma}}e^{-4\varphi}({}^{(m)}\widetilde{R}+{}^{(m)}R^{\varphi})\beta^{\lambda}(\widetilde{D}_{\lambda}\mathcal{S})-\frac{1}{2\widetilde{\gamma}}(\widetilde{D}_{\lambda}\beta^{\rho})e^{-4\varphi}(\widetilde{D}_{\rho}\widetilde{D}^{\lambda}\widetilde{\mathcal{S}})\\
&+\frac{1}{2\widetilde{\gamma}}(\widetilde{D}_{\lambda}\beta^{\omega})\,{}^{(m)}\widetilde{\Gamma}^{\lambda\nu}{}_{\omega}e^{-4\varphi}(\widetilde{D}_{\nu}\widetilde{\mathcal{S}})-\frac{1}{m\widetilde{\gamma}}(\widetilde{D}_{\lambda}\beta^{\omega})e^{-4\varphi}\widetilde{\mathcal{G}}^{\lambda}(\widetilde{D}_{\omega}\widetilde{\mathcal{S}})\\
&+\frac{18m-17}{2m\widetilde{\gamma}^{3}}\beta^{\omega}e^{-4\varphi}(\widetilde{D}^{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\omega}\widetilde{\mathcal{S}})+\frac{-9m+10}{2m\widetilde{\gamma}^{2}}\beta^{\omega}e^{-4\varphi}(\widetilde{D}_{\lambda}\widetilde{D}^{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\omega}\widetilde{\mathcal{S}})\\
&+\frac{15m+16}{2m\widetilde{\gamma}^{2}}\beta^{\rho}e^{-4\varphi}(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\rho}\widetilde{D}^{\lambda}\widetilde{\mathcal{S}})+\frac{3m-4}{m\widetilde{\gamma}}\beta^{\rho}e^{-4\varphi}(\widetilde{D}_{\lambda}\widetilde{D}_{\rho}\widetilde{D}^{\lambda}\widetilde{\mathcal{S}})\\
&-\frac{1}{2\widetilde{\gamma}^{2}}\beta^{\omega}e^{-4\varphi\,(m)}\widetilde{\Gamma}^{\lambda\nu}{}_{\omega}(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\nu}\widetilde{\mathcal{S}})+\frac{1}{2\widetilde{\gamma}}\beta^{\omega}e^{-4\varphi}(\widetilde{D}_{\lambda}\,{}^{(m)}\widetilde{\Gamma}^{\lambda\nu}{}_{\omega})(\widetilde{D}_{\nu}\widetilde{\mathcal{S}})\\
&+\frac{1}{2\widetilde{\gamma}}\beta^{\omega}e^{-4\varphi\,(m)}\widetilde{\Gamma}^{\lambda\nu}{}_{\omega}(\widetilde{D}_{\lambda}\widetilde{D}_{\nu}\widetilde{\mathcal{S}})+\frac{1}{m\widetilde{\gamma}^{2}}\beta^{\omega}e^{-4\varphi}\widetilde{\mathcal{G}}^{\lambda}(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\omega}\widetilde{\mathcal{S}})\\
&-\frac{1}{m\widetilde{\gamma}}\beta^{\omega}e^{-4\varphi}\widetilde{\mathcal{G}}^{\lambda}(\widetilde{D}_{\lambda}\widetilde{D}_{\omega}\widetilde{\mathcal{S}})+\frac{1}{2\widetilde{\gamma}}(\widetilde{D}_{\nu}\beta^{\rho})e^{-4\varphi}\widetilde{\Gamma}^{\nu}(\widetilde{D}_{\rho}\widetilde{\mathcal{S}})-\frac{1}{2\widetilde{\gamma}^{2}}\beta^{\sigma}e^{-4\varphi}\widetilde{\mathcal{G}}^{\omega}(\widetilde{D}_{\omega}\widetilde{\mathcal{S}})(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})\\
&+\frac{1}{2\widetilde{\gamma}}\beta^{\rho}e^{-4\varphi}\widetilde{\Gamma}^{\nu}(\widetilde{D}_{\nu}\widetilde{D}_{\rho}\widetilde{\mathcal{S}})+\frac{1}{\widetilde{\gamma}^{2}}\alpha e^{-4\varphi}\widetilde{A}^{\lambda\omega}(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\omega}\widetilde{\mathcal{S}})-\frac{1}{\widetilde{\gamma}}\alpha e^{-4\varphi}\widetilde{A}^{\lambda\omega}(\widetilde{D}_{\lambda}\widetilde{D}_{\omega}\widetilde{\mathcal{S}})\\
&-\frac{1}{\widetilde{\gamma}}\alpha e^{-4\varphi\,(m)}\widetilde{\Gamma}^{\mu}{}_{\lambda\omega}\widetilde{A}^{\lambda\omega}(\widetilde{D}_{\mu}\widetilde{\mathcal{S}})-\frac{1}{2m\widetilde{\gamma}^{2}}\beta^{\rho}e^{-4\varphi\,(m)}\widetilde{\Gamma}^{\mu\lambda}{}_{\lambda}(\widetilde{D}_{\mu}\widetilde{\mathcal{S}})(\widetilde{D}_{\rho}\widetilde{\mathcal{S}})\\
&-\frac{1}{m\widetilde{\gamma}}\beta^{\sigma}e^{-4\varphi\,(m)}\widetilde{\Gamma}^{\lambda\rho\nu\,(m)}\widetilde{\Gamma}_{\nu\lambda\rho}(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})-\frac{1}{2\widetilde{\gamma}}(\widetilde{D}^{\omega}\widetilde{D}_{\omega}\beta^{\sigma})e^{-4\varphi}(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})\\
&-\frac{1}{\widetilde{\gamma}}(\widetilde{D}_{\omega}\beta^{\sigma})e^{-4\varphi}(\widetilde{D}^{\omega}\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})-\frac{1}{2\widetilde{\gamma}}(\widetilde{D}_{\rho}\beta^{\sigma})e^{-4\varphi\,(m)}\widetilde{\Gamma}^{\rho\lambda}{}_{\lambda}(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})\\
&-\frac{m-2}{2m\widetilde{\gamma}}\beta^{\sigma}e^{-4\varphi\,(m)}\widetilde{\Gamma}^{\rho\lambda}{}_{\lambda}(\widetilde{D}_{\rho}\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})+\frac{1}{m\widetilde{\gamma}}(\widetilde{D}_{\mu}\beta^{\sigma})e^{-4\varphi\,(m)}\widetilde{\Gamma}^{\mu\nu}{}_{\nu}(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})\\
&+\frac{4(m-1)}{m\widetilde{\gamma}}\beta^{\sigma}e^{-4\varphi}(\widetilde{D}^{\lambda}\widetilde{D}_{\lambda}\varphi)(\widetilde{D}_{\sigma}\mathcal{S})+\frac{4(m-1)}{m\widetilde{\gamma}}\beta^{\sigma}e^{-4\varphi\,(m)}\widetilde{\Gamma}^{\rho\lambda}{}_{\lambda}(\widetilde{D}_{\rho}\varphi)(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})\\
&+\frac{4(m-1)}{m\widetilde{\gamma}}e^{-4\varphi}(\widetilde{D}^{\nu}\widetilde{D}_{\nu}\beta^{\lambda})(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})-\frac{8(m-1)}{m\widetilde{\gamma}^{2}}(\widetilde{D}^{\nu}\beta^{\lambda})e^{-4\varphi}(\widetilde{D}_{\nu}\widetilde{\mathcal{S}})(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})\\
&+\frac{8(m-1)}{m\widetilde{\gamma}}(\widetilde{D}^{\nu}\beta^{\lambda})e^{-4\varphi}(\widetilde{D}_{\nu}\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})-4(m-1)e^{-4\varphi}(\widetilde{D}_{\lambda}\varphi)\Big\{\frac{1}{m\widetilde{\gamma}}\beta^{\rho}\widetilde{\Gamma}^{\lambda}(\widetilde{D}_{\rho}\widetilde{\mathcal{S}})\\
&\quad-\frac{1}{m\widetilde{\gamma}}\beta^{\rho}\widetilde{\mathcal{G}}^{\lambda}(\widetilde{D}_{\rho}\widetilde{\mathcal{S}})+\frac{2m-3}{m\widetilde{\gamma}^{2}}\beta^{\rho}(\widetilde{D}^{\lambda}\widetilde{\mathcal{S}})(\widetilde{D}_{\rho}\widetilde{\mathcal{S}})-\frac{3(m-2)}{2m\widetilde{\gamma}}\beta^{\rho}(\widetilde{D}^{\lambda}\widetilde{D}_{\rho}\widetilde{\mathcal{S}})\Big\}\\
&+\frac{4(m-2)(m-1)}{m\widetilde{\gamma}}\beta^{\sigma}e^{-4\varphi}(\widetilde{D}^{\omega}\varphi)(\widetilde{D}_{\omega}\varphi)(\widetilde{D}_{\sigma}\widetilde{\mathcal{S}})\\
&+\frac{6(m-2)(m-1)}{m\widetilde{\gamma}}(\widetilde{D}_{\mu}\beta^{\lambda})e^{-4\varphi}(\widetilde{D}^{\mu}\varphi)(\widetilde{D}_{\lambda}\widetilde{\mathcal{S}})-\frac{1}{\widetilde{\gamma}^{2}}\alpha e^{-4\varphi}\widetilde{A}^{\mu\nu}(\widetilde{D}_{\mu}\widetilde{\mathcal{S}})(\widetilde{D}_{\nu}\widetilde{\mathcal{S}})\\
&+\frac{1}{\widetilde{\gamma}}\alpha e^{-4\varphi}\widetilde{A}^{\mu\nu}(\widetilde{D}_{\mu}\widetilde{D}_{\nu}\widetilde{\mathcal{S}})+\frac{1}{\widetilde{\gamma}}\alpha e^{-4\varphi\,(m)}\widetilde{\Gamma}^{\rho}{}_{\nu\mu}\widetilde{A}^{\mu\nu}(\widetilde{D}_{\rho}\widetilde{\mathcal{S}}).
\end{aligned}
\tag{C.53}
$$

Appendix D

Some Convenient Relations

In this appendix, we denote the some convenient relations for calculating of this paper.

For the connection,

$$\partial_\nu g^{\mu\nu} = -{}^{(m)}\Gamma^{\mu}{}_{\omega\lambda} g^{\omega\lambda} - g^{\mu\nu}(\partial_\nu \log\sqrt{g}), \tag{D.1}$$

$$g^{\mu\nu}\Gamma^{\omega}{}_{\mu\nu} + {}^{(m)}\Gamma^{\lambda\omega}{}_{\lambda} = -\partial_\lambda g^{\omega\lambda}, \tag{D.2}$$

$${}^{(m)}\Gamma_{(\mu\nu)\lambda} = \frac{1}{2}\partial_\lambda g_{\mu\nu}, \tag{D.3}$$

$${}^{(m)}\Gamma_{[\mu\nu]\lambda} = -\partial_{[\mu} g_{\nu]\lambda}. \tag{D.4}$$

For the Lie derivative operator and covariant derivative,
$\forall T \in \mathcal{T}^0_0(\mathcal{M}^m)$, $\forall T^\nu \in \mathcal{T}^1_0(\mathcal{M}^m)$, $\forall T_\nu \in \mathcal{T}^0_1(\mathcal{M}^m)$, $\forall T_{\mu\nu} \in \mathcal{T}^0_2(\mathcal{M}^m)$,

$$D_\mu(\mathcal{L}_\beta(T)) = \mathcal{L}_\beta(D_\mu T), \tag{D.5}$$

$$D_\mu(\mathcal{L}_\beta(T^\nu)) = \mathcal{L}_\beta(D_\mu T^\nu) + \beta^\lambda(D_\mu D_\lambda T^\nu) - \beta^\lambda(D_\lambda D_\mu T^\nu) - T^\lambda(D_\mu D_\lambda \beta^\nu), \tag{D.6}$$

$$D_\mu(\mathcal{L}_\beta(T_\nu)) = \mathcal{L}_\beta(D_\mu T_\nu) + \beta^\lambda(D_\mu D_\lambda T_\nu) - \beta^\lambda(D_\lambda D_\mu T_\nu) + T_\lambda(D_\mu D_\nu \beta^\lambda), \tag{D.7}$$

$$\begin{aligned} D_\lambda(\mathcal{L}_\beta(T_{\mu\nu})) = {} & \mathcal{L}_\beta(D_\lambda T_{\mu\nu}) + \beta^\omega(D_\lambda D_\omega T_{\mu\nu}) - \beta^\omega(D_\omega D_\lambda T_{\mu\nu}) + T_{\omega\nu}(D_\lambda D_\mu \beta^\omega) \\ & + T_{\omega\mu}(D_\lambda D_\nu \beta^\omega). \end{aligned} \tag{D.8}$$

For the second order partial derivative and the second order covariant derivative,

$$\begin{aligned} \partial_\lambda\partial_\omega T_{\mu\nu} = {} & D_\lambda D_\omega T_{\mu\nu} + 2(\partial_\lambda {}^{(m)}\Gamma^{\rho}{}_{\omega(\mu)} T_{\nu)\rho} + 2(D_\lambda T_{\rho(\nu)})\,{}^{(m)}\Gamma^{\rho}{}_{\mu)\omega} + 2\,{}^{(m)}\Gamma^{\sigma}{}_{\lambda\rho}\,{}^{(m)}\Gamma^{\rho}{}_{\omega(\mu} T_{\nu)\sigma} \\ & + 2\,{}^{(m)}\Gamma^{\rho}{}_{\omega(\mu}\,{}^{(m)}\Gamma^{\sigma}{}_{\nu)\lambda} T_{\rho\sigma} + {}^{(m)}\Gamma^{\rho}{}_{\lambda\omega}(D_\rho T_{\mu\nu}) + {}^{(m)}\Gamma^{\rho}{}_{\lambda\mu}(D_\omega T_{\rho\nu}) \\ & + {}^{(m)}\Gamma^{\rho}{}_{\lambda\nu}(D_\omega T_{\mu\rho}). \end{aligned} \tag{D.9}$$

For the Riemann tensor and the Ricci tensor,

$$2D_{[\mu}D_{\nu]}D_\lambda T^\omega = -{}^{(m)}R^\rho{}_{\lambda\mu\nu}(D_\rho T^\omega) + {}^{(m)}R^\omega{}_{\rho\mu\nu}(D_\lambda T^\rho), \tag{D.10}$$

$$\begin{aligned} {}^{(m)}R_{\mu\nu} &= \partial_\lambda\, {}^{(m)}\Gamma^\lambda{}_{\mu\nu} - \partial_{(\nu}\, {}^{(m)}\Gamma^\lambda{}_{\mu)\lambda} + {}^{(m)}\Gamma^\lambda{}_{\omega\lambda}\, {}^{(m)}\Gamma^\omega{}_{\mu\nu} - \Gamma^\lambda{}_{\omega\nu}\, {}^{(m)}\Gamma^\omega{}_{\mu\lambda} && \text{(D.11)}\\ &= \partial_\lambda\, {}^{(m)}\Gamma^\lambda{}_{\mu\nu} - (\partial_{(\nu}g_{\mu)\omega})\Gamma^{\lambda\omega}{}_\lambda - g_{\omega(\mu}(\partial_{\nu)}\Gamma^{\lambda\omega}{}_\lambda) + {}^{(m)}\Gamma^{\lambda\omega}{}_\lambda\, {}^{(m)}\Gamma_{\omega\mu\nu} \\ &\quad - {}^{(m)}\Gamma^\lambda{}_{\omega\nu}\, {}^{(m)}\Gamma^\omega{}_{\mu\lambda} && \text{(D.12)}\\ &= \partial_\lambda\, {}^{(m)}\Gamma^\lambda{}_{\mu\nu} + (\partial_{(\nu}g_{\mu)\omega})(\partial_\lambda g^{\omega\lambda} + {}^{(m)}\Gamma^\omega{}_{\rho\sigma}g^{\rho\sigma}) \\ &\quad + g_{\omega(\mu}\partial_{\nu)}(\partial_\lambda g^{\omega\lambda} + {}^{(m)}\Gamma^\omega{}_{\rho\sigma}g^{\rho\sigma}) \\ &\quad - (\partial_\lambda g^{\omega\lambda} + {}^{(m)}\Gamma^\omega{}_{\rho\sigma}g^{\rho\sigma})\, {}^{(m)}\Gamma_{\omega\mu\nu} - {}^{(m)}\Gamma^\lambda{}_{\omega\nu}\, {}^{(m)}\Gamma^\omega{}_{\mu\lambda} && \text{(D.13)}\\ &= g_{\omega(\mu}\partial_{\nu)}(\, {}^{(m)}\Gamma^\omega{}_{\rho\sigma}g^{\rho\sigma}) + (\partial_{(\nu}g_{\mu)\omega})\, {}^{(m)}\Gamma^\omega{}_{\rho\sigma}g^{\rho\sigma} - {}^{(m)}\Gamma_{\omega\mu\nu}\, {}^{(m)}\Gamma^\omega{}_{\sigma\rho}g^{\sigma\rho} \\ &\quad + \partial_\lambda\, {}^{(m)}\Gamma^\lambda{}_{\mu\nu} + (\partial_{(\nu}g_{\mu)\omega})(\partial_\lambda g^{\omega\lambda}) + g_{\omega(\mu}\partial_{\nu)}\partial_\lambda g^{\omega\lambda} - (\partial_\lambda g^{\omega\lambda})\, {}^{(m)}\Gamma_{\omega\mu\nu} \\ &\quad - {}^{(m)}\Gamma^\lambda{}_{\omega\nu}\, {}^{(m)}\Gamma^\omega{}_{\mu\lambda} && \text{(D.14)}\\ &= g_{\omega(\mu}\partial_{\nu)}(\, {}^{(m)}\Gamma^\omega{}_{\sigma\rho}g^{\sigma\rho}) + {}^{(m)}\Gamma_{(\mu\nu)\omega}\, {}^{(m)}\Gamma^\omega{}_{\sigma\rho}g^{\sigma\rho} + g^{\lambda\omega}\partial_\lambda]\, {}^{(m)}\Gamma_{\omega\mu\nu} \\ &\quad - (\partial_\lambda\partial_{(\nu}g_{\mu)\omega})g^{\omega\lambda} - (\partial_\lambda g_{\omega(\mu})(\partial_{\nu)}g^{\omega\lambda}) - {}^{(m)}\Gamma^{\lambda\omega}{}_\nu(-{}^{(m)}\Gamma_{\lambda\omega\mu} + \partial_\mu g_{\lambda\omega}) && \text{(D.15)}\\ &= g_{\omega(\mu}\partial_{\nu)}(\, {}^{(m)}\Gamma^\omega{}_{\sigma\rho}g^{\sigma\rho}) + {}^{(m)}\Gamma_{(\mu\nu)\omega}\, {}^{(m)}\Gamma^\omega{}_{\sigma\rho}g^{\sigma\rho} - \frac{1}{2}g^{\lambda\omega}\partial_\lambda\partial_\omega g_{\mu\nu} \\ &\quad + {}^{(m)}\Gamma^{\lambda\omega}{}_\nu\, {}^{(m)}\Gamma_{\lambda\omega\mu} + 2\, {}^{(m)}\Gamma^{\lambda\omega}{}_{(\mu}\Gamma_{\nu)\lambda\omega} && \text{(D.16)} \end{aligned}$$

$$D_\lambda D_\omega T_{\mu\nu} - D_\omega D_\lambda T_{\mu\nu} = {}^{(m)}R^\rho{}_{\mu\omega\lambda}T_{\nu\rho} + {}^{(m)}R^\rho{}_{\nu\omega\lambda}T_{\mu\rho}. \tag{D.17}$$

In the BSSN formulation, we use these relations;

$$\widetilde{D}_\mu\widetilde{D}_\nu T^\mu - \widetilde{D}_\nu\widetilde{D}_\mu T^\mu = \widetilde{R}_{\mu\nu}T^\mu - \widetilde{D}_{(\nu}\widetilde{\mathcal{G}}_{\mu)}T^\mu. \tag{D.18}$$

$${}^{(m)}\widetilde{R}_{\mu\nu} = \widetilde{D}_{(\mu}\widetilde{\Gamma}_{\nu)} - \widetilde{D}_\lambda\, {}^{(m)}\widetilde{\Gamma}_{(\nu\mu)}{}^\lambda - {}^{(m)}\widetilde{\Gamma}^{\sigma\lambda}{}_\lambda\, {}^{(m)}\widetilde{\Gamma}_{(\nu\mu)\sigma} + {}^{(m)}\widetilde{\Gamma}^{\sigma\omega}{}_{(\mu}\, {}^{(m)}\widetilde{\Gamma}_{\nu)\omega\sigma}, \tag{D.19}$$

$${}^{(m)}\widetilde{R} = \widetilde{D}_\mu\widetilde{\Gamma}^\mu - \widetilde{D}_\lambda\, {}^{(m)}\widetilde{\Gamma}^{\mu\lambda}{}_\mu - {}^{(m)}\widetilde{\Gamma}^{\sigma\lambda}{}_\lambda\, {}^{(m)}\widetilde{\Gamma}^\mu{}_{\mu\sigma} + {}^{(m)}\widetilde{\Gamma}^{\sigma\omega\mu}\, {}^{(m)}\widetilde{\Gamma}_{\mu\omega\sigma}. \tag{D.20}$$

Appendix E

Mathematica Program of CAFs

We denote the Mathematica code for calculating the CAFs of the C^2-adjusted ADM and BSSN formulation in flat spacetime.

E.1 CAFs of C^2-adjusted ADM Formulation

```
ClearAll["\[<Global`*\>"]
kk = k1*k1+k2*k2+k3*k3;
(*ADM*)
(* A is the original coefficient matrix of Fourier
transformed standard ADM *)
A={{0,-2*I*k1,-2*I*k2,-2*I*k3},{-1/2*I*k1,0,0,0},
  {-1/2*I*k2,0,0,0},{-1/2*I*k3,0,0,0}};
(* B is the coefficient matrix of additional term
by C2-adjustment *)
B={{-4*kk*kk,0,0,0},{0,-kk-3*k1*k1,-3*k1*k2,-3*k1*k3},
  {0,-3*k2*k1,-kk-3*k2*k2,-3*k2*k3},
  {0,-3*k3*k1,-3*k2*k3,-kk-3*k3*k3}};
ev=Simplify[Eigenvalues[A+kappa*B]];
Print[ev];
```

E.2 CAFs of C^2-adjusted BSSN Formulation

```
ClearAll["\[<Global`*\>"]
k1 = 1; k2 = 0; k3 = 0;
kk = k1*k1 + k2*k2 + k3*k3;
lambdaPhi = lambda;
lambdaK = lambda;
lambdaGamma = lambda;
lambdaA = lambda;
lambdaCGamma = lambda;
hH = 1;
hM = 1;
A = {{0, 0, 0, 0, 0, 0, 0, -kk, 0}, {1/6*I*k1, 0, 0, 0, -1/2*kk, 0, 0,
```

```
    0, 0}, {1/6*I*k2, 0, 0, 0, 0, -1/2*kk, 0, 0, 0}, {1/6*I*k3, 0, 0,
    0, 0, 0, -1/2*kk, 0, 0}, {0, 2, 0, 0, 0, 0, 0, -I*k1, 0}, {0, 0,
    2, 0, 0, 0, 0, -I*k2, 0}, {0, 0, 0, 2, 0, 0, 0, -I*k3, 0}, {0, 0,
    0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, -2, 0}};
B = {{hH*(-128*lambdaPhi*kk*kk - 3/2*lambdaGamma*kk*kk -
    2*lambdaCGamma*kk), 0, 0, 0,
    hG*(1/2*I*lambdaGamma*kk*k1 - 2*I*lambdaCGamma*k1),
    hG*(1/2*I*lambdaGamma*kk*k2 - 2*I*lambdaCGamma*k2),
    hG*(1/2*I*lambdaGamma*kk*k3 - 2*I*lambdaCGamma*k3),
    0, -3*hS*lambdaGamma*kk}, {0,
    hM*(-8/9*lambdaK*k1*k1 - lambdaA*kk - lambdaA*k1*k1),
    hM*(-8/9*lambdaK*k1*k2 - lambdaK*k1*k2),
    hM*(-8/9*lambdaK*k1*k3 - lambdaK*k1*k3), 0, 0,
    0, -2*I*hA*lambdaA*k1, 0}, {0,
    hM*(-8/9*lambdaK*k2*k1 - lambdaK*k2*k1),
    hM*(-8/9*lambdaK*k2*k2 - lambdaA*kk - lambdaK*k2*k2),
    hM*(-8/9*lambdaK*k2*k3 - lambdaK*k2*k3), 0, 0,
    0, -2*I*hA*lambdaA*k2, 0}, {0,
    hM*(-8/9*lambdaK*k3*k1 - lambdaK*k3*k1),
    hM*(-8/9*lambdaK*k3*k2 - lambdaK*k3*k2),
    hM*(-8/9*lambdaK*k3*k3 - lambdaA*kk - lambdaK*k3*k3), 0, 0,
    0, -2*I*hA*lambdaA*k3,
    0}, {hH*(-1/2*I*lambdaGamma*k1*kk + 2*I*lambdaCGamma*k1), 0, 0, 0,
    hG*(-lambdaGamma*kk - 1/2*lambdaGamma*k1*k1 - 2*lambdaCGamma),
    hG*(-1/2*lambdaGamma*k1*k2), hG*(-1/2*lambdaGamma*k1*k3),
    0, -I*lambdaGamma*hS*
    k1}, {hH*(-1/2*I*lambdaGamma*k2*kk + 2*I*lambdaCGamma*k2), 0, 0,
    0, hG*(-1/2*lambdaGamma*k2*k1),
    hG*(-lambdaGamma*kk - 1/2*lambdaGamma*k2*k2 - 2*lambdaGamma),
    hG*(-1/2*lambdaGamma*k2*k3),
    0, -I*lambdaGamma*hS*
    k2}, {hH*(-1/2*I*lambdaGamma*k3*kk + 2*I*lambdaCGamma*k3), 0, 0,
    0, hG*(-1/2*lambdaGamma*k3*k1), hG*(-1/2*lambdaGamma*k3*k2),
    hG*(-lambdaGamma*kk - 1/2*lambdaGamma*k3*k3 - 2*lambdaGamma),
    0, -I*lambdaGamma*hS*k3}, {0, 2*hM*lambdaA*I*k1,
    2*hM*lambdaA*I*k2, 2*hM*lambdaA*I*k3, 0, 0, 0, -6*hA*lambdaA,
    0}, {-3 hH*lambdaGamma*kk, 0, 0, 0, hG*lambdaGamma*I*k1,
    hG*lambdaGamma*I*k2, hG*lambdaGamma*I*k3, 0, -6 hS*lambdaGamma}};
ev = Simplify[Eigenvalues[A + B]];
Print["Change the Coefficient of the G-Constraint"]
gG1 = Plot3D[
  Re[ev[[1]] /. {hS -> 1, hA -> 1}], {lambda, 0, 1}, {hG, 0, 10}]
gG2 = Plot3D[
  Re[ev[[2]] /. {hS -> 1, hA -> 1}], {lambda, 0, 1}, {hG, 0, 10}]
gG3 = Plot3D[
  Re[ev[[3]] /. {hS -> 1, hA -> 1}], {lambda, 0, 1}, {hG, 0, 10}]
gG4 = Plot3D[
```

```
  Re[ev[[4]] /. {hS -> 1, hA -> 1}], {lambda, 0, 1}, {hG, 0, 10}]
gG5 = Plot3D[
  Re[ev[[5]] /. {hS -> 1, hA -> 1}], {lambda, 0, 1}, {hG, 0, 10}]
gG6 = Plot3D[
  Re[ev[[6]] /. {hS -> 1, hA -> 1}], {lambda, 0, 1}, {hG, 0, 10}]
gG7 = Plot3D[
  Re[ev[[7]] /. {hS -> 1, hA -> 1}], {lambda, 0, 1}, {hG, 0, 10}]
gG8 = Plot3D[
  Re[ev[[8]] /. {hS -> 1, hA -> 1}], {lambda, 0, 1}, {hG, 0, 10}]
gG9 = Plot3D[
  Re[ev[[9]] /. {hS -> 1, hA -> 1}], {lambda, 0, 1}, {hG, 0, 10}]
Print["Change the Coefficient of the A-Constraint"]
gA1 = Plot3D[
  Re[ev[[1]] /. {hS -> 1, hG -> 1}], {lambda, 0, 1}, {hA, 0, 10}]
gA2 = Plot3D[
  Re[ev[[2]] /. {hS -> 1, hG -> 1}], {lambda, 0, 1}, {hA, 0, 10}]
gA3 = Plot3D[
  Re[ev[[3]] /. {hS -> 1, hG -> 1}], {lambda, 0, 1}, {hA, 0, 10}]
gA4 = Plot3D[
  Re[ev[[4]] /. {hS -> 1, hG -> 1}], {lambda, 0, 1}, {hA, 0, 10}]
gA5 = Plot3D[
  Re[ev[[5]] /. {hS -> 1, hG -> 1}], {lambda, 0, 1}, {hA, 0, 10}]
gA6 = Plot3D[
  Re[ev[[6]] /. {hS -> 1, hG -> 1}], {lambda, 0, 1}, {hA, 0, 10}]
gA7 = Plot3D[
  Re[ev[[7]] /. {hS -> 1, hG -> 1}], {lambda, 0, 1}, {hA, 0, 10}]
gA8 = Plot3D[
  Re[ev[[8]] /. {hS -> 1, hG -> 1}], {lambda, 0, 1}, {hA, 0, 10}]
gA9 = Plot3D[
  Re[ev[[9]] /. {hS -> 1, hG -> 1}], {lambda, 0, 1}, {hA, 0, 10}]
Print["Change the Coefficient of the S-Constraint"]
gS1 = Plot3D[
  Re[ev[[1]] /. {hG -> 1, hA -> 1}], {lambda, 0, 1}, {hS, 0, 10}]
gS2 = Plot3D[
  Re[ev[[2]] /. {hG -> 1, hA -> 1}], {lambda, 0, 1}, {hS, 0, 10}]
gS3 = Plot3D[
  Re[ev[[3]] /. {hG -> 1, hA -> 1}], {lambda, 0, 1}, {hS, 0, 10}]
gS4 = Plot3D[
  Re[ev[[4]] /. {hG -> 1, hA -> 1}], {lambda, 0, 1}, {hS, 0, 10}]
gS5 = Plot3D[
  Re[ev[[5]] /. {hG -> 1, hA -> 1}], {lambda, 0, 1}, {hS, 0, 10}]
gS6 = Plot3D[
  Re[ev[[6]] /. {hG -> 1, hA -> 1}], {lambda, 0, 1}, {hS, 0, 10}]
gS7 = Plot3D[
  Re[ev[[7]] /. {hG -> 1, hA -> 1}], {lambda, 0, 1}, {hS, 0, 10}]
gS8 = Plot3D[
  Re[ev[[8]] /. {hG -> 1, hA -> 1}], {lambda, 0, 1}, {hS, 0, 10}]
```

```
gS9 = Plot3D[
  Re[ev[[9]] /. {hG -> 1, hA -> 1}], {lambda, 0, 1}, {hS, 0, 10}]
```

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