# Studies of coupled nonlinear Schrödinger equations and nonlinear scalar field equations via variational methods

# 変分法を用いた非線型連立シュレディンガー方程式系，非線型スカラー場方程式の研究

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# Contents

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# Chapter 1

# Introduction

A main subject in this thesis is an analysis of coupled nonlinear Schrödinger equations (CNLS) and nonlinear scalar field equations (NSF) by variational methods. More precisely, we consider the following nonlinear partial differential equations:

$$\text{(CNLS)} \qquad \begin{cases} -\varepsilon^2\Delta u_1 + V_1(x)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbf{R}^N, \\ -\varepsilon^2\Delta u_2 + V_2(x)u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 & \text{in } \mathbf{R}^N, \\ u_1, u_2 \in H^1(\mathbf{R}^N) \end{cases}$$

and

$$\text{(NSF)} \qquad \begin{cases} -\Delta u = g(|x|, u) & \text{in } \Omega, \\ u \in H^1(\Omega). \end{cases}$$

Here $V_1, V_2 : \mathbf{R}^N \to \mathbf{R}$ are potential functions, $\varepsilon > 0, \mu_1, \mu_2, \beta \in \mathbf{R}$ constants, $1 \leq N \leq 3$ in (CNLS), $N \geq 2$ in (NSF), $\Omega \subset \mathbf{R}^N$ and we denote the Sobolev space by $H^1(\Omega)$:

$$H^1(\Omega) := \{u : \mathbf{R}^N \to \mathbf{R} \ : \ \|u\|_{H^1(\Omega)}^2 := \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 < \infty\}.$$

This thesis consists of two parts. In Part I, we treat (CNLS) and in Part II, we consider (NSF). Part I includes from Chapters 2 to 4 and Part II from Chapters 5 to 7. In what follows, we give a summary and main results of this thesis.

## 1.1 Part I: Introduction to (CNLS)

In Part I, we consider (CNLS). The equation (CNLS) appears when we consider the existence of standing waves of the following time-depend coupled nonlinear Schrödinger equations (TCNLS):

$$\text{(TCNLS)} \quad \begin{cases} i\hbar\dfrac{\partial\psi_1}{\partial t} + \dfrac{\hbar^2}{2m}\Delta\psi_1 + \tilde{V}_1(x)\psi_1 + (\mu_1|\psi_1|^2 + \beta|\psi_2|^2)\psi_1 = 0 & \text{in } (0,\infty)\times\mathbf{R}^N, \\ i\hbar\dfrac{\partial\psi_2}{\partial t} + \dfrac{\hbar^2}{2m}\Delta\psi_2 + \tilde{V}_2(x)\psi_2 + (\beta|\psi_1|^2 + \mu_2|\psi_2|^2)\psi_2 = 0 & \text{in } (0,\infty)\times\mathbf{R}^N \end{cases}$$

where $\hbar, m$ are positive constants, $\tilde{V}_1, \tilde{V}_2 : \mathbf{R}^N \to \mathbf{R}$ given functions and $\psi_1, \psi_2 : (0, \infty) \times \mathbf{R}^N \to \mathbf{C}$ unknown functions. The standing wave solutions of (TCNLS) are solutions of the form $\psi_j(t, x) = \exp(i\lambda_j t/\hbar) u_j(x)$ $(j = 1, 2)$ where $\lambda_j \in \mathbf{R}$ and $u_j(x)$ is a real valued function. Substituting this form into (TCNLS), then we obtain (CNLS) with $\varepsilon^2 = \hbar^2/2m$ and $V_j(x) = \lambda_j - \tilde{V}_j(x)$ $(j = 1, 2)$.

The equations (CNLS) and (TCNLS) appear in nonlinear optics and the theory of Bose-Einstein condensates. Recently, a lot of researchers have studied (CNLS) and (TC-NLS) not only in physics but also in mathematics. For example, see [36, 39, 44, 45, 81, 83, 89, 102, 111] for physical treatments, [9, 27, 58, 59, 101] for numerical treatments, and [3, 4, 10, 12, 13, 26, 29, 30, 31, 37, 43, 46, 47, 49, 50, 51, 65, 66, 67, 68, 69, 70, 71, 73, 74, 75, 76, 77, 79, 80, 82, 85, 86, 87, 94, 100, 105, 106, 107, 108, 109] for mathematical treatments. See also references therein.

In this thesis, we concentrate on (CNLS) and consider the case $\mu_1, \mu_2 > 0$. The constant $\beta$ in (CNLS) plays an important role. It stands for the strength of interactions between $u_1$ and $u_2$. We call the interaction *repulsive* if $\beta < 0$ and *attractive* if $\beta > 0$. In the articles mentioned above, they consider both cases, namely the repulsive and the attractive case. Through Chapters 2 – 4, we focus on the attractive case, i.e., $\beta > 0$.

We also remark that (CNLS) has a *semitrivial solution*. Here we call solution $u = (u_1, u_2)$ *semitrivial solution* if $u$ solves (CNLS) and either $u_1 \equiv 0$ or $u_2 \equiv 0$. If $u_2 \equiv 0$, (CNLS) becomes the following scalar nonlinear Schrödinger equation:

$$\text{(SNLS)} \qquad -\varepsilon^2 \Delta u_1 + V_1(x) u_1 = \mu_1 u_1^3 \qquad \text{in } \mathbf{R}^N.$$

Under some suitable conditions for $V_1(x)$, (SNLS) has a nontrivial solution. We refer to [2, 5, 8, 11, 21, 22, 23, 24, 32, 33, 34, 38, 42, 56, 63, 93, 103, 104] for more precise results. Therefore, under some suitable conditions, (CNLS) has a semitrivial solution.

On the other hand, we call $u = (u_1, u_2)$ *nontrivial solution* if $u$ solves (CNLS) and $u_1, u_2 \not\equiv 0$. Furthermore, if both of $u_1$ and $u_2$ are positive function, then we say it as *nontrivial positive solution*. In this thesis, we are interested in the existence of nontrivial positive solutions of (CNLS).

From now on, we suppose that $\mu_1, \mu_2, \beta > 0$. In the following, we state a summary of from Chapter 2 to Chapter 4.

### 1.1.1 Summary of Chapter 2

Chapter 2 is based on a work of [50]. In Chapter 2, we study (CNLS) in the following setting: $\varepsilon = 1$. Namely we consider the following equations:

$$(1.1.1) \qquad \begin{cases} -\Delta u_1 + V_1(x) u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbf{R}^N, \\ -\Delta u_2 + V_2(x) u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 & \text{in } \mathbf{R}^N, \\ u_1, u_2 \in H^1(\mathbf{R}^N). \end{cases}$$

First we state known results in this setting. The most simple case is that $V_j(x) \equiv V_j > 0$ $(j = 1, 2)$, namely all coefficients are positive constants. Many researchers have considered this case and we refer to [3, 4, 10, 12, 13, 26, 29, 30, 31, 46, 47, 65, 66, 75, 76, 94, 107]. In particular, Ambrosetti and Colorado [3, 4], Lin and Wei [65], and Sirakov [94] showed that for $N = 2, 3$, there exist positive constants $0 < \tilde{\beta}_1 \leq \tilde{\beta}_2 < \infty$ such that

(i) If $0 < \beta < \tilde{\beta}_1$, then (1.1.1) has a nontrivial positive solution.

(ii) If $\tilde{\beta}_2 < \beta$, then (1.1.1) has a nontrivial positive solution.

Moreover, if $V_1(x) \equiv V_2(x) \equiv V > 0$, then Sirakov [94] showed that $\tilde{\beta}_1 = \min\{\mu_1, \mu_2\}$, $\tilde{\beta}_2 = \max\{\mu_1, \mu_2\}$ and there is no nontrivial positive solution of (1.1.1) if $\tilde{\beta}_1 \le \beta \le \tilde{\beta}_2$ and $\tilde{\beta}_1 < \tilde{\beta}_2$. A similar result is also obtained in Bartsch and Wang [12]. In these articles, they used essentially the compactness of the embedding $H^1_r(\mathbf{R}^N) \subset L^p(\mathbf{R}^N)$ for $2 < p < 2N/(N-2)$ if $N \ge 3$ and for $2 < p < \infty$ if $N = 2$. Here we denote $H^1_r(\mathbf{R}^N)$ as the space of all radially symmetric functions in $H^1(\mathbf{R}^N)$. This compactness property plays an important role to show the existence of nontrivial positive solutions.

In contrast with constant coefficients case, the existence problem becomes delicate if $V_1(x)$ or $V_2(x)$ is not equal to a positive constant identically. In fact, we will prove the following nonexistence result.

**Theorem 1.1.1** (Theorem 2.1.4 in Chapter 2)**.** *Let $N = 1, 2, 3$ and $V_1, V_2$ satisfy the following conditions*:

(2–V1') *For $j = 1, 2$, $V_j \in C^1(\mathbf{R}^N, \mathbf{R})$ and $\nabla V_j \in L^\infty(\mathbf{R}^N, \mathbf{R}^N)$.*

(2–V2') *For $j = 1, 2$, $0 < \inf_{x \in \mathbf{R}^N} V_j(x) \le \sup_{x \in \mathbf{R}^N} V_j(x) < \infty$.*

(2–V3') *There exists a $\nu \in \mathbf{R}^N \backslash \{0\}$ such that $(\partial V_j / \partial \nu)(x) = \nabla V_j(x) \cdot \nu \ge 0$ for $j = 1, 2$.*

(2–V4') *It holds that either $\partial V_1 / \partial \nu \not\equiv 0$ or $\partial V_2 / \partial \nu \not\equiv 0$.*

*Then* (1.1.1) *has no nontrivial positive solution for any $\beta > 0$.*

We note that if $V_1(x)$ and $V_2(x)$ satisfy the conditions (2-V1')–(2-V4'), then $V_1(\varepsilon x)$ and $V_2(\varepsilon x)$ also satisfy (2-V1')–(2-V4'). Therefore from Theorem 1.1.1, we see that (CNLS) has no nontrivial positive solution for any $\beta > 0$ under (2-V1')–(2-V4').

Theorem 1.1.1 suggests that even if graphs of $V_1$ and $V_2$ are close to positive constants, (1.1.1) does not have any nontrivial positive solution. Therefore, if the coefficients in (1.1.1) depend on the space variable $x \in \mathbf{R}^N$, the existence problem may turn out to be delicate.

Next we recall the result of G. Wei [105]. He considered (1.1.1) under the conditions: For $j = 1, 2$,

$$(1.1.2) \qquad \begin{aligned} &V_j \in C^\infty(\mathbf{R}^N), \quad 0 < V_{0,j} \le V_j(x) \quad \text{for all } \ x \in \mathbf{R}^N, \\ &\text{meas}\,\{x \in \mathbf{R}^N : V_j(x) < M\} < \infty \quad \text{for all } M > 0. \end{aligned}$$

The condition (1.1.2) appeared in Bartsch and Wang [11] and this is a generalization of the condition in Rabinowitz [93]. Under these conditions, G. Wei [105] proved that for $N = 1, 2, 3$, there exists a $\tilde{\beta} > 0$ such that if $\tilde{\beta} < \beta$, then (1.1.1) has a nontrivial positive solution. His method also depends on the compactness of the embedding. Indeed, in [105], he worked in the following space instead of the usual $H^1(\mathbf{R}^N)$ space:

$$\mathcal{H} := \left\{ u = (u_1, u_2) \in (H^1(\mathbf{R}^N))^2 : \int_{\mathbf{R}^N} V_1(x) u_1^2(x) dx < \infty, \ \int_{\mathbf{R}^N} V_2(x) u_2^2(x) dx < \infty \right\}.$$

In this case, the embedding $\mathcal{H} \subset (L^p(\mathbf{R}^N))^2$ is compact for $2 \le p < 2N/(N-2)$ if $N \ge 3$ and $2 \le p < \infty$ if $N = 1, 2$. Therefore the situation is similar to ones in the constant coefficient case.

Now we explain our setting in Chapter 2. We will consider (1.1.1) under the following conditions:

(2–V1) For $j = 1, 2$, $V_j(x) \in C^1(\mathbf{R}^N, \mathbf{R})$.

(2–V2) For $j = 1, 2$, $0 < \inf_{x \in \mathbf{R}^N} V_j(x) \le \sup_{x \in \mathbf{R}^N} V_j(x) \equiv V_{\infty,j} < \infty$.

(2–V3) For $j = 1, 2$, $V_j(x) \to V_{\infty,j}$ as $|x| \to \infty$.

In this setting we shall prove the following existence result.

**Theorem 1.1.2** (Theorem 2.1.1 in Chapter 2). *Suppose that $N = 1, 2, 3$ and $V_1, V_2$ satisfy* (2–V1)– (2–V3). *Then there exist $0 < \beta_1 \le \beta_2$ such that*

(i) *If $0 < \beta < \beta_1$, then* (1.1.1) *has a nontrivial positive solution.*

(ii) *If $\beta_2 < \beta$, then* (1.1.1) *has a nontrivial positive solution.*

In Chapter 2, we will also show the characterization of solutions found in Theorem 1.1.2. In order to state a result, we need the following notion.

**Definition 1.1.3.** The solution $u = (u_1, u_2)$ is said a *least energy solution* of (1.1.1) if $u$ satisfies the following equality:

$$I(u) := \inf\{I(v) : v \text{ is a solution of (1.1.1) and } v \not\equiv (0,0)\} \tag{1.1.3}$$

where

$$\begin{aligned} I(u) :=& \frac{1}{2} \int_{\mathbf{R}^N} |\nabla u_1|^2 + V_1(x) u_1^2 dx + |\nabla u_2|^2 + V_2(x) u_2^2 dx \\ &- \frac{1}{4} \int_{\mathbf{R}^N} \mu_1 u_1^4 + 2\beta u_1^2 u_2^2 + \mu_2 u_2^4 dx \in C^1((H^1(\mathbf{R}^N))^2, \mathbf{R}). \end{aligned}$$

We remark that the functional $I$ corresponds to (1.1.1), which means that solutions of (1.1.1) are equivalent to critical points of $I$. Therefore if the minimizer of (1.1.3) exists, then the minimizer has the least energy among all solutions have except for the trivial solution $(0, 0)$.

The following theorem gives a characterization of solutions found in Theorem 1.1.2.

**Theorem 1.1.4** (Theorem 2.1.2 in Chapter 2). *Suppose that $V_1$ and $V_2$ satisfy* (2–V1)–(2–V3).

(i) *There exists a $\beta_3 > 0$ such that if $\beta \in (0, \beta_3)$, then the nontrivial positive solution obtained in Theorem* 1.1.2 (i) *is not a least energy solution.*

(ii) *If $\beta_2 < \beta$, then the nontrivial positive solution obtained in Theorem* 1.1.2 (ii) *is a least energy solution.*

### 1.1.2 Summary of Chapter 3

Chapter 3 is based on a work of [49]. In Chapter 3, we observe a uniqueness of nontrivial positive solutions of (1.1.1). As mentioned in subsection 1.1.1, the existence of positive solution is well-studied. However, the uniqueness of nontrivial positive solutions of (1.1.1) is not studied very well even for the constant coefficients case. We will split into the following two cases:

(I) constant coefficients case ($V_j(x) \equiv V_j > 0, j = 1, 2$) or

(II) either $V_1(x)$ or $V_2(x)$ depends on $x \in \mathbf{R}^N$.

More precisely, in the case (II), we assume the following conditions:

(3–V1) For $j = 1, 2$, $V_j \in C^2(\mathbf{R}^N)$, $V_j(x) = V_j(|x|)$ and $\limsup_{|x|\to\infty} V_j(x) > 0$.

(3–V2) For $j = 1, 2$,

$$\inf \left\{ \int_{\mathbf{R}^N} |\nabla u|^2 + V_j(x)u^2 dx : \ \|u\|_{H^1(\mathbf{R}^N)} = 1 \right\} > 0.$$

(3–V3) For $j = 1, 2$ and $r \geq 0$, $V_j'(r) \geq 0$.

(3–V4) There exist $C > 0$ and $M > 0$ such that $|V_j(r)| \leq C(1 + r)^M$ for $j = 1, 2$ and $r \geq 0$.

(3–V5) When $N = 3$, the function

$$H_j(r) := \frac{4}{3} r^2 V_j(r) + r^3 V'(r) - \frac{4}{27}$$

has a unique simple zero in $(0, \infty)$.

First we give a remark about symmetry property of nontrivial positive solutions of (1.1.1).

*Remark* 1.1.5. (i) The conditions (3–V1)–(3–V5) include the constant coefficient case, namely, $V_1(x) \equiv V_1 > 0$, $V_2(x) \equiv V_2 > 0$ satisfy (3–V1)–(3–V5).
(ii) It is easy to see that $V_j(x) = |x|^\alpha$ with $\alpha \geq 2$ satisfies (3–V1)–(3–V5). Therefore, the conditions (3–V1)–(3–V5) include unbounded potentials.
(iii) Under the conditions (3–V1), (3–V3) and $N \geq 2$, by the result of Busca and Sirakov [20], any nontrivial positive solution of (1.1.1) is radially symmetric with respect to some point in $\mathbf{R}^N$. Furthermore, it is nonincreasing with respect to $r = |x|$. However, this symmetry property holds even for $N = 1$. We will prove it at section 3.4 in Chapter 3.

By (ii) in Remark 1.1.5, it is sufficient to consider the uniqueness of nontrivial positive solutions which are radially symmetric.

Before stating main results in Chapter 3, we need the following notions.

**Definition 1.1.6.**
(i) For $V_1$ and $V_2$ satisfying (3–V1)–(3–V5), we define $\mathcal{H}_{V_1,V_2,r} \subset H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N)$ as follows:

$$\mathcal{H}_{V_1,V_2,r} := \left\{ u = (u_1, u_2) \in (H^1_r(\mathbf{R}^N))^2 : \int_{\mathbf{R}^N} V_1(x) u_1^2 dx < \infty, \ \int_{\mathbf{R}^N} V_2(x) u_2^2 dx < \infty \right\}$$

(ii) A solution $\omega = (\omega_1, \omega_2) \in \mathcal{H}_{V_1,V_2,r}$ is said *nondegenerate in* $\mathcal{H}_{V_1,V_2,r}$ if the linearized equation of (1.1.1) at $\omega$

$$\begin{cases} -\Delta u_1 + V_1(x) u_1 = 3\mu_1 \omega_1^2 u_1 + \beta \omega_2^2 u_1 + 2\beta \omega_1 \omega_2 u_2 & \text{in } \mathbf{R}^N, \\ -\Delta u_2 + V_2(x) u_2 = 2\beta \omega_1 \omega_2 u_1 + \beta \omega_1^2 u_2 + 3\mu_2 \omega_2^2 u_2 & \text{in } \mathbf{R}^N, \\ \qquad (u_1, u_2) \in \mathcal{H}_{V_1,V_2,r} \end{cases}$$

has only trivial solution $u_1 \equiv u_2 \equiv 0$.

First we consider the case (I) (constant coefficient case).

**Theorem 1.1.7** (Theorem 3.1.1 in Chapter 3)**.** *Suppose that* $N = 1, 2, 3$ *and* $V_j(x) \equiv V_j > 0$ *for* $j = 1, 2$*. Then there exists a* $\beta_1 > 0$ *such that if* $0 < \beta < \beta_1$*, then the nontrivial positive solution of* (1.1.1) *is unique up to translations. Furthermore, the unique nontrivial positive solution of* (1.1.1) *is nondegenerate in* $H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N)$ *for* $0 < \beta < \beta_1$*.*

Here we should mention a work of Wei and Yao [107]. In [107], they obtained a similar result to Theorem 1.1.7.

Next we consider the case (II) (variable coefficient case). In this case, a uniqueness result is the following:

**Theorem 1.1.8** (Theorem 3.1.4 in Chapter 3)**.** *Suppose that* $N = 2, 3$ *and* $V_1(x), V_2(x)$ *satisfy the conditions* (3–V1)–(3–V5)*. Then there exists a* $\beta_2 > 0$ *such that if* $0 < \beta < \beta_2$*, then* (1.1.1) *has a unique nontrivial radially symmetric positive solution in* $\mathcal{H}_{V_1,V_2,r}$*. Furthermore, the unique nontrivial positive solution of* (1.1.1) *is nondegenerate in* $\mathcal{H}_{V_1,V_2,r}$*.*

### 1.1.3 Summary of Chapter 4

Chapter 4 is based on a work of [51]. In Chapter 4, we treat (CNLS), and observe the existence of nontrivial positive solutions and the asymptotic behavior as $\varepsilon \to 0$. Throughout in Chapter 4, we assume $N = 2, 3$.

In general, there are 4 kinds of asymptotic behaviors of a family of solutions of (CNLS). Let $u_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}) \neq (0, 0)$ be a family of solution of (CNLS). Then

(I) The function $u_{\varepsilon,1}$ concentrates at a critical point $P_1 \in \mathbf{R}^N$ of $V_1(x)$ and $u_{\varepsilon,2}$ converges to 0.

(II) The function $u_{\varepsilon,2}$ concentrates at a critical point $P_2 \in \mathbf{R}^N$ of $V_2(x)$ and $u_{\varepsilon,1}$ converges to 0.

(III) The function $u_{\varepsilon,j}$ concentrates at a critical point $P_j \in \mathbf{R}^N$ respectively and $P_1 \neq P_2$.

(IV) Both of components concentrate to the same point.

The aim of Chapter 4 is to show the existence of a family of solutions which is type (IV).

As to (I) or (II), we can easily construct such a family of solutions of (CNLS) from the scalar nonlinear Schrödinger equation (SNLS). Indeed, many researchers have studied (SNLS). We refer to [2, 5, 21, 22, 23, 24, 32, 33, 34, 38, 42, 56, 63, 103, 104] and references therein. In these articles, it is shown that under the suitable assumptions, (SNLS) has a family of solutions $(u_{\varepsilon,j})$ concentrating to a critical point $P_j$. Setting $u_\varepsilon := (u_{\varepsilon,1}, 0)$ or $u_\varepsilon := (0, u_{\varepsilon,2})$, we can obtain a family of solutions of (CNLS) which is type (I) or (II).

Before stating known results concerning the existence result of type (III) and (IV), we state a main result of Chapter 4. After the main result, we explain known results.

The aim of Chapter 4 is to show the existence of a family of solutions which is type (IV). In order to state the main result, we prepare some notations.

First, to analyze the asymptotic behavior of (CNLS), it is important to study the following constant coefficient problem: For $P \in \mathbf{R}^N$,

$$(1.1.4)\qquad \begin{cases} -\Delta v_1 + V_1(P)v_1 = \mu_1 v_1^3 + \beta v_1 v_2^2 & \text{in } \mathbf{R}^N, \\ -\Delta v_2 + V_2(P)v_2 = \beta v_1^2 v_2 + \mu_2 v_2^3 & \text{in } \mathbf{R}^N, \\ \qquad v_1, v_2 \in H^1(\mathbf{R}^N). \end{cases}$$

We define a functional $J_P$ corresponding to (1.1.4) and the least energy $m(P)$ among nontrivial solutions as follows:

$$\begin{aligned} J_{V_1(P),V_2(P)}(v) :=& \frac{1}{2}\int_{\mathbf{R}^N} |\nabla v_1|^2 + V_1(P)v_1^2 + |\nabla v_2|^2 + V_2(P)v_2^2 dx \\ & - \frac{1}{4}\int_{\mathbf{R}^N} \mu_1 v_1^4 + 2\beta v_1^2 v_2^2 + \mu_2 v_2^4 dx, \\ \mathcal{M}(V_1(P), V_2(P)) :=& \{v \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N) : v_1, v_2 \neq 0, \\ & \qquad J'_{V_1(P),V_2(P)}(v)(v_1, 0) = J'_{V_1(P),V_2(P)}(v)(0, v_2) = 0\}, \\ m(P) :=& \inf_{v \in \mathcal{M}(V_1(P),V_2(P))} J_{V_1(P),V_2(P)}(v). \end{aligned}$$

Next, we state our settings in Chapter 4. First we assume

$$(1.1.5)\qquad 0 < \beta < \sqrt{\mu_1\mu_2}.$$

Secondly, we suppose the following two conditions:

**Assumption (4–A1)** There exists a set $A = [a_{10}, a_{11}] \times [a_{20}, a_{21}] \subset (0, \infty) \times (0, \infty)$ with the following properties:

(i) For any $(\lambda_1, \lambda_2) \in A$, it holds that operators $-\Delta + \lambda_1 - \beta\hat{\omega}_2^2$ and $-\Delta + \lambda_2 - \beta\hat{\omega}_1^2$ are positive definite on $H_r^1(\mathbf{R}^N)$. This means that

$$\int_{\mathbf{R}^N} |\nabla\varphi|^2 + \lambda_1\varphi^2 - \beta\hat{\omega}_2^2\varphi^2 dx, \quad \int_{\mathbf{R}^N} |\nabla\varphi|^2 + \lambda_2\varphi^2 - \beta\hat{\omega}_1^2\varphi^2 dx > 0$$

for all $\varphi \in H_r^1(\mathbf{R}^N)\backslash\{0\}$. Here $\hat{\omega}_i \in H_r^1(\mathbf{R}^N)$ is a unique positive solution of $-\Delta\omega_i + \lambda_i\omega_i = \mu_i\omega_i^3$ in $\mathbf{R}^N$ (See Kwong [60]).

(ii) $(V_1(P), V_2(P)) \in A$ for all $P \in \mathbf{R}^N$.

**Assumption (4–A2)** There exists a bounded open set $\Lambda \subset \mathbf{R}^N$ such that

$$\inf_{P\in\Lambda} m(P) < \inf_{P\in\partial\Lambda} m(P). \tag{1.1.6}$$

Finally, we define the following value and set:

$$m_0 := \inf_{P\in\Lambda} m(P),$$
$$K := \{P \in \Lambda : m(P) = m_0\}.$$

Now we can state our main result.

**Theorem 1.1.9** (Theorem 4.1.3 in Chapter 4)**.** *Let $V_1, V_2 \in C(\mathbf{R}^N)$ and suppose that* (1.1.5) *and Assumptions* (4–A1), (4–A2) *hold. Then there exists an $\varepsilon_0 > 0$ such that* (CNLS) *has a family of nontrivial positive solutions $(u_{1\varepsilon}(x), u_{2\varepsilon}(x))_{0<\varepsilon<\varepsilon_0}$ satisfying the following properties: after taking a subsequence $\varepsilon_j \to 0$ there exists a sequence $(P_{\varepsilon_j}) \subset \Lambda$ such that*

$$P_{\varepsilon_j} \to P_0 \in K, \tag{1.1.7}$$

$$(u_{1\varepsilon_j}(\varepsilon_j x + P_{\varepsilon_j}), u_{2\varepsilon_j}(\varepsilon_j x + P_{\varepsilon_j})) \to (w_1(x), w_2(x)) \quad \textit{strongly in } H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N). \tag{1.1.8}$$

*Here $(w_1(x), w_2(x))$ is a nontrivial radial positive solution of the limit problem:*

$$\begin{cases} -\Delta w_1 + V_1(P_0)w_1 = \mu_1 w_1^3 + \beta w_1 w_2^2 & \text{in } \mathbf{R}^N, \\ -\Delta w_2 + V_2(P_0)w_2 = \beta w_1^2 w_2 + \mu_2 w_2^3 & \text{in } \mathbf{R}^N \end{cases}$$

*and it satisfies $J_{V_1(P_0),V_2(P_0)}(w_1, w_2) = m(P_0) = m_0$.*

Theorem 1.1.9 suggests that if the conditions (1.1.5), (4–A1) and (4–A2) hold, then there exists a family of solutions which is type (IV). About when the condition (4–A2) is satisfied, see Remark 4.1.5 (i). By Remark 4.1.5, we can see that there are many examples in which the condition (4–A2) is satisfied.

Next we compare our results to known results. As to the existence of a family of solutions which is type (III) or (IV), we refer to Lin and Wei [68], Montefusco, Pellacci and Squassina [79], Pomponio [87] and G. Wei [105, 106]. Here we only state the results of Lin and Wei [68].

Lin and Wei [68] studied the existence of least energy solutions among nontrivial solutions and its asymptotic behavior. More precisely, we define

$$\begin{aligned} I_\varepsilon(u) :=& \frac{1}{2}\int_{\mathbf{R}^N} \varepsilon^2|\nabla u_1|^2 + V_1(x)u_1^2 + \varepsilon^2|\nabla u_2|^2 + V_2(x)u_2^2 dx \\ &- \frac{1}{4}\int_{\mathbf{R}^N} \mu_1 u_1^4 + 2\beta u_1^2 u_2^2 + \mu_2 u_2^4 dx, \\ \mathcal{H} :=& \left\{u \in (H^1(\mathbf{R}^N))^2 : \int_{\mathbf{R}^N} V_1(x)u_1^2(x)dx < \infty,\ \int_{\mathbf{R}^N} V_2(x)u_2^2(x)dx < \infty\right\}, \\ \mathcal{M}_\varepsilon :=& \{u \in \mathcal{H} : u_1, u_2 \neq 0,\ I'_\varepsilon(u)(u_1, 0) = I'_\varepsilon(u)(0, u_2) = 0\}, \\ b_\varepsilon :=& \inf_{u\in\mathcal{M}_\varepsilon} I_\varepsilon(u). \end{aligned}$$

In [68], they proved that $b_\varepsilon$ is attained by nontrivial positive solution $u_\varepsilon$ and studied its behavior under $0 < \beta < \beta_0$ and some conditions concerning behaviors of $V_1(x)$ and $V_2(x)$ as $|x| \to \infty$. Especially, their result about asymptotic behaviors is the following: Let $(u_\varepsilon) \subset \mathcal{M}_\varepsilon$ be a family of minimizer. Then they proved that

$$I_\varepsilon(u_\varepsilon) = b_\varepsilon \to \min \left\{ \inf_{P \in \mathbf{R}^N} m(P), \ \inf_{P_1 \in \mathbf{R}^N} e_1(P_1) + \inf_{P_2 \in \mathbf{R}^N} e_2(P_2) \right\} \quad \text{as } \varepsilon \to 0.$$

Here $e_i(P_i)$ stands for the least energy value for $-\Delta w_i + V_i(P_i) w_i = \mu_i w_i^3$ in $\mathbf{R}^N$ and it has the following formula:

$$e_i(P_i) = \frac{V_i(P_i)^{(4-N)/2}}{\mu_i} e_0$$

where $e_0 > 0$ is the least energy value for $-\Delta w + w = w^3$ in $\mathbf{R}^N$. Moreover, Lin and Wei showed that if

$$\inf_{P \in \mathbf{R}^N} m(P) < \inf_{P_1 \in \mathbf{R}^N} e_1(P_1) + \inf_{P_2 \in \mathbf{R}^N} e_2(P_2),$$

then the behavior of $u_\varepsilon$ is type (IV) and a concentration point of both components of $u_\varepsilon$ is a global minimum point $P_0$ of $m(P)$. On the other hand, if the opposite inequality

$$\inf_{P_1 \in \mathbf{R}^N} e_1(P_1) + \inf_{P_2 \in \mathbf{R}^N} e_2(P_2) < \inf_{P \in \mathbf{R}^N} m(P)$$

holds, then $(u_\varepsilon)$ is type (III), which means that each component concentrates at $P_i$ respectively and $P_1 \neq P_2$. Here $P_i$ is a global minimum point of $e_i(P)$.

We remark that we can construct the following example. The function $m(P)$ has a global minimizer, however, the minimizer on $\mathcal{M}_\varepsilon$ is type (III), which means each component of the minimizer concentrates at different point. On the other hand, applying our theorem 1.1.9, we can find a family of solution which concentrates at a global minimum point of $m$. This is a type (IV) solution.

## 1.2 Part II: Introduction to (NSF)

In Part II, we treat nonlinear scalar field equations (NSF). The equation (NSF) appears in various research fields.

First we consider the following nonlinear Klein–Gordon equation:

$$\psi_{tt} - \Delta\psi + m^2\psi - f(\psi) = 0 \quad \text{in } \mathbf{R} \times \mathbf{R}^N \tag{1.2.1}$$

where $m > 0$. We assume that $f$ satisfies $f(e^{i\theta}s) = e^{i\theta} f(s)$ for all $\theta, s \in \mathbf{R}$. We look for a standing wave solution, which is a form of $\psi(t,s) = e^{i\omega t} u(x)$ where $\omega \geq 0$ and $u$ is a real valued function. Substitute this form into (1.2.1), then we obtain

$$-\Delta u + (m^2 - \omega^2) u = f(u) \quad \text{in } \mathbf{R}^N$$

This is a special case of (NSF) with $g(r,s) = -(m^2 - \omega^2)s + f(s)$ and $\Omega = \mathbf{R}^N$.

Secondly, we consider a travelling wave solution of (1.2.1). This is a solution which has the following form: $\psi(t,x) = u(x - ct)$ where $u$ is a real valued function and $c \in \mathbf{R}^N$. Then (1.2.1) is reduced to the following equation:

$$(1.2.2) \qquad -\sum_{i,j=1}^{N} (\delta_{ij} - c_i c_j) \frac{\partial^2 u}{\partial x_i \partial x_j} + m^2 u = f(u) \quad \text{in } \mathbf{R}^N.$$

We note that the equation (1.2.2) is elliptic provided $|c| < 1$. In fact, let $A = (A_{ij})_{ij} = (\delta_{ij} - c_i c_j)_{ij}$, then we have

$$\xi \cdot A\xi = |\xi|^2 - |c \cdot \xi|^2 \geq (1 - |c|^2)|\xi|^2 \quad \text{for all } \xi \in \mathbf{R}^N.$$

Thus (1.2.2) is elliptic. Using the change of coordinates, (1.2.2) is equivalent to

$$-\Delta u + \tilde{m} u = \tilde{f}(u) \quad \text{in } \mathbf{R}^N.$$

Thus this equation is also a special case of (NSF).

We can also derive (NSF) from the following nonlinear Schrödinger equation:

$$(1.2.3) \qquad i\frac{\partial \psi}{\partial t} + \Delta\psi + V(|x|)\psi + f(\psi) = 0 \quad \text{in } (0,\infty) \times \mathbf{R}^N,$$

where $f(e^{i\theta}s) = e^{i\theta} f(s)$ for all $\theta, s \in \mathbf{R}$. When we look for a standing wave solution of the form $\psi(t,s) = \exp(i\omega t)u(x)$, then (1.2.3) becomes

$$-\Delta u + (\omega - V(|x|))u = f(u) \quad \text{in } \mathbf{R}^N.$$

This is (NSF) with $g(r,s) = -(\omega - V(r))s + f(s)$.

Thus the equation (NSF) is a generalization of the nonlinear Klein–Gordon equations, the nonlinear Schrödinger equations and so on.

The equation (NSF) has been extensively studied by many authors [7, 11, 14, 15, 16, 17, 18, 19, 28, 55, 61, 62, 95].

As to (NSF) with $\Omega = \mathbf{R}^N$ and $g(r,s) = g(s)$, Strauss [95] showed the existence of at least one radial positive solution for $N \geq 2$. He also treated the existence of infinitely many radial possibly sign changing solutions. Colemann, Glaser and Martin [28] studied the existence of a least energy solution for $N \geq 3$. Berestycki, Gallouët and Kavian [14] ($N = 2$) and Berestycki and Lions [15, 16] ($N \geq 3$) gave an almost necessary and sufficient conditions for the existence of nontrivial solutions. In [14, 15, 16], they assume the following conditions for $g(s)$:

(5-g0) The function $g \in C(\mathbf{R},\mathbf{R})$ and $g$ is odd: $g(-\xi) = -g(\xi)$.

(5-g1) For $N \geq 3$,

$$\limsup_{\xi\to\infty} \frac{g(\xi)}{\xi^{(N+2)/(N-2)}} \leq 0.$$

For $N = 2$,

$$\limsup_{\xi\to\infty} \frac{g(\xi)}{e^{\alpha\xi^2}} \leq 0 \quad \text{for any } \alpha > 0.$$

(5-g2) For $N \geq 3$,

$$-\infty < \liminf_{\xi \to 0} \frac{g(\xi)}{\xi} \leq \limsup_{\xi \to 0} \frac{g(\xi)}{\xi} < 0. \tag{1.2.4}$$

For $N = 2$,

$$-\infty < \lim_{\xi \to 0} \frac{g(\xi)}{\xi} < 0. \tag{1.2.5}$$

(5-g3) There exists a $\zeta_0 > 0$ such that $G(\zeta_0) > 0$, where $G(\xi) \equiv \int_0^\xi g(\tau) d\tau$.

Under the conditions (5-g1)–(5-g3), they proved the existence of at least one radial positive solution and infinitely many radial possibly sign changing radial solutions.

Brezis and Lieb [18] ($N \geq 2$) and Brüning [19] ($N = 2$) considered the existence at least one positive solution in not only scalar case but also vector case.

In [55], Jeanjean and Tanaka studied the relationship between a mountain pass solution and a least energy solution of (NSF) . They proved that a mountain pass solution is actually a least energy solution under (5-g1)–(5-g3).

Next we consider the case where $g(r, s)$ depends on $r$. First, Li [62] studied such a problem. He treated (NSF) with $N \geq 3$ and $\Omega = \mathbf{R}^N$ or $\Omega = \{x \in \mathbf{R}^N : |x| > R\}$. In the case where $\Omega = \{x \in \mathbf{R}^N : |x| > R\}$, he consider (NSF) under the Dirichlet boundary condition. In [62], it is shown that (NSF) has at least one radial positive solution and infinitely many radial possibly sign changing solutions under some conditions of $g$. In particular, they assumed that $g(r, s)$ is a monotone function with respect to $r$. In [61], Li and Li treated (NSF) in the case $N = 2$, and proved the same result.

On the other hand, Azzollini and Pomponio [7] considered the following nonlinear Schrödinger equation:

$$-\Delta u + V(|x|)u = \tilde{g}(u) \quad \text{in } \mathbf{R}^N \tag{1.2.6}$$

where $N \geq 3$ and $\tilde{g}$ satisfies (5-g1)–(5-g3). In [7], they showed the existence of at least one positive solution without the monotonicity condition about $V(r)$, but they assumed

$$\|(x \cdot \nabla V(|x|))^+\|_{L^{\frac{N}{2}}(\mathbf{R}^N)} < 2S_N$$

where

$$(x \cdot \nabla V(|x|))^+ := \max\{0, x \cdot \nabla V(|x|)\} \quad \text{and} \quad S_N := \inf_{u \in H^1(\mathbf{R}^N) \setminus \{0\}} \frac{\|\nabla u\|^2_{L^2(\mathbf{R}^N)}}{\|u\|^2_{L^{2^*}(\mathbf{R}^N)}}.$$

Part II consists of three chapters. In Chapter 5, we treat (NSF) with $\Omega = \mathbf{R}^N$ and $g(r, s) = g(s)$. Chapter 6 is devoted to (NSF) where $g(r, s)$ does depend on $r$ and $\Omega = \mathbf{R}^N$ or $\Omega = \{x \in \mathbf{R}^N : |x| > R\}$. Chapter 7 is an appendix of Chapters 5 and 6, and we prove that a sequence of minimax values defined in Chapters 5 and 6 tend to infinity.

### 1.2.1 Summary of Chapter 5

Chapter 5 is based on a work of [48]. In Chapter 5, we consider the following equation:

$$\begin{cases} -\Delta u = g(u) \quad \text{in } \mathbf{R}^N, \\ \quad u \in H^1(\mathbf{R}^N). \end{cases} \tag{1.2.7}$$

Next we state settings in Chapter 5. We suppose the conditions (5-g0), (5-g1) and (5-g3). Moreover, we assume

(5-g2')
$$-\infty < \liminf_{\xi \to 0} \frac{g(\xi)}{\xi} \le \limsup_{\xi \to 0} \frac{g(\xi)}{\xi} < 0.$$

We remark that there is a slight difference between the case $N \ge 3$ and $N = 2$ in [14, 15, 16] (cf. (1.2.4) and (1.2.5)).

Now we state our main result in Chapter 5.

**Theorem 1.2.1** (Theorem 5.1.3 in Chapter 5)**.** *Assume $N \ge 2$ and* (5-g0), (5-g1), (5-g3) *and* (5-g2')*. Then* (1.2.7) *has a least energy positive solution and infinitely many radially symmetric* (*possibly sign changing*) *solutions, which are characterized by the mountain pass and symmetric mountain pass minimax arguments in $H^1_r(\mathbf{R}^N)$.*

As mentioned above, Theorem 1.2.1 is a slight extension of results in [14]. Furthermore, our idea to prove Theorem 1.2.1 is different from ones in [14, 15, 16]. Indeed, in [14, 15, 16], they considered the following problem:

- ($N \ge 3$) Find critical points in $H^1_r(\mathbf{R}^N)$ of

$$\int_{\mathbf{R}^N} |\nabla u|^2 dx \qquad \text{subject to } \int_{\mathbf{R}^N} G(u) dx = 1$$

or

- ($N = 2$) Find critical points in $H^1_r(\mathbf{R}^2)$ of

$$\int_{\mathbf{R}^2} |\nabla u|^2 dx \qquad \text{subject to } \int_{\mathbf{R}^2} G(u) dx \ge 0 \quad \text{and} \quad \|u\|_{L^2(\mathbf{R}^2)} = 1.$$

If we find a critical point $v(x)$ of the above problems, then there exists a $\lambda > 0$ such that $-\Delta v = \lambda g(v)$ in $\mathbf{R}^N$. Thus by setting $u(x) = v(x/\sqrt{\lambda})$, $u(x)$ is a solution of (1.2.7).

On the other hand, our approach is the following. We consider an unconstrained functional

$$I(u) := \frac{1}{2} \int_{\mathbf{R}^N} |\nabla u|^2 dx - \int_{\mathbf{R}^N} G(u) dx.$$

Then we shall find critical points of $I$ directly. Our approach is based on symmetric mountain pass arguments in Ambrosetti and Rabinowitz [6] and Rabinowitz [92]. Moreover, in order to obtain bounded Palais–Smale sequences, we use an idea in Jeanjean [53]. Namely, we use the following augmented functional

$$\tilde{I}(\theta, u) := I(u(e^{-\theta} x)) = \frac{e^{(N-2)\theta}}{2} \int_{\mathbf{R}^N} |\nabla u|^2 dx - e^{N\theta} \int_{\mathbf{R}^N} G(u) dx \in C^1(\mathbf{R} \times H^1_r(\mathbf{R}^N), \mathbf{R}).$$

This functional is based on the scale properties and gives us bounded Palais–Smale sequences.

### 1.2.2 Summary of Chapter 6

Chapter 6 is based on a work of [52] and devoted to study the inhomogeneous case:

$$-\Delta u = g(|x|, u) \quad \text{in } \Omega. \tag{1.2.8}$$

Here $\Omega = \mathbf{R}^N$ or $\Omega = \{x \in \mathbf{R}^N : |x| > R\}$ and $N \geq 2$. Moreover, if $\Omega = \{x \in \mathbf{R}^N : |x| > R\}$, then we consider (1.2.8) with the Dirichlet or the Neumann boundary condition:

$$\text{(D)} \qquad u = 0 \qquad \text{on } \partial\Omega,$$

$$\text{(N)} \qquad \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega$$

where $\nu$ stands for the outward vector of $\partial\Omega$. Namely, we consider the following equations:

$$(P_{\mathbf{R}^N}) \quad -\Delta u = g(|x|, u) \quad \text{in } \mathbf{R}^N, \quad u \in H^1(\mathbf{R}^N).$$

$$(P_D) \quad -\Delta u = g(|x|, u) \quad \text{in } \{|x| > R\}, \quad u = 0 \quad \text{on } |x| = R, \quad u \in H^1(\{|x| > R\}).$$

$$(P_N) \quad -\Delta u = g(|x|, u) \quad \text{in } \{|x| > R\}, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } |x| = R, \quad u \in H^1(\{|x| > R\}).$$

For the above problems, we assume the following conditions for $g(r, s)$. In what follows, we regard $R = 0$ if $\Omega = \mathbf{R}^N$.

(6-g1) $g \in C([R, \infty) \times \mathbf{R}, \mathbf{R})$ and $g(r, -s) = -g(r, s)$ for all $r \geq R$ and $s \in \mathbf{R}$.

(6-g2) If $R \leq r_1 \leq r_2 < \infty$ and $s \geq 0$, then $g(r_1, s) \leq g(r_2, s)$.

(6-g3) As $r \to \infty$, $g(r, s) \to g_\infty(s)$ in $L^\infty_{\mathrm{loc}}(\mathbf{R})$.

(6-g4) There exists an $m_1 > 0$ such that

$$-\infty < \liminf_{s \to 0} \inf_{r \geq R} \frac{g(r, s)}{s} \leq \limsup_{s \to 0} \sup_{r \geq R} \frac{g(r, s)}{s} \leq -m_1$$

(6-g5) For $N \geq 3$,

$$\lim_{s \to \infty} \sup_{r \geq R} \frac{|g(r, s)|}{s^{2^*-1}} = 0 \quad \text{where} \quad 2^* = 2N/(N-2).$$

For $N = 2$,

$$\lim_{s \to \infty} \sup_{r \geq R} \frac{|g(r, s)|}{\exp(\alpha s^2)} = 0 \quad \text{for any } \alpha > 0.$$

(6-g6) There exist $\zeta_0 > 0$ and $R \geq R_0$ such that

$$\inf_{r \geq R_0} G(r, \zeta_0) > 0 \quad \text{where} \quad G(r, s) := \int_0^s g(r, \tau) d\tau$$

For $(P_{\mathbf{R}^N})$ and $(P_D)$, we do not need the following condition (6-g7), however for the Neumann problem $(P_N)$, in addition to (6-g1)–(6-g6), we assume

(6-g7) $-\infty < \inf_{s \in \mathbf{R}} G(R,s)$.

Now we state one of main results in Chapter 6.

**Theorem 1.2.2** (Theorems 6.2.1 and 6.2.2 in Chapter 6)**.** *Suppose that* $N \geq 2$ *and* $g$ *satisfies* (6-g1)–(6-g6)*. Then*

(i) *The problem* $(P_{\mathbf{R}^N})$ *( resp.* $(P_D)$ *) has at least one radial positive solution and infinitely many radial possibly sign changing solutions.*

(ii) *In addition to* (6-g1)–(6-g6)*, assume* (6-g7)*. Then* $(P_N)$ *has at least one radial positive solution and infinitely many radial possibly sign changing solutions.*

Here we give a remark. Li and Li [61] and Li [62] considered $(P_{\mathbf{R}^N})$ and $(P_D)$ under the similar conditions to (6-g1)–(6-g6). However, they also assume $g(r,s) = -s + o(s)$ as $s \to 0$ uniformly with respect to $r$. Besides, they did not treat the Neumann problem $(P_N)$.

In Chapter 6, we also consider the nonlinear Schrödinger equation without the monotonicity condition about $g(r,s)$. Namely, we treat the following equation:

$$-\Delta u + V(|x|)u = \tilde{g}(u) \quad \text{in } \Omega. \tag{1.2.9}$$

Here $\Omega = \mathbf{R}^N$ or $\Omega = \{x \in \mathbf{R}^N : |x| > R\}$ and $N \geq 3$.

Azzollini and Pomponio [7] studied the case where $\Omega = \mathbf{R}^N$ and showed the existence of at least one radial positive solution. We note that if we set $g(r,s) = -V(r)s + \tilde{g}(s)$, then (NSF) becomes (1.2.9). As in the above, we consider both of the Dirichlet and the Neumann problems when $\Omega = \{x \in \mathbf{R}^N : |x| > R\}$.

For $(P_{\mathbf{R}^N})$, $(P_D)$ and $(P_N)$ with $g(r,s) = -V(r)s + \tilde{g}(s)$, we assume the following conditions:

(6-g8) $\tilde{g}$ satisfies (5-g0)–(5-g3).

(6-g9) $-\infty < \inf_{s \in \mathbf{R}} \left( -\frac{1}{2} V(R) s^2 + \tilde{G}(s) \right)$ where $\tilde{G}(s) = \int_0^s \tilde{g}(t)dt$.

(6-V1) $V \in C^1([R,\infty))$ and $V(r) \geq 0$ for all $r \geq R$.

(6-V2) $\lim_{r \to \infty} V(r) = 0$.

(6-V3) $\|(x \cdot \nabla V(|x|))^+\|_{L^{\frac{N}{2}}(|x|>R)} < 2S_N$ where

$$(x \cdot \nabla V(|x|))^+ := \max\{0, x \cdot \nabla V(|x|)\} \quad \text{and} \quad S_N := \inf_{u \in H^1(\mathbf{R}^N) \setminus \{0\}} \frac{\|\nabla u\|^2_{L^2(\mathbf{R}^N)}}{\|u\|^2_{L^{2^*}(\mathbf{R}^N)}}.$$

Now we state a result concerning the nonlinear Schrödinger equations (1.2.9).

**Theorem 1.2.3** (Theorem 6.2.4 in Chapter 6)**.** *Suppose that* $N \geq 3$ *and* $g(r,s) = -V(r)s + \tilde{g}(s)$ *satisfies* (6-g8) *and* (6-V1)–(6-V3)*. Then the following hold:*

(i) $(P_{\mathbf{R}^N})$ *(resp.* $(P_D)$*) admits at least one radial positive solution and infinitely many radial possibly sign changing solutions.*

(ii) *Assume* (6-g9) *in addition to* (6-g8) *and* (6-V1)–(6-V3)*. Then* $(P_N)$ *admits at least one radial positive solution and infinitely many radial possibly radial sign changing solutions.*

In Theorem 1.2.3, we treat both of the Dirichlet problem and the Neumann problem. Furthermore, we establish the existence of infinitely many solutions.

# Part I

# Coupled nonlinear Schrödinger equations

# Chapter 2

# Existence of standing waves for variable coefficient problems

## 2.1 Introduction and main result

In this chapter, we consider the existence of standing waves for (TCNLS) with $\varepsilon = 1$. Namely, we consider

$$\text{(2.1.1)} \qquad \begin{cases} i\dfrac{\partial \psi_1}{\partial t} + \Delta_x \psi_1 + \lambda_1(x)\psi_1 + (\mu_1|\psi_1|^2 + \beta|\psi_2|^2)\psi_1 = 0 & \text{in} \quad (0,\infty)\times \mathbf{R}^N, \\ i\dfrac{\partial \psi_2}{\partial t} + \Delta_x \psi_2 + \lambda_2(x)\psi_2 + (\beta|\psi_1|^2 + \mu_2|\psi_2|^2)\psi_2 = 0 & \text{in} \quad (0,\infty)\times \mathbf{R}^N, \end{cases}$$

where $\mu_1, \mu_2, \beta > 0$ are constants and the dimension $N = 1, 2, 3$. The system (2.1.1) appears in many physical problems, especially in the Hartree–Fock theory and nonlinear optics. We refer to [3, 4, 12, 36, 45, 65, 75, 94, 105, 111] and references therein for more physical treatments.

In order to obtain standing waves, we substitute $\psi_j(t,x) = e^{i\tilde{\lambda}_j t}u_j(x)$ into (2.1.1). Then $u_1(x), u_2(x)$ solve

$$\text{(2.1.2)} \qquad \begin{cases} -\Delta u_1 + V_1(x)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in} \quad \mathbf{R}^N, \\ -\Delta u_2 + V_2(x)u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 & \text{in} \quad \mathbf{R}^N, \\ u_1, u_2 \in H^1(\mathbf{R}^N), \end{cases}$$

where $V_j(x) = \tilde{\lambda}_j - \lambda_j(x)$. In particular we are interested in a nontrivial positive solution of (2.1.2). Here, we say $u = (u_1, u_2)$ is a *nontrivial positive solution* of (2.1.2) if $u$ solves (2.1.2) and both $u_1, u_2$ are positive in $\mathbf{R}^N$.

Our aim of this chapter is to study the existence of a nontrivial positive solution for the system with variable coefficients. Our work is motivated by Sirakov [94], and Ambrosetti and Colorado [4]. They consider (2.1.2) in constant coefficient case, which means that $V_j(x) \equiv \text{const.} > 0$. Roughly speaking, they proved that there exist positive constants $\tilde{\beta}_1$ and $\tilde{\beta}_2$ such that if $0 \le \beta < \tilde{\beta}_1$ or $\tilde{\beta}_2 < \beta$ holds, then (2.1.2) has a nontrivial positive solution. We remark that the existence problem becomes delicate when the coefficient depends on the space variable $x$. In Theorem 2.1.4 we give an example even if $V_j(x)$ is very close to constant, (2.1.2) does not have any nontrivial positive solutions.

In this chapter, except for the nonexistence result (Theorem 2.1.4), we assume that $V_j(x)$ satisfies the following conditions:

(2–V1) For $j = 1, 2$, $V_j(x) \in C^1(\mathbf{R}^N, \mathbf{R})$.

(2–V2) For $j = 1, 2$, $0 < \inf_{x\in\mathbf{R}^N} V_j(x) \le \sup_{x\in\mathbf{R}^N} V_j(x) \equiv V_{\infty,j} < \infty$.

(2–V3) For $j = 1, 2$, $V_j(x) \to V_{\infty,j}$ as $|x| \to \infty$.

Here we introduce some terminologies. We call $u = (u_1, u_2)$ *nontrivial solution* if $u$ solves (2.1.2) and $u_1, u_2 \not\equiv 0$. On the other hand, we call $u$ *semitrivial solution* if $u$ solves (2.1.2) and $u_1 \equiv 0$ or $u_2 \equiv 0$. We remark that if $V_j(x)$ satisfies (2–V1)–(2–V3), then (2.1.2) has a semitrivial solution. Indeed, the equation

$$\begin{cases} -\Delta u_1 + V_1(x)u_1 = \mu_1 u_1^3 \quad \text{in} \quad \mathbf{R}^N, \\ \qquad\qquad u_1 \in H^1(\mathbf{R}^N) \end{cases}$$

or

$$\begin{cases} -\Delta u_2 + V_2(x)u_2 = \mu_2 u_2^3 \quad \text{in} \quad \mathbf{R}^N, \\ \qquad\qquad u_2 \in H^1(\mathbf{R}^N) \end{cases}$$

has a nontrivial solution (for instance, see Willem [110]). Then $u = (u_1, 0)$ or $u = (0, u_2)$ is a semitrivial solution of (2.1.2).

Hereafter, we fix $\mu_1, \mu_2 > 0, V_1(x), V_2(x)$ and consider the range of $\beta > 0$ in which (2.1.2) has a nontrivial positive solution. Here we state the main theorem in this paper.

**Theorem 2.1.1.** *Suppose that* $N = 1, 2, 3$ *and* $V_j(x)$ *satisfies* (2–V1)–(2–V3)*. Then there exist* $\beta_1 > 0$ *and* $\beta_2 > \beta_1$ *such that*

(i) *If* $0 < \beta < \beta_1$*, then* (2.1.2) *has a nontrivial positive solution.*

(ii) *If* $\beta_2 < \beta$*, then* (2.1.2) *has a nontrivial positive solution.*

Next, we consider whether the solutions obtained in Theorem 2.1.1 is a least energy solution or not. We say a solution $u = (u_1, u_2)$ of (2.1.2) is *a least energy solution* if $u$ satisfies the equality

$$I(u_1, u_2) = \inf \left\{ I(v_1, v_2) : (v_1, v_2) \not\equiv (0, 0) \text{ solves } (2.1.2) \right\}.$$

Here, we use notation: for $v = (v_1, v_2) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$,

$$I(v) = \frac{1}{2} \int_{\mathbf{R}^N} |\nabla v_1|^2 + V_1(x)v_1^2 + |\nabla v_2|^2 + V_2(x)v_2^2 \, dx - \frac{1}{4} \int_{\mathbf{R}^N} \mu_1 v_1^4 + 2\beta v_1^2 v_2^2 + \mu_2 v_2^4 \, dx.$$

**Theorem 2.1.2.** *The following hold*:

(i) *There exists a* $\beta_3 \in (0, \beta_2]$ *such that if* $\beta \in [0, \beta_3)$*, then the nontrivial positive solution obtained in Theorem* 2.1.1 (i) *is not a least energy solution.*

(ii) *If* $\beta > \beta_2$*, then the nontrivial positive solution obtained in Theorem* 2.1.1 (ii) *is a least energy solution. Here* $\beta_2$ *is given in Theorem* 2.1.1.

*Remark* 2.1.3. Ambrosetti and Colorado [4] obtained a nontrivial positive solution of (2.1.2) in the constant coefficient case with the mountain pass argument on the Nehari manifold. When $\beta > 0$ is small, they showed that the nontrivial positive solution of (2.1.2) has a higher energy than the semitrivial positive solutions.

Next, we give the nonexistence result. We assume that $V_j(x)$ satisfies the following conditions:

(2–V1') For $j = 1, 2$ and $i = 1, \ldots, N$, $V_j \in C^1(\mathbf{R}^N, \mathbf{R})$, $\nabla V_j \in L^\infty(\mathbf{R}^N)$.

(2–V2') For $j = 1, 2$, $0 < \inf_{x \in \mathbf{R}^N} V_j(x) \leq \sup_{x \in \mathbf{R}^N} V_j(x) < \infty$.

(2–V3') There exists a $\nu \in \mathbf{R}^N \backslash \{0\}$ such that $(\partial V_j / \partial \nu)(x) = \nabla V_j(x) \cdot \nu \geq 0$ for $j = 1, 2$.

(2–V4') It holds that either $\partial V_1 / \partial \nu \not\equiv 0$ or $\partial V_2 / \partial \nu \not\equiv 0$.

Here we state the nonexistence result.

**Theorem 2.1.4.** *Let* $V_j(x)$ *satisfy* (2–V1')–(2–V4')*. Then* (2.1.2) *has no nontrivial positive solution for any* $\beta > 0$.

*Remark* 2.1.5. There is a function which is close to a constant and satisfies (2–V1')–(2–V4'). For instance, setting $V_j(x) = \varepsilon \arctan(x_1) + \pi$, then $V_j(x)$ satisfies (2–V1')–(2–V4') and (2.1.2) has no nontrivial positive solution for any $\varepsilon \in (0, 2)$. This fact implies that the existence of nontrivial positive solution is a delicate problem and we need some conditions concerning the behavior of $V_j(x)$ at infinity in order to show the existence of .

We prove Theorem 2.1.1 by variational methods. To obtain a nontrivial solution of (2.1.2), we introduce the Nehari manifold $\mathcal{N}$ and the Nehari type manifold $\mathcal{M}$:

$$\mathcal{N} := \left\{ u \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N) : u \not\equiv (0,0),\ I'(u)u = 0 \right\},$$
$$\mathcal{M} := \left\{ u \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N) : u_1, u_2 \not\equiv 0,\ I'(u)(u_1, 0) = I'(u)(0, u_2) = 0 \right\}.$$

When $\beta > 0$ is large, which implies the setting of Theorem 2.1.1(ii), a nontrivial solution will be obtained as a minimizer of $I$ on $\mathcal{N}$ (see section 2.5).

When $\beta > 0$ is small, which is dealt in Theorem 2.1.1(i), we will also observe that $\inf_{\mathcal{N}} I$ is also attained. However the minimizer turns out to be a semitrivial function and the Nehari type manifold $\mathcal{M}$ plays a role to find a nontrivial solution. In section 2.2, we will prove that $\mathcal{M}$ is a smooth Hilbert manifold with codimension 2 under the condition $0 < \beta < \sqrt{\mu_1 \mu_2}$ and a nontrivial solution will be obtained as a minimizer of $I$ on $\mathcal{M}$ (see section 2.6).

We remark that for problems with constant coefficients Sirakov [94] introduced manifolds in the space of radially symmetric functions:

$$\mathcal{N}_r := \left\{ u \in H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N) : u \not\equiv (0,0),\ I'(u)u = 0 \right\},$$
$$\mathcal{M}_r := \left\{ u \in H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N) : u_1, u_2 \not\equiv 0,\ I'(u)(u_1, 0) = I'(u)(0, u_2) = 0 \right\}.$$

He obtained a nontrivial solution as a minimizer of $I$ on $\mathcal{N}_r$ ($\mathcal{M}_r$ respectively) when $\beta > 0$ is large ($\beta > 0$ is small respectively). We remark that when $\beta > 0$ is small Ambrosetti and Colorado [4] develops a mountain pass argument in $\mathcal{N}_r$ to find a nontrivial solution. We also remark that in these works, the compactness of the embedding $H_r^1(\mathbf{R}^N) \hookrightarrow L^4(\mathbf{R}^N)$ is very important to get the Palais–Smale condition ((PS) condition).

In our setting, we cannot work in the space of radially symmetric functions and due to noncompactness of the embedding $H^1(\mathbf{R}^N) \hookrightarrow L^4(\mathbf{R}^N)$, the corresponding functional $I$ does not satisfy the (PS) condition. To solve this difficulty we will develop a concentration compactness type result and give the estimates of critical value of $I$.

Finally, we give a mention to a work of Wei [105]. Wei considered (2.1.2) with variable coefficients, but under different conditions of $V_j(x)$ from ours. He considered the case where $V_j(x)$ is smooth, positive and $V_j(x) \to \infty$ as $|x| \to \infty$. The functional $I$ is considered on

$$\mathcal{H} = \left\{ u \in H \ : \ \int_{\mathbf{R}^N} V_j(x) u_j^2 \, dx < \infty \quad \text{for} \quad j = 1, 2 \right\}.$$

In this case, the embedding $\mathcal{H} \hookrightarrow L^4(\mathbf{R}^N) \times L^4(\mathbf{R}^N)$ is compact (See Rabinowitz [93], and Bartsch and Wang [11]), which implies that $I$ satisfies the (PS) condition on $\mathcal{H}$.

This chapter is organized as follows: In sections 2.2 and 2.3, we give some preliminaries: especially we give functional frameworks and introduce our variational settings. In section 2.4, we prove the achievement of $\inf_{\mathcal{N}} I$ for all $\beta > 0$. It is important to determine whether the minimizer is nontrivial or not. In sections 2.5 and 2.6, we give a proof to Theorems 2.1.1 and 2.1.2. In section 2.5, we deal with the case where $\beta$ is large and it turns out that the minimizer of $\inf_{\mathcal{N}} I$ is a nontrivial solution. In section 2.6, we study the case where $\beta$ is small. In this case the Nehari type manifold $\mathcal{M}$ plays a role. Moreover we will show that for sufficiently small $\beta$, a least energy solution of (2.1.2) is a semitrivial solution. In section 2.7, we prove Theorem 2.1.4.

## 2.2 Preliminaries

In this section, we prove some preliminary results to prove Theorem 2.1.1.

### 2.2.1 Function spaces and functionals

We set $H := H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ and denote elements of $H$ by $u = (u_1, u_2)$. For $u = (u_1, u_2)$ and $v = (v_1, v_2) \in H$, we define inner products and norms in $H^1(\mathbf{R}^N)$ and $H$ as follows:

$$
\begin{aligned}
\langle u_j, v_j\rangle_j &:= \int_{\mathbf{R}^N} \nabla u_j\cdot\nabla v_j + V_j(x)u_jv_j\,dx \quad (j=1,2),\\
\langle u_j, v_j\rangle_{\infty,j} &:= \int_{\mathbf{R}^N} \nabla u_j\cdot\nabla v_j + V_{\infty,j}u_jv_j\,dx \quad (j=1,2),
\end{aligned}
$$

$$
\begin{aligned}
\langle u, v\rangle &:= \langle u_1, v_1\rangle_1 + \langle u_2, v_2\rangle_2, & \langle u, v\rangle_\infty &:= \langle u_1, v_1\rangle_{\infty,1} + \langle u_2, v_2\rangle_{\infty,2},\\
\|u_j\|_j^2 &:= \langle u_j, u_j\rangle_j, & \|u_j\|_{\infty,j}^2 &:= \langle u_j, u_j\rangle_{\infty,j} \quad (j=1,2),\\
\|u\|^2 &:= \|u_1\|_1^2 + \|u_2\|_2^2, & \|u\|_\infty^2 &:= \|u_1\|_{\infty,1}^2 + \|u_2\|_{\infty,2}^2.
\end{aligned}
$$

We remark that $\|\cdot\|_j$, $\|\cdot\|_{\infty,j}$ are equivalent to the standard $H^1(\mathbf{R}^N)$ norm under the conditions (2–V1)–(2–V2). We define the functional $I : H \to \mathbf{R}$ as follows:

$$
I(u) := \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbf{R}^N} \mu_1 u_1^4 + 2\beta u_1^2 u_2^2 + \mu_2 u_2^4\,dx.
$$

Differentiating $I$, we have

$$
I'(u)v = \langle u, v\rangle - \int_{\mathbf{R}^N} \mu_1 u_1^3 v_1 + \beta u_1 u_2^2 v_1 + \beta u_1^2 u_2 v_2 + \mu_2 u_2^3 v_2\,dx \quad \text{for all } v \in H.
$$

It is easily seen that any critical point of $I$ is a solution of (2.1.2). We also use a notation $\nabla I(u) \in H$, where $\nabla I(u)$ is a unique element satisfying $I'(u)v = \langle \nabla I(u), v\rangle$ for all $v \in H$. We also define the functional $I_\infty : H \to \mathbf{R}^N$ as follows:

$$
I_\infty(u) := \frac{1}{2}\|u\|_\infty^2 - \frac{1}{4}\int_{\mathbf{R}^N} \mu_1 u_1^4 + 2\beta u_1^2 u_2^2 + \mu_2 u_2^4\,dx.
$$

The functional $I_\infty$ corresponds to the problem 'at infinity':

$$
\begin{cases}
-\Delta u_1 + V_{\infty,1}u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbf{R}^N,\\
-\Delta u_2 + V_{\infty,2}u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 & \text{in } \mathbf{R}^N,\\
u_1, u_2 \in H^1(\mathbf{R}^N).
\end{cases} \tag{2.2.1}
$$

Any critical point of $I_\infty$ is also a solution of (2.2.1).

It is easily seen that the following equalities hold:

$$
\begin{aligned}
I'(u)u &= \|u\|^2 - \mu_1\|u_1\|_{L^4}^4 - 2\beta\|u_1u_2\|_{L^2}^2 - \mu_2\|u_2\|_{L^4}^4,\\
I'(u)(u_1, 0) &= \|u_1\|_1^2 - \mu_1\|u_1\|_{L^4}^4 - \beta\|u_1u_2\|_{L^2}^2,\\
I'(u)(0, u_2) &= \|u_2\|_2^2 - \beta\|u_1u_2\|_{L^2}^2 - \mu_2\|u_2\|_{L^4}^4.
\end{aligned}
$$

### 2.2.2 Nehari manifold and Nehari type manifold

In this subsection we introduce the Nehari manifold $\mathcal{N}$ and the Nehari type manifold $\mathcal{M}$ and state some properties of $\mathcal{N}$ and $\mathcal{M}$.

We define $J,\ J_1,\ J_2 : H \to \mathbf{R}$ as follows:

$$
J(u) := I'(u)u, \quad J_1(u) := I'(u)(u_1, 0), \quad J_2(u) := I'(u)(0, u_2).
$$

**Definition 2.2.1.** We define the Nehari manifold $\mathcal{N}$ and the Nehari type manifold $\mathcal{M}$ as follows:

$$\mathcal{N} := \{u \in H : u \not\equiv (0,0),\ J(u) = 0\},$$
$$\mathcal{M} := \{u \in H : u_1 \not\equiv 0,\ u_2 \not\equiv 0,\ J_1(u) = J_2(u) = 0\}.$$

We also define $\mathcal{N}_\infty$ and $\mathcal{M}_\infty$ corresponding to (2.2.1):

$$\mathcal{N}_\infty := \{u \in H : u \not\equiv (0,0),\ J_\infty(u) = 0\},$$
$$\mathcal{M}_\infty := \{u \in H : u_1 \not\equiv 0,\ u_2 \not\equiv 0,\ J_{\infty,1}(u) = J_{\infty,2}(u) = 0\}.$$

*Remark* 2.2.2. (i) $\mathcal{M} \subset \mathcal{N}$ and $\mathcal{M}_\infty \subset \mathcal{N}_\infty$.

(ii) Except for $(0,0)$, any solution of (2.1.2) belongs to $\mathcal{N}$.

(iii) If $u$ is a nontrivial solution of (2.1.2), then $u \in \mathcal{M}$.

*Remark* 2.2.3. We set $|u| := (|u_1|, |u_2|)$, then the following hold:

(i) If $u \in \mathcal{N}$, then $|u| \in \mathcal{N}$.

(ii) If $u \in \mathcal{M}$, then $|u| \in \mathcal{M}$.

Next, we state the fundamental properties of $\mathcal{N}$ and $\mathcal{N}_\infty$.

**Proposition 2.2.4.** *The following properties hold*:

(i) *For each $u \in H$ with $u \not\equiv (0,0)$, there exist unique $\theta_0 > 0$ and $\theta_{\infty,0} > 0$ such that $\theta_0 u \in \mathcal{N},\ \theta_{\infty,0} u \in \mathcal{N}_\infty$.*

(ii) $I(u) = \dfrac{1}{4}\|u\|^2$ on $\mathcal{N}$, $I_\infty(u) = \dfrac{1}{4}\|u\|_\infty^2$ on $\mathcal{N}_\infty$.

(iii) *There exist $\delta_0 > 0$ and $\delta_\infty > 0$ such that*

$$\|u\| \geq \delta_0 \quad \text{for all } u \in \mathcal{N}, \quad \|v\|_\infty \geq \delta_\infty \quad \text{for all } v \in \mathcal{N}_\infty.$$

*Proof.* We only prove for $\mathcal{N}$.
(i) Suppose that $u \in H, u \not\equiv (0,0)$ and set

$$f(\theta) := I(\theta u) = \frac{\theta^2}{2}\|u\|^2 - \frac{\theta^4}{4}\int_{\mathbf{R}^N} \mu_1 u_1^4 + 2\beta u_1^2 u_2^2 + \mu_2 u_2^4 \, dx.$$

Then we see

$$f'(\theta) = I'(\theta u)u = \theta\left\{\|u\|^2 - \theta^2(\mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_1 u_2\|_{L^2}^2 + \mu_2\|u_2\|_{L^4}^4)\right\}.$$

Thus $f'(\theta) = 0$ holds if and only if $\theta = \theta_0$, where

$$\theta_0 := \frac{\|u\|}{\sqrt{\mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_1 u_2\|_{L^2}^2 + \mu_2\|u_2\|_{L^4}^4}} > 0.$$

(ii) Let $u \in \mathcal{N}$. Then it follows that

$$\|u\|^2 = \mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_1u_2\|_{L^2}^2 + \mu_2\|u_2\|_{L^4}^4.$$

From the above equality, we obtain

$$I(u) = \frac{\|u\|^2}{2} - \frac{\|u\|^2}{4} = \frac{\|u\|^2}{4}.$$

(iii) Let $u \in \mathcal{N}$. By using Hölder's inequality and Sobolev's embedding, we have

$$\begin{aligned}\|u\|^2 &= \mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_1u_2\|_{L^2}^2 + \mu_2\|u_2\|_{L^4}^4 \\ &\le \mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_1\|_{L^4}^2\|u_2\|_{L^4}^2 + \mu_2\|u_2\|_{L^4}^4 \\ &\le C(\mu_1\|u_1\|_1^4 + 2\beta\|u_1\|_1^2\|u_2\|_2^2 + \mu_2\|u_2\|_2^4) \\ &\le C(\|u_1\|_1^2 + \|u_2\|_2^2)^2 = C\|u\|^4.\end{aligned}$$

Therefore it follows that

$$\frac{1}{C} \le \|u\|^2.$$

□

Next, we prove that $\mathcal{N}$ and $\mathcal{M}$ are smooth Hilbert manifolds.

**Lemma 2.2.5.** *It holds that*

(i) *For each $\beta > 0$, $\mathcal{N}$ and $\mathcal{N}_\infty$ are smooth Hilbert manifolds with codimension* 1.

(ii) *If $0 < \beta < \sqrt{\mu_1\mu_2}$, then $\mathcal{M}$ and $\mathcal{M}_\infty$ are smooth Hilbert manifolds with codimension* 2.

(iii) $T_u\mathcal{N} = \{v \in H : J'(u)v = 0\}$.

(iv) $T_u\mathcal{M} = \{v \in H : J_1'(u)v = J_2'(u)v = 0\}$.

The above lemma will be derived from the following well-known lemma. For example, see Ambrosetti and Malchiodi [5].

**Lemma 2.2.6.** *Let $O \subset H$ be an open set. Suppose $G, G_1, G_2 \in C^m(O, \mathbf{R})$ and set $M := G^{-1}(0), \tilde{M} := G_1^{-1}(0) \cap G_2^{-1}(0)$. Then the following hold*:

(i) *If $G'(p) \neq 0$ for each $p \in M$, then $M$ is a $C^m$ Hilbert manifold with codimension* 1.

(ii) *If $G_1'(p)$ and $G_2'(p)$ are linearly independent for each $p \in \tilde{M}$, then $\tilde{M}$ is a $C^m$ Hilbert manifold with codimension* 2.

(iii) $T_pM = \{q \in H : G'(p)q = 0\}$.

(iv) $T_p\tilde{M} = \{q \in H : G_1'(p)q = G_2'(p)q = 0\}$.

We prove Lemma 2.2.5 with the aid of Lemma 2.2.6.

*Proof of Lemma 2.2.5.* We only prove (i) and (ii) since (iii) and (iv) are directly derived from Lemma 2.2.6.
(i) For $u \in \mathcal{N}$, we have

$$J'(u)u = 2\|u\|^2 - 4(\mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_1u_2\|_{L^2}^2 + \mu_2\|u_2\|_{L^4}^4) = -2\|u\|^2 < 0.$$

In particular, we have $J'(u) \neq 0$ for any $u \in \mathcal{N}$. Thus applying Lemma 2.2.6 to $J : H\backslash\{0\} \to \mathbf{R}$, we have (i) of Lemma 2.2.5.
(ii) Next we apply (ii) of Lemma 2.2.6 to $J_1, J_2 : H\backslash\{u_1 = 0 \text{ or } u_2 = 0\} \to \mathbf{R}$. For $u \in \mathcal{M}$, we have

$$J_1'(u)(u_1, 0) = -2\mu_1\|u_1\|_{L^4}^4, \quad J_2'(u)(0, u_2) = -2\mu_2\|u_2\|_{L^4}^4,$$
$$J_1'(u)(0, u_2) = J_2'(u)(u_1, 0) = -2\beta\|u_1u_2\|_{L^2}^2.$$

Define $A(u)$ by

$$A(u) := \begin{pmatrix} J_1'(u)(u_1, 0) & J_1'(u)(0, u_2) \\ J_2'(u)(u_1, 0) & J_2'(u)(0, u_2) \end{pmatrix} = \begin{pmatrix} -2\mu_1\|u_1\|_{L^4}^4 & -2\beta\|u_1u_2\|_{L^2}^2 \\ -2\beta\|u_1u_2\|_{L^2}^2 & -2\mu_2\|u_2\|_{L^4}^4 \end{pmatrix},$$

and we see

$$\det A(u) = 4(\mu_1\mu_2\|u_1\|_{L^4}^4\|u_2\|_{L^4}^4 - \beta^2\|u_1u_2\|_{L^2}^4) \geq 4(\mu_1\mu_2 - \beta^2)\|u_1\|_{L^4}^4\|u_2\|_{L^4}^4 > 0.$$

The above inequality implies that $J_1'(u)$ and $J_2'(u)$ are linearly independent. Thus by Lemma 2.2.6 we infer that $\mathcal{M}$ is a smooth Hilbert manifold with codimension 2. □

Lastly we state some properties of the level sets of $\mathcal{N}$ and $\mathcal{M}$. For each $\alpha > 0$, we define $\mathcal{N}^\alpha$ and $\mathcal{M}^\alpha$ as follows:

$$\mathcal{N}^\alpha := \{u \in \mathcal{N} : I(u) \leq \alpha\}, \quad \mathcal{M}^\alpha := \{u \in \mathcal{M} : I(u) \leq \alpha\}.$$

**Proposition 2.2.7** (Properties of $\mathcal{N}$)**.** *The following properties hold*:

(i) *The set $\mathcal{N}$ is a closed subset of $H$ and $\mathcal{N}^\alpha$ is a bounded closed subset of $H$. In particular,*

$$0 < \delta_0 \leq \|u\| \leq 2\sqrt{\alpha} \quad \text{for all } u \in \mathcal{N}^\alpha,$$

*where $\delta_0$ is given in Proposition* 2.2.4.

(ii) *For each $\alpha > 0$, it holds*

$$0 < 2\delta_0 \leq \|\nabla J(u)\| \leq c_1(\alpha) \quad \text{for all } u \in \mathcal{N}^\alpha,$$

*where $c_1(\alpha)$ depends on $\alpha$ but not on $u \in \mathcal{N}^\alpha$.*

*Proof.* (i) It is clear from Proposition 2.2.4 (ii) and (iii).
(ii) Since $J'(u)u = -2\|u\|^2$ and $\|u\| \geq \delta_0$, we have $2\delta_0 \leq \|J'(u)\|$. On the other hand, since $J' : H \to H^*$ maps bounded sets to bounded sets and $\mathcal{N}^\alpha$ is bounded, we infer the conclusion of Proposition 2.2.7. □

We define $T_u\mathcal{N}^\perp$ and $T_u\mathcal{M}^\perp$ as the orthonormal complement of $T_u\mathcal{N}$ and $T_u\mathcal{M}$, respectively:

$$T_u\mathcal{N}^\perp := \{v \in H : \langle v, h\rangle = 0 \text{ for } h \in T_u\mathcal{N}\},$$
$$T_u\mathcal{M}^\perp := \{v \in H : \langle v, h\rangle = 0 \text{ for } h \in T_u\mathcal{M}\}.$$

We also define $P_{T_u\mathcal{N}^\perp}$ and $P_{T_u\mathcal{M}^\perp}$ as the projections from $H$ to $T_u\mathcal{N}^\perp$ and $T_u\mathcal{M}^\perp$, respectively:

$$P_{T_u\mathcal{N}^\perp} : H \to T_u\mathcal{N}^\perp, \quad P_{T_u\mathcal{M}^\perp} : H \to T_u\mathcal{M}^\perp.$$

By Lemma 2.2.5, we have $T_u\mathcal{N}^\perp = \operatorname{span}\{\nabla J(u)\}$. Thus

$$P_{T_u\mathcal{N}^\perp} u = \left\langle \frac{\nabla J(u)}{\|\nabla J(u)\|}, u \right\rangle \frac{\nabla J(u)}{\|\nabla J(u)\|}.$$

By Lemma 2.2.5 and Proposition 2.2.7, we have the following corollary.

**Corollary 2.2.8.** *For each $\alpha > 0$, there holds*

$$0 < c_1(\alpha) \le \|P_{T_u\mathcal{N}^\perp} u\| \le c_2(\alpha) \quad \text{for all } u \in \mathcal{N}^\alpha,$$

*where $c_1(\alpha)$, $c_2(\alpha)$ are positive constants and depend on $\alpha$.*

Next we state the properties of $\mathcal{M}$.

**Proposition 2.2.9** (Properties of $\mathcal{M}$). *Let $\alpha > 0$.*

(i) *There exist $\beta_1(\alpha) \in (0, \sqrt{\mu_1\mu_2})$, $c_1(\alpha)$, $c_2(\alpha) > 0$ such that for each $\beta \in (0, \beta_1(\alpha))$ and $u \in \mathcal{M}^\alpha$,*

$$c_1(\alpha) \le \|u_j\|_{L^4} \le c_2(\alpha), \ c_1(\alpha) \le \|u_j\|_j \le c_2(\alpha),$$
$$c_1(\alpha) \le \|\nabla J_j(u)\| \le c_2(\alpha) \quad (j = 1, 2).$$

(ii) *If $\beta \in (0, \beta_1(\alpha))$, then $\mathcal{M}^\alpha$ is a closed subset of $H$.*

(iii) *There exists an $\varepsilon_1(\alpha) > 0$ such that for each $u \in \mathcal{M}^\alpha$ and $\beta \in (0, \beta_1(\alpha))$,*

$$|\langle \nabla J_1(u), \nabla J_2(u)\rangle| \le (1 - \varepsilon_1(\alpha))\|\nabla J_1(u)\|\|\nabla J_2(u)\|.$$

(iv) *There exist $c_3(\alpha) > 0$ and $c_4(\alpha) > 0$ such that for each $\beta \in (0, \beta_1(\alpha))$ and $u = (u_1, u_2) \in \mathcal{M}^\alpha$,*

$$0 < c_3(\alpha) \le \|P_{T_u\mathcal{M}^\perp} U_j\| \le c_4(\alpha) \quad (j = 1, 2),$$

*where $U_1 = (u_1, 0)$ and $U_2 = (0, u_2)$. Moreover, there exists an $\varepsilon_2(\alpha) > 0$ such that*

$$|\langle P_{T_u\mathcal{M}^\perp} U_1, P_{T_u\mathcal{M}^\perp} U_2\rangle| \le (1 - \varepsilon_2(\alpha))\|P_{T_u\mathcal{M}^\perp} U_1\|\|P_{T_u\mathcal{M}^\perp} U_2\|$$

*for all $u \in \mathcal{M}^\alpha$.*

*Proof.* (i) Since $\mathcal{M}^\alpha \subset \mathcal{N}^\alpha$, $\mathcal{N}^\alpha$ is a bounded set in $H$ and $J_j'$ maps bounded sets to bounded sets, it is sufficient to show that

$$0 < c_1(\alpha) \le \|u_j\|_{L^4},\ 0 < c_1(\alpha) \le \|u_j\|_j,\ 0 < c_1(\alpha) \le \|\nabla J_j(u)\|$$

for each $u \in \mathcal{M}^\alpha$. We only show the statements for $u_1$ and $\nabla J_1$ since the same argument is valid for $u_2$ and $\nabla J_2$.

Since

$$\|u_1\|_1^2 = \mu_1 \|u_1\|_{L^4}^4 + \beta \|u_1 u_2\|_{L^2}^2,$$

using Hölder's inequality and Sobolev's embedding theorem, it follows that

$$\|u_1\|_{L^4}^2 \le C\|u_1\|_1^2 \le C(\mu_1 \|u_1\|_{L^4}^4 + \beta \|u_1\|_{L^4}^2 \|u_2\|_{L^4}^2).$$

This implies that

$$\frac{1}{C} - \beta \|u_2\|_{L^4}^2 \le \mu_1 \|u_1\|_{L^4}^2.$$

Since $\|u_j\|_j$ are bounded, there exists a $\beta(\alpha) > 0$ such that if $\beta \in (0, \beta(\alpha))$, then

$$0 < c_1(\alpha) \le \|u_1\|_{L^4}.$$

By Sobolev's embedding, we have

$$c_1(\alpha) \le \|u_1\|_{L^4} \le C\|u_1\|_1.$$

Since $J_1'(u)(u_1, 0) = -2\mu_1 \|u_1\|_{L^4}^4$, we have $c_1(\alpha) \le \|\nabla J_1(u)\|$.
(ii) By (i) and the continuity of $J_j(u)$, it is easy to check that (ii) holds.
(iii) Let $u \in \mathcal{M}^\alpha$ and set

$$\xi_1 := \frac{\nabla J_1(u)}{\|\nabla J_1(u)\|}, \quad \xi_2 := \frac{\nabla J_2(u)}{\|\nabla J_2(u)\|}, \quad \tilde{\xi}_2 := \xi_2 - \langle \xi_1, \xi_2 \rangle \xi_1, \quad \xi_3 := \frac{\tilde{\xi}_2}{\|\tilde{\xi}_2\|}.$$

Since $\mathcal{M}^\alpha$ is bounded and $\nabla J_1,\ \nabla J_2$ map bounded sets into bounded sets, we only prove that there exists a $c(\alpha) = c > 0$ such that

$$0 < c \le \|\tilde{\xi}_2\|^2 \quad \text{for all } u \in \mathcal{M}^\alpha. \tag{2.2.2}$$

Indeed, since

$$\|\tilde{\xi}_2\|^2 = 1 - \langle \xi_1, \xi_2 \rangle^2 = \frac{\|\nabla J_1(u)\|^2 \|\nabla J_2(u)\|^2 - \langle \nabla J_1(u), \nabla J_2(u) \rangle^2}{\|\nabla J_1(u)\|^2 \|\nabla J_2(u)\|^2},$$

(iii) follows from (2.2.2).

Set $U_1 := (u_1, 0)$, $U_2 := (0, u_2)$ and define $A(u)$ as follows:

$$A(u) := \begin{pmatrix} \langle U_1, \xi_1 \rangle & \langle U_1, \xi_3 \rangle \\ \langle U_2, \xi_1 \rangle & \langle U_2, \xi_3 \rangle \end{pmatrix}.$$

Since

$$\det A(u) = \frac{1}{\|\tilde{\xi}_2\|} \det \begin{pmatrix} \langle U_1, \xi_1 \rangle & \langle U_1, \tilde{\xi}_2 \rangle \\ \langle U_2, \xi_1 \rangle & \langle U_2, \tilde{\xi}_2 \rangle \end{pmatrix} = \frac{1}{\|\tilde{\xi}_2\|} \det \begin{pmatrix} \langle U_1, \xi_1 \rangle & \langle U_1, \xi_2 - \langle \xi_1, \xi_2 \rangle \xi_1 \rangle \\ \langle U_2, \xi_1 \rangle & \langle U_2, \xi_2 - \langle \xi_1, \xi_2 \rangle \xi_1 \rangle \end{pmatrix}$$
$$= \frac{1}{\|\tilde{\xi}_2\|} \det \begin{pmatrix} \langle U_1, \xi_1 \rangle & \langle U_1, \xi_2 \rangle \\ \langle U_2, \xi_1 \rangle & \langle U_2, \xi_2 \rangle \end{pmatrix}$$

and

$$\langle U_1, \xi_1 \rangle = -\frac{2\mu_1 \|u_1\|_{L^4}^4}{\|\nabla J_1(u)\|}, \qquad \langle U_1, \xi_2 \rangle = -\frac{2\beta \|u_1 u_2\|_{L^2}^2}{\|\nabla J_2(u)\|},$$
$$\langle U_2, \xi_1 \rangle = -\frac{2\beta \|u_1 u_2\|_{L^2}^2}{\|\nabla J_1(u)\|}, \quad \langle U_2, \xi_2 \rangle = -\frac{2\mu_2 \|u_2\|_{L^4}^4}{\|\nabla J_2(u)\|},$$

we have

$$\det A(u) = \frac{4(\mu_1\mu_2 \|u_1\|_{L^4}^4 \|u_2\|_{L^4}^4 - \beta^2 \|u_1 u_2\|_{L^2}^4)}{\|\tilde{\xi}_2\| \|\nabla J_1(u)\| \|\nabla J_2(u)\|} \geq \frac{4(\mu_1\mu_2 - \beta^2)\|u_1\|_{L^4}^4 \|u_2\|_{L^4}^4}{\|\tilde{\xi}_2\| \|\nabla J_1(u)\| \|\nabla J_2(u)\|}.$$

By (i) and the assumption of $\beta$,

$$\det A(u) \geq \frac{C(\alpha)}{\|\tilde{\xi}_2\|} \quad \text{for all } u \in \mathcal{M}^\alpha. \tag{2.2.3}$$

On the other hand, the components of $A(u)$ are bounded, which implies that there exists a $C_1 = C_1(\alpha) > 0$ such that

$$\det A(u) \leq C_1(\alpha) \quad \text{for all } u \in \mathcal{M}^\alpha. \tag{2.2.4}$$

From (2.2.3) and (2.2.4), there exists a $c = c(\alpha) > 0$ such that

$$0 < c \leq \|\tilde{\xi}_2\| \quad \text{for all } u \in \mathcal{M}^\alpha.$$

(iv) Since

$$P_{T_u\mathcal{M}^\perp} U_1 = \langle U_1, \xi_1 \rangle \xi_1 + \langle U_1, \xi_3 \rangle \xi_3, \tag{2.2.5}$$

where $\xi_j$ are given in (iii), it follows that

$$\|U_1\|^2 \geq \|P_{T_u\mathcal{M}^\perp} U_1\|^2 = \langle U_1, \xi_1 \rangle^2 + \langle U_1, \xi_3 \rangle^2 \geq \langle U_1, \xi_1 \rangle^2 = \frac{4\mu_1^2 \|u_1\|_{L^4}^8}{\|\nabla J_1(u)\|^2}.$$

By (i), it follows that there exist $c_3(\alpha) > 0$ and $c_4(\alpha) > 0$ such that

$$c_3(\alpha) \leq \|P_{T_u\mathcal{M}^\perp} U_1\| \leq c_4(\alpha) \quad \text{for all } u \in \mathcal{M}^\alpha. \tag{2.2.6}$$

Similarly we have (2.2.6) for $U_2$. Since (2.2.5) and

$$P_{T_u\mathcal{M}^\perp} U_2 = \langle U_2, \xi_1 \rangle \xi_1 + \langle U_2, \xi_3 \rangle \xi_3,$$

we have

$$\|P_{T_u\mathcal{M}^\perp} U_1\|^2 \|P_{T_u\mathcal{M}^\perp} U_2\|^2 - |\langle P_{T_u\mathcal{M}^\perp} U_1, P_{T_u\mathcal{M}^\perp} U_2 \rangle|^2 = (\det A(u))^2.$$

By (2.2.3) and the boundedness of $(P_{T_u\mathcal{M}^\perp} U_j)$, for sufficiently small $\varepsilon_2(\alpha) > 0$, the conclusion of (iv) holds. □

**Corollary 2.2.10.** (i) *Let $u \in \mathcal{N}$ satisfy $I'(u)h = 0$ for all $h \in T_u\mathcal{N}$. Then $I'(u) = 0$.*

(ii) *Let $u \in \mathcal{M}$ satisfy $I'(u)h = 0$ for all $h \in T_u\mathcal{M}$. Then $I'(u) = 0$.*

*Remark* 2.2.11. Similar results hold for $\mathcal{N}_\infty$ and $\mathcal{M}_\infty$.

*Proof.* We only treat (ii) since (i) can be shown similarly. Let $u \in \mathcal{M}$ satisfy $I'(u)h = 0$ for all $h \in T_u\mathcal{M}$. It is sufficient to prove $I'(u) = 0$ on $T_u\mathcal{M}^\perp$. Let $U_1 = (u_1, 0)$ and $U_2 = (0, u_2)$ as in the proof of Proposition 2.2.9. Then from the proof of Proposition 2.2.9 (iv), we see that $T_u\mathcal{M}^\perp = \mathrm{span}\,\{P_{T_u\mathcal{M}^\perp}U_1, P_{T_u\mathcal{M}^\perp}U_2\}$. Hence we show $I'(u)P_{T_u\mathcal{M}^\perp}U_1 = I'(u)P_{T_u\mathcal{M}^\perp}U_2 = 0$. However, this is easily follows from the fact that $I'(u)U_1 = I'(u)U_2 = 0$ and $I'(u)h = 0$ for all $h \in T_u\mathcal{M}$. Therefore we have $I'(u) = 0$. □

### 2.2.3 $(\mathrm{PS})_c$ sequence

First, we introduce important values to obtain a nontrivial solution of (2.1.2).

We define $b_\mathcal{N}, \hat{b}_\mathcal{M}, b_{\mathcal{N}_\infty}, \hat{b}_{\mathcal{M}_\infty}$ as follows.

$$b_\mathcal{N} := \inf_{u\in\mathcal{N}} I(u), \quad \hat{b}_\mathcal{M} := \inf_{u\in\mathcal{M}} I(u), \quad b_{\mathcal{N}_\infty} := \inf_{u\in\mathcal{N}_\infty} I_\infty(u), \quad \hat{b}_{\mathcal{M}_\infty} := \inf_{u\in\mathcal{M}_\infty} I_\infty(u).$$

*Remark* 2.2.12. By Remark 2.2.2, it follows that

$$0 < b_\mathcal{N} \le \hat{b}_\mathcal{M}, \quad 0 < b_{\mathcal{N}_\infty} \le \hat{b}_{\mathcal{M}_\infty}.$$

To obtain a solution of (2.1.2), we see that $b_\mathcal{N}$ or $\hat{b}_\mathcal{M}$ is attained. So it is important to see the behavior of the minimizing sequence on $\mathcal{N}$ or $\mathcal{M}$.

**Definition 2.2.13.** Let $c \in \mathbf{R}$.

(i) A sequence $(u_n) \subset H$ is said to be a Palais–Smale sequence of $I$ on $H$ at level $c$ (in short $(\mathrm{PS})_{c,\,H}$ sequence), if it satisfies

$$I(u_n) \to c, \quad \|I'(u_n)\|_{H^*} \to 0,$$

where

$$\|I'(u)\|_{H^*} := \sup_{h\in H,\, \|h\|=1} I'(u)h.$$

(ii) A sequence $(u_n) \subset \mathcal{N}$ is said to be a $(\mathrm{PS})_{c,\,\mathcal{N}}$ sequence of $I$, if it satisfies

$$I(u_n) \to c, \quad \|I'(u_n)\|_{T_{u_n}\mathcal{N}^*} \to 0,$$

where

$$\|I'(v)\|_{T_v\mathcal{N}^*} := \sup_{h\in T_v\mathcal{N},\, \|h\|=1} I'(v)h.$$

(iii) Let $\beta < \sqrt{\mu_1\mu_2}$. A sequence $(u_n) \subset \mathcal{M}$ is said to be a $(\mathrm{PS})_{c,\,\mathcal{M}}$ sequence of $I$ on $\mathcal{M}$, if it satisfies

$$I(u_n) \to c, \quad \|I'(u_n)\|_{T_{u_n}\mathcal{M}^*} \to 0,$$

where

$$\|I'(w)\|_{T_w\mathcal{M}^*} := \sup_{h\in T_w\mathcal{M},\, \|h\|=1} I'(w)h.$$

Next we see the relationships between a $(\mathrm{PS})_{c,\ H}$, $(\mathrm{PS})_{c,\ \mathcal{N}}$ and $(\mathrm{PS})_{c,\ \mathcal{M}}$ sequence.

**Lemma 2.2.14.** (i) *Any* $(\mathrm{PS})_{c,\ H}$ *sequence* $(u_n)$ *is a bounded sequence on* $H$.

(ii) *Any* $(\mathrm{PS})_{c,\ \mathcal{N}}$ *sequence* $(u_n)$ *is a* $(\mathrm{PS})_{c,\ H}$ *sequence.*

(iii) *If* $c < \alpha$ *and* $\beta \in (0, \beta_1(\alpha))$, *then any* $(\mathrm{PS})_{c,\ \mathcal{M}}$ *sequence is a* $(\mathrm{PS})_{c,\ H}$ *sequence, where* $\beta_1(\alpha)$ *appeared in Proposition* 2.2.9.

*Proof.* (i) Let $(u_n)$ be a $(\mathrm{PS})_{c,\ H}$ sequence. Since $\|I(u_n)\|_{H^*} \to 0$, there exists an $n_1 \in \mathbf{N}$ such that

$$|I'(u_n)u_n| \le \|u_n\| \quad \text{for all } n \ge n_1.$$

On the other hand, it holds

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{1}{4}(\mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_{n,1}u_{n,2}\|_{L^2}^2 + \mu_2\|u_{n,2}\|_{L^4}^4),$$
$$I'(u_n)u_n = \|u_n\|^2 - (\mu_1\|u_1\|_{L^4}^4 + 2\beta\|u_{n,1}u_{n,2}\|_{L^2}^2 + \mu_2\|u_{n,2}\|_{L^4}^4),$$

which implies

$$I(u_n) - \frac{1}{4}I'(u_n)u_n = \frac{1}{4}\|u_n\|^2.$$

Thus we conclude that for sufficiently large $n$,

$$\frac{1}{4}\|u_n\|^2 \le \|u_n\| + c + o(1),$$

which implies that $(u_n)$ is a bounded sequence.
(ii) By $\|I'(u_n)\|_{T_{u_n}\mathcal{N}^*} \to 0$ and $I(u_n) \to c$, it is sufficient to prove $\|I'(u_n)\|_{H^*} \to 0$. Since we may assume that $(u_n) \subset \mathcal{N}^\alpha$ for some $\alpha > 0$, $(u_n)$ is a bounded sequence. By Lemma 2.2.5, $H = \operatorname{span}\{\nabla J(u_n)\} \oplus T_{u_n}\mathcal{N}$. So we prove that $I'(u_n)\zeta_n \to 0$ where $\zeta_n = \nabla J(u_n)/\|\nabla J(u_n)\|$ and it is equivalent to

$$I'(u_n)\left[\frac{P_{T_{u_n}\mathcal{N}^\perp}u_n}{\|P_{T_{u_n}\mathcal{N}^\perp}u_n\|}\right] \to 0. \tag{2.2.7}$$

First, we prove that $I'(u_n)[P_{T_{u_n}\mathcal{N}^\perp}u_n] \to 0$. Since $I'(u_n)u_n = J(u_n) = 0$ and $u_n - P_{T_{u_n}\mathcal{N}^\perp}u_n \in T_{u_n}\mathcal{N}$, it follows that

$$|I'(u_n)[P_{T_{u_n}\mathcal{N}^\perp}u_n]| = |I'(u_n)u_n - I'(u_n)[u_n - P_{T_{u_n}\mathcal{N}^\perp}u_n]| = |I'(u_n)[u_n - P_{T_{u_n}\mathcal{N}^\perp}u_n]|$$
$$\le \|I'(u_n)\|_{T_{u_n}\mathcal{N}^*}\|u_n - P_{T_{u_n}\mathcal{N}^\perp}u_n\| \to 0.$$

By Corollary 2.2.8, $(\|P_{T_{u_n}\mathcal{N}^\perp}u_n\|)$ is bounded below away from 0. Thus (2.2.7) holds.
(iii) Let $(u_n)$ be a $(\mathrm{PS})_{c,\ \mathcal{M}}$ sequence and $c < \alpha$. We remark that $(u_n)$ is bounded in $H$ and $(U_{n,j})$ also. As in (ii), by Lemma 2.2.5 and Proposition 2.2.9, we prove that

$$I'(u_n)\xi_{n,1} \to 0, \quad I'(u_n)\xi_{n,3} \to 0, \tag{2.2.8}$$

where $(\xi_{n,1})$ and $(\xi_{n,3})$ are given in the proof of Proposition 2.2.9. Since $I'(u_n)U_{n,1} = I'(u_n)U_{n,2} = 0$, $U_{n,j} - P_{T_{u_n}\mathcal{M}^\perp}U_{n,j} \in T_{u_n}\mathcal{M}$ and $\|I'(u_n)\|_{T_{u_n}\mathcal{M}} \to 0$, we have

$$\begin{aligned} |I'(u_n)[P_{T_{u_n}\mathcal{M}^\perp}U_{n,j}]| &= |I'(u_n)U_{n,j} - I'(u_n)[U_{n,j} - P_{T_{u_n}\mathcal{M}^\perp}U_{n,j}]| \\ &= |I'(u_n)[U_{n,j} - P_{T_{u_n}\mathcal{M}^\perp}U_{n,j}]| \\ &\le \|I'(u_n)\|_{T_{u_n}\mathcal{M}^*}\|U_{n,j} - P_{T_{u_n}\mathcal{M}^\perp}U_{n,j}\| \to 0. \end{aligned}$$

By Proposition 2.2.9, $\|P_{T_{u_n}\mathcal{M}^\perp}U_{n,j}\|$ are bounded below away from 0, it follows that

$$I'(u_n)\xi_{n,1} \to 0, \quad I'(u_n)\xi_{n,2} \to 0.$$

Using Proposition 2.2.9 again, it follows that $\|\xi_{n,2} - \langle \xi_{n,2}, \xi_{n,1}\rangle \xi_{n,1}\|$ is bounded below away from 0, which implies (2.2.8). □

The following lemma tells us that we can obtain a $(\mathrm{PS})_{b_\mathcal{N},\ H}$ sequence and a $(\mathrm{PS})_{\hat{b}_\mathcal{M},\ H}$ sequence from the minimizing sequence, respectively.

**Lemma 2.2.15.** (i) *For each $\beta > 0$, there exists a $(\mathrm{PS})_{b_\mathcal{N},\ H}$ sequence.*

(ii) *Suppose that $\alpha > \hat{b}_\mathcal{M}$ for all $\beta \in (0, \sqrt{\mu_1\mu_2})$. Then there exists a $0 < \tilde{\beta}(\alpha) \le \sqrt{\mu_1\mu_2}$ such that if $\beta \in (0, \tilde{\beta}(\alpha))$, then there exists a $(\mathrm{PS})_{\hat{b}_\mathcal{M},\ H}$ sequence.*

*Remark* 2.2.16. We remark that $\hat{b}_\mathcal{M}$ depends on $\beta$. In Proposition 2.6.1, we will prove $\sup_{\beta\in[0,\infty)} \hat{b}_\mathcal{M} < \infty$. In particular, there exists an $\alpha$ which satisfies the assumption of Lemma 2.2.15 (ii).

We can prove Lemma 2.2.15 by applying Ekeland's variational principle (See Ekeland [35] and Mahwin and Willem [78]). So we omit the proof.

The following lemma is so-called Concentration Compactness Lemma (cf. Lions [72]). This lemma plays an important role in analysing a $(\mathrm{PS})_{c,\ H}$ sequence.

**Lemma 2.2.17** (Concentration Compactness Lemma)**.** *Let $(u_n)$ be a $(\mathrm{PS})_{c,\ H}$ sequence. Then there exist a subsequence $(u_{n_k})$, an $\ell \in \mathbf{N}$, a critical point $u_0$ of $I$, critical points $\omega_i (1 \le i \le \ell)$ of $I_\infty$, $(y_k^i) \subset \mathbf{R}^N (1 \le i \le \ell)$ satisfying the following:*

(i) $|y_k^i| \to \infty \ (1 \le i \le \ell), \quad |y_k^i - y_k^j| \to \infty \ (i \ne j).$

(ii) $\displaystyle \left\| u_{n_k} - u_0 - \sum_{i=1}^{\ell} \omega_i(x - y_k^i) \right\| \to 0.$

(iii) $\displaystyle I(u_{n_k}) \to c = I(u_0) + \sum_{i=1}^{\ell} I_\infty(\omega_i).$

See Bahri and Lions [8] and Jeanjean and Tanaka [56] for a proof of Lemma 2.2.17.

*Remark* 2.2.18. If $\ell = 0$ in the above lemma, then $u_{n_k}$ converges to $u_0$ strongly in $H$.

## 2.3 Semitrivial solutions

Here, we consider some properties of semitrivial solutions, i.e., the solution of a form $(u_1, 0)$ or $(0, u_2)$.

The functionals $u_1 \mapsto I(u_1, 0)$ and $u_2 \mapsto I(0, u_2)$ corresponds to

$$\text{(2.3.1)} \qquad \begin{cases} -\Delta u_1 + V_1(x)u_1 = \mu_1 u_1^3 \quad \text{in} \quad \mathbf{R}^N, \\ \qquad\qquad u_1 \in H^1(\mathbf{R}^N), \end{cases}$$

$$\text{(2.3.2)} \qquad \begin{cases} -\Delta u_2 + V_2(x)u_2 = \mu_2 u_2^3 \quad \text{in} \quad \mathbf{R}^N, \\ \qquad\qquad u_2 \in H^1(\mathbf{R}^N). \end{cases}$$

We define $d_1$, $d_2$ as the least energy of (2.3.1), (2.3.2), respectively:

$$d_1 := \inf_{(u_1,0)\in\mathcal{N}} I(u_1, 0), \quad d_2 := \inf_{(0,u_2)\in\mathcal{N}} I(0, u_2).$$

Similarly, we set

$$d_{\infty,1} := \inf_{(u_1,0)\in\mathcal{N}_\infty} I_\infty(u_1, 0), \quad d_{\infty,2} := \inf_{(0,u_2)\in\mathcal{N}_\infty} I_\infty(0, u_2).$$

*Remark* 2.3.1. By the definition of $d_j$, we have

$$\text{(2.3.3)} \qquad b_{\mathcal{N}} \le \min\{d_1, d_2\}.$$

If the inequality (2.3.3) is strict, we can see the critical point corresponding to $b_{\mathcal{N}}$ is nontrivial. We will see in section 2.5 that this is the case when $\beta$ is large.

The following lemma shows that $d_j$ is attained and (2.1.2) has a semitrivial solution.

**Lemma 2.3.2.** *Suppose $N = 1, 2, 3$ and $V_1, V_2$ satisfy* (2–V1)–(2–V3). *Then,*

(i) *The equation* (2.3.1) ( *resp.* (2.3.2) ) *has a least energy solution which is positive in* $\mathbf{R}^N$.

(ii) *It holds that $d_j \le d_{\infty,j}$. Moreover if $V_j(x) \not\equiv V_{\infty,j}$, then $d_j < d_{\infty,j}$.*

A proof of Lemma 2.3.2 is standard, so we omit it. For example, see Willem [110].

## 2.4 Achievements of $b_{\mathcal{N}}$, $b_{\mathcal{N}_\infty}$

In this section, we prove that $b_{\mathcal{N}}$ and $b_{\mathcal{N}_\infty}$ are attained for each $\beta > 0$. These facts are useful to prove the existence of nontrivial solutions of (2.1.2) in section 2.5.

First we recall the following result.

**Proposition 2.4.1** (Ambrosetti and Colorado [4], and Sirakov [94])**.** *It holds that*

(i) *For each $\beta > 0$, $b_{\mathcal{N}_\infty}$ is attained.*

(ii) *There exists a* $\beta_0 > 0$ *such that if* $\beta > \beta_0$, *then* $b_{\mathcal{N}_\infty}$ *is attained by a nontrivial positive solution of* (2.2.1).

This proposition is proved in Ambrosetti and Colorado [4] and Sirakov [94] in the case $N = 2, 3$. For reader's convenience, we shall give a proof of Theorem 2.4.1 (i) for the case $N = 2, 3$ as well as the case $N = 1$. To prove Proposition 2.4.1 (i), we need the Schwarz symmetrization. We denote $u^*$ the Schwarz symmetrization of $u$:

$$u^* = (u_1^*, u_2^*).$$

It is well-known that the Schwarz symmertization satisfies the following: (See Lieb and Loss [64])

$$\|u_j^*\|_{L^4} = \|u_j\|_{L^4}, \quad \|\nabla u_j^*\|_{L^2} \le \|\nabla u_j\|_{L^2}, \quad \|u_1^* u_2^*\|_{L^2} \ge \|u_1 u_2\|_{L^2}.$$

*Proof of Proposition 4.1.* We consider two cases.

**Case 1:** $N = 2, 3$.

Suppose that $(u_n) \subset \mathcal{N}_\infty$ satisfies $I_\infty(u_n) \to b_{\mathcal{N}_\infty}$. Then Proposition 2.2.4 implies that $(u_n)$ is a bounded sequence. By the above properties of $u^*$, $(u_n^*)$ is also a bounded sequence. Let $H_r^1(\mathbf{R}^N)$ be the space of radially symmetric functions in $H^1(\mathbf{R}^N)$. Since the embedding $H_r^1(\mathbf{R}^N) \hookrightarrow L^4(\mathbf{R}^N)$ is compact, there exists a subsequence (write still $(u_n)$) such that

$$\begin{aligned} u_n^* &\rightharpoonup u_0 \quad \text{weakly} \quad \text{in} \quad H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N), \\ u_n^* &\to u_0 \quad \text{strongly} \quad \text{in} \quad L^4(\mathbf{R}^N) \times L^4(\mathbf{R}^N). \end{aligned}$$

Then it follows that

$$\begin{aligned} \|u_0\|_\infty^2 &\le \liminf_{n\to\infty} \|u_n^*\|_\infty^2 \le \liminf_{n\to\infty} \|u_n\|_\infty^2 \\ &= \liminf_{n\to\infty} (\mu_1 \|u_{n,1}\|_{L^4}^4 + 2\beta \|u_{n,1}u_{n,2}\|_{L^2}^2 + \mu_2 \|u_{n,2}\|_{L^4}^4) \\ &\le \liminf_{n\to\infty} (\mu_1 \|u_{n,1}^*\|_{L^4}^4 + 2\beta \|u_{n,1}^* u_{n,2}^*\|_{L^2}^2 + \mu_2 \|u_{n,2}^*\|_{L^4}^4) \\ &= \mu_1 \|u_{0,1}\|_{L^4}^4 + 2\beta \|u_{0,1}u_{0,2}\|_{L^2}^2 + \mu_2 \|u_{0,2}\|_{L^4}^4. \end{aligned}$$

Hence there exists a unique $\theta_0 \in (0, 1]$ such that $\theta_0 u_0 \in \mathcal{N}_\infty$. Thus we see

$$b_{\mathcal{N}_\infty} \le \frac{\theta_0^2}{4} \|u_0\|_\infty^2 \le \liminf_{n\to\infty} \frac{\theta_0^2}{4} \|u_n^*\|_\infty^2 \le \liminf_{n\to\infty} \frac{\theta_0^2}{4} \|u_n\|_\infty^2 = \theta_0^2 b_{\mathcal{N}_\infty},$$

which implies $\theta_0 = 1$, $u_0 \in \mathcal{N}_\infty$ and $I_\infty(u_0) = b_{\mathcal{N}_\infty}$.

**Case 2:** $N = 1$

By Lemma 2.2.15, there exists a $(\mathrm{PS})_{b_{\mathcal{N}},\,H}$ sequence. We denote it by $(u_n)$. Then Proposition 2.2.4 implies that $(u_n)$ is bounded. Furthermore, by Lemma 2.2.17, there exist a subsequence $(u_{n_k})$, an $\ell \in \mathbf{N}$, critical points $\omega_i\,(0 \le i \le \ell)$ of $I_\infty$ and sequences $(y_k^i) \subset \mathbf{R}^N\ (1 \le i \le \ell)$ such that

$$
\begin{aligned}
&|y_k^i| \to \infty \quad \text{if } 1 \le i \le \ell, \qquad\qquad |y_k^i - y_k^j| \to \infty \quad \text{if } 1 \le i < j \le \ell, \\
&\left\| u_{n_k} - \omega_0 - \sum_{i=1}^{\ell} \omega_i(x - y_k^i) \right\| \to 0, \quad I_\infty(u_{n_k}) \to \sum_{i=0}^{\ell} I_\infty(\omega_i).
\end{aligned} \tag{2.4.1}
$$

Moreover, we may assume that if $\ell \ge 1$, then $\omega_i \ne 0$ for all $1 \le i \le \ell$.

If $\omega_0 \ne 0$, then we can conclude $\ell = 0$ and $I_\infty(\omega_0) = b_{\mathcal{N}_\infty}$. Indeed, since $\omega_0 \ne 0$, $I_\infty(\omega_0) \ge b_{\mathcal{N}_\infty}$ holds. Since it holds that $I_\infty(\omega_i) \ge b_{\mathcal{N}_\infty}$ for $1 \le i \le \ell$, it follows from (2.4.1) that $\ell = 0$ and $I_\infty(\omega_0) = b_{\mathcal{N}_\infty}$.

If $\omega_0 \equiv 0$, then we infer that $\ell = 1$. In fact, if $\ell = 0$, then from (2.4.1) we have $u_{n_k} \to 0$ strongly in $H$. This contradicts Proposition 2.2.4 (ii). On the other hand, if $\ell \ge 2$, then from (2.4.1) we have $\liminf_{k\to\infty} I_\infty(u_{n_k}) \ge 2b_{\mathcal{N}_\infty} > b_{\mathcal{N}_\infty}$. This is a contradiction and we obtain $\ell = 1$. We set $v_k(x) = u_{n_k}(x + y_k^1)$. Then we have $I_\infty(v_k) = I_\infty(u_{n_k})$, $v_k \in \mathcal{N}_\infty$ and $v_k \to \omega_1$ strongly in $H$. Hence we obtain $I_\infty(\omega_1) = b_{\mathcal{N}_\infty}$, which completes a proof.

□

Next we prove that $b_{\mathcal{N}}$ is attained.

**Proposition 2.4.2.** *For each $\beta > 0$, $b_{\mathcal{N}}$ is attained.*

*Proof.* First, we prove the inequality $b_{\mathcal{N}} \le b_{\mathcal{N}_\infty}$. By Proposition 2.4.1, there exists a $u_\infty \in \mathcal{N}_\infty$ such that $I_\infty(u_\infty) = b_{\mathcal{N}_\infty}$. With the assumption of $V_j(x)$ we obtain

$$\|u_\infty\|^2 \le \|u_\infty\|_\infty^2 = \mu_1\|u_{\infty,1}\|_{L^4}^4 + 2\beta\|u_{\infty,1}u_{\infty,2}\|_{L^2}^2 + \mu_2\|u_{\infty,2}\|_{L^4}^4,$$

which implies that there exists a $\theta_\infty \in (0,1]$ such that $\theta_\infty u_\infty \in \mathcal{N}$. Then it follows that

$$b_{\mathcal{N}} \le I(\theta_\infty u_\infty) = \frac{\theta_\infty^2}{4}\|u_\infty\|^2 \le \frac{1}{4}\|u_\infty\|_\infty^2 = I_\infty(u_\infty) = b_{\mathcal{N}_\infty}. \tag{2.4.2}$$

Thus we obtain $b_{\mathcal{N}} \le b_{\mathcal{N}_\infty}$.

Next we consider two cases: $b_{\mathcal{N}} = b_{\mathcal{N}_\infty}$ and $b_{\mathcal{N}} < b_{\mathcal{N}_\infty}$.

If $b_{\mathcal{N}} = b_{\mathcal{N}_\infty}$ takes place, then by (2.4.2), we have $\theta_\infty = 1$. This implies that $u_\infty \in \mathcal{N}$ and $I(u_\infty) = b_{\mathcal{N}}$. This is our conclusion.

If $b_{\mathcal{N}} < b_{\mathcal{N}_\infty}$ takes place, then by Lemma 2.2.15, there exists a $(\mathrm{PS})_{b_{\mathcal{N}},\,H}$ sequence $(u_n)$. By Lemma 2.2.17, there exist subsequence $(u_{n_k})$, $\ell \in \mathbf{N}$, $u_0$ with $I'(u_0) = 0$, $\omega_i \ne 0$ with $I'_\infty(\omega_i) = 0$ and $(y_k^i) \subset \mathbf{R}^N$ such that

$$\left\| u_{n_k} - u_0 - \sum_{i=1}^{\ell} \omega_i(x - y_k^i) \right\| \to 0, \quad I(u_{n_k}) \to b_{\mathcal{N}} = I(u_0) + \sum_{i=1}^{\ell} I_\infty(\omega_i).$$

Since $\omega_i \ne (0,0)$, we have $b_{\mathcal{N}_\infty} \le I_\infty(\omega_i)$. By $b_{\mathcal{N}} < b_{\mathcal{N}_\infty}$, it follows that $\ell = 0$, which implies

$$u_{n_k} \to u_0 \quad \text{strongly} \quad \text{in} \quad H.$$

This shows that $u_0 \in \mathcal{N}$ and $I(u_0) = b_{\mathcal{N}}$. □

*Remark* 2.4.3. We consider the situation $b_{\mathcal{N}} = b_{\mathcal{N}_\infty}$ more precisely. We deal with the two cases. (a) $b_{\mathcal{N}_\infty}$ is attained by nontrivial functions $u_0$. In this case, we can show that both of $V_j(x)$ are constant functions. (b) $b_{\mathcal{N}_\infty}$ is attained by semitrivial functions $u_0$. We may assume that $u_0 = (u_1, 0)$. Then we can show that $V_1(x)$ is a constant function. Moreover, we can prove the equality $b_{\mathcal{N}} = b_{\mathcal{N}_\infty} = d_{\infty,1} = d_1$.

## 2.5 Proof of Theorem 1.1 (ii) (when $\beta$ is large)

In this section, we prove the existence of a nontrivial positive solution of (2.1.2) when $\beta$ is large. By Proposition 2.4.2, there exists a $u_0 = (u_{0,1}, u_{0,2}) \in \mathcal{N}$ such that $I(u_0) = b_{\mathcal{N}}$. Moreover, Corollary 2.2.10 implies that $u_0$ is a solution of (2.1.2). Hence we need to prove $u_{0,1},\ u_{0,2} \not\equiv 0$.

Following Ambrosetti and Colorado [4], let us define constants which are related to the stability of semitrivial solutions on $\mathcal{N}$.

**Definition 2.5.1.** We define $\hat{\beta}_1$ and $\hat{\beta}_2$ as follows:

$$\hat{\beta}_1 := \inf_{(u_1,0)\in S_1} \inf_{\varphi_2 \in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\|\varphi_2\|_2^2}{\int_{\mathbf{R}^N} u_1^2\varphi_2^2 dx}, \quad \hat{\beta}_2 := \inf_{(0,u_2)\in S_2} \inf_{\varphi_1 \in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\|\varphi_1\|_1^2}{\int_{\mathbf{R}^N} u_2^2\varphi_1^2\ dx}.$$

Here, $S_1$ and $S_2$ are defined by

$$S_1 := \{(u_1, 0) \in \mathcal{N} : I(u_1, 0) = d_1\}, \quad S_2 := \{(0, u_2) \in \mathcal{N} : I(0, u_2) = d_2\}.$$

A main result in this section is the following:

**Theorem 2.5.2.** *If $\beta > \max\{\hat{\beta}_1,\ \hat{\beta}_2\}$, then both components of any minimizer of $I$ on $\mathcal{N}$ are not zero, i.e.,*

$$I(u_0) = b_{\mathcal{N}}, \quad u_0 \in \mathcal{N} \quad \Rightarrow \quad u_{0,1},\ u_{0,2} \not\equiv 0.$$

*Proof.* It suffices to prove $b_{\mathcal{N}} < \min\{d_1, d_2\}$. Since $\beta > \max\{\hat{\beta}_1, \hat{\beta}_2\}$, there exist $(u_1, 0) \in S_1, (0, u_2) \in S_2, \varphi_1, \varphi_2 \in H^1(\mathbf{R}^N)$ such that

$$\frac{\|\varphi_1\|_1^2}{\int_{\mathbf{R}^N} u_2^2\varphi_1^2\ dx} < \beta, \quad \frac{\|\varphi_2\|_2^2}{\int_{\mathbf{R}^N} u_1^2\varphi_2^2\ dx} < \beta.$$

We remark that $\{0\} \times H^1(\mathbf{R}^N) \subset T_{(u_1,0)}\mathcal{N}$ and $H^1(\mathbf{R}^N) \times \{0\} \subset T_{(0,u_2)}\mathcal{N}$. In fact, for each $\psi_1, \psi_2 \in H^1(\mathbf{R}^N)$, we have

$$J'(u_1, 0)[(0, \psi_2)] = 0, \quad J'(0, u_2)[(\psi_1, 0)] = 0.$$

Thus, it holds that $\{0\} \times H^1(\mathbf{R}^N) \subset T_{(u_1,0)}\mathcal{N}$ and $H^1(\mathbf{R}^N) \times \{0\} \subset T_{(0,u_2)}\mathcal{N}$ by Lemma 2.2.5.

Next let $\gamma_1, \gamma_2 \in C^2((-\varepsilon, \varepsilon), \mathcal{N})$ satisfy

$$\gamma_1(0) = (u_1, 0), \quad \gamma_1'(0) = (0, \varphi_2), \quad \gamma_2(0) = (0, u_2), \quad \gamma_2'(0) = (\varphi_1, 0).$$

By the Taylor expansion of $I(\gamma_j(t))$ and $I'(u_1, 0) = I'(0, u_2) = 0$, we obtain

$$I(\gamma_j(t)) = I(\gamma_j(0)) + \frac{1}{2} I''(\gamma_j(0))[\gamma_j'(0), \gamma_j'(0)]t^2 + o(t^2).$$

Since

$$I''(u_1, 0)[(0, \varphi_2), (0, \varphi_2)] = \|\varphi_2\|_2^2 - \beta \int_{\mathbf{R}^N} u_1^2 \varphi_2^2 \, dx < 0,$$
$$I''(0, u_2)[(\varphi_1, 0), (\varphi_1, 0)] < 0,$$

it follows that for sufficiently small $t > 0$

$$I(\gamma_j(t)) - I(\gamma_j(0)) < 0.$$

Thus we have $b_{\mathcal{N}} < \min\{d_1, d_2\}$. □

Next, we give a proof of Theorem 2.1.1 (ii).

*Proof of Theorem 2.1.1 (ii).* By Theorem 2.5.2, there exists a $u_0$ such that $b_{\mathcal{N}} = I(u_0)$, $u_{0,1} \not\equiv 0, u_{0,2} \not\equiv 0$. By Remark 2.2.3, we have

$$|u_0| = (|u_{0,1}|, |u_{0,2}|) \in \mathcal{N}, \quad b_{\mathcal{N}} = I(u_0) = I(|u_0|),$$

which implies that $|u_0|$ is also a minimizer of $I$ on $\mathcal{N}$. Thus we may assume that $u_{0,1} \geq 0, u_{0,1} \not\equiv 0, u_{0,2} \geq 0, u_{0,2} \not\equiv 0$. Moreover, it hold that $I'(u_0) = 0$ and $u_{0,1}, u_{0,2} > 0$ by Corollary 2.2.10 and the maximum principle. □

## 2.6 Proofs of Theorem 2.1.1 (i) and Theorem 2.1.2. (when $\beta$ is small)

### 2.6.1 Proof of Theorem 2.1.1(i).

The aim of this subsection is to prove the existence of a nontrivial positive solution of (2.1.2) when $\beta$ is small.

The following two propositions give some estimates of $\hat{b}_{\mathcal{M}}$.

**Proposition 2.6.1.** *For each $\beta > 0$,*

(i) $\hat{b}_{\mathcal{M}} < \min\{d_1 + d_{\infty,2}, d_{\infty,1} + d_2\}$.

(ii) $\hat{b}_{\mathcal{M}_\infty} < d_{\infty,1} + d_{\infty,2}$.

*Remark* 2.6.2. $\hat{b}_{\mathcal{M}}$ depends on $\beta$ but $d_1, d_2, d_{\infty,1}, d_{\infty,2}$ are independent of $\beta$.

**Proposition 2.6.3.** *There exists a $\tilde{\beta}_1 > 0$ such that for each $\beta \in (0, \tilde{\beta}_1)$*

$$\hat{b}_{\mathcal{M}} < \hat{b}_{\mathcal{M}_\infty}.$$

Proofs of Propositions 2.6.1 and 2.6.3 will be given in subsection 2.6.2.

**Theorem 2.6.4.** *There exists a $\tilde{\beta}_2 > 0$ such that for each $\beta \in (0, \tilde{\beta}_2)$, $\hat{b}_{\mathcal{M}}$ is attained.*

*Proof of Theorem 2.6.4.* Set $\alpha_0 = \min\{d_1 + d_{\infty,2}, d_{\infty,1} + d_2\}$. By Proposition 2.6.1, $\mathcal{M}^{\alpha_0} \neq \emptyset$ for all $\beta \in (0, \sqrt{\mu_1 \mu_2})$. By Proposition 2.2.9, there exist $\hat{\beta}_0 > 0$ and $\delta_1 > 0$ such that for each $u \in \mathcal{M}^{\alpha_0}$ and $\beta \in (0, \hat{\beta}_0)$,

$$\|u_1\|_1 \geq \delta_1, \ \|u_2\|_2 \geq \delta_1. \tag{2.6.1}$$

Suppose $0 < \beta < \min\{\tilde{\beta}_1, \hat{\beta}_0\}$. Then we remark that there exists a $(\mathrm{PS})_{\hat{b}_{\mathcal{M}},\ H}$ sequence $(u_n)$ by Lemmas 2.2.14 and 2.2.15. Then by Lemma 2.2.17, we have

$$\left\| u_n - u_0 - \sum_{i=1}^{\ell} \omega_i(x - y_n^i) \right\| \to 0, \tag{2.6.2}$$

$$I(u_n) \to \hat{b}_{\mathcal{M}} = I(u_0) + \sum_{i=1}^{\ell} I_\infty(\omega_i). \tag{2.6.3}$$

We shall show that $u_0 = (u_{0,1}, u_{0,2}), u_{0,1} \neq 0, u_{0,2} \neq 0$ and $\ell = 0$. We divide our argument into three steps.

**Step 1.** $u_0 \not\equiv (0,0)$.

We prove indirectly and we assume that $u_0 \equiv (0,0)$. By (2.6.3), it follows that

$$\hat{b}_{\mathcal{M}} = \sum_{i=1}^{\ell} I_\infty(\omega_i).$$

By $\hat{b}_{\mathcal{M}} > 0$, we obtain $\ell \neq 0$. Since $\hat{b}_{\mathcal{M}} < \hat{b}_{\mathcal{M}_\infty}$, we conclude that one of the components of $\omega_i$ equals 0. Moreover if $\ell \geq 2$, we have

$$\omega_{i,1} \equiv 0 \quad (1 \leq i \leq \ell) \quad \text{or} \quad \omega_{i,2} \equiv 0 \quad (1 \leq i \leq \ell). \tag{2.6.4}$$

Otherwise, we have $\hat{b}_{\mathcal{M}} \geq d_{\infty,1} + d_{\infty,2}$, which contradicts Proposition 2.6.1.

Suppose that $\omega_{i,1} \equiv 0 \quad (1 \leq i \leq \ell)$. By (2.6.2), we obtain $\|u_{n,1}\|_1 \to 0$, which contradicts (2.6.1). In a similar way, $\omega_{i,2} \equiv 0$ $(1 \leq i \leq \ell)$ does not take place. This implies that $u_0 \not\equiv (0,0)$.

**Step 2.** $u_0 \notin (H^1(\mathbf{R}^N) \times \{0\}) \cup (\{0\} \times H^1(\mathbf{R}^N))$.

We prove indirectly and we assume that $u_0 \in H^1(\mathbf{R}^N) \times \{0\}$. By (2.6.3) we have

$$\hat{b}_{\mathcal{M}} = I(u_0) + \sum_{i=1}^{\ell} I_\infty(\omega_i).$$

Since $\hat{b}_{\mathcal{M}} < \hat{b}_{\mathcal{M}_\infty}$, one of the components of $\omega_i$ is equal to 0 for $1 \leq i \leq \ell$. Since $\hat{b}_{\mathcal{M}} < d_1 + d_{\infty,2}$ and $d_1 \leq I(u_0)$, we have

$$\omega_{i,2} \equiv 0 \quad \text{for} \quad 1 \leq i \leq \ell. \tag{2.6.5}$$

From (2.6.2) and (2.6.5), it follows that $\|u_{n,2}\|_2 \to 0$, which contradicts (2.6.1). So, we conclude $u_0 \notin H^1(\mathbf{R}^N) \times \{0\}$. In a similar way, we can prove that $u_0 \notin \{0\} \times H^1(\mathbf{R}^N)$.

**Step 3.** Conclusion.

Now we complete a proof of Theorem 2.6.4. By Steps 1 and 2, it follows that $u_{0,1},\ u_{0,2} \neq 0$. Since $\hat{b}_{\mathcal{M}} \leq I(u_0)$ and $I_\infty(\omega_i) > 0$, we have $\ell = 0$. By Remark 2.2.18, $(u_n)$ converges to $u_0$ strongly in $H$, so $I(u_0) = \inf_{\mathcal{M}} I = \hat{b}_{\mathcal{M}}$. □

We give a proof of Theorem 2.1.1 (i).

*Proof of Theorem 2.1.1 (i).* As in the proof of Theorem 2.1.1 (ii), we obtain a nontrivial positive solution of (2.1.2) by Theorem 2.6.4, Corollary 2.2.10 and the maximum principle. □

### 2.6.2 Proofs of Propositions 2.6.1 and 2.6.3.

Before proving Propositions 2.6.1 and 2.6.3, we state a useful lemma. For $u \in H, u_1 \neq 0, u_2 \neq 0$, we set

$$\begin{aligned} f_u(s_1, s_2) &:= I(\sqrt{s_1}u_1, \sqrt{s_2}u_2) \\ &= \frac{s_1}{2}\|u_1\|_1^2 + \frac{s_2}{2}\|u_2\|_2^2 - \frac{s_1^2}{4}\mu_1\|u_1\|_{L^4}^4 - \frac{s_1 s_2}{2}\beta\|u_1 u_2\|_{L^2}^2 - \frac{s_2^2}{4}\mu_2\|u_2\|_{L^4}^4. \end{aligned}$$

**Lemma 2.6.5.** *Let $u \in H$, $u_1 \not\equiv 0, u_2 \not\equiv 0$. Then the following hold.*

(i) *Let $0 \leq \beta < \sqrt{\mu_1\mu_2}$. Then $f_u(s_1, s_2)$ is strictly concave in $[0, \infty) \times [0, \infty)$.*

(ii) *Let $u \in \mathcal{M}$ and $0 \leq \beta < \sqrt{\mu_1\mu_2}$. Then $(1, 1)$ is a unique maximum point of $f_u(s_1, s_2)$. Namely, it follows*

$$I(u) = f_u(1, 1) = \max_{[0,\infty)\times[0,\infty)} I(\sqrt{s_1}u_1, \sqrt{s_2}u_2).$$

(iii) *Let $\beta \geq 0$ and $(s_{0,1}, s_{0,2}) \in (0, \infty) \times (0, \infty)$ be a maximum point of $f_u(s_1, s_2)$. Then $(\sqrt{s_{0,1}}u_1, \sqrt{s_{0,2}}u_2) \in \mathcal{M}$.*

*Remark* 2.6.6. Similar results hold for $I_\infty$ and $\mathcal{M}_\infty$.

*Proof.* This lemma is proved in Lin and Wei [65], however, for reader's convenience, we give a proof.
(i) Differentiating $f_u(s_1, s_2)$, we have

$$\begin{aligned} \frac{\partial f_u}{\partial s_1} &= \frac{1}{2}\|u_1\|_1^2 - \frac{s_1}{2}\mu_1\|u_1\|_{L^4}^4 - \frac{s_2}{2}\beta\|u_1u_2\|_{L^2}^2, \\ \frac{\partial f_u}{\partial s_2} &= \frac{1}{2}\|u_2\|_2^2 - \frac{s_1}{2}\beta\|u_1u_2\|_{L^2}^2 - \frac{s_2}{2}\mu_2\|u_2\|_{L^4}^4, \\ \frac{\partial^2 f_u}{\partial s_j^2} &= -\frac{1}{2}\mu_j\|u_j\|_{L^4}^4 \ (j = 1, 2), \qquad \frac{\partial^2 f_u}{\partial s_1 \partial s_2} = -\frac{1}{2}\beta\|u_1u_2\|_{L^2}^2. \end{aligned} \tag{2.6.6}$$

Since $0 \leq \beta < \sqrt{\mu_1\mu_2}$, the matrix

$$\begin{pmatrix} \frac{\partial^2 f_u}{\partial s_1^2}(s_1,s_2) & \frac{\partial^2 f_u}{\partial s_1 \partial s_2}(s_1,s_2) \\ \frac{\partial^2 f_u}{\partial s_1 \partial s_2}(s_1,s_2) & \frac{\partial^2 f_u}{\partial s_2^2}(s_1,s_2) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\mu_1\|u_1\|_{L^4}^4 & -\beta\|u_1u_2\|_{L^2}^2 \\ -\beta\|u_1u_2\|_{L^2}^2 & -\mu_2\|u_2\|_{L^4}^4 \end{pmatrix}$$

is negative definite. Thus $f_u(s_1,s_2)$ is strictly concave in $[0,\infty)\times[0,\infty)$.
(ii) Suppose $u \in \mathcal{M}$. By (2.6.6) and $\beta \in [0,\sqrt{\mu_1\mu_2})$, we have

$$\nabla f_u(s_1,s_2) = (0,0) \ \Leftrightarrow \ (s_1,s_2) = (1,1).$$

Since $f_u(s_1,s_2)$ is strictly concave, $(1,1)$ is an unique maximum point and

$$I(u) = f_u(1,1) = \max_{[0,\infty)\times[0,\infty)} I(\sqrt{s_1}u_1, \sqrt{s_2}u_2).$$

(iii) Suppose $(s_{0,1},s_{0,2}) \in (0,\infty)\times(0,\infty)$ is a maximum point of $f_u(s_1,s_2)$. Since $\nabla f_u(s_{0,1},s_{0,2}) = (0,0)$, we have

$$\begin{aligned} s_{0,1}\|u_1\|_1^2 &= s_{0,1}^2\mu_1\|u_1\|_{L^4}^4 + s_{0,1}s_{0,2}\beta\|u_1u_2\|_{L^2}^2, \\ s_{0,2}\|u_2\|_2^2 &= s_{0,1}s_{0,2}\beta\|u_1u_2\|_{L^2}^2 + s_{0,2}^2\mu_2\|u_2\|_{L^4}^4. \end{aligned}$$

Thus this implies $(\sqrt{s_{0,1}}u_{0,1}, \sqrt{s_{0,2}}u_{0,2}) \in \mathcal{M}$. □

First, we prove Proposition 2.6.1.

*Proof of Proposition 2.6.1.* We only prove $\hat{b}_{\mathcal{M}} < d_1 + d_{\infty,2}$ since we can prove other inequality in a similar way. By Lemma 2.3.2, we suppose that $(\varphi_{0,1},0) \in \mathcal{N}$, $(0,\varphi_{\infty,2}) \in \mathcal{N}_\infty$ satisfy

$$I(\varphi_{0,1},0) = d_1, \quad I_\infty(0,\varphi_{\infty,2}) = d_{\infty,2}, \quad \varphi_{0,1} > 0, \quad \varphi_{\infty,2} > 0.$$

We remark that for a $k \in \mathbf{N}$, it follows $\|\varphi_{0,1}(x)\varphi_{\infty,2}(x-ke_1)\|_{L^2}^2 \to 0$ as $k \to \infty$ where $e_1 = (1,0,\ldots,0)$. Thus we have

$$g_k(s_1,s_2) := I(\sqrt{s_1}\varphi_{0,1}(x), \sqrt{s_2}\varphi_{\infty,2}(x-ke_1)) \to g(s_1,s_2) \quad \text{in} \quad C^2_{\text{loc}}((0,\infty)^2)$$

where

$$g(s_1,s_2) := \frac{s_1}{2}\|\varphi_{0,1}\|_1^2 - \frac{s_1^2}{4}\mu_1\|\varphi_{0,1}\|_{L^4}^4 + \frac{s_2}{2}\|\varphi_{\infty,2}\|_{\infty,2}^2 - \frac{s_2^2}{4}\|\varphi_{\infty,2}\|_{L^4}^4.$$

Since $g(s_1,s_2)$ has a unique maximum point $(1,1)$ and $g_k(s_1,s_2) \leq g(s_1,s_2)$, $g_k(s_1,s_2)$ has a maximum point $(s_{k,1},s_{k,2}) \in (0,\infty)\times(0,\infty)$ for a sufficiently large $k$. By Lemma 2.6.5 we have $(\sqrt{s_{k,1}}\varphi_{0,1}(x), \sqrt{s_{k,2}}\varphi_{\infty,2}(x-ke_1)) \in \mathcal{M}$.

Thus we have

$$
\begin{aligned}
\hat{b}_{\mathcal{M}} &\le I(\sqrt{s_{k,1}}\varphi_{0,1}, \sqrt{s_{k,2}}\varphi_{\infty,2}(x-ke_1)) \\
&= \frac{1}{2}s_{k,1}\|\varphi_{0,1}\|_1^2 + \frac{1}{2}s_{k,2}\|\varphi_{\infty,2}(x-ke_1)\|_2^2 - \frac{1}{4}s_{k,1}^2\mu_1\|\varphi_{0,1}\|_{L^4}^4 \\
&\quad - \frac{1}{2}\beta s_{k,1}s_{k,2}\|\varphi_{0,1}\varphi_{\infty,2}(x-ke_1)\|_{L^2}^2 - \frac{1}{4}s_{k,2}^2\mu_2\|\varphi_{\infty,2}\|_{L^4}^4 \\
&\le \frac{1}{2}s_{k,1}\|\varphi_{0,1}\|_1^2 + \frac{1}{2}s_{k,2}\|\varphi_{\infty,2}\|_{\infty,2}^2 - \frac{1}{4}s_{k,1}^2\mu_1\|\varphi_{0,1}\|_{L^4}^4 \\
&\quad - \frac{1}{2}\beta s_{k,1}s_{k,2}\|\varphi_{0,1}\varphi_{\infty,2}(x-ke_1)\|_{L^2}^2 - \frac{1}{4}s_{k,2}^2\mu_2\|\varphi_{\infty,2}\|_{L^4}^4 \\
&= d_1 + d_{\infty,2} + \frac{1}{2}(s_{k,1}-1)\|\varphi_{0,1}\|_1^2 + \frac{1}{2}(s_{k,2}-1)\|\varphi_{\infty,2}\|_{\infty,2}^2 \\
&\quad + \frac{\mu_1}{4}(1-s_{k,1}^2)\|\varphi_{0,1}\|_{L^4}^4 + \frac{\mu_2}{4}(1-s_{k,2}^2)\|\varphi_{\infty,2}\|_{L^4}^4 - \frac{1}{2}\beta s_{k,1}s_{k,2}\|\varphi_{0,1}\varphi_{\infty,2}(x-ke_1)\|_{L^2}^2.
\end{aligned}
$$

Since

$$\|\varphi_{0,1}\|_1^2 = \mu_1\|\varphi_{0,1}\|_{L^4}^4, \quad \|\varphi_{\infty,2}\|_{\infty,2} = \mu_2\|\varphi_{\infty,2}\|_{L^4}^4,$$

we obtain

$$
\begin{aligned}
&\frac{1}{2}(s_{k,1}-1)\|\varphi_{0,1}\|_1^2 + \frac{\mu_1}{4}(1-s_{k,1}^2)\|\varphi_{0,1}\|_{L^4}^4 = \frac{\|\varphi_{0,1}\|_1^2}{4}(-s_{k,1}^2+2s_{k,1}-1) \\
=&-\frac{\|\varphi_{0,1}\|_1^2}{4}(s_{k,1}-1)^2 \le 0, \\
&\frac{1}{2}(s_{k,2}-1)\|\varphi_{\infty,2}\|_{\infty,2}^2 + \frac{\mu_2}{4}(1-s_{k,2}^2)\|\varphi_{\infty,2}\|_{L^4}^4 \le 0.
\end{aligned}
$$

Moreover, since $\varphi_{0,1},\ \varphi_{\infty,2} > 0$, it follows that $\|\varphi_{0,1}\varphi_{\infty,2}(x-ke_1)\|_{L^2}^2 > 0$. Hence we have

$$\hat{b}_{\mathcal{M}} < d_1 + d_{\infty,2}.$$

□

The following lemma is related to the existence of minimizer for $\hat{b}_{\mathcal{M}_\infty}$, which is due to Lin and Wei [65], and Sirakov [94] in the case $N = 2, 3$.

**Lemma 2.6.7** (Lin and Wei [65], and Sirakov [94]). *There exists a $\bar{\beta} \in (0, \sqrt{\mu_1\mu_2}]$ such that if $\beta \in (0, \bar{\beta})$, then $\hat{b}_{\mathcal{M}_\infty}$ is attained by a nontrivial positive solution $\omega = (\omega_1, \omega_2)$ of* (2.2.1).

*Proof.* We only consider the case $N = 1$ since the other case is proved in [65] and [94]. We use the arguments in the proof of Theorem 2.6.4. Set $\alpha_0 = d_{\infty,1} + d_{\infty,2}$. Since $\hat{b}_{\mathcal{M}_\infty} < \alpha_0$ by Proposition 2.6.1, $\mathcal{M}^{\alpha_0} \neq \emptyset$ for all $\beta \in (0, \sqrt{\mu_1\mu_2})$. By Proposition 2.2.9, there exist $\hat{\beta}_0 > 0$ and $\delta_1 > 0$ such that for each $u \in \mathcal{M}^{\alpha_0}$ and $\beta \in (0, \hat{\beta}_0)$,

$$\|u_1\|_1 \ge \delta_1,\ \|u_2\|_2 \ge \delta_1. \tag{2.6.7}$$

Suppose $0 < \beta < \min\{\tilde{\beta}_1, \hat{\beta}_0\}$ and let $(u_n)$ be a $(PS)_{\hat{b}_{\mathcal{M}_\infty},H}$ sequence. Then by Lemma 2.2.17, we have

$$\left\| u_n - u_0 - \sum_{i=1}^{\ell} \omega_i(x - y_n^i) \right\| \to 0, \quad I_\infty(u_n) \to \hat{b}_{\mathcal{M}_\infty} = I_\infty(u_0) + \sum_{i=1}^{\ell} I_\infty(\omega_i). \tag{2.6.8}$$

We shall show that $u_0 = (u_{0,1}, u_{0,2}), u_{0,1} \neq 0, u_{0,2} \neq 0$ and $\ell = 0$.

First we prove $\ell \leq 1$ indirectly and assume $\ell \geq 2$. Since $I_\infty(\omega_i) > 0$ for all $1 \leq i \leq \ell$, we may suppose $\omega_{i,1} \equiv 0$ or $\omega_{i,2} \equiv 0$ for all $1 \leq i \leq \ell$. Otherwise, we have a contradiction: $\lim_{n\to\infty} I_\infty(u_n) > \hat{b}_{\mathcal{M}_\infty}$. Suppose $\omega_{i,1} \equiv 0$ for all $1 \leq i \leq \ell$. In this case, it follows from (2.6.8) that $\|u_{n,1}\|_1 \to 0$, which contradicts (2.6.7). In the other case, we can lead a contradiction. Therefore we obtain $\ell \leq 1$.

If $\ell = 0$, then since $u_n \to u_0$ strongly in $H$, it follows from (2.6.7) and (2.6.8) that $u_0 \in \mathcal{M}_\infty$ and $I_\infty(u_0) = \hat{b}_{\mathcal{M}_\infty}$. This is our conclusion.

If $\ell = 1$ and $\omega_{1,1} \equiv 0$, then we can infer that $u_{0,2} \neq 0$. In fact, if $u_{0,2} \equiv 0$, then from (2.6.8), it holds that $\|u_{n,2}\|_2 \to 0$. This is a contradiction. Thus we have $u_{0,2} \neq 0$. However, in this case, from (2.6.8) we obtain $\hat{b}_{\mathcal{M}_\infty} \geq d_{\infty,1} + d_{\infty,2}$, which contradicts to $\hat{b}_{\mathcal{M}_\infty} < d_{\infty,1} + d_{\infty,2}$. In a similar way, the case $\ell = 1$ and $\omega_{1,2} \equiv 0$ does not take place.

Hence we have $\ell = 1$ and $\omega_{1,j} \neq 0$ for $j = 1, 2$. Since $I_\infty(\omega_1) \geq \hat{b}_{\mathcal{M}_\infty}$, we obtain $u_0 \equiv 0$ and $I_\infty(\omega_1) = \hat{b}_{\mathcal{M}_\infty}$, which completes a proof. □

Now we prove Proposition 2.6.3.

*Proof of Proposition 2.6.3.* Set $\tilde{\beta}_1 := \bar{\beta}$ where $\bar{\beta}$ is given in Lemma 2.6.7. By Lemma 2.6.7, there exists an $\omega \in \mathcal{M}_\infty$ such that $I_\infty(\omega) = \hat{b}_{\mathcal{M}_\infty}$ and $\omega_j > 0$ in $\mathbf{R}^N$. By Lemma 2.6.5 a function

$$h(s_1, s_2) := I_\infty(\sqrt{s_1}\omega_1, \sqrt{s_2}\omega_2)$$

has a unique maximum point $(1, 1)$. Let $h_k(s_1, s_2) := I(\omega_1(x - ke_1), \omega_2(x - ke_1))$. Since $h_k(s_1, s_2) \leq h(s_1, s_2)$ and

$$h_k(s_1, s_2) = I(\omega_1(x - ke_1), \omega_2(x - ke_1)) \to h(s_1, s_2) \quad \text{in} \quad C^2_{\text{loc}}((0, \infty)^2),$$

the function $h_k(s_1, s_2)$ has a maximum point $(s_{k,1}, s_{k,2}) \in (0, \infty) \times (0, \infty)$ for a sufficiently large $k$. By Lemma 2.6.5, we have

$$(\sqrt{s_{k,1}}\omega_1(x - ke_1), \sqrt{s_{k,2}}\omega_2(x - ke_1)) \in \mathcal{M}.$$

Since we can suppose that $V_1 \not\equiv$ const. or $V_2 \not\equiv$ const., by Lemma 2.6.5 again, we have

$$\begin{aligned} \hat{b}_{\mathcal{M}} &\leq I(\sqrt{s_{k,1}}\omega_1(x - ke_1), \sqrt{s_{k,2}}\omega_2(x - ke_1)) \\ &< I_\infty(\sqrt{s_{k,1}}\omega_1(x - ke_1), \sqrt{s_{k,2}}\omega_2(x - ke_1)) = I_\infty(\sqrt{s_{k,1}}\omega_1(x), \sqrt{s_{k,2}}\omega_2(x)) \\ &\leq \max_{(s_1,s_2)\in[0,\infty)\times[0,\infty)} I_\infty(\sqrt{s_1}\omega_1(x), \sqrt{s_2}\omega_2(x)) = \hat{b}_{\mathcal{M}_\infty}, \end{aligned}$$

which completes the proof. □

### 2.6.3 Proof of Theorem 2.1.2.

In this subsection, we prove Theorem 2.1.2. When $\beta > 0$ is large, in other words, Theorem 2.1.2(ii) follows from the construction of a positive solution of (2.1.2). So we only prove (i). A main result in this section is the following.

**Proposition 2.6.8.** *For each sufficiently small $\beta > 0$, it holds*

$$b_{\mathcal{N}} < \hat{b}_{\mathcal{M}}. \tag{2.6.9}$$

We remark that (2.6.9) shows that the minimizer of $\inf_{\mathcal{N}} I$ is a semitrivial solution and a proof of Theorem 2.1.2 easily follows.

*Proof of Proposition 2.6.8.* We prove (2.6.9) indirectly. So we assume that there exists a sequence $(\beta_n)$ such that $\beta_n \to 0$ and $\hat{b}_{\mathcal{M}_n} = b_{\mathcal{N}_n}$, where

$$\begin{aligned} I_n(u) &:= \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbf{R}^N} \mu_1 u_1^4 + 2\beta_n u_1^2 u_2^2 + \mu_2 u_2^4 \, dx, \\ \mathcal{N}_n &:= \{u \in H : u \not\equiv 0,\ I_n'(u)u = 0\}, \\ \mathcal{M}_n &:= \{u \in H : u_1, u_2 \not\equiv 0,\ I_n'(u)(u_1, 0) = I_n'(u)(0, u_2) = 0\}, \\ b_{\mathcal{N}_n} &:= \inf_{u \in \mathcal{N}_n} I_n(u), \quad b_{\mathcal{M}_n} := \inf_{u \in \mathcal{M}_n} I_n(u). \end{aligned}$$

By Theorem 2.6.4, there exists a $(u_n) \subset \mathcal{M}_n$ such that $I_n(u_n) = \hat{b}_{\mathcal{M}_n} = b_{\mathcal{N}_n}$. It is obvious that $(u_n)$ is a bounded sequence. So we assume that $u_n \rightharpoonup u_0$ weakly in $H$. Since

$$\begin{cases} -\Delta u_{n,1} + V_1(x)u_{n,1} = \mu_1 u_{n,1}^3 + \beta_n u_{n,1} u_{n,2}^2 & \text{in} \quad \mathbf{R}^N, \\ -\Delta u_{n,2} + V_2(x)u_{n,2} = \beta_n u_{n,1}^2 u_{n,2} + \mu_2 u_{n,2}^3 & \text{in} \quad \mathbf{R}^N, \end{cases}$$

we have

$$\begin{cases} -\Delta u_{0,1} + V_1(x)u_{0,1} = \mu_1 u_{0,1}^3 & \text{in} \quad \mathbf{R}^N, \\ -\Delta u_{0,2} + V_2(x)u_{0,2} = \mu_2 u_{0,2}^3 & \text{in} \quad \mathbf{R}^N. \end{cases} \tag{2.6.10}$$

We prove the following claim.

**Claim** $u_{0,1} \equiv 0$ or $u_{0,2} \equiv 0$.

*Proof of Claim.* We assume that $u_{0,1} \not\equiv 0$ and $u_{0,2} \not\equiv 0$. From (2.6.10), we have $d_1 + d_2 \le I_0(u_0)$. On the other hand, since $I_n(u_n) = \|u_n\|^2/4$ and $u_n \rightharpoonup u_0$ weakly in $H$, it follows that

$$I_0(u_0) \le \liminf_{n\to\infty} I_n(u_n) = \liminf_{n\to\infty} b_{\mathcal{N}_n} \le \min\{d_1, d_2\}.$$

This is a contradiction, hence $u_{0,1} \equiv 0$ or $u_{0,2} \equiv 0$. □

Suppose that $u_{0,2} \equiv 0$. By Proposition 2.2.9, there exists a $\delta_1 > 0$ such that $\|u_{n,j}\|_{L^4} \ge \delta_1$ $(j = 1, 2)$. Developing a concentration–compactness type argument, we can find a sequence $(y_n) \subset \mathbf{R}^N$ such that

$$|y_n| \to \infty, \quad \|u_{n,2}\|_{L^4(Q+y_n)} \to c > 0, \quad u_{n,2}(x + y_n) \rightharpoonup \omega_2 \quad \text{weakly} \quad \text{in} \quad H^1(\mathbf{R}^N),$$

where $Q = [0,1]^N$. Moreover $\omega_2$ satisfies that $\omega_2 \not\equiv 0$ and

$$-\Delta\omega_2 + V_{\infty,2}\omega_2 = \mu_2\omega_2^3.$$

Since

$$I_n(u_n) = \frac{1}{4}\int_{\mathbf{R}^N} \mu_1 u_{n,1}^4 + 2\beta_n u_{n,1}^2 u_{n,2}^2 + \mu_2 u_{n,2}^4\,dx \geq \frac{\mu_1}{4}\int_{\mathbf{R}^N} u_{n,1}^4\,dx + \frac{\mu_2}{4}\int_{\mathbf{R}^N} u_{n,2}^4(x+y_n)\,dx,$$

we have

$$\begin{aligned} I_{\infty,0}(0,\omega_2) = \frac{\mu_2}{4}\int_{\mathbf{R}^N} \omega_2^4 dx &< \frac{\mu_1}{4}\delta_1^4 + \liminf_{n\to\infty} \frac{\mu_2}{4}\int_{\mathbf{R}^N} u_{n,2}^4(x+y_n)dx \\ &\leq \liminf_{n\to\infty} I_n(u_n) = \lim_{n\to\infty} b_{\mathcal{N}_n} \leq \min\{d_1, d_2\}, \end{aligned}$$

which implies that $d_{\infty,2} < \min\{d_1, d_2\} \leq d_2$. This is a contradiction. The situation $u_{0,1} \equiv 0$ can be treated similarly. Thus we have (2.6.9). □

## 2.7 Proof of Theorem 2.1.4.

In this section, we prove Theorem 2.1.4 and follow the idea in Tanaka [99] (cf. Wang [103]).

*Proof of Theorem 2.1.4.* We prove indirectly and we assume that (2.1.2) has a positive solution $u$. Since $V_j(x) \in C^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, we remark $u_j \in H^2(\mathbf{R}^N)$. Without loss of generality we may assume that $\nu = e_1 = (1, 0, \ldots, 0)$. Since $I'(u)\left[\left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1}\right)\right] = 0$, we have

$$\sum_{j=1}^{2} \left\langle u_j, \frac{\partial u_j}{\partial x_1} \right\rangle_j = \sum_{j=1}^{2} \int_{\mathbf{R}^N} \mu_j u_j^3 \frac{\partial u_j}{\partial x_1}\,dx + \beta \int_{\mathbf{R}^N} u_1 u_2^2 \frac{\partial u_1}{\partial x_1} + u_1^2 u_2 \frac{\partial u_2}{\partial x_1}\,dx. \tag{2.7.1}$$

Here, we claim that

$$\int_{\mathbf{R}^N} \nabla u_j \cdot \nabla \left(\frac{\partial u_j}{\partial x_1}\right) dx = 0, \quad \int_{\mathbf{R}^N} \mu_j u_j^3 \frac{\partial u_j}{\partial x_1}\,dx = 0, \tag{2.7.2}$$

$$\int_{\mathbf{R}^N} u_1 u_2^2 \frac{\partial u_1}{\partial x_1} + u_1^2 u_2 \frac{\partial u_2}{\partial x_1}\,dx = 0, \tag{2.7.3}$$

$$\int_{\mathbf{R}^N} V_j(x) u_j \frac{\partial u_j}{\partial x_1}\,dx = -\frac{1}{2}\int_{\mathbf{R}^N} \frac{\partial V_j}{\partial x_1} u_j^2\,dx. \tag{2.7.4}$$

Assuming (2.7.2)–(2.7.4). It follows from (2.7.1) that

$$-\frac{1}{2}\sum_{j=1}^{2} \int_{\mathbf{R}^N} \frac{\partial V_j}{\partial x_1} u_j^2\,dx = 0.$$

By (2–V3'), (2–V4') and $u_j > 0$, this is a contradiction, so (2.1.2) has no positive solution.

Next we show (2.7.2)–(2.7.4). We only prove (2.7.3) since proofs of other cases are similar. For $\varphi_1, \varphi_2 \in C_0^\infty(\mathbf{R}^N)$, we have

$$\varphi_1\varphi_2^2\frac{\partial\varphi_1}{\partial x_1} + \varphi_1^2\varphi_2\frac{\partial\varphi_2}{\partial x_1} = \frac{1}{2}\frac{\partial}{\partial x_1}\left(\varphi_1^2\varphi_2^2\right).$$

Thus

$$\begin{aligned}\int_{\mathbf{R}^N} \varphi_1\varphi_2^2\frac{\partial\varphi_1}{\partial x_1} + \varphi_1^2\varphi_2\frac{\partial\varphi_2}{\partial x_1}\,dx &= \int_{\mathbf{R}^N} \frac{\partial}{\partial x_1}\left(\varphi_1^2\varphi_2^2\right)dx \\ &= \int_{\mathbf{R}^{N-1}}\int_{-\infty}^{\infty} \frac{\partial}{\partial x_1}\left(\varphi_1^2\varphi_2^2\right)dx_1 dx' = 0.\end{aligned}$$

Since $C_0^\infty(\mathbf{R}^N)$ is dense in $H^2(\mathbf{R}^N)$ and the functional

$$(u_1, u_2) \mapsto \int_{\mathbf{R}^N} u_1 u_2^2\frac{\partial u_1}{\partial x_1} + u_1^2 u_2\frac{\partial u_2}{\partial x_1}\,dx : H^2(\mathbf{R}^N)\times H^2(\mathbf{R}^N) \to \mathbf{R}$$

is continuous, (2.7.3) holds. □

# Chapter 3

# Uniqueness of nontrivial positive solutions

## 3.1 Introduction

In this chapter, we consider the uniqueness of nontrivial positive solution of (CNLS) with $\varepsilon = 1$, namely,

$$-\Delta u_1 + V_1(x)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 \quad \text{in } \mathbf{R}^N, \tag{3.1.1}$$

$$-\Delta u_2 + V_2(x)u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 \quad \text{in } \mathbf{R}^N, \tag{3.1.2}$$

$$u_1(x), u_2(x) \to 0 \text{ as } |x| \to \infty, \tag{3.1.3}$$

$$u_1(x), u_2(x) > 0 \quad \text{in } \mathbf{R}^N, \tag{3.1.4}$$

where $\mu_1, \mu_2, \beta > 0$ and $N = 1, 2, 3$.

Recently the existence of nontrivial positive solutions has been studied extensively in [4, 31, 50, 75, 65, 94, 105]. In particular, the case where $V_1(x)$, $V_2(x)$ are positive and independent of $x$ is well studied and it is shown in [4, 31, 75, 65, 94] that there exist positive constants $\beta_2 \geq \beta_1 > 0$ such that for $\beta \in [0, \beta_1) \cup (\beta_2, \infty)$, (3.1.1)–(3.1.4) has a nontrivial positive solution. And it has been extended to $x$-dependent situations in [50, 105]. However the uniqueness of nontrivial positive solutions is not studied and the main purpose of this chapter is to establish the uniqueness for small $\beta > 0$.

First we consider the uniqueness of nontrivial positive solutions in the constant coefficient case:

$$\begin{cases} -\Delta u_1 + V_1 u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbf{R}^N, \\ -\Delta u_2 + V_2 u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 & \text{in } \mathbf{R}^N, \\ u_1(x), u_2(x) > 0 & \text{in } \mathbf{R}^N, \\ u_1(x), u_2(x) \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{3.1.5}$$

Here $V_1, V_2$ are positive constants and $N = 1, 2, 3$.

Now we state our result in the constant coefficient case.

**Theorem 3.1.1.** *Suppose that $N = 1, 2, 3$. Then there exists a $\beta_0 > 0$ such that if $\beta \in (0, \beta_0)$, then nontrivial positive solutions of* (3.1.5) *are unique up to translation.*

*Remark* 3.1.2. Wei and Yao [107] also obtains similar results concerning (3.1.5).

Next we deal with the variable coefficient case. Here we assume that $N = 2, 3$ and $V_j(x)$ satisfies the following condition:

(3–V1) For $j = 1, 2$, $V_j \in C^2(\mathbf{R}^N)$, $V_j(x) = V_j(|x|)$ and $\lim_{|x|\to\infty} V_j(x) > 0$.

(3–V2) For $j = 1, 2$,

$$\inf\left\{\int_{\mathbf{R}^N} |\nabla u|^2 + V_j(x)u^2 dx : \ \|u\|_{H^1(\mathbf{R}^N)} = 1\right\} > 0.$$

(3–V3) For $j = 1, 2$ and $r \geq 0$, $V_j'(r) \geq 0$.

(3–V4) There exist $C > 0$ and $M > 0$ such that $|V_j(r)| \leq C(1 + r)^M$ for $j = 1, 2$ and $r \geq 0$.

(3–V5) When $N = 3$, the function

$$H_j(r) := \frac{4}{3}r^2 V_j(r) + r^3 V'(r) - \frac{4}{27}$$

has a unique simple zero in $(0, \infty)$.

As in Remark 1.1.5, the function $V_j(r) = r^\alpha$ ( $\alpha \geq 2$ ) satisfies (3–V1)–(3–V5). Therefore, we restrict ourselves in the following function space.

**Definition 3.1.3.** For $V_1$ and $V_2$ satisfying (3–V1)–(3–V5), we define $\mathcal{H}_{V_1,V_2,r} \subset H^1_r(\mathbf{R}^N)\times H^1_r(\mathbf{R}^N)$ as follows:

$$\mathcal{H}_{V_1,V_2,r} := \left\{u \in H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N) : u(x) = u(|x|), \int_{\mathbf{R}^N} V_j(x)u_j^2 dx < \infty \text{ for } j = 1, 2\right\}.$$

Under (3–V1)–(3–V5), we have the following uniqueness result.

**Theorem 3.1.4.** *Suppose $N = 2, 3$ and $V_j(x)$ satisfies* (3–V1)–(3–V5)*. Then there exists a $\beta_0 > 0$ such that for $\beta \in (0, \beta_0)$* (3.1.1)–(3.1.4) *has a unique nontrivial positive solution in $\mathcal{H}_{V_1,V_2,r}$.*

*Remark* 3.1.5. (i) When $N = 2, 3$, by the result of Busca and Sirakov [20], it is known that any nontrivial positive solution of (3.1.1)–(3.1.4) is radially symmetric and monotone decreasing under (3–V1) and (3–V3), and radially symmetric with respect to some point $x_0 \in \mathbf{R}^N$ if $V_j(x)$ is independent of $x$ and positive.
(ii) We remark that the argument in [20] also works for $N = 1$ after suitable modification. However, it is not clearly stated in [20], we give the symmetry and monotonicity result for $N = 1$ in section 3.4 for the sake of readers.
(iii) The uniqueness of nontrivial positive solution of (3.1.1)–(3.1.4) does not hold for some $\beta > 0$. Indeed, the following example is given in Sirakov [94], and Montefusco, Pellacci and Squassina [79]; let $V_1(x) \equiv V_2(x) \equiv 1$, $\mu_1 = \mu_2 = \beta = 1$ and $\omega_1$ be a positive solution of

$$-\Delta u_1 + u_1 = u_1^3 \quad \text{in } \mathbf{R}^N.$$

Then $u(x) = (u_1(x), u_2(x)) = (\omega_1(x)\cos\theta, \omega_1(x)\sin\theta)$ is a nontrivial positive solution of (3.1.1)–(3.1.4) for any $\theta \in (0, \pi/2)$.

We say that a nontrivial positive solution $u = (u_1, u_2) \in \mathcal{H}_{V_1,V_2,r}$ of (3.1.1)–(3.1.4) is *nondegenerate* in the function space $\mathcal{H}_{V_1,V_2,r}$ if and only if the linearized system

$$\begin{cases} -\Delta w_1 + V_1(x)w_1 = 3\mu_1 u_1^2 w_1 + \beta(u_2^2 w_1 + 2u_1u_2w_2) & \text{in } \mathbf{R}^N, \\ -\Delta w_2 + V_2(x)w_2 = \beta(2u_1u_2w_1 + u_1^2 w_2) + 3\mu_2 u_2^2 w_2 & \text{in } \mathbf{R}^N, \\ \qquad (w_1, w_2) \in \mathcal{H}_{V_1,V_2,r} \end{cases}$$

has only trivial solution $w_1 \equiv w_2 \equiv 0$.

As to the nondegeneracy of our unique radial positive solution, we have

**Theorem 3.1.6.** *The unique nontrivial radial positive solution of* (3.1.1)–(3.1.4) *is nondegenerate in* $\mathcal{H}_{V_1,V_2,r}$ *for* $\beta \in (0, \beta_0)$.

To prove our Theorems 3.1.1, 3.1.4 and 3.1.6, a priori estimates and uniform exponential decays of nontrivial radial positive solutions are important. In section 3.2, we will obtain a priori $L^\infty$ estimates and uniform exponential decays of

$$\mathcal{S}_{\bar\beta} := \{u \in \mathcal{H}_{V_1,V_2,r} : u \text{ is a nontrivial positive solution of (3.1.1)–(3.1.4)} \\ \text{for some } \beta \in [0, \bar\beta]\}.$$

Here Liouville type result (Lemma 3.2.1) and monotonicity of solutions play roles. In section 3.3, we prove Theorems 3.1.1, 3.1.4 and 3.1.6. The behavior of nontrivial positive solutions as $\beta \to 0$ is a key of our argument.

## 3.2 A priori bounds for nontrivial positive solutions

In this section, we assume that $1 \le N \le 3$ and $V_j(x)$ satisfies (3–V1)–(3–V5).

### 3.2.1 $L^\infty$ bound of nontrivial positive solutions

First, we introduce an inner product in $\mathcal{H}_{V_1,V_2,r}$: for $u, v \in \mathcal{H}_{V_1,V_2,r}$, we define $\langle \cdot, \cdot \rangle_{\mathcal{H}_{V_1,V_2,r}}$ by

$$\langle u, v \rangle_{\mathcal{H}_{V_1,V_2,r}} := \int_{\mathbf{R}^N} \nabla u_1 \cdot \nabla v_1 + V_1(x)u_1v_1 + \nabla u_2 \cdot \nabla v_2 + V_2(x)u_2v_2 dx.$$

Then it is easily seen that $\mathcal{H}_{V_1,V_2,r}$ is a Hilbert space with $\langle \cdot, \cdot \rangle_{\mathcal{H}_{V_1,V_2,r}}$ and we denote its norm by $\| \cdot \|_{\mathcal{H}_{V_1,V_2,r}}$. Next, for $\bar\beta > 0$ we define $\mathcal{S}_{\bar\beta}$ as follows:

$$\mathcal{S}_{\bar\beta} := \{u \in \mathcal{H}_{V_1,V_2,r} : u \text{ is a nontrivial positive solution of (3.1.1)–(3.1.4)} \\ \text{for some } \beta \in [0, \bar\beta],$$

The following lemma is essential to prove uniform $L^\infty$ estimates and uniform exponential decays of $u \in \mathcal{S}_{\bar\beta}$.

**Lemma 3.2.1.** *Let* $1 < \alpha \le 3$ *if* $N = 3$ *and* $1 < \alpha < \infty$ *if* $N = 1, 2$. *Then there is no positive function such that*

$$-\Delta u \ge |u|^{\alpha-1}u \text{ in } \mathbf{R}^N. \tag{3.2.1}$$

*Proof.* When $N = 3$, we refer to Theorem 8.4 in Quittner and Souplet [90]. When $N = 2$, the conclusion of Lemma 3.2.1 follows from Liouville's theorem. For example, see Protter and Weinberger [88]. When $N = 1$, Lemma 3.2.1 can be shown easily. $\square$

The following proposition gives us a uniform a priori $L^\infty$–bound of $\mathcal{S}_{\bar\beta}$.

**Proposition 3.2.2.** *For any $\bar\beta > 0$ there exists an $M_{\bar\beta} > 0$ such that the following inequality*

$$\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty} \le M_{\bar\beta}$$

*holds for all $u \in \mathcal{S}_{\bar\beta}$.*

*Proof.* We prove indirectly and assume that there exist $(u_k) \subset \mathcal{S}_{\bar\beta}$ and $(\beta_k)$ such that $\beta_k \to \beta_0$ and $\|u_{k,1}\|_{L^\infty} + \|u_{k,2}\|_{L^\infty} \to \infty$. Furthermore, without loss of generality, we may assume $\|u_{k,2}\|_{L^\infty} \le \|u_{k,1}\|_{L^\infty} \to \infty$. For $j = 1, 2$, we set

$$\eta_k := \frac{1}{\|u_{k,1}\|_{L^\infty}}, \ v_{k,j}(x) := \eta_k u_{k,j}(\eta_k x).$$

By Remark 3.1.5, $(u_k)$ is radially symmetric and monotone decreasing, and we have $v_{k,1}(0) = 1$ and $\|v_{k,j}\|_{L^\infty} \le 1$ $(j = 1, 2)$. Moreover, since $u_k$ is a solution of (3.1.1)–(3.1.2), $v_k$ satisfies

$$\begin{cases} -\Delta v_{k,1} + \eta_k^2 V_1(\eta_k x) v_{k,1} = \mu_1 v_{k,1}^3 + \beta_k v_{k,1} v_{k,2}^2, \\ -\Delta v_{k,2} + \eta_k^2 V_2(\eta_k x) v_{k,2} = \beta_k v_{k,1}^2 v_{k,2} + \mu_2 v_{k,2}^3. \end{cases}$$

By the standard elliptic argument, if necessary take a subsequence, it holds that $v_{k,j} \to v_{0,j}$ in $C^2_{\mathrm{loc}}(\mathbf{R}^N)$ where $v_{0,j}$ is a nonnegative solution of

$$\begin{cases} -\Delta v_{0,1} = \mu_1 v_{0,1}^3 + \beta_0 v_{0,1} v_{0,2}^2, \\ -\Delta v_{0,2} = \beta_0 v_{0,1}^2 v_{0,2} + \mu_2 v_{0,2}^3. \end{cases}$$

Since $v_{0,1}(0) = 1$ and the maximum principle, $v_{0,1} > 0$ in $\mathbf{R}^N$. So $v_{0,1}(x)$ satisfies the following differential inequality:

$$-\Delta v_{0,1} \ge \mu_1 v_{0,1}^3 \text{ in } \mathbf{R}^N.$$

This contradicts Lemma 3.2.1, so the conclusion of Proposition 3.2.2 holds. $\square$

### 3.2.2 Uniform exponential decay estimates in $\mathcal{S}_{\bar\beta}$.

As pointed in Remark 3.1.5, any nontrivial positive solution of (3.1.1)–(3.1.4) is a radially symmetric function. So we rewrite (3.1.1)–(3.1.2) as follows:

$$\begin{cases} -u_1''(r) - \dfrac{N-1}{r} u_1'(r) + V_1(r) u_1(r) = \mu_1 u_1^3(r) + \beta u_1(r) u_2^2(r), \\ -u_2''(r) - \dfrac{N-1}{r} u_2'(r) + V_2(r) u_2(r) = \beta u_1^2(r) u_2(r) + \mu_2 u_2^3(r). \end{cases} \tag{3.2.2}$$

The following proposition gives us a uniform exponential decay of $\mathcal{S}_{\bar\beta}$.

**Proposition 3.2.3.** *For any $\bar{\beta} > 0$ there exist $C_1 = C_1(\bar{\beta})$ and $C_2 = C_2(\bar{\beta}) > 0$ such that*

$$|u_1(r)| + |u_2(r)| \leq C_1 \exp(-C_2 r), \quad |u_1'(r)| + |u_2'(r)| \leq C_1 \exp(-C_2 r)$$

*for all $u \in \mathcal{S}_{\bar{\beta}}$.*

*Proof.* The proof is divided into 2 steps.

**Step 1** *Let $\beta \geq 0$ and $u(r) \in \mathcal{H}_{V_1,V_2,r}$ be a nontrivial positive solution of* (3.1.1)–(3.1.4)*. There exist $C_3 = C_3(u),\ C_4 = C_4(u) > 0$ such that*

$$|u_1(r)| + |u_2(r)| \leq C_3 \exp(-C_4 r), \quad |u_1'(r)| + |u_2'(r)| \leq C_3 \exp(-C_4 r). \tag{3.2.3}$$

Let $\beta \geq 0$ and $u \in \mathcal{H}_{V_1,V_2,r}$ be a nontrivial positive solution of (3.2.2). We prove that $u_1(r)$ and $u_1'(r)$ decay exponentially. We follow arguments in Tanaka [99]. Since $u_j(r) \to 0$ as $r \to \infty$ and (3–V1) holds, there exists an $r_0 \geq 1$ such that for $r \geq r_0$, we have

$$u_1(r),\ u_2(r) \leq 1, \quad 0 < \frac{V_1(r_0)}{2} \leq V_1(r) - \mu_1 u_1^2(r) - \beta u_2^2(r). \tag{3.2.4}$$

Thus it holds

$$\begin{aligned} 0 &= -u_1''(r) - \frac{N-1}{r} u_1'(r) + \left(V_1(r) - \mu_1^2 u_1^2(r) - \beta u_2^2(r)\right) u_1(r) \\ &\geq -u_1''(r) - \frac{N-1}{r} u_1'(r) + \frac{V_1(r_0)}{2} u_1(r) \quad \text{for } r \geq r_0. \end{aligned} \tag{3.2.5}$$

Let $\delta > 0$ satisfy $\max\{\delta^2, (N-1)\delta\} < V_1(r_0)/4$ and for any $R > r_0$ we set

$$\varphi_{R,\delta}(r) := \exp(-\delta(r - r_0)) + \exp(-\delta(R - r)).$$

Then we have

$$-\varphi_{R,\delta}''(r) - \frac{N-1}{r}\varphi_{R,\delta}'(r) + \frac{V_1(0)}{2}\varphi_{R,\delta}(r) \geq 0 \qquad \text{for all } r \in [r_0, R]. \tag{3.2.6}$$

Since $\varphi_{R,\delta}(r_0) \geq 1,\ \varphi_{R,\delta}(R) \geq 1$ and (3.2.4)–(3.2.6) hold, by the comparison theorem, it follows

$$u_1(r) \leq \varphi_{R,\delta}(r) \quad \text{for all } r \in [r_0, R]. \tag{3.2.7}$$

Since $R > r_0$ is arbitrary, let $R \to \infty$, then we obtain

$$u_1(r) \leq \exp(-\delta(r - r_0)) \quad \text{for all } r \geq r_0.$$

Thus $u_1(r)$ has an exponential decay. In a similar way, we can show that $u_2(r)$ has an exponential decay.

Next we will show that $u_1'(r)$ has an exponential decay. We follow the arguments in Berestycki and Lions [15]. By (3.2.2), it holds that

$$(r^{N-1}u_1'(r))' = r^{N-1}u_1(r)(V_1(r) - \mu_1 u_1^2(r) - \beta u_2^2(r)). \tag{3.2.8}$$

Hence the function $(r^{N-1}u_1'(r))'$ has an exponential decay, which implies that the limit $\lim_{r\to\infty} r^{N-1}u_1'(r)$ exists. Since $u_1$ has an exponential decay, it holds $r^{N-1}u_1'(r) \to 0$ as $r \to \infty$. Otherwise, $u_1$ does not have an exponential decay. Therefore we obtain

$$r^{N-1}u_1'(r) = -\int_r^\infty (s^{N-1}u_1'(s))'ds,$$

which implies that $u_1'(r)$ has an exponential decay. In a similar way, it follows that $u_2'(r)$ has an exponential decay.

Next we prove that for any $\bar{\beta} > 0$ we can choose $C_3$, $C_4$ in (3.2.3) uniformly with respect to $u \in \mathcal{S}_{\bar{\beta}}$, that is, we complete the proof of Proposition 3.2.3.

**Step 2** *Conclusion.*

We prove that there exist $C_3 = C_3(\bar{\beta}), C_4 = C_4(\bar{\beta}) > 0$ such that

$$|u_1(r)| + |u_2(r)| \le C_3 \exp(-C_4 r) \tag{3.2.9}$$

for all $u \in \mathcal{S}_{\bar{\beta}}$. From (3.2.8) and (3.2.9) it follows that

$$|u_1'(r)| + |u_2'(r)| \le C_3 \exp(-C_4 r)$$

for all $u \in \mathcal{S}_{\bar{\beta}}$.

We prove (3.2.9) indirectly and assume that there exist $(u_k) \subset \mathcal{S}_{\bar{\beta}}$ and $(\beta_k)$ such that $\beta_k \to \beta_0$ and

$$u_{k,1}(r_k) + u_{k,2}(r_k) > k\exp(-r_k/k) \quad \text{for some } r_k \ge 0. \tag{3.2.10}$$

Since $(u_k)$ satisfies (3.2.2) with $\beta = \beta_k$ and $\|u_k\|_{L^\infty}$ is bounded by Proposition 3.2.2, we can also assume that $u_{k,j} \to u_{0,j}$ in $C^2_{\mathrm{loc}}(\mathbf{R}^N)$, where $u_0$ is a nonnegative solution of

$$\begin{cases} -u_{0,1}''(r) - \dfrac{N-1}{r}u_{0,1}'(r) + V_1(r)u_{0,1}(r) = \mu_1 u_{0,1}^3(r) + \beta_0 u_{0,1}(r)u_{0,2}^2(r), \\ -u_{0,2}''(r) - \dfrac{N-1}{r}u_{0,2}'(r) + V_2(r)u_{0,2}(r) = \beta_0 u_{0,1}^2(r)u_{0,2}(r) + \mu_2 u_{0,2}^3(r). \end{cases} \tag{3.2.11}$$

We set

$$E_k(r) := \sum_{j=1}^2 \left(\frac{1}{2}(u_{k,j}'(r))^2 + \frac{\mu_j}{4}u_{k,j}^4 - \frac{1}{2}V_j(r)u_{k,j}^2(r)\right) + \frac{\beta_k}{2}u_{k,1}^2(r)u_{k,2}^2(r),$$

$$E_0(r) := \sum_{j=1}^2 \left(\frac{1}{2}(u_{0,j}'(r))^2 + \frac{\mu_j}{4}u_{0,j}^4 - \frac{1}{2}V_j(r)u_{0,j}^2(r)\right) + \frac{\beta_0}{2}u_{0,1}^2(r)u_{0,2}^2(r).$$

Since $u_k \to u_0$ in $C^2_{\rm loc}(\mathbf{R}^N)$, $E_k \to E_0$ in $C^1_{\rm loc}(\mathbf{R}^N)$. By (3–V3) and (3.2.2) with $\beta = \beta_k$, we have

$$\begin{aligned} E_k'(r) &= \sum_{j=1}^{2} \left(u_{k,j}''(r) + \mu_j u_{k,j}^3(r) - V_j(r)u_{k,j}(r)\right) u_{k,j}'(r) - \sum_{j=1}^{2} \frac{1}{2} V_j'(r) u_{k,j}^2(r) \\ &\quad + \beta_k u_{k,1} u_{k,2}^2(r) u_{k,1}'(r) + \beta_k u_{k,1}^2(r) u_{k,2}(r) u_{k,2}'(r) \\ &= -\frac{N-1}{r} \sum_{j=1}^{2} (u_{k,1}'(r))^2 - \sum_{j=1}^{2} \frac{1}{2} V_j'(r) u_{k,j}^2(r) \le 0. \end{aligned}$$

Since $u_k$ and $u_k'$ have an exponential decay for each $k \ge 1$ by Step 1, it holds that $\lim_{r\to\infty} E_k(r) = 0$. By the monotonicity of $E_k$, it follows that $E_k(r) \ge 0$ for $r \ge 0$, which implies that $E_0(r) \ge 0$ for $r \ge 0$.

By [20], $u_k(r)$ is a decreasing function when $N = 2, 3$. We remark that this result holds even when $N = 1$. See Theorem 3.4.1 in section 3.4. Thus $u_0(r)$ is also a nonincreasing function. So we set

$$\lim_{r\to\infty} u_{0,j}(r) =: u_{0,\infty,j} \ge 0 \quad (j = 1, 2). \tag{3.2.12}$$

and next we claim

$$u_{0,\infty,1} = u_{0,\infty,2} = 0.$$

Since $E_0(r)$ and $u_{0,j}(r)$ are bounded in $[0, \infty)$, $u_0'(r)$ is also bounded in $[0, \infty)$. Furthermore, $u_{0,j}'(r)$ is bounded since $E_0(r)$ and $u_{0,j}(r)$ converge as $r \to \infty$.

Here we consider 3 cases.

**Case 1** $V_{j,\infty} := \lim\limits_{r\to\infty} V_j(r) \in (0, \infty)$ for $j = 1, 2$.

By (3.2.11), $u_{0,\infty,1}$ and $u_{0,\infty,2}$ satisfy

$$\begin{aligned} u_{0,\infty,1}(\mu_1 u_{0,\infty,1}^2 + \beta_0 u_{0,\infty,2}^2 - V_{1,\infty}) = 0, \\ u_{0,\infty,2}(\beta_0 u_{0,\infty,1}^2 + \mu_2 u_{0,\infty,2}^2 - V_{2,\infty}) = 0. \end{aligned} \tag{3.2.13}$$

Then we have

$$(u_{0,\infty,1}, u_{0,\infty,2}) \in \left\{ (0,0), \left(\sqrt{\frac{V_{1,\infty}}{\mu_1}}, 0\right), \left(0, \sqrt{\frac{V_{2,\infty}}{\mu_2}}\right), (u_1^*, u_2^*) \right\}$$

where $(u_1^*, u_2^*)$ satisfies

$$\begin{aligned} V_{1,\infty} = \mu_1 (u_1^*)^2 + \beta_0 (u_2^*)^2 \\ V_{2,\infty} = \beta_0 (u_1^*)^2 + \mu_2 (u_2^*)^2. \end{aligned}$$

If $(u_{0,\infty,1}, u_{0,\infty,2}) \ne (0,0)$, then it is easily seen that $\lim\limits_{r\to\infty} E_0(r) < 0$, which contradicts to $E_0(r) \ge 0$. Thus $u_{0,\infty,j} = 0$ $(j = 1, 2)$.

**Case 2** $V_{j,\infty} = \infty$ for $j = 1, 2$.

In this case, $u_{0,\infty,j} = 0$ follows easily. In fact, Since $E_0(r) \geq 0$ for all $r \geq 0$, it must be $u_{0,\infty,j} = 0$ for $j = 1, 2$.

**Case 3** $V_{i_0,\infty} = \infty$ and $V_{j_0,\infty} < \infty$. Here $\{i_0, j_0\} = \{1, 2\}$.

Without loss of generality, we can assume $i_0 = 1$ and $j_0 = 2$. As in Case 2, we may infer that $u_{0,\infty,1} = 0$. Furthermore, by (3.2.11), it follows that

$$u_{0,\infty,2}(\mu_2 u_{0,\infty,2}^2 - V_{2,\infty}) = 0.$$

Now we suppose that $u_{0,\infty,2} = \sqrt{V_{2,\infty}/\mu_2}$. Since $V_1(r) \to \infty$ as $r \to \infty$ and $u_{0,2}$ is bounded, we can choose $r_1 > 0$ such that

$$0 < \inf_{r \geq r_1} \left\{ V_1(r) - \mu_1 u_{0,1}^2(r) - \beta u_{0,2}^2(r) \right\}.$$

Hence, as in Step 1, it holds that there exist $C_1, C_2 > 0$ such that

$$u_{0,1}(r) \leq C_1 \exp(-C_2 r) \quad \text{for all } r \geq 0.$$

By (3–V4), we obtain

$$\lim_{r \to \infty} E_0(r) = -\frac{V_{\infty,2}^2}{4\mu_2} < 0,$$

which contradicts that $E_0(r) \geq 0$ for all $r \geq 0$. Therefore we have $u_{0,\infty,2} = 0$.

Bylim$_{r\to\infty} u_{0,j}(r) = 0$, as in Step 1, we can prove that there exist $\bar{C}_1, \bar{C}_2 > 0$ such that

$$|u_{0,1}(r)| + |u_{0,2}(r)| \leq \bar{C}_1 \exp(-\bar{C}_2 r).$$

Moreover, since $u_{k,j}(r)$ is a decreasing function and $u_{k,j} \to u_{0,j}$ in $C^2_{\text{loc}}(\mathbf{R}^N)$, there exist $r_2 > 0$, $k_0 \in \mathbf{N}$ such that

$$\begin{aligned} &\sup_{r \geq r_2} \{u_{k,1}(r) + u_{k,2}(r)\} \leq 1 && \text{for all } k \text{ with } k \geq k_0, \\ &0 < \inf_{r \geq r_2} \left\{ V_1(r) - \mu_1 u_{k,1}^2(r) - \beta_k u_{k,2}^2(r) \right\} && \text{for all } k \text{ with } k \geq k_0. \end{aligned}$$

Hence, we can show that there exist $\bar{C}_3$, $\bar{C}_4 > 0$ such that

$$|u_{k,1}(r)| + |u_{k,2}(r)| \leq \bar{C}_3 \exp(-\bar{C}_4 r)$$

for all $k \geq 1$ and $r \geq 0$. However this is a contradiction to (3.2.10) and (3.2.9) holds. □

As a corollary to Proposition 3.2.3, we have

**Corollary 3.2.4.** *For any $\bar{\beta} > 0$, $\mathcal{S}_{\bar{\beta}}$ is bounded in $\mathcal{H}_{V_1,V_2,r}$. Moreover it has the following compactness property : for any sequence $(u_k) \subset \mathcal{S}_{\bar{\beta}}$, there exists a strongly convergent subsequence $(u_{k_m})$ in $\mathcal{H}_{V_1,V_2,r}$.*

*Proof.* By Proposition 3.2.3 and (3–V4), it is clear that $(u_k)$ is bounded in $\mathcal{H}_{V_1,V_2,r}$ and for any $\varepsilon > 0$ there exists an $R_\varepsilon > 0$ such that

$$\sum_{j=1}^{2} \int_{|x| \geq R_\varepsilon} |\nabla u_{k,j}(x)|^2 + V_j(x) u_{k,j}^2(x) dx < \varepsilon \quad \text{for } k \geq 1.$$

Moreover, since $(u_k)$ is bounded in $\mathcal{H}_{V_1,V_2,r}$, there exists a subsequence $(u_{k_m})$ such that $u_{k_m} \rightharpoonup u_0$ weakly in $\mathcal{H}_{V_1,V_2,r}$. Since $(u_k)$ satisfies (3.2.2) and is bounded in $L^\infty$, we can assume $u_{k_m} \to u_0$ in $C^2_{\text{loc}}(\mathbf{R}^N)$. Therefore we have $u_{k_m} \to u_0$ strongly in $\mathcal{H}_{V_1,V_2,r}$. □

## 3.3 Proof of Theorems 3.1.1, 3.1.4 and 3.1.6

In this section, we assume that $V_j(x)$ is a positive constant function if $N = 1$ and $V_j(x)$ satisfies (3–V1)–(3–V5) if $N = 2, 3$.

### 3.3.1 Nondegeneracy of solutions when $\beta = 0$

In this subsection, we consider the following limit equations as $\beta \to 0$ in (3.1.1)–(3.1.4):

$$\begin{cases} -\Delta v_j + V_j(x) v_j = \mu_j v_j^3 & \text{in } \mathbf{R}^N, \\ v_j \in \mathcal{H}_{V_j,r}, \ v_j > 0. \end{cases} \tag{3.3.1}$$

Here

$$\mathcal{H}_{V_j,r} := \left\{ u_j \in H^1(\mathbf{R}^N) : u_j(x) = u_j(|x|), \ \int_{\mathbf{R}^N} V_j(x) u_j^2 dx < \infty \right\}.$$

**Definition 3.3.1.** A solution $\omega_j$ of (3.3.1) is *nondegenerate* in $\mathcal{H}_{V_j,r}$ if the following equation has only a trivial solution $\psi_j \equiv 0$:

$$\begin{cases} -\Delta \psi_j + V_j(x) \psi_j = 3\mu_j \omega_j^2 \psi_j & \text{in } \mathbf{R}^N, \\ \psi_j \in \mathcal{H}_{V_j,r}. \end{cases}$$

The following proposition is due to Byeon and Ohshita [25], and Kabeya and Tanaka [57].

**Proposition 3.3.2.** *If $V_j(x)$ satisfies (3–V1)–(3–V5) and $N = 2, 3$, then (3.3.1) has a unique radial positive solution $\omega_j(x) \in \mathcal{H}_{V_j,r}$. Moreover, $\omega_j(x)$ is nondegenerate in $\mathcal{H}_{V_j,r}$.*

The following proposition is well–known. See Willem [110] and Kwong [60].

**Proposition 3.3.3.** *Suppose that $1 \le N \le 3$, and $V_j(x)$ is positive and independent of $x$. Then (3.3.1) has a unique positive solution $\omega_j(x)$. Moreover, $\omega_j(x)$ is nondegenerate in $H_r^1(\mathbf{R}^N)$.*

By Propositions 3.3.2 and 3.3.3, we have

**Corollary 3.3.4.** *Under the assumptions in Theorem 3.1.1 or 3.1.4, (3.1.1)–(3.1.4) with $\beta = 0$ has a unique nontrivial positive solution $\omega(x) := (\omega_1(x), \omega_2(x))$ in $\mathcal{H}_{V_1,V_2,r}$ and $\omega(x)$ is nondegenerate in $\mathcal{H}_{V_1,V_2,r}$.*

*Remark* 3.3.5. We set

$$I_j(\psi_j) := \frac{1}{2} \int_{\mathbf{R}^N} |\nabla \psi_j|^2 + V_j(x) \psi_j^2 dx - \frac{\mu_j}{4} \int_{\mathbf{R}^N} \psi_j^4 dx,$$
$$I_0(\psi) := I_0(\psi_1, \psi_2) = I_1(\psi_1) + I_2(\psi_2).$$

Then solutions $\omega_j \in \mathcal{H}_{V_j,r}$ of (3.3.1) and $\omega(x)$ of (3.1.1)–(3.1.4) with $\beta = 0$ are nondegenerate in $\mathcal{H}_{V_j,r}$ and $\mathcal{H}_{V_1,V_2,r}$ if and only if the mapping

$$\psi_j \mapsto I_j''(\omega_j)[\psi_j, \cdot] : \ \mathcal{H}_{V_j,r} \to (\mathcal{H}_{V_j,r})^*,$$
$$\psi \mapsto I_0''(\omega)[\psi, \cdot] : \ (\mathcal{H}_{V_1,V_2,r})^2 \to (\mathcal{H}_{V_1,V_2,r})^*$$

are invertible.

### 3.3.2 Proof of Theorems 3.1.1, 3.1.4 and 3.1.6

We prove Theorems 3.1.1, 3.1.4 and 3.1.6.

*Proof of Theorems 3.1.1, 3.1.4 and 3.1.6.* Let $(u_\beta) = (u_{\beta,1}, u_{\beta,2})$ be a nontrivial positive solution of (3.1.1)–(3.1.4). By Corollary 3.2.4, there exist $(u_k)$ and $(\beta_k)$ such that $u_k \to u_0$ in $\mathcal{H}_{V_1,V_2,r}$ and $\beta_k \to 0$. Since $(u_k)$ satisfies (3.1.1)–(3.1.4) with $\beta = \beta_k$, we have

$$\|u_{k,1}\|^2_{\mathcal{H}_{V_1,r}} = \mu_1 \|u_{k,1}\|^4_{L^4} + \beta_k \|u_{k,1}u_{k,2}\|^2_{L^2}.$$

Using Sobolev's embedding, Hölder's inequality and (3–V2), it holds that

$$\|u_{k,1}\|_{\mathcal{H}_{V_1,r}} \leq C_1(\|u_{k,1}\|^4_{\mathcal{H}_{V_1,r}} + \beta_k \|u_{k,1}\|^2_{\mathcal{H}_{V_1,r}} \|u_{k,2}\|^2_{\mathcal{H}_{V_2,r}}).$$

Since $\beta_k \to 0$ and $(u_k)$ is bounded in $\mathcal{H}_{V_1,V_2,r}$, there exists a $C_2 > 0$ such that

$$C_2 \leq \|u_{k,1}\|_{\mathcal{H}_{V_1,r}} \quad \text{for all } k \in \mathbf{N}.$$

Similar arguments lead that

$$0 < C_3 \leq \|u_{k,2}\|_{\mathcal{H}_{V_2,r}} \quad \text{for all } k \in \mathbf{N}.$$

Since $u_{k,j} \to u_{0,j}$ in $C^2_{\mathrm{loc}}(\mathbf{R}^N)$ and $\mathcal{H}_{V_1,V_2,r}$, we deduce that $u_{0,j} \geq 0$ and $u_{0,j} \not\equiv 0$ for $j = 1, 2$. By the maximum principle, we have $u_{0,j}(x) > 0$ in $\mathbf{R}^N$. By Propositions 3.3.2 and 3.3.3, we have $u_{0,j} = \omega_j$. This implies that

$$u_\beta \to \omega = (\omega_1, \omega_2) \text{ strongly in } \mathcal{H}_{V_1,V_2,r} \text{ as } \beta \to 0.$$

Thus, for any $\varepsilon > 0$ there exists a $\tilde{\beta} = \tilde{\beta}(\varepsilon) > 0$ such that for any $\beta \in [0, \tilde{\beta}]$ and any nontrivial positive solution $u \in \mathcal{H}_{V_1,V_2,r}$ of (3.1.1)–(3.1.4) satisfies

$$\|u_j - \omega_j\|_{\mathcal{H}_{V_1,V_2,r}} < \varepsilon. \tag{3.3.2}$$

Set

$$I_\beta(u) := I_1(u_1) + I_2(u_2) - \frac{\beta}{2}\int_{\mathbf{R}^N} u_1^2 u_2^2 dx,$$
$$\Phi(\beta, u) := I'_\beta(u) : \ \mathbf{R} \times \mathcal{H}_{V_1,V_2,r} \to (\mathcal{H}_{V_1,V_2,r})^*.$$

Then it is clear that $\Phi(0, \omega) = 0$. Moreover, by Corollary 3.3.4 and Remark 3.3.5, $\Phi_u(0, \omega) = I''_0(\omega)$ is invertible. By the implicit function theorem, there exist $\beta_0 > 0$, $r_0 > 0$ and $\phi : (-\beta_0, \beta_0) \to B_{r_0}(\omega)$ such that

1. $\phi(0) = \omega$,
2. $\Phi(\beta, \phi(\beta)) = 0$ for all $\beta \in (-\beta_0, \beta_0)$,
3. For any $\beta \in (-\beta_0, \beta_0)$, $\Phi(\beta, u) = 0$ has a unique solution $u = \phi(\beta)$ in $B_{r_0}(\omega)$.
4. The mapping $\psi \mapsto I''_0(u)[\psi, \cdot] : \mathcal{H}_{V_1,V_2,r} \to (\mathcal{H}_{V_1,V_2,r})^*$ is invertible in $B_{r_0}(\omega)$.

By (3.3.2) and the above properties, we conclude that (3.1.1)–(3.1.4) has a unique radial nontrivial solution for any $\beta \in [0, \beta_0)$. The nondegeneracy of solution also follows from the above properties. □

## 3.4 Symmetry and monotonicity of nontrivial positive solutions ($N = 1$)

In this section, we consider the symmetry and monotonicity of nontrivial positive solutions when $N = 1$. We consider the following equations:

$$\begin{cases} u_1'' + f_1(x, u_1(x), u_2(x)) = 0 & \text{in } \mathbf{R}, \\ u_2'' + f_2(x, u_1(x), u_2(x)) = 0 & \text{in } \mathbf{R}, \\ u_1(x), u_2(x) > 0 & \text{in } \mathbf{R}, \\ u_1(x), u_2(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.4.1}$$

Here we assume that $f_i$ satisfies the following conditions:

(f0) $f_i(x, s_1, s_2) \in C^1(\mathbf{R} \times (0, \infty)^2)$ and $f_i(-x, s_1, s_2) = f_i(x, s_1, s_2)$ for any $x \in \mathbf{R}$, $(s_1, s_2) \in (0, \infty)^2$ and $i = 1, 2$.

(f1) $\dfrac{\partial f_i}{\partial x}(x, s_1, s_2) \leq 0$ for all $x \geq 0$, $(s_1, s_2) \in (0, \infty)^2$ and $i = 1, 2$.

(f2) $\dfrac{\partial f_1}{\partial s_2}(x, s_1, s_2), \dfrac{\partial f_2}{\partial s_1}(x, s_1, s_2) \geq 0$ for all $x \in \mathbf{R}$, $(s_1, s_2) \in (0, \infty)^2$ and $i = 1, 2$.

(f3) There exist $R_1 > 0$, $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that if $|x| \geq R_1$ and $s_1^2 + s_2^2 < \varepsilon_1^2$, then

$$\frac{\partial f_i}{\partial s_i}(x, s_1, s_2) < -\delta_1 \text{ for } i = 1, 2, \quad 0 < \frac{\partial f_i}{\partial s_j}(x, s_1, s_2) \leq \delta_1 \text{ for } i \neq j.$$

In this section we will prove the following theorem:

**Theorem 3.4.1.** *Suppose that $f_i$ satisfies* (f0)–(f3). *Then any nontrivial solution of* (3.4.1) *is even and monotone decreasing.*

*Remark* 3.4.2. We can apply Theorem 3.4.1 to (3.1.1)–(3.1.4) with $N = 1$ under (3–V1)–(3–V5).

In [20], they considered the symmetry and monotonicity of nontrivial positive solutions for $N \geq 2$. In the following, we show the argument in [20] still works for $N = 1$ after modification and give an outline of the proof of Theorem 3.4.1.

We use the moving plane method to prove the symmetry and monotonicity of nontrivial positive solutions of (3.4.1). Let $u(x) = (u_1(x), u_2(x))$ be a solution of (3.4.1) and we set

$$x^\lambda := 2\lambda - x,$$
$$U_i^\lambda(x) := u_i(x^\lambda) - u_i(x) = u_i(2\lambda - x) - u_i(x)$$

for $\lambda \geq 0$ and $x \geq \lambda$.

*Outline of the proof of Theorem 3.4.1.*
First, we observe that $U_i^\lambda(x)$ satisfies

$$(U_i^\lambda)'' + \sum_{k=1}^{2} c_{ik}(x) U_k^\lambda \le 0 \text{ in } (\lambda, \infty), \tag{3.4.2}$$

where

$$\begin{aligned} c_{ik}(x) &:= \int_0^1 \frac{\partial f_i}{\partial u_k}(|x| + t(|x^\lambda| - |x|), u(x) + tU^\lambda(x))dt, \\ U^\lambda(x) &:= (U_1^\lambda(x), U_2^\lambda(x)). \end{aligned}$$

These inequalities are corresponding to the inequality (13) in [20] and we can show (3.4.2) in a similar way to [20].

Next we define

$$A := \{\lambda \ge 0 : U_i^\mu \ge 0 \text{ in } (\mu, \infty) \text{ for all } \mu \ge \lambda,\ i = 1, 2\},$$

and a key step of the proof of Theorem 3.4.1 is to show $A \ne \emptyset$.

Here we need to modify the argument in [20]. In [20], they introduced the auxiliary function $\bar{U}^\lambda(x) := U^\lambda(x)/g(x)$, where

$$g(x) := \begin{cases} |x|^{-(N-2)/2} + 1 & \text{if} \quad N \ge 3, \\ \log(\log(|x| + 27)) & \text{if} \quad N = 2. \end{cases}$$

When $N = 1$, $g(x)$ is not given in [20].

Under our assumption (f3), which is stronger than those in [20], we can show there exist $R_2 \ge R_1$ and $\lambda_1 > R_2$ such that

$$\begin{aligned} & u_1^2(x) + u_2^2(x) < \varepsilon_1 \qquad \text{for } |x| \ge R_2, \\ & \max_{[2\lambda - R_2, 2\lambda + R_2]} u_i < \min_{[-R_2, R_2]} u_i \qquad \text{for } \lambda \ge \lambda_1, \\ & c_{11}(x), c_{22}(x) < -\delta_1, \quad 0 < c_{12}(x), c_{21}(x) \le \delta_1 \\ & \qquad \text{for } \lambda \ge \lambda_1 \text{ and } x, x^\lambda \notin [-R_2, R_2]. \end{aligned} \tag{3.4.3}$$

From (3.4.3), we have

$$\det \begin{pmatrix} c_{11}(x) & c_{12}(x) \\ c_{21}(y) & c_{22}(y) \end{pmatrix} > 0 \qquad \text{for } \lambda \ge \lambda_1 \text{ and } x, x^\lambda, y, y^\lambda \notin [-R_2, R_2]. \tag{3.4.4}$$

We will see that (3.4.4) enables us to show $A \ne \emptyset$ without introducing $\bar{U}^\lambda(x)$.

Indeed, we show $\lambda_1 \in A$ to prove $A \ne \emptyset$. We show indirectly and assume $\lambda_1 \notin A$. Then there exist a $\lambda_2 \ge \lambda_1$ and $x_1, x_2 \in [\lambda_2, \infty)$ and one of the following three cases takes place:

$$\begin{aligned} &\text{(i) } U_1^{\lambda_2}(x_1) = \min_{[\lambda_2, \infty)} U_1^{\lambda_2} < 0, \quad \min_{[\lambda_2, \infty)} U_2^{\lambda_2} = 0, \\ &\text{(ii) } \min_{[\lambda_2, \infty)} U_1^{\lambda_2} = 0, \quad U_2^{\lambda_2}(x_2) = \min_{[\lambda_2, \infty)} U_2^{\lambda_2} < 0, \\ &\text{(iii) } U_1^{\lambda_2}(x_1) = \min_{[\lambda_2, \infty)} U_1^{\lambda_2} < 0, \quad U_2^{\lambda_2}(x_2) = \min_{[\lambda_2, \infty)} U_2^{\lambda_2} < 0. \end{aligned}$$

We consider the case (i). Since $x_1$ is a minimum point of $U_1^{\lambda_2}$ in $(\lambda_2, \infty)$ and the inequality (3.4.2), $U_2^{\lambda_2}(x_1)c_{12}(x_1) \geq 0$ hold, we have

$$c_{11}(x_1)U_1^{\lambda_2}(x_1) \leq 0. \tag{3.4.5}$$

On the other hand, by (3.4.3), $U_1^{\lambda_2}(x_1) < 0$ implies that $x_1, x_1^{\lambda_2} \notin [-R_2, R_2]$ and $c_{11}(x_1) < -\delta_1$, which contradicts (3.4.5). Thus the case (i) never occurs. Similarly, the case (ii) cannot occur.

In the case (iii), Since $x_i$ is a minimum point of $U_i^{\lambda_2}$, it follows

$$\begin{pmatrix} c_{11}(x_1) & c_{12}(x_1) \\ c_{21}(x_2) & c_{22}(x_2) \end{pmatrix} \begin{pmatrix} U_1^{\lambda_2}(x_1) \\ U_2^{\lambda_2}(x_2) \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \qquad \text{where } \xi_1, \xi_2 \leq 0.$$

By (3.4.3), we have

$$x_1, x_2, x_1^{\lambda_2}, x_2^{\lambda_2} \notin [-R_2, R_2], \quad c_{ii}(x_i) < -\delta_1, \quad 0 \leq c_{ij}(x_i) < \delta_1 \quad (j \neq i). \tag{3.4.6}$$

By (3.4.4) and (3.4.6), we have

$$U_1^{\lambda_2}(x_1) = \frac{c_{22}(x_2)\xi_1 - c_{12}(x_1)\xi_2}{c_{11}(x_1)c_{22}(x_2) - c_{12}(x_1)c_{21}(x_2)} \geq 0.$$

However, this is a contradiction. Thus $\lambda_1 \in A$.

Therefore $A \neq \emptyset$ and $\lambda_* = \inf A$ is well–defined. Using $U^{\lambda_*}(x)$, we can show the symmetry and monotonicity of $u(x)$ along the argument in [20] after slight modification. □

# Chapter 4

# Existence of concentration solutions

## 4.1 Introduction

In this chapter we consider a singular perturbation problem for (CNLS), namely

$$-\varepsilon^2\Delta v_1 + V_1(x)v_1 = \mu_1 v_1^3 + \beta v_1 v_2^2 \quad \text{in } \mathbf{R}^N, \tag{4.1.1}$$

$$-\varepsilon^2\Delta v_2 + V_2(x)v_2 = \beta v_1^2 v_2 + \mu_2 v_2^3 \quad \text{in } \mathbf{R}^N, \tag{4.1.2}$$

$$v_1(x),\ v_2(x) > 0 \quad \text{in } \mathbf{R}^N, \tag{4.1.3}$$

$$v_1(x),\ v_2(x) \in H^1(\mathbf{R}^N), \tag{4.1.4}$$

where $N = 2, 3$, $\mu_1,\ \mu_2 > 0$, $\beta \in \mathbf{R}$ are constants, $V_1(x)$, $V_2(x) : \mathbf{R}^N \to \mathbf{R}$ are bounded continuous positive functions, and $\varepsilon > 0$ is a small perturbation parameter.

One of the difficulties in the study of (4.1.1)–(4.1.4) is that it has semitrivial solutions of type $(v_1(x), 0)$ or $(0, v_2(x))$, where $v_1(x)$ or $v_2(x)$ solves

$$-\varepsilon^2\Delta v_i + V_i(x)v_i = \mu_i v_i^3 \quad \text{in } \mathbf{R}^N.$$

We call solutions $(v_1(x), v_2(x))$ with $v_1(x) \not\equiv 0$ and $v_2(x) \not\equiv 0$ by *nontrivial solutions.*

(4.1.1)–(4.1.4) is studied in Lin and Wei [68], Pomponio [87], Montefusco, Pellacci and Squassina [79] and G. Wei [105], [106]. In [68], Lin and Wei studied (4.1.1)–(4.1.4) by analyzing least energy nontrivial solutions. They studied both of attractive interaction (i.e., $\beta > 0$) and repulsive interaction (i.e., $\beta < 0$). Especially, when $\beta > 0$, they showed the existence of a least energy nontrivial solution for a small $\varepsilon > 0$ under suitable conditions on the behavior of $V_1(x)$ and $V_2(x)$ as $|x| \to \infty$. Moreover they showed that if

$$\inf_{P\in\mathbf{R}^N} m(P) < \inf_{P_1\in\mathbf{R}^N} e_1(P_1) + \inf_{P_2\in\mathbf{R}^N} e_2(P_2)$$

(see (4.1.12) and (4.1.18) for notation), then both components of the least energy nontrivial solution $(v_{\varepsilon 1}(x), v_{\varepsilon 2}(x))$ concentrate to the some point $P_0$ satisfying $m(P_0) = \inf_{P\in\mathbf{R}^N} m(P)$ as $\varepsilon \to 0$ after taking a subsequence. See Remark 4.1.6 below. We also refer to Lin and Wei [66] for study of a singularly perturbed system of nonlinear Schrödinger equations in a bounded domain.

In [79], Montefusco, Pellacci and Squassina studied the case $\beta > 0$. They consider concentration of solutions around the local minimum (possibly degenerate) point of the potentials; they assume that $z \in \mathbf{R}^N$ and $r > 0$ satisfy

$$\min_{|x-z|\le r} V_i(x) < \min_{|x-z|=r} V_i(x) \quad (i = 1, 2),$$

and they showed for small $\varepsilon > 0$ (4.1.1)–(4.1.4) has a nonzero solution $(v_{\varepsilon 1}(x), v_{\varepsilon 2}(x))$ such that $v_{\varepsilon 1}(x) + v_{\varepsilon 2}(x)$ has exactly one global maximum point in $\{x; |x - z| < r\}$. However, when $\beta > 0$ is small, one component of $(v_{\varepsilon 1}(x), v_{\varepsilon 2}(x))$ converges to 0 as $\varepsilon \to 0$ (see Theorem 2.1 (ii) in [79]). We also refer to [68, 87, 105] for the study of (4.1.1)–(4.1.4) when $\beta < 0$.

We consider the case where the interaction parameter $\beta$ is positive and the aim in this chapter is to construct a family of solutions of (4.1.1)–(4.1.4) which concentrates to a nontrivial positive solution.

In the study of (4.1.1)–(4.1.4), the following constant coefficient problem plays an important role:

$$-\Delta u_1 + V_1 u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 \quad \text{in } \mathbf{R}^N, \tag{4.1.5}$$
$$-\Delta u_2 + V_2 u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 \quad \text{in } \mathbf{R}^N, \tag{4.1.6}$$
$$u_1(x),\ u_2(x) > 0 \quad \text{in } \mathbf{R}^N, \tag{4.1.7}$$
$$u_1(x),\ u_2(x) \in H^1(\mathbf{R}^N), \tag{4.1.8}$$

where $V_1$, $V_2 > 0$ are positive constants. We remark that (4.1.5)–(4.1.8) appears as a limit problem after a suitable rescaling. There are many works on the existence of nontrivial positive solutions of (4.1.5)–(4.1.8). See [4, 10, 12, 13, 29, 31, 47, 49, 50, 65, 75, 94, 107, 108]. Sign and size of $\beta$ are important in the study of (4.1.5)–(4.1.8) and various situations are studied in the above papers.

Here we consider the case where the interaction $\beta$ is positive and relatively small and treat the existence of a nontrivial radially symmetric positive solution, which is characterized as a critical point of

(4.1.9)
$$J_{(V_1,V_2)}(u_1, u_2) = \int_{\mathbf{R}^N} \frac{1}{2}(|\nabla u_1|^2 + V_1 u_1^2 + |\nabla u_2|^2 + V_2 u_2^2) - \frac{1}{4}(\mu_1 u_1^4 + 2\beta u_1^2 u_2^2 + \mu_2 u_2^4)\, dx$$
$$: H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N) \to \mathbf{R}.$$

Here $H_r^1(\mathbf{R}^N) := \{u \in H^1(\mathbf{R}^N) : u(x) = u(|x|)\}$. We assume

$$0 < \beta < \sqrt{\mu_1 \mu_2} \tag{4.1.10}$$

and

**Condition (AC)** Let $\hat\omega_i(x) \in H_r^1(\mathbf{R}^N)$ be the least energy solution of $-\Delta\hat\omega_i + V_i\hat\omega_i = \mu_i\hat\omega_i^3$, in $\mathbf{R}^N$. Then the operators $-\Delta + V_1 - \beta\hat\omega_2^2$, $-\Delta + V_2 - \beta\hat\omega_1^2$ are positive definite on $H_r^1(\mathbf{R}^N)$, that is,

$$\int_{\mathbf{R}^N} |\nabla\varphi|^2 + (V_1 - \beta\hat\omega_2^2)\varphi^2\, dx,\ \int_{\mathbf{R}^N} |\nabla\varphi|^2 + (V_2 - \beta\hat\omega_1^2)\varphi^2\, dx > 0$$

for all $\varphi(x) \in H_r^1(\mathbf{R}^N) \setminus \{0\}$.

We will show the following existence result in Section 4.2.

**Proposition 4.1.1.** *Assume* (4.1.10) *and* (AC). *Then* (4.1.5)–(4.1.8) *has a nontrivial radially symmetric positive solution* $(u_{01}, u_{02}) \in H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N)$, *which can be characterized as*

$$J_{(V_1,V_2)}(u_{01}, u_{02}) = b(V_1, V_2), \tag{4.1.11}$$

*where*

$$\begin{aligned} b(V_1, V_2) &:= \inf_{(u_1,u_2)\in\mathcal{M}_r(V_1,V_2)} J_{(V_1,V_2)}(u_1, u_2), \\ \mathcal{M}_r(V_1, V_2) &:= \{(u_1, u_2) \in H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N) : u_1 \neq 0,\ u_2 \neq 0, \\ &\qquad J'_{(V_1,V_2)}(u_1, u_2)(u_1, 0) = 0,\ J'_{(V_1,V_2)}(u_1, u_2)(0, u_2) = 0\}. \end{aligned}$$

*Moreover, suppose that there exists a set* $A \subset (0,\infty) \times (0,\infty)$ *such that (AC) holds for all* $(V_1, V_2) \in A$. *Then*

(i) $(V_1, V_2) \mapsto b(V_1, V_2);\ A \to \mathbf{R}$ *is continuous.*

(ii) $V_1 \mapsto b(V_1, V_2)$ (*resp.* $V_2 \mapsto b(V_1, V_2)$) *is strictly increasing for a fixed* $V_2$ (*resp.* $V_1$).

*Remark* 4.1.2. (i) In [20], Busca and Sirakov showed that when $\beta > 0$, any nontrivial positive solution of (4.1.5)–(4.1.8) is radially symmetric with respect to some point $P_0 \in \mathbf{R}^N$.
(ii) Condition (AC) is introduced in Ambrosetti and Colorado [4] and they showed the existence of nontrivial positive solutions. We give another proof of their existence result as well as the characterization (4.1.11) and some additional compactness properties. We also refer to Lin and Wei [65], Sirakov [94] for the existence of least energy nontrivial solutions.

For (4.1.1)–(4.1.4), we assume the following

**Assumption (4–A1)** There exists a set $A = [a_{10}, a_{11}] \times [a_{20}, a_{21}] \subset (0,\infty) \times (0,\infty)$ with the following properties:

(i) For any $(V_1, V_2) \in A$, the constant coefficient problem (4.1.5)–(4.1.6) satisfies the condition (AC).

(ii) $(V_1(P), V_2(P)) \in A$ for all $P \in \mathbf{R}^N$.

We set

$$m(P) := b(V_1(P), V_2(P)) :\ \mathbf{R}^N \to \mathbf{R}. \tag{4.1.12}$$

As the second assumption, we assume

**Assumption (4–A2)** There exists a bounded open set $\Lambda \subset \mathbf{R}^N$ such that

$$\inf_{P\in\Lambda} m(P) < \inf_{P\in\partial\Lambda} m(P). \tag{4.1.13}$$

We set

$$m_0 := \inf_{P\in\Lambda} m(P),$$
$$K := \{P \in \Lambda : m(P) = m_0\}.$$

Now we can state our main result.

**Theorem 4.1.3.** *Suppose that* (4.1.10) *and Assumptions* (4–A1), (4–A2) *hold. Then there exists an* $\varepsilon_0 > 0$ *such that* (4.1.1)–(4.1.4) *has a family of nontrivial positive solutions* $(v_{1\varepsilon}(x), v_{2\varepsilon}(x))_{0<\varepsilon<\varepsilon_0}$ *satisfying the following properties: after taking a subsequence* $\varepsilon_j \to 0$ *there exists a sequence* $(P_{\varepsilon_j}) \subset \Lambda$ *such that*

$$P_{\varepsilon_j} \to P_0 \in K, \tag{4.1.14}$$

$$(v_{1\varepsilon_j}(\varepsilon_j x + P_{\varepsilon_j}), v_{2\varepsilon_j}(\varepsilon_j x + P_{\varepsilon_j})) \to (w_1(x), w_2(x)) \quad \textit{strongly in } H^1(\mathbf{R}^N)\times H^1(\mathbf{R}^N). \tag{4.1.15}$$

*Here* $(w_1(x), w_2(x))$ *is a nontrivial positive solution of the limit problem:*

$$\begin{cases} -\Delta w_1 + V_1(P_0)w_1 = \mu_1 w_1^3 + \beta w_1 w_2^2 & \text{in } \mathbf{R}^N, \\ -\Delta w_2 + V_2(P_0)w_2 = \beta w_1^2 w_2 + \mu_2 w_2^3 & \text{in } \mathbf{R}^N \end{cases}$$

*and it satisfies* $J_{(V_1(P_0),V_2(P_0))}(w_1, w_2) = m_0$.

*Remark* 4.1.4. If we assume $V_1(x)$, $V_2(x) \in C^1(\mathbf{R}^N)$ in addition to the assumptions of Theorem 4.1.3, we have

$$K \subset \{P \in \Lambda;\ \lambda_1 \nabla V_1(P) + \lambda_2 \nabla V_2(P) = 0 \quad \text{for some } \lambda_1, \lambda_2 > 0\}. \tag{4.1.16}$$

See Lemma 4.2.9 in section 4.2 for a proof of (4.1.16).

*Remark* 4.1.5. (i) We remark that $b(V_1, V_2)$ and $m(P)$ also depend on $\beta$. We write dependence on $\beta$ explicitly and use notation $b_\beta(V_1, V_2)$, $m_\beta(P)$ in this remark. We also remark that if (4.1.10) and (AC), (4–A1) hold for $\beta = \beta_0 > 0$, then they also hold for all $\beta \in (0, \beta_0]$. Concerning a behavior of $b_\beta(V_1, V_2)$ as $\beta \to 0$, we have

$$b_\beta(V_1, V_2) \to e_1(V_1) + e_2(V_2) \quad \text{as } \beta \to 0, \tag{4.1.17}$$

where

$$e_i(V_i) = \frac{V_i^{(4-N)/2}}{\mu_i} e_0$$

is the least energy level for $-\Delta u + V_i u = \mu_i u^3$. Here $e_0 > 0$ is the least energy level for $-\Delta \omega + \omega = \omega^3$, that is, $e_0 = \int_{\mathbf{R}^N} \frac{1}{2}(|\nabla \omega_0|^2 + \omega_0^2) - \frac{1}{4}\omega_0^4\, dx$ where $\omega_0(x)$ is the unique radial positive solution of $-\Delta\omega + \omega = \omega^3$. See Lemma 4.2.10 in section 4.2 for a proof of (4.1.17).

In particular, from Proposition 4.1.1, (4–A1) and Remark 4.2.11, we have

$$m_\beta(P) \to e_1(P) + e_2(P) \quad \text{in } C_{\mathrm{loc}}(\mathbf{R}^N) \quad \text{as } \beta \to 0,$$

where

$$e_i(P) = e_i(V_i(P)) = \frac{V_i(P)^{(4-N)/2}}{\mu_i} e_0. \tag{4.1.18}$$

Thus if a bounded open set $\Lambda \subset \mathbf{R}^N$ satisfies

$$\inf_{P \in \Lambda} (e_1(P) + e_2(P)) < \inf_{P \in \partial\Lambda} (e_1(P) + e_2(P)), \tag{4.1.19}$$

then (4–A2) also holds for $\beta > 0$ small.
(ii) We can easily construct an example of $V_1(x)$, $V_2(x)$ and $\Lambda$, where $V_i(x)$ $(i = 1, 2)$ has no critical points in $\Lambda$ but (4.1.19) holds. Thus Theorem 4.1.3 can be applied for $\beta > 0$ small to find a concentrating family of solutions in $\Lambda$.
(iii) When $\beta = 0$, there is no interaction between two equations. Then concentration points must be critical points of $V_i(x)$'s. See Wang [103] and Wang and Zeng [104]. Thus, even if a bounded open set $\Lambda \subset \mathbf{R}^N$ satisfies (4.1.19), there does not exist a family of concentrating solutions in $\Lambda$ in general and thus the positivity of $\beta$ is necessary to find a concentrating solution in $\Lambda$.

*Remark* 4.1.6. Under suitable conditions on the behavior of $V_1(x)$ and $V_2(x)$ as $|x| \to \infty$, Lin and Wei [68] showed the existence of a least energy nontrivial solution $U_\varepsilon(x) = (u_{1\varepsilon}(x), u_{2\varepsilon}(x))$ which can be characterized as

$$I_\varepsilon(U_\varepsilon) = \inf_{(u_1,u_2) \in \mathcal{M}_\varepsilon} I_\varepsilon(u_1, u_2).$$

Here $I_\varepsilon(U)$ is a functional and $\mathcal{M}_\varepsilon$ is a Nehari type manifold corresponding to (4.1.1)–(4.1.4). See section 4.3 (especially (4.3.5) and (4.3.8)). They showed

$$I_\varepsilon(U_\varepsilon) \to \min \left\{ \inf_{P \in \mathbf{R}^N} m(P),\ \inf_{P_1 \in \mathbf{R}^N} e_1(P_1) + \inf_{P_2 \in \mathbf{R}^N} e_2(P_2) \right\}$$

and if

$$\inf_{P \in \mathbf{R}^N} m(P) > \inf_{P_1 \in \mathbf{R}^N} e_1(P_1) + \inf_{P_2 \in \mathbf{R}^N} e_2(P_2),$$

then, after taking a subsequence $u_{1\varepsilon}(x)$ and $u_{2\varepsilon}(x)$ concentrate to different points $Q_1$ and $Q_2$ in general. Here $Q_i$ satisfies $V_i(Q_i) = \inf_{P \in \mathbf{R}^N} V_i(P)$. Thus, even if $P_0 \in \mathbf{R}^N$ is a global minimizer of $m(P)$, i.e., $m(P_0) = \inf_{P \in \mathbf{R}^N} m(P)$, the minimizer $U_\varepsilon(x) = (u_{1\varepsilon}(x), u_{2\varepsilon}(x))$ of $I_\varepsilon$ in $\mathcal{M}_\varepsilon$ does not have the desired behavior (4.1.14)–(4.1.15) in general. We remark that in a singular perturbation problem for a nonlinear Schrödinger equation: $-\varepsilon^2 \Delta u + V(x)u = u^3$ in $\mathbf{R}^N$, the situation is simpler and we can find a family of solutions concentrating to a global minimum of $V(x)$ via global minimization of the functional on the Nehari manifold $\mathcal{N} := \{u \in H^1(\mathbf{R}^N) \setminus \{0\} : \int_{\mathbf{R}^N} |\nabla u|^2 + V(\varepsilon x)u^2\, dx = \int_{\mathbf{R}^N} |u|^4\, dx\}$.

The following sections are devoted to proofs of our Proposition 4.1.1 and Theorem 4.1.3. As stated in Remark 4.1.6, one of the difficulties in proving Theorem 4.1.3 is that the global minimization method on the Nehari type manifold $\mathcal{M}_\varepsilon$ does not work even for a global minimizer $P_0$ of $m(P)$. Another difficulty is that uniqueness and nondegeneracy of solutions of the limit equation (4.1.5)–(4.1.8) are not known and thus classical Liapunov-Schmidt reduction approach seems to be difficult to apply in our setting. To overcome these difficulties, we use an idea from Byeon and Jeanjean [22, 23] (c.f. [24]). In [22, 23], Byeon and Jeanjean developed a new variational approach to find a localized positive solution for nonlinear Schrödinger equations with a wide class of nonlinearities. We also refer to [2, 32, 33, 34, 38, 42, 56, 63, 93, 103, 104] and references therein for preceding results on nonlinear Schrödinger equations. Here, adapting the idea in [22, 23] on the Nehari type manifold $\mathcal{M}_\varepsilon$ and developing new estimates, we find a subset of $\mathcal{M}_\varepsilon$ in which the corresponding functional has a local minimizer with the desired property (4.1.14)–(4.1.15).

## 4.2 Constant coefficient problems

In this section we study the existence of a nontrivial radially symmetric positive solution of (4.1.5)–(4.1.8).

The main purpose of this section is to give a proof of Proposition 4.1.1 as well as additional compactness properties.

### 4.2.1 Preliminaries

In the following sections, we denote for $D \subset \mathbf{R}^N$

$$\begin{aligned} \|u\|_{L^p(D)} &:= \left(\int_D |u|^p\,dx\right)^{1/p} && \text{for } u \in L^p(D),\\ \|u\|_{L^\infty(D)} &:= \operatorname{ess\,sup}_{x\in D}|u(x)| && \text{for } u \in L^\infty(D),\\ \|u\|_{H^1(D)} &:= \sqrt{\|\nabla u\|_{L^2(D)}^2 + \|u\|_{L^2(D)}^2} && \text{for } u \in H^1(D). \end{aligned}$$

When $D = \mathbf{R}^N$, we also use abbreviation: $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbf{R}^N)}$ $(p \in [1,\infty])$, $\|\cdot\|_{H^1} = \|\cdot\|_{H^1(\mathbf{R}^N)}$. We also write

$$(u,v)_2 := \int_{\mathbf{R}^N} uv\,dx \qquad \text{for } u, v \in L^2(\mathbf{R}^N).$$

For $u_1$, $u_2 \in H^1(\mathbf{R}^N)$ we write $U = (u_1, u_2)$ and

$$\|U\|_{H^1}^2 := \|u_1\|_{H^1}^2 + \|u_2\|_{H^1}^2.$$

For $(V_1, V_2) \in (0,\infty)^2$ and $U \in H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N)$, we define $J_{(V_1,V_2)}(U)$ as in (4.1.9), that is,

$$J_{(V_1,V_2)}(U) := \frac{1}{2}\|U\|_{H^1,(V_1,V_2)}^2 - \int_{\mathbf{R}^N} W(U)\,dx,$$

where

$$\begin{aligned} \|U\|_{H^1,(V_1,V_2)}^2 &:= \|\nabla u_1\|_2^2 + V_1\|u_1\|_2^2 + \|\nabla u_2\|_2^2 + V_2\|u_2\|_2^2,\\ W(\xi_1,\xi_2) &:= \frac{1}{4}(\mu_1\xi_1^4 + 2\beta\xi_1^2\xi_2^2 + \mu_2\xi_2^4). \end{aligned}$$

We can easily see that $J_{(V_1,V_2)} \in C^2(H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N), \mathbf{R})$ and

$$
\begin{aligned}
& J'_{(V_1,V_2)}(u_1,u_2)(h_1,h_2) \\
= \ & (\nabla u_1, \nabla h_1)_2 + V_1(u_1,h_1)_2 + (\nabla u_2, \nabla h_2)_2 + V_2(u_2,h_2)_2 \\
& \qquad - \int_{\mathbf{R}^N} \nabla W(u_1,u_2)(h_1,h_2)\, dx \\
= \ & \int_{\mathbf{R}^N} \nabla u_1 \nabla h_1 + V_1 u_1 h_1 + \nabla u_2 \nabla h_2 + V_2 u_2 h_2 \\
& \qquad -\mu_1 u_1^3 h_1 - \beta u_1 u_2^2 h_1 - \mu_2 u_2^3 h_2 - \beta u_1^2 u_2 h_2 \, dx
\end{aligned}
$$

for all $(u_1,u_2)$, $(h_1,h_2) \in H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N)$. Thus critical points of $J_{(V_1,V_2)}$ are radially symmetric solutions of (4.1.5)–(4.1.8).

We have the following lemma.

**Lemma 4.2.1.** $J_{(V_1,V_2)} \in C^2(H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N), \mathbf{R})$ *satisfies the Palais-Smale compactness condition.*

*Proof.* Lemma 4.2.1 follows from the compactness of the embedding $H_r^1(\mathbf{R}^N) \subset L^4(\mathbf{R}^N)$ and the fact that $\nabla W(U)U = 4W(U)$ in a rather standard way. □

One of the difficulties in the studying (4.1.5)–(4.1.8) is to distinguish nontrivial solutions from semitrivial solutions. We remark that (4.1.5)–(4.1.8) has 2 semitrivial solutions:

$$
\begin{aligned}
\hat{\Omega}_1(V_1;x) &:= (\hat{\omega}_1(V_1;x), 0) = \left( \sqrt{\frac{V_1}{\mu_1}}\omega_0(\sqrt{V_1}x), 0 \right), \\
\hat{\Omega}_2(V_2;x) &:= (0, \hat{\omega}_2(V_2;x)) = \left( 0, \sqrt{\frac{V_2}{\mu_2}}\omega_0(\sqrt{V_2}x) \right),
\end{aligned}
$$

where $\omega_0(x)$ is the unique radially symmetric positive symmetric solution of $-\Delta u + u = u^3$ in $H_r^1(\mathbf{R}^N)$. We note that for $i = 1, 2$

$$
J_{(V_1,V_2)}(\hat{\Omega}_i) = e_i(V_i) = \frac{V_i^{(4-N)/2}}{\mu_i} e_0,
$$

where $e_0 = \frac{1}{2}\|\omega_0\|_{H^1}^2 - \frac{1}{4}\|\omega_0\|_4^4$.

### 4.2.2 Ambrosetti and Colorado's condition

In [4], Ambrosetti and Colorado introduced the condition (AC) and they showed the existence of nontrivial positive solutions through a mountain pass argument on the Nehari manifold:

$$
\mathcal{N}_r(V_1,V_2) := \{U \in H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N) : U \neq (0,0),\ J'_{(V_1,V_2)}(U)U = 0\}.
$$

We give another proof to their existence result. Since estimates, which is uniform in $(V_1,V_2)$, are important for the study of the singular perturbation problem, we assume (4.1.10) and the following condition:

(C1) There exists a set $A = [a_{10}, a_{11}] \times [a_{20}, a_{21}] \subset (0, \infty)^2$ such that (AC) holds for $(V_1, V_2) \in A$.

The condition (AC) ensures the positivity of the following bilinear forms:

$$
\begin{aligned}
J''_{(V_1,V_2)}(\hat{\Omega}_2)(h_1,0)(h_2,0) &= (\nabla h_1, \nabla h_2)_2 + V_1(h_1,h_2)_2 - \beta \int_{\mathbf{R}^N} \hat{\omega}_2^2 h_1 h_2 \, dx \\
&\quad : (H_r^1(\mathbf{R}^N) \times \{0\})^2 \to \mathbf{R}, \\
J''_{(V_1,V_2)}(\hat{\Omega}_1)(0,h_1)(0,h_2) &= (\nabla h_1, \nabla h_2)_2 + V_2(h_1,h_2)_2 - \beta \int_{\mathbf{R}^N} \hat{\omega}_1^2 h_1 h_2 \, dx \\
&\quad : (\{0\} \times H_r^1(\mathbf{R}^N))^2 \to \mathbf{R}.
\end{aligned} \tag{4.2.1}
$$

It is easily seen that $\mathcal{N}_r(V_1, V_2)$ is a Hilbert manifold with codimension 1 and the critical points of the constraint functional $J_{(V_1,V_2)}|_{\mathcal{N}_r(V_1,V_2)} : \mathcal{N}_r(V_1, V_2) \to \mathbf{R}$ are nonzero critical points of $J_{(V_1,V_2)}$.

Under the condition (C1) we have

**Lemma 4.2.2.** *Assume* (C1). *Then there exist* $\rho_0 > 0$ *and* $0 < r_0 < \min\{\|\hat{\Omega}_1(V_1;x)\|_{H^1,(V_1,V_2)}, \|\hat{\Omega}_2(V_2;x)\|_{H^1,(V_1,V_2)} : (V_1,V_2) \in A\}$ *such that for each* $(V_1, V_2) \in A$*, the following hold*:

(i) $J_{(V_1,V_2)}(U) \geq e_1(V_1) + \rho_0$ *for all* $U \in \mathcal{N}_r(V_1,V_2) \cap \{\|U - \hat{\Omega}_1\|_{H^1,(V_1,V_2)} = r_0\}$.

(ii) $J_{(V_1,V_2)}(U) \geq e_2(V_2) + \rho_0$ *for all* $U \in \mathcal{N}_r(V_1,V_2) \cap \{\|U - \hat{\Omega}_2\|_{H^1,(V_1,V_2)} = r_0\}$.

*Proof.* We prove just (i). (ii) can be shown in a similar way. We set

$$
\begin{aligned}
\mathcal{N}_{1,r}(V_1) :=& \{u_1 \in H_r^1(\mathbf{R}^N) \setminus \{0\} : J'_{(V_1,V_2)}(u_1,0)(u_1,0) = 0\} \\
=& \{u_1 \in H_r^1(\mathbf{R}^N) \setminus \{0\} : \|\nabla u_1\|_2^2 + V_1\|u_1\|_2^2 = \mu_1\|u_1\|_4^4\},
\end{aligned}
$$

which is the Nehari manifold for the scalar equation: $-\Delta u + V_1 u = \mu_1 u^3$. We remark that $\mathcal{N}_{1,r}(V_1) \times \{0\} \subset \mathcal{N}_r(V_1, V_2)$ and

$$
J_{(V_1,V_2)}(\hat{\Omega}_1) = e_1(V_1) = \inf_{u_1 \in \mathcal{N}_{1,r}(V_1)} J_{(V_1,V_2)}(u_1, 0).
$$

Since $\hat{\omega}(V_1;x)$ is a nondegenerate critical point of $J_{(V_1,V_2)}|_{\mathcal{N}_{1,r}(V_1)\times\{0\}}$, the conclusion (i) follows from the positivity of the bilinear form (4.2.1). □

Now we introduce our minimax method to find a nontrivial solution. We set

$$
\begin{aligned}
\Gamma := \{\gamma(s,t) \in C([0,\infty)^2, H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N)) :& \\
\text{For some } u_{01}(x), u_{02}(x) \geq 0 \, (\not\equiv 0) \text{ and } R > 0,\ \gamma(s,t) \text{ satisfies}& \\
\gamma(s,0) = (\sqrt{s}\, u_{01}, 0) \quad \text{for } s \geq 0,& \\
\gamma(0,t) = (0, \sqrt{t}\, u_{02}) \quad \text{for } t \geq 0,& \\
\gamma(s,t) = (\sqrt{s}\, u_{01}, \sqrt{t}\, u_{02}) \quad \text{for } s^2 + t^2 \geq R^2 \ \}&
\end{aligned}
$$

and

$$(4.2.2) \qquad b(V_1,V_2) := \inf_{\gamma\in\Gamma} \sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2)}(\gamma(s,t)).$$

Our main result in this subsection is

**Proposition 4.2.3.** *Suppose* (4.1.10) *and* (C1) *hold. Then for any* $(V_1,V_2)\in A$,

(i) $\max\{e_1(V_1)+\rho_0,\, e_2(V_2)+\rho_0\} \le b(V_1,V_2) \le e_1(V_1)+e_2(V_2)$.

(ii) *The value* $b(V_1,V_2)$ *is attained by a nontrivial radially symmetric positive solution of* (4.1.5)–(4.1.8).

*Proof.* Setting $\gamma_0(s,t) := (\sqrt{s}\hat\omega_1(V_1;x), \sqrt{t}\hat\omega_2(V_2;x))$ and

$$J_{(V_1,V_2),0}(u_1,u_2) := \frac{1}{2}(\|\nabla u_1\|_2^2 + V_1\|u_1\|_2^2 + \|\nabla u_2\|_2^2 + V_2\|u_2\|_2^2) - \frac{1}{4}(\mu_1\|u_1\|_4^4 + \mu_2\|u_2\|_4^4),$$

we have

$$\begin{aligned} b(V_1,V_2) &\le \sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2)}(\gamma_0(s,t)) \\ &\le \sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2),0}(\gamma_0(s,t)) = e_1(V_1)+e_2(V_2). \end{aligned}$$

Thus we obtain $b(V_1,V_2)\le e_1(V_1)+e_2(V_2)$. To prove the remaining parts, we set

$$\begin{aligned} \widetilde\Gamma(V_1,V_2) :=& \{\gamma(s,t)\in C([0,\infty)^2, H_r^1(\mathbf{R}^N)\times H_r^1(\mathbf{R}^N));\\ &\text{For some } R>0,\ \gamma(s,t) \text{ satisfies}\\ &\quad \gamma(s,0) = (\sqrt{s}\,\hat\omega_1(V_1;x),0) \quad \text{for } s\ge 0,\\ &\quad \gamma(0,t) = (0,\sqrt{t}\,\hat\omega_2(V_2;x)) \quad \text{for } t\ge 0,\\ &\quad \gamma(s,t) = (\sqrt{s}\,\hat\omega_1(V_1;x),\sqrt{t}\,\hat\omega_2(V_2;x)) \quad \text{for } s^2+t^2\ge R^2\ \}. \end{aligned}$$

It is clear that $\widetilde\Gamma(V_1,V_2)\subset\Gamma$. First we show

**Step 1:** $b(V_1,V_2) = \inf_{\gamma\in\widetilde\Gamma(V_1,V_2)} \sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2)}(\gamma(s,t))$.

It suffices to show that for any $\gamma\in\Gamma$ there exists $\widetilde{\widetilde\gamma}\in\widetilde\Gamma(V_1,V_2)$ such that

$$\sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2)}(\widetilde{\widetilde\gamma}(s,t)) = \sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2)}(\gamma(s,t)).$$

Let $\gamma(s,t)\in\Gamma$ be a given path and suppose $\gamma(s,t) = (\sqrt{s}\,u_{01},\sqrt{t}\,u_{02})$ for $(s,t)\in(\mathbf{R}\times\{0\})\cup(\{0\}\times\mathbf{R})\cup\{(s,t);\ s^2+t^2\ge R^2\}$. Moreover, we may assume $u_{01}\in\mathcal{N}_{1,r}(V_1)$ and $u_{02}\in\mathcal{N}_{2,r}(V_2)$ without loss of generality. We remark that $\{\sqrt{s}\,u_{01};\ s\in[0,\infty)\}$ and $\mathcal{N}_{1,r}(V_1)$ intersects at exactly one point $u_{01}$. Take a curve $\zeta_1(\tau)\in C([0,1],\mathcal{N}_{1,r}(V_1))$ such that

$$\begin{aligned} &\zeta_1(0)=\hat\omega_1,\ \zeta_1(1)=u_{01},\ \zeta_1(\tau)(x)\ge 0 \quad \text{for all } \tau\in[0,1],\\ &\tau\mapsto J_{(V_1,V_2)}(\zeta_1(\tau),0) \text{ is nondecreasing.} \end{aligned}$$

Such a curve exists since $J_{(V_1,V_2)}(\hat{\omega}_1, 0) = \inf_{u\in\mathcal{N}_{1,r}(V_1)} J_{(V_1,V_2)}(u, 0)$ and there are no other nonnegative critical points on $\mathcal{N}_{1,r}(V_1)$ other than $\hat{\omega}_1$.

Similarly, we take a curve $\zeta_2(\tau) \in C([0, 1], \mathcal{N}_{2,r}(V_2))$ joining $\hat{\omega}_2(V_2; x)$ and $u_{02}$. We set

$$\tilde{\gamma}(s,t) := \begin{cases} \gamma(s-1, t-1) & \text{if } s \geq 1 \text{ and } t \geq 1, \\ \big(\sqrt{s-t}\,\zeta_1(t), 0\big) & \text{if } t \in [0,1) \text{ and } t \leq s, \\ \big(0, \sqrt{t-s}\,\zeta_2(s)\big) & \text{if } s \in [0,1) \text{ and } s < t. \end{cases}$$

Then we can see that

$$\sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2)}(\tilde{\gamma}(s,t)) = \sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2)}(\gamma(s,t)).$$

For sufficiently large $\tilde{R} > R$, modifying $\tilde{\gamma}(s,t)$ in a suitable way, we can get $\tilde{\tilde{\gamma}}(s,t) \in \widetilde{\Gamma}(V_1, V_2)$ with the desired properties.

Next we show

**Step 2:** *For any $\gamma \in \widetilde{\Gamma}(V_1, V_2)$,*

$$\gamma([0,\infty)^2) \cap \mathcal{N}_r(V_1, V_2) \cap \{\|U - \hat{\Omega}_1(V_1)\|_{H^1,(V_1,V_2)} = r_0\} \neq \emptyset.$$

*where $r_0 > 0$ is given in Lemma 4.2.2.*

For a given $\gamma(s,t) \in \widetilde{\Gamma}(V_1, V_2)$ we set

$$F(s,t) := (J'_{(V_1,V_2)}(\gamma(s,t))\gamma(s,t), \|\gamma(s,t) - \hat{\Omega}_1(V_1)\|^2_{H^1,(V_1,V_2)} - r_0^2).$$

We compute $\deg(F, [0,R]^2, (0,0))$ for sufficiently large $R > 0$. Set $G(s,t) = \nabla g(s,t)$ where $g(s,t) = -(s-1)^2 + (t-\varepsilon)^2$ for sufficiently small $\varepsilon > 0$ and consider the following homotopy between $F$ and $G$: For $\theta \in [0, 1]$,

$$F_\theta(s,t) = (1-\theta)F(s,t) + \theta G(s,t)$$

For sufficiently large $R > 0$ we can see that $F_\theta(s,t) \neq (0,0)$ for all $(s,t) \in \partial[0,R]^2$ and $\theta \in [0,1]$. Therefore we obtain $\deg(F, [0,R]^2, (0,0)) = \deg(G, [0,R]^2, (0,0)) = -1$. Thus there exists $(s_0, t_0) \in (0,R)^2$ such that $F(s_0,t_0) = (0,0)$. That is, $U_0 = \gamma(s_0,t_0)$ satisfies $U_0 \in \mathcal{N}_r(V_1, V_2)$ and $\|U_0 - \hat{\Omega}_1(V_1)\|_{H^1,(V_1,V_2)} = r_0$.

**Step 3:** $b(V_1, V_2) \geq \max\{e_1(V_1) + \rho_0, e_2(V_2) + \rho_0\}$.

By Step 2 and Lemma 4.2.2 (i), we have

$$\begin{aligned} \sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2)}(\gamma(s,t)) &\geq \inf_{U\in\mathcal{N}_r(V_1,V_2)\cap\{\|U-\hat{\Omega}_1\|_{H^1}=r_0\}} J_{(V_1,V_2)}(U) \\ &\geq e_1(V_1) + \rho_0 \end{aligned}$$

for all $\gamma \in \widetilde{\Gamma}(V_1, V_2)$. Thus by Step 1, we have $b(V_1, V_2) \geq e_1(V_1) + \rho_0$. In a similar way, we can show $b(V_1, V_2) \geq e_2(V_2) + \rho_0$. Therefore we get the conclusion of Step 3.

**Step 4:** *The value $b(V_1, V_2)$ is attained by a nontrivial positive solution of (4.1.5)–(4.1.8).*

Since $\sup_{s\geq 0} J_{(V_1,V_2)}(\gamma(s,0)) \leq e_1(V_1)$, $\sup_{t\geq 0} J_{(V_1,V_2)}(\gamma(0,t)) \leq e_2(V_2)$ for $\gamma \in \widetilde{\Gamma}(V_1, V_2)$, we can see from Step 3 that $\widetilde{\Gamma}(V_1, V_2)$ is stable under deformation. Thus, by Lemma 4.2.1, $b(V_1, V_2) = \inf_{\gamma\in\widetilde{\Gamma}(V_1,V_2)} \sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2)}(\gamma(s,t))$ is achieved.

Next we show that $b(V_1, V_2)$ is attained by a nontrivial positive solution. For $\gamma(s,t) = (\gamma_1(s,t), \gamma_2(s,t))$, we set $|\gamma|(s,t) := (|\gamma_1(s,t)|, |\gamma_2(s,t)|)$. Since $J_{(V_1,V_2)}(\gamma(s,t)) = J_{(V_1,V_2)}(|\gamma|(s,t))$, we can conclude that there exists a nonnegative critical point corresponding to $b(V_1, V_2)$. By Step 3, we can see that the corresponding critical point is a nontrivial positive solution. □

### 4.2.3 Minimizing property

The aim of this subsection is to give characterizations to $b(V_1, V_2)$ using the Nehari type manifolds. In what follows, we define $J_{(V_1,V_2)}(U)$ by (4.1.9) also for $U \in H^1(\mathbf{R}^N)\times H^1(\mathbf{R}^N)$.

We consider 2 type of manifolds:

$$\begin{aligned}\mathcal{M}_r(V_1, V_2) :=\{&U = (u_1, u_2) \in H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N) :\ u_1 \neq 0,\ u_2 \neq 0, \\ &J'_{(V_1,V_2)}(u_1, u_2)(u_1, 0) = 0,\ J'_{(V_1,V_2)}(u_1, u_2)(0, u_2) = 0\}, \\ \mathcal{M}(V_1, V_2) :=\{&U = (u_1, u_2) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N) :\ u_1 \neq 0,\ u_2 \neq 0, \\ &J'_{(V_1,V_2)}(u_1, u_2)(u_1, 0) = 0,\ J'_{(V_1,V_2)}(u_1, u_2)(0, u_2) = 0\}.\end{aligned}$$

Such a type of manifolds were introduced in Lin and Wei [65] and Sirakov [94] and they studied the existence of a minimizer of $J_{(V_1,V_2)}$ on $\mathcal{M}_r(V_1, V_2)$ and $\mathcal{M}(V_1, V_2)$.

We have

**Lemma 4.2.4** (cf. Lin and Wei [65])**.** *Assume* (4.1.10)*. Then we have*

(i) *The set $\mathcal{M}_r(V_1, V_2)$ (resp. $\mathcal{M}(V_1, V_2)$) is a Hilbert submanifold of $H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N)$ (resp. $H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$) with codimension* 2*.*

(ii) *For $(u_1, u_2) \in H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N)$ (resp. $H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$) with $u_1 \not\equiv 0$, $u_2 \not\equiv 0$ and $s$, $t > 0$,*

$$(\sqrt{s}\, u_1, \sqrt{t}\, u_2) \in \mathcal{M}_r(V_1, V_2)\ (\textit{resp. } \mathcal{M}(V_1, V_2))$$

*if and only if*

$$\begin{cases} \mu_1\|u_1\|_4^4 s + \beta\|u_1u_2\|_2^2 t = \|\nabla u_1\|_2^2 + V_1\|u_1\|_2^2, \\ \beta\|u_1u_2\|_2^2 s + \mu_2\|u_2\|_4^4 t = \|\nabla u_2\|_2^2 + V_2\|u_2\|_2^2. \end{cases} \tag{4.2.3}$$

*Proof.* We show for $\mathcal{M}_r(V_1, V_2)$. We can show for $\mathcal{M}(V_1, V_2)$ in a similar way.
(i) Set

$$F_1(u_1, u_2) := J'_{(V_1,V_2)}(u_1, u_2)(u_1, 0), \quad F_2(u_1, u_2) := J'_{(V_1,V_2)}(u_1, u_2)(0, u_2).$$

Then $\mathcal{M}_r(V_1,V_2) = \{(u_1,u_2) \in H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N) : u_1 \neq 0, u_2 \neq 0, F_1(u_1,u_2) = F_2(u_1,u_2) = 0\}$. For $U = (u_1,u_2) \in \mathcal{M}_r(V_1,V_2)$ we have

$$\begin{bmatrix} F_1'(u_1,u_2)(u_1,0) & F_1'(u_1,u_2)(0,u_2) \\ F_2'(u_1,u_2)(u_1,0) & F_2'(u_1,u_2)(0,u_2) \end{bmatrix} = -2 \begin{bmatrix} \mu_1\|u_1\|_4^4 & \beta\|u_1u_2\|_2^2 \\ \beta\|u_1u_2\|_2^2 & \mu_2\|u_2\|_4^4 \end{bmatrix}$$

and

$$\begin{aligned} \det \begin{bmatrix} \mu_1\|u_1\|_4^4 & \beta\|u_1u_2\|_2^2 \\ \beta\|u_1u_2\|_2^2 & \mu_2\|u_2\|_4^4 \end{bmatrix} &= \mu_1\mu_2\|u_1\|_4^4\|u_2\|_4^4 - \beta^2\|u_1u_2\|_2^4 \\ &= (\mu_1\mu_2 - \beta^2)\|u_1\|_4^4\|u_2\|_4^4 > 0. \end{aligned} \tag{4.2.4}$$

Thus $F_1'(u_1,u_2)$ and $F_2'(u_1,u_2)$ are linearly independent for all $(u_1,u_2) \in \mathcal{M}_r(V_1,V_2)$, and $\mathcal{M}_r(V_1,V_2)$ is a submanifold of $H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N)$ with codimension 2.
(ii) Since

$$\begin{aligned} F_1(\sqrt{s}\,u_1, \sqrt{t}\,u_2) &= s(\|\nabla u_1\|_2^2 + V_1\|u_1\|_2^2 - \mu_1\|u_1\|_4^4 s - \beta\|u_1u_2\|_2^2 t), \\ F_2(\sqrt{s}\,u_1, \sqrt{t}\,u_2) &= t(\|\nabla u_2\|_2^2 + V_2\|u_2\|_2^2 - \beta\|u_1u_2\|_2^2 s - \mu_2\|u_2\|_4^4 t), \end{aligned}$$

we see that (ii) holds. □

The aim of this subsection is to show the following proposition.

**Proposition 4.2.5.** *Assume* (4.1.10) *and* (C1). *Then*

$$b(V_1,V_2) = \inf_{U \in \mathcal{M}_r(V_1,V_2)} J_{(V_1,V_2)}(U) = \inf_{U \in \mathcal{M}(V_1,V_2)} J_{(V_1,V_2)}(U). \tag{4.2.5}$$

Since all nontrivial positive solutions of (4.1.5)–(4.1.8) lie in $\mathcal{M}_r(V_1,V_2)$ after a suitable shift by the result of Busca and Sirakov [20], we can see that the critical point corresponding to $b(V_1,V_2)$ has the least energy among nontrivial solutions. Thus we call the solution corresponding to $b(V_1,V_2)$ the *least energy nontrivial solution.* We also call $b(V_1,V_2)$ the *least energy level for nontrivial solution.*

*Proof.* First we remark that $\mathcal{M}_r(V_1,V_2) \subset \mathcal{M}(V_1,V_2)$ implies

$$\inf_{U \in \mathcal{M}_r(V_1,V_2)} J_{(V_1,V_2)}(U) \geq \inf_{U \in \mathcal{M}(V_1,V_2)} J_{(V_1,V_2)}(U). \tag{4.2.6}$$

For $U = (u_1,u_2) \in \mathcal{M}(V_1,V_2)$, we set $\gamma(s,t) = (\sqrt{s}\,u_1, \sqrt{t}\,u_2)$. Since

$$\begin{aligned} J_{(V_1,V_2)}(\gamma(s,t)) =& \frac{1}{2}(\|\nabla u_1\|_2^2 + V_1\|u_1\|_2^2)s + \frac{1}{2}(\|\nabla u_2\|_2^2 + V_2\|u_2\|_2^2)t \\ &- \frac{1}{4}(\mu_1\|u_1\|_4^4 s^2 + 2\beta\|u_1u_2\|_2^2 st + \mu_2\|u_2\|_4^4 t^2) \end{aligned}$$

and (4.2.3) holds with $(s,t) = (1,1)$, we can see

$$\sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2)}(\gamma(s,t)) = J_{(V_1,V_2)}(\gamma(1,1)) = J_{(V_1,V_2)}(U).$$

We also set $\gamma^*(s,t) = (\sqrt{s}\, u_1^*, \sqrt{t}\, u_2^*) \in \Gamma$, where $u_1^*$ (resp. $u_2^*$) is the Schwarz symmetrization of $u_1$ (resp. $u_2$). We can easily see that

$$J_{(V_1,V_2)}(\gamma^*(s,t)) \le J_{(V_1,V_2)}(\gamma(s,t)) \quad \text{for all } (s,t) \in [0,\infty)^2.$$

Thus

$$\begin{aligned} b(V_1,V_2) &\le \sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2)}(\gamma^*(s,t)) \le \sup_{(s,t)\in[0,\infty)^2} J_{(V_1,V_2)}(\gamma(s,t)) \\ &= J_{(V_1,V_2)}(U). \end{aligned}$$

Since $U \in \mathcal{M}(V_1,V_2)$ is arbitrary, we have

$$b(V_1,V_2) \le \inf_{U\in\mathcal{M}(V_1,V_2)} J_{(V_1,V_2)}(U). \tag{4.2.7}$$

On the other hand, by Proposition 4.2.3, $b(V_1,V_2)$ is achieved by a nontrivial positive solution. Denoting the corresponding nontrivial solution $(u_{01},u_{02})$, we have $(u_{01},u_{02}) \in \mathcal{M}_r(V_1,V_2)$ and thus

$$b(V_1,V_2) = J_{(V_1,V_2)}(u_{01},u_{02}) \ge \inf_{U\in\mathcal{M}_r(V_1,V_2)} J_{(V_1,V_2)}(U). \tag{4.2.8}$$

Therefore we get (4.2.5) from (4.2.6)–(4.2.8). □

*Remark* 4.2.6. In [4], Ambrosetti and Colorado showed the existence of nontrivial positive solutions via a mountain pass argument in $\mathcal{N}_r(V_1,V_2)$. More precisely they showed the following minimax value is corresponding to a nontrivial solution.

$$b_{mp}(V_1,V_2) = \inf_{\gamma\in\Gamma_{mp}(V_1,V_2)} \max_{t\in[0,1]} J_{(V_1,V_2)}(\gamma(t)).$$

Here $\Gamma_{mp}(V_1,V_2)$ is the class of continuous curves in $\mathcal{N}_r(V_1,V_2)$ which join $\hat{\Omega}_1(V_1;x)$ and $\hat{\Omega}_2(V_2;x)$. We remark that $b_{mp}(V_1,V_2) = b(V_1,V_2)$ holds.

To show this fact, for any $U = (u_1,u_2) \in \mathcal{M}_r(V_1,V_2)$ we need to find a path $\gamma(t) \in \Gamma_{mp}(V_1,V_2)$ such that

$$\max_{t\in[0,1]} J_{(V_1,V_2)}(\gamma(t)) = J_{(V_1,V_2)}(U). \tag{4.2.9}$$

To construct such a path, we set

$$\gamma(t) := r(t)((1-t)u_1, tu_2),$$

where $r(t) > 0$ is uniquely determined so that $\gamma(t) \in \mathcal{N}_r(V_1,V_2)$. We can easily see that $\gamma(0) \in H_r^1(\mathbf{R}^N)\times\{0\}$, $\gamma(\frac{1}{2}) = U$, $\gamma(1) \in \{0\}\times H_r^1(\mathbf{R}^N)$ and $J_{(V_1,V_2)}(\gamma(t)) \le J_{(V_1,V_2)}(U)$ for all $t \in [0,1]$. Using paths $\zeta_1(\tau)$ and $\zeta_2(\tau)$ in the proof of Proposition 4.2.3, we can find a path $\gamma(t) \in \Gamma_{mp}(V_1,V_2)$ satisfying (4.2.9).

### 4.2.4 Some compactness properties

We use the following notation:

$$\begin{aligned}\mathcal{S}_r(V_1,V_2) :=& \{\Omega=(\omega_1,\omega_2)\in H^1_r(\mathbf{R}^N)\times H^1_r(\mathbf{R}^N):\ J'_{(V_1,V_2)}(\Omega)=0,\\ &\omega_1(x)>0,\ \omega_2(x)>0,\ J_{(V_1,V_2)}(\Omega)=b(V_1,V_2)\}\quad \text{for } (V_1,V_2)\in A,\\ \mathcal{S}_{r,A} :=& \{((V_1,V_2),\Omega)\in \mathbf{R}^2\times H^1_r(\mathbf{R}^N)\times H^1_r(\mathbf{R}^N):\ (V_1,V_2)\in A,\quad \Omega\in\mathcal{S}_r(V_1,V_2)\}.\end{aligned}$$

The sets $\mathcal{S}_r(V_1,V_2)$ and $\mathcal{S}_{r,A}$ have the following compactness properties.

**Proposition 4.2.7.** *Assume* (4.1.10) *and* (C1). *Then*

(i) *There exist $C_0$, $C_1$, $C_2$, $C_3>0$ such that for all $((V_1,V_2),\Omega)\in\mathcal{S}_{r,A}$, it follows that*

$$\|\Omega\|_{H^1}\le C_0, \tag{4.2.10}$$

$$\|\omega_1\|_4,\ \|\omega_2\|_4\ge C_1, \tag{4.2.11}$$

$$\omega_1(x),\ \omega_2(x),|\nabla\omega_1(x)|,\ |\nabla\omega_2(x)|\le C_2e^{-C_3|x|}\quad \text{for all } x\in\mathbf{R}^N. \tag{4.2.12}$$

*Here we write $\Omega=(\omega_1,\omega_2)$.*

(ii) *The set $\mathcal{S}_{r,A}$ is compact in $\mathbf{R}^2\times H^1_r(\mathbf{R}^N)\times H^1_r(\mathbf{R}^N)$. More precisely, any sequence $((V_{1j},V_{2j}),\Omega_j)_{j=1}^\infty\subset\mathcal{S}_{r,A}$ has a subsequence $((V_{1j_n},V_{2j_n}),\Omega_{j_n})$ and $((V_{10},V_{20}),\Omega_0)\in\mathcal{S}_{r,A}$ such that*

$$(V_{1j_n},V_{2j_n})\to(V_{10},V_{20})\ \textit{and}\ \Omega_{j_n}\to\Omega_0\ \textit{strongly in}\ H^1_r(\mathbf{R}^N)\times H^1_r(\mathbf{R}^N).$$

(iii) *The function $b(V_1,V_2):\ A\to\mathbf{R}$ is continuous.*

(iv) *The function $b(V_1,V_2)$ is strictly increasing in $V_1$ (resp. $V_2$) for fixed $V_2$ (resp. $V_1$).*

*Proof.* First we prove (iv). By the definition (4.2.2) of $b(V_1,V_2)$, it is clear that $b(V_1,V_2)$ is nondecreasing in $V_1$ for fixed $V_2$. Since $b(V_1,V_2)$ is achieved by $\gamma(s,t)=(\sqrt{s}u_1,\sqrt{t}u_2)$ with $(u_1,u_2)\in\mathcal{S}_r(V_1,V_2)$, it is also easy to see $b(V_1,V_2)$ is strictly increasing in $V_1$ for fixed $V_2$. In a similar way, we can see that $b(V_1,V_2)$ is strictly increasing in $V_2$ for fixed $V_1$.

Next we show (i). By (iv) we have

$$b(a_{10},a_{20})\le b(V_1,V_2)\le b(a_{11},a_{21})\quad \text{for all } (V_1,V_2)\in A.$$

Since $\|\Omega\|^2_{H^1,(V_1,V_2)}=4b(V_1,V_2)$ for $\Omega\in\mathcal{S}_r(V_1,V_2)$, we have (4.2.10) for some constant $C_0$ independent of $(V_1,V_2)\in A$. We recall that for some constant $C_N>0$, it holds that

$$|u(x)|\le\frac{C_N}{|x|^{(N-1)/2}}\|u\|_{H^1}\quad \text{for } u\in H^1_r(\mathbf{R}^N) \text{ and } |x|\ge 1.$$

(For example, see Lemma A.II of Berestycki and Lions [15].) Thus $\Omega=(\omega_1,\omega_2)\in\mathcal{S}_r(V_1,V_2)$ satisfies

$$\omega_i(x)\le\frac{C_NC_0}{|x|^{(N-1)/2}}.$$

Since $\Omega$ satisfies (4.1.5)–(4.1.6), we can see (4.2.12) holds for some constants $C_2$, $C_3 > 0$ independent of $(V_1, V_2) \in A$.

Next we show (4.2.11). We argue indirectly and we assume that there exists a sequence $((V_{1j}, V_{2j}), \Omega_j) \in \mathcal{S}_{r,A}$ such that $\Omega_j = (\omega_{1j}, \omega_{2j})$ satisfies

$$\|\omega_{2j}\|_4 \to 0. \tag{4.2.13}$$

(we can deal with the case $\|\omega_{1j}\|_4 \to 0$ in a similar way.) We may also assume $(V_{1j}, V_{2j}) \to (V_{10}, V_{20}) \in A$. Since $\Omega_j$ solves (4.1.5)–(4.1.8) and the embedding $H^1_r(\mathbf{R}^N) \subset L^4(\mathbf{R}^N)$ is compact, we can see that $\Omega_j$ has a strongly convergent subsequence. Extracting a subsequence if necessary, we may assume $\Omega_j \to \Omega_0 = (\omega_{10}, \omega_{20})$. It is easily seen that $\Omega_0$ is a critical point of $J_{(V_{10},V_{20})}(U)$ and

$$b(V_{1j}, V_{2j}) = J_{(V_{1j},V_{2j})}(\Omega_j) \to J_{(V_{10},V_{20})}(\Omega_0). \tag{4.2.14}$$

On the other hand, it follows from $J'_{(V_{1j},V_{2j})}(\omega_{1j}, \omega_{2j})(0, \omega_{2j}) = 0$ that

$$\|\nabla\omega_{2j}\|_2^2 + \|\omega_{2j}\|_2^2 = \mu_2\|\omega_{2j}\|_4^4 + \beta\|\omega_{1j}\omega_{2j}\|_2^2 \to 0 \quad \text{as } j \to \infty$$

under the condition (4.2.13). That is, $\omega_{20} = 0$ and $\Omega_0 = (\omega_{10}, 0)$ is a semitrivial positive solution. Thus,

$$J_{(V_{10},V_{20})}(\Omega_0) \le e_1(V_{10}).$$

By (4.2.14), we have $\limsup_{j\to\infty} b(V_{1j}, V_{2j}) \le e_1(V_{10})$, which is in contradiction with (i) of Proposition 4.2.3. Thus (4.2.11) holds.

Next we show (ii). Suppose $((V_{1j}, V_{2j}), \Omega_j) \in \mathcal{S}_{r,A}$ $(j = 1, 2, \cdots)$ and $(V_{1j}, V_{2j}) \to (V_{10}, V_{20}) \in A$. We may also assume $\Omega_j \to \Omega_0$ strongly in $H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N)$ and (4.2.14) holds. Here, by (4.2.11), $\Omega_0$ is a nontrivial positive solution of (4.1.5)–(4.1.8) and we have

$$J_{(V_{10},V_{20})}(\Omega_0) \ge b(V_{10}, V_{20}). \tag{4.2.15}$$

On the other hand, by the definition of $b(V_1, V_2)$, we can see that $b(V_1, V_2)$ is upper semi-continuous, that is,

$$\limsup_{j\to\infty} b(V_{1j}, V_{2j}) \le b(V_{10}, V_{20}). \tag{4.2.16}$$

Thus by (4.2.14), (4.2.15) and (4.2.16), we have $J_{(V_{10},V_{20})}(\Omega_0) = b(V_{10}, V_{20})$. Thus $\Omega_0 \in \mathcal{S}_r(V_{10}, V_{20})$, that is, $((V_{10}, V_{20}), \Omega_0) \in \mathcal{S}_{r,A}$.

(iii) also follows from the proof of (ii). □

**Corollary 4.2.8.** *For any $\delta > 0$ there exists $\rho_1(\delta) > 0$ such that if $(V_1, V_2)$ and $(\tilde V_1, \tilde V_2) \in A$ satisfy*

$$|(V_1, V_2) - (\tilde V_1, \tilde V_2)| < \rho_1(\delta),$$

*then*

$$\operatorname{dist}(\Omega, \mathcal{S}_r(\tilde V_1, \tilde V_2)) \left( = \inf_{\tilde\Omega \in \mathcal{S}_r(\tilde V_1, \tilde V_2)} \|\Omega - \tilde\Omega\|_{H^1} \right) < \delta$$

*for any $\Omega \in \mathcal{S}_r(V_1, V_2)$.*

Now Proposition 4.1.1 easily follows.

*Proof of Proposition 4.1.1.* Proposition 4.1.1 follows from Propositions 4.2.3, 4.2.5, 4.2.7. □

Finally in this section, we prove (4.1.16) and (4.1.17).

For (4.1.16), we study the behavior of $m(P) = b(V_1(P), V_2(P))$ in the setting of Remark 4.1.4. Clearly (4.1.16) holds from the following lemma.

**Lemma 4.2.9.** *Suppose that $V_1(x)$, $V_2(x) \in C^1(\mathbf{R}^N)$ in addition to the assumption of Theorem 4.1.3. If $m(P)$ takes a local minimum at $P_0 \in \mathbf{R}^N$, that is, for some $r > 0$*

$$m(P) \geq m(P_0) \qquad \text{for all } |P - P_0| \leq r,$$

*then there exist $\lambda_1$ and $\lambda_2 > 0$ such that*

$$\lambda_1 \nabla V_1(P_0) + \lambda_2 \nabla V_2(P_0) = 0.$$

*Proof.* It suffices to show that if $P_0 \in \mathbf{R}^N$ satisfies

$$\lambda_1 \nabla V_1(P_0) + \lambda_2 \nabla V_2(P_0) \neq 0 \quad \text{for all } \lambda_1, \lambda_2 > 0, \tag{4.2.17}$$

then $m(P)$ does not take a local minimum at $P_0$.

First we remark that (4.2.17) implies that at least one of $\nabla V_1(P_0)$, $\nabla V_2(P_0)$ is not 0. We consider the case $\nabla V_1(P_0) \neq 0$. In this case, (4.2.17) implies $\nabla V_2(P_0) \notin \{-\lambda \nabla V_1(P_0) : \lambda > 0\}$ and we can find a vector $h_0 \in \mathbf{R}^N \setminus \{0\}$ such that

$$\nabla V_1(P_0) h_0 < 0 \quad \text{and} \quad \nabla V_2(P_0) h_0 \leq 0. \tag{4.2.18}$$

We will show that

$$m(P_0 + \tau h_0) < m(P_0) \qquad \text{for small } \tau > 0. \tag{4.2.19}$$

Let $(u_{01}, u_{02}) \in \mathcal{M}_r(V_1(P_0), V_2(P_0))$ be a nontrivial solution corresponding to $m(P_0) = b(V_1(P_0), V_2(P_0))$ and set $\gamma_0(s,t) := (\sqrt{s} u_{01}, \sqrt{t} u_{02}) \in \Gamma$. Then we have

$$m(P_0) = b(V_1(P_0), V_2(P_0)) = \sup_{(s,t) \in [0,\infty)^2} J_{(V_1(P_0), V_2(P_0))}(\gamma_0(s,t)).$$

We also remark that $(s,t) \mapsto J_{(V_1(P_0), V_2(P_0))}(\gamma_0(s,t))$ takes its maximum only at $(s,t) = (1,1)$. Thus there exist $\delta_0$ and $\tau_0 > 0$ such that for $0 < \tau < \tau_0$

$$(s,t) \mapsto J_{(V_1(P_0 + \tau h_0), V_2(P_0 + \tau h_0))}(\gamma_0(s,t))$$

takes its maximum in $\{(s,t); \, |s-1| + |t-1| \leq \delta_0\}$. Thus we have

$$\begin{aligned}
&m(P + \tau h_0) - m(P_0) \\
\leq& \max_{|s-1|+|t-1| \leq \delta_0} J_{(V_1(P_0+\tau h_0), V_2(P_0+\tau h_0))}(\gamma_0(s,t)) - \max_{|s-1|+|t-1| \leq \delta_0} J_{(V_1(P_0), V_2(P_0))}(\gamma_0(s,t)) \\
\leq& \max_{|s-1|+|t-1| \leq \delta_0} \Big[ J_{(V_1(P_0+\tau h_0), V_2(P_0+\tau h_0))}(\gamma_0(s,t)) - J_{(V_1(P_0), V_2(P_0))}(\gamma_0(s,t)) \Big] \\
=& \frac{1}{2} \max_{|s-1|+|t-1| \leq \delta_0} \Big[ (V_1(P_0 + \tau h_0) - V_1(P_0)) \|u_{01}\|_2^2 s + (V_2(P_0 + \tau h_0) - V_2(P_0)) \|u_{02}\|_2^2 t \Big].
\end{aligned}$$

We remark that (4.2.18) implies

$$m(P+\tau h_0)-m(P_0) \le \frac{\tau}{2}\Big[(\nabla V_1(P_0)h_0+o(1))\|u_{01}\|_2^2(1-\delta_0)+o(1)\|u_{02}\|_2^2(1+\delta_0)\Big]$$
$$< 0 \qquad \text{for small } \tau > 0,$$

which implies (4.2.19) and $m(P)$ does not take a local minimum at $P_0$. The case where $\nabla V_2(P_0) \neq 0$ can be treated in a similar way. □

Next we deal with (4.1.17). In the following lemma we write dependence on $\beta$ explicitly and use notation $b_\beta(V_1, V_2)$ for $b(V_1, V_2)$ as in Remark 4.1.5

**Lemma 4.2.10.** *As $\beta \to 0$, it follows that $b_\beta(V_1, V_2) \to e_1(V_1) + e_2(V_2)$.*

*Proof.* We also use notation $J_{(V_1,V_2),\beta}$ for $J_{(V_1,V_2)}$. We remark that $b_\beta(V_1, V_2)$ and $J_{(V_1,V_2),\beta}$ are nonincreasing in $\beta$ and if (4.1.10) and (AC) hold for $\beta_0 > 0$, then (4.1.10) and (AC) hold for all $\beta \in (0, \beta_0]$.

By Proposition 4.2.3, there exists a nontrivial positive solution $U_\beta(x) \in H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N)$ such that for each $\beta \in (0, \beta_0]$

$$J_{(V_1,V_2),\beta}(U_\beta) = b_\beta(V_1, V_2) \in [\max\{e_1(V_1)+\rho_0, e_2(V_2)+\rho_0\}, e_1(V_1)+e_2(V_2)]. \tag{4.2.20}$$

Here $\rho_0 > 0$ is independent of $\beta$. As in the proof of Proposition 4.2.7, we can show that $(U_\beta)_{\beta\in(0,\beta_0]}$ is bounded in $H^1_r(\mathbf{R}^N) \times H^1_r(\mathbf{R}^N)$ and, after extracting a subsequence $\beta_j \to 0$, $U_{\beta_j}(x)$ converges strongly to some critical point $U_0(x)$ of $J_{(V_1,V_2),0}(U)$. We remark that $J_{(V_1,V_2),0}(U)$ is corresponding to two equations without interaction:

$$-\Delta u_1 + V_1 u_1 = \mu_1 u_1^3 \quad \text{in } \mathbf{R}^N,$$
$$-\Delta u_2 + V_2 u_2 = \mu_2 u_2^3 \quad \text{in } \mathbf{R}^N.$$

By (4.2.20),

$$\max\{e_1(V_1)+\rho_0, e_2(V_2)+\rho_0\} \le J_{(V_1,V_2),0}(U_0).$$

Since $U_0(x)$ is nonnegative, from (4.2.11), we have $U_0(x) = (\hat\omega_1(V_1; x), \hat\omega_2(V_2; x))$ and $b_{\beta_j}(V_1, V_2) \to e_1(V_1)+e_2(V_2)$. Since the limit does not depend on the choice of subsequence $\beta_j$, we have the conclusion of Lemma 4.2.10. □

*Remark* 4.2.11. We can also show that if $(V_{1j}, V_{2j}) \to (V_{10}, V_{20})$ where $V_{10}, V_{20} > 0$, then $b_\beta(V_{1j}, V_{2j}) \to e_1(V_{10}) + e_2(V_{20})$ as $\beta \to 0$.

## 4.3 Nehari manifolds and the Palais-Smale condition

### 4.3.1 A singular perturbation problem

In the following sections we study a singular perturbation problem (4.1.1)–(4.1.4). From now on we assume (A1) and (A2).

We set $u_i(x) = v_i(\varepsilon x)$ and we try to find a solution $U(x) = (u_1(x), u_2(x))$ of

$$-\Delta u_1 + V_1(\varepsilon x)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 \quad \text{in } \mathbf{R}^N, \tag{4.3.1}$$
$$-\Delta u_2 + V_2(\varepsilon x)u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 \quad \text{in } \mathbf{R}^N, \tag{4.3.2}$$
$$u_1(x),\ u_2(x) > 0 \quad \text{in } \mathbf{R}^N, \tag{4.3.3}$$
$$u_1(x),\ u_2(x) \in H^1(\mathbf{R}^N). \tag{4.3.4}$$

A functional corresponding to (4.3.1)–(4.3.4) is

$$\begin{aligned} I_\varepsilon(U) :=& I_\varepsilon(u_1, u_2) \\ =& \frac{1}{2}(\|\nabla u_1\|_2^2 + \int_{\mathbf{R}^N} V_1(\varepsilon x)u_1^2\,dx) + \frac{1}{2}(\|\nabla u_2\|_2^2 + \int_{\mathbf{R}^N} V_2(\varepsilon x)u_2^2\,dx) \\ &- \frac{1}{4}(\mu_1\|u_1\|_4^4 + 2\beta\|u_1u_2\|_2^2 + \mu_2\|u_2\|_4^4) \\ &\in C^1(H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N), \mathbf{R}). \end{aligned}$$

We use the following notation:

$$\begin{aligned} \|u_i\|_{H^1,i,\varepsilon}^2 &:= \|\nabla u_i\|_2^2 + \int_{\mathbf{R}^N} V_i(\varepsilon x)u_i^2\,dx \quad \text{for } i = 1, 2, \\ \|U\|_{H^1,\varepsilon}^2 &:= \|u_1\|_{H^1,1,\varepsilon}^2 + \|u_2\|_{H^1,2,\varepsilon}^2. \end{aligned}$$

With this notation, we can write

$$I_\varepsilon(U) = \frac{1}{2}\|U\|_{H^1,\varepsilon}^2 - \frac{1}{4}\int_{\mathbf{R}^N} W(U)\,dx. \tag{4.3.5}$$

We remark that under (A1) there exist constants $a_1$, $a_2 > 0$

$$a_1\|U\|_{H^1} \le \|U\|_{H^1,\varepsilon} \le a_2\|U\|_{H^1} \quad \text{for all } U \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N). \tag{4.3.6}$$

For $d > 0$ we set

$$K^d := \{x \in \mathbf{R}^N :\ \mathrm{dist}\,(x, K) \le d\}.$$

We remark that there exists $d_1 > 0$ such that

$$K^{d_1} \subset \Lambda.$$

Under the conditions (A1)–(A2), we introduce for $P \in \mathbf{R}^N$ and $\delta > 0$

$$\begin{aligned} \mathcal{S}_{r,P} &:= \mathcal{S}_r(V_1(P), V_2(P)), \\ \mathcal{S}_{r,P}^\delta &:= \{\Omega_P + \Phi :\ \Omega_P \in \mathcal{S}_{r,P},\ \Phi \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N),\ \|\Phi\|_{H^1} < \delta\}, \\ m(P) &:= b(V_1(P), V_2(P)). \end{aligned}$$

It follows from Proposition 4.2.7 that $\mathcal{S}_{r,P}$ ($P \in \mathbf{R}^N$) and $\bigcup_{P\in K^d} \mathcal{S}_{r,P}$ are compact in $H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N)$. We set for $\varepsilon$, $d$, $\delta > 0$,

$$\begin{aligned} \mathcal{S}_{\varepsilon,d} :=& \{\Omega_P(x - P/\varepsilon) :\ P \in K^d,\ \Omega_P \in \mathcal{S}_{r,P}\}, \\ \mathcal{S}_{\varepsilon,d}^\delta :=& \{\Omega_P(x - P/\varepsilon) + \Phi(x) :\ P \in K^d,\ \Omega_P \in \mathcal{S}_{r,P}, \\ & \qquad \Phi \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N),\ \|\Phi\|_{H^1} < \delta\}. \end{aligned}$$

Such a type of sets are introduced in Byeon and Jeanjean [22, 23] (cf. Byeon, Jeanjean and Tanaka [24]) for nonlinear Schrödinger equations $-\varepsilon^2\Delta u + V(x)u = f(u)$ and used successfully to construct a family of solutions, which concentrates at a local minimum of $V(x)$ without assumptions of uniqueness or nondegeneracy of solutions of the limit problems.

*Remark* 4.3.1. In [22, 23, 24], they introduced a class of sets which is slightly different from our $\mathcal{S}^\delta_{\varepsilon,d}$. Their class of sets is, in our setting,

$$\begin{aligned}\mathcal{X}_{\varepsilon,d} :=&\{\Omega_P(x-\tilde{P}/\varepsilon) :\, P\in K,\, |P-\tilde{P}|\le d,\, \Omega_P\in\mathcal{S}_{r,P}\},\\ \mathcal{X}^\delta_{\varepsilon,d} :=&\{\Omega_P(x-\tilde{P}/\varepsilon)+\Phi(x) :\, \Omega_P(x-\tilde{P}/\varepsilon)\in\mathcal{X}_{\varepsilon,d},\\ &\qquad \Phi\in H^1(\mathbf{R}^N)\times H^1(\mathbf{R}^N),\, \|\Phi\|_{H^1}<\delta\}.\end{aligned}$$

We remark that

$$\mathcal{X}_{\varepsilon,d}\subset\mathcal{S}^\delta_{\varepsilon,d},\quad \mathcal{X}^\delta_{\varepsilon,d}\subset\mathcal{S}^{2\delta}_{\varepsilon,d} \tag{4.3.7}$$

for sufficiently small $d$. In fact, by Corollary 4.2.8, for any $\delta>0$ there exists a $\rho_2(\delta)>0$ such that

$$\mathcal{S}_{r,P}\subset\mathcal{S}^\delta_{r,\tilde{P}}\quad\text{for any } P,\ \tilde{P}\in\overline{\Lambda}\text{ with } |P-\tilde{P}|\le\rho_2(\delta).$$

Thus (4.3.7) holds for $d\in(0,\rho_2(\delta)]$.

In [22, 23, 24], the desired solution is obtained through mountain pass arguments in $H^1(\mathbf{R}^N)$. Here we deal with the constraint problem $I_\varepsilon|_{\mathcal{M}_\varepsilon} : \mathcal{M}_\varepsilon\to\mathbf{R}$, where

$$\begin{aligned}\mathcal{M}_\varepsilon := \{U=(u_1,u_2)\in H^1(\mathbf{R}^N)\times H^1(\mathbf{R}^N) :\, u_1\ne 0,\, u_2\ne 0,\\ I'_\varepsilon(U)(u_1,0)=0,\ I'_\varepsilon(U)(0,u_2)=0\},\end{aligned} \tag{4.3.8}$$

and we try to find a critical point corresponding to a local minimum of $m:\,\Lambda\to\mathbf{R}$.

Here we fix $d_0\in(0,d_1]$ arbitrary and, in what follows, we try to find a critical point of $I_\varepsilon(U)$ in $\mathcal{S}^\delta_{\varepsilon,d_0}$. We use the following abbreviation:

$$\mathcal{S}_\varepsilon=\mathcal{S}_{\varepsilon,d_0}\quad\text{and}\quad\mathcal{S}^\delta_\varepsilon=\mathcal{S}^\delta_{\varepsilon,d_0}.$$

We start with the following

**Lemma 4.3.2.** *There exist $\varepsilon_1$ and $\delta_1>0$ which have the following properties*:

(i) *If $\varepsilon\in(0,\varepsilon_1]$ and $U(x)\in\mathcal{S}^{\delta_1}_\varepsilon$ satisfy $I'_\varepsilon(U)=0$, then $U(x)$ is a nontrivial positive solution of* (4.3.1)–(4.3.4).

(ii) *If $P\in K^{d_0}$ and $U\in\mathcal{S}^{\delta_1}_{r,P}$ satisfy $J'_{(V_1(P),V_2(P))}(U)=0$, then $U(x)$ is a nontrivial positive solution of* (4.1.5)–(4.1.8) *with $V_1=V_1(P)$ and $V_2=V_2(P)$.*

*Proof.* We give just an outline of a proof of (i).

It suffices to show that both components of $U(x)$ are positive. Via the standard regularity argument for solutions of elliptic equations, we can show that for any $\nu>0$ there exist $\varepsilon_\nu$ and $\delta_\nu>0$ such that

$$\|\Phi\|_\infty<\nu$$

for all $\varepsilon \in (0, \varepsilon_\nu]$ and $U(x) = \Omega_P(x - P/\varepsilon) + \Phi(x - P/\varepsilon) \in \mathcal{S}_\varepsilon^{\delta_\nu}$ satisfying $I'_\varepsilon(U) = 0$.

For any $R > 0$ we choose $\nu > 0$ small so that the following inequality holds for all $\varepsilon \in (0, \varepsilon_\nu]$ and $U(x) = (u_1(x), u_2(x))(= \Omega_P(x - P/\varepsilon) + \Phi(x - P/\varepsilon)) \in \mathcal{S}_\varepsilon^{\delta_\nu}$ satisfying $I'_\varepsilon(U) = 0$:

$$u_1(x),\ u_2(x) > 0 \quad \text{for } |x - P/\varepsilon| \le R.$$

In particular, we have $\operatorname{supp} u_{1-}$, $\operatorname{supp} u_{2-} \subset \{x \in \mathbf{R}^N : |x - P/\varepsilon| \ge R\}$. Setting $U_- = (u_{1-}, u_{2-})$, it follows from $I'_\varepsilon(U)U_- = 0$ that

$$\|U_-\|^2_{H^1,\varepsilon} - \int_{|x-P/\varepsilon|\ge R} \nabla W(U)U_-\, dx = 0. \tag{4.3.9}$$

When $R > 0$ is sufficiently large, $|\nabla W(U(x))| \ll 1$ for $|x - P/\varepsilon| \ge R$ and (4.3.9) implies $U_- \equiv 0$, that is, both components of $U(x)$ are nonnegative.

By the maximal principle we have $u_1(x)$, $u_2(x) > 0$ for all $x \in \mathbf{R}^N$ and $U(x)$ is a nontrivial positive solution. Setting $\varepsilon_1 = \varepsilon_\nu$, $\delta_1 = \delta_\nu$, we have (i). □

In what follows, we always assume $\varepsilon \in (0, \varepsilon_1]$ and $\delta \in (0, \delta_1]$.

### 4.3.2 Nehari type manifold $\mathcal{M}_\varepsilon$ and a projection $\mathcal{P}_\varepsilon : \mathcal{S}_\varepsilon^\delta \to \mathcal{M}_\varepsilon$

We can show the following lemma as in Lemma 4.2.4.

**Lemma 4.3.3.** *Assume* (4.1.10). *Then we have*

(i) *The set $\mathcal{M}_\varepsilon$ is a submanifold of $H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ with codimension* 2.

(ii) *For $(u_1, u_2) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ with $u_1 \not\equiv 0$, $u_2 \not\equiv 0$ and $s$, $t > 0$,*

$$(\sqrt{s}\, u_1, \sqrt{t}\, u_2) \in \mathcal{M}_\varepsilon$$

*holds if and only if*

$$\begin{cases} \mu_1\|u_1\|_4^4 s + \beta\|u_1u_2\|_2^2 t = \|\nabla u_1\|_2^2 + \int_{\mathbf{R}^N} V_1(\varepsilon x)u_1^2\, dx, \\ \beta\|u_1u_2\|_2^2 s + \mu_2\|u_2\|_4^4 t = \|\nabla u_2\|_2^2 + \int_{\mathbf{R}^N} V_2(\varepsilon x)u_2^2\, dx. \end{cases} \tag{4.3.10}$$

For $U = (u_1, u_2) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ with $u_1 \neq 0$ and $u_2 \neq 0$, we set

$$A(U) = \begin{bmatrix} \mu_1\|u_1\|_4^4 & \beta\|u_1u_2\|_2^2 \\ \beta\|u_1u_2\|_2^2 & \mu_2\|u_2\|_4^4 \end{bmatrix}.$$

By (4.2.4), we remark that $A(U)$ is invertible and the system (4.3.10) has a unique solution — we denote it $(s_\varepsilon(U), t_\varepsilon(U))$ —.

For $P \in \mathbf{R}^N$, we also consider

$$A(U)\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \|\nabla u_1\|_2^2 + V_1(P)\|u_1\|_2^2 \\ \|\nabla u_2\|_2^2 + V_2(P)\|u_2\|_2^2 \end{bmatrix},$$

which is equivalent to $(\sqrt{s}\, u_1, \sqrt{t}\, u_2) \in \mathcal{M}(V_1(P), V_2(P))$. We denote the unique solution by $(s_P(U), t_P(U))$. We have the following

**Lemma 4.3.4.** *For any $\nu \in (0, \frac{1}{2})$ there exist $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for each $\varepsilon \in (0, \varepsilon_2]$ and $\delta \in (0, \delta_2]$*

(i) *For all $U \in \mathcal{S}_\varepsilon^\delta$, it holds that $s_\varepsilon(U)$, $t_\varepsilon(U) \in (1-\nu, 1+\nu)$.*

(ii) *It holds that $s_P(U)$, $t_P(U) \in (1-\nu, 1+\nu)$ for all $U(x) = \Omega_P(x - P/\varepsilon) + \Phi(x)$ satisfying $P \in \overline{\Lambda}$, $\Omega_P \in \mathcal{S}_{r,P}$ and $\|\Phi\|_{H^1} < \delta$.*

*Proof.* We can see that

$$\begin{aligned}
\begin{bmatrix} s_\varepsilon(U) \\ t_\varepsilon(U) \end{bmatrix} &= A(U)^{-1} \begin{bmatrix} \|u_1\|_{H^1,1,\varepsilon}^2 \\ \|u_2\|_{H^1,2,\varepsilon}^2 \end{bmatrix}, \\
\begin{bmatrix} s_P(U) \\ t_P(U) \end{bmatrix} &= A(U)^{-1} \begin{bmatrix} \|\nabla u_1\|_2^2 + V_1(P)\|u_1\|_2^2 \\ \|\nabla u_2\|_2^2 + V_2(P)\|u_2\|_2^2 \end{bmatrix}.
\end{aligned}$$

Since $\Omega_P = (\omega_{P1}, \omega_{P2}) \in \mathcal{S}_{r,P} \subset \mathcal{M}_r(V_1(P), V_2(P))$, we have (4.2.3) with $(s,t) = (1,1)$, that is,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = A(\Omega_P)^{-1} \begin{bmatrix} \|\nabla \omega_{P1}\|_2^2 + V_1(P)\|\omega_{P1}\|_2^2 \\ \|\nabla \omega_{P2}\|_2^2 + V_2(P)\|\omega_{P2}\|_2^2 \end{bmatrix}.$$

We remark that

(4.3.11)

$$\|\omega_{Pi}(x - P/\varepsilon)\|_{H^1,i,\varepsilon}^2 = \|\nabla \omega_{Pi}\|_2^2 + \int_{\mathbf{R}^N} V_1(\varepsilon x + P)\omega_{Pi}^2\, dx \to \|\nabla \omega_{Pi}\|_2^2 + V_i(P)\|\omega_{Pi}\|_2^2$$

for $i = 1, 2$ as $\varepsilon \to 0$ uniformly in $P \in \overline{\Lambda}$ and $\Omega_P \in \mathcal{S}_{r,P}$.

Thus the conclusions (i) and (ii) follow from the continuity of $U \mapsto A(U)^{-1}$ in a neighborhood of $\mathcal{S}_\varepsilon$ and compactness of $\bigcup_{P \in \overline{\Lambda}} \mathcal{S}_{r,P}$. □

By Lemma 4.3.4, we can see the projections

$$\begin{aligned}
&\mathcal{P}_\varepsilon : \mathcal{S}_\varepsilon^\delta \to \mathcal{M}_\varepsilon;\ U = (u_1, u_2) \mapsto (\sqrt{s_\varepsilon(U)}\, u_1, \sqrt{t_\varepsilon(U)}\, u_2), \\
&\mathcal{P}_P : \{\Omega_P(x - P/\varepsilon) + \Phi(x) : \Omega_P \in \mathcal{S}_{r,P},\ \|\Phi\|_{H^1} < \delta\} \to \mathcal{M}(V_1(P), V_2(P)); \\
&\qquad U = (u_1, u_2) \mapsto (\sqrt{s_P(U)}\, u_1, \sqrt{t_P(U)}\, u_2)
\end{aligned}$$

are well-defined and continuous for $\varepsilon \in (0, \varepsilon_2]$ and $\delta \in (0, \delta_2]$.

The projection $\mathcal{P}_\varepsilon$ has the following properties:

**Lemma 4.3.5.** *There exist $\varepsilon_3 \in (0, \varepsilon_2)$, $\delta_3 \in (0, \delta_2)$, $L_0 > 0$ and a nondecreasing function $\rho_3(\varepsilon) \in C([0, \varepsilon_3], [0, \infty))$ satisfying $\rho_3(0) = 0$ such that*

(i) *For all $\varepsilon \in (0, \varepsilon_3]$, $P \in \overline{\Lambda}$ and $\Omega_P \in \mathcal{S}_{r,P}$,*

$$\|\mathcal{P}_\varepsilon(\Omega_P(x - P/\varepsilon)) - \Omega_P(x - P/\varepsilon)\|_{H^1 \times H^1} \le \rho_3(\varepsilon).$$

(ii) *For all $\varepsilon \in (0, \varepsilon_3]$ and $U$, $\tilde{U} \in \mathcal{S}_\varepsilon^{\delta_3}$,*

$$\|\mathcal{P}_\varepsilon(U) - \mathcal{P}_\varepsilon(\tilde{U})\|_{H^1 \times H^1} \le L_0 \|U - \tilde{U}\|_{H^1 \times H^1} \tag{4.3.12}$$

(iii) *For all* $\varepsilon \in (0, \varepsilon_3]$ *and* $\delta \in (0, \delta_3]$, *it holds that* $\mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon \subset \mathcal{P}_\varepsilon(\mathcal{S}_\varepsilon^\delta) \subset \mathcal{S}_\varepsilon^{L_0\delta+\rho_3(\varepsilon)}$.

*Proof.* Noting the convergence (4.3.11) is uniform in $P \in \overline{\Lambda}$ and $\Omega_P \in \mathcal{S}_{r,P}$, we have (i). Since $U \mapsto A(U)^{-1}$, $\|\nabla u_i\|^2_{H^1,i,\varepsilon}$, $\cdots$ are Lipschitz continuous in $\mathcal{S}_\varepsilon^\delta$, (ii) holds. (iii) follows from (i) and (ii). □

Choosing $\varepsilon_3$, $\delta_3 > 0$ smaller if necessary, we may assume

(i) It follows that

$$\delta_3 \le \frac{1}{4}\min\{\|\omega_1\|_{H^1},\ \|\omega_2\|_{H^1} : (\omega_1,\omega_2) \in \bigcup_{P\in\overline{\Lambda}} \mathcal{S}_{r,P}\}. \tag{4.3.13}$$

(ii) For $a_1 > 0$ appeared in (4.3.6),

$$\int_{\mathbf{R}^N} \nabla W(U)U\,dx \le \frac{a_1^2}{2}\|U\|^2_{H^1} \quad \text{for} \quad \|U\|_{H^1} \le 2\delta_3. \tag{4.3.14}$$

(iii) There exist $C_4$, $C_5$, $C_6 > 0$ such that for any $\varepsilon \in (0, \varepsilon_3]$, $\delta \in (0, \delta_3]$ and $U \in \mathcal{S}_\varepsilon^\delta$

$$\|U\|_{H^1} \le C_4, \tag{4.3.15}$$

$$\|A(U)\|_{M(2,2)} \le C_5, \tag{4.3.16}$$

$$\|A(U)^{-1}\|_{M(2,2)} \le C_6, \tag{4.3.17}$$

$$A(U)^{-1}\begin{bmatrix} \|u_1\|^2_{H^1,1,\varepsilon} \\ \|u_2\|^2_{H^1,2,\varepsilon} \end{bmatrix} \in \left\{\begin{bmatrix} s \\ t \end{bmatrix} : s, t \in \left[\frac{1}{2}, \frac{3}{2}\right]\right\}. \tag{4.3.18}$$

Here we use notation: $\|A\|_{M(2,2)} := \sup_{\xi\in\mathbf{R}^2, |\xi|=1} |A\xi|$ for $2\times 2$ matrix $A$.

### 4.3.3 The Palais-Smale condition in $\mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon$

The behavior of the Palais-Smale sequence is important for the proof of our main result. Here we consider its behavior in the set $\mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon$.

For $U \in \mathcal{M}_\varepsilon$ we use the following notation:

$$\|I'_\varepsilon(U)\|_{(T_U\mathcal{M}_\varepsilon)^*} := \sup_{\Phi\in T_U\mathcal{M}_\varepsilon,\, \|\Phi\|_{H^1}\le 1} |I'_\varepsilon(U)\Phi|.$$

Here $T_U\mathcal{M}_\varepsilon$ is the tangent space of $\mathcal{M}_\varepsilon$ at $U$:

$$T_U\mathcal{M}_\varepsilon := \{\Phi \in H^1(\mathbf{R}^N)\times H^1(\mathbf{R}^N) : F'_{1\varepsilon}(U)\Phi = F'_{2\varepsilon}(U)\Phi = 0\},$$

where $F_{1\varepsilon}(u_1,u_2) = I'_\varepsilon(u_1,u_2)(u_1,0)$, $F_{2\varepsilon}(u_1,u_2) = I'_\varepsilon(u_1,u_2)(0,u_2)$.

We show 2 types of the concentration-compactness results.

**Proposition 4.3.6.** *Assume* $\delta \in (0, \delta_3]$. *Furthermore, for a fixed* $\varepsilon \in (0, \varepsilon_3]$, *suppose a sequence* $(U_j)_{j=1}^\infty \subset \mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon$ *satisfies for some* $c_0 \in \mathbf{R}$

$$I_\varepsilon(U_j) \to c_0,$$
$$\|I'_\varepsilon(U_j)\|_{(T_{U_j}\mathcal{M}_\varepsilon)^*} \to 0 \quad \text{as } j \to \infty.$$

*Then* $U_j$ *has a strongly convergent subsequence in* $H^1(\mathbf{R}^N)\times H^1(\mathbf{R}^N)$ *and its limit* $U_0$ *is a critical point of* $I_\varepsilon(U)$ *which satisfies* $I_\varepsilon(U_0) = c_0$ *and* $U_0(x) > 0$.

**Proposition 4.3.7.** *Assume $\delta \in (0, \delta_3]$. Furthermore suppose that sequences $(\varepsilon_j)_{j=1}^\infty \subset (0, \varepsilon_3]$ and $(U_j)_{j=1}^\infty \subset H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ satisfy for some $c_0 \le m_0$*

$$\varepsilon_j \to 0, \tag{4.3.19}$$
$$U_j \in \mathcal{S}^\delta_{\varepsilon_j} \cap \mathcal{M}_{\varepsilon_j}, \tag{4.3.20}$$
$$I_{\varepsilon_j}(U_j) \to c_0 (\le m_0), \tag{4.3.21}$$
$$\|I'_{\varepsilon_j}(U_j)\|_{(T_{U_j}\mathcal{M}_{\varepsilon_j})^*} \to 0. \tag{4.3.22}$$

*Then $c_0 = m_0$ and there exists a subsequence — we still denote it by $j$ — $(Q_j)_{j=1}^\infty \subset \mathbf{R}^N$, $Q_0 \in K$ and $\Omega_0 \in \mathcal{S}_{r,Q_0}$ such that*

$$Q_j \to Q_0 \in K, \tag{4.3.23}$$
$$U_j(x + Q_j/\varepsilon_j) \to \Omega_0(x) \quad \textit{strongly in } H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N). \tag{4.3.24}$$

In what follows we give a proof of Proposition 4.3.7 and an outline of a proof of Proposition 4.3.6.

*Proof of Proposition 4.3.7.* Suppose $(\varepsilon_j)_{j=1}^\infty$ and $(U_j)_{j=1}^\infty$ satisfy (4.3.19)–(4.3.22).

**Step 1:** $\|I'_{\varepsilon_j}(U_j)\|_{H^{-1}} \to 0$ as $j \to \infty$.

We remark that

$$\|I'_\varepsilon(U)\|_{(T_U\mathcal{M}_\varepsilon)^*} = \min_{c_1,c_2\in\mathbf{R}} \|I'_\varepsilon(U) - c_1F'_{1\varepsilon}(U) - c_2F'_{2\varepsilon}(U)\|_{H^{-1}}.$$

Thus, under the assumption (4.3.22), there exists a sequence $(c_{1j}, c_{2j}) \in \mathbf{R}^2$ such that

$$\|I'_{\varepsilon_j}(U_j) - c_{1j}F'_{1\varepsilon_j}(U_j) - c_{2j}F'_{2\varepsilon_j}(U_j)\|_{H^{-1}} \to 0.$$

In particular, writing $U_j := (u_{1j}, u_{2j})$, we have

$$(I'_{\varepsilon_j}(U_j) - c_{1j}F'_{1\varepsilon_j}(U_j) - c_{2j}F'_{2\varepsilon_j}(U_j))(u_{1j}, 0) \to 0, \tag{4.3.25}$$
$$(I'_{\varepsilon_j}(U_j) - c_{1j}F'_{1\varepsilon_j}(U_j) - c_{2j}F'_{2\varepsilon_j}(U_j))(0, u_{2j}) \to 0. \tag{4.3.26}$$

For $U = (u_1, u_2) \in \mathcal{M}_\varepsilon$, we note that $I'_\varepsilon(U)(u_1, 0) = I'_\varepsilon(U)(0, u_2) = 0$ and

$$\begin{aligned} F'_{1\varepsilon}(U)(u_1, 0) &= -2\mu_1\|u_1\|_4^4, & F'_{2\varepsilon}(U)(u_1, 0) &= -2\beta\|u_1u_2\|_2^2, \\ F'_{1\varepsilon}(U)(0, u_2) &= -2\beta\|u_1u_2\|_2^2, & F'_{2\varepsilon}(U)(0, u_2) &= -2\mu_2\|u_2\|_4^4. \end{aligned}$$

Thus (4.3.25)–(4.3.26) implies

$$A(U_j)\begin{bmatrix} c_{1j} \\ c_{2j} \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{as } j \to \infty.$$

By (4.3.17), $A(U)^{-1}$ is bounded in $\mathcal{S}^\delta_\varepsilon$. Thus we have $c_{1j}$, $c_{2j} \to 0$ and

$$\|I'_{\varepsilon_j}(U_j)\|_{H^{-1}} \le |c_{1j}|\|F'_{1\varepsilon_j}(U_j)\|_{H^{-1}} + |c_{2j}|\|F'_{2\varepsilon_j}(U_j)\|_{H^{-1}} + o(1) \to 0$$

as $j \to \infty$. Thus Step 1 is proved.

Since $U_j \in \mathcal{S}^\delta_{\varepsilon_j}$, we can write $U_j(x) = \Omega_{P_j}(x - P_j/\varepsilon_j) + \Phi_j(x - P_j/\varepsilon_j)$ with $P_j \in K^{d_0}$, $\Omega_{P_j} \in \mathcal{S}_{r,P_j}$, $\|\Phi_j\|_{H^1} < \delta$. We set

$$\tilde{U}_j(x) := U_j(x + P_j/\varepsilon_j) = \Omega_{P_j}(x) + \Phi_j(x) \in \mathcal{S}^\delta_{r,P_j}$$

and suppose

$$\begin{aligned}
&P_j \to P_0 \in K^{d_0}, \\
&\Omega_{P_j}(x) \to \Omega_{P_0}(x) \in \mathcal{S}_{r,P_0} \quad \text{strongly in } H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N), \\
&\Phi_j(x) \rightharpoonup \Phi_0(x) \qquad\qquad\quad \text{weakly in } H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N).
\end{aligned}$$

Here we used the compactness of $\mathcal{S}_{r,A}$. We also set $\tilde{U}_0(x) := \Omega_{P_0}(x) + \Phi_0(x)$ and $\Psi_j(x) := \tilde{U}_j(x) - \tilde{U}_0(x)$. It follows from (4.3.13) that $\tilde{U}_0 = (\tilde{u}_{01}, \tilde{u}_{02})$ satisfies $\tilde{u}_{01} \neq 0$, $\tilde{u}_{02} \neq 0$. We also have

$$\begin{aligned}
&\limsup_{j\to\infty} \|\Psi_j\|_{H^1} \le 2\delta, \\
&\Psi_j(x) \rightharpoonup 0 \qquad \text{weakly in } H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N), \\
&\tilde{U}_j(x) = \tilde{U}_0(x) + \Psi_j(x) \rightharpoonup \tilde{U}_0(x) \quad \text{weakly in } H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N).
\end{aligned} \tag{4.3.27}$$

Next we show

**Step 2:** $\|\Psi_j\|_{H^1} \to 0$ *as* $j \to \infty$. *In particular,* $\tilde{U}_j(x) \to \tilde{U}_0(x)$ *strongly in* $H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$.

Since $I'_{\varepsilon_j}(U_j) \to 0$, we have

$$I'_{\varepsilon_j}(U_j)\Psi_j(x - P_j/\varepsilon_j) \to 0.$$

Thus, writing $\Psi_j = (\psi_{1j}, \psi_{2j})$ and noting $\tilde{U}_j = \tilde{U}_0 + \Psi_j = (\tilde{u}_{10} + \psi_{1j}, \tilde{u}_{20} + \psi_{2j})$, we have

$$\begin{aligned}
&\int_{\mathbf{R}^N} \nabla(\tilde{u}_{10} + \psi_{1j})\nabla\psi_{1j} + V_1(\varepsilon_j x + P_j)(\tilde{u}_{1j} + \psi_{1j})\psi_{1j}\, dx \\
&+ \int_{\mathbf{R}^N} \nabla(\tilde{u}_{20} + \psi_{2j})\nabla\psi_{2j} + V_2(\varepsilon_j x + P_j)(\tilde{u}_{2j} + \psi_{2j})\psi_{2j}\, dx \\
&- \int_{\mathbf{R}^N} \nabla W(\tilde{U}_0 + \Psi_j)\Psi_j\, dx \to 0.
\end{aligned}$$

It follows from (4.3.27) that for $i = 1, 2$, $\int_{\mathbf{R}^N} \nabla\tilde{u}_{i0}\nabla\psi_{ij}\, dx$, $\int_{\mathbf{R}^N} V_i(\varepsilon_j x + P_j)\tilde{u}_{i0}\psi_{ij}\, dx$, $\int_{\mathbf{R}^N} \tilde{u}_{10}^3\psi_{ij}\, dx$, $\int_{\mathbf{R}^N} \tilde{u}_{10}^2\psi_{ij}^2\, dx$, $\cdots \to 0$. Thus,

$$\|\Psi_j\|^2_{H^1,\varepsilon_j} - \int_{\mathbf{R}^N} \nabla W(\Psi_j)\Psi_j\, dx \to 0.$$

Thus by (4.3.6) and (4.3.14),

$$\frac{a_1^2}{2}\|\Psi_j\|^2_{H^1} \le \|\Psi_j\|^2_{H^1,\varepsilon_j} - \int_{\mathbf{R}^N} \nabla W(\Psi_j)\Psi_j\, dx \to 0.$$

That is, the conclusion of Step 2 holds.

Next we show

**Step 3:** *$P_0 \in K$ and $\tilde{U}_0(x-\ell_0) \in \mathcal{S}_{r,P_0}$ for some $\ell_0 \in \mathbf{R}^N$.*

We remark that Step 2 implies $\tilde{U}_j \to \tilde{U}_0$ strongly in $H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$, from which it follows

$$(4.3.28) \qquad \begin{aligned} &J_{(V_1(P_0),V_2(P_0))}(\tilde{U}_0) = \lim_{j\to\infty} I_{\varepsilon_j}(\tilde{U}_j(x-P_j/\varepsilon_j)) = c_0(\le m_0), \\ &J'_{(V_1(P_0),V_2(P_0))}(\tilde{U}_0) = 0. \end{aligned}$$

Thus, by Lemma 4.3.2, $\tilde{U}_0(x) \in \overline{\mathcal{S}^{\delta}_{r,P_0}} \subset \mathcal{S}^{\delta_1}_{r,P_0}$ is a nontrivial positive critical point of $J_{(V_1(P_0),V_2(P_0))}(U)$. By the result of Busca and Sirakov [20], $\tilde{U}_0(x)$ is radially symmetric after a suitable shift, that is, $\tilde{U}_0(x-\ell_0) \in \mathcal{M}_r(V_1(P_0),V_2(P_0))$ for some $\ell_0 \in \mathbf{R}^N$.

On the other hand, we have by Proposition 4.2.5

$$J_{(V_1(P_0),V_2(P_0))}(\tilde{U}_0) \ge b(V_1(P_0),V_2(P_0)) = m(P_0).$$

Thus, by (4.3.28), $m(P_0) \le c_0 \le m_0$, which implies $c_0 = m_0$, $P_0 \in K$ and $\tilde{U}_0(x-\ell_0) \in \mathcal{S}_{r,P_0}$ for some $\ell_0 \in \mathbf{R}^N$.

**Step 4:** *Conclusion*

Setting $Q_j = P_j + \varepsilon_j \ell_0$, we have the conclusion of Proposition 4.3.7 with $Q_0 = P_0$, $\Omega_0(x) = \tilde{U}_0(x-\ell_0)$. □

*Proof of Proposition 4.3.6.* As in Step 1 of the proof of Proposition 4.3.7, we can show $\|I'_\varepsilon(U_j)\|_{H^{-1}} \to 0$. We assume that $U_j \rightharpoonup U_0$ weakly in $H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$. As in Step 2, we can show that $U_j \to U_0$ strongly. A positivity of $U_0(x)$ follows from Lemma 4.3.2. □

As a corollary to Proposition 4.3.7, we have the following corollary, which will play an important role in the proof of our main Theorem 4.1.3.

**Corollary 4.3.8.** *For any $\delta \in (0, \delta_3/4]$, there exist $\nu_0$, $h_0$, $\varepsilon_0 > 0$ such that*

$$\|I'_\varepsilon(U)\|_{(T_U\mathcal{M}_\varepsilon)^*} \ge \nu_0$$

*for all $\varepsilon \in (0, \varepsilon_0]$ and $U \in (\mathcal{S}^{4\delta}_\varepsilon \setminus \mathcal{S}^{\delta}_\varepsilon) \cap \mathcal{M}_\varepsilon$ satisfying $I_\varepsilon(U) \le m_0 + h_0$.*

*Proof.* We argue indirectly and assume that there exists $\bar{\delta} \in (0, \delta_3/4]$ which satisfies the following property:

For any $j \in \mathbf{N}$, there exist $\varepsilon_j \in (0, \frac{1}{j}]$ and $U_j \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ such that

$$(4.3.29) \qquad \begin{aligned} &U_j \in (\mathcal{S}^{4\bar{\delta}}_{\varepsilon_j} \setminus \mathcal{S}^{\bar{\delta}}_{\varepsilon_j}) \cap \mathcal{M}_{\varepsilon_j}, \\ &I_{\varepsilon_j}(U_j) \le m_0 + \frac{1}{j}, \\ &\|I'_{\varepsilon_j}(U_j)\|_{(T_U\mathcal{M}_{\varepsilon_j})^*} < \frac{1}{j}. \end{aligned}$$

Applying Proposition 4.3.7, there exist a sequence $(Q_j)_{j=1}^\infty \subset \mathbf{R}^N$, $Q_0 \in K$ and $\Omega_0 \in \mathcal{S}_{r,Q_0}$ such that

$$Q_j \to Q_0, \quad U_j(x + Q_j/\varepsilon_j) \to \Omega_0(x) \quad \text{strongly in } H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N).$$

Thus $\operatorname{dist}(U_j, \mathcal{S}_{\varepsilon_j}) \to 0$, which is in contradiction to (4.3.29). Therefore Corollary 4.3.8 holds. □

## 4.4 An estimate of $I_\varepsilon(U)$ on $\mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon$

The aim of this section is to show the following estimate which is a key of the proof of our main Theorem 4.1.3.

**Proposition 4.4.1.** *For sufficiently small $\delta > 0$,*

$$\lim_{\varepsilon \to 0} \inf_{U \in \mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon} I_\varepsilon(U) = m_0.$$

Here we explain our idea to get the above estimate. It is rather easy to show the upper estimate $\lim_{\varepsilon\to 0} \inf_{U \in \mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon} \le m_0$. The harder part is to show the lower estimate.

First we show any $U(x) \in \mathcal{S}_\varepsilon^\delta$ can be approximated uniformly by a function of a form: $\Xi(x - P/\varepsilon) + \Phi(x - P/\varepsilon)$, where $P \in K^{d_0}$ and

$$\operatorname{supp} \Xi(x) \subset B(0, \overline{R}), \tag{4.4.1}$$

$$\|\Phi\|_{H^1} < 4\delta, \tag{4.4.2}$$

$$\operatorname{supp} \Xi(x) \cap \operatorname{supp} \Phi(x) = \emptyset. \tag{4.4.3}$$

By (4.4.3),

$$I_\varepsilon(U) \sim I_\varepsilon(\Xi(x - P/\varepsilon) + \Phi(x - P/\varepsilon)) = I_\varepsilon(\Xi(x - P/\varepsilon)) + I_\varepsilon(\Phi(x - P/\varepsilon)). \tag{4.4.4}$$

By (4.4.2), we have $I_\varepsilon(\Phi(x - P/\varepsilon)) \ge 0$ for $\delta > 0$ small. Thus

$$I_\varepsilon(U) \sim I_\varepsilon(\Xi(x - P/\varepsilon) + \Phi(x - P/\varepsilon)) \ge I_\varepsilon(\Xi(x - P/\varepsilon)). \tag{4.4.5}$$

By (4.4.1), we also have

$$I_\varepsilon(\Xi(x - P/\varepsilon)) \to J_{(V_1(P), V_2(P))}(\Xi) \quad \text{as } \varepsilon \to 0. \tag{4.4.6}$$

(4.4.4)–(4.4.6) are useful to estimate $I_\varepsilon(U)$ from below. We remark that to prove our Proposition 4.4.1 we need to deal with $I_\varepsilon(U)$ on $\mathcal{M}_\varepsilon$ and develop more precise arguments which involve the projection $\mathcal{P}_\varepsilon : \mathcal{S}_\varepsilon^\delta \to \mathcal{M}_\varepsilon$.

For a proof of Proposition 4.4.1, we choose $\delta > 0$ and $\varepsilon > 0$ small so that

$$8\delta < \delta_3, \quad 8L_0\delta + \rho_3(\varepsilon) < \delta_3.$$

Under these conditions we have

$$\begin{aligned} &\mathcal{P}_\varepsilon : \mathcal{S}_\varepsilon^{8\delta} \to \mathcal{M}_\varepsilon \quad \text{is well defined for small } \varepsilon, \\ &\mathcal{P}_\varepsilon(\mathcal{S}_\varepsilon^{8\delta}) \subset \mathcal{S}_\varepsilon^{8L_0\delta + \rho_3(\varepsilon)} \subset \mathcal{S}_\varepsilon^{\delta_3}. \end{aligned}$$

We also assume that $\delta > 0$ satisfies

$$a_1^2 - 4a_2^2 {C_4}^4 {C_6}^2 C_7 (4\delta)^2 > 0, \tag{4.4.7}$$

where $C_4$, $C_6 > 0$ are constants appeared in (4.3.15), (4.3.17) and $C_7 > 0$ is a constant such that

$$\left\| \begin{bmatrix} \mu_1 \|\Phi_1\|_4^4 & \beta \|\Phi_1\Phi_2\|_2^2 \\ \beta \|\Phi_1\Phi_2\|_2^2 & \mu_2 \|\Phi_2\|_4^4 \end{bmatrix} \right\|_{M(2,2)} \leq C_7 \|\Phi\|_{H^1}^4 \tag{4.4.8}$$

for all $\Phi = (\Phi_1, \Phi_2) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$.

*Proof of Proposition 4.4.1.* Proof of Proposition 4.4.1 consists of several steps. First we show that any $U \in \mathcal{S}_\varepsilon^\delta$ can be approximated by a function of form: $\Xi(x - P/\varepsilon) + \Phi(x - P/\varepsilon)$.

**Step 1:** *For any $\nu > 0$ there exists $\bar{R}_{\delta,\nu} > 0$ independent of $\varepsilon$ and $U \in \mathcal{S}_\varepsilon^\delta$ such that the following property holds: for any $U(x) \in \mathcal{S}_\varepsilon^\delta$ there exist $P \in K^{d_0}$ and $\Xi(x)$, $\Phi(x) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ such that*

$$\Xi(x - P/\varepsilon) + \Phi(x - P/\varepsilon) \in \mathcal{S}_\varepsilon^{8\delta}, \tag{4.4.9}$$

$$\Xi(x) \in \mathcal{S}_{r,P}^{4\delta}, \ \|\Phi\|_{H^1} < 4\delta, \tag{4.4.10}$$

$$\|U(x) - (\Xi(x - P/\varepsilon) + \Phi(x - P/\varepsilon))\|_{H^1} < \nu, \tag{4.4.11}$$

$$\operatorname{supp} \Xi \subset B(0, \bar{R}_{\delta,\nu}), \tag{4.4.12}$$

$$\operatorname{supp} \Phi(x) \cap \operatorname{supp} \Xi(x) = \emptyset. \tag{4.4.13}$$

*Moreover, there exists $C_8 > 0$ such that for $U(x) \in \mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon$*

$$I_\varepsilon(U) \geq I_\varepsilon(\mathcal{P}_\varepsilon(\Xi(x - P/\varepsilon) + \Phi(x - P/\varepsilon))) - C_8\nu. \tag{4.4.14}$$

We choose $\underline{R}_\delta > 0$ such that

$$\|\Omega\|_{H^1(|x| \geq \underline{R}_\delta)} < \delta \quad \text{for all } \Omega \in \mathcal{S}_{r,A}. \tag{4.4.15}$$

Let $n_\nu \in \mathbf{N}$ be an integer such that

$$\left(\frac{\nu}{2}\right)^2 n_\nu \geq 4\delta^2 \tag{4.4.16}$$

and we set

$$\bar{R}_{\delta,\nu} := \underline{R}_\delta + 3n_\nu.$$

We show that $\bar{R}_{\delta,\nu}$ has the desired property.

In fact, let $U(x) = \Omega_P(x - P/\varepsilon) + \Psi(x - P/\varepsilon) \in \mathcal{S}_\varepsilon^\delta$, where $P \in K^{d_0}$, $\Omega_P \in \mathcal{S}_{r,P}$ and $\|\Psi\|_{H^1} < \delta$. We set $\tilde{U}(x) := U(x + P/\varepsilon) = \Omega_P(x) + \Psi(x)$. By (4.4.15), we have $\|\tilde{U}(x)\|_{H^1(|x| \geq \underline{R}_\delta)} = \|\Omega_P(x) + \Psi(x)\|_{H^1(|x| \geq \underline{R}_\delta)} < 2\delta$. We remark that

$$\sum_{j=0}^{n_\nu - 1} \|\tilde{U}\|^2_{H^1(|x| \in [\underline{R}_\delta + 3j, \underline{R}_\delta + 3(j+1)])} \leq \|\tilde{U}\|^2_{H^1(|x| \geq \underline{R}_\delta)} < 4\delta^2.$$

By (4.4.16), there exists $n \in \{0, 1, 2, \cdots, n_\nu - 1\}$ such that

$$\|\tilde{U}\|_{H^1(|x| \in [\underline{R}_\delta + 3n, \underline{R}_\delta + 3(n+1)])} \le \frac{\nu}{2}.$$

We choose two functions $\zeta_1(r)$, $\zeta_2(r)$ such that

$$\zeta_1(r) = \begin{cases} 1 & \text{for } r \in [0, \underline{R}_\delta + 3n], \\ 0 & \text{for } r \in [\underline{R}_\delta + 3n + \frac{4}{3}, \infty), \end{cases}$$
$$\zeta_2(r) = \begin{cases} 0 & \text{for } r \in [0, \underline{R}_\delta + 3n + \frac{5}{3}], \\ 1 & \text{for } r \in [\underline{R}_\delta + 3n + 3, \infty), \end{cases}$$
$$\zeta_1(r),\, \zeta_2(r) \in [0, 1],\ \zeta_1'(r) \in [-1, 0],\ \zeta_2'(r) \in [0, 1] \text{ for all } r \in \mathbf{R}$$

and we set

$$\Xi(x) = \zeta_1(|x|)\tilde{U}(x),\ \ \Phi(x) = \zeta_2(|x|)\tilde{U}(x).$$

(4.4.12)–(4.4.13) clearly hold. We note that $\|\zeta_1\varphi\|_{H^1} \le 2\|\varphi\|_{H^1(|x| \in [0, \underline{R}_\delta + 3n + \frac{4}{3}])}$ for all $\varphi \in H^1$ and similar inequalities hold for $\|(1-\zeta_1)\varphi\|_{H^1}$, $\|\zeta_2\varphi\|_{H^1}$, $\|(1-\zeta_2)\varphi\|_{H^1}$.

Thus we have

$$\begin{aligned}
&\|\Xi - \Omega_P\|_{H^1} = \|\zeta_1(\Omega_P + \Psi) - \Omega_P\|_{H^1} \le \|(\zeta_1 - 1)\Omega_P\|_{H^1} + \|\zeta_1\Psi\|_{H^1} \\
&\qquad \le 2\|\Omega_P\|_{H^1(|x| \ge \underline{R}_\delta + 3n)} + 2\|\Psi\|_{H^1} < 4\delta, \\
&\|\Phi\|_{H^1} = \|\zeta_2(\Omega_P + \Psi)\|_{H^1} \le \|\zeta_2\Omega_P\|_{H^1} + \|\zeta_2\Psi\|_{H^1} < 4\delta, \\
&\|\tilde{U} - (\Xi + \Phi)\|_{H^1} = \|(1 - \zeta_1 - \zeta_2)\tilde{U}\|_{H^1} \\
&\qquad \le 2\|\tilde{U}\|_{H^1(|x| \in [\underline{R}_\delta + 3n, \underline{R}_\delta + 3(n+1)])} < \nu.
\end{aligned}$$

Thus (4.4.9)–(4.4.11) hold.

Next we suppose $U \in \mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon$. Then $\mathcal{P}_\varepsilon(U) = U$ and by (4.3.12)

$$\|U - \mathcal{P}_\varepsilon(\Xi + \Phi)\|_{H^1} = \|\mathcal{P}_\varepsilon(U) - \mathcal{P}_\varepsilon(\Xi + \Phi)\|_{H^1} \le L_0\nu.$$

Thus we have

$$I_\varepsilon(U) \ge I_\varepsilon(\mathcal{P}_\varepsilon(\Xi + \Phi)) - CL_0\nu$$

and (4.4.14) holds.

Next we show a property corresponding (4.4.5).

**Step 2:** *For any given $U \in \mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon$, let $\Xi(x) \in \mathcal{S}_{r,P}^{4\delta}$ and $\Phi \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ ($\|\Phi\|_{H^1} < 4\delta$) be given in Step 1. Then*

$$I_\varepsilon(\mathcal{P}_\varepsilon(\Xi(x - P/\varepsilon))) \le I_\varepsilon(\mathcal{P}_\varepsilon(\Xi(x - P/\varepsilon) + \Phi(x - P/\varepsilon))).$$

It suffices to show that

$$\theta \mapsto I_\varepsilon(\mathcal{P}_\varepsilon(\Xi(x - P/\varepsilon) + \theta\Phi(x - P/\varepsilon))) :\ [0, 1] \to \mathbf{R}$$

is a nondecreasing function. We write $\Xi = (\Xi_1, \Xi_2)$, $\Phi = (\Phi_1, \Phi_2)$ and

$$\begin{aligned} & \mathcal{P}_\varepsilon((\Xi + \theta\Phi)(x - P/\varepsilon)) \\ = \; & (\sqrt{\lambda_1(\theta)}(\Xi_1 + \theta\Phi_1)(x - P/\varepsilon), \sqrt{\lambda_2(\theta)}(\Xi_2 + \theta\Phi_2)(x - P/\varepsilon)), \end{aligned}$$

where $(\lambda_1(\theta), \lambda_2(\theta))$ satisfies

$$A(\Xi + \theta\Phi) \begin{bmatrix} \lambda_1(\theta) \\ \lambda_2(\theta) \end{bmatrix} = \begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta) \end{bmatrix}. \tag{4.4.17}$$

Here for $i = 1, 2$

$$G_{\varepsilon i}(\theta) := \|\Xi_i + \theta\Phi_i\|^2_{H^1,i,\varepsilon} = \|\Xi_i\|^2_{H^1,i,\varepsilon} + \theta^2\|\Phi_i\|^2_{H^1,i,\varepsilon}.$$

It follows from (4.4.17) that

$$\begin{bmatrix} \lambda_1(\theta) \\ \lambda_2(\theta) \end{bmatrix} = A(\Xi + \theta\Phi)^{-1} \begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta) \end{bmatrix}.$$

We also have

$$\begin{aligned} \begin{bmatrix} \lambda_1'(\theta) \\ \lambda_2'(\theta) \end{bmatrix} = \; & A(\Xi + \theta\Phi)^{-1} \begin{bmatrix} G_{\varepsilon 1}'(\theta) \\ G_{\varepsilon 2}'(\theta) \end{bmatrix} \\ & - A(\Xi + \theta\Phi)^{-1} \left( \frac{d}{d\theta} A(\Xi + \theta\Phi) \right) A(\Xi + \theta\Phi)^{-1} \begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta) \end{bmatrix}. \end{aligned}$$

Since $I_\varepsilon'(\mathcal{P}_\varepsilon((\Xi + \theta\Phi)(x - P/\varepsilon)))(\mathcal{P}_\varepsilon((\Xi + \theta\Phi)(x - P/\varepsilon))) = 0$, we have

$$\begin{aligned} & I_\varepsilon(\mathcal{P}_\varepsilon((\Xi + \theta\Phi)(x - P/\varepsilon))) = \frac{1}{4}\|\mathcal{P}_\varepsilon((\Xi + \theta\Phi)(x - P/\varepsilon))\|^2_{H^1,\varepsilon} \\ = \; & \frac{1}{4}\left(\lambda_1(\theta)G_{\varepsilon 1}(\theta) + \lambda_2(\theta)G_{\varepsilon 2}(\theta)\right) = \frac{1}{4}\left( \begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta) \end{bmatrix}, \begin{bmatrix} \lambda_1(\theta) \\ \lambda_2(\theta) \end{bmatrix} \right). \end{aligned}$$

Thus

$$\begin{aligned} & 4\frac{d}{d\theta} I_\varepsilon(\mathcal{P}_\varepsilon((\Xi + \theta\Phi)(x - P/\varepsilon))) \\ = & \left( \begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta) \end{bmatrix}, \begin{bmatrix} \lambda_1'(\theta) \\ \lambda_2'(\theta) \end{bmatrix} \right) + \left( \begin{bmatrix} G_{\varepsilon 1}'(\theta) \\ G_{\varepsilon 2}'(\theta) \end{bmatrix}, \begin{bmatrix} \lambda_1(\theta) \\ \lambda_2(\theta) \end{bmatrix} \right) \\ = & \Bigg( \begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta) \end{bmatrix}, A(\Xi + \theta\Phi)^{-1} \begin{bmatrix} G_{\varepsilon 1}'(\theta) \\ G_{\varepsilon 2}'(\theta) \end{bmatrix} \\ & \qquad - A(\Xi + \theta\Phi)^{-1} \left( \frac{d}{d\theta} A(\Xi + \theta\Phi) \right) A(\Xi + \theta\Phi)^{-1} \begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta) \end{bmatrix} \Bigg) \\ & + \left( \begin{bmatrix} G_{\varepsilon 1}'(\theta) \\ G_{\varepsilon 2}'(\theta) \end{bmatrix}, A(\Xi + \theta\Phi)^{-1} \begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta) \end{bmatrix} \right) \\ = & 2\left( \begin{bmatrix} G_{\varepsilon 1}'(\theta) \\ G_{\varepsilon 2}'(\theta) \end{bmatrix}, A(\Xi + \theta\Phi)^{-1} \begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta) \end{bmatrix} \right) \\ & - \left( \begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta) \end{bmatrix}, A(\Xi + \theta\Phi)^{-1} \left( \frac{d}{d\theta} A(\Xi + \theta\Phi) \right) A(\Xi + \theta\Phi)^{-1} \begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta) \end{bmatrix} \right). \end{aligned} \tag{4.4.18}$$

By (4.4.13), we remark that

$$A(\Xi+\theta\Phi)=\begin{bmatrix} \mu_1(\|\Xi_1\|_4^4+\theta^4\|\Phi_1\|_4^4) & \beta(\|\Xi_1\Xi_2\|_2^2+\theta^4\|\Phi_1\Phi_2\|_2^2) \\ \beta(\|\Xi_1\Xi_2\|_2^2+\theta^4\|\Phi_1\Phi_2\|_2^2) & \mu_2(\|\Xi_2\|_4^4+\theta^4\|\Phi_2\|_4^4) \end{bmatrix}.$$

Thus

$$\frac{d}{d\theta}A(\Xi+\theta\Phi)=4\theta^3\begin{bmatrix} \mu_1\|\Phi_1\|_4^4 & \beta\|\Phi_1\Phi_2\|_2^2 \\ \beta\|\Phi_1\Phi_2\|_2^2 & \mu_2\|\Phi_2\|_4^4 \end{bmatrix}.$$

By (4.4.8), we have

$$\left\|\frac{d}{d\theta}A(\Xi+\theta\Phi)\right\|_{M(2,2)}\le 4C_7\theta^3\|\Phi\|_{H^1}^4. \tag{4.4.19}$$

Since $\Xi+\theta\Phi\in\mathcal{S}_\varepsilon^{8\delta}$, we have by (4.3.18) that $\lambda_1(\theta)$, $\lambda_2(\theta)\in[\frac{1}{2},\frac{3}{2}]$ for all $\theta\in[0,1]$. Thus

$$\begin{aligned} &2\left(\begin{bmatrix} G'_{\varepsilon 1}(\theta) \\ G'_{\varepsilon 2}(\theta)\end{bmatrix}, A(\Xi+\theta\Phi)^{-1}\begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta)\end{bmatrix}\right) \\ =&2\left(\lambda_1(\theta)G'_{\varepsilon 1}(\theta)+\lambda_2(\theta)G'_{\varepsilon 2}(\theta)\right)\ge G'_{\varepsilon 1}(\theta)+G'_{\varepsilon 2}(\theta) \\ \ge&2\theta\left(\|\Phi_1\|_{H^1,1,\varepsilon}^2+\|\Phi_2\|_{H^1,2,\varepsilon}^2\right)=2\theta\|\Phi\|_{H^1,\varepsilon}^2\ge 2a_1^2\theta\|\Phi\|_{H^1}^2. \end{aligned} \tag{4.4.20}$$

By (4.3.15) we remark that

$$\left|\begin{bmatrix} G_{\varepsilon 1}(\theta) \\ G_{\varepsilon 2}(\theta)\end{bmatrix}\right|\le\sqrt{2}\|\Xi+\theta\Phi\|_{H^1,\varepsilon}^2\le\sqrt{2}a_2^2\|\Xi+\theta\Phi\|_{H^1}^2\le\sqrt{2}a_2^2{C_4}^2.$$

Thus, by (4.4.18)–(4.4.20) and (4.3.16)–(4.3.17), we have

$$\begin{aligned} 4\frac{d}{d\theta}I_\varepsilon(\mathcal{P}_\varepsilon(\Xi+\theta\Phi)) &\ge 2a_1^2\theta\|\Phi\|_{H^1}^2-(\sqrt{2}a_2^2{C_4}^2)^2{C_6}^2\cdot 4C_7\theta^3\|\Phi\|_{H^1}^4. \\ &= 2\theta\|\Phi\|_{H^1}^2\left(a_1^2-4a_2^2{C_4}^4{C_6}^2C_7\theta^2\|\Phi\|_{H^1}^2\right). \end{aligned}$$

By (4.4.7),

$$\frac{d}{d\theta}I_\varepsilon(\mathcal{P}_\varepsilon(\Xi+\theta\Phi))\ge 0 \quad \text{for } \|\Phi\|_{H^1}\le 4\delta,$$

which implies the conclusion of Step 2.

The following step shows a property corresponding to (4.4.6).

**Step 3:** *For any $\nu>0$ there exists an $\varepsilon_4>0$ such that*

$$\left|I_\varepsilon(\mathcal{P}_\varepsilon(\Xi(x-P/\varepsilon)))-J_{(V_1(P),V_2(P))}(\mathcal{P}_P(\Xi))\right|<\nu$$

*for any $\varepsilon\in(0,\varepsilon_4]$ and $\Xi(x)\in\mathcal{S}_{r,P}^{4\delta}$ with* $\operatorname{supp}\Xi\subset B(0,\bar{R}_{\delta,\nu})$.

We have

$$\begin{aligned} &\left|\int_{\mathbf{R}^N}V_i(\varepsilon x+P)\Xi_i(x)^2\,dx-V_i(P)\|\Xi_i\|_2^2\right|\le\max_{|x|\le\bar{R}_{\delta,\nu}}|V_i(\varepsilon x+P)-V_i(P)|\,\|\Xi_i\|_2^2 \\ &\to 0 \quad \text{as } \varepsilon\to 0 \text{ uniformly in } P \text{ and } \Xi\in\mathcal{S}_{r,P}^{4\delta} \text{ with } \operatorname{supp}\Xi\subset B(0,\bar{R}_{\delta,\nu}). \end{aligned}$$

Thus,

$$\|\Xi_i(x - P/\varepsilon)\|^2_{H^1,i,\varepsilon} \to \|\nabla \Xi_i\|_2^2 + V_i(P)\|\Xi_i\|_2^2$$

uniformly. From the definition of $\mathcal{P}_\varepsilon$, $\mathcal{P}_P$, the conclusion of Step 3 holds.

**Step 4:** *Conclusion*

By Step 3, we have for $\varepsilon \in (0, \varepsilon_4]$

$$I_\varepsilon(\mathcal{P}_\varepsilon(\Xi(x - P/\varepsilon))) \geq J_{(V_1(P,V_2(P))}(\mathcal{P}_P(\Xi)) - \nu \geq m(P) - \nu.$$

Thus by Steps 1–2, we have for any $U(x) \in \mathcal{S}_\varepsilon^\delta$

$$\begin{aligned} I_\varepsilon(U) \geq & I_\varepsilon(\mathcal{P}_\varepsilon((\Xi + \Phi)(x - P/\varepsilon))) - C_8\nu \geq I_\varepsilon(\mathcal{P}_\varepsilon(\Xi(x - P/\varepsilon)) - C_8\nu \\ \geq & m(P) - (C_8 + 1)\nu \geq \inf_{P \in K^{d_0}} m(P) - (C_8 + 1)\nu \\ = & m_0 - (C_8 + 1)\nu. \end{aligned}$$

Since $U \in \mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon$ and $\nu > 0$ are arbitrary, we have

$$\liminf_{\varepsilon \to 0} \inf_{U \in \mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon} I_\varepsilon(U) \geq m_0. \tag{4.4.21}$$

On the other hand, we have for $P \in K$ and $\Omega_P \in \mathcal{S}_{r,P}$

(i) $\mathcal{P}_\varepsilon(\Omega_P(x - P/\varepsilon)) \in \mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon$ for small $\varepsilon$.

(ii) $I_\varepsilon(\mathcal{P}_\varepsilon(\Omega_P(x - P/\varepsilon))) \to J_{(V_1(P),V_2(P))}(\Omega_P) = m(P) = m_0$.

Thus we have

$$\limsup_{\varepsilon \to 0} \inf_{U \in \mathcal{S}_\varepsilon^\delta \cap \mathcal{M}_\varepsilon} I_\varepsilon(U) \leq \lim_{\varepsilon \to 0} I_\varepsilon(\mathcal{P}_\varepsilon(\Omega_P(x - P/\varepsilon))) = m_0.$$

Together with (4.4.21), we complete the proof of Proposition 4.4.1. □

Corollary 4.3.8 and Proposition 4.4.1 imply the following:

**Proposition 4.4.2.** *Choose $\delta > 0$ small so that Corollary* 4.3.8 *and Proposition* 4.4.1 *hold. Then there exist $\nu_1 > 0$ and $\varepsilon_5 > 0$ such that*

$$I_\varepsilon(U) \geq m_0 + \nu_1 \tag{4.4.22}$$

*for all $\varepsilon \in (0, \varepsilon_5]$ and $U \in (\mathcal{S}_\varepsilon^{3\delta} \setminus \mathcal{S}_\varepsilon^{2\delta}) \cap \mathcal{M}_\varepsilon$.*

*Proof.* By Corollary 4.3.8, there exist $\nu_0$ and $h_0 > 0$ such that for small $\varepsilon > 0$

$$\|I'_\varepsilon(U)\|_{(T_U\mathcal{M}_\varepsilon)^*} \geq \nu_0 \quad \text{for any } U \in (\mathcal{S}_\varepsilon^{4\delta} \setminus \mathcal{S}_\varepsilon^\delta) \cap \mathcal{M}_\varepsilon \text{ with } I_\varepsilon(U) \leq m_0 + h_0. \tag{4.4.23}$$

By Proposition 4.4.1, we have $\lim_{\varepsilon \to 0} \inf_{U \in \mathcal{S}_\varepsilon^{4\delta} \cap \mathcal{M}_\varepsilon}(U) = m_0$. Thus for sufficiently small $\varepsilon > 0$, it holds that

$$\inf_{U \in \mathcal{S}_\varepsilon^{4\delta} \cap \mathcal{M}_\varepsilon} I_\varepsilon(U) \geq m_0 - \frac{1}{4}\nu_0\delta. \tag{4.4.24}$$

We will show that (4.4.22) holds with $\nu_1 = \min\{h_0, \frac{1}{4}\nu_0\delta\}$.

By (4.4.23), we can construct a pseudo-gradient vector field $\mathcal{V}(U)$ on $(\mathcal{S}_\varepsilon^{4\delta} \setminus \mathcal{S}_\varepsilon^{\delta}) \cap \mathcal{M}_\varepsilon \cap \{U : I_\varepsilon(U) \le m_0 + h_0\}$ such that for all $U \in (\mathcal{S}_\varepsilon^{4\delta} \setminus \mathcal{S}_\varepsilon^{\delta}) \cap \mathcal{M}_\varepsilon$ with $I_\varepsilon(U) \le m_0 + h_0$, it follows that

$$\begin{aligned}
&\mathcal{V}(U) \in T_U\mathcal{M}_\varepsilon, \\
&\|\mathcal{V}(U)\|_{H^1\times H^1} \le 1, \\
&I'_\varepsilon(U)\mathcal{V}(U) \ge \frac{1}{2}\nu_0.
\end{aligned}$$

We consider the following ODE in $\mathcal{M}_\varepsilon$:

$$\frac{d\eta}{dt} = -\mathcal{V}(\eta), \quad \eta(0) = U_0. \tag{4.4.25}$$

We can easily see that

$$\begin{aligned}
&\left\|\frac{d\eta}{dt}(t)\right\|_{H^1\times H^1} \le 1, \\
&\frac{d}{dt}I_\varepsilon(\eta(t)) = -I'_\varepsilon(\eta)\mathcal{V}(\eta) \le -\frac{1}{2}\nu_0.
\end{aligned}$$

as long as $\eta(t) \in (\mathcal{S}_\varepsilon^{4\delta} \setminus \mathcal{S}_\varepsilon^{\delta}) \cap \mathcal{M}_\varepsilon$ and $I_\varepsilon(\eta(t)) \le m_0 + h_0$.

Now let $U_0 \in (\mathcal{S}_\varepsilon^{3\delta} \setminus \mathcal{S}_\varepsilon^{2\delta}) \cap \mathcal{M}_\varepsilon$ and we show (4.4.22) holds for $U_0$. Suppose that $I_\varepsilon(U_0) \le m_0 + h_0$ otherwise (4.4.22) holds. We consider the solution $\eta(t)$ of (4.4.25). By the above argument we have

$$\eta(t) \in (\mathcal{S}_\varepsilon^{4\delta} \setminus \mathcal{S}_\varepsilon^{\delta}) \cap \mathcal{M}_\varepsilon \quad \text{for } t \in [0, \delta], \tag{4.4.26}$$

$$I_\varepsilon(\eta(\delta)) \le I_\varepsilon(U_0) - \frac{1}{2}\nu_0\delta. \tag{4.4.27}$$

It follows from (4.4.26)–(4.4.27) that

$$I_\varepsilon(U_0) \ge I_\varepsilon(\eta(\delta)) + \frac{1}{2}\nu_0\delta \ge \inf_{U\in\mathcal{S}_\varepsilon^{4\delta}\cap\mathcal{M}_\varepsilon} I_\varepsilon(U) + \frac{1}{2}\nu_0\delta.$$

By (4.4.24), we have

$$I_\varepsilon(U_0) \ge m_0 + \frac{1}{4}\nu_0\delta.$$

Since $U_0 \in (\mathcal{S}_\varepsilon^{3\delta} \setminus \mathcal{S}_\varepsilon^{2\delta}) \cap \mathcal{M}_\varepsilon$ is arbitrary, we have (4.4.22). □

*End of the proof of Theorem 4.1.3.* We fix $\delta > 0$ small so that Propositions 4.4.1 and 4.4.2 hold and let $\varepsilon_5 > 0$ be given in Proposition 4.4.2. Then we have for $\varepsilon \in (0, \varepsilon_5]$

$$\inf_{U\in(\mathcal{S}_\varepsilon^{3\delta}\setminus\mathcal{S}_\varepsilon^{2\delta})\cap\mathcal{M}_\varepsilon} I_\varepsilon(U) \ge m_0 + \nu_1.$$

On the other hand, by Proposition 4.4.1, it follows that

$$\lim_{\varepsilon\to 0} \inf_{U\in\mathcal{S}_\varepsilon^{3\delta}\cap\mathcal{M}_\varepsilon} I_\varepsilon(U) = m_0. \tag{4.4.28}$$

Since $I_\varepsilon|_{\mathcal{M}_\varepsilon}$ satisfies the Palais-Smale condition in $\mathcal{S}_\varepsilon^{3\delta} \cap \mathcal{M}_\varepsilon$ by Proposition 4.3.6, there exists a minimizer $U_\varepsilon(x) \in \mathcal{S}_\varepsilon^{2\delta} \cap \mathcal{M}_\varepsilon$ via Ekeland principle. $U_\varepsilon(x)$ satisfies

$$\begin{aligned} I_\varepsilon(U_\varepsilon) &= \inf_{U \in \mathcal{S}_\varepsilon^{3\delta} \cap \mathcal{M}_\varepsilon} I_\varepsilon(U), \\ I'_\varepsilon(U_\varepsilon) &= 0. \end{aligned}$$

By Lemma 4.3.2 (i), $U_\varepsilon(x)$ is a nontrivial positive solution of (4.3.1)–(4.3.4). Since $I_\varepsilon(U_\varepsilon) \to m_0$ by (4.4.28), Proposition 4.3.7 implies the existence of a subsequence $\varepsilon_j \to 0$ and $(Q_j)_{j=1}^\infty \subset \mathbf{R}^N$, $Q_0 \in K$ and $\Omega_0 \in \mathcal{S}_{r,Q_0}$ that satisfies (4.3.23)–(4.3.24). Thus, denoting $U_\varepsilon(x) := (u_{1\varepsilon}(x), u_{2\varepsilon}(x))$ and setting $v_{1\varepsilon}(x) := u_{1\varepsilon}(x/\varepsilon)$, $v_{2\varepsilon}(x) := u_{2\varepsilon}(x/\varepsilon)$ and $P_{\varepsilon_j} := Q_j$, $P_0 := Q_0$, we have (4.1.14)–(4.1.15). Thus the proof of Theorem 4.1.3 is completed. $\square$

# Part II

# Nonlinear scalar field equations

# Chapter 5

# Existence of positive and infinitely many solutions: homogeneous case

## 5.1 Introduction

In this chapter, we study (NSF) with $\Omega = \mathbf{R}^N$ and $g(r,s) = g(s)$. Namely, we consider the existence of radially symmetric solutions of the following nonlinear scalar field equations:

$$-\Delta u = g(u) \quad \text{in } \mathbf{R}^N, \tag{5.1.1}$$

$$u \in H^1(\mathbf{R}^N). \tag{5.1.2}$$

Here $N \geq 2$ and $g : \mathbf{R} \to \mathbf{R}$ is a continuous function. This type of problem appears in many models in mathematical physics etc. and almost necessary and sufficient conditions for the existence of nontrivial solutions are obtained by Berestycki and Lions [15, 16] for $N \geq 3$ and Berestycki, Gallouët and Kavian [14] for $N = 2$. See also Strauss [95] and Coleman, Glaser and Martin [28] for earlier works.

In [14, 15, 16], they assume:

(5–g0) The function $g \in C(\mathbf{R}, \mathbf{R})$ and $g$ is odd: $g(-\xi) = -g(\xi)$.

(5–g1) For $N \geq 3$,

$$\limsup_{\xi\to\infty} \frac{g(\xi)}{\xi^{(N+2)/(N-2)}} \leq 0.$$

For $N = 2$,

$$\limsup_{\xi\to\infty} \frac{g(\xi)}{e^{\alpha\xi^2}} \leq 0 \quad \text{for any } \alpha > 0.$$

(5–g2) For $N \geq 3$,

$$-\infty < \liminf_{\xi\to 0} \frac{g(\xi)}{\xi} \leq \limsup_{\xi\to 0} \frac{g(\xi)}{\xi} < 0. \tag{5.1.3}$$

For $N = 2$,

$$-\infty < \lim_{\xi\to 0} \frac{g(\xi)}{\xi} < 0. \tag{5.1.4}$$

(5–g3) There exists a $\zeta_0 > 0$ such that $G(\zeta_0) > 0$, where $G(\xi) \equiv \int_0^\xi g(\tau)d\tau$.

Under the above conditions, they show the existence of a *radially symmetric positive solution* and *infinitely many radially symmetric* (*possibly sign changing*) *solutions.*

*Remark* 5.1.1. For the existence of a positive solution, it is sufficient to assume (5–g0)–(5–g3) just for $\xi > 0$. Namely we assume

(5–g0') $g \in C([0,\infty))$, $g(0) = 0$

and (5–g1), (5–g3) and (5–g2) just for a limit as $\xi \to +0$.

*Remark* 5.1.2. (a) We refer to Berestycki and Lions [17] (see also Section 11, Chapter II of Struwe [98]) for the study of *zero mass case*, where $N \geq 3$. In particular, they assume

$$\limsup_{\xi\to 0} \frac{G(\xi)}{|\xi|^{2N/(N-2)}} \leq 0$$

instead of (5–g2) and they show the existence of infinitely many solutions in $\mathcal{D}^{1,2}(\mathbf{R}^N)$.
(b) For the study of the existence of at least one solution, especially the existence of a least energy solution, we also refer to Brezis and Lieb [18], in which they study the system of equations

$$-\Delta u_i = g^i(u) \quad \text{in } \mathbf{R}^d,\ i = 1, \dots, n, \tag{5.1.5}$$

where $d \geq 2$ with $u : \mathbf{R}^d \to \mathbf{R}^n$ and $g^i(u) = \partial G/\partial u_i$. Here we call $u = (u_1, \dots, u_n)$ a *least energy solution of* (5.1.5) if $u$ satisfies

$$J(u) = \inf\{J(v) : v \text{ is a nontrivial solution of (5.1.5) }\} \tag{5.1.6}$$

where

$$J(v) = \frac{1}{2}\sum_{i=1}^{n} \int_{\mathbf{R}^d} |\nabla v_i(x)|^2 dx - \int_{\mathbf{R}^d} G(v(x))dx.$$

Under suitable conditions on $G$ (which differ between $d = 2$ and $d \geq 3$), they prove that (5.1.5) admits a nontrivial solution $u$ which satisfies (5.1.6). We also refer to Brüning [19] for a generalization when $d = 2$.

(5–g0)–(5–g3) are natural conditions for the existence of solutions. However we can see a difference between cases $N \geq 3$ and $N = 2$ in the condition (5–g2). We remark that when $N = 2$, the existence of a limit $\lim_{\xi\to 0} g(\xi)/\xi \in (-\infty, 0)$ is used essentially to show that the Palais–Smale compactness condition for the corresponding functional under suitable constraint ([14]).

The aim of this chapter is to extend the result of [14] slightly and we prove the existence of radially symmetric positive solution and infinitely many radially symmetric solutions under the conditions (5–g0), (5–g1), (5–g3) and (5.1.3) (not (5.1.4)).

We also remark that in [14, 15, 16] (cf. [18, 19]), they constructed solutions of (5.1.1)–(5.1.2) through constraint problems in the space of radially symmetric functions:

- find critical points of

$$\left\{ \int_{\mathbf{R}^N} |\nabla u|^2 dx : \int_{\mathbf{R}^N} G(u)dx = 1 \right\} \quad (N \geq 3), \tag{5.1.7}$$

or

• find critical points of

$$\left\{ \int_{\mathbf{R}^2} |\nabla u|^2 dx : \int_{\mathbf{R}^2} G(u)dx = 0, \ \int_{\mathbf{R}^2} u^2 dx = 1 \right\} \quad (N=2). \tag{5.1.8}$$

In fact, if $v(x)$ is a critical point of (5.1.7) or (5.1.8), then for a suitable $\lambda > 0$, $u(x) = v(x/\lambda)$ is a solution of (5.1.1)–(5.1.2). On the other hand, solutions of (5.1.1)–(5.1.2) are also characterized as critical points of the functional $I \in C^1(H_r^1(\mathbf{R}^N), \mathbf{R})$ defined by

$$I(u) := \frac{1}{2} \int_{\mathbf{R}^N} |\nabla u|^2 dx - \int_{\mathbf{R}^N} G(u)dx.$$

Here we denote by $H_r^1(\mathbf{R}^N)$ the space of radially symmetric $H^1$-functions defined on $\mathbf{R}^N$. It is natural to ask whether it is possible to find critical points through the unconstraint functional $I$.

Our second aim is to give another proof of the results of [14, 15, 16] using mountain pass and symmetric mountain pass arguments to $I$.

Before stating our main result in this chapter, we prepare one notation. We say that a nontrivial solution $u$ of (5.1.1)–(5.1.2) is a *least energy solution* if and only if $u$ satisfies

$$I(u) = \inf\{I(v) : v \in H^1(\mathbf{R}^N) \text{ is a nontrivial solution of (5.1.1)}\}.$$

Now we can state our main result.

**Theorem 5.1.3.** *Assume* $N \geq 2$, (5–g0), (5–g1), (5–g3) *and*

$$-\infty < \liminf_{\xi \to 0} \frac{g(\xi)}{\xi} \leq \limsup_{\xi \to 0} \frac{g(\xi)}{\xi} < 0. \tag{g2'}$$

*Then* (5.1.1)–(5.1.2) *has a least energy positive solution and infinitely many radially symmetric* (*possibly sign changing*) *solutions, which are characterized by the mountain pass and symmetric mountain pass minimax arguments in* $H_r^1(\mathbf{R}^N)$ (*see* (5.3.1)–(5.3.2) *and* (5.6.1)–(5.6.3) *below*).

*Remark* 5.1.4. (a) When $N \geq 3$, the existence of solutions of (5.1.1)–(5.1.2) is obtained in [15, 16], and we provide another proof and give a minimax characterization of infinitely many solutions using the functional $I$.
(b) When $N = 2$, our existence result extends the result of [14] slightly. Indeed, we show the existence under the condition (5–g2') not (5.1.4).

In Jeanjean and Tanaka [55], they give a mountain pass characterization to a least energy solution of (5.1.1)–(5.1.2) under the conditions (5–g0)–(5–g3). More precisely, let $b$ be the mountain pass minimax value for $I$ and $m$ the least energy level. To show $b = m$, we argued in [55] as follows: To show $b \leq m$, for any solution $u(x)$ we constructed a path $\gamma \in C([0,1], H_r^1(\mathbf{R}^N))$ such that $u \in \gamma([0,1])$, $\gamma(0) = 0$, $I(\gamma(1)) < 0$ and $\max_{t\in[0,1]} I(\gamma(t)) = I(u)$. To show $b \geq m$, the existence of a minimizer of the minimization problems (5.1.7) or (5.1.8) is essential and we relied on the arguments in [14, 15].

We will take mountain pass and symmetric mountain pass approaches to prove Theorem 5.1.3. In section 5.3, we will observe that $I$ is an even functional with a mountain pass geometry and it is possible to define a mountain pass minimax values $b_{mp}$ and symmetric mountain pass values $b_n$ ($n \in \mathbf{N}$) for $I$. By Ekeland's principle, we can find a Palais–Smale sequence $(u_j)_{j=1}^{\infty} \subset H_r^1(\mathbf{R}^N)$ at levels $b_{mp}$ and $b_n$, that is, $(u_j)_{j=1}^{\infty}$ satisfies

$$I(u_j) \to b_{mp} \quad (\text{or } b_n), \tag{5.1.9}$$

$$I'(u_j) \to 0 \quad \text{strongly in } (H_r^1(\mathbf{R}^N))^*. \tag{5.1.10}$$

However one of the difficulty is a lack of the Palais–Smale compactness condition and it seems difficult to show the existence of strongly convergent subsequence merely under the conditions (5.1.9)–(5.1.10). A key of our argument is to find a Palais–Smale sequence with an extra property related to the Pohozaev Identity. We recall that if $u$ is a critical point of $I$, then $u$ satisfies

$$P(u) = 0, \quad \text{where} \quad P(u) := \frac{N-2}{2} \int_{\mathbf{R}^N} |\nabla u|^2 dx - N \int_{\mathbf{R}^N} G(u) dx.$$

The above equality is called the Pohozaev Identity. It is natural to ask the existence of a Palais–Smale sequence $(u_j)_{j=1}^{\infty}$ satisfying (5.1.9)–(5.1.10) and $P(u_j) \to 0$. For this purpose, in section 5.4, we introduce an auxiliary functional:

$$\tilde{I}(\theta, u) := \frac{e^{(N-2)\theta}}{2} \int_{\mathbf{R}^N} |\nabla u|^2 dx - e^{N\theta} \int_{\mathbf{R}^N} G(u) dx : \mathbf{R} \times H_r^1(\mathbf{R}^N) \to \mathbf{R}.$$

We will find a Palais–Smale sequence $(\theta_j, u_j)$ in the augmented space $\mathbf{R} \times H_r^1(\mathbf{R}^N)$ satisfying

$$\theta_j \to 0, \tag{5.1.11}$$

$$\tilde{I}(\theta_j, u_j) \to b_{mp} \quad (\text{or } b_n),, \tag{5.1.12}$$

$$\tilde{I}'(\theta_j, u_j) \to 0 \quad \text{strongly in } (H_r^1(\mathbf{R}^N))^*, \tag{5.1.13}$$

$$\frac{N-2}{2} e^{(N-2)\theta_j} \int_{\mathbf{R}^N} |\nabla u_j|^2 dx - N e^{N\theta_j} \int_{\mathbf{R}^N} G(u_j) dx \to 0. \tag{5.1.14}$$

*Remark* 5.1.5. We remark that this type of auxiliary functionals was first used in Jeanjean [54] for a nonlinear eigenvalue problem. It should be compared with a monotonicity method due to Struwe [96, 97] and Jeanjean [54]. We expect that this type of auxiliary functionals can be applied to other problems.

We remark that our auxiliary functional $\tilde{I}(\theta, u)$ satisfies

$$\tilde{I}(0, u) = I(u),$$
$$\tilde{I}(\theta, u) = I(u(e^{-\theta} x)) \quad \text{for all } \theta \in \mathbf{R} \text{ and } u \in H_r^1(\mathbf{R}^N).$$

Properties (5.1.11)–(5.1.14) enable us to obtain the boundedness and the existence of strongly convergent subsequence of $(u_j)$.

## 5.2 Preliminaries

We will deal with the cases $N = 2$ and $N \geq 3$ in a unified way. In what follows we assume $N \geq 2$ and $g$ satisfies (5–g0), (5–g1), (5–g2') and (5–g3).

### 5.2.1 Modification of $g$

To give a proof of Theorem 5.1.3, we modify the nonlinearity $g$. First we remark that we can assume

$$\text{(g1')} \qquad \begin{aligned} &\text{when } N \geq 3, \quad \lim_{\xi\to\infty} \frac{g(\xi)}{\xi^{(N+2)/(N-2)}} = 0, \\ &\text{when } N = 2, \quad \lim_{\xi\to\infty} \frac{g(\xi)}{e^{\alpha\xi^2}} = 0 \quad \text{for any } \alpha > 0. \end{aligned}$$

In fact, if $g$ satisfies $g(\xi) > 0$ for $\xi \geq \zeta_0$, (5–g1') clearly follows from (5–g1). If there exists a $\zeta_1 > \zeta_0$ such that $g(\zeta_1) = 0$, we set

$$\tilde{g}(\xi) := \begin{cases} g(\xi) & \text{for } 0 \leq \xi \leq \zeta_1, \\ 0 & \text{for } \xi > \zeta_1, \\ -g(-\xi) & \text{for } \xi < 0. \end{cases}$$

Then $\tilde{g}$ satisfies (5–g0), (5–g1'), (5–g2'), (5–g3) and all solutions of $-\Delta u = \tilde{g}(u)$ in $\mathbf{R}^N$ which belong to $H^1(\mathbf{R}^N)$ satisfy $-\zeta_1 \leq u(x) \leq \zeta_1$ for all $x \in \mathbf{R}^N$, that is, $u$ also solves (5.1.1). Thus we may replace $g$ by $\tilde{g}$ and assume (5–g1').

In what follows, we assume that $g$ satisfies (5–g0), (5–g1'), (5–g2') and (5–g3).

Next we set

$$m_0 := -\frac{1}{2} \limsup_{\xi\to 0} \frac{g(\xi)}{\xi} \in (0, \infty)$$

and rewrite (5.1.1) as

$$-\Delta u + m_0 u = m_0 u + g(u) \quad \text{in } \mathbf{R}^N.$$

We introduce $h \in C(\mathbf{R}, \mathbf{R})$ by

$$h(\xi) := \begin{cases} \max\{m_0\xi + g(\xi), 0\} & \text{for } \xi \geq 0, \\ -h(-\xi) & \text{for } \xi < 0. \end{cases}$$

Furthermore, we choose $p_0 \in (1, (N+2)/(N-2))$ if $N \geq 3$, $p_0 \in (1, \infty)$ if $N = 2$ and set

$$\overline{h}(\xi) := \begin{cases} \xi^{p_0} \sup\limits_{0<\tau\leq\xi} \dfrac{h(\tau)}{\tau^{p_0}} & \text{for } \xi > 0, \\ 0 & \text{for } \xi = 0, \\ -\overline{h}(-\xi) & \text{for } \xi < 0. \end{cases}$$

We also set

$$H(\xi) := \int_0^\xi h(\tau)d\tau, \quad \overline{H}(\xi) := \int_0^\xi \overline{h}(\tau)d\tau.$$

From the definition of $h, \overline{h}$ and $m_0$, we have

**Lemma 5.2.1.** *The following hold*:

(a) *For all* $\xi \geq 0$, $m_0\xi + g(\xi) \leq h(\xi) \leq \overline{h}(\xi)$.

(b) *For all* $\xi \geq 0$, $h(\xi) \geq 0$ *and* $\overline{h}(\xi) \geq 0$.

(c) *There exists a* $\delta_0 > 0$ *such that* $h(\xi) = \overline{h}(\xi) = 0$ *for all* $\xi \in [0, \delta_0]$.

(d) *There exists a* $\xi_0 > 0$ *such that* $0 < h(\xi_0) \leq \overline{h}(\xi_0)$.

(e) *The map* $\xi \mapsto \overline{h}(\xi)/\xi^{p_0}; (0, \infty) \to \mathbf{R}$ *is non-increasing.*

(f) *The functions* $h$, $\overline{h}$ *satisfy* (5–g1').

*Proof.* (a), (b) and (e) follow from the definitions of $h$ and $\overline{h}$.
(c) By the definition of $m_0$, we can easily see that $\xi g(\xi) \leq -m_0\xi^2$ in a neighborhood of $\xi = 0$. Thus (c) holds for small $\delta_0 > 0$.
(d) By (5–g3), there exists a $\xi_0 \in (0, \zeta_0)$ such that $g(\xi_0) > 0$. Thus $\overline{h}(\xi_0) \geq h(\xi_0) \geq m_0\xi_0 + g(\xi_0) > 0$ and (d) holds.
(f) It is easy to see that $h$ satisfies (5–g1') and we will show (f) for $\overline{h}$. We consider the case $N \geq 3$ first. We remark that

$$\frac{\overline{h}(\xi)}{\xi^{(N+2)/(N-2)}} = \xi^{-((N+2)/(N-2)-p_0)} \sup_{0<\tau\leq\xi} \frac{h(\tau)}{\tau^{p_0}} = \sup_{0<\tau\leq\xi} \frac{h(\tau)}{\tau^{(N+2)/(N-2)}} \frac{\tau^{(N+2)/(N-2)-p_0}}{\xi^{(N+2)/(N-2)-p_0}}.$$

Since $h$ satisfies (5–g1'), for any $\varepsilon > 0$ there exists a $\tau_\varepsilon > 0$ such that

$$\left| \frac{h(\tau)}{\tau^{(N+2)/(N-2)}} \right| < \varepsilon \quad \text{for all } \tau \geq \tau_\varepsilon.$$

Thus denoting $C_\varepsilon := \sup_{0<\tau\leq\tau_\varepsilon} |h(\tau)/\tau^{(N+2)/(N-2)}|$, we have

$$\begin{aligned}\frac{\overline{h}(\xi)}{\xi^{(N+2)/(N-2)}} &\leq \max\left\{ \sup_{0<\tau\leq\tau_\varepsilon} \left| \frac{h(\tau)}{\tau^{(N+2)/(N-2)}} \right| \frac{\tau_\varepsilon^{(N+2)/(N-2)-p_0}}{\xi^{(N+2)/(N-2)-p_0}}, \sup_{\tau_\varepsilon\leq\tau\leq\xi} \left| \frac{h(\tau)}{\tau^{(N+2)/(N-2)}} \right| \right\} \\ &\leq \max\left\{ \frac{C_\varepsilon \tau_\varepsilon^{(N+2)/(N-2)-p_0}}{\xi^{(N+2)/(N-2)-p_0}}, \varepsilon \right\}.\end{aligned}$$

Therefore we have

$$\limsup_{\xi\to\infty} \frac{\overline{h}(\xi)}{\xi^{(N+2)/(N-2)}} \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\lim_{\xi\to\infty} \overline{h}(\xi)/\xi^{(N+2)/(N-2)} = 0$.

Next we deal with the case $N = 2$. It suffices to show

$$\lim_{\xi\to\infty} \frac{\overline{h}(\xi)}{\xi^{p_0} e^{\alpha\xi^2}} = 0 \quad \text{for any } \alpha > 0. \tag{5.2.1}$$

Since

$$\frac{\overline{h}(\xi)}{\xi^{p_0} e^{\alpha\xi^2}} = \frac{1}{e^{\alpha\xi^2}} \sup_{0<\tau\leq\xi} \frac{h(\tau)}{\tau^{p_0}} = \sup_{0<\tau\leq\xi} \frac{h(\tau)}{\tau^{p_0} e^{\alpha\tau^2}} \frac{e^{\alpha\tau^2}}{e^{\alpha\xi^2}}$$

and $h$ satisfies $\lim_{\xi\to\infty} h(\xi)/\xi^{p_0} e^{\alpha\xi^2} = 0$, we can show (5.2.1) in a similar way. □

**Corollary 5.2.2.** *The following hold*:

(a) *For all* $\xi \in \mathbf{R}$, $m_0\xi^2/2 + G(\xi) \le H(\xi) \le \overline{H}(\xi)$.

(b) *For all* $\xi \in \mathbf{R}$, $H(\xi), \overline{H}(\xi) \ge 0$.

(c) *There exists a* $\delta_0 > 0$ *such that* $H(\xi) = \overline{H}(\xi) = 0$ *for* $|\xi| \le \delta_0$.

(d) *It holds that* $\overline{H}(\zeta_0) - m_0\zeta_0^2/2 > 0$.

(e) *For all* $\xi \in \mathbf{R}$, $0 \le (p_0+1)\overline{H}(\xi) \le \xi\overline{h}(\xi)$.

(f) *The functions* $H$ *and* $\overline{H}$ *satisfy*

$$\lim_{|\xi|\to\infty} \frac{H(\xi)}{|\xi|^{2N/(N-2)}} = \lim_{|\xi|\to\infty} \frac{\overline{H}(\xi)}{|\xi|^{2N/(N-2)}} = 0 \quad \text{when } N \ge 3,$$
$$\lim_{|\xi|\to\infty} \frac{H(\xi)}{e^{\alpha\xi^2}} = \lim_{|\xi|\to\infty} \frac{\overline{H}(\xi)}{e^{\alpha\xi^2}} = 0 \qquad \text{for any } \alpha > 0 \text{ when } N = 2.$$

*Proof.* (a)–(c) easily follow from (a)–(c) of Lemma 5.2.1.

By (a) and (5–g3), it follows that

$$\overline{H}(\zeta_0) \ge H(\zeta_0) \ge \frac{1}{2}m_0\zeta_0^2 + G(\zeta_0) > 0.$$

Thus (d) holds.

Since the map $\xi \mapsto \overline{h}(\xi)/\xi^{p_0}; (0,\infty) \to \mathbf{R}$ is nondecreasing, we have for $\xi > 0$

$$\begin{aligned}\xi\overline{h}(\xi) - (p_0+1)\overline{H}(\xi) &= \int_0^\xi \overline{h}(\xi) - (p_0+1)\overline{h}(\tau)d\tau = \int_0^\xi \xi^{p_0}\frac{\overline{h}(\xi)}{\xi^{p_0}} - (p_0+1)\tau^{p_0}\frac{\overline{h}(\tau)}{\tau^{p_0}}d\tau \\ &\ge \int_0^\xi \xi^{p_0}\frac{\overline{h}(\xi)}{\xi^{p_0}} - (p_0+1)\tau^{p_0}\frac{\overline{h}(\xi)}{\xi^{p_0}}d\tau = 0.\end{aligned}$$

Therefore (e) holds.

The statement (f) also follows from (f) of Lemma 5.2.1. □

### 5.2.2 Fundamental properties of $H_r^1(\mathbf{R}^N)$

In what follows, we use the following notation: for $u \in H_r^1(\mathbf{R}^N)$ and $1 \le p < \infty$,

$$\|u\|_p := \left(\int_{\mathbf{R}^N} |u|^p dx\right)^{1/p}, \quad \|u\|_\infty := \operatorname*{ess\,sup}_{x\in\mathbf{R}^N} |u(x)|,$$
$$\|u\|_{H^1} := (\|\nabla u\|_2^2 + m_0\|u\|_2^2)^{1/2}.$$

We also write

$$(u,v)_2 := \int_{\mathbf{R}^N} uv dx, \quad (u,v)_{H^1} := \int_{\mathbf{R}^N} \nabla u\cdot\nabla v + m_0 uv dx.$$

We remark that $H^1_r(\mathbf{R}^N)$ equips the norm $\|\cdot\|_{H^1}$ and is a closed subspace of $H^1(\mathbf{R}^N)$.

The following properties are well-known (see [1, 15]).

(i) *For $N \geq 2$, there exists a $C_N > 0$ such that*

$$|u(x)| \leq C_N |x|^{-(N-1)/2} \|u\|_{H^1} \quad \text{for } u \in H^1_r(\mathbf{R}^N) \text{ and } |x| \geq 1. \tag{5.2.2}$$

(ii) *The embedding $H^1_r(\mathbf{R}^N) \subset L^p(\mathbf{R}^N)$ is continuous for $2 \leq p \leq 2N/(N-2)$ if $N \geq 3$, $2 < p < \infty$ if $N = 2$ and it is compact for $2 < p < 2N/(N-2)$ if $N \geq 3$, $2 < p < \infty$ if $N = 2$.*

(iii) *Set $\Phi(s) := e^s - 1$. When $N = 2$, for any $\beta \in (0, 4\pi)$ there exists a $\tilde{C}_\beta < 0$ such that*

$$\int_{\mathbf{R}^2} \Phi\left(\frac{\beta u^2}{\|\nabla u\|_2^2}\right) dx \leq \tilde{C}_\beta \frac{\|u\|_2^2}{\|\nabla u\|_2^2} \quad \text{for all } u \in H^1(\mathbf{R}^2)/\{0\}. \tag{5.2.3}$$

(iv) *In particular, for any $M > 0$*

$$\int_{\mathbf{R}^2} \Phi\left(\frac{\beta u^2}{M^2}\right) dx \leq \tilde{C}_\beta \frac{\|u\|_2^2}{M^2} \quad \text{for all } u \in H^1(\mathbf{R}^2) \text{ with } \|\nabla u\|_2 \leq M. \tag{5.2.4}$$

In fact, if $\|\nabla u\|_2 \leq M$ holds, then we have

$$\begin{aligned} M^2 \Phi\left(\frac{\beta u^2}{M^2}\right) &= M^2 \sum_{j=1}^\infty \frac{1}{j!}\left(\frac{\beta u^2}{M^2}\right)^j = \sum_{j=1}^\infty \frac{1}{j!} \frac{\beta^j u^{2j}}{M^{2j-2}} \leq \sum_{j=1}^\infty \frac{1}{j!} \frac{\beta^j u^{2j}}{\|\nabla u\|_2^{2j-2}} \\ &= \|\nabla u\|_2^2 \Phi\left(\frac{\beta u^2}{\|\nabla u\|_2^2}\right). \end{aligned}$$

Thus (5.2.4) follows from (5.2.3) (see also Byeon, Jeanjean and Tanaka [24]).

Let $\delta_0 > 0$ be a number given in Lemma 5.2.1 (c) and Corollary 5.2.2 (c). By (5.2.2), for any $M > 0$ there exists an $R_M > 0$ such that

$$|u(x)| \leq \delta_0 \quad \text{for all } |x| \geq R_M \text{ and } u \in H^1_r(\mathbf{R}^N) \text{ with } \|u\|_{H^1} \leq M. \tag{5.2.5}$$

In particular, it follows that from (5.2.5) that

$$h(u(x)),\ \overline{h}(u(x)),\ H(u(x)),\ \overline{H}(u(x)) = 0 \quad \text{for } |x| \geq R_M \text{ and } \|u\|_{H^1} \leq M. \tag{5.2.6}$$

From (5.2.6) and the compactness of the embedding $H^1_r(\mathbf{R}^N) \to L^p(\mathbf{R}^N)$, we have

**Lemma 5.2.3.** *Let $N \geq 2$ and suppose that $(u_j)_{j=1}^\infty \subset H^1_r(\mathbf{R}^N)$ converges to $u_0 \in H^1_r(\mathbf{R}^N)$ weakly in $H^1_r(\mathbf{R}^N)$. Then*

(a) $\int_{\mathbf{R}^N} H(u_j) dx \to \int_{\mathbf{R}^N} H(u_0) dx$ *and* $\int_{\mathbf{R}^N} \overline{H}(u_j) dx \to \int_{\mathbf{R}^N} \overline{H}(u_0) dx.$

(b) $h(u_j) \to h(u_0)$ *and* $\overline{h}(u_j) \to \overline{h}(u_0)$ *strongly in* $(H^1_r(\mathbf{R}^N))^*$.

*Proof.* We show only $h(u_j) \to h(u_0)$ strongly in $(H^1_r(\mathbf{R}^N))^*$ and deal with the case $N=2$. Other cases can be treated similarly.

Suppose that $\|u_j\|_{H^1} \le M$ for all $j \in \mathbf{N}$. By (5.2.4), we have

$$\int_{\mathbf{R}^N} \Phi\left(\frac{u_j^2}{M^2}\right) dx \le \frac{\tilde{C}_1}{M^2}\|u_j\|_2^2 \le \tilde{C}_1.$$

Since $h$ satisfies (5–g1'), for any $\varepsilon > 0$ there exists an $\ell_\varepsilon$ $(\ge \delta_0 > 0)$ such that

$$|h(\xi)| \le \varepsilon \Phi\left(\frac{\xi^2}{2M^2}\right) \quad \text{for } |\xi| \ge \ell_\varepsilon.$$

We set

$$\tilde{h}(\xi) := \begin{cases} h(\xi) & \text{for } |\xi| \le \ell_\varepsilon, \\ h(\ell_\varepsilon) & \text{for } \xi > \ell_\varepsilon, \\ -h(\ell_\varepsilon) & \text{for } \xi < -\ell_\varepsilon. \end{cases}$$

Then we have

$$|h(\xi) - \tilde{h}(\xi)| \le 2\varepsilon \Phi\left(\frac{\xi^2}{2M^2}\right) \quad \text{for all } \xi \in \mathbf{R}.$$

Since the embedding $H^1_r(\mathbf{R}^N) \subset L^2(|x| \le R_M)$ is compact, we have $u_j \to u_0$ strongly in $L^2(|x| \le R_M)$, which implies

$$\tilde{h}(u_j) \to \tilde{h}(u_0) \quad \text{strongly in } L^2(|x| \le R_M).$$

Thus, by (5.2.6) and the definition of $\tilde{h}$, we have $\tilde{h}(u_j(x)) = 0$ for $|x| \ge R_M$ and

$$\|\tilde{h}(u_j) - \tilde{h}(u_0)\|_2 \to 0 \quad \text{as } j \to \infty.$$

On the other hand,

$$\|h(u_j) - \tilde{h}(u_j)\|_2^2 \le 4\varepsilon^2 \int_{\mathbf{R}^2} \Phi\left(\frac{u_j^2}{2M^2}\right)^2 dx \le 4\varepsilon^2 \int_{\mathbf{R}^2} \Phi\left(\frac{u_j^2}{M^2}\right) dx \le 4\varepsilon^2 \tilde{C}_1.$$

Here we used the fact that $\Phi(s/2)^2 \le \Phi(s)$ for all $s \ge 0$. Similarly we also have $\|h(u_0) - \tilde{h}(u_0)\|_2^2 \le 4\varepsilon^2 \tilde{C}_1$. Thus

$$\begin{aligned} \|h(u_j) - h(u_0)\|_2 &\le \|h(u_j) - \tilde{h}(u_j)\|_2 + \|\tilde{h}(u_j) - \tilde{h}(u_0)\|_2 + \|\tilde{h}(u_0) - h(u_0)\|_2 \\ &\le \|\tilde{h}(u_j) - \tilde{h}(u_0)\|_2 + 4\varepsilon\sqrt{\tilde{C}_1} \to 4\varepsilon\sqrt{\tilde{C}_1} \quad \text{as } j \to \infty. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\|h(u_j) - h(u_0)\|_2 \to 0$. We remark that $H^1_r(\mathbf{R}^N) \subset L^2(\mathbf{R}^N)$ implies $L^2(\mathbf{R}^N) \subset (H^1_r(\mathbf{R}^N))^*$ and thus $h(u_j) \to h(u_0)$ strongly in $(H^1_r(\mathbf{R}^N))^*$. □

### 5.2.3 A comparison functional $J$

We define two functionals $I$ and $J; H^1_r(\mathbf{R}^N) \to \mathbf{R}$ by

$$I(u) := \frac{1}{2}\|\nabla u\|_2^2 - \int_{\mathbf{R}^N} G(u)dx = \frac{1}{2}\|u\|_{H^1}^2 - \int_{\mathbf{R}^N} \frac{1}{2}m_0u^2 + G(u)dx,$$
$$J(u) := \frac{1}{2}\|u\|_{H^1}^2 - \int_{\mathbf{R}^N} \overline{H}(u)dx.$$

Critical points of $I$ are solutions of our original problem (5.1.1)–(5.1.2) and critical points of $J$ are solutions of the following equation: $-\Delta u + m_0 u = \overline{h}(u)$ in $\mathbf{R}^N$. For $I$ and $J$, we have the following lemma.

**Lemma 5.2.4.** *The following hold*:

(a) *The functionals $I, J \in C^1(H^1_r(\mathbf{R}^N), \mathbf{R})$ and for all $u, \varphi \in H^1_r(\mathbf{R}^N)$, it holds that*

$$I'(u)\varphi = (u, \varphi)_{H^1} - \int_{\mathbf{R}^N} m_0u\varphi + g(u)\varphi dx,$$
$$J'(u)\varphi = (u, \varphi)_{H^1} - \int_{\mathbf{R}^N} \overline{h}(u)\varphi dx.$$

(b) *For all $u \in H^1_r(\mathbf{R}^N)$, $I(u) \geq J(u)$.*

(c) *There exist $r_0 > 0$ and $\rho_0 > 0$ such that*

$$I(u),\ J(u) \geq 0 \quad \text{for each } u \in H^1_r(\mathbf{R}^N) \text{ with } \|u\|_{H^1} \leq r_0,$$
$$I(u),\ J(u) \geq \rho_0 \quad \text{for each } u \in H^1_r(\mathbf{R}^N) \text{ with } \|u\|_{H^1} = r_0.$$

(d) *For any $n \in \mathbf{N}$, there exists an odd continuous mapping $\gamma_{0n} : S^{n-1} := \{ \sigma = (\sigma_1, \dots, \sigma_n) \in \mathbf{R}^n;\ |\sigma| = 1\} \to H^1_r(\mathbf{R}^N)$ such that*

$$J(\gamma_{0n}(\sigma)) \leq I(\gamma_{0n}(\sigma)) < 0 \quad \text{for all } \sigma \in S^{n-1}.$$

*Proof.* The statement (a) follows from (5–g1') and (5–g2'), and (b) follows from (a) of Corollary 5.2.2.

(c) By Corollary 5.2.2, for any $\varepsilon > 0$ there exists a $C_\varepsilon$ such that $\overline{H}(\xi) \leq C_\varepsilon|\xi|^{p_0} + \varepsilon\Psi(\xi)$ for all $\xi \in \mathbf{R}$ where $p$ appearing in the definition of $\overline{h}$ and $\Psi(\xi) = |\xi|^{2N/(N-2)}$ if $N \geq 3$, $\Psi(\xi) = e^{\xi^2} - 1$ if $N = 2$. Thus by the Sobolev's inequality and (5.2.4), for all $u \in H^1_r(\mathbf{R}^N)$ with $\|u\|_{H^1} \leq 1$, we have

$$\int_{\mathbf{R}^N} \overline{H}(u)dx \leq C_\varepsilon\|u\|_{H^1}^{p_0} + \varepsilon\int_{\mathbf{R}^N} \Psi(u)dx = \begin{cases} C_\varepsilon\|u\|_{H^1}^{p_0} + C_1\varepsilon\|u\|_2^2 & \text{if } N = 2, \\ C_\varepsilon\|u\|_{H^1}^{p_0} + C\varepsilon\|u\|_{2N/(N-2)}^{2N/(N-2)} & \text{if } N \geq 3. \end{cases}$$

Therefore, for $\|u\|_{H^1} \leq 1$, it follows that

$$I(u) \geq J(u) \geq \begin{cases} \frac{1}{2}\|u\|_{H^1}^2 - C_\varepsilon\|u\|_{H^1}^{p_0} - C_1\varepsilon\|u\|_2^2 & \text{if } N = 2, \\ \frac{1}{2}\|u\|_{H^1}^2 - C_\varepsilon\|u\|_{H^1}^{p_0} - C\varepsilon\|u\|_{2N/(N-2)}^{2N/(N-2)} & \text{if } N \geq 3. \end{cases}$$

Thus choosing $\varepsilon > 0$ and $r_0 > 0$ small enough, (c) holds.
(d) Since $\overline{h}$ is an odd function and satisfies $\overline{H}(\zeta_0) - m_0\zeta_0^2/2 \geq G(\zeta_0) > 0$, we can argue as in Theorem 10 of [16] and find for any $n \in \mathbf{N}$ an odd continuous mapping $\pi_n : S^{n-1} \to H_r^1(\mathbf{R}^N)$ such that

$$0 \notin \pi_n(S^{n-1}), \quad \int_{\mathbf{R}^N} G(\pi_n(\sigma))dx \geq 1 \quad \text{for all } \sigma \in S^{n-1}.$$

For $\ell \geq 1$, set

$$\gamma_{0n}(\sigma)(x) := \pi_n(\sigma)(x/\ell) : S^{n-1} \to H_r^1(\mathbf{R}^N).$$

Then

$$I(\gamma_{0n}(\sigma)) = \frac{\ell^{N-2}}{2}\|\nabla\pi_n(\sigma)\|_2^2 - \ell^N \int_{\mathbf{R}^N} G(\pi_n(\sigma))dx \leq \frac{\ell^{N-2}}{2}\|\nabla\pi_n(\sigma)\|_2^2 - \ell^N.$$

Thus for sufficiently large $\ell = \ell_n \geq 1$, $\gamma_{0n}$ has the desired property. □

By the above lemma, $I$ and $J$ have symmetric mountain pass geometry and we can define symmetric mountain pass values. We will give them in section 5.3.

One of the virtue of our comparison functional $J$ is the following:

**Lemma 5.2.5.** *The functional $J$ satisfies the Palais–Smale compactness condition.*

*Proof.* Since $\overline{h}$ satisfies the global Ambrosetti–Rabinowitz condition (see Corollary 5.2.2 (e)), we can easily verify the Palais–Smale condition. Indeed, let $(u_j)_{j=1}^\infty \subset H_r^1(\mathbf{R}^N)$ be a sequence satisfying

(5.2.7) $$J(u_j) \to b,$$

(5.2.8) $$\|J'(u_j)\|_{(H_r^1(\mathbf{R}^N))^*} \to 0.$$

From Corollary 5.2.2 (e), we have

(5.2.9)
$$\begin{aligned} J(u_j) - \frac{1}{p_0+1}J'(u_j)u_j &= \left(\frac{1}{2} - \frac{1}{p_0+1}\right)\|u_j\|_{H^1}^2 - \int_{\mathbf{R}^N} \overline{H}(u_j) - \frac{1}{p_0+1}\overline{h}(u_j)u_j dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p_0+1}\right)\|u_j\|_{H^1}^2. \end{aligned}$$

Thus we can get the boundedness of $(u_j)_{j=1}^\infty$ in $H_r^1(\mathbf{R}^N)$ from (5.2.7)–(5.2.9) and extract a subsequence such that $u_{j_k} \rightharpoonup u_0$ weakly in $H_r^1(\mathbf{R}^N)$. By Lemma 5.2.3 (b), we have $\overline{h}(u_{j_k}) \to \overline{h}(u_0)$ strongly in $(H_r^1(\mathbf{R}^N))^*$, thus by (5.2.8), $u_{j_k}$ converges to $u_0$ strongly in $H_r^1(\mathbf{R}^N)$, which completes the proof. □

## 5.3 Minimax arguments

By Lemma 5.2.4, $I$ and $J$ have a symmetric mountain pass geometry and we can define mountain pass and symmetric mountain pass values. Here we follow Rabinowitz [92] essentially and set for $n \in \mathbf{N}$

(5.3.1) $$b_n := \inf_{\gamma\in\Gamma_n} \max_{\sigma\in D_n} I(\gamma(\sigma)), \quad c_n := \inf_{\gamma\in\Gamma_n} \max_{\sigma\in D_n} J(\gamma(\sigma)).$$

Here $D_n := \{\sigma \in \mathbf{R}^n : \ |\sigma| \le 1\}$ and a family of mappings $\Gamma_n$ is defined by

$$\Gamma_n := \{\gamma \in C(D_n, H^1_r(\mathbf{R}^N)) : \ \gamma_n \text{ is odd and } \gamma_n(\sigma) = \gamma_{0n}(\sigma) \text{ on } \sigma \in \partial D_n\}, \tag{5.3.2}$$

where $\gamma_{0n} : \partial D_n = S^{n-1} \to H^1_r(\mathbf{R}^N)$ is given in Lemma 5.2.4. We remark that

$$\gamma(\sigma) := \begin{cases} |\sigma| \gamma_{0n}\left(\dfrac{\sigma}{|\sigma|}\right) & \text{for } \sigma \in D_n \backslash \{0\}, \\ 0 & \text{for } \sigma = 0, \end{cases}$$

belongs to $\Gamma_n$ and $\Gamma_n \neq \emptyset$ for all $n \in \mathbf{N}$.

*Remark* 5.3.1. We can define mountain pass values $b_{mp}, c_{mp}$ for $I, J$ by

$$b_{mp} := \inf_{\gamma \in \Gamma_{mp}} \max_{0 \le t \le 1} I(\gamma(t)), \quad c_{mp} := \inf_{\gamma \in \Gamma_{mp}} \max_{0 \le t \le 1} J(\gamma(t)), \tag{5.3.3}$$

where $\Gamma_{mp} := \{\gamma \in C([0,1], H^1_r(\mathbf{R}^N)) : \gamma(0) = 0, \ \gamma(1) = e_0\}$ and $e_0 \in H^1_r(\mathbf{R}^N)$ is chosen so that $I(e_0) < 0$. We will show in section 5.6 that $b_{mp}$ and $c_{mp}$ do not depend on the choice of $e_0$ (see Lemma 5.6.1). Thus, recalling $S^0 = \{\pm 1\}$ and choosing $e_0 = \gamma_{01}(1)$, we can see $b_{mp} = b_1, c_{mp} = c_1$. We will also show that $b_{mp}$ is corresponding to a positive least energy solution of (5.1.1)–(5.1.2) in section 5.6.

We can easily see that $\gamma(D_n) \cap \{u \in H^1_r(\mathbf{R}^N) : \|u\|_{H^1} = r_0\} \neq \emptyset$ for all $\gamma \in \Gamma_n$. Thus it follows from Lemma 5.2.4 (b) and (c) that

$$b_n \ge c_n \ge \rho_0 > 0. \tag{5.3.4}$$

Moreover, we have

**Lemma 5.3.2.** *The following hold*:

(a) *The value $c_n$ is a critical value of $J$ for all $n \in \mathbf{N}$.*

(b) *As $n \to \infty$, $c_n \to \infty$.*

*Proof.* (a) By Lemma 5.2.5, $J$ satisfies the Palais–Smale condition. Thus (a) holds (see for example [92]).
(b) By Theorem 7.1.1 in chapter 7, we can see that $c_n \to \infty$ as $n \to \infty$. □

By (5.3.4) and Lemma 5.3.2, the minimax values $b_n$ satisfy

$$b_n > 0 \ (n \in \mathbf{N}), \quad b_n \to \infty \text{ as } n \to \infty.$$

In the following sections we will see that the value $b_n$ is critical value of $I$ for all $n \in \mathbf{N}$.

## 5.4 Functional $\tilde{I}(\theta, u)$

It seems difficult to show the Palais–Smale compactness condition for $I$ directly and it is a main difficulty in showing that $b_n$ is a critical value of $I$.

As stated in section 5.1, we introduce an auxiliary functional $\tilde{I} \in C^1(\mathbf{R} \times H_r^1(\mathbf{R}^N), \mathbf{R})$ by

$$\tilde{I}(\theta, u) := \frac{1}{2} e^{(N-2)\theta} \int_{\mathbf{R}^N} |\nabla u|^2 dx - e^{N\theta} \int_{\mathbf{R}^N} G(u) dx.$$

The functional $\tilde{I}$ is introduced based on the scaling properties of $\|\nabla u\|_2^2$ and $\int_{\mathbf{R}^N} G(u)dx$, and has the following properties:

$$\tilde{I}(0, u) = I(u), \tag{5.4.1}$$

$$\tilde{I}(\theta, u) = I(u(e^{-\theta}x)) \quad \text{for all } \theta \in \mathbf{R} \text{ and } u \in H_r^1(\mathbf{R}^N). \tag{5.4.2}$$

We equip a standard product norm $\|(\theta, u)\|_{\mathbf{R}\times H^1} \equiv (\theta^2 + \|u\|_{H^1}^2)^{1/2}$ to $\mathbf{R} \times H_r^1(\mathbf{R}^N)$.

We define a minimax value $\tilde{b}_n$ for $\tilde{I}$ by

$$\begin{aligned}
\tilde{b}_n &:= \inf_{\tilde{\gamma}\in\tilde{\Gamma}_n} \max_{\sigma\in D_n} \tilde{I}(\tilde{\gamma}(\sigma)), \\
\tilde{\Gamma}_n &:= \{\tilde{\gamma} \in C(D_n, \mathbf{R} \times H_r^1(\mathbf{R}^N)) : \tilde{\gamma}(\sigma) = (\theta(\sigma), \eta(\sigma)) \text{ satisfies} \\
&\qquad (\theta(-\sigma), \eta(-\sigma)) = (\theta(\sigma), -\eta(\sigma)) \quad \text{for all } \sigma \in D_n, \\
&\qquad (\theta(\sigma), \eta(\sigma)) = (0, \gamma_{0n}(\sigma)) \quad \text{for all } \sigma \in \partial D_n\}.
\end{aligned}$$

Then we have

**Lemma 5.4.1.** *For each $n \in \mathbf{N}$, $\tilde{b}_n = b_n$ holds.*

*Proof.* For any $\gamma \in \Gamma_n$, we can see that $(0, \gamma) \in \tilde{\Gamma}_n$ and we may regard $\Gamma_n \subset \tilde{\Gamma}_n$. Thus by the definitions of $b_n, \tilde{b}_n$ and (5.4.1), we have $\tilde{b}_n \leq b_n$.

Next, for any given $\tilde{\gamma}(\sigma) = (\theta(\sigma), \eta(\sigma)) \in \tilde{\Gamma}_n$, we set $\gamma(\sigma) := \eta(\sigma)(e^{-\theta(\sigma)}x)$. We can verify that $\gamma \in \Gamma_n$ and by (5.4.2), $I(\gamma(\sigma)) = \tilde{I}(\tilde{\gamma}(\sigma))$ for all $\sigma \in D_n$. Thus we also have $\tilde{b}_n \geq b_n$. □

As a virtue of $\tilde{I}(\theta, u)$, we can obtain a Palais–Smale sequence $(\theta_j, u_j)_{j=1}^{\infty}$ in the augmented space $\mathbf{R} \times H_r^1(\mathbf{R}^N)$ with an additional property (d) in Proposition 5.4.2 below. Namely we have:

**Proposition 5.4.2.** *For any $n \in \mathbf{N}$, there exists a sequence $(\theta_j, u_j) \subset \mathbf{R} \times H_r^1(\mathbf{R}^N)$ such that :*

(a) $\theta_j \to 0$.

(b) $\tilde{I}(\theta_j, u_j) \to b_n$.

(c) $\tilde{I}'(\theta_j, u_j) \to 0$ *strongly in* $(H_r^1(\mathbf{R}^N))^*$.

(d) $\dfrac{\partial}{\partial\theta}\tilde{I}(\theta_j, u_j) \to 0$.

To prove Proposition 5.4.2, we need the following lemma, which is a version of Ekeland's principle. We use the following notation:

$$D\tilde{I}(\theta,u) := \left(\frac{\partial \tilde{I}}{\partial\theta}(\theta,u), \tilde{I}'(\theta,u)\right),$$

$$\mathrm{dist}_{\mathbf{R}\times H^1_r(\mathbf{R}^N)}((\theta,u),A) := \inf_{(\tau,v)\in A}(|\theta-\tau|^2+\|u-v\|^2_{H^1})^{1/2} \quad \text{for} \quad A\subset \mathbf{R}\times H^1_r(\mathbf{R}^N).$$

**Lemma 5.4.3.** *Let $n\in\mathbf{N}$ and $\varepsilon>0$. Suppose $\tilde{\gamma}\in\tilde{\Gamma}_n$ satisfies*

$$\max_{\sigma\in D_n}\tilde{I}(\tilde{\gamma}(\sigma))\le \tilde{b}_n+\varepsilon.$$

*Then there exists $(\theta,u)\in\mathbf{R}\times H^1_r(\mathbf{R}^N)$ such that:*

(a) $\mathrm{dist}_{\mathbf{R}\times H^1_r(\mathbf{R}^N)}((\theta,u),\tilde{\gamma}(D_n))\le 2\sqrt{\varepsilon}$.

(b) $\tilde{I}(\theta,u)\in[\tilde{b}_n-\varepsilon,\tilde{b}_n+\varepsilon]$.

(c) $\|D\tilde{I}(\theta,u)\|_{\mathbf{R}\times(H^1_r(\mathbf{R}^N))^*}\le 2\sqrt{\varepsilon}$.

*Proof.* Since $\tilde{I}$ satisfies

$$\tilde{I}(\theta,-u)=\tilde{I}(\theta,u) \quad \text{for all } (\theta,u)\in\mathbf{R}\times H^1_r(\mathbf{R}^N),$$

we can see that the family $\tilde{\Gamma}_n$ is stable under the pseudo-deformation flow generated by $\tilde{I}$. Moreover, since $\tilde{b}_n=b_n>0$, $\max_{\sigma\in\partial D_n}\tilde{I}(0,\gamma_{0n}(\sigma))<0$, we can show Lemma 5.4.3 in a standard way. □

*Proof of Proposition 5.4.2.* For any $j\in\mathbf{N}$, we can find a $\gamma_j\in\Gamma_n$ such that

$$\max_{\sigma\in D_n} I(\gamma_j(\sigma))\le b_n+\frac{1}{j}.$$

Since $\tilde{b}_n=b_n$, $\tilde{\gamma}_j(\sigma):=(0,\gamma_j(\sigma))\in\tilde{\Gamma}_n$ satisfies $\max_{\sigma\in D_n}\tilde{I}(\tilde{\gamma}_j(\sigma))\le\tilde{b}_n+1/j$. Applying Lemma 5.4.3, we can find a $(\theta_j,u_j)$ such that

$$\mathrm{dist}_{\mathbf{R}\times H^1_r(\mathbf{R}^N)}((\theta_j,u_j),\tilde{\gamma}_j(D_n))\le\frac{2}{\sqrt{j}}, \tag{5.4.3}$$

$$\tilde{I}(\theta_j,u_j)\in\left[b_n-\frac{1}{j},b_n+\frac{1}{j}\right], \tag{5.4.4}$$

$$\|D\tilde{I}(\theta_j,u_j)\|_{\mathbf{R}\times H^1}\le\frac{2}{\sqrt{j}}. \tag{5.4.5}$$

Since $\tilde{\gamma}(D_n)\subset\{0\}\times H^1_r(\mathbf{R}^N)$, (5.4.3) implies $|\theta_j|\le 2/\sqrt{j}$, in particular, (a). Clearly (5.4.4) implies (b) and (5.4.5) implies (c) and (d). Thus the proof of Proposition 5.4.2 is completed. □

In the following section, we consider the boundedness and compactness properties of the sequence $(\theta_j,u_j)_{j=1}^\infty$ satisfying (a)–(d) of Proposition 5.4.2.

## 5.5 Boundedness and compactness of $(\theta_j, u_j)$

Let $(\theta_j, u_j) \subset \mathbf{R} \times H^1_r(\mathbf{R}^N)$ be a sequence given in Proposition 5.4.2. In particular, $u_j$ satisfies (a)–(d) of Proposition 5.4.2. First we observe that (b) and (d) imply the following

$$\begin{aligned}
&\frac{1}{2}e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 - e^{N\theta_j}\int_{\mathbf{R}^N} G(u_j)dx \to b_n,\\
&\frac{N-2}{2}e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 - Ne^{n\theta_j}\int_{\mathbf{R}^N} G(u_j)dx \to 0 \quad \text{as } j \to \infty.
\end{aligned}$$

Thus we have

$$\|\nabla u_j\|_2^2 \to Nb_n, \tag{5.5.1}$$

$$\int_{\mathbf{R}^N} G(u_j)dx \to \frac{N-2}{2}b_n. \tag{5.5.2}$$

First we show the boundedness of $(u_j)$ in $H^1_r(\mathbf{R}^N)$.

**Proposition 5.5.1.** *Let $(\theta_j, u_j)$ be a sequence satisfying* (a)–(d) *of Proposition* 5.4.2. *Then $(u_j)$ is bounded in $H^1_r(\mathbf{R}^N)$.*

*Proof.* (cf. Proof of Proposition 5.5 of Jeanjean and Tanaka [56]). We set

$$\varepsilon_j := \|\tilde{I}'(\theta_j, u_j)\|_{(H^1_r(\mathbf{R}^N))^*}.$$

By Proposition 5.4.2 (c), we have $\varepsilon_j \to 0$ and for any $\psi \in H^1_r(\mathbf{R}^N)$,

$$|\tilde{I}'(\theta_j, u_j)\psi| \le \varepsilon_j \|\psi\|_{H^1},$$

that is,

$$\left| e^{(N-2)\theta_j}\int_{\mathbf{R}^N} \nabla u_j \cdot \nabla\psi dx - e^{N\theta_j}\int_{\mathbf{R}^N} g(u_j)\psi dx \right| \le \varepsilon_j\sqrt{\|\nabla\psi\|_2^2 + m_0\|\psi\|_2^2}. \tag{5.5.3}$$

We argue indirectly and assume $\|u_j\|_2 \to \infty$. We remark that $\|\nabla u_j\|_2$ is bounded by (5.5.1). We set $t_j := \|u_j\|_2^{-2/N} \to 0$ and $v_j(y) := u_j(y/t_j)$. Then we have

$$\|v_j\|_2 = 1 \quad \text{and} \quad \|\nabla v_j\|_2^2 = t_j^{N-2}\|\nabla u_j\|_2^2. \tag{5.5.4}$$

In particular, $(v_j)$ is bounded in $H^1_r(\mathbf{R}^N)$ and we can extract a subsequence $v_j \rightharpoonup v_0$ weakly in $H^1_r(\mathbf{R}^N)$. First we claim:

**Step 1:** $v_0 = 0$.

Let $\varphi \in H^1_r(\mathbf{R}^N)$ be a function with compact support. Setting $\psi := \varphi(t_j x)$ in (5.5.3), we have

$$\left| e^{(N-2)\theta_j} t_j^{-(N-2)}(\nabla v_j, \nabla\varphi)_2 - e^{N\theta_j}t_n^{-N}\int_{\mathbf{R}^N} g(v_j)\varphi dy \right| \le \varepsilon_j\sqrt{t_j^{-(N-2)}\|\nabla\varphi\|_2^2 + m_0 t_j^{-N}\|\varphi\|_2^2}.$$

Multiplying $t_j^N$,

$$\left| e^{(N-2)\theta_j} t_j^2 (\nabla v_j, \nabla\varphi) - e^{N\theta_j} \int_{\mathbf{R}^N} g(v_j)\varphi dy \right| \le \varepsilon_j t_j^{N/2} \sqrt{t_j^2 \|\nabla\varphi\|_2^2 + m_0 \|\varphi\|_2^2} \to 0.$$

Thus $v_0 \in H_r^1(\mathbf{R}^N)$ satisfies

$$\int_{\mathbf{R}^N} g(v_0)\varphi dy = 0 \quad \text{for all } \varphi \in H_r^1(\mathbf{R}^N) \text{ with compact support,} \tag{5.5.5}$$

which implies $g(v_0) \equiv 0$. Since $\xi = 0$ is an isolated solution of $g(\xi) = 0$ by (5–g2'), it follows from (5.5.5) that $v_0 \equiv 0$.

**Step 2:** *Conclusion*

Next we set $\psi(x) := u_j(x)$ in (5.5.3). We have

$$\left| e^{(N-2)\theta_j} t_j^{-(N-2)} \|\nabla v_j\|_2^2 - e^{N\theta_j} t_j^{-N} \int_{\mathbf{R}^N} g(v_j) v_j dx \right| \le \varepsilon_j \sqrt{t_j^{-(N-2)} \|\nabla v_j\|_2^2 + m_0 t_j^{-N} \|v_j\|_2^2}.$$

Again, multiplying $t_j^N$, it follows that

$$\delta_j := e^{(N-2)\theta_j} t_j^2 \|\nabla v_j\|_2^2 - e^{N\theta_j} \int_{\mathbf{R}^N} g(v_j) v_j dx \to 0.$$

Thus

$$\begin{aligned} e^{(N-2)\theta_j} t_j^2 \|\nabla v_j\|_2^2 + m_0 e^{N\theta_j} \|v_j\|_2^2 &= e^{N\theta_j} \int_{\mathbf{R}^N} m_0 v_j^2 + g(v_j) v_j dx + \delta_j \\ &\le e^{N\theta_j} \int_{\mathbf{R}^N} h(v_j) v_j dx + \delta_j. \end{aligned} \tag{5.5.6}$$

Here we used Lemma 5.2.1 (a). Since $v_j \rightharpoonup 0$ weakly in $H_r^1(\mathbf{R}^N)$, Lemma 5.2.3 (b) implies $\int_{\mathbf{R}^N} h(v_j) v_j dx \to 0$. Thus (5.5.6) implies $\|v_j\|_2 \to 0$, which is in contradiction to (5.5.4). Therefore $(u_j)$ is bounded in $H_r^1(\mathbf{R}^N)$. □

*Remark* 5.5.2. When $N \ge 3$, we can prove Proposition 5.5.1 in a direct way. Indeed, by the definition of $h$, we have for some constant $C > 0$

$$|h(\xi)| \le C|\xi|^{(N+2)/(N-2)} \quad \text{for all } \xi \in \mathbf{R}.$$

It follows from $\varepsilon_j = \|\tilde{I}'(\theta_j, u_j)\|_{(H_r^1(\mathbf{R}^N))^*} \to 0$ that $|\tilde{I}'(\theta_j, u_j) u_j| \le \varepsilon_j \|u_j\|_{H^1}$. Thus

$$\begin{aligned} e^{(N-2)\theta_j} \|\nabla u_j\|_2^2 + m_0 e^{N\theta_j} \|u_j\|_2^2 &\le e^{N\theta_j} \int_{\mathbf{R}^N} m_0 u_j^2 + g(u_j) u_j dx + \varepsilon_j \|u_j\|_{H^1} \\ &\le e^{N\theta_j} \int_{\mathbf{R}^N} h(u_j) u_j dx + \varepsilon_j \|u_j\|_{H^1} \\ &\le C e^{N\theta_j} \|u_j\|_{2N/(N-2)}^{2N/(N-2)} + \varepsilon_j \|u_j\|_{H^1}. \end{aligned} \tag{5.5.7}$$

Since $\|\nabla u_j\|_2$ is bounded, we can observe that $\|u_j\|_{2N/(N-2)}$ is also bounded. Thus (5.5.7) implies the boundedness of $\|u_j\|_2$, that is, $(u_j)$ is bounded in $H_r^1(\mathbf{R}^N)$.

Lastly in this section, we prove that $(u_j)$ has a strongly convergent subsequence in $H^1_r(\mathbf{R}^N)$.

**Proposition 5.5.3.** *Let* $(\theta_j, u_j)$ *be a sequence satisfying* (a)–(d) *of Proposition* 5.4.2. *Then* $(\theta_j, u_j)$ *has a strongly convergent subsequence in* $\mathbf{R} \times H^1_r(\mathbf{R}^N)$.

*Proof.* It suffices to prove $(u_j)$ has a strongly convergent subsequence in $H^1_r(\mathbf{R}^N)$. By Proposition 5.5.1, $(u_j)$ is bounded in $H^1_r(\mathbf{R}^N)$ and we may assume $u_j \rightharpoonup u_0$ weakly in $H^1_r(\mathbf{R}^N)$ as $j \to \infty$.

It follows from Proposition 5.4.2 (c) that $\tilde{I}'(\theta_j, u_j)\varphi \to 0$ as $j \to \infty$ for any $\varphi \in H^1_r(\mathbf{R}^N)$, that is,

$$\int_{\mathbf{R}^N} e^{(N-2)\theta_j} \nabla u_j \cdot \nabla \varphi - e^{N\theta_j} g(u_j)\varphi dx \to 0 \quad \text{as } j \to \infty. \tag{5.5.8}$$

Thus $u_0$ satisfies $\int_{\mathbf{R}^N} \nabla u_0 \cdot \nabla \varphi - g(u_0)\varphi dx = 0$ for all $\varphi \in H^1_r(\mathbf{R}^N)$ and $u_0$ is a solution of (5.1.1)–(5.1.2). In particular, we have $\|\nabla u_0\|_2^2 - \int_{\mathbf{R}^N} g(u_0)u_0 dx = 0$, that is,

$$\|u_0\|_{H^1}^2 - \int_{\mathbf{R}^N} m_0 u_0^2 + g(u_0)u_0 dx = 0. \tag{5.5.9}$$

Setting $\varphi := u_j$ in (5.5.8), we have $e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 - e^{N\theta_j} \int_{\mathbf{R}^N} g(u_j)u_j dx \to 0$. Thus

$$\begin{aligned} e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 + m_0 e^{N\theta_j}\|u_j\|_2^2 &= e^{N\theta_j} \int_{\mathbf{R}^N} m_0 u_j^2 + g(u_j)u_j dx + o(1) \\ &= e^{N\theta_j} \int_{\mathbf{R}^N} h(u_j)u_j dx - e^{N\theta_j} \int_{\mathbf{R}^N} h(u_j)u_j - m_0 u_j^2 - g(u_j)u_j dx + o(1) \\ &= e^{N\theta_j}(\mathrm{I}) - e^{N\theta_j}(\mathrm{II}) + o(1) \quad \text{as } j \to \infty. \end{aligned} \tag{5.5.10}$$

By Lemma 5.2.3 (b), we have

$$(\mathrm{I}) \to \int_{\mathbf{R}^N} h(u_0)u_0 dx. \tag{5.5.11}$$

On the other hand, by Lemma 5.2.1 (a), we have

$$h(u_j(x))u_j(x) - m_0 u_j(x)^2 - g(u_j(x))u_j(x) \ge 0 \quad \text{for all } j \in \mathbf{N} \text{ and } x \in \mathbf{R}.$$

Thus by Fatou's lemma,

$$\liminf_{j\to\infty}(\mathrm{II}) \ge \int_{\mathbf{R}^N} h(u_0)u_0 - m_0 u_0^2 - g(u_0)u_0 dx. \tag{5.5.12}$$

It follows from (5.5.10)–(5.5.12) that

$$\limsup_{j\to\infty} \|u_j\|_{H^1}^2 = \limsup_{j\to\infty} \left(e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 + m_0 e^{N\theta_j}\|u_j\|_2^2\right) \le \int_{\mathbf{R}^N} m_0 u_0^2 + g(u_0)u_0 dx.$$

Thus by (5.5.9), we have $\limsup_{j\to\infty} \|u_j\|_{H^1} \le \|u_0\|_{H^1}$, which implies $u_j \to u_0$ strongly in $H^1_r(\mathbf{R}^N)$. □

Now we can prove

**Theorem 5.5.4.** *Assume $N \geq 2$ and* (5–g0), (5–g1'), (5–g2'), (5–g3). *Then $b_n (n \in \mathbf{N})$ defined in* (5.3.1)–(5.3.2) *is a critical value of $I$. That is , for any $n \in \mathbf{N}$, there exists a critical point $u_{0n} \in H^1_r(\mathbf{R}^N)$, which is a solution of* (5.1.1)–(5.1.2), *such that*

$$I(u_{0n}) = b_n, \quad I'(u_{0n}) = 0. \tag{5.5.13}$$

*Proof.* Let $(\theta_j, u_j)$ be a sequence obtained in Proposition 5.4.2. By Proposition 5.5.3, we may assume $u_j \to u_{0n}$ strongly in $H^1_r(\mathbf{R}^N)$. Then $u_{0n}$ satisfies

$$\tilde{I}(0, u_{0n}) = b_n, \quad \tilde{I}'(0, u_{0n}) = 0,$$

that is nothing but (5.5.13). Thus $b_n$ is a critical value of $I$ which completes the proof. □

## 5.6 Least energy solutions

In this section, we show that a mountain pass value $b_{mp}$ is corresponding to a positive solution of (5.1.1)–(5.1.2), which has the least energy among all nontrivial solutions.

We start with the following lemma.

**Lemma 5.6.1.** *Suppose $N \geq 2$ and assume* (5–g0), (5–g1'), (5–g2') *and* (5–g3). *Let $O = \{u \in H^1_r(\mathbf{R}^N) : I(u) < 0\}$. Then the set $O$ is arcwise connected.*

We will give a proof of Lemma 5.6.1 in the section 5.7. By Lemma 5.6.1, we can easily see that the mountain pass value $b_{mp}$ given in (5.3.3) does not depend on the end point $e_0$ and we may write

$$b_{mp} = \inf_{\gamma \in \Gamma_{mp}} \max_{0 \leq t \leq 1} I(\gamma(t)), \tag{5.6.1}$$

$$\Gamma_{mp} := \{\gamma \in C([0,1], H^1_r(\mathbf{R}^N)) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}. \tag{5.6.2}$$

This fact is also used in Remark 5.3.1.

*Remark* 5.6.2. Lemma 5.6.1 is also obtained in Byeon [21] (but with a different proof). We learned [21] from Professor J. Byeon and the referee after a submission of [48].

Our main result in this section is the following.

**Theorem 5.6.3.** *Suppose $N \geq 2$ and assume* (5–g0), (5–g1'), (5–g2'), (5–g3). *Then for $b_{mp}$ defined in* (5.6.1)–(5.6.2) *it holds that*:

(a) *There exists a positive solution $u_0$ of* (5.1.1)–(5.1.2) *such that*

$$I(u_0) = b_{mp}. \tag{5.6.3}$$

(b) *For any nontrivial solution $v$ of* (5.1.1)–(5.1.2), *we have*

$$b_{mp} \leq I(v), \tag{5.6.4}$$

*that is, $u_0$ is a least energy solution of* (5.1.1)–(5.1.2) *and the value $b_{mp}$ is the least energy level.*

*Proof.* (a) We argue as in previous sections and for any $\gamma_j \in \Gamma_{mp}$ satisfying

$$\max_{0\le t\le 1} I(\gamma_j(t)) \le b_{mp} + \frac{1}{j} \tag{5.6.5}$$

we can find a $(\theta_j, u_j) \in \mathbf{R} \times H^1_r(\mathbf{R}^N)$ such that

$$\mathrm{dist}_{\mathbf{R}\times H^1_r(\mathbf{R}^N)}((\theta_j, u_j), \{0\} \times \gamma_j([0,1])) \le \frac{2}{\sqrt{j}}, \tag{5.6.6}$$

$$u_j \to u_0 \quad \text{strongly in } H^1_r(\mathbf{R}^N). \tag{5.6.7}$$

Here $u_0$ is a critical point of $I$ satisfying $I(u_0) = b_{mp}$. Since $I(u) = I(|u|)$ for all $u \in H^1_r(\mathbf{R}^N)$, we may assume $\gamma_j \in \Gamma_{mp}$ in (5.6.5) satisfies

$$\gamma_j(t)(x) \ge 0 \quad \text{for all } t \in [0,1] \text{ and } x \in \mathbf{R}^N.$$

Then it follows from (5.6.6) that

$$\|(u_j)_-\|_{H^1} \le \mathrm{dist}_{\mathbf{R}\times H^1_r(\mathbf{R}^N)}((\theta_j, u_j), \{0\} \times \gamma_j([0,1])) \to 0,$$

where $u_-(x) = \max\{0, -u(x)\}$. Thus we have $(u_0)_- = 0$ and by the maximum principle, $u_0(x) > 0$ in $\mathbf{R}^N$, and (a) is proved.
(b) To see (5.6.4), we can use argument in [55] and for any given nontrivial solution $v \in H^1_r(\mathbf{R}^N)$, we can construct a path $\gamma \in \Gamma_{mp}$ such that

$$v \in \gamma([0,1]), \quad \max_{0\le t\le 1} I(\gamma(t)) = I(v).$$

Thus we have (b) and the proof of Theorem 5.6.3 is completed. □

## 5.7 Proof of Lemma 5.6.1

The aim of this section is to give a proof of Lemma 5.6.1. We will show that for any $u_0, u_1 \in O$, there exists a continuous path $\gamma$ in $O$ joining $u_0$ and $u_1$.

In this section, we write $r = |x|$ and we identify $u(r)$ and a radially symmetric function $u(x) = u(|x|)$. We set for $R \ge 1$, $t \ge 0$,

$$\eta(R,t;r) := \begin{cases} 0 & \text{if } r \in [0,R], \\ \zeta_0(r-R) & \text{if } r \in [R, R+1], \\ \zeta_0 & \text{if } r \in [R+1, R+1+t], \\ \zeta_0(R+2+t-r) & \text{if } r \in [R+1+t, R+2+t], \\ 0 & \text{if } r \in [R+2+t, \infty). \end{cases}$$

Here $\zeta_0 > 0$ is given in (5–g3). In particular, we have $G(\zeta_0) > 0$.

We will see that $\eta(R,T;r) \in O$ for large $R, T$ and there exist continuous curves joining $u_i$ $(i = 0, 1)$ and $\eta(R,T;r)$ in $O$. Clearly this proves our Lemma 5.6.1.

We start with the following lemma.

**Lemma 5.7.1.** *There exist $R_0 \geq 1$ and $C_0, C_1 > 0$ which do not depend on $R$ and $t$ such that*:

(a) *For all $(R,t)$ with $t \geq R \geq R_0$, $I(\eta(R,t;r)) \leq -C_0 G(\zeta_0) t^N$.*

(b) *For all $R \geq R_0$, $\sup_{0 \leq t < \infty} I(\eta(R,t;r)) \leq C_1 R^{N-1}$.*

(c) *For all $R \geq R_0$, $\max_{0 \leq s \leq 1} I(s\eta(R,0;r)) \leq C_1 R^{N-1}$.*

*Proof.* For $R \geq 1$, $t \geq 0$, a direct computation gives us

$$\begin{aligned}
&I(\eta(R,t;r)) \\
=&\omega_{N-1}\left(\int_R^{R+1} + \int_{R+1}^{R+1+t} + \int_{R+1+t}^{R+2+t}\right)\left(\frac{1}{2}|\eta_r(R,t;r)|^2 - G(\eta(R,t;r))\right) r^{N-1} dr \\
\leq&\frac{\omega_{N-1}}{N} B((R+1)^N - R^N + (R+2+t)^N - (R+1+t)^N) \\
&\quad - \frac{\omega_{N-1}}{N} G(\zeta_0)((R+1+t)^N - (R+1)^N),
\end{aligned}$$

where $\omega_{N-1}$ is the surface area of the unit sphere in $\mathbf{R}^N$ and $B$ is defined by

$$B = \frac{1}{2}\zeta_0^2 + \max_{\xi \in [0,\zeta_0]} |G(\xi)|. \tag{5.7.1}$$

We remark for $R \geq 1$ and $t \geq 0$

$$\begin{aligned}
&(R+1)^N - R^N = {}_N C_1 R^{N-1} + {}_N C_2 R^{N-2} + \ldots + {}_N C_N \leq ({}_N C_1 + \ldots + {}_N C_N) R^{N-1} \\
&\qquad\qquad\qquad = (2^N - 1) R^{N-1}, \\
&(R+2+t)^N - (R+1+t)^N \leq (2^N - 1)(R+1+t)^{N-1} \leq 2^{N-1}(2^N - 1)(R+t)^{N-1}, \\
&(R+1+t)^N - (R+1)^N \geq t^N.
\end{aligned}$$

Thus there exists a constant $C_2 > 0$ independent of $R \geq 1$, $t \geq 0$ such that

$$I(\eta(R,t;r)) \leq C_2(R^{N-1} + (R+t)^{N-1}) - \frac{\omega_{N-1}}{N} G(\zeta_0) t^N. \tag{5.7.2}$$

(a)–(c) follow from (5.7.2). Indeed, if $t \geq R$, it follows from (5.7.2) that

$$I(\eta(R,t;r)) \leq C_2(t^{N-1} + (2t)^{N-1}) - \frac{\omega_{N-1}}{N} G(\zeta_0) t^N.$$

Thus for sufficiently large $R_0 \geq 1$, (a) holds.

By (a), for each $R \geq R_0$, we have $\sup_{0 \leq t < \infty} I(\eta(R,t;r)) = \max_{0 \leq t \leq R} I(\eta(R,t;r))$. From (5.7.2), we have

$$I(\eta(R,t;r)) \leq C_2(R^{N-1} + (2R)^{N-1}) \quad \text{for } t \in [0,R].$$

Thus we have (b).

For (c), recalling (5.7.1), we have

$$\begin{aligned}
I(s\eta(R,0;r)) &\leq \omega_{N-1} \int_R^{R+2} \left(\frac{1}{2}|s\eta_r(R,0;r)|^2 - G(s\eta(R,0;r))\right) r^{N-1} dr \\
&\leq \frac{\omega_{N-1}}{N} B\left((R+2)^N - R^N\right) \quad \text{for } s \in [0,1].
\end{aligned}$$

Thus, choosing $C_1 > 0$ larger if necessary, we get (c). □

Now, suppose $u_0, u_1 \in O$ and we try to join $u_0$ and $u_1$ through $\eta(R_1, T_1; r)$ ($T_1 \geq R_1 \gg 1$) in $O$. We remark that we may assume that $u_0, u_1$ have compact supports and

$$\operatorname{supp} u_0,\ \operatorname{supp} u_1 \subset [0, L_0] \quad \text{for some constant } L_0 > 0.$$

We consider the following curves:

$$\begin{aligned}
&\gamma_1 : [L_0, R_1] \to H^1_r(\mathbf{R}^N), && R \mapsto u_0(L_0 r/R),\\
&\gamma_2 : [0, 1] \to H^1_r(\mathbf{R}^N), && s \mapsto u_0(L_0 r/R_1) + s\eta(R_1, 0; r),\\
&\gamma_3 : [0, T_1] \to H^1_r(\mathbf{R}^N), && t \mapsto u_0(L_0 r/R_1) + \eta(R_1, t; r),\\
&\gamma_4 : [0, 1] \to H^1_r(\mathbf{R}^N), && s \mapsto (1-s)u_0(L_0 r/R_1) + \eta(R_1, T_1; r).
\end{aligned}$$

Joining these curves, we get the desired path joining $u_0$ and $\eta(R_1, T_1; r)$. We need to show with suitable choices of $R_1, T_1$, our path is included in $O$.

**Lemma 5.7.2.** *It holds that*

(a) *For all $R \in [L_0, \infty)$, $I(u_0(L_0 r/R)) < 0$.*

(b) *There exists an $R_1 \geq R_0$ such that*

$$I(u_0(L_0 r/R_1) + s\eta(R_1, 0; r)) < 0 \quad \text{for all } s \in [0, 1], \tag{5.7.3}$$
$$I(u_0(L_0 r/R_1) + \eta(R_1, t; r)) < 0 \quad \text{for all } t \in [0, \infty). \tag{5.7.4}$$

(c) *There exists a $T_1 \geq R_1$ such that*

$$I((1-s)u_0(L_0 r/R_1) + \eta(R_1, T_1; r)) < 0 \quad \text{for all } s \in [0, 1]. \tag{5.7.5}$$

*Proof.* (a) Since $u_0 \in O$, we have $\int_{\mathbf{R}^N} G(u_0)dx > 0$ and we can see that the map $R \mapsto I(u_0(r/R)) : [1, \infty) \to H^1_r(\mathbf{R}^N)$ is strictly decreasing. Thus (a) holds.
(b) We mainly deal with (5.7.4). Suppose $R_1 \geq R_0$, where $R_0 \geq 1$ is given in Lemma 5.7.1. We remark

$$\operatorname{supp} u_0(L_0 r/R_1) \subset [0, R_1], \quad \operatorname{supp} \eta(R_1, t; r) \subset [R_1, R_1 + 2 + t].$$

Thus for all $t \geq 0$, $R_1 \geq R_0$,

$$\begin{aligned}
I(u_0(L_0 r/R_1) + \eta(R_1, t; r)) &= I(u_0(L_0 r/R_1)) + I(\eta(R_1, t; r))\\
&\leq \frac{1}{2}\left(\frac{R_1}{L_0}\right)^{N-2} \|\nabla u_0\|_2^2 - \left(\frac{R_1}{L_0}\right)^N \int_{\mathbf{R}^N} G(u_0)dx + C_1 R_1^{N-1}.
\end{aligned}$$

Here we used Lemma 5.7.1 (b). Thus for sufficiently large $R_1 \geq R_0$, we have (5.7.4). Using Lemma 5.7.1 (c), we also get (5.7.3).
(c) As in the proof of (b), for $T_1 \geq R_1$, we have from Lemma 5.7.1 (a)

$$\begin{aligned}
I((1-s)u_0(L_0 r/R_1) + \eta(R_1, T_1; r)) &= I((1-s)u_0(L_0 r/R_1)) + I(\eta(R_1, T_1; r))\\
&\leq I((1-s)u_0(L_0 r/R_1)) - C_0 T_1^N.
\end{aligned}$$

Taking $T_1 \geq R_1$ large, we have (5.7.5).

□

*Proof of Lemma 5.6.1.* We choose $R_1 \geq R_0$ and $T_1 \geq R_1$ as in Lemma 5.7.2. We can see $\gamma_1([L_0, R_1])$, $\gamma_2([0, 1])$, $\gamma_3([0, T_1])$, $\gamma_4([0, 1]) \subset O$ and thus $u_0$ and $\eta(R_1, T_1; r)$ are connected by a continuous path in $O$. We can also join $u_1$ and $\eta(R_1, T_1; r)$ in $O$ in a similar way. Thus Lemma 5.6.1 is proved. $\square$

# Chapter 6

# Existence of positive and infinitely many solutions: inhomogeneous case

## 6.1 Introduction

In this chapter, we are concerned with the following nonlinear scalar field equation:

$$\begin{cases} -\Delta u = g(|x|, u) & \text{in } \Omega, \\ u \in H^1(\Omega). \end{cases} \tag{6.1.1}$$

Here $\Omega \subset \mathbf{R}^N$ is either the whole space $\Omega = \mathbf{R}^N$ or the exterior domain of the ball $B_R(0)$ with radius $R > 0$ ($\Omega = \{x \in \mathbf{R}^N : \ |x| > R\}$) and the function $g(r,s) : [R, \infty) \times \mathbf{R} \to \mathbf{R}$ is continuous in both variables and odd with respect to $s \in \mathbf{R}$. In the case where $\Omega$ is the exterior domain, we consider (6.1.1) under the homogeneous Dirichlet or Neumann boundary condition:

$$u = 0 \text{ on } \partial\Omega, \tag{D}$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{N}$$

where $\nu$ is the outward normal vector of $\partial\Omega$. Namely, we consider the following equations:

$(P_{\mathbf{R}^N})$ $\quad -\Delta u = g(|x|, u) \quad \text{in } \mathbf{R}^N, \quad u \in H^1(\mathbf{R}^N).$

$(P_D)$ $\quad -\Delta u = g(|x|, u) \quad \text{in } \{|x| > R\}, \quad u = 0 \quad \text{on } |x| = R, \quad u \in H^1(\{|x| > R\}).$

$(P_N)$ $\quad -\Delta u = g(|x|, u) \quad \text{in } \{|x| > R\}, \quad \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } |x| = R, \quad u \in H^1(\{|x| > R\}).$

When $\Omega = \mathbf{R}^N$ and $g(r,s)$ does not depend on $r$, that is $g(r,s) = g(s)$, $(P_{\mathbf{R}^N})$ has been studied by many researchers. For example, we refer to [14, 15, 16, 18, 19, 48, 55, 95] and references therein.

On the other hand, when $g(r,s)$ depends on $r$ in a monotone decreasing way, Li and Li [61] and Li [62] studied $(P_{\mathbf{R}^N})$ and $(P_D)$. They showed the existence of a radial positive solution and infinitely many radial possibly sign changing solutions for a suitable class of nonlinearities (see Remark 6.2.3 for a precise statement).

One of the aims in this chapter is to deal with the Neumann boundary problem $(P_N)$ as well as $(P_{\mathbf{R}^N})$ and $(P_D)$, and give a generalization of the results of [61, 62]. Especially, we relax the conditions on the behavior of $g(r,s)$ near $s=0$. In [61, 62], they assumed $\lim_{s\to 0} g(r,s)/s = -1$ uniformly with respect to $r$ (see Remark 6.2.3). However, our main results (Theorems 6.2.1 and 6.2.2 below) enable us to deal with the following case: $-\infty < \liminf_{s\to 0} \inf_{r\geq R} g(r,s)/s \leq \limsup_{s\to 0} \sup_{r\geq R} g(r,s)/s < 0$. Therefore we can treat the following example: $-\Delta u = -(V(|x|)+a(|x|)\sin^2(1/u))u + b(|x|)f(u)$ in $\Omega$ where $V(r)$, $a(r)$, $b(r)$ are monotone functions and $f(s)$ is superlinear near $s=0$.

Another aim of this chapter is to deal with nonlinear Schrödinger type problems without a monotonicity assumption on $g(r,s)$ with respect to $r$. Namely, setting $g(r,s) := -V(r)s + \tilde{g}(s)$ in (6.1.1), we consider the following equation:

$$(6.1.2) \qquad \begin{cases} -\Delta u + V(|x|)u = \tilde{g}(u) & \text{in } \Omega, \\ \quad u \in H^1(\Omega). \end{cases}$$

When $\Omega = \mathbf{R}^N$, Azzollini and Pomponio [7] studied (6.1.2) and obtained the existence of at least one radial positive solution. We give an extension of their result to the exterior problems $(P_D)$ and $(P_N)$. Moreover, we show the existence of infinitely many solutions. See Theorem 6.2.4 for a precise statement (see also Remark 6.2.5).

We will prove our theorems by variational methods and use the monotonicity method due to Struwe [96], and developed by Jeanjean [54] and Rabier [91]. With the monotonicity method, a newly developed Pohozaev type inequality (see Propositions 6.5.5 and 6.5.7) will play important roles in our argument.

This chapter is organized as follows. We state our main results in section 6.2. In section 6.3, we introduce an auxiliary functional $J$ and prepare some lemmas. Proofs of lemmas in section 6.3 will be given in section 6.6. In section 6.4, we define minimax values based on the symmetric mountain pass arguments. Section 6.5 is devoted to prove Theorems 6.2.1, 6.2.2 and 6.2.4. In section 6.6, we prove some lemmas.

## 6.2 Statement of main results

In this section, we state our main results of this chapter.

### 6.2.1 Results for the equation (6.1.1)

First we consider the equation (6.1.1). We assume that $g(r,s) : [R,\infty)\times\mathbf{R}\to\mathbf{R}$ satisfies the following conditions. In what follows, we regard $R=0$ if $\Omega=\mathbf{R}^N$.

(6–g1) $g \in C([R,\infty)\times\mathbf{R},\mathbf{R})$ and $g(r,-s) = -g(r,s)$ for all $r\geq R$ and $s\in\mathbf{R}$.

(6–g2) If $R\leq r_1\leq r_2<\infty$ and $s\geq 0$, then $g(r_1,s)\leq g(r_2,s)$.

(6–g3) As $r\to\infty$, $g(r,s)\to g_\infty(s)$ in $L^\infty_{\text{loc}}(\mathbf{R})$.

(6–g4) There exists an $m_1 > 0$ such that

$$\infty < \liminf_{s\to 0} \inf_{r\geq R} \frac{g(r,s)}{s} \leq \limsup_{s\to 0} \sup_{r\geq R} \frac{g(r,s)}{s} \leq -m_1$$

(6–g5) For $N \geq 3$,

$$\lim_{s\to\infty} \sup_{r\geq R} \frac{|g(r,s)|}{s^{2^*-1}} = 0 \quad \text{where} \quad 2^* = 2N/(N-2).$$

For $N = 2$,

$$\lim_{s\to\infty} \sup_{r\geq R} \frac{|g(r,s)|}{\exp(\alpha s^2)} = 0 \quad \text{for any } \alpha > 0.$$

(6–g6) There exist $\zeta_0 > 0$ and $R \geq R_0$ such that

$$\inf_{r\geq R_0} G(r,\zeta_0) > 0 \quad \text{where} \quad G(r,s) := \int_0^s g(r,\tau)d\tau$$

Except for (6–g3) and (6–g4), the above conditions are same to the ones in [61, 62]. As for (6–g4), this type of condition is used in [15, 16, 48, 95] when $g(r,s)$ does not depend on $r$, i.e., $g(r,s) = g(s)$ (cf. see also ($\tilde{\mathrm{g}}$2) below). We remark that in [61, 62], they suppose $\lim_{s\to 0} g(r,s)/s = -1$ uniformly with respect to $r$, which is stronger than (6–g4).

For the Neumann problem $(P_N)$, in addition to (6–g1)–(6–g6), we assume

(6–g7) $-\infty < \inf_{s\in\mathbf{R}} G(R,s)$.

Our main results are as follows. First we state a result for $(P_N)$.

**Theorem 6.2.1.** *Suppose that* $\Omega = \{|x| > R\}$ *and* (6–g1)–(6–g7) *are satisfied. Then* $(P_N)$ *has at least one radial positive solution and infinitely many radial possibly sign changing solutions.*

For $(P_{\mathbf{R}^N})$ and $(P_D)$, we assume (6–g1)–(6–g6) and we do not need (6–g7).

**Theorem 6.2.2.** *Suppose that* $\Omega = \mathbf{R}^N$ *(resp.* $\Omega = \{|x| > R\}$*) and* (6–g1)–(6–g6) *are satisfied. Then* $(P_{\mathbf{R}^N})$ *(resp.* $(P_D)$*) has at least one radial positive solution and infinitely many radial possibly sign changing solutions.*

*Remark* 6.2.3. In [61, 62], in addition to (6–g1), (6–g2), (6–g4)–(6–g6), they suppose that the function $g$ has a form $g(r,s) = -s + f(r,s)$ where $f(r,s) = o(1)$ uniformly with respect to $r$ as $s \to 0$ (cf. (6–g4)). Under these conditions, they proved the existence of one radial positive solution and infinitely many radial possibly sign changing solutions to $(P_{\mathbf{R}^N})$ and $(P_D)$. However Theorem 6.2.2 enables us to deal with the following type of equations: $-\Delta u = -(V(|x|) + a(|x|)\sin^2(1/u))u + b(|x|)f(u)$ where $V, a, b$ are monotone functions and $f(s)$ is superlinear near $s = 0$.

### 6.2.2 Results for the equation (6.1.2)

Next we consider (6.1.2) for $N \geq 3$. We write $g(r,s) = -V(r)s + \tilde{g}(s)$ and assume the following conditions:

(g̃1) $\tilde{g} \in C(\mathbf{R},\mathbf{R})$ and $\tilde{g}(-s) = -\tilde{g}(s)$ for all $s \in \mathbf{R}$.

(g̃2) There exists an $m_1 > 0$ such that $-\infty < \liminf_{s\to 0} \frac{\tilde{g}(s)}{s} \leq \limsup_{s\to 0} \frac{\tilde{g}(s)}{s} \leq -\tilde{m}_1$.

(g̃3) $\limsup_{s\to\infty} \frac{\tilde{g}(s)}{s^{2^*-1}} \leq 0$.

(g̃4) There exists a $\tilde{\zeta}_0 > 0$ such that $\tilde{G}(\tilde{\zeta}_0) > 0$ where $\tilde{G}(s) := \int_0^s \tilde{g}(\tau)d\tau$.

(g̃5) $-\infty < \inf_{s\in\mathbf{R}} \left( -\frac{1}{2}V(R)s^2 + \tilde{G}(s) \right)$.

The conditions (g̃1)-(g̃4) are same to the ones in [15, 16, 48]. The condition (g̃5) corresponds to (6–g7) above and is only needed for $(P_N)$. For $V$, we assume the following:

(6–V1) $V \in C^1([R,\infty))$ and $V(r) \geq 0$ for all $r \geq R$.

(6–V2) $\lim_{r\to\infty} V(r) = 0$.

(6–V3) $\|(x\cdot\nabla V(|x|))^+\|_{L^{\frac{N}{2}}(|x|>R)} < 2S_N$ where

$$(x\cdot\nabla V(|x|))^+ := \max\{0, x\cdot\nabla V(|x|)\} \quad \text{and} \quad S_N := \inf_{u\in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\mathbf{R}^N)}}{\|u\|^2_{L^{2^*}(\mathbf{R}^N)}}.$$

When $\Omega = \mathbf{R}^N$, the above conditions (g̃1)–(g̃4) and (6–V1)–(6–V3) are same to the ones in [7]. Next we give a remark about (6–V3). If $g(r,s) = -V(r)s + \tilde{g}(s)$ satisfies (6–g2), then we can see $x\cdot\nabla V(|x|) \leq 0$, which implies (6–V3). Therefore, we can relax the monotonicity condition (6–g2) by (6–V3) for the equation (6.1.2).

Now we state a result for (6.1.2).

**Theorem 6.2.4.** *Suppose that $N \geq 3$ and $g(r,s) = -V(r)s + \tilde{g}(s)$ satisfies* (g̃1)–(g̃4) *and* (6–V1)–(6–V3)*. Then the following hold*:

(i) $(P_{\mathbf{R}^N})$ *(resp. $(P_D)$) admits at least one radial positive solution and infinitely many possibly radial sign–changing solutions.*

(ii) *Assume* (g̃5) *in addition to* (g̃1)–(g̃4) *and* (6–V1)–(6–V3)*. Then $(P_N)$ admits at least one radial positive solution and infinitely many possibly radial sign changing solutions.*

*Remark* 6.2.5. In [7], they showed the existence of one radially symmetric positive solution to $(P_{\mathbf{R}^N})$ with $g(r,s) = -V(r)s + \tilde{g}(s)$ under the conditions (g̃1)–(g̃4) and (6–V1)–(6–V3).

In the following, we give an idea of proofs of Theorems 6.2.1, 6.2.2, 6.2.4.

We will prove Theorems 6.2.1, 6.2.2 and 6.2.4 by variational methods, and find critical points of

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(|x|, u)dx.$$

One of difficulties is to show the boundedness of Palais–Smale (for short (PS)) sequences.

In [61, 62], they introduced the following parametrized functional in order to obtain bounded (PS) sequences: (cf. Remark 6.2.3)

$$\hat{I}_\lambda(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 dx - \int_\Omega F(|x|, u)dx - \lambda \int_\Omega q(|x|)B(u)dx, \qquad \lambda \in [0, 1].$$

Here $F(r, s) := \int_0^s f(r, t)dt$, and $B(s)$ and $q(r)$ are suitable penalty functions. The virtue of their penalty functions is that $\hat{I}_\lambda$ satisfies the (PS) condition. However, the construction is rather complicated.

In our proofs, we consider another parametrized functional to obtain bounded (PS) sequences:

$$I_\lambda(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(|x|, u)dx - \lambda \int_\Omega H(u)dx \qquad \lambda \in [0, 1].$$

Here $H(s)$ is also a penalty function which is different from $B(s)$ in $\hat{I}_\lambda$ and we can construct the function $H(s)$ in a simply way (see the definition of $H(s)$ in section 6.3). To obtain critical points of $I$, we will apply the monotonicity method to $I_\lambda$. Here, we apply a version of Rabier [91] (see Propositions 6.5.1 and 6.5.2), and obtain sequences $(\lambda_k)$ and $(u_k)$ such that

$$\lambda_k \to 0, \quad -\Delta u_k = g(|x|, u_k) + \lambda_k h(u_k) \quad \text{in } \Omega,$$

where $h(s) := H'(s)$. To show that $(u_k)$ has a strongly convergent subsequence, we use the Pohozaev type inequality (6.5.2), (6.5.8), (6.5.9). Here we remark that in [61, 62] they used the Pohozaev Identity (for instance, see (6.5.7), (6.5.10), (6.5.11)) which includes the term $x \cdot \nabla G(|x|, u)$ and they need to approximate $g(r, s)$ with a function of class $C^1$ in $r$. However, in this paper, we introduce a new Pohozaev type inequality, which enables us to argue without introducing approximations.

Our proofs can also be applied for the equation (6.1.2), namely $g(r, s) = -V(r)s + \tilde{g}(s)$ in (6.1.1). By virtue of our proofs of Theorems 6.2.1 and 6.2.2, we will be able to show that not only $(P_{\mathbf{R}^N})$ but also $(P_D)$ and $(P_N)$ admit at least one radial positive solution and infinitely many radial possibly sign changing solutions under the conditions (g̃1)–(g̃4), (6–V1)–(6–V3) or (g̃1)–(g̃5), (6–V1)–(6–V3).

## 6.3 Preliminaries

In this section, we introduce an auxiliary functional $J$ and state some lemmas. A Proof of Lemma 6.3.2 will be given in section 6.6.

First, we remark that when we consider $(P_D)$ or $(P_N)$ under the assumptions of Theorems 6.2.1, 6.2.2 or 6.2.4 we may assume $R = 1$ without loss of generality. Indeed, set

$\Omega = \{|x| > R\}$, $v(x) := u(Rx)$ and $g_R(r,s) := R^2 g(Rr,s)$. Then (6.1.1) is equivalent to the following equation:

$$-\Delta v = g_R(r,v) \quad \text{in } \{|x| > 1\}.$$

Moreover, it is easily seen that $g$ satisfies (6–g1)–(6–g7) in $\{|x| > R\}$ if and only if $g_R$ satisfies (6–g1)–(6–g7) in $\{|x| > 1\}$. In the case where $g(r,s) = -V(r)s + \tilde{g}(s)$, set $V_R(r) := R^2 V(Rr)$ and $\tilde{g}_R(s) := R^2 \tilde{g}(s)$. Then it is also clear that $V$ and $\tilde{g}$ satisfy (6–V1)–(6–V3), ($\tilde{g}$1)–($\tilde{g}$5) in $\{|x| > R\}$ if and only if $V_R$ and $\tilde{g}_R$ satisfy (6–V1)–(6–V3), ($\tilde{g}$1)–($\tilde{g}$5) in $\{|x| > 1\}$. Therefore to prove Theorems 6.2.1, 6.2.2 and 6.2.4, we may $R = 1$ without loss of generality.

Hereafter we mainly consider $(P_N)$ and let $\Omega = \{x \in \mathbf{R}^N : |x| > 1\}$. Furthermore we assume the following condition in this section:

$(\mathcal{H}_1)$ The conditions (6–g1) and (6–g3)–(6–g5) are satisfied.

In order to obtain radial solutions, we consider the following function space:

$$E := H^1_r(\Omega) = \{u \in H^1(\Omega) : u \text{ is a radial function }\}.$$

The following properties hold (For (i) and (ii), see Berestycki and Lions [15], Strauss [95]):

(i) *There exists a $C > 0$ such that for all $u \in E$ and $|x| \geq 1$,*

$$|u(x)| \leq C|x|^{-\frac{N-1}{2}} \|u\|_{H^1(\Omega)}. \tag{6.3.1}$$

(ii) *The embedding $E \subset L^q(\Omega)$ is continuous for $2 \leq q \leq 2^*$ if $N \geq 3$ and $2 \leq q < \infty$ if $N = 2$ and it is compact for $2 < q < 2^*$ if $N \geq 3$ and $2 < q < \infty$ if $N = 2$.*

(iii) *For each $s \in (0,1]$, we define the extension operator $T_s : H^1_r(\{|x| > s\}) \to H^1_r(\mathbf{R}^N)$ by*

$$(T_s u)(x) := (T_s u)(|x|) = \begin{cases} u(|x|) & \text{if } |x| \geq s, \\ u(2s - |x|) & \text{if } |x| < s. \end{cases} \tag{6.3.2}$$

*Then, for each $s \in (0,1]$ and $u \in H^1_r(\{|x| > s\})$, it holds that*

$$\|T_s u\|_{L^2(\mathbf{R}^N)} \leq \sqrt{2}\|u\|_{L^2(\{|x|>s\})}, \quad \|\nabla T_s u\|_{L^2(\mathbf{R}^N)} \leq \sqrt{2}\|\nabla u\|_{L^2(\{|x|>s\})}. \tag{6.3.3}$$

*Using (6.3.3), we have the following Sobolev inequality holds for $N \geq 3$:*

$$\|u\|_{L^{2^*}(\{|x|>s\})} \leq C\|\nabla u\|_{L^2(\{|x|>s\})} \quad \text{for all } u \in H^1_r(\{|x| > s\}),\ s \in (0,1]. \tag{6.3.4}$$

We define the following functional:

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(|x|,u)dx : E \to \mathbf{R}.$$

We note that $I \in C^1(E,\mathbf{R})$ under the condition $(\mathcal{H}_1)$ and the functional $I$ corresponds to $(P_N)$. So, we will find critical points of $I$.

Following chapter 5, we prepare a penalty function to construct an auxiliary functional. For $s \geq 0$, we define $f(s)$ and $h(s)$ as follows:

$$f(s) := \max \left\{ 0, \frac{1}{2} m_1 s + \sup_{r \geq 1} g(r,s) \right\}, \quad h(s) := s^p \sup_{0<\tau\leq s} \frac{f(\tau)}{\tau^p}.$$

Here $m_1$ is a constant appearing in (6–g4) and $p$ is a positive number satisfying $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $1 < p < \infty$ if $N = 2$. Note that by (6–g3) and (6–g4), $f$ and $h$ are well–defined. We extend $h$ as an odd function on $\mathbf{R}$ and set

$$H(s) := \int_0^s h(t)dt.$$

Then $h$ and $H$ have the following properties.

**Lemma 6.3.1** (cf. Lemma 5.2.1 and Corollary 5.2.2 in Chapter 5)**.** *The following properties hold*:

(i) $h \in C(\mathbf{R})$, $0 \leq h(s)$ *and* $h(-s) = -h(s)$ *for all* $s \in [0, \infty)$.

(ii) *There exists an* $s_0 > 0$ *such that* $h = H = 0$ on $[-s_0, s_0]$.

(iii) *For all* $s \in \mathbf{R}$, *it follows that*

$$\frac{1}{2} m_1 s^2 + \sup_{r\geq 1} g(r,s)s \leq h(s)s, \qquad \frac{1}{4} m_1 s^2 + \sup_{r \geq 1} G(r,s) \leq H(s).$$

(iv) *It holds that*

$$\lim_{s\to\infty} \frac{h(s)}{\exp(\alpha s^2)} = 0 \quad \text{for all } \alpha > 0 \quad \text{if } N = 2,$$
$$\lim_{s\to\infty} \frac{h(s)}{s^{2^*-1}} = 0 \quad \text{if } N \geq 3.$$

(v) *The function* $h$ *satisfies a global Ambrosetti–Rabinowitz condition:*

$$0 \leq (p+1)H(s) \leq h(s)s \quad \text{for all } s \in \mathbf{R}.$$

*Here* $p$ *appears in the definition of* $h$.

Since we can prove Lemma 6.3.1 as in Lemma 5.2.1 and Corollary 5.2.2 in Chapter 5, we omit a proof.

Next we rewrite the functional $I$ as follows:

$$I(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \int_\Omega G(|x|, u)dx = \frac{1}{2}\|u\|_E^2 - \int_\Omega \frac{m_1}{4}u^2 + G(|x|,u)dx$$

where

$$\|u\|_E^2 := \|\nabla u\|_{L^2}^2 + \frac{m_1}{2}\|u\|_{L^2}^2.$$

We remark that $\|\cdot\|_E$ and the standard $H^1$-norm are equivalent.

Next, we define a parametrized functional $I_\lambda$ ( $\lambda \in [0,1]$ ) and an auxiliary functional $J$ which gives us lower bounds of minimax values $b_n(\lambda)$ defined in section 6.4:

$$
\begin{aligned}
I_\lambda(u) &:= \frac{1}{2}\|u\|_E^2 - \int_\Omega \frac{m_1}{4}u^2 + G(|x|,u) + \lambda H(u)dx \in C^1(E,\mathbf{R}),\\
J(u) &:= \frac{1}{2}\|u\|_E^2 - 2\int_\Omega H(u)dx \in C^1(E,\mathbf{R}),
\end{aligned}
$$

Note that if $\lambda = 0$, then $I_0(u) = I(u)$. Furthermore, by Lemma 6.3.1, $I_\lambda$ and $J$ satisfy the following: for any $0 \le \lambda_1 \le \lambda_2 \le 1$ and $u \in E$,

$$
J(u) \le I_1(u) \le I_{\lambda_2}(u) \le I_{\lambda_1}(u) \le I_0(u). \tag{6.3.5}
$$

Now we state properties of $I_\lambda$ and $J$. Similar properties are obtained in [7, 48]

**Lemma 6.3.2.** (cf. Lemma 3.5 in [7], Lemmas 5.2.3, 5.2.5, Proposition 5.5.3 in Chapter 5) *Set $K(u) := \int_\Omega H(u)dx$. Then,*

(i) *The maps $K : E \to \mathbf{R}$ and $K' : E \to E^*$ are weakly continuous.*

(ii) *Any bounded* (PS) *sequence $(u_k) \subset E$ for $I_\lambda$ has a strongly convergent subsequence.*

(iii) *The functional $J$ satisfies the* (PS) *condition.*

## 6.4 Minimax arguments

In this section, we define minimax values $b_n(\lambda)$ of $I_\lambda$ based on the arguments of symmetric mountain pass theorem (cf. [48] and Rabinowitz [92]). In this section, we assume the following conditions:

$(\mathcal{H}_2)$ The conditions (6–g1) and (6–g3)–(6–g6) are satisfied.

First of all, we prove that $I_\lambda$ and $J$ have a symmetric mountain pass geometry under the condition $(\mathcal{H}_2)$. More precisely, we have

**Lemma 6.4.1.** *The following hold*:

(i) *There exist $\delta > 0$ and $\rho > 0$ such that*

$$
\begin{aligned}
&0 < \delta \le J(u) && \text{for all } u \in E \text{ with } \|u\|_E = \rho,\\
&0 \le J(u) && \text{for all } u \in E \text{ with } \|u\|_E \le \rho.
\end{aligned}
$$

(ii) *For each $n \in \mathbf{N}$, there exists an odd continuous map $\gamma_n : S^{n-1} \to H^1_{0,r}(\Omega)$ such that*

$$
I_0(\gamma_n(\sigma)) < 0 \quad \text{for all } \sigma \in S^{n-1}.
$$

*Here*

$$
S^{n-1} := \{\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbf{R}^n : |\sigma| = 1\}, \quad H^1_{0,r}(\Omega) := \{u \in E : u(1) = 0\}.
$$

*Remark* 6.4.2. By (6.3.5), we see that $I_\lambda$ and $J$ have a symmetric mountain pass geometry.

*Proof.* We only prove (i). (ii) will be proven in section 6.6.

First, we show for $N \geq 3$. By Lemma 6.3.1, there exists a $C > 0$ such that

$$H(s) \leq C|s|^{2^*} \quad \text{for all } s \in \mathbf{R}.$$

Using Sobolev's embedding, we obtain

$$J(u) \geq \|u\|_E^2 - C\|u\|_{L^{2^*}(\Omega)}^{2^*} \geq \|u\|_E^2(1 - C\|u\|_E^{2^*-2}).$$

Thus (i) holds for $N \geq 3$.

Next we consider the case $N = 2$. By Lemma 6.3.1, there exists a $C_1 > 0$ such that

$$H(s) \leq C_1\Phi(s^2/2) \quad \text{where } \Phi(s) := \exp(s) - 1 - s.$$

By Lemma 6.6.2 (iii), we have

$$\int_\Omega H(u)dx \leq C_2\|u\|_E^4 \quad \text{for all } u \in E \text{ with } \|u\|_E \leq 1.$$

Thus it follows that if $\|u\|_E \leq 1$, then

$$J(u) \geq \|u\|_E^2 - C_2\|u\|_E^4,$$

which completes the proof of (i). □

Next, we define minimax values of $I_\lambda$ and $J$ using mappings $(\gamma_n)$ appearing in Lemma 6.4.1.

**Definition 6.4.3.** For each $n \in \mathbf{N}$ and $\lambda \in [0, 1]$, we define $b_n(\lambda)$ and $c_n$ as follows:

$$b_n(\lambda) := \inf_{\gamma\in\Gamma_n} \max_{\sigma\in D_n} I_\lambda(\gamma(\sigma)), \quad c_n := \inf_{\gamma\in\Gamma_n} \max_{\sigma\in D_n} J(\gamma(\sigma)),$$

where

$$D_n := \{\sigma \in \mathbf{R}^n : |\sigma| \leq 1\}, \quad \Gamma_n := \{\gamma \in C(D_n, E) : \gamma \text{ is odd and } \gamma = \gamma_n \text{ on } S^{n-1}\}.$$

The values $b_n(\lambda)$ and $c_n$ have the following properties.

**Lemma 6.4.4.** *The following properties hold*:

(i) $\Gamma_n \neq \emptyset$ *for all* $n \in \mathbf{N}$.

(ii) *For each* $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, $b_n(\lambda_2) \leq b_n(\lambda_1)$.

(iii) *For each* $n \in \mathbf{N}$ *and* $\lambda \in [0, 1]$, *it holds that* $0 < \delta \leq c_n \leq b_n(\lambda)$ *where* $\delta$ *appears in Lemma* 6.4.1 (i).

*Proof.* (i) We define $\tilde{\gamma}_n$ as follows: for $\sigma \in D^n$, $\tilde{\gamma}_n(\sigma) := |\sigma|\gamma_n(\sigma/|\sigma|)$. Then $\tilde{\gamma}_n \in \Gamma_n$.
(ii) By (6.3.5), (ii) holds.
(iii) By (6.3.5) and (i), it holds $c_n \le b_n(\lambda)$ for each $\lambda \in [0,1]$. The property $\delta \le c_n$ follows from the fact

$$\{u \in E : \|u\|_E = \rho\} \cap \gamma(D_n) \neq \emptyset \quad \text{for all } \gamma \in \Gamma_n.$$

□

Since $J$ satisfies the (PS) condition by Lemma 6.3.2, we can show the following lemma by Theorem 7.1.1 in Chapter 7.

**Lemma 6.4.5.** (c.f. Lemma 5.3.2 in Chapter 5) *The following hold:*

(i) *The values $c_n$is a critical value of $J$.*

(ii) *As $n \to \infty$, $c_n \to \infty$.*

# 6.5 proofs of Theorems 6.2.1, 6.2.2 and 6.2.4

In this section, we prove Theorems 6.2.1, 6.2.2 and 6.2.4 by using the monotonicity method and the Pohozaev type inequality (Propositions 6.5.5 and 6.5.7).

## 6.5.1 Monotonicity method

First, we will recall Rabier's result [91]. Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{A} : X \to \mathbf{R}$, $\mathcal{B} : [0,1] \times X \to \mathbf{R}$ be $C^1$ functionals and set $\mathcal{I}_\lambda(u) := \mathcal{A}(u) - \mathcal{B}(\lambda, u)$. We assume that $\mathcal{A}$ and $\mathcal{B}$ satisfy the following:

(BPS1) $\mathcal{B}(\cdot, u)$ is nondecreasing on $[0,1]$ for every $u \in X$.

(BPS2) $\displaystyle\lim_{\mathcal{B}(\lambda,u)\to\infty} \frac{\partial \mathcal{B}}{\partial \lambda}(\lambda, u) = \infty.$

(BPS3) $\displaystyle\lim_{\|u\|\to\infty} \mathcal{A}(u) = \infty.$

Moreover, we suppose that there exist $e_1, e_2 \in X$ such that

(BSP4) $\max\{\mathcal{I}_\lambda(e_1), \mathcal{I}_\lambda(e_2)\} < c_\lambda$ for all $\lambda \in [0,1]$.

Here

$$c_\lambda := \inf_{\gamma\in\Gamma^*} \max_{0\le t\le 1} \mathcal{I}_\lambda(\gamma(t)),$$
$$\Gamma^* := \{\gamma \in C([0,1], X) \mid \gamma(0) = e_1,\ \gamma(1) = e_2\}.$$

Then the following proposition holds.

**Proposition 6.5.1** (Rabier [91])**.** *Under the conditions* (BPS1)–(BPS4)*, for almost every $\lambda \in [0,1]$, $\mathcal{I}_\lambda$ has a bounded* (PS) *sequence at level $c_\lambda$.*

We will apply the above proposition for the functional which satisfies the symmetric mountain pass structure. Assume the following conditions in addition to (BSP1)–(BSP3):

(BPS5) $A(-u) = A(u)$ and $B(\lambda, -u) = B(\lambda, u)$ for all $u \in X$ and $\lambda \in [0, 1]$.

(BPS6) For each $n \in \mathbf{N}$, there exists a continuous odd map $\gamma_n^* : S^{n-1} \to H_r^1(\Omega)$ such that

$$\max_{\sigma \in S^{n-1}} \mathcal{I}_\lambda(\gamma_n^*(\sigma)) < d_n(\lambda).$$

Here

$$d_n(\lambda) := \inf_{\gamma \in \Gamma_n^*} \max_{\sigma \in D^n} \mathcal{I}_\lambda(\gamma(\sigma)),$$
$$\Gamma_n^* := \{\gamma \in C(D_n, E) : \gamma \text{ is odd and } \gamma = \gamma_n^* \text{ on } S^{n-1}\}$$

The following proposition holds from the arguments in [91].

**Proposition 6.5.2.** *Suppose* (BPS1)–(BPS3), (BPS5)–(BPS6). *Then, for almost every* $\lambda \in [0, 1]$, *there exists a bounded* (PS) *sequence of* $\mathcal{I}_\lambda$ *at level* $d_n(\lambda)$ *for all* $n \in \mathbf{N}$.

Next, we show that we can apply Proposition 6.5.2 for $I_\lambda$ to obtain a bounded (PS) sequence of $I_\lambda$.

**Lemma 6.5.3.** *Under the assumption* $(\mathcal{H}_2)$, *for almost every* $\lambda \in [0, 1]$, $I_\lambda$ *has a bounded* (PS) *sequence at level* $b_n(\lambda)$ *for all* $n \in \mathbf{N}$.

*Proof.* Set $X := E$, $\gamma_n^* := \gamma_n$,

$$\mathcal{A}(u) := \frac{1}{2}\|u\|_E^2, \quad \mathcal{B}(\lambda, u) := \int_\Omega \frac{m_1}{4}u^2 + G(|x|, u) + \lambda H(u)dx.$$

It is easily seen that (BPS1), (BPS3) and (BPS5) are satisfied. Moreover, by Lemmas 6.4.1 and 6.4.4, (BPS6) holds. As to (BPS2), by Lemma 6.3.1, we have

$$\mathcal{B}(\lambda, u) \le (1 + \lambda) \int_\Omega H(u)dx.$$

On the other hand, it follows that

$$\frac{\partial \mathcal{B}}{\partial \lambda}(\lambda, u) = \int_\Omega H(u)dx,$$

which implies (BPS2). Then by Proposition 6.5.2, for almost every $\lambda \in [0, 1]$, $I_\lambda$ has a bounded (PS) sequence at level $b_n(\lambda)$ for all $n \in \mathbf{N}$. □

Combining Lemmas 6.3.2 and 6.5.3, we have the following:

**Proposition 6.5.4.** *Suppose that* $(\mathcal{H}_2)$ *is satisfied. Then for almost every* $\lambda \in (0, 1]$, *there is a critical point* $u_{\lambda,n} \in E$ *such that* $I_\lambda(u_{\lambda,n}) = b_n(\lambda)$ *for all* $n \in \mathbf{N}$.

From Proposition 6.5.4, it follows that for each $n \in \mathbf{N}$, there exist $(\lambda_{n,k}) \subset [0, 1]$, $(u_{n,k}) \subset E$ such that $\lambda_{n,k} \to 0$ and

$$I_{\lambda_{n,k}}(u_{n,k}) = b_n(\lambda_{n,k}), \quad I'_{\lambda_{n,k}}(u_{n,k}) = 0. \tag{6.5.1}$$

### 6.5.2 Pohozaev type inequality

To show that $(u_{n,k})$ in (6.5.1) is bounded, we introduce the following Pohozaev type inequality.

**Proposition 6.5.5.** *Assume that the conditions* (6–g1)–(6–g6) *are satisfied. Let* $u_N \in E$ *be a solution of*

$$-\Delta u = g(|x|, u) + \lambda h(u) \quad \text{in } \Omega, \qquad \frac{\partial u_N}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

*where* $\nu$ *is the outward normal vector of* $\partial\Omega$. *Then* $u_N$ *satisfies the following*:

$$\frac{N-2}{2}\|\nabla u_N\|_{L^2}^2 - N\int_\Omega \hat{G}_\lambda(|x|, u_N)dx \geq \int_{\partial\Omega} \hat{G}_\lambda(|x|, u_N)dS. \tag{6.5.2}$$

*Here* $\hat{G}_\lambda(|x|, s) := G(|x|, s) + \lambda H(s)$.

*Proof.* Note that under the conditions (6–g1)–(6–g6), $u_N$ has an exponential decay:

$$|u_N(r)| + |u_N'(r)| + |u_N''(r)| \leq C_1 \exp(-C_2 r) \quad \text{for all } r \geq 1.$$

Therefore $x \cdot \nabla u_N \in H^1(\Omega)$ and the curve $\eta(t) := u_N(tx) : [1,2] \to H^1(\Omega)$ is of class $C^1$. Since $I_\lambda'(u_N) = 0$, we have

$$\frac{d}{dt} I_\lambda(\eta(t))\Big|_{t=1} = I_\lambda'(u_N(x))(x \cdot \nabla u_N(x)) = 0. \tag{6.5.3}$$

On the other hand, it holds that

$$I_\lambda(\eta(t)) = \frac{t^{-N+2}}{2}\int_{|x|\geq t} |\nabla u_N(x)|^2 dx - t^{-N}\int_{|x|\geq t} G\left(\frac{|x|}{t}, u_N(|x|)\right) + \lambda H(u_N(|x|))dx. \tag{6.5.4}$$

By (6–g2), it follows that

$$I_\lambda(\eta(t)) \geq \hat{I}_\lambda(t) := \frac{t^{-N+2}}{2}\int_{|x|\geq t} |\nabla u_N(x)|^2 dx - t^{-N}\int_{|x|\geq t} G(|x|, u_N(|x|)) + \lambda H(u_N(|x|))dx. \tag{6.5.5}$$

Noting that $I(\eta(1)) = \hat{I}_\lambda(1)$, from (6.5.5), we infer

$$\frac{I_\lambda(\eta(t)) - I_\lambda(\eta(1))}{t-1} \geq \frac{\hat{I}_\lambda(t) - \hat{I}_\lambda(1)}{t-1} \quad \text{for all } t \in (1,2]. \tag{6.5.6}$$

By (6.5.3),

$$\frac{I_\lambda(\eta(t)) - I_\lambda(\eta(1))}{t-1} \to 0 \quad \text{as } t \to 1+0.$$

On the other hand, since $\partial u_N / \partial \nu = 0$ on $\partial\Omega$, it is easily seen that

$$\frac{\hat{I}_\lambda(t) - \hat{I}_\lambda(1)}{t-1}$$
$$\to -\frac{N-2}{2}\|\nabla u_N\|_{L^2}^2 + N\int_\Omega G(|x|, u_N) + \lambda H(u_N)dx + \int_{\partial\Omega} G(|x|, u_N) + \lambda H(u_N)dS$$

as $t \to 1+0$. Thus, from (6.5.6), we conclude that

$$\int_{\partial\Omega} G(|x|, u_N) + \lambda H(u_N) dS \leq \frac{N-2}{2} \|\nabla u_N\|_{L^2}^2 - N \int_{\Omega} G(|x|, u_N) + \lambda H(u_N) dx.$$

□

*Remark* 6.5.6. If we suppose that $g$ is of class $C^1$ with respect to $r$ in addition to (6–g1)–(6–g6), then the Pohozaev Identity holds:
(6.5.7)

$$\frac{N-2}{2} \|\nabla u_N\|_{L^2(\Omega)}^2 - N \int_{\Omega} \hat{G}_\lambda(|x|, u_N) dx = \int_{\Omega} x \cdot \nabla G(|x|, u_N) dx + \int_{\partial\Omega} \hat{G}_\lambda(|x|, u_N) dS.$$

Thus from (6.5.7) and (6–g2), we can see that (6.5.2) holds. Noting that the right hand side of (6.5.4) is differentiable with respect to $t$ and combining (6.5.3), we can obtain (6.5.7). See also Lemma 1.4 in Chapter III of Struwe [98].

Here we also state the Pohozaev type inequality for $(P_{\mathbf{R}^N})$ and $(P_D)$.

**Proposition 6.5.7.** *Assume that* (6–g1)–(6–g6) *are satisfied. Let* $u_D \in H^1_{0,r}(\Omega)$ ( resp. $u_{\mathbf{R}^N} \in H^1_r(\mathbf{R}^N)$ ) *be a solution of*

$$-\Delta u = g(|x|, u) + \lambda h(u) \quad \text{in } \Omega, \ u = 0 \text{ on } \partial\Omega \quad (\text{resp. } -\Delta u = g(|x|, u) + \lambda h(u) \quad \text{in } \mathbf{R}^N).$$

*Then* $u_D$ ( resp. $u_{\mathbf{R}^N} \in H^1_r(\mathbf{R}^N)$ ) *satisfies the following*:

$$\frac{N-2}{2} \|\nabla u_D\|_{L^2(\Omega)}^2 - N \int_{\Omega} \hat{G}_\lambda(|x|, u_D) dx \geq \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u_D}{\partial \nu}\right)^2 dS. \tag{6.5.8}$$

$$\left( \text{resp. } \frac{N-2}{2} \|\nabla u_{\mathbf{R}^N}\|_{L^2(\mathbf{R}^N)}^2 - N \int_{\mathbf{R}^N} \hat{G}_\lambda(|x|, u_{\mathbf{R}^N}) dx \geq 0 \right). \tag{6.5.9}$$

*Remark* 6.5.8. As in Remark 6.5.6, if $g(r, s)$ is of class $C^1$ with respect to $r$, then the following Pohozaev identity holds:

(6.5.10)

$$\frac{N-2}{2} \|\nabla u_D\|_{L^2}^2 - N \int_{\Omega} \hat{G}_\lambda(|x|, u_D) dx = \int_{\Omega} x \cdot \nabla G(|x|, u_D) dx + \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u_D}{\partial \nu}\right)^2 dS,$$

(6.5.11)

$$\left( \text{resp. } \frac{N-2}{2} \|\nabla u_{\mathbf{R}^N}\|_{L^2(\mathbf{R}^N)}^2 - N \int_{\mathbf{R}^N} \hat{G}_\lambda(|x|, u_{\mathbf{R}^N}) dx = \int_{\mathbf{R}^N} x \cdot \nabla G(|x|, u_{\mathbf{R}^N}) dx \right).$$

By (6–g2), we can show (6.5.8) and (6.5.9) from (6.5.10) and (6.5.11).

*Proof of Proposition 6.5.7.* We only show for $u_D$ since a proof for $u_{\mathbf{R}^N}$ is similar to the one of Proposition 6.5.5.

For the Dirichlet problem, critical points of $I_\lambda \in C^1(H^1_0(\Omega), \mathbf{R})$ corresponds to solutions. However, for technical reasons, we regard $I_\lambda \in C^1(H^1(\Omega), \mathbf{R})$ in this proof. We set $\tilde{\eta}(t) := u_D(tx) \in C^1([1,2], H^1(\Omega))$ and as in the proof of Proposition 6.5.5, we shall calculate

$$\frac{d}{dt} I_\lambda(\tilde{\eta}(t))\big|_{t=1}.$$

Since $u_D$ satisfies $-\Delta u_D = g(|x|, u_D) + \lambda h(u_D)$ in $\Omega$, $u_D = 0$ on $\partial\Omega$, using integration by parts, for any $\varphi \in H^1(\Omega) \cap C^1(\overline{\Omega})$, we have

$$I'_\lambda(u_D)\varphi = \int_\Omega \nabla u_D \cdot \nabla \varphi dx - \int_\Omega (g(|x|, u_D) + \lambda h(u_D))\varphi dx = -\int_{\partial\Omega} \nabla u_D \cdot x \varphi dS$$

Noting $\tilde{\eta}'(1) = x \cdot \nabla u_D(x) \in H^1(\Omega) \cap C^1(\overline{\Omega})$, it follows that

$$\frac{d}{dt} I_\lambda(\tilde{\eta}(t))\big|_{t=1} = -\int_{\partial\Omega} \left(\frac{\partial u_D}{\partial \nu}\right)^2 dS. \tag{6.5.12}$$

On the other hand, set

$$\tilde{I}_\lambda(t) := \frac{t^{-N+2}}{2} \int_{|x|\geq t} |\nabla u_D(x)|^2 dx - t^{-N} \int_{|x|\geq t} G(|x|, u_D) + \lambda H(u_D) dx,$$

then we have

$$\tilde{I}'_\lambda(1) = -\frac{N-2}{2}\|\nabla u_D\|^2_{L^2} - \frac{1}{2}\int_{\partial\Omega} \left(\frac{\partial u_D}{\partial \nu}\right)^2 dS + N\int_\Omega G(|x|, u_D) + \lambda H(u_D) dx. \tag{6.5.13}$$

Since $I(\tilde{\eta}(t)) \geq \tilde{I}_\lambda(t)$ and $I(\tilde{\eta}(1)) = \tilde{I}_\lambda(1)$, by (6.5.12) and (6.5.13), it follows that

$$\frac{1}{2}\int_{\partial\Omega} \left(\frac{\partial u_D}{\partial \nu}\right)^2 dS \leq \frac{N-2}{2}\|\nabla u_D\|^2_{L^2} - N\int_\Omega G(|x|, u_D) + \lambda H(u_D) dx.$$

□

### 6.5.3 Proof of Theorem 6.2.1

Now we prove Theorem 6.2.1. Suppose that the conditions (6–g1)–(6–g7) are satisfied. Let $(u_{n,k})$ be a sequence satisfying (6.5.1) and set

$$b_{n,0} := \lim_{\lambda \to 0} b_n(\lambda) = \lim_{k\to\infty} I_{\lambda_{n,k}}(u_{n,k}) \in [b_n(1), b_n(0)]$$

**Proposition 6.5.9.** *There exists a $C_n > 0$ such that $\|u_{n,k}\|_E \leq C_n$ for all $k \in \mathbf{N}$.*

*Proof.* First, we prove that $(\nabla u_{n,k})_{k=1}^\infty$ is bounded in $L^2(\Omega)$. Since $I'_{\lambda_{n,k}}(u_{n,k}) = 0$, by Proposition 6.5.5, we have

$$\begin{aligned} -\int_\Omega G(|x|, u_{n,k}) + \lambda_{n,k} H(u_{n,k}) dx \geq &-\frac{N-2}{2N}\|\nabla u_{n,k}\|^2_{L^2(\Omega)} \\ &+ \frac{1}{N}\int_{\partial\Omega} G(1, u_{n,k}) + \lambda_{n,k} H(u_{n,k}) dx. \end{aligned} \tag{6.5.14}$$

From (6.5.14), we obtain

$$
\begin{aligned}
b_n(\lambda_{n,k}) =& \frac{1}{2}\|\nabla u_{n,k}\|_{L^2(\Omega)}^2 - \int_\Omega G(|x|, u_{n,k}) + \lambda_{n,k} H(u_{n,k}) dx \\
\geq& \frac{1}{N}\|\nabla u_{n,k}\|_{L^2(\Omega)}^2 + \frac{1}{N}\int_{\partial\Omega} G(1, u_{n,k}) + \lambda_{n,k} H(u_{n,k}) dx.
\end{aligned} \tag{6.5.15}
$$

Noting that $H(s) \geq 0$ for all $s \in \mathbf{R}$, $\lim_{k\to\infty} b_n(\lambda_{n,k}) = b_{n,0} \leq b_n(0)$ and (6–g7), we deduce from (6.5.15) that there exists a $C_n > 0$ such that $\|\nabla u_{n,k}\|_{L^2(\Omega)} \leq C_n$ for all $k \in \mathbf{N}$.

Next, we show $\|u_{n,k}\|_E \leq C_n$ for all $k \in \mathbf{N}$. First, we consider the case $N \geq 3$. By Lemma 6.3.1, it holds

$$
\begin{aligned}
b_n(\lambda_{n,k}) = I_{\lambda_{n,k}}(u_{n,k}) = \frac{1}{2}\|u_{n,k}\|_E^2 - \int_\Omega \frac{m_1}{4} u_{n,k}^2 + G(|x|, u_{n,k}) + \lambda_{n,k} H(u_{n,k}) dx \\
\geq \frac{1}{2}\|u_{n,k}\|_E^2 - (1 + \lambda_{n,k}) \int_\Omega H(u_k) dx \geq \frac{1}{2}\|u_{n,k}\|_E^2 - C\|u_{n,k}\|_{L^{2^*}(\Omega)}^{2^*}.
\end{aligned} \tag{6.5.16}
$$

From (6.3.4) and (6.5.16), it holds that

$$
b_n(\lambda_{n,k}) \geq \frac{1}{2}\|u_{n,k}\|_E^2 - C\|\nabla u_{n,k}\|_{L^2(\Omega)}^{2^*} \tag{6.5.17}
$$

Since $b_n(\lambda_{n,k})$ and $(\|\nabla u_{n,k}\|_{L^2(\Omega)})_{k=1}^\infty$ are bounded, taking $C_n$ sufficiently large, $\|u_{n,k}\|_E \leq C_n$ follows from (6.5.17).

Next we consider the case $N = 2$. Following the arguments in [61] (cf. Proof of Proposition 5.5 in [56]), we prove indirectly. Assume that $r_k := \|u_{n,k}\|_{L^2(\Omega)}^{-1} \to 0$. Set

$$
v_k(x) := (T_{r_k}\tilde{v}_k)(x), \quad \tilde{v}_k(x) := u_{n,k}\left(\frac{x}{r_k}\right), \quad \Omega_k := \{x \in \mathbf{R}^N : |x| > r_k\},
$$

where $T_{r_k}$ defined by (6.3.2). From $\|\nabla \tilde{v}_k\|_{L^2(\Omega_k)} = \|\nabla u_k\|_{L^2(\Omega)}$, $\|\tilde{v}_k\|_{L^2(\Omega_k)} = 1$ and (6.3.3), $(v_k)$ is bounded in $H^1(\mathbf{R}^2)$. Therefore, we may assume

$$
v_k \rightharpoonup v_0 \quad \text{weakly in } H^1(\mathbf{R}^2) \quad \text{and} \quad v_k(x) \to v_0(x) \quad \text{a.a. } x \in \mathbf{R}^2.
$$

Next, we show $v_0 = 0$. We remark that since $v_k(x) = \tilde{v}_k(x)$ in $\Omega_k$, $v_k$ satisfies

$$
\begin{cases}
-r_k^2 \Delta v_k = g\left(\frac{|x|}{r_k}, v_k\right) + \lambda_{n,k} h(v_k) \quad \text{in } \Omega_k, \\
v_k'(r_k) = 0.
\end{cases} \tag{6.5.18}
$$

By the boundedness of $(v_k)$ in $H^1(\mathbf{R}^2)$, for any $\varphi \in C_0^\infty(\mathbf{R}^2)$ with $\operatorname{supp}\varphi \subset \mathbf{R}^2 \setminus \{0\}$, we can show

$$
\int_{\Omega_k} h(v_k)\varphi dx \to \int_{\mathbf{R}^2} h(v_0)\varphi dx, \quad \int_{\Omega_k} g\left(\frac{|x|}{r_k}, v_k\right)\varphi dx \to \int_{\mathbf{R}^2} g_\infty(v_0)\varphi dx. \tag{6.5.19}
$$

By (6.5.18) and (6.5.19), we obtain

$$
\int_{\mathbf{R}^2} g_\infty(v_0)\varphi dx = 0 \quad \text{for any } \varphi \in C_0^\infty(\mathbf{R}^2) \text{ with } \operatorname{supp}\varphi \subset \mathbf{R}^2 \setminus \{0\},
$$

which implies

$$g_\infty(v_0(x)) = 0 \quad \text{a.a. } x \in \mathbf{R}^2. \tag{6.5.20}$$

Since $v_0 \in H^1_r(\mathbf{R}^2) \subset C(\mathbf{R}^2\backslash\{0\})$, (6–g4) and (6.5.20), we infer that $v_0 \equiv 0$.

On the other hand, by (6.5.18), we have

$$r_k^2 \int_{\Omega_k} |\nabla v_k|^2 dx = \int_{\Omega_k} g\left(\frac{|x|}{r_k}, v_k\right) v_k + \lambda_{n,k} h(v_k) v_k dx. \tag{6.5.21}$$

Therefore it follows from (6.5.21), $1 = \|\tilde{v}_k\|_{L^2(\Omega_k)} = \|v_k\|_{L^2(\Omega_k)}$ and Lemma 6.3.1 that

$$\begin{aligned} 0 < \frac{m_1}{2} &= \frac{m_1}{2}\|v_k\|^2_{L^2(\Omega_k)} \le r_k^2 \|\nabla v_k\|^2_{L^2(\Omega_k)} + \frac{m_1}{2}\|v_k\|^2_{L^2(\Omega_k)} \\ &\le \int_{\Omega_k} \frac{m_1}{2} v_k^2 + g\left(\frac{|x|}{r_k}, v_k\right) v_k + \lambda_k h(v_k) v_k dx \le (1+\lambda_{n,k}) \int_{\Omega_k} h(v_k) v_k dx. \end{aligned}$$

Since $\lambda_{n,k} \le 1$ and $h(s)s \ge 0$ for all $s \in \mathbf{R}$, we obtain

$$\frac{m_1}{2} \le 2 \int_{\mathbf{R}^2} h(v_k) v_k dx. \tag{6.5.22}$$

On the other hand, since $v_k \rightharpoonup 0$ weakly in $H^1(\mathbf{R}^2)$, by Lemma 6.3.2 (i), we have

$$\int_{\mathbf{R}^2} h(v_k) v_k dx \to 0.$$

This contradicts to (6.5.22), therefore it holds that $\|u_{n,k}\|_{L^2(\Omega)} \le C_n$, which completes the proof. □

By virtue of Proposition 6.5.9, we have

**Corollary 6.5.10.** *The sequence* $(u_{n,k})_{k=1}^\infty$ *is a bounded* (PS) *sequence at level* $b_{n,0}$ *for* $I_0$.

*Proof.* We remark that it holds that

$$|I_0(u_{n,k}) - I_{\lambda_{n,k}}(u_{n,k})| \le \lambda_{n,k} K(u_{n,k}), \quad |I'(u_{n,k})\varphi - I'_{\lambda_{n,k}}(u_{n,k})\varphi| \le \lambda_{n,k} \|K'(u_{n,k})\|_{E^*} \|\varphi\|_E.$$

By Lemma 6.3.2 and $\lambda_{n,k} \to 0$, we can prove $I_0(u_{n,k}) \to b_{n,0}$ and $I_0'(u_{n,k}) \to 0$ as $k \to \infty$. Thus $(u_{n,k})_{k=1}^\infty$ is a bounded (PS) sequence at level $b_{n,0}$ for $I$. □

Now we complete a proof of Theorem 6.2.1.

*Proof of Theorem 6.2.1.* For each $n \in \mathbf{N}$, by Corollary 6.5.10, there exists a bounded sequence $(u_{n,k})_{k=1}^\infty \subset E$

$$I_0(u_{n,k}) \to b_{n,0}, \quad I_0'(u_{n,k}) \to 0 \quad \text{as } k \to \infty.$$

Thus by Lemma 6.3.2 (ii), there exists a $u_{n,0} \in E$ such that

$$I_0(u_{n,0}) = b_{n,0}, \quad I_0'(u_{n,0}) = 0.$$

On the other hand, by Lemmas 6.4.4 and 6.4.5, $b_{n,0} \to \infty$ as $n \to \infty$. Therefore we can show the existence of infinitely many radial solutions.

In order to obtain positive solutions, we modify $g(r,s)$ as follows:

$$g_+(r,s) := \begin{cases} g(r,s) & \text{if } s \geq 0, \\ 0 & \text{if } s \leq 0. \end{cases}$$

Then any nontrivial radial solution of

$$-\Delta u = g_+(|x|, u) \quad \text{in } \Omega, \quad u'(1) = 0$$

is positive on $\{|x| \geq 1\}$ by the maximum principle. Thus we will find a critical point of

$$I_+(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \int_\Omega G_+(|x|, u)dx.$$

We can prove that $I_+$ has a mountain pass geometry as in Lemma 6.4.1. Moreover, using the monotonicity method as before, we can show that $I_+$ has a nontrivial critical point. Thus we complete a proof. □

### 6.5.4 Outline of proof of Theorem 6.2.2

In this subsection, we give an outline of proof of Theorem 6.2.2. Throughout this subsection, we assume the conditions (6–g1)–(6–g6).

As in the Neumann case, we define the following functionals: for each $\lambda \in [0,1]$,

$$\begin{aligned} I_{D,\lambda}(v) &:= \frac{1}{2}\|\nabla v\|_{L^2(\Omega)}^2 - \int_\Omega G(|x|, v) + \lambda H(v)dx \in C^1(H_{0,r}^1(\Omega), \mathbf{R}), \\ I_{\mathbf{R}^N,\lambda}(w) &:= \frac{1}{2}\|\nabla w\|_{L^2(\mathbf{R}^N)}^2 - \int_{\mathbf{R}^N} G(|x|, w) + \lambda H(w)dx \in C^1(H_r^1(\mathbf{R}^N), \mathbf{R}), \\ J_{\mathbf{R}^N}(w) &:= \frac{1}{2}\|\nabla w\|_{L^2(\mathbf{R}^N)}^2 - 2\int_{\mathbf{R}^N} H(w)dx \in C^1(H_r^1(\mathbf{R}^N), \mathbf{R}). \end{aligned}$$

Then, noting $H_{0,r}^1(\Omega) \subset H_r^1(\mathbf{R}^N)$, we can see that

$$J_{\mathbf{R}^N}(v) \leq I_{D,\lambda}(v), \quad J_{\mathbf{R}^N}(w) \leq I_{\mathbf{R}^N,\lambda}(w)$$

for all $\lambda \in [0,1]$, $v \in H_{0,r}^1(\Omega)$, $w \in H_r^1(\mathbf{R}^N)$. Furthermore $I_{D,\lambda}$, $I_{\mathbf{R}^N,\lambda}$ satisfy (6.3.5).

Let $\gamma_n \in C(S^{n-1}, H_{0,r}^1(\Omega))$ appear in Lemma 6.4.1. Then $\gamma_n \in C(S^{n-1}, H_r^1(\mathbf{R}^N))$ and we can define minimax values for $I_{D,\lambda}$, $I_{\mathbf{R}^N,\lambda}$ and $J_{\mathbf{R}^N}$:

$$\begin{aligned} b_{n,D}(\lambda) &:= \inf_{\gamma \in \Gamma_{n,D}} \max_{\sigma \in D_n} I_{D,\lambda}(\gamma(\sigma)), \quad b_{n,\mathbf{R}^N}(\lambda) := \inf_{\gamma \in \Gamma_{n,\mathbf{R}^N}} \max_{\sigma \in D_n} I_{\mathbf{R}^N,\lambda}(\gamma(\sigma)), \\ c_{n,\mathbf{R}^N} &:= \inf_{\gamma \in \Gamma_{n,\mathbf{R}^N}} \max_{\sigma \in D_n} J_{\mathbf{R}^N}(\gamma(\sigma)), \end{aligned}$$

where

$$\begin{aligned} \Gamma_{n,D} &:= \{\gamma \in C(D_n, H_{0,r}^1(\Omega)) : \gamma = \gamma_n \text{ on } S^{n-1}\}, \\ \Gamma_{n,\mathbf{R}^N} &:= \{\gamma \in C(D_n, H_r^1(\mathbf{R}^N)) : \gamma = \gamma_n \text{ on } S^{n-1}\}. \end{aligned}$$

It is easily seen that all lemmas in sections 6.3 and 6.4 hold if we replace $I_\lambda$, $J$, $b_n(\lambda)$, $c_n$ by $I_{D,\lambda}$, $I_{\mathbf{R}^N,\lambda}$, $b_{n,D}(\lambda)$, $b_{n,\mathbf{R}^N}(\lambda)$, $c_{n,\mathbf{R}^N}$. Moreover, we can apply the monotonicity method for $I_{D,\lambda}$ and $I_{\mathbf{R}^N,\lambda}$ (cf. Lemma 6.5.3). Therefore for each $n \in \mathbf{N}$ there are sequences $(\lambda_{n,k}) \subset [0,1]$, $(v_{n,k}) \subset H^1_{0,r}(\Omega)$, $(w_{n,k}) \subset H^1_r(\mathbf{R}^N)$ such that $\lambda_{n,k} \to 0$ and

$$\begin{aligned} I_{D,\lambda_{n,k}}(v_{n,k}) &= b_{n,D}(\lambda_{n,k}), \quad & I'_{D,\lambda_{n,k}}(v_{n,k}) &= 0, \\ I_{\mathbf{R}^N,\lambda_{n,k}}(w_{n,k}) &= b_{n,\mathbf{R}^N}(\lambda_{n,k}), \quad & I'_{\mathbf{R}^N,\lambda_{n,k}}(w_{n,k}) &= 0. \end{aligned}$$

As in the Neumann case, it is sufficient to show that $(v_{n,k})_{k=1}^\infty$ (resp. $(w_{n,k})_{k=1}^\infty$) is bounded in $H^1_{0,r}(\Omega)$ (resp. $H^1_r(\mathbf{R}^N)$). Using (6.5.8) and (6.5.9) instead of (6.5.2), it is easily seen that $(v_{n,k})_{k=1}^\infty$ (resp. $(w_{n,k})_{k=1}^\infty$) is bounded in $H^1_{0,r}(\Omega)$ (resp. $H^1_r(\mathbf{R}^N)$) in a similar way to the proof of Proposition 6.5.9.

The remaining part of proof of Theorem 6.2.2 is the same as the proof of Theorem 6.2.1, so we omit it.

### 6.5.5 Proof of Theorem 6.2.4

In this subsection, we prove Theorem 6.2.4 and let $g(r,s) = -V(r)s + \tilde{g}(s)$. We only consider $(P_N)$, since proofs in other cases are similar. As mentioned before, we can suppose $\Omega := \{x \in \mathbf{R}^N : |x| > 1\}$. Furthermore, as in [7, 15, 16, 48], instead of ($\tilde{g}$3), we can assume

$$(\tilde{g}3') \qquad \lim_{s\to\infty} \frac{\tilde{g}(s)}{s^{2^*-1}} = 0.$$

Indeed, set

$$\tilde{\zeta}_1 := \inf \left\{ s \in [\tilde{\zeta}_0, \infty) : \tilde{g}(s) = 0 \right\}$$

where $\tilde{\zeta}_0 > 0$ appearing in ($\tilde{g}$4). If $\tilde{g}(s) > 0$ for all $s \ge \tilde{\zeta}_0$, then we set $\tilde{\zeta}_1 = \infty$. We define $\bar{g}(s)$ as follows:

$$\bar{g}(s) := \begin{cases} \tilde{g}(s) & \text{if } |s| \le \tilde{\zeta}_1, \\ 0 & \text{if } |s| > \tilde{\zeta}_1. \end{cases}$$

Then $\bar{g}$ satisfies ($\tilde{g}$1), ($\tilde{g}$2), ($\tilde{g}3'$) and ($\tilde{g}$4). Moreover, any solution of

$$(6.5.23) \qquad -\Delta u + V(|x|)u = \bar{g}(u) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

satisfies $\|u\|_{L^\infty(\Omega)} \le \tilde{\zeta}_1$ by the maximum principle. Therefore any solution of (6.5.23) satisfies $(P_N)$ with $g(r,s) = -V(r)s + \tilde{g}(s)$, which implies that we can assume ($\tilde{g}3'$) instead of ($\tilde{g}$3) without loss of generality.

As stated in the above, we prove Theorem 6.2.4 under

($\mathcal{H}_3$) $N \ge 3$, the conditions ($\tilde{g}$1), ($\tilde{g}$2), ($\tilde{g}3'$), ($\tilde{g}$4), ($\tilde{g}$5), (6–V1)–(6–V3) are satisfied.

Under the condition $(\mathcal{H}_3)$ we will find infinitely many critical points of

$$\begin{aligned}\tilde{I}(u) &:= \frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_\Omega \left(V(|x|) + \frac{\tilde{m}_1}{2}\right) u^2 dx - \int_{\Omega_R} \frac{\tilde{m}_1}{4}u^2 + \tilde{G}(u)dx \\ &= \frac{1}{2}\|u\|^2 - \int_\Omega \frac{\tilde{m}_1}{4}u^2 + \tilde{G}(u)dx,\end{aligned}$$

where $\tilde{m}_1$ appears in ($\tilde{g}2$) and $\tilde{G}(s) = \int_0^s \tilde{g}(t)dt$.

In this case, we can define $h \in C(\mathbf{R})$ satisfying Lemma 6.3.1. Thus we define an auxiliary functional $\tilde{J}$ and parametrized functional $\tilde{I}_\lambda$ for each $\lambda \in [0,1]$. We note that all lemmas and propositions in section 6.4 hold for these functionals. Moreover, noting $V(r) \to 0$ as $r \to \infty$ and a proof of Proposition 6.6.1 in subsection 6.6.1, we can also prove that $\tilde{I}_\lambda$, $\tilde{J}$ have a symmetric mountain pass structure and define $\tilde{b}_n(\lambda)$ and $\tilde{c}_n$ as in Definition 6.4.3. Furthermore, we see that all lemmas in section 6.3 hold. By Proposition 6.5.2, for each $n \in \mathbf{N}$ there exist $(\tilde{\lambda}_{n,k})_{k=1}^\infty$ and $(\tilde{u}_{n,k})_{k=1}^\infty \subset H_r^1(\Omega)$ such that $\tilde{\lambda}_{n,k} \to 0$ as $k \to \infty$ and

$$\tilde{I}_{\tilde{\lambda}_{n,k}}(\tilde{u}_{n,k}) = \tilde{b}_n(\tilde{\lambda}_{n,k}), \quad \tilde{I}'_{\tilde{\lambda}_{n,k}}(\tilde{u}_{n,k}) = 0$$

Next, we show that $(\tilde{u}_{n,k})_{k=1}^\infty$ is bounded in $H_r^1(\Omega)$.

**Lemma 6.5.11.** *There exists a $C_n > 0$ such that $\|\tilde{u}_{n,k}\| \leq C_n$ for all $k \geq 1$.*

*Proof.* As in Proposition 6.5.9, firstly we show that $(\nabla\tilde{u}_{n,k})_{k=1}^\infty$ is bounded in $L^2(\Omega)$. By Remark 6.5.6, $\tilde{u}_{n,k}$ satisfies

$$\begin{aligned}&\frac{1}{2}\int_\Omega V(|x|)\tilde{u}_{n,k}^2 dx - \int_\Omega \tilde{G}(\tilde{u}_{n,k}) + \tilde{\lambda}_{n,k}H(\tilde{u}_{n,k})dx \\ =& -\frac{1}{2^*}\|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}^2 - \frac{1}{2N}\int_\Omega x\cdot\nabla V(|x|)\tilde{u}_{n,k}^2 dx \\ &+ \int_{\partial\Omega} -\frac{1}{2}V(1)\tilde{u}_{n,k}^2 + \tilde{G}(\tilde{u}_{n,k}) + \tilde{\lambda}_{n,k}H(\tilde{u}_{n,k})dS.\end{aligned}$$

By ($\tilde{g}5$) and Hölder's inequality, there exists a $C > 0$ such that

$$\begin{aligned}\tilde{I}_{\tilde{\lambda}_{n,k}}(\tilde{u}_{n,k}) &= \frac{1}{2}\|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_\Omega V(|x|)\tilde{u}_{n,k}^2 dx - \int_\Omega \tilde{G}(\tilde{u}_{n,k}) + \tilde{\lambda}_{n,k}H(\tilde{u}_{n,k})dx \\ &= \frac{1}{N}\|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}^2 - \frac{1}{2N}\int_\Omega x\cdot\nabla V(|x|)\tilde{u}_{n,k}^2 dx \\ &+ \int_{\partial\Omega} -\frac{1}{2}V(1)\hat{u}_{n,k}^2 + \tilde{G}(\tilde{u}_{n,k}) + \tilde{\lambda}_{n,k}H(\tilde{u}_{n,k})dS \\ &\geq \frac{1}{N}\|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}^2 - \frac{1}{2N}\|(x\cdot\nabla V(|x|))^+\|_{L^{\frac{N}{2}}(\Omega)}\|\tilde{u}_{n,k}\|_{L^{2^*}(\Omega)}^2 - C.\end{aligned}$$

We extend $\tilde{u}_{n,k}$ as follows:

$$\hat{u}_{n,k}(x) := \begin{cases} \tilde{u}_{n,k}(|x|) & \text{if } |x| \geq 1, \\ \tilde{u}_{n,k}(1) & \text{if } |x| < 1. \end{cases}$$

Then it is clear that $\hat{u}_{n,k} \in H^1_r(\mathbf{R}^N)$, $\|\nabla \hat{u}_{n,k}\|_{L^2(\mathbf{R}^N)} = \|\nabla \tilde{u}_{n,k}\|_{L^2(\Omega)}$ and $\|\tilde{u}_{n,k}\|_{L^{2^*}(\Omega)} \leq \|\hat{u}_{n,k}\|_{L^{2^*}(\mathbf{R}^N)}$. Furthermore, since $\|\hat{u}_{n,k}\|^2_{L^{2^*}(\mathbf{R}^N)} \leq \|\nabla \hat{u}_{n,k}\|^2_{L^2(\mathbf{R}^N)}/S_N$ holds, we obtain $\|\tilde{u}_{n,k}\|^2_{L^{2^*}(\Omega)} \leq \|\nabla \tilde{u}_{n,k}\|^2_{L^2(\Omega)}/S_N$. Here, from (6–V3), we can take an $\varepsilon_0 > 0$ such that

$$\|(x \cdot \nabla V(|x|))^+\|_{L^{\frac{N}{2}}(\Omega)} < 2S_N - \varepsilon_0.$$

Then we have

$$\tilde{I}_{\tilde{\lambda}_{n,k}}(\tilde{u}_{n,k}) \geq \frac{1}{N}\|\nabla \tilde{u}_{n,k}\|^2_{L^2(\Omega)} - \frac{1}{N}\frac{2S_N - \varepsilon_0}{2S_N}\|\nabla \tilde{u}_{n,k}\|^2_{L^2(\Omega)} - C \geq \varepsilon_1 \|\nabla \tilde{u}_{n,k}\|^2_{L^2(\Omega)} - C$$

for some $\varepsilon_1 > 0$. Thus there exists a $C_n > 0$ such that $\|\nabla \tilde{u}_{n,k}\|_{L^2(\Omega)} \leq C_n$ for all $k \in \mathbf{N}$.

Since a proof of the boundedness of $(\tilde{u}_{n,k})_{k=1}^\infty$ in $L^2(\Omega)$ is similar to the one of Proposition 6.5.9, we omit it. □

Now we complete a proof of Theorem 6.2.4.

*Proof of Theorem 6.2.4.* From Lemma 6.5.11, we see that $(\tilde{u}_{n,k})_{k=1}^\infty$ is a bounded (PS) sequence for $\tilde{I}$ as in Corollary 6.5.10. Therefore we can show the existence of infinitely many solutions as in Theorem 6.2.1. For the existence of at least one positive solution, we replace $\tilde{I}$ by

$$\tilde{I}_+(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + V(|x|)u^2 dx - \int_\Omega \tilde{G}_+(u)dx$$

where $\tilde{G}_+(s) = \int_0^s \tilde{g}_+(\tau)\tau$. In this case, we can show the existence of nontrivial critical point in the similar way to the proof of Theorem 6.2.1. □

## 6.6 Proofs of Proposition 6.4.1 (ii) and Lemma 6.3.2, and technical lemma

In this section, we prove Proposition 6.4.1 (ii) and Lemma 6.3.2. Moreover, we state a useful lemma. First, we give a proof of Proposition 6.4.1 (ii).

### 6.6.1 Proof of Proposition 6.4.1 (ii)

In this subsection, we prove the following proposition.

**Proposition 6.6.1.** *Let $\Omega = \{x \in \mathbf{R}^N : |x| > 1\}$ and $(\mathcal{H}_2)$ be satisfied. Then for each $n \in \mathbf{N}$, there exists a continuous odd map $\gamma_n : S^{n-1} \to H^1_{0,r}(\Omega)$ such that*

$$I(\gamma_n(\sigma)) < 0 \qquad \text{for all } \sigma \in S^{n-1}.$$

Before proving Proposition 6.6.1, we introduce some notations. First, we define $\underline{G}(s)$ for $s \geq 0$ as follows:

$$\underline{G}(s) := \inf_{r \geq R_0} G(r, s),$$

where $R_0$ appears in (6–g6). By (6–g3) and (6–g6), $\underline{G}(s)$ is well-defined and satisfies $\underline{G}(\zeta_0) > 0$. We also set

$$\underline{I}(u) := \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbf{R}^N)} - \int_{\mathbf{R}^N} \underline{G}(u)dx \in C(H^1_r(\mathbf{R}^N), \mathbf{R}).$$

Note that if $u \in H^1_r(\Omega)$ and $\operatorname{supp} u \subset \{|x| > R_0\}$, then $I(u) \le \underline{I}(u)$. Therefore it is sufficient to prove that there exists a continuous odd map $\gamma_n : S^{n-1} \to H^1_r(\Omega)$ such that

$$\text{(6.6.1)} \qquad \underline{I}(\gamma_n(\sigma)) < 0, \quad \operatorname{supp}\gamma_n(\sigma) \subset \{|x| > R_0\} \qquad \text{for all } \sigma \in S^{n-1}.$$

*Proof of Proposition 6.6.1.* By the arguments of Theorem 10 in [16], for each $n \in \mathbf{N}$, there exists a $\pi_n \in C(S^{n-1}, H^1_r(\mathbf{R}^N))$ such that

$$\pi_n(-\sigma) = -\pi_n(\sigma), \quad \|\pi_n(\sigma)\|_{L^\infty(\mathbf{R}^N)} = \zeta_0, \quad \int_{\mathbf{R}^N} \underline{G}(\pi_n(\sigma))dx \ge 1 \quad \text{for all } \sigma \in S^{n-1}.$$

We modify $\pi_n$ to obtain $\gamma_n$ satisfying the property (6.6.1). Let $\varphi \in C^\infty([0,\infty))$ be a cut-off function such that

$$0 \le \varphi(t) \le 1, \quad \varphi(t) = \begin{cases} 0 & \text{if } t \le 1, \\ 1 & \text{if } t \ge 2, \end{cases}$$

and set $\varphi_k(t) := \varphi(kt)$ and $\eta_k(\sigma)(x) := \varphi_k(|x|)\pi_n(\sigma)(x)$ for $k \in \mathbf{N}$. Then it holds $\operatorname{supp}\eta_k(\sigma) \subset \{|x| \ge 1/k\}$ for all $\sigma \in S^{n-1}$ and

$$\int_{\mathbf{R}^N} \underline{G}(\eta_k(\sigma))dx \to \int_{\mathbf{R}^N} \underline{G}(\pi_n(\sigma))dx \quad \text{as } k \to \infty \quad \text{uniformly w.r.t. } \sigma \in S^{n-1},$$

since $\pi_n(S^{n-1})$ is bounded in $L^\infty(\mathbf{R}^N)$. Therefore for a large $k_0 \in \mathbf{N}$, we have

$$\text{(6.6.2)} \qquad \int_{\mathbf{R}^N} \underline{G}(\eta_{k_0})dx \ge \frac{1}{2} \quad \text{for all } \sigma \in S^{n-1}.$$

We consider $\eta_{k_0}(\sigma)(x/t)$ for $t \ge 1$. By (6.6.2), we see that $\operatorname{supp}\eta_{k_0}(\sigma)(\cdot/t) \subset \{|x| \ge t/k_0\}$ and

$$\begin{aligned}\underline{I}(\eta_{k_0}(\sigma)(\cdot/t)) &= t^{N-2}\left(\frac{1}{2}\|\nabla\eta_{k_0}(\sigma)\|^2_{L^2(\mathbf{R}^N)} - t^2\int_{\mathbf{R}^N}\underline{G}(\eta_{k_0})dx\right) \\ &\le t^{N-2}\left(\frac{1}{2}\|\nabla\eta_{k_0}(\sigma)\|^2_{L^2(\mathbf{R}^N)} - \frac{t^2}{2}\right).\end{aligned}$$

Since $\|\nabla\eta_{k_0}(\sigma)\|_{L^2(\mathbf{R}^N)}$ is uniformly bounded with respect to $\sigma \in S^{n-1}$, we can choose a $t_0 \ge 1$ satisfying $t_0/k_0 > R_0$ and

$$\underline{I}(\eta_{k_0}(\sigma)(\cdot/t_0)) < 0 \quad \text{for all } \sigma \in S^{n-1}.$$

Set $\gamma_n(\sigma)(x) := \eta_{k_0}(\sigma)(x/t_0)$, then $\gamma_n$ satisfies (6.6.1). The oddness and continuity of $\gamma_n$ follows from the ones of $\eta_{k_0}$, which completes a proof. $\square$

### 6.6.2 Proof of Lemma 6.3.2

Next we give a proof of Lemma 6.3.2.

*Proof of Lemma 6.3.2.* (i) First we show that $K$ is weakly continuous. Let $u_k$ satisfy $u_k \rightharpoonup u_0$ weakly in $E$. Without loss of generality, we may assume

$$u_k(x) \to u_0(x) \quad \text{a.a. } x \in \Omega, \quad \|u_k\|_E \le M.$$

Since $(u_k)$ is bounded, by (6.3.1) and Lemma 6.3.1, there exists an $R_1 > 0$ such that if $|x| \ge R_1$, then $H(u_k(x)) = H(u_0(x)) = 0$ for all $k \ge 1$. Therefore, it is sufficient to show

$$\int_{\Omega \cap B_{R_1}} |H(u_k) - H(u_0)| dx \to 0.$$

We set $Q(s) := |s|^{2^*}$ $(N \ge 3)$, $Q(s) := \exp(s^2/(2M^2)) - 1 - s^2/(2M^2)$ $(N = 2)$. Then by Lemma 6.3.1, for each $\varepsilon > 0$ there exists an $s_\varepsilon \ge 0$ such that if $|s| \ge s_\varepsilon$, then $H(s) \le \varepsilon Q(s)$. We define $\hat{H}(s)$ as follows:

$$\hat{H}(s) := \begin{cases} H(s) & \text{if } |s| \le s_\varepsilon, \\ H(s_\varepsilon) & \text{if } |s| > s_\varepsilon. \end{cases}$$

Since $\hat{H}$ is bounded, it is easy to see that

$$\hat{H}(u_k) \to \hat{H}(u_0) \quad \text{in } L^1(\Omega \cap B_{R_1}).$$

On the other hand, since $|\hat{H}(s) - H(s)| \le \varepsilon Q(s)$ we have

$$\begin{aligned} &\int_{\Omega \cap B_{R_1}} |H(u_k) - H(u_0)| dx \\ \le &\int_{\Omega \cap B_{R_1}} |H(u_k) - \hat{H}(u_k)| + |\hat{H}(u_k) - \hat{H}(u_0)| + |\hat{H}(u_0) - H(u_0)| dx \\ \le &\varepsilon \int_{\Omega \cap B_{R_1}} Q(u_k) + Q(u_0) dx + \|\hat{H}(u_k) - \hat{H}(u_0)\|_{L^1(\Omega \cap B_{R_1})} \end{aligned}$$

Thus to prove the weak continuity of $K$, it is sufficient to prove

$$\sup_{k \ge 1} \int_\Omega Q(u_k) dx < \infty. \tag{6.6.3}$$

In the case $N \ge 3$, (6.6.3) follows from Sobolev's inequality and in the case $N = 2$, (6.6.3) holds by Lemma 6.6.2 (iii). Therefore $K$ is weakly continuous.

Next we prove that $K'(u_k) \to K'(u_0)$ strongly in $E^*$. Since

$$K'(u_k)\varphi = \int_\Omega h(u_k)\varphi \, dx \quad \text{for all } \varphi \in E,$$

if we can show

$$(6.6.4) \qquad h(u_k) \to h(u_0) \quad \text{strongly in } L^{p_N}(\Omega), \quad p_N := \begin{cases} 2 & \text{if } N = 2, \\ 2N/(N+2) & \text{if } N \geq 3, \end{cases}$$

then $K'(u_k) \to K'(u_0)$ strongly in $E^*$.

We prove (6.6.4). As in the above, there exists an $R_1 \geq 1$ such that if $|x| \geq R_1$, then $h(u_k(x)) = h(u_0(x)) = 0$ for all $k \geq 1$. Therefore we only show

$$h(u_k) \to h(u_0) \quad \text{strongly in } L^{p_N}(\Omega \cap B_{R_1}).$$

Set $Q(s) := \exp(s^2/(8M^2)) - 1 - s^2/(8M^2)$ $(N = 2)$, $Q(s) := |s|^{(N+2)/(N-2)}$ $(N \geq 3)$. By Lemma 6.3.1, for each $\varepsilon > 0$, there exists an $s_\varepsilon \geq 0$ such that if $|s| \geq s_\varepsilon$, then $|h(s)| \leq \varepsilon Q(s)$. Define $\hat{h}(s)$ as follows:

$$\hat{h}(s) := \begin{cases} h(s) & \text{if } |s| \leq s_\varepsilon, \\ h(s_\varepsilon) & \text{if } s > s_\varepsilon, \\ h(-s_\varepsilon) & \text{if } s < -s_\varepsilon. \end{cases}$$

Then we have $\hat{h}(u_k) \to \hat{h}(u_0)$ strongly in $L^{p_N}(\Omega \cap B_{R_1})$. Therefore, to prove (6.6.4), it is sufficient to show

$$(6.6.5) \qquad \sup_{k \geq 1} \int_\Omega Q(u_k)^{p_N} dx < \infty.$$

In the case $N \geq 3$, by Sobolev's inequality and $p_N(2^* - 1) = 2^*$, (6.6.5) holds. In the case $N = 2$, we remark that $Q(s)^2 \leq Q(2s)$ for all $s \in \mathbf{R}$. By Lemma 6.6.2 (iii), we have

$$\sup_{k \geq 1} \int_\Omega Q(u_k)^2 dx \leq \sup_{k \geq 1} \int_\Omega Q(2u_k) dx \leq C \sup_{k \geq 1} \|u_k\|_E^4 < \infty,$$

which implies (6.6.5). Therefore $K'(u_k) \to K'(u_0)$ strongly in $E^*$.

(ii) Let $(u_k) \subset E$ be a (PS) sequence at level $c$ for $I_\lambda$ and $\|u_k\|_E \leq M$. Since $(u_k)$ is bounded, there exist $u_0 \in E$ and subsequence $(u_{k_\ell})$ such that

$$u_{k_\ell} \rightharpoonup u_0 \quad \text{weakly in } E, \quad u_{k_\ell}(x) \to u_0(x) \quad \text{a.a. } x \in \Omega.$$

Let $\varphi \in C^\infty_{0,r}(\{|x| \geq 1\}) := \{\phi \in C^\infty(\{|x| \geq 1\}) : \phi(x) = \phi(|x|) \text{ and } \operatorname{supp}\phi \text{ is compact}\}$. Set $p_N := 2$ $(N = 2)$, $p_N := 2N/(N+2)$ $(N \geq 3)$. Applying the similar arguments in the above, we can show

$$\begin{aligned} &g(|x|, u_{k_\ell}) \to g(|x|, u_0(x)) && \text{strongly in } L^{p_N}(\Omega \cap B_{\hat{R}}), \\ &h(u_{k_\ell}) \to h(u_0) && \text{stronlgy in } L^{p_N}(\Omega) \end{aligned}$$

for all $\hat{R} > 1$. Therefore we obtain

$$(6.6.6) \qquad \int_\Omega g(|x|, u_{k_\ell})\varphi dx \to \int_\Omega g(|x|, u_0)\varphi dx, \quad \int_\Omega h(u_{k_\ell}) u_{k_\ell} dx \to \int_\Omega h(u_0) u_0 dx.$$

Noting that $I'_\lambda(u_{k_\ell}) \to 0$, by (6.6.6), we see that $I'_\lambda(u_0)\varphi = 0$ for all $\varphi \in C^\infty_{0,r}(\{|x| \geq 1\})$. Since $C^\infty_{0,r}(\{|x| \geq 1\})$ is dense in $E$, $I'_\lambda(u_0)u_0 = 0$ holds, that is,

$$\|u_0\|_E^2 = \int_\Omega \frac{m_1}{2}u_0^2 + g(|x|, u_0)u_0 + \lambda h(u_0)u_0 dx. \tag{6.6.7}$$

On the other hand, since $(u_{k_\ell})$ is bounded, we have $I'_\lambda(u_{k_\ell})u_{k_\ell} \to 0$, which implies

$$\|u_{k_\ell}\|_E^2 - \int_\Omega \frac{m_1}{2}u_{k_\ell}^2 + g(|x|, u_{k_\ell})u_{k_\ell} + \lambda h(u_{k_\ell})u_{k_\ell} dx \to 0. \tag{6.6.8}$$

Next, we rewrite

$$\begin{aligned}&\int_\Omega \frac{m_1}{2}u_{k_\ell}^2 + g(|x|, u_{k_\ell})u_{k_\ell} + \lambda h(u_{k_\ell})u_{k_\ell} dx\\ =&(1+\lambda)\int_\Omega h(u_{k_\ell})u_{k_\ell}dx - \int_\Omega h(u_{k_\ell})u_{k_\ell} - \frac{m_1}{2}u_{k_\ell}^2 - g(|x|, u_{k_\ell})u_{k_\ell}dx.\end{aligned}$$

By Lemma 6.3.1 and Fatou's lemma, we have
(6.6.9)

$$\liminf_{\ell\to\infty}\int_\Omega h(u_{k_\ell})u_{k_\ell} - \frac{m_1}{2}u_{k_\ell}^2 - g(|x|, u_{k_\ell})u_{k_\ell}dx \geq \int_\Omega h(u_0)u_0 - \frac{m_1}{2}u_0^2 - g(|x|, u_0)u_0 dx.$$

By (6.6.6)–(6.6.9), we obtain

$$\begin{aligned}\limsup_{\ell\to\infty}\|u_{k_\ell}\|_E^2 &\leq (1+\lambda)\int_\Omega h(u_0)u_0 dx - \int_\Omega h(u_0)u_0 - \frac{m_1}{2}u_0^2 - g(|x|, u_0)u_0 dx\\ &= \int_\Omega \frac{m_1}{2}u_0^2 + g(|x|, u_0)u_0 + \lambda h(u_0)u_0 dx = \|u_0\|_E^2.\end{aligned}$$

Thus $u_{k_\ell}$ converges to $u_0$ strongly in $E$, which completes the proof.
(iii) Next we prove that $J$ satisfies the (PS) condition. Let $(u_k) \subset E$ be a (PS) sequence at level $c$ of $J$, i.e., $J(u_k) \to c$ and $J'(u_k) \to 0$ strongly in $E^*$. Since $h(s)$ satisfies a global Ambrosetti–Rabinowitz condition, we can infer that $(u_k)$ is bounded. Indeed, the boundedness of $(u_k)$ in $E$ comes from

$$\begin{aligned}J(u_k) - \frac{J'(u_k)u_k}{p+1} &= \left(1 - \frac{2}{p+1}\right)\|u_k\|_E^2 - \int_{|x|>1} H(u_k) - \frac{1}{p+1}h(u_k)u_k dx\\ &\geq \left(1 - \frac{2}{p+1}\right)\|u_k\|_E^2.\end{aligned}$$

Thus we may assume that taking a subsequence if necessary,

$$u_k \rightharpoonup u_0 \quad \text{weakly in } E.$$

By (i), we have $K'(u_k) \to K'(u_0)$ strongly in $E^*$. Therefore by standard arguments we can conclude that $(u_k)$ has a strongly convergent subsequence and this completes the proof. □

### 6.6.3 A technical lemma

The following lemma is useful and we use it in proofs of Lemmas 6.3.2 and 6.4.1.

**Lemma 6.6.2** (cf. Adachi–Tanaka [1], Byeon, Jeanjean and Tanaka [24], Ogawa [84])**.**

(i) *Let* $\Phi(s) := \exp(s) - 1 - s$ *and* $\beta \in (0, 4\pi)$*. Then there exists a* $\tilde{C}_\beta > 0$ *such that*

$$\int_{\mathbf{R}^2} \Phi\left(\beta \frac{u^2}{\|\nabla u\|^2_{L^2(\mathbf{R}^2)}}\right) dx \le \tilde{C}_\beta \frac{\|u\|^4_{L^4(\mathbf{R}^2)}}{\|\nabla u\|^4_{L^2(\mathbf{R}^2)}} \quad \text{for all } u \in H^1(\mathbf{R}^2)\backslash\{0\}.$$

(ii) *For any* $M > 0$ *and* $\beta \in (0, 4\pi)$*, there exists a* $\tilde{C}_{\beta,1} > 0$ *such that*

$$\int_{\mathbf{R}^2} \Phi\left(\frac{\beta u^2}{M^2}\right) dx \le \tilde{C}_{\beta,1} \frac{\|u\|^4_{L^4(\mathbf{R}^2)}}{M^4} \quad \text{for all } u \in H^1(\mathbf{R}^2) \text{ with } \|\nabla u\|_{L^2(\mathbf{R}^2)} \le M.$$

(iii) *For any* $M > 0$ *and* $\beta \in (0, 4\pi)$*, there exists a* $\tilde{C}_{\beta,2} > 0$ *such that*

$$\int_{\Omega} \Phi\left(\frac{\beta u^2}{2M^2}\right) dx \le \tilde{C}_{\beta,2} \frac{\|u\|^4_{E}}{M^4} \quad \text{for all } u \in H^1_r(\Omega) \text{ with } \|\nabla u\|_{L^2(\Omega)} \le M,$$

*where* $\Omega := \{x \in \mathbf{R}^2 : |x| > 1\}$.

*Proof.* The inequality in (i) can be proven in the same way to [1]. (ii) is a direct consequence of (i). Indeed, since for each $x \in \mathbf{R}^2$ it follows that

$$\begin{aligned} M^4\Phi\left(\frac{\beta u^2(x)}{M^2}\right) &= M^4 \sum_{j=2}^{\infty} \frac{(\beta u^2(x))^j}{j! M^{2j}} = \sum_{j=2}^{\infty} \frac{(\beta u^2(x))^j}{j! M^{2j-4}} \\ &\le \sum_{j=2}^{\infty} \frac{(\beta u^2(x))^j}{j! \|\nabla u\|^{2j-4}_{L^2}} = \|\nabla u\|^4_{L^2} \Phi\left(\frac{\beta u^2(x)}{\|\nabla u\|^2_{L^2}}\right), \end{aligned}$$

(ii) holds by (i). As to (iii), Using the operator $T_1$ (see (6.3.2)), by (ii) and Sobolev's inequality, we can easily obtain (iii). □

# Chapter 7

# Appendix

## 7.1 Assumptions and main statement

The aim of this chapter is to prove that minimax values defined in Chapters 5 and 6 diverge to infinity.

In this chapter, we slightly generalize our settings. Namely, let $E$ be a Banach space, $\| \cdot \|$ denote its norm and $I \in C^1(E, \mathbf{R})$ satisfy the Palais–Smale condition. Moreover, $I$ satisfies the following conditions:

$$(I_0) \qquad I(-u) = I(u) \quad \text{for all u} \in \text{E}.$$

$$(I_1) \qquad \begin{array}{l} \text{There exist } \rho \text{ and } \alpha > 0 \text{ such that} \\ I(u) \geq \alpha \quad \text{for all } u \in E \text{ with } \|u\| = \rho, \\ I(u) \geq 0 \quad \text{for all } u \in E \text{ with } \|u\| \leq \rho. \end{array}$$

$$(I_2) \qquad \begin{array}{l} \text{For each } n \in \mathbf{N} \text{ there exists an } h_n \in C(S^{n-1}, E) \text{ such that} \\ h_n(-\sigma) = -h_n(\sigma), \quad I(h_n(\sigma)) < 0 \qquad \text{for all } \sigma \in S^{n-1}, \\ \text{where } S^{n-1} := \{x \in \mathbf{R}^n \ : \ |x| = 1\}. \end{array}$$

Next, we define minimax values of $I$: For each $n \in \mathbf{N}$, we set

$$c_n := \inf_{\gamma \in \tilde{\Gamma}_n} \max_{\sigma \in D_n} I(\gamma(\sigma)), \tag{7.1.1}$$

where $D_n := \{x \in \mathbf{R}^n : |x| \leq 1\}$ and

$$\tilde{\Gamma}_n := \{\gamma \in C(D_n, E) : \gamma(-\sigma) = -\gamma(\sigma) \text{ for all } \sigma \in D_n, \ \gamma_(\sigma) = \gamma_n(\sigma) \text{ for all } \sigma \in S^{n-1}\}.$$

Note that the auxiliary functionals in Chapters 5 and 6 satisfy $(I_0)$–$(I_2)$. Moreover, the minimax values defined in Chapters 5 and 6 correspond to $c_n$ in (7.1.1).

Now we will prove the following theorem:

**Theorem 7.1.1.** *As $n \to \infty$, $c_n \to \infty$.*

## 7.2 Proof of Theorem 7.1.1

We will prove Theorem 7.1.1 using similar arguments in Rabinowitz [92] and need some preparations.

**Definition 7.2.1.** For each $n, j \in \mathbf{N}$, we define $\mathcal{E}$, $\mathcal{E}_{\mathbf{R}^n}$ and $\Gamma_j$ as follows:

$$\begin{aligned} \mathcal{E} &:= \{A \subset E\backslash\{0\} : A \text{ is closed}, -A = A\}, \\ \mathcal{E}_{\mathbf{R}^n} &:= \{B \subset \mathbf{R}^n\backslash\{0\} : B \text{ is closed}, -B = B\}, \\ \Gamma_j &:= \{h(\overline{D_n\backslash Y}) : n \geq j,\ h \in \tilde{\Gamma}_n,\ Y \in \mathcal{E}_{\mathbf{R}^n},\ \text{genus}\,(Y) \leq n-j\}. \end{aligned}$$

Here genus $(Y)$ is Krasnoselski's genus.

First, we state properties of $\Gamma_j$.

**Proposition 7.2.2.** *The following hold*:

(i) *For all* $j \in \mathbf{N}$, $\Gamma_j \neq \emptyset$.

(ii) *If* $\varphi \in C(E,E)$ *satisfies* $\varphi(-u) = -\varphi(u)$ *for all* $u \in E$ *and* $\varphi = \mathrm{id}_E$ *on* $h_n(S^{n-1})$ *for all* $n \in \mathbf{N}$, *then* $\varphi : \Gamma_j \to \Gamma_j$ *for each* $j \in \mathbf{N}$.

(iii) *If* $B \in \Gamma_j$, $Z \in \mathcal{E}$ *and* genus $(Z) \leq s < j$, *then* $\overline{B\backslash Z} \in \Gamma_{j-s}$.

*Proof.* (i) For each $n \in \mathbf{N}$, we set $\tilde{h}_n(\sigma) := |\sigma| h_n(\sigma/|\sigma|)$. Then it is easily seen $\tilde{h}_n \in \tilde{\Gamma}_n$, which implies $\Gamma_j \neq \emptyset$ for all $j \in \mathbf{N}$.
(ii) Let $B \in \Gamma_j$, $B = h(\overline{D_n\backslash Y})$, $h \in \tilde{\Gamma}_n$, $n \geq j$, $Y \in \mathcal{E}_{\mathbf{R}^n}$, genus $(Y) \leq n-j$. Then the composite map $\varphi \circ h : D_n \to E$ is odd and continuous. Furthermore it holds that $\varphi \circ h = h_n$ on $S^{n-1}$. Thus $\varphi \circ h \in \tilde{\Gamma}_n$ and $\varphi(B) = (\varphi \circ h)(\overline{D_n\backslash Y}) \in \Gamma_j$.
(iii) Let $B = h(\overline{D_n\backslash Y}) \in \Gamma_j$, $Z \in \mathcal{E}$, genus $(Z) \leq s < j$. If the following equality

$$\overline{B\backslash Z} = h(\overline{D_n\backslash(Y \cup h^{-1}(Z))}) \tag{7.2.1}$$

holds, then we can prove (iii). Indeed, we assume (7.2.1) and prove (iii). By $h^{-1}(Z) \in \mathcal{E}_{\mathbf{R}^n}$ and $Y \in \mathcal{E}_{\mathbf{R}^n}$, we see $Y \cup h^{-1}(Z) \in \mathcal{E}_{\mathbf{R}^n}$. By the mapping property and subadditivity of the genus, it follows that

$$\begin{aligned} \text{genus}\,(Y \cup h^{-1}(Z)) &\leq \text{genus}\,(Y) + \text{genus}\,(h^{-1}(Z)) \leq \text{genus}\,(Y) + \text{genus}\,(Z) \\ &\leq m - j + s = m - (j-s). \end{aligned} \tag{7.2.2}$$

By (7.2.1) and (7.2.2), we can conclude $\overline{B\backslash Z} \in \Gamma_{j-s}$.
Now we prove (7.2.1). Let $b \in h(D_n\backslash(Y \cup h^{-1}(Z)))$. Then, we have

$$b \in h(D_n\backslash Y)\backslash Z \subset B\backslash Z \subset \overline{B\backslash Z}.$$

Hence $\overline{h(D_n\backslash(Y \cup h^{-1}(Z)))} \subset \overline{B\backslash Z}$.
On the other hand, let $b \in B\backslash Z$ and $w \in \overline{D_n\backslash Y}$ satisfy $b = h(w)$. Since $b \notin Z$, $w \notin h^{-1}(Z)$ holds. Thus we have

$$w \in \left(\overline{D_n\backslash Y}\right)\backslash h^{-1}(Z) \subset \overline{D_n\backslash(Y \cup h^{-1}(Z))},$$

which implies $B\backslash Z \subset h(\overline{D_n\backslash(Y \cup h^{-1}(Z))})$. Therefore we can infer

$$\overline{B\backslash Z} \subset h(\overline{D_n\backslash(Y \cup h^{-1}(Z))}).$$

This completes the proof. □

Next we define another minimax values of $I$.

**Definition 7.2.3.** For each $j \in \mathbf{N}$, we define $d_j$ by

$$d_j := \inf_{B \in \Gamma_j} \max_{u \in B} I(u).$$

Note that since genus $(\emptyset) = 0$, it is easily seen that $d_j \leq c_j$ for all $j \in \mathbf{N}$. Thus in order to prove Theorem 7.1.1, it is sufficient to show $d_j \to \infty$ as $j \to \infty$.

The following proposition shows that $d_j$ is well-defined and $0 < \alpha \leq d_j$ for all $j \in \mathbf{N}$. Here $\alpha > 0$ appears in $(I_1)$.

**Proposition 7.2.4.** *For each $j \in \mathbf{N}$ and $B \in \Gamma_j$, $B \cap \partial B_\rho \neq \emptyset$ holds.*

*Proof.* Let $B = h(\overline{D_n\backslash Y})$, $n \geq j$, genus $(Y) \leq n-j$. Set $\hat{\mathcal{O}} := \{x \in \mathrm{Int}\,(D_n) : h(x) \in B_\rho\}$, then $0 \in \hat{\mathcal{O}}$, $-\hat{\mathcal{O}} = \hat{\mathcal{O}}$ and $\hat{\mathcal{O}}$ is open. Let $\mathcal{O}$ denote a component of $\hat{\mathcal{O}}$ containing 0, then we have genus $(\partial\mathcal{O}) = n$. We claim

$$h(\partial\mathcal{O}) \subset \partial B_\rho. \tag{7.2.3}$$

Assuming (7.2.3) for the moment, set $W = \{x \in D_n : h(x) \in \partial B_\rho\}$. By (7.2.3), we can see $\partial\mathcal{O} \subset W$, which implies genus $(W) = n$. Therefore we have

$$\text{genus}\,(\overline{W\backslash Y}) \geq n - (n-j) = j.$$

and

$$\text{genus}\,(h(\overline{W\backslash Y})) \geq \text{genus}\,(\overline{W\backslash Y}) \geq j > 0.$$

Thus it follows that $h(\overline{W\backslash Y}) \neq \emptyset$. On the other hand, since $h(\overline{W\backslash Y}) \subset B \cap \partial B_\rho$, it holds that $B \cap \partial B_\rho \neq \emptyset$.

Now we prove (7.2.3). By the property of $h_n$, it holds that $I(u) < 0$ for all $u \in h_n(\partial D_n)$. On the other hand, by $(I_1)$, we have $I(u) \geq 0$ for all $u \in \overline{B_\rho}$. Therefore we have $h_n(\partial D_n) \subset (\overline{B_\rho})^c$. Let $x \in \partial\mathcal{O}$. If $x \in D_n$ and $h(x) \in B_\rho$, then there exists an $r_0 > 0$ such that $h(y) \in B_\rho$ for all $y \in D_n$ satisfying $|y - x| < r_0$. However, this contradicts to $x \in \partial\mathcal{O}$. Therefore $x \in \partial D_n$ when $h(x) \in B_\rho$. In this case, since $h = h_n$ on $\partial D_n$ and $h_n(\partial D_n) \cap B_\rho = \emptyset$, this is also a contradiction. This implies $h(\partial\mathcal{O}) \subset \partial B_\rho$. □

Next, we will show that $d_j$ is a critical value of $I$ and estimate genus $(K_{d_j})$ where

$$K_c := \{u \in E : I(u) = c,\ I'(u) = 0\}.$$

**Proposition 7.2.5.**
(i) *For each $j \in \mathbf{N}$, $d_j$ is a critical value of $I$.*
(ii) *If there exist $p_1 < p_2 < \ldots < p_k$ such that $d := d_j = d_{j+p_1} = \cdots = d_{j+p_k}$, then* genus $(K_d) \geq p_k + 1$.

*Proof.* (i) Since $I$ satisfies the (PS) condition, it is standard to show that $d_j$ is a critical value of $I$ and we omit its proof.
(ii) First we note that $K_d \in \mathcal{E}$. Indeed, since $I(0) = 0$ and $d \geq \alpha > 0$ hold, we have $0 \notin K_d$, which implies $K_d \in \mathcal{E}$. Furthermore, $K_d$ is compact since $I$ satisfies the (PS) condition.

We shall prove indirectly and assume genus $(K_d) \leq p_k$. Then there exists a $\delta > 0$ such that genus $(N_\delta(K_d)) \leq p_k$ where $N_\delta(K_d) := K_d + \overline{B_\delta(0)}$. By applying the Deformation Theorem, there exist $\varepsilon \in (0, \alpha/2)$ and $\eta \in C([0,1] \times E, E)$ such that

$$\eta(1, -u) = -\eta(1, u) \quad \text{for all } u \in E \qquad \text{and} \qquad \eta(1, A_{d+\varepsilon} \backslash \mathcal{O}) \subset A_{d-\varepsilon}, \tag{7.2.4}$$

where

$$\mathcal{O} := N_\delta(K_d), \quad A_c := \{u \in E : I(u) \leq c\}.$$

Here we choose a $B \in \Gamma_{j+p_k}$ satisfying $\max_{u \in B} I(u) < d - \varepsilon$. Then by Proposition 7.2.2 (iii), we obtain $\overline{B \backslash \mathcal{O}} \in \Gamma_j$. On the other hand, for each $n \in \mathbf{N}$ and $u \in h_n(\partial D_n)$, $I(u) < 0$ holds, which implies $\eta(1, \cdot) = \text{id}$ on $h_n(\partial D_n)$. By Proposition 7.2.2 (ii), it follows that $\eta(1, \overline{B \backslash \mathcal{O}}) \in \Gamma_j$. However, by (7.2.4), we obtain

$$\max_{u \in \eta(1, \overline{B \backslash \mathcal{O}})} I(u) < d - \varepsilon,$$

which contradicts to the definition of $d_j$. Therefore genus $(K_d) \geq p_k + 1$ holds. □

In the following proposition, we will prove that $d_j \to \infty$ as $j \to \infty$, which completes a proof of Theorem 7.1.1.

**Proposition 7.2.6.** *As $j \to \infty$, $d_j \to \infty$.*

*Proof.* We prove indirectly and assume that $(d_j)$ is bounded. We may assume that there exists a subsequence $(d_{j_k})$ such that $d_{j_k} \to d_0$.

If there exists a sequence $(k_m)$ such that $k_m \to \infty$ and $d_0 = d_{j_{k_1}} = \cdots = d_{j_{k_m}} = \cdots$, then we obtain genus $(K_{d_0}) = \infty$ by Proposition 7.2.5 (ii). However, since $K_{d_0}$ is compact by the (PS) condition, genus $(K_{d_0}) < \infty$ holds. This is a contradiction.

Therefore we may suppose $d_0 \neq d_{j_k}$ for all $k \in \mathbf{N}$. For each $\xi > 0$, we set

$$\tilde{K}_\xi := \{u \in E : d_0 - \xi \leq I(u) \leq d_0 + \xi, \ I'(u) = 0\}.$$

By the (PS) condition, $\tilde{K}_\xi$ is compact. We choose a $\delta > 0$ such that

$$s = \text{genus}\,(K_\xi) = \text{genus}\,(N_\delta(K_\xi)), \quad \frac{\alpha}{2} < d_0 - \xi.$$

Since $d_{j_k}$ is a critical value of $I$ by Proposition 7.2.5 (i), $K_\xi \neq \emptyset$ for each $\xi > 0$. This implies $s \geq 1$. Set $\bar{\varepsilon} := d_0 - \xi/2$ and $\mathcal{O} := N_\delta(K_\xi)$. Applying the Deformation Theorem, we obtain $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ such that $\eta(1, A_{d_0+\varepsilon} \backslash \mathcal{O}) \subset A_{d_0-\varepsilon}$.

Next we take $j_k, j_\ell$ and $B \in \Gamma_{j_k+j_\ell}$ satisfying

$$s < j_\ell, \quad d_{j_k} \in (d_0 - \varepsilon, d_0 + \varepsilon), \quad \max_{u \in B} I(u) < d_0 + \varepsilon.$$

Since $s < j_\ell$, we have $\overline{B \backslash \mathcal{O}} \in \Gamma_{j_k}$ by Proposition 7.2.2 (iii). Furthermore, for each $n \in \mathbf{N}$, it follows that

$$I(u) \leq 0 \quad \text{on } h_n(\partial D_n), \quad 0 < \frac{\alpha}{2} < d_0 - \bar{\varepsilon} < d_{j_k},$$

which implies $\eta(1, \cdot) = \text{id}$ on $h_n(\partial D_n)$. Therefore $\eta(1, \overline{B \backslash \mathcal{O}}) \in \Gamma_{j_k}$.

On the other hand, it follow from the property of $\eta$ that

$$d_{j_k} \leq \max_{u \in \eta(1, \overline{B \backslash \mathcal{O}})} I(u) \leq d_0 - \varepsilon < d_{j_k}.$$

This is a contradiction and we obtain $d_j \to \infty$ as $j \to \infty$. □

*Proof of Theorem 7.1.1.* As we note in the above, it holds $d_j \leq c_j$ for each $j \in \mathbf{N}$. By Proposition 7.2.6, we can see that $c_j \to \infty$ as $j \to \infty$, which completes a proof. □

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# List of original papers


1. Uniqueness of positive solutions for a nonlinear elliptic system. *NoDEA Nonlinear Differential Equations Appl.* **16** (2009), no. 5, 555–567.

2. Existence of standing waves for coupled nonlinear Schrödinger equations. *Tokyo J. Math.* **33** (2010), 89–116.

3. (with J. Hirata and K. Tanaka) Nonlinear scalar field equations in $\mathbf{R}^N$: mountain pass and symmetric mountain pass approaches. *Topol. Methods Nonlinear Anal.* **35** (2010), no. 2, 253–276.

4. (with K. Tanaka) A local mountain pass type result for a system of nonlinear Schrödinger equations. *Calc. Var. Partial Differential Equations* **40** (2011), no. 3-4, 449–480.