早稲田大学大学院 基幹理工学研究科

博 士 論 文 概 要

論 文 題 目

Arithmetic of Quaternion Orders and its Applications
四元数環の整環に関する数論研究とその応用

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数学応用数理専攻 整数論研究

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The research interest of the applicant is in number theory, especially the areas related to the arithmetic properties of modular forms on modular curves, automorphic forms on Shimura curves, and the arithmetic of quaternion orders. Such majors are all based on the arithmetic of quaternion orders. In this dissertation, what we are concerned with are the arithmetic and applications of certain quaternion orders.

Arithmetic of Quaternion Orders

A quaternion algebra B over a base field K is a 4-dimensional central simple K-algebra. If K is a field of fractions of a Dedekind domain R, an order of B is a complete R-lattice, and a ring with unity. For a global field K, we denote K_v the completion of K at the place v. According to the local-global correspondence, the arithmetic of global orders in B is closely related to the arithmetic of local orders at finite places. In fact, for almost all finite places, the localization of the quaternion algebra has a structure as the matrix algebra $M(2,K_v)$. Therefore, a main goal for studying the arithmetic of quaternion orders is "To classify all the orders in $M(2,K_v)$".

On Orders of $\mathbf{M(2,K)}$ over Non-Archimedean Local Fields. For a non-Archimedean local field K, it is known that every maximal order in $M(2,K)$ is isomorphic to the maximal order $M(2,R)$, where R is the valuation ring of K. Also, the so-called split order (or Eichler order), which contains $R\oplus R$ as a subring, has been studied from 1950 by numerous mathematicians, such as Ehichler, Hijikata, Pizer, Shemanske, Shimura and so on. In 1974, Hijikata gave a complete characterization of split orders. He showed that the split orders can be uniquely determined by the intersections of two maximal orders, and they are isomorphic to the orders $\begin{pmatrix} R & R \\ \pi^n R & R \end{pmatrix}$, $n \geq 0$, where π is the uniformizer. Therefore, up to now, most studies of quaternion orders are related to the split orders.

However, there still exist non-split orders in $M(2,K)$ that have not been characterized completely yet, so it is natural to ask the questions

(1) Can we classify all orders that obtained by maximal orders?

(2) Is there any order that is not the intersection of maximal orders?

In this thesis, we will give an answer for the first problem and two examples to show the existence of non-intersection orders.

Theorem 1. Given finite number of maximal orders $O_1,\cdots,O_r$ in $M(2,K)$, there exist at most 3 maximal orders O_{j_1}, O_{j_2}, and O_{j_3} among them so that

$$\cap_{i=1}^{r} O_i = O_{j_i} \cap O_{j_2} \cap O_{j_3}.$$

The theorem shows that it is always enough to determine an intersection order by 3 suitable maximal orders. Besides, we also give a precisely way to find such 3 maximal orders. According to the result and the properties of the action of $GL(2,K)$ on the graph of maximal orders, we can obtain a complete classification for intersection orders.

Theorem 2. (Classification of Intersection Orders of $M(2,K)$)

If an order in $M(2,K)$ is the intersection of finitely many maximal orders, then it is isomorphic to exactly one of the following orders

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in R\right\}, (O_{j_1} = O_{j_2} = O_{j_3})$$

$$\left\{\begin{pmatrix} a & b \\ \pi^n c & d \end{pmatrix} \middle| a,b,c,d \in R\right\}, (n > 0, \quad O_{j_2} = O_{j_3})$$

$$\left\{\begin{pmatrix} a & b \\ \pi^k c & a + \pi^\ell d \end{pmatrix} \middle| a,b,c,d \in R\right\}, (k \geq 2\ell > 0).$$

Applications

For the applications described in the thesis, one is using quaternion ideals to construct lattices having highest known densities in Euchlidean spaces, which is related to the lattice packing problem; the other is to obtain defining equations of modular curve $X_0(2^{2n})$.

Lattice Packing form Quaternion Algebras. The bilinear form defined by $tr(x\bar{y})$ on a definite quaternion algebra B over a totally real number field K is nondegenerate and symmetric, where tr is the reduced trace on B. For a chosen ideal I of a definite quaternion algebra B, by a suitable scaled trace construction via a totally positive integral element in K, the map $tr_{\mathbb{Q}}^{K}(\alpha\, tr(x\bar{y}))$ gives a positive definite symmetric $\mathbb{Z}$-bilinear form on the ideal I. In this case, we denote (I, α) the ideal lattice. Then one has a determinant formula $\det(I) = {d_K}^4 N_{\mathbb{Q}}^{K}({d_B}^4 \alpha^4 N(I)^4)$, where d_K is the discriminant of the field K, d_B is the discriminant of the quaternion algebra B, and N(I) is the reduced norm of the ideal I of B. Using the information, we successfully constructed ideal lattices which have best known densities in dimension 4, 8, 12, 16, 24, and 32. In particular, these lattices are isomorphic to the well-known root lattices D_4, E_8, Coxeter-Todd lattice K_{12}, the laminated Λ_{16}, and the Leech lattice Λ_{24}, respectively.

Defining equations of modular curves $X_0(2^{2n})$. The classical modular curves $X_0(N) := \Gamma_0(N)\backslash\mathfrak{H}^*$, associated to the order $\Gamma_0(N) = \left\{\gamma \in SL(2,\mathbb{Z}) \middle| \gamma = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N\right\}$

of the quaternion algebra $M(2,\mathbb{Q})$, is a compact Riemann surface. The polynomial defining the surface is called the defining equation for X0(N). Note that there is a natural covering map from $X_0(2^{2n+2})$ to $X_0(2^{2n})$. Once the defining equation of $X_0(2^{2n})$ is known, one may deduce an equation of $X_0(2^{2n+2})$ from the covering map. However, in general, it is not easy to find a precise description of the map $X(\Gamma_1) \longrightarrow X(\Gamma_2)$ if $\Gamma_1 < \Gamma_2$ are congruence subgroups of $SL(2,\mathbb{Z})$. The key point of our method is to obtain relations between the generators of the function fields of the modular curves. These relations then give rise a recursive polynomials which define the equations.

Theorem 3. Let $P_6(x,y) = y^4 - x^3 - 4x$. For $n \geq 7$, define polynomials $P_n(x,y)$ recursively by

$$P_n(x,y) = P_{n-1}\left(\frac{\sqrt{x^2+4}}{\sqrt{x}}, \frac{y}{\sqrt{x}}\right) P_{n-1}\left(-\frac{\sqrt{x^2+4}}{\sqrt{x}}, \frac{y}{\sqrt{x}}\right) x^{2^{n-5}}.$$

Then $P_{2n}(x,y) = 0$ is a defining equation of the modular curve $X_0(2^{2n})$ for $n \geq 3$.

In particular, for $n \geq 1$, we define

$$x_n = \frac{2\theta_3(2^{n-1}\tau)}{\theta_2(2^{n-1}\tau)} \text{ and } y_n = \frac{\theta_2(8\tau)}{\theta_2(2^{n-1}\tau)},$$

where

$$\theta_2 = \frac{2\eta(2\tau)^2}{\eta(\tau)} \text{ and } \theta_3 = \frac{\eta(\tau)^5}{\eta(\tau/2)^2\eta(2\tau)^2}$$

are Jacobi theta functions,

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

is the Dedekind eta function, and $q = e^{2\pi i\tau}$ with $\mathrm{Im}\tau > 0$. Then one can show that

(1) For $n \geq 2$, we have $x_{n-1} = \sqrt{(x_n^2+4)/x_n}$ and $y_{n-1} = y_n/\sqrt{x_n}$;

(2) For $n \geq 6$, $P_n(x_n, y_n) = 0$, and $P_n(x,y)$ is irreducible over $\mathbb{C}$;

(3) For $n \geq 2$, x_{2n} and y_{2n} are modular functions on $\Gamma_0(2^{2n})$ that are holomorphic everywhere except for a pole of order 2^{2n-4} and $2^{2n-4}-1$, respectively, at ∞.

Therefore, the modular functions x_{2n} and y_{2n} generate the field of modular functions on $X_0(2^{2n})$ and the relation $P_{2n}(x_{2n}, y_{2n}) = 0$ between them is a defining equation for $X_0(2^{2n})$.

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研究発表	1. Defining Equations of X_0(2^{2n}), XXVth Journées Arithmétiques, Edinburgh, July 5, 2007. 2. On Orders of M(2;K) over a Non-Archimedean Local Field, Joint Workshop on Number Theory between Japan and Taiwan, Japan, March 18, 2010. 3. On Orders of M(2;K) over a Non-Archimedean Local Field, Taiwan-Korean Workshop on Number Theory, Taiwan, July 5, 2010. 4. Algebraic Transformations of Hypergeometric Functions and Automorphic Forms on Shimura Curves, PMI-NCTS Number Theory Workshop, Korea, July 25, 2011. 5. Algebraic Transformations of Hypergeometric Functions and Automorphic Forms on Shimura Curves, 京都数理解析研究所研究集会「代数的整数論とその周辺」 Program of RIMS Workshop “Algebraic Number Theory and Related Topics” Japan, November 28, 2011. <u>Seminar Talks</u> 1. Finite Graph and Orders in M(2, K) over Local Fields, NCTS Seminar on Number Theory, Taiwan, March 24, 2010. 2. Lattice Packings from Quaternion Algebras, NCTS Working/ Student Seminar on Number Theory, Taiwan, August 5, September 7, 2010. 3. Algebraic Transformations of Hypergeometric Functions and Automorphic Forms on Shimura Curves. NCTS Seminar on Number Theory, Taiwan, December 15, 2010. 4. Equations of Shimura Curves of Genus One, NCTS Seminar on Number Theory, Taiwan, March 9, 2011. 5. On Orders of M(2;K) over a Non-Archimedean Local Field, 早稲田大学整数論セミナー, Japan, May 20, 2011.