The Effects of (m,n)th-Degree Stochastic Dominant Shifts in a Distribution of Portfolio Returns

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Abstract

This paper explains the binary choices of agents and portfolio selection under higher-degree risk conditions. First, we re-examine the equivalence of (m,n)th-degree stochastic dominance and preference order for agents exhibiting (m,n)th-degree mixed risk aversion. Next, we demonstrate sufficient conditions for increased demand for a risky asset when the return distribution of portfolios undergoes (m,n)th-degree stochastic dominant shift. This shift includes the $(1,n)$th and $(2,n)$th-degree stochastic dominance as well as mean-preserving spread, downside risk, and outer risk. The results depend on an upper bound of higher-degree relative risk aversion coefficients, and we investigate their implications.

1 Introduction

Analyzing how changes in endogenous risk parameters affect agents' behavior is one of the problems in comparative statics. Specifically, the effect of stochastic dominant shifts on equilibrium prices of portfolios and agent behavior in the risky asset market has been reported by Hadar and Seo (1990), Gollier (2001), Ohnishi and Osaki (2007), and Jokung (2013). In this paper, we focus on (m,n)th-degree stochastic dominance introduced by Wong (2019) to characterize agent behavior under higher-degree risk. (m,n)th-degree stochastic dominance encompasses various stochastic dominance criteria and higher-degree risks, such as mean-preserving spread by Rosthchild and Stiglitz (1970), increase in higher-degree risks by Menezes, Geiss, and Tressler (1980) and Menezes and Wang (2005), and $(1,n)$th and $(2,n)$th-degree stochastic dominance by Denuit and Eeckhoudt (2013).

This paper obtains the following results: First, we re-evaluate the equivalence of (m,n)th-degree stochastic dominance and preference order for agents exhibiting (m,n)th-degree risk-aversion by modifying the condition as proposed by Wong (2019). Second, we establish sufficient conditions for an increase in optimal demand for a risky asset due to (m,n)th-degree stochastic dominant shift in the return distribution of portfolios. These conditions impose an upper bound of the agent's higher-degree relative risk aversion. Finally, we provide examples illustrating the implications of the increase in optimal demand for the risky asset and the upper bound on higher-degree relative risk aversion.

This paper is structured as follows: Section 2 revisits the relationship between (m, n)th-degree stochastic dominance and (m, n)th-degree mixed risk aversion. Section 3 presents the model and results conditions for increased demand in the risky asset. Section 4 illustrates the implications of the conditions for the increase in optimal demand for the risky asset. The final section concludes.

2 (m, n)th-Degree Stochastic Dominance Criteria

2.1 Preliminaries

Consider random variables $\tilde{x}_i$ in some interval $[\underline{x}, \overline{x}]$ for each $i = 1, 2$. Denote each cumulative distribution function as $F_i(x)$, where $F_i(\underline{x}) = 0$ and $F_i(\overline{x}) = 1$. Let $F_i^n(x) = \int_{\underline{x}}^{x} F_i^{n-1}(y)dy$ be the sucssesive integration of the cumulative distribution function. We introduce (m, n)th-degree stochastic dominance defined by Wong (2019) as follows.

Definition 1 (Wong, 2019). For all two integers, n and m, such that $n \geq m \geq 1$, and any two random variables, $\tilde{x}_1$ and $\tilde{x}_2$, we say $\tilde{x}_1$ is riskier than $\tilde{x}_2$ via (m, n)th-degree stochastic dominance if

$$
\begin{aligned}
&F_1^s(\overline{x}) = F_2^s(\overline{x}) \text{ for } s = 2, \ldots, m, \\
&F_1^k(\overline{x}) \geq F_2^k(\overline{x}) \text{ for } k = m+1, \ldots, n, \\
&F_1^n(x) \geq F_2^n(x) \text{ for all } x \in [\underline{x}, \overline{x}].
\end{aligned}
$$

Denote the (m, n)th-degree stochastic dominance as follows: given two random variables $\tilde{x}_1$ and $\tilde{x}_2$, $\tilde{x}_2$ dominates $\tilde{x}_1$ in the sense of (m, n)th-degree stochastic dominance, denoted by $\tilde{x}_1 \preccurlyeq_{(m,n)SD} \tilde{x}_2$ when $\tilde{x}_1$ is riskier than $\tilde{x}_2$ via (m, n)th-degree stochastic dominance. If $m = n$, (n, n)th-degree stochastic dominance is equivalent to nth increase in risk introduced by Ekern (1980). Define this order as $\preccurlyeq_{nIR}$ corresponds to $\preccurlyeq_{(n,n)SD}$.

In particular, the mean-preserving spread by Rothschild and Stiglitz (1970) is identical to the second increase in risk. Menezes, Geiss, and Tressler (1980) defined an increase in downside risk, which shifts the risk to the left while preserving the mean and variance. A third increase in risk is equivalent to downside risk. As defined by Menezes and Wang (2005), an increase in outer risk indicates that the shift has higher peaks and longer tails with a constant mean, variance, and third central moment, corresponding to a fourth increase in risk.

2.2 (m,n)-Concave Order

Denote vNM utility function as $u\colon \mathbb{R} \to \mathbb{R}, u(x)$, which is differentiable at least n times. Let $u^{(s)}(x)$ be sth successive derivative of $u(x)$ for $s = 1, 2, \ldots, n$. Consider a class of (m,n)-concave utility functions for two integers, n and m, such that $n \geq m \geq 1$, as follows.

$$\mathcal{U}_{(m,n)-cv} = \{u(x) \mid (-1)^{s+1} u^{(s)}(x) \geq 0 \text{ for } s = m, \ldots, n\}.$$

Same as Wong (2019), we say $u(x)$ exhibits (m,n)th-degree mixed risk aversion if $u(x) \in \mathcal{U}_{(m,n)-cv}$. When $m = 1$, $\mathcal{U}_{(1,n)-cv}$ is equivalent to the class of s-increasing concave utility function, $\mathcal{U}_{s-icv}$, such that $(-1)^{s+1} u^{(s)}(x) \geq 0$ for $s = 1, \ldots, n$. Caballé and Pomansky (1996) showed that allowing n to approach infinity results in all odd derivatives of $u(x)$ being positive and all even derivatives being negative, making $u(x)$ completely monotone. In this case, $(1,n)$th-degree mixed risk aversion becomes mixed risk aversion. When $m = n$, $\mathcal{U}_{(n,n)-cv}$ becomes the class of s-concave utility function, $\mathcal{U}_{s-cv}$, where $(-1)^{s+1} u^{(s)}(x) \geq 0$. Thus, (n,n)th-degree mixed risk aversion degenerates the nth-degree risk aversion by Ekern (1980).

Consider (m,n)-concave order as follows: given two random variables $\tilde{x}_1$ and $\tilde{x}_2$, $\tilde{x}_1$ is said to be smaller than $\tilde{x}_2$ in the (m,n)-concave order, denoted by $\tilde{x}_1 \preccurlyeq_{(m,n)-cv} \tilde{x}_2$ when $E[u(\tilde{x}_2)] \geq E[u(\tilde{x}_1)]$ for all $u(x) \in \mathcal{U}_{(m,n)-cv}$. Similarly, let $\preccurlyeq_{s-icv}$ and $\preccurlyeq_{s-cv}$ be the orders which satisfies $E[u(\tilde{x}_2)] \geq E[u(\tilde{x}_1)]$ for all $u(x) \in \mathcal{U}_{s-icv}$ and $u(x) \in \mathcal{U}_{s-cv}$, respectively. It is clear that $\preccurlyeq_{(1,n)-cv} = \preccurlyeq_{n-icv}$ and $\preccurlyeq_{(n,n)-cv} = \preccurlyeq_{n-cv}$.

Wong (2019) mentions the equivalence of (m,n)-concave order and (m,n)th-degree stochastic dominance. For the sake of completeness in the proof, we revise Lemma 1 of Wong (2019) as follows.

Proposition 1. The following holds for all two integers, n and m, such that $n \geq m \geq 1$:

$$\begin{cases} \tilde{x}_1 \preccurlyeq_{(m,n)-cv} \tilde{x}_2, \\ E[\tilde{x}_1^{s-1}] = E[\tilde{x}_2^{s-1}] \text{ for } s = 1, \ldots, m \end{cases}$$

$$\Leftrightarrow \tilde{x}_1 \preccurlyeq_{(m,n)SD} \tilde{x}_2$$

$$\Leftrightarrow \begin{cases} E[(\bar{x}-\tilde{x}_1)^{s-1}] = E[(\bar{x}-\tilde{x}_2)^{s-1}] \text{ for } s = 1,\ldots,m, \\ E[(\bar{x}-\tilde{x}_1)^{k-1}] \geq E[(\bar{x}-\tilde{x}_2)^{k-1}] \text{ for } k = m+1,\ldots,n, \\ E[(x-\tilde{x}_1)_+^{n-1}] \geq E[(x-\tilde{x}_2)_+^{n-1}] \text{ for all } x \in [\underline{x},\bar{x}], \end{cases}$$

where $(\cdot)_+ = \max[\cdot, 0]$.

Proof. See Appendix A.1.

Proposition 1 means that (m,n)-concave order can be characterized by (m,n)th-degree stochastic dominance as well as $E[(x-\tilde{x}_i)_+^{s-1}]$ for $s \in \mathbb{N}$ when non-central moments equal for all $s = 1,\ldots,m-1$. $E[(x-\tilde{x}_i)_+^{n-1}]$ is referred to as the nth-degree Lower Partial Moment (LPM) by Bawa (1975), Fishburn (1977), Ingersoll (1987), and Harlow and Rao (1989). LPM becomes larger when smaller values of $\tilde{x}_i$ occur, which can be interpreted as a nonnegative index with the left tail on the distribution. Therefore, the random variables with a larger LPM are also riskier in the sense of stochastic dominance criteria.

For instance, the following holds for $(1,3)$th-degree stochastic dominance.

$$\tilde{x}_1 \preccurlyeq_{3-icv} \tilde{x}_2 \Leftrightarrow x_1 \preccurlyeq_{(1,3)SD} x_2$$
$$\Leftrightarrow E[\bar{x}-\tilde{x}_1] \geq E[\bar{x}-\tilde{x}_2],\ E[(x-\tilde{x}_1)_+^2] \geq E[(x-\tilde{x}_2)_+^2]\ \forall x \in [\underline{x},\bar{x}].$$

In general, $(1,n)$th-degree stochastic dominance corresponds to nth-degree stochastic dominance. In $(2,3)$th-degree case,

$$\tilde{x}_1 \preccurlyeq_{(2,3)-cv} \tilde{x}_2, E[\tilde{x}_2] = E[\tilde{x}_1]$$
$$\Leftrightarrow \tilde{x}_1 \preccurlyeq_{(2,3)SD} \tilde{x}_2$$
$$\Leftrightarrow E[\bar{x}-\tilde{x}_1] = E[\bar{x}-\tilde{x}_2],\ E[(x-\tilde{x}_1)_+^2] \geq E[(x-\tilde{x}_2)_+^2]\ \forall x \in [\underline{x},\bar{x}].$$

Similarly, $(2,n)$th-degree stochastic dominance becomes mean-preserving nth-degree stochastic dominance as in Denuit and Eeckhoudt (2013). This way, Proposition 1 enables the demonstration of equivalence with commonly used stochastic dominance criteria.

3 The Impact of Demand for Higher Degree Stochastic Dominant Shift

In this section, we consider a standard portfolio problem, the same as Gollier (2001), and explain the conditions for the optimal demand on the risky asset to increase when the return distribution of the risky asset shifts due to (m,n)th-degree stochastic dominance or nth increase in risk.

3.1 The Model

Consider an economy in which one risk-free asset, one risky asset, and an agent with a concave vNM utility function. The agent faces the problem of determining the optimal consumption $(W-\alpha_1,\alpha_1)$ of his consumption, where W is the agent's initial wealth in $\mathbb{R}_+,\alpha_1$ is the amount invested in the risky asset, and $W-\alpha_1$ is the amount invested in the risk-free asset. The portfolio amount at the end of the period can be given as

$$(W-\alpha_1)(1+r)+\alpha_1(1+\tilde{x}_0)=w_0+\alpha_1\tilde{x}_1, \tag{1}$$

where

r is the return of the risk-free asset in $\mathbb{R}_+$,

$\tilde{x}_0$ is the return of the risky asset over the period,

$w_0=W(1+r)$ is future risk-free wealth in the portfolio,

$\tilde{x}_1=\tilde{x}_0-r$ is the excess return of the risky asset in $[\underline{x},\overline{x}]$.

Let $F_1(x)$ be the distribution function of the excess return on the risky asset. Assume the agent is not allowed to short-sell each asset. For the existence of a positive solution, additionally assume $E[\tilde{x}_1]>0$ and $\tilde{x}_1$ alternates in sign. For equation (1), the optimization problem for the agent to choose the demand of risky asset may be written by

$$\max_{\alpha_1} E[u(w_0+\alpha_1\tilde{x}_1)]. \tag{2}$$

The first-order condition for problem (2) is

$$E[\tilde{x}_1u'(w_0+\alpha_1^*\tilde{x}_1)]=0,$$

where α_1^* is the optimal demand for the risky asset.

We examine the following question: Under what conditions does a distribution change lead to an increase in the optimal demand for the risky asset? To explore this question, we define the excess return of the risky asset as the change in the distribution of $\tilde{x}_1$, denoted by $\tilde{x}_2\in[\underline{x},\overline{x}]$. Let $F_2(x)$ be the distribution function of $\tilde{x}_2$. By concavity of objective function for α_1, the change in the distribution increases the optimal amount in the risky asset if and only if

$$E[\tilde{x}_1u'(w_0+\tilde{x}_1)]=0 \Rightarrow E[\tilde{x}_2u'(w_0+\tilde{x}_2)]\geq 0, \tag{3}$$

where α_1^* is unity. Condition (3) means that when the optimal exposure to risky asset $\tilde{x}_1$ is unity, the optimal demand of $\tilde{x}_2$ as the change in the distribution is larger than unity. This condition is equivalent to the following condition as

$$E[\tilde{x}_1 u'(w_0+\tilde{x}_1)]=0 \Rightarrow \int_{\underline{x}}^{\overline{x}} xu'(w_0+x)[f_2(x)-f_1(x)]\,dx \geq 0. \tag{4}$$

Therefore, the condition for increased demand for the risky asset can be formulated as the difference in excess return densities, with this difference signifying the distributional shift. Subsequent analyses will explore higher-degree distributional shifts and elucidate their effect on optimal demand.

3.2 (m,n)th-Degree Stochastic Dominant Shift

This section elucidates the impact of (m,n)th-degree stochastic dominance shifts on the demand for the risky asset. Consider a distributional shift where $\tilde{x}_2$ dominates $\tilde{x}_1$ in the sense of (m,n)th-degree stochastic dominance, i.e., $\tilde{x}_1 \preccurlyeq_{(m,n)SD} \tilde{x}_2$. Subsequently, we provide the condition for an increase in the demand for the risky asset when subjected to the (m,n)th-degree stochastic dominant shift in the distribution.

Proposition 2. Suppose the agent's vNM utility function $u\colon \mathbb{R} \to \mathbb{R}, u(x)$ is concave. The shift in the return distribution increases the optimal demand for the risky asset for all $u(x) \in \mathcal{U}_{(m,n+1)-cv}$ if the following conditions hold for all two integers n and m where $n \geq m \geq 1$:

$$\begin{cases} \tilde{x}_1 \preccurlyeq_{(m,n)SD} \tilde{x}_2, \\ -x\dfrac{u^{(k+1)}(x)}{u^{(k)}(x)} \leq k \text{ for all } x \in [\underline{x},\overline{x}], \text{and } k=m,\ldots,n. \end{cases}$$

Proof. See Appendix A.2. □

Proposition 2 specifies the sufficient condition for an increase in demand for the risky asset under the assumption of $\tilde{x}_1 \preccurlyeq_{(m,n)SD} \tilde{x}_2$. Note that this sufficient condition places constraints not only on the agent, who exhibits $(m.n)$th-degree mixed risk aversion but also sets an upper bound on $-x\dfrac{u^{(k+1)}(x)}{u^{(k)}(x)}$ for $k=m,\ldots,n$. This broader constraint arises because the condition for the increase in optimal demand for the risky asset relies on the derivative of the objective function rather than the utility function.

For each $s=1,\ldots,n$, let

$$\mathcal{R}_{s+1}(x) := -x\frac{u^{(s+1)}(x)}{u^{(s)}(x)} \text{ and } \mathcal{A}_{s+1}(x) := -\frac{u^{(s+1)}(x)}{u^{(s)}(x)}$$

be $s+1$th-degree relative risk aversion and $s+1$th-degree absolute risk aversion, respectively. In their work, Jindapon and Neilson (2007) introduced Arrow-Pratt and Ross risk aversion measures for higher degrees and elucidated the impact of comparative higher-degree risk aversion. Additionally, Kimball (1990) demonstrated the relationship between precautionary savings and $\mathcal{A}_3(x)$, which is referred to as absolute prudence. Similarly, Kimball (1993) termed $\mathcal{A}_4(x)$ as absolute temperance. Consequently, the condition for an increase in the demand for the risky asset imposes an upper bound on the $k+1$th-degree relative risk aversion for each $k = m, \ldots, n$.

For some examples, in the case of $(1, 3)$th-degree stochastic dominant shift, the further condition on concave vNM utility function is

$$(-1)^4 u'''(x) \geq 0,\ (-1)^5 u^{(4)}(x) \geq 0,\ \mathcal{R}_2(x) \leq 1,\ \mathcal{R}_3(x) \leq 2,$$

and

$$\mathcal{R}_4(x) \leq 3 \text{ for all } x \in [\underline{x}, \overline{x}].$$

While the condition encompasses $u(x) \in \mathcal{U}_{4-icv}$, it's worth noting that $(1, 3)$th-degree stochastic dominance implies 3-increasing concave order. As a generalization, the $(1, n)$th-degree stochastic dominance shift provides insight into the risk-taking behavior of agents with $u(x) \in \mathcal{U}_{n+1-icv}$ and $\mathcal{R}_{k+1}(x) \leq k$ for $k = m, \ldots, n$ and all $x \in [\underline{x}, \overline{x}]$. On the other hand, it can be associated with the binary choice behavior of agents with $u(x) \in \mathcal{U}_{n-icv}$.

Similarly, with respect to the (2,3)th-degree stochastic dominant shift,

$$(-1)^4 u'''(x) \geq 0,\ (-1)^5 u^{(4)}(x) \geq 0,\ \mathcal{R}_3(x) \leq 2,$$

and

$$\mathcal{R}_4(x) \leq 3 \text{ for all } x \in [\underline{x}, \overline{x}].$$

Therefore, in the case of distribution changes where the means are equal, it is not relative risk aversion but higher-degree relative risk aversion that affects the agent's risk-taking.

3.3 nth Increase in Risk

In this section, we investigate a distributional shift where $\tilde{x}_2$ dominates $\tilde{x}_1$ in the sense of nth increase in risk, $\tilde{x}_1 \preccurlyeq_{nIR} \tilde{x}_2$. As discussed in the previous section, $\preccurlyeq_{nIR}$ is equivalent to (n, n)th-degree stochastic dominance. Consequently, the condition for an increase in the optimal demand for the risky asset becomes immediately apparent.

Corollary 1. Suppose the agent's vNM utility function $u\colon \mathbb{R} \to \mathbb{R}, u(x)$ is concave. The shift in the return distribution increases the optimal demand for the risky asset for all $u(x) \in \mathcal{U}_{(n,n+1)-cv}$ if the following conditions hold for all integer n where $n \geq 1$:

$$\begin{cases} \tilde{x}_1 \preccurlyeq_{nIR} \tilde{x}_2, \\ \mathcal{R}_{n+1}(x) \leq n \text{ for all } x \in [\underline{x}, \overline{x}]. \end{cases}$$

Proof. Substituting $m = n$ in Proposition 2 satisfies the above statement. □

Since the distribution shift enforces the constraint that the kth non-central moments become equal for $k = 1, \ldots, n-1$, the constraint on higher-degree e risk aversion is reduced to one. For instance, with mean-preserving spread, the sufficient condition for an increase in demand for the risky asset is

$$(-1)^4 u'''(x) \geq 0 \text{ and } \mathcal{R}_3(x) \leq 2 \text{ for all } x \in [\underline{x}, \overline{x}].$$

In the case of a third increase in risk (downside risk or mean-variance-preserving spread),

$$(-1)^4 u'''(x) \geq 0, (-1)^5 u^{(4)}(x) \geq 0 \text{ and } \mathcal{R}_4(x) \leq 3 \text{ for all } x \in [\underline{x}, \overline{x}].$$

When the agent faces outer risk (mean-variance-skewness-preserving spread),

$$(-1)^5 u^{(4)}(x) \geq 0, (-1)^6 u^{(5)}(x) \geq 0, \text{ and } \mathcal{R}_5(x) \leq 4 \text{ for all } x \in [\underline{x}, \overline{x}].$$

From the examples, Proposition 2 is comprehensive, including the results related to nth increase in risk.

4 An Interpretation of the Condition for Increase in Demand

This section considers the higher-degree utility premium and binary lotteries to provide the interpretation of the effect of agents' risk-taking on the higher-degree distributional shift. Following Courbage, Loubergé, and Rey (2018), as well as Wong (2019), we introduce a non-monetary measure of risk premium, referred to as the $(m.n)$th-degree utility premium. This measure considers both the initial wealth and the demand for the risky asset and is defined as follows:

$$\pi_u(w_0, \alpha) := E[u(w_0 + \alpha\tilde{x}_2)] - E[u(w_0 + \alpha\tilde{x}_1)].$$

To represent the condition for an increase in demand by the $(m.n)$th-degree utility premium, condition (4) can be rewritten as follows.

$$\lim_{\alpha\to 0}\frac{E[u(w_0+(\alpha_1^*+\alpha)\tilde{x}_2)-u(w_0+\alpha_1^*\tilde{x}_2)]}{\alpha}\geq\lim_{\alpha\to 0}\frac{E[u(w_0+(\alpha_1^*+\alpha)\tilde{x}_1)-u(w_0+\alpha_1^*\tilde{x}_1)]}{\alpha}$$

$$\Leftrightarrow \lim_{\alpha\to 0}\frac{\pi_u(w_0,\alpha_1^*+\alpha)}{\alpha}\geq\lim_{\alpha\to 0}\frac{\pi_u(w_0,\alpha_1^*)}{\alpha} \tag{5}$$

Thus, the distributional shift increases the optimal amount in the risky asset if and only if

$$\pi_u(w_0,\alpha_1^*+\alpha)\geq\pi_u(w_0,\alpha_1^*). \tag{6}$$

Let's assume $\alpha^*=0$ for the sake of simplification. Under this simplification, condition (6) degenerates into $\pi_u(w_0,\alpha)\geq 0$. According to Proposition 1, this implies that $\tilde{x}_2$ dominates $\tilde{x}_1$ in the sense of (m,n)th-degree stochastic dominance when $u(x)\in\mathcal{U}_{(m,n)-cv}$ and $E[\tilde{x}_1^{s-1}]=E[\tilde{x}_2^{s-1}]$ for $s=1,\ldots,m$, and the converse is also true.

Similar to Magnani (2017) and Wong (2019), we consider the two $50-50$ lottelies given initial wealth w_0. Lottery A provides either $(\alpha_1^*+\alpha)\tilde{x}_2$ or $\alpha_1^*\tilde{x}_1$, and Lottery B provides either $(\alpha_1^*+\alpha)\tilde{x}_1$ or $\alpha_1^*\tilde{x}_2$. Then, lottery A is equally or more favored than lottery B if and only if

$$\begin{aligned}&\frac{1}{2}E[u(w_0+(\alpha_1^*+\alpha)\tilde{x}_2)]+\frac{1}{2}E[u(w_0+\alpha_1^*\tilde{x}_1)]\\ \geq\ &\frac{1}{2}E[u(w_0+(\alpha_1^*+\alpha)\tilde{x}_1)]+\frac{1}{2}E[u(w_0+\alpha_1^*\tilde{x}_2)],\end{aligned} \tag{7}$$

which satisfies necessary and sufficient condition for condition (4). When risk size α_1^* is assigned to $\tilde{x}_1$ and $\tilde{x}_2$ in each lottery, lottery A has risk size α assigned to risk $\tilde{x}_2$, while lottery B has risk size α assigned to risk $\tilde{x}_1$. In this case, lottery A has better risk apportionment than lottery B if and only if the distribution change increases the optimal amount in the risky asset.

To examine the effect of the upper bound for higher-degree relative risk aversion in Proposition 2, we recall condition (5) as follows:

$$\lim_{\alpha\to 0}\frac{\pi_u(w_0,\alpha_1^*+\alpha)-\pi_u(w_0,\alpha_1^*)}{\alpha}\geq 0. \tag{8}$$

If $\tilde{x}_1\preccurlyeq_{(m,n)SD}\tilde{x}_2$, then $E[(\bar{x}-\tilde{x}_1)^{s-1}]=E[(\bar{x}-\tilde{x}_2)^{s-1}]$ for $s=2,\ldots,m$. Consequently, by expanding the nth-order Taylor series for $\pi_u(w_0,\alpha_1^*+\alpha)$ and $\pi_u(w_0,\alpha_1^*)$ around $w_0+(\alpha_1^*+\alpha)\tilde{x}_1=w_0+(\alpha_1^*+\alpha)\bar{x}$ and $w_0+\alpha_1^*\tilde{x}_2=w_0+\alpha_1^*\bar{x}$, respectively, we obtain[(1)]

(1) See appendix A.3.

$$\lim_{\alpha\to 0}\frac{\pi_u(w_0,\alpha_1^*+\alpha)-\pi_u(w_0,\alpha_1^*)}{\alpha}$$
$$\approx\sum_{k=m}^{n}\frac{(-1)^{k+1}u^{(k)}(w_0+\alpha_1^*\bar{x})\Delta_k(\alpha_1^*)}{k!\,\alpha_1^*}[\eta_{\Delta_k,\alpha_1^*}+\eta_{u^{(k)},\alpha_1^*}]$$
$$=\sum_{k=m}^{n}\frac{(-1)^{k+1}u^{(k)}(w_0+\alpha_1^*\bar{x})\Delta_k(\alpha_1^*)}{k!\,\alpha_1^*}[k-\alpha_1^*\bar{x}\mathcal{A}_{k+1}(w_0+\alpha_1^*\bar{x})]\geq 0, \quad (9)$$

where $\Delta_k(\alpha):=\alpha^k[E[(\bar{x}-\tilde{x}_1)^k]-E[(\bar{x}-\tilde{x}_2)^k]]$, $\eta_{\Delta_k,\alpha_1^*}:=\frac{\alpha_1^*}{\Delta_k(\alpha_1^*)}\frac{d\Delta_k(\alpha_1^*)}{d\alpha_1^*}$, $\eta_{u^{(k)},\alpha_1^*}:=\frac{\alpha_1^*}{u^{(k)}(w_0+\alpha_1^*\bar{x})}$ $\frac{\partial u^{(k)}(w_0+\alpha_1^*\bar{x})}{\partial\alpha_1^*}$.

Therefore, the condition for an increase in the optimal demand for the risky asset can be approximated using two demand elasticities: $\eta_{\Delta_k,\alpha_1^*}$ and $\eta_{u^{(k)},\alpha_1^*}$. The demand elasticity of $\Delta_k(\alpha)$ equals a positive constant k. When $\tilde{x}_1\preccurlyeq_{(m,n)SD}\tilde{x}_2$, $\Delta_k(\alpha)$ represents the scaling factor that hedges the $(k+1)$th-degree risk at $\bar{x}$ relative to the original distribution. Thus, $\eta_{\Delta_k,\alpha_1^*}$ can be interpreted as the rate of change in the scaling factor for hedging $(k+1)$th-degree risk at $\bar{x}$ when increasing the demand of the risky asset by one unit. In contrast, $\eta_{u^{(k)},\alpha_1^*}$ represents the demand elasticity of $u^{(k)}$, signifying the rate of change in the agent's kth-degree risk preference at $\bar{x}$ when the demand of the risky asset increases by one unit. Note that if $u(x)\in\mathcal{U}_{(m,n+1)-cv}$, then $\eta_{u^{(k)},\alpha_1^*}$ is always nonpositive.

From appendix A.2, the upper bound for higher-degree relative risk aversion ensures that $\eta_{\Delta_k,\alpha_1^*}\geq-\eta_{u^{(k)},\alpha_1^*}$ and the nonnegativity of condition (9) when $\tilde{x}_1\preccurlyeq_{(m,n)SD}\tilde{x}_2$ and $u(x)\in\mathcal{U}_{(m,n+1)-cv}$. Hence, this upper bound condition implies that the increase in the agent's kth-degree risk preference does not exceed the increase in the scaling factor for hedging $(k+1)$th-degree risk when the risky asset's demand increases by one unit. In this case, it approximately leads to the increase in the optimal demand for the risky asset for all $u(x)\in\mathcal{U}_{(m,n+1)-cv}$ when $\tilde{x}_1\preccurlyeq_{(m,n)SD}\tilde{x}_2$.

5 Conclusion

In this paper, we examined how the behavior of agents with various risk preferences can be characterized through (m,n)th-degree stochastic dominance. As a result, we demonstrated the equivalence of (m,n)-concave order augmented with non-central moments and (m,n)th-degree stochastic dominance, which can be reformulated using LPM. This reformulation suggests that all the stochastic dominances encompassed within (m,n)th-degree stochastic dominance are applicable to empirical analysis. Furthermore, we showed that when the portfolio return distribution shifts according to (m,n)th-degree stochastic dominance, upper bounds are imposed on all $(m+1)$ to $(n+1)$th degree relative risk aversions as the sufficient condition for an increase in optimal demand for the risky asset. These upper bound conditions are expressed as demand elasticities of risk hedging scale and risk preferences, highlighting an instance of the implications of higher-degree relative risk aversions.

Appendix: Proofs of the Results

A.1 Proof of Proposition 1

First, we will prove the following:

$$\tilde{x}_1 \preccurlyeq_{(m,n)SD} \tilde{x}_2 \Leftrightarrow \begin{cases} E[(\overline{x}-\tilde{x}_1)^{s-1}] = E[(\overline{x}-\tilde{x}_2)^{s-1}] \text{ for } s = 1,\dots,m, \\ E[(\overline{x}-\tilde{x}_1)^{k-1}] \geq E[(\overline{x}-\tilde{x}_2)^{k-1}] \text{ for } k = m+1,\dots,n, \\ E[(x-\tilde{x}_1)_+^{n-1}] \geq E[(x-\tilde{x}_2)_+^{n-1}] \text{ for all } x \in [\underline{x},\overline{x}] \end{cases}$$

for all two integers, n and m, such that $n \geq m \geq 1$.

Proof. Integration by parts on $F_i^2(x)$ yields

$$\begin{aligned} F_i^2(x) &= \int_{\underline{x}}^{x} F_i(y)dy = xF_i(x) - \int_{\underline{x}}^{x} yf_i(y)dy \\ &= \int_{\underline{x}}^{x} (x-y)f_i(y)dy. \end{aligned} \tag{10}$$

Using equation (10) and Fubini's theorem, $F_i^3(x)$ can be rewritten as follows:

$$\begin{aligned} F_i^3(x) &= \int_{\underline{x}}^{x} F_i^2(y)dy = \int_{\underline{x}}^{x}\int_{\underline{x}}^{y} (x-z)f_i(z)dzdy \\ &= \int_{\underline{x}}^{x}\left[\int_{z}^{x} (x-y)dy\right] f_i(z)dz \\ &= \frac{1}{2}\int_{\underline{x}}^{x} (x-z)^2 f_i(z)dz. \end{aligned}$$

Similarly, for $F_i^4(x)$,

$$\begin{aligned} F_i^4(x) &= \int_{\underline{x}}^{x} F_i^3\left(y\right)dy = \frac{1}{2}\int_{\underline{x}}^{x}\int_{\underline{x}}^{y} (x-z)^2 f_i(z)dzdy \\ &= \frac{1}{2}\int_{\underline{x}}^{x}\left[\int_{z}^{x} (x-y)^2 dy\right] f_i(z)dz \\ &= \frac{1}{3!}\int_{\underline{x}}^{x} (x-z)^3 f_i(z)dz. \end{aligned}$$

Thus, by induction, the following holds.

$$F_i^n(x) = \frac{1}{(n-1)!}\int_{\underline{x}}^{x}(x-z)^{n-1}f_i(z)dz \tag{11}$$

for all integer n such that $n \geq 2$.

Define $1[\tilde{x}_i \leq x]$ as the indicator function on $\tilde{x}_i$ such that

$$1[\tilde{x}_i \leq x] = \begin{cases} 1 & \text{if} \quad \tilde{x}_i \leq x \\ 0 & \text{if} \quad \tilde{x}_i > x \end{cases}.$$

Then, equation (11) can be rewritten as

$$\begin{aligned} F_i^n(x) &= \frac{1}{(n-1)!}\int_{\underline{x}}^{x}(x-z)^{n-1}f_i(z)dz \\ &= \frac{1}{(n-1)!}\int_{\underline{x}}^{\overline{x}}(x-z)^{n-1}1[z \leq x]f_i(z)dz \\ &= \frac{1}{(n-1)!}E[(x-\tilde{x}_i)^{n-1}1[\tilde{x}_i \leq x]] \\ &= \frac{1}{(n-1)!}E[(x-\tilde{x}_i)_+^{n-1}]. \end{aligned}$$

When $x = \overline{x}$, we have

$$\begin{aligned} F_i^n(\overline{x}) &= \frac{1}{(n-1)!}\int_{\underline{x}}^{\overline{x}}(\overline{x}-z)^{n-1}f_i(z)dz \\ &= \frac{1}{(n-1)!}E[(\overline{x}-\tilde{x}_i)^{n-1}]. \end{aligned}$$

Therefore, $\tilde{x}_1 \preccurlyeq_{(m,n)SD} \tilde{x}_2$ if and only if

$$\begin{cases} E[(\overline{x}-\tilde{x}_1)^{s-1}] = E[(\overline{x}-\tilde{x}_2)^{s-1}] \text{ for } s = 1,\dots,m, \\ E[(\overline{x}-\tilde{x}_1)^{k-1}] \geq E[(\overline{x}-\tilde{x}_2)^{k-1}] \text{ for } k = m+1,\dots,n, \\ E[(x-\tilde{x}_1)_+^{n-1}] \geq E[(x-\tilde{x}_2)_+^{n-1}] \text{ for all } x \in [\underline{x},\overline{x}], \end{cases}$$

for all two integers, n and m, such that $n \geq m \geq 1$. □

Next, we will prove the following: $\tilde{x}_1 \preccurlyeq_{(m,n)-cv} \tilde{x}_2$ and $E[\tilde{x}_1^{s-1}] = E[\tilde{x}_2^{s-1}]$ for $s = 1,\dots,m$ if and only if one of the following equivalent conditions is satisfied:

$$\tilde{x}_1 \preccurlyeq_{(m,n)SD} \tilde{x}_2 \Leftrightarrow \begin{cases} E[(\overline{x}-\tilde{x}_1)^{s-1}] = E[(\overline{x}-\tilde{x}_2)^{s-1}] \text{ for } s = 1,\dots,m, \\ E[(\overline{x}-\tilde{x}_1)^{k-1}] \geq E[(\overline{x}-\tilde{x}_2)^{k-1}] \text{ for } k = m+1,\dots,n, \\ E[(x-\tilde{x}_1)_+^{n-1}] \geq E[(x-\tilde{x}_2)_+^{n-1}] \text{ for all } x \in [\underline{x},\overline{x}]. \end{cases} \tag{12}$$

Proof. Sufficiency. Assume that $\tilde{x}_2$ dominates $\tilde{x}_1$ in the sense of (m,n)th-degree stochastic dominance for all two integers, n and m, such that $n \geq m \geq 1$. Repeatedly integrating by parts for $E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)]$ n times yields

$$\begin{aligned} E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)] &= \int_{\underline{x}}^{\overline{x}} u(x) dF_2(x) - \int_{\underline{x}}^{\overline{x}} u(x) dF_1(x) \\ &= \sum_{k=m}^{n-1} (-1)^k u^{(k)}(\overline{x})[F_2^{k+1}(\overline{x}) - F_1^{k+1}(\overline{x})] \\ &\quad + (-1)^n \int_{\underline{x}}^{\overline{x}} u^{(n)}(x)[F_2^n(x) - F_1^n(x)] dx \\ &= \sum_{k=m}^{n-1} \frac{(-1)^k}{k!} u^{(k)}(\overline{x})[E[(\overline{x} - \tilde{x}_2)^k] - E[(\overline{x} - \tilde{x}_1)^k]] \\ &\quad + \frac{(-1)^n}{(n-1)!} \int_{\underline{x}}^{\overline{x}} u^{(n)}(x)[E[(\overline{x} - \tilde{x}_2)^{n-1}] - E[(\overline{x} - \tilde{x}_1)^{n-1}]] dx. \end{aligned} \tag{13}$$

Thus, we can verify that $E[u(\tilde{x}_2)] \geq E[u(\tilde{x}_1)]$ for all $u(x) \in \mathcal{U}_{(m,n)-cv}$ using condition (12). In addition, integration by parts on $E[\tilde{x}_2] - E[\tilde{x}_1]$ yields

$$\begin{aligned} E[\tilde{x}_2] - E[\tilde{x}_1] &= \int_{\underline{x}}^{\overline{x}} x dF_2(x) - \int_{\underline{x}}^{\overline{x}} x dF_1(x) \\ &= -[F_1^2(\overline{x}) - F_2^2(\overline{x})] \\ &= -[E[(\overline{x} - \tilde{x}_2)] - E[(\overline{x} - \tilde{x}_1)]] = 0. \end{aligned} \tag{14}$$

Similarly, by repeatedly integrating by parts on $E[\tilde{x}_2^{s-1}] - E[\tilde{x}_1^{s-1}]$ $s-1$ times for $s \geq 2$, we obtain the following:

$$\begin{aligned} E[\tilde{x}_2^{s-1}] - E[\tilde{x}_1^{s-1}] &= \int_{\underline{x}}^{\overline{x}} x^{s-1} dF_2(x) - \int_{\underline{x}}^{\overline{x}} x^{s-1} dF_1(x) \\ &= \sum_{k=1}^{s-1} \frac{(-1)^k k!}{(s-k-1)!} \overline{x}^{s-k-1}[F_2^{k+1}(\overline{x}) - F_1^{k+1}(\overline{x})] \\ &= \sum_{k=1}^{s-1} \frac{(-1)^k}{(s-k-1)!} \overline{x}^{s-k-1}[E[(\overline{x} - \tilde{x}_2)^k] - E[(\overline{x} - \tilde{x}_1)^k]] = 0. \end{aligned}$$

Therefore, $\tilde{x}_1 \preccurlyeq_{(m,n)-cv} \tilde{x}_2$ and $E[\tilde{x}_1^{s-1}] = E[\tilde{x}_2^{s-1}]$ for $s = 1, \ldots, m$ if condition (12) is satisfied.

Necessity. Assume that $\tilde{x}_1 \preccurlyeq_{(m,n)-cv} \tilde{x}_2$ and $E[\tilde{x}_1^{s-1}] = E[\tilde{x}_2^{s-1}]$ for $s = 1, \ldots, m$. It is clear that $F_1^2(\overline{x}) = F_2^2(\overline{x})$ and $E[(\overline{x} - \tilde{x}_1)] = E[(\overline{x} - \tilde{x}_2)]$ based on equation (14) when $s \geq 2$. In $s \geq 3$ case, by repeatedly Integrating by parts on $F_1^3(\overline{x}) - F_2^3(\overline{x})$ 2 times, we obtain the following:

$$
\begin{aligned}
F_1^3(\overline{x}) - F_2^3(\overline{x}) &= \int_{\underline{x}}^{\overline{x}} [F_1^2(x) - F_2^2(x)]dx \\
&= \overline{x}[F_1^2(\overline{x}) - F_2^2(\overline{x})] + \frac{1}{2}[E[\tilde{x}_1^2] - E[\tilde{x}_2^2]] \\
&= \overline{x}[E[(\overline{x} - \tilde{x}_1)] - E[(\overline{x} - \tilde{x}_2)]] + \frac{1}{2}[E[\tilde{x}_1^2] - E[\tilde{x}_2^2]].
\end{aligned}
$$

Thus, it satisfies that $F_1^3(\overline{x}) = F_2^3(\overline{x})$ and $E[(\overline{x} - \tilde{x}_1)^2] = E[(\overline{x} - \tilde{x}_2)^2]$ since $F_1^2(\overline{x}) = F_2^2(\overline{x})$, $E[(\overline{x} - \tilde{x}_1)] = E[(\overline{x} - \tilde{x}_2)]$, and $E[\tilde{x}_1^2] = E[\tilde{x}_2^2]$. Hence, by induction, $F_1^s(\overline{x}) - F_2^s(\overline{x})$ satisfies the following for each $s \geq 2$.

$$
\begin{aligned}
F_1^s(\overline{x}) - F_2^s(\overline{x}) &= \frac{1}{(s-1)!}[E[(\overline{x} - \tilde{x}_1)^{s-1}] - E[(\overline{x} - \tilde{x}_2)^{s-1}]] \\
&= \frac{(-1)^{s-1}}{(s-1)!}[E[\tilde{x}_1^{s-1}] - E[\tilde{x}_2^{s-1}]] = 0.
\end{aligned}
\tag{15}
$$

Next, define the following utility function related to Levy (2015).

$$
u(x) = \begin{cases} -\exp(-\lambda x) & \text{for } x \in [\underline{x}, \delta], \\ -\exp(-\lambda \delta) & \text{for } x \in (\delta, \overline{x}], \end{cases}
\tag{16}
$$

where λ is a positive constant, and $\delta \in [\underline{x}, \overline{x}]$. This utility funcition is included in $\mathcal{U}_{(m,n)-cv}$ because

$$
(-1)^{s+1} u^{(s)}(x) = (-1)^{2(s+1)} \lambda^s \exp(-\lambda x) \geq 0
$$

for $s = m, \ldots, n$. For $x \in [\underline{x}, \delta]$, there are strict inequalities in all derivatives. Also $u^{(s)}(\overline{x}) = 0$ for $s = m, \ldots, n$. Repeatedly integrating by parts for $E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)]$ $n-1$ times yields

$$
\begin{aligned}
E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)] &= \sum_{k=1}^{n-2} (-1)^k u^{(k)}(\overline{x})[F_2^{k+1}(\overline{x}) - F_1^{k+1}(\overline{x})] \\
&\quad + (-1)^{n-1} \int_{\underline{x}}^{\overline{x}} u^{(n-1)}(x)[F_2^{n-1}(x) - F_1^{n-1}(x)]dx \geq 0.
\end{aligned}
\tag{17}
$$

Based on equations (15) and (16), all terms of the right-hand side in equation (17), except the last one, vanish and we are left with:

$$
E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)] = (-1)^{n-1} \int_{\underline{x}}^{\overline{x}} u^{(n-1)}(x)[F_2^{n-1}(x) - F_1^{n-1}(x)]dx \geq 0.
$$

As $u^{(n)}(x) = 0$ for $x \in (\delta, \overline{x}]$, we have

$$
E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)] = (-1)^{n-1} \int_{\underline{x}}^{\delta} u^{(n-1)}(x)[F_2^{n-1}(x) - F_1^{n-1}(x)]dx \geq 0.
$$

By the weighted mean value theorem for integrals, there exists $\xi_1 \in (\underline{x}, \delta)$ such that

$$E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)] = (-1)^{n-1} u^{(n-1)}(\xi_1)[F_2^n(\delta) - F_1^n(\delta)]$$
$$= \frac{(-1)^{n-1}}{(n-1)!} u^{(n-1)}(\xi_1)[E[(\delta - \tilde{x}_2)^{n-1}] - E[(\delta - \tilde{x}_1)^{n-1}]] \geq 0.$$

Therefore, by arbitrarily choosing $\delta \in [\underline{x}, \overline{x})$, the conditions of $\tilde{x}_1 \preccurlyeq_{(m,n)-cv} \tilde{x}_2$ and $E[\tilde{x}_1^{s-1}] = E[\tilde{x}_2^{s-1}]$ for $s = 1, \ldots, m$ imply that one of the following equivalent conditions is satisfied:

$$F_1^n(x) \geq F_2^n(x) \text{ for all } x \in [\underline{x}, \overline{x}] \tag{18}$$
$$\Leftrightarrow E[(x - \tilde{x}_1)^{n-1}] \geq E[(x - \tilde{x}_2)^{n-1}] \text{ for all } x \in [\underline{x}, \overline{x}]. \tag{19}$$

Now, let us turn to the other condition. Choosing $\delta = \overline{x}$ to obtain $u(x) = -\exp(-\lambda x)$ for $x \in [\underline{x}, \overline{x}]$. By repeatedly integration by parts for $E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)]$ $n-2$ times, we obtain

$$E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)] = \sum_{k=m}^{n-3} (-1)^{2k+1} \lambda^k \exp(-\lambda \overline{x})[F_2^{k+1}(\overline{x}) - F_1^{k+1}(\overline{x})]$$
$$+ (-1)^{2n-3} \lambda^{n-2} \int_{\underline{x}}^{\overline{x}} \exp(-\lambda x)[F_2^{n-2}(x) - F_1^{n-2}(x)]dx \geq 0.$$

By the weighted mean value theorem for integrals, there exists $\xi_2 \in (\underline{x}, \overline{x})$ such that

$$\begin{aligned} E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)] &= \sum_{k=m}^{n-3} (-1)^{2k+1} \lambda^k \exp(-\lambda \overline{x})[F_2^{k+1}(\overline{x}) - F_1^{k+1}(\overline{x})] \\ &+ (-1)^{2n-3} \lambda^{n-2} \exp(-\lambda \xi_2)[F_2^{n-1}(\overline{x}) - F_1^{n-1}(\overline{x})] \geq 0. \end{aligned} \tag{20}$$

Since $\overline{x} > \xi_2$, condition (20) can be rewritten as follows using a positive constant ϵ_1 that satisfies $\epsilon_1 = \overline{x} - \xi_2$.

$$\sum_{k=m}^{n-3} (-1)^{2k+1} \lambda^{k-(n-2)} \exp(-\lambda \epsilon_1)[F_2^{k+1}(\overline{x}) - F_1^{k+1}(\overline{x})] + (-1)^{2n-3}[F_2^{n-1}(\overline{x}) - F_1^{n-1}(\overline{x})] \geq 0.$$

Taking a value $\lambda \to \infty$, we have

$$F_1^{n-1}(\overline{x}) \geq F_2^{n-1}(\overline{x})$$
$$\Leftrightarrow E[(\overline{x} - \tilde{x}_1)^{n-2}] \geq E[(\overline{x} - \tilde{x}_2)^{n-2}]$$

because $\lim_{\lambda \to \infty} \lambda^{k-(n-2)} \exp(-\lambda \epsilon_1) = \lim_{\lambda \to \infty} \lambda^{k-(n-2)} \lim_{\lambda \to \infty} \exp(-\lambda \epsilon_1) = 0$.

Similarly, repeatedly integration by parts for $E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)]$ $n-3$ times yields

$$E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)] = \sum_{k=m}^{n-4} (-1)^{2k+1} \lambda^k \exp(-\lambda \overline{x})[F_2^{k+1}(\overline{x}) - F_1^{k+1}(\overline{x})]$$
$$+ (-1)^{2n-5} \lambda^{n-3} \int_{\underline{x}}^{\overline{x}} \exp(-\lambda x)[F_2^{n-3}(x) - F_1^{n-3}(x)]dx \geq 0.$$

By the weighted mean value theorem for integrals, there exists $\xi_3 \in (\underline{x}, \overline{x})$ such that

$$E[u(\tilde{x}_2)] - E[u(\tilde{x}_1)] = \sum_{k=m}^{n-4} (-1)^{2k+1} \lambda^k \exp(-\lambda \overline{x})[F_2^{k+1}(\overline{x}) - F_1^{k+1}(\overline{x})]$$
$$+ (-1)^{2n-5} \lambda^{n-3} \exp(-\lambda \xi_3)[F_2^{n-2}(\overline{x}) - F_1^{n-2}(\overline{x})] \geq 0.$$

Using a positive constant ϵ_2 that satisfies $\epsilon_2 = \overline{x} - \xi_3$ to obtain

$$\sum_{k=m}^{n-4} (-1)^{2k+1} \lambda^{k-(n-3)} \exp(-\lambda \epsilon_2)[F_2^{k+1}(\overline{x}) - F_1^{k+1}(\overline{x})] + (-1)^{2n-5}[F_2^{n-2}(\overline{x}) - F_1^{n-2}(\overline{x})] \geq 0.$$

Taking a value $\lambda \to \infty$, we have

$$F_1^{n-2}(\overline{x}) \geq F_2^{n-2}(\overline{x})$$
$$\Leftrightarrow E[(\overline{x} - \tilde{x}_1)^{n-3}] \geq E[(\overline{x} - \tilde{x}_2)^{n-3}].$$

Therefore, by induction, the following is satisfied.

$$F_1^k(\overline{x}) \geq F_2^k(\overline{x}) \text{ for all } k = m+1, \ldots, n-1 \tag{21}$$
$$\Leftrightarrow E[(\overline{x} - \tilde{x}_1)^{k-1}] \geq E[(\overline{x} - \tilde{x}_2)^{k-1}] \text{ for all } k = m+1, \ldots, n-1. \tag{22}$$

The proof is completed by combining conditions (15), (18), (19), (21), and (22). □

A.2 Proof of Proposition 2

Proof. For the right term in condition (4), repeatedly integrating by parts n times yields

$$\int_{\underline{x}}^{\overline{x}} xu'(w_0 + x)dF_2(x) - \int_{\underline{x}}^{\overline{x}} xu'(w_0 + x)dF_1(x)$$
$$= \sum_{k=1}^{n-1} (-1)^k [ku^{(k)}(w_0 + \overline{x}) + \overline{x}u^{(k+1)}(w_0 + \overline{x})][F_2^{k+1}(\overline{x}) - F_1^{k+1}(\overline{x})]$$
$$+ (-1)^n \int_{\underline{x}}^{\overline{x}} [nu^{(n)}(w_0 + x) + xu^{(n+1)}(w_0 + x)][F_2^n(x) - F_1^n(x)]dx.$$

When $\tilde{x}_1 \preccurlyeq_{(m,n)SD} \tilde{x}_2$, the direction of the sign in condition (4) is determined by the direction of the sign in

$$(-1)^{n+1}[nu^{(n)}(w_0 + x) + xu^{(n+1)}(w_0 + x)] \geq 0$$

for all $x \in [\underline{x}, \overline{x}]$ as well as

$$(-1)^{k+1}[ku^{(k)}(w_0 + \overline{x}) + \overline{x}u^{(k+1)}(w_0 + \overline{x})] \geq 0$$

for $k = m, \ldots, n-1$. These inequalities for all $u(x) \in \mathcal{U}_{(m,n+1)-cv}$ are equivalent to the following conditions:

$$-\overline{x}\,\frac{u^{(k+1)}(w_0+\overline{x})}{u^{(k)}(w_0+\overline{x})} \le k \tag{23}$$

for $k = m, \ldots, n-1$, and

$$-x\,\frac{u^{(n+1)}(w_0+x)}{u^{(n)}(w_0+x)} \le n \tag{24}$$

for all $x \in [\underline{x}, \overline{x}]$. If $-x\,\dfrac{u^{(k+1)}(x)}{u^{(k)}(x)} \le k$ for all $x \in [\underline{x}, \overline{x}]$, and $k = m, \ldots, n$, then

$$-w_0\,\frac{u^{(k+1)}(w_0+x)}{u^{(k)}(w_0+x)} - x\,\frac{u^{(k+1)}(w_0+x)}{u^{(k)}(w_0+x)} \le k \tag{25}$$

for all $x \in [\underline{x}, \overline{x}]$, and $k = m, \ldots, n$. Since $w_0 \in \mathbb{R}_+$, condition (25) has the property that $-x\,\dfrac{u^{(k+1)}(w_0+x)}{u^{(k)}(w_0+x)} \le k$ for all $x \in [\underline{x}, \overline{x}]$, for $k = m, \ldots, n$, and for all $u(x) \in \mathcal{U}_{(m,n)-cv}$. Therefore, condition (25) implies both condition (23) and condition (24) are satisfied. □

A.3 Proof of Condition (9)

Proof. Assume that $\tilde{x}_2$ dominates $\tilde{x}_1$ in the sense of (m,n)th-degree stochastic dominance. Using equation (10) and Fubini's theorem, $-F_i^3(x)$ can be rewritten as

$$\begin{aligned}
-F_i^3(x) &= -\int_{\underline{x}}^{x} F_i^2(y)dy = \int_{\underline{x}}^{x}\int_{\underline{x}}^{y}(z-x)f_i(z)dzdy \\
&= \int_{\underline{x}}^{x}\left[\int_{z}^{x}(y-x)dy\right]f_i(z)dz \\
&= -\frac{1}{2}\int_{\underline{x}}^{x}(z-x)^2 f_i(z)dz.
\end{aligned}$$

Similarly, for $-F_i^4(x)$,

$$\begin{aligned}
-F_i^4(x) &= -\int_{\underline{x}}^{x} F_i^3(y)dy = -\frac{1}{2}\int_{\underline{x}}^{x}\int_{\underline{x}}^{y}(z-x)^2 f_i(z)dzdy \\
&= -\frac{1}{2}\int_{\underline{x}}^{x}\left[\int_{z}^{x}(y-x)^2 dy\right]f_i(z)dz \\
&= \frac{(-1)^2}{3!}\int_{\underline{x}}^{x}(z-x)^3 f_i(z)dz.
\end{aligned}$$

Thus, by induction, the following holds.

$$\begin{aligned} -F_i^n(x) &= \frac{(-1)^{n-2}}{(n-1)!}\int_{\underline{x}}^{x}(z-x)^{n-1}f_i(z)dz \\ &= \frac{(-1)^{n-2}}{(n-1)!}\int_{\underline{x}}^{\overline{x}}(z-x)^{n-1}1[z\le x]f_i(z)dz \\ &= \frac{(-1)^{n-2}}{(n-1)!}E[(\tilde{x}_i-x_+^{n-1})] \end{aligned} \tag{26}$$

for all integer n such that $n\ge 2$. Combining equations (11) and (26), we have $E[(\tilde{x}_i-x)_+^n]=(-1)^n E[(x-\tilde{x}_i)_+^n]$ for $x\in[\underline{x},\overline{x}]$ and for all integer n such that $n\ge 1$.

By nth-order approximations for $\pi_u(w_0,\alpha_1^*+\alpha)$ and $\pi_u(w_0,\alpha_1^*)$ around $w_0+(\alpha_1^*+\alpha)\tilde{x}_1=w_0+(\alpha_1^*+\alpha)\overline{x}$ and $w_0+\alpha_1^*\tilde{x}_2=w_0+\alpha_1^*\overline{x}$, respectively, we have

$$\begin{aligned} \pi_u(w_0,\alpha_1^*+\alpha) &\approx \sum_{k=1}^{n}\frac{1}{k!}u^{(k)}(w_0+(\alpha_1^*+\alpha)\overline{x})(\alpha_1^*+\alpha)^k[E[(\tilde{x}_2-\overline{x})^k]-E[(\tilde{x}_1-\overline{x})^k]] \\ &= \sum_{k=m}^{n}\frac{(-1)^{k+1}}{k!}u^{(k)}(w_0+(\alpha_1^*+\alpha)\overline{x})\Delta_k(\alpha_1^*+\alpha), \end{aligned} \tag{27}$$

$$\begin{aligned} \pi_u(w_0,\alpha_1^*) &\approx \sum_{k=1}^{n}\frac{1}{k!}u^{(k)}(w_0+\alpha_1^*\overline{x})\alpha_1^{*k}[E[(\tilde{x}_2-\overline{x})^k]-E[(\tilde{x}_1-\overline{x})^k]] \\ &= \sum_{k=m}^{n}\frac{(-1)^{k+1}}{k!}u^{(k)}(w_0+\alpha_1^*\overline{x})\Delta_k(\alpha_1^*). \end{aligned} \tag{28}$$

Substituting equations (27) and (28) into condition (8) to obtain

$$\begin{aligned} &\lim_{\alpha\to 0}\frac{\pi_u(w_0,\alpha_1^*+\alpha)-\pi_u(w_0,\alpha_1^*)}{\alpha} \\ &\approx \lim_{\alpha\to 0}\sum_{k=m}^{n}\frac{(-1)^{k+1}}{k!}\left[\frac{u^{(k)}(w_0+(\alpha_1^*+\alpha)\overline{x})\Delta_k(\alpha_1^*+\alpha)-u^{(k)}(w_0+\alpha_1^*\overline{x})\Delta_k(\alpha_1^*)}{\alpha}\right] \\ &= \sum_{k=m}^{n}\frac{(-1)^{k+1}}{k!}\left[\frac{d\Delta_k(\alpha_1^*)}{d\alpha_1^*}u^{(k)}(w_0+\alpha_1^*\overline{x})+\frac{\partial u^{(k)}(w_0+\alpha_1^*\overline{x})}{\partial\alpha_1^*}\Delta_k(\alpha_1^*)\right] \\ &= \sum_{k=m}^{n}\frac{(-1)^{k+1}u^{(k)}(w_0+\alpha_1^*\overline{x})\Delta_k(\alpha_1^*)}{k!\,\alpha_1^*}\left[\frac{\alpha_1^*}{\Delta_k(\alpha_1^*)}\frac{d\Delta_k(\alpha_1^*)}{d\alpha_1^*}+\frac{\alpha_1^*}{u^{(k)}(\alpha_1^*\overline{x})}\frac{\partial u^{(k)}(w_0+\alpha_1^*\overline{x})}{\partial\alpha_1^*}\right] \\ &= \sum_{k=m}^{n}\frac{(-1)^{k+1}u^{(k)}(w_0+\alpha_1^*\overline{x})\Delta_k(\alpha_1^*)}{k!\,\alpha_1^*}\left[k+\alpha_1^*\overline{x}\frac{u^{(k+1)}(w_0+\alpha_1^*\overline{x})}{u^{(k)}(w_0+\alpha_1^*\overline{x})}\right]\ge 0. \end{aligned}$$

Hence, condition (8) can be approximated by condition (9). □

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