Graduate School of Fundamental Science and Engineering
Waseda University

博　士　論　文　概　要
Doctoral Dissertation Synopsis

論　文　題　目
Dissertation Title

Solutions of the tt*-Toda Equations with Integer Stokes Data and Quantum Cohomology of Minuscule Flag Manifolds

整数Stokesデータをもつtt*戸田方程式の解とminusculeな旗多様体の量子コホモロジー

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January, 2023

The tt* equations were introduced by Cecotti and Vafa in 1991. They defined the tt* equations in the context of N=2 superconformal field theory, in physics. They proposed to classify superconformal field theories by studying their deformations, which are given by solutions of these equations.

In mathematics, Guest-Its-Lin solved the tt*-Toda equations of type A_n, a special case of the tt* equations. We have three aspects of the global solutions of the tt*-Toda equations. The first one is the holomorphic data. This is a certain holomorphic 1-form, from which a solution can be constructed by the so-called DPW construction. The second one is the asymptotic data of the global solutions, as the variable goes to zero. The third one is the Stokes data. The tt*-Toda equations are equivalent to the conditions of isomonodromic deformation of a certain O.D.E., from which it follows that a solution corresponds to a collection of monodromy data. This data reduces to the entries of the Stokes matrices of the O.D.E. at t=0.
In this thesis, we focus on solutions corresponding to integer Stokes data of the tt*-Toda equations. According to Cecotti and Vafa, such solutions are expected to correspond to physically realistic superconformal field theories.

Our first result is related to properties of the integer Stokes data. Guest-Lin found that the holomorphic 1-forms of certain solutions corresponding to integer Stokes data can be identified with the Dubrovin connection forms of algebraic varieties such as projective space, weighted projective space, or some intersections of hypersurfaces in projective spaces. Thus solutions corresponding to integer Stokes data may have important geometrical, as well as physical, significance.
It is difficult to determine the integer Stokes data, even in the special case of the tt*-Toda equations of type A_n. Guest-Its-Lin showed that the asymptotic data of the global solutions correspond to the points of (a subspace of) the Fundamental Weyl Chamber of type A_n. Using their results we investigated the solutions with integer Stokes data from the point of view of Lie theory. In particular we computed the number of solutions with integer Stokes data, and we found all solutions with integer Stokes data where the asymptotic data lies in a certain line in the Fundamental Weyl Chamber.

More precisely, let us assume that the asymptotic data lies on the ρ-line, where ρ is the sum of the basic weights.
Our first main theorem is:

Theorem. For n≧3, there are only four points on the ρ-line which correspond to solutions with integer Stokes data. One point corresponds to the trivial solution of tt*-Toda equations. The other three points have well-known geometrical interpretations: the quantum coholomogy of complex projective space $\mathbb{C}P^n$, the quantum coholomogy of the weighted projective space $\mathbb{C}P^{1,n}$, the unfolding of the A_{n+1} singularity.

Our second result concerns solutions which are related to quantum cohomology of flag manifolds, not just in the A_n case, but more generally in the case of flag manifolds of complex simple Lie groups. We focus on the particular solution whose asymptotic data is given by the point corresponds to the point -ρ (which corresponds to the origin of the Fundamental Weyl Chamber). (Strictly speaking this is a family of solutions, but it is expected that exactly one member of this family will be a global solution.) It is known that this point corresponds to integer Stokes data, for any complex simple Lie group. This data has the following geometrical interpretation:

Theorem. The holomorphic 1-forms of the solution whose asymptotic data is given by the point $-\rho$ can be identified with the Dubrovin connection form of the quantum coholomogy of (any) minuscule flag manifold of the Lie group.

The quantum cohomology of a minuscule flag manifold can be calculated by the quantum Chevalley formula. Using this, the Dubrovin connection form was calculated by Lam and Templier, and this turns out to agree with the holomorphic 1-form of the solution whose asymptotic data is given by the point $-\rho$.

In the A_n case, our result shows that the quantum cohomology of (any) complex Grassmannian corresponds to the same solution of the tt*-Toda equations. This is consistent with the quantum Satake isomorphism of Golyshev and Manivel, which describes a relation between the quantum cohomology algebras of these spaces. For Lie groups of types D_n or E_6, i.e. those Lie groups which admit more than one minuscule flag manifold, a similar relation can be expected. We give such a relation in the D_n case.

Our third result compares the classification method of Cecotti-Vafa with that of Guest-Its-Lin. The Cecotti-Vafa method applies to triangular matrices with integer entries, which they regard as candidates for integer Stokes matrices of

solutions to the tt* equations. When such solutions exist, the corresponding Stokes matrices are not unique (as they depend on various choices), but they must be related by the action of a (signed) braid group. Thus, Cecotti-Vafa formulate the problem of classifying braid group orbits of triangular matrices with integer entries. On the other hand, the Guest-Its-Lin method simply counts solutions of the tt*-Toda equations; these solutions are known to exist, from a construction which produces a particular choice of Stokes matrix. Thus, the Cecotti-Vafa classification gives an upper bound on the number of equivalence classes of solutions of the tt*-Toda equations, and the Guest-Its-Lin classification gives a lower bound.
Theorem. For n=1,2,3, the Guest-Its-Lin classification of solutions of the tt*-Toda equations coincides with the Cecotti-Vafa classification.

We prove this theorem by making use of two invariants of the braid group action (as the braid group action is difficult to study directly). One is the conjugacy class of monodromy matrix $S \cdot S^{-T}$, and the other is the signature of the matrix $S + S^T$ where S is (any) Stokes matrix. For n=1,2,3 these invariants suffice to distinguish the braid group orbits, as we show by a case-by-case argument. Our theorem confirms the validity of the classification method of Cecotti-Vafa, in the case of the tt*-Toda equations, for n=1,2,3.

List of papers

[1] Yoshiki Kaneko, Solutions of the tt*-Toda equations from the quantum cohomology of minuscule flag manifolds, Nagoya Mathematical Journal, 248 (2022) 990-1004.

[2] Yudai Hateruma and Yoshiki Kaneko, On some Lie-theoretic solutions of the tt*-Toda equations with integer Stokes data, preprint.

List of research achievements for application of Doctor of Science, Waseda University

Full Name：金子　吉樹　seal or signature

Date Submitted(yyyy/mm/dd): 2023/2/25

種類別 (By Type)	題名、 発表・発行掲載誌名、 発表・発行年月、 連名者（申請者含む） (theme, journal name, date & year of publication, name of authors inc. yourself)
論文[1]	◯Yoshiki Kaneko, (2022.6). SOLUTIONS OF THE tt*-TODA EQUATIONS AND QUANTUM COHOMOLOGY OF MINUSCULE FLAG MANIFOLDS. Nagoya Mathematical Journal, 1-15. doi:10.1017/nmj.2022.17
発表[1]	金子吉樹 , “Solutions of the tt*-Toda Equations with integer Stokes data”, Poisson geometry and related topics 22, 東京理科大学 神楽坂キャンパス, 2022/12.
発表[2]	Yoshiki Kaneko, “Solutions of the tt*-Toda Equations from Minuscule Flag Manifolds”, The international workshop“Geometry of submanifolds and integrable systems”, 大阪市立大学(zoom), 2022/2.
発表[3]	金子吉樹, “tt*幾何の展開”, Koriyama Geometry and Physics Days, 日本大学郡山キャンパス, 2021/11.
発表[4]	金子吉樹, “量子ドリンフェルト-ソコロフ簡約”, Koriyama Geometry and Physics Days, 日本大学郡山キャンパス, 2021/11.
発表[5]	金子吉樹, “tt*戸田方程式の局所解とminusculeな旗多様体の量子コホモロジーについて”, 幾何学シンポジウム, 北海道大学(zoom), 2021/8.
発表[6]	Yoshiki Kaneko, “Solutions of the tt*-Toda equations and quantum cohomology of minuscule flag manifolds”, Nonabelian Hodge theory, 早稲田大学(zoom), 2021/7.
発表[7]	金子吉樹, “tt*-戸田方程式の解と旗多様体の量子コホモロジーについて”, 日本数学会年会, 慶應大学理工学部, 2021/3.
発表[8]	Yoshiki kaneko, "Solutions of the tt*-Toda Equations and Quantum Cohomology of Flag Manifolds", International Workshop on Multiphase Flows: Analysis, Modelling and Numerics, Oxford-Waseda(zoom), 2020/12.
発表[9]	金子吉樹, "Pseudodifferential symbol と Hamiltonian equations", Koriyama Geometry and Physics Days 2020, 日本大学郡山キャンパス, 2020/2.

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種類別 (By Type)	題名、 発表・発行掲載誌名、 発表・発行年月、 連名者（申請者含む） (theme, journal name, date & year of publication, name of authors inc. yourself)
発表[10]	金子吉樹, "Virasoro algebra と coadjoint action", Koriyama Geometry and Physics Days 2020, 日本大学郡山キャンパス, 2020/2.
発表[11]	金子吉樹, "旗多様体から得られるtt*戸田方程式の解", 異分野異業種交流会, 東京大学, 2019/10.
発表[12]	金子吉樹, "一般旗多様体から得られるtt*戸田方程式の局所解について", 関東若手幾何セミナー, 首都大学東京(現、都立大学), 2019/6.
発表[13]	金子吉樹, "Local solutions of the tt*-Toda equations from flag manifolds", 早稲田大学数学若手異分野交流会, 早稲田大学, 2019/3.
発表[14]	Yoshiki Kaneko, "Solutions of The tt*-Toda Equations Corresponding to Quantum Cohomology of Flag Manifolds", UK-Japan Winter School 2019, Leeds University, 2019/1.
発表[15]	Yoshiki Kaneko, "Introduction to orbifolds", Korimaya Geometry and Physics Days 2018, 日本大学郡山キャンパス, 2018/2.
発表[16]	Yoshiki Kaneko, "Construction of representations of compact Lie groups and their loop groups", Koriyama Geometry and Physics Days 2016, 日本大学郡山キャンパス, 2016/2.