Solutions of the tt*-Toda Equations with Integer Stokes Data and Quantum Cohomology of Minuscule Flag Manifolds

整数Stokesデータをもつtt*戸田方程式の解とminusculeな旗多様体の量子コホモロジー

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Chapter 1

Introduction

1.1 History of the tt*-Toda equations

The tt* equations were introduced by Ceccoti and Vafa in 1991 [CV1]. This equation is suggested in the context of the classification of deformations of $N = 2$ super conformal field theory. In this article, they considered the finite numbers of vacuum and chiral field, which defined a chiral ring. In 1991, Dubrovin found that the tt* equations could be written by the zero curvature equations on the tangent bundle over a Frobenius manifold and the pluriharmonic map equation for maps into a symmetric space [Dub].

The zero curvature equations are integrable. One can solve these equations by using the DPW method [GIL3]. In this method, we consider the holomorphic data (we call this the DPW potential) and then express the solutions as a loop group factorization of a function obtained from this data. We also have an asymptotic expansion of solutions and their coefficients (we call this the asymptotic data).

We focus on the "Toda" types of the tt* equations. We call these equations the tt*-Toda equations. The tt*-Toda equations are well studied in the series of works by M.Guest, A.Its and C.Lin [GIL1][GIL2][GIL3][GIL4]. They focus on the tt*-Toda equations with type A_n as follows.

$$2(w_i)_{t\bar{t}} = -e^{2(w_{i+1}-w_i)} + e^{2(w_i-w_{i-1})} \tag{1.1.1}$$

where $w_i : U \to \mathbb{R}$ (U is an open subset of $\mathbb{C} = \mathbb{R}^2$), $i \in \mathbb{Z}$, $w_i = w_{n+1+i}$ and $\sum_{i=0}^{n} w_i = 0$ and where we assume the "Frobenius condition" (in some articles, this is called “anti-symmetry condition”)

$$w_i + w_{n-i} = 0 \text{ (for } 0 \leqslant i \leqslant n)$$

and the radial condition $w_i = w_i(|t|)$. We consider the following connection form α on the trivial principal $SL(n+1, \mathbb{C})$ bundle over $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$

$$\alpha = (w_t + \frac{1}{\lambda} W^T)dt + (-w_{\bar{t}} + \lambda W)d\bar{t} \tag{1.1.2}$$

where $w = \operatorname{diag}(w_0, \cdots, w_n)$ and where

$$W = \begin{pmatrix} 0 & e^{w_1 - w_0} & & \\ & \ddots & \ddots & \\ & & \ddots & e^{w_n - w_{n-1}} \\ e^{w_0 - w_n} & & & 0 \end{pmatrix}.$$

Then the zero curvature equation $d\alpha + \alpha \wedge \alpha = 0$ for ${}^\forall \lambda \in \mathbb{C}^\times$ is equivalent to (1.1.1). By direct calculation, we know that the equation (1.1.1) is equivalent to $2w_{t\bar{t}} = [W^T, W]$ where $w = \operatorname{diag}(w_0, w_1, \cdots, w_n)$. Thus this equation is equivalent to the compatibility condition $(\Psi_t)_{\bar{t}} = (\Psi_{\bar{t}})_t$ for the linear system

$$\begin{cases} \Psi_t = \left(w_t + \frac{1}{\lambda} W\right)\Psi \\ \Psi_{\bar{t}} = \left(-w_{\bar{t}} + \lambda W^T\right)\Psi. \end{cases} \tag{1.1.3}$$

For the holomorphic potential

$$\omega = \frac{1}{\lambda} \begin{pmatrix} 0 & & & z^{k_0} \\ z^{k_1} & \ddots & & \\ & \ddots & \ddots & \\ & & z^{k_n} & 0 \end{pmatrix} dz$$

where $z \in U$ and $k_i \geqslant -1$ (${}^\forall i$), one can construct a global solution w by using the DPW-method through the Iwasawa decomposition of loop groups. Then we obtain global solutions whose asymptotic expansions are

$$w_i(|t|) \sim -m_i \log |t|.$$

Here m_i are defined by follows:

$$m_{i-1} - m_i = \frac{n+1}{N}(k_i + 1) - 1$$

where $N = n + 1 + \sum_{i=0}^n k_i$. Thus we obtain the following ([GIL3],[Mo1]).

Theorem 1. *For fixed $N > 0$, the global solutions of* (1.1.1) *are in one to one correspondence with the following $\Lambda\mathfrak{sl}_{n+1}\mathbb{C}$-valued one forms $\frac{1}{\lambda}\eta(z)dz$ on $\mathbb{C}^*$, where*

$$\eta(z) = \left(\begin{array}{c|c|c|c} & & & z^{k_0} \\ \hline z^{k_1} & & & \\ \hline & \ddots & & \\ \hline & & z^{k_n} & \end{array}\right).$$

Here $k_i \in [-1, \infty)$, $n + 1 + \sum_{i=0}^{n} k_i = N$ and $k_i = k_{n-i+1}$ for any i. The variable z means $t = \frac{n+1}{N} z^{\frac{n+1}{N}}$.

From the radial condition $w_i(x) = w_i(|t|)$ $(x = |t|)$, we have

$$\left(x\frac{d}{dx} - t\frac{d}{dt} - \bar{t}\frac{d}{d\bar{t}}\right) w = 0, \ \left(t\frac{d}{dt} - \bar{t}\frac{d}{d\bar{t}}\right) w = 0.$$

Then we have the radial version of (1.1.1) as $(xw_x)_x = 2x[W^T, W]$. This is equivalent to the compatibility condition $\Psi_{x\mu} = \Psi_{\mu x}$ for the linear system

$$\begin{cases} \Psi_\mu = \left(-\frac{1}{\mu^2}xW - \frac{1}{\mu}xw_x + xW^T\right)\Psi \\ \Psi_x = (\frac{1}{\mu}W + \mu W^T)\Psi. \end{cases} \tag{1.1.4}$$

where $\mu = \frac{\lambda x}{t}$, $x = |t|$. By Chapter 4 in [FIKN], this compatibility condition (1.1.4) is equivalent to the isomonodromy deformation with x of the first differential equation

$$\Psi_\mu = \left(-\frac{1}{\mu^2}xW - \frac{1}{\mu}xw_x + xW^T\right)\Psi. \tag{1.1.5}$$

Therefore for solutions w of the tt*-Toda equations, equations (1.1.5) has the monodromy data which is independent of x. Hence we have a correspondence between a solution w and the Stokes matrices of (1.1.5).

By changing variable $\zeta = \frac{\mu}{x}$, the equation (1.1.5) is equivalent to

$$\Psi_\zeta = \left(-\frac{1}{\zeta^2}W - \frac{1}{\zeta}xw_x + x^2W^T\right)\Psi. \tag{1.1.6}$$

By Proposition 1.1 of [FIKN], we have the unique formal solution around $\zeta = 0$ as

$$\tilde{\Psi}_f^{(0)} = \tilde{P}_0\left(I + \sum_{k\geqslant -1}\tilde{\Psi}_k\zeta^k\right)e^{\frac{1}{\zeta}d_{n+1}}$$

where $-W = \tilde{P}_0(-d_{n+1})\tilde{P}_0^{-1}$, $\tilde{P}_0 = e^{-w}\,\Omega\,\mathrm{diag}(1,\omega^{\frac{1}{2}},\cdots,\omega^{\frac{n+1}{2}})$, Ω is the Vandermonde matrix $(\omega^{ij})_{0\leqslant i,j\leqslant n}$, $\omega = e^{\frac{2\pi\sqrt{-1}}{n+1}}$ and $d_{n+1} = \mathrm{diag}(1,\omega,\cdots,\omega^n)$. We take Stokes sectors at $\zeta = 0$ as follows: for the $n+1 = 2m$ case,

$$\Omega_1^{(0)} = \{\zeta \in \mathbb{C}^\times \mid -\frac{\pi}{2} - \frac{\pi}{n+1} < \arg\zeta < \frac{\pi}{2}\} \tag{1.1.7}$$

and for the $n+1 = 2m+1$ case,

$$\Omega_1^{(0)} = \{\zeta \in \mathbb{C}^\times \mid -\frac{\pi}{2} - \frac{\pi}{2(n+1)} < \arg\zeta < \frac{\pi}{2} + \frac{\pi}{2(n+1)}\} \tag{1.1.8}$$

and $\Omega^{(0)}_{k\frac{1}{n+1}} := e^{-\frac{\pi}{n+1}}\Omega_k$ for $k \in \frac{1}{n+1}\mathbb{Z}$. We have the canonical solutions $\tilde{\Psi}_k^{(0)}$ on Ω_k such that $\tilde{\Psi}_k^{(0)} \sim \tilde{\Psi}_f$ as $\zeta \to 0$. We define $\tilde{Q}_k$ and $\tilde{S}_k$ by

$$\tilde{\Psi}^{(0)}_{k+\frac{1}{n+1}} = \tilde{\Psi}^{(0)}_k\tilde{Q}_k,\ \tilde{\Psi}^{(0)}_{k+1} = \tilde{\Psi}^{(0)}_k\tilde{S}_k$$

for $k \in \frac{1}{n+1}\mathbb{Z}$. Using the symmetries of $\tilde{Q}_k$ in page 7 and 12 of [GH1], we have the monodromy around $\zeta = 0$

$$\tilde{S}_1^{(0)}\tilde{S}_2^{(0)} = \begin{cases} -(\tilde{Q}_1\tilde{Q}_{1\frac{1}{n+1}}\tilde{\Pi})^{n+1} & (n+1 = 2m) \\ (\tilde{Q}_1\tilde{Q}_{1\frac{1}{n+1}}\Pi)^{n+1} & (n+1 = 2m+1) \end{cases} \tag{1.1.9}$$

where

$$\Pi = \begin{pmatrix} & I_n \\ 1 & \end{pmatrix},\ \tilde{\Pi} = \begin{pmatrix} & I_n \\ -1 & \end{pmatrix}$$

and we also have $\tilde{S}_2^{(0)} = (\tilde{S}_1^{(0)})^{-T}$. We can write the monodromy around $\zeta = 0$ as $\tilde{S}_1^{(0)}(\tilde{S}_1^{(0)})^{-T}$.

We have the characteristic polynomial of $\tilde{Q}_1\tilde{Q}_{1\frac{1}{n+1}}\tilde{\Pi}$ as

$$P(x) = x^{n+1} - s_1x^n + s_2x^{n-1} - \cdots - s_nx + 1$$

for $n+1$ even and the characteristic polynomial of the semisimple part of $\tilde{Q}_1\tilde{Q}_{1\frac{1}{n+1}}\Pi$ as

$$P(x) = x^{n+1} - s_1x^n + s_2x^{n-1} - \cdots + s_nx - 1$$

for $n+1$ odd where s_i is the i-th symmetric function of the $n+1$ entries of $e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)}$.

Theorem 2 ([GIL3],[Mo1]). *For each $n \in \mathbb{N}$ and $N > 0$. The global solutions of (1.1.1) correspond to n-tuples of parameters*

$$\mathcal{S} = (s_1, \ldots, s_n) \in \mathbb{R}^n$$

with $s_i = s_{n-i+1}$. We call this $\mathcal{S} = (s_1, \cdots, s_n)$ the Stokes data.

From the above discussion, we have correspondences among holomorphic data k_i, asymptotic data m_i and Stokes data s_i:

$$k_i \leftrightarrow m_i \leftrightarrow s_i.$$

For these global solutions, there are several interesting interpretations as follows.

1. Quantum cohomology

In [DGR], Dorfmeister-Guest-Rossman showed that the quantum cohomology of $\mathbb{C}P^1$ corresponds to a global solution. By considering the quantum product by the second cohomology, we have the Dubrovin connection on the trivial vector bundle.

Let $1 \in H^0(\mathbb{C}P^1, \mathbb{C})$ and $x \in H^2(\mathbb{C}P^1, \mathbb{C})$ be generators. Then we have the quantum product $\circ_q$ as

$$1 \circ_q 1 = 1,\ 1 \circ_q x = x,\ x \circ_q x = 1 \cdot q.$$

From this product, we have the Dubrovin connection on the trivial vector bundle $H^2(\mathbb{C}P^1, \mathbb{C}) \times H^*(\mathbb{C}P^1, \mathbb{C}) \to H^2(\mathbb{C}P^1, \mathbb{C})$ as follows:

$$\frac{1}{\lambda}\begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}\frac{dq}{q}$$

where $q \in \mathbb{C}^\times \cong H^2(\mathbb{C}P^1, \mathbb{C})$. This is a connection form of the same type as ω in the tt*-Toda equation in the $SL(2, \mathbb{C})$ case.

For other cases, Iritani [Ir1],[Ir2] studied the tt* structure on the quantum cohomology of orbifold or algebraic curves. Guest-Lin [GL1] obtained some quantum cohomology of projective weighted spaces whose Dubrovin connections correspond to holomorphic data.

2. Generalization of the tt*-Toda equations

In 2019, the tt*-Toda equations were generalized from the $SL(n + 1, \mathbb{C})$

type to the general complex Lie group types by Guest-Ho [GH2]. This work plays a fundamental role for this thesis.

The equation (1.1.1) is the tt*-Toda equation with $G = SL(n+1, \mathbb{C})$. Guest and Ho used the notations of Lie theory and representation theory. They generalized the equations, holomorphic data, asymptotic data and Stokes data. They used several remarkable works by Kostant [Ko], Hichtin [Hit], Steinberg [Ste] and Boalch [Boa]. We will give details in Chapter 2.

3. Integer Stokes data

By integer Stokes data, we mean the condition that all Stokes data take integer values. As we shall see in Section 2, we have finite many solutions with integer Stokes data, for each n. In [CV2], they use characteristic polynomials of monodromy matrices to classify models of quantum fields. We will compare the classification by Cecotti and Vafa and the classification by Guest-Its-Lin in Chapter 4.

For such Stokes data, there are some remarkable results. In [GL1] and [GL2], several quantum cohomology correspond to some solutions with integer Stokes data, e.g. quantum cohomology algebras of $\mathbb{C}P^n$ or weighted projective spaces, and A_n singularities of unfolding also corresponds to special integer Stokes datum on ρ-line in FWA. In [Mo1] and [Mo2], Mochizuki studied $\mathbb{Z}$-structure on parabolic vector bundles.

Another aspect is the action of the braid group Br_{n+1} on Stokes matrices. In [CV2], when Cecotti and Vafa classified quantum field models, they considered Stokes matrices modulo the braid group action. In [BH], Balnojan and Hertling studied the orbits of the braid group action on Stokes matrices.

4. Other researches

In [FN], Fredrickson and Neitzke found the surprising fact that the set of $SL(n+1, \mathbb{C})$-Higgs bundle on $\mathbb{C}P^1$ corresponding to connection forms of the same type as ω correspond to certain representations of W-algebras (Remark 5.4 in [FN]). Following this result, Guest-Otofuji studied Stokes data from the view point of representation theory of affine Lie algebras in [GO]. They found that a certain integer Stokes point corresponds to a fundamental representation of a Kac-Moody Lie algebra. In [CV1], Cecotti and Vafa defined the tt* equations in the context of superconformal Virasoro algebra, i.e. Vertex algebras, so relationships among them can be expected.

1.2 Main results

In this thesis we investigate the integer Stokes data. As above, we know that the integer Stokes data contain rich information in terms of physics and mathematics. First we obtain properties of the integer Stokes data by considering cyclotomic polynomials. It is written in Section 3.1.

Proposition 3. *[HK] If m is asymptotic data corresponding to an integer Stokes data, $P(x)$ satisfies the following conditions.*

$$1.\ \sum_{d} \nu(d)\varphi(d) = n+1$$

$$2.\ \nu(1) = n+1\ (mod\, 2).$$

Conversely if $Q(x) = \prod_{d\in\mathbb{N}} (\Phi_d)^{\nu(d)}$ satisfies 1 and 2, then the roots of $Q(x)$ come from an integer Stokes point.

By using this proposition, we obtain a formula for the total number of the integer Stokes data.

$$\sum_{\tau\in\mathcal{I}} H_{\alpha_1}^{\xi(a_1)} \cdot \cdots \cdot H_{\alpha_l}^{\xi(a_l)} \cdot (m+1).$$

As the second topic, we focus on the integer Stokes data on ρ-line. The points on this line are parametrized by real numbers $\lambda \in [-1, \frac{1}{n}]$. The first main theorem is as follows:

Theorem 4. *[HK] Assume $n \geqslant 3$ and $\lambda \in [-1, \frac{1}{n}]$. Then the Stokes data s of the corresponding solution of the tt*-Toda equations is integral if and only if*

$$\lambda = -1,\ -\frac{1}{n+2},\ 0,\ \frac{1}{n}.$$

We know that we have a solution of the tt*-Toda equation with A_n type from the Dubrovin connection of the quantum cohomology of $\mathbb{C}P^n$. Furthermore, we know that the tt*-Toda equations are defined for general complex simple Lie groups. From this generalization, we can ask:
For general complex simple Lie groups G, can we obtain solutions with the asymptotic data $m = -\rho$ which correspond to the Dubrovin connection of the quantum cohomology of some homogeneous space ?
We give an answer of this question in Section 3.3. This is the second main result.

Theorem 5. *([GIL3],[Mo1],[GH2],[GM],[LT],[K])*
For a complex simple Lie group G and a minuscule weight λ_i, there is a natural correspondence between (i) the asymptotic data

$$m = -\rho \in \mathfrak{h}_\sharp$$

and (ii) the holomorphic data

$$\omega = \frac{1}{\lambda}\left(\sum_{j=1}^{n} e_{-\alpha_j} + q e_{-\alpha_0}\right)\frac{dq}{q}$$

for solutions of the tt-Toda equations. The asymptotic data $m = -\rho$ corresponds to the unique global solution when $\mathfrak{g}$ has type A_n. The holomorphic data correspond to the Dubrovin connection for the quantum cohomology of G/P_i.*

After that we compare the classification by Cecotti and Vafa in Section 6 of [CV2] with the results of Guest-Its-Lin of [GIL1]. These approaches are different, but we found that the results are surprisingly close. We obtain the third main theorem as follows. It is written in Section 3.4.

Theorem 6. *[HK] For $n = 1, 2, 3$, the classification of Cecotti-Vafa coincides with the classification of Guest-Its-Lin.*

The following is the list of contents of this thesis. In Chapter 2, we prepare the notations and theory for this thesis. We consider the general complex simple Lie groups and we define the tt*-Toda equations, holomorphic data, asymptotic data and Stokes data. This part follows [GH2]. In Chapter 3, we focus on the integer Stokes data. In Section 3.1, we see properties of the integer Stokes data. Then we calculate the total number of the integer Stokes data. Then we determine the integer Stokes data on the ρ-line in Section 3.2. In Section 3.3, we obtain solutions of the tt*-Toda equations with the asymptotic data $m = -\rho$ from the quantum cohomology of the minuscule flag manifolds. Finally we compares the classification by Cecotti and Vafa with the classification by Guest-Its-Lin in Section 3.4.

Chapter 2

The tt*-Toda equations

In this chapter, we review notations of Lie groups and Lie algebras. Then we define the tt*-Toda equations.

2.1 Definition of the tt*-Toda equations

2.1.1 Notations

First we review the notations of Lie groups and Lie algebras. Let G be a complex simple simply-connected Lie group and $\mathfrak{g}$ be its Lie algebra. We take a Cartan subalgebra $\mathfrak{h}$. We decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \triangle} g_\alpha$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid ad(H)(X) = \alpha(H)X,\ {}^\forall h \in \mathfrak{h}\}$ is nonzero, α is called a root and the set of roots denoted by $\triangle$.

We choose a set of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ such that $\{\alpha_j\}_{j \in J}$ span $\mathfrak{h}$ where $J = \{1, \cdots, n\}$. We call this n the rank of G. Then we define the set of positive roots $\triangle^+$ by $\triangle^+ = \{\alpha \in \triangle \mid \alpha = b_1\alpha_1 + \cdots + b_n\alpha_n,\ b_j \in \mathbb{Z}_{\geqslant 0}\}$ and the set of negative roots $\triangle^-$ by $-\triangle^+$. These satisfy $\triangle = \triangle^+ \cup \triangle^-$.

Let $(,)$ be any positive scalar multiple of the Killing form. If $\mathfrak{g}$ is simple, then the Killing form is nondegenerate. This form induces the form on $\mathfrak{h}^*$ and we denote this by the same notation $(,)$. We denote the coroot $\frac{2\alpha}{(\alpha,\alpha)}$ of a root α by $\alpha^\vee$. We define an ordering of the roots by $\alpha < \beta$ if $\beta - \alpha$ is positive. In terms of this ordering, the highest one is called the highest root. We denote the highest root by $-\alpha_0$. If $-\alpha_0 = \sum_{j=1}^n a_j\alpha_j$, the Coxeter number h is defined by $h = 1 + \sum_{j=1}^n a_j$.

We define H_α in $\mathfrak{h}$ by $(H_\alpha, H) = \alpha^\vee(H)$ for all $H \in \mathfrak{h}$. Then we obtain a basis $\{H_{\alpha_1}, \cdots, H_{\alpha_n}\}$ of $\mathfrak{h}$. We choose eigenvectors $e_\alpha \in \mathfrak{g}_\alpha$ such that $(e_\alpha, e_{-\alpha}) = 2/(\alpha,\alpha)$ for all $\alpha \in \triangle$. Then we have

$$[e_\alpha, e_\beta] = \begin{cases} 0 & \text{if } \alpha+\beta \notin \triangle \\ H_\alpha & \text{if } \alpha+\beta = 0 \\ N_{\alpha+\beta}e_{\alpha+\beta} & \text{if } \alpha+\beta \in \triangle - \{0\} \end{cases}$$

where $N_{\alpha+\beta}$ is a nonzero complex number. We define $\{\varpi_j\}_{j\in J}$ as the dual basis in $\mathfrak{h}$ to $\{\alpha_j\}_{j\in J}$, that is $\alpha_i(\varpi_j) = \delta_{ij}$.

Example 1.
The case of $G = SL(n+1,\mathbb{C})$

We have $SL(n+1,\mathbb{C}) = \{X \in GL(n+1,\mathbb{C}) \mid det(X) = 1\}$ and $\mathfrak{sl}(n+1,\mathbb{C}) = \{X \in M(n+1,\mathbb{C}) \mid tr(X) = 0\}$. We take the Cartan subalgebra $\mathfrak{h} = \{\text{diag}(x_1, \cdots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i = 0\}$.

Then the roots are $\triangle = \{\pm(\epsilon_i - \epsilon_j) \mid 1 \leqslant i < j \leqslant n+1\}$. When we take the simple roots as $\Pi = \{\epsilon_i - \epsilon_{i+1} \mid i = 1, \cdots, n\}$, then the highest root is $-\alpha_0 = \epsilon_1 - \epsilon_{n+1}$. Thus the Coxeter number is $h = n+1$ because $-\alpha_0 = (\epsilon_1 - \epsilon_2) + \cdots + (\epsilon_n - \epsilon_{n+1})$.

We define $(,)$ by $(X,Y) = tr(XY)$. Then we have the induced form $(,)$ on $\mathfrak{h}^*$. For a simple root $\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(1 \leqslant i \leqslant n)$, we have $\frac{2\alpha_i}{(\alpha_i,\alpha_i)} = \alpha_i$. Thus $H_{\alpha_i} = E_{i,i} - E_{i+1,i+1}$ where $E_{i,j}$ is the matrix which (i,j) component is one and the others are zero. We have $e_{\alpha_i} = E_{i,i+1}$, $e_{-\alpha_i} = E_{i+1,i}$ $(1 \leqslant i \leqslant n)$ and $e_{-\alpha_0} = E_{1,n+1}$. $\varpi_i = \frac{1}{2}(\sum_{j=1}^i E_{j,j} - \sum_{j=i+1}^{n+1} E_{j,j})$.

2.1.2 Connection form α

We consider the following $\mathfrak{g}$-valued 1-form α on the trivial principal bundle $\mathbb{C} \times G \to \mathbb{C}$,

$$\alpha = (w_t + \frac{1}{\lambda}\tilde{E}_-)dt + (-w_{\bar{t}} + \lambda\tilde{E}_+)d\bar{t} =: \alpha' dt + \alpha'' d\bar{t}$$

where $\tilde{E}_{\pm} = Ad(e^{\mp w})(\sum_{j=0}^{n} c_j^{\pm} e_{\pm\alpha_j})$ for $c_i^{\pm} \in \mathbb{C}^{\times}$. Then we consider the zero curvature equation for all $\lambda \in \mathbb{C}$. We have

$$\begin{aligned}
d\alpha + \alpha \wedge \alpha =& \frac{d}{d\bar{t}}(w_t + \frac{1}{\lambda}\tilde{E}_-)d\bar{t} \wedge dt + \frac{d}{dt}(-w_{\bar{t}} + \lambda\tilde{E}_+)dt \wedge d\bar{t} \\
&+ [w_t + \frac{1}{\lambda}\tilde{E}_-, -w_{\bar{t}} + \lambda\tilde{E}_+]dt \wedge d\bar{t} \\
=& (-w_{t\bar{t}} - \frac{1}{\lambda}\mathrm{ad}(w_{\bar{t}})\tilde{E}_- - w_{t\bar{t}} - \lambda\mathrm{ad}(w_t)\tilde{E}_+)dt \wedge d\bar{t} \\
&+ (\frac{1}{\lambda}\mathrm{ad}(w_{\bar{t}})\tilde{E}_- + \lambda\mathrm{ad}(w_t)\tilde{E}_+ + [\tilde{E}_-, \tilde{E}_+])dt \wedge d\bar{t} \\
=& (-2w_{t\bar{t}} + [\tilde{E}_-, \tilde{E}_+])dt \wedge d\bar{t} \\
=& (-2w_{t\bar{t}} + [\sum_{i=1}^{n} c_i^- e^{-\alpha_i(w)} e_{-\alpha_i}, \sum_{i=1}^{n} c_i^+ e^{-\alpha_i(w)} e_{\alpha_i}])dt \wedge d\bar{t}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
d\alpha + \alpha \wedge \alpha =& (-2w_{t\bar{t}} + \sum_{i,j=1}^{n} c_i^- c_j^+ [e^{-\alpha_i(w)} e_{-\alpha_i}, e^{-\alpha_j(w)} e_{\alpha_j}])dt \wedge d\bar{t} \\
=& (-2w_{t\bar{t}} - \sum_{i=1}^{n} d_i e^{-2\alpha_i(w)} H_{\alpha_i})dt \wedge d\bar{t}
\end{aligned}$$

where $d_i = c_i^- c_i^+$. Therefore we obtain the following proposition.

Proposition 7. *The connection $d + \alpha$ is flat, i.e. $d\alpha + \alpha \wedge \alpha = 0$ ($^{\forall}\lambda \in \mathbb{C}^{\times}$) if and only if $2w_{t\bar{t}} = -\sum_{j=0}^{n} d_j e^{-2\alpha_j(w)} H_{\alpha_j}$ holds.*

Definition 8. *We call this equation*

$$2w_{t\bar{t}} = -\sum_{j=0}^{n} d_i e^{-2\alpha_i(w)} H_{\alpha_i} \tag{2.1.1}$$

the two dimensional Toda equation for a complex simple Lie group G.

Example 2. *($G = SL(n+1, \mathbb{C})$ case)*
We assume the same setting as for $SL(n+1, \mathbb{C})$ in Example 1 and that all $d_i = 1$. We have the connection form (1.1.2) and the zero-curvature equation is (1.1.1).

2.1.3 Three involutions θ, σ, χ

We add the condition coming from tt* geometry. tt* geometry means topological-antitopological fusion and it was introduced by Cecotti and Vafa [CV1]. To add the tt* condition, we define a $\mathbb{C}$-linear involution σ and two conjugate-linear involution θ, χ.

We define a conjugate linear Lie algebra homomorphism θ by

$$\theta(e_\alpha) = -e_\alpha,\ \theta(H_\alpha) = -H_\alpha\ ({}^\forall\alpha \in \triangle).$$

This θ defines the compact real form. For example, if $G = SL(n+1, \mathbb{C})$, the fixed set of θ is $SU(n+1)$.

The $\mathbb{C}$-linear involution σ needs some preparation (see [Hit]). We consider a three dimensional subalgebra $\mathfrak{g}_{TDS}$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. We introduce

$$\rho = \sum_{i=1}^{n} \varpi_i = \sum_{i=1}^{n} r_i H_{\alpha_i},\ e_0 = \sum_{i=1}^{n} \sqrt{r_i} e_{\alpha_i},\ f_0 = \sum_{i=1}^{n} \sqrt{r_i} e_{-\alpha_i}$$

where the real numbers $r_1, \cdots, r_n$ are determined by a choice of simple roots. Then these satisfy

$$[\rho, e_0] = e_0,\ [\rho, f_0] = -f_0,\ [e_0, f_0] = \rho.$$

Therefore span$\{e_0, f_0, \rho\} \cong \mathfrak{sl}(2, \mathbb{C})$. This subalgebra is called the principal three dimensional subalgebra and denoted by $\mathfrak{g}_{TDS}$. When we consider the adjoint action by ρ, we can decompose $\mathfrak{g}$ into irreducible representations by $\mathfrak{g} = \bigoplus_{i=1}^{n} V_i$. Let u_i be a highest weight vector of V_i and m_i its weight. m_i are called exponents. We may take $u_1 = e_0$ and $u_n = e_{-\alpha_0}$. These exponents satisfy $1 = m_1 < m_2 < \cdots < m_n = h-1$. It is known that $\dim V_i = 2m_i + 1$ $(i = 1, \cdots, n)$. With respect to this decomposition $\mathfrak{g} = \bigoplus_{i=1}^{n} V_i$, we can take a basis of $\mathfrak{g}$ where each V_i has $\{ad(f_0)^j(u_i) |\ j = 0, \cdots, 2m_i\}$. We define a Lie algebra homomorphism σ by

$$\sigma(u_i) = -u_i,\ \sigma(f_0) = -f_0.$$

The conjugate-linear involution χ is defined by $\chi := \sigma\theta$. Here we have $\sigma\theta = \theta\sigma$. We define the real form of the Toda equation as follows.

Definition 9. *Given a real form of the Lie algebra $\mathfrak{g}$, the corresponding real form of the Toda equations is defined by the following conditions.*

(R1) $\alpha_i(w) \in \mathbb{R}$ *for all* i.
(R2) $\alpha'(z,\bar{z},\lambda) \mapsto \alpha''(z,\bar{z},1/\bar{\lambda})$ *under the involution which defines the real form of* $\mathfrak{g}$.

We define the tt*-Toda equations.

Definition 10. *(The tt*-Toda equations) The tt*-Toda equations are the Toda equations which are the reality conditions (with respect to* χ*) where* $w : \mathbb{C}^\times \to \mathfrak{h}_\sharp$ *satisfies the additional conditions*
(F) $\sigma(w) = w$ *(anti-symmetry condition)*
(R) $w = w(|t|)$ *(radial condition)*

From this definition, we have

$$c_i^- = \bar{c}_i^{\,+} \text{ and } d_i \in \mathbb{R}.$$

Example 3. *(*$G = SL(n+1,\mathbb{C})$ *case)*
We assume Example 1. Then σ, χ *are given by*

$$\sigma(X) = -\triangle X^T \triangle,\ \chi(X) = \triangle \bar{X} \triangle,\ \textit{where } \triangle = \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{pmatrix}.$$

Then we have the tt-Toda equation in* A_n *type as*

$$2(w_i)_{t\bar{t}} = -e^{2(w_{i+1}-w_i)} + e^{2(w_i - w_{i-1})},\ w_i : \mathbb{C}^\times \to \mathbb{R},\ i \in \{0,\cdots,n\}$$

Furthermore, the w_i*'s satisfy the following two conditions:*

1. $w_i + w_{n-i} = 0$ *(anti-symmetry condition)*
2. $w_i = w_i(|t|)$ *(radial condition).*

2.1.4 tt* equation

We review some tt* geometry. We refer to the articles by Cecotti and Vafa [CV1] and Dubrovin [Dub]. We define the tt* equations on a Frobenius manifold.

Definition 11. *Let $\boldsymbol{M}$ be a complex manifold. Its tangent space $T_x\boldsymbol{M}$ ($^\forall x \in \boldsymbol{M}$) has a structure of Frobenius algebra $(T_x\boldsymbol{M}, \cdot, e, \langle, \rangle)$ if $\cdot$ is a multiplication on $T_x\boldsymbol{M}$, e is the unit vector and $\langle, \rangle$ is a nondegenerate inner product on $T_x\boldsymbol{M}$ such that $\langle a \cdot b, c\rangle = \langle a, b \cdot c\rangle$. Then we call this $\boldsymbol{M}$ a quasi-Frobenius manifold.*
A quasi-Frobenius manifold is called Frobenius manifold if the curvature $\nabla^\lambda_X Y = \nabla_X Y + \lambda(X \cdot Y)$ vanishes ($^\forall \lambda \in \mathbb{C}^\times$) where ∇ is the Levi-Civita connection with respect to $\langle, \rangle$.

Let η be the matrix such that $\langle X, Y\rangle = \sum_{a,b} \eta_{ab} X_a Y_b$ where X_a, Y_b are basis. Let C be the matrix such that $X \cdot Y = \sum_{a,b} C_{ab} X_a Y_b$. Let g be a Hermitian positive definite form i.e. $g = \sum_{a,b} g_{\bar{a}b} d\bar{z}_a dz_b$.

Definition 12. *The pair (η, g) is called compatible if there exists a complex connection D where for any complex vector field $X = \sum_a X_a \partial_a$*

$$D_{\partial_c} X_a = \partial_c X_a + \sum_b \Gamma^a_{cb} X_b$$
$$D_{\bar{\partial}_c X_a} = \bar{\partial}_c X_a, \ D_{\bar{\partial}_c} = \bar{D}_{\partial_c}$$

such that

$$D_{\partial_c} \eta_{ab} = \partial_c \eta_{ab} - \sum_b \Gamma^d_{cb} \eta_{db} - \sum_a \Gamma^d_{ca} \eta_{da} = 0$$
$$D_{\partial_c} g_{\bar{a}b} = \partial_c g_{\bar{a}b} - \sum_d \Gamma^d_{cb} g_{\bar{a}d} = 0$$

From this definition we have $\Gamma_c = g^{-1}\partial_c g$. Let $M = g\eta^{-1}$. We call M normalized if M satisfies $M\bar{M} = 1$. For these data (g, M, C) (or (η, g, C)) we define the tt* equation as follows.

Definition 13. *The tt* equations are*

$$D_a C_b = D_b C_a$$
$$[D_a, D_{\bar{b}}] + [C_a, C_{\bar{b}}] = 0$$

where $C_{\bar{b}} = M\bar{C}_b\bar{M}$.

Now we consider these equations on $\mathbb{C} \times \mathbb{C}^n \to \mathbb{C}$ (First we consider the tt* equations on $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ and then we restricted the base space into $\mathbb{C}$). Then we have only one equation

$$[D_z, D_{\bar{z}}] + [C_z, C_{\bar{z}}] = 0.$$

We have

$$\partial_{\bar{z}}(g^{-1}\partial_z g) - [C, g^{-1}\bar{C}^t g] = 0.$$

When we consider

$$C = \tilde{E}_-,\ \bar{C}^t = \tilde{E}_+,\ g = e^{2w},$$

then we have the equation (2.1.1).

2.2 Local solutions of the tt*-Toda equations

In this section, following [GH2], we describe some local solutions near $t = 0$. We consider the following connection form ω.

$$\omega = \frac{1}{\lambda}\left(\sum_{j=0}^{n} z^{k_j} e_{-\alpha_j}\right) dz$$

(i.e. from any $k_0, \cdots, k_n \geqslant -1$). Here z is a complex variable related to t by

$$t = sz^{\frac{1}{h}}.$$

First of all we solve the differential equation

$$\frac{dL}{dz} = \frac{1}{\lambda}\left(\sum_{j=0}^{n} z^{k_j} e_{-\alpha_j}\right) L$$

where L is a G-valued function on $\mathbb{C}^\times$. We have this L as follows.

$$L(z, \lambda) = e^{\frac{1}{\lambda}\mathcal{N}\log z}\left(I + \sum_{1\leqslant i} \lambda^{-i} S_i(z)\right)$$

where $\mathcal{N}$ is nilpotent and $S_i(0) = 0$ $({}^\forall i)$. From Section 6 in [GH2], we know that there exist a loop group element $\gamma \in LG$ and gauge transformation G_h such that

$$\alpha = (\gamma L_{\mathbb{R}} G_h)^{-1} d(\gamma L_{\mathbb{R}} G_h)$$

where $\gamma L_{\mathbb{R}}$ is a part of Iwasawa decomposition of γL i.e.

$$\gamma L = (\gamma L)_{\mathbb{R}} (\gamma L)_+.$$

Then we obtain local solutions w whose asymptotic is

$$w \sim -m \log |t| \text{ as } t \to 0$$

where m is defined by

$$\alpha_i(m) = \frac{s}{N}(k_i + 1) - 1 \ (0 \leqslant i \leqslant n)$$

where $N = h + \sum_{i=0}^{n} a_i k_i$ and $a_0 = 1$. From Proposition 6.1 in [GH2], we have

Proposition 14. *Let $m \in \mathfrak{h}_\sharp$. There exists a local solution near zero of the tt*-Toda equations such that $w \sim -m \log |t|$ as $t \to 0$ if and only if $\alpha_i(m) \geqslant -1$ for $i = 0, \cdots, n$.*

Let $\mathcal{A}$ be the set of asymptotic data of local solutions near $z = 0$ i.e

$$\mathcal{A} = \{m \in \mathfrak{h}_\sharp | \ \alpha_i(m) \geqslant -1 \ (i = 0, \cdots, n)\}.$$

for $G \neq A_n, D_{4n+2}, E_6$. We define $\mathcal{A}^\sigma = \{m \in \mathcal{A} | \ \sigma(m) = m\}$ for $G = A_n, D_{4n+2}, E_6$. We consider for $G \neq A_n, D_{4n+2}, E_6$, the fundamental Weyl alcove $\mathfrak{A}$ i.e.

$$\mathfrak{A} = \{X \in \sqrt{-1}\mathfrak{h}_\sharp | \ 0 \leqslant \alpha_i^{real}(X), \ -\alpha_0^{real}(y) \leqslant 1\}$$

where $\alpha^{real} = \frac{1}{2\pi\sqrt{-1}}\alpha$. We define $\mathfrak{A}^\sigma := \{X \in \mathfrak{A} | \ \sigma(X) = X\}$ for $G = A_n, D_{4n+2}, E_6$. From Theorem 6.9 in [GH2], we have

Theorem 15. *We have a bijection map from $\mathcal{A}$ (or $\mathcal{A}^\sigma$) to $\mathfrak{A}$ (or $\mathfrak{A}^\sigma$) by*

$$m \mapsto X = \frac{2\pi\sqrt{-1}}{h}(m + \rho).$$

From this theorem, we can parametrize local solutions with its asymptotic $w \sim -m \log |t|$ by the fundamental Weyl alcove.

Example 4. *($G = SL(n+1, \mathbb{C})$ case) We have the set of asymptotic data in the $SL(n+1, \mathbb{C})$ case as*

$$\mathcal{A}_{2k}^\sigma = \{m = diag(m_0, \cdots, m_k, -m_k, \cdots, -m_0) | \ m_i - m_{i+1} + 1 \geqslant 0 \ (0 \leqslant i \leqslant 2k)\}$$

for $n+1=2k$ *or*

$$\mathcal{A}^{\sigma}_{2k+1}=\{m=diag(m_0,\cdots,m_k,0,-m_k,\cdots,-m_0)|\ m_i-m_{i+1}+1\geqslant 0\ (0\leqslant i\leqslant 2k+1)\}$$

for $n+1=2k+1$. *We have the fundamental Weyl alcove (invariant by the anti-symmetry) as*

$$\mathfrak{A}^{\sigma}_{2k}=\{X=diag(x_0,\cdots,x_k,-x_k,\cdots,-x_0)|\ 0\leqslant x_k\leqslant\cdots\leqslant x_0\leqslant\frac{1}{2}\}$$

for the $n+1=2k$ *case or*

$$\mathfrak{A}^{\sigma}_{2k+1}=\{X=diag(x_0,\cdots,x_k,0,-x_k,\cdots,-x_0)|\ 0\leqslant x_k\leqslant\cdots\leqslant x_0\leqslant\frac{1}{2}\}$$

for the $n+1=2k+1$ *case.*

As the same of the A_n case, we define Stokes data as

$$\mathcal{S}=(s_1,\cdots,s_n)$$

where s_i is the value on $e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)}$ of the character of the i-th fundamental representation of G. For the A_n case, we have correspondence among the global solutions of the tt*-Toda equations, the holomorphic data, the asymptotic data and the Stokes data by Theorem 2.

Finally we introduce an equivalence relation on asymptotic data in $\mathcal{A}^{\sigma}_{n+1}$ for the A_n case. Let $n+1=2k$ and $\theta_k=\begin{pmatrix} & I_k\\ I_k & \end{pmatrix}$. We can define an operator on connections as follows.

$$\Theta(\alpha)=\theta_k\alpha\theta_k^{-1}=\theta_k(w_t+\frac{1}{\lambda}W^T)\theta_k dt+\theta_k(-w_{\bar{t}}+\lambda W)\theta_k d\bar{t}. \tag{2.2.1}$$

We find that the gauge-equivalent connection $\Theta(\alpha)$ has the same form as α, but with w replaced by

$$\theta_k w\theta_k=\mathrm{diag}(-w_{k-1},\cdots,-w_0,w_0,\cdots,w_{k-1}).$$

We define a map $\Theta_{n+1}:\mathcal{A}^{\sigma}_{n+1}\to\mathcal{A}^{\sigma}_{n+1}$ by

$$\Theta_{n+1}(m)=\begin{cases}\theta_k m\theta_k & \text{if } k=\frac{n+1}{2}\in\mathbb{Z}\ ,\\ m & \text{if } k=\frac{n+1}{2}\notin\mathbb{Z}\ .\end{cases}$$

Then, up to permutations, m and $\Theta_{n+1}(m)$ correspond to the same solution of the tt*-Toda equations. If $\Theta_{n+1}(m) = m$ and $k = \frac{n+1}{2} \in \mathbb{Z}$, then we have a solution of the tt*-Toda equations for $w_0, \cdots w_{k-1}$. If k is even, we can repeat this process using $\Theta_{\frac{n+1}{2}}$.

Definition 16. *We define an equivalence relation $\sim_{GIL}$ on $\mathcal{A}_{n+1}$ in the following way. First, $\Theta_{n+1}(m) \sim_{GIL} m$ for all $m \in \mathcal{A}_{n+1}$. Next, if $n+1$ is even, and $\Theta_{n+1}(m) = m$ (i.e. $m = diag(m', m')$ for $m' \in \mathcal{A}_{\frac{n+1}{2}}$), $diag(\Theta_{\frac{n+1}{2}}(m'), \Theta_{\frac{n+1}{2}}(m')) \sim_{GIL} m$. After a finite number of steps, this process terminates and generates an equivalence relation.*

We refer to the resulting classification of asymptotic data as the GIL classification. By this classification, we also classify the solutions of the tt*-Toda equations because the asymptotic data one-to-one correspond to the global solutions.

Example 5. *(The $n = 3$ case) Let $m = diag(m_0, m_1, -m_1, -m_0)$. We have $(m_0, m_1, -m_1, -m_0) \sim_{GIL} (-m_1, -m_0, m_0, m_1)$. If $m_0 + m_1 = 0$, then we have the further identificaion $(m_0, -m_0, m_0, -m_0) \sim_{GIL} (-m_0, m_0, -m_0, m_0)$.*

2.3 Integer Stokes problem

In this section, we review the integer Stokes problem. We consider the A_n case for Section 3.1, Section 3.2 and Section 3.4. We consider the general complex simple cases for Section 3.3.

If $G = SL(n+1, \mathbb{C})$, we know that a solution w of the tt*-Toda equations satisfies the isomonodromic deformation (1.1.6) as we see in Section 1.1. Then we take Stokes sectors as (1.1.7) or (1.1.8) and we have the monodromy around $\zeta = \infty$ as

$$\tilde{S}_1^{(\infty)} \tilde{S}_2^{(\infty)} = \begin{cases} -(\tilde{Q}_1 \tilde{Q}_{1_{\frac{1}{n+1}}} \tilde{\Pi})^{n+1} & (n+1 = 2k) \\ (\tilde{Q}_1 \tilde{Q}_{1_{\frac{1}{n+1}}} \Pi)^{n+1} & (n+1 = 2k+1) \end{cases}$$

Then we have the characteristic polynomial of $\tilde{Q}_1 \tilde{Q}_{1_{\frac{1}{n+1}}} \tilde{\Pi}$ as

$$\lambda^{n+1} - s_1 \lambda^n + s_2 \lambda^{n-1} - \cdots - s_n \lambda + 1$$

for $n+1$ even and the characteristic polynomial of $\tilde{Q}_1 \tilde{Q}_{1\frac{1}{n+1}} \Pi$ as

$$\lambda^{n+1} - s_1 \lambda^n + s_2 \lambda^{n-1} - \cdots + s_n \lambda - 1$$

for $n+1$ odd where s_i is the i-th symmetric function of the $n+1$ entries of $e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)}$.

We define

$$S_{\mathrm{Hor}} = \begin{pmatrix} 1 & -s_n & s_{n-1} & \dots & (-1)^n s_1 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & s_{n-1} \\ \vdots & \ddots & \ddots & 1 & -s_n \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \tag{2.3.1}$$

where $s_i = s_{n-i+1}$. S_{Hor} is related to $\tilde{S}_1^{(\infty)}$ by $S_{\mathrm{Hor}} = F \tilde{S}_1^{(\infty)} F^T$, where F is the matrix in Proposition 3.4 of [Hor]. Hence $s \in \mathbb{Z}^n$ is if and only if $S_{\mathrm{Hor}} \in SL(n+1, \mathbb{Z})$. From the form of F, it follows that this condition is equivalent to $\tilde{S}_1^{(\infty)} \in SL(n+1, \mathbb{Z})$.

We consider the condition that all s_i are integers. This is called the integer Stokes condition. We call finding solutions satisfying the integer Stokes condition the integer Stokes problem. In [GL1], Guest-Lin calculated all integer Stokes data in the 4×4 case and they found several examples from the quantum cohomologies of Kähler varieties (Table 4 in [GL1]). From the results of [GL1], some questions occur.

Question 1 Can we determine all integer Stokes data and count the number of the solutions with integer Stokes data ?

Question 2 Do there exist integer Stokes data of Lie-theoretic origin?

These questions are considered in Section 3.1 and Section 3.2. After these sections, we find new interpretations of solutions of the tt*-Toda equations with the asymptotic data $m = -\rho$. Finally we compare the classification by Cecotti and Vafa [CV2] and the classification by Guest-Its-Lin [GIL1] and [GL2] for the A_n case. We will give details in Section 3.4.

2.4 Quantum cohomology and the Dubrovin connection

In this section, we review briefly the definition of (small) quantum cohomology and the corresponding Dubrovin connection. As we need only the case of compact Kähler homogeneous spaces, we can use a naive definition of Gromov-Witten invariants. We use the same notation in [G1]. We set the coefficients of homology groups and cohomology rings to be $\mathbb{C}$.

Let M be such a complex manifold. Let p, q, r be three distinct points in $\mathbb{C}P^1$. Let A, B, C be homology classes of $H_*(M)$ and D be an element of $H_2(M;\mathbb{Z}) = \pi_2(M)$. We define

$$\mathrm{Hol}_D^{A,p} = \{\text{holomorphic maps } f : \mathbb{C}P^1 \to M \mid f(p) \in A,\ [f] = D\}$$

where $[f]$ means the homotopy class of f. $\mathrm{Hol}_D^{B,q}, \mathrm{Hol}_D^{C,r}$ are defined in the same way.

Definition 17. *Gromov-Witten invariants are defined by*

$$\langle A|B|C\rangle_D = \sharp Hol_D^{A,p} \cap Hol_D^{B,q} \cap Hol_D^{C,r}.$$

We define the quantum product for M as follows.

Definition 18. *For $C \in H_*(M)$ and $t \in H^2(M)$, $a \circ_t b$ is defined by*

$$\langle a \circ_t b, C\rangle = \sum_{D \in H_2(M)} \langle A|B|C\rangle_D e^{\langle t, D\rangle}$$

where A, B are the dual homology classes to a, b and $\langle , \rangle$ is the pairing between $H^*(M)$ and $H_*(M)$.

We call $(H^*(M), \circ_t)$ the quantum cohomology algebra of M and denote it by $QH^*(M)$. Finally we define the Dubrovin connection. We take a basis $b_1, \cdots, b_r$ of $H^*(M;\mathbb{C})$ where let r be the dimension of $H^*(M;\mathbb{C})$. Let $t = t_1 b_1 + \cdots + t_r b_r$ where $t_i \in \mathbb{C}$. We change the coordinate from $b_i \in H^2(M;\mathbb{C})$ to $q_i \in H^2(M;\mathbb{C}^\times)$ by $e^{b_i} = q_i$.

Definition 19. *The Dubrovin connection on the trivial vector bundle $H^2(M;\mathbb{C}^\times) \times H^*(M;\mathbb{C}) \to H^2(M;\mathbb{C}^\times)$ is defined by*

$$\nabla = d + \frac{1}{\lambda} \sum_{i=1}^{r} A_i(q) \frac{dq_i}{q_i}$$

where $A_i(q)$ are the operators given by the quantum product $b_i \circ_t$.

We seek flag manifolds whose Dubrovin connection form coincides with the connection form ω, i.e. $\omega = \frac{1}{\lambda}A(q)\frac{dq}{q}$ (in our case $r = 1$).

From the article by Iritani [Ir2], we know that Dubrovin connections of the quantum cohomology algebras of orbifolds satisfy the tt* equations. We can say that the Dubrovin connection of the quantum cohomology of $\mathbb{C}P^n$ is the form of the tt*-Toda equations.

Example 6. *(The quantum cohomology of $\mathbb{C}P^n$)*
This example is important for the tt*-Toda equations. It was suggested by Cecotti and Vafa. We have the quantum product by the second cohomology $x \in H(\mathbb{C}P^n, \mathbb{C})$ as follows.

$$x \circ x^i = x^{i+1} \ (i = 0, \cdots, n), \ x \circ x^n = q.$$

Then we obtain the Dubrovin connection by the second cohomology as

$$\begin{aligned}\frac{1}{\lambda}(x\circ)\frac{dq}{q} = \frac{1}{\lambda}\begin{pmatrix} 0 & & & q \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}\frac{dq}{q} &= \frac{1}{\lambda}\left(\sum_{i=1}^{n} z^0 e_{-\alpha_i} + z^1 e_{-\alpha_0}\right)\frac{dz}{z} \\ &= \frac{1}{\lambda}\left(\sum_{i=1}^{n} z^{-1} e_{-\alpha_i} + z^0 e_{-\alpha_0}\right)dz\end{aligned}$$

where we change notation by $q = z$. From this form, we know that the Dubrovin connection of the quantum cohomology of $\mathbb{C}P^n$ is related to the tt*-Toda equation where $G = SL(n+1, \mathbb{C})$. We obtain $k_i = -1$ for $i = 1, \cdots, n$ and $k_0 = 0$. Thus we have $\alpha_i(m) = -1$ and $m = -\rho$. Therefore we have a local solution which corresponds to the origin of $\mathcal{A}^\sigma$.

Chapter 3

Integer Stokes problems

In this chapter, we focus on the integer Stokes data. Section 3.1, Section 3.2 and Section 3.4 are joint works with Yudai Hateruma. We establish the following results concerning integer Stokes data.

1. Properties of the integer Stokes data.

2. There are only four points with integer Stokes data on the ρ-line

3. Solutions of the tt*-Toda equations with the asymptotic data $m = -\rho$ corresponding the quantum cohomology

4. Comparison of the CV classification with the GIL classification.

Then we compare the classification by Ceccotti and Vafa [CV2] with the classification by Guest-Its-Lin [GIL1]. The motivations of these classifications are different. However the results are surprisingly almost the same. In addition, the classification by GIL is simpler and clearer than by CV for the tt*-Toda equations. Finally we see that the symmetry (2.2.1) is a sign group action $G^{\pm}_{n+1}$ on integer Stokes data.

3.1 Properties of the integer Stokes data

Recall that each Stokes data s_i is the i-th symmetric function of the $n+1$ numbers of $e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)}$. We need to introduce some arithmetic settings. We have the following fundamental theorem by Kronecker.

Theorem 20. *[Kro] If the roots of a monic polynomial over $\mathbb{Z}$ are on the unit circle, then they have to be roots of unity.*

The important corollary we will use in Chapter 4 is the following:

Corollary 21. *If a monic polynomial over $\mathbb{Z}$ has all roots on the unit circle, then it has to be a product of cyclotomic polynomials.*

Proof. By Kronecker's theorem, all roots of the polynomial $f(x)$ have the form $e^{\frac{a}{n}2\pi\sqrt{-1}}$ with $1 \leqslant a \leqslant m-1$, $(a,m)=1$. It is known that the minimal polynomial $p(x) \in \mathbb{Q}[x]$ such that $p(e^{\frac{a}{n}2\pi\sqrt{-1}}) = 0$ is the cyclotomic polynomial $\Phi_n(x)$. Hence the statement is proved. $\square$

Let the roots of $P(x)$ be the entries of $e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)}$. Then we have

$$P(x) = \begin{cases} x^{n+1} - s_1 x^n + \cdots - s_n x + 1 & (n+1 \text{ even}) \\ x^{n+1} - s_1 x^n + \cdots + s_n x - 1 & (n+1 \text{ odd}) \end{cases}$$

When we consider the integer Stokes data, we have

$$P(x) = \prod_{m \in \mathbb{Z}_{>0}} (\Phi_m(x))^{\mu(m)}. \tag{3.1.1}$$

by Corollary 21. Here for each positive integer n, let $\Phi_n(x)$ be a n-th cyclotomic polynomial, that is,

$$\Phi_n(x) = \prod_{(1 \leqslant j \leqslant n, \gcd(j,n)=1)} (x - \zeta_n^j)$$

where $\zeta_n^j = e^{\frac{j}{n}2\pi\sqrt{-1}}$, be primitve n-th roots of unity.

Let us give a general formula to calculate the number of all integer Stokes data in $SL(n+1,\mathbb{C})$ case. For

$$\frac{1}{n+1}(m+\rho) = \mathrm{diag}(x_0, \cdots, x_k, -x_k, \cdots, -x_0) \in \mathfrak{A}^{\sigma}_{n+1} \ (n+1 \text{ is even})$$

or

$$\frac{1}{n+1}(m+\rho) = \mathrm{diag}(x_0, \cdots, x_k, 0, -x_k, \cdots, -x_0) \in \mathfrak{A}^{\sigma}_{n+1} \ (n+1 \text{ is odd}),$$

we consider a polynomial $P(x)$ as

$$P(x) = \prod_{0 \leqslant i \leqslant k} (x - e^{\pm 2\pi\sqrt{-1}x_i}) \ (n+1 \text{ even}) \tag{3.1.2}$$

or

$$P(x) = (x-1) \cdot \prod_{0 \leqslant i \leqslant k} (x - e^{\pm 2\pi\sqrt{-1}x_i}) \ (n+1 \text{ odd}). \tag{3.1.3}$$

When we consider an integer Stokes data $s_i(e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)}) \in \mathbb{Z}$ $(i = 1, \cdots, n)$, $P(x)$ is a monic polynomial over $\mathbb{Z}$ with roots in S^1. Thus $P(x)$ has to be a product of cyclotomic polynomials $P(x) = \prod_{d \in \mathbb{N}} (\Phi_d)^{\nu(d)}$ with degree $n+1$ by Corollary 21. We have the following proposition.

Proposition 22. *If m is an asymptotic data with an integer Stokes data, $P(x)$ satisfies the following conditions.*

$$1. \ \sum_d \nu(d)\varphi(d) = n+1$$

$$2. \ \nu(1) = n+1 \ (mod\, 2).$$

Conversely if $Q(x) = \prod_{d \in \mathbb{N}} (\Phi_d)^{\nu(d)}$ satisfies 1 and 2, then the roots of $Q(x)$ give an integer Stokes point.

Proof. First we show that $P(x)$ satisfies the conditions 1 and 2. The condition 1 holds because the degree of $P(x)$ is $n+1$. Because $\Phi_i(0) = 1$ for all $i \in \mathbb{Z}_{\geqslant 2}$, we have

$$P(0) = (\Phi_1(0))^{\nu(1)} = (-1)^{\nu(1)}.$$

If $n+1$ is even, then we have $P(0) = 1$ by (3.1.2). If $n+1$ is odd, then we have $P(0) = -1$ by (3.1.3). Therefore $\nu(1) = n+1$ (mod 2). Thus $P(x)$ satisfies the condition 2.

We show the converse. From the condition 1, $Q(x)$ has solutions $e^{2\pi\sqrt{-1}y_i}$ $(i = 1, \cdots, n+1)$ where $-\frac{1}{2} \leqslant y_{n+1} \leqslant \cdots \leqslant y_1 \leqslant \frac{1}{2}$ because roots of cyclotomic polynomials is in S^1.

If $n+1 = 2k$, from the condition 2, $Q(x)$ has an even number of Φ_1. It is well-known that degree Φ_i $(i \geqslant 3)$ is even. Hence we know $Q(x)$ has an even number of Φ_2. It is also well-known that for Φ_i $(i \geqslant 3)$, if $e^{2\pi\sqrt{-1}w}$

$(0 < w < \frac{1}{2})$ is a solution of Φ_i, then $e^{-2\pi\sqrt{-1}w}$ is also a solution of Φ_i. Thus we have the roots of Φ_i $(i \geqslant 3)$ as

$$e^{2\pi\sqrt{-1}w_1}, \cdots, e^{2\pi\sqrt{-1}w_\ell}, e^{-2\pi\sqrt{-1}w_\ell}, \cdots, e^{-2\pi\sqrt{-1}w_1}$$

for some $\ell \in \mathbb{N}$. Therefore we can reorder y_i's as

$$y_{k+1} = -y_k, \cdots, y_{2k} = -y_1$$

where $0 \leqslant y_k \leqslant \cdots \leqslant y_0 \leqslant \frac{1}{2}$ and we have solutions of $Q(x)$ as

$$e^{2\pi\sqrt{-1}y_0}, \cdots, e^{2\pi\sqrt{-1}y_k}, e^{-2\pi\sqrt{-1}y_k}, \cdots, e^{-2\pi\sqrt{-1}y_0}.$$

Thus we know that the point

$$y = \mathrm{diag}(y_0, \cdots, y_k, -y_k, \cdots, -y_0)$$

is in $\mathfrak{A}^\sigma_{n+1}$ and this point corresponds to an integer Stokes data because $Q(x)$ given by y has integer coefficients.

If $n + 1 = 2k + 1$, from the condition 2, $Q(x)$ has an odd number of Φ_1. Thus in this case, as the same reason above, $Q(x)$ has an even number of Φ_2. Therefore we have solutions of $Q(x)$ as

$$e^{2\pi\sqrt{-1}y_0}, \cdots, e^{2\pi\sqrt{-1}y_k}, 1, e^{-2\pi\sqrt{-1}y_k}, \cdots, e^{-2\pi\sqrt{-1}y_0}$$

where $0 \leqslant y_k \leqslant \cdots \leqslant y_0 \leqslant \frac{1}{2}$. Thus we know that the point

$$y = \mathrm{diag}(y_0, \cdots, y_k, 0, -y_k, \cdots, -y_0)$$

is in $\mathfrak{A}^\sigma_{n+1}$ and this point corresponds to an integer Stokes data as the same reason in the case that $n + 1$ is even. $\square$

From Proposition 22, we only have to count combination numbers of the product of cyclotomic polynomials which satisfy 1 and 2. If we know the total number in $SL(2k, \mathbb{C})$ case, then we also know the total number in $SL(2k + 1, \mathbb{C})$ by the bijective map

$$\begin{aligned} \iota : &\mathfrak{A}^\sigma_{2k} \to \mathfrak{A}^\sigma_{2k+1} \\ &\mathrm{diag}(x_0, \cdots, x_k, -x_k, \cdots, -x_0) \mapsto \mathrm{diag}(x_0, \cdots, x_k, 0, -x_k, \cdots, -x_0). \end{aligned}$$

Thus we obtain the same total numbers in $SL(2k,\mathbb{C})$ and $SL(2k+1,\mathbb{C})$ cases.

Let $n+1$ be an even number. Consider a partition

$$\tau = (a_1, \cdots, a_1, a_2, \cdots, a_2, \cdots, a_l, \cdots, a_l, 1, \cdots, 1)$$

of $n+1$ with each $2 \leqslant a_i \leqslant n+1$ is an even positive integer and $a_i < a_j$ for $i < j$. These a_i correspond to each degree of Φ_j ($j \leqslant 3$) and 1 corresponds to the dimensions of Φ_1 and Φ_2. Let α_i be the number of a_i in a partition and α_0 be $2m$ for 1. Let $\mathcal{I}$ be the set of all such partition. Set $\xi(a_i)$ as a number of elements of a set $\{x \in \mathbb{Z}_{>0} \mid \varphi(x) = a_i\}$ where φ is the Euler function. Then the number of all integer Stokes data is given by the formula:

$$\sum_{\tau \in \mathcal{I}} H_{\alpha_1}^{\xi(a_1)} \cdot \cdots \cdot H_{\alpha_l}^{\xi(a_l)} \cdot (m+1)$$

where $H_j^i = \binom{i+j-1}{j}$.

Recall the set of the interior integer Stokes data ($n+1 = 2k$) as

$$(\mathfrak{A}_{2k}^{\sigma})^{\circ} = \{x = (x_0, \cdots, x_k, -x_k, \cdots, -x_0) \in \mathfrak{A}_{2k}^{\sigma} |\ 0 < x_k < \cdots < x_0 < \frac{1}{2}\}.$$

Then we consider a partition

$$\tau = (a_1, \cdots, a_1, a_2, \cdots, a_2, \cdots, a_l, \cdots, a_l)$$

of $n+1$ with each $2 \leqslant a_i \leqslant n+1$ is an even positive integer and $a_i < a_j$ for $i < j$. When we count the number of the interior integer Stokes data, then we have to remove overlaps of the same cyclotomic polynomials. Thus we have the fomula:

$$\sum_{\tau \in \mathcal{I}} \binom{\xi(a_1)}{\alpha_1} \cdot \cdots \cdot \binom{\xi(a_l)}{\alpha_l}$$

then we obtain the number of interior points.

Example 7. *(The $SL(4,\mathbb{C})$ case) We consider the above partitions of $n+1 = 4$ as follows:*

$$(1,1,1,1), (2,1,1), (2,2), (4).$$

For $(1,1,1,1)$, we have $\alpha_0 = 2 \cdot 2$ and $m = 2$. We have the three product of cyclotomic polynomials

$$\Phi_1^4,\ \Phi_1^2\Phi_2^2,\ \Phi_2^4.$$

For $(2,1,1)$*, we have* $a_1 = 2$*,* $\alpha_1 = 1$*,* $\alpha_0 = 2 \cdot 1$ *and* $m = 1$*. Then we have* $\xi(a_1) = 3$ *and the six* $P(x)$ *as follows:*

$$\Phi_3\Phi_1^2,\ \Phi_4\Phi_1^2,\ \Phi_6\Phi_1^2,\ \Phi_3\Phi_2^2,\ \Phi_4\Phi_2^2,\ \Phi_6\Phi_2^2.$$

For $(2,2)$*, we have* $a_1 = 2$*,* $\alpha_1 = 2$*,* $\alpha_0 = 2 \cdot 0$ *and* $m = 0$*. Then we have* $\xi_1 = 3$*. In this case, multiplicative combinations happen. We have the six* $P(x)$ *as follows:*

$$\Phi_3^2,\ \Phi_4^2,\ \Phi_6^2,\ \Phi_3\Phi_4,\ \Phi_4\Phi_6,\ \Phi_3\Phi_6.$$

For (4)*, we have* $a_1 = 4$*,* $\alpha_1 = 1$*,* $\alpha_0 = 2 \cdot 0$ *and* $m = 0$*. Then we the four* $P(x)$ *as follows:*

$$\Phi_5,\ \Phi_8,\ \Phi_{10},\ \Phi_{12}.$$

Thus we have 19 *integer Stokes data in the* $SL(4,\mathbb{C})$ *case. The polynomials correspond to the interior integer Stokes data are* $\Phi_3\Phi_4,\ \Phi_4\Phi_6,\ \Phi_3\Phi_6,\ \Phi_5,\ \Phi_8,\ \Phi_{10},\ \Phi_{12}$*.*

Example 8. *(The* $SL(6,\mathbb{C})$ *case)*

We calculate the total number of integer Stokes points in the $SL(6,\mathbb{C})$ *case. Let* $n+1 = 6$ *and consider all partitions of* 6*. We have*

$$(1,1,1,1,1,1),(2,1,1,1,1),(2,2,1,1),(4,1,1),(2,2,2),(4,2),(6).$$

Then the total number of the integer Stokes points is

$$\begin{aligned}&\sum_{\tau\in\mathcal{I}} H_{\alpha_1}^{\xi(a_1)}\cdot\cdots\cdot H_{\alpha_l}^{\xi(a_l)}\cdot(m+1)\\ =&4+3\cdot 3+(9-3)\cdot 2+4\cdot 2+(27-2\cdot 6-5)+12+4\\ =&4+9+12+8+10+12+4\\ =&59\end{aligned}$$

We have the total number of the interior integer Stokes points as

$$\begin{aligned}&\sum_{\tau\in\mathcal{I}}\binom{\xi(a_1)}{\alpha_1}\cdot\cdots\cdot\binom{\xi(a_l)}{\alpha_l}\\ =&1+12+4\\ =&17\end{aligned}$$

3.2 Integer Stokes data on the ρ-line

In the following we will introduce some examples based on [GL2] and [GO]. Let us consider the global solution with asymptotic data $m = -\rho$. By the Theorem 1, the corresponding holomorphic data is $k_0 = 0$, $k_1 = \cdots = k_n = -1$. In this case the $\Lambda\mathfrak{sl}_{n+1}\mathbb{C}$-valued 1-form is

$$\eta(z) = \frac{1}{\lambda}\left(\begin{array}{c|c|c|c} & & & 1 \\ \hline z^{-1} & & & \\ \hline & \ddots & & \\ \hline & & z^{-1} & \end{array}\right) dz.$$

This connection form is the Dubrovin connection associated to the quantum cohomology of $\mathbb{C}P^n$. As $e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)} = I$ in this case, we obtain $s_i = \binom{n+1}{i}$.

When $m = -\frac{1}{n+2}\rho$ the corresponding holomorphic data are $k_0 = 1$, $k_1 = \cdots = k_n = 0$. This data is related to the unfolding of A_{n+1} singularities. All s_i are equal to 1.

When $m = \frac{1}{n}\rho$ the corresponding holomorphic data is $k_0 = -1, k_1 = \cdots = k_n = \frac{1-n}{n}$. This data is related to the quantum cohomology of weighted projective space $\mathbb{P}^{1,n}$. The corresponding Stokes data is $s_0 = -1, s_2 = \cdots = s_{n-1} = 0, s_n = -1$.

When $m = 0$ the situation is trivial. That is, all $s_i = 0$ and the corresponding solution of (1.1.1) in given by $w_i = 0$ for all i.

Solutions which correspond to integer Stokes data are interesting objects for both mathematics and physics. However computing such examples is a difficult problem. In this section we will find some examples on the line $\mathbb{R}\rho$. We call this line the ρ-line. It is natural to expect that there are some relations between the Lie-theoretic objects and integer Stokes data. We will show that (when $n > 2$) the only solutions on the ρ-line with integer Stokes are the 4 examples described above.

Theorem 23. *Assume $n \geqslant 3$ and $\lambda \in [-1, \frac{1}{n}]$. The Stokes data $\mathcal{S} = (s_1, \cdots, s_n)$ corresponding to the solution of the tt*-Toda equation with $m = \lambda\rho$ is in $\mathbb{Z}^n$ if and only if*

$$\lambda = \frac{1}{n},\ 0,\ -\frac{1}{n+2},\ -1.$$

3.2.1 Proof of Theorem 23

To show Theorem 23, we first show the following lemma. The following lemma is mainly proved by the author of this Ph.D thesis.

Lemma 24. *Assume $n \geqslant 3$ and $\lambda = \frac{1-l}{n+l}$, $l \in \mathbb{Z}_{\geqslant 0}$. Stokes data s is in $\mathbb{Z}^n$ if and only if*

$$l = 0, 1, 2, \infty.$$

Proof. We assume $n \geqslant 3$ and $\ell \geqslant 3$. We consider

$$P(x) := \Pi_{i=0}^{n}(x - e^{\frac{n-2i}{2(n+l)}2\pi\sqrt{-1}}).$$

$P(x)$ has to be $P(x) = \prod_{m \in \mathbb{Z}_{>0}} (\Phi_m(x))^{\mu(m)}$.

When n is even, $P(x)$ has a solution $e^{\frac{1}{n+\ell}2\pi\sqrt{-1}}$. Therefore it should be $\Phi_{n+\ell}|P(x)$. For the same reason, it should be $\Phi_{2(n+\ell)}|P(x)$ when n is odd. To show Lemma 24, we only have to show that $P(x)$ does not have some roots of $\Phi_{n+\ell}$ or $\Phi_{2(n+\ell)}$ because we can not construct Φ_d when some roots of Φ_d are missing.

First we consider the case that n is even. Let $n = 2a$ where $a \geqslant 2$ and $a \in \mathbb{Z}$. Then $P(x)$ has the roots

$$e^{\frac{a}{n+\ell}2\pi\sqrt{-1}}, \cdots, e^{\frac{1}{n+\ell}2\pi\sqrt{-1}}, 1, e^{-\frac{1}{n+\ell}2\pi\sqrt{-1}}, \cdots, e^{-\frac{a}{n+\ell}2\pi\sqrt{-1}}$$

When $\ell = 2b + 1$ where $b \geqslant 1$, $P(x)$ does not have

$$e^{\pm\frac{a+b}{n+\ell}2\pi\sqrt{-1}}, \cdots, e^{\pm\frac{a+1}{n+\ell}2\pi\sqrt{-1}}.$$

If $\ell = 2b + 1$, then we have $gcd(n + \ell, a + b) = 1$ because we have

$$(2(a + b) + 1) \cdot 1 + (a + b) \cdot (-2) = 1.$$

It follows that $e^{\frac{a+b}{n+\ell}2\pi\sqrt{-1}}$ is a root of $\Phi_{n+\ell}$. However $P(x)$ does not have this root. This is a contradiction.

When $\ell = 2b$ where $b \geqslant 2$, $P(x)$ does not have

$$e^{\pm\frac{a+b-1}{n+\ell}2\pi\sqrt{-1}}, \cdots, e^{\pm\frac{a+1}{n+\ell}2\pi\sqrt{-1}}, -1.$$

If $\ell = 2b$, then we consider two cases, $a + b$ is odd and $a + b$ is even.

If $a + b$ is odd, then we consider not $\Phi_{n+\ell}$ but $\Phi_{\frac{1}{2}(n+\ell)} = \Phi_{a+b}$. Because $a \geqslant 2$, $P(x)$ has $e^{\frac{2}{n+\ell}2\pi\sqrt{-1}} = e^{\frac{1}{a+b}2\pi\sqrt{-1}}$. This is a root of $\Phi_{a+b}(x)$ because $gcd(a+b, 1) = 1$. Thus $\Phi_{a+b}|P(x)$. However, we have $gcd(a+b, \frac{1}{2}(a+b-1)) = 1$ because

$$(a + b) \cdot 1 + \frac{1}{2}(a + b - 1) \cdot (-2) = 1.$$

Thus $e^{\frac{a+b-1}{2(a+b)}2\pi\sqrt{-1}}$ is a root of Φ_{a+b} but $P(x)$ does not have it as a root. This is a contradiction.

If $a + b$ is even, then we have $gcd(2(a + b), a + b - 1) = 1$. We show this by the following lemma.

Lemma 25. *We consider a positive coprime pair (m, n) ($m > n$, and $m, n \in \mathbb{Z}_{\geqslant 0}$). Then $(2m + n, m)$ is also a positive coprime pair.*

Proof. If (m, n) is a positive coprime pair, then there exist $x, y \in \mathbb{Z}$ such that $mx + ny = 1$. Then we have

$$(2m + n)y + m(x - 2y) = 1.$$

$x - 2y$ is also integer number. Thus $(2m + n, m)$ is also a positive coprime pair. □

Because $a+b-1$ is odd, $(a+b-1, 2)$ is a coprime pair. Therefore we obtain $gcd(2(a + b), a + b - 1) = 1$ by Lemma 25. So $\Phi_{n+\ell}$ has a root $e^{\frac{a+b-1}{2(a+b)}2\pi\sqrt{-1}}$. However $P(x)$ does not have this as a root. This is a contradiction.

We consider the case that n is odd. Let $n = 2a + 1$ where $a \geqslant 1$. $P(x)$ has roots

$$e^{\frac{2a+1}{2(n+\ell)}2\pi\sqrt{-1}}, \cdots, e^{\frac{1}{2(n+\ell)}2\pi\sqrt{-1}}, e^{-\frac{1}{2(n+\ell)}2\pi\sqrt{-1}}, \cdots, e^{-\frac{2a+1}{2(n+\ell)}2\pi\sqrt{-1}}$$

Thus these are roots of $2(n + \ell)$-th of unity. Because $P(x)$ has a solution $e^{\frac{1}{2(n+\ell)}2\pi\sqrt{-1}}$, it should be $\Phi_{2(n+\ell)}|P(x)$. We consider the two cases $\ell = 2b + 1$ where $b \geqslant 1$ or $\ell = 2b$ where $b \geqslant 2$.

If $\ell = 2b + 1$, then $n + \ell$ is even. $P(x)$ does not have a root $e^{\frac{n+\ell-1}{2(n+\ell)}2\pi\sqrt{-1}}$ because $\ell \geqslant 3$. However we can find that $\Phi_{2(n+\ell)}$ has this root. Since $(n + \ell - 1, 2)$ is a coprime pair, we have $gcd(2(n + \ell), n + \ell - 1) = 1$ by Lemma 25. So $\Phi_{2(n+\ell)}$ has a root $e^{\frac{n+\ell-1}{2(n+\ell)}2\pi\sqrt{-1}}$. This is a contradiction.

If $\ell = 2b$, then $n+\ell$ is odd. Then $P(x)$ does not have a root $e^{\frac{n+\ell-2}{2(n+\ell)}2\pi\sqrt{-1}}$ because $\ell \geqslant 4$. However we can find that $\Phi_{2(n+\ell)}$ has this root. Since $(n+\ell-2, 2)$ is a coprime pair, we have $gcd(2(n+\ell), n+\ell-2) = 1$ by Lemma 25. Thus $\Phi_{2(n+\ell)}$ has a root $e^{\frac{n+\ell-2}{2(n+\ell)}2\pi\sqrt{-1}}$. This is a contradiction. □

Finally we prove Theorem 23.

Proof. (of Theorem 23) Let us consider $P(x)$ as (3.1.2). Assume all coefficients are integer numbers. $P(x)$ has to be a product of cyclotomic polynomials as (3.1.3). This fact implies that λ is a rational number. Indeed if λ is not a rational number, the diagonal entries of $e^{\frac{2\pi}{n+1}\rho(\lambda+1)}$ is not an m-th root of unity. Thus the integer Stokes problem for general λ reduces to the problem:
Let $\lambda = \frac{1-l}{n+l}$, $l \in \mathbb{Q}_{\geqslant 0}$, then

$$s \in \mathbb{Z}^n \Leftrightarrow l = 0, 1, 2, \infty.$$

It is easily seen that the above l correspond to the λ in the statement of Theorem 23 respectively.

Let $l = \frac{q}{p}$ such that $\gcd(p,q) = 1$. Recall that for any cyclotomic polynomial $\Phi_m(x)$, the roots with the lowest angle have the form $e^{\frac{1}{*}2\pi\sqrt{-1}}$ ($*$ is some integer). So similarly the roots of $P(x)$ with the lowest (except 0) angle have to be such a form. Hence, depending on the parity of n, we get the following conditions:

1. $e^{\frac{1}{2(n+l)}2\pi\sqrt{-1}} = e^{\frac{1}{2(n+\frac{q}{p})}2\pi\sqrt{-1}} = e^{\frac{1}{*}2\pi\sqrt{-1}}$ n: odd,
2. $e^{\frac{2}{2(n+l)}2\pi\sqrt{-1}} = e^{\frac{1}{n+\frac{q}{p}}2\pi\sqrt{-1}} = e^{\frac{1}{*}2\pi\sqrt{-1}}$ n: even.

Now gcd $(p,q) = 1$, so $\gcd(p, np+q) = 1$. Then $p = 2$ or $p = 1$ when n is an odd number, $p = 1$ when n is an even number, respectively. Not depend on n, the case $p = 1$, is proved by Lemma 24.

Consider the case in which n is odd and $p = 2$. Now the all roots of $P(x)$ is given by

$$e^{\frac{n-2i}{2n+q}2\pi\sqrt{-1}}, \quad i = 0, 1, \ldots, n. \tag{3.2.1}$$

So $e^{\frac{1}{2n+q}2\pi\sqrt{-1}}$ is one of the roots of $P(x)$. By the minimality of a cyclotomic polynomial $\Phi_{2n+q}|P(x)$. Now q is an odd number so gcd $(2n+q, 2) = 1$.

Thus $e^{\frac{2}{2n+q}2\pi\sqrt{-1}}$ is a root of $\Phi_{2n+q}(x)$. These facts implies that $e^{\frac{2}{2n+q}2\pi\sqrt{-1}}$ is one of the roots of $P(x)$. However $e^{\frac{2}{2n+q}2\pi\sqrt{-1}}$ does not appear in the list (3.2.1) because n is odd. Hence $P(x)$ must not be a product of cyclotomic polynomials. $\square$

3.2.2 Another proof of Theorem 23

We give another proof of Lemma 24. This is mainly proved by Y.Hateruma. To give another proof of Lemma 24, we need some preparation. Set $D_k = \{z \in \mathbb{C}^\times \mid \frac{\pi}{2}(k-1) < \arg z < \frac{\pi}{2}k\}$, $k \in \{1,2,3,4\}$.

Definition 26. *Let $P(x)$ be a polynomial. If there exists at least one root of $P(x)$ in each quadrant D_k, we say that the roots of the $P(x)$ are balanced.*

Then we have the following proposition.

Lemma 27. *The roots of $\Phi_n(x)$ are not balanced if and only if $n \in \{1,2,3,4,6\}$.*

To obtain this proposition, we define the Möbius function $\mu : \mathbb{Z}_{>0} \to \{-1,0,1\}$ which is defined by

$$\mu(n) = \begin{cases} 0 & \text{(if } n \text{ can be divided by a square number excluding 1)} \\ (-1)^k & \text{(if } n = p_1 \cdots p_k \text{ where } p_i \text{ are different prime numbers)} \end{cases}$$

Then we have

Proposition 28. *The sum of all primitive n-th roots of unity equals $\mu(n)$.*

Proof. Let $f(n)$ be the sum of all primitive n-th roots of unity.

$$g(n) := \sum_{d|n} f(d) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

because the sum means the sum of all roots of $x^n - 1$. By using Möbius inversion formula, $f(n)$ can be computed as

$$\begin{aligned} f(n) &= \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d) \\ &= \mu(n)g(1) + \mu\left(\frac{n}{d_1}\right) g(d_1) + \cdots + \mu\left(\frac{n}{d_i}\right) g(d_i) + \mu(1)g(n) \\ &= \mu(n) \end{aligned}$$

where ${}^\forall d_i|n$. $\square$

Proof. (of Lemma 27) When n is one of the above five numbers, it is easily found that the roots of $\Phi_n(x)$ are not balanced.

Obviously the roots of $\Phi_5(x)$ are balanced, so it remains to consider a case $n > 6$. Assume that the roots of $\Phi_n(x)$ are not balanced. Observe that $e^{\frac{1}{n}2\pi\sqrt{-1}}$ and $e^{\frac{n-1}{n}2\pi\sqrt{-1}}$ are roots of $\Phi_n(x)$ in D_1, D_4 respectively. Then all roots of $\Phi_n(x)$ must be in $D_1 \cup D_4$. This situation implies that the sum of all roots of $\Phi_n(x)$ is bigger than 1 because $e^{\frac{1}{n}2\pi\sqrt{-1}} + e^{\frac{n-1}{n}2\pi\sqrt{-1}} > e^{\frac{1}{6}2\pi\sqrt{-1}} + e^{\frac{5}{6}2\pi\sqrt{-1}} = 1$. On the other hand, sum of all primitive n-th roots of unity equals a value of Möbius function μ at n, i.e.,

$$\mu(n) = \sum_{1 \leqslant i \leqslant n, \gcd(n,i)=1} \zeta_n^i$$

As $\mu(n) \in \{-1, 0, 1\}$. This is a contradiction. Hence the roots of $\Phi_n(x)$, $n > 6$ are balanced. □

Proof. (Proof of Lemma 24) As the same of (41), let

$$P(x) := \Pi_{i=0}^{n}(x - e^{\frac{n-2i}{2(n+l)}2\pi\sqrt{-1}}) = \prod_{m \in \mathbb{Z}_{>0}} (\Phi_m(x))^{\mu(m)}.$$

Consider the case $n \leqslant l$. In this case the roots of $P(x)$ are not balanced. $e^{\frac{1}{(n+l)}2\pi\sqrt{-1}}$ or $e^{\frac{1}{2(n+l)}2\pi\sqrt{-1}}$ is one of the roots of $P(x)$ when n is even or odd respectively. Hence, by the minimality of a cyclotomic polynomial, $\Phi_{n+l}(x)|P(x)$ (or $\Phi_{2(n+l)}(x)|P(x)$). (In short we represent $\Phi_\alpha|P(x)$.) Now α is larger than 6. By Lemma 27, the roots of Φ_α are balanced. This is a contradiction. So we need $n > l$.

In this situation, all roots of $P(x)$ have the form $e^{\frac{k}{2(n+l)}2\pi\sqrt{-1}}$ where k is even or odd when n is even or odd respectively. So let us consider the following polynomial factorization:

$$x^{2(n+l)} - 1 = P(x)Q(x)R(x)$$

where $Q(x)$ is $x^{n+l} + 1$ or $x^{n+l} - 1$ when n is even or odd respectively and $R(x)$ has the form $R(x) = \prod(x - \xi)$. Note that all roots of $Q(x)$ have the form $e^{\frac{k}{2(n+l)}2\pi\sqrt{-1}}$ where k is odd or even when n is even or odd respectively. By Corollary 21, $x^{2(n+l)} - 1$ and $Q(x)$ have to be products of cyclotomic polynomials. Moreover $x^{2(n+l)} - 1 = \prod_{m|2(n+l)} \Phi_m(x)$ and $P(x)$ and $Q(x)$ have no common roots. Then $R(x) = \frac{x^{2(n+l)}-1}{P(x)Q(x)}$ also becomes a product of cyclotomic polynomials. Because of the distribution of the roots of $P(x)$ and $Q(x)$, the roots of $R(x)$ satisfy $\frac{n}{2(n+l)}2\pi < \arg \xi < \frac{n+2l}{2(n+l)}2\pi$, i.e., the roots of $R(x)$ are in $D_2 \cup D_3 \cup \{z \in \mathbb{C} | \operatorname{Re}(z) < 0, \operatorname{Im}(z) = 0\}$. These imply that $R(x)$ has to be $\Phi_2(x)$, $\Phi_3(x)$ or $\Phi_2(x)\Phi_3(x)$. It is easily seen that $\deg R(x) = l-1$. Hence these three cases correspond to $l = 2$, $l = 3$, or $l = 4$ respectively. Now $l > 2$, so we need to observe the cases $l = 3$ and $l = 4$. In both cases $R(x)$ has the root $e^{\pm\frac{n+2}{n+l}2\pi}$ and these roots have to be $e^{\pm\frac{1}{3}2\pi\sqrt{-1}}$. Now we conclude that if $l = 3$ or $l = 4$ we need $n = 0$ or $n = 2$ respectively. Therefore $P(x)$ must not be a product of cyclotomic polynomials. □

3.3 Minuscule flag manifolds

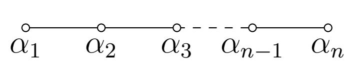
In this section, we focus on the solution of the tt*-Toda equations with the asymptotic data $m = -\rho$. In [GIL] and [CV], in the case $\mathbb{C}P^n$ the Dubrovin connection on $H^2(\mathbb{C}P^n; \mathbb{C})$ is identified with the global solution with asymptotic data $m = -\rho$ for the A_n case. For other types, we have no example of solutions with $m = -\rho$ corresponding to the quantum cohomology. We can find that the quantum cohomology of minuscule flag manifolds are examples of solutions of the tt*-Toda equations with $m = -\rho$.

3.3.1 Minuscule weights

We review some properties of minuscule weights. We refer to the article [CMP]. For a complex simple Lie algebra, we define the weight lattice I as the $\mathbb{Z}$-module spanned by $\lambda_1, \cdots, \lambda_n$ where λ_i is defined by $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$. These λ_i are called the fundamental weights.

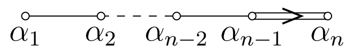
Definition 29. *We call a non-zero weight λ a dominant weight if $(\lambda, \alpha_i^\vee) > 0$ for all $\alpha_i \in \Pi$. We call a dominant weight λ a minuscule weight if $(\lambda, \alpha^\vee) \leqslant 1$ for all $\alpha \in \triangle^+$.*

It is well-known that the minuscule weights are a subset of the fundamental weights. In the following table of fundamental weights, the minuscule weights are marked.

A_n $(n \geqslant 1)$:


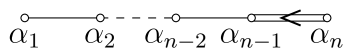
fund. weight	λ_1	λ_2	λ_3		λ_{n-1}	λ_n
minuscule	✓	✓	✓		✓	✓

B_n $(n \geqslant 2)$:



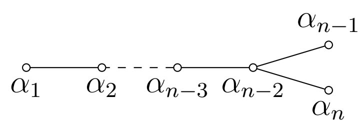
fund. weight	λ_1	λ_2		λ_{n-2}	λ_{n-1}	λ_n
minuscule						✓

C_n $(n \geqslant 2)$:



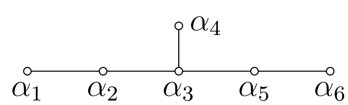
fund. weight	λ_1	λ_2		λ_{n-2}	λ_{n-1}	λ_n
minuscule	✓					

D_n $(n \geqslant 3)$:



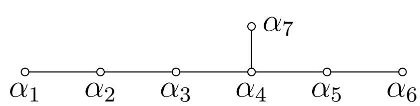
fund. weight	λ_1	λ_2		λ_{n-3}	λ_{n-2}	λ_{n-1}	λ_n
minuscule	✓					✓	✓

E_6 :



fund. weight	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
minuscule	✓					✓

E_7 :



fund. weight	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
minuscule	✓						

It is known that G_2, F_4 and E_8 have no minuscule weight. G/P_{λ_i} can be described conveniently as a quotient of compact groups as follows.

$$
\begin{array}{ll}
(A_n \text{ case}) & G/P_i \cong SU(n+1)/S(U(i)\times U(n+1-i)) \cong Gr(k,n+1) \\
(B_n \text{ case}) & G/P_n \cong SO(2n+1)/U(n) \cong OG(n,2n+1) \\
(C_n \text{ case}) & G/P_1 \cong Sp(n)/U(1)\times Sp(n-1) \cong \mathbb{C}P^{2n-1} \\
(D_n \text{ case}) & G/P_1 \cong SO(2n)/U(1)\times SO(2n-2) \cong Q_{2n-2}, \\
 & G/P_{n-1} \cong SO(2n)/U(n) \cong S_+, G/P_n \cong SO(2n)/U(n) \cong S_- \\
(E_6 \text{ case}) & G/P_1 \cong G/P_6 \cong E_6/SO(10)\times U(1) \cong \mathbb{O}P^2 \\
(E_7 \text{ case}) & G/P_1 \cong E_7/E_6 \times U(1)
\end{array}
$$

Here $OG(k,n)$ is the set of k-dimensional isotropic subspaces of n-dimensional complex vector space V with a nondegenerate quadratic form. This is called the orthogonal Grassmannian. For D_n, $OG(n,2n)$ has two components S_+ and S_-. These are called varieties of pure spinors (or spinor varieties) and these are isomorphic to each other [Ma].

For A_n, B_n, C_n and D_n, the minuscule representations are familiar (see Section 6.5 in [BD]). For A_n, V_{λ_i} is the exterior power $\bigwedge^i V_{\lambda_1}$ $(1 \leqslant i \leqslant n)$ where V_{λ_1} is the standard representation on $\mathbb{C}^{n+1}$. For B_n, V_n is the half-spin representation. For C_n, V_{λ_1} is the standard representation on $\mathbb{C}^{2n}$. For D_n, V_{λ_1} is the standard representation on $\mathbb{C}^{2n}$. $V_{\lambda_{n-1}}$ and V_{λ_n} are the half-spin representations. We denote these two representations by Δ_+ and Δ_-.

For exceptional groups, the minuscule representations are given in the Section 5 of [Gec]. For E_6, V_{λ_1} and V_{λ_6} are 27 dimensional representations. For E_7, V_{λ_1} is a 56 dimensional representation.

3.3.2 Representation V_{λ_i} and $H^*(G/P_i)$

Let W be the Weyl group of G. We denote the weight orbit of λ_i by $W \cdot \lambda_i$. That is $W \cdot \lambda_i = \{x(\lambda_i) |\ x \in W\}$. When we write x as a product of simple reflections, we denote by $\ell(x)$ the minimal length of x in W. The following fact holds for any parabolic subgroup P of G. Let $\triangle_P$ be the subset of $\triangle$ such that $\mathrm{Lie}(P) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \triangle_P} \mathfrak{g}_\alpha$ We denote the subset of the simple roots which belong to $\triangle_P$ by Π_P. Let W_P be the subgroup of W generated by the elements of Π_P.

Proposition 30. *(see Section 1.10 in [Hum]) For $x \in W$, there exist unique* *elements* $u \in W^P$ *and* $v \in W_P$ *such that*

$$x = uv$$

where $W^P = \{x \in W | \ \ell(xs_\alpha) > \ell(x) \ ^\forall\alpha \in \Pi_P\}$.

By this fact, u is a representative of $[x] \in W/W_P$. We have $W \cdot \lambda_i = W^{P_i} \cdot \lambda_i$.

We shall consider the cohomology ring of G/P_i. The following fact is well-known.

Theorem 31. *(Bruhat decomposition)[Hil] For a parabolic subgroup* P *of* G*, we have a decomposition*

$$G = \coprod_{u \in W^P} BuP.$$

We define the Schubert varieties of G/P_i by $X_u := \overline{BuP_i/P_i}$. We also define the opposite Schubert varieties by $Y_u := \overline{x_0Bx_0uP_i/P_i} = x_0X_{x_0u}$ where x_0 is the longest element of W. Then $[Y_u] \in H_{2n-2l(u)}(G/P_i)$ and these classes form an additive basis. By the Poincaré duality theorem, we have a basis of $H^{2l(u)}(G/P_i)$. We denote this generator by σ_u.

Now we obtain the correspondence between $W^{P_i} \cdot \lambda_i$ and an additive basis of the cohomology $H^*(G/P_i)$ by

$$u(\lambda_i) \longleftrightarrow \sigma_u.$$

In this subsection, we observe relationships between minuscule weight orbits and the simple roots. Let λ_i be a minuscule weight.

Proposition 32. *The set of all weights of* V_{λ_i} *is the* W*-orbit of* λ_i *and the multiplicities of all weights of* V_{λ_i} *are one.*

Proof. It is obvious that $\sharp W/W_{P_i} \leqslant \dim(W \cdot v_{\lambda_i}) \leqslant \dim(V_{\lambda_i})$. If there is a weight which has multiplicity more than one, then $\sharp W/W_{P_i} < \dim V_{\lambda_i}$. Therefore by contraposition when we show that $\sharp W/W_{P_i}$ coincides with $\dim_{\mathbb{C}} V_{\lambda_i}$, we obtain the statement of Proposition 32.

We justify the above claim in each case. We have the orders of all Weyl groups from the table 2 in Section 2.11 of [Hum].

For type A_n, we have $\dim_{\mathbb{C}} \bigwedge^i \mathbb{C}^{n+1} = \binom{n+1}{i}$ $(1 \leqslant i \leqslant n)$. On the other hand, for this representation we have $W/W_{P_i} = \mathfrak{S}_{n+1}/(\mathfrak{S}_i \times \mathfrak{S}_{n+1-i})$. Therefore we obtain $\sharp W/W_{P_i} = \frac{(n+1)!}{i!(n+1-i)!} = \binom{n+1}{i}$.

For type B_n, a minuscule representation is the half-spin representation and its dimension is 2^n. Then $W/W_{P_n} = \mathfrak{S}_n \cdot (\mathbb{Z}_2)^n/\mathfrak{S}_n$. Hence $\sharp W/W_{P_n} = 2^n \cdot n!/n! = 2^n$.

For type C_n, a minuscule representation is the standard representation $\mathbb{C}^{2n}$ and its dimension is $2n$. The corresponding $W/W_{P_1} = \mathfrak{S}_n \cdot (\mathbb{Z}_2)^n/\mathfrak{S}_{n-1} \cdot (\mathbb{Z}_2)^{n-1}$. Hence $\sharp W/W_{P_1} = 2^n \cdot n!/2^{n-1} \cdot (n-1)! = 2n$.

For type D_n, there are three minuscule representations. These are the standard representations and the two half-spin representations. These dimensions are $2n$, 2^{n-1}, 2^{n-1} respectively. The corresponding W/W_{P_i} $(i = 1, n-1, n)$ are $\mathfrak{S}_n \cdot (\mathbb{Z}_2)^{n-1}/\mathfrak{S}_{n-1} \cdot (\mathbb{Z}_2)^{n-2}$, $\mathfrak{S}_n \cdot (\mathbb{Z}_2)^{n-1}/\mathfrak{S}_n$, $\mathfrak{S}_n \cdot (\mathbb{Z}_2)^{n-1}/\mathfrak{S}_n$, and $\sharp W/W_{P_i}(i = 1, n-1, n)$ are $2n$, 2^{n-1}, 2^{n-1} respectively.

For type E_6, there are two minuscule representations. These representations are both 27 dimensional representations. The corresponding W/W_{P_1} and W/W_{P_6} are both $W_{E_6}/\mathfrak{S}_5 \cdot (\mathbb{Z}_2)^4$ where W_{E_6} is the Weyl group of E_6. Then $\sharp W_{E_6}/\mathfrak{S}_5 \cdot (\mathbb{Z}_2)^4 = 2^7 \cdot 3^4 \cdot 5/2^4 \cdot 5! = 27$.

For type E_7, the minuscule representation is a 56 dimensional representation. The corresponding W/W_{P_1} is W_{E_7}/W_{E_6} where W_{E_7} is the Weyl group of E_7. Then $\sharp W/W_{P_1} = 2^{10} \cdot 3^4 \cdot 5 \cdot 7/2^7 \cdot 3^4 \cdot 5 = 56$. This completes the proof. $\square$

From Proposition 32, we have the weights of V_{λ_i} as $\{v_{u(\lambda_i)} |\ u \in W^{P_i}\}$ and the multiplicities of these weights are all one. In addition, we know that the Weyl group is generated by the simple reflections $\{s_{\alpha_j} |\ j \in \{1, \cdots, n\}\}$. Therefore all weights can be obtained from λ_i by applying $\{s_{\alpha_j} |\ j \in \{1, \cdots, n\}\}$ to λ_i repeatedly. Thus we obtain the following isomorphism.

$$V_{\lambda_i} \to H^*(G/P_i),\ v_{u(\lambda_i)} \mapsto \sigma_u.$$

3.3.3 Theorem 35

From Section 3.2, we obtain the following diagram.

$$\begin{array}{ccc} \mathbb{C} \times V_{\lambda_i} & \xrightarrow{\cong} & H^2(G/P_i) \times H^*(G/P_i) \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{\cong} & H^2(G/P_i) \end{array}$$

By using the representation V_{λ_i} of G, we obtain the connection form on the trivial vector bundle $\mathbb{C} \times V_{\lambda_i} \to \mathbb{C}$ from ω. In [GM], Golyshev and Manivel showed a relationship between the representations of minuscule weights and the quantum product by the second cohomology for G of type A_n, D_n, E_6, E_7.

Theorem 33. *[GM] For G of type A_n, D_n, E_6, E_7 and λ_i is a minuscule weight, we have*

$$\sum_{j=1}^{n} e_{-\alpha_j} + qe_{-\alpha_0} = \sigma_{s_{\alpha_i}} \circ . \tag{3.3.1}$$

They showed Theorem 33 case by case. After that, in [LT], Lam and Templier uniformly showed the equation (3.3.1) for $G = A_n, B_n, C_n, D_n, E_6, E_7$.

Theorem 34. *[LT] For a complex simple Lie group G and a minuscule weight λ_i, we have the equation (3.3.1).*

From this equation, we obtain new interpretations of solutions of the tt*-Toda equations with the asymptotic data $m = -\rho$, corresponding to the quantum cohomology for types B_n, C_n, D_n, E_6, E_7 as in the case of type A_n in Section 2.4.

By Theorem 34, we have

$$\frac{1}{\lambda}\left(\sum_{j=1}^{n} q^{-1} e_{-\alpha_j} + e_{-\alpha_0}\right) dq = \frac{1}{\lambda}\left(\sigma_{s_{\alpha_i}} \circ\right) \frac{dq}{q}.$$

We identify the left hand side of this equation with the holomorphic data ω of the tt*-Toda equations with $k_0 = 0$ and $k_1 = \cdots = k_n = -1$ by putting $q = z$. Then we obtain

$$\alpha_j(m) = \frac{h}{N}(k_j + 1) - 1 = -1 \ \ (1 \leqslant j \leqslant n).$$

Thus $m = -\rho$ and m satisfies $\alpha_0(m) = h - 1 > -1$. By Proposition 14, this corresponds to the solution of the tt*-Toda equations with the asymptotic data $m = -\rho$.

The Stokes data corresponding to the asymptotic data $m = -\rho$ is given by

$$\mathcal{S}(e^{\frac{2\pi\sqrt{-1}}{h}(-\rho+\rho)}) = (s_1(I), \cdots, s_n(I))$$

where I is the identity element of G. Since s_j is the character of j-th fundamental representation V_{λ_j}, we have $s_j(I) = \dim V_{\lambda_j} \in \mathbb{Z}$ $(1 \leqslant {}^{\forall} j \leqslant n)$.

Therefore the asymptotic data $m = -\rho$ corresponds to integer Stokes data.

By Theorem 15, the asymptotic data $m = -\rho$ corresponds to the origin of $\mathfrak{A}$ (or $\mathfrak{A}^\sigma$).

As a conclusion, we obtain the following.

Theorem 35. *([GIL3],[Mo1],[GH2],[GM],[LT],[K])*
For a complex simple Lie group G and a minuscule weight λ_i, there is a natural correspondence between (i) the asymptotic data

$$m = -\rho \in \mathfrak{h}_\sharp$$

and (ii) the holomorphic data

$$\omega = \frac{1}{\lambda}\left(\sum_{j=1}^{n} e_{-\alpha_j} + q e_{-\alpha_0}\right)\frac{dq}{q}$$

for solutions of the tt-Toda equations. The asymptotic data $m = -\rho$ corresponds to the unique global solution when $\mathfrak{g}$ has type A_n.*[1] *The holomorphic data correspond to the Dubrovin connection for the quantum cohomology of G/P_i.*

From Theorem 35, we obtain new interpretations of solutions of the tt*-Toda equations which correspond to quantum cohomology. In the A_n case, the quantum cohomology and the σ-model of Grassmannians is discussed in detail in the context of the tt*-Toda equations in [G3]. In addition, we obtain the first examples of solutions of the tt*-Toda equations which correspond to the quantum cohomology in the other Lie group types, i.e. B_n, C_n, D_n, E_6, E_7. According to Cecotti and Vafa, we can expect that there are relationships between these new quantum cohomology example and physics models in the context of tt* geometry.

3.3.4 Direct calculation of (3.3.1)

In this subsection, we calculate the equation (3.3.1). We consider the irreducible representations V_{λ_i} whose highest weights are minuscule weights λ_i (see table in Section 3.1). In this section we use results on quantum cohomology to prove that the quantum multiplication by the generator of the

[1] For any $\mathfrak{g}$, it is conjectured that $m = -\rho$ corresponds to a unique global solution of the tt*-Toda equations.

second cohomology coincides with the endomorphism $\sum_{j=1}^{n} e_{-\alpha_j} + qe_{-\alpha_0}$ for a minuscule representation V_{λ_i}. To show this statement, we use the quantum Chevalley formula and the canonical basis.

We define the canonical basis of V_{λ_i} in Section $5A.1$ of the article [Ja] with the following properties:

$$e_{-\alpha_j}(v_{u(\lambda_i)}) = \begin{cases} v_{u(\lambda_i)-\alpha_j} & (u(\lambda_i), \alpha_j^\vee) = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.2}$$

$$e_{\alpha_j}(v_{u(\lambda_i)}) = \begin{cases} v_{u(\lambda_i)+\alpha_j} & (u(\lambda_i), \alpha_j^\vee) = -1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.3}$$

$$H_{\alpha_j}(v_{u(\lambda_i)}) = (u(\lambda_i), \alpha_j^\vee) v_{u(\lambda_i)},$$

for all weights $u(\lambda_i)$ and all $j \in \{1, \cdots, n\}$. As a consequence of (3.3.3), we have

$$e_{-\alpha_0}(v_{u(\lambda_i)}) = \begin{cases} v_{u(\lambda_i)+(-\alpha_0)} & (u(\lambda_i), -\alpha_0^\vee) = -1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.4}$$

Example 9. *(canonical basis for classical groups)*
$\underline{\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})}$ *For a minuscule representatioin $\bigwedge^k \mathbb{C}^{n+1}$ ($1 \leqslant k \leqslant n$), we consider the standard basis $e_{i_1} \wedge e_{i_{i+1}} \wedge \cdots \wedge e_{i_k}$ ($i_1 < i_2 < \cdots < i_k$) where e_i is the standard basis of $\mathbb{C}^{n+1}$. Then this basis satisfies (3.3.2), (3.3.3) and (3.3.4).*

For the quantum products, we use the quantum Chevalley formula.

Theorem 36. *([FW]) For $\beta \in \Pi \backslash \Pi_{P_i}$ and $u \in W^{P_i}$, we have the quantum product $\circ$ by σ_β as*

$$\begin{aligned} \sigma_{s_\beta} \circ \sigma_u = & \sum_{\ell(us_\alpha) = \ell(u)+1} (\lambda_\beta, \alpha^\vee) \sigma_{us_\alpha} \\ & + \sum_{l(us_\alpha) = l(u) - n_\alpha + 1} (\lambda_\beta, \alpha^\vee) \sigma_{us_\alpha} \cdot q^{d(\alpha)} \end{aligned}$$

where α ranges over $\triangle^+ \backslash \triangle^+_{P_i}$, λ_β is the fundamental weight corresponding to β,

$$n_\alpha = (\sum_{\gamma \in \triangle^+ \backslash \triangle^+_{P_i}} \gamma, \alpha^\vee)$$

and

$$d(\alpha) = \sum_{\beta \in \Pi \backslash \Pi_{P_i}} (\lambda_\beta, \alpha^\vee)\sigma(s_\beta),$$

and where $\sigma(s_\beta)$ is the homology class of $H_2(G/P_i)$ which is Poincaré dual to σ_{s_β}.

In our situation, $\Pi \backslash \Pi_{P_i} = \{\alpha_i\}$. Therefore the generator of the second cohomology is only $\sigma_{s_{\alpha_i}}$ and $\lambda_\beta = \lambda_i$. We have $d(\alpha) = (\lambda_i, \alpha^\vee)\sigma(s_{\alpha_i}) = \sigma(s_{\alpha_i})$ for $\alpha \in \triangle^+ \backslash \triangle^+_{P_i}$ because λ_i is a minuscule weight. We consider $q^{\sigma(s_\beta)}$ only as a complex parameter q in $\mathbb{C}$.

From Lemma 3.5 in [FW], the first Chern class of G/P_i is n_α times a generator of $H^2(G/P_i)$. Then we have the following proposition

Proposition 37. *n_α is the Coxeter number h. Explicitly, we have $n_\alpha = n+1$ (A_n type), $n_\alpha = 2n$ (B_n type), $n_\alpha = 2n$ (C_n type), $n_\alpha = 2n-2$ (D_n type), $n_\alpha = 12$ (E_6 type), $n_\alpha = 18$ (E_7 type) for all $\alpha \in \triangle^+ \backslash \triangle^+_{P_i}$.*

Proof. For A_n case, fix a minuscule weight $\lambda = \lambda_k$ $(1 \leqslant k \leqslant n)$. Then we have $\triangle^+ \backslash \triangle^+_{P_k} = \{\epsilon_i - \epsilon_j \ (1 \leqslant i \leqslant k, \ k+1 \leqslant j \leqslant n+1)\}$ and $\sum_{\gamma \in \triangle^+ \backslash \triangle^+_{P_k}} \gamma = \sum_{i=1}^{k}(n+1-k)\epsilon_i - \sum_{i=k+1}^{n+1} k\epsilon_i$. For all $\alpha \in \triangle^+ \backslash \triangle^+_{P_k}$, we have $2/\langle \alpha, \alpha \rangle = 1$. Therefore we obtain

$$n_\alpha = (\sum_{\gamma \in \triangle^+ \backslash \triangle^+_{P_k}} \gamma, \alpha^\vee) = (\sum_{\gamma \in \triangle^+ \backslash \triangle^+_{P_k}} \gamma, \alpha) = n+1.$$

For B_n case, we consider the minuscule weight λ_n. Then $\triangle^+ \backslash \triangle^+_{P_n} = \{\epsilon_i + \epsilon_j \ (1 \leqslant i < j \leqslant n), \ \epsilon_i \ (1 \leqslant i \leqslant n)\}$ and $\sum_{\gamma \in \triangle^+ \backslash \triangle^+_{P_n}} \gamma = \sum_{i=1}^{n} n\epsilon_i$. For $\epsilon_i + \epsilon_j$ $(1 \leqslant i < j \leqslant n)$, we have $2/\langle \alpha, \alpha \rangle = 1$. Therefore we obtain

$$n_\alpha = (\sum_{\gamma \in \triangle^+ \backslash \triangle^+_{P_n}} \gamma, \alpha^\vee) = (\sum_{\gamma \in \triangle^+ \backslash \triangle^+_{P_n}} \gamma, \alpha) = 2n.$$

For ϵ_i $(1 \leqslant i \leqslant n)$, we have $2/\langle \alpha, \alpha \rangle = 2$. Therefore we obtain

$$n_\alpha = (\sum_{\gamma \in \triangle^+ \backslash \triangle^+_{P_n}} \gamma, \alpha^\vee) = 2(\sum_{\gamma \in \triangle^+ \backslash \triangle^+_{P_n}} \gamma, \alpha) = 2n.$$

For C_n case, we consider the minuscule weight λ_1. Then $\triangle^+ \backslash \triangle^+_{P_1} = \{\epsilon_1 - \epsilon_i, \ \epsilon_1 + \epsilon_i \ (1 \leqslant i \leqslant n), \ 2\epsilon_1\}$ and $\sum_{\gamma \in \triangle^+ \backslash \triangle^+_{P_1}} \gamma = 2n\epsilon_1$. For $\epsilon_1 - \epsilon_i, \ \epsilon_1 + \epsilon_i$ $(1 \leqslant$

$i \leqslant n)$, we have $2/\langle\alpha,\alpha\rangle = 1$. Therefore we obtain

$$n_\alpha = (\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_1}} \gamma, \alpha^\vee) = (\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_1}} \gamma, \alpha) = 2n.$$

For $2\epsilon_i$ $(1 \leqslant i \leqslant n)$, we have $2/\langle\alpha,\alpha\rangle = 1/2$. Therefore we obtain

$$n_\alpha = (\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_1}} \gamma, \alpha^\vee) = \frac{1}{2}(\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_1}} \gamma, \alpha) = 2n.$$

For D_n case, fix a minuscule weight λ_1. Then $\triangle^+\backslash\triangle^+_{P_1} = \{\epsilon_1 - \epsilon_i,\ \epsilon_1 + \epsilon_i\ (1 \leqslant i \leqslant n)\}$ and $\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_1}} \gamma = 2(n-1)\epsilon_1$. For $\epsilon_1 - \epsilon_i,\ \epsilon_1 + \epsilon_i\ (1 \leqslant i \leqslant n)$, we have $2/\langle\alpha,\alpha\rangle = 1$. Therefore we obtain

$$n_\alpha = (\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_1}} \gamma, \alpha^\vee) = (\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_1}} \gamma, \alpha) = 2n-2.$$

When we consider a minuscule weight λ_{n-1}, then we have $\triangle^+\backslash\triangle^+_{P_{n-1}} = \{\epsilon_i - \epsilon_n\ (1 \leqslant i \leqslant n),\ \epsilon_i + \epsilon_j\ (1 \leqslant i < j \leqslant n-1)\}$ and $\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_{n-1}}} \gamma = \sum_{i=1}^{n-1}(n-1)\epsilon_i - (n-1)\epsilon_n$. For $\epsilon_i - \epsilon_n\ (1 \leqslant i \leqslant n-1),\ \epsilon_i + \epsilon_j\ (1 \leqslant i < j \leqslant n-1)$, we have $2/\langle\alpha,\alpha\rangle = 1$. Therefore we obtain

$$n_\alpha = (\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_{n-1}}} \gamma, \alpha^\vee) = (\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_{n-1}}} \gamma, \alpha) = 2n-2.$$

When we consider a minuscule weight λ_n, then we have $\triangle^+\backslash\triangle^+_{P_n} = \{\epsilon_i + \epsilon_j\ (1 \leqslant i < j \leqslant n)\}$ and $\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_n}} \gamma = \sum_i^n (n-1)\epsilon_i$. For $\epsilon_i + \epsilon_j\ (1 \leqslant i < j \leqslant n)$, we have $2/\langle\alpha,\alpha\rangle = 1$. Therefore we obtain

$$n_\alpha = (\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_n}} \gamma, \alpha^\vee) = (\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_n}} \gamma, \alpha) = 2n-2.$$

For E_6 case, fix a minuscule weight λ_1. In E_6 case and E_7 case, we use the same notations in [Yo]. Then the set $\triangle^+\backslash\triangle^+_{P_1}$ is listed as follows.

$$\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6,$$
$$\alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6,$$
$$\alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5,$$
$$\alpha_1+2\alpha_2+3\alpha_3+\alpha_4+2\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+2\alpha_3+\alpha_4+2\alpha_5+\alpha_6,$$
$$\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+2\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+\alpha_3+\alpha_5,$$
$$\alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5,$$
$$\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5,\ \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+2\alpha_5+\alpha_6.$$

We have $\sum_{\gamma\in\triangle^+\backslash\triangle^+_{P_1}}\gamma = 16\alpha_1+20\alpha_2+24\alpha_3+12\alpha_4+16\alpha_5+8\alpha_6,$. For all $\alpha\in\triangle^+\backslash\triangle^+_{P_1}$, we have $2/\langle\alpha,\alpha\rangle = 24$ and obtain $n_\alpha = 12$. In the case of a minuscule weight λ_6, we also obtain $n_\alpha = 12$ by the same calculation.

For E_7 case, we consider the minuscule weight λ_1. Then the set $\triangle^+\backslash\triangle^+_{P_1}$ is listed as follows.

$$\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7,$$
$$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7,$$
$$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7,$$
$$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5,$$
$$\alpha_1+2\alpha_2+3\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7,$$
$$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7,$$
$$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7,$$
$$\alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7,$$
$$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7,$$
$$\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7,$$
$$\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7,$$
$$\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+2\alpha_5+\alpha_6+2\alpha_7,$$
$$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_7,$$
$$\alpha_1+2\alpha_2+3\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+2\alpha_7,$$
$$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+2\alpha_7,$$
$$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+2\alpha_7,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_7,$$
$$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_7,\ \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_7,$$
$$\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_7,\ \alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+\alpha_6+2\alpha_7.$$

We have $\sum_{\gamma \in \triangle^+ \backslash \triangle^+_{P_1}} \gamma = 27\alpha_1 + 36\alpha_2 + 45\alpha_3 + 54\alpha_4 + 36\alpha_5 + 18\alpha_6 + 27\alpha_7,$. For all $\alpha \in \triangle^+ \backslash \triangle^+_{P_1}$, we have $2/\langle \alpha, \alpha \rangle = 36$ and obtain $n_\alpha = 18$. □

We can simplify the quantum Chevalley formula as follows.

$$\begin{aligned} \sigma_{s_{\alpha_i}} \circ \sigma_u = & \sum_{\ell(us_\alpha) = \ell(u)+1} (\lambda_i, \alpha^\vee) \sigma_{us_\alpha} \\ & + \sum_{\ell(us_\alpha) = \ell(u)-(h-1)} (\lambda_i, \alpha^\vee) \sigma_{us_\alpha} \cdot q \end{aligned}$$

where $\alpha \in \triangle^+ \backslash \triangle^+_{P_i}$.

To replace the conditions of these summations, the following lemma, corollary and proposition are key ingredients.

Lemma 38. *Let λ_i be a minuscule weight. For $u \in W^{P_i}$ and $\alpha \in \Pi$, we have the three following situations.*
(I) $(u(\lambda_i), \alpha^\vee) = 1 \Leftrightarrow \ell(s_\alpha u) = \ell(u) + 1$.
(II) $(u(\lambda_i), \alpha^\vee) = 0 \Leftrightarrow \ell(s_\alpha u) = \ell(u)$.
(III) $(u(\lambda_i), \alpha^\vee) = -1 \Leftrightarrow \ell(s_\alpha u) = \ell(u) - 1$.
Here we consider the length function $l(u)$ in W^{P_i}.

Proof. (a) First we show the implication ($\Rightarrow$), for each of (I), (II), (III). Here we do not use the minuscule condition.
(I) We assume $(u(\lambda_i), \alpha^\vee) = 1$. We show $s_\alpha u \in W^{P_i}$. If $(u(\lambda_i), \alpha^\vee) = 1$, $(\lambda_i, u^{-1}(\alpha)^\vee) = 1$ and $u^{-1}(\alpha)$ is a positive root. Therefore $\ell(s_\alpha u) = \ell(u) + 1$ in W (see Section 1.6 in [Hum]). For $\beta \in \Pi_{P_i}$, we have

$$\begin{aligned} (\lambda_i, (us_\beta)^{-1}(\alpha)^\vee) > 0 & \Leftrightarrow (\lambda_i, s_\beta u^{-1}(\alpha)^\vee) > 0 \\ & \Leftrightarrow (s_\beta(\lambda_i), u^{-1}(\alpha)^\vee) > 0 \\ & \Leftrightarrow (\lambda_i, u^{-1}(\alpha)^\vee) > 0 \end{aligned}$$

Therefore we have $(\lambda_i, (us_\beta)^{-1}(\alpha)^\vee) > 0$. Hence $\ell(s_\alpha us_\beta) > \ell(us_\beta)$. On the other hand, for all $\beta \in \Pi_{P_i}$, we have $\ell(us_\beta) > \ell(u)$ because u is in W^{P_i}. Hence $\ell(us_\beta) = \ell(u) + 1$ in W. Thus we have

$$\ell(s_\alpha u) = \ell(us_\beta) < \ell(s_\alpha us_\beta).$$

This means that $s_\alpha u \in W^{P_i}$. Therefore we obtain $\ell(s_\alpha u) = \ell(u) + 1$ in W^{P_i}.
(II) We assume $(u(\lambda_i), \alpha^\vee) = 0$. We show $s_{u^{-1}(\alpha)} \in W_{P_i}$. Let $u^{-1}(\alpha)^\vee = b_1\alpha_1^\vee + \cdots + b_l\alpha_l^\vee$ $(b_i \in \mathbb{R})$. Then we have

$$(\lambda_i, u^{-1}(\alpha)^\vee) = b_i = 0$$

Therefore $u^{-1}(\alpha) \in \triangle_{P_i}$ and $s_{u^{-1}(\alpha)} \in W_{P_i}$. We obtain

$$\ell(s_\alpha u) = \ell(us_{u^{-1}(\alpha)}) = \ell(u) \text{ in } W^{P_i}.$$

(III) We assume $(u(\lambda_i), \alpha^\vee) = -1$. We show $s_\alpha u \in W^{P_i}$. If $(u(\lambda_i), \alpha^\vee) = -1$, then $(\lambda_i, u^{-1}(\alpha)^\vee) = -1 < 0$. $u^{-1}(\alpha)$ is a negative root. Hence we have

$$\ell(s_\alpha u) = \ell(u) - 1 < \ell(u) < \ell(us_\beta) \text{ in } W$$

for $\beta \in \Pi_{P_i}$. Now we have

$$\ell(us_\beta) = \ell(u) + 1 = \ell(s_\alpha u) + 2 \text{ in } W.$$

Let $\ell(us_\beta) = r$. Then $\ell(s_\alpha us_\beta) = r-1, r+1$ and $\ell(s_\alpha u) = r-2$. So $\ell(s_\alpha us_\beta) > \ell(s_\alpha u)$. This means that $s_\alpha u \in W^{P_i}$. Thus we obtain $\ell(s_\alpha u) = \ell(u) - 1$ in W^{P_i}.
(b) Next we show the implication ($\Leftarrow$), for each of (I), (II), (III). For (I), we assume $\ell(s_{\alpha u}) = \ell(u) + 1$. Since λ_i is minuscule, $(u(\lambda_i), \alpha^\vee)$ takes only the values $1, 0, -1$. If $(u(\lambda_i), \alpha^\vee)$ is 0 or -1, we obtain a contradiction, by part (a). The proofs in the case (II), (III) are similar. □

Now we have the weights of V_{λ_i} as $\lambda_i - \sum_{j=1}^n n_j\alpha_j$ where $n_j \in \mathbb{Z}_{\geqslant 0}$. From this lemma, we obtain the following corollary.

Corollary 39. *For $u \in W^{P_i}$ such that $u(\lambda_i) = \lambda_i - \sum_{j=1}^n n_j\alpha_j$, we have $\ell(u) = \sum_{j=1}^n n_j$.*

Proof. We have

$$\begin{aligned} \ell(s_{\alpha_j} u) = \ell(u) + 1 &\Leftrightarrow (u(\lambda_i), \alpha_j^\vee) = 1 \\ &\Leftrightarrow s_{\alpha_j}(u(\lambda_i)) = u(\lambda_i) - \alpha_j \end{aligned}$$

by Lemma 38. The elements of W^{P_i} are described by a product of simple reflections. Thus $\ell(u) = \sum_{j=1}^n n_j$. □

We have the following proposition.

Proposition 40. *(I) If there exist $\alpha \in \triangle^+$ such that $\ell(s_\alpha u) = \ell(u) + 1$ for $u \in W^{P_i}$, then $\alpha \in \Pi$ and $(u(\lambda_i), \alpha^\vee) = 1$.*
(II) If there exist $\alpha \in \triangle^+$ such that $\ell(s_\alpha u) = \ell(u) - (s-1)$ for $u \in W^{P_i}$, then $\alpha = -\alpha_0$ and $(u(\lambda_i), -\alpha_0^\vee) = -1$.

Proof. (I) For $\alpha \in \triangle^+$ such that $\ell(s_\alpha u) = \ell(u) + 1$, we have

$$s_\alpha u(\lambda_i) = u(\lambda_i) - (u(\lambda_i), \alpha^\vee)\alpha.$$

By the assumption that $\ell(s_\alpha u) > \ell(u)$, we have $(u(\lambda_i), \alpha^\vee) = 1$ and α must be a simple root by Corollary 39.
(II) For $\alpha \in \triangle^+$ such that $\ell(s_\alpha u) = \ell(u) - (s-1)$. Then we have

$$s_\alpha u(\lambda_i) = u(\lambda_i) - (u(\lambda_i), \alpha^\vee)\alpha.$$

By the assumption $\ell(s_\alpha u) < \ell(u)$, we have $(u(\lambda_i), \alpha^\vee) = -1$. When $\alpha = \sum_{j=1}^n q_j \alpha_j$, then α must be $-\alpha_0$ because there is only one positive root which has the height $\sum_{j=1}^n q_j = h - 1$. □

By using the relation $us_\alpha = s_{u(\alpha)} u = s_{-u(\alpha)} u$, Corollary 39 and Proposition 40, we can replace the conditions of the summation in the quantum Chevalley formula.

We show that we can simplify the first summation to

$$\sum_{(u(\lambda_i), \alpha'^\vee) = 1, \alpha' \in \Pi} \sigma_{s_{\alpha'} u}$$

by setting $\alpha' = u(\alpha)$. Then we shall show that α' is a positive root. In fact, if α' is a negative root, then $(u(\lambda_i), \alpha'^\vee) = -1$ satisfies $\ell(s_{\alpha'} u) = \ell(u) + 1$. However this contradicts $\alpha \in \triangle^+ \backslash \triangle^+_{P_i}$ because we have

$$(u(\lambda_i), \alpha'^\vee) = -1 \Leftrightarrow (u(\lambda_i), u(\alpha^\vee)) = -1 \Leftrightarrow (\lambda_i, \alpha^\vee) = -1.$$

Thus α' is in $\triangle^+$. By Proposition 40, we have $\alpha' \in \Pi \subset \triangle^+$. Hence we have

$$\sum_{\ell(us_\alpha) = \ell(u) + 1} (\lambda_i, \alpha^\vee) \sigma_{us_\alpha} = \sum_{(u(\lambda_i), \alpha'^\vee) = 1, \alpha' \in \Pi} \sigma_{s_{\alpha'} u}$$

as the first summation of $\sigma_{s_{\alpha_i}} \circ \sigma_u$.

For the second summation, let $\alpha' = -u(\alpha)$. Then we shall show that α' is also a positive root. In fact, if α' is a negative root, then $(u(\lambda_i), \alpha'^\vee) = 1$ satisfies $\ell(s_{\alpha'} u) = \ell(u) - (h-1)$. However this contradicts $\alpha \in \triangle^+ \backslash \triangle^+_{P_i}$ because we have

$$(u(\lambda_i), \alpha'^\vee) = 1 \Leftrightarrow (u(\lambda_i), -u(\alpha^\vee)) = 1 \Leftrightarrow (\lambda_i, \alpha^\vee) = -1.$$

Thus $\alpha' = -u(\alpha)$ is in $\triangle^+$ for $\alpha \in \triangle^+ \backslash \triangle^+_{P_i}$. By Proposition 40, we have $\alpha' = -\alpha_0$ and $(u(\lambda_i), -\alpha_0^\vee) = -1$. Hence for the second summation of $\sigma_{s_{\alpha_i}} \circ \sigma_u$ we have

$$\begin{aligned}
&\sum_{\ell(us_\alpha)=\ell(u)-(h-1)} (\lambda_i, \alpha^\vee)\sigma_{us_\alpha} \cdot q \\
=& \sum_{\ell(s_{\alpha'}u)=\ell(u)-(h-1)} (\lambda_i, -u^{-1}(\alpha'^\vee))\sigma_{s_{\alpha'}u} \cdot q \\
=& \begin{cases} q\sigma_{s_{-\alpha_0}u} & (u(\lambda_i), -\alpha_0^\vee) = -1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus we obtain

$$\sigma_{s_{\alpha_i}} \circ \sigma_u = \begin{cases} \displaystyle\sum_{(u(\lambda_i),\alpha_j^\vee)=1} \sigma_{s_{\alpha_j}u} + q\sigma_{s_{-\alpha_0}u} & (u(\lambda_i), -\alpha_0^\vee) = -1 \\ \displaystyle\sum_{(u(\lambda_i),\alpha_j^\vee)=1} \sigma_{s_{\alpha_j}u} & \text{otherwise.} \end{cases}$$

On the other hand, for $v_{u(\lambda_i)}$ we have

$$\begin{aligned}
&(\sum_{j=1}^n e_{-\alpha_j} + qe_{-\alpha_0}) \cdot v_{u(\lambda_i)} \\
=& \begin{cases} \displaystyle\sum_{(u(\lambda_i),\alpha_j^\vee)=1} v_{u(\lambda_i)-\alpha_j} + qv_{u(\lambda_i)+(-\alpha_0)} & (u(\lambda_i), -\alpha_0^\vee) = -1 \\ \displaystyle\sum_{(u(\lambda_i),\alpha_j^\vee)=1} v_{u(\lambda_i)-\alpha_j} & \text{otherwise} \end{cases} \\
=& \begin{cases} \displaystyle\sum_{(u(\lambda_i),\alpha_j^\vee)=1} v_{s_{\alpha_j}u(\lambda_i)} + qv_{s_{-\alpha_0}u(\lambda_i)} & (u(\lambda_i), -\alpha_0^\vee) = -1 \\ \displaystyle\sum_{(u(\lambda_i),\alpha_j^\vee)=1} v_{s_{\alpha_j}u(\lambda_i)} & \text{otherwise} \end{cases}
\end{aligned}$$

by using the definitions of (3.3.2) and (3.3.4). Therefore we obtain

$$\sum_{j=1}^n e_{-\alpha_j} + qe_{-\alpha_0} = \sigma_{s_{\alpha_i}} \circ .$$

3.3.5 Quantum Satake isomorphism

When $\mathfrak{g}$ is of type A_n (or, conjecturally, of type D_n,E_6), the same global solution corresponds to the Dubrovin connection of any minuscule weight. This suggests a relation between the quantum cohomology algebra of the corresponding flag manifolds. In the A_n case this can be stated as

$$\bigwedge^k QH^*(\mathbb{C}P^n) \cong QH^*Gr(k, n+1)$$

(see [GM] for further explanation).

In the D_n case, the analogous relation is:

$$\bigwedge\nolimits_{\pm}^{half} QH^*(Q_{2n-2}) \cong \mathrm{End}_{\mathbb{C}}(QH^*(S_\pm)). \tag{3.3.5}$$

This is an isomorphism of D_n-modules and it preserves the operation of quantum product by the generator of the second cohomology. This follows from Theorem 34 when we identify $H^*(Q_{2n-2};\mathbb{C})$ with $\mathbb{C}^{2n}$ and $H^*(S_\pm;\mathbb{C})$ with $\Delta_\pm$, because (3.3.5) corresponds to the well known relation

$$\bigwedge\nolimits_{\pm}^{half} \mathbb{C}^{2n} \cong \mathrm{End}_{\mathbb{C}}(\Delta_\pm).$$

In order to explain the notation, we recall the relation here. We denote a positively oriented orthonormal basis of $\mathbb{C}^{2n}$ by $e_1, \cdots, e_{2n}$. We define the isomorphism $\star : \bigwedge^i \mathbb{C}^{2n} \to \bigwedge^{2n-i} \mathbb{C}^{2n}$ by

$$\star(e_{\xi(1)} \wedge e_{\xi(2)} \wedge \cdots \wedge e_{\xi(i)}) = \mathrm{sign}(\xi) e_{\xi(i+1)} \wedge e_{\xi(i+2)} \wedge \cdots \wedge e_{\xi(2n)}$$

for any permutation ξ. Then we obtain $\star \cdot \star = (-1)^{i(2n-i)}\mathrm{id}$. We define $\iota := (-i)^n \star : \bigwedge^n \mathbb{C}^{2n} \to \bigwedge^n \mathbb{C}^{2n}$. Then $\iota \cdot \iota = \mathrm{id}$. Thus we have the canonical eigenspace decomposition $\bigwedge^n \mathbb{C}^{2n} \cong \bigwedge^n_+ \mathbb{C}^{2n} \oplus \bigwedge^n_- \mathbb{C}^{2n}$. If $n = 2m+1$, then we define $\bigwedge^{half}_\pm \mathbb{C}^{2n}$ by

$$\bigwedge^0 \mathbb{C}^{4m+2} \bigoplus \bigwedge^2 \mathbb{C}^{4m+2} \bigoplus \cdots \bigoplus \bigwedge^{2m} \mathbb{C}^{4m+2}.$$

If $n = 2m$, then we define $\bigwedge^{half}_+ \mathbb{C}^{2n}$ by

$$\bigwedge^0 \mathbb{C}^{4m} \bigoplus \bigwedge^2 \mathbb{C}^{4m} \bigoplus \cdots \bigoplus \bigwedge\nolimits^{2m}_+ \mathbb{C}^{4m}.$$

and $\bigwedge^{half}_- \mathbb{C}^{2n}$ by

$$\bigwedge^0 \mathbb{C}^{4m} \bigoplus \bigwedge^2 \mathbb{C}^{4m} \bigoplus \cdots \bigoplus \bigwedge\nolimits^{2m}_- \mathbb{C}^{4m}.$$

From Theorem (6.2) of [BD], we have

$$\begin{aligned}
\Delta_+ \otimes \Delta_+ &= \bigwedge\nolimits_+^n + \bigwedge\nolimits^{n-2} + \cdots \\
\Delta_+ \otimes \Delta_- &= \bigwedge\nolimits^{n-1} + \bigwedge\nolimits^{n-3} + \cdots \\
\Delta_- \otimes \Delta_- &= \bigwedge\nolimits_-^n + \bigwedge\nolimits^{n-2} + \cdots
\end{aligned}$$

as $\mathfrak{spin}(2n)$ representations where the last terms are $\bigwedge^4 + \bigwedge^2 + \bigwedge^0$ or $\bigwedge^3 + \bigwedge^1$. If $n = 2m+1$, then we have

$$\begin{aligned}
\mathrm{End}_{\mathbb{C}}(\Delta_+) &\cong \Delta_+^* \otimes \Delta_+ \cong \Delta_+ \otimes \Delta_- \\
&\cong \bigwedge\nolimits^{2m} + \bigwedge\nolimits^{2m-2} + \cdots + \bigwedge\nolimits^2 + \bigwedge\nolimits^0 \\
&= \bigwedge\nolimits_{\pm}^{half} \mathbb{C}^{4m+2}.
\end{aligned}$$

If $n = 2m$, then we have

$$\begin{aligned}
\mathrm{End}_{\mathbb{C}}(\Delta_+) &\cong \Delta_+^* \otimes \Delta_+ \cong \Delta_+ \otimes \Delta_+ \\
&\cong \bigwedge\nolimits_+^{2m} + \bigwedge\nolimits^{2m-2} + \cdots + \bigwedge\nolimits^2 + \bigwedge\nolimits^0 \\
&= \bigwedge\nolimits_+^{half} \mathbb{C}^{4m}.
\end{aligned}$$

When we consider the minuscule Δ_- and the corresponding homogeneous space S_-, we obtain

$$\bigwedge\nolimits_-^{half} QH^*(Q_{2n-2}) \cong \mathrm{End}_{\mathbb{C}}(QH^*(S_-))$$

as in the case of Δ_+.

3.4 The GIL classification and the CV classification

In this section, we compare the classifications by Cecotti-Vafa and by Guest-Its-Lin. First we review the classification by Cecotti-Vafa. It is written in Section 6 of [CV2].

3.4.1 The classification by Cecotti and Vafa

Their classification method is purely algebraic and based on the classification of 'Stokes matrices' S of hypothetical solutions of the tt* equations and their hypothetical 'monodromy matrices' $H = SS^{-T}$. Let S be an upper triangular matrix with 1 on the diagonal. According to Cecotti and Vafa, the Stokes matrices coming from physically realistic solutions of the tt* equations have to satisfy:

$$S \in SL(n+1,\mathbb{Z}), \text{ and} \tag{3.4.1}$$

$$\text{eigenvalues of } H \text{ are in } S^1. \tag{3.4.2}$$

By the same discussion as in Section 3, $P(x)$ can be written as a product of cyclotomic polynomials

$$P(x) = \Pi_{m\in\mathbb{Z}_{>0}}(\Phi_m(x))^{\nu(m)}. \tag{3.4.3}$$

Note that, in Section 3, we studied the properties of characteristic polynomials of $\tilde{Q}_1^{(\infty)}\tilde{Q}_{1\frac{1}{n+1}}^{(\infty)}\tilde{\Pi}$ and these properties were proved by the property of its solutions. On the other hand, in the following lemma, we prove the properties of characteristic polynomials of $H = SS^{-T}$. They will be purely proved by its definition and the above assumptions.

Lemma 41. *The above $P(x)$ satisfies the following conditions.*

(1) $\sum_m \nu(m)\phi(m) = n+1$, $\phi(m)$ is the Euler function.

(2) $\nu(1) = n+1$ mod 2.

(3) For n odd, either $\nu(1) > 0$ or $\sum_{k\geqslant 1}\nu(p^k) = 0$ mod 2 for all primes p.

Proof. 1: This is trivial since $\deg \Phi_m(z) = \phi(m)$.
2: $P(0) = \det(-H) = \det(-S)\det(S^{-T}) = (-1)^(n+1)$. On the other hand, $\Phi_m(0) = 1$ for all m but for $m = 1$ where $\Phi_1(0) = -1$. Hence $\nu(1) = n+1$ mod 2.
3: We have

$$\begin{aligned} P(1) &= \det(1-H) = \det(1-SS^{-T}) \\ &= \det(S^T - S)/\det(S^T) \\ &= (\mathrm{pf}(S^T-S))^2 \end{aligned}$$

where pf means the Pfaffian of a matrix. Note that $S^T - S$ is a skew-symmetric matrix. Hence

$$\Pi_{m\in\mathbb{N}}(\Phi_m(1))^{\nu(m)} = (\mathrm{pf}(S^T-S))^2.$$

On the other hand,

$$\Phi_m(1) = \begin{cases} 0 & \text{if } m = 1 \\ p & \text{if } m = p^k,\ p \text{ prime } k \geqslant 1 \\ 1 & \text{otherwise.} \end{cases}$$

If n is even the condition is equivalent to the condition (2), because from (2) $\nu(1) = 1 \bmod 2$ then the left side of the equation vanishes and Pfaffian is identically 0. If n is odd, there are two possibilities. When the Pfaffian is 0 the left hand side has to satisfy $\nu(1) > 0$. When the Pfaffian is not 0, it is easily seen that $\sum_{k\geqslant 1} \nu(p^k) = 0 \bmod 2$ for all primes p because the left side must be a square number. □

By Lemma 41, we obtain all possible characteristic polynomials. Comparing both sides of the equation (3.4.3), we have a system of Diophantine equations. Solving the systems we obtain all hypothetical Stokes matrices S.

Let $\mathcal{U}_{n+1}$ be the set of $(n+1)\times(n+1)$ upper triangular matrices with 1 on the diagonal with two conditions (3.4.1) and (3.4.2). Let $S = (a_{ij})_{1\leqslant i\leqslant j\leqslant n} \in \mathcal{U}_{n+1}$. A braid group action is generated by

$$\sigma_i : S \mapsto P_i S P_i^T \ \ (1 \leqslant i \leqslant n)$$

where $P_i = \begin{pmatrix} I_i & & \\ & \sigma_i & \\ & & I_{n-i-1} \end{pmatrix}$, I_j is the $j \times j$ identity matrix and $\sigma_i = \begin{pmatrix} -a_{i,i+1} & 1 \\ 1 & 0 \end{pmatrix}$. A product of elements is defined by

$$\sigma_i \circ \sigma_j(S) = \sigma_i(P_j S P_j^T)\sigma_j(S)$$

where on the right hand side we use the matrix multiplication. Then these σ_i satisfy

$$\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i \ \ (|i-j| \leqslant 2)$$
$$\sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1}$$

The braid group is generated by these generators σ_i and we denote them by Br_{n+1}. We consider the sign group $G_{n+1}^{\pm}$:

$$G_{n+1}^{\pm} = \left\{ \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \right\}.$$

We denote the element whose i-th entry is -1 and other entries are 1 by τ_i. $G^{\pm}_{n+1}$ also acts on $\mathcal{U}_{n+1}$ by

$$S \mapsto \tau_i S \tau_i^T.$$

We consider the semidirect product $Br_{n+1} \ltimes G^{\pm}_{n+1}$. Here we have $\sigma_i \circ \tau_j = \tau_j \circ \sigma_i$ for $i \neq j$ and $\sigma_i \circ \tau_i = \tau_{i+1} \circ \sigma_i$.

Definition 42. *[CV2, Bon] We define an equivalence relation on $\mathcal{U}_{n+1}$ by $S_1 \sim_{CV} S_2$ if and only if S_1, S_2 are in same orbit of $Br_{n+1} \ltimes G^{\pm}_{n+1}$.*

We refer to the resulting classification of Stokes matrices as the CV classification.

In fact we shall also use a coarser equivalence relation on $\mathcal{U}_{n+1}$ by making use of two invariants. The conjugacy class of the monodromy matrix is an invariant of actions of $Br_n \ltimes G^{\pm}_{n+1}$. Indeed, the above action induces the action on monodromy matrices by

$$SS^{-T} \mapsto P_i SS^{-T} P_i^{-1}.$$

Next the braid group action on matrices $S + S^T$ is given by

$$S + S^T \mapsto P(S + S^T)P^T.$$

By the Sylvester's law of inertia, the signature of the matrix $S + S^T$ is also invariant under the action. For the elements of $G^{\pm}_{n+1}$, the conjugacy classes of SS^{-T} and the signature of $S + S^T$ are also invariant. Hence we can introduce equivalence relations

$$\begin{aligned} &S \sim_{con} \hat{S} :\Leftrightarrow [SS^{-T}] = [\hat{S}\hat{S}^{-T}] \\ &S \sim_{con,sig} \hat{S} :\Leftrightarrow [SS^{-T}] = [\hat{S}\hat{S}^{-T}] \text{ and } sig(S + S^T) = sig(\hat{S} + \hat{S}^T) \end{aligned}$$

where $[*]$ means the conjugacy class and $sig(*)$ is the signature. We use these invariance to show that the GIL classification coincides with the CV classification for $n = 1, 2, 3$.
We have a natural projection as follows.

$$\mathcal{U}_{n+1}/\sim_{CV} \to \mathcal{U}_{n+1}/\sim_{con,sig} \to \mathcal{U}_{n+1}/\sim_{con}$$

by

$$[S]_{CV} \mapsto [S]_{con,sig} \mapsto [S]_{con}.$$

In the following we summarize the CV classification in [CV2] and compute the Jordan normal forms of the monodromy matrices for use in the next section.
$n = 1$: Set

$$S = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of H is

$$P(x) = x^2 + (a^2 - 2)x + 1.$$

By Lemma 41, there are three possibilities for the form of $P(x)$. In Table

Table 3.1:

$P(x)$	b
$\Phi_1(x)^2$	-2
$\Phi_6(x)$	-1
$\Phi_2(x)^2$	2

3.1, b means the coefficient of x^1 and $[H]$ is the Jordan normal form of H. We have

$$a^2 - 2 = b.$$

Except for $b = -2$, we have two solutions $a = \pm\sqrt{b+2}$. However the corresponding Stokes matrices are equivalent under the action of the braid group. Indeed,

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} -a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -a & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}. \tag{3.4.4}$$

Thus if suffices to take $b < 0$.

- For $b = -2$, the solution is $a = 0$. This case corresponds to the trivial model.
- For $b = -1$, the solution is $a = -1$. This solution corresponds to the Landau-Ginzburg model with superpotential $W(X) = X^3 - X$ (the A_2 minimal model).

Table 3.2:

a	S	$[H]$
0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
-1	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} e^{\frac{\pi\sqrt{-1}}{3}} & 0 \\ 0 & e^{-\frac{\pi\sqrt{-1}}{3}} \end{pmatrix}$
-2	$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

- For $b = 2$, the solution is $a = -2$. This solution corresponds to the $\mathbb{C}P^1$ σ-model.

$n = 2$: Set

$$S = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of H is

$$P(x) = x^3 + \alpha(x_i)x^2 - \alpha(x_i)x - 1$$

where $\alpha(x_i) = x_1^2+x_2^2+x_3^2-x_1x_2x_3$. By Lemma 41, there are five possibilities for the form of $P(x)$. In Table 3.3, b means the coefficient of x^2. These lead

Table 3.3:

$P(x)$	b
$\Phi_1(x)^3$	0
$\Phi_1(x)\Phi_2(x)^2$	4
$\Phi_1(x)\Phi_3(x)$	3
$\Phi_1(x)\Phi_4(x)$	2
$\Phi_1(x)\Phi_6(x)$	1

to the Diophantine equations

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = b.$$

This equation is called the Markov type Diophantine equation, and its complete solutions were studied in [Mor]. The important facts are

Table 3.4:

(x_1, x_2, x_3)	S	$[H]$
$(0,0,0)$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
$(3,3,3)$	$\begin{pmatrix}1&3&3\\0&1&3\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$
$(-1,-1,-1)$	$\begin{pmatrix}1&-1&-1\\0&1&-1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&1&0\\0&-1&0\\0&0&1\end{pmatrix}$
$(2,2,2)$	$\begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$
$(1,1,1)$	$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}e^{\frac{\pi}{2}}&0&0\\0&e^{-\frac{\pi}{2}}&0\\0&0&1\end{pmatrix}$

- for a generic $b \in \mathbb{Z}$ there are infinitely many solutions of the equations, and
- these solutions are generated by three transformations from so-called fundamental solutions[2](see [CV2] or [Mor]). The three transformations contains the generators of the action of the braid group B_3

$$(x_1, x_2, x_3) \mapsto (-x_1, x_3, x_2 - x_1x_3)$$
$$(x_1, x_2, x_3) \mapsto (x_2, x_1 - x_2x_3, -x_3).$$

By using these facts, Cecotti-Vafa obtained four physically realistic models corresponding to the braid group orbits.

- For $b = 0$, a fundamental solutions is $(3, 3, 3)$. This solution corresponds to the $\mathbb{C}P^3$ σ-model.
- For $b = 4$, there are two fundamental solutions, $(1, 1, 2)$ and $(2, 2, 2)$. The solution $(2, 2, 2)$ corresponds to the Ising 3-point function. The

[2] The fundamental solution (x_1, x_2, x_3) is a nontrivial solution satisfying $0 < x_1 \leqslant x_2 \leqslant x_3$ and $x_1 + x_2 + x_3$ is minimal. Triviality means that at least two x_i's vanish.

physics model corresponding to $(1,1,2)$ can be explained as an $N=2$ Toda theory related to the $\hat{A}_2$ root system, and $(1,1,2)$ can be transformed into $(-1,-1,-1)$.

- For $b=2$, the fundamental solution is $(1,1,1)$. This corresponds to the A_3 minimal model.
- For $b=1,\,3$, there are no fundamental solutions.

3.4.2 The comparison of classifications

In this section, we compare the classification of Guest-Its-Lin with the classification of Cecotti-Vafa in the $n=1,2,3$ cases. To compare them, we calculate the monodromy matrices from the asymptotic data (or Stokes data). For the $n=1,2$ cases, it turns out to be necessary to consider only the conjugacy classes. For the $n=3$ case, we have to consider not only the conjugacy classes but also the signatures.

We consider the asymptotic data with the integer Stokes data. We denote the set $\{m\in\mathcal{A}^{\sigma}_{n+1}|\ s_i\in\mathbb{Z},\ i=1,\cdots,n\}$ by $\mathcal{A}^{\sigma}_{n+1,\mathbb{Z}}$. As we have seen in Section 3, by (2.3.1) we can define the map

$$\mathscr{S}:\ \mathcal{A}^{\sigma}_{n+1,\mathbb{Z}}\to\mathcal{U}_{n+1},\ m\mapsto S_{\mathrm{Hor}}(e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)}).$$

We denote $S_{\mathrm{Hor}}(e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)})$ by $S_{\mathrm{Hor}}(m)$.

Proposition 43. *The map $\mathscr{S}$ is well-defined.*

Proof. For $m\in\mathcal{A}^{\sigma}_{n+1,\mathbb{Z}}$, $S_{\mathrm{Hor}}(m)$ is in $SL(n+1,\mathbb{Z})$ from (2.3.1). Thus $S_{\mathrm{Hor}}(m)$ satisfies (3.4.1).

As we have seen in Section 2, the monodromy matrix of the solution corresponding to the asymptotic data m (or the Stokes data $s_1,\cdots,s_n$) is

$$\tilde{S}^{(\infty)}_1\tilde{S}^{(\infty)}_2=\tilde{S}^{(\infty)}_1(\tilde{S}^{(\infty)}_1)^{-T}=\begin{cases}-(\tilde{Q}^{(\infty)}_1\tilde{Q}^{(\infty)}_{1_{\frac{1}{n+1}}}\tilde{\Pi})^{n+1} & (n+1\text{ even})\\ (\tilde{Q}^{(\infty)}_1\tilde{Q}^{(\infty)}_{1_{\frac{1}{n+1}}}\Pi)^{n+1} & (n+1\text{ odd})\end{cases}$$

In addition, for $m\in\mathcal{A}^{\sigma}_{n+1}$, we know from the discussion in Section 2 that $\tilde{Q}^{(\infty)}_1\tilde{Q}^{(\infty)}_{1_{\frac{1}{n+1}}}\tilde{\Pi}$ has a Jordan normal form as

$$\tilde{Q}^{(\infty)}_1\tilde{Q}^{(\infty)}_{1_{\frac{1}{n+1}}}\tilde{\Pi}=g^{-1}e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)}e^{N}g$$

where $g \in SL(n+1,\mathbb{C})$ and N is a nilpotent element which commutes with $m+\rho$. Then we have the monodromy matrices

$$\begin{cases} -(\tilde{Q}_1^{(\infty)}\tilde{Q}_{1\frac{1}{n+1}}^{(\infty)}\tilde{\Pi})^{n+1} = -g^{-1}e^{2\pi\sqrt{-1}(m+\rho)}e^{(n+1)N}g & (n+1 \text{ is even}) \\ (\tilde{Q}_1^{(\infty)}\tilde{Q}_{1\frac{1}{n+1}}^{(\infty)}\Pi)^{n+1} = g^{-1}e^{2\pi\sqrt{-1}(m+\rho)}e^{(n+1)N}g & (n+1 \text{ is odd}). \end{cases}$$

Then we have

$$\begin{aligned} [S_{\mathrm{Hor}}(m)S_{\mathrm{Hor}}(m)^{-T}]_{con} &= [F\tilde{S}_1^{(\infty)}F^TF^{-T}(\tilde{S}_1^{(\infty)})^{-T}F^{-1}]_{con} \\ &= [\tilde{S}_1^{(\infty)}(\tilde{S}_1^{(\infty)})^{-T}]_{con} \\ &= \begin{cases} [-(\tilde{Q}_1^{(\infty)}\tilde{Q}_{1\frac{1}{n+1}}^{(\infty)}\tilde{\Pi})^{n+1}]_{con} & (n+1 \text{ even}) \\ [(\tilde{Q}_1^{(\infty)}\tilde{Q}_{1\frac{1}{n+1}}^{(\infty)}\Pi)^{n+1}]_{con} & (n+1 \text{ odd}) \end{cases} \\ &= \begin{cases} [-e^{2\pi\sqrt{-1}(m+\rho)}e^{(n+1)N}]_{con} & (n+1 \text{ even}) \\ [e^{2\pi\sqrt{-1}(m+\rho)}e^{(n+1)N}]_{con} & (n+1 \text{ odd}). \end{cases} \end{aligned}$$

Therefore all eigenvalues of $S_{\mathrm{Hor}}(m)S_{\mathrm{Hor}}(m)^{-T}$ are in S^1. Thus $S_{\mathrm{Hor}}(m)$ satisfies (3.4.2). □

We consider the set of equivalence classes of $\sim_{GIL}$ for $\mathcal{A}^\sigma_{n+1,\mathbb{Z}}$ and denote them by $\mathcal{A}^\sigma_{n+1,\mathbb{Z}}/\sim_{GIL}$. Then we define the map

$$\widetilde{\mathscr{S}}:\ \mathcal{A}^\sigma_{n+1,\mathbb{Z}}/\sim_{GIL} \to \mathcal{U}_{n+1}/\sim_{CV},\ [m]_{GIL} \mapsto [S_{\mathrm{Hor}}(m)]_{CV}.$$

We see that the map $\widetilde{\mathscr{S}}$ is well-defined. First, we have

Lemma 44. *For $k = 2\ell$ ($\ell \in \mathbb{Z}$) and $m \in \mathcal{A}^\sigma_k$, we have*

$$s_i(e^{\frac{2\pi\sqrt{-1}}{k}(\Theta_k(m)+\rho)}) = \begin{cases} -s_i(e^{\frac{2\pi\sqrt{-1}}{k}(m+\rho)}) & \textit{if } i \textit{ is odd} \\ s_i(e^{\frac{2\pi\sqrt{-1}}{k}(m+\rho)}) & \textit{if } i \textit{ is even} \end{cases} \tag{3.4.5}$$

Proof. We have

$$\begin{aligned} \Theta_k(m)+\rho &= \mathrm{diag}(-m_\ell,\cdots,-m_0,m_0,\cdots,m_\ell) + \mathrm{diag}(\frac{k-1}{2},\cdots,\frac{1}{2},-\frac{1}{2},\cdots,-\frac{k-1}{2}) \\ &= \mathrm{diag}(-m_k+\frac{k-1}{2},\cdots,-m_0+\frac{1}{2},m_0-\frac{1}{2},\cdots,m_k-\frac{k-1}{2}) \\ &= \mathrm{diag}(-m_k-\frac{1}{2},\cdots,-m_0-\frac{k-1}{2},m_0+\frac{k-1}{2},\cdots,m_k+\frac{1}{2}) + \frac{k-1}{2}E_\ell \end{aligned}$$

where $E_\ell = \begin{pmatrix} I_\ell & \\ & -I_\ell \end{pmatrix}$. Therefore we have

$$\begin{aligned} e^{\frac{2\pi\sqrt{-1}}{k}(\Theta_k(m)+\rho)} &= e^{\frac{2\pi\sqrt{-1}}{k}\mathrm{diag}(-m_k-\frac{1}{2},\cdots,-m_0-\frac{k-1}{2},m_0+\frac{k-1}{2},\cdots,m_k+\frac{1}{2})}e^{\pi\sqrt{-1}E_\ell} \\ &= (-1)e^{\frac{2\pi\sqrt{-1}}{k}\mathrm{diag}(-m_k-\frac{1}{2},\cdots,-m_0-\frac{k-1}{2},m_0+\frac{k-1}{2},\cdots,m_k+\frac{1}{2})}. \end{aligned}$$

Since $e^{\frac{2\pi\sqrt{-1}}{k}\mathrm{diag}(-m_k-\frac{1}{2},\cdots,-m_0-\frac{k-1}{2},m_0+\frac{k-1}{2},\cdots,m_k+\frac{1}{2})}$ has the same entries of $e^{\frac{2\pi\sqrt{-1}}{k}(m+\rho)}$, we have

$$s_i(e^{\frac{2\pi\sqrt{-1}}{k}(\Theta_k(m)+\rho)}) = \begin{cases} -s_i(e^{\frac{2\pi\sqrt{-1}}{k}(m+\rho)}) & \text{if } i \text{ is odd} \\ s_i(e^{\frac{2\pi\sqrt{-1}}{k}(m+\rho)}) & \text{if } i \text{ is even} \end{cases}$$

because s_i is the i-th elementary symmetric function. $\square$

For $k = 2\ell$ and $m \in \mathcal{A}_\ell^\sigma$, we have (cf. Section 9 in [GIL4])

$$s_i(e^{\frac{2\pi\sqrt{-1}}{k}(\mathrm{diag}(m,m)+\rho_k)}) = \begin{cases} s_{\frac{i}{2}}(e^{\frac{2\pi\sqrt{-1}}{\ell}(m+\rho_\ell)}) & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases} \tag{3.4.6}$$

where we denote ρ in $\mathcal{A}_i^\sigma$ by ρ_i. Then we have

Lemma 45. *Let $n+1$ be even. If $2^k|n+1$ ($k \in \mathbb{Z}_{\geqslant 0}$), for $m \in \mathcal{A}^\sigma_{\frac{n+1}{2^k}}$ we have*

$$s_i(e^{\frac{2\pi\sqrt{-1}}{n+1}(diag(m,\cdots,m)+\rho_{n+1})}) = \begin{cases} s_{\frac{i}{2^k}}(e^{\frac{2\pi\sqrt{-1}}{\frac{n+1}{2^k}}(m+\rho_{\frac{n+1}{2^k}})}) & \text{if } i = 0 \ (mod\ 2^k) \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.7}$$

where $diag(m,\cdots,m) \in \mathcal{A}^\sigma_{n+1}$.

Proof. We show this statement by induction. The $k = 0$ case of (3.4.7) trivially holds. Then we assume that the $k = \ell$ case of (3.4.7) holds. If $2^{\ell+1}|n+1$, for $m \in \mathcal{A}^\sigma_{\frac{n+1}{2^{\ell+1}}}$ we have

$$\begin{aligned} s_i(e^{\frac{2\pi\sqrt{-1}}{n+1}(\mathrm{diag}(m,\cdots,m)+\rho_{n+1})}) &= \begin{cases} s_{\frac{i}{2^\ell}}(e^{\frac{2\pi\sqrt{-1}}{\frac{n+1}{2^\ell}}(\mathrm{diag}(m,m)+\rho_{\frac{n+1}{2^\ell}})}) & \text{if } i = 0 \ (\mathrm{mod}\ 2^\ell) \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} s_{\frac{i}{2^{\ell+1}}}(e^{\frac{2\pi\sqrt{-1}}{\frac{n+1}{2^{\ell+1}}}(m+\rho_{\frac{n+1}{2^{\ell+1}}})}) & \text{if } i = 0 \ (\mathrm{mod}\ 2^{\ell+1}) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here we use (3.4.6). Thus the $k = \ell + 1$ case of (3.4.7) also holds. $\square$

Proposition 46. *The map $\widetilde{\mathscr{S}}$ is well-defined.*

Proof. If $n+1$ is odd, Θ_{n+1} is identity. Thus we have to only consider the case that $n+1$ is even.

If $2^k | n+1$ ($k \in \mathbb{Z}_{\geqslant 0}$) and $2^{k+1} | n+1$, for $\mathrm{diag}(m, \cdots, m) \in \mathcal{A}^{\sigma}_{n+1}$ where $m \in \mathcal{A}^{\sigma}_{\frac{n+1}{2^k}}$, we have

$$s_i(e^{\frac{2\pi\sqrt{-1}}{n+1}(\mathrm{diag}(\Theta_k(m),\cdots,\Theta_k(m))+\rho_{n+1})}) = \begin{cases} s_{\frac{i}{2^k}}(e^{\frac{2\pi\sqrt{-1}}{\frac{n+1}{2^k}}(\Theta_k(m)+\rho_{\frac{n+1}{2^k}})}) & \text{if } i = 0 \pmod{2^k} \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} -s_{\frac{i}{2^k}}(e^{\frac{2\pi\sqrt{-1}}{\frac{n+1}{2^k}}(m+\rho_{\frac{n+1}{2^k}})}) & \text{if } i = 2^k \pmod{2^{k+1}} \\ s_{\frac{i}{2^k}}(e^{\frac{2\pi\sqrt{-1}}{\frac{n+1}{2^k}}(m+\rho_{\frac{n+1}{2^k}})}) & \text{if } i = 0 \pmod{2^{k+1}} \\ 0 & \text{otherwise.} \end{cases}$$

by Lemma 44 and Lemma 45. Then we have for $m \in \mathcal{A}^{\sigma}_{\frac{n+1}{2^k}}$

$$S_{\mathrm{Hor}}(e^{\frac{2\pi\sqrt{-1}}{n+1}(\mathrm{diag}(m,\cdots,m)+\rho_{n+1})}) = \begin{pmatrix} I_{\frac{n+1}{2^k}} & s_1 I_{\frac{n+1}{2^k}} & s_2 I_{\frac{n+1}{2^k}} & \cdots & s_{\frac{n+1}{2^k}-1} I_{\frac{n+1}{2^k}} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & s_2 I_{\frac{n+1}{2^k}} \\ \vdots & \ddots & \ddots & \ddots & s_1 I_{\frac{n+1}{2^k}} \\ 0 & \cdots & \cdots & 0 & I_{\frac{n+1}{2^k}} \end{pmatrix}$$

and

$$S_{\mathrm{Hor}}(e^{\frac{2\pi\sqrt{-1}}{n+1}(\mathrm{diag}(\Theta_{\frac{n+1}{2^k}}(m),\cdots,\Theta_{\frac{n+1}{2^k}}(m))+\rho_{n+1})}) = \begin{pmatrix} I_{\frac{n+1}{2^k}} & -s_1 I_{\frac{n+1}{2^k}} & s_2 I_{\frac{n+1}{2^k}} & \cdots & -s_{\frac{n+1}{2^k}-1} I_{\frac{n+1}{2^k}} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & s_2 I_{\frac{n+1}{2^k}} \\ \vdots & \ddots & \ddots & \ddots & -s_1 I_{\frac{n+1}{2^k}} \\ 0 & \cdots & \cdots & 0 & I_{\frac{n+1}{2^k}} \end{pmatrix}$$

Thus we have

$$S_{\mathrm{Hor}}(e^{\frac{2\pi\sqrt{-1}}{n+1}(\mathrm{diag}(\Theta_{\frac{n+1}{2^k}}(m),\cdots,\Theta_{\frac{n+1}{2^k}}(m))+\rho_{n+1})}) = h^T S_{\mathrm{Hor}}(e^{\frac{2\pi\sqrt{-1}}{n+1}(\mathrm{diag}(m,\cdots,m)+\rho_{n+1})})h$$

where

$$h = \begin{pmatrix} -I_{\frac{n+1}{2^k}} & & & & \\ & I_{\frac{n+1}{2^k}} & & & \\ & & \ddots & & \\ & & & -I_{\frac{n+1}{2^k}} & \\ & & & & I_{\frac{n+1}{2^k}} \end{pmatrix}.$$

This h is an element of $G^{\pm}_{n+1}$. Therefore we obtain

$$[S_{\mathrm{Hor}}(e^{\frac{2\pi\sqrt{-1}}{n+1}(\mathrm{diag}(\Theta_{\frac{n+1}{2^k}}(m),\cdots,\Theta_{\frac{n+1}{2^k}}(m))+\rho_{n+1})})]_{CV} = [S_{\mathrm{Hor}}(e^{\frac{2\pi\sqrt{-1}}{n+1}(\mathrm{diag}(m,\cdots,m)+\rho_{n+1})})]_{CV}.$$

This means the map $\widetilde{\mathscr{S}}$ is well-defined. $\square$

The aim of this section is to show that the map $\widetilde{\mathscr{S}}$ is injective. By the sequence of maps in Section 4, we have the following diagram.

$$\begin{tikzcd}
\mathcal{A}^{\sigma}_{n+1,\mathbb{Z}}/\sim_{GIL} \arrow[r, "\widetilde{\mathscr{S}}"] \arrow[dr] \arrow[ddr] & \mathcal{U}_{n+1}/\sim_{CV} \arrow[d] \\
& \mathcal{U}_{n+1}/\sim_{con,sig} \arrow[d] \\
& \mathcal{U}_{n+1}/\sim_{con}
\end{tikzcd} \tag{3.4.8}$$

We show the following result.

Theorem 47. *For $n = 1, 2, 3$, the map $\widetilde{\mathscr{S}}$ is injective, that is, the GIL classification coincides with the CV classification.*

Proof. To show the statement we show that the map

$$\mathcal{A}^{\sigma}_{n+1,\mathbb{Z}}/\sim_{GIL} \to \mathcal{U}_{n+1}/\sim_{con} \tag{3.4.9}$$

for the integer Stokes data is injective if $n = 1, 2$ and that the map

$$\mathcal{A}^{\sigma}_{n+1,\mathbb{Z}}/\sim_{GIL} \to \mathcal{U}_{n+1}/\sim_{con,sig} \tag{3.4.10}$$

for the integer Stokes data is injective if $n = 3$. It follows that the map $\widetilde{\mathscr{S}}$ is injective.

$\underline{n = 1}$ We have

$$\mathcal{A}_2^\sigma = \{m = \mathrm{diag}(m_0, -m_0) \mid -\frac{1}{2} \leqslant m_0 \leqslant \frac{1}{2}\}$$

and

$$\mathcal{A}_2^\sigma / \sim_{GIL} = \{[m]_{GIL} \mid -\frac{1}{2} \leqslant m \leqslant 0\}.$$

We obtain

$$\mathcal{A}_{n+1,\mathbb{Z}}^\sigma / \sim_{GIL} = \{[(-\frac{1}{2}, \frac{1}{2})], [(-\frac{1}{6}, \frac{1}{6})], [(0,0)]\}$$

as Table 3.5.[3] From Table 3.5, we know that the monodromy for each $[m]_{GIL}$

Table 3.5: $n + 1 = 2$

$m = (m_0, -m_0)$	$S_{\mathrm{Hor}}(m)$	$[-e^{2\pi\sqrt{-1}(m+\rho)}e^{(n+1)N}]_{con}$
$(-\frac{1}{2}, \frac{1}{2})$	$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$
$(-\frac{1}{6}, \frac{1}{6})$	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} e^{\frac{\pi\sqrt{-1}}{3}} & 0 \\ 0 & e^{-\frac{\pi\sqrt{-1}}{3}} \end{pmatrix}$
$(0, 0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

are distinct. Therefore the map (3.4.9) is injective.
$\underline{n = 2}$ We have

$$\mathcal{A}_3^\sigma = \{m = \mathrm{diag}(m_0, 0, -m_0) \mid -1 \leqslant m_0 \leqslant \frac{1}{2}\}.$$

In the $n = 2$ case, the relation in Definition 16 is trivial, that is $\mathcal{A}_3^\sigma = \mathcal{A}_3^\sigma / \sim_{GIL}$. We obtain the elements of $\mathcal{A}_{3,\mathbb{Z}}^\sigma / \sim_{GIL}$ as Table 3.6. Therefore the map (3.4.9) is injective.
$\underline{n = 3}$ We show that the map (3.4.10) is injective. We have

$$\mathcal{A}_4^\sigma = \{m = \mathrm{diag}(m_0, m_1, -m_1, -m_0) \mid -\frac{1}{2} \leqslant m_1 \leqslant \frac{3}{2},\ -\frac{3}{2} \leqslant m_0 \leqslant \frac{1}{2},\ m_1 \leqslant m_0 + 1\}.$$

[3] In tables, we omit the words "diag" of diagonal matrices m.

Table 3.6: $n+1=3$

$m=(m_0,0,-m_0)$	$S_{\mathrm{Hor}}(m)$	$[e^{2\pi\sqrt{-1}(m+\rho)}e^{(n+1)N}]_{con}$
$(-1,0,1)$	$\begin{pmatrix}1&-3&3\\0&1&-3\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$
$(-\frac{1}{2},0,\frac{1}{2})$	$\begin{pmatrix}1&-2&2\\0&1&-2\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$
$(-\frac{1}{4},0,\frac{1}{4})$	$\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}\sqrt{-1}&0&0\\0&-\sqrt{-1}&0\\0&0&1\end{pmatrix}$
$(0,0,0)$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
$(\frac{1}{2},0,-\frac{1}{2})$	$\begin{pmatrix}1&1&-1\\0&1&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&1&0\\0&-1&0\\0&0&1\end{pmatrix}$

By the relation in $n=3$ (see Example 5), we have

$$\begin{aligned}\mathcal{A}_4^\sigma/\sim_{GIL}=&\{[m]_{GIL}\mid -\frac{1}{2}\leqslant m_1\leqslant\frac{3}{2},\ -\frac{3}{2}\leqslant m_0\leqslant\frac{1}{2},\ m_1\leqslant m_0+1,\ m_0+m_1\leqslant 0\}\\ &-\{[m]_{GIL}\mid m_0+m_1=0, 0<m_0\leqslant\frac{1}{2}\}.\end{aligned}$$

Then we obtain the elements of $\mathcal{A}_{4,\mathbb{Z}}^\sigma/\sim_{GIL}$ as Table 3.7. By Theorem 3.5 in [Hor], we know that $S_{\mathrm{Hor}}(m)+S_{\mathrm{Hor}}(m)^T$ has the same signature as

$$\mathrm{diag}((-1)^{-3}p(\pi_0),\cdots,(-1)^3p(\pi_3)) \tag{3.4.11}$$

where $p(x)=x^4-s_1x^3+s_2x^2-s_1x+1$, $\pi_k=e^{\frac{1}{2}(k+\frac{1}{2})\pi\sqrt{-1}}$. From Table 3.7, Then we find that the map (3.4.10) is injective. □

Remark 48. *For $n=1,2$, we obtain a more powerful result. We can define a map*

$$\mathcal{A}_{n+1}^\sigma/\sim_{GIL}\to\mathcal{U}_{n+1}/\sim_{Br_n},\ [m]_{GIL}\mapsto[S_{Hor}(m)]_{brd} \tag{3.4.12}$$

and show that this map is injective.

For $n = 1$, we have the braid group action for $S_{Hor}(m)$ by (3.4.4) as

$$\begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} -s_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -s_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -s_1 \\ 0 & 1 \end{pmatrix}.$$

For $[m]_{GIL} = [m']_{GIL}$, we have $[S_{Hor}(m)] = [S_{Hor}(m')]$. Thus the map (3.4.12) is well-defined. From Table 3.5, we obtain that for the integer Stokes data the map (3.4.12) is injective.

For $n = 2$, the relation in Definition 16 is trivial. Thus we can define the map for the integer Stokes data

$$\mathcal{A}_3^\sigma \to \mathcal{U}_3/\sim_{brd},\ m \mapsto [S_{Hor}(m)]_{brd}. \tag{3.4.13}$$

From Table 3.6, we have the five integer Stokes data in $\mathcal{A}_3^\sigma$. On the other hand, we find that there are five elements $[S_{Hor}(m)]_{brd}$ in $\mathcal{U}_3/\sim_{brd}$ because the conjugacy classes of the their monodromy are different. Then the map (3.4.13) is injective.

Table 3.7: $n+1=4$

m	S_{Hor}	$[-e^{2\pi\sqrt{-1}(m+\rho)}e^{(n+1)N}]$	$sig(S_{\text{Hor}}+S_{\text{Hor}}^T)$
$(-\frac{1}{6},\frac{1}{6},-\frac{1}{6},\frac{1}{6})$	$\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}e^{\frac{\pi\sqrt{-1}}{3}}&0&0&0\\0&e^{\frac{-\pi\sqrt{-1}}{3}}&0&0\\0&0&e^{\frac{\pi\sqrt{-1}}{3}}&0\\0&0&0&e^{\frac{-\pi\sqrt{-1}}{3}})\end{pmatrix}$	$(+,+,+,+)$
$(0,0,0,0)$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$(+,+,+,+)$
$(-\frac{1}{2},\frac{1}{6},-\frac{1}{6},\frac{1}{2})$	$\begin{pmatrix}1&-1&2&-1\\0&1&-1&2\\0&0&1&-1\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0&0\\0&e^{\frac{-\pi\sqrt{-1}}{3}}&0&0\\0&0&e^{\frac{\pi\sqrt{-1}}{3}}&0\\0&0&0&-1\end{pmatrix}$	$(+,+,0,0)$
$(-\frac{3}{10},-\frac{1}{10},\frac{1}{10},\frac{3}{10})$	$\begin{pmatrix}1&-1&1&-1\\0&1&-1&1\\0&0&1&-1\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}e^{\frac{-3\pi\sqrt{-1}}{5}}&0&0&0\\0&e^{\frac{-\pi\sqrt{-1}}{5}}&0&0\\0&0&e^{\frac{\pi\sqrt{-1}}{5}}&0\\0&0&0&e^{\frac{3\pi\sqrt{-1}}{5}}\end{pmatrix}$	$(+,+,+,+)$
$(-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2})$	$\begin{pmatrix}1&-4&6&-4\\0&1&-4&6\\0&0&1&-4\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&1&0&0\\0&-1&1&0\\0&0&-1&1\\0&0&0&-1\end{pmatrix}$	$(+,0,-,-)$
$(-\frac{5}{6},-\frac{1}{2},\frac{1}{2},\frac{5}{6})$	$\begin{pmatrix}1&-3&4&-3\\0&1&-3&4\\0&0&1&-3\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&1&0&0\\0&-1&0&0\\0&0&e^{\frac{-\pi\sqrt{-1}}{3}}&0\\0&0&0&e^{\frac{\pi\sqrt{-1}}{3}}\end{pmatrix}$	$(+,0,-,-)$
$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\begin{pmatrix}1&-2&2&-2\\0&1&-2&2\\0&0&1&-2\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&1&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$	$(+,0,0,0)$
$(-\frac{1}{6},-\frac{1}{2},\frac{1}{2},\frac{1}{6})$	$\begin{pmatrix}1&-1&0&-1\\0&1&-1&0\\0&0&1&-1\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&1&0&0\\0&-1&0&0\\0&0&e^{\frac{-\pi\sqrt{-1}}{3}}&0\\0&0&0&e^{\frac{\pi\sqrt{-1}}{3}}\end{pmatrix}$	$(+,+,+,0)$
$(-\frac{5}{6},\frac{1}{6},-\frac{1}{6},\frac{5}{6})$	$\begin{pmatrix}1&-2&3&-2\\0&1&-2&3\\0&0&1&-2\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}e^{\frac{-\pi\sqrt{-1}}{3}}&1&0&0\\0&e^{\frac{-\pi\sqrt{-1}}{3}}&0&0\\0&0&e^{\frac{\pi\sqrt{-1}}{3}}&1\\0&0&0&e^{\frac{\pi\sqrt{-1}}{3}}\end{pmatrix}$	$(+,+,-,-)$
$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	$\begin{pmatrix}1&0&2&0\\0&1&0&2\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}-1&1&0&0\\0&-1&0&0\\0&0&-1&1\\0&0&0&-1\end{pmatrix}$	$(+,+,0,0)$

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List of Research Achievement by Yoshiki Kaneko

1. Published Article

1 [K] Yoshiki Kaneko, *Solutions of the tt*-Toda equations from the quantum cohomology of minuscule flag manifolds*, Nagoya Mathematical Journal, 248, 990-1004.

2. Research Presentations

1 金子吉樹 , “Solutions of the tt*-Toda Equations with integer Stokes data”, Poisson geometry and related topics 22, 東京理科大学 神楽坂キャンパス, 2022/12.

2 Yoshiki Kaneko, “Solutions of the tt*-Toda Equations from Minuscule Flag Manifolds”, The international workshop “Geometry of submanifolds and integrable systems”, 大阪市立大学 (zoom), 2022/2.

3 金子吉樹, “tt*幾何の展開”, Koriyama Geometry and Physics Days, 日本大学郡山キャンパス, 2021/11.

4 金子吉樹, “量子ドリンフェルト-ソコロフ簡約”, Koriyama Geometry and Physics Days, 日本大学郡山キャンパス, 2021/11.

5 金子吉樹, “tt*戸田方程式の局所解と minuscule な旗多様体の量子コホモロジーについて”, 幾何学シンポジウム, 北海道大学 (zoom), 2021/8.

6 Yoshiki Kaneko, “Solutions of the tt*-Toda equations and quantum cohomology of minuscule flag manifolds”, Nonabelian Hodge theory, 早稲田大学 (zoom), 2021/7.

7 金子吉樹, “tt*-戸田方程式の解と旗多様体の量子コホモロジーについて”, 日本数学会年会, 慶應大学理工学部, 2021/3.

8 Yoshiki kaneko, ”Solutions of the tt*-Toda Equations and Quantum Cohomology of Flag Manifolds”, International Workshop on Multiphase Flows: Analysis, Modelling and Numerics, Oxford-Waseda(zoom), 2020/12.

9 金子吉樹, ”Pseudodifferential symbol と Hamiltonian equations”, Koriyama Geometry and Physics Days 2020, 日本大学郡山キャンパス, 2020/2.

10 金子吉樹, ”Virasoro algebra と coadjoint action”, Koriyama Geometry and Physics Days 2020, 日本大学郡山キャンパス, 2020/2.

11 金子吉樹, ”旗多様体から得られる tt*戸田方程式の解”, 異分野異業種交流会, 東京大学, 2019/10.

12 金子吉樹, ”一般旗多様体から得られる tt*戸田方程式の局所解について”, 関東若手幾何セミナー, 首都大学東京 (現、都立大学), 2019/6.

13 金子吉樹, ”Local solutions of the tt*-Toda equations from flag manifolds”, 早稲田大学数学若手異分野交流会, 早稲田大学, 2019/3.

14 Yoshiki Kaneko, ”Solutions of The tt*-Toda Equations Corresponding to Quantum Cohomology of Flag Manifolds”, UK-Japan Winter School 2019, Leeds University, 2019/1.

15 Yoshiki Kaneko, ”Introduction to orbifolds”, Korimaya Geometry and Physics Days 2018, 日本大学郡山キャンパス, 2018/2.

16 Yoshiki Kaneko, ”Construction of representations of compact Lie groups and their loop groups”, Koriyama Geometry and Physics Days 2016, 日本大学郡山キャンパス, 2016/2.