Graduate School of Fundamental Science and Engineering
Waseda University

博 士 論 文 審 査 報 告 書
Doctoral Dissertation Review Report

論　文　題　目
Dissertation Title

Solutions of the tt*-Toda Equations with Integer Stokes Data and Quantum Cohomology of Minuscule Flag Manifolds

整数Stokesデータをもつtt*戸田方程式の解とminusculeな旗多様体の量子コホモロジー

申　請　者
(Applicant Name)
Yoshiki KANEKO
金子　吉樹

Department of Pure and Applied Mathematics　Research on Differential Geometry

March, 2023

The tt*-Toda equations are a special case of the Toda equations, a well-known integrable system with many applications in differential geometry. They are also a special case of the tt* equations, which were introduced by the physicists Cecotti and Vafa in 1991 in order to describe massive (non-conformal) deformations of SUSY CFT (supersymmetric conformal field theory). Cecotti and Vafa conjectured that there exist special solutions of the tt* equations which have deep relations to geometry as well as physics.

In the case of the tt*-Toda equations, the existence and basic properties of these special solutions were verified by Guest-Its-Lin in the period 2010-2020. The results relevant to this thesis are as follows.

(i) Solutions "near 0" of the tt*-Toda equations, for G=SL(n+1,C), correspond to points of a compact convex polytope. For global solutions, this is a 1:1 correspondence.
(ii) Each solution has a corresponding Stokes matrix, which arises from the ordinary differential equation whose isomonodromic deformations are described by that solution.
(iii) The solutions related to supersymmetric conformal field theories are the ones whose Stokes matrices have integer entries. These solutions are expected to correspond to geometric structures, such as quantum cohomology algebras of Kaehler manifolds or unfoldings of isolated singularities. For example, one point (a vertex) of the polytope corresponds to the quantum cohomology of CP^n. Another point (an interior point) corresponds to an unfolding of the A_n singularity.
(iv) It was shown by Guest-Ho that the polytope is the Fundamental Weyl Alcove of G=SL(n+1,C) (more precisely, a subset given by the fixed points of an isometry sigma), and that the vertex corresponding to the quantum cohomology of CP^n (complex projective space) is the origin of the Fundamental Weyl Alcove.
(v) There is a version of the tt*-Toda equations for each complex simple Lie group G. It was shown by Guest-Ho that solutions "near 0", of the tt*-Toda equations for G, correspond to points of the Fundamental Weyl Alcove of G (more precisely, a subset given by the fixed points of a certain isometry). For global solutions, this is conjectured to be a 1:1 correspondence.

Yoshiki Kaneko's research is concerned with the special solutions which have integral Stokes matrices. There is (at present) no practical description of the points of the polytope corresponding to solutions of the tt*-Toda equations which have integral Stokes matrices. For this reason, it is necessary to study examples, case-by-case. The thesis consists of several contributions to this study.

The thesis is arranged as follows. Chapter 1 gives some history and motivation for studying the tt*-Toda equations, and summarizes the results of Guest-Its-Lin. A brief summary of the main results of the thesis is given. Chapter 2 gives more details of the tt*-Toda equations and their relation with Stokes data and quantum cohomology. Chapter 3 starts with a description of the "integer Stokes problem", then sections 3.2,3.3,3.4 present the results of the thesis.

Section 3.2 Kaneko's first result is a complete description of the solutions with integral Stokes matrices which lie in a certain Lie-theoretic 1-parameter family (intersected with the polytope), in the case G=SL(n+1,C). (This is a joint work with D3 student Yudai Hateruma.) For n>2 he proved that there are exactly 4 such solutions. One of these is the special solution whose underlying geometric object is the quantum cohomology algebra of CP^n. The proof uses only elementary number-theoretic calculations, but skill and ingenuity were needed to achieve the desired result. The result is interesting because it suggests that Lie-theoretic methods may lead to a better understanding of the structure of the solutions of the tt*-Toda equations which have integral Stokes matrices.

Section 3.3 Kaneko's second result concerns the tt*-Toda equations for general G. For general G it is natural to ask whether the origin of the Fundamental Weyl Alcove corresponds to the quantum cohomology of some Kaehler manifold (as in the case G=SL(n+1,C)). It is not obvious what answer to expect. (Only very few points of the FWA correspond to quantum cohomology for G=SL(n+1,C), and some correspond to quantum cohomology of non-manifolds, for example orbifolds.) Kaneko discovered the answer: precisely the G/P which are highest weight orbits of minuscule representations.

The proof uses a description of the Stokes matrices of the G-version of the tt*-Toda equations due to Guest-Ho, as well as the work of Guest-Ho on solutions "near 0", of the tt*-Toda equations. It also uses a description of the quantum cohomology algebra of a minuscule flag manifold of G due to Golyshev-Manivel (in special cases) and Lam-Templier (for general G). The proof is not difficult, but the result was not easy to find. It is a worthwhile contribution to research on the tt* equations. Again, the result suggests problems and conjectures for further investigation.

Section 3.4 Kaneko's third result is a comparison of two different methods of classification of solutions of the tt*-Toda equations with integral Stokes matrices. (This is a joint work with D3 student Yudai Hateruma.) One method is the original method of Cecotti and Vafa. This is a "top down" approach, which applies to the general tt* equations and the (hypothetical) Stokes matrices of their (hypothetical) solutions. The other is the "bottom up" method of Guest-Its-Lin, which applies only to the tt*-Toda equations, but uses the actual solutions whose existence has been proved. In the work of Kaneko these methods are compared for n=1,2,3. The classification method of Guest-Its-Lin is based on counting solutions of the tt*-Toda equations; the solutions themselves are nontrivial, but the counting procedure is elementary. The classification method of Cecotti and Vafa is based on counting orbits of Stokes matrices under the action of a braid group; the objects (integer matrices) are elementary, but the counting procedure (enumeration of braid group orbits) is nontrivial, and still an open problem. Although the two methods are quite different, he showed that the classifications agree. These calculations were carried out only for low values of n, but they suggest interesting directions for further study.

Both topics involve rather non-standard ingredients; they are not simply a matter of extending or generalizing known results and methods. In this sense the results are truly original and contribute to an important research area. A wide spectrum of knowledge was required to work on these problems: integrable systems theory, classical o.d.e. theory, differential geometry, Lie groups and representations, as well as some algebraic topology.

It is my impression that Kaneko has worked hard to study a difficult topic, and he is able to communicate his research to other mathematicians. I consider that his PhD research is at a suitable level for the PhD degree.

2023 年 2 月

審査員

主査　早稲田大学　教授　D.Phil.（Oxford University）Martin GUEST

副査　早稲田大学　教授　博士（理学）　早稲田大学　本間　泰史

副査　早稲田大学　教授　博士（理学）　東北大学　池田　岳

副査　日本大学　准教授　博士（理学）　東京工業大学　乙藤　隆史