Symplectic Geometry of the tt*-Toda Equations

tt*-戸田方程式のシンプレクティック幾何

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Acknowledgements

I sincerely acknowledge my supervisor, Professor Martin Guest for his kind support throughout my graduate studies, his patient guidance in this field and giving me many opportunities of various kinds of research activities.

I am happy to acknowledge Professor Alexander Its for his friendly advice and his suggestions regarding the constant problem. I am also grateful to him for inviting me to Indiana University-Purdue University Indianapolis for the first five months of 2020 and two months in 2022.

I am glad to acknowledge the financial support from Mathematics and Physics Unit "Multiscale Analysis, Modelling and Simulation", Top Global University Project, Waseda University. My visits to Indianapolis would not be possible without this support.

I also acknowledge the financial support from Japan Student Services Organization.

Finally, I thank my friends for their emotional support.

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CHAPTER 1

Introduction

The tt* (topological–anti topological fusion) equations arose in the work of Cecotti and Vafa on $d = 2$, $N = 2$ supersymmetric quantum field theories. The tt*-Toda equations are a special case of these equations of "Toda type" (also in the work of Cecotti and Vafa). These equations are not only important in physics but also in mathematics, e.g., quantum cohomology and harmonic maps. The tt*-equations are the equations for a Hermitian metric defined by a real structure of a Frobenius manifold. The quantum cohomology ring is a Frobenius manifold. The product structure (quantum product) of the quantum cohomology of a complex manifold X induces a connection on the trivial bundle $H^{2*}(X) \times H^2(X)$ over $H^2(X)$. This connection is called the Givental connection, a special case of the Dubrovin connection, which is defined for a Frobenius manifold in general. In the case $X = \mathbb{C}P^n$ the flatness condition of the Givental connection and a decomposition of a loop group (DPW method) relate the quantum cohomology to the tt*-Toda equations. The solutions can be considered as (special kinds of) harmonic maps from $H^2(\mathbb{C}P^n) \setminus \{0\} \simeq \mathbb{C}^*$ to the symmetric space $\mathrm{SL}(n+1, \mathbb{R})/\mathrm{SO}(n+1)$.

The simplest nontrivial case of the tt*-Toda equations is the radial reduction of the sinh-Gordon equation, which is a special case of the Third Painlevé equation. It was investigated by McCoy-Tracy-Wu [**26**], and their work has far-reaching consequences. More recently, the tt*-Toda equations were investigated in general and in detail by Guest-Its-Lin [**17**, **18**] and by Mochizuki [**27**, **28**]. Our motivation is to investigate these theories from the symplectic point of view. There are three kinds of important data which characterise the solutions of the tt*-Toda equations [**18**, **14**]: asymptotic data (asymptotics of the solutions), holomorphic data (DPW potential), and monodromy data (connection matrix and Stokes matrix). The asymptotic data and the holomorphic data are essentially the same, but their relation with the monodromy data is highly transcendental, and it is given by the Riemann-Hilbert correspondence. We consider separately the space of asymptotic/holomorphic data and the space of monodromy data, for solutions of the tt*-Toda equations which are smooth in a neighbourhood of 0. We find that each spaces has a symplectic structure.

Our first main result is the following:

Theorem. *The Riemann-Hilbert map is a symplectomorphism.*

In fact we construct explicit canonical coordinates for each symplectic structure, and an explicit 'generating function' which gives rise to the Riemann-Hilbert map.

We also explain the geometrical origin of these symplectic structures. For the asymptotic/holomorphic data, the symplectic structure is of Kirillov-Kostant-Souriau type, for an adjoint orbit of a suitable Lie group (a finite-dimensional subgroup of a loop group). For the monodromy data the symplectic structure is closely related to the symplectic structure on the space of the based rational maps due to Atiyah-Hitchin [**3**].

As in the well-studied case of the Painlevé equations, the monodromy data arises from the isomonodromic property of the tt*-Toda equations. The (holomorphic) symplectic geometry of isomonodromic families of meromorphic connections in general has been thoroughly investigated by Hitchin, Boalch and Yamakawa. It is a complex analogue of the theory of moduli spaces of the flat connections investigated by Atiyah, Bott, Alekseev, Malkin, Meinrenken and other researchers. These authors have shown that the Riemann-Hilbert correspondence is a symplectomorphism in many situations, so our result is another example of this phenomenon. We emphasize, however, the case of the tt*-Toda equations is not covered by the existing theory. Furthermore, our results are more explicit than those of the above authors.

We make essential use of this explicit approach to study the particular family of solutions which are globally smooth. These solutions, and their asymptotic expansions at 0 and infinity, are of particular importance in physics. We observe first that the tt*-Toda equations have a Hamiltonian formulation, with respect to the above symplectic structure. It follows that each solution has an associated 'tau-function', defined up to a multiplicative constant.

Our second main result is the solution of the 'constant problem' for such global solutions of the tt*-Toda equations. This is the problem of evaluating the constant which relates the short-distance and long-distance expansions of the tau function. The constant problem for Painlevé equations has been investigated and solved by several researchers. The problem also has a long history in physics, for example, the strong Szegő theorem by Onsager and Kaufman, the constant in the scaling theory of the two-dimensional Ising model by Tracy, and Dyson's constant.

In some cases these arguments lacked rigour. In the case of the (radial) sine-Gordon equation, a rigorous and direct proof was given recently by Its and Prokhorov [**23**]. Our second main result makes use of their method. Although the formula for the constant is very complicated, we find (as in [23]) that our explicit generating function is the key ingredient in the formula (Theorem 68).

We conclude this introduction with a brief description of the layout of the thesis. In Chapter 2 we review some basic theory for meromorphic o. d. e. (or meromorphic connections) with irregular singularities. We explain how this theory applies to the tt*-Toda equations. In particular we define the asymptotic/holomorphic data and the monodromy data. Finally we present the Hamiltonian formulation of the tt*-Toda equations. In Chapter 3 we review some basic results in symplectic geometry, and we prove our first main theorem, that the Riemann-Hilbert correspondence is a symplectomorphism. We also provide geometric explanations for the symplectic forms used here. In Chapter 4 we describe our solution of the constant problem for global solutions of the tt*-Toda equations.

CHAPTER 2

The tt*-Toda equations

2.1. Basic ODE theory

In this section, we will review the Stokes phenomenon, the Riemann-Hilbert correspondence and isomonodromic deformations. Examples can be seen in the next section for the auxiliary linear ordinary differential equation of the tt*-Toda equations. This section refers mainly to [**6**, **12**].

Let us consider the following local linear ordinary differential equation

$$\frac{d\Psi}{d\zeta} = \left(\sum_{k=-r-1}^{\infty} A_{k+1}\zeta^k \right) \Psi, \zeta \in D_0 \setminus \{0\},\ r > 0, \tag{2.1.1}$$

where D_0 is a disk centered at 0 and ζ is a coordinate on D_0. In what follows, we assume that the leading coefficient A_{-r} has distinct eigenvalues. Let $\Lambda_{-r} = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ and $P^{-1}A_{-r}P = \Lambda_{-r}$.

If we suitably fix the normal form, the formal solution will be unique:

THEOREM 1 ([**12**, **24**]). *Under the above assumptions, equation (2.1.1) has a unique formal solution*

$$\Psi_f(\zeta) = P \left(\sum_{k=0}^{\infty} \Psi_k \zeta^k \right) \zeta^{\Lambda_0} \exp\left(\frac{\Lambda_{-r}}{-r}\zeta^{-r} + \cdots + \frac{\Lambda_{-1}}{-1}\zeta^{-1} \right)$$

where $\Lambda_{-r+1}, \ldots, \Lambda_0$ are diagonal matrices.

This can be proved by substitution and recursively determining the coefficients from the lowest degree, in the following way.

PROOF. The above formal solution corresponds to the following normal form

$$\Psi_f(\zeta) = P \left(\sum_{k=0}^{\infty} Y_k \zeta^k \right) \zeta^{\Lambda_0} \exp\left(\frac{\Lambda_{-r}}{-r}\zeta^{-r} + \cdots + \frac{\Lambda_{-1}}{-1}\zeta^{-1} + \sum_{k=1}^{\infty} \frac{\Lambda_k}{k}\zeta^k \right),$$

where the matrices Y_k, $k > 0$ are off-diagonal, and the matrices Λ_k are diagonal. (For example, $\Psi_1 = Y_1 + \Lambda_1$, $\Psi_2 = Y_1\Lambda_1 + Y_2 + \frac{1}{2}\Lambda_1^2 + \Lambda_2$.) Substituting the above expansion, we have the following recurrence relation

$$\Lambda_{-r+k} + [Y_k, \Lambda_{-r}] = F_{-r+k},\ k > 0,$$

where F_{-r+k} are defined by

$$F_{-r+1} = P^{-1}A_{-r+1}P,$$

$$F_{-r+k} = P^{-1}A_{-r+k}P + H_{-r+k} + \sum_{m=1}^{k-1}\left(P^{-1}A_{-r+k-m}PY_m - Y_m\Lambda_{-r+k-m}\right),$$

and

$$H_{-r+k} = \begin{cases} 0 & k = 2, \dots, r \\ -(k-r)Y_{-r+k} & k = r+1, r+2, \dots. \end{cases}$$

□

Definition 2. The rays in $\mathbb{C}P^1$ given by

$$\arg\zeta = \frac{1}{r}\arg(\alpha_j - \alpha_i) + \frac{\pi}{r}\left(n + \frac{1}{2}\right),\ n = 0, \dots, 2r-1$$

are called Stokes rays and denoted by $l_n^{(i,j)}$ for $i < j$. The sector Ω is called a Stokes sector when Ω contains exactly one Stokes ray for each pair (i, j) with $i < j$.

Theorem 3. *Let Ω be a Stokes sector. Let the principal branch of* $\log$ *in the formal solution be chosen. There exists a unique fundamental solution $\Psi(\zeta)$ of equation (2.1.1) which is asymptotic to the formal solution $\Psi_f(\zeta)$ in Ω, i.e.,*

$$\Psi(\zeta) \sim \Psi_f(\zeta),\ \zeta \to 0,\ \zeta \in \Omega.$$

Definition 4. Fix a Stokes sector Ω. Then the sectors $\Omega_n := e^{i\frac{\pi}{r}(n-1)}\Omega$ are also Stokes sectors. From the above theorem, for each n there exists a unique solution $\Psi_n(\zeta)$ satisfying

$$\Psi_n(\zeta) \sim \Psi_f(\zeta),\ \zeta \to 0,\ \zeta \in \Omega_n.$$

We call the matrix S_n defined by $\Psi_{n+1}(\zeta) = \Psi_n(\zeta)S_n$ a Stokes matrix. (Since the equation is linear, there exists such a matrix. Since Ψ_n is a fundamental solution, the matrix S_n is independent of ζ.)

Definition 5. Let ∇ be a meromorphic connection on a degree 0 holomorphic vector bundle $V \to \mathbb{C}P^1$ with poles $a_1, \dots, a_m$ of respective orders $k_1, \dots, k_m$. And let $\nabla = d - A^i$ in a local coordinate ζ of $\mathbb{C}P^1$ vanishing at a_i where

$$A^i = \sum_{k=-r_i-1}^{\infty} A^i_{k+1}\zeta^k d\zeta$$
$$= \left(A^i_{-(k_i-1)}\frac{1}{\zeta^{k_i}} + A^i_{-(k_i-2)}\frac{1}{\zeta^{k_i-1}} + \cdots + A^i_0\frac{1}{\zeta} + A^i_1 + A^i_2\zeta\cdots \right) d\zeta$$

where $r_i = k_i - 1$. The connection ∇ is said to be nice if the following conditions hold:

- If $k_i \geq 2$, the leading coefficient $A^i_{-(k_i-1)}$ is diagonalisable with distinct eigenvalues.
- If $k_i = 1$, the leading coefficient is diagonalisable with eigenvalues which are distinct mod $\mathbb{Z}$.

A nice system A is a germ of a nice meromorphic connection ∇. A compatibly framed nice connection $(V, \nabla, (P_1, \ldots, P_m))$ consists of a degree 0 holomorphic vector bundle $V \to \mathbb{C}P^1$, a nice meromorphic connection ∇ on V with poles $a_1, \ldots, a_m$, and $P_i \in \mathrm{GL}_n(\mathbb{C})$ diagonalises a local expression A^i ($\nabla = d - A^i$) at a_i for some local trivialisation.

The m-tuple $(\hat{A}^1, \ldots, \hat{A}^m)$ of nice systems is said to be associated to a compatibly framed nice connection $(V, \nabla, (P_1, \ldots, P_m))$ if

- there exists $F \in \mathrm{GL}(\mathbb{C}\,[\![\zeta]\!])$ such that the irregular part of $F[A^i]$ is equal to the irregular part of $\hat{A}^i$ and $F(0) = P_i$ where $F[A] := (dF)F^{-1} + FAF^{-1}$ and $\nabla = d - A^i$ in a local coordinate ζ vanishing at a_i for each i
- ∇ has only poles at a_i.

We denote by $\mathcal{M}_{\mathrm{ext}}(\hat{A})$ (resp. $\mathcal{M}^*_{\mathrm{ext}}(\hat{A}) \subset \mathcal{M}_{\mathrm{ext}}(\hat{A})$) the set of the isomorphism classes of compatibly framed nice connections with a degree 0 (resp. trivialisable) holomorphic vector bundle which have the associated m-tuple $\hat{A}$ of nice systems.

Definition 6. Let $\hat{A}$ be an m-tuple $(\hat{A}^1, \ldots, \hat{A}^m)$ of nice systems. Let $\tilde{C}_i := \mathrm{GL}_n(\mathbb{C}) \times (U_- \times U_+)^{k_i-1} \times \mathfrak{t}$ for each $i = 1, \ldots, m$. If $k_i = 1$, we replace $\mathfrak{t}$ by the subset $\mathfrak{t}' \subset \mathfrak{t}$ of the matrices with eigenvalues distinct mod $\mathbb{Z}$. The extended monodromy manifold of $\hat{A}$ is the set

$$M_{\mathrm{ext}}(\hat{A}) := \left\{ (\mathbf{C}, \mathbf{S}, \mathbf{\Lambda}) \in \tilde{C}_1 \times \cdots \times \tilde{C}_m \middle| \rho(\mathbf{C}, \mathbf{S}, \mathbf{\Lambda}) = 1, \sum_{i=1}^{m} \mathrm{Tr}(\Lambda_i) = 0 \right\} / \mathrm{GL}_n(\mathbb{C})$$

where $\rho : \tilde{C_1} \times \cdots \times \tilde{C_m} \to \mathrm{GL}_n(\mathbb{C})\,;\ \rho = \rho_m \cdot \cdots \cdot \rho_1$ and

$$\rho_i\left(C_i, \mathbf{S}^i, \mathbf{\Lambda}^i\right) = C_i^{-1} S^i_{2k_i-2} \cdots S^i_1 \exp\left(2\pi\sqrt{-1}\Lambda^i\right) C_i$$

and $\mathrm{GL}_n(\mathbb{C})$ acts on $\tilde{C_1} \times \cdots \times \tilde{C_m}$ as the diagonal action of the action on $\tilde{C_i}$ by $g \cdot (C_i, \mathbf{S}^i, \mathbf{\Lambda}^i) = (C_i g^{-1}, \mathbf{S}^i, \mathbf{\Lambda}^i)$.

THEOREM 7 ([6]). *If we fix singular points $a_1, \ldots, a_m \in \mathbb{C}P^1$ and an m-tuple $\hat{A} = (\hat{A}^1, \ldots, \hat{A}^m)$ of nice systems, there is a bijection ν between $\mathcal{M}_{ext}(\hat{A})$ and $M_{ext}(\hat{A})$.*

DEFINITION 8. We call the restriction $\nu|_{\mathcal{M}^*_{\mathrm{ext}}(\hat{A})} : \mathcal{M}^*_{\mathrm{ext}}(\hat{A}) \to \nu\left(\mathcal{M}^*_{\mathrm{ext}}(\hat{A})\right)$ the Riemann-Hilbert-Birkhoff (or briefly Riemann-Hilbert) correspondence.

DEFINITION 9. Let m be a fixed positive integer. Let X be the manifold of the deformation parameters, i.e., the pairs $(\mathbf{a}, \hat{A})$ of m-distinct points $\mathbf{a} = (a_1 = \infty, a_2, \ldots, a_m)$ and a nice system $\hat{A}$. The extended moduli bundle $\mathcal{M}^*_{\mathrm{ext}}$ is the set of isomorphism classes of $(V, \nabla, \mathbf{P}, \mathbf{a})$ consisting of a compatibly framed nice connection $(V, \nabla, \mathbf{P})$ on a rank $n+1$ holomorphic vector bundle V with the poles $a_1, \ldots, a_m$. The isomonodromic deformation is the map $s \circ f : U \to \mathcal{M}^*_{\mathrm{ext}}$ where $f : U \to X$ is a holomorphic map from an open subset $U \subset \mathbb{C}^q$ to X and s is a section of $\mathcal{M}^*_{\mathrm{ext}}$ with respect to the natural projection $\mathcal{M}^*_{\mathrm{ext}} \to X$ which satisfies $\nu \circ s$ is a constant (flat) section of $\nu\left(\mathcal{M}^*_{\mathrm{ext}}\right)$.

Let $\Theta(\zeta, t)$ be the logarithmic derivative $d_U\Psi \cdot \Psi^{-1}$ of the fundamental solution $\Psi(\zeta, t)$ of ∇ entry of $s \circ f(t)$. Let us locally write $s \circ f(t)$ by $\nabla = d - A(\zeta, t)d\zeta$. The map $s \circ f$ is an isomonodromic deformation if and only if the following deformation equations hold:

$$\frac{\partial A}{\partial t_j} = \frac{\partial \Theta_j}{\partial \zeta} + [\Theta_j, A]\,,\ j = 1, \ldots, q \tag{2.1.2}$$

where $\Theta(\zeta, t) = \sum_{j=1}^q \Theta_j(\zeta, t)dt_j$.

Since it follows by definition that

$$\frac{\partial \Theta_k}{\partial t_j} - \frac{\partial \Theta_j}{\partial t_k} = [\Theta_j, \Theta_k]\,,\ j, k = 1, \ldots, q,$$

system (2.1.2) is equivalent to the flatness condition of

$$d - \left(A(\zeta, t)d\zeta + \sum_{j=1}^q \Theta_j(\zeta, t)dt_j\right).$$

2.2. Basics of the tt*-Toda equations

2.2.1. The tt*-Toda equations. Let a positive integer n be fixed. The tt*-Toda equations [17, 18] are

$$2\,(w_i)_{t\bar{t}} = -e^{2(w_{i+1}-w_i)} + e^{2(w_i-w_{i-1})},\ w_i : \mathbb{C}^* \to \mathbb{R},\ i \in \mathbb{Z}, \tag{2.2.1}$$

where, for all i, $w_i = w_{i+n+1}$, $w_i = w_i(|t|)$ $(t \in \mathbb{C}^*)$, and

$$w_0 + w_n = 0, w_1 + w_{n-1} = 0,\ \ldots \text{ (anti-symmetry condition)}.$$

The equations (2.2.1) are equivalent to the flatness of $\nabla := d + \alpha$, i.e. the zero curvature equation $d\alpha + \alpha \wedge \alpha = 0$, where

$$\alpha := \left(w_t + \frac{1}{\lambda}W^T\right)dt + \left(-w_{\bar{t}} + \lambda W\right)d\bar{t},$$

$$w = \begin{pmatrix} w_0 & & \\ & \ddots & \\ & & w_n \end{pmatrix},\ W = \begin{pmatrix} 0 & e^{w_1-w_0} & & \\ & 0 & \ddots & \\ & & \ddots & e^{w_n-w_{n-1}} \\ e^{w_0-w_n} & & & 0 \end{pmatrix}.$$

They are equivalent to $2w_{t\bar{t}} = \left[W^T, W\right]$, which is the compatibility condition for the linear system

$$\begin{cases} \Psi_t = \left(w_t + \frac{1}{\lambda}W\right)\Psi \\ \Psi_{\bar{t}} = \left(-w_{\bar{t}} + \lambda W^T\right)\Psi. \end{cases}$$

The connection form α has three important symmetries:

Proposition 10. *Cyclic symmetry:*

$$d_{n+1}^{-1}\alpha(\lambda)d_{n+1} = \alpha(e^{2\pi\sqrt{-1}/(n+1)}\lambda)$$

Anti-symmetry:

$$-\Delta\alpha(\lambda)^T\Delta = \alpha(-\lambda)$$

Reality:

$$\Delta\overline{\alpha'(\lambda)}\Delta = \alpha''(1/\bar{\lambda}),$$

where $\alpha = \alpha' dt + \alpha'' d\bar{t}$,

$$\Delta = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix},$$

and

$$d_{n+1} = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^n \end{pmatrix}.$$

The tt*-Toda equations are the radial version $(xw_x)_x = 2x\left[W^T, W\right]$ of the above compatibility condition, where $x := |t|$. This is equivalent to the compatibility condition for the linear system

$$\begin{cases} \Psi_\mu = \left(-\frac{1}{\mu^2}xW - \frac{1}{\mu}xw_x + xW^T\right)\Psi \\ \Psi_x = \left(\frac{1}{\mu}W + \mu W^T\right)\Psi, \end{cases}$$

where $\mu = \lambda x/t$. They are equivalent to the isomonodromy condition for the following ordinary differential equation

$$\frac{d\Psi}{d\zeta} = \left(-\frac{1}{\zeta^2}W - \frac{1}{\zeta}xw_x + x^2W^T\right)\Psi, \tag{2.2.2}$$

where $\zeta = \mu/x = \lambda/t$. If we define

$$\hat{\alpha} := \left(-\frac{t}{\lambda^2}W - \frac{1}{\lambda}xw_x + \bar{t}W^T\right)^T d\lambda,$$

the above linear ordinary differential equation is equivalent to $\left(d - \hat{\alpha}^T\right)\Psi = 0$.[1] As we saw in the previous section, the isomonodromy condition for (2.2.2) (equivalently the tt*-Toda equations) is equivalent to the flatness condition of $d + \alpha + \hat{\alpha}$.

2.2.2. The Stokes data at zero and infinity.

PROPOSITION 11 ([18]). *At $\zeta = 0$ we have a formal solution $\Psi_f^{(0)} = e^{-w}\Omega(I + \sum_{i\geq 1}\Psi_i^{(0)}\zeta^i)e^{\frac{1}{\zeta}d_{n+1}}$ of (2.2.2), where*

$$\Omega = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^n \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^n & \omega^{2n} & \cdots & \omega^{n^2} \end{pmatrix},$$

and $\omega := e^{2\pi\sqrt{-1}/(n+1)}$. Note that we have no formal monodromy for equation (2.2.2).

[1] This hat is unrelated to the hat in the previous section.

Proof. Let us calculate by the same way as Theorem (1). In the notation of Theorem (1), $r = 1$, $A_{-1}^{(0)} = -W$, $A_0^{(0)} = -xw_x$, $A_1^{(0)} = x^2 W^T$. Let $P^{(0)} = e^{-w}\Omega$. Then it follows that $P^{(0)-1}A_{-1}^{(0)}P^{(0)} = P^{(0)-1}(-W)P^{(0)} = -d_{n+1} = \Lambda_{-1}^{(0)}$. Let us substitute the formal solution

$$P^{(0)}\left(\sum_{k=0}^{\infty} Y_k^{(0)}\zeta^k\right)\zeta^{\Lambda_0^{(0)}}\exp\left(\frac{\Lambda_{-1}^{(0)}}{-1}\zeta^{-1} + \sum_{k=1}^{\infty}\frac{\Lambda_k}{k}\zeta^k\right)$$

to equation (2.2.2). Comparing the coefficients of $1/\zeta$, we obtain that

$$\Lambda_0^{(0)} + \left[Y_1^{(0)}, \Lambda_{-1}^{(0)}\right] = P^{(0)-1}A_0^{(0)}P^{(0)} = -\Omega^{-1}xw_x\Omega.$$

The diagonal entries of $-\Omega^{-1}xw_x\Omega$ are zero. □

We define the sector

$$\Omega_1^{(0)} := \begin{cases} (-(\frac{1}{n+1}+\frac{1}{2})\pi, \frac{\pi}{2}) & (n+1 \in 2\mathbb{Z}) \\ (-(\frac{1}{2(n+1)}+\frac{1}{2})\pi, (\frac{1}{2(n+1)}+\frac{1}{2})\pi) & (n+1 \in 2\mathbb{Z}+1) \end{cases},$$

where we use the notation $(a, b) := \{\zeta \in \mathbb{C}^* | a < \arg\zeta < b\}$.

We let $\Omega_{k+\frac{1}{n+1}}^{(0)} = e^{-\frac{\pi}{n+1}\sqrt{-1}}\Omega_k^{(0)}$ $(k \in \frac{1}{n+1}\mathbb{Z})$ in the universal covering $\tilde{\mathbb{C}}^*$. These are Stokes sectors of (2.2.2) around $\zeta = 0$. (The eigenvalues of the leading coefficient $-W$ are $1, \omega, \dots, \omega^n$, so the Stokes rays are the rays $l_n^{(i,j)}$ determined by $\arg\zeta = \arg(\omega^j - \omega^i) + \pi(n + 1/2)$, $n = 0, 1$. The sectors $\Omega_k^{(0)}$ $(k \in \frac{1}{n+1}\mathbb{Z})$ contain exactly one Stokes ray for each pair (i, j) with $i < j$.)

Let $\Psi_k^{(0)}$ be the fundamental solution such that $\Psi_k^{(0)} \sim \Psi_f^{(0)}$ on $\Omega_k^{(0)}$.

Similarly, at $\zeta = \infty$, we have the formal solution

$$\Psi_f^{(\infty)} = e^w\Omega^{-1}(I + \sum_{i\geq 1}\Psi_i^{(\infty)}\zeta^{-i})e^{x^2\zeta d_{n+1}}$$

and the sectors

$$\Omega_1^{(\infty)} := \begin{cases} (-\frac{\pi}{2}, (\frac{1}{n+1}+\frac{1}{2})\pi) & (n+1 \in 2\mathbb{Z}) \\ (-(\frac{1}{2(n+1)}+\frac{1}{2})\pi, (\frac{1}{2(n+1)}+\frac{1}{2})\pi) & (n+1 \in 2\mathbb{Z}+1) \end{cases}$$

$$\Omega_{k+\frac{1}{n+1}}^{(\infty)} := e^{\frac{\pi}{n+1}\sqrt{-1}}\Omega_k^{(\infty)}.$$

The sectors $\Omega_k^{(\infty)}$ $(k \in \frac{1}{n+1}\mathbb{Z})$ are Stokes sectors.

Let $\Psi_k^{(\infty)}$ be the fundamental solution such that $\Psi_k^{(\infty)} \sim \Psi_f^{(\infty)}$ on $\Omega_k^{(\infty)}$.

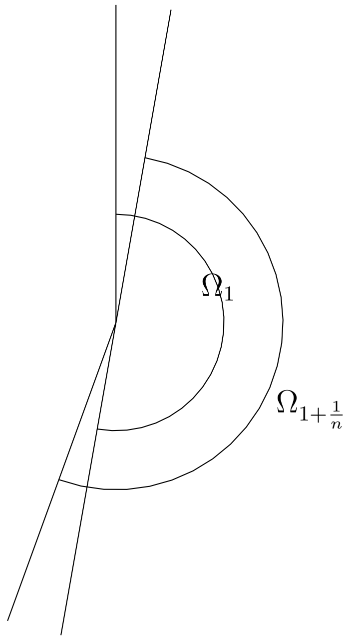

FIGURE 2.2.1. Stokes sectors at $\zeta = 0$ in the case of $n + 1 \in 2\mathbb{Z}$

We define the Stokes matrices $S_k^{(0)}$, $S_k^{(\infty)}$ by $\Psi_{k+1}^{(0)} = \Psi_k^{(0)} S_k^{(0)}$, $\Psi_{k+1}^{(\infty)} = \Psi_k^{(\infty)} S_k^{(\infty)}$.

We define the Stokes factors $Q_k^{(0)}$, $Q_k^{(\infty)}$ by $\Psi_{k+\frac{1}{n+1}}^{(0)} = \Psi_k^{(0)} Q_k^{(0)}$, $\Psi_{k+\frac{1}{n+1}}^{(\infty)} = \Psi_k^{(\infty)} Q_k^{(\infty)}$.

Stokes matrices, factors satisfy the following symmetries:

PROPOSITION 12 ([17]). *Cyclic symmetry:*

$$Q_{k+\frac{2}{n+1}}^{(0)} = \Pi Q_k^{(0)} \Pi^{-1}$$

$$Q_{k+\frac{2}{n+1}}^{(\infty)} = \Pi Q_k^{(\infty)} \Pi^{-1}$$

Anti-symmetry:

$$Q_{k+1}^{(0)} = d_{n+1} Q_k^{(0)-T} d_{n+1}^{-1}$$

$$Q_{k+1}^{(\infty)} = d_{n+1}^{-1} Q_k^{(\infty)-T} d_{n+1}$$

Reality:

$$Q_k^{(0)} = A \bar{Q}_{\frac{2n}{n+1}-k}^{(0)-1} A$$

$$Q_k^{(\infty)} = A \bar{Q}_{\frac{2n}{n+1}-k}^{(\infty)-1} A,$$

where

$$A = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & \iddots & \\ & 1 & & \end{pmatrix}.$$

Stokes factors at zero and infinity are related to each other:

PROPOSITION 13 ([17]).

$$Q_k^{(0)} = d_{n+1} Q_k^{(\infty)} d_{n+1}^{-1}$$

DEFINITION 14. Let $M^{(0)} := Q_1^{(0)} Q_{1+\frac{1}{n+1}}^{(0)} \Pi$ where

$$\Pi = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}.$$

Since the formal monodromy near $\zeta = 0$ of equation (2.2.2) is trivial, the monodromy matrix of $\Psi_1^{(0)}(\zeta)$ around $\zeta = 0$ is $S_1^{(0)} S_2^{(0)}$. From the cyclic symmetry, it holds that

$$\left(M^{(0)}\right)^{n+1} = S_1^{(0)} S_2^{(0)}.$$

The eigenvalues of $M^{(0)}$ are $\omega^{m_i - i}$. It is proved in [17, 18] that the m_i determine all $Q_k^{(0)}$ (and $Q_k^{(\infty)}$). The definition and details of the m_i will be explained in subsubsection 2.2.4.

2.2.3. The connection matrix. We define the connection matrices E_k by $\Psi_k^{(\infty)} = \Psi_k^{(0)} E_k$. Let E_1^{global} be $\frac{1}{n+1} A Q_{\frac{n}{n+1}}^{(\infty)}$.

Many important facts about the connection matrix E_1 are proved in [18]:

THEOREM 15 ([18]). $E_1 = E_1^{global}$ *if the solutions* w_i *are global (defined and smooth on* $\mathbb{C}^*$*). The eigenvalues* $e_i^{\mathbb{R}}$ *of* $E := E_1 \left(E_1^{global}\right)^{-1}$ *determine* E_1*. Thus, the* m_i *and the* $e_i^{\mathbb{R}}$ *determine* E_1*.*

As in the case of the Stokes matrices, the connection matrices also have symmetries:

PROPOSITION 16 ([17]). *Cyclic symmetry:*

$$d_{n+1}^{-1} E_k = \omega \left(Q_k^{(\infty)} Q_{k+\frac{1}{n+1}}^{(\infty)} \Pi \right) d_{n+1}^{-1} E_k \left(Q_k^{(\infty)} Q_{k+\frac{1}{n+1}}^{(\infty)} \Pi \right)$$

Anti-symmetry:

$$d_{n+1}^{-1}E_k = -\frac{1}{(n+1)^2}\left(d_{n+1}^{-1}\Pi^2\right)\left(d_{n+1}^{-1}E_k\right)^{-T}\left(d_{n+1}^{-1}\Pi^2\right)^{-1}$$

Reality:

$$E_k = A\bar{E}_{\frac{2n}{n+1}-k}A$$

PROPOSITION 17.

$$EM^{(0)} = M^{(0)}E$$

PROOF. Noting that $Q_k^{(0)} = d_{n+1}Q_k^{(\infty)}d_{n+1}^{-1}$, from the cyclic symmetry of the connection matrix E_1 we obtain

$$E_1 = \omega\left(Q_1^{(0)}Q_{1+\frac{1}{n+1}}^{(0)}\Pi\right)E_1\left(Q_1^{(\infty)}Q_{1+\frac{1}{n+1}}^{(\infty)}\Pi\right).$$

Since the matrix E_1^{global} is a special value of E_1,

$$E_1^{\text{global}} = \omega\left(Q_1^{(0)}Q_{1+\frac{1}{n+1}}^{(0)}\Pi\right)E_1^{\text{global}}\left(Q_1^{(\infty)}Q_{1+\frac{1}{n+1}}^{(\infty)}\Pi\right).$$

Then

$$\begin{aligned}E &= E_1\left(E_1^{\text{global}}\right)^{-1}\\ &= \left(Q_1^{(0)}Q_{1+\frac{1}{n+1}}^{(0)}\Pi\right)E_1\left(E_1^{\text{global}}\right)^{-1}\left(Q_1^{(0)}Q_{1+\frac{1}{n+1}}^{(0)}\Pi\right)^{-1}\\ &= M^{(0)}E\left(M^{(0)}\right)^{-1}.\end{aligned}$$

□

2.2.4. Solutions near zero. Let us review the DPW (Dorfmeister-Pedit-Wu) method.

PROPOSITION 18 ([13]). *Let $\alpha^\lambda = \frac{1}{2}\left(1-\frac{1}{\lambda}\right)\alpha_1 dz_1 + \frac{1}{2}(1-\lambda)\alpha_2 dz_2$, where α_1, $\alpha_2 : \mathbb{C}^2 \to \mathfrak{g}^{\mathbb{C}}$. If $d + \alpha^\lambda$ is flat for all $\lambda \in \mathbb{C}^*$, then there exists $\phi : \mathbb{C}^2 \to G^{\mathbb{C}}$ such that $\alpha_1 = \phi^{-1}\partial_1\phi$, $\alpha_2 = \phi^{-1}\partial_2\phi$, and we have $\partial_1\left(\phi^{-1}\partial_2\phi\right) + \partial_2\left(\phi^{-1}\partial_1\phi\right) = 0$. Conversely, let $\phi : \mathbb{C}^2 \to G^{\mathbb{C}}$ satisfy this equation. Then $\alpha^\lambda = \frac{1}{2}\left(1-\frac{1}{\lambda}\right)\left(\phi^{-1}\partial_1\phi\right)dz_1 + \frac{1}{2}(1-\lambda)\,\phi^{-1}\partial_2\phi dz_2$ satisfies $d\alpha^\lambda + \alpha^\lambda \wedge \alpha^\lambda = 0$ for any $\lambda \in \mathbb{C}^*$.*

DEFINITION 19. Let $\gamma : S^1 \to \mathrm{GL}_n\,\mathbb{C}$, $\lambda \mapsto \gamma(\lambda) = \sum_{i=-\infty}^{\infty} A_i\lambda^i$ be a smooth map, i.e., an element of the smooth loop group of $\mathrm{GL}_n\,\mathbb{C}$. For any such γ sufficiently near to the identity of the loop group, we have $\gamma = \gamma_-\gamma_+$, where $\gamma_-, \gamma_+ : S^1 \to \mathrm{GL}_n\,\mathbb{C}$ are smooth maps and $\gamma_- = \sum_{i\geq 0} B_i\lambda^{-i}$, $\gamma_+ = \sum_{i\geq 0} C_i\lambda^i$. The factorisation is unique if we impose a basepoint condition, for example $\gamma_-(\infty) = I$ (equivalently $B_0 = I$). We call this decomposition the Birkhoff factorisation of γ. (Of course, we can similarly have a decomposition $\gamma = \check{\gamma}_+ \check{\gamma}_-$. We call this the opposite factorisation.)

PROPOSITION 20 (Krichever, Dorfmeister-Pedit-Wu, Segal). *Let*

$$\alpha^\lambda = \frac{1}{2}\left(1 - \frac{1}{\lambda}\right)\alpha_1 dz_1 + \frac{1}{2}\left(1-\lambda\right)\alpha_2 dz_2,$$

where $\alpha_1, \alpha_2 : \mathbb{C}^2 \to \mathrm{GL}_n\mathbb{C}$. *Take a map* $F = F(z_1, z_2, \lambda) \in \mathrm{GL}_n\mathbb{C}$ *such that* $F^{-1}dF = \alpha^\lambda$. *Let* $F = F_- F_+$ *be the Birkhoff factorisation of* F, *and let* $F = \check{F}_+ \check{F}_-$ *be the opposite factorisation . Then*

(1) $F_- = F_-(z_1, \lambda)$
(2) $\check{F}_+ = \check{F}_+(z_2, \lambda)$
(3) $F = \check{F}_+\left(\check{F}_+^{-1}F_-\right)_-$

PROOF. Let us prove (3) first. Since $F = F_-F_+ = \check{F}_+\check{F}_-$, we obtain $\check{F}_- = \check{F}_+^{-1}F_-F_+ = \left(\check{F}_+^{-1}F_-F_+\right)_- = \left(\check{F}_+^{-1}F_-\right)_-$. Thus $F = \check{F}_+\check{F}_- = \check{F}_+\left(\check{F}_+^{-1}F_-\right)_-$.

Next we will show (2).

$$\begin{aligned}\check{F}_+^{-1}\partial_1\check{F}_+ &= \left(F\check{F}_-^{-1}\right)^{-1}\partial_1\left(F\check{F}_-^{-1}\right)\\ &= \check{F}_-F^{-1}\left((\partial_1 F)\check{F}_-^{-1} + F\partial_1\check{F}_-^{-1}\right)\\ &= \check{F}_-\left(F^{-1}\partial_1 F\right)\check{F}_-^{-1} + \check{F}_-\partial_1\check{F}_-^{-1}.\end{aligned}$$

The left hand side of the last equality is of the form $\sum_{i\geq 1} M_i\lambda^i$, and the right hand side is of the form $\sum_{i\geq 1} N_i\lambda^i$. Thus the both side are zero, so $\partial_1\check{F}_+ = 0$. The proof of (1) is similar. □

PROPOSITION 21 (Krichever). *The maps* F_-, $\check{F}_+$ *satisfy*

$$\begin{cases} F_-^{-1}\frac{dF_-}{dz_1} = \frac{1}{\lambda}\omega_1(z_1)\\ \check{F}_+^{-1}\frac{d\check{F}_+}{dz_2} = \lambda\omega_2(z_2)\end{cases}$$

PROOF. It holds that

$$F_-^{-1}\frac{\partial F_-}{\partial z_1} = \sum_{i\geq 1} K_i\lambda^{-i},$$

where K_i only depends on z_1. We will show that $K_i = 0$ for $i \geq 2$. From $F = F_- F_+$ we obtain

$$\begin{aligned} F_-^{-1}\partial_1 F_- &= \left(FF_+^{-1}\right)^{-1}\partial_1\left(FF_+^{-1}\right) \\ &= F_+F^{-1}\left(\partial_1 F F_+^{-1} + F\partial_1 F_+^{-1}\right) \\ &= F_+F^{-1}\partial_1 F F_+^{-1} + F_+\partial_1 F_+^{-1} \\ &= \sum_{i\geq -1} D_i\lambda^i \quad \left(F^{-1}\partial_1 F = \frac{1}{2}\left(1-\frac{1}{\lambda}\right)\alpha_1,\ F_+ = \sum_{i\geq 0} B_i\lambda^i\right). \end{aligned}$$

Hence $F_-^{-1}\partial_1 F_- = K_1\lambda^{-1}$. Putting $K_1 = \omega_1$ completes the proof. □

Similar arguments hold if we replace z_1 and z_2 by z and $\bar{z}$.

As a summary of the above discussion, we obtain the following theorem:

THEOREM 22 ([**13**]). *The following correspondences are essentially one-to-one:*

$$\phi : \mathbb{R}^2 \to G \longleftrightarrow F(z,\bar{z},\lambda) \longleftrightarrow F_-(z,\lambda)(\mathit{or}\ \check{F}_+(\bar{z},\lambda)) \longleftrightarrow \omega(z) \in \mathfrak{g}^{\mathbb{C}}$$

with $\phi = F|_{\lambda=-1}$, $\phi^{-1}d\phi = \alpha^{-1}$, $F^{-1}dF = \alpha^{\lambda}$ *and* $\omega = F_-^{-1}\frac{\partial F_-}{\partial z}$.

We call $\frac{1}{\lambda}\omega(z)dz$ *the normalised DPW potential.*

We can obtain $\omega = \frac{1}{\lambda}\eta dz$ from α by using some of the above correspondences, where

$$\eta = \begin{pmatrix} & & & p_0 \\ p_1 & & & \\ & \ddots & & \\ & & p_n & \end{pmatrix}$$

and $z = c^{\frac{1}{N}}\left(\frac{N}{n+1}t\right)^{\frac{n+1}{N}}$, $N = \sum_{i=0}^{n}(k_i+1)$, $p_i(z) = c_i z^{k_i}$ with $c_i > 0$, $k_i \geq -1$ and $p_i = p_{n-i+1}$. We call the data c_i, k_i the holomorphic data. We refer to the case $k_i > -1$ for all i as the generic case.

Let us briefly see how the Iwasawa factorisation relates ω to α. (See subsection 2.2 of [**18**] for detail.) Let $L = L(z,\lambda)$ be the unique local holomorphic solution of the ordinary differential equation

$$L^{-1}L_z = \frac{1}{\lambda}\eta,\ \ L|_{z=z_0} = I.$$

Let $L = L_{\mathbb{R}}L_+$ be the Iwasawa factorisation of L, where $c\left(L_{\mathbb{R}}(z,\bar{z},1/\bar{\lambda})\right) = L_{\mathbb{R}}(z,\bar{z},\lambda)$, and $L_+(z,\bar{z},\lambda) = \sum_{i=0}^{\infty} L_i(z,\bar{z})\lambda^i$, $L_0 = \mathrm{diag}(b_0,\dots,b_n)$, $b_i > 0$. Such C^∞ maps $L_{\mathbb{R}}$ and L_+ exist and are unique in some neighbourhood of $z = z_0$. We can construct a family of solutions by using

$$\alpha = (L_{\mathbb{R}} G_h)^{-1} d (L_{\mathbb{R}} G_h),$$

where $G_h = \mathrm{diag}(|h_0|/h_0, \ldots, |h_n|/h_n)$ and the h_i are holomorphic functions with the following conditions:

- $p_0 h_0/h_n = p_1 h_1/h_0 = \cdots = p_n h_n/h_{n-1}$
- $h_0 h_n = 1, h_1 h_{n-1} = 1, \ldots$

Let us define $m_0, \ldots, m_n$ by

$$m_{i-1} - m_i = -1 + \frac{n+1}{N}(k_i + 1).$$

In the generic case we may take $z_0 = 0$, and then the local solutions near $x = 0$ of the tt*-Toda equations are parametrised by real numbers γ_i, ρ_i as follows [18, 27, 28]:

$$2w_i(x) = \gamma_i \log x + \rho_i + o(1) \quad \text{as } x \to 0. \tag{2.2.3}$$

We call the parameters γ_i, ρ_i the asymptotic data. It is proved in [18] that $\gamma_i = -2m_i$ and that the ρ_i can be expressed in terms of the m_i and c_i (see Theorem 26 and Example 27). In the generic case we have $-2 < \gamma_{i+1} - \gamma_i < 2$; the general case has $-2 \le \gamma_{i+1} - \gamma_i \le 2$. We assume the generic condition from now on.

As in the case of $\hat{\alpha}$, we can define $\hat{\omega}$ by

$$\hat{\omega} := \left(-\frac{n+1}{N} \frac{z}{\lambda^2} \eta + \frac{1}{\lambda} m \right) d\lambda.$$

The connection $d + \omega + \hat{\omega}$ is also flat. We can also define the associated linear ordinary differential equation

$$\frac{d\Phi}{d\lambda} = \left(-\frac{n+1}{N} \frac{z}{\lambda^2} \eta^T + \frac{1}{\lambda} m \right) \Phi,$$

a formal solution $\Phi_f^{(0)} = h\Omega (I + O(\lambda)) e^{\frac{t}{\lambda} d_{n+1}}$ and fundamental solutions $\Phi_k^{(0)}$ such that $\Phi_k^{(0)} \sim \Phi_f^{(0)}$ as $\lambda \to 0$ in $\Omega_k^{(0)}$. The sectors $\Omega_k^{(0)}$ can be taken to be the same as for equation (2.2.2) since $-\frac{n+1}{N} \frac{z}{\lambda^2} \eta^T = -\frac{t}{\lambda^2} h\Omega d_{n+1} (h\Omega)^{-1}$. Putting $\zeta = t/\lambda$ gives

$$\frac{d\Phi}{d\zeta} = \left(-\frac{1}{\zeta^2} \eta^T + \frac{1}{\zeta} m \right) \Phi, \tag{2.2.4}$$

$\Phi_f^{(0)} = h\Omega (I + O(\zeta)) e^{\frac{1}{\zeta} d_{n+1}}$. Equation (2.2.4) has a regular singularity at $\zeta = \infty$. Since we assume the genericity condition $m_{i-1} - m_i + 1 > 0$, equation (2.2.4) has a unique solution of the form

$$\Phi^{(\infty)}(\zeta) = \left(I + O(\zeta^{-1}) \right) \zeta^m.$$

If $m_{i-1} - m_i + 1 \geq 0$, it is possible that $\Phi^{(\infty)}(\zeta) = (I + O(\zeta^{-1}))\,\zeta^m \zeta^M$ with some nilpotent matrix M. We define the connection matrices D_k by $\Phi^{(\infty)} = \Phi_k^{(0)} D_k$.

The following theorem is an important consequence of the Iwasawa factorisation:

THEOREM 23 ([18]). *The Stokes factors of equation (2.2.4) around $\zeta = 0$ are the same as those of equation (2.2.2), i.e.,* $\Phi^{(0)}_{k+\frac{1}{n+1}} = \Phi_k^{(0)} Q_k^{(0)}$.

COROLLARY 24 ([18]). $D_k^{-1}\left(Q_k^{(0)} Q^{(0)}_{k+\frac{1}{n+1}} \Pi\right) D_k = d_{n+1}^{-1} e^{2\pi\sqrt{-1}m/(n+1)}$.

PROOF. By direct calculation, we obtain

$$\begin{aligned}
\Phi^{(\infty)}(\zeta) d_{n+1} e^{-2\pi\sqrt{-1}m/(n+1)} &= d_{n+1}\Phi^{(\infty)}(\omega^{-1}\zeta) \\
&= d_{n+1}\Phi_k^{(0)}(\omega^{-1}\zeta) D_k \\
&= \Phi^{(0)}_{k-\frac{2}{n+1}}(\zeta)\Pi^{-1} D_k \\
&= \Phi^{(\infty)}(\zeta) D_k^{-1} Q^{(0)-1}_{k-\frac{1}{n+1}} Q^{(0)-1}_{k-\frac{2}{n+1}} \Pi^{-1} D_k.
\end{aligned}$$

The cyclic symmetry of Stokes factors $Q_k^{(0)}$ gives

$$\begin{aligned}
\left(Q^{(0)-1}_{k-\frac{1}{n+1}} Q^{(0)-1}_{k-\frac{2}{n+1}} \Pi^{-1}\right)^{-1} &= \Pi Q^{(0)}_{k-\frac{2}{n+1}} Q^{(0)}_{k-\frac{1}{n+1}} \\
&= \Pi Q^{(0)}_{k-\frac{2}{n+1}} \Pi^{-1} \Pi Q^{(0)}_{k-\frac{1}{n+1}} \Pi^{-1} \Pi \\
&= Q_k^{(0)} Q^{(0)}_{k+\frac{1}{n+1}} \Pi.
\end{aligned}$$

□

The corollary shows the eigenvalues of $M^{(0)} = Q_1^{(0)} Q^{(0)}_{1+\frac{1}{n+1}} \Pi$ are $\omega^{m_i - i}$.

THEOREM 25 ([18, 19]). *The matrix E is diagonalisable, i.e., conjugate to*

$$\operatorname{diag}\left(e_0^{\mathbb{R}}, \dots, e_n^{\mathbb{R}}\right), \; (e_i^{\mathbb{R}} e_{n-i}^{\mathbb{R}} = 1).$$

The data m_i, $e_i^{\mathbb{R}}$ are called the monodromy data. The proof in [18] is for the case $n = 3$, but exactly the same method provides the results of Theorem 26 and 28 below for general n.

THEOREM 26. [**18**, **19**] *The monodromy data m_i, $e_i^{\mathbb{R}}$ may be expressed in terms of the asymptotic data as follows:*

$$m_i = -\frac{1}{2}\gamma_i,$$

$$e_i^{\mathbb{R}} = \begin{cases} e^{\rho_i}(n+1)^{\gamma_i}\dfrac{X_{n-i}(\gamma_0,\dots,\gamma_{(n-1)/2},-\gamma_{(n-1)/2},\dots,-\gamma_0)}{X_i(\gamma_0,\dots,\gamma_{(n-1)/2},-\gamma_{(n-1)/2},\dots,-\gamma_0)} & n: odd \\ e^{\rho_i}(n+1)^{\gamma_i}\dfrac{X_{n-i}(\gamma_0,\dots,\gamma_{(n-2)/2},0,-\gamma_{(n-2)/2},\dots,-\gamma_0)}{X_i(\gamma_0,\dots,\gamma_{(n-2)/2},0,-\gamma_{(n-2)/2},\dots,-\gamma_0)} & n: even \end{cases},$$

where

$$X_k(\gamma_0,\dots,\gamma_n) := \prod_{j=1}^{n}\Gamma(\frac{-\gamma_k+\gamma_{k+j}+2j}{2(n+1)})\ (\gamma_{j+n+1}=\gamma_j).$$

EXAMPLE 27. We can translate the above formulae in the case $n = 3$ as follows.

$$\rho_0 = \log e_0^{\mathbb{R}} + (4\log 2)m_0 - \log\frac{\Gamma(\frac{-2m_0+1}{4})\Gamma(\frac{-m_0-m_1+2}{4})\Gamma(\frac{-m_0+m_1+3}{4})}{\Gamma(\frac{m_0-m_1+1}{4})\Gamma(\frac{m_0+m_1+2}{4})\Gamma(\frac{2m_0+3}{4})} \tag{2.2.5}$$

$$\rho_1 = \log e_1^{\mathbb{R}} + (4\log 2)m_1 - \log\frac{\Gamma(\frac{-m_1+m_0+1}{4})\Gamma(\frac{-m_1-m_0+2}{4})\Gamma(\frac{-2m_1+3}{4})}{\Gamma(\frac{2m_1+1}{4})\Gamma(\frac{m_1+m_0+2}{4})\Gamma(\frac{m_1-m_0+3}{4})} \tag{2.2.6}$$

Global solutions can be parametrised only by the γ_i (or only by the m_i), that is, for global solutions the ρ_i are determined by the γ_i:

THEOREM 28. [**18**] *For global solutions (i.e. solutions which are smooth for $0 < x < \infty$) we have*

$$\rho_i = -\gamma_i\log(n+1) + \log(X_i/X_{n-i}),$$

i.e. $e_i^{\mathbb{R}} = 1$.

2.2.5. The Hamiltonian formulation. Next, we introduce a Hamiltonian function and a symplectic form.

PROPOSITION 29. *Let $\lfloor x\rfloor := \max\{n \in \mathbb{Z} : n \le x\}$ for $x \in \mathbb{R}$. The tt*-Toda equations can be written as a non-autonomous Hamiltonian system*

$$(w_i)_x = \frac{\partial H}{\partial \tilde{w}_i} = \frac{\tilde{w}_i}{x} \tag{2.2.7}$$

$$(\tilde{w}_i)_x = -\frac{\partial H}{\partial w_i} = -2x\left(e^{2(w_{i+1}-w_i)} - e^{2(w_i-w_{i-1})}\right) \tag{2.2.8}$$

on the phase space

$$\mathbb{R}^{2\lfloor(n-1)/2\rfloor+2} = \{(w,\tilde{w})\}\,(w = (w_0,\dots,w_{\lfloor(n-1)/2\rfloor}), \tilde{w} = (\tilde{w}_0,\dots,\tilde{w}_{\lfloor(n-1)/2\rfloor}))$$

equipped with the symplectic structure

$$\theta := \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} dw_i \wedge d\tilde{w}_i,$$

where the Hamiltonian H is defined by

$$\begin{aligned} H(w,\tilde{w};x) :=& \frac{1}{2x} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \tilde{w}_i^2 - x \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} e^{2(w_i - w_{i-1})} \\ & - \frac{x}{2}\left(e^{4w_0} + \left(1+\delta_{(n-1)/2,\lfloor (n-1)/2 \rfloor}\right) e^{-2\left(1+\delta_{(n-1)/2,\lfloor (n-1)/2 \rfloor}\right) w_{\lfloor (n-1)/2 \rfloor}} \right) \\ =& \begin{cases} \frac{1}{2x} \sum_{i=0}^{(n-1)/2} \tilde{w}_i^2 - x \sum_{i=1}^{(n-1)/2} e^{2(w_i - w_{i-1})} - \frac{x}{2}\left(e^{4w_0} + e^{-4w_{(n-1)/2}}\right) & n : odd \\ \frac{1}{2x} \sum_{i=0}^{(n-2)/2} \tilde{w}_i^2 - x \sum_{i=1}^{(n-2)/2} e^{2(w_i - w_{i-1})} - x e^{-2w_{(n-2)/2}} - \frac{x}{2} e^{4w_0} & n : even \end{cases} \end{aligned}$$

Proof. We consider a curve $x \mapsto (w(x), \tilde{w}(x))$ in $\mathbb{R}^{2\lfloor (n-1)/2 \rfloor + 2}$. From radial symmetry, we have

$$4\left(w_i\right)_{t\bar{t}} = \left(w_i\right)_{xx} + \frac{1}{x}\left(w_i\right)_x .$$

Then we obtain

$$4x\left(w_i\right)_{t\bar{t}} = \left(x\left(w_i\right)_x\right)_x = \left(\tilde{w}_i\right)_x = -2x\left(e^{2(w_{i+1}-w_i)} - e^{2(w_i - w_{i-1})}\right).$$

Hamilton's equations (2.2.7) and (2.2.8) follow immediately from this. □

Remark 30. The Hamiltonian system may be written in terms of $X := \log x$ as follows:

$$\begin{aligned} H(w,\tilde{w};X) :=& \frac{1}{2e^X} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \tilde{w}_i^2 - e^X \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} e^{2(w_i - w_{i-1})} \\ & - \frac{e^X}{2}\left(e^{4w_0} + \left(1+\delta_{(n-1)/2,\lfloor (n-1)/2 \rfloor}\right) e^{-2\left(1+\delta_{(n-1)/2,\lfloor (n-1)/2 \rfloor}\right) w_{\lfloor (n-1)/2 \rfloor}} \right) \end{aligned}$$

$$\begin{aligned} \left(w_i\right)_X &= \frac{\partial e^X H}{\partial \tilde{w}_i} = \tilde{w}_i \\ \left(\tilde{w}_i\right)_X &= -\frac{\partial e^X H}{\partial w_i} = -2e^{2X}\left(e^{2(w_{i+1}-w_i)} - e^{2(w_i - w_{i-1})}\right) \end{aligned}$$

This is the form which is usually stated in textbooks on symplectic geometry.

CHAPTER 3

Symplectic geometry

3.1. Preliminaries

3.1.1. Generating function. First, we will review the physical, Euclidean formulation of the notion of generating function of a symplectomorphism.

DEFINITION 31 (Generating function). Let $F_3 : \mathbb{R}^{2n} \to \mathbb{R}$,

$$(p, Q) = (p_1, \ldots, p_n, Q_1, \ldots, Q_n) \mapsto F_3(p, Q)$$

be a smooth function with

$$\det \left(\frac{\partial^2 F_3}{\partial p_i \partial Q_j} \right)_{i,j} \neq 0.$$

Let $q_i := -\frac{\partial F_3}{\partial p_i}$, $P_i := -\frac{\partial F_3}{\partial Q_i}$. Then by the implicit function theorem, we obtain a local diffeomorphism

$$(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n) \mapsto (Q, P) = (Q_1, \ldots, Q_n, P_1, \ldots, P_n).$$

This is a canonical transformation since $0 = ddF_3 = d\sum_{i=1}^n \left(\frac{\partial F_3}{\partial p_i} dp_i + \frac{\partial F_3}{\partial Q_i} dQ_i \right) = d\sum_{i=1}^n (-q_i dp_i - P_i dQ_i) = \sum_{i=1}^n (-dq_i \wedge dp_i - dP_i \wedge dQ_i)$. The function F_3 is called a generating function. There are many types of generating functions. The following four types are basic:

assumption	derivatives
$\det \left(\frac{\partial^2 F_1(q,Q)}{\partial q_i \partial Q_j} \right)_{i,j} \neq 0$	$p_i = \frac{\partial F_1}{\partial q_i}$, $P_i = -\frac{\partial F_1}{\partial Q_i}$
$\det \left(\frac{\partial^2 F_2(q,P)}{\partial q_i \partial P_j} \right)_{i,j} \neq 0$	$p_i = \frac{\partial F_2}{\partial q_i}$, $Q_i = \frac{\partial F_2}{\partial P_i}$
$\det \left(\frac{\partial^2 F_3(p,Q)}{\partial p_i \partial Q_j} \right)_{i,j} \neq 0$	$q_i = -\frac{\partial F_3}{\partial p_i}$, $P_i = -\frac{\partial F_3}{\partial Q_i}$
$\det \left(\frac{\partial^2 F_4(p,P)}{\partial p_i \partial P_j} \right)_{i,j} \neq 0$	$q_i = -\frac{\partial F_4}{\partial p_i}$, $Q_i = \frac{\partial F_4}{\partial P_i}$

If we obtain one of the above generating functions, we can calculate other types of generating function using $F_1 = F_2 - QP = F_3 + qp = F_4 + qp - QP$. We will only use the type F_3 in what follows.

Next, let us review the geometric point of view.

Let (M_1, ω_1) and (M_2, ω_2) be exact symplectic manifolds, i.e., ω_1 and ω_2 are nondegenerate exact 2-forms. Let $\varphi : M_1 \to M_2$ be a diffeomorphism. Let $\alpha \in \Omega^1(M_1 \times M_2)$ with $-d\alpha = -\operatorname{pr}_1^* \omega_1 + \operatorname{pr}_2^* \omega_2$. Let $\operatorname{gr}_\varphi : M_1 \to M_1 \times M_2, p \mapsto (p, \varphi(p))$ be the graph of φ.

Definition 32. A function $S : M_1 \to \mathbb{R}$ is said to be an α-generating function of φ if $dS = \operatorname{gr}_\varphi^* \alpha$.

Proposition 33. *If there exists an α-generating function S of φ, φ is a symplectomorphism.*

Proof. By direct calculation, we obtain

$$\begin{aligned}
\operatorname{id}_{M_1}^* \omega_1 - \varphi^* \omega_2 &= \left(\operatorname{pr}_1 \circ \operatorname{gr}_\varphi\right)^* \omega_1 - \left(\operatorname{pr}_2 \circ \operatorname{gr}_\varphi\right)^* \omega_2 \\
&= \operatorname{gr}_\varphi^* \left(\operatorname{pr}_1^* \omega_1 - \operatorname{pr}_2^* \omega_2\right) \\
&= d \operatorname{gr}_\varphi^* \alpha \\
&= ddS \\
&= 0.
\end{aligned}$$

□

3.1.2. Quasi-Poisson manifold and Quasi-Hamiltonian space. Let $G = K^{\mathbb{C}}$ be a complex reductive Lie group, i.e., the complexification of a compact real Lie group K. Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras $\operatorname{Lie}(G), \operatorname{Lie}(K)$ of G, K respectively. Let β be a nondegenerate ad-invariant symmetric $\mathbb{C}$-bilinear form on $\mathfrak{g}$ such that $\beta(X, Y) \in \mathbb{R}$ for any $X, Y \in \mathfrak{k}$.

Definition 34. A triple (M, A, π) is a quasi-Poisson G-manifold if M is a complex manifold, $A : G \times M \to M$ is a holomorphic action of G on M, and $\pi \in \Gamma(\bigwedge^2 TM)$ is a G-invariant bivector field such that $[\pi, \pi] = \varphi_M$, where the bracket is the Schouten-Nijenhuis bracket and φ_M is the 3-vector field induced by the Cartan 3-vector $\varphi \in \wedge^3 \mathfrak{g}$, which is the dual of the invariant trilinear form $\eta \in \wedge^3 \mathfrak{g}^*$ defined by $\eta(x, y, z) = \frac{1}{12}\beta(x, [y, z])$ for any $x, y, z \in \mathfrak{g}$.

Remark 35. If G is abelian, then $\varphi_M = 0$, so a quasi-Poisson G-manifold (M, A, π) is a Poisson G-manifold, i.e., π is a G-invariant Poisson structure.

Definition 36. A quasi-Poisson G-manifold (M, A, π) is said to be nondegenerate if $\pi^\sharp \oplus \rho : T^*M \oplus \mathfrak{g} \to TM$, $(\alpha, X) \mapsto \pi^\sharp(\alpha) + \rho(X)$ is surjective, where $\pi^\sharp : T^*M \to TM$, $\alpha \mapsto \pi(\alpha, \cdot)$ and ρ is the infinitesimal action $\rho : M \times \mathfrak{g} \to TM$, $(p, X) \mapsto (X_M)_p$ (X_M is the fundamental vector field of X).

Let us introduce complex (respectively real) quasi-Hamiltonian G-space (respectively K-space) for a complex reductive Lie group G (respectively a compact real Lie group K).

Definition 37 ([2]). A quadruple (M, A, ω, μ) is a complex quasi-Hamiltonian G-space if M is a complex manifold, $A : G \times M \to M$ is a holomorphic action of G on M, $\mu : M \to G$ is a G-equivariant moment map, ω is a G-invariant holomorphic 2-form on M such that

(1)
$$d\omega = -\mu^*\chi,$$

(2) for any $X \in \mathfrak{g}$,
$$\omega(v_X, \cdot) = \frac{1}{2}\mu^*\beta(\theta_G^L + \theta_G^R, X),$$

(3) for any $x \in M$,
$$\operatorname{Ker}\omega_x^\flat = \left\{ (v_X)_x \middle| X \in \operatorname{Ker}\left(\operatorname{Ad}_{\mu(x)} + \operatorname{id}_{\mathfrak{g}}\right)\right\},$$

where
$$\chi := \frac{1}{12}\beta(\theta_G^L, [\theta_G^L \wedge \theta_G^L]) = \frac{1}{12}\beta(\theta_G^R, [\theta_G^R \wedge \theta_G^R]) \in \Omega^3(G),$$
$$\omega_x^\flat : T_xM \to T_x^*M,\ v \mapsto \omega_x(v, \cdot),$$
and θ_G^L (resp. θ_G^R) is the left (resp. right) Maurer-Cartan form of G. If we replace G by K, complex by real, and holomorphic by smooth, we call the quadruple a real quasi-Hamiltonian K-space.

As any nondegenerate Poisson G-manifold can be considered as a Hamiltonian G-space, any nondegenerate Hamiltonian quasi-Poisson manifold can be considered as a quasi-Hamiltonian space:

Theorem 38 ([1]). *For any nondegenerate Hamiltonian quasi-Poisson G-manifold (M, A, π, Φ) there is a unique 2-form $\omega \in \Omega^2(M)$ such that (M, A, ω, Φ) is a quasi-Hamiltonian G-space with*

$$\pi^\sharp \circ \omega^\flat = \operatorname{id}_{TM} - \frac{1}{4}\sum_a (e_a)_M \circ \Phi^*\left(\theta_{G,a}^L - \theta_{G,a}^R\right) =: C, \tag{3.1.1}$$

where $\{e_a\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to β and $\theta_{G,a}^L, \theta_{G,a}^R$ are components of θ_G^L, θ_G^R respectively. Conversely, for any quasi-Hamiltonian G-space (M, A, ω, Φ) there is a unique bivector field π such that (M, A, π, Φ) is a nondegenerate Hamiltonian quasi-Poisson G-manifold with equation (3.1.1).

DEFINITION 39. Let V be a finite dimensional vector space. A linear Dirac structure on V is a maximally isotropic subspace $L \subset V \oplus V^*$ with respect to the symmetric pairing

$$\langle (X,\alpha), (Y,\beta)\rangle := \alpha(Y) + \beta(X).$$

An almost Dirac structure on a smooth manifold M is a subbundle $L \subset TM \oplus T^*M$ defining a linear Dirac structure on T_pM at each point $p \in M$. Let $\phi \in \Omega^3(M)$ be a closed 3-form on M. A ϕ-twisted Dirac structure on M is an almost Dirac structure $L \subset TM \oplus T^*M$ on M whose space $\Gamma(L)$ of sections is closed under the ϕ-twisted Courant bracket

$$[(X,\alpha), (Y,\beta)]_\phi := \left([X,Y], \mathcal{L}_X\beta - \iota_Y d\alpha + \phi(X,Y,\cdot)\right).$$

THEOREM 40 ([4]). *A Hamiltonian quasi-Poisson structure π on G-manifold M corresponds to the $-\Phi^*\eta$-twisted Dirac structure*

$$L_M = \left\{ \left(\pi^\sharp(\alpha) + \rho(X), C^*(\alpha) + \Phi^*\sigma(X)\right) \middle|\, \alpha \in T^*M, \xi \in \mathfrak{g} \right\} \subset TM \oplus T^*M,$$

where $C = \pi^\sharp \circ \omega^\flat$ (more precisely, the right hand side of (3.1.1)) and σ is defined by

$$\begin{aligned} \sigma :& \mathfrak{g} \to \Gamma(T^*G) \\ & X \mapsto \frac{1}{2}\left(X^L + X^R\right)^\vee. \end{aligned}$$

The dual in the definition of σ is under the isomorphism $TG \cong T^*G$ induced by β.

DEFINITION 41. Let T be a maximal torus of G. Let $c \in N(T)/T$ be a Coxeter element. (A Coxeter element is a product of all simple reflections, which depends on the order of the simple reflections and is unique up to conjugation.) Let $\dot{c} \in N(T)$ be a representative of c. Let B_+, B_- be a pair of opposite Borel subgroups containing T and U_+, U_- be their corresponding unipotent radicals. We call

$$\Sigma := U_+\dot{c} \cap \dot{c}U_- \subset G$$

a Steinberg cross-section.

THEOREM 42 ([4]). *Let H and G be complex Lie groups. Let G be simply-connected and reductive. Let Σ be a Steinberg cross-section of G. Let (M, A, π, Φ) be a Hamiltonian quasi-Poisson $H \times G$-manifold. Then the preimage $M_{H,\Sigma} := \Phi_G^{-1}(\Sigma)$ of the G-component Φ_G of Φ is a smooth submanifold of M. The pullback of the twisted Dirac structure L_M to $M_{H,\Sigma}$ is quasi-Poisson for the action of H with the group valued moment map $\Phi_H|_{M_{H,\Sigma}}$. Any connected component of $M_{H,\Sigma} \cap S$ for any nondegenerate leaf S of M is a nondegenerate leaf of $M_{H,\Sigma}$, and any nondegenerate leaf of $M_{H,\Sigma}$ is such. The quasi-Hamiltonian 2-form of any nondegenerate leaf is the restriction of the quasi-Hamiltonian 2-form ω_S of S.*

Applying the above theorem to the case of $H = 1$, we obtain the following corollary:

COROLLARY 43 ([4]). *Let (M, π, Φ) be a quasi-Poisson G-manifold. Then $M_\Sigma := \Phi^{-1}(\Sigma)$ is a smooth submanifold of M. The pullback of the twisted Dirac structure L_M to M_Σ is Poisson. Any connected component of $M_\Sigma \cap S$ for any nondegenerate leaf S of M is a symplectic leaf of M_Σ, and any symplectic leaf of M_Σ is such. The symplectic form of any symplectic leaf is the restriction of the quasi-Hamiltonian 2-form ω_S of S.*

3.2. Riemann-Hilbert correspondence

Both the asymptotic data γ_i, ρ_i and the monodromy data m_i, $\log e_i^{\mathbb{R}}$ can be considered as defining local charts of the moduli space of solutions. From Theorem 26 we shall show that the transformation between two charts via the Riemann-Hilbert correspondence is symplectic with respect to the natural symplectic structures. The symplectic form 2θ we define in section 2.2.5 is equal to the left hand side of the equality below as $x \to 0$.

THEOREM 44.

$$-\frac{1}{2}\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} d\gamma_i \wedge d\rho_i = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} dm_i \wedge d\log e_i^{\mathbb{R}}.$$

REMARK 45. The left hand side is related to the Kirillov-Kostant form on a coadjoint orbit, and the right hand side is related to the Atiyah-Hitchin form on the space of the based rational maps of degree $n+1$ from $\mathbb{C}P^1$ to itself. Thus, both symplectic forms arise naturally from geometry. We shall present details of these facts in section 3.3 and 3.4.

Theorem 44 can be verified by direct calculation, but we prefer to give a proof by showing the existence of a generating function. The generating function will play an important role later.

DEFINITION 46. Let

$$F(\rho_0, \dots, \rho_{\lfloor (n-1)/2 \rfloor}, m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}) : = -\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \rho_i m_i + (\log(n+1)) \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} m_i^2 + \frac{n+1}{2}\sum_{k=0}^{n}\sum_{j=1}^{n} \psi^{(-2)}\left(\frac{m_{k-j} - m_k + j}{n+1}\right) \tag{3.2.1}$$

where $m_{j+n+1} = m_j$ and $m_j = -m_{n-j}$. Here $\psi^{(-2)}(z) = \int_0^z \log\Gamma(x)dx = \frac{z(1-z)}{2} + \frac{z}{2}\log 2\pi + z\log\Gamma(z) - \log G(1+z)$, and G is the Barnes G-function. The Barnes G-function G satisfies

$$G(z+1) = \Gamma(z)G(z),$$
$$G(1) = 1.$$

THEOREM 47. *The function F is a generating function of the transformation*

$$(m_0, \ldots, m_{\lfloor(n-1)/2\rfloor}, \rho_0, \ldots, \rho_{\lfloor(n-1)/2\rfloor}) \mapsto (m_0, \ldots, m_{\lfloor(n-1)/2\rfloor}, \log e_0^{\mathbb{R}}, \ldots, \log e_{\lfloor(n-1)/2\rfloor}^{\mathbb{R}})$$

with respect to the given symplectic forms. More precisely, F satisfies

$$m_i = -\frac{\partial F}{\partial \rho_i},\ \log e_i^{\mathbb{R}} = -\frac{\partial F}{\partial m_i}.$$

PROOF. The first identity is obvious. We show the second identity. Let

$$\tilde{K}(m_0, \ldots, m_n) := (n+1)\sum_{i=0}^{n}\sum_{j=1}^{n}\psi^{(-2)}(\frac{m_i - m_{i+j} + j}{n+1}),$$

where $m_{j+n+1} = m_j$. Let

$$K(m_0, \ldots, m_{\lfloor(n-1)/2\rfloor}) := \frac{1}{2}\tilde{K}(m_0, \ldots, m_{\lfloor(n-1)/2\rfloor}, -m_{\lfloor(n-1)/2\rfloor}, \ldots, -m_0).$$

This K is the last term of F in (3.2.1). From the formula for $\log e_i^{\mathbb{R}}$ in Theorem 26 and the definition of F, it suffices to show that

$$\frac{\partial K}{\partial m_k}(m_0, \ldots, m_{\lfloor(n-1)/2\rfloor})$$
$$= \log\left(\frac{X_k(m_0, \ldots .m_{\lfloor(n-1)/2\rfloor}, -m_{\lfloor(n-1)/2\rfloor}, \ldots, -m_0)}{X_{n-k}(m_0, \ldots .m_{\lfloor(n-1)/2\rfloor}, -m_{\lfloor(n-1)/2\rfloor}, \ldots, -m_0)}\right).$$

We can easily obtain that

$$X_{n-k}(-m_n, \ldots, -m_0) = \prod_{j=1}^{n}\Gamma(\frac{-m_k + m_{k-j} + j}{n+1}).$$

Then by direct calculation we obtain

$$\frac{\partial\tilde{K}}{\partial m_k} = \log(X_k(m_0, \ldots, m_n)/X_{n-k}(-m_n, \ldots, -m_0)).$$

Hence we have

$$\begin{aligned}
&\frac{\partial K}{\partial m_k}(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}) \\
=&\frac{1}{2}\left(\frac{\partial \tilde{K}}{\partial m_k}(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0) \right. \\
&\left. - \frac{\partial \tilde{K}}{\partial m_{n-k}}(m_0, \dots, m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0) \right) \\
=&\frac{1}{2}(\log X_k(m_0, \dots . m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0) \\
&- \log X_{n-k}(m_0, \dots . m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0) \\
&- \log X_{n-k}(m_0, \dots . m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0) \\
&+ \log X_k(m_0, \dots . m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0)) \\
=&\log X_k(m_0, \dots . m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0) \\
&- \log X_{n-k}(m_0, \dots . m_{\lfloor (n-1)/2 \rfloor}, -m_{\lfloor (n-1)/2 \rfloor}, \dots, -m_0).
\end{aligned}$$

This completes the proof. □

Let us write out the above proof in the case of $n = 3$.

EXAMPLE 48. In the $n = 3$ case, we have the following formulae:

$$X_0(m_0, m_1, m_2, m_3) = \Gamma\left(\frac{m_0 - m_1 + 1}{4}\right)\Gamma\left(\frac{m_0 - m_2 + 2}{4}\right)\Gamma\left(\frac{m_0 - m_3 + 3}{4}\right)$$

$$X_1(m_0, m_1, m_2, m_3) = \Gamma\left(\frac{m_1 - m_2 + 1}{4}\right)\Gamma\left(\frac{m_1 - m_3 + 2}{4}\right)\Gamma\left(\frac{m_1 - m_0 + 3}{4}\right)$$

$$X_2(m_0, m_1, m_2, m_3) = \Gamma\left(\frac{m_2 - m_3 + 1}{4}\right)\Gamma\left(\frac{m_2 - m_0 + 2}{4}\right)\Gamma\left(\frac{m_2 - m_1 + 3}{4}\right)$$

$$X_3(m_0, m_1, m_2, m_3) = \Gamma\left(\frac{m_3 - m_0 + 1}{4}\right)\Gamma\left(\frac{m_3 - m_1 + 2}{4}\right)\Gamma\left(\frac{m_3 - m_2 + 3}{4}\right)$$

$$
\begin{aligned}
&\tilde{K}(m_0,m_1,m_2,m_3)\\
=&4\Bigg(\psi^{(-2)}\left(\frac{m_0-m_1+1}{4}\right)+\psi^{(-2)}\left(\frac{m_0-m_2+2}{4}\right)+\psi^{(-2)}\left(\frac{m_0-m_3+3}{4}\right)\\
&+\psi^{(-2)}\left(\frac{m_1-m_2+1}{4}\right)+\psi^{(-2)}\left(\frac{m_1-m_3+2}{4}\right)+\psi^{(-2)}\left(\frac{m_1-m_0+3}{4}\right)\\
&+\psi^{(-2)}\left(\frac{m_2-m_3+1}{4}\right)+\psi^{(-2)}\left(\frac{m_2-m_0+2}{4}\right)+\psi^{(-2)}\left(\frac{m_2-m_1+3}{4}\right)\\
&+\psi^{(-2)}\left(\frac{m_3-m_0+1}{4}\right)+\psi^{(-2)}\left(\frac{m_3-m_1+2}{4}\right)+\psi^{(-2)}\left(\frac{m_3-m_2+3}{4}\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&K(m_0,m_1)\\
=&2\Bigg(\psi^{(-2)}\left(\frac{m_0-m_1+1}{4}\right)+\psi^{(-2)}\left(\frac{m_0+m_1+2}{4}\right)+\psi^{(-2)}\left(\frac{2m_0+3}{4}\right)\\
&+\psi^{(-2)}\left(\frac{2m_1+1}{4}\right)+\psi^{(-2)}\left(\frac{m_1+m_0+2}{4}\right)+\psi^{(-2)}\left(\frac{m_1-m_0+3}{4}\right)\\
&+\psi^{(-2)}\left(\frac{-m_1+m_0+1}{4}\right)+\psi^{(-2)}\left(\frac{-m_1-m_0+2}{4}\right)+\psi^{(-2)}\left(\frac{-2m_1+3}{4}\right)\\
&+\psi^{(-2)}\left(\frac{-2m_0+1}{4}\right)+\psi^{(-2)}\left(\frac{-m_0-m_1+2}{4}\right)+\psi^{(-2)}\left(\frac{-m_0+m_1+3}{4}\right)\Bigg)
\end{aligned}
$$

$$
F(\rho_0,\rho_1,m_0,m_1)=-(\rho_0m_0+\rho_1m_1)+(2\log 2)(m_0^2+m_1^2)+K(m_0,m_1)
$$

Proof of the above theorem for $n = 3$. For $n = 3$, we have

$$
\begin{aligned}
&\frac{\partial K}{\partial m_0}(m_0, m_1)\\
=&\frac{1}{2}\log\Bigg(\Gamma\left(\frac{m_0 - m_1 + 1}{4}\right)\Gamma\left(\frac{m_0 + m_1 + 2}{4}\right)\Gamma\left(\frac{2m_0 + 3}{4}\right)^2\Gamma\left(\frac{m_1 + m_0 + 2}{4}\right)\\
&\times\Gamma\left(\frac{m_1 - m_0 + 3}{4}\right)^{-1}\Gamma\left(\frac{-m_1 + m_0 + 1}{4}\right)\Gamma\left(\frac{-m_1 - m_0 + 2}{4}\right)^{-1}\\
&\times\Gamma\left(\frac{-2m_0 + 1}{4}\right)^{-2}\Gamma\left(\frac{-m_0 - m_1 + 2}{4}\right)^{-1}\Gamma\left(\frac{-m_0 + m_1 + 3}{4}\right)^{-1}\Bigg)\\
=&\frac{1}{2}\log\Bigg(X_0(m_0, m_1, -m_1, -m_0)\Gamma\left(\frac{2m_0 + 3}{4}\right)\Gamma\left(\frac{m_1 + m_0 + 2}{4}\right)\\
&\times\Gamma\left(\frac{m_1 - m_0 + 3}{4}\right)^{-1}\Gamma\left(\frac{-m_1 + m_0 + 1}{4}\right)\Gamma\left(\frac{-m_1 - m_0 + 2}{4}\right)^{-1}\\
&\times\Gamma\left(\frac{-2m_0 + 1}{4}\right)^{-1}X_3(m_0, m_1, -m_1, -m_0)^{-1}\Bigg)\\
=&\frac{1}{2}(\log X_0(m_0, m_1, -m_1, -m_0)^2 X_3(m_0, m_1, -m_1, -m_0)^{-2})\\
=&\log(X_0(m_0, m_1, -m_1, -m_0)/X_3(m_0, m_1, -m_1, -m_0)).
\end{aligned}
$$

Similarly, we have

$$
\frac{\partial K}{\partial m_1}(m_0, m_1) = \log(X_1(m_0, m_1, -m_1, -m_0)/X_2(m_0, m_1, -m_1, -m_0)).
$$

This completes the proof. □

We can also prove the equality of the two symplectic forms without using a generating function.

Proof. If we write $dm_0 \wedge d\log e_0^{\mathbb{R}} + dm_1 \wedge d\log e_1^{\mathbb{R}}$ in terms of $\gamma_0, \rho_0, \gamma_1, \rho_1$, using (2.2.5) and (2.2.6), the coefficient of $d\gamma_0 \wedge d\gamma_1$ is

$$
\begin{aligned}
\frac{1}{4}\Bigg(&\frac{\partial \log X_3(m_0, m_1, -m_1, -m_0)/X_0(m_0, m_1, -m_1, -m_0)}{\partial m_1}\\
&-\frac{\partial \log X_2(m_0, m_1, -m_1, -m_0)/X_1(m_0, m_1, -m_1, -m_0)}{\partial m_0}\Bigg),
\end{aligned}
$$

where $X_k(m_0, m_1, m_2, m_3) = \prod_{j=1}^{3} \Gamma\left(\frac{m_k - m_{k+j} + j}{4}\right)$ $(m_{j+4} = m_j)$. And we have

$$4\frac{\partial \log X_1(m_0, m_1, -m_1, -m_0)}{\partial m_0} = \psi^{(0)}\left(\frac{m_1 + m_0 + 2}{4}\right) - \psi^{(0)}\left(\frac{m_1 - m_0 + 3}{4}\right)$$

$$4\frac{\partial \log X_2(m_0, m_1, -m_1, -m_0)}{\partial m_0} = \psi^{(0)}\left(\frac{-m_1 + m_0 + 1}{4}\right) - \psi^{(0)}\left(\frac{-m_1 - m_0 + 2}{4}\right)$$

$$4\frac{\partial \log X_0(m_0, m_1, -m_1, -m_0)}{\partial m_1} = -\psi^{(0)}\left(\frac{m_0 - m_1 + 1}{4}\right) + \psi^{(0)}\left(\frac{m_0 + m_1 + 2}{4}\right)$$

$$4\frac{\partial \log X_3(m_0, m_1, -m_1, -m_0)}{\partial m_1} = -\psi^{(0)}\left(\frac{-m_0 - m_1 + 2}{4}\right) + \psi^{(0)}\left(\frac{-m_0 + m_1 + 3}{4}\right).$$

Then we can conclude the coefficient of $d\gamma_0 \wedge d\gamma_1$ is zero. □

3.3. Atiyah-Hitchin structure and the universal centraliser

3.3.1. The universal centraliser and the space of the based rational maps. Next, we will see that the monodromy data can be considered as a canonical coordinate system of the space $\mathrm{Rat}_{n+1}(\mathbb{C}P^1)$ of the rational maps p/q of degree $n+1$ from $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ to itself which send ∞ to 0. Here q is a monic polynomial of degree $n+1$, p is a polynomial of degree less than $n+1$, and p and q have no common root. If the roots β_i of q are distinct, $(\beta_0, \ldots, \beta_n, p(\beta_0), \ldots, p(\beta_n))$ gives an open dense chart of $\mathrm{Rat}_{n+1}(\mathbb{C}P^1)$. There is a natural symplectic form $\sum_{i=0}^{n} \frac{dp(\beta_i)}{p(\beta_i)} \wedge d\beta_i$ on this chart, which extends to $\mathrm{Rat}_{n+1}(\mathbb{C}P^1)$. For the details, we refer to chapter 5 of Atiyah-Hitchin [3]. We shall give another interpretation of this space, related to Lie theory.

Recall our assumptions:

- $G = K^{\mathbb{C}}$: a complex reductive Lie group
- $\mathfrak{g}, \mathfrak{k}$: the Lie algebras of G, K respectively.
- β: a nondegenerate ad-invariant symmetric $\mathbb{C}$-bilinear form on $\mathfrak{g}$ such that $\beta(X, Y) \in \mathbb{R}$ for any $X, Y \in \mathfrak{k}$.

We identify $G \times \mathfrak{g}$ with the tangent bundle T^*G of G through the following isomorphisms

$$G \times \mathfrak{g} \cong G \times \mathfrak{g}^* \cong T^*G,$$

which are induced by β and the left trivialisation. The symplectic form ϖ on $G \times \mathfrak{g}$ corresponding to the canonical symplectic form on T^*G is given by the following equality [25]:

$$\varpi_{(g,x)}((A_1, B_1), (A_2, B_2)) = \beta(A_1, B_2) - \beta(A_2, B_1) - \beta(x, [A_1, A_2]),$$

where $(g, x) \in G \times \mathfrak{g}$, $A_1, A_2 \in \mathfrak{g} \cong T_g G$ and $B_1, B_2 \in \mathfrak{g} \cong T_x \mathfrak{g}$. The action $(G \times G) \times (G \times \mathfrak{g}) \to G \times \mathfrak{g}$ defined by

$$(g_1, g_2) \cdot (g, x) := (g_1 g g_2^{-1}, \mathrm{Ad}_{g_2} x)$$

preserves the 2-form ϖ.

Let $(\xi, h, \eta) \in \mathfrak{g}^{\oplus 3}$ be a principal $\mathfrak{sl}_2\mathbb{C}$ triple, i.e., suppose that ξ, h and η are regular elements and that the Lie subalgebra of $\mathfrak{g}$ generated by $\{\xi, h, \eta\}$ is isomorphic to $\mathfrak{sl}_2\mathbb{C}$ as Lie algebras. Let $\mathcal{S} := \xi + \mathfrak{g}_\eta \subset \mathfrak{g}$, where $\mathfrak{g}_\eta$ is the annihilator of η in $\mathfrak{g}$. It is known as the Slodowy slice (or Kostant section). The universal centraliser is

$$\mathcal{Z}_{\mathfrak{g}} := \{ (g, x) \in G \times \mathcal{S} | \, g \in G_x \} ,$$

where G_x is the stabiliser of x with respect to the adjoint action of G.

Proposition 49 ([9]). *The universal centraliser $\mathcal{Z}_{\mathfrak{g}}$ is a symplectic submanifold of $G \times \mathfrak{g}$, where $G \times \mathfrak{g}$ is identified with T^*G.*

Proposition 50 ([5]). *In the case $G = \mathrm{GL}_{n+1}\mathbb{C}$ and $\beta(A, B) = \mathrm{Tr}(AB)$, the symplectic form of the universal centraliser $\mathcal{Z}_{\mathfrak{g}}$ induced from the symplectic form of $G \times \mathfrak{g}$ agrees with the symplectic form of $\mathrm{Rat}_{n+1}(\mathbb{C}P^1)$.*

Remark 51. Bielawski's proof in [5] is obtained by combining his results and Donaldson's result [10] on the moduli space of framed SU(2) monopoles of charge m from $\mathbb{C}P^1$ to itself.

Definition 52. Let $\mathcal{L}^{\mathrm{zero}}_{n+1}$ be the space of all pairs $(E, M^{(0)})$ of the matrices in Theorem 15 and Definition 14 for any local solution near $x = 0$.

Let us consider the open subset $\mathrm{Rat}^*_{n+1}(\mathbb{C}P^1)$ of $\mathrm{Rat}_{n+1}(\mathbb{C}P^1)$ consisting of p/q, with the same condition as in the definition of $\mathrm{Rat}_{n+1}(\mathbb{C}P^1)$, with the extra condition that the roots of q are never zero. There is another natural symplectic form $\sum_{i=0}^{n} p(\beta_i)^{-1} dp(\beta_i) \wedge \beta_i^{-1} d\beta_i$ on $\mathrm{Rat}^*_{n+1}(\mathbb{C}P^1)$. We conjecture the restriction of this form to $\mathcal{L}^{\mathrm{zero}}_{n+1}$ is equal to the right hand side $\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} dm_i \wedge d \log e_i^{\mathbb{R}}$ of Theorem 44.

3.3.2. The multiplicative universal centraliser. Next, we will review the multiplicative version of the universal centraliser, which will give another interpretation of the form $\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} dm_i \wedge d \log e_i^{\mathbb{R}}$.

DEFINITION 53 ([2]). Let $D(G) := G \times G$. $D(G)$ is called the double of G when it is equipped with the 2-form ω_D defined by

$$\omega_{D,(g,h)} = \frac{1}{2}\beta(g^*\theta_G^L, h^*\theta_G^R) + \frac{1}{2}\beta(g^*\theta_G^R, h^*\theta_G^L).$$

The action $A : G \times G \times D(G) \to D(G)$ of $G \times G$ on $D(G)$ defined by

$$(s,t) \cdot (g,h) := (sgt^{-1}, ths^{-1})$$

preserves the 2-form ω_D. The quasi-Hamiltonian moment map $\mu_D : D(G) \to G \times G$ is given by $\mu_D(g,h) = (gh, g^{-1}h^{-1})$.

PROPOSITION 54 ([2]). *The quadruple $(D(G), A, \mu_D, \omega_D)$ is a quasi-Hamiltonian $G \times G$-space.*

EXAMPLE 55 ([1]). The image of the moment map μ_D is

$$\left\{ (s,t) \in G \times G \middle|\, s \text{ is conjugate to } t^{-1} \right\}.$$

(This holds since $(g^{-1}h^{-1})^{-1} = hg$ and $gh = g(hg)g^{-1}$.) Let

$$\Sigma_\Delta := \left\{ (h,h^{-1}) \in G \times G \middle|\, h \in \Sigma \right\},$$

where Σ is a Steinberg cross-section. Any two elements of Σ are conjugate if and only if they are equal, so we have

PROPOSITION 56.

$$\mu_D^{-1}(\Sigma_\Delta) = \mu_D^{-1}(\Sigma \times \Sigma^{-1}).$$

PROOF. Since $\Sigma_\Delta \subset \Sigma \times \Sigma^{-1}$, $\mu_D^{-1}(\Sigma_\Delta) \subset \mu_D^{-1}(\Sigma \times \Sigma^{-1})$. If $(g,h) \in \mu_D^{-1}(\Sigma \times \Sigma^{-1})$, there exists $f \in \Sigma$ such that $g^{-1}h^{-1} = f^{-1}$. Since (gh, f^{-1}) is in the image of μ_D, gh is conjugate to f. Two elements gh and f are both in Σ, so they are equal. This means $(g,h) \in \mu_D^{-1}(\Sigma_\Delta)$. □

EXAMPLE 57. Since $\Sigma \times \Sigma^{-1}$ is a Steinberg cross-section of $G \times G$, it follows from Corollary 43 that $\mu_D^{-1}(\Sigma_\Delta)$ is a smooth submanifold of $D(G)$ and the restriction of the quasi-Hamiltonian 2-form ω_D to $\mu_D^{-1}(\Sigma_\Delta)$ is symplectic.

On the other hand, it holds that

$$\mu_D^{-1}(\Sigma_\Delta) = \left\{ (g,h) \in G \times \Sigma \middle|\, ghg^{-1} = h \right\}.$$

This is the multiplicative version of the universal centraliser. Let us write

$$\mathcal{Z}_G = \mu_D^{-1}(\Sigma_\Delta).$$

In the case $G = \mathrm{GL}_{n+1}\mathbb{C}$, we conjecture the following explicit coordinate expression of $\iota^*_{\mathcal{Z}_G}\omega_D$, where $\iota_{\mathcal{Z}_G} : \mathcal{Z}_G \to D(G)$ is the natural inclusion.

CONJECTURE 58. *Let $G = \mathrm{GL}_{n+1}\mathbb{C}$ and $\beta(A, B) = \mathrm{Tr}(AB)$. Let $(X, Y) \in \mu_D^{-1}(\Sigma_\Delta)$. Let x_i, y_i $(i = 0, \ldots, n)$ be the eigenvalues of X and Y respectively. It holds that*

$$\iota^*_{\mathcal{Z}_G}\omega_D = \sum_{i=0}^{n} x_i^{-1} dx_i \wedge y_i^{-1} dy_i.$$

Before introducing the relation between the multiplicative universal centraliser and monodromy matrices, let us give an explicit expression of a Steinberg cross-section. Let $G = \mathrm{SL}_{n+1}\mathbb{C}$. Let $\Gamma = \{\alpha_{1,0}, \ldots, \alpha_{n,n-1}\}$ be the standard choice of simple roots, $E_{\alpha_{i.i-1}}$ the corresponding root vectors, and σ_i a representative in $N(T)$ of the corresponding element in the Weyl group $N(T)/T = \mathfrak{S}_{n+1}$. Let

$$\mathcal{R}^\Gamma_{n+1} = \{ e_1(t_1)\sigma_1 \cdots e_n(t_n)\sigma_n \in \mathrm{SL}_{n+1}\mathbb{C} |\, (t_1, \ldots, t_n) \in \mathbb{C}^n \},$$

where $e_i(t_i) = \exp(t_i E_{\alpha_{i,i-1}})$. It is shown in [15] that $\mathcal{R}^\Gamma_{n+1}$ is a Steinberg cross-section of G.

In our case, $M^{(0)}$ commutes with E, and we have the following fact:

THEOREM 59 ([15]). *Let $M^{(0)}$ be the matrix defined in Definition 14 for any local solution near $x = 0$. The set of all such matrices is equal to a subset of $\mathcal{R}^\Gamma_{n+1}$.*

From this theorem and Conjecture 58, we obtain the following conclusion.

CONJECTURE 60. *The restriction of ω_D on $D(\mathrm{GL}_{n+1}\mathbb{C})$ to $\mathcal{L}^{zero}_{n+1}$ is equal to*

$$2 \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} dm_i \wedge d\log e_i^{\mathbb{R}}.$$

(As mentioned before, the eigenvalues of $M^{(0)}$ are $\omega^{m_i - i}$, and the eigenvalues of E are $e_i^{\mathbb{R}}$.)

The above symplectic form is equal to the symplectic form in Theorem 44 up to a multiplicative constant.

3.4. Adjoint orbit

Let us see how the natural symplectic form related to the asymptotic data can be realised as the Kiriillov-Kostant-Souriau symplectic form.

In this section, we will consider the case of $n = 3$. Let G be the set of all matrices

$$\begin{pmatrix} x_0 & & & \\ & x_1 & & \\ & & x_1^{-1} & \\ & & & x_0^{-1} \end{pmatrix} + \begin{pmatrix} & dx_0x_1 & & \\ & & c & \\ & & & d \\ a & & & \end{pmatrix} \zeta \in \mathrm{GL}_4\left(\mathbb{C}[\zeta]/(\zeta^2)\right)$$

such that $x_0, x_1 \in \mathbb{R}\backslash\{0\},\ a, c, d \in \mathbb{R}$. Let $\mathfrak{g}$ be the Lie algebra of G. It follows that

$$\mathfrak{g} = \left\{ \begin{pmatrix} q & & & \\ & r & & \\ & & -r & \\ & & & -q \end{pmatrix} + \begin{pmatrix} & t & & \\ & & u & \\ & & & t \\ s & & & \end{pmatrix} \zeta \,\middle|\, q, r, s, t, u \in \mathbb{R} \right\}.$$

Let $O := \mathrm{Ad}^*(G)p$ where

$$p := \left(\frac{1}{\zeta^2} \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix} + \frac{1}{\zeta} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \right) d\zeta.$$

This orbit can be explicitly expressed in the following way.

Proposition 61. *The orbit O is equal to the set of all equivalence classes of 1-forms*

$$\left(\frac{1}{\zeta^2} \begin{pmatrix} & & & c_0 \\ c_1 & & & \\ & \frac{1}{c_0c_1^2} & & \\ & & c_1 & \end{pmatrix} + \frac{1}{\zeta} \begin{pmatrix} m_0 & & & \\ & m_1 & & \\ & & -m_1 & \\ & & & -m_0 \end{pmatrix} \right) d\zeta$$

such that $c_0 \in \mathbb{R}_{>0}, c_1 \in \mathbb{R}\backslash\{0\}, m_0, m_1 \in \mathbb{R}$.

Proof. Let

$$g := \begin{pmatrix} x_0 & & & \\ & x_1 & & \\ & & x_1^{-1} & \\ & & & x_0^{-1} \end{pmatrix} + \begin{pmatrix} & dx_0x_1 & & \\ & & c & \\ & & & d \\ a & & & \end{pmatrix} \zeta$$

and

$$A := \left(\frac{1}{\zeta^2}\begin{pmatrix} & & & c_0 \\ c_1 & & & \\ & \frac{1}{c_0c_1^2} & & \\ & & c_1 & \end{pmatrix} + \frac{1}{\zeta}\begin{pmatrix} m_0 & & & \\ & m_1 & & \\ & & -m_1 & \\ & & & -m_0 \end{pmatrix}\right) d\zeta.$$

Then

$$gAg^{-1} = \left(\frac{1}{\zeta^2}\begin{pmatrix} & & & x_0^2c_0 \\ x_0^{-1}x_1c_1 & & & \\ & \frac{1}{c_0c_1^2x_1^2} & & \\ & & x_0^{-1}x_1c_1 & \end{pmatrix} + \frac{1}{\zeta}\begin{pmatrix} \diamondsuit & & & \\ & \clubsuit & & \\ & & \spadesuit & \\ & & & \heartsuit \end{pmatrix}\right) d\zeta,$$

where $\diamondsuit = m_0 + c_1dx_1 - ac_0x_0$, $\clubsuit = m_1 + \frac{c}{c_0c_1^2x_1} - dx_1c_1$, $\spadesuit = -m_1 - \frac{c}{c_0c_1^2x_1} + dx_1c_1$, $\heartsuit = -m_0 - c_1dx_1 + ac_0x_0$.

Putting $c_0 = c_1 = 1$ and $m_0 = m_1 = 0$ completes this proof. □

Let

$$A := \left(\frac{1}{\zeta^2}\begin{pmatrix} & & & c_0 \\ c_1 & & & \\ & \frac{1}{c_0c_1^2} & & \\ & & c_1 & \end{pmatrix} + \frac{1}{\zeta}\begin{pmatrix} m_0 & & & \\ & m_1 & & \\ & & -m_1 & \\ & & & -m_0 \end{pmatrix}\right) d\zeta \in O.$$

Let

$$C_0 := \frac{1}{2c_0}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} + \begin{pmatrix} & 0 & & \\ & & 0 & \\ & & & 0 \\ 0 & & & \end{pmatrix}\zeta,$$

$$C_1 := \frac{1}{c_1}\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} & 0 & & \\ & & 0 & \\ & & & 0 \\ 0 & & & \end{pmatrix}\zeta,$$

$$M_0 := \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} - \frac{1}{c_0}\begin{pmatrix} & 0 & & \\ & & 0 & \\ & & & 0 \\ 1 & & & \end{pmatrix}\zeta,$$

and

$$M_1 := \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \left(-\frac{1}{c_1} \begin{pmatrix} & 1 & & \\ & & 0 & \\ & & & 1 \\ 0 & & & \end{pmatrix} - \frac{1}{c_0} \begin{pmatrix} & 0 & & \\ & & 0 & \\ & & & 0 \\ 1 & & & \end{pmatrix} \right) \zeta.$$

Then their fundamental vector fields $\underline{C_i}, \underline{M_i}$ satisfy the following equations:

$$\left.\frac{\partial}{\partial c_0}\right|_A = \underline{C_0}_A$$
$$\left.\frac{\partial}{\partial c_1}\right|_A = \underline{C_1}_A$$
$$\left.\frac{\partial}{\partial m_0}\right|_A = \underline{M_0}_A$$
$$\left.\frac{\partial}{\partial m_1}\right|_A = \underline{M_1}_A$$

Let ω be the Kirillov-Kostant symplectic form on O. It follows that

$$\omega_A(\left.\frac{\partial}{\partial c_0}\right|_A, \left.\frac{\partial}{\partial c_1}\right|_A) = \omega_A(\left.\frac{\partial}{\partial m_0}\right|_A, \left.\frac{\partial}{\partial m_1}\right|_A) = \omega_A(\left.\frac{\partial}{\partial c_1}\right|_A, \left.\frac{\partial}{\partial m_0}\right|_A) = 0$$

since $[C_0, C_1] = [M_0, M_1] = [C_1, M_0] = 0$. ($[C_1, M_0] = 0$ follows from

$$\left[\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} & 0 & & \\ & & 0 & \\ & & & 0 \\ 1 & & & \end{pmatrix} \right] = 0.)$$

Since

$$[C_0, M_0] = -\frac{1}{2c_0^2} \left[\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \begin{pmatrix} & 0 & & \\ & & 0 & \\ & & & 0 \\ 1 & & & \end{pmatrix} \right] \zeta = \frac{1}{c_0^2} \begin{pmatrix} & 0 & & \\ & & 0 & \\ & & & 0 \\ 1 & & & \end{pmatrix} \zeta,$$

$$\omega_A(\left.\frac{\partial}{\partial c_0}\right|_A, \left.\frac{\partial}{\partial m_0}\right|_A) = \left\langle A, \frac{1}{c_0^2} \begin{pmatrix} & 0 & & \\ & & 0 & \\ & & & 0 \\ 1 & & & \end{pmatrix} \zeta \right\rangle = c_0 \frac{1}{c_0^2} = \frac{1}{c_0}.$$

Since

$$[C_0, M_1] = \frac{1}{c_0^2}\begin{pmatrix} & 0 & & \\ & & 0 & \\ & & & 0 \\ 1 & & & \end{pmatrix}\zeta - \frac{1}{2c_0c_1}\left[\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \begin{pmatrix} & 1 & & \\ & & 0 & \\ & & & 1 \\ 0 & & & \end{pmatrix}\right]\zeta$$

and

$$\left[\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \begin{pmatrix} & 1 & & \\ & & 0 & \\ & & & 1 \\ 0 & & & \end{pmatrix}\right] = 0,$$

$$\omega_A(\left.\frac{\partial}{\partial c_0}\right|_A, \left.\frac{\partial}{\partial m_1}\right|_A) = \frac{1}{c_0}.$$

Since

$$[C_1, M_1] = \frac{1}{c_1^2}\begin{pmatrix} & 1 & & \\ & & 0 & \\ & & & 1 \\ 0 & & & \end{pmatrix}\zeta,$$

$$\omega_A(\left.\frac{\partial}{\partial c_1}\right|_A, \left.\frac{\partial}{\partial m_1}\right|_A) = \frac{2}{c_1}.$$

We obtain:

THEOREM 62.

$$\omega = \frac{1}{c_0}dc_0 \wedge dm_0 + \frac{1}{c_0}dc_0 \wedge dm_1 + \frac{2}{c_1}dc_1 \wedge dm_1.$$

Note that $m_0 = \frac{1}{2}(-3\alpha_0 + 2\alpha_1 + \alpha_2) = \frac{1}{2}(-3k_0 + 2k_1 + k_2 - 3 + 2 + 1) = \frac{1}{2}(-4k_0 - 3)$ if we suppose $N = \alpha_0 + \cdots + \alpha_3 = 1$.

Similarly $m_1 = \frac{1}{2}(-4k_0 - 8k_1 - 9)$. Then we obtain the following corollary:

COROLLARY 63.

$$\omega = 4dk_0 \wedge \left(\frac{1}{c_0}dc_0\right) + 4dk_1 \wedge \left(\frac{1}{c_0}dc_0\right) + 4dk_0 \wedge \left(\frac{1}{c_1}dc_1\right) + 8dk_1 \wedge \left(\frac{1}{c_1}dc_1\right).$$

And noting that $\frac{1}{c_2}dc_2 = -\frac{1}{c_0}dc_0 - \frac{2}{c_1}dc_1$ ($c_2 = \frac{1}{c_0c_1^2}$), we obtain the following corollary:

Corollary 64.
$$\omega = \frac{1}{2}(d\gamma_0 \wedge d\rho_0 + d\gamma_1 \wedge d\rho_1).$$

This means that the asymptotic data gives canonical coordinates with respect to the Kirillov-Kostant-Souriau structure ω of an adjoint orbit of the subgroup G of $\mathrm{GL}_4\left(\mathbb{C}[\zeta]/(\zeta^2)\right)$.

CHAPTER 4

The constant problem

In this section, we assume for simplicity that $n = 3$, so $w = (w_0, w_1, w_2, w_3)$ with $w_2 = -w_1$, $w_3 = -w_0$. For general n the same method applies. We consider only the global solutions, i.e., we assume $\log e_i^{\mathbb{R}} = 0$, which means that the ρ_i's are determined by the γ_j's, as in Theorem 28. The following calculation is motivated by Theorem 1 in [23]. See also [29] for further details.

DEFINITION 65. Let us define the tau function of a global solution w by

$$\log \tau^w(x_1, x_2) = \int_{x_1}^{x_2} H(w_i(x), \tilde{w}_i(x), x) dx$$

where H is the Hamiltonian function.

REMARK 66. Usually the tau function is defined (up to a multiplicative constant) by $\log \tau^w(x) = \int^x H(w_i(x), \tilde{w}_i(x), x)dx$. In that notation we have $\tau^w(x_1, x_2) = \tau^w(x_2)/\tau^w(x_1)$.

The Hamiltonian function is

$$H(x, w_0, w_1, \tilde{w}_0, \tilde{w}_1) = \frac{1}{2x}(\tilde{w}_0^2 + \tilde{w}_1^2) - xe^{2(w_1 - w_0)} - \frac{x}{2}\left(e^{-4w_1} + e^{4w_0}\right)$$

and is quasihomogeneous, that is,

$$H(w, \lambda\tilde{w}; \lambda x) = \lambda H(w, \tilde{w}; x) \text{ for any } \lambda > 0.$$

It follows that

$$\sum_{i=0}^{1} \tilde{w}_i \frac{\partial H}{\partial \tilde{w}_i} + x\frac{\partial H}{\partial x} = H.$$

For the solution $(w_0(x), w_1(x), \tilde{w}_0(x), \tilde{w}_1(x))$ of (2.2.7) and (2.2.8), we have

$$\sum_{i=0}^{1} \tilde{w}_i(x) \frac{\partial H}{\partial \tilde{w}_i}(x, w_0(x), w_1(x), \tilde{w}_0(x), \tilde{w}_1(x)) = \sum_{i=0}^{1} \tilde{w}_i(x) \left(w_i\right)_x(x)$$

$$x\frac{\partial H}{\partial x}(x, w_0(x), w_1(x), \tilde{w}_0(x), \tilde{w}_1(x)) = -H(w_0(x), w_1(x), \tilde{w}_0(x), \tilde{w}_1(x)) + \frac{d}{dx}\Big(xH(x, w_0(x), w_1(x), \tilde{w}_0(x), \tilde{w}_1(x))\Big).$$

The first equality is obvious. The second equality follows from

$$\frac{d}{dx}\Big(xH\Big) = x\frac{dH}{dx} + H$$

and

$$\begin{aligned}
&\frac{d}{dx}\Big(H(x, w_0(x), w_1(x), \tilde{w}_0(x), \tilde{w}_1(x))\Big)\\
=&\frac{\partial H}{\partial x}(x, w_0(x), w_1(x), \tilde{w}_0(x), \tilde{w}_1(x)) + \sum_{i=0}^{1}\left(w_i\right)_x(x)\frac{\partial H}{\partial w_i}(x, w_0(x), w_1(x), \tilde{w}_0(x), \tilde{w}_1(x))\\
&+\sum_{i=0}^{1}\left(\tilde{w}_i\right)_x(x)\frac{\partial H}{\partial \tilde{w}_i}(x, w_0(x), w_1(x), \tilde{w}_0(x), \tilde{w}_1(x))\\
=&\frac{\partial H}{\partial x}(x, w_0(x), w_1(x), \tilde{w}_0(x), \tilde{w}_1(x))\\
&+\sum_{i=0}^{1}\left(-\left(w_i\right)_x(x)\left(\tilde{w}_i\right)_x(x) + \left(\tilde{w}_i\right)_x(x)\left(w_i\right)_x(x)\right)\\
=&\frac{\partial H}{\partial x}(x, w_0(x), w_1(x), \tilde{w}_0(x), \tilde{w}_1(x)).
\end{aligned}$$

Then it follows that

Proposition 67.

$$H = \tilde{w}_0\left(w_0\right)_x + \tilde{w}_1\left(w_1\right)_x - H + \frac{d}{dx}\Big(xH\Big).$$

Let

$$S(x_1, x_2) := \int_{x_1}^{x_2}\left(\sum_{i=0}^{1}\tilde{w}_i\left(w_i\right)_x - H\right)dx,$$

which is called the classical action, the functional from which we can derive the Euler-Lagrange equation using the fundamental lemma of calculus of variations. We obtain

$$\begin{aligned}
&\frac{\partial S(x_1,x_2)}{\partial \gamma_j}\\
&=\int_{x_1}^{x_2}\left(\sum_{i=0}^{1}\left(\left(\tilde{w}_i\right)_{\gamma_j}\left(w_i\right)_x+\tilde{w}_i\left(\left(w_i\right)_x\right)_{\gamma_j}\right)-\left(H\right)_{\gamma_j}\right)dx\\
&=\int_{x_1}^{x_2}\left(\sum_{i=0}^{1}\left(\left(\tilde{w}_i\right)_{\gamma_j}\frac{\partial H}{\partial \tilde{w}_i}+\tilde{w}_i\left(\left(w_i\right)_x\right)_{\gamma_j}\right)-\sum_{i=0}^{1}\left(\frac{\partial H}{\partial w_i}\left(w_i\right)_{\gamma_j}+\frac{\partial H}{\partial \tilde{w}_i}\left(\tilde{w}_i\right)_{\gamma_j}\right)\right)dx\\
&=\int_{x_1}^{x_2}\left(\sum_{i=0}^{1}\left(\left(\tilde{w}_i\right)_{\gamma_j}\frac{\partial H}{\partial \tilde{w}_i}-\left(\tilde{w}_i\right)_x\left(w_i\right)_{\gamma_j}\right)-\sum_{i=0}^{1}\left(\frac{\partial H}{\partial w_i}\left(w_i\right)_{\gamma_j}+\frac{\partial H}{\partial \tilde{w}_i}\left(\tilde{w}_i\right)_{\gamma_j}\right)\right)dx\\
&\quad+\left(\sum_{i=0}^{1}\tilde{w}_i\left(w_i\right)_{\gamma_j}\right)\Bigg|_{x_1}^{x_2}\\
&=\left(\tilde{w}_0\left(w_0\right)_{\gamma_j}+\tilde{w}_1\left(w_1\right)_{\gamma_j}\right)\Big|_{x_1}^{x_2}.
\end{aligned}$$

The second equality follows from (2.2.7) and the chain rule, the third from integration by parts, and the fourth from (2.2.8).

From Proposition 67 and the definition of the τ function, we obtain

$$\begin{aligned}
\frac{\partial}{\partial \gamma_j}\log\tau^w(x_1,x_2)&=\frac{\partial}{\partial \gamma_j}\int_{x_1}^{x_2}\left(\sum_{i=0}^{1}\tilde{w}_i\left(w_i\right)_x-H+\frac{d}{dx}(xH)\right)dx\\
&=\frac{\partial S(x_1,x_2)}{\partial \gamma_j}+\left(x_2H(x_2)-x_1H(x_1)\right)_{\gamma_j}\\
&=\left(\sum_{i=0}^{1}\tilde{w}_i\left(w_i\right)_{\gamma_j}\right)\Bigg|_{x_1}^{x_2}+\left(x_2H(x_2)-x_1H(x_1)\right)_{\gamma_j}. \qquad (4.0.1)
\end{aligned}$$

At $x=0$ a more precise form of (2.2.3) is

$$w_i(x)=\frac{\gamma_i}{2}\log x+\frac{\rho_i}{2}+O(x^{\varepsilon_i}),\quad x\to 0$$

for some $\varepsilon_i > 0$ (which depends on γ_0 and γ_1); this can be shown as in Theorem 14.1 of [12] for the case $n = 1$. This formula is differentiable in x and the γ_i's. Therefore

$$\begin{aligned}\tilde{w}_i &= \frac{\gamma_i}{2} + O(x^{\varepsilon_i}),\\ (w_i)_{\gamma_j} &= \frac{\delta_{i,j}}{2}\log x + \frac{1}{2}(\rho_i)_{\gamma_j} + O(x^{\varepsilon_i}\log x),\\ (\tilde{w}_i)_{\gamma_j} &= \frac{\delta_{i,j}}{2} + O(x^{\varepsilon_i}\log x)\end{aligned}$$

as $x \to 0$.

At $x = \infty$, from [17], if $s_1^{\mathbb{R}} \neq 0$,

$$w_i(x) = -s_1^{\mathbb{R}} 2^{-\frac{7}{4}}(\pi x)^{-\frac{1}{2}} e^{-2\sqrt{2}x} + O(x^{-1}e^{-2\sqrt{2}x}) \quad \text{as } x \to \infty, \tag{4.0.2}$$

where $s_1^{\mathbb{R}} = -2\cos\frac{\pi}{4}(\gamma_0 + 1) - 2\cos\frac{\pi}{4}(\gamma_1 + 3)$. If $s_1^{\mathbb{R}} = 0$, we have

$$\begin{aligned}w_0(x) &= s_2^{\mathbb{R}} 2^{-\frac{5}{2}}(\pi x)^{-\frac{1}{2}} e^{-4x} + O(x^{-1}e^{-4x}) \sim O(x^{-1}e^{-2\sqrt{2}x})\\ w_1(x) &= -s_2^{\mathbb{R}} 2^{-\frac{5}{2}}(\pi x)^{-\frac{1}{2}} e^{-4x} + O(x^{-1}e^{-4x}) \sim O(x^{-1}e^{-2\sqrt{2}x}),\end{aligned}$$

where $s_2^{\mathbb{R}} = -2 - 4\cos\frac{\pi}{4}(\gamma_0 + 1)\cos\frac{\pi}{4}(\gamma_1 + 3)$, so equation (4.0.2) holds for any generic (γ_0, γ_1). The equation (4.0.2) is also differentiable in x and the γ_i's, so

$$\begin{aligned}\tilde{w}_i(x) &= s_1^{\mathbb{R}} 2^{-\frac{1}{4}}\sqrt{\pi} x^{\frac{1}{2}} e^{-2\sqrt{2}x} + O(e^{-2\sqrt{2}x}),\\ (w_i)_{\gamma_j} &= -\left(s_1^{\mathbb{R}}\right)_{\gamma_j} 2^{-\frac{7}{4}}(\pi x)^{-\frac{1}{2}} e^{-2\sqrt{2}x} + O(x^{-1}e^{-2\sqrt{2}x}),\\ (\tilde{w}_i)_{\gamma_j} &= \left(s_1^{\mathbb{R}}\right)_{\gamma_j} 2^{-\frac{1}{4}}\sqrt{\pi} x^{\frac{1}{2}} e^{-2\sqrt{2}x} + O(e^{-2\sqrt{2}x})\end{aligned}$$

as $x \to \infty$.

By substituting the above asymptotic expansions into (4.0.1) we obtain

$$\frac{\partial}{\partial\gamma_i}\log\tau^w(x_1, x_2) = -\frac{\gamma_i}{4}\log x_1 - \sum_{k=0}^{1}\frac{\gamma_k}{4}(\rho_k)_{\gamma_i} - \frac{\gamma_i}{4} + O(x_1^{\varepsilon_i}\log x_1) + O(x_2^{\frac{3}{2}} e^{-2\sqrt{2}x_2})$$

as $x_1 \to 0$, $x_2 \to \infty$.

In our situation we have:

$$\tau^w(1, x) = C_0 x^{\frac{1}{8}(\gamma_0^2 + \gamma_1^2)}(1 + O(x^{\varepsilon})), \quad x \to 0,$$

$$\tau^w(1, x) = C_\infty e^{-x^2}(1 + O(x^{1/2}e^{-2\sqrt{2}x})), \quad x \to \infty.$$

Then

$$\log \tau^w(x_1, x_2) = \log \frac{C_\infty}{C_0} - x_2^2 - \frac{1}{8}(\gamma_0^2 + \gamma_1^2) \log x_1 + O(x_1^\varepsilon) + O(x_2^{1/2} e^{-2\sqrt{2}x_2}).$$

Let

$$C := \log \frac{C_\infty}{C_0} = \lim_{\substack{x_1 \to 0 \\ x_2 \to \infty}} \left(\log \tau^w(x_1, x_2) + x_2^2 + \frac{\gamma_0^2 + \gamma_1^2}{8} \log x_1 \right).$$

Then we obtain

$$\frac{\partial C}{\partial \gamma_i} = \lim_{\substack{x_1 \to 0 \\ x_2 \to \infty}} \left(\frac{\partial}{\partial \gamma_i} \left(\log \tau^w(x_1, x_2) + x_2^2 + \frac{\gamma_0^2 + \gamma_1^2}{8} \log x_1 \right) \right) = -\frac{\gamma_i}{4} - \sum_{k=0}^{1} \frac{\gamma_k}{4} (\rho_k)_{\gamma_i}$$

that is,

$$C = -\sum_{i=0}^{1} \frac{\gamma_i^2}{8} - \frac{1}{4} \sum_{k=0}^{1} \gamma_k \rho_k + \frac{1}{4} \int \sum_{k=0}^{1} \rho_k d\gamma_k. \tag{4.0.3}$$

Note that $\frac{\partial K}{\partial m_i} = -2 \frac{\partial K}{\partial \gamma_i}$, where K is the function defined in the proof of Theorem 47. Using the formula in Theorem 28, we obtain

$$\int \sum_{k=0}^{1} \rho_k d\gamma_k = -(\log 2) \sum_{k=0}^{1} \gamma_k^2 - 2K + \text{const}.$$

The constant above is independent of the γ_i's. By substituting $\gamma_0 = \gamma_1 = 0$, which corresponds to the trivial solution $w_0 \equiv w_1 \equiv 0$, into (4.0.3), we obtain $C = -4\left(\psi^{(-2)}(1/4) + \psi^{(-2)}(2/4) + \psi^{(-2)}(3/4)\right) + \text{const}$. On the other hand, the tau function $\tau^w(x_1, x_2)$ corresponding to the trivial solution is $\exp(x_1^2 - x_2^2)$, so $C = 0$ in this case. In conclusion we have the following result:

Theorem 68.

$$C = -\frac{1}{8}\left(\gamma_0^2 + \gamma_1^2\right) - \frac{1}{2}\left(\gamma_0 \rho_0 + \gamma_1 \rho_1\right) - \frac{1}{2} F + 4\left(\psi^{(-2)}(1/4) + \psi^{(-2)}(2/4) + \psi^{(-2)}(3/4)\right).$$

The function F in the theorem, which is the generating function, is given in Definition 46.

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Ryosuke Odoi, *Symplectic aspects of the tt*-Toda equations*, J. Phys. A **55** (2022), no. 16, article 165201

List of research achievements for application of Doctor of Science, Waseda University

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Date Submitted(yyyy/mm/dd): 2022/12/23

種類別 (By Type)	題名、 発表・発行掲載誌名、 発表・発行年月、 連名者（申請者含む） (theme, journal name, date & year of publication, name of authors inc. yourself)
論文○	Ryosuke Odoi, Symplectic aspects of the tt*-Toda equations, J. Phys. A **55** (2022), no.16, 165201
発表	Ryosuke Odoi, Symplectic geometry of the tt*-Toda equations and its application to the constant problem, Symplectic geometry and its application, 早稲田大学, 2022/10
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