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博　士　論　文　概　要
Doctoral Dissertation Synopsis

論　文　題　目
Dissertation Title

Parameter estimation for stochastic differential equations with small Lévy noise

小分散レヴィノイズを伴う確率微分方程式におけるパラメータ推定

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In this thesis, we study two types of parametric estimations for unknown parameters in the coefficient functions of stochastic differential equations (SDEs) with small Lévy noise, and we establish our estimators from a discrete sample path derived from the SDE with true parameters in the coefficient functions, called discretely observed case. Problems of parametric estimation for discretely observed stochastic processes with small diffusion have been studied by various authors (e.g., Genon-Catalot (1990), Laredo (1990), Sorensen and Uchida (2003) and so on) and problems of ones with small Lévy noise have been studied by Long et al. (2013, 2017) and references therein.

In the first half of this thesis, we focus on the parametric estimation for drift parameter only, and we propose a new type of least square estimator (LSE) based on the Adams method, which is well-known as a numerical computation method for calculating numerical solutions to ordinary differential equations (ODEs). Then, we prove the consistency and the asymptotic normality of the proposed estimator, and we show that the LSEs based on the Adams method can be better than the usual LSE based on the Euler method in the finite sample performance.

In order to say more precisely about our estimators in the first half of this thesis, let us introduce the Adams method, which is a numerical method for ODEs approximate solutions (see, e.g., Butcher (2016), Hairer et al. (1993), Hairer and Wanner (2010) and Iserles (2008)). The Adams method contains two linear multistep methods, *i.e.*, the Adams-Bashforth method and the Adams-Moulton method. When we consider numerical solutions to the following ODE by using the Adams method:

$$\frac{dx_t}{dt} = a(x_t) \tag{1}$$

with the initial condition $x_0 \in \mathbb{R}$, where a is a Lipschitz function, we first calculate an approximate value $x^*_{t_l}$ of the solution of (1) at $t = t_l$ by using the known past values $x_t,\ t = t_0, \dots, t_{l-1}$ $\left(t_k - t_{k-1} = \frac{1}{n},\ k = 1, \dots, l\right)$ as the following:

$$x^*_{t_l} = x_{t_{l-1}} + \frac{1}{n}\sum_{j=0}^{l-1} \gamma_j\, a(x_{t_j})$$

with some constant $\gamma_0, \dots, \gamma_{l-1} \in \mathbb{R}$, which is called the Adams-Bashforth method. Then, we next modify this approximate value $x^*_{t_l}$ as the following:

$$\hat{x}_{t_l} = x_{t_{l-1}} + \frac{1}{n}\sum_{j=0}^{l-1} \beta_j\, a(x_{t_j}) + \frac{1}{n}\beta_l\, a(x^*_{t_l})$$

with some constant $\beta_0, \dots, \beta_l$. Some values of the coefficients γ_j's and β_j's can be found in Table 244 in Butcher (2016). We remark that for any $g: \mathbb{R} \to \mathbb{R}$, the coefficients γ_j's and β_j's satisfy

$$\int_{t_{l-1}}^{t_l} P(s; g, t_0, \dots, t_{l-1})\, ds = \frac{1}{n} \sum_{j=0}^{l-1} \gamma_j\, g\left(x_{t_j}\right),$$

and

$$\int_{t_{l-1}}^{t_l} P(s; g, t_0, \dots, t_l)\, ds = \frac{1}{n} \sum_{j=0}^{l} \beta_j\, g\left(x_{t_j}\right),$$

where $s \mapsto P(s; g, t_0, \dots, t_l)$ is the Lagrange interpolating polynomial through the points $(s, g(s))$, $s = t_0, \dots, t_l$ (see, e.g., Section III.1 in Hairer et al. (1993)). By substituting $g \equiv 1$ in the above equations, we obtain

$$\sum_{j=0}^{l-1} \gamma_j = \sum_{j=0}^{l} \beta_j = 1.$$

Here, we suppose that we have a discrete sample $\{X_{t_k}^{\epsilon}\}_{k=0,\dots,n}$ derived by

$$dX_t^{\epsilon} = a(X_t^{\epsilon}, \theta_0)\, dt + \epsilon\, dL_t, \quad X_0^{\epsilon} = x_0 \in \mathbb{R}^d,$$

where Θ_0 is a smooth bounded open convex set in $\mathbb{R}^p$ with $p \in \mathbb{N}$, $\theta_0 \in \Theta_0$, $\epsilon > 0$, a is a function from $\mathbb{R}^d \times \overline{\Theta}_0$ to $\mathbb{R}^d$, and $L = (L_t)_{t \geq 0}$ is a d-dimensional Lévy process. As in the usual literature (e.g., Long et al. (2013)), we use the following contrast function for LSE based on Euler method:

$$\Psi_{n,\epsilon}(\theta) = \frac{n}{\epsilon^2} \sum_{k=1}^{n} \left| X_{t_k}^{\epsilon} - X_{t_{k-1}}^{\epsilon} - \frac{1}{n} a\left(X_{t_{k-1}}^{\epsilon}, \theta\right) \right|^2,$$

though the Euler method sometimes fails to approximate the solution of ODEs (e.g., $a(x, \theta) = -\theta\, x$ for $x, \theta > 0$ and $\theta / n \notin (0,2)$, in Section 4.2 in Iserles (2008)) and is less accurate than the Runge-Kutta method, the Adams method, etc. Of course, these numerical approximation methods except for the Euler method do not work to calculate numerical solutions to SDEs, while they are available for the ODE given in the limit $\epsilon \to 0$. Thus, we employ the Adams method instead of the Euler method and define the Adams-Moulton type contrast function $\Psi_{n,\epsilon,l}(\theta)$ as

$$\Psi_{n,\epsilon,l}(\theta) = \frac{n}{\epsilon^2} \sum_{k=l}^{n} \left| X_{t_k}^{\epsilon} - X_{t_{k-1}}^{\epsilon} - \frac{1}{n} \sum_{j=0}^{l} \beta_j\, a\left(X_{t_{k-l+j}}^{\epsilon}, \theta\right) \right|^2.$$

We can also define the Adams-Moulton type contrast function as we shall see in Chapter 1 of this thesis, and will discuss only the Adams-Moulton type contrast function, since the proof for Adams-Bashforth type is analogous.

In the second half, we expand our scope to the joint estimation of the parameters in the drift, diffusion and jump terms, while we restrict Lévy noise to compound Poisson process. In the ergodic case, such joint estimation for SDEs with Lévy noise is proposed in Shimizu and Yoshida (2006), and it has been considered so far by various researchers (references are given in Amorino and Gloter (2021)). On the other hand, in the small noise case, no one has succeeded in giving a proof for such joint threshold estimation of the parameter relative to drift, diffusion and jumps. So, the aim of the second half of this thesis is to give a framework and a proof for the threshold estimation in the small noise case. As an essential part of our framework for estimation, we suppose not only $n \to \infty$ and $\epsilon \to 0$ but $\lambda \to \infty$ (λ is the intensity of the Lévy noise), while the intensity λ is fixed in the usual small noise case (see, *e.g.*, Sørensen and Uchida (2003)). The asymptotics with $\lambda \to \infty$ would be the first and new attempt in many works of literature, and it enables us to deal with the joint estimation of the parameters relative to drift, diffusion and jumps, while the previous works in the small noise case deal with only the estimation of drift and diffustion parameters (or in some papers drift parameter only). Note that the assumption $\lambda \to \infty$ seems natural when we deal with data obtained in the long term with the pitch of observations shortened, which is familiar in both cases of ergodic and small noise.

Furthermore, in the second half of this thesis, we aim to give a proof by using localization argument (as in, e.g., Remark 1 in Sørensen and Uchida (2003)) in the entire context, and to simplify the contrast functions used in earlier works in the ergodic case. Then, we can establish many examples of the class of jump size densities, which includes unbounded densities (e.g., log-normal distribution) and are not included in the earlier works in the ergodic case.

List of research achievements for application of Doctor of Science, Waseda University

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