無限次元線形行列不等式に基づく制御系の解析設計法に関する研究

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まえがき

パラメータを持つ線形系としてモデル化できる制御対象のクラスは広く，例えば外生信号によって変化する特性をもつ線形系，状態の一部をパラメータにとることにより線形形となる非線形系，さらには時間変数以外の独立変数を持つ分布定数系を含んでいる．パラメータを持つ線形系としてモデル化できれば，パラメータを固定する毎に線形系に対する方法を適用して解析設計が出来る．スケジューリングパラメータを用いて非線形系を線形系として記述することから出発するゲインスケジューリングの方法は，その代表的な例である．近年，線形制御系に対する多くの解析設計問題が線形行列不等式の解を求める問題に帰着されることがわかってきた．さらに，線形行列不等式の効率的な解法が開発され，信頼性の高いソフトウェアパッケージが入手できるようになっている．このような背景から，パラメータを持つ線形系に対しても線形行列不等式に基づく解析設計法の確立が望まれる．

本研究では，スケジューリングパラメータを持つ線形系に対して本研究者によって展開されてきた解析設計手法を発展させ，不確かなパラメータを持つロバスト／適応系，むだ時間や空間変数などのパラメータを持つ分布定数系の解析設計にも適用できる手法を確立することを目的とした．

まず，状態の一部分をパラメータにとることで線形形となる非線形系，むだ時間や空間パラメータを持つ分布定数系，不確かなパラメータを持つ系について，モデリング手法およびモデリングの実例の調査，および関連の研究のサーベイから開始した．次に，パラメータを持つ線形系のモデリングをおこない制御問題を定式化した．具体的には，想定する 3 種類のパラメータを用いた線形系によるモデリングの有効性を詳細に調べ，モデルに基づいて構成される制御則によってどのような制御性能が保証できるのかを，パラメータの性質に基づいて理論的に，また計算機シミュレーションを用いて検討し，制御問題を定式化した．

本研究の前半での成果は次のように要約される．すなわち，定式化された制御問題において，無限次元線形行列不等式によって記述される条件の導出をおこないゲインスケジューリングにおける隠れループの問題の解法を提案したことである．最初に，

定式化された制御問題に対する条件を，パラメータに依存する無限次元線形行列不等式の形式で導出した．引き続き，非線形形をパラメータを持つ線形系としてモデル化した場合に焦点を絞り，パラメータが状態に依存してしまう（すなわち隠れループが存在する）ことを事前に考慮するために，可到達領域の評価法を導入して，閉ループ系の特性のみならずパラメータの特性も含めた解析設計のための無限次元線形行列不等式条件を導出した．

本研究の後半では，むだ時間や空間パラメータを持つ分布定数系を重点的に検討した．まず，むだ時間／空間パラメータを持つ分布定数系や不確かなパラメータを持つ系のパラメータを用いた線形系によるモデリングの有効性を詳細に検討した．また，モデルに基づいて構成される制御則によってどのような制御性能が保証できるのかを，パラメータの性質に基づいて理論的に，また計算機シミュレーションを用いて検討した．以上の検討結果を踏まえて，むだ時間／分布定数系およびロバスト／適応系の解析設計の無限次元線形行列不等式による解法を提案した．定式化された問題を，むだ時間／空間変数をパラメータとした場合，不確かなパラメータも含めたパラメータの場合について検討し，それぞれの場合の固有の問題点の解決を図りながら，無限次元線形行列不等式の形式で制御系の解析設計条件を導出した．

研究組織

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出力可到達集合解析に基づいたアンチワインドアップコントローラの解析†

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Analysis of Anti-Windup Controllers Based on Analysis of Output Reachable Sets†

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The actual control systems possess some kind of restrictions on the control inputs like saturation, rate limit, and so on. It is known that these restrictions usually cause the large overshoot of the controlled variables for step reference signals. This overshoot phenomenon is called the windup phenomenon. Though the windup phenomenon crucially causes the undesirable performance of the actual control systems, formulation of the windup phenomenon from a control-theoretical point of view has not been proposed yet.

In this paper, we characterize the windup phenomenon via output reachable sets and propose analysis technique for the windup based on reachable set analysis. Then we propose a new framework for anti-windup technique and characterize its anti-windup performance via output reachable sets. We also propose analysis technique for anti-windup performance based on reachable set analysis.

Key Words: windup, anti-windup, output reachable set, gain scheduling

1. はじめに

実際の制御系では「アクチュエーターに飽和特性が存在する」「制御対象を保護する」等の理由から，その制御入力は何らかの形で制限されている．このような制御系では，多くの場合にステップ応答等が大きくオーバーシュートするワインドアップ[4),8),15)]と呼ばれる現象が発生することが知られている．ワインドアップは実際の制御系においてその制御性能を大きく低下させる要因であるにもかかわらず，これまでに与えられている記述は，安定性からの解釈や特定のコントローラと特定の飽和要素に対する定性的な説明がほとんどで，何をもってワインドアップと呼ぶかという制御理論的な定式化はなされていない．

ここで，制御入力に対する制限が飽和要素である場合を考えてみる．この場合，制御系に加わる指令値が充分小さければそれに応じて制御入力の大きさも減少，飽和要素は制御系に何ら影響を与えない．このことは，制御入力に対する制限が制御系に与える影響は，指令値や内部信号 (制御入力 etc.) のピーク値に深く依存していることを示唆している．そこで，本稿ではピーク値が制限された指令値に対して制御系の出力が到達可能な領域，出力可到達集合[20)]，を考え，これに基づいてワインドアップを特徴づけることを試みる．

一方，ワインドアップの発生は実際の制御における古くからの問題であったことから，その抑制を目的とした手法が現在までに幾つか提案されている[1),4),5),8)~11),14),16),18),22)]．その中の一つに，制限が存在しないものとして設計されたコントローラにワインドアップの抑制を目的とするフィードバックループを付け加える，という手法がある．この手法はアンチワインドアップ手法[4),8)]と呼ばれ，その適用の容易さも手伝い現在広く用いられている．本稿では，従来提案されているアンチワインドアップ手法に対し，ゲインスケジューリング[13),21)]の観点からそれらを包含する新しい枠組を提案する．また，アンチワインドアップ手法におけるコントローラであるアンチワインドアップコントローラの性能を，ワインドアップと同様に出力可到達集合に基づいて特徴づけることを試みる．

以下，2 節では出力可到達集合に基づいてワインドアップを定式化し，3 節では出力可到達集合解析を用いたワインドアップの解析手法を与える．4 節では従来のアンチワインドアップ手法をゲインスケジューリングの観点からとらえた枠組を示し，そのアンチワインドアップ性能をワインドアップと同様に出力可到達集合に基づいて定式化する．5 節では出力可到達集合解析を用いたアンチワインドアップ性能の解析手法を与える．

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† 第 25 回制御理論シンポジウムで発表 (1996・5)

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なお，本稿で用いる記法は次の通りである．行列 P の転置を P' で表す．$P > (<)0$ は行列 P が正定 (負定) であることを，$P \geq (\leq)0$ は行列 P が半正定 (半負定) であることを表す．また，$P > 0$ に対し，$\mathcal{E}(P)$ を次で定義する．

$$\mathcal{E}(P) = \{\zeta |\ \zeta' P \zeta \leq 1\}.$$

2. ワインドアップ

2.1 制御入力に対する制限

コントローラの設計は通常 **Fig. 1** の枠組でおこなわれるが，実際の制御系では「アクチュエーターに飽和特性が存在する」「制御対象を保護する」等の理由から，その制御入力に何らかの制限が存在する．多くの場合，このような制御入力 u に対する制限は，**Fig. 2** に示される δ_a として表現することが可能である．ここで，Fig.2 における u_a は次で与えられるものとする．

$$u_a = \delta_a(u)u.$$

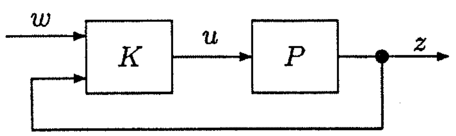

Fig. 1 Framework of controller design

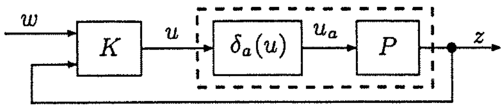

Fig. 2 Constraint on control input

さらに，実際の制御系では $u \neq u_a$ を回避するために，**Fig. 3** に示されるコントローラの出力 u_c に対する制限 δ をあらかじめ設けておくことが多い．この場合，δ を $u = u_a$ がつねに成立するように設定することで，Fig. 3 の制御系を **Fig. 4** の制御系と見なすことができる．

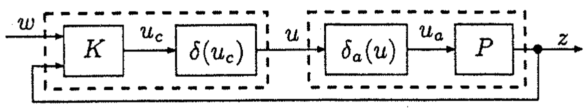

Fig. 3 Actual control system

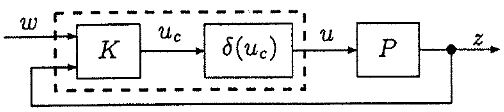

Fig. 4 Equivalent control system

(注意) 制御入力に対する制限 δ_a の値は必ずしもオンラインで入手できるとは限らないが，コントローラの出力に対する制限 δ は δ_a とは異なりオンラインでつねに入手可能なパラメータである．

2.2 ワインドアップ現象

Fig.1 の枠組で設計されたコントローラを Fig.4 の制御系に直接適用すると，出力 z のステップ応答が Fig.1 の制御系と比べて大きくオーバーシュートするワインドアップ[8)15)]と呼ばれる現象を生ずることがある．ワインドアップを生ずる制御系の例を次に示す．

〔例〕[17)] 制御対象 P，コントローラ K，およびコントローラの出力に対する制限 δ が次で与えられているものとする．

制御対象

$$\begin{bmatrix} \dot{x}_p \\ z \end{bmatrix} = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] \begin{bmatrix} x_p \\ u \end{bmatrix}$$

コントローラ

$$\begin{bmatrix} \dot{x}_c \\ u_c \end{bmatrix} = \left[\begin{array}{c|cc} 0 & 3 & -3 \\ \hline 30 & 0 & -30 \end{array}\right] \begin{bmatrix} x_c \\ w \\ z \end{bmatrix}$$

コントローラの出力に対する制限

$$\delta(u_c) = \begin{cases} 1, & |u_c| \leq 2.25 \\ 2.25/|u_c|, & |u_c| > 2.25 \end{cases}$$

ここで，δ は飽和要素を表していることに注意する．上記の P, K, δ を用いた場合の Fig.1，および Fig.4 の制御系の応答を **Fig. 5** に示す．Fig.5 において，実線が Fig.1 の制御系の応答を，破線が Fig.4 の制御系の応答を示す．

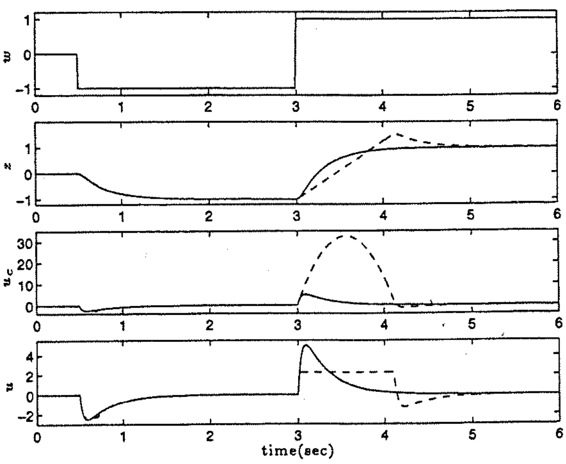

Fig. 5 Step responses

この例からも明らかなように，制御系の性能はワインドアップの発生によって大きく低下する．ワインドアップの発生に関する共通の認識として，δ によって引き起こされるコントローラの出力と制御入力の不一致がその原因であること，コントローラが積分要素等の収束の遅いダイナミクスを

有する場合に発生し易いこと等があげられるが，これまでに与えられているワインドアップに関する記述は，安定性からの解釈 (例えば文献 9)) と，特定のコントローラと特定の飽和要素に対する定性的な説明 (例えば文献 15)) がほとんどで，何をもってワインドアップと呼ぶかという制御理論的な定式化はなされていないのが現状である．

2.3　ワインドアップの定式化

ワインドアップの定式化を議論するにあたり，Fig.5 における $t=0$ から $t=3$ までの応答に注目する．この範囲では $u_c \approx u$，また，ワインドアップは発生していない．その理由として，Fig.4 の制御系に加わる指令値 w のピーク値が小さければ，それに応じてコントローラの出力，u_c のピーク値も減少，δ が制御系に与える影響が小さくなることがあげられる．このことは，δ が制御系に与える影響は，指令値 w や制御系の内部信号のピーク値に深く依存していることを示唆している．

そこで，出力 z に対する出力可到達集合 (与えられた有界閉集合に含まれる指令値 w に対して出力 z が到達可能な領域) に基づいたワインドアップの特徴づけを考える．なお，以降では Fig.4 の制御系を $\Sigma(K,P,\delta)$ と呼ぶことにする．このとき，Fig.1 の制御系は $\Sigma(K,P,I)$ で表されることに注意する．

制御系 $\Sigma(K,P,\delta)$ に次を仮定する．

［仮定 1］　制御系 $\Sigma(K,P,\delta)$ の状態 x の初期値は $x(0)=0$ であるものとする．また，任意の指令値 w に対してコントローラの出力 u_c，制御入力 u，制御対象の出力 z の初期値はそれぞれ $u_c(0)=0$, $u(0)=0$, $z(0)=0$ であるものとする．

［仮定 2］　制御系 $\Sigma(K,P,\delta)$ における指令値 w は，あらかじめ与えられた有界閉集合 W に対して，次の条件を満たしているものとする．

$$w(t) \in W, \quad \forall t \in [0,\infty). \tag{1}$$

制御系 $\Sigma(K,P,\delta)$ の出力 z に対する出力可到達集合を以下で定義する．

【定義】　制御系 $\Sigma(K,P,\delta)$ において，時間 $T \in [0,\infty)$ で $z(T)=\zeta$ を実現する (1) 式の条件を満たす外乱 w が存在するとき，ζ は $z(0)$ から出力可到達であるという．

【定義】　制御系 $\Sigma(K,P,\delta)$ の出力 z に対する出力可到達集合 $\mathcal{R}_z(\Sigma(K,P,\delta))$ を次で定義する．

$$\mathcal{R}_z(\Sigma(K,P,\delta)) = \{\zeta : \zeta \text{ は } z(0) \text{ から出力可到達}\}.$$

制御系 $\Sigma(K,P,\delta)$ におけるワインドアップの発生を，出力可到達集合に基づいて定式化する．

【定義】　制御系 $\Sigma(K,P,\delta)$ の出力 z に対する出力可到達集合 $\mathcal{R}_z(\Sigma(K,P,I))$ と制御系 $\Sigma(K,P,I)$ の出力 z に対する出力可到達集合 $\mathcal{R}_z(\Sigma(K,P,\delta))$ に対して次が成立しているとき，制御系 $\Sigma(K,P,\delta)$ には ζ 方向にワインドアップが発生するという．

$$\exists\zeta \in \mathcal{R}_z(\Sigma(K,P,\delta)) \quad \text{s.t.} \quad \zeta \notin \mathcal{R}_z(\Sigma(K,P,I)).$$

3.　ワインドアップの解析

2.3 節で与えたワインドアップの定式化に従って議論を進めるためには，Fig.1 に示される制御系 $\Sigma(K,P,I)$ の出力可到達集合を実際に解析することが重要な問題となる．本節では，制御対象 P，およびコントローラ K が線形時不変の場合に，制御系 $\Sigma(K,P,I)$ の出力可到達集合を実際に評価する手法を与え、そのワインドアップ解析への具体的な適用を示す．

3.1　制御系 $\Sigma(K,P,I)$ の出力可到達集合

Fig.1 の制御系 $\Sigma(K,P,I)$ において，制御対象 P，およびコントローラ K の状態変数表現が次で与えられているものとする．

制御対象

$$\begin{bmatrix} \dot{x}_p \\ z \end{bmatrix} = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & 0 \end{array}\right] \begin{bmatrix} x_p \\ u \end{bmatrix}. \tag{2}$$

コントローラ

$$\begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \left[\begin{array}{c|cc} A_c & B_{cw} & B_{cz} \\ \hline C_c & 0 & D_{cz} \end{array}\right] \begin{bmatrix} x_c \\ w \\ z \end{bmatrix}. \tag{3}$$

このとき，制御系 $\Sigma(K,P,I)$ の状態変数表現は次で与えられる．

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_z & 0 \end{array}\right] \begin{bmatrix} x \\ w \end{bmatrix}.$$

ここで，$x=[x_p', x_c']'$，また，A，B，および C_z は次で与えられるものとする．

$$A = \begin{bmatrix} A_p + B_p D_{cz} C_p & B_p C_c \\ B_{cz} C_p & A_c \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ B_{cw} \end{bmatrix},$$

$$C_z = \begin{bmatrix} C_p & 0 \end{bmatrix}.$$

また，制御系 $\Sigma(K,P,I)$ に対する指令値 w は次の条件を満たしていると仮定する．

$$w'(t)w(t) \le 1, \quad \forall t \in [0,\infty).$$

このとき，制御系 $\Sigma(K,P,I)$ の出力 z に対する出力可到達集合 $\mathcal{R}_z(\Sigma(K,P,I))$ に関して次が成立する．

《定理 1》[20]　$P>0$, $S>0$, および $\alpha>0$ が存在し，次の条件を満たしているとする．

$$\begin{bmatrix} A'P + PA + \alpha P & PB \\ B'P & -\alpha I \end{bmatrix} \le 0,$$

$$\begin{bmatrix} P & C_z' \\ C_z & S \end{bmatrix} \ge 0.$$

このとき，制御系 $\Sigma(K,P,I)$ の出力 z に対する出力可到達集合 $\mathcal{R}_z(\Sigma(K,P,I))$ は次を満たす．

$$\mathcal{R}_z(\Sigma(K,P,I)) \subset \mathcal{E}(R), \quad R=S^{-1}.$$

（注意） 証明に関しては文献 3), 20), 19) を参照のこと．また，定理 1 に基づいた出力可到達集合の具体的な評価手順については，文献 20), 19), 2), 12) を参照のこと．

3.2 数値例

コントローラ K，制御対象 P，およびコントローラの出力に対する制限 δ は 2.2 節の例と同様とする．このとき，制御系 $\Sigma(K,P,I)$ の出力 z に対する出力可到達集合 $\mathcal{R}_z(\Sigma(K,P,I))$ の評価は，定理 1 に基づき次で与えられる．

$$\mathcal{R}_z(\Sigma(K,P,I)) \subset \{\zeta\ :\ |\zeta| \leq 1.0310\}. \tag{4}$$

制御系 $\Sigma(K,P,I)$ の応答，Fig.4 の制御系 $\Sigma(K,P,\delta)$ の応答，および制御系 $\Sigma(K,P,I)$ の出力 z に対する出力可到達集合の上限を **Fig. 6** に示す．Fig.6 において，実線が制御系 $\Sigma(K,P,I)$ の応答を，破線が制御系 $\Sigma(K,P,\delta)$ の応答を，一点鎖線が制御系 $\Sigma(K,P,I)$ の出力 z に対する出力可到達集合の上限を表す．

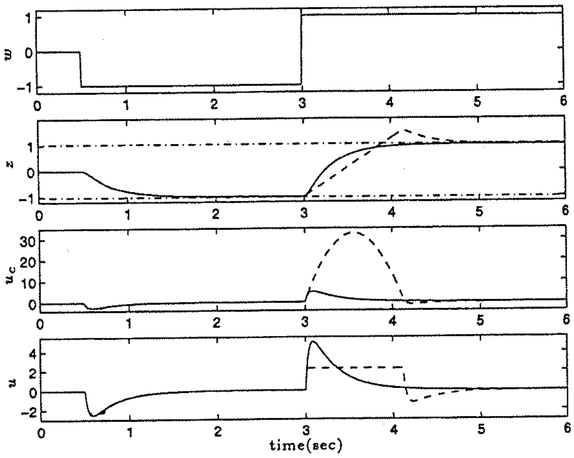

Fig. 6 Step responses

Fig.6 の応答における制御系 $\Sigma(K,P,\delta)$ の出力 z の最大値を ζ とすると，ζ は次で与えられる．

$$\zeta = 1.5097$$

上記の ζ は $\zeta \in \mathcal{R}_z(\Sigma(K,P,\delta))$ である一方，(4) 式から

$$\zeta \notin \mathcal{R}_z(\Sigma(K,P,I)).$$

従って，制御系 $\Sigma(K,P,\delta)$ には 2.3 節で与えた定式化の意味でワインドアップが発生することが結論される．

4. アンチワインドアップ

4.1 アンチワインドアップコントローラ

2.2 節の注意で述べたように，コントローラの出力に対する制限 δ は，制御入力に対する制限 δ_a の値とは異なりオンラインでつねに入手可能なパラメータである．このことを鑑み，δ をスケジューリングパラメータ[13),21)] とするコントローラ $K(\delta)$ で，Fig.1 の枠組で設計されたコントローラ K に対して次の条件を満たすものを考える．

$$K(I) = K. \tag{5}$$

(5) 式は，コントローラの出力に対する制限 δ が存在しない場合，つまり $\delta = I$ である場合に，$K(\delta)$ が K と一致することを要求する条件である．コントローラ $K(\delta)$ を用いた制御系を **Fig. 7** に示す．(5) 式の条件を満たすコントローラには 2.3 節の意味でのワインドアップを抑制する可能性が存在することから，本稿ではこれをアンチワインドアップコントローラと呼ぶことにする．

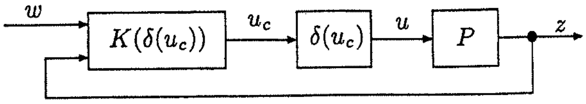

Fig. 7 Anti-windup controller

次に，従来提案されているアンチワインドアップコントローラが上記の $K(\delta)$ の形で表現されることを，Campo らによって提案されたアンチワインドアップコントローラ[4)] を例に示す．なお，Campo らのアンチワインドアップコントローラは，従来提案されている多くのアンチワインドアップコントローラを含んでいることが文献 4) において示されている．

〔例〕 Campo らによって提案されたアンチワインドアップコントローラを用いた制御系を **Fig. 8** に示す．

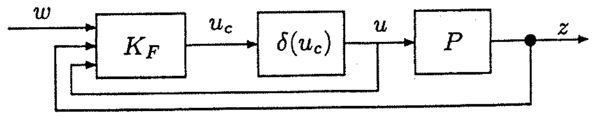

Fig. 8 Campo's anti-windup controller

Fig.1 の枠組で設計されたコントローラ K の状態変数表現が (3) 式で与えられているとき，Fig.8 におけるアンチワインドアップコントローラ K_F の状態変数表現は次で与えられる．

$$\begin{bmatrix} \dot{x}_c \\ u_c \end{bmatrix} = \left[\begin{array}{c|ccc} A_c - FC_c & B_{cw} & B_{cz} - FD_{cz} & F \\ \hline C_c & 0 & D_{cz} & 0 \end{array}\right] \begin{bmatrix} x_c \\ w \\ z \\ u \end{bmatrix}$$

ここで，F は K_F の設計パラメータを表す．上記の K_F を Fig.7 における $K(\delta)$ の形で書き直した場合，その状態変数表現は次で与えられる．

$$\begin{bmatrix} \dot{x}_c \\ u_c \end{bmatrix} = \left[\begin{array}{c|cc} A_c(\delta(u_c)) & B_{cw} & B_{cz}(\delta(u_c)) \\ \hline C_c & 0 & D_{cz} \end{array}\right] \begin{bmatrix} x_c \\ w \\ z \end{bmatrix} \tag{6}$$

ここで

$$A_c(\delta) = A_c - (1-\delta)FC_c,$$
$$B_{cz}(\delta) = B_{cz} - (1-\delta)FD_{cz}.$$

なお，この $K(\delta)$ が (5) 式の条件を満たすことは，容易に確かめられる．

4.2　アンチワインドアップの定式化

2.3 節で定式化したワインドアップと同様に，コントローラ $K(\delta)$ のアンチワインドアップ性能を出力可到達集合に基づいて定式化する．

【定義】 $K(\delta)$ をアンチワインドアップコントローラとする．制御系 $\Sigma(K,P,\delta)$ の出力可到達集合 $\mathcal{R}_z(\Sigma(K,P,\delta))$ と制御系 $\Sigma(K(\delta),P,\delta)$ の出力可到達集合 $\mathcal{R}_z(\Sigma(K(\delta),P,\delta))$ に対して，次が成立しているとき，$K(\delta)$ は ζ 方向にアンチワインドアップ性能を有するという．

$$\exists\zeta \in \mathcal{R}_z(\Sigma(K,P,\delta)) \quad \text{s.t.} \quad \zeta \notin \mathcal{R}_z(\Sigma(K(\delta),P,\delta)).$$

5.　アンチワインドアップの解析

4.2 節で与えたアンチワインドアップの定式化に従って議論を進めるためには，Fig.7 の制御系 $\Sigma(K(\delta),P,\delta)$ の出力可到達集合を実際に解析することが重要な問題となる．本節では，コントローラの出力に対する制限 δ が飽和要素，制御対象 P が線形時不変，およびコントローラ $K(\delta)$ が δ をスケジューリングパラメータとする線形システム [13),21)] の場合に，制御系 $\Sigma(K(\delta),P,\delta)$ の出力可到達集合を実際に評価する手法を与え，そのアンチワインドアップ解析への具体的な適用を示す．なお，記述の繁雑さ避けるために本節における制御対象は SISO とする．

5.1　制御系 $\Sigma(K(\delta),P,\delta)$ の出力可到達集合

Fig.7 の制御系 $\Sigma(K(\delta),P,\delta)$ における制御対象 P の状態変数表現が (2) 式で，コントローラ $K(\delta)$ の状態変数表現が次で与えられているものとする．

$$\begin{bmatrix}\dot{x}_c\\ u_c\end{bmatrix} = \left[\begin{array}{c|cc} A_c(\delta(u_c)) & B_{cw}(\delta(u_c)) & B_{cz}(\delta(u_c))\\ \hline C_c(\delta(u_c)) & 0 & D_{cz}(\delta(u_c))\end{array}\right]\begin{bmatrix}x_c\\ w\\ z\end{bmatrix}.$$

また，制御対象 P は SISO とする．このとき，制御系 $\Sigma(K(\delta),P,\delta)$ の状態変数表現は次で与えられる．

$$\begin{bmatrix}\dot{x}\\ z\\ u_c\end{bmatrix} = \left[\begin{array}{c|c} A(\delta(u_c)) & B(\delta(u_c))\\ \hline C_z & 0\\ C_u(\delta(u_c)) & 0\end{array}\right]\begin{bmatrix}x\\ w\end{bmatrix}.$$

ここで，$x=[x_p', x_c']'$，また $A(\delta)$, $B(\delta)$, C_z, $C_u(\delta)$ は次で与えられるものとする．

$$A(\delta) = \begin{bmatrix}A_p + B_p\delta D_{cz}(\delta)C_p & B_p\delta C_c(\delta)\\ B_{cz}(\delta)C_p & A_c(\delta)\end{bmatrix},$$
$$B(\delta) = \begin{bmatrix}0\\ B_{cw}(\delta)\end{bmatrix},$$
$$C_z = \begin{bmatrix}C_p & 0\end{bmatrix},$$
$$C_u(\delta) = \begin{bmatrix}D_{cz}(\delta) & C_c(\delta)\end{bmatrix}.$$

$\delta(u)$ は次で与えられるものとする．

$$\delta(u_c) = \begin{cases}1, & |u_c| \le u_{max},\\ u_{max}/|u_c|, & |u_c| > u_{max}.\end{cases}.$$

また，制御系 $\Sigma(K(\delta),P,\delta)$ に対する指令値 w は次の条件を満たしていると仮定する．

$$w'(t)w(t) \le 1, \quad \forall t \in [0,\infty). \tag{7}$$

このとき，制御系 $\Sigma(K(\delta),P,\delta)$ の出力 z に対する出力可到達集合 $\mathcal{R}_z(\Sigma(K(\delta),P,\delta))$ について次が成立する．

《定理 2》 ある $\delta_{min} \in [0,1]$ に対して $P_u > 0$, $P_z > 0$, $S_u > 0$, $S_z > 0$ が存在し，任意の $\delta \in [\delta_{min}, 1]$ に対して次の条件を満たしているものとする．

$$\begin{bmatrix}A'(\delta)P_u + P_uA(\delta) + \alpha P_u & P_uB(\delta)\\ B'(\delta)P_u & -\alpha I\end{bmatrix} \le 0, \tag{8}$$

$$\begin{bmatrix}P_u & C_u'(\delta)\\ C_u(\delta) & S_u\end{bmatrix} \ge 0, \tag{9}$$

$$\frac{u_{max}}{\sqrt{S_u}} \ge \delta_{min}, \tag{10}$$

$$\begin{bmatrix}A'(\delta)P_z + P_zA(\delta) + \alpha P_z & P_zB(\delta)\\ B'(\delta)P_z & -\alpha I\end{bmatrix} \le 0, \tag{11}$$

$$\begin{bmatrix}P_z & C_z'\\ C_z & S_z\end{bmatrix} \ge 0. \tag{12}$$

このとき，制御系 $\Sigma(K(\delta),P,\delta)$ の出力 z に対する出力可到達集合 $\mathcal{R}_z(\Sigma(K(\delta),P,\delta))$ は次を満たす．

$$\mathcal{R}_z(\Sigma(K(\delta),P,\delta)) \subset \mathcal{E}(R_z), \quad R_z = S_z^{-1}. \tag{13}$$

（証明） (8) 式，(9) 式を満たす P_u,S_u が存在するならば，$\delta(u_c)$ の最小値を δ_{min} と仮定した場合の u_c に対する出力可到達集合は，付録定理 5 より次のように評価される．

$$\mathcal{R}_{u_c} \subset \{\nu \,:\, |\nu| \le \sqrt{S_u}\}.$$

これは，(7) 式の条件を満たす任意の外乱 w に対し，$\delta(u_c)$ が次を満たすことを意味している．

$$\delta(u_c) \ge \frac{u_{max}}{\sqrt{S_u}}. \tag{14}$$

(14) 式と (10) 式から次をえる．

$$\delta(u_c) \ge \frac{u_{max}}{\sqrt{S_u}} \ge \delta_{min}. \tag{15}$$

(15) 式は，(7) 式の条件を満たす任意の外乱 w に対する $\delta(u_c)$ の範囲を次で与えたことの妥当性を示している．

$$\delta(u_c) \in [\delta_{min}, 1]. \tag{16}$$

一方，(11) 式，(12) 式を満たす P_z, S_z が存在するならば，$\delta(u_c)$ の範囲が (16) 式で与えられると仮定した場合における出力 z に対する出力可到達集合は，付録定理 5 より (13) 式で評価される．

（注意） 定理 2 に基づいた出力可到達集合の具体的な評価手順については文献 20), 2), 12), 19) を参照のこと．

5.2 数値例

アンチワインドアップコントローラ $K(\delta)$ の状態変数表現は (6) 式で与えられているものとする．制御対象 P，Fig.1 の枠組で設計されたコントローラ K，およびコントローラの出力に対する制限 $\delta(u_c)$ は 2.2 節の例 2.1，および 3.2 節の数値例と同様とする．また，$K(\delta)$ の設計パラメータ F は $F = 0.3$ であるものとする．

定理 5.1 に基づいた制御系 $\Sigma(K(\delta), P, \delta)$ の出力可到達集合を解析するにあたり，$\delta_{min} = 0.11$ とした．このとき得られた $\sqrt{S_u}$ は次の通り．

$$\sqrt{S_u} = 20.285$$

ここで，次が成立することに注意する．

$$\frac{u_{max}}{\sqrt{S_u}} = \frac{2.25}{20.285} = 0.11092 \geq 0.11 = \delta_{min}.$$

従って，上記の $\sqrt{S_u}$ は u_{max}, δ_{min} に対して (10) 式の条件を満たす．一方，出力 z に対する出力可到達集合に関して得られた評価は次の通りである．

$$\mathcal{R}_z(\Sigma(K(\delta), P, \delta)) \subset \{z \in R \mid |z| \leq 1.3382\}. \tag{17}$$

制御系 $\Sigma(K, P, \delta)$ の応答，制御系 $\Sigma(K(\delta), P, \delta)$ の応答，および制御系 $\Sigma(K(\delta), P, \delta)$ の出力 z に対する出力可到達集合の上限を **Fig. 9** に示す．Fig.9 において，実線が制御系 $\Sigma(K, P, \delta)$ の応答を，破線が制御系 $\Sigma(K(\delta), P, \delta)$ の応答を，一点鎖線が制御系 $\Sigma(K(\delta), P, \delta)$ に対する出力可到達集合の上限を表す．

Fig.9 における制御系 $\Sigma(K, P, \delta)$ の出力 z の最大値を ζ とすると，ζ は次で与えられる．

$$\zeta = 1.5097$$

上記 ζ は $\zeta \in \mathcal{R}_z(\Sigma(K, P, \delta))$ である一方，(17) 式から

$$\zeta \notin \mathcal{R}_z(\Sigma(K(\delta), P, \delta)).$$

従って，コントローラ $K(\delta)$ は 4.2 節で与えた定式化の意味でアンチワインドアップ性能を有することが結論される．

6. おわりに

本稿では，ワインドアップを可到達集合の考え方に基づいて定式化，可到達集合解析に基づいたワインドアップの解析法を提案した．また，ワインドアップの抑制を実現するコントローラとして従来提案されていたアンチワインドアップコントローラを，ゲインスケジューリングの観点からとらえた枠組を示した．さらに，ワインドアップと同様にアンチワインドアップコントローラの性能を可到達集合の考え方に基づいて定式化，可到達集合解析に基づいたアンチワインドアップ性能の解析手法を提案した．数値例では，提案したワインドアップの解析手法，およびアンチワインドアップ性能の解析手法を与えられた問題に実際に適用，その有効性を確認した．

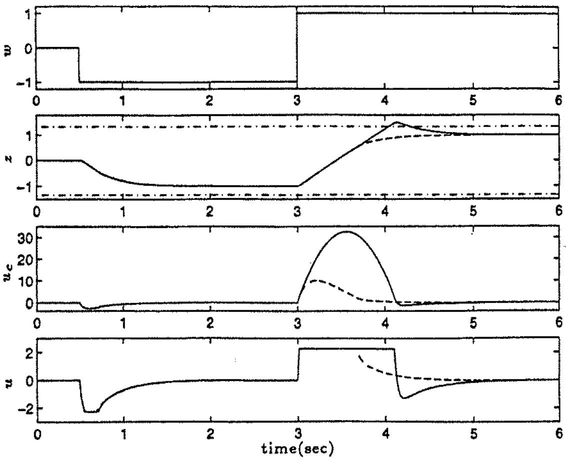

Fig. 9 Step responses

なお，本稿で試みた出力可到達集合の観点からのワインドアップ，およびアンチワインドアップの特徴づけは，ワインドアップ現象に対する定量的な定式化への第一歩であり，応答性等の他の観点も考慮に入れた特徴づけをおこなうことがワインドアップ現象のさらなる理解へ向けた課題としてあげられる．

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《付　　録》

A.　状態可到達集合

次の非線形システム Σ_x を考える．

$$\dot{x} = f(x, w(t)), \quad x(0) = x_0. \tag{A.1}$$

ここで，$x \in R^n$ は状態，$w \in R^m$ は外乱とする．また，外乱 w は予め与えられた集合 $W \subset R^m$ に対し，次のように拘束されているものとする．

$$w(t) \in W, \quad \forall t \in [0, \infty) \tag{A.2}$$

【定義】 $\xi \in R^n$ に対し，(A.2) 式の条件を満たす外乱 w で時間 $T \in [0, \infty)$ で $x(T) = \xi$ となるものが存在するとき，ξ は x_0 から状態可到達であるという．

【定義】 Σ_x の状態可到達集合 $\mathcal{R}_x(\Sigma_x)$ を次で定義する．

$$\mathcal{R}_x(\Sigma_x) = \{\xi \in R^n : \xi \text{は} x_0 \text{から状態可到達}\}.$$

B.　出力可到達集合

次の非線形システム Σ_z を考える．

$$\dot{x} = f(x, w(t)), \quad x(0) = x_0, \tag{B.1}$$

$$z = g(x). \tag{B.2}$$

(B.1) 式において，$x \in R^n$ は状態，$w \in R^m$ は外乱を表すものとする．(B.2) 式において，$z \in R^l$ は出力を表すものとする．また，外乱 w は，$W \subset R^m$ に対して次の拘束を受けているものとする．

$$w(t) \in W, \quad \forall t \in [0, \infty) \tag{B.3}$$

【定義】 $\zeta \in R^l$ に対し，時間 $T \in [0, \infty)$ で $z(T) = \zeta$ を実現する (B.3) 式の条件を満たす外乱 w が存在するとき，ζ は $g(x_0)$ から出力可到達であるという．

【定義】 Σ_z の出力可到達集合 $\mathcal{R}_z(\Sigma_z)$ を次で定義する．

$$\mathcal{R}_z(\Sigma_z) = \{\zeta \in R^l : \zeta \text{は} g(x_0) \text{から出力可到達}\}.$$

C.　非線形システムに対する状態可到達集合の解析

《定理 3》[7]　C^1 関数 $V : R^n \to R$ が存在し，次の条件を満たしているものとする．

1) $V(x_0) \le 1$,

2) $\{\xi : V(\xi) \le 1\} \subseteq \{\xi : V(\xi) \le k\}, \ \forall k > 1$,

3) $\{\xi : \dot{V}(\xi, \omega) \ge 0, \omega \in W\} \subset \{\xi : V(\xi) \le 1\}$.

このとき，(A.1) 式，および (A.2) 式で定義される非線形システム Σ_x の状態可到達集合 $\mathcal{R}_x(\Sigma_x)$ は次を満たす．

$$\mathcal{R}_x(\Sigma_x) \subset \{\xi \in R^n : V(\xi) \le 1\}.$$

ここで $\dot{V}(x, w)$ は次で与えられるものとする．

$$\dot{V}(x, w) = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots \frac{\partial V}{\partial x_n}\right] f(x, w).$$

D.　スケジューリングパラメータを持つ線形システムに対する状態可到達集合の解析

状態変数表現が次で与えられるスケジューリングパラメータを持つ線形システム $\Sigma_x(\theta)$ を考える．

$$\dot{x} = A(\theta(t))x + B(\theta(t))w(t), \quad x(0) = 0,$$

ここで，$x(t) \in R^n$ は状態，$w(t) \in R^m$ は外乱とする．$\theta(t)$ は上記システムのスケジューリングパラメータで，次の条件を満たしているものとする．

$$\theta(t) \in [\theta_{min}, \theta_{max}], \quad \forall t \in [0, \infty).$$

外乱 w は次の条件を満たしていると仮定する．

$$w'(t)w(t) \le 1, \quad \forall t \in [0, \infty).$$

《定理 4》[2) 19)]　正定行列 $P \in R^{n \times n}$ と非負の実数 α が存在し，任意の $\theta \in [\theta_{min}, \theta_{max}]$ に対して次の条件を満たしているとする．

$$\begin{bmatrix} A'(\theta)P + PA(\theta) + \alpha P & PB(\theta) \\ B'(\theta)P & -\alpha I \end{bmatrix} \le 0, \tag{D.1}$$

このとき，$\Sigma_x(\theta)$ の状態可到達集合 $\mathcal{R}_x(\Sigma_x(\theta))$ は次を満たす．

$$\mathcal{R}_x(\Sigma_x(\theta)) \subset \mathcal{E}(P).$$

（証明） $P > 0$ に対し，連続微分可能な関数 $V(x)$ を $V(x) = x'Px$ で定義する．このとき定理 3 の条件 3) は次で与えられる．

$$x'[A'(\theta)P + PA(\theta)]x + x'PB(\theta)w + w'B'(\theta)Px < 0,$$
$$\forall x \in \{\xi \mid \xi'P\xi \geq 1\}, \quad \forall w \in \{\omega \mid w'w \leq 1\},$$
$$\forall \theta \in [\theta_{min}, \theta_{max}].$$

この時，上記に $\mathcal{S}$-procedure[3] を適用することで，条件 (D.1) を導くことができる．詳細については，文献2),19), 3) を参照のこと．

E. スケジューリングパラメータを持つ線形システムに対する出力可到達集合の解析

状態変数表現が次で与えられるスケジューリングパラメータを持つ線形システム $\Sigma_z(\theta)$ を考える．

$$\dot{x} = A(\theta(t))x + B(\theta(t))w(t), \qquad x(0) = 0,$$
$$z = C(\theta(t))x.$$

ここで，$x(t) \in R^n$ は状態，$w(t) \in R^m$，$z(t) \in R^l$ は出力を表すものとする．$\theta(t)$ は上記システムのスケジューリングパラメータで，次の条件を満たしているものとする．

$$\theta(t) \in [\theta_{min}, \theta_{max}], \quad \forall t \in [0, \infty).$$

外乱 w は次の条件を満たしていると仮定する．

$$w'(t)w(t) \leq 1, \quad \forall t \in [0, \infty).$$

［補題 1］[20) 2)] 正定対称行列 $P \in R^{n\times n}$ と行列 $C \in R^{l\times n}$ に対し，正定対称行列 $R \in R^{l\times l}$ が存在し次の条件を満たしているものとする．

$$P - C'RC \geq 0.$$

このとき，次の条件が $\zeta = C\xi$ に対して成立する．

$$\zeta \in \mathcal{E}(R), \qquad \forall \xi \in \mathcal{E}(P). \tag{E.1}$$

（証明） $F_0(\xi)$ と $F_1(\xi)$ を次で定義する．

$$F_0(\xi) = \xi'C'RC\xi - 1, \quad F_1(\xi) = \xi'P\xi - 1.$$

正の実数 α が存在して次の条件

$$F_0(\xi) - \alpha F_1(\xi) \leq 0, \quad \forall \xi,$$

を満足すれば，$\mathcal{S}$-procedure によって (E.1) が成立する．つまり

$$\xi'(C'RC - \alpha P)\xi - 1 + \alpha \leq 0, \quad \forall \xi. \tag{E.2}$$

これより，明かに $\alpha \leq 1$. 次に，条件 (E.2) がある $\alpha \leq 1$ に対して成立すれば，任意の α', $\alpha \leq \alpha' \leq 1$ に対して成立することに注意する．これより，一般性を失わずに $\alpha = 1$ とおくことが可能で，この時条件 (E.2) は次で与えられる．

$$\xi(C'RC - P)\xi \leq 0, \quad \forall \xi.$$

《定理 5》[20) 2)] 正定対称行列 $Q \in R^{n\times n}$，$S \in R^{l\times l}$，および非負の実数 α が存在し，任意の $\theta \in [\theta_{min}, \theta_{max}]$ に対して次の条件を満たしているものとする．

$$\begin{bmatrix} A'(\theta)P + PA(\theta) + \alpha P & PB(\theta) \\ B'(\theta)P & -\alpha I \end{bmatrix} \leq 0,$$

$$\begin{bmatrix} P & C'(\theta) \\ C(\theta) & S \end{bmatrix} \geq 0,$$

このとき，$\Sigma_z(\theta)$ の出力可到達集合 $\mathcal{R}_z(\Sigma_z(\theta))$ は次を満たす．

$$\mathcal{R}_z(\Sigma_z(\theta)) \subset \mathcal{E}(R), \quad R = S^{-1}.$$

（証明） 定理 4，および補題 1 より明らか

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論　文

自己スケジューリングパラメータを持つ線形システムの準大域的な L^2 ゲイン解析*

東　剛人**・渡辺　亮***・内田　健康**

Semi-Global L^2 Gain Analysis of Linear Systems with Self-Scheduling Parameters*

Takehito AZUMA**, Ryo WATANABE*** and Kenko UCHIDA**

In this paper, we consider semi-global L^2 gain analysis for nonlinear systems described as linear systems with self-scheduling parameters. First we show a method to convert linear systems with self-scheduling parameters into linear systems with scheduling parameters based on evaluation of the domain of the self-scheduling parameters. Second, using the tools for linear systems with scheduling parameters, we discuss semi-global L^2 gain analysis for the nonlinear systems and propose an approach together with feasible formulas of computation, which provides a solution to the so-called hidden loop problem in gain scheduling. Finally, we show numerical examples.

1. はじめに

スケジューリングパラメータを持つ線形システムに対する解析・設計問題では，数多くの手法が提案されている[1),3),7),11),12)]．これらの手法は，スケジューリングパラメータ（およびその時間微分値）が有界であり，かつその許容領域が事前に得られているという仮定に基づいている．状態に依存した関数をパラメータとみなすことでパラメータを持つ線形システムとして記述できる非線形システムに対しては，パラメータが状態に関して有界な関数であれば，これらの手法をそのまま適用することができる[8)]．しかし，パラメータとして状態（またはその一部）をとるような場合には，パラメータは状態に関して非有界な関数となり，事前にその許容領域を得ることは困難である．このような場合，パラメータの許容領域を適当に定め解析・設計を行うことになるが，実際にパラメータがその定めた領域に含まれているかが大きな問題となる．(これは，Hidden Loop 問題と呼ばれる.) 従来の研究では，この Hidden Loop 問題に対してはシミュレーションによりパラメータがあらかじめ定めた領域に含まれていることを確認しているが，この場合の理論的な保証はない．

本論文では，この問題に対する一つのアプローチとして，自己スケジューリングパラメータを持つ線形システム

$$\dot{x} = A(\theta_x(x))x + B(\theta_x(x))w(t),\ x(0) = 0$$
$$z = C(\theta_x(x))x$$
$$\theta_x(x) = hx$$

として記述できる非線形システムを考え，このシステムに対して L^2 ゲイン解析を行う．このシステムにおいて，$\theta_x(x)$ が自己スケジューリングパラメータであり，状態 x の要素の線形結合として与えられるものを考える．本稿では，このシステムが非線形システムであることを考慮して，あらかじめ外乱を拘束する集合を与え，その拘束された外乱に対する準大域的な L^2 ゲイン解析を行う．

ところで，一般に非線形システムに対する L^2 ゲイン解析として Hamilton-Jacobi 方程式に基づいた手法（以下では Hamilton-Jacobi アプローチと呼ぶ）がある[14)]．Hamilton-Jacobi アプローチでは，非線形システムに対する原点近傍での局所的な領域での L^2 ゲイン解析が可能である．もちろん，自己スケジューリングパラメータ

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を持つ線形システムとして記述される非線形システムに対しても Hamilton-Jacobi アプローチは適用可能であるが，本論文で考察するようなあらかじめその大きさが与えられた外乱に対する L^2 ゲイン解析は困難である．また，Hamilton-Jacobi アプローチは非線形項を厳密に取り扱っているため，非線形システムに対して非常に有効な解析手法であるが，Hamilton-Jacobi 方程式の解を実際に求めることも困難であるといった問題がある．

準大域的な L^2 ゲイン解析に対する本提案手法では，外乱がある集合に拘束されることから出力可到達集合解析[16]の手法に基づいて自己スケジューリングパラメータの許容領域を求め，システムをスケジューリングパラメータを持つ線形システムとして表す．つぎに，このスケジューリングパラメータを持つ線形システムに対してパラメータ依存リアプノフ関数を用いた L^2 ゲイン解析[11]を行うことにより，もとの自己スケジューリングパラメータを持つ線形システムの準大域的な L^2 ゲインを評価する．ここでは，準大域的な L^2 ゲイン解析をパラメータに依存した線形行列不等式（Linear Matrix Inequalities: LMIs）条件の解の存在に帰着させる．このパラメータ依存 LMI 条件は有限個のパラメータに依存しない LMI 条件に還元することができるため[2]，実際にその解を求めることができる．しかし，本アプローチでは，非線形のパラメータをスケジューリングパラメータとしてとらえてしまうため，非線形項を陽に考慮した Hamilton-Jacobi アプローチと比較すると保守的な結果しか得られない可能性があるので注意が必要である．

本論文の構成は以下のようになっている．**2.** で自己スケジューリングパラメータを持つ線形システムを導入し，**3.** で自己スケジューリングパラメータの許容領域を評価する手法を提案する．**4.** では，**3.** の結果に基づいた準大域的な L^2 ゲインの解析手法を提案する．**5.** で数値例を示す．以下，対称行列 M に対して，$M>0(<0)$ は M が正定（負定）行列であることを意味する．

2. 自己スケジューリングパラメータを持つ線形システム

2.1 対象システム

本稿では，以下の非線形システムを考える．

$$\Sigma: \begin{array}{l} \dot{x} = A(hx)x + B(hx)w(t),\ x(0)=0 \\ z = C(hx)x \end{array}$$

ここで，$x\in R^n$ は状態，$z\in R^l$ は観測出力，$w\in R^m$ は外乱を表し，$h\in R^{1\times n}$ は定数行列である．h が $1\times n$ の行列であるので，hx はスカラである．また，$A(hx), B(hx), C(hx)$ は以下のように与えられているものとする．

$$\begin{array}{l} A(hx) = A_0 + a_1(hx)A_1 + \cdots + a_{r_a}(hx)A_{r_a} \\ B(hx) = B_0 + b_1(hx)B_1 + \cdots + b_{r_b}(hx)B_{r_b} \\ C(hx) = C_0 + c_1(hx)C_1 + \cdots + c_{r_c}(hx)C_{r_c} \end{array} \quad (1)$$

ここで，$a_i,b_i,c_i : R\rightarrow R$ は連続関数であり，A_i,B_i,C_i は適当な次元を持つ定数行列である．

本稿では，非線形システム Σ を以下の自己スケジューリングパラメータを持つ線形システム（Linear systems with Self-Scheduling parameters : LSS）として表す．

$$\Sigma_{LSS}: \begin{array}{l} \dot{x} = A(\theta_x(x))x + B(\theta_x(x))w(t),\ x(0)=0 \\ z = C(\theta_x(x))x \\ \theta_x(x) = hx \end{array}$$

ここで，$\theta_x(x)$ がシステム Σ_{LSS} の自己スケジューリングパラメータである．

また，$\theta_x(x(t))$ とその時間微分値 $\dot{\theta}_x(x(t))$ がそれぞれ任意の時間 $t\in[0,\infty)$ で領域 Θ, Ψ に含まれるとき，システム Σ_{LSS} は以下のスケジューリングパラメータを持つ線形システム（Linear systems with Scheduling parameters : LS）として表される．

$$\Sigma_{LS}: \begin{array}{l} \dot{x} = A(\theta(t))x + B(\theta(t))w(t),\ x(0)=0 \\ z = C(\theta(t))x \\ \theta(t)\in\Theta,\ \dot{\theta}(t)\in\Psi,\ {}^\forall t\in[0,\infty) \end{array}$$

このスケジューリングパラメータを持つ線形システムに対しては，従来から数多くの解析・設計手法が提案されている[1),3),7),11),12)]．本稿では，Σ_{LSS} を Σ_{LS} として表し，この Σ_{LS} に対しパラメータ依存リアプノフ関数を用いる手法[11]を適用することで，Σ_{LSS}（すなわち Σ）に対する L^2 ゲイン解析を行う．

2.2 外乱に対する拘束

非線形システム $\Sigma(\Sigma_{LSS})$ に対して，外乱に対する大域的な制御性能について議論することは一般に困難である．そこで，本稿では外乱 w に以下の拘束条件を与える．

$$\begin{array}{l} w(t)\in\mathcal{W},\ {}^\forall t\in[0,\infty) \\ \mathcal{W} = \{w\in R^m \mid w'W^{-1}w\leq 1\} \end{array} \quad (2)$$

ただし，W はあらかじめ与えられる正定な行列である．外乱 w があらかじめ与えられる集合 $\mathcal{W}$ で拘束されていることから，以下では外乱が集合 $\mathcal{W}$ で拘束される場合の Σ_{LSS} に対する L^2 ゲインを準大域的な L^2 ゲインと呼ぶ．

3. 自己スケジューリングパラメータの許容領域

ここでは，以下で与えられる自己スケジューリングパラメータを持つ線形システム $\hat{\Sigma}_{LSS}$ に対し，自己スケ

ジューリングパラメータの許容領域を評価するための手法を提案する．

$$\hat{\Sigma}_{LSS}: \begin{array}{l} \dot{x} = A(\theta_x(x))x + B(\theta_x(x))w(t),\ x(0)=0 \\ \theta_x(x) = hx \end{array}$$

ただし，外乱 w は (2) 式で拘束されているものとする．

はじめに，$\theta_x(x(t))$ の許容領域を評価する手法を与える．$\theta_x(x)=hx$ を $\hat{\Sigma}_{LSS}$ の出力方程式とみなすことで，出力可到達集合解析[16]の手法を直接適用して，$\theta_x(x(t))$ の許容領域を評価することができる．

【補題 1】 以下を満たす行列 $Q_x>0$，スカラ $\alpha>0$ および $q_\theta>0$ が存在するとする．

$$\begin{bmatrix} A(\theta)Q_x + Q_xA'(\theta) + \alpha Q_x & B(\theta) \\ B'(\theta) & -\alpha W^{-1} \end{bmatrix} < 0 \tag{3}$$

$$\begin{bmatrix} Q_x & Q_xh' \\ hQ_x & q_\theta \end{bmatrix} > 0 \tag{4}$$

$$\forall\theta \in [-\sqrt{q_\theta},\ \sqrt{q_\theta}\,]$$

このとき，$\theta_x(x(t))$ の許容領域 Θ は次のように評価される．

$$\Theta \subset [-\sqrt{q_\theta},\ \sqrt{q_\theta}\,]$$

（証明）参考文献16) の定理 5.2 において，出力方程式として$\theta_x(x)=hx$ を考えればよい． □

つぎに，自己スケジューリングパラメータの時間微分値 $\dot{\theta}_x(x(t))$ の許容領域を評価する手法を与える．自己スケジューリングパラメータが $\theta_x(x)=hx$ で与えられることから，$\theta_x(x(t))$ の時間微分値は以下のように与えられる．

$$\begin{aligned} \dot{\theta}_x(x(t)) &= h\dot{x}(t) \\ &= hA(\theta_x(x(t)))x(t) + hB(\theta_x(x(t)))w(t) \end{aligned} \tag{5}$$

しかし，自己スケジューリングパラメータの時間微分値 $\dot{\theta}_x(x(t))$ が w に依存しているため，補題 **1** のように出力可到達集合解析[16]を直接適用することができない．そこで，まず，以下の補題を示す．

【補題 2】 あらかじめ与えられた行列 $C,D,Q>0,W>0$ に対し，以下を満たす行列 $Y>0$ が存在するとする．

$$\begin{bmatrix} Q & 0 & QC' \\ 0 & W & WD' \\ CQ & DW & \frac{1}{2}Y \end{bmatrix} > 0 \tag{6}$$

このとき，以下が成立する．

$$y'Y^{-1}y<1,\ y=Cx+Dw \tag{7}$$

ただし，x,w は以下を満たすものとする．

$$\begin{array}{l} x \in \{x\in R^n \mid x'Q^{-1}x \le 1\} \\ w \in \{w\in R^m \mid w'W^{-1}w \le 1\} \end{array} \tag{8}$$

（証明）(6) 式は以下と等価である．

$$0 < \begin{bmatrix} Q & 0 \\ 0 & W \end{bmatrix} -2\begin{bmatrix} Q & 0 \\ 0 & W \end{bmatrix}\begin{bmatrix} C' \\ D' \end{bmatrix}Y^{-1}\begin{bmatrix} C & D \end{bmatrix}\begin{bmatrix} Q & 0 \\ 0 & W \end{bmatrix}$$

これは，以下のように書くことができる．

$$\frac{1}{2}\begin{bmatrix} Q^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} - \begin{bmatrix} C' \\ D' \end{bmatrix}Y^{-1}\begin{bmatrix} C & D \end{bmatrix} > 0$$

したがって，以下が成立する．

$$\begin{array}{l} \frac{1}{2}\xi_1'Q^{-1}\xi_1 + \frac{1}{2}\xi_2'W^{-1}\xi_2 \\ \quad -\{C\xi_1+D\xi_2\}'Y^{-1}\{C\xi_1+D\xi_2\} > 0 \\ \forall\xi_1 \in R^n,\ \forall\xi_2 \in R^m \end{array}$$

ここで，$\xi_1=x, \xi_2=w$ とすると，以下が得られる．

$$1 > \{Cx+Dw\}'Y^{-1}\{Cx+Dw\}$$

ただし，x,w は (8) 式を満たすものとする． □

これより，自己スケジューリングパラメータ $\theta_x(x(t))$ およびその時間微分値 $\dot{\theta}_x(x(t))$ の許容領域は以下のように評価することができる．

【定理 1】 以下を満たす行列 $Q_x>0$，スカラ $\alpha>0, q_\theta>0$ および $q_\psi>0$ が存在するとする．

$$\begin{bmatrix} A(\theta)Q_x + Q_xA'(\theta) + \alpha Q_x & B(\theta) \\ B'(\theta) & -\alpha W^{-1} \end{bmatrix} < 0 \tag{9}$$

$$\begin{bmatrix} Q_x & Q_xh' \\ hQ_x & q_\theta \end{bmatrix} > 0 \tag{10}$$

$$\begin{bmatrix} Q_x & 0 & Q_xA'(\theta)h' \\ 0 & W & WB'(\theta)h' \\ hA(\theta)Q_x & hB(\theta)W & \frac{1}{2}q_\psi \end{bmatrix} > 0 \tag{11}$$

$$\forall\theta \in [-\sqrt{q_\theta},\ \sqrt{q_\theta}\,]$$

このとき，$\theta_x(x(t))$ および $\dot{\theta}_x(x(t))$ の許容領域 Θ および Ψ は以下のように評価される．

$$\Theta \subset [-\sqrt{q_\theta},\ \sqrt{q_\theta}\,],\ \Psi \subset [-\sqrt{q_\psi},\ \sqrt{q_\psi}\,]$$

（証明）(5) 式に対し補題 **2** を適用し（$Y=q_\psi$ とする），補題 **1** を組み合わせればよい． □

$\theta_x(x(t))$ および $\dot{\theta}_x(x(t))$ の許容領域は定理 **1** により

求められるが，定理 **1** の条件式には以下のような問題があり，直接解くことは困難である．

a). (9) 式に未知変数に関する双線形項 αQ_x がある．

b). (9) 式，(11) 式がパラメータ θ に依存した条件になっている．

c). パラメータ θ の許容領域が未知変数 q_θ に依存している．

これらの問題に対し，問題 **a)** と **c)** に対しては以下の系 **1** を用い，問題 **c)** に対しては系 **2** の手法を用いることにより解決する．これにより，定理 **1** の条件式の解は系 **1** を繰り返し用いることにより求めることができる．

【系 1】 与えられた行列 $\tilde{Q}_x>0$，スカラ $\tilde{q}_\theta>0$ および $\tilde{q}_\psi>0$ に対し，以下を満たす行列 $Q_x>0$，スカラ $\alpha>0, q_\theta>0$ および $q_\psi>0$ が存在するとする．

$$\begin{bmatrix} A(\theta)Q_x+Q_xA'(\theta)+\alpha\tilde{Q}_x & B(\theta) \\ B'(\theta) & -\alpha W^{-1} \end{bmatrix}<0 \quad (12)$$

$$\begin{bmatrix} Q_x & Q_xh' \\ hQ_x & q_\theta \end{bmatrix}>0 \quad (13)$$

$$\begin{bmatrix} Q_x & 0 & Q_xA'(\theta)h' \\ 0 & W & WB'(\theta)h' \\ hA(\theta)Q_x & hB(\theta)W & \frac{1}{2}q_\psi \end{bmatrix}>0 \quad (14)$$

$$Q_x<\tilde{Q}_x \quad (15)$$

$$q_\theta<\tilde{q}_\theta \quad (16)$$

$$q_\psi<\tilde{q}_\psi \quad (17)$$

$$\forall\theta\in[-\sqrt{\tilde{q}_\theta},\ \sqrt{\tilde{q}_\theta}\,]$$

このとき，$\theta_x(x(t))$ および $\dot{\theta}_x(x(t))$ の許容領域（Θ および Ψ）は以下のように評価される．

$$\Theta\subset[-\sqrt{q_\theta},\ \sqrt{q_\theta}\,],\quad \Psi\subset[-\sqrt{q_\psi},\ \sqrt{q_\psi}\,] \quad (18)$$

（証明）参考文献 16) の系 5.3，補題 **1** および補題 **2** より，証明される．□

（注意 1）系 **1** は，$[-\sqrt{q_\theta},\ \sqrt{q_\theta}\,]$ および $[-\sqrt{q_\psi},\ \sqrt{q_\psi}\,]$ がそれぞれ $[-\sqrt{\tilde{q}_\theta},\ \sqrt{\tilde{q}_\theta}\,]$, $[-\sqrt{\tilde{q}_\psi},\ \sqrt{\tilde{q}_\psi}\,]$ に含まれていることを示している．したがって，系 **1** を繰り返し用いることにより，より小さい許容領域を得ることができる．

（注意 2）システムパラメータ A,B が (1) 式で与えられることを考慮すると，(12) 式，(14) 式は次の形式のパラメータ依存 LMI 条件として記述できる．

$$F_0(Q_u)+f_1(\hat{\theta})F_1(Q_u)+\cdots+f_r(\hat{\theta})F_r(Q_u)<0$$

ただし，$Q_u=(Q_x,\alpha,q_\theta,q_\psi)$ であり，$\hat{\theta}$ は以下を満たすパラメータ（系 **1** では，$s=1$）である．

$$\hat{\theta}\in\hat{\Theta}=\{\left[\hat{\theta}_1\ \hat{\theta}_2\ \cdots\ \hat{\theta}_s\right]'\mid\hat{\theta}_i\in\left[\hat{\theta}_i^{min},\hat{\theta}_i^{max}\right]\}$$

$f_i:R^s\rightarrow R$ は $\hat{\theta}$ に関する連続関数で，F_i は対称かつ未知行列 Q_u に関してアフィンな行列関数である．この形式のパラメータ依存 LMI 条件の解の存在は，有限個のパラメータに依存しない LMI 条件の解の存在に帰着される[2]．したがって，系 **1** の解は，LMILAB[5] などを用いて実際に計算することができる．

4. 準大域的な L^2 ゲイン解析

ここでは，外乱が (2) 式で拘束される場合において，システム Σ_{LSS}（すなわち Σ）が内部安定であり，かつ準大域的な L^2 ゲインが有界となるための十分条件を導出する．Σ_{LSS} の内部安定性と準大域的な L^2 ゲインを以下で定義する．

【定義 1】 以下で与えられるシステムを $\tilde{\Sigma}_{LSS}$ とする．

$$\tilde{\Sigma}_{LSS}:\begin{array}{l} \dot{x}=A(\theta_x(x))x \\ \theta_x(x)=hx \end{array}$$

平衡点 $x=0$ を含む領域 $\mathcal{M}$ の任意の点から出発する $\tilde{\Sigma}_{LSS}$ の軌跡が平衡点に収束するとき，Σ_{LSS} は領域 $\mathcal{M}$ で内部安定であるという．

【定義 2】 Σ_{LSS} が内部安定であるとき，Σ_{LSS} の L^2 ゲインを以下で定義する．

$$G(\Sigma_{LSS})=\sup_{w\in L^2\cap\mathcal{W},w\neq0}\frac{\|\ z\ \|_{L^2}}{\|\ w\ \|_{L^2}}$$

このとき，外乱 w が集合 $\mathcal{W}$ で拘束されていることから，これを準大域的な L^2 ゲインと呼ぶ．

本稿では，Σ_{LSS} の準大域的な L^2 ゲインを，以下に示すようなスケジューリングパラメータを持つ線形システムに対する L^2 ゲイン解析[11),12)] の手法に基づいて求める．

【補題 3】[11),12)] 以下で与えられるスケジューリングパラメータを持つ線形システム $\hat{\Sigma}_{LS}$ を考える．

$$\hat{\Sigma}_{LS}:\begin{array}{l} \dot{x}=A(\theta(t))x+B(\theta(t))w(t), x(0)=0 \\ z=C(\theta(t))x \\ \theta(t)\in[\alpha_1,\ \alpha_2],\ \forall t\in[0,\ \infty) \\ \dot{\theta}(t)\in[\beta_1,\ \beta_2],\ \forall t\in[0,\ \infty) \end{array}$$

システム $\hat{\Sigma}_{LS}$ に対し，以下を満たす連続微分可能な行列関数 $Q(\theta)$ が存在するとする．

$$Q(\theta)>0 \quad (19)$$

$$\begin{bmatrix} \begin{pmatrix} A'(\theta)Q(\theta)+Q(\theta)A(\theta) \\ +\rho\dfrac{dQ}{d\theta}(\theta) \end{pmatrix} & Q(\theta)B(\theta) & C'(\theta) \\ B'(\theta)Q(\theta) & -\gamma^2 I & 0 \\ C(\theta) & 0 & -I \end{bmatrix}<0 \quad (20)$$

$$\forall\theta\in[\alpha_1,\ \alpha_2],\quad \rho=\beta_1,\ \beta_2$$

このとき，システム $\tilde{\Sigma}_{LS}$ は内部安定であり，かつ γ 以下の L^2 ゲインを有する．

（注意 3）補題 **3** の条件式も，解 $Q(\theta)$ の形式を限定することにより注意 **2** で示した形式のパラメータ依存LMI条件として記述することができる[2]．したがって，任意の $\theta \in [\alpha_1, \alpha_2]$ に対し，不等式条件 (19) 式，(20) 式を満たすことを保証する解 $Q(\theta)$ を数値計算により求めることができる．

補題 **3** と定理 **1** より，システム Σ_{LSS} が内部安定であり，かつ準大域的な L^2 ゲインが有界となるための条件として以下が得られる．

【定理 2】 以下を満たす行列 $Q_x > 0$，スカラ $\alpha > 0, q_\theta > 0, q_\psi > 0$ および連続微分可能な行列関数 $Q(\theta)$ が存在するとする．

$$\begin{bmatrix} A(\theta)Q_x + Q_xA'(\theta) + \alpha Q_x & B(\theta) \\ B'(\theta) & -\alpha W \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} Q_x & Q_xh' \\ hQ_x & q_\theta \end{bmatrix} > 0 \tag{22}$$

$$\begin{bmatrix} Q_x & 0 & Q_xA'(\theta)h' \\ 0 & W^{-1} & W^{-1}B'(\theta)h' \\ hA(\theta)Q_x & hB(\theta)W^{-1} & \frac{1}{2}q_\psi \end{bmatrix} > 0 \tag{23}$$

$$Q(\theta) > 0 \tag{24}$$

$$\begin{bmatrix} \begin{pmatrix} A'(\theta)Q(\theta) + Q(\theta)A(\theta) \\ \pm\left(\sqrt{q_\psi}\dfrac{dQ}{d\theta}(\theta)\right) \end{pmatrix} & Q(\theta)B(\theta) & C'(\theta) \\ B'(\theta)Q(\theta) & -\gamma^2 I & 0 \\ C(\theta) & 0 & -I \end{bmatrix} < 0 \tag{25}$$

$$\forall\theta \in [-\sqrt{q_\theta},\ \sqrt{q_\theta}\ ]$$

ただし，$\pm(\cdot)$ は，$+(\cdot)$ と $-(\cdot)$ の場合の 2 本の不等式条件を意味する．

このとき，システム Σ_{LSS} は領域 $\mathcal{M}$

$$\mathcal{M} = \{x \mid x'Q_x^{-1}x \le 1\}$$

で内部安定であり，γ 以下の準大域的な L^2 ゲインを有する．

（証明）定理 **1** より，不等式条件 (21) 式，(22) 式，(23) 式はシステム Σ_{LSS} が以下のスケジューリングパラメータを持つ線形システム $\tilde{\Sigma}_{LS}$ として記述されることを示している．

$$\tilde{\Sigma}_{LS}: \begin{array}{l} \dot{x} = A(\theta(t))x + B(\theta(t))w(t), x(0) = 0 \\ z = C(\theta(t))x \\ \theta(t) \in [-\sqrt{q_\theta},\ \sqrt{q_\theta}\ ],\ \forall t \in [0,\ \infty) \\ \dot{\theta}(t) \in [-\sqrt{q_\psi},\ \sqrt{q_\psi}\ ],\ \forall t \in [0,\ \infty) \end{array}$$

また，補題 **3** より，不等式条件 (24) 式および (25) 式は，$\tilde{\Sigma}_{LS}$ が内部安定でありかつ γ 以下の L^2 ゲインを有することを示している．したがって，Σ_{LSS} が $\tilde{\Sigma}_{LS}$ として表されることから，Σ_{LSS} の準大域的な L^2 ゲインは γ 以下となる．

つぎに，Σ_{LSS} が領域 $\mathcal{M}$ で内部安定であることを示す．$\tilde{\Sigma}_{LS}$ が内部安定であることから，汎関数 V_{LS}

$$V_{LS}(x,t) = x'Q(\theta(t))x \tag{26}$$

は，$\tilde{\Sigma}_{LS}$ のリアプノフ関数である．ここで，Σ_{LSS} に対し，次の汎関数 V

$$V(x) = x'Q(hx)x \tag{27}$$

を考える．$\theta_x(x(t))$ および $\dot{\theta}_x(x(t))$ の許容領域が $[-\sqrt{q_\theta},\ \sqrt{q_\theta}\ ]$，$[-\sqrt{q_\psi},\ \sqrt{q_\psi}\ ]$ で与えられるときの状態 $x(t)$ の許容領域が $\mathcal{M}$ で与えられることから，Σ_{LSS} で $w \equiv 0$ としたシステムの解軌道に沿った (27) 式の時間微分は以下を満たす．

$$\begin{aligned} \dot{V}(x(t)) = x(t)'\{&A'(\theta_x(x(t)))Q(\theta_x(x(t))) \\ &+ Q(\theta_x(x(t)))A(\theta_x(x(t))) \\ &+ \dot{\theta}_x(x(t))\frac{dQ}{d\theta_x}(\theta_x(x(t)))\}x(t) \\ < 0& \\ \theta_x(x(t)) = hx(t),&\ \forall x(t) \in \mathcal{M} \end{aligned}$$

また，$V(x) > 0, \forall x \in \mathcal{M}$ も成立する．したがって，(27) 式は領域 $\mathcal{M}$ 上で Σ_{LSS} のリアプノフ関数となり，システム Σ_{LSS} は領域 $\mathcal{M}$ で内部安定である．□

また，系 **1** と同様に，定理 **2** に対して次の系が導かれる．

【系 2】 与えられた行列 $\tilde{Q}_x > 0$，スカラ $\tilde{q}_\theta > 0$ および $\tilde{q}_\psi > 0$ に対し，以下を満たす行列 $Q_x > 0$，スカラ $\alpha > 0, q_\theta > 0, q_\psi > 0$ および連続微分可能な行列関数 $Q(\theta)$ が存在するとする．

$$\begin{bmatrix} A(\theta)Q_x + Q_xA'(\theta) + \alpha\tilde{Q}_x & B(\theta) \\ B'(\theta) & -\alpha W \end{bmatrix} < 0 \tag{28}$$

$$\begin{bmatrix} Q_x & Q_xh' \\ hQ_x & q_\theta \end{bmatrix} > 0 \tag{29}$$

$$\begin{bmatrix} Q_x & 0 & Q_xA'(\theta)h' \\ 0 & W^{-1} & W^{-1}B'(\theta)h' \\ hA(\theta)Q_x & hB(\theta)W^{-1} & \frac{1}{2}q_\psi \end{bmatrix} > 0 \tag{30}$$

$$Q_x < \tilde{Q}_x \tag{31}$$

$$q_\theta < \tilde{q}_\theta \tag{32}$$

$$q_\psi < \tilde{q}_\psi \tag{33}$$

$$Q(\theta) > 0 \tag{34}$$

$$\begin{bmatrix} \begin{pmatrix} A'(\theta)Q(\theta)+Q(\theta)A(\theta) \\ \pm\left(\sqrt{\tilde{q}_\psi}\dfrac{dQ}{d\theta}(\theta)\right) \end{pmatrix} & Q(\theta)B(\theta) & C'(\theta) \\ B'(\theta)Q(\theta) & -\gamma^2 I & 0 \\ C(\theta) & 0 & -I \end{bmatrix} < 0 \quad (35)$$

$$^\forall\theta \in [-\sqrt{\tilde{q}_\theta},\ \sqrt{\tilde{q}_\theta}\ ]$$

ただし，$\pm(\cdot)$ は，$+(\cdot)$ と $-(\cdot)$ の場合の 2 本の不等式条件を意味する．

このとき，システム Σ_{LSS} は領域 $\mathcal{M}$

$$\mathcal{M}=\{x \mid x'Q_x^{-1}x \le 1\}$$

で内部安定であり，γ 以下の準大域的な L^2 ゲインを有する．

5.　数値例

5.1　数値例 1

ここでは，以下の自己スケジューリングパラメータを持つ線形システムに対し，提案手法を用いて準大域的な L^2 ゲインを評価する．

$$\dot{x}=\begin{bmatrix} -1-0.5\exp(\theta_x(x)) & 1+0.1\exp(\theta_x(x)) \\ -2 & -1+0.1\theta_x(x) \end{bmatrix}x + \begin{bmatrix} 1 \\ 0.1\exp(\theta_x(x)) \end{bmatrix}w(t),\quad x(0)=0$$

$$z=[2\ \ 1]x,\quad \theta_x(x)=[1\ \ 0]x$$

ただし，外乱 w には $w(t)\in\{w \mid w'w \le 1\}$, $^\forall t\in[0,\infty)$ の拘束が与えられているとする．系 **2** の解 $Q(\theta)$ として，次の二つを考える．

Case 1) $Q(\theta)=Q_0+e^\theta Q_1$

Case 2) $Q(\theta)=Q_0$

また，系 **2** を用いる際の初期値として $Q_{x0}=10I$, $\tilde{q}_{\theta 0}=\tilde{q}_{\psi 0}=10$ を用いた．

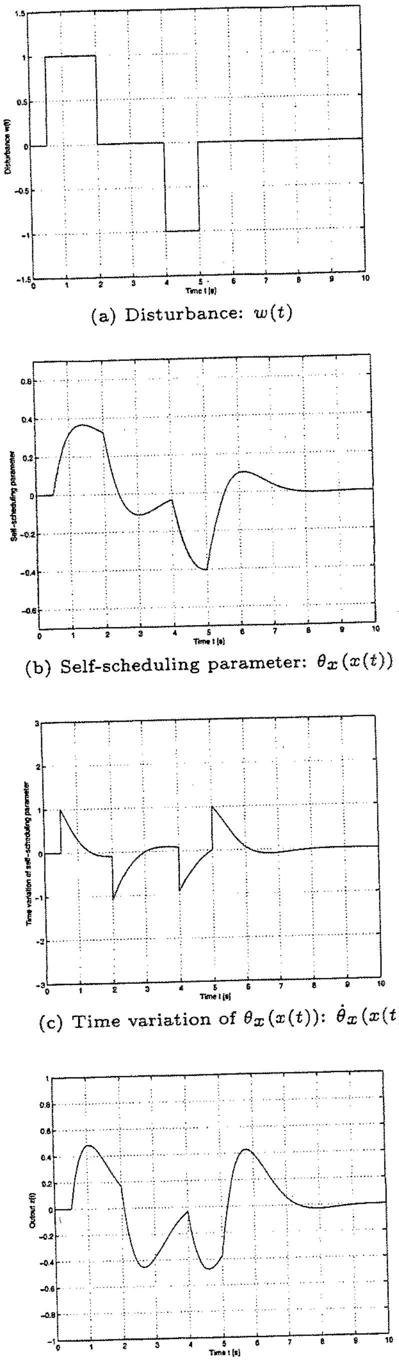
Case 1:　準大域的な L^2 ゲイン γ，自己スケジューリングパラメータの許容領域 Θ およびその時間微分値の許容領域 Ψ は次のように評価された．

$$\gamma=0.98,\quad \Theta=[-0.71\ 0.71],\quad \Psi=[-3.06,\ 3.06]$$

Case 2:　この場合，準大域的な L^2 ゲインの評価は Ψ に依存しないので，γ および Θ を評価した．

$$\gamma=1.04,\quad \Theta=[-0.71,\ 0.71]$$

これらの結果は，**Case 2** よりも **Case 1**，すなわち自己スケジューリングパラメータの時間微分値を考慮したほうが，準大域的な L^2 ゲインの評価に関する保守性が改善されることを示している．また，Fig. 1(a) の外乱に対しシミュレーションを行うと，自己スケジューリングパラメータおよびその微分値の時間応答は Fig. 1 (b), (c) のようになり，上記で評価した領域 Θ および Ψ に含まれている．



(a) Disturbance: $w(t)$

(b) Self-scheduling parameter: $\theta_x(x(t))$

(c) Time variation of $\theta_x(x(t))$: $\dot{\theta}_x(x(t))$

(d) Output: $z(t)$

Fig. 1　Simulation results (Numerical Example 1)

5.2　数値例 2

ここでは，以下の自己スケジューリングパラメータを持つ線形システムに対して，スケジューリング制御手法を用いてコントローラを設計し，提案手法を用いてそのコントローラの妥当性を検証する．

$$\Sigma: \begin{cases} \dot{x} = 2u, \quad x(0) = 0 \\ y = \exp(\theta_x(x)/3)x \\ \theta_x(x) = x \end{cases}$$

このシステムはジェットエンジンの燃料流量制御弁[15]のモデルを表している．このシステムに対し，Fig. 2 の閉ループ系を構成し,「定常状態において出力 y と指令値 r の偏差が 0」（ただし，指令値は $r'r \leq 1$ を満たす）を満たす状態フィードバックコントローラ $K(\theta_x(x))$ を設計する．

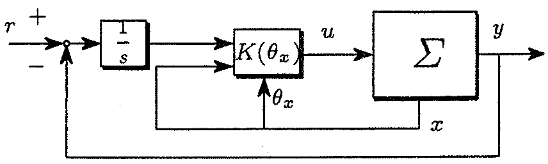


Fig. 2　Closed loop system

まずはじめに，自己スケジューリングパラメータをスケジューリングパラメータ（許容領域は $[-1.48, 1.48]$ と仮定する）とみなしスケジューリング制御手法[13]を用いてコントローラを設計する．一般化プラントとして Fig. 3 を考え，w から z までの L^2 ゲインを 1 以下とするようなスケジューリングコントローラを設計し，次のコントローラを得た．

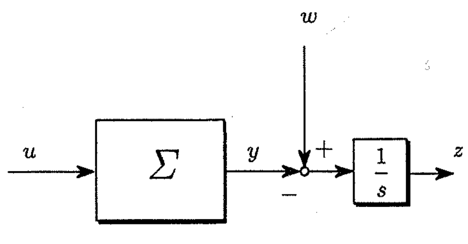


Fig. 3　Generalized plant

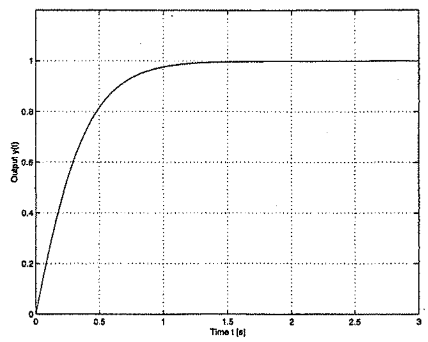
$$K(\theta_x(x)) = [9.13\times10^3 \ 3.55\times10^3] + \exp(\theta_x(x)/3)[3.21 \ 1.25]$$

つぎに，上記コントローラを Fig. 3 の一般化プラントに適用し w から z まで準大域的な L^2 ゲインを系 **2**（指令値が $r'r \leq 1$ を満たすので，外乱に対しては $w'w \leq 1$ を考える）を用いて評価すると，以下のように評価された．ここで，γ は準大域的な L^2 ゲインであり，Θ は自己スケジューリングパラメータの許容領域である．

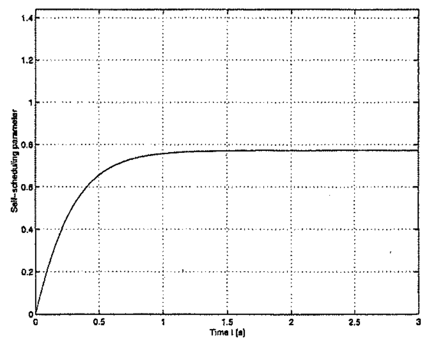
$$\gamma = 0.63, \quad \Theta = [-1.44 \ \ 1.44]$$

なお，系 **2** を用いる際には，初期値を $Q_{x0} = 100I, \tilde{q}_{\theta 0} = 4$ とし，$Q(\theta) = Q_0$ として求めた．これより，上記コントローラは非線形システム Σ に対して適用可能であることがわかる．

また，Fig. 4 にシミュレーション結果（指令値 $r = 1$）を示す．出力 $y(t)$ は指令値に追従し，自己スケジューリングパラメータの時間応答が許容領域 Θ に含まれていることがわかる．



(a) Output: $y(t)$



(b) Self-scheduling parameter: $\theta_x(x(t))$

Fig. 4　Simulation results（Numerical Example 2）

（注意 4）数値例 1, 2 ともに，自己スケジューリングパラメータの許容領域の評価が保守的になっている．これは，本提案手法を用いた場合においては非線形項を上限と下限が既知である時変パラメータととらえてしまうこと，およびリアプノフ関数として $V(x) = x'Q(hx)x$ のような特別な形をしたものを用いることに起因すると考えられる．

6. おわりに

本稿では，自己スケジューリングパラメータを持つ線形システムとして記述できる非線形システムに対し，外乱がある集合に拘束される場合の準大域的な L^2 ゲイン解析を行った．まず，自己スケジューリングパラメータとその時間微分値の許容領域を評価する手法を提案した．そして，この手法に基づき，自己スケジューリングパラメータを持つ線形システムに対する準大域的な L^2 ゲイン解析を，パラメータ依存LMI条件の解の存在に帰着させる手法を提案した．また，数値例では，実際に自己スケジューリングパラメータを持つ線形システムの準大域的な L^2 ゲインを求めた．さらに，非線形システムに対してスケジューリング制御手法を用いてコントローラを設計し，そのコントローラの妥当性を提案手法を用いて検証した．本稿の結果は，ゲインスケジューリング手法による非線形システムの解析・設計における Hidden Loop 問題への一つのアプローチを与えるものである．

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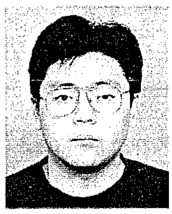

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LQG Control for Systems with Scheduling Parameter

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Abstract

We discuss control synthesis through a quadratic performance index for linear stochastic systems with scheduling parameter, which we call LQG control synthesis for systems with scheduling parameter. First, modifying the optimal LQG control synthesis for time-varying systems such that it leads to the causal dependence on the scheduling parameter, we propose a synthesis method based on a Riccati differential inequality and a forward Riccati differential equation. Being suggested by a relation between L^2 gain control for linear systems with scheduling parameter and that for linear time-varying systems, second, we propose another synthesis method based on two Riccati differential inequalities which correspond to two Riccati differential equations in the optimal LQG control synthesis for time-varying systems. To evaluate performance levels of the synthesized LQG controls, we also discuss some bounding techniques of the quadratic performance index.

Notation: (') denotes the transpose of a matrix or vector. I denotes the identity matrix. $X > 0$ ($X \geq 0$) denotes that the matrix X is positive definite (positive semi-definite), and $X < 0$ ($X \leq 0$) denotes that the matrix X is negative definite (negative semi-definite).

1 Introduction

Gain scheduling is a practical approach to nonlinear control problems. Control theory for linear systems with scheduling parameter, called linear parameter-varying systems control theory, has developed recently, and provides gain scheduling theoretical bases and systematic synthesis procedures. So far L^2 gain analysis and synthesis procedures have been discussed in most of the previous works (e.g., [7], [1], [6], [3], [9], [11], [10]) and the work [7]which discusses LQG (Linear-Quadratic-Gaussian) type analysis and synthesis procedure is an exception. Remembering the fundamental role of the optimal LQG control in linear control theory, it is desired to establish LQG control methods also for linear systems with scheduling parameter.

In this paper, we discuss LQG control synthesis for systems with scheduling parameter. The scheduling parameter is measured only on-line, so that system parameters which depend on the scheduling parameter are not known a priori over the whole interval. This makes LQG control synthesis for systems with scheduling parameter different from that for time-varying systems and requires some other consideration. The frozen parameter technique (see, e.g., [7]) is a feasible approach to the problem, but can not generally assure stability and any performance levels of closed-loop systems. We propose two synthesis methods: One is based on a Riccati differential inequality and a forward Riccati differential equation, which may be regarded as a modification of the optimal LQG control synthesis for time-varying systems such that the synthesized control depends causally on the scheduling parameter; The other is based on two Riccati differential inequalities, which is suggested by the correspondence between L^2 gain control synthesis for linear systems with scheduling parameter and that for linear time-varying systems and the fact that L^2 gain control (H^∞ control) is reduced to the optimal LQG control by taking the L^2 gain level infinity. To evaluate performance levels of the synthesized LQG controls, we introduce some bounding procedures of the expected quadratic performance index. We also discuss describing the synthesis and/or evaluation methods with parameter-dependent LMIs (Linear Matrix Inequalities) instead of Riccati differential inequalities.

2 Problem Formulation

The linear system with scheduling parameter we consider is described, over the interval $[0, \infty)$, as follows:

$$(\Sigma): \ \frac{d}{dt}x(t) = A(\theta(t))x(t) + B(\theta(t))u(t) + D(\theta(t))v(t)$$
$$y(t) = C(\theta(t))x(t) + w(t), \quad x(0) = x_0$$

$$z(t) = F(\theta(t))x(t).$$

$x(t)$ is the state vector; $u(t)$ is the control input vector; $y(t)$ is the measurement vector; $g(t) := (v(t)', w(t)')'$ is the disturbance vector; $h(t) := (z(t)', u(t)')'$ is the controlled vector. The scheduling parameter $\theta(t)$ over $[0, \infty)$ is a continuously differentiable function such that

$$\theta(t) \in [\theta_{min},\ \theta_{max}],\ \ \frac{d}{dt}\theta(t) \in [\nu_{min}, \nu_{max}]$$

for all t in $[0, \infty)$. Let the class of these scheduling parameter $\theta(\cdot)$ be denoted by Θ. $A(\theta), B(\theta), C(\theta), D(\theta)$, and $F(\theta)$ are matrix functions whose elements are bounded continuous functions of θ in $[\theta_{min}, \theta_{max}]$. In the case when the disturbance g is a square-integrable(L^2) function, we call the system (Σ) the *deterministic* system; In the case when the disturbance g is a white-Gaussian random process, the initial state x_0 is a Gaussian variable with zero-mean and variance $P_0 \geq 0$, and g and x_0 are independent, we call the system (Σ) the *stochastic* system.

The scheduling parameter $\theta(\cdot)$ is unknown a priori except it belongs to the class Θ, but $\theta(t)$ can be measured online. This requires that the control input depends causally on the parameter, and implies that the control synthesis problem differs from the standard control synthesis problem for time-varying systems where all system parameters are assumed to be known a priori over the whole interval. To synthesize control inputs which have the causal parameter-dependence, we assume the structure of admissible controllers in the following, parameter-dependent, linear form:

$$(\Gamma): \ \frac{d}{dt}x_c(t) = A_c(\theta(t))x_c(t) + H_c(\theta(t))y(t)$$
$$u(t) = K_c(\theta(t))x_c(t), \quad x_c(0) = 0$$

where $A_c(\theta)$, $H_c(\theta)$ and $K_c(\theta)$ are matrix functions whose elements are bounded continuous functions of θ in $[\theta_{min}, \theta_{max}]$. Our problem, which we call LQG control synthesis for systems with scheduling parameter, is to find a controller of the form(Γ) such that I) for the stochastic system(Σ), it assures a certain (possibly minimum) level of the performance index of the quadratic form

$$J = \mathrm{E}\left[x(T)'M_T x(T) + \int_0^T h(t)'h(t)dt\right] \qquad (1)$$

for $0 < T < \infty$, where E denotes expectation and M_T is a constant matrix and $M_T \geq 0$; II) for the deterministic system(Σ), it makes the closed-loop system internally stable. The constant T in the performance index (1) is a design parameter and could be chosen to specify an initial interval where transient responses are concerned.

3 Preliminaries

3.1 L^2 Gain Control for Linear Systems with Scheduling Parameter and Limiting Form

Here we pay attention to the following synthesis result for the deterministic case([9], [8], and also see [5]): For a positive number γ, let matrix functions $M(\theta)$ and $P(\theta)$, whose elements are continuously differentiable functions, satisfy that $M(\theta) > 0, P(\theta) > 0$,

$$-\Omega_M(\theta, \nu, M; \gamma) := \nu\frac{d}{d\theta}M(\theta) + A(\theta)'M(\theta) + M(\theta)A(\theta)$$
$$-M(\theta)\left[B(\theta)B(\theta)' - \frac{1}{\gamma^2}D(\theta)D(\theta)'\right]M(\theta) + F(\theta)'F(\theta) < 0$$
$$-\Omega_P(\theta, \nu, P; \gamma) := -\nu\frac{d}{d\theta}P(\theta) + P(\theta)A(\theta)' + A(\theta)P(\theta)$$
$$-P(\theta)\left[C(\theta)'C(\theta) - \frac{1}{\gamma^2}F(\theta)'F(\theta)\right]P(\theta) + D(\theta)D(\theta)' < 0$$

$$\gamma^2 M(\theta)^{-1}\Omega_M(\theta, \nu, M; \gamma)M(\theta)^{-1} > \Omega_P(\theta, \nu, P; \gamma) > 0$$

and $\gamma^2 M(\theta)^{-1} > P(\theta)$ for all (θ, ν) in $[\theta_{min}, \theta_{max}] \times [\nu_{min}, \nu_{max}]$. Then, the controller synthesized in the following form,

$$\frac{d}{dt}x_c(t) = [A(\theta(t)) - B(\theta(t))B(\theta(t))'S(\theta(t))$$
$$-P(\theta(t))C(\theta(t))'C(\theta(t))$$
$$+\frac{1}{\gamma^2}P(\theta(t))F(\theta(t))'F(\theta(t))]x_c(t)$$
$$+P(\theta(t))C(\theta(t))'y(t)$$
$$u(t) = -B(\theta(t))'S(\theta(t))x_c(t), \quad x_c(0) = 0$$

where $S(\theta) := M(\theta)[I - (1/\gamma^2)P(\theta)M(\theta)]^{-1}$, is causally parameter-dependent and yields the closed-loop system internally stable, and the L^2 gain from the disturbance g to the controlled output h when $x_0 = 0$ is less than γ for all scheduling parameters $\theta(\cdot)$ in Θ. That is, this is an L^2 gain controller (H^∞ controller) for the linear system with scheduling parameter. Consider an extreme case of this result: Letting $\gamma \to \infty$ reduces the above condition for $M(\theta)$ and $P(\theta)$ to that $M(\theta) > 0, P(\theta) > 0$,

$$-\Omega_M(\theta, \nu, M) := \nu\frac{d}{d\theta}M(\theta) + A(\theta)'M(\theta) + M(\theta)A(\theta)$$
$$-M(\theta)B(\theta)B(\theta)'M(\theta) + F(\theta)'F(\theta) < 0 \qquad (2)$$
$$-\Omega_P(\theta, \nu, P) := -\nu\frac{d}{d\theta}P(\theta) + P(\theta)A(\theta)' + A(\theta)P(\theta)$$
$$-P(\theta)C(\theta)'C(\theta)P(\theta) + D(\theta)D(\theta)' < 0 \qquad (3)$$

for all (θ, ν) in $[\theta_{min}, \theta_{max}] \times [\nu_{min}, \nu_{max}]$, and reduces the above controller to the limiting form

$$\frac{d}{dt}x_c(t) = [A(\theta(t)) - B(\theta(t))B(\theta(t))'M(\theta(t))]x_c(t)$$
$$+P(\theta(t))C(\theta(t))'[y(t) - C(\theta(t))x_c(t)]$$
$$u(t) = -B(\theta(t))'M(\theta(t))x_c(t), \quad x_c(0) = 0. \qquad (4)$$

In parallel with the relation between H^∞ control and optimal LQG control in linear time-invariant or time-varying systems cases, we expect that the limiting form (4) induced from the H^∞ controller will be a reasonable LQG controller for the linear stochastic system with scheduling parameter and the performance index of expected quadratic form.

3.2 LQG control for Linear Time-Varying Systems and Riccati Differential Inequality

To simplify descriptions, we introduce two notations. Let

$$u(t) = \Xi(t, y(\cdot), X(\cdot), Y(\cdot))$$

denote the control input given by

$$\begin{aligned}\frac{d}{dt}x_c(t) &= [A(\theta(t)) - B(\theta(t))B(\theta(t))'X(t)]\,x_c(t)\\ &\quad + Y(t)C(\theta(t))'\,[y(t) - C(\theta(t))x_c(t)]\\ u(t) &= -B(\theta(t))'X(t)x_c(t), \quad x_c(0) = 0\end{aligned}$$

and define $\Delta_M(t,X)$ and $\Delta_P(t,Y)$ as

$$\begin{aligned}-\Delta_M(t,X) &:= \frac{d}{dt}X(t) + A(\theta(t))'X(t) + X(t)A(\theta(t))\\ &\quad - X(t)B(\theta(t))B(\theta(t))'X(t) + F(\theta(t))'F(\theta(t))\\ -\Delta_P(t,Y) &:= -\frac{d}{dt}Y(t) + Y(t)A(\theta(t))' + A(\theta(t))Y(t)\\ &\quad - Y(t)C(\theta(t))'C(\theta(t))Y(t) + D(\theta(t))D(\theta(t))'.\end{aligned}$$

In this section, supposing a scheduling parameter $\theta(\cdot)$ over $[0,T]$ is known a priori, we regard the system (Σ) a linear time-varying stochastic system defined over $[0,T]$. For the linear time-varying stochastic system (Σ), it is well-known that the optimal LQG controller, which minimizes the performance index (1), is given as (see, e.g., Chapter 14 of [4])

$$u(t) = \Xi(t, y(\cdot), M^*(\cdot), P^*(\cdot)) \tag{5}$$

and the optimal value of performance index is expressed as

$$\begin{aligned}J(M^*,P^*) &= \mathrm{tr}[M_T P^*(T) + \int_0^T \{F(\theta(t))'F(\theta(t))P^*(t)\\ &\quad + M^*(t)P^*(t)C(\theta(t))'C(\theta(t))P^*(t)\}\,dt]\\ &= \mathrm{tr}[M^*(0)P_0 + \int_0^T \{M^*(t)D(\theta(t))D(\theta(t))'\\ &\quad + M^*(t)B(\theta(t))B(\theta(t))'M^*(t)P^*(t)\}\,dt]\end{aligned} \tag{6}$$

where $M^*(t)$ and $P^*(t)$ are the unique solutions to the Riccati differential equations:

$$-\Delta_M(t,M^*) = 0, \quad M^*(T) = M_T \tag{7}$$

$$-\Delta_P(t,P^*) = 0, \quad P^*(0) = P_0. \tag{8}$$

Now we consider three modifications of the optimal LQG controller (5):

$$u(t) = \Xi(t, y(\cdot), M^*(\cdot), P(\cdot)) \tag{9}$$

$$u(t) = \Xi(t, y(\cdot), M(\cdot), P^*(\cdot)) \tag{10}$$

$$u(t) = \Xi(t, y(\cdot), M^*(\cdot), P^*(\cdot)) \tag{11}$$

where $M(t)$ and $P(t)$ are continuously differentiable matrix functions which satisfy the Riccati differential inequalities

$$-\Delta_M(t,M) \le 0, \quad M(T) \ge M_T \tag{12}$$

$$-\Delta_P(t,P) \le 0, \quad P(0) \ge P_0. \tag{13}$$

Here note the following fact:

Lemma 1. *$M^*(t) \le M(t)$ and $P^*(t) \le P(t)$ for all t in $[0,T]$.*

Proof: From (7) and (12), we can see that

$$\begin{aligned}&\frac{d}{dt}M^*(t) + A(\theta(t))'M^*(t) + M^*(t)A(\theta(t))\\ &\quad - M^*(t)B(\theta(t))B(\theta(t))'M^*(t)\\ &\quad + F(\theta(t))'F(\theta(t)) = 0\\ &\frac{d}{dt}M(t) + A(\theta(t))'M(t) + M(t)A(\theta(t))\\ &\quad - M(t)B(\theta(t))B(\theta(t))'M(t)\\ &\quad + F(\theta(t))'F(\theta(t)) + \Delta_M(t,M) = 0.\end{aligned}$$

Now, letting $\Pi(t) := M(t) - M^*(t)$, and $\Lambda(t) := [A(\theta(t)) - (1/2)B(\theta(t))B(\theta(t))'\{M(t) + M^*(t)\}]$, we have

$$\frac{d}{dt}\Pi(t) + \Lambda(t)'\Pi(t) + \Pi(t)\Lambda(t) + \Delta_M(t,M) = 0$$

and, denoting by $\Psi(\cdot,\cdot)$ the transition matrix associated with the time-varying matrix $-\Lambda(t)$, we have

$$\begin{aligned}\Pi(t) &= \Psi(t,T)\Pi(T)\Psi(t,T)'\\ &\quad + \int_t^T \Psi(t,s)\Delta_M(s,M)\Psi(t,s)'\,ds\end{aligned}$$

where $\Pi(T) = M(T) - M_T \ge 0$ and $\Delta_M(s,M) \ge 0$ for each s, which implies the first inequality. The second inequality follows from the dual argument. □

Let $J(M^*,P)$, $J(M,P^*)$ and $J(M,P)$ denote the values of the performance indices of the three controllers (9), (10) and (11) respectively. Then, we obtain the following evaluation:

Lemma 2. *$J(M^*,P^*) \le \{J(M,P^*), J(M^*,P)\} \le J(M,P)$ where $J(M^*,P^*)$ is given by (6), and $J(M,P^*)$, $J(M^*,P)$ and $J(M,P)$ are expressed as*

$$\begin{aligned}J(M,P^*) &= \mathrm{tr}[M(T)P^*(T)\\ &\quad + \int_0^T \{(F(\theta(t))'F(\theta(t)) + \Delta_M(t,M))\,P^*(t)\end{aligned}$$

$$
\begin{aligned}
&+M(t)P^*(t)C(\theta(t))'C(\theta(t))P^*(t)\}\,dt] \\
&= \mathrm{tr}[M(0)P^*(0) + \int_0^T \{M(t)D(\theta(t))D(\theta(t))' \\
&+M(t)B(\theta(t))B(\theta(t))'M(t)P^*(t)\}\,dt] \quad (14)
\end{aligned}
$$

$$
\begin{aligned}
J(M^*, P) &= \mathrm{tr}[M^*(T)P(T) + \int_0^T \{F(\theta(t))'F(\theta(t))P(t) \\
&+M^*(t)P(t)C(\theta(t))'C(\theta(t))P(t)\}\,dt] \\
&= \mathrm{tr}[M^*(0)P(0) \\
&+\int_0^T \{M^*(t)\,(D(\theta(t))D(\theta(t))' + \Delta_P(t,P)) \\
&+M^*(t)B(\theta(t))B(\theta(t))'M^*(t)P(t)\}\,dt] \quad (15)
\end{aligned}
$$

$$
\begin{aligned}
J(M, P) &= \mathrm{tr}[M(T)P(T) \\
&+\int_0^T \{(F(\theta(t))'F(\theta(t)) + \Delta_M(t,M))\,P(t) \\
&+M(t)P(t)C(\theta(t))'C(\theta(t))P(t)\}\,dt] \\
&= \mathrm{tr}[M(0)P(0) \\
&+\int_0^T \{M(t)\,(D(\theta(t))D(\theta(t))' + \Delta_P(t,P)) \\
&+M(t)B(\theta(t))B(\theta(t))'M(t)P(t)\}\,dt]. \quad (16)
\end{aligned}
$$

Proof: The expressions of the performance indices (14), (15) and (16) are modifications of the optimal LQG performance expressions (6) with the definitions of $\Delta_M(t, M)$ and $\Delta_P(t, P)$. The inequalities among three performance indices are shown by applying the inequalities of **Lemma 1** and the inequalities (12) and (13) to the expressions (14), (15) and (16). □

4 LQG Control Synthesis for Systems with Scheduling Parameter

4.1 Parameterization

We begin to check feasibility of the four controllers (5), (9), (10) and (11) as controllers for linear systems with scheduling parameter. We should note that from the definition, to construct the control input $u(t) = \Xi(t, y(\cdot), X(\cdot), Y(\cdot))$ at time t, we need data $\{(\theta(s), y(s), X(s), Y(s)), 0 \le s \le t\}$, where $(\theta(s), y(s))$ is measured on-line at each time s, and also remember that, as stated in Section 1, controllers for linear systems with scheduling parameter are required to have causal dependence on scheduling parameter $\theta(t)$. Then, we see immediately that the optimal LQG controller (5) and the controller (9) are not feasible, because both need $M^*(\cdot)$ which is the solution of the backward Riccati differential equation (7) and so has anti-causal dependence on scheduling parameter. From now on, we focus on the remaining two controllers

$$
\begin{aligned}
u(t) &= \Xi(t, y(\cdot), M(\cdot), P^*(t)) \\
u(t) &= \Xi(t, y(\cdot), M(\cdot), P(t)).
\end{aligned}
$$

First we consider the controller $u(t) = \Xi(t, y(\cdot), M(\cdot), P^*(t))$. This controller needs $M(\cdot)$ and $P^*(\cdot)$. Let $\underline{M}(\theta)$ be a continuously differentiable solution to the Riccati differential inequality (1), and set $M(t) = \underline{M}(\theta(t))$ for a given scheduling parameter $\theta(\cdot)$. Then, if $M(T) = \underline{M}(\theta(T)) \ge M_T$ is satisfied, it is shown by direct substitution that $M(t) = \underline{M}(\theta(t))$ satisfies the Riccati differential inequalities (12). $P^*(\cdot)$ is the solution of the forward Riccati differential equation (8) and so, corresponding to on-line measurement of scheduling parameter $\theta(\cdot)$, can be calculated on-line. Thus we obtain a feasible controller $u(t) = \Xi(t, y(\cdot), M(\cdot), P^*(\cdot))$. Noting that the inequalities (2) for all ν in $[\nu_{min}, \nu_{max}]$ can be replaced equivalently with those at $\nu = \nu_{min}$ and $\nu = \nu_{max}$ due to the linear dependence of the inequality (2) on ν, we summarize our discussion as follows:

Step A: *Find off-line a continuously differentiable solution $\underline{M}(\theta)$ of the Riccati differential inequalities*

$$
\begin{aligned}
&-\Omega_{\underline{M}}(\theta, \nu_{min}, \underline{M}) < 0 \\
&-\Omega_{\underline{M}}(\theta, \nu_{max}, \underline{M}) < 0, \quad \underline{M}(\theta) > M_T \qquad (17)
\end{aligned}
$$

for all θ in $[\theta_{min}, \theta_{max}]$. Measure $\theta(t)$ and set $M(t) = \underline{M}(\theta(t))$ on-line.

Step B: *Measure $\theta(t)$ and solve on-line, with respect to $P^*(t)$, the forward Riccati differential equation*

$$
-\Delta_P(t, P^*) = 0, \quad P^*(0) = P_0.
$$

Through **Step A** and **Step B**, we have $u(t) = \Xi(t, y(\cdot), M(\cdot), P^*(\cdot))$ on-line. Here note that, since the inequalities (17) in **Step A** allow non-unique solutions $\underline{M}(\theta)$, $\theta_{min} \le \theta \le \theta_{max}$, this controller is parameterized with these solutions.

Next we consider the controller $u(t) = \Xi(t, y(\cdot), M(\cdot), P(t))$. This controller needs $M(\cdot)$ and $P(\cdot)$. Let $\underline{M}(\theta)$ and $\underline{P}(\theta)$ be continuously differentiable solutions to the Riccati differential inequalities (2) and (3) respectively, and set $M(t) = \underline{M}(\theta(t))$ and $P(t) = \underline{P}(\theta(t))$ for a given scheduling parameter $\theta(\cdot)$. Then, if $M(T) = \underline{M}(\theta(T)) \ge M_T$ and $P(0) = \underline{P}(\theta(0)) \ge P_0$ are satisfied, it is shown by direct substitution that $M(t) = \underline{M}(\theta(t))$ and $P(t) = \underline{P}(\theta(t))$ satisfies the Riccati differential inequalities (12) and (13) respectively. Thus we obtain a feasible controller $u(t) = \Xi(t, y(\cdot), M(\cdot), P(\cdot))$, which is just the controller (4). In this case, the construction procedure is summarized as follows:

Step C: *Find off-line a continuously differentiable solution $\underline{P}(\theta)$ of the Riccati differential inequalities*

$$
\begin{aligned}
&-\Omega_{\underline{P}}(\theta, \nu_{min}, \underline{P}) < 0 \\
&-\Omega_{\underline{P}}(\theta, \nu_{max}, \underline{P}) < 0, \quad \underline{P}(\theta) > P_0 \qquad (18)
\end{aligned}
$$

for all θ in $[\theta_{min}, \theta_{max}]$. Measure $\theta(t)$ and set $P(t) = \underline{P}(\theta(t))$ on-line.

Through **Step A** and **Step C**, we have $u(t) = \Xi(t,y(\cdot),M(\cdot),P(\cdot))$ on-line. Note that this controller is parameterized by the solutions $M(\theta)$, $\theta_{min} \leq \theta \leq \theta_{max}$ of the inequalities (17) in **Step A** and the solutions $P(\theta)$, $\theta_{min} \leq \theta \leq \theta_{max}$ of the inequalities (18) in **Step C**.

Thus, let the controller $u(t) = \Xi(t,y(\cdot),M(\cdot),P^*(\cdot))$ be constructed through **Step A** and **Step B**, and the controller $u(t) = \Xi(t,y(\cdot),M(\cdot),P^*(\cdot))$ be constructed through **Step A** and **Step C**. Then, both controllers are feasible and have the following properties:

Proposition 1.

(a) *For the linear stochastic system with scheduling parameter* (Σ), *the controller* $u(t) = \Xi(t,y(\cdot),M(\cdot),P^*(\cdot))$ *is superior to the controller* $u(t) = \Xi(t,y(\cdot),M(\cdot),P(\cdot))$ *in the sense that* $J(M,P^*) \leq J(M,P)$, *where* $J(M,P^*)$ *and* $J(M,P)$ *are given by* (14) *and* (16) *respectively.*

(b) *For the linear deterministic system with scheduling parameter* (Σ), *if there exist positive numbers* ε *and* η *such that* $\varepsilon I \leq P^*(t) \leq \eta I$ *for all* t *in* $[0,\infty)$, *the controller* $u(t) = \Xi(t,y(\cdot),M(\cdot),P^*(\cdot))$ *makes the closed-loop system internally (exponentially) stable.*

(c) *For the linear deterministic system with scheduling parameter* (Σ), *the controller* $u(t) = \Xi(t,y(\cdot),M(\cdot),P(\cdot))$ *makes the closed-loop system internally (exponentially) stable.*

Note that these properties hold for any given scheduling parameter $\theta(\cdot)$ in Θ. The part **(a)** follows directly from **Lemma 2**. The parts **(b)** and **(c)** are proved by applying the Lyapunov stability results for linear time-varying systems to the closed-loop systems, since the boundedness of $P^*(\cdot)$ is assumed as in **(b)** and similar boundednesses of $M(\cdot)$ and $P(\cdot)$ are automatically satisfied.

The on-line operation in **Step B** may be difficult in practice but is possible in principle (see [7] where the same idea is used in improving the frozen parameter LQG method). The off-line calculations in **Step A** and **Step C** can be done by using the techniques for solving parameter-dependent LMIs (see, e.g., [10],[2]).

4.2 Optimization

The two controllers obtained above are parameterized by $M(\theta)$, $\theta_{min} \leq \theta \leq \theta_{max}$ and $P(\theta)$, $\theta_{min} \leq \theta \leq \theta_{max}$ which are solutions to the Riccati inequalities (17) and (18) respectively. To obtain better controllers from the viewpoint of the performance index (1), we are led to the following optimization problems:

For the controller $u(t) = \Xi(t,y(\cdot),M(\cdot),P^*(\cdot))$, *minimize* $J(M,P^*)$ *with respect to* $M(\theta)$, $\theta_{min} \leq \theta \leq \theta_{max}$ *subject to* (17) *where* $J(M,P^*)$ *is given by* (14);

For the controller $u(t) = \Xi(t,y(\cdot),M(\cdot),P(\cdot))$, *minimize* $J(M,P)$ *with respect to* $(M(\theta), P(\theta))$, $\theta_{min} \leq \theta \leq \theta_{max}$ *subject to* (17) *and* (18) *where* $J(M,P)$ *is given by* (16).

However these optimization problem are generally difficult to solve, and so we propose an ad hoc procedure which minimizes some upper bounds of the performance indexes.

For given positive numbers α and β, consider the Riccati differential inequalities

$$-\Omega_M(\theta,\nu_{min},M) + \alpha I > 0$$
$$-\Omega_M(\theta,\nu_{max},M) + \alpha I > 0, \quad M(\theta) < M_T + \alpha I \quad (19)$$
$$-\Omega_P(\theta,\nu_{min},P) + \beta I > 0$$
$$-\Omega_P(\theta,\nu_{max},P) + \beta I > 0, \quad P(\theta) < P_0 + \beta I \quad (20)$$

for all θ in $[\theta_{min},\theta_{max}]$.

Let $M(\theta)$ which is a solution to (17) satisfy (19) and $P(\theta)$ which is a solution to (18) satisfy (20), and set $M(t) = M(\theta(t))$ and $P(t) = P(\theta(t))$. Then,

Lemma 3. (a) $\Delta_M(t,M) < \alpha I$ *and* $M(t) < M_T + \alpha I$ *for all* t *in* $[0,T]$.
(b) $\Delta_P(t,P) < \beta I$ *and* $P(t) < P_0 + \beta I$ *for all* t *in* $[0,T]$.

Proof: The inequalities of (a) are rewrites of (19) and the inequalities of (b) are rewrites of (20). □

Using the inequalities of **Lemma 3** in the expressions of $J(M,P^*)$ and $J(M,P)$ of **Lemma 2**, we obtain the following upper bounds of $J(M,P^*)$ and $J(M,P)$:

Proposition 2.
$J(M,P^*) \leq U(\alpha,P^*)$ *and* $J(M,P) \leq \{U(\alpha,P),V(\beta,M)\}$ *where the functions* $U(\alpha,Y)$ *and* $V(\beta,X)$ *are defined by*

$$U(\alpha,Y) := \mathrm{tr}[Y(T) + \int_0^T \{Y(t) + Y(t)C(\theta(t))'C(\theta(t))Y(t)\}\,dt]\alpha + \mathrm{tr}[M_T Y(T) + \int_0^T \{F(\theta(t))'F(\theta(t))Y(t) + M_T Y(t)C(\theta(t))'C(\theta(t))Y(t)\}\,dt]$$

$$V(\beta,X) := \mathrm{tr}[X(0) + \int_0^T \{X(t) + X(t)B(\theta(t))B(\theta(t))'X(t)\}\,dt]\beta + \mathrm{tr}[X(0)P_0 + \int_0^T \{X(t)D(\theta(t))D(\theta(t))' + X(t)B(\theta(t))B(\theta(t))'X(t)P_0\}\,dt].$$

We see from **Proposition 2** that the upper bounds have linear dependence on either ε or η. Therefore, to obtain better controllers based on the upper bounds, it is reasonable to consider the following optimization problems:

For the controller $u(t) = \Xi(t,y(\cdot),M(\cdot),P^*(\cdot))$, *minimize* α *with respect to* $M(\theta)$, $\theta_{min} \leq \theta \leq \theta_{max}$ *subject to* (17) *and* (19);

For the controller $u(t) = \Xi(t,y(\cdot),M(\cdot),P(\cdot))$, *minimize* α *with respect to* $M(\theta)$, $\theta_{min} \leq \theta \leq \theta_{max}$

subject to (17) *and* (19), *and minimize* β *with respect to* $P(\theta)$, $\theta_{min} \leq \theta \leq \theta_{max}$ *subject to* (18) *and* (20).

The Riccati differential inequalities (17) and (19) can be rewritten in the form of parameter-dependent LMIs with a rank condition, though the detail is omitted. Similarly, the Riccati differential inequalities (18) and (20) can be rewritten in the form of parameter-dependent LMIs with a rank condition. Thus, our problems are reduced to parameter-dependent LMIs optimizations, and again the techniques for solving parameter-dependent LMIs (e.g., [10], [2]) can be applied.

5 Conclusion

In this paper, we discussed LQG control synthesis problems for systems with scheduling parameter, and proposed synthesis methods based on a forward Riccati differential equation and Riccati differential inequalities. The procedures proposed are described by parameter-dependent LMIs, and can be executed by some promising techniques recently developed.

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INFINITE-DIMENSIONAL LMI APPROACH TO H^∞ CONTROL SYNTHESIS FOR LINEAR SYSTEMS WITH TIME DELAY

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Abstract

This paper considers a synthesis problem of H^∞ controllers for linear systems with time delay in the form of delay-dependent memory state feedback, and develops a new Linear Matrix Inequalities (LMIs) approach. First, we present an existence condition and an explicit formula of H^∞ controllers in terms of infinite-dimensional LMIs. This result is rather general in the sense that it covers, as special cases, some known results for the cases of delay-independent/dependent and memoryless/memory controllers, while the infinity dimensionality of the LMIs makes the result difficult to apply. Second, we introduce a technique to reduce the infinite-dimensional LMIs to a finite number of LMIs, and present a feasible algorithm for synthesis of H^∞ controllers based on the finite-dimensional LMIs.

1 Introduction

For linear systems with time delay, delay-independent memoryless state feedback controllers and delay-dependent memory state feedback controllers can be considered. We can expect that memory controllers achieve better performances than memoryless controllers. Thus some results focus on this memory controller synthesis problem [12] [11] [2] [8].

For LMI approach to linear systems with time delay, many result focus on memoryless controller synthesis problems [9] [4] [10] [8]. Memory controller synthesis problems via LMI approach are discussed in some results [2] [8].

In this paper, we consider a synthesis problem of H^∞ memory state feedback controllers for linear systems with time delay via LMI approach. First we show a result of L^2 gain analysis of linear systems with time delay and make a comparison with some previous works. Next we discuss a H^∞ controller synthesis problem based on this result of L^2 gain analysis. Here we also consider a synthesis problem of H^∞ controllers with constrained feedback gains. We derive results of L^2 gain analysis and H^∞ controller synthesis in the form of infinite-dimensional LMIs. Next we reduce the infinite-dimensional LMIs to a finite number of LMIs. Finally we show a numerical example.

2 System description

Consider a time-delay system,

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-h) + Bu(t) + Dw(t), \\ z(t) &= Cx(t), \\ x(\beta) &= \phi(\beta) = 0,\ -h \le \beta \le 0, \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^{m_u}$ is the input, $w(t) \in R^{m_w}$ is the disturbance, $z(t) \in R^l$ is the output and $\phi(\beta) \in R^n$ is a continuous initial function. $A_0 \in R^{n\times n}$, $A_1 \in R^{n\times n}$, $B \in R^{n\times m_u}$, $C \in R^{l\times n}$ and $D \in R^{n\times m_w}$ are constant matrices. The parameter h denotes the time delay of this system and $h > 0$.

We consider the following state feedback controller,

$$u(t) = K_0x(t) + \int_{-h}^{0} K_{01}(\beta)x(t+\beta)d\beta, \tag{2}$$

where $K_0 \in R^{m_u\times n}$ is a constant matrix and $K_{01}(\beta) \in L^2([-h,0]; R^{m_u\times n})$ is a continuous matrix function.

In this paper, we use a notation,

$$\begin{bmatrix} P_0 & P_1(\beta) \\ P_1'(\alpha) & P_2(\alpha,\beta) \end{bmatrix} > 0,$$
$$\forall\alpha \in [-h,0], \forall\beta \in [-h,0],$$

in the sense that

$$\begin{bmatrix} P_0 & \frac{1}{2}(P_1(\alpha)+P_1(\beta)) \\ \frac{1}{2}(P_1'(\alpha)+P_1'(\beta)) & \frac{1}{2}(P_2(\alpha,\beta)+P_2(\beta,\alpha)) \end{bmatrix} > 0,$$
$$P_0' = P_0,\ P_2'(\alpha,\beta) = P_2(\beta,\alpha),$$
$$\forall\alpha \in [-h,0],\ \forall\beta \in [-h,0],$$

where ">" denotes positive definiteness of matrix and "′" denotes transposition of vector and matrix.

3 L^2 gain analysis

First we consider the following linear system with a time delay,

$$\begin{aligned} \dot{x}(t) &= \tilde{A}_0 x(t) + \tilde{A}_1 x(t-h) \\ &\quad + \int_{-h}^{0} \tilde{A}_{01}(\beta) x(t+\beta) d\beta + Dw(t), \\ z(t) &= Cx(t), \end{aligned} \tag{3}$$

and show the result of L^2 gain analysis for this system. Here we define the L^2 gain of (3) as follows,

$$G \quad \sup_{w \in L^2, w \neq 0} \frac{\|z\|_{L^2}}{\|w\|_{L^2}}.$$

Now we introduce the following functional V,

$$\begin{aligned} V(x_t) &= x'(t)Px(t) \\ &\quad + \int_{-h}^{0} x'(t+\beta)Qx(t+\beta)d\beta \\ &\quad + x'(t)\int_{-h}^{0} R(\beta)x(t+\beta)d\beta \\ &\quad + \int_{-h}^{0} x'(t+\alpha)R'(\alpha)d\alpha x(t) \\ &\quad + \int_{-h}^{0}\int_{-h}^{0} x'(t+\alpha)S(\alpha,\beta)x(t+\beta)d\alpha d\beta, \end{aligned} \tag{4}$$

where

$$\begin{aligned} & x_t = \{x(t+\beta) \mid -h \le \beta \le 0\}, \\ & P, Q \in R^{n\times n}, \\ & R(\beta) \in L^2([-h,0]; R^{n\times n}), \\ & S(\alpha,\beta) \in L^2([-h,0]\times[-h,0]; R^{n\times n}). \end{aligned}$$

By using this functional, we have a result for L^2 gain analysis of the time delay system (3).

Theorem 3.1 *If there exist P, Q and continuously differentiable matrix functions $R(\beta)$, $S(\alpha,\beta)$ which satisfy the following inequalities,*

$$L_3(\alpha,\beta) =$$

$$\left[\begin{array}{cccc} \left(\begin{array}{c} \tilde{A}_0'P + P\tilde{A}_0 + Q \\ +R'(0) + R(0) + C'C \end{array}\right) & P\tilde{A}_1 - R(-h) & \left(\begin{array}{c} P\tilde{A}_{01}(\beta) + \tilde{A}_0'R(\beta) \\ -\frac{\partial}{\partial\beta}R(\beta) + S(0,\beta) \end{array}\right) & PD \\ \tilde{A}_1'P - R'(-h) & -Q & \tilde{A}_1'R(\beta) - S(-h,\beta) & 0 \\ \left(\begin{array}{c} \tilde{A}_{01}'(\alpha)P + R'(\alpha)\tilde{A}_0 \\ -\frac{\partial}{\partial\alpha}R'(\alpha) + S(\alpha,0) \end{array}\right) & R(\alpha)'\tilde{A}_1 - S(\alpha,-h) & \left(\begin{array}{c} R'(\alpha)\tilde{A}_{01}(\beta) + \tilde{A}_{01}'(\alpha)R(\beta) \\ -(\frac{\partial}{\partial\beta} + \frac{\partial}{\partial\alpha})S(\alpha,\beta) \end{array}\right) & R'(\alpha)D \\ D'P & 0 & D'R(\beta) & -\gamma^2 I \end{array}\right] < 0, \tag{5}$$

$$L_4(\alpha,\beta) = \begin{bmatrix} P & R(\beta) \\ R'(\alpha) & S(\alpha,\beta) \end{bmatrix} > 0, \tag{6}$$

$$Q > 0, \tag{7}$$

$$\forall\alpha \in [-h,\ 0],\ \forall\beta \in [-h,\ 0],$$

then the time delay system (3) is asymptotically stable and the L^2 gain of (3) is less than γ. □

Remark 3.2 *We observe that, for particular choices of structure of the solution $(P, Q, R(\beta), S(\alpha,\beta))$, LMI conditions of Theorem 3.1 is reduced to the well known condition of delay-independent types [9] or delay-dependent types [8]. To simplify the discussion, we focus on the case of the following system,*

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-h) + Dw(t), \\ z(t) &= Cx(t). \end{aligned}$$

First note that the positive definiteness of inequalities (6) and (7) in Theorem 3.1, which are required for (4) to be a Lyapunov functional of this system, can be relaxed to positive semidefiniteness except $P > 0$. In view of this, let $R(\beta) \equiv 0$ and $S(\alpha,\beta) \equiv 0$ in the inequality (5), we can rewrite (5) as

$$\begin{bmatrix} A_0'P + PA_0 + Q + C'C & PA_1 & PD \\ A_1'P & -Q & 0 \\ D'P & 0 & -\gamma^2 I \end{bmatrix} < 0. \tag{8}$$

and obtain the next Corollary 3.3 from Theorem 3.1.

Corollary 3.3 *If there exists positive definite P and Q which satisfy the LMI condition (8), then the time delay system is asymptotically stable and the L^2 gain is less than γ.*

The LMI condition (8) is equivalent to the Riccati inequality condition derived by Lee et. al. in [9].

Next let $R(\beta) = PU(\beta)$ and $S(\alpha,\beta) = U'(\alpha)PU(\beta)$, where $U(\beta)$ is a matrix function defined by the following functional differential equation,

$$\begin{aligned} \tfrac{d}{d\beta}U(\beta) &= (A_0 + U(0))U(\beta), \\ U(-h) &= A_1, \ -h \le \beta \le 0. \end{aligned} \tag{9}$$

We have a sufficient condition for the inequality (5), which is given by

$$\begin{bmatrix} M + C'C & M & PD \\ M & M & PD \\ D'P & D'P & -\gamma^2 I \end{bmatrix} < 0, \tag{10}$$

where $M = (A_0 + U(0))'P + P(A_0 + U(0))$. Thus we can obtain the next Corollary 3.4 from Theorem 3.1.

Corollary 3.4 *If there exist a positive definite matrix P and a matrix function $U(\beta)$ which satisfy the inequality (9) and the LMI condition (10), then the time delay system is asymptotically stable and the L^2 gain is less than γ.*

Corollary 3.4 is the result derived by J. He et. al. in [8] where the LMI condition (10) is expressed in the equivalent Riccati inequality form.

The LMI condition (8) is independent of the time-delay h and is finite-dimensional. On the other hand, the LMI condition (10), which seems the finite-dimensional one at first sight, is infinite-dimensional in actual, since it requires to solve the infinite-dimensional equation (9) that depends on the time-delay h. □

Remark 3.5 *As shown in Theorem 3.1 and observed in Remark 3.2, the Lyapunov functional (4) leads generally to infinite-dimensional and delay-dependent conditions or finite-dimensional and delay-independent conditions. In some special cases, however, our approach with a generalization of the functional (4) leads us to finite-dimensional and delay-dependent conditions. To illustrate this fact, consider the system with distributed delay and no control input,*

$$\dot{x}(t) = A_0 x(t) + \int_{-h}^{0} A_{01}(\beta) x(t+\beta) d\beta, \quad z(t) = Cx(t), \tag{11}$$

and consider the following Lyapunov functional,

$$V(x_t) = x'(t)Px(t) + \int_{-h}^{0} x'(t+\beta)Q(\beta)x(t+\beta)d\beta. \tag{12}$$

Note that $Q(\beta)$ is here allowed to depend on β. Then calculating the time derivative of (12) and rearranging terms as in the proof of Theorem 3.1, we have a sufficient condition for $\frac{d}{dt}V(x_t) + z'(t)z(t) - \gamma^2 w'(t)w(t) < 0$, which is given as $Q(-h) \geq 0$ and

$$\begin{bmatrix} A_0'P + PA_0 + Q(0) & PA_{01}(\beta) & PD \\ A_{01}'(\beta)P & -h^{-1}\frac{d}{d\beta}Q(\beta) & 0 \\ D'P & 0 & -\gamma^2 I \end{bmatrix} < 0,$$
$$\forall \beta \in [-h,\ 0].$$

This LMI condition is the infinite-dimensional one. However, in the special case of $A_{01}(\beta) = A_{01}$, setting $Q(\beta) = (\beta + h)I$ yields the following finite-dimensional LMI condition of delay-dependence,

$$\begin{bmatrix} A_0'P + PA_0 + hI & PA_{01} & PD \\ A_{01}'P & -h^{-1}I & 0 \\ D'P & 0 & -\gamma^2 I \end{bmatrix} < 0. \tag{13}$$

Thus we obtain Corollary 3.6.

Corollary 3.6 *If there exists the positive definite matrix P which satisfies the LMI condition (13), then the time delay system (11) with $A_{01}(\beta) = A_{01}$ is asymptotically stable and the L^2 gain is less than γ.*

In [10], X. Li and C.E. de Souza derived a finite-dimensional and delay-dependent LMI condition for robust stability and stabilization based on a Lyapunov functional which has essentially the same structure as the Lyapunov functional (12) has. □

4 H^∞ Control Synthesis

From (1) and (2), the closed loop system can be written in the following form,

$$\begin{aligned} \dot{x}(t) &= \tilde{A}_0 x(t) + \tilde{A}_1 x(t-h) \\ &\quad + \int_{-h}^{0} \tilde{A}_{01}(\beta)x(t+\beta)d\beta + Dw(t), \\ z(t) &= Cx(t), \end{aligned} \tag{14}$$

where

$$\tilde{A}_0 = A_0 + BK_0,\ \tilde{A}_1 = A_1, \quad \tilde{A}_{01}(\beta) = BK_{01}(\beta).$$

Thus considering the previous result of L^2 gain analysis, we have the next theorem for H^∞ control synthesis problem.

Theorem 4.1 *If there exist W, X, Z_0 and continuously differentiable matrix function $Z_{01}(\beta)$ and $Y(\alpha,\beta)$ which satisfy the following inequalities,*

$$L_5(\alpha,\beta) = \left[\begin{matrix} \begin{pmatrix} WA_0' + WA_0 \\ +X + 2W \\ +BZ_0 + Z_0'B' \end{pmatrix} & A_1W - W \\ WA_1' - W & -X \\ \begin{pmatrix} WA_{01}(\alpha)' + Z_{01}(\alpha)'B' \\ +A_0W + BZ_0 \\ +Y(\alpha,0) \end{pmatrix} & A_1W - Y(\alpha,-h) \\ CW & 0 \\ D' & 0 \end{matrix}\right.$$
$$\left.\begin{matrix} \begin{pmatrix} A_{01}(\beta)W + BZ_{01}(\beta) \\ +WA_0' + Z_0'B' \\ +Y(0,\beta) \end{pmatrix} & WC' & D \\ WA_1' - Y(-h,\beta) & 0 & 0 \\ \begin{pmatrix} A_{01}(\beta)W + WA_{01}(\alpha)' \\ +BZ_{01}(\beta) + Z_{01}(\alpha)'B' \\ -(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta})Y(\alpha,\beta) \end{pmatrix} & 0 & D \\ 0 & -I & 0 \\ D' & 0 & -\gamma^2 I \end{matrix}\right] < 0, \tag{15}$$

$$L_6(\alpha,\beta) = \begin{bmatrix} W & W \\ W & Y(\alpha,\beta) \end{bmatrix} > 0, \tag{16}$$

$$X > 0, \tag{17}$$
$$\forall \alpha \in [-h,\ 0],\ \forall \beta \in [-h,\ 0],$$

then the time-delay system (1) with the state feedback controller (2)

$$K_0 = Z_0 W^{-1},\quad K_{01}(\beta) = Z_{01}(\beta)W^{-1}, \tag{18}$$

is asymptotically stable and the L^2 gain is less than γ.

At the next remark, we make a comment on robust stability of the closed loop system, which is formed by the controller of Theorem 4.1, against variations of time-delay.

Remark 4.2 *Let $\tilde{h}$ be the actual time delay and $\tilde{h} \neq h$. Consider the functional (4) along the trajectory of the closed loop system with $w(t) \equiv 0$. We obtain the following inequality, for a positive number λ,*

$$\begin{aligned}\frac{d}{dt}V(x_t) \leq\ & -\lambda x'(t)x(t) - z'(t)z(t) \\ & -\gamma^{-2}\eta'(t)D'D\eta(t) \\ & -2\{x(t-\tilde{h}) - x(t-h)\}'A_1\eta(t), \quad (19)\end{aligned}$$

where $\eta(t) = W^{-1}\{x(t) + \int_{-h}^{0} x(t+\beta)d\beta\}$. Now integrating both sides of (19) over a large time-interval and using Parseval equality, we obtain a sufficient condition for robust stability as follows: If there exists a positive number ρ such that, for a sufficiently large positive number N,

$$\begin{aligned}&\lambda I + C'C + \gamma^{-2}|1 + \int_{-h}^{0} e^{j\omega\beta}d\beta|^2 W^{-1}D'DW^{-1} \\ &+(e^{j\omega\tilde{h}} - e^{j\omega h})A_1'W^{-1}(1 + \int_{-h}^{0} e^{j\omega\beta}d\beta) \\ &+(1 + \int_{-h}^{0} e^{-j\omega\beta}d\beta)W^{-1}A_1(e^{-j\omega\tilde{h}} - e^{-j\omega h}) \\ &\geq \rho I,\ ^\forall\omega \in [-N,\ N], \quad (20)\end{aligned}$$

then the closed loop system is asymptotically stable. It is obvious that asymptotical stability is maintained for sufficiently small variations of time-delay. □

Finally we consider H^∞ control synthesis problem with the constrained feedback gain. We constrain the feedback gain as follows,

$$K_0'K_0 < \gamma_1 I,\ K_{01}'K_{01}(\beta) < \gamma_2 I,\ ^\forall\beta \in [-h,\ 0], \quad (21)$$

where γ_1 and γ_2 are given in advance.

Utilizing Theorem 4.1, we have the following theorem.

Theorem 4.3 *For given positive numbers p_1, p_2 and q, if there exist W, X, Z_0 and continuously differentiable matrix function $Z_{01}(\beta)$ and $Y(\alpha,\beta)$ which satisfy the following inequalities,*

$$L_5(\alpha,\beta) < 0,\ L_6(\alpha,\beta) < 0,\ X > 0, \quad (22)$$

$$\begin{bmatrix} p_1 I & Z_0' \\ Z_0 & I \end{bmatrix} > 0, \quad (23)$$

$$\begin{bmatrix} p_2 I & Z_{01}'(\beta) \\ Z_{01}(\beta) & I \end{bmatrix} > 0, \quad (24)$$

$$\begin{bmatrix} qI & I \\ I & W \end{bmatrix} > 0, \quad (25)$$

$$^\forall\alpha \in [-h,\ 0],\ ^\forall\beta \in [-h,\ 0],$$

where $L_5(\alpha,\beta)$ and $L_6(\alpha,\beta)$ are given as (15) and (16) respectively, then the time-delay system (1) with the state feedback controller (2)

$$K_0 = Z_0W^{-1},\quad K_{01}(\beta) = Z_{01}(\beta)W^{-1}, \quad (26)$$

is asymptotically stable and the L^2 gain is less than γ. Here K_0 and $K_{01}(\beta)$ are constrained as follows,

$$K_0'K_0 < p_1q^2I,\quad K_1(\beta)'K_1(\beta) < p_2q^2I.$$

PROOF:
(23), (24) and (25) are equivalent to the following conditions respectively,

$$\begin{aligned}Z_0'Z_0 &< p_1 I \\ Z_{01}(\beta)'Z_{01}(\beta) &< p_2 I \\ W^{-1} &< qI\end{aligned}$$

By using the above conditions, we have the following results,

$$\begin{aligned}K_0'K_0 &= W^{-1}Z_0'Z_0W^{-1} \\ &< p_1W^{-1}W^{-1} \\ &< p_1q^2I \\ K_{01}(\beta)'K_{01}(\beta) &= W^{-1}Z_{01}(\beta)'Z_{01}(\beta)W^{-1} \\ &< p_2W^{-1}W^{-1} \\ &< p_2q^2I.\end{aligned}$$

Q.E.D.

Thus by using this theorem and choosing p_1, p_2 and q appropriately, we can obtain H^∞ controllers with feedback gains constrained as (21). Next we show an algorithm to choose p_1, p_2 and q.

Algorithm:

Step 1: Let p_{10}, p_{20} and q_0 be initial values of p_1, p_2 and q respectively.

Step 2: Solve inequalities in Theorem 4.3 and the following inequalities,

$$p_1 < p_{10},\quad p_2 < p_{20},\quad q < q_0.$$

Step 3: Check the next conditions for p_1, p_2 and q of **Step 2**.

$$p_1q^2 < \gamma_1,\quad p_2q^2 < \gamma_2. \quad (27)$$

- If (27) is satisfied, the algorithm is finished. The controller designed in **Step 2** satisfies (21).
- If (27) is not satisfied, go back to **Step 1**.

5 Reduction to a Finite Number of LMI Conditions

Inequalities in Theorem 4.1 depend on parameters α and β. It seems difficult to solve these infinite-dimensional (parameter dependent) inequalities directly. In our approach, we reduce these infinite-dimensional inequalities to a finite number of LMIs by using the technique in [1], and obtain the solution of the infinite-dimensional inequalities by computing the finite number of LMIs.

Here we restrict solutions in Theorem 4.1 to the following forms,

$$\begin{aligned} Y(\alpha,\beta) &= Y_0 + g_1(\alpha,\beta)Y_1 + g_2(\alpha,\beta)Y_2 \\ &\quad + \cdots + g_{l_Y}(\alpha,\beta)Y_{l_Y}, \\ Z_{01}(\beta) &= Z_0^{01} + h_1(\beta)Z_1^{01} + h_2(\beta)Z_2^{01} \\ &\quad + \cdots + h_{l_z}(\beta)Z_{l_z}^{01}, \end{aligned} \tag{28}$$

where $g_i : R^2 \to R$ is a continuous differentiable function of α and β such that

$$g_i(\alpha,\beta) = g_i(\beta,\alpha),$$

$h_i : R \to R$ is a continuous differentiable function of β, and the unknown matrices satisfy

$$\begin{aligned} &Y_i \in R^{n\times n},\ Y_i' = Y_i\ (i = 0,1,\cdots,l_Y), \\ &Z_i^{01} \in R^{m\times n}\ (i = 0,1,\cdots,l_Z). \end{aligned}$$

Note that (28) satisfies matrix inequalities (15) (16). Then inequalities in *Theorem 4.1* can be written in the form of the following parameter dependent LMI condition,

$$F_0(M) + f_1(\theta)F_1(M) + \cdots + f_r(\theta)F_r(M) < 0, \tag{29}$$

where

$$\theta \in \Theta = \{[\alpha\ \beta]' \mid \alpha \in [-h,\ 0], \beta \in [-h,\ 0]\},$$

and $f_i : R^2 \to R$ is a continuous function of α and β, and a symmetric matrix function F_i depends affinely on the unknown matrix $M = [Y_0, \cdots, Y_{l_Y}, Z_0^{01}, \cdots, Z_{l_Z}^{01}]$. The parameter dependent LMI condition (29) can be reduced to a finite number of LMI conditions as follows.

Theorem 5.1 *[1] Let $\{p_1, p_2, \cdots, p_q\}$ be vertices of a convex polyhedron which includes the curved surface T,*

$$T = \{[f_1(\theta)\ f_2(\theta)\ \cdots\ f_r(\theta)]' \mid \theta \in \Theta\}. \tag{30}$$

Assume that there exists M which satisfies the following LMI condition for all $p_i (i = 1, 2, \cdots, q)$,

$$F_0(M) + p_{i1}F_1(M) + \cdots + p_{ir}F_r(M) < 0, \tag{31}$$

where p_{ij} is the jth element of p_i. Then M satisfies (29) for all $\theta \in \Theta$.

The techniques to construct the convex polyhedron which includes the curved surface T are proposed in [1].

Remark 5.2 *Our approach to reduction is dependent on choice of functions $h_i(\beta)$ $i = 0, 1, \cdots, l_Y$ and $g_i(\alpha,\beta), i = 0, 1, \cdots, l_Z$. Unfortunately, we have no general guideline for the choice.*

If we adopt the technique derived by Azuma et. al. [1] to construct the convex polyhedron including the curved surface T, we recommend that $h_i(\beta)$ should be a monotone (decreasing or increasing) function of β and $g_i(\alpha,\beta)$ should be also a monotone function of α and β at each α and β, e.g.,

$$\begin{aligned} h_i(\beta) &= \beta^k,\ e^{\beta},\ \cdots \\ g_i(\alpha,\beta) &= \alpha^p\beta^q,\ e^{(\alpha+\beta)},\ \cdots \end{aligned}$$

which makes the technique by Azuma et. al. easy to apply.

6 Numerical Example

Consider the next time delay system,

$$\Sigma_p : \begin{aligned} &\dot{x}(t) = x(t) + 0.3x(t-1) + u(t), \\ &y(t) = x(t), \\ &x(\beta) = \phi(\beta) = 0,\ -h \le \beta \le 0. \end{aligned} \tag{32}$$

Now we design the memory state feedback controller K such that the error, $r - y$, is asymptotically zero (Figure 1). When we use the technique of Section 5, we restrict solutions of Theorem 4.1 and Theorem 4.3 as follows,

$$\begin{aligned} Z_{01}(\beta) &= Z_0 + \beta Z_1 + \beta^2 Z_2, \\ Y(\alpha,\beta) &= Y_0 + (\alpha+\beta)Y_1 + (\alpha^2+\beta^2)Y_2. \end{aligned}$$

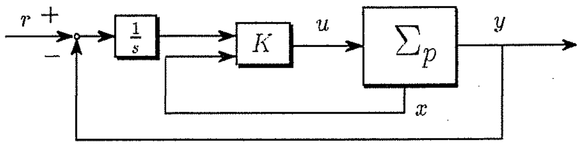


Figure 1: The closed loop system

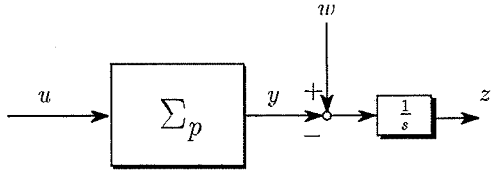


Figure 2: Generalized plant

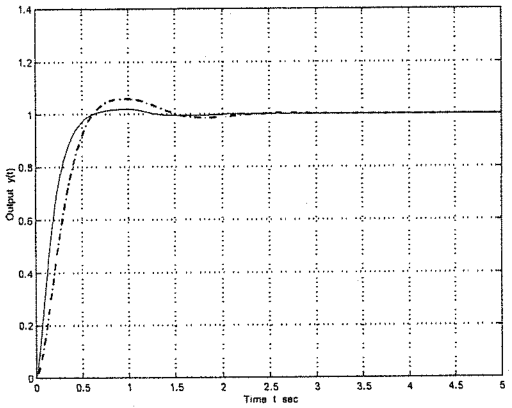
First we apply Theorem 4.1 to Figure 2 and obtain the state feedback controller (2) with the next feedback gains,

$$\begin{aligned} K_0 &= [\ 115.48\quad -24.94\ ], \\ K_{01}(\beta) &= [\ 75.79\quad -12.45\ ] \\ &\quad +\beta\,[\ -9.31\quad -3.09\ ] \\ &\quad +\beta^2\,[\ 15.14\quad -3.53\ ]. \end{aligned} \tag{33}$$

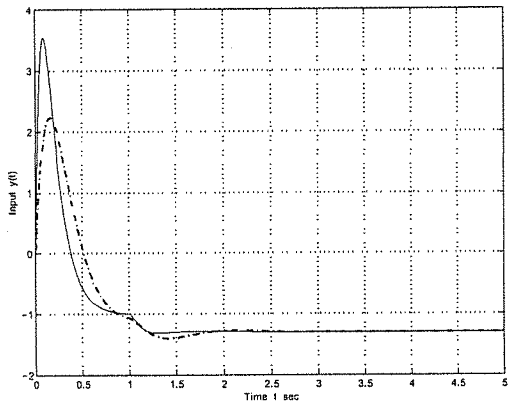
Second setting $p_1 = 3.49 \times 10^4$, $p_2 = 1.28 \times 10^2$, $q = 2.56$ and using Theorem 4.3, we obtain the state feedback controller (2) with the next feedback gains,

$$\begin{aligned} K_0 &= [\ 36.16\quad -11.74\ ], \\ K_{01}(\beta) &= [\ 23.49\quad -4.01\ ] \\ &\quad +\beta\,[\ 1.71\quad -0.53\ ] \\ &\quad +\beta^2\,[\ -0.07\quad -0.19\ ]. \end{aligned} \tag{34}$$

The simulation results are shown in Figure 3, where the reference is 1 ($r = 1$). In this figure, the solid line and the dashdot line denote the simulation result of the case (33) and (34) respectively. The error $r - y$ is asymptotically zero at both cases. By using Theorem 4.3, we can make the maximum of the control input small.



(a) Output y



(b) Input u

Figure 3: Simulation result

7 Conclusion

In this paper, we considered a synthesis problem of H^∞ memory state feedback controllers for linear systems with time delay via LMI approach. We derived an existence condition of H^∞ controllers in the form of infinite-dimensional LMIs and showed a technique to reduce the infinite-dimensional LMIs to a finite number of LMIs which is feasible formulas. Finally we demonstrated the efficacy of our approach by a numerical example.

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NONLINEAR OUTPUT FEEDBACK CONTROL OF PWM INVERTER WITH INTERNAL RESISTANCE

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Abstract

Nonlinear output feedback controllers are derived to achieve active harmonic reduction at ac (alternating current) port current in PWM (Pulse-Width-Modulated) inverter with internal resistance in dc (direct current) port. The controller design is carried out on the basis of state averaged model which is a bilinear system. The control problems are treated as nonlinear H_∞ output feedback control problems for the bilinear system via Lyapunov-based game theory approach. Convex programming technique gives concretely the controllers. Computer simulations show efficiency of the control system design approach.

1 Introduction

There has been a steady growth of interest in control of power electronics circuits (eg. [2, 6]). Many works (eg. [2]) discuss linear feedback control problems for power electronics circuits on the basis of linearized state averaged model of the circuits. A work [6] discusses passivity-based feedback control problems for dc/dc power converters on the basis of Euler-Lagrange systems.

This paper discusses nonlinear output feedback control problems for PWM inverter with internal resistance in dc port. The resistance causes dc port voltage drop, and so distorts ac port, current whom we would like to be sinusoidal. The control problems are to achieve active harmonic reduction at ac port current in the inverter. The controller design is carried out on the basis of state averaged model which is a bilinear system [4].

Our control system design for the problems is given as follows. First, a switched model of PWM inverter with internal resistance in dc port is derived. Second, a state averaged model is derived from the switched model [2], which is a bilinear system [4]. State variables of the model consist of averaged variables of an output inductor current and two input capacitor voltages. In this control system, the output current is an only available state. The averaged model is treated as a bilinear system with uncertainties because our control system design approach regards unmeasurable nonlinear states as uncertainties. Third, the control problems are treated as nonlinear H_∞ output feedback control problems of the bilinear system. Finally, a convex programming technique [5] concretely solves the problems. Computer simulations show efficiency of the control system design approach.

Notations: I and 0 denote an identity matrix and a zero matrix of suitable dimensions, respectively. For a vector $\sigma \in \mathbb{R}^n$ with positive elements, $\mathcal{B}_\sigma^n$ denotes $\{x \,|\, |x_i| \leq \sigma_i, i = 1, \ldots, n\}$, where x_i and σ_i stands for the i-th element of the vector, respectively. When there exists a domain $\mathcal{B}_\sigma^n \subseteq \mathbb{R}^m$ which contains origin, for a continuously differentiable symmetric matrix-valued function $P(y) : \mathcal{B}_\sigma^m \to \mathbb{R}^{n \times n}$, a notation $P(y) > 0, \forall y \in \mathcal{B}_\sigma^m$ means $x^T P(y) x > 0, \forall y \in \mathcal{B}_\sigma^m, \forall x \in \mathbb{R}^n, x \neq 0$.

2 PWM Inverter with Internal Resistance

This section derives a state averaged model of PWM inverter with internal resistance in dc port. The resistance causes dc port voltage drop, and so distorts ac port current whom we would like to be sinusoidal. Figure 2.1 shows a control system of voltage source PWM inverter with internal resistance, which shall be constructed in this paper.

The figure gives differential equations of the circuit for two switch states; in a switch state where a switch SW_1 is on-state and a SW_2 off-state,

$$\frac{d}{dt}\begin{bmatrix} i \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_L}{L} & \frac{1}{L} & 0 \\ -\frac{1}{C} & -\frac{1}{R_E C} & 0 \\ 0 & 0 & -\frac{1}{R_E C} \end{bmatrix} \begin{bmatrix} i \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{E}{R_E C}, \tag{2.1}$$

in the other switch state where the SW_1 off-state and the

$$\frac{d}{dt}\begin{bmatrix} i \\ v_1 \\ v_2 \end{bmatrix} \text{ on-state, } = \begin{bmatrix} -\frac{R_L}{L} & 0 & -\frac{1}{L} \\ & -\frac{1}{R_E C} & 0 \\ \frac{1}{C} & & -\frac{1}{R_E C} \end{bmatrix} \begin{bmatrix} i \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{E}{R_E C}, \quad (2.2)$$

where i denotes an output inductor current, and v_1 and v_2 denote input capacitor voltages. Es denote voltages of dc voltage sources, R_E and R_L resistances, C a capacitance, and L an inductance. Especially, R_E means an internal resistance in the voltage sources.

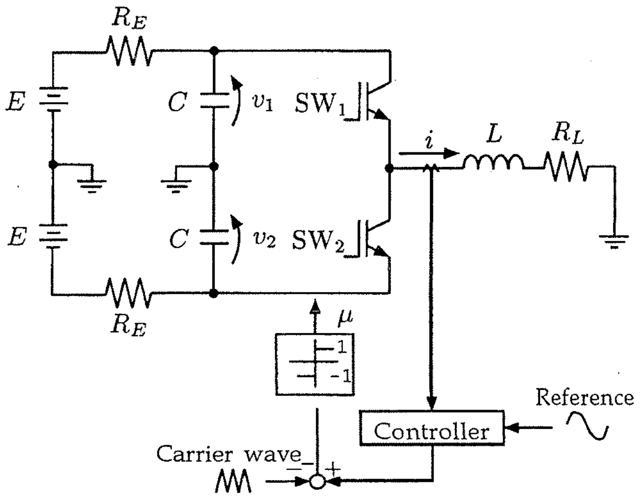

Figure 2.1: Control system of PWM inverter with internal resistance in dc port,

A switch state function μ, which takes 1 for the first switch state (2.1) and -1 for the second switch state (2.2), gives an equation of the form

$$\frac{d}{dt}\begin{bmatrix} i \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_L}{L} & \frac{1}{2L} & -\frac{1}{2L} \\ -\frac{1}{2C} & -\frac{1}{R_E C} & 0 \\ \frac{1}{2C} & 0 & -\frac{1}{R_E C} \end{bmatrix} \begin{bmatrix} i \\ v_1 \\ v_2 \end{bmatrix} + \left\{ i \begin{bmatrix} 0 \\ -\frac{1}{2C} \\ -\frac{1}{2C} \end{bmatrix} + v_1 \begin{bmatrix} \frac{1}{2L} \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} \frac{1}{2L} \\ 0 \\ 0 \end{bmatrix} \right\} \mu + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{E}{R_E C} \quad (2.3)$$

from the two equations (2.1) and (2.2). The function μ is used as a control input. From the model (2.3), a state averaging approach, which is introduced in [2], derives a state averaged model of the form

$$\frac{d}{dt}\begin{bmatrix} \bar{i} \\ \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} \approx \begin{bmatrix} -\frac{R_L}{L} & \frac{1}{2L} & -\frac{1}{2L} \\ -\frac{1}{2C} & -\frac{1}{R_E C} & 0 \\ \frac{1}{2C} & 0 & -\frac{1}{R_E C} \end{bmatrix} \begin{bmatrix} \bar{i} \\ \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} + \left\{ \bar{i} \begin{bmatrix} 0 \\ -\frac{1}{2C} \\ -\frac{1}{2C} \end{bmatrix} + \bar{v}_1 \begin{bmatrix} \frac{1}{2L} \\ 0 \\ 0 \end{bmatrix} + \bar{v}_2 \begin{bmatrix} \frac{1}{2L} \\ 0 \\ 0 \end{bmatrix} \right\} \bar{\mu} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{E}{R_E C} \quad (2.4)$$

where $\bar{i}, \bar{v}_1, \bar{v}_2$ and $\bar{\mu}$ denote moving averaged variables of i, v_1, v_2 and μ, respectively. In the following section, the state averaged model (2.4) is used to design the averaged control input, $\bar{\mu}$.

Remark 2.1 In inverter systems, an internal resistance in dc port, causes dc port voltage drop, and so distorts ac port current whom we would like to be sinusoidal. The reason is that dc port currents across the resistances vary capacitor voltages whom we would like to assume to be constant. It results in the varying that, the dc port current drawn by the inverter first vary as well as sinusoidal inverter output and second have a. second harmonic component of a fundamental frequency at the invcrter output (in addition to high switching frequency component) [3]. The model (2.4) has those characteristics.

3 Nonlinear Control System Design

This section treats the active harmonic reduction problems as nonlinear H_∞ output feedback control problems. First, a generalized plant, is constructed to meet a control system design specification on the basis of the averaged model (2.4). Next, a nonlinear H_∞ output feedback controller is derived for the generalized plant.

3.1 Control System Design Specification

The work [5] rewrites the averaged model (2.4) into an averaged model, which is a bilinear system [4], around an equilibrium point $[\bar{i}, \bar{v}_1, \bar{v}_2, \bar{\mu}]$ = *[0, E, E, 0]* of the form

$$\frac{d}{dt}\begin{bmatrix} \tilde{i} \\ \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_L}{L} & \frac{1}{2L} & -\frac{1}{2L} \\ -\frac{1}{2C} & -\frac{1}{R_E C} & 0 \\ \frac{1}{2C} & 0 & -\frac{1}{R_E C} \end{bmatrix} \begin{bmatrix} \tilde{i} \\ \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} + \left\{ \begin{bmatrix} \frac{E}{L} \\ 0 \\ 0 \end{bmatrix} + \tilde{i} \begin{bmatrix} 0 \\ -\frac{1}{2C} \\ -\frac{1}{2C} \end{bmatrix} + \tilde{v}_1 \begin{bmatrix} \frac{1}{2L} \\ 0 \\ 0 \end{bmatrix} + \tilde{v}_2 \begin{bmatrix} \frac{1}{2L} \\ 0 \\ 0 \end{bmatrix} \right\} \tilde{\mu}, \quad (3.1)$$

where $[\ \tilde{i}\ \tilde{v}_1\ \tilde{v}_2\ \tilde{\mu}\ ] = [\ \bar{i}\ \bar{v}_1\ \bar{v}_2\ \bar{\mu}\ ] - [\ 0\ E\ E\ 0\ ]$. In the control problems, $\tilde{i}$ is an only measurable state. A nonlinear H_∞ control system design technique in the work [5] treats unmeasurable nonlinear states as uncertainties, and so treats the model (3.1) as a model with uncertainties of the form

$$\dot{x}_p = A_p x_p + B_{p1} w_p + \{B_{p20} + y_p B_{p21}\}\, \tilde{\mu}, \quad (3.2)$$

$$z_p = D_{p21} \tilde{\mu}, \quad (3.3)$$

$$y_p = C_p x_p, \quad (3.4)$$

$$w_p = \begin{bmatrix} \delta_1 & \\ 0 & \delta_2 \end{bmatrix} z_p, \quad (3.5)$$

where

$$x_p = \begin{bmatrix} \tilde{i} & \tilde{v}_1 & \tilde{v}_2 \end{bmatrix}^T, \quad A_p = \begin{bmatrix} -\frac{R_L}{L} & \frac{1}{2L} & -\frac{1}{2L} \\ -\frac{1}{2C} & -\frac{1}{R_E C} & 0 \\ \frac{1}{2C} & 0 & -\frac{1}{R_E C} \end{bmatrix},$$

$$B_{p1} = \begin{bmatrix} \frac{1}{2L} & \frac{1}{2L} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\gamma V_{b1}}{W_{u1}} & 0 \\ 0 & \frac{\gamma V_{b2}}{W_{u2}} \end{bmatrix}, \quad B_{p20} = \begin{bmatrix} \frac{E}{L} \\ 0 \\ 0 \end{bmatrix},$$

$$B_{p21} = \begin{bmatrix} 0 \\ -\frac{1}{2C} \\ -\frac{1}{2C} \end{bmatrix}, \quad D_{p21} = \begin{bmatrix} W_{u1} & 0 \\ 0 & W_{u2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$C_p = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad |\delta_1| < 1, \ |\delta_2| < 1.$$

The second term in the right hand side of equation (3.2) presents uncertainties which are affected by the control input $\tilde{\mu}$. $\gamma, V_{b1}, V_{b2}, W_{u1}, W_{u2}$ are suitable weighting parameters which are specified in the following discussion.

The active harmonic reduction of ac port current in the PWM inverter is achieved by constructing a control system to meet the following control system design specification.

(S1) An output inductor current $\tilde{i}$ should track a sinewave reference ($r \rightarrow z_e$).

(S2) A stability of closed-loop system should be kept against varying dc port voltages ($w_p \rightarrow z_p$).

The specification gives a generalized plant, as shown in Figure 3.1 of the form

$$\dot{x} = Ax + B_1 w + B_2(y)u, \tag{3.6}$$
$$z = C_1 x + D_{12}u, \tag{3.7}$$
$$y = C_2 x, \tag{3.8}$$

where

$$x = \begin{bmatrix} x_w^T & x_p^T \end{bmatrix}^T, \ w = \begin{bmatrix} r & w_p^T \end{bmatrix}^T, \ u = \tilde{\mu}, \ z = \begin{bmatrix} z_e & z_p^T \end{bmatrix}^T,$$

$$A = \begin{bmatrix} A_w & A_{12} \\ 0 & A_p \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -k\omega & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} B_w & 0 \\ 0 & B_{p1} \end{bmatrix}, \ B_2(y) = \begin{bmatrix} 0 \\ B_{p20} \end{bmatrix} + y_4 \begin{bmatrix} 0 \\ B_{p21} \end{bmatrix},$$

$$C_1 = \begin{bmatrix} W_e C_w & 0 \\ 0 & 0 \end{bmatrix}, \ D_{12} = \begin{bmatrix} 0 \\ D_{p21} \end{bmatrix}, \ C_2 = \begin{bmatrix} I & 0 \end{bmatrix},$$

and x_w denotes state of weighting function for the specification (S1), which is

$$W(s) = \frac{k\omega}{s(s^2 + \omega^2)} \tag{3.9}$$

and so whose state space description is

$$x_w = A_w x_w + B_w(r - C_p x_p), \tag{3.10}$$
$$y_w = C_w x_w, \tag{3.11}$$

where

$$A_w = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ 0 \\ k\omega \end{bmatrix}, \quad C_w = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

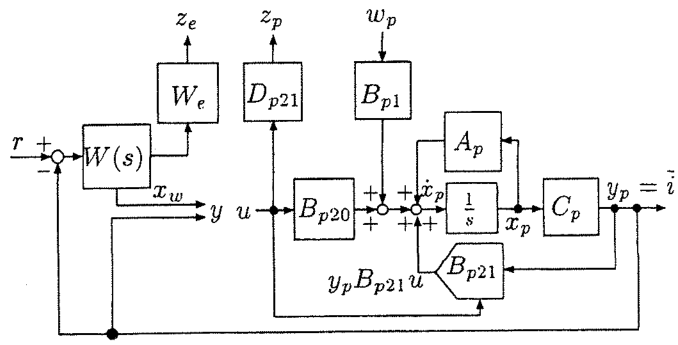

Figure 3.1: Generalized plant,

Note that a controller in Figure 2.1 finally consists of the weighting function $W(s)$ and a controller designed for the generalized plant in the following discussion.

3.2 Lyapunov-based game theory approach

Nonlinear H_∞ output feedback controllers are derived via Lyapunov-based game theory approach in the work [5]. The work [5] considers input-affine polynomial-type nonlinear systems (C) of the form

$$\dot{x} = A(y)x + B_1(y)w + B_2(y)u, \tag{3.12}$$
$$z = C_1(y)x + D_{12}(y)u, \tag{3.13}$$
$$y = C_2 x, \tag{3.14}$$

where $x \in \mathbb{R}^{n_x}$ is the state, and $y \in \mathbb{R}^{n_y}$ is the measured output which is directly measurable states in the state x. The generalized plant in the previous subsection is in a class of the systems (C). Therefore, the work [5] can treat the control problems. This section introduces a tractable sufficient condition for the nonlinear H_∞ output feedback control problems to be solvable, which is shown in the work [5].

Here, in order to simplify notations in the following theorem, matrix-valued functions are defined as

$$Q(\lambda) := \begin{bmatrix} Q_a(\lambda) & Q_b \\ Q_b^T & Q_c \end{bmatrix} \in \mathbb{R}^{n_x \times n_x},$$

$$Y(Q(\lambda), \lambda) := \begin{bmatrix} \hat{Q}_a(\lambda) & Q_b \\ Q_b^T & Q_c \end{bmatrix},$$

where,

$$\lambda \in \mathbb{R}^{n_y}, \quad Q_a(\lambda) \in \mathbb{R}^{n_y \times n_y},$$

$$Q_b \in \mathbb{R}^{n_y \times (n_x - n_y)}, \quad Q_c \in \mathbb{R}^{(n_x - n_y) \times (n_x - n_y)},$$

$$\hat{Q}_a(\lambda) := Q_a(\lambda) + \frac{1}{2} \begin{bmatrix} \frac{\partial Q_a(\lambda)}{\partial \lambda_1} \lambda & \cdot & \frac{\partial Q_a(\lambda)}{\partial \lambda_{n_y}} \lambda \end{bmatrix}$$

Theorem 3.1 [5] Considcr a system (C). For a given positive constant γ, if there exist a domain $(\mathcal{B}_{\sigma}^{n_y} \times \mathbb{R}^{n_x - n_y}) \times (\mathcal{B}_{\hat{\sigma}}^{n_y} \times \mathbb{R}^{n_x - n_y}) \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$, a matrix $X \in \mathbb{R}^{n_x \times n_x}$ and a matrix-valued function $Q(y - y_\xi) \in \mathbb{R}^{n_x \times n_x}$ such that

$$(1) \begin{bmatrix} F_1(X, y) & XC_1^T(y) & B_1(y) \\ C_1(y)X & -I & 0 \\ B_1^T(y) & 0 & -\gamma^2 I_1 \end{bmatrix} < 0, \ \forall y \in \mathcal{B}_{\sigma}^{n_y}, \tag{3.15}$$

$$(2) \begin{bmatrix} F_2(Y(Q(y - y_\xi), y - y_\xi), y) & Y(Q(y - y_\xi), y - y_\xi)B_1(y) & C_1^T(y) \\ B_1^T(y)Y(Q(y - y_\xi), y - y_\xi) & -I & 0 \\ C_1(y) & 0 & -\gamma^2 I \end{bmatrix} < 0,$$
$$\forall (y, y_\xi) \in \mathcal{B}_{\sigma}^{n_y} \times \mathcal{B}_{\hat{\sigma}}^{n_y}, \tag{3.16}$$

$$(3) \begin{bmatrix} Y(Q(y - y_\xi), y - y_\xi) & I \\ I & \gamma^2 X \end{bmatrix} > 0,$$
$$\forall (y, y_\xi) \in \mathcal{B}_{\sigma}^{n_y} \times \mathcal{B}_{\hat{\sigma}}^{n_y}, \tag{3.17}$$

where

$$F_1(X,y) := XA^T(y) + A(y)X - B_2(y)R^{-1}(y)B_2^T(y),$$
$$F_2(Y(Q(y - y_\xi), y - y_\xi), y) := Y(Q(y - y_\xi), y - y_\xi)A(y) + A^T(y)Y(Q(y - y_\xi), y - y_\xi) - C_2^T C_2,$$
$$y_\xi := C_2 \xi,$$

then an output feedback controller (Γ) solving the problem is given as

$$\begin{aligned} \dot{\hat{x}} = & A(y)\hat{x} - B_2(y)R^{-1}(y)B_2^T(y)X^{-1}\hat{x} \\ & + \gamma^{-2}B_1(y)B_1^T(y)X^{-1}\hat{x} \\ & + [Y(Q(y - \hat{y}), y - \hat{y}) - \gamma^{-2}X^{-1}]^{-1}C_2^T(y - C_2\hat{x}) \\ & + \gamma^{-2}[Y(Q(y - \hat{y}), y - \hat{y}) \\ & \quad - \gamma^{-2}X^{-1}]^{-1}X^{-1}\mathrm{Eqn}(X, y)X^{-1}\hat{x}, \end{aligned} \tag{3.18}$$

$$u = -R^{-1}(y)B_2^T(y)X^{-1}\hat{x}, \tag{3.19}$$

where

$$\hat{y} := C_2\hat{x},$$
$$\mathrm{Eqn}(X, y) := XA^T(y) + A(y)X - (B_2(y)R^{-1}(y)B_2^T(y) - \gamma^{-2}B_1(y)B_1^T(y)) + XC_1^T(y)C_1(y)X.$$

Moreover, the closed-loop system (Σ, Γ) is internally stable in the maximum supersolid

$$\Omega_{of}(\alpha_{of}) := \{(x, \xi) \mid x^T X^{-1} x + \gamma^2 (x - \xi)^T T(x - \xi)(x - \xi) \le \alpha_{of}\}$$

that is contained in the domain $(\mathcal{B}_{\sigma}^{n_y} \times \mathbb{R}^{n_x - n_y}) \times (\mathcal{B}_{\hat{\sigma}}^{n_y} \times \mathbb{R}^{n_x - n_y})$, where

$$T(x - \xi) := Q(C_2(x - \xi)) - \gamma^{-2}X^{-1}.$$

3.3 Convex Programming Technique

The condition in Theorem 3.1 consists of state-depended matrix inequalities. A convex programming technique [5] can give the solution. The steps are that first we enclose a domain of state of the system in a convex hull, and second solve linear matrix inequalities which are given at vertices of the convex hull [1]. The details are in the work [5] which applies a result in [7] and gives extensions to solve nonlinear control problems.

4 Computer Simulations

This section shows efficiency of the control system design approach through computer simulations. The simulations use MATLAB, Simulink and LMI Control Toolbox. In the simulations, system parameters are used as

$$R_L = 10\ [\Omega], \quad L = 5 \times 10^{-3}[\mathrm{H}],$$
$$R_E = 1\ [\Omega], \quad C = 1 \times 10^{-3}\ [\mathrm{F}], \quad E = 141\ [\mathrm{V}].$$

Control system design parameters are given as

$$\gamma = 1.2, \quad V_{b1} = V_{b2} = 9, \quad W_{u1} = W_{u2} = 0.5,$$
$$W_e = 1, \quad w = 100\pi, \quad k = 120.$$

Parameters $V_{b1} = V_{b2} = 9$ mean that input, capacitor voltages are varying as $|\tilde{v}_1| \le 9$ and $|\tilde{v}_2| \le 9$. In order to enclose a domain of state for the convex programming technique, it is also considered that ac port current $\tilde{i}$ is varying as $|\tilde{i}| \le 11$. Then, the Lyapunov-based game theory approach in Section 3 gives a quadratic Lyapunov function as a solution of Theorem 3.1, whose coefficients are shown in (4.1)(4.2).

The controller, which is constructed on the basis of the solution, gives simulation results for the averaged model (2.4) as shown in Figure 4.1. The control system design specification is very well satisfied. The controller also gives simulation results for the switched model (2.3) as shown in Figure 4.2. Control system block diagram of this case is shown in Figure 2.1 where a frequency of carrier wave is 5 kHz. The figure 4.2 shows that the controller also acts very well for the switched model (2.3). The averaged control system, which is constructed on the basis of averaged model, satisfactorily captures behaviors of the switched control system.

5 Conclusion

Nonlinear output feedback controllers were derived to achieve active harmonic reduction of ac port current in PWM inverter with internal resistance in dc port. The controller design was carried out on the basis of state averaged model which was a bilinear system. The control problems were treated as nonlinear H_∞ output feedback control problems for the bilinear system via Lyapunov-based game theory approach. A convex programming technique gave the solution. Computer simulations showed efficiency of

$$X = \begin{vmatrix} 2.43 \times 10'' & -1.52 \times 10^{1} & -1.57 \times 10'' & -2.64 \times 10' & 1.13 \times 10^{3} & -1.13 \times 10^{3} \\ -1.52 \times 10' & 5.27 \times 10'' & -6.51 \times 10'' & 1.14 \times 10^{4} & -3.05 \times 10'' & 3.05 \times 10^{5} \\ -1.57 \times 10^{3} & -6.51 \times 10^{5} & 7.02 \times 10^{8} & 2.05 \times 10^{7} & 4.92 \times 10^{7} & -4.92 \times 10^{7} \\ -2.64 \times 10^{1} & 1.14 \times 10'' & 2.05 \times 10^{7} & 2.62 \times 10^{6} & -9.06 \times 10^{6} & 9.06 \times 10^{6} \\ 1.13 \times 10^{3} & -3.05 \times 10^{5} & 4.92 \times 10^{7} & -9.06 \times 10^{6} & 3.31 \times 10^{8} & 1.49 \times 10^{8} \\ -1.13 \times 10'' & 3.05 \times 10^{5} & -4.92 \times 10^{7} & 9.06 \times 10^{6} & 1.49 \times 10^{8} & 3.31 \times 10^{8} \end{vmatrix} \quad (4.1)$$

$$Q = \begin{bmatrix} 8.79 \times 10^{-1} & -2.36 \times 10^{-1} & 2.77 \times 10^{-6} & 5.29 \times 10^{-6} & -1.82 \times 10^{-6} & 2.23 \times 10^{-6} \\ -2.36 \times 10^{-1} & 1.42 \times 10^{0} & 5.37 \times 10^{-5} & -2.18 \times 10^{-4} & -3.63 \times 10^{-3} & 4.06 \times 10^{-3} \\ 2.77 \times 10^{-6} & 5.37 \times \mathrm{lo}\text{-}'' & 1.17 \times \mathrm{lo}\text{-}'' & 2.30 \times 10^{-5} & -3.78 \times 10^{-6} & 4.85 \times 10^{-6} \\ 5.29 \times 10^{-6} & -2.18 \times 10^{-4} & 2.30 \times 10^{-5} & 5.33 \times 10^{-4} & 4.40 \times 10^{-4} & -5.27 \times 10^{-4} \\ -1.82 \times 10^{-6} & -3.63 \times 10^{-3} & -3.78 \times 10^{-6} & 4.40 \times 10^{-4} & 2.61 \times 10^{8} & 2.61 \times 10^{8} \\ 2.23 \times 10^{-6} & 4.06 \times 10^{-3} & 4.85 \times 10^{-6} & -5.27 \times 10^{-4} & 2.61 \times 10^{8} & 2.61 \times 10^{8} \end{bmatrix} \quad (4.2)$$

the control system design approach. The averaged nonlinear control system, which was constructed on the basis of averaged model, satisfactorily captured behaviors of the switched control system.

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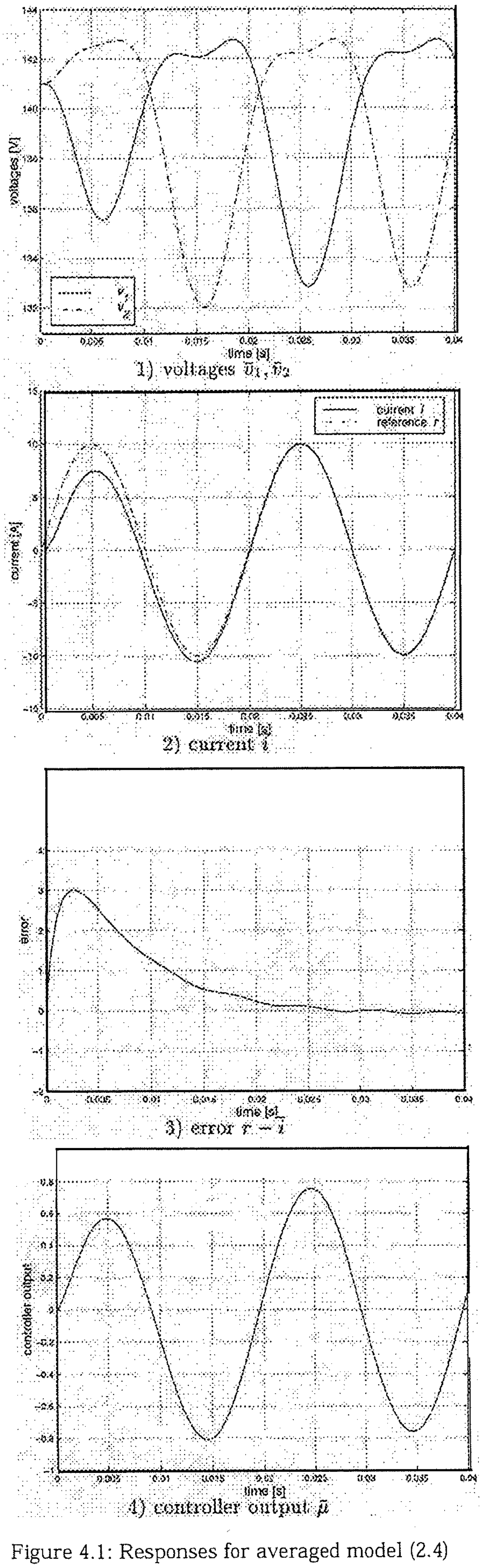

Figure 4.1: Responses for averaged model (2.4)

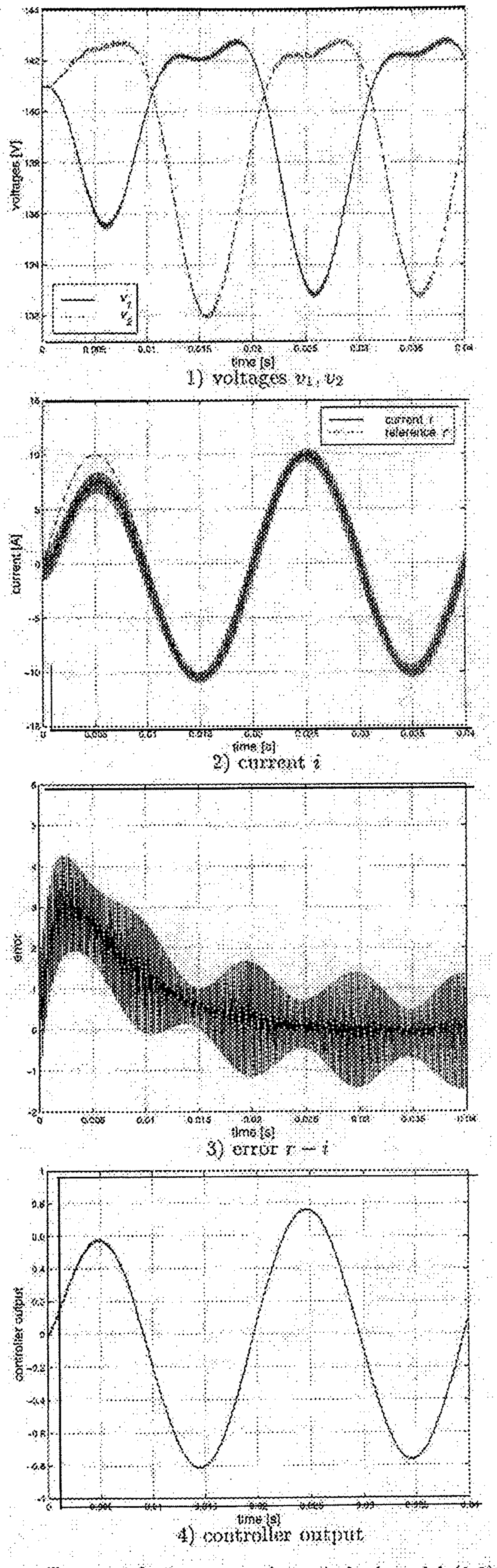

Figure 4.2: Responses for switched model (2.3)

A New LMI Approach to Analysis of Linear Systems Depending on Scheduling Parameter in Polynomial Forms

Takehito Azuma, Ryo Watanabe, Kenko Uchida, Masayuki Fujita

Dedicated to Prof. Dr.-Ing. Dr.-Ing. E. h. Günther Schmidt on the occasion of his 65th birthday

This paper proposes a new LMI approach to analysis of linear systems depending on scheduling parameter in polynomial forms: we first propose a method to reduce the parameter dependent LMI condition, which characterizes internal stability and L^2 gain, to the finite number of LMI conditions by introducing a convex polyhedron which includes a polynomial curve parameterized by scheduling parameter; next we propose a systematic procedure to construct the convex polyhedron. Our approach enable us to analyze L^2 gain of linear systems with scheduling parameter in polynomial forms through computation of the finite number of LMIs. To show efficacy of our approach, we finally make a numerical experiment of L^2 gain analysis for a gasturbine engine model which is described as a linear system with a scheduling parameter in polynomial form of two degree.

1 Introduction

Recently, many results on analysis and design of scheduled control have been presented for linear systems with scheduling parameters [1; 3; 5; 6; 7; 10; 11; 12; 13]. In these results, analysis and design are characterized by solutions to algebraic/differential Riccati inequalities which depends on the scheduling parameter [1; 3; 5; 10; 11; 12; 13]. Using Schur complement [4], the parameter dependent algebraic/differential Riccati inequality can be described as a parameter dependent Linear Matrix Inequality (LMI). Except for the particular case (e.g. dependence on the scheduling parameter is affine [3] or quadratic [5; 13]), the infinite number of computation is required to solve this parameter dependent LMI directly.

An adhoc method (gridding method) to reduce this parameter dependent LMI to the finite number of LMIs is discussed in [12]. Although this method can be applied to the general parameter dependent LMI, it provides, in general, an approximate solution which satisfies the LMI only at the finite number of gridding points in the parameter space, and requires the very large number of computation to guarantee the feasibility of the approximate solution. In the case of linear systems depending affinely on the scheduling parameter, the specific convexifying methods are proposed which reduce the parameter dependent LMI to the fixed finite number of LMIs [3; 5; 13]. The work [3] develops the method to obtain the parameter independent solution to the affinely parameter dependent LMI. In [5], the authors propose the method to solve the quadratically parameter dependent LMI in the affinely parameter dependent set of solutions. In [13], a convex covering technique is applied to quadratically parameter dependent LMI with the affinely parameter dependent solutions.(The same case is discussed in [5].) However, it seems difficult to extend these method to more general cases such as general polynomial dependence cases discussed in this paper.

In this paper, we focus on the parameter dependent LMI, which characterize internal stability and L^2 gain for linear systems depending on the scheduling parameter in polynomial forms, and propose a method to reduce this condition to the finite number of LMIs by introducing a convex polyhedron. Generally volume of the convex polyhedron must be small to get less conservative result. We also propose a systematic procedure to construct such small convex polyhedron.Thus by taking our approach, the polynomial parameter dependent LMI is solved through the finite number of computations, and the L^2 gain analysis can be performed with CAD like `LMILAB` [8]. It is noted however that we must solve the larger number of LMIs to obtain less conservative solutions, because in the procedure of constructing convex polyhedron the smaller volume leads to solving the larger number of LMIs.

This paper is organized as follows. In section 2, we present a sufficient condition on internal stability and L^2 gain for linear systems with scheduling parameter. We formulate our problem in section 3, and show the main result of this note in section 4. In section 5, we demonstrate the proposed method by a numerical example.

2 Analysis of L^2 Gain

We first summarize results of analysis of the following linear systems with a scalar scheduling parameter.

$$\Sigma: \begin{array}{l} \dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))w(t),\ x(0) = 0, \\ z(t) = C(\theta(t))x(t), \end{array}$$

where $x(t) \in R^n$ is the state, $z(t) \in R^l$ is the observed output, $w(t) \in R^m$ is the disturbance, and $\theta(t) \in R$ is the scheduling parameter of Σ which is a continuously differentiable function of t. We assume the following properties on $\theta(t)$,

i) $\theta(t) \in [0,\ 1],\ ^\forall t \in [0, \infty)$,

ii) $|\dot{\theta}(t)| \le \nu_{max},\ \nu_{max} > 0,\ ^\forall t \in [0, \infty)$.

Each of $A(\theta)$, $B(\theta)$ and $C(\theta)$ is the continuous matrix function of θ.

Internally stability and L^2gain of Σ are defined as follows.

Definition 1 The system Σ is said to be internally stable if the trivial solution $x \equiv 0$ of the following ordinary differential equation is exponentially stable.

$$\dot{x}(t) = A(\theta(t))x(t).$$

Definition 2 Let Σ be an internally stable linear system with scheduling parameter. Then the L^2 gain of Σ is defined by

$$G(\Sigma) = \sup_{w \in L^2, w \neq 0} \frac{\|z\|_{L^2}}{\|w\|_{L^2}},$$

where $\|\cdot\|_{L^2}$ denotes L^2 norm.

Then we have the next theorem for L^2 gain analysis of Σ by applying Schur Complement [4] to the result in [10].

Theorem 1 The system Σ is internally stable and $G(\Sigma)$ is less than γ if there exists a matrix function $Q(\theta)$ defined on [0, 1] such that

$$\begin{bmatrix} -Q(\theta) & 0 & 0 & 0 & 0 \\ 0 & \delta\frac{dQ}{d\theta}(\theta) & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} A'(\theta)Q(\theta) + Q(\theta)A(\theta) \\ -\delta\nu_{max}\frac{dQ}{d\theta}(\theta) \end{pmatrix} & C'(\theta) & Q(\theta)B(\theta) \\ 0 & 0 & C(\theta) & -I & 0 \\ 0 & 0 & B'(\theta)Q(\theta) & 0 & -\gamma^2 I \end{bmatrix} < 0, \quad (1)$$

for all $\theta \in [0, 1]$, where $\delta = 1$ or -1.

Remark 1 Using the parameter dependent Lyapunov function $V = x'Q(\theta(t))x$, the following inequalities are derived for the result of the L^2 gain analysis.

$$Q(\theta) > 0, \quad (2)$$

$$\begin{array}{l} A'(\theta)Q(\theta) + Q(\theta)A(\theta) + \dot{\theta}\frac{dQ}{d\theta}(\theta) \\ \quad + C'(\theta)C(\theta) + \gamma^{-2}Q(\theta)B(\theta)B'(\theta)Q(\theta) < 0, \\ ^\forall\theta \in [0,\ 1],\ ^\forall\dot{\theta} \in [-\nu_{max},\ \nu_{max}]. \end{array} \quad (3)$$

The condition (1) is a sufficient condition for (2) and (3). In [12], another sufficient condition which is less conservative is derived, but the dimension of the condition is larger than that of the condition (1).

3 Problem Formulation

We assume that $A(\theta)$, $B(\theta)$ and $C(\theta)$ are described as the polynomial function of θ (See Remark 2),

$$A(\theta) = \sum_{i=0}^{L_a} \theta^i A_i,\ B(\theta) = \sum_{i=0}^{L_b} \theta^i B_i,\ C(\theta) = \sum_{i=0}^{L_c} \theta^i C_i.$$

And we restrict $Q(\theta)$ such that

$$Q(\theta) = \sum_{i=0}^{L_q} \theta^i Q_i,\ Q_i' = Q_i.$$

The parameter dependent LMI condition (1) is described as follows,

$$F(\theta) = F_0(Q_s) + \theta F_1(Q_s) + \cdots + \theta^r F_r(Q_s) < 0, \quad (4)$$

where Q_s denotes $(Q_0,\ Q_1,\ \cdots,\ Q_{L_q})$ and $r = L_q + \max(L_a,\ L_b,\ L_c)$. F_i is a symmetric matrix function and depends affinely on the unknown matrix Q_s.

Our problem is to describe the sufficient condition for existence of solutions to the following polynomial parameter dependent LMI condition,

$$F_0(Q_s) + \theta F_1(Q_s) + \cdots + \theta^r F_r(Q_s) < 0,\ ^\forall\theta \in [0,\ 1],$$

as the finite number of parameter independent LMI conditions.

Remark 2 The assumption on system matrices of the system Σ is only continuity on $\theta \in [0, 1]$, so there exist finite matrices E_a, F_a, E_b, F_b, E_c and F_c such that the following system includes the system Σ for any L_a, L_b and L_c.

$$\begin{aligned} \dot{x}(t) &= \left[\sum_{k=0}^{L_a} \theta^k(t)A_k + E_a\Delta_a(t)F_a\right]x(t) \\ &\quad + \left[\sum_{k=0}^{L_b} \theta^k(t)B_k + E_b\Delta_b(t)F_b\right]w(t), \\ z(t) &= \left[\sum_{k=0}^{L_c} \theta^k(t)C_k + E_c\Delta_c(t)F_c\right]x(t), \end{aligned}$$

where $\Delta_a(t)$, $\Delta_b(t)$, and $\Delta_c(t)$ satisfies

$$\Delta_a'(t)\Delta_a(t) \leq I,$$
$$\Delta_b'(t)\Delta_b(t) \leq I,$$
$$\Delta_c'(t)\Delta_c(t) \leq I.$$

This observation motivates us to focus the case of system matrices of polynomial function, $A(\theta)$, $B(\theta)$, $C(\theta)$.

4 Main Results

We show main results of this paper. First we propose a method to reduce the polynomial parameter dependent LMI condition (4) to the finite number of LMI conditions. In this method, a convex polyhedron plays an important role, which includes a curve characterized by the scheduling parameter. Next we propose a procedure to construct this convex polyhedron.

The proposed method is given as follows.

Theorem 2 Let H be a convex polyhedron which includes a curve T defined as

$$T = \left\{ \begin{bmatrix} \theta & \theta^2 & \cdots & \theta^r \end{bmatrix}' \mid \theta \in [0,\ 1] \right\}.$$

And let vertices of H be $\{p_1, p_2, \cdots, p_q\}$, where $p_i \in R^r$. Assume that there exists Q_s which satisfies the following LMI condition for all p_i $(i = 1, 2, \cdots, q)$,

$$F_0(Q_s) + p_{i1}F_1(Q_s) + \cdots + p_{ir}F_r(Q_s) < 0, \quad (5)$$

where p_{ij} is the jth element of p_i. Then Q_s satisfies (4) for any $\theta \in [0,\ 1]$.

Proof (5) is equivalent to

$$\xi' F_0(Q_s)\xi + \sum_{j=1}^{r} p_{ij}\xi' F_j(Q_s)\xi < 0, \quad (6)$$

for all $\xi \in R^n$, $\xi \neq 0$. We express (6) as

$$\xi' F_0(Q_s)\xi + \langle f(\xi, Q_s), p_i \rangle < 0, \quad (7)$$

where $\langle *, * \rangle$ denotes inner product and

$$f(\xi, Q_s) = \begin{bmatrix} \xi' F_1(Q_s)\xi & \xi' F_2(Q_s) & \cdots & \xi' F_r(Q_s)\xi \end{bmatrix}'.$$

(7) is satisfied for all vertices of H and for all $\xi \in R^n$, so the following condition is obtained

$$0 > \xi' F_0(Q_s)\xi + \sum_{i=1}^{q} \lambda_i \langle f(\xi, Q_s), p_i \rangle$$
$$= \xi' F_0(Q_s)\xi + \langle f(\xi, Q_s), \sum_{i=1}^{q} \lambda_i p_i \rangle, \quad (8)$$

for all $\lambda_i \geq 0$, $\sum_{i=1}^{r} \lambda_i = 1$. This implies that Q_s satisfies the parameter dependent LMI condition (4) for all $\theta \in [0,\ 1]$, because any point on T can be described as linear combination of $\{p_1, p_2, \cdots, p_q\}$.

Remark 3 Theorem 2 shows that Q_s in Theorem 2 satisfies (5) for any point in H. Thus volume of the convex polyhedron H is closely related to the conservativeness of the evaluation. It is necessary to construct small size H for sharp evaluation.

Now we consider the construction of a small size convex polyhedron H and propose the following procedure (See Figure 1):

(Step 1) We divide the domain of θ into $\{\Theta_1, \Theta_2, \cdots, \Theta_d\}$. We call d Division Number.

$$\Theta_i = \left\{ \theta \mid \theta \in \left[\frac{i-1}{d},\ \frac{i}{d} \right] \right\}. \quad (9)$$

We define a curve T_i as

$$T_i = \{[\theta, \theta^2, \cdots, \theta^r]' \mid \theta \in \Theta_i\}.$$

(Step 2) For each T_i, we define a polyhedron H_i whose vertices are $\{p_i^0, p_i^1, \cdots, p_i^r\}$.

$$p_i^0 = \left[\frac{i-1}{d}, \left(\frac{i-1}{d}\right)^2, \cdots, \left(\frac{i-1}{d}\right)^r \right]',$$
$$p_i^j = p_i^{j-1} + \alpha(i,j)\, e_j \ (j = 1, 2, \cdots, r-1),$$
$$p_i^r = p_i^{r-1} + \alpha(i,r) e_r$$
$$= \left[\frac{i}{d}, \left(\frac{i}{d}\right)^2, \cdots, \left(\frac{i}{d}\right)^r \right]', \quad (10)$$

where

$$\alpha(i,j) = \left(\frac{i}{d}\right)^j - \left(\frac{i-1}{d}\right)^j,$$

and $e_1, e_2, \cdots, e_r$ are basis vector on R^r. Note that H_i is a convex polyhedron on R^r because $\{p_i^j - p_i^0\}$ $(j = 1, 2, \cdots, r)$ are linearly independent.

(Step 3) We define the convex hull H as

$$H = Co(H_1 \cup H_2 \cup \cdots \cup H_d), \quad (11)$$

where $Co(S)$ denotes the convex hull of S.

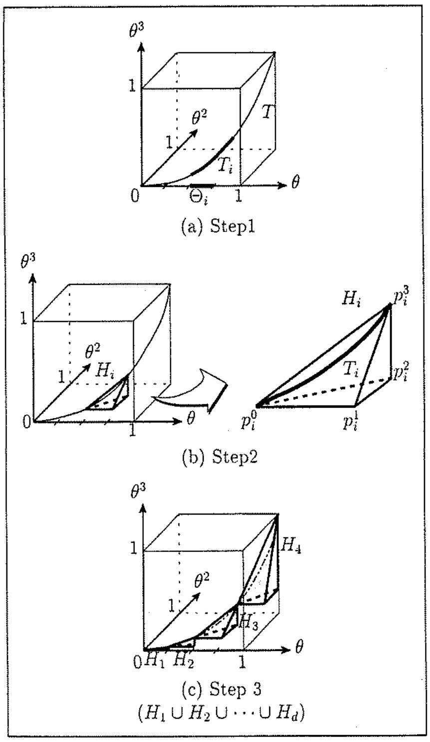

Figure 1: Construction of convex hull.

Theorem 3 The convex hull H given by (11) is a convex polyhedron which includes the curve T defined in *Theorem 2*.

Proof It is obvious that H is a convex polyhedron including H_i ($i = 1, 2, \cdots, d$), because H is the convex hull of $H_1 \cup H_2 \cup \cdots \cup H_d$. So we show that H_i includes T_i. In this proof, we first describe $T(\theta)$ as linear combination of $\{p_i^0, p_i^1, \cdots, p_i^r\}$, where $T(\theta)$ is a point on T_i and is defined as

$$T(\theta) = [\theta\ \theta^2\ \cdots\ \theta^r]',\ \theta \in \Theta_i.$$

Next we show that each coefficient of its linear combination is not less than 0 and the sum of all coefficients is equal to 1.

$T(\theta)$ can be expressed as follows,

$$\begin{aligned} T(\theta) &= \theta e_1 + \theta^2 e_2 + \cdots + \theta^r e_r \\ &= p_i^0 - p_i^0 + \theta e_1 + \theta^2 e_2 + \cdots + \theta^r e_r \\ &= p_i^0 + \left\{\theta - \frac{i-1}{d}\right\} e_1 + \left\{\theta^2 - \left(\frac{i-1}{d}\right)^2\right\} e_2 \\ &\quad + \cdots + \left\{\theta^r - \left(\frac{i-1}{d}\right)^r\right\} e_r. \end{aligned}$$

By using (10), $T(\theta)$ is described as linear combination of p_i^j ($j = 0, 1, \cdots, r$).

$$\begin{aligned} T(\theta) &= p_i^0 + \sum_{j=1}^{r} \left\{\theta^j - \left(\frac{i-1}{d}\right)^j\right\} \alpha(i,j)^{-1} \{p_i^j - p_i^{j-1}\} \\ &= \left[1 - \left\{\theta - \frac{i-1}{d}\right\} \alpha(i,1)^{-1}\right] p_i^0 \\ &\quad + \sum_{j=1}^{r-1} \left[\left\{\theta^j - \left(\frac{i-1}{d}\right)^j\right\} \alpha(i,j)^{-1} \right. \\ &\quad \left. - \left\{\theta^{j+1} - \left(\frac{i-1}{d}\right)^{j+1}\right\} \alpha(i,j+1)^{-1}\right] p_i^j \\ &\quad + \left\{\theta^r - \left(\frac{i-1}{d}\right)^r\right\} \alpha(i,r)^{-1} p_i^r. \end{aligned} \tag{12}$$

Let be

$$T(\theta) = \sum_{j=0}^{r} \lambda_j(\theta) p_i^j,$$

$$\lambda_0(\theta) = 1 - \left\{\theta - \frac{i-1}{d}\right\} \alpha(i,1)^{-1}$$

$$\begin{aligned} \lambda_j(\theta) &= \left\{\theta^j - \left(\frac{i-1}{d}\right)^j\right\} \alpha(i,j)^{-1} \\ &\quad - \left\{\theta^{j+1} - \left(\frac{i-1}{d}\right)^{j+1}\right\} \alpha(i,j+1)^{-1} \\ &\quad (j = 1, 2, \cdots, r-1) \end{aligned}$$

$$\lambda_r(\theta) = \left\{\theta^r - \left(\frac{i-1}{d}\right)^r\right\} \alpha(i,r)^{-1}$$

It is obvious that the sum of $\lambda_j(\theta)$ ($j = 0, 1, \cdots, r$) is equal to 1. It is easy to show that $\lambda_j(\theta)$ ($j = 0, \cdots, r$) is not less than 0 for all $\theta \in \Theta_i$.

Remark 4 As the division number increases, conservativeness improves but the number of LMIs also increases. (If the division number is d, the number of LMIs is $d(r+1)$ where r is the degree of polynomial.) Using our approach, we must consider such trade-off between improvement of conservativeness and increase of the number of LMIs.

5 Numerical Example

Consider the following linear system with scheduling parameter,

$$\begin{aligned} \dot{x}(t) &= \left[A_0 + \theta(t)A_1 + \theta^2(t)A_2\right]x(t) \\ &\quad + \left[B_0 + \theta(t)B_1 + \theta^2(t)B_2\right]w(t), \end{aligned} \tag{13}$$

$$z(t) = Cx(t), (14)$$

where $[A_0 \mid A_1 \mid A_2]$, $[B_0 \mid B_1 \mid B_2]$, C are given as follows,

$$\left[A_0 \mid A_1 \mid A_2\right] = \left[\begin{array}{ccc} -4.3650 & -0.67230 & -0.33630 \\ 7.0880 & -6.5570 & -4.6010 \\ -2.4100 & 7.5840 & -14.310 \\ \hline -0.56081 & 0.85534 & 0.58923 \\ 0.66981 & -1.3750 & -0.99093 \\ 3.1917 & 1.7971 & -2.5887 \\ \hline 0.66981 & -1.3750 & -0.99093 \\ -2.8963 & -1.5292 & 10.516 \\ -3.5777 & 2.8389 & 1.9087 \end{array}\right],$$

$$\left[B_0 \mid B_1 \mid B_2\right] = \left[\begin{array}{cc} 2.3740 & 0.74850 \\ 1.3660 & 3.4440 \\ 0.94610 & -9.6190 \\ \hline -0.16023 & -0.35209 \\ 0.11622 & -2.4839 \\ -0.11058 & -4.6057 \\ \hline 0.15623 & 0.13063 \\ -0.49582 & 4.0379 \\ -0.030616 & 0.89473 \end{array}\right],$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\theta(t) \in [0,\ 1],\ {}^\forall t \in [0,\ \infty).$$

This is the model of the gasturbine engine [9] described as a linear system with a scalar scheduling parameter, where x_1, x_2 and x_3 are the compressor speed, the fan speed and the outlet pressure respectively, and θ is the normalized compressor speed. The scheduled system matrices, $A(\theta) = A_0 + \theta A_1 + \theta^2 A_2$, $B(\theta) = B_0 + \theta B_1 + \theta^2 B_2$, are given as interpolation of the three system matrices, which are identified by using maximum likelihood method at three equilibrium points of the nonlinear simulation model, as shown in Figure 2. In this figure, $*$ denotes the equilibrium point.

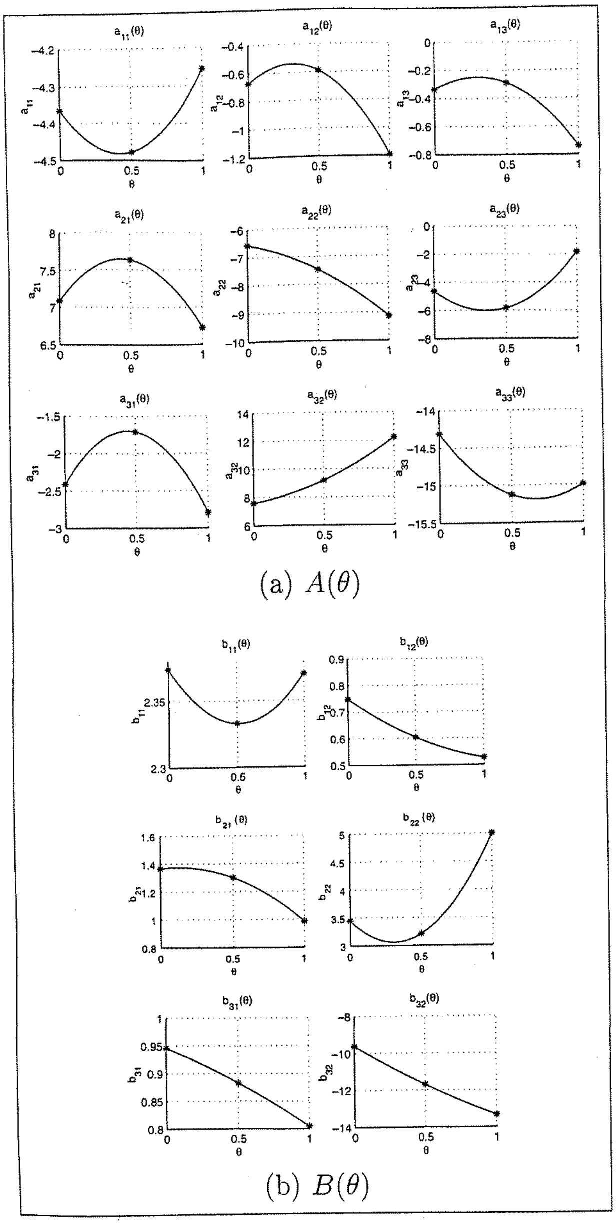

Figure 2: System parameters.

We evaluate L^2gain of the above system for the next three cases,

Case 1: $Q(\theta) = Q_0$,

Case 2: $Q(\theta) = Q_0 + \theta Q_1 + \theta^2 Q_2$ and $v_{max} = 10.0$,

Case 3: $Q(\theta) = Q_0 + \theta\ Q_1 + \theta^2\ Q_2 + \theta^3\ Q_3$ and $v_{max} = 10.0$.

Note that, in these cases, we cannot evaluate L^2 gain by using directly the technique in [3] or [5], because the system parameters $A(\theta)$ and $B(\theta)$ are quadratic function of θ.

The result is shown in Figure 3. In this figure, the vertical axis denotes L^2gain evaluated by the proposed technique and the horizontal axis denotes the division number. $+$, $\times$ and $*$ are L^2 gain evaluated for Case 1, Case 2 and Case 3 respectively. Results for Case 1 and Case 2 are almost identical. This result shows that conservativeness of the evaluation is improved as the division number increases; conservativeness is greatly improved while the division number is small; conservativeness is improved very little at the division number larger than some number (e.g. 20 in this example). Thus it seems that we can obtain good evaluation of L^2 gain even if the division number is not enough large.

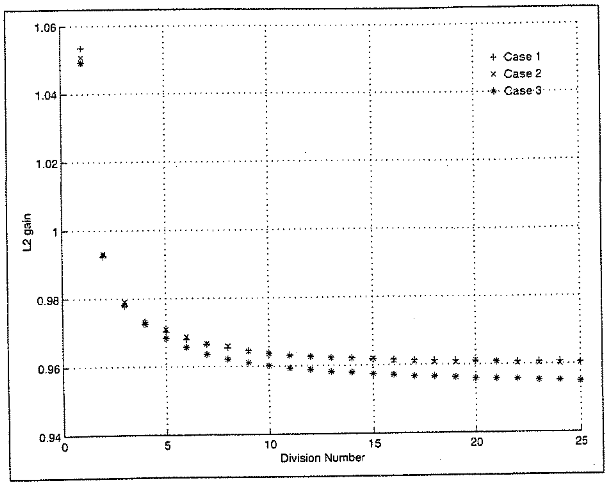

Figure 3: Evaluation of L^2 gain.

If the interdependence of the parameters, $\theta_1(t) = \theta(t)$, $\theta_2(t) = \theta^2(t)$, is neglected, the system (13) is reduced to the linear systems with two, independent, scheduling parameters,

$$\begin{aligned} \dot{x}(t) = {} & [A_0 + \theta_1(t)A_1 + \theta_2(t)A_2]x(t) \\ & + [B_0 + \theta_1(t)B_1 + \theta_2(t)B_2]w(t), \end{aligned} \tag{15}$$

where $\theta_1(t)$ and $\theta_2(t)$ satisfy

$$\theta_1(t) \in [0,\ 1],\ \theta_2(t) \in [0,\ 1],\ {}^\forall t \in [0,\ \infty),$$
$$\dot{\theta}_1(t) \in [-10,\ 10],\ \dot{\theta}_2(t) \in [-20,\ 20],\ {}^\forall t \in [0,\ \infty).$$

Since the system (15) depends affinely on scheduling parameters, the technique of [5] can be applied. That is, taking the solution to the vector form of (3) in the affine form,

$$Q(\theta_1, \theta_2) = Q_0 + \theta_1 Q_1 + \theta_2 Q_2.$$

and using the technique of [5], we evaluate the L^2 gain as

$$\gamma_1 = 1.5764.$$

The corresponding result of the L^2 gain analysis using our technique is that of Case 2, and the value is evaluated from Figure 3 as

$$\gamma_2 = 0.960.$$

Thus we can see less conservative evaluation of the L^2 gain by considering interdependence of scheduling parameters.

6 Conclusions

We proposed a new LMI approach to analysis of linear systems depending on scheduling parameter in polynomial forms. In the numerical case study, we evaluated actual L^2 gain for a given linear system with scheduling parameter and verified its efficacy. Though we considered the polynomial case in this paper, the proposed approach can be extended to more general cases [2].

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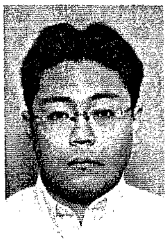

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H_∞ Output Feedback Control Problems for Bilinear Systems

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Abstract: In this paper, we consider H_∞ output feedback control problems for bilinear systems, and present a design example of artificial rubber muscle actuator control system. First we derive two types of H_∞ output feedback controller via differential game approach. The controllers are characterized in terms of the solutions satisfying two Riccati inequalities depending on the state of generalized plant or controller. Second, we propose algorithms to solve the Riccati inequalities via specifying a domain of the state and solving constant coefficient Riccati inequalities. The proposed algorithms include an evaluation method for the domain of internal stability. Finally, we demonstrate efficiency of the proposed algorithm through a numerical example.

Key Words: bilinear system, H_∞ control, Riccati inequality

1 Introduction

Bilinear systems comprise perhaps the simplest class of nonlinear systems. However, the linearization of bilinear systems easily lose the essential nature of the problem for the systems. Moreover, bilinear systems are nonlinear systems that have a lot of practical applications in various fields. Many researchers have studied various aspects of bilinear systems for the past thirty years [11]. However, there is few research that considers exogenous inputs, e.g. disturbance, to bilinear systems. It is important for practical applications to consider the exogenous inputs for guaranteeing a good performance.

H_∞ theory is a control theory that considers explicitly the exogenous inputs for guaranteeing a good performance [3]. Since the time-domain methodology has been developed for the linear H_∞ control problem, H_∞ theory has been generalized to nonlinear systems [7, 6]. In particular, the differential game theory [1] can generalize the linear H_∞ theory in the time domain to the nonlinear H_∞ theory. For the linear H_∞ control problem, we obtain the solution by solving the Riccati inequality. On the other hand, for the nonlinear H_∞ control problem, we obtain the solution by solving the partial differential inequality called "Hamilton-Jacobi-Isaacs (HJI) inequality". But until now, it seems to us that there does not exist a true effective method to solve the HJI inequality. It is just the same with H_∞ control problem for bilinear systems that are a special class of nonlinear systems. So far, there exists the formal tensor series solution of the HJI inequality for bilinear H_∞ state feedback control problem [2]. The work [13] solves finite time horizon H_∞ output feedback control problems for bilinear systems via information state approach.

In this paper, we consider (infinite time horizon) H_∞ output feedback control problems for bilinear systems. Generalizing the approach of the work [10] that discusses the linear H_∞ output feedback control problems, we derive two types of H_∞ output feedback controller. To construct the controllers needs solutions satisfying two Riccati inequalities that depend on the state of a generalized plant or a controller. But it is difficult to obtain the solutions satisfying the inequalities. We propose algorithms to obtain the solutions satisfying the Riccati inequalities via specifying a domain of the state and solving constant coefficient Riccati inequalities. The proposed algorithms include an evaluation method for the domain of internal stability. Finally, we demonstrate efficiency of the proposed algorithm through a design example of artificial rubber muscle actuator control system.

The paper is organized as follows. Section 2 gives a statement of bilinear H_∞ output feedback control problem. Section 3 gives sufficient conditions for the existence of controllers solving the problem. Section 4 gives algorithms that obtain a solution satisfying the sufficient conditions. Final section demonstrates efficiency of the proposed algorithm through a numerical example.

Notations: For a vector x, $\|x\|$ denotes the Euclidean norm. For a matrix X, $\|X\|$ denotes the norm induced by the Euclidean norm (largest singular value). $\|\cdot\|_2$ denotes the norm of the space of square integrable signals, and is defined as

$$\|x\|_2 := \left(\int_0^\infty \|x(t)\|^2 dt\right)^{\frac{1}{2}}, x \in L_2.$$

$L_{2\rho}$ denotes the set of bounded functions with $\|x(t)\| < \rho$ for all $t \in [0,\infty)$. For a real matrix P, $P > 0 (P \geq 0)$ means P is symmetric and positive(positive-semi) definite. I denotes the identity matrix of appropriate dimensions.

2 Bilinear H_∞ Control Problem

We consider a nonlinear system (Σ) :

$$\dot{x} = Ax + B_1 w + B(x)u, \qquad (2.1)$$

$$z = C_1 x \qquad + D_{12} u, \qquad (2.2)$$

$$y = C_2 x + D_{21} w, \qquad (2.3)$$

where,

$$B(x) = B_2 + \{xN\}, \qquad (2.4)$$

$$\{xN\} := \sum_{i=1}^{n} x_i N_i, \quad N_i \in \mathbb{R}^{n \times r}, i = 1, \ldots, n. \qquad (2.5)$$

x_i stands for the i-th element of x.

$x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^r$ is the control input, $y(t) \in \mathbb{R}^m$ is the measured output, $z(t) \in \mathbb{R}^q$ is the controlled output and $w(t) \in \mathbb{R}^p$ is the exogenous input. A, B_1, B_2, C_1, C_2, D_{12}, D_{21} are coefficient matrices of appropriate dimensions, and satisfy the following "Orthogonality Condition":

$$\text{(AO)} \quad D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}, \qquad (2.6)$$

$$D_{21} \begin{bmatrix} B_1^T & D_{21}^T \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}. \qquad (2.7)$$

For given matrices $N_i, i = 1, \ldots, n$, there exist matrices $M_j, j = 1, \ldots, r$ such that

$$\{xN\}u = \{uM\}x, \qquad (2.8)$$

for all x, u, where $\{uM\}$ is defined as

$$\{uM\} := \sum_{j=1}^{r} u_j M_j, \quad M_j \in \mathbb{R}^{n \times n}, j = 1, \ldots, r. \qquad (2.9)$$

u_j stands for the j-th element of u.

For the system (Σ), we consider an output feedback controller (Γ) of the form

$$\dot{\xi} = \eta_1(\xi) + \eta_2(\xi)u + \eta_3(\xi)y, \tag{2.10}$$

$$u = \theta_1(\xi), \tag{2.11}$$

where $\eta_1(\xi)$, $\eta_2(\xi)$, $\eta_3(\xi)$, $\theta_1(\xi)$ are sufficiently smooth functions with $\eta_1(0) = 0$, $\theta_1(0) = 0$.

For the closed-loop system (Σ, Γ), an internal stability is defined as follows.

Definition (Internal stability)

Let $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a domain that contains the equilibrium point $(0,0)$. Consider the closed-loop system (Σ, Γ) with $w = 0$. If the equilibrium point $(0,0)$ is (locally) asymptotically stable and the solution $(x(t), \xi(t))$ starting in the domain Ω approaches to the point $(0,0)$ as $t \to \infty$, we say the closed-loop system (Σ, Γ) is internally stable in the domain Ω.

In this paper, we consider the following problem.

Bilinear H_∞ Control Problem [P]

Consider a system (Σ). Given γ (> 0), find an output feedback controller (Γ) satisfying the following conditions (P_1) and (P_2), and characterize the domain Ω satisfying the condition (P_1):

(P_1) The closed-loop system (Σ, Γ) is internally stable in a domain $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ that contains the equilibrium point $(0,0)$;

(P_2) Whenever $(x(0), \xi(0)) = (0,0)$, there exists some $\rho(> 0)$ and $\|z\|_2 \leq \gamma \|w\|_2$ for all $w \in L_{2\rho}$.

3 Characterizations of Controllers

Now assume that the structures $\eta_1(\xi)$, $\eta_2(\xi)$, $\eta_3(\xi)$ of the controller (Γ) are already designed. Then, the differential game approach [7, 1] leads us to that the problem [P] is solved by obtaining a solution $V(x_a)$ satisfying the Hamilton-Jacobi-Isaacs (HJI) inequality,

$$\min_{u=\theta_1(\xi)} \max_{w=w(x,\xi,u)} \left[\frac{\partial V}{\partial x_a}(x_a)\{f_1(x_a) + f_2(x_a)w + f_3(x_a)u\} + \|z\|^2 - \gamma^2 \|w\|^2 \right] \leq 0, \tag{3.1}$$

for the augmented system that consists of the system (Σ) and the controller (Γ), given as

$$\dot{x}_a = f_1(x_a) + f_2(x_a)w + f_3(x_a)u, \tag{3.2}$$

$$z = C_1 x + D_{12} u, \tag{3.3}$$

where

$$x_a = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad f_1(x_a) = \begin{bmatrix} Ax \\ \eta_1(\xi) + \eta_3(\xi) C_2 x \end{bmatrix},$$

$$f_2(x_a) = \begin{bmatrix} B_1 \\ \eta_3(\xi) D_{21} \end{bmatrix}, \quad f_3(x_a) = \begin{bmatrix} B(x) \\ \eta_2(\xi) \end{bmatrix}.$$

It is not easy to obtain the solution $V(x_a)$ satisfying the HJI inequality (3.1). In this paper, by restricting a structure of the solution $V(x_a)$ to two particular types, we obtain two solutions presented in Theorems 3.1 and 3.2. The solution $V(x_a)$ for Theorem 3.1 has the structure

$$V(x_a) := \xi^T S \xi + \gamma^2 (x - \xi)^T Y^{-1} (x - \xi), \tag{3.4}$$

and the solution for Theorem 3.2 the structure

$$V(x_a) := x^T X x + \gamma^2 (x-\xi)^T T^{-1} (x-\xi). \tag{3.5}$$

These structures of the solutions are the same as those of the linear case [10, 3]. We might think that the structures (3.4),(3.5) are introduced as the second order approximation of $V(x_a)$ with respect to the linearization model in neighborhood of the equilibrium point $(x,\xi) = (0,0)$ of the bilinear system (2.1) and controller (2.10). This is not true, because the linearization model of bilinear system (2.1) is generally useless because of lacking the control input. In this paper, we regard the state x in the bilinear term as an unknown parameter, and the bilinear system (2.1) as a linear system with an unknown parameter. We can prove the following theorems with the same technique "completing the square" as in the linear case [10].

Theorem 3.1 *Consider the system* (Σ) *satisfying the condition (AO). If there exists a domain* $\Phi_1 \subseteq \mathbb{R}^n \times \mathbb{R}^n$ *that contains the origin, and for all* $(x,\xi) \in \Phi_1, (x,\xi) \neq (0,0)$, *there exist* $S > 0, Y > 0$ *such that*
(C11) $\mathrm{Eqn}_1(S, B(\xi)) :=$

$$S(A + \gamma^{-2} Y C_1^T C_1) + (A + \gamma^{-2} Y C_1^T C_1)^T S - S(B(\xi)B^T(\xi) - \gamma^{-2} Y C_2^T C_2 Y)S + C_1^T C_1 < 0; \tag{3.6}$$

(C12) $\mathrm{Eqn}_2(Y) :=$

$$YA^T + AY - Y(C_2^T C_2 - \gamma^{-2} C_1^T C_1)Y + B_1 B_1^T < 0; \tag{3.7}$$

(C13)

$$\xi^T \mathrm{Eqn}_1(S, B(\xi))\xi + \gamma^2 (x-\xi)^T Y^{-1} [\mathrm{Eqn}_2(Y) + Y\{\underline{\nu}(\xi)M\}^T + \{\underline{\nu}(\xi)M\}Y] Y^{-1}(x-\xi) < 0, \tag{3.8}$$

where $\underline{\nu}(\xi) := -B^T(\xi)S\xi$,
then, the controller (Γ) *solving the problem [P] is given as*

$$\dot{\underline{x}} = A\underline{x} - B(\underline{x})B^T(\underline{x})S\underline{x} + \gamma^{-2} Y C_1^T C_1 \underline{x} + Y C_2^T (y - C_2 \underline{x}), \tag{3.9}$$

$$u = -B^T(\underline{x})S\underline{x}. \tag{3.10}$$

Moreover, the closed-loop system (Σ, Γ) *is internally stable in the maximum hyper-ellipsoid*

$$\Omega_1(\sigma_1) := \{\, (x,\xi) \,|\, \xi^T S \xi + \gamma^2 (x-\xi)^T Y^{-1} (x-\xi) \leq \sigma_1 \,\}, \tag{3.11}$$

that is contained in the domain Φ_1.

Theorem 3.2 *Consider the system* (Σ) *satisfying the condition (AO). If there exists a domain* $\Phi_2 \subseteq \mathbb{R}^n \times \mathbb{R}^n$ *that contains the origin, and for all* $(x,\xi) \in \Phi_2, (x,\xi) \neq (0,0)$, *there exist* $X > 0, T > 0$ *such that*
(C21) $\mathrm{Eqn}_3(X, B(x)) :=$

$$XA + A^T X - X(B(x)B^T(x) - \gamma^{-2} B_1 B_1^T)X + C_1^T C_1 < 0; \tag{3.12}$$

(C22) $\mathrm{Eqn}_4(T, B(x)) :=$

$$T(A + \gamma^{-2} B_1 B_1^T X)^T + (A + \gamma^{-2} B_1 B_1^T X)T - T(C_2^T C_2 - \gamma^{-2} X B(x) B^T(x) X)T + B_1 B_1^T < 0; \tag{3.13}$$

(C23)

$$x^T \mathrm{Eqn}_3(X, B(x))x + \gamma^2(x-\xi)^T T^{-1}[\mathrm{Eqn}_4(T, B(x)) + T\{\hat{\nu}(\xi)M\}^T + \{\hat{\nu}(\xi)M\}T]T^{-1}(x-\xi)$$
$$+\|B^T(x)Xx - B^T(\xi)X\xi\|^2 - \|B^T(x)X(x-\xi)\|^2 < 0, \qquad (3.14)$$

where $\hat{\nu}(\xi) := -B^T(\xi)X\xi$,
then, the controller (Γ) *solving the problem [P] is given as*

$$\dot{\hat{x}} = A\hat{x} - B(\hat{x})B^T(\hat{x})X\hat{x} + \gamma^{-2}B_1B_1^TX\hat{x} + TC_2^T(y - C_2\hat{x}), \qquad (3.15)$$
$$u = -B^T(\hat{x})X\hat{x}. \qquad (3.16)$$

Moreover, the closed-loop system (Σ, Γ) *is internally stable in the maximum hyper-ellipsoid*

$$\Omega_2(\sigma_2) := \{\, (x, \xi) \mid x^TXx + \gamma^2(x-\xi)^TT^{-1}(x-\xi) \le \sigma_2 \,\}, \qquad (3.17)$$

that is contained in the domain Φ_2.

Remark 3.1 Theorem 3.2 corresponds to a case which we let $V(x) = x^TXx$ and $Q(x-\xi) = \gamma^2(x-\xi)^TT^{-1}(x-\xi)$ in Theorem 3.1 of the work [7]. In the work [2], however, there is not a result to which Theorem 3.1 of this paper corresponds. This paper gives another type of controller as shown for linear systems in the work [3, 10].

Remark 3.2 We use the condition (C23) (or (C13)) to obtain the largest domain as possible. In a extreme neighborhood of the origin (for example, in [7]), the higher-order terms of the condition (C23)(or (C13)) with respect to the state are contained in a "gap" of the inequality (3.13) (or (3.7)) and the condition (C23)(or (C13)) becomes useless. The paper [7] does not discuss the evaluation of a domain in which a system is internally stable.

Remark 3.3 In Theorem 3.1, first we find a solution satisfying the conditions (C11) and (C12), and construct the controller (Γ). Next, we use the condition (C13) to evaluate the domain in which the closed-loop system (Σ, Γ) is internally stable. In the same way, in Theorem 3.2, first we find a solution satisfying the conditions (C21) and (C22), and construct the controller (Γ). Next, we use the condition (C23) to evaluate the domain in which the closed-loop system (Σ, Γ) is internally stable. The conditions (C11)(C12) and (C21)(C22) correspond to two Riccati inequalities in linear H_∞ control problems [3, 10], respectively. We see those in the following algorithms. In Remark 4.3, we give more constructive characterization of domains Ω_1 and Ω_2.

4 Controller Synthesis Algorithms

In Section 3, we showed two theorems which present the sufficient conditions to obtain the output feedback controller (Γ) for the problem [P]. To construct the controller (Γ), from the sufficient conditions, we must solve the Riccati inequalities that depend on the state x (or ξ) of the generalized plant (or the controller). In this section, we propose algorithms which consist of considering the admissible domain of the state x (or ξ) and solving constant coefficient Riccati inequalities. The algorithms give also the domain Ω_1 (or Ω_2) in which the closed-loop system is internally stable. Algorithm 1 and 2 proposed here correspond to Theorem 3.1 and Theorem 3.2, respectively.

Algorithm 1
Step1. Set a domain $\Phi_\xi^{11} \subseteq \mathbb{R}^n$ that contains the origin. For all $\xi \in \Phi_\xi^{11}$, find a matrix $\underline{B}$ such that

$$\underline{B}\underline{B}^T \leq B(\xi)B^T(\xi), \tag{4.1}$$

and $S > 0, Y > 0, \underline{\Delta}_1 > 0, \underline{\Delta}_2 > 0$ such that

$$\mathrm{Eqn}_1(S, \underline{B}) + \underline{\Delta}_1 = 0, \tag{4.2}$$
$$\mathrm{Eqn}_2(Y) + \underline{\Delta}_2 = 0. \tag{4.3}$$

At this time, the obtained S, Y give a controller (Γ) in the form of (3.9), (3.10).
Step2. Find σ_1 such that

$$\Omega_1 = \{(x,\xi)\,|\xi^T S\xi + \gamma^2(x-\xi)^T Y^{-1}(x-\xi) \leq \sigma_1 \} \subseteq \Phi_1. \tag{4.4}$$

Ω_1 is a domain in which the closed-loop system (Σ, Γ) is internally stable, where Φ_1 is an admissible domain that contains Ω_1, defined as

$$\Phi_1 := (\mathbb{R}^n \times \Phi_\xi^{11}) \cap \Phi^{12}. \tag{4.5}$$

Here, Φ^{12} is defined as

$$\begin{aligned}\Phi^{12} \;:=\; &\{(x,\xi)| - \xi^T[S(B(\xi)B^T(\xi) - \underline{B}\underline{B}^T)S + \underline{\Delta}_1]\xi \\ &+\gamma^2(x-\xi)^T Y^{-1}[Y\{\underline{\nu}(\xi)M\}^T + \{\underline{\nu}(\xi)M\}Y - \underline{\Delta}_2]Y^{-1}(x-\xi) < 0\}.\end{aligned} \tag{4.6}$$

Algorithm 2
Step1. Set a domain $\Phi_x^{21} \subseteq \mathbb{R}^n$ that contains the origin. For all $x \in \Phi_x^{21}$, find matrices $\hat{B}_u, \hat{B}_l$ such that

$$\hat{B}_l\hat{B}_l^T \leq B(x)B^T(x) \leq \hat{B}_u\hat{B}_u^T, \tag{4.7}$$

and $X > 0, T > 0, \hat{\Delta}_1 > 0, \hat{\Delta}_2 > 0$ such that

$$\mathrm{Eqn}_3(X, \hat{B}_l) + \hat{\Delta}_1 = 0, \tag{4.8}$$
$$\mathrm{Eqn}_4(T, \hat{B}_u) + \hat{\Delta}_2 = 0. \tag{4.9}$$

At this time, the obtained X, T give a controller (Γ) in the form of (3.15), (3.16).
Step2. Find σ_2 such that

$$\Omega_2 = \{(x,\xi)\,|x^T Xx + \gamma^2(x-\xi)^T T^{-1}(x-\xi) \leq \sigma_2\} \subseteq \Phi_2. \tag{4.10}$$

Ω_2 is a domain in which the closed-loop system (Σ, Γ) is internally stable, where Φ_2 is an admissible domain that contains Ω_2, defined as

$$\Phi_2 := (\Phi_x^{21} \times \mathbb{R}^n) \cap \Phi^{22}. \tag{4.11}$$

Here, Φ^{22} is defined as

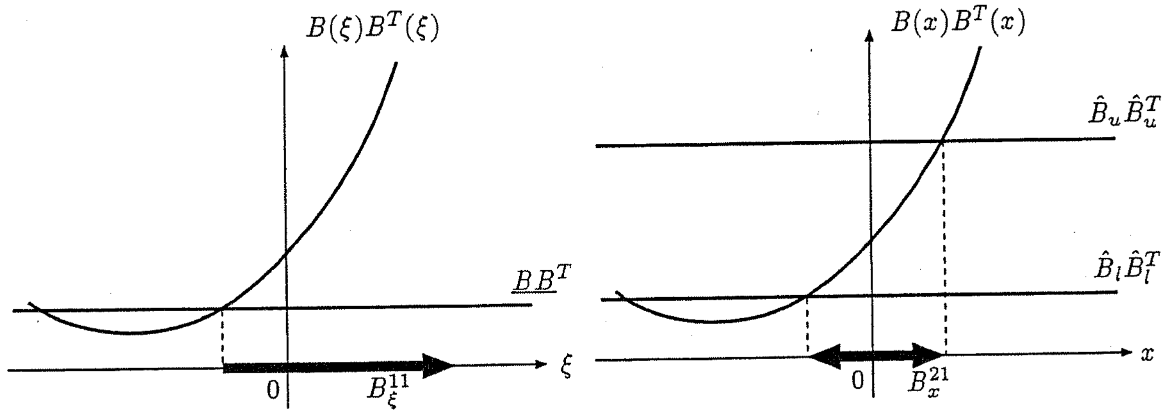
$$\begin{aligned}\Phi^{22} := &\{(x,\xi)| - x^T[X(B(x)B^T(x) - \hat{B}_l\hat{B}_l^T)X + \hat{\Delta}_1]x \\ &-\gamma^2(x-\xi)^T X(\hat{B}_u\hat{B}_u^T - B(x)B^T(x))X(x-\xi) \\ &+\gamma^2(x-\xi)^T T^{-1}[T\{\hat{\nu}(\xi)M\}^T + \{\hat{\nu}(\xi)M\}T - \hat{\Delta}_2]T^{-1}(x-\xi) + \|B^T(x)Xx - B^T(\xi)X\xi\|^2 \\ &-\|B^T(x)X(x-\xi)\|^2 < 0\}.\end{aligned} \tag{4.12}$$

In Algorithm 1 and 2, (4.6) is given by substituting (3.6)(3.7)(4.2)(4.3) into (3.8), and (4.12) is given by substituting (3.12)(3.13)(4.8)(4.9) into (3.14).

Remark 4.1 A basic idea of the algorithms is to solve the constant coefficient Riccati inequalities instead of the Riccati inequalities that depend on the state x (or ξ), by setting the admissible domain for x (or ξ) and evaluating the nonlinear term with respect to x (or ξ) in the admissible domain. It is a key point of the algorithms how large admissible domain we set for x (or ξ):

For example, in Algorithm 1, if there exists $\underline{B}$ such that $\underline{B}^T\underline{B} \leq B(\xi)B^T(\xi)$ for all $\xi \in \mathbb{R}^n$, then we get $\text{Eqn}_1(S, B(\xi)) \leq \text{Eqn}_1(S, \underline{B})$. Therefore, by solving the inequality $\text{Eqn}_1(S, \underline{B}) < 0$, we can solve the inequality $\text{Eqn}_1(S, B(\xi)) < 0$. Note that the smaller $\|\underline{BB}^T\|$ is, the more difficult it is to obtain the solution; conversely, the larger $\|\underline{BB}^T\|$ is, the smaller the admissible domain is (see Figure 4.1).

Remark 4.2 In Algorithm 1, we consider a domain of ξ to solve Riccati inequalities. In Algorithm 2, we consider a domain of x to do so. At this time, we restrict a lower bound of the nonlinear term in Algorithm 1, and both upper and lower bounds in Algorithm 2. Therefore, in those algorithms, the structure (3.4) of solution $V(x_a)$ gives an easier task than another structure (3.5) does.



1) Algorithm 1 2) Algorithm 2

Figure 4.1: Bilinear term vs. Admissible domain

Remark 4.3 In Step 2, to obtain a domain in which the system is internally stable (that is, to get σ_1 in Algorithm 1, or σ_2 in Algorithm 2), we solve a nonlinear optimization problem given as

$$\sigma_1 = \min_{(x,\xi)\in\partial\Phi_1} \left[\xi^T S\xi + \gamma^2(x-\xi)^T Y^{-1}(x-\xi)\right], \tag{4.13}$$

or

$$\sigma_2 = \min_{(x,\xi)\in\partial\Phi_2} \left[x^T Xx + \gamma^2(x-\xi)^T T^{-1}(x-\xi)\right]. \tag{4.14}$$

Section 5.2.2 shall discuss the problem in detail.

5 Numerical Example

In this section, we present a design example of artificial rubber muscle actuator control system. We demonstrate efficiency of the proposed algorithm through the numerical example. We model a single-link manipulator with paralleled artificial rubber muscle actuators as a bilinear system, and construct a nonlinear H_∞ output feedback controller which controls the joint angle.

5.1 Bilinear Model of Rubber Muscle Actuators

We consider a single-link manipulator with paralleled artificial rubber muscle actuators shown in Figure 5.1.

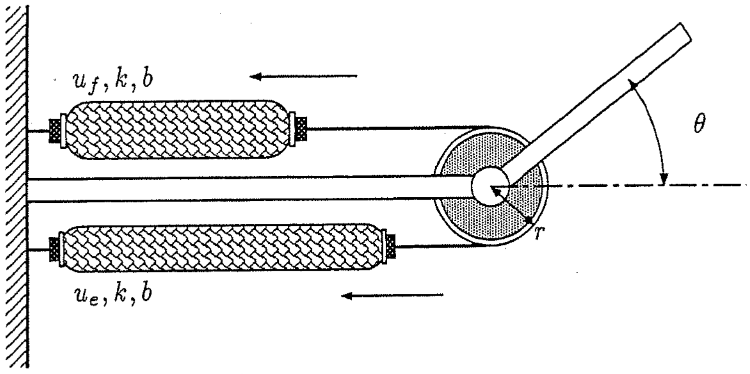


Figure 5.1: Single-link manipulator with paralleled artificial rubber muscle actuators

By transforming difference in shrinking forces of the actuators into a rotation force, we use a pair of actuators like a human muscle. On the basis of the works [8, 9], we model it as a bilinear system of the form

$$\dot{x}_n = A_n x_n + (B_n + x_{n1} N_{n1} + x_{n2} N_{n2}) u, \tag{5.1}$$

$$y_n = C_n x_n \tag{5.2}$$

where $x_n, u, A_n, B_n, C_n, N_{n1}$ and N_{n2} are given as

$$x_n = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \quad u = \begin{bmatrix} u_f \\ u_e \end{bmatrix},$$

$$A_n = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_n = \frac{r}{I}\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, C_n = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$N_{n1} = -\frac{r}{I}\begin{bmatrix} 0 & 0 \\ k & k \end{bmatrix}, \quad N_{n2} = -\frac{r}{I}\begin{bmatrix} 0 & 0 \\ b & b \end{bmatrix}.$$

Here, θ denotes a joint angle, u_f, u_e shrinking-forces, I an inertial moment, r a radius of joint part and k, b constant coefficients. The following discussion uses $I/r = 0.03, k = 0.2, b = 0.05$ [8].

5.2 Control System Design

We design a control system for the bilinear model (5.1)(5.2). First, we construct a generalized plant meeting our specifications. Then, for the generalized plant, we construct an H_∞ output feedback controller, and at the same time, we evaluate a domain of stability.

5.2.1 Generalized Plant Design

We shall design a control system meeting the following specifications:

(S1) A joint angle θ should track output signals of a reference model ($w_1 \rightarrow z_1$), (S2) Sensitivity from process noises to a control input should be reduced ($w_2 \to z_2$).

Thus, we obtain a generalized plant (Figure 5.2) of the form

$$\frac{d}{dt}\begin{bmatrix} x_n \\ x_p \end{bmatrix} = \begin{bmatrix} A_n & 0 \\ 0 & -a_p \end{bmatrix}\begin{bmatrix} x_n \\ x_p \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ b_p & 0 & 0 \end{bmatrix} w + \left(\begin{bmatrix} B_n \\ 0 \end{bmatrix} + x_n \begin{bmatrix} N_n \\ 0 \end{bmatrix} + x_p \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) u, \tag{5.3}$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -C_n & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_n \\ x_p \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{5.4}$$

$$y = \begin{bmatrix} -K_2 C_n & K_2 \end{bmatrix}\begin{bmatrix} x_n \\ x_p \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} w \tag{5.5}$$

where x_p denotes a state of reference model $M(s)$ of the form

$$M(s) = \frac{b_p}{s + a_p} = \frac{1}{s+1}.$$

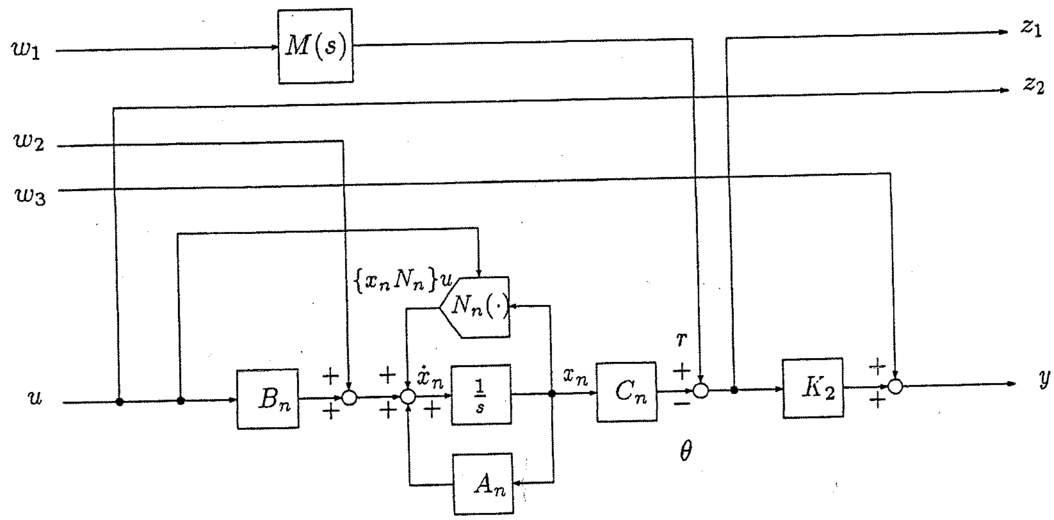


Figure 5.2: Generalized plant

The model $M(s)$ does not reflect on any particular actual problem. We consider an external input w_3 only for the orthogonality condition to hold. Correspondingly, we add a gain block $K_2 = 10$ to the nominal plant in order to attenuate an influence of w_3 on constructing a controller.

5.2.2 Controller Design and Evaluation of Domain of Stability

We construct an output feedback controller by using Algorithm 1 given in Section 4. The following numerical calculations use MATLAB/LMITOOL [4]/Optimization Toolbox [5].

Algorithm 1
Step 1. We shall construct a controller. The bilinear term of (5.3) gives

$$B(\xi) = \begin{bmatrix} B_n \\ 0 \end{bmatrix} + \xi_1 \begin{bmatrix} N_{n1} \\ 0 \end{bmatrix} + \xi_2 \begin{bmatrix} N_{n2} \\ 0 \end{bmatrix} + \xi_3 \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where ξ denotes a state of controller. Figures 5.3 and 5.4 illustrate maximum and minimum eigenvalues of $B(\xi)B^T(\xi)$ as functions of states of controller $(\xi_1, \xi_2) \in ([-100, 100], [-100, 100])$, respectively.

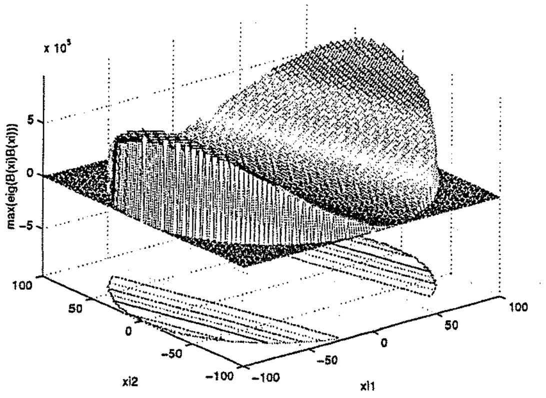


Figure 5.3: Max-eigenvalues of $B(\xi)B^T(\xi)$

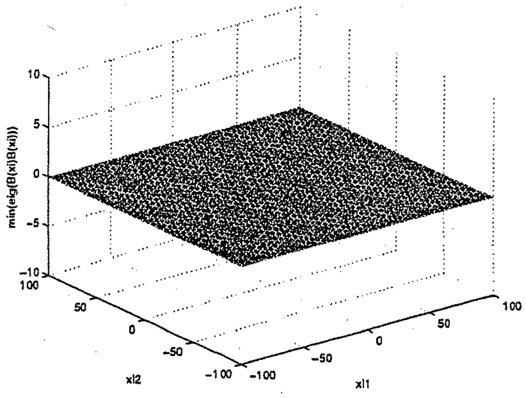


Figure 5.4: Min-eigenvalues of $B(\xi)B^T(\xi)$

From these figures, we easily obtain an inequality $B(0)B^T(0) \leq B(\xi)B^T(\xi)$ for all $\xi \in \mathbb{R}^3$. Thus, the matrix $\underline{BB}^T$, which satisfies

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2222.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underline{BB}^T < B(0)B^T(0),$$

assures that the inequality $\underline{BB}^T < B(\xi)B^T(\xi)$ holds globally, i.e. $\Phi_\xi^{11} = \mathbb{R}^3$. By using this matrix $\underline{BB}^T$, we obtain solutions

$$S = \begin{bmatrix} 1.0828 & 0.11208 & -0.97144 \\ 0.11208 & 0.015301 & -0.097308 \\ -0.97144 & -0.097308 & 0.87464 \end{bmatrix}, \tag{5.6}$$

$$Y = \begin{bmatrix} 1.6254 & 0.46810 & 1.3757 \\ 0.46810 & 0.61939 & 0.29732 \\ 1.3757 & 0.29732 & 1.3460 \end{bmatrix} \tag{5.7}$$

which satisfy Riccati equations (4.2) and (4.3) with $\gamma = 0.8$, and a controller.

Step 2. We shall evaluate a domain of stability $\Omega_1(\sigma_1)$, which is done by finding σ_1. To obtain σ_1 (or a lower bound of σ_1), we solve two constrained nonlinear optimization problems

$$\min_{(x,\xi)\in\partial B_r} V(x,\xi) \leq \sigma_1 = \min_{(x,\xi)\in\partial\Phi_1} V(x,\xi)$$

where $V(x,\xi) = \xi^T S\xi + \gamma^2(x-\xi)^T Y^{-1}(x-\xi)$, and B_r is a maximum hyperball contained in $\Phi_1 = (\mathbb{R}^3 \times \mathbb{R}^3) \cap \Phi^{12}$. Then we obtain an inequality (see Figure 5.5)

$$4.0057 \times 10^{-5} < \sigma_1 = 1.1177.$$

Here, if the value $\sigma_1 = 1.1177$ is a local optimal solution, then the value is only an upper bound of the true solution. In this problem, we obtain fairly small lower bound because there is difference of $1 \sim 10^5$ between magnitudes of eigenvalues of S and Y.

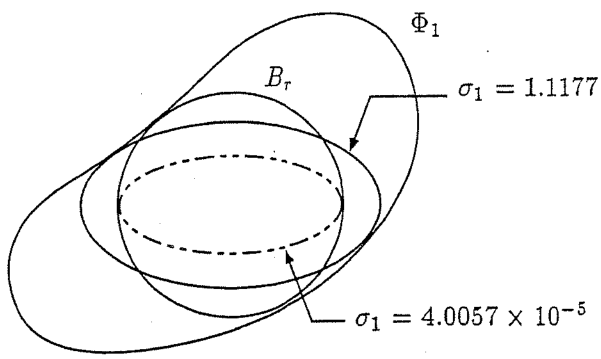


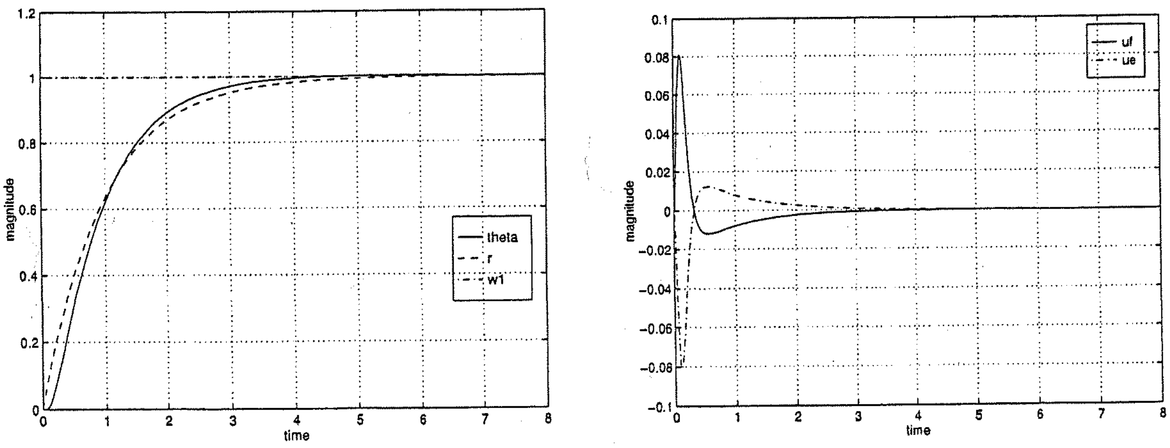
Figure 5.5: Domain of stability

5.3 Computer Simulations

We show efficiency of the obtained controller through computer simulations. Consider four cases: Case 1 (R1+D1), Case 2 (R1+D2), Case 3 (R2+D1) and Case 4 (R2+D2), where reference input w_1 and process noise w_2 are given as follows:

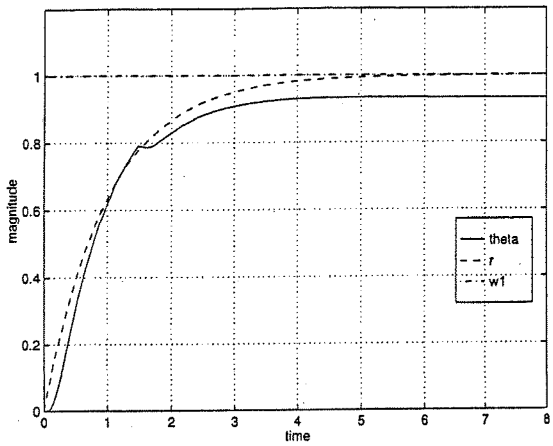
R1: $w_1 = 1 \; (0 \le t \le 8)$, D1: $w_2 = 0$,
R2: $w_1 = \sin(0.5\pi t) \; (0 \le t \le 8)$, D2: $w_2 = -0.3 \; (1.5 \le t \le 8)$.

Results for the four cases are shown in Figures 5.6-5.9. Figures 5.6 and 5.8 show that a joint angle θ tracks reference output signals r, and so mean that the obtained controller satisfies the specification (S1). Figures 5.7 and 5.9 show that the obtaining controller attenuates influence of process noises w_2 and satisfies the specification (S2).

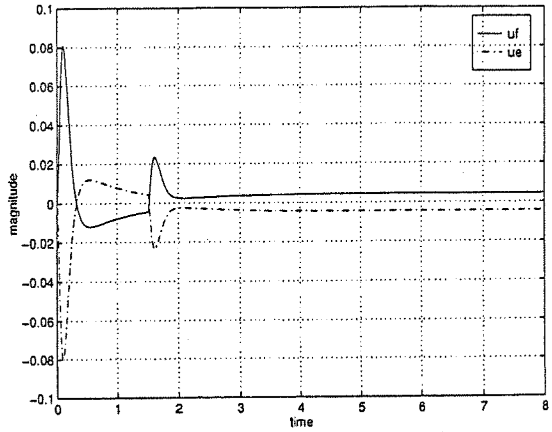


a) Joint angle (θ) b) Control input (u)

Figure 5.6: Case 1 (R1+D1)

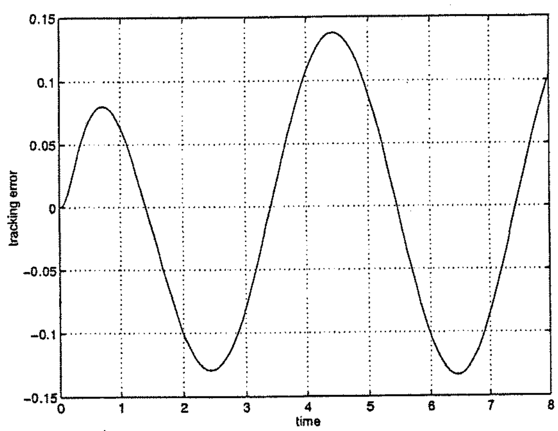


a) Joint angle (θ)

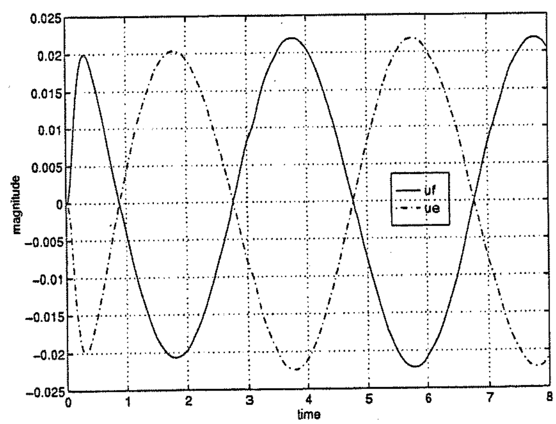


b) Control input (u)

Figure 5.7: Case 2 (R1+D2)

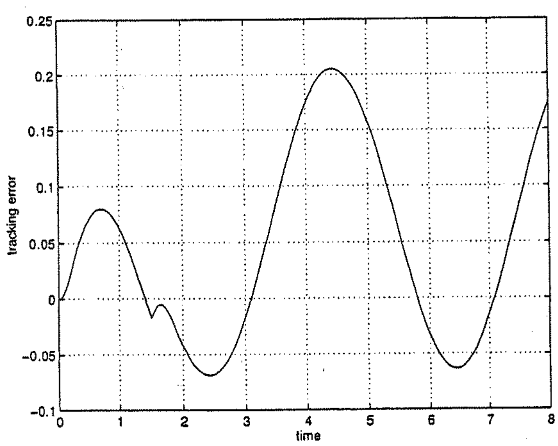


a) Tracking Error ($\theta - r$)

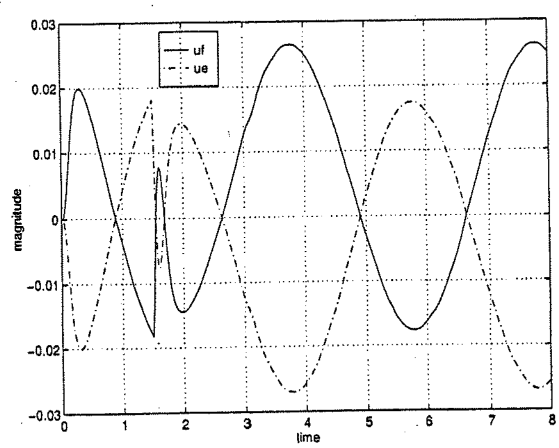


b) Control input (u)

Figure 5.8: Case 3 (R2+D1)



a) Tracking Error ($\theta - r$)



b) Control input (u)

Figure 5.9: Case 4 (R2+D2)

6 Conclusion

We considered H_∞ output feedback control problems for bilinear systems, and presented a design example of artificial rubber muscle actuator control system. We derived two types of H_∞ output feedback controller via differential game approach, and proposed algorithms to construct the controllers based on Riccati inequalities. We also demonstrated efficiency of the proposed algorithm. The algorithm has also been examined through other numerical examples. Details can be found in [12].

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Finite-Dimensional Characterizations of Analysis and Synthesis for Time-Delay Systems

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Abstract. Focusing L^2 gain analysis and H^∞ state feedback control synthesis, we present two approaches to finite-dimensional characterizations of analysis and synthesis for linear time-delay systems. One is based on the spectrum decomposition and the other is based on a convex covering technique.

1 Introduction

The fact that the state space of time-delay systems is infinite-dimensional leads generally to infinite-dimensional characterizations for analysis and control synthesis in time-delay systems. For example, stability of linear time-delay systems is completely analyzed with the infinite-dimensional Lyapunov equations. As for control synteses, it is well known that the optimal LQ control for time-delay systems is given in the memory, i.e. infinite-dimensional, state feedback form whose feedback gains are characterized by infinite-dimensional Riccati equations (see, e.g. [3]); we could say that the memory state feedback form is general and natural for time-delay systems, and can expect that the memory state feedback controllers achieve better performance than memoryless, i.e. finite-dimensional, state feedback controllers. Of course, the infinite-dimensional characterizations contrary give us hard problems in computations and implementations. Our concern is to find a feasible and effective approach to such infinite-dimensional tasks in analysis and control synthesis for linear time-delay systems.

In this paper, we focus L^2 gain analysis and H^∞ state feedback control synthesis, and present two approaches in which the main steps of analysis and synthesis require only finite-dimensional computations: (Approach I) The first approach is based on focusing, by using the spectrum decomposition technique [3], only a finite number of specific modes of the system; controlled outputs are generalized such that they include the history of state in the delay interval, so that the approach makes it possible to control the finite number of modes, and the feedback gains are characterized by finite-dimensional Riccati equations. (Approach II) The second one is based on a reduction technique which was originally developed for solving parameter-dependent linear matrix inequalities (LMIs) for analysis and control synthesis of LPV systems [1]; we first derive infinite-dimensional LMI conditions for analysis and control synthesis in linear time-delay systems, and, applying the reduction technique to the infinite-dimansional LMIs, characterize the analysis and control synthesis by finite-dimensional LMIs.

2 Problem Formulation

Consider the following linear time-delay system defined on the time interval $[0,\infty)$,

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t) + Dw(t) \tag{1}$$

with the initial condition such that $x(\beta) = 0, -h \le \beta \le 0$. Here, $x(t)$ is the n-dimensional state vector, $u(t)$ is the m_u-dimansional control input vector, and $w(t)$ is the m_w-dimensional disturbance vector. The positive number $h > 0$ is the length of time-delay. The system parameters A_0, A_1, B, D are constant matrices. The control vector is given in the following form of state feedback with memory:

$$u(t) = K_0 x(t) + \int_{-h}^{0} K_{01}(\beta) x(t+\beta) d\beta \tag{2}$$

where K_0 is a constant matrix and $K_{01}(\beta)$ is a matrix function whose elments are square integrable functions, i.e. $K_{01} \in L^2([-h,0]; R^{m_u \times n})$. As the controlled output vector $z(t)$ of l-dimension, consider the following two types: The first type is given by

$$z(t) = \begin{bmatrix} F_0 x(t) + \int_{-h}^{0} F_{01}(\beta) x(t+\beta) d\beta \\ u(t) \end{bmatrix} \tag{3}$$

where F_0 is a constant matrix and $F_{01} \in L^2([-h,0]; R^{(l-m_u)\times n})$, and the second type is given by

$$z(t) = Cx(t) \tag{4}$$

where C is a constant matrix.

The analysis problem discussed in this paper is to check whether I) the closed loop system formed by the system (1) and the feedback control (2) is asymptotically stable, and II) the closed loop system has the L^2 gain defined by

$$\int_0^{\infty} z(t)' z(t) dt \le \int_0^{\infty} w(t)' w(t) dt, \qquad \forall w \in L([0,\infty); R^{m_w}), \tag{5}$$

where (') denotes transposition. On the other hand, the synthesis problem of this paper is to find a gain $(K_0, K_{01}(\beta))$ of the controller (2) such that the closed loop system satisfies I) and II), and such a controller is called an H^∞ controller.

In the following discussions, we use a notation,

$$L(\alpha,\beta)=\begin{bmatrix} P_0 & P_1(\beta) \\ P_1'(\alpha) & P_2(\alpha,\beta) \end{bmatrix} > (<) 0, \quad \forall(\alpha,\beta)\in[-h,0]\times[-h,0]$$

which means that P_0 and $P_2(\alpha,\beta)$ are symmetric, that is $P_0' = P_0$ and $P_2'(\alpha,\beta) = P_2(\beta,\alpha)$, and the symmetrized matrix defined by

$$\frac{1}{2}(L(\alpha,\beta)+L'(\alpha,\beta))=\begin{bmatrix} P_0 & \frac{1}{2}(P_1(\alpha)+P_1(\beta)) \\ \frac{1}{2}(P_1'(\alpha)+P_1'(\beta)) & \frac{1}{2}(P_2(\alpha,\beta)+P_2(\beta,\alpha)) \end{bmatrix}$$

is positive definite (negative definite) for each $(\alpha,\beta)\in[-h,0]\times[-h,0]$. The notation, $L(\alpha,\beta)\geq(\leq)0$, is similary defined. Note that, if each elements of a matrix function $L(\alpha,\beta)>0$ is continuous in (α,β), there exists a positive number λ such that $L(\alpha,\beta)\geq\lambda I$ for all $(\alpha,\beta)\in[-h,0]\times[-h,0]$, where I denotes identity matrix.

3 Approach I (Based on Spectrum Decomposition)

3.1 Infinite-Dimensional Characterization of Control Synthesis

We develop Approach I in the control synthesis problem for the controlled output (3). The form of the controlled output (3) seems rather specific compared with the (standard) form of the controlled output (4). From the viewpoint of the state space theory of time-delay systems, however, the type (3) is a general linear functional of the state $(x(t), x(t+\beta), -h\leq\beta\leq 0)$ and the control input $u(t)$. As we will see in the next section, this gereral form makes it possible to obtain a finite-dimansional characterization of the control synthesis besed on spectrum decomposition. We start to present an infinite-dimensional characterization.

Theorem 1 Suppose that there exist a constant matrix M_0 and continuously differentable matrix functions $M_1(\beta)$ and $M_2(\alpha,\beta)$ which satisfy

$$\begin{bmatrix} M_0 & M_1(\beta) \\ M_1'(\alpha) & M_2(\alpha,\beta) \end{bmatrix}\geq 0, \quad \forall(\alpha,\beta)\in[-h,0]\times[-h,0], \tag{6}$$

with $M_0 = M_1(-h)$ and $M_1(\beta) = M_2(-h,\beta), -h\leq\beta\leq 0$, and

$$\begin{bmatrix} \Omega_0 & \Omega_1(\beta) \\ \Omega_1'(\alpha) & \Omega_2(\alpha,\beta) \end{bmatrix} \leq 0, \quad \forall(\alpha,\beta) \in [-h,0]\times[-h,0] \tag{7}$$

where Ω_0, $\Omega_1(\beta)$ and $\Omega_2(\alpha,\beta)$ are defined as

$$\Omega_0 = A_0'M_0 + M_0A_0 + A_1'M_1'(0) + M_1(0)A_1 + F_0'F_0$$
$$- M_0(BB'-DD')M_0,$$
$$\Omega_1(\beta) = -\frac{\partial}{\partial\beta}M_1(\beta) + A_0'M_1(\beta) + A_1'M_2(0,\beta) + F_0'F_1(\beta)$$
$$- M_0(BB'-DD')M_1(\beta),$$
$$\Omega_2(\alpha,\beta) = -(\frac{\partial}{\partial\partial} + \frac{\partial}{\partial\beta})M_2(\alpha,\beta) + F_1'(\alpha)F_1(\beta)$$
$$- M_1'(\alpha)(BB'-DD')M_1(\beta).$$

Then, if $M_0 > 0$ and $\Omega_0 < 0$ in addition, an H^∞ controller is given by

$$u(t) = -B'M_0x(t) - \int_{-h}^{0} B'M_1(\beta)A_1x(t+\beta)d\beta. \tag{8}$$

This theorem can be proved by using a standard argument of completing the square, and the details are omitted. Note that the main step of the synthesis based on this theorem is to solve the infinite-dimensional Riccati inequlaity (7).

3.2 Spectrum Decomposition and Finite-Dimensional Characterization

An appropriate discretization may be a practical approach to solving the infinite-dimensional Riccati inequality (7), but provides generally only approximate solutions which have no theoretical guarantee of sufficiency. In this section, we present another approach which requires only finite-dimensional computations but guarantees necessary properties of solutions.

Here, we summarize necesarry definitions and notations concerning the spectrum decomposition: R^q ($R^{q}{}'$) is the q-dimensional space of colum (row) vectors over the field of real numbers. $H^1([a,b];R^q)$ is the space of absolutely continuous R^q-valued functions on $[a,b]$ with square integrable derivatives. Whenever necessary the space $H^1([a,b];R^q)$ is to be interpreted as its complex extension and correspondingly the transposition ($'$) is to be interpreted as the complex-conjugate transposition. The unique solution of the system (1) with zero-input defines a semigroup on the state space $R^n \times L^2([-h,0];R^n)$. The semigroup defines the infinitesimal generator $\mathbf{A}$ and the formal adjoint $\mathbf{A}'$, given by

$$\mathbf{A}\begin{bmatrix}\phi(0)\\ \phi\end{bmatrix}=\begin{bmatrix}A_0\phi(0)+A_1\phi(-h)\\ \dfrac{d}{d\xi}\phi(\xi)\end{bmatrix} \quad \text{and} \quad \mathbf{A}'\begin{bmatrix}\psi(0)\\ \psi\end{bmatrix}=\begin{bmatrix}\psi(0)A_0+\psi(h)A_1\\ -\dfrac{d}{d\varsigma}\psi(\varsigma)\end{bmatrix}$$

where $\phi \in H^1([-h,0];R^n)$ and $\psi \in H^1([0,h];R^{n'})$. Let λ_i, $i=1,\cdots,p$ be eigenvalues of **A**, i.e. $\det\Delta(\lambda_i)=0$, $i=1,\cdots,p$, where $\Delta(s)=sI-A_0-A_1e^{-sh}$, and define a finite set $\Lambda=\{\lambda_1,\cdots,\lambda_p\}$. If λ_i is a zero of $\det\Delta(s)=0$ of order d_i and a pole of $\Delta^{-1}(s)$ of order m_i, then the generalized eigenspace of λ_i is the d_i-dimensional space $Ker(\lambda_i I-\mathbf{A})^{m_i}$, where I is the identity operator; let $\Phi_i \in H^1([-h,0];R^{n\times d_i})$ be a basis of this eigenspace. λ_i, $i=1,\cdots,p$ are also eigenvalues of $\mathbf{A}'$ and the eigenspace of λ_i is the d_i-dimensional space $Ker(\lambda_i I-\mathbf{A}')^{m_i}$; let $\Psi_i \in H^1([0,h];R^{d_i\times n})$ be a basis of this eigenspace. These bases are chosen such that

$$\Psi_i(0)\Phi_j(0)+\int_{-h}^{0}\Psi_i(\beta+h)A_1\Phi_j(\beta)d\beta=\delta_{ij}I$$

where δ_{ij} is the Kronecker's delta, which is always possible. Note that there exists a $d_i\times d_i$-dimensional matrix A^i such that $\mathbf{A}\Phi_i=\Phi_iA^i$ or $\mathbf{A}'\Psi_i=A^i\Psi_i$. Let $\Psi=(\Psi_1'\cdots\Psi_p')'$ and $d=d_1+\cdots+d_p$. Define a $d\times d$-dimensional matrix A_Λ, a $d\times m_u$-dimensional matrix B_Λ and a $d\times m_w$-dimensional matrix D_Λ such that

$$A_\Lambda=diag(A^1\cdots A^p), \quad B_\Lambda=\Psi(0)B \quad \text{and} \quad D_\Lambda=\Psi(0)D.$$

Now we can state the result.

Theorem 2 Suppose that the coefficient matrix F_0 and the coefficient matrix function $F_{01}(\beta)$ in the controlled output (3) given by

$$\begin{aligned}&F_0=F_\Lambda\Psi(0),\\ &F_{01}(\beta)=F_\Lambda\Psi(\beta+h)A_1\end{aligned} \tag{9}$$

for a constant matrix F_Λ, and that there exists a constant matrix $M>0$ satisfying

$$A_\Lambda'M+MA_\Lambda+F_\Lambda'F_\Lambda-M(B_\Lambda B_\Lambda'-D_\Lambda D_\Lambda')M<0. \tag{10}$$

Then, a solution $(M_0, M_1(\beta), M_2(\alpha,\beta))$ to the inequalities (6) and (7) is given by

$$\begin{aligned}&M_0=\Psi'(0)M\Psi(0),\\ &M_1(\beta)=\Psi'(0)M\Psi(\beta+h),\\ &M_2(\alpha,\beta)=\Psi'(\alpha+h)M\Psi(\beta+h).\end{aligned} \tag{11}$$

Furthermore, if Λ contains all the unstable zeros (i.e. the zeros with non-negative real parts) of the characteristic equation $\det\Delta(s)=0$, the controller (8) given by (11) is an H^∞ controller.

The first part of this theorem can be proved by direct substitution. For the proof of the second part, note that the condition (9) implies that we intend to control only a finite number of specific modes corresponding to the finite set Λ. Therefore, the condition that Λ contains all the unstable characteristic roots of the open loop system is required for the asymptotic stability of the closed loop system. The details are omitted.

Approach I, which is based on Theorem 2, consists of the two steps: The first step is the spectrum decomposition, where a finite number of spectra Λ to be controlled are chosen and matrices $A_\Lambda, B_\Lambda, D_\Lambda$ are calculated; The second step is to find a solution M to the finite-dimensional Riccati inequality (10); Then, an H^∞ controller is synthesized by (8) and (11).

4 Approach II (Based on Convex Covering Technique)

4.1 Infinite-Dimensional Characterization of Analysis and Synthesis

We develop Approach II in the analysis and synthesis problem for the controlled output (4). For the analysis problem, let us describe the closed loop system formed by (1) and (2) as

$$\begin{aligned}\dot{x}(t) &= \tilde{A}_0 x(t) + \tilde{A}_1 x(t-h) + \int_{-h}^{0} \tilde{A}_{01}(\beta) x(t+\beta) d\beta + Dw(t),\\ z(t) &= Cx(t)\end{aligned} \tag{12}$$

where $\tilde{A}_0 = A_0 + BK_0$, $\tilde{A}_1 = A_1$ and $\tilde{A}_{01}(\beta) = BK_{01}(\beta)$. We have an infinite-dimensional charcterization for the L^2 gain analysis of the closed loop system in the following form [2].

Theorem 3 Suppose that there exist constant matrices P, Q and continuously differentable matrix functions $R(\beta)$, $S(\alpha,\beta)$ which satisfy the linear matrix inequalities (LMIs):

$$Q > 0, \tag{13}$$

$$\begin{bmatrix} P & R(\beta) \\ R'(\alpha) & S(\alpha,\beta) \end{bmatrix} > 0, \tag{14}$$

$$\begin{bmatrix} \Pi_{11} & P\tilde{A}_1 - R(-h) & \Pi_{13}(\beta) & PD \\ \tilde{A}_1' P - R'(-h) & -Q & \tilde{A}_1' R(\beta) - S(-h,\beta) & 0 \\ \Pi_{13}'(\alpha) & R'(\alpha)\tilde{A}_1 - S(\alpha,-h) & \Pi_{33}(\alpha,\beta) & R'(\alpha)D \\ D'P & 0 & D'R(\beta) & -I \end{bmatrix} < 0, \tag{15}$$

$$\forall(\alpha,\beta) \in [-h,0]\times[-h,0],$$

where Π_{11}, $\Pi_{13}(\beta)$ and $\Pi_{33}(\alpha,\beta)$ are defined by

$$\Pi_{11} = \tilde{A}_0'P + P\tilde{A}_0 + Q + R'(0) + R(0) + C'C,$$
$$\Pi_{13}(\beta) = -\frac{\partial}{\partial\beta}R(\beta) + P\tilde{A}_{01}(\beta) + \tilde{A}_0'R(\beta) + S(0,\beta),$$
$$\Pi_{33}(\alpha,\beta) = -(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta})S(\alpha,\beta) + R'(\alpha)\tilde{A}_{01}(\beta) + \tilde{A}_{01}'(\alpha)R(\beta).$$

Then, the closed loop system (12) satisfies the conditions I) and II).

This theorem can be proved by using the standard argument of completing the square together with Schur complement. The details are omitted. Now we consider the synthesis of H^∞ controllers and provide an infinite-dimensional characterization [2]. The problem is to find a controller gain $(K_0, K_{01}(\beta))$ based on the analysis result of Theorem 3.

Theorem 4 Suppose that there exist constant matrices W, X, Z_0 and continuously differentable matrix functions $Z_{01}(\beta)$, $Y(\alpha,\beta)$ which satisfy the LMIs:

$$X > 0, \tag{16}$$
$$\begin{bmatrix} W & W \\ W & Y(\alpha,\beta) \end{bmatrix} > 0, \tag{17}$$
$$\begin{bmatrix} \Sigma_{11} & A_1W - W & \Sigma_{13}(\beta) & \Sigma_{14} \\ WA_1'-W & -X & WA_1'-Y(-h,\beta) & 0 \\ \Sigma_{13}'(\alpha) & A_1W - Y(\alpha,-h) & \Sigma_{33}(\alpha,\beta) & \Sigma_{34} \\ \Sigma_{14}' & 0 & \Sigma_{34}' & -I \end{bmatrix} < 0, \quad \forall(\alpha,\beta) \in [-h,0]\times[-h,0], \tag{18}$$

where Σ_{11}, $\Sigma_{13}(\beta)$, Σ_{14}, $\Sigma_{33}(\alpha,\beta)$ and Σ_{34} are defined by

$$\Sigma_{11} = WA_0'+WA_0 + X + 2W + BZ_0 + Z_0'B',$$
$$\Sigma_{13}(\beta) = BZ_{01}(\beta) + WA_0'+Z_0'B'+Y(0,\beta),$$
$$\Sigma_{14} = [WC' \;\; D],$$
$$\Sigma_{33}(\alpha,\beta) = -(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta})Y(\alpha,\beta) + BZ_{01}(\beta) + Z_{01}'(\beta)B',$$
$$\Sigma_{34} = [0 \;\; D].$$

Then, an H^∞ controller of the form (2) is given by

$$K_0 = Z_0W^{-1} \quad \text{and} \quad K_{01}(\beta) = Z_{01}(\beta)W^{-1}. \tag{19}$$

The analysis result (Theorem 3) requires to solve the LMIs (13), (14) and (15), and the synthesis result (Theorem 4) requires to solve the LMIs (16), (17) and (18). Note that these LMIs are infinite-dimensional.

4.2 Finite-Dimensional Characterization by Convex Convering Tecnique

The main difficulty in solving the LMIs in Theorem 3 and Theorem 4 comes from their infinite dimensionality, i.e. the dependence on parameters α and β. In Approach II, we reduce a infinite-dimensional (parameter-dependent) LMIs to a finite number of LMIs by using a convex covering technique [1], and obtain the solution of the infinite-dimensional LMIs by computing the finite number of LMIs.

Here let us focus the infinite-dimensional LMIs in Theorem 4. We start to restrict the solutions $Y(\alpha,\beta)$ and $Z_{01}(\beta)$ to the following forms:

$$\begin{aligned} Y(\alpha,\beta) &= Y_0 + g_1(\alpha,\beta)Y_1 + g_2(\alpha,\beta)Y_2 + \cdots + g_{l_Y}(\alpha,\beta)Y_{l_Y}, \\ Z_{01}(\beta) &= Z_0^{01} + h_1(\beta)Z_1^{01} + h_2(\beta)Z_2^{01} + \cdots + h_{l_Z}(\beta)Z_{l_Z}^{01}, \end{aligned} \tag{20}$$

where $g_i(\alpha,\beta)$ is a continuously differentiable scalar function such that $g_i(\alpha,\beta) = g_i(\beta,\alpha)$ and $h_i(\beta)$ is a continuously differentiable scalar function, and Y_i is the $n \times n$-dimensional unknown matrix such that $Y_i = Y_i'$ and Z_i^{01} is the $m_u \times n$-dimensional unknown matrix. Then, substituting the forms of solutions (20) into the LMIs (16), (17) and (18), we have the following form of parameter dependent LMI:

$$L_0(M) + f_1(\alpha,\beta)L_1(M) + \cdots + f_r(\alpha,\beta)L_r(M) < 0, \tag{21}$$

where $f_i(\alpha,\beta)$ is a continuous scalar function and $L_i(M)$ is an affine matrix function of the unknown matrix $M = \left(Y_0, \cdots, Y_{l_Y}, Z_0^{01}, \cdots, Z_{l_Z}^{01}\right)$. The parameter dependent LMI (21) can be reduced to a finite-dimensional LMI, by the convex covering technique [1], as follows.

Theorem 5 Let $\{p_1, p_2, \cdots, p_q\}$ be vertices of a convex polyhedron which includes the curved surface T defined by

$$T = \{[f_1(\alpha,\beta)\ f_2(\alpha,\beta)\ \cdots\ f_r(\alpha,\beta)]' \mid (\alpha,\beta) \in [-h,0] \times [-h,0]\}.$$

If there exists a matrx M which satisfies the finite-dimensional LMIs:

$$L_0(M) + p_{i1}L_1(M) + \cdots + p_{ir}L_r(M) < 0, \quad i = 1, \cdots, q, \tag{22}$$

where p_{ij} is the j th element of p_i, then the matrix M is a solution to the infinite-dimensional LMI.

Thus we obtain the finite-dimensional characterization (22) for the controller synthesis problem. A general technique to construct a convex polyhedron which includes the curved surface T is proposed in [1]. To make the volume of convex polyhedrons smaller for less conservative solutions, we may derive the interval $[-h,0]$ into sub-intervals and repeat the same argument of convex covering on each sub-interval.

5 Conclusion

Focusing L^2 gain analysis and H^∞ state feedback control synthesis, we presented two approaches to finite-dimensional characterizations of analysis and synthesis for linear time-delay systems. One was based on the spectrum decomposition and the other was based on a convex covering technique.

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Resolution Improvement of the Actuator which Has Giant Magnetrostrictive Material by Gain Scheduled Controller

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Abstract

Hysteresis of GMM (Giant Magnetrostrictive Material) declines the actuator performance, especially for small input signal range. Here, gain scheduled controller is examined and applied to the actuator position control system.

Plant model is obtained by the system identification experiment. Since plant characteristics depends on the magnitude of input, which is defined to be a scheduling parameter, a set of the models is acquired to various magnitude of the M-sequence (Maximum-length linear shift register sequence) signals.

A gain scheduled controller synthesis is presented by solving LMI constraints and applied to experimental actuator.

1 Introduction

The giant magnetrostrictive material(later, be called GMM) has the nature, which distorts itself in the magnetic field. The distortion is quite small, about 1200 ppm in the 500 Oe magnetic fields. However there are some merits, such as
1) It has the bandwidth of tens of KHz.
2) The generated driving force is relatively large.
3) Environment capability is better than that of piezoelectric device.
4) Actuator structure becomes simple.
For these reasons, mentioned above, a variety of study has been carried out. The actuator structure is shown in Fig 1.

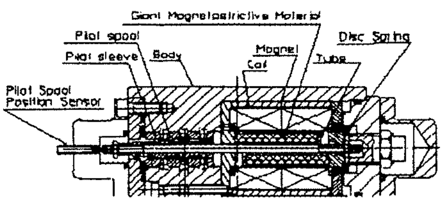


Figure 1: GMM Actuator Structure

This actuator shows hysteresis characteristic in Fig 2. Addition to this, there is an area where the output does not follow to the electric current when it turns around. Because of this feature, the distortion quantity reduces along with the electric current gets smaller. Then, the dead zone appears by decreasing the electric current more and more.

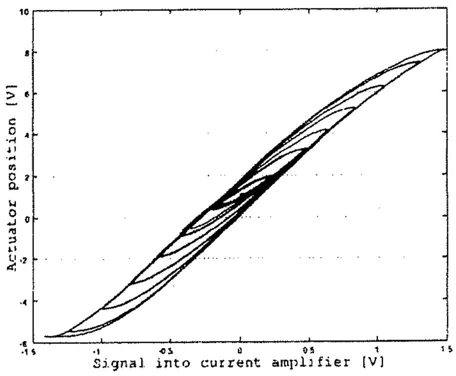


Figure 2: Hysteresis Characteristics

Applying single linear controller that is based on the specific operation point of the plant, we cannot expect control ability successfully in whole operation range. On the other hand, for the dead zone, an inverse compensator is often applied. But, there is a problem to cause unstable oscillation phenomenon when drift occurs by the temperature variation.

Here, we attempt to apply gain scheduled controller with the scheduling parameter for the resolution improvement purpose in a frequency range where we have concern. We compose the generalized LPV (Linear Parameter Varying) system, which contains plant and weighting functions. In addition to value of scheduling parameter, we assume that the maximum value of changing rate of the scheduling parameter is known so that we could take it consideration into controller design. Gain scheduled controller is applied in experimental actuator and verified usefulness.

2 Modeling

Considering that scheduling parameter $\theta(t)$ (later $\theta(t)$ will be presented as θ for simplicity) which is equal to control variable in the closed loop system, is a normalized magnitude of input signal, plant model is obtained as equation (3) by means of system identification experiment as follows.

2.1 System Identification Condition

1) Adopt six resister M-sequence, as identification signal.
2) Amplitude are ± 2,3,5,10,20,30,40,50,60,70,80,90 and 100%.
3) Sampling time is 0.08msec.

2.2 Parameter Calculation

1) Remove bias and trend from experimental results.
2) Used ARMAX model. After some try and error, order of denominator and numerator are five and four with one step time-delay.
3) Continuous time models are acquired (order of denominator and numerator are four and three) by means of curve-fitting of discrete model to have minimum inverse under-shoot model.
4) Finally, add time-delay as one order Pade model to above models.

2.3 Transfer Function Representation

The transfer function of continuous time model is

$$G_p(s) = \frac{b_4(\theta)s^4 + \cdots + b_1(\theta)s + b_0}{s^5 + a_4(\theta)s^4 + \cdots + a_1(\theta)s + a_0} \quad (1)$$

Here, each coefficient is supposed to be represented in the third polynomial approximation of the scheduling parameter.

$$a_i(\theta) = a_{i,0} + a_{i,1}\theta + a_{i,2}\theta^2 + a_{i,3}\theta^3$$
$$b_i(\theta) = b_{i,0} + b_{i,1}\theta + b_{i,2}\theta^2 + b_{i,3}\theta^3$$
$$i = 0, 1, 2, 3, 4 \quad (2)$$

Coefficients of polynomial are shown in Table 1.

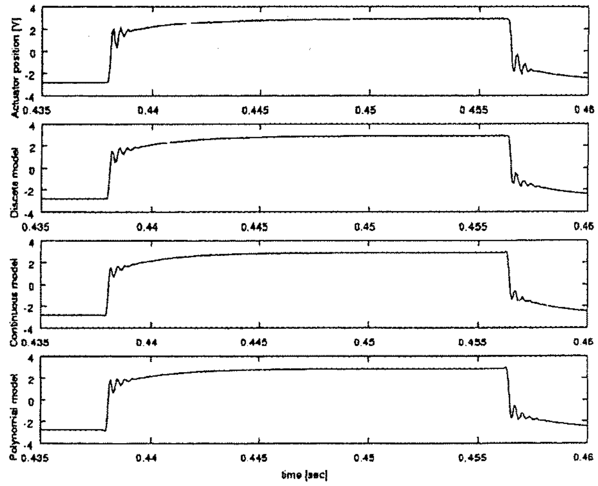
In Fig 3 (a), (b) shows ± 50%, ± 2% response of various plant models. In order from top,
1) Output of identification experiment
2) Output of discrete time model
3) Output of continuous time model
4) Output of polynomial model
The input signal for calculation is identical with experimental input signal.

Table 1: Coefficients of Polynomial

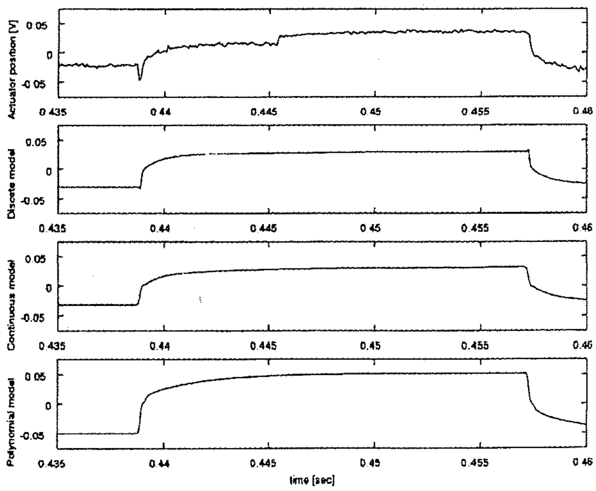
$b_i(\theta)$	$b_{i,0}$	$b_{i,1}$	$b_{i,2}$	$b_{i,3}$
$b_4(\theta)$	0.0332	-2.4263	4.5279	-3.2769
$b_3(\theta)$	-0.4526	-26.6971	57.1425	-28.2858
$b_2(\theta)$	6.7079	-16.8666	16.1603	-13.5530
$b_1(\theta)$	9.1597	142.4606	-266.5099	146.0958
$b_0(\theta)$	0.4913	12.7101	-27.1006	16.0988

$a_i(\theta)$	$a_{i,0}$	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$
$a_4(\theta)$	4.8881	-2.7046	2.7018	-0.8250
$a_3(\theta)$	13.2543	-16.3460	15.0593	-3.9602
$a_2(\theta)$	22.8389	-14.3179	-5.3850	11.7323
$a_1(\theta)$	9.16472	27.3741	-71.8940	44.8869
$a_0(\theta)$	0.3126	2.1733	-5.5099	3.4476

Note: Time axis is scaled by 9425.



(a) ±50% Response



(b) ±2% Response

Figure 3: Various Model Response Comparison

In Fig 3 (b), there is somewhat difference in gain of experiment result and polynomial model output. But, a sufficiently good enough model is obtained as a whole.

2.4 State Space Representation

Equation (1) and (2) are described as state space representation below,

$$\Sigma_p(\theta):$$
$$\begin{cases} \dot{x}_p = (A_{p,0} + A_{p,1}\theta + A_{p,2}\theta^2 + A_{p,3}\theta^3)x_p + B_p u \\ y_p = (C_{p,0} + C_{p,1}\theta + C_{p,2}\theta^2 + C_{p,3}\theta^3)x_p \end{cases} \quad (3)$$

Here, $x_p \in \mathcal{R}^{n_p}$ is plant state variable, $u \in \mathcal{R}^m$ is control variable, $y \in \mathcal{R}^l$ is measured variable. In this case, n_p=5,m=1, and l=1. Notice that when we fix the scheduling parameter, plant becomes a time-invariant linear system.

【Assumption】
Scheduling parameter satisfies following conditions.

1) $\theta(t) \in [\theta_{min}, \theta_{max}] \ \forall t \in [0, \infty)$
2) $\theta(t) \in \mathcal{C}^1$
3) $|\dot{\theta}(t)| \leq v_{max} \quad \forall t \in [0, \infty)$

Then the system which is described equation (3), is called LPV system.

3 Gain Scheduled Controller Synthesis

3.1 Problem Description

We consider full-rank output feedback controller, which has same order as that of plant. It is natural that controller also depends on scheduling parameter. Then, state space representation is

$$\Sigma_K : \begin{cases} \dot{x}_K = A_K(\theta)x_K + B_K(\theta)y, \quad x_K(0) = 0 \\ u = C_K(\theta)x_K + D_K(\theta)y \end{cases} \quad (4)$$

x_K is controller state variable.

For the controller design, construct generalized LPV system ($\Sigma_g(\theta)$) with additional weighting functions which reflect closed loop specification. The state space representation of the generalized LPV system that has an available scheduling parameter on-line, as follows,

$$\Sigma_g(\theta) : \begin{cases} \dot{x} = A(\theta)x + B_1(\theta)w + B_2(\theta)u \\ z = C_1(\theta)x + D_{11}(\theta)w + D_{12}(\theta)u \\ y = C_2(\theta)x + D_{21}(\theta)w \end{cases} \quad (5)$$

Again, $x \in \mathcal{R}^n$ is state variable, $w \in \mathcal{R}^{m_w}$ is disturbance and $z \in \mathcal{R}^l$ is controlled variable.

Define $\mathcal{L}_{gain}$ as a gain from w to z, such as

$$\mathcal{L}_{gain} = \sup_{w \in L_2, w \neq 0} \frac{\|z\|_2}{\|w\|_2} \quad (6)$$

Then, the controller design problem is that

- make the closed loop system internally stable.
- make the upper bond of $\mathcal{L}_{gain}$ less than or equal to a positive constant γ in the whole range of scheduling parameter for zero initial state of the controller.

3.2 Controller Characterization

Consider generalized LPV system in equation (5) and supposes assumptions 1) 2) and 3) are satisfied. Then, controller design problem is solvable when constant matrices K_B, K_C and continuous positive definite symmetric matrices $Y(\theta), X(\theta)$ exist and satisfy following three LMI constraints (7), (8), (9) and BMI (10).

$$\mathcal{A}'Y(\theta) + Y(\theta)\mathcal{A} + C_2'(\theta)K_B + (C_2'(\theta)K_B)' - v_{max}\frac{dY(\theta)}{d\theta} + \begin{pmatrix} Y(\theta)\mathcal{B} + K_B'D_{21}(\theta) & \mathcal{C}' \end{pmatrix} \Delta_{cl}^{-1} \times \begin{pmatrix} (Y(\theta)\mathcal{B} + K_B'D_{21}(\theta))' \\ \mathcal{C} \end{pmatrix} < 0 \quad (7)$$

$$X(\theta)\mathcal{A}' + \mathcal{A}X(\theta) + B_2(\theta)K_C + (B_2(\theta)K_C)' + v_{max}\frac{dX(\theta)}{d\theta} + \begin{pmatrix} \mathcal{B} & (\mathcal{C}X(\theta) + D_{12}(\theta)K_C)' \end{pmatrix} \Delta_{cl}^{-1} \times \begin{pmatrix} \mathcal{B}' \\ \mathcal{C}X + D_{12}(\theta)K_C \end{pmatrix} < 0 \quad (8)$$

$$\begin{pmatrix} Y(\theta) & I \\ I & X(\theta) \end{pmatrix} > 0 \quad (9)$$

$$\begin{pmatrix} \frac{dY(\theta)}{d\theta} & -X^{-1}(\theta)\frac{dX(\theta)}{d\theta} \\ -\frac{dX(\theta)}{d\theta}X^{-1}(\theta) & -\frac{dX(\theta)}{d\theta} \end{pmatrix} \leq 0 \quad (10)$$

Where,

$$\Delta_{cl} = \begin{pmatrix} \gamma I & -D_{cl}'(\theta) \\ -D_{cl}(\theta) & \gamma I \end{pmatrix} \quad (11)$$

$$\mathcal{A} := A(\theta) + B_2(\theta)D_K(\theta)C_2(\theta) \quad (12)$$

$$\mathcal{B} := B_1(\theta) + B_2(\theta)D_K(\theta)D_{21}(\theta) \quad (13)$$

$$\mathcal{C} := C_1(\theta) + D_{12}(\theta)D_K(\theta)C_2(\theta) \quad (14)$$

and D_K is chosen so that $\Delta_{cl} > 0$.

3.3 Controller Formula

When $X(\theta), Y(\theta), K_B$ and K_C matrices which satisfy above constraints are given, the full-rank output feedback controller is obtained by following calculation. B_K, C_K are

$$B_K = Z(\theta)^{-1}K_B' \quad (15)$$

$$C_K = -K_C X(\theta)^{-1} \quad (16)$$

Here, $Z(\theta) = Y(\theta) - X(\theta)^{-1}$. Then A_K is

$$
\begin{aligned}
A_K = Z^{-1}(\theta)\Big[&\mathcal{A}' + Y(\theta)\mathcal{A}X(\theta) + K_B'C_2(\theta)X(\theta) \\
&+ Y(\theta)B_2(\theta)K_C + v_{max}X^{-1}(\theta)\frac{dX(\theta)}{d\theta} \\
&+ \begin{pmatrix} Y(\theta)\mathcal{B} + K_B'D_{21}(\theta) & \mathcal{C}' \end{pmatrix}\Delta_{cl}^{-1} \\
&\times \begin{pmatrix} \mathcal{B}' \\ \mathcal{C}X + D_{12}(\theta)K_C \end{pmatrix}\Big]X^{-1}(\theta)
\end{aligned} \qquad (17)
$$

When $\dot{\theta}$ is available, BMI constraint (10) disappears and controller design synthesis is proposed in [1]. It is a specific feature in this paper that on-line $\dot{\theta}$ information is not required for controller implementation.

But, BMI constraint (10) can not be solve right away. Therefore, we introduce following procedure. We take $\frac{dX(\theta)}{d\theta} > 0$. Then, from expression (10), we have $\frac{dY(\theta)}{d\theta} < 0$. Thus, we use these two constraints instead of (10) to have solution candidates with appropriate $v_{max}(> 0)$. Then, examine if obtained solution satisfies BMI constraint (10) or not.

(for the details, see[2])

4 Design Conditions

In this section, we design controller according to the procedure mentioned before, with given weighting function of the generalized LPV system and $Y(\theta), X(\theta)$ which minimize γ. Further, here $Y(\theta), X(\theta)$ are depends on θ as polynomial of order three.

4.1 Weighting Function

Fig 4 shows the generalized LPV system ($\Sigma_g(\theta)$). An integrator is added at the plant input in order to minimize steady-state error. $W_s(s)$ and $W_t(s)$ are selected so that closed loop characteristics meet requirements. $W_s(s)$ is decided as

$$W_s(s) = \frac{s + 1.5}{2(s + 0.001)} \qquad (18)$$

and state space representation is

$$\Sigma_{W_s} : \begin{cases} \dot{x}_s = A_s x_s + B_p u_s \\ z_1 = C_s x_s + D_s u_s \end{cases}$$

Similarly,

$$W_t(s) = \frac{2s}{s + 2} \qquad (19)$$

$$\Sigma_{W_t} : \begin{cases} \dot{x}_t = A_t x_t + B_t u \\ z_2 = C_t x_t + D_t u \end{cases}$$

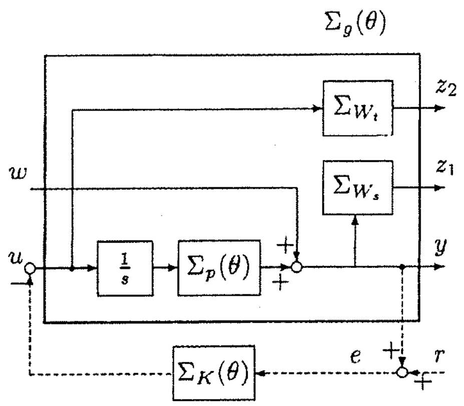


Figure 4: Generalized LPV System

4.2 Reducing to finite number of LMI constraints

Thus far LMI constraints depend on continuous scheduling parameter θ. As a result we have to solve infinite number of LMI whenever we fix the scheduling parameter. For reducing to finite number of constraints, a technique that proposed by Azuma et al[3], to construct a convex hull that covers the model, is introduced.

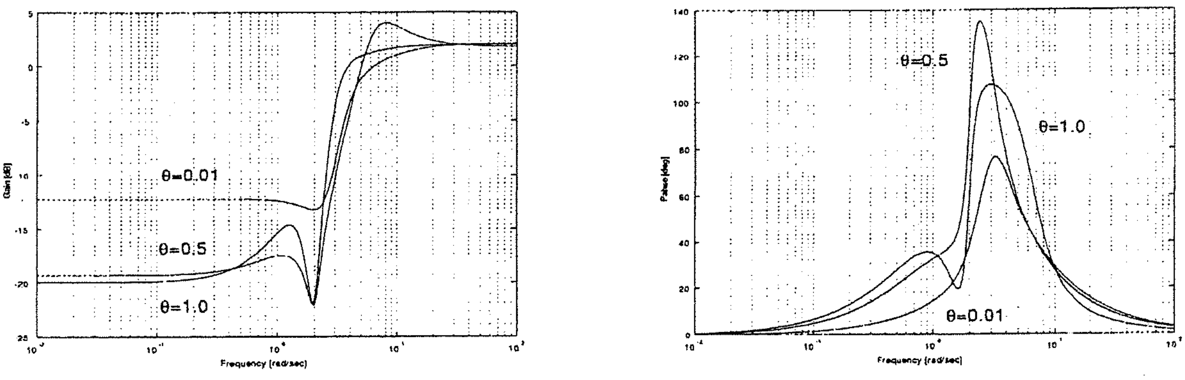
4.3 value of v_{max}

Based on plant physical conditions, candidates of v_{max} = 15,150,1500 are selected. Then, $v_{max} = 150$ is chosen such that it minimizes γ.

4.4 Gain Scheduled Controller

According to the controller synthesis described above, first we soluve $Y(\theta)$ and $X(\theta)$. In this case, γ_{min} is 6.9. Next, $Y(\theta)$ and $X(\theta)$ are examined if they satisfy constraint (10) which cannot be considered in the design procedure.

To understand the controller features, bode plots of the controller under the condition of $\theta = 0.01, 0.5, 1.0$ are shown in Fig 5. From Fig 5, we know that controller characteristics varies corresponding to the magnitude of the control signal (here, integrator characteristics is not included). For the smaller control signal, controller has larger gain in low frequency range, comparing with larger control signal. From these aspects, actuator performance improvement is expected in whole operating range.



(a) Gain Plot

(b) Phase Plot

Figure 5: Gain Scheduled Controller Bode Plot

5 Test Results

On the experiment, gain scheduled controller that corrspond to nine scheduling parameters, is prepared beforehand because of the throughput performance of the digital hardware. Then, switch them by scheduling parameter.

5.1 Low Frequency Range Linearity and Response

In Fig 6 shows closed loop GMM actuator output characteristics in low frequency. In a whole operating range, hysteresis is suppressed successfully and linearity of input-output characteristic is achieved. Response to the 0.2% peak to peak amplitude of sinusoidal command is shown in Fig 7. At the open loop configuration, there is not any observable output in this command range. The linearlization and resolution improvement are obtained sufficiently.

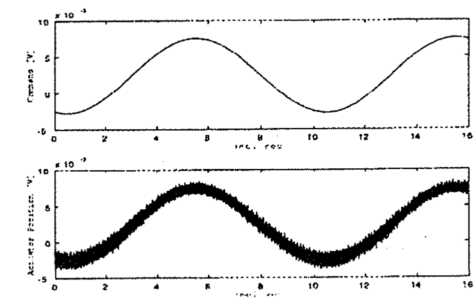


Figure 7: Sinusoidal Response

5.2 Response Characteristics

Comparing with results that obtained by the fixed and linear H_∞ controller which is designed based on a typical operating point, rise time of step response by the gain scheduled controller becomes half. And looking at the frequency response, the bandwidth shifts to higher frequency range by 1.5 times.

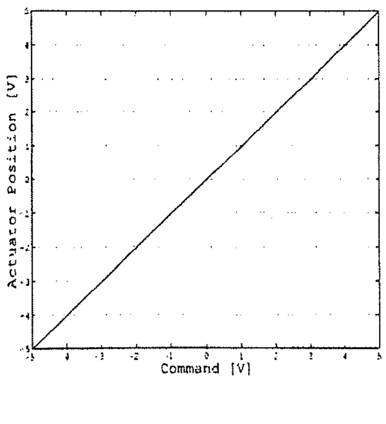


Figure 6: Linearized Characteristic

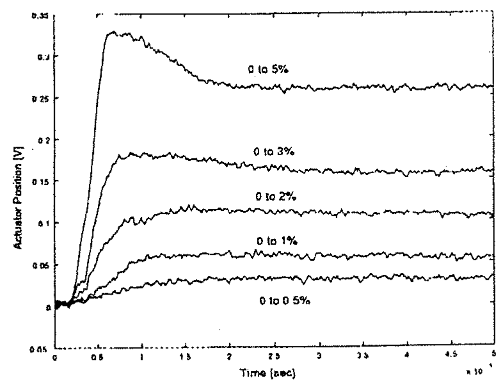
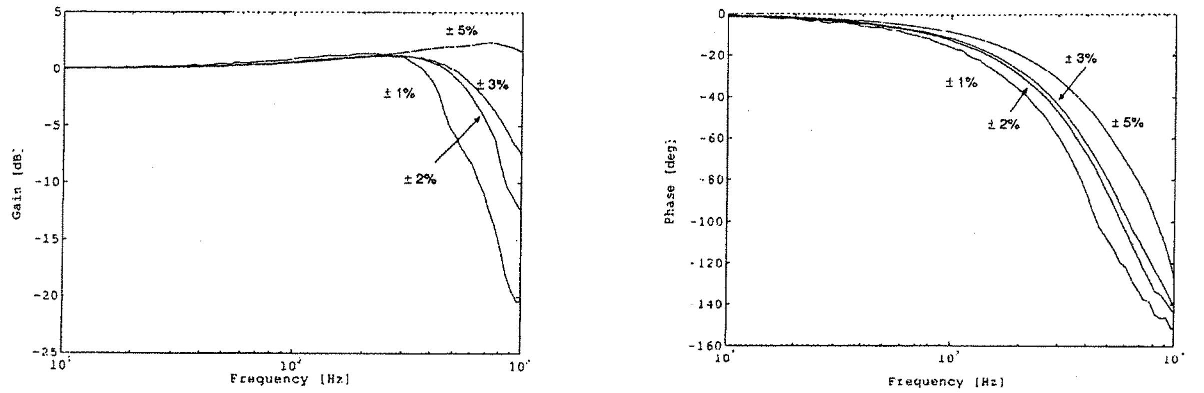


Figure 8: Closed Loop Step Response



(a) Gain Plot

(b) Phase Plot

Figure 9: Closed Loop Frequency Response

6 Conclusion

Because of hysteresis characteristics of GMM actuator, there is a difficulty to have sufficient resolution for small command signal in the frequency range where we have concern.

In this paper, we propose a controller design procedure for gain scheduling control that depends on magnitude of control variable as a scheduling parameter with pre-information of maximum value of scheduling parameter changing rate.

We have obtained sufficient results where command is close to zero range and confirmed the usefulness of gain scheduling control technique, except undesirable windup phenomena for larger than 5% of command signal. To overcome this windup problem is our future subject.

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Constrained State Feedback H^∞ Control of Nonlinear Systems

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Abstract

This paper considers a synthesis problem of constrained state feedback H^∞ control for nonlinear systems described as linear systems with self-scheduling parameters. The controllers assure that the closed loop system is asymptotically stable, the semi-global L^2 gain is less than a positive value and the input is constrained to a given set. First we show a condition to construct constrained state feedback H^∞ controllers in terms of infinite-dimensional Linear Matrix Inequalities(LMIs). Second we introduce a technique to reduce infinite-dimensional LMIs to a finite number of LMIs, and present a feasible algorithm for synthesis of controllers based on the finite-dimensional LMIs.

1 Introduction

If we consider to apply gain scheduling techniques [9][11] to analysis or controller synthesis problems of nonlinear systems whose system parameters depend on the state as follows,

$$\begin{aligned}\dot{x} &= A(x)x + B_1(x)w(t) + B_2(x)u(t),\\ z &= C(x)x,\end{aligned}$$

it is important to evaluate the domain of the state. If the domain of the state is obtained, the nonlinear systems can be described as linear systems with scheduling parameters (called the linear parameter varying systems: LPV systems). Thus the gain scheduling techniques are applicable to analysis or synthesis problems of the nonlinear systems.

In this article, we consider constrained state feedback H^∞ control of nonlinear systems whose system parameters depend on the state. The controllers designed by using our technique assure that the closed loop system is asymptotically stable, the semi-global L^2 gain is less than a positive value and the input is constrained to a given set. We derive conditions in the form of infinite-dimensional Linear Matrix Inequlities. We show a technique to reduce the infinite-dimensional Linear Matrix Inequlities to the finite-dimensional Linear Matrix Inequalities which are feasible formulas.

2 Problem Formulation

In this paper, we consider the following nonlinear system,

$$\Sigma: \begin{aligned}\dot{x} &= A(x)x + B_1(x)w(t) + B_2(x)u(t),\\ z &= C(x)x,\\ x(0) &= 0,\end{aligned}$$

where $x \in R^n$ is the state, $z \in R^l$ is the output, $w \in R^{m_w}$ is the disturbance and $u \in R^m$ is the input.

For the general nonlinear system, it is difficult to discuss system performances (e.g., L^2 gain performance) globally against the disturbance. We assume that the disturbance $w(t)$ is constrained to the following ellipse,

$$w(t) \in \mathcal{W},\ {}^\forall t \in [0,\infty),\quad \mathcal{W} = \{w \mid w'Ww \le 1\}.$$

Since the disturbance $w(t)$ is constrained to a given set $\mathcal{W}$, we consider semi-global L^2 gain for Σ.

The purpose of this article is to design the state feedback controller Γ,

$$\Gamma : u(t) = K(x(t))x(t), \tag{1}$$

such that

- the closed loop system $\Sigma\Gamma$ is asymptotically stable,
- the semi-global L^2 gain $G(\Sigma\Gamma)$ of the closed loop system is less than γ,

$$G(\Sigma\Gamma) = \sup_{\substack{w \in L^2 \cap \mathcal{W} \\ w \neq 0}} \frac{\| z \|_{L^2}}{\| w \|_{L^2}},$$

- the input $u(t)$ is included in a given set $\mathcal{U}$ for any time $t \in [0, \infty)$,

$$\mathcal{U} = \{u \mid u'Q_u^{-1}u \leq 1\}.$$

Remark 1. We call the nonlinear system Σ as the "*linear system with self-scheduling parameters*", because this system can be described as linear system with scheduling parameters if the domain of the self-scheduling parameter $x(t)$ is obtained.

3 Controller Synthesis

First we show the result of synthesis problem of state feedback controllers which assure that the closed loop system $\Sigma\Gamma$ is asymptotically stable and the semi-global L^2 gain of the closed loop system is less than γ.

Theorem 2. Assume that there exist a scalar α, a matrix Q and a continuous matrix function $Y(\theta)$ which satisfy the next inequalities,

$$\alpha > 0, \tag{2}$$

$$Q > 0, \tag{3}$$

$$\begin{bmatrix} \begin{pmatrix} A(\theta)Q + QA'(\theta) + \alpha Q \\ +B_2(\theta)Y(\theta) + Y'(\theta)B_2'(\theta) \end{pmatrix} & B_1(\theta) \\ B_1'(\theta) & -\alpha W \end{bmatrix} < 0, \tag{4}$$

$$\begin{bmatrix} \begin{pmatrix} A(\theta)Q + QA'(\theta) \\ +B_2(\theta)Y(\theta) \\ +Y'(\theta)B_2'(\theta) \end{pmatrix} & B_1(\theta) & QC'(\theta) \\ B_1'(\theta) & -\gamma^2 I & 0 \\ C(\theta)Q & 0 & -I \end{bmatrix} < 0, \tag{5}$$

$$^\forall\theta \in \{\theta \in R^n \mid \theta'Q^{-1}\theta \leq 1\}$$

Then the closed loop system $\Sigma\Gamma$ is asymptotically stable and the semi-global L^2gain is less than γ. The state feedback gain $K(x)$ is given as

$$K(x) = Y(x)Q^{-1}. \tag{6}$$

Proof. This theorem is derived by using the result of [3]. **Q.E.D.**

Considering that the reachable set of the state $x(t)$ in the closed loop system $\Sigma\Gamma$ is given as Ω_x,

$$\Omega_x = \{x \mid x'Q^{-1}x \leq 1\},$$

our main theorem is given as follows.

Theorem 3. Assume that there exist a scalar α, a matrix Q and a continuous matrix function $Y(\theta)$ which satisfy the next inequalities,

$$\alpha > 0, \tag{7}$$

$$Q > 0, \tag{8}$$

$$\begin{bmatrix} \begin{pmatrix} A(\theta)Q + QA'(\theta) + \alpha Q \\ +B_2(\theta)Y(\theta) + Y'(\theta)B_2'(\theta) \end{pmatrix} & B_1(\theta) \\ B_1'(\theta) & -\alpha W \end{bmatrix} < 0, \tag{9}$$

$$\begin{bmatrix} \begin{pmatrix} A(\theta)Q + QA'(\theta) \\ +B_2(\theta)Y(\theta) \\ +Y'(\theta)B_2'(\theta) \end{pmatrix} & B_1(\theta) & QC'(\theta) \\ B_1'(\theta) & -\gamma^2 I & 0 \\ C(\theta)Q & 0 & -I \end{bmatrix} < 0, \tag{10}$$

$$\begin{bmatrix} Q & Y'(\theta) \\ Y(\theta) & Q_u \end{bmatrix} > 0 \tag{11}$$

$$^\forall\theta \in \{\theta \in R^n \mid \theta'Q^{-1}\theta \leq 1\}.$$

Then the closed loop system $\Sigma\Gamma$ is asymptotically stable, the semi-global L^2gain is less than γ and the input $u(t)$ is included in $\mathcal{U}$. The state feedback gain $K(x)$ is given as

$$K(x) = Y(x)Q^{-1}. \tag{12}$$

Proof. From (1) and (12), the next condition is obtained.

$$u'Q_u^{-1}u = x'Q^{-1}Y'(x)Q_u^{-1}Y(x)Q^{-1}x.$$

Now considering that the state reachable set Ω_x is given as

$$\Omega_x = \{x \mid x'Q^{-1}x \leq 1\},$$

the input is included in the set $\mathcal{U}$ if the following condition is satisfied,

$$Q^{-1} - Q^{-1}Y'(x)Q_u^{-1}Y(x)Q^{-1} > 0, \quad {}^\forall x \in \Omega_x.$$

From (8), (11) and $Q_u > 0$, the next is satisfied for any $x \in \Omega_x$.

$$\begin{bmatrix} Q & Y'(x) \\ Y(x) & Q_u \end{bmatrix} > 0$$
$$\Leftrightarrow Q - Y'(x)Q_u^{-1}Y(x) > 0$$
$$\Leftrightarrow Q^{-1} - Q^{-1}Y'(x)Q_u^{-1}Y(x)Q^{-1} > 0$$

Thus it is assured that the input $u(t)$ is included in $\mathcal{U}$ for any $t \in [0, \infty)$. **Q.E.D.**

In Theorem 3, it is difficult to obtain solutions which satisfy inequalities (9), (10) and (11). The main difficulties are as follows.

P1 (9) is a bilinear matrix inequality about α and Q.

P2 (9), (10) and (11) depend on the parameter θ.

P3 The domain of the parameter θ depends on the unknown matrix Q.

As for the problems **P1** and **P3**, we put aside these problems by adopting recursive calculation of the following corollary. We consider the problem **P2** in the next section.

Corollary 4. Assume that there exist a scalar α, a matrix Q and a continuous matrix function $Y(\theta)$ which satisfy the next inequalities for a given matrix Q_1.

$$\alpha > 0, \tag{13}$$
$$Q_1 > Q > 0 \tag{14}$$
$$\begin{bmatrix} \begin{pmatrix} A(\theta)Q + QA'(\theta) + \alpha Q_1 \\ +B_2(\theta)Y(\theta) + Y'(\theta)B_2'(\theta) \end{pmatrix} & B_1(\theta) \\ B_1'(\theta) & -\alpha W \end{bmatrix} < 0 \tag{15}$$

$$\begin{bmatrix} \begin{pmatrix} A(\theta)Q + QA'(\theta) \\ +B_2(\theta)Y(\theta) \\ +Y'(\theta)B_2'(\theta) \end{pmatrix} & B_1(\theta) & QC'(\theta) \\ B_1'(\theta) & -\gamma^2 I & 0 \\ C(\theta)Q & 0 & -I \end{bmatrix} < 0 \tag{16}$$

$$\begin{bmatrix} Q & Y'(\theta) \\ Y(\theta) & Q_u \end{bmatrix} > 0 \tag{17}$$

$${}^\forall \theta \in \{\theta \in R^n \mid \theta' Q_1^{-1} \theta \le 1\}$$

Then the closed loop system $\Sigma\Gamma$ is asymptotically stable, the semi-global L^2gain is less than γ and the input $u(t)$ is included in $\mathcal{U}$. The state feedback gain $K(x)$ is given as

$$K(x) = Y(x)Q^{-1}. \tag{18}$$

Proof. The proof is omitted. **Q.E.D.**

4 Reduction to Finite Dimensional Conditions

Conditions (15) (16) (17) in Corollary 4 are infinite-dimensional conditions depending on the parameter θ. In this section, a technique to reduce this infinite-dimensional conditions to finite-dimensional conditions is shown.

First we restrict system parameters as follows,

$$\begin{aligned} A(x) &= A_0 + a_1(x)A_1 + \cdots + a_{r_a}(x)A_{r_a}, \\ B_1(x) &= B_{10} + b_{11}(x)B_{11} + \cdots + b_{1r_{b1}}(x)B_{1r_{b1}}, \\ B_2(x) &= B_{20} + b_{21}(x)B_{21} + \cdots + b_{2r_{b2}}(x)B_{2r_{b2}}, \\ C(x) &= C_0 + c_1(x)C_1 + \cdots + c_{r_c}(x)C_{r_c}, \end{aligned} \tag{19}$$

where $a_i, b_{1i}, b_{2i}, c_i : R^n \to R$ are continuous functions and A_i, B_{1i}, B_{2i}, C_i are constant matrices with adequate dimensions. The solution $Y(\theta)$ is also restricted as follows,

$$Y(\theta) = Y_0 + \xi_1(\theta)Y_1 + \cdots + \xi_{r_\xi}(\theta)Y_{r_\xi}$$

where $\xi_i : R^n \to R$ is a continuous function and Y_i is a matrix with an adequate dimension. $\xi_i(\theta)$ is given by us and Y_i is the unknown matrix.

Then inequalities (15) (16) (17) are described as the form of the following parameter dependent LMI(Linear Matrix Inequality).

$$F_0(Q_u) + f_1(\theta)F_1(Q_u) + \cdots + f_r(\theta)F_r(Q_u) < 0, \tag{20}$$
$$\theta \in \Theta = \{\theta \mid \theta' Q_1^{-1} \theta \le 1\}$$

$f_i : R^n \to R$ is the continuous function of θ and a symmetric matrix function F_i depends affinely on the unknown matrix Q_u,

$$Q_u = [Q, Y_0, \cdots, Y_{r_\xi}].$$

The parameter dependent LMI in this form can be reduced to a finite number of LMI conditions, which are finite-dimensional, by using the next theorem.

Theorem 5. [1] Let $\{p_1, p_2, \cdots, p_q\}(q \geq r+1)$ be vertices of a convex polyhedron which includes the curved surface T,

$$T = [f_1(\theta)\ f_2(\theta)\ \cdots\ f_r(\theta)]'\ ,\ \theta \in \Theta. \quad (21)$$

Assume that there exists Q_u which satisfies the following LMI condition for all $p_i (i = 1, 2, \cdots, q)$.

$$F_0(Q_u) + p_{i1}F_1(Q_u) + \cdots + p_{ir}F_r(Q_u) < 0. (22)$$

where p_{ij} is the jth element of p_{ij}. Then Q_u satisfies (20) for all $\theta \in \Theta$.

Remark 6. The convex polyhedron which includes the curved surface T is important in Theorem 5. It is easy to construct this convex polyhedron if the following two points r_1, r_2 are obtained, because the hyper rectangular(a convex polyhedron) whose diagonal points are r_1 and r_2 includes the curved surface T.

$$r_1 = \left[\max_{\theta\in\Theta} f_1(\theta)\quad \max_{\theta\in\Theta} f_2(\theta)\quad \cdots\quad \max_{\theta\in\Theta} f_r(\theta)\right]'$$
$$r_2 = \left[\min_{\theta\in\Theta} f_1(\theta)\quad \min_{\theta\in\Theta} f_2(\theta)\quad \cdots\quad \min_{\theta\in\Theta} f_r(\theta)\right]'$$

But we need to compute the maximum and the minimum of $f_i(\theta)$ for $\theta \in \Theta$. This is a difficult problem in some cases. We show a convenient method to get r_1, r_2.

1. We construct a hyper rectangular Θ_R which includes the ellipse Θ. (If m_i, M_i in the next step are obtained for Θ, it is not needed to construct Θ_R.)

2. We compute the following m_i and M_i for $\theta \in \Theta_R$(or Θ).

$$m_i \leq f_i(\theta) \leq M_i$$

3. We use the next two points as r_1, r_2.

$$q_1 = [M_1\ \ M_2\ \ \cdots\ \ M_r]'$$
$$q_2 = [m_1\ \ m_2\ \ \cdots\ \ m_r]'$$

5 Conclusion

In this article, we consider nonlinear systems which can be described as linear systems with self-scheduling parameters and propose a method to design the state feedback controllers which assure that the closed loop system is asymptotically stable, the semi-global L^2gain of the closed loop system is less than γ and the input $u(t)$ is included in a given set.

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FINITE-DIMENSIONAL CHARACTERIZATIONS OF H^∞ CONTROL FOR LINEAR SYSTEMS WITH DELAYS IN CONTROL INPUT AND CONTROLLED OUTPUT

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Abstract: We formulate an H^∞ control problem for linear systems with delays in control input and controlled output, and discuss possibility of finite-dimensional characterizations of the solution. First, we derive a state feedback H^∞ control formula constructed by using any solution to an infinite-dimensional Riccati matrix equation or an infinite-dimensional Riccati matrix inequality. Second, we show that, if the controlled output is chosen such that it satisfies the "prediction condition", the solution to the infinite-dimensional Riccati equation can be calculated by solving a finite-dimensional Riccati equation and similarly the infinite-dimensional Riccati inequality can be solved with a solution to a finite-dimensional Riccati inequality. We provide a system theoretic interpretation for the prediction condition, and show finally that, if the prediction condition is satisfied, there is an H^∞ control problem for finite-dimensional linear systems which is equivalent to the problem formulated in this paper for linear systems with delays in control input and controlled output.

Keywords: Time-delay, Linear systems, H-infinity control, Riccati equations, Prediction

1. NTRODUCTION

The infinite-dimensional characteristics is an intrinsic feature of systems with delay, and makes analysis or synthesis problems for systems with delay difficult to handle. To overcome or bypass this difficulty, a lot of approaches have been proposed. For example, we mention the spectrum decomposition and/or prediction approach, which guarantees finite-dimensional characterizations of the solutions, to LQ control problems (Uchida *et al.*, 1988a, 1988b) and H^∞ control problems (Kojima *et al.*, 1994,); the memoryless feedback control synthesis via linear matrix inequality (LMI) may be one of such approaches (Shen *et al.*, 1991, Lee *et al.*, 1994, Dugard and Verriest, 1997); the present authors have recently proposed the infinite-dimensional LMI characterization of the solutions and an finite-dimensional LMI algorithm (Azuma *et al.*, 1998, Azuma *et al.*, 1999); we also note the discretization technique of Lyapunov functional (Gu, 1997)

In this paper we concern ourselves also with finite-dimensional solutions to synthesis problem in linear systems with delay. We formulate an H^∞ control problem for linear systems with delays in control input and controlled output, and discuss possibility of finite-dimensional characterizations of the solution along the same line developed in (Uchida *et al.*, 1988b). First, we derive a state feedback H^∞ control formula constructed by using any solution to an infinite-dimensional Riccati matrix equation or an infinite-dimensional Riccati matrix inequality. Second, we show that, if the controlled output is chosen such that it satisfies the "prediction condition", the solution to the infinite-dimensional Riccati equation can be calculated by solving a finite-dimensional Riccati equation and similarly the infinite-dimensional Riccati inequality can be solved with a solution to a finite-dimensional Riccati inequality. We provide a system theoretic interpretation for the prediction condition, and show finally that, if the prediction condition is satisfied, there is an H^∞ control problem for finite-dimensional linear systems which is equivalent to the problem formulated in this paper for linear systems with delays in control input and controlled output.

2. SYSTEM DESCRIPTION AND PROBLEM STATEMENT

Consider a linear system with delays in control input and controlled output. The system is defined over the interval $[0,\infty)$ and described by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 w(t) + B_{20}u(t) + B_{21}u(t-h)\\ z(t) &= C_{10}x(t) + \int_{-h}^{0} C_{11}(\beta)B_{21}u(t+\beta)d\beta + D_1 u(t)\\ y(t) &= C_2 x(t) + D_2 w(t)\end{aligned} \tag{1}$$

with the initial condition such that $x(0)=0$ and $u(\beta)=0, -h\le\beta\le 0$. Here, $x(t)$ is the internal variable vector of the system; $w(t)$ is the disturbance vector; $u(t)$ is the control input vector; $z(t)$ is the controlled output vector; $y(t)$ is the measurement output vector. The number h denotes the length of time delay and $h>0$. The parameters $A, B_1, B_{20}, B_{21}, C_{10}, D_1, C_2, D_2$ are constant matrices and the parameter $C_{11}(\beta)$ is a matrix function whose elements are bounded continuous functions. It is noted that the future trajectory of the internal variable $x(\tau), \tau\ge t$ is uniquely determined by the value $x(t)$ and the function $B_{21}u_t$, where $u_t := (u(t+\beta), -h\le\beta\le 0)$, if the future control input $u(\xi), t\le\xi\le\tau$ and the future disturbance $w(\xi), t\le\xi\le\tau$ are given. From this viewpoint, we call the pair $(x(t), B_{21}u_t)$ the state of the system (1). Note that the controlled output $z(t)$ is a general form of linear function of the state $(x(t), B_{21}u_t)$ and the control input $u(t)$.

In addition to the measurement output $y(t)$, a part of the state, that is u_t, is available at each time t, because u_t is a past history of the control input. Thus, we consider the pair $(y(t), u_t)$ as the measurement output, and define each admissible control input $u(t)$ as a bounded linear causal function of the measurement output $(y(t), u_t)$. H^∞ control problem discussed in this paper is to find an admissible control $u(t)$ which controls the system (1) such that Ⅰ) the closed loop system is asymptotically stable, and Ⅱ) the closed loop system satisfies the inequality:

$$\int_0^\infty z(t)^T z(t)dt \le \int_0^\infty w(t)^T w(t)dt, \quad \forall w\in L_2(0,\infty) \tag{2}$$

where "^{T}" indicates transposition. Here we note that the inequality (2) is equivalent to $\|T_{zw}(s)\|_\infty \le 1$, where $T_{zw}(s)$ denotes the transfer function from the disturbance w to the controlled output z. An admissible control which satisfies Ⅰ) and Ⅱ) is called an H^∞ control.

3. STATE FEEDBACK H^∞ CONTROL AND PREDICTION CONDITION

We consider here a special case of the problem formulated in Section 2. Let the system (1) be given in the following form:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 w(t) + B_{20}u(t) + B_{21}u(t-h)\\ z(t) &= \begin{bmatrix} F_0 x(t) + \int_{-h}^{0} F_1(\beta)B_{21}u(t+\beta)d\beta \\ u(t)\end{bmatrix}\\ y(t) &= x(t)\end{aligned} \tag{3}$$

The special form of the controlled output implies that the state and the control input are evaluated separately, and the special form of the measurement output implies that the state $(x(t), B_{21}u_t)$ is available. Now we prepare a notation for defining a quadratic form of the state. Denote by $\{S_0, S_1, S_2\}$ a triplet of three matrices S_0, $S_1(\beta)$ and $S_2(\alpha,\beta)$ with the same dimensions such that S_0 is a constant matrix, $S_1(\beta)$ is a matrix function whose elements are in $L_2[-h,0]$ and $S_2(\alpha,\beta)$ is a matrix function whose elements are in $L_2([-h,0]\times[-h,0])$. A triplet $\{S_0,S_1,S_2\}$ is called symmetric if $S_0^T = S_0$ and $S_2^T(\alpha,\beta) = S_2(\beta,\alpha)$. For a given symmetric triplet $\{S_0,S_1,S_2\}$, a quadratic form associated with this triplet is defined as follows:

$$(\phi_0,\phi_1)^T\{S_0,S_1,S_2\}(\phi_0,\phi_1) := \phi_0^T S_0 \phi_0 + 2\phi_0^T \int_{-h}^{0} S_1(\beta)\phi_1(\beta)d\beta + \int_{-h}^{0}\int_{-h}^{0} \phi_1^T(\alpha)S_2(\alpha,\beta)\phi_1(\beta)d\alpha d\beta \quad (4)$$

for a vector ϕ_0 and a vector function ϕ_1 in $L_2[-h,0]$. A symmetric triplet $\{S_0,S_1,S_2\}$ is called positive semi-definite if $\phi^T\{S_0,S_1,S_2\}\phi \ge 0$ for all $\phi := (\phi_0,\phi_1)$ and, in particular, called positive definite if there exists a positive number ε such that $\phi^T\{S_0,S_1,S_2\}\phi \ge \phi^T\{\varepsilon I, 0, \varepsilon I\}\phi$ for all ϕ, where I denotes identity matrix. We denote $\{S_0,S_1,S_2\} \ge 0\ (>0)$ when $\{S_0,S_1,S_2\}$ is positive semi-definite (positive definite). Negative semi-definiteness and negative definiteness are similarly defined.

3.1. Infinite-dimensional formula.

For a triplet $\{M_0,M_1,M_2\}$, introduce a triplet $\{\Omega_0,\Omega_1,\Omega_2\}$ and a matrix function Π defined by

$$\Omega_0(M_0,M_1) := A^T M_0 + M_0 A + F_0^T F_0 + M_0 B_1 B_1^T M_0 - \{M_0 B_{20} + M_1(0)B_{21}\}\{B_{20}^T M_0 + B_{21}^T M_1(0)\}$$

$$\Omega_1(\beta; M_0,M_1,M_2) := -\frac{\partial}{\partial\beta}M_1(\beta) + A^T M_1(\beta) + F_0^T F_1(\beta) + M_0 B_1 B_1^T M_1(\beta) - \{M_0 B_{20} + M_1(0)B_{21}\}\{B_{20}^T M_1(\beta) + B_{21}^T M_2(0,\beta)\}$$

$$\Omega_2(\alpha,\beta; M_0,M_1,M_2) := -(\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta})M_2(\alpha,\beta) + F_1^T(\alpha)F_1(\beta) + M_1^T(\alpha)B_1 B_1^T M_1(\beta) - \{M_1^T(\alpha)B_{20} + M_2(\alpha,0)B_{21}\} \times \{B_{20}^T M_1(\beta) + B_{21}^T M_2(0,\beta)\}$$

$$\Pi(\beta; M_0,M_1,M_2) := \begin{bmatrix} M_1(-h) - M_0 & M_2(-h,\beta) - M_1(\beta) \end{bmatrix}$$

Theorem 1. a) Suppose that (A, F_0) is detectable. If there exists a positive semi-definite solution $\{M_0,M_1,M_2\}$ to the infinite-dimensional Riccati equation:

$$\begin{aligned} &\Omega_0(M_0,M_1) = 0, \\ &\Omega_1(\beta; M_0,M_1,M_2) = 0, \\ &\Omega_2(\alpha,\beta; M_0,M_1,M_2) = 0, \\ &\qquad -h \le \alpha \le 0,\ -h \le \beta \le 0 \end{aligned} \quad (5)$$

with the boundary condition $\Pi(\beta; M_0,M_1,M_2) = 0,\ -h \le \beta \le 0$, then an H^∞ control is given by

$$u(t) = -\{B_{20}^T M_0 + B_{21}^T M_1^T(0)\}x(t) - \int_{-h}^{0}\{B_{20}^T M_1(\beta) + B_{21}^T M_2(0,\beta)\}B_{21}u(t+\beta)d\beta \quad (6)$$

b) If there exists a positive definite solution $\{M_0,M_1,M_2\}$ to the infinite-dimensional Riccati inequality $\{\Omega_0,\Omega_1,\Omega_2\} < 0$ with the boundary condition $\Pi(\beta; M_0,M_1,M_2) = 0,\ -h \le \beta \le 0$, then an H^∞ control is given in the form (6).

The above characterizations are shown by following the standard argument of completing the square with respect to the quadratic form of the state $(x(t), B_{21}u_t)^T\{M_0,M_1,M_2\}(x(t), B_{21}u_t)$. The details are omitted here.

Remark. A solution $\{M_0,M_1,M_2\}$ to the inequality $\{\Omega_0,\Omega_1,\Omega_2\} < 0$ is given by solving the infinite-dimensional matrix inequality

$$\frac{1}{2}\begin{bmatrix} \Omega_0(M_0,M_1) & \Omega_1(\beta;M_0,M_1,M_2) \\ \Omega_1^T(\alpha;M_0,M_1,M_2) & \Omega_2(\alpha,\beta;M_0,M_1,M_2) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \Omega_0(M_0,M_1) & \Omega_1(\beta;M_0,M_1,M_2) \\ \Omega_1^T(\alpha;M_0,M_1,M_2) & \Omega_2(\alpha,\beta;M_0,M_1,M_2) \end{bmatrix}^T < 0,$$

$$\forall \alpha,\beta \in [-h,0]$$

This reformulation is used in another finite-dimensional characterization of analysis/synthesis problems for time-delay systems (Azuma *et al.*, 1998, Azuma *et al.*, 1999).

3.2. Prediction condition and finite-dimensional formula.

For the controlled output of the system (3), we consider further a special structure described by the following condition:

$$\text{(C1)} \quad F_1(\beta) = F_0 e^{-A(\beta+h)}, \quad -h \le \beta \le 0$$

Theorem 2. a) Suppose that the condition (C1) is satisfied and (A, F_0) is detectable. If there exists a positive semi-definite solution M to the finite-dimensional Riccati equation:

$$A^T M + MA + F_0^T F_0 - M(B_h B_h^T - B_1 B_1^T)M = 0 \quad (7)$$

where $B_h := B_{20} + e^{-Ah} B_{21}$, then, a semi-definite solution $\{M_0, M_1, M_2\}$ to the infinite-dimensional Riccati equation (5) with the boundary condition is given by

$$\begin{aligned} &M_0 := M \\ &M_1(\beta) := M e^{-A(\beta+h)} \\ &M_2(\alpha,\beta) := e^{-A^T(\alpha+h)} M e^{-A(\beta+h)} \end{aligned} \quad (8)$$

and an H^∞ control is given in the form (6).

b) Suppose that the condition (C1) is satisfied. If there exists a positive definite solution M to the finite-dimensional inequality:

$$A^T M + MA + F_0^T F_0 - M(B_h B_h^T - B_1 B_1^T)M < 0 \quad (9)$$

where B_h is defined as above in a), then, a positive definite solution $\{M_0, M_1, M_2\}$ to the infinite-dimensional Riccati inequality $\{\Omega_0, \Omega_1, \Omega_2\} < 0$ with the boundary condition is given in the form (8), and an H^∞ control is given in the form (6).

Theorem 2 is shown by direct substitution of the formula (8) and using the results of Theorem 1. Here we present a system theoretic interpretation of the condition. When the condition (C1) is satisfied, the controlled output in the system (3) is reduced to the form:

$$z(t) = \begin{bmatrix} F_0 p(t) \\ u(t) \end{bmatrix} \quad (10)$$

$$p(t) := x(t) + \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(t+\beta) d\beta \quad (11)$$

From the formula (11), we can see that $p(t)$ (more precisely, $e^{Ah} p(t)$) corresponds to the predictive value $x(t+h)$ of the internal variable $x(t)$. Thus, if the condition (C1) is satisfied, the problem becomes to control the predictive value of the internal variable, and we call (C1) the "prediction condition". It is also noted that $p(t)$ defined by (11) satisfies a finite-dimensional linear system given as

$$\dot{p}(t) = Ap(t) + B_1 w(t) + B_h u(t) \quad (12)$$

As discussed in the next section, the finite-dimensional characterizations presented in Theorem 2 come from the resultant descriptions of (10) and (12).

4. PREDICTION CONDITION AND FINITE-DIMENSIONAL PROBLEM

Here we come back to the output feedback H^∞ control problem formulated in Section 2. For the controlled output in the system (1), we consider the following condition:

$$\text{(C2)} \quad C_{11}(\beta) B_{21} = C_{10} e^{-A(\beta+h)} B_{21}, \quad -h \le \beta \le 0$$

The implication of the condition (C2) is the same to that of the condition (C1), although (C2) is more general than (C1). So we call (C2) also the prediction condition.

Theorem 3. Suppose that the condition (C2) is satisfied. Then, the output feedback H^∞ control problem described by the criterion (2) and the system

(1) is equivalent to the output feedback H^∞ control problem described by the criterion (2) and the following finite-dimensional system:

$$\begin{aligned} \dot{p}(t) &= Ap(t) + B_1 w(t) + B_h u(t) \\ z(t) &= C_{10} p(t) + D_1 u(t) \\ q(t) &= C_2 p(t) + D_2 w(t) \end{aligned} \qquad (13)$$

where $p(t)$ is the internal variable with $p(0) = 0$, $q(t)$ is the measurement output and $B_h = B_{20} + e^{-Ah} B_{21}$.

We can verify Theorem 3 as follows: Let $u(t) = \Gamma(t, q(\cdot), u_.)$ be a solution to the finite-dimensional H^∞ control problem described by (2) and (13). Defining

$$\begin{aligned} x(t) &:= p(t) - \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(t+\beta) d\beta \\ y(t) &:= q(t) - C_2 \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(t+\beta) d\beta \end{aligned} \qquad (14)$$

and using (C2), we can show from (13) that $x(t)$ and $y(t)$ defined above obey the system (1) and the control

$u(t) = \Gamma(t, y(\cdot) + C_2 \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(\cdot+\beta), u_.)$ is an solution to the H^∞ control problem described by (2) and (1). Conversely, let $u(t) = \Delta(t, y(\cdot), u_.)$ be a solution to the H^∞ control problem described by (2) and (1). Defining

$$\begin{aligned} p(t) &:= x(t) + \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(t+\beta) d\beta \\ q(t) &:= y(t) + C_2 \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(t+\beta) d\beta \end{aligned} \qquad (15)$$

and using (C2), we can show from (1) that $p(t)$ and $q(t)$ defined above obey the system (13) and the control

$u(t) = \Delta(t, q(\cdot) - C_2 \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(\cdot+\beta), u_.)$ is an solution to the finite-dimensional H^∞ control problem described by (2) and (13). Thus, we obtain the conclusion of Theorem 3.

Controlled outputs are chosen correspondingly to purposes of control design. Finally we comment briefly on meanings of the prediction condition (C2) (or (C1)) from the viewpoint of control design. If it is required to control the predictive value of the internal variable, the prediction condition will be satisfied automatically. In the previous work (Kojima et al., 1994), the authors proved the same equivalence as in Theorem 3 under the assumption that the condition

$$\text{(C3)} \quad \begin{aligned} &C_{11}(\beta) B_{21} = 0, \quad -h \le \beta \le 0, \\ &C_{10} A^i B_{21} = 0, \quad i = 0, 1, 2, \cdots \end{aligned}$$

and showed that the framework of H^∞ control problems for input delayed systems satisfying (C3) includes the robust stabilization problems against additive or multiplicative perturbations (including uncertain delay case). We can verify immediately that the condition (C3) is a sufficient condition for the prediction condition (C2) to hold, and therefore see that such robust stabilization problems can be handled also in the framework of this paper. To clarify more general meanings of the prediction condition from the viewpoint of control design is still an open problem of interest.

5. CONCLUSION

We discussed the state/output feedback H^∞ control problem for linear systems with delays in control input and controlled output, and showed that the finite-dimensional characterization of the solution is possible if the prediction condition on the controlled output is satisfied. We also discussed the meaning of the prediction condition from the viewpoint of control design.

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論　文

超磁歪材アクチュエータを有する直動型サーボ弁のモデリングとゲインスケジューリングによるスプール位置制御*

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Modeling of Direct-Drive Servovalve which Has Giant Magnetrostrictive Material and Spool Position Control by Gain Scheduling*

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Because of the reasons of favorable features such as high driving force,, fast response, fine environmental durability and big distortion about 10 times more than that of PZT, GMM (Giant Magnetrostrictive Material) is suitable for the actuation structure of servovalve. But, GMM distortion depends on the magnitude of coil current and the output-gain declines according to decreasing current magnitude. Then, the dead-zone appears in small current range. One of the important characteristics of servovalve is the fine tracking performance to the reference signal. So far, we attempted to apply linear controllers, such as PI or H^∞ controller. However GMM tracking performance is inferior to the traditional driving device for reference signal around null area, and this flaw remains as an important matter.

In this paper, adopting LPV (Linear Parameter-Varying) system modeling in which we regard the magnitude of input signal as a scheduling parameter, we design a gain scheduling controller and attempt to solve the problem that is caused by the nonlinearities mentioned above. Since it is hard to implement the controller which uses the information on the varying rate of scheduling parameters, we propose a new approach for output-feedback controller design. First, we describe the plant modeling as an LPV system, then present an outline of the controller design synthesis without any information on the varying rate of scheduling parameter and the design process. Lastly, we show the usefulness of the proposed controller by experimental results.

1.　はじめに

高出力・高応答を特徴とする電気油圧サーボシステムにおいて，電気油圧変換のインタフェースとなるサーボ弁には，応答性と高い分解能が求めらる．近年の省資源とランニングコスト低減要求などに対し，サーボ弁スプールの新しい駆動方式の導入が行われている [1,2].

超磁歪材は磁界中で歪む性質を持ち，高推力，高応答，耐環境性に優れ，歪み率は圧電素子より一桁程度大きい．磁石やコイルなどで発生する磁界中へ超磁歪材を配置するだけで歪みを取り出せるので，スプール駆動機構が簡潔になるなどの利点から，超磁歪材駆動によるサーボ弁の開発を行っている．

サーボ弁に求められる特性は目標信号への良好な追従性であり，これまでに代表的動作点に基づく PI 制御や H^∞ 制御の適用を試みたが，満足のいく結果は得られていない．これは超磁歪材の歪み量がコイル電流の大きさに依存し，電流値が小さくなるに従い出力ゲインが低減する性質を持つこと，微小電流領域で不感帯が現われるなどによる．したがって一つの動作点に対する固定・線形な制御器を実装しても全動作領域に対して十分な制御性が達成されず，サーボ弁性能で重要な中立点付近での制御性が従来の駆動方式より劣り問題となっている．このような制御対象に対しても，より高い制御性を達成するために，従来その影響が十分小さいとされてきた非線形特性を考慮した制御器設計手法が求められる．

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Key Words: gain scheduling, servovalve, gaint magnetrostrictive material.

本論文では，ゲインスケジューリングにより非線形性を考慮した制御器設計を行い，この問題の克服を試みる．具体的には操作量の大きさをスケジューリングパラメータとし，これが与えられるごとに制御器パラメータを変更するゲインスケジューリング制御を適用することで小振幅応答の改善を行う．ゲインスケジューリング制御器の構成は，制御対象の LPV モデルに基づく方法 [3–5] を採用する．

本論文の制御問題では状態変数を測定することが困難であるため，出力フィードバック制御の構成が必要である．標準的な出力フィードバックの構成法を与える参考文献 [6] に従えば，スケジューリングパラメータの変化量の測定と測定値を用いた制御器をオンラインで算出する必要があり，本制御問題の性質からこれも困難である．そこで本論文では，スケジューリングパラメータの変化量を必要としない (変化量の最大値は制御器設計の評価に必要) 出力フィードバック制御の構成法を新たに展開し，その構成法を用いた制御器の設計を行う．

最初に制御対象の LPV システムとしてのモデリングを述べ，次にスケジューリングパラメータの変化量を必要としない出力フィードバック制御構成法の概要と設計手順を示す．最後に実機試験結果より本手法の有効性を示す．

2. 直動型サーボ弁概要とモデリング

2.1 サーボ弁構造と特性

Fig. 1 に超磁歪材を有するサーボ弁の構造図を示す．超磁歪材のまわりに，コイルと永久磁石が配置されている．コイル電流を印加することで歪みを発生させ，永久磁石によりバイアスを与えている．小型化のため超磁歪材は円周上に配置されている．同時に曲げモーメントの回避，熱膨張による中立点変動を低減する構造となっている．スプール端には，位置検出用の差動トランス (LVDT) が取り付けられている．

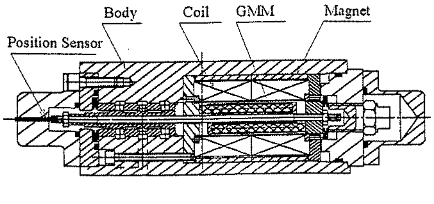

Fig. 1 Servovalve structure with GMM

Fig. 2 に開ループの入出力特性を示した．横軸は電流増幅器への入力電圧，縦軸はスプール位置信号である．出力飽和が顕著に表れない範囲で極力大きな変位が得られる領域を利用し，ここでは入力電圧換算で ±1.25[V]，スプール位置電圧で ±5[V] を定格値としている．図からは，制御対象がヒステリシス特性を持つこと，入力が小さくなるに従い出力ゲインが低減し，ヒステリシスループの内側ではこの傾向が顕著に現れること，入力信号換算でおよそ 1 %以下では不感帯を持つことなどの特性が読み取られる．

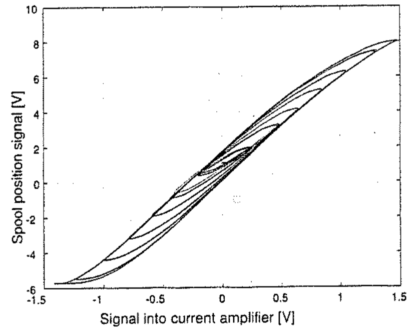

Fig. 2 GMM actuator input-output characteristic

3. システム同定実験と制御対象モデル

このような制御対象を LPV システムとして数式表現するため，幾通りか大きさの異なる入力を加え同定実験を行う．同定実験では，次のような条件を設定する．

- 同定入力は周期 63 ステップの M 系列信号．
- データ採取サンプリング時間は 0.02 [ms]．
- 同定入力信号の大きさは定格の ± 2, 3, 5, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100%.

実験後，一括処理同定アルゴリズムにより制御対象のパラメータを求めるが，次のような前処理を行っている．

- 入出力信号からバイアスやトレンドの影響を取り除くため平均値をデータから差し引く．
- 採取データから 4 ステップごとに抽出したものに ARMAX モデル

$$A(z)y(k) = B(z)u(k) + C(z)w(k)$$

を採用．ここで $u(k)$ は同定入力，$w(k)$ は雑音，$y(k)$ は出力である．

いくつかの試行錯誤の結果，制御対象の離散時間モデル次数は 5，雑音のモデル次数を 2，むだ時間は 1 ステップとする．

連続時間モデルへの変換では，逆応答が小さくなるよう次のような方法を用いる．まず離散時間モデルからむだ時間を除き，周波数応答線図を計算する．注目している周波数領域においてゲイン曲線を近似する 4 次の連続時間伝達関数をカーブフィットにより求める．最後にむだ時間要素を 1 次パーデ近似し，カーブフィットで得た伝達関数へ加えて連続時間モデルとする．ここでは分子式の最高次係数が十分に小さいことを確かめたうえで，この項を取り除き厳密にプロパーなモデルとしている．

このようにして得られた 13 通りのモデルに対して，伝達関数の各係数がスケジューリングパラメータ θ の 3 次式多項式で表せるものと仮定して，近似誤差を最小とする係数を求めると Table 1 に示す値となる．これより制御対象の伝達関数表現は，次式のようになる．

Table 1 Coefficients of polynomial

$b_{s,i}(\theta)$	b_{i0}	b_{i1}	b_{i2}	b_{i3}
$b_{s,4}(\theta)$	0.0332	−2.4263	4.5279	−3.2769
$b_{s,3}(\theta)$	−0.4526	−26.6971	57.1425	−28.2858
$b_{s,2}(\theta)$	6.7079	−16.8666	16.1603	−13.5530
$b_{s,1}(\theta)$	9.1597	142.4606	−266.5099	146.0958
$b_{s,0}(\theta)$	0.4913	12.7101	−27.1006	16.0988

$a_{s,i}(\theta)$	a_{i0}	a_{i1}	a_{i2}	a_{i3}
$a_{s,4}(\theta)$	4.8881	−2.7046	2.7018	−0.8250
$a_{s,3}(\theta)$	13.2543	−16.3460	15.0593	−3.9602
$a_{s,2}(\theta)$	22.8389	−14.3179	−5.3850	11.7323
$a_{s,1}(\theta)$	9.1647	27.3741	−71.8940	44.8869
$a_{s,0}(\theta)$	0.3126	2.1733	−5.5099	3.4476

注) ここでは時間軸を 9425 でスケーリングしている.

$$F_p(s)=\frac{b_{s,4}(\theta)s^4+\ldots+b_{s,1}(\theta)s+b_{s,0}(\theta)}{s^5+a_{s,4}(\theta)s^4+\ldots+a_{s,1}(\theta)s+a_{s,0}(\theta)} \quad (1)$$

ここで,

$$a_{s,i}(\theta)=\sum_{j=0}^{3}a_{ij}\theta^j,\ \ b_{s,i}(\theta)=\sum_{j=0}^{3}b_{ij}\theta^j \quad (i=0,1,2,3,4)$$

Figs. 3, 4 に同定から得た連続時間モデルのパラメータ値と 3 次多項式近似曲線とのばらつきの様子を示す.

スケジューリングパラメータ θ による状態空間表現は (2) 式から

$$\Sigma_p(\theta):\begin{cases}\dot{x}_p(t)=A_p(\theta)x_p(t)+B_pu(t)\\ y_p(t)=C_p(\theta)x_p(t)\end{cases} \quad (2)$$

ここで $A_p(\theta)=\sum_{i=0}^{3}A_{p,i}\theta^i$, $C_p(\theta)=\sum_{i=0}^{3}C_{p,i}\theta^i$ と表され, 上の (2) 式のようにスケジューリングパラメータが固定されると線形システムとなる. なお, スケジュリングパラメータを定格値で規格化し, θ は仮定

1) $\theta \in [0\ \ 1]$, $\forall t \in [0\ \infty)$
2) $\theta \in \mathcal{C}^1$
3) $|\dot{\theta}| \le v_{max}$, $\forall t \in [0\ \infty)$

を満たすものとする. v_{max} については **5.1** で触れる.

ここで前述したモデリングの過程を Figs. 5~8 に示す. Fig. 5 は ±80% ステップ応答で, Figs. 6~8 は ±50%, ±10%, ±2% 応答である. 各図は上から順に

1. 同定実験によるアクチュエータ位置信号
2. 離散系制御対象の応答
3. 連続時間系モデルによる応答
4. θ の 3 次多項式モデルによる応答

となっている. これらモデルの応答計算から, 離散系モデルは実応答をよく再現しており, 連続時間系モデルでは振動的な様子が低減するものの立ち上がりから静定までの全体的な応答は良好であること, 3 次多項で表されるモデルも制御器設計に十分なものであることがわかる.

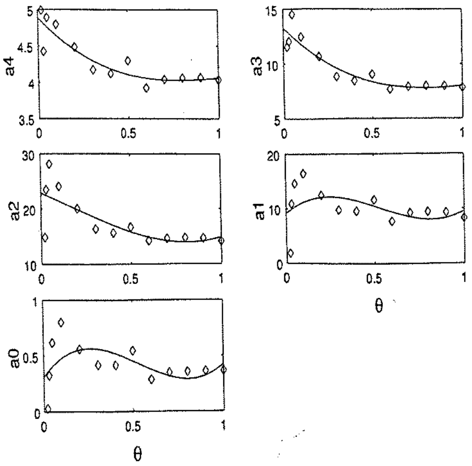

Fig. 3 Coefficients of denominator

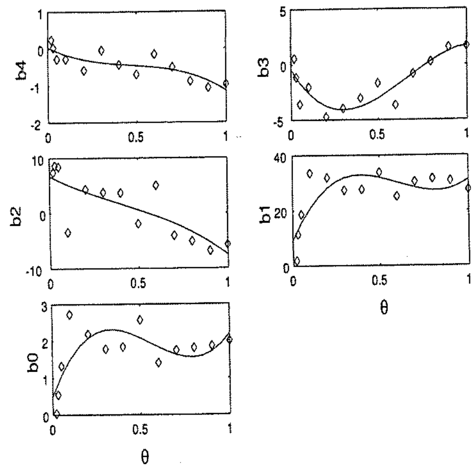

Fig. 4 Coefficients of numerator

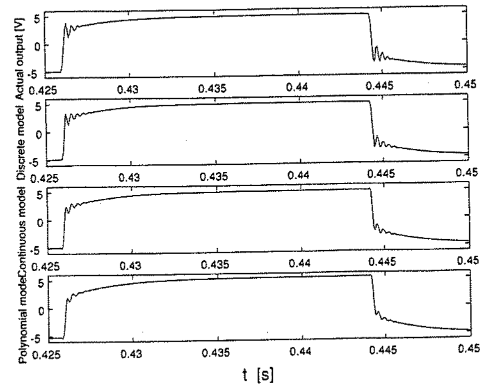

Fig. 5 ±80% Step response

4. ゲインスケジューリング制御器設計

LPV 制御対象へ設計仕様を表す重み関数を加えたスケジューリングパラメータ θ を持つ一般化システムの状

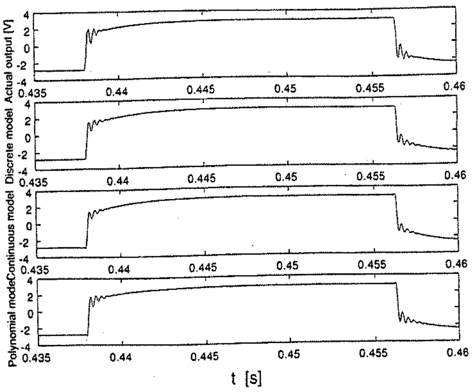

Fig. 6 ±50% Step response

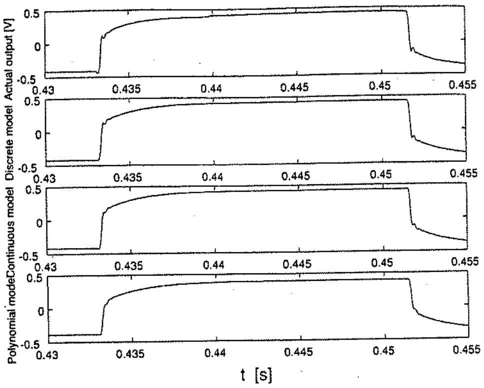

Fig. 7 ±10% Step response

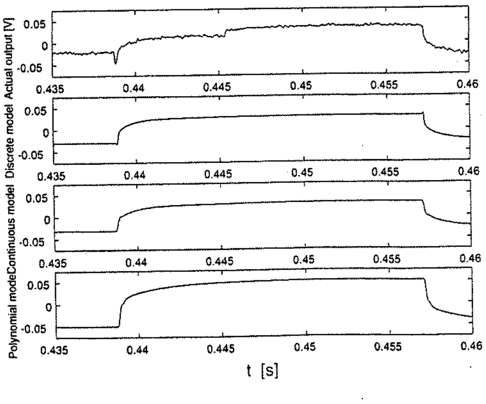

Fig. 8 ±2% Step response

態変数表現を

$$\Sigma_g(\theta): \begin{cases} \dot{x}(t) = A(\theta)x(t) + B_1(\theta)w(t) + B_2(\theta)u(t) \\ z(t) = C_1(\theta)x(t) + D_{11}(\theta)w(t) + D_{12}(\theta)u(t) \\ y(t) = C_2(\theta)x(t) + D_{21}(\theta)w(t) \end{cases} \quad (3)$$

で表す．$x(t) \in \mathcal{R}^n$ は状態，$u(t) \in \mathcal{R}^m$ は操作量，$w(t) \in \mathcal{R}^{m_w}$ は外乱入力，$z(t) \in \mathcal{R}^l$ は被制御量，$y(t) \in \mathcal{R}^{p_y}$ は観測量である．ここで (3) 式の行列の要素は θ の多項式で表される連続な関数とする．

システム $\Sigma_g(\theta)$ が内部安定のとき，このシステムの L_2 ゲイン G を次のように定義する．

$$G = \sup_{w \in L_2, w \neq 0} \frac{\|z\|_2}{\|w\|_2} \quad (4)$$

一般化 LPV システム $\Sigma_g(\theta)$ は，スケジューリングパラメータ θ を固定すると一つの線形システムとなるから，この線形システムを制御するためには θ に依存して決まる線形制御器を考える．制御器の状態変数表現を

$$\Sigma_K: \begin{cases} \dot{x}_K(t) = A_K(\theta)x_K(t) + B_K(\theta)y(t), \ x_K(0) = 0 \\ u(t) = C_K(\theta)x_K(t) + D_K(\theta)y(t) \end{cases} \quad (5)$$

で表す．ゲインスケジューリング制御器設計問題は，パラメータ θ の変化しうるすべての領域で閉ループ系を内部安定とし，L_2 ゲインをその上限 γ 以下とする制御器 Σ_K を求めるものである．なお本論文では $A(\theta)$ と $A_K(\theta)$ の次数が同じフルオーダ制御器を考える．

4.1 制御器の可解条件

一般化 LPV システム (3) 式を考え，仮定 1)，2)，3) が満たされるものとする．この時，制御器設計問題は定数行列 K_B，K_C と連続で正定対称行列 $Y(\theta)$，$X(\theta)$ が存在し，これらが次の LMI 制約式 (6)，(7)，(8) と BMI 制約式 (9) を満足すれば可解という．

$$\mathcal{A}'Y(\theta) + Y(\theta)\mathcal{A} + C_2'(\theta)K_B + (C_2'(\theta)K_B)' - v_{max}\frac{dY(\theta)}{d\theta} + \left(Y(\theta)\mathcal{B} + K_B'D_{21}(\theta) \ \ \mathcal{C}'\right)\triangle_{cl}^{-1} \times \begin{pmatrix} (Y(\theta)\mathcal{B} + K_B'D_{21}(\theta))' \\ \mathcal{C} \end{pmatrix} < 0 \quad (6)$$

$$X(\theta)\mathcal{A}' + \mathcal{A}X(\theta) + B_2(\theta)K_C + (B_2(\theta)K_C)' + v_{max}\frac{dX(\theta)}{d\theta} + \left(\mathcal{B} \ \ (\mathcal{C}X(\theta) + D_{12}(\theta)K_C)'\right)\triangle_{cl}^{-1} \times \begin{pmatrix} \mathcal{B}' \\ \mathcal{C}X(\theta) + D_{12}(\theta)K_C \end{pmatrix} < 0 \quad (7)$$

$$\begin{pmatrix} Y(\theta) & I \\ I & X(\theta) \end{pmatrix} > 0 \quad (8)$$

$$\begin{pmatrix} \frac{dY(\theta)}{d\theta} & -X^{-1}(\theta)\frac{dX(\theta)}{d\theta} \\ -\frac{dX(\theta)}{d\theta}X^{-1}(\theta) & -\frac{dX(\theta)}{d\theta} \end{pmatrix} \leq 0 \quad (9)$$

ここで，

$$\mathcal{A} := A(\theta) + B_2(\theta)D_K(\theta)C_2(\theta) \quad (10)$$

$$\mathcal{B} := B_1(\theta) + B_2(\theta)D_K(\theta)D_{21}(\theta) \quad (11)$$

$$\mathcal{C} := C_1(\theta) + D_{12}(\theta)D_K(\theta)C_2(\theta) \quad (12)$$

$$\triangle_{cl} := \begin{pmatrix} \gamma I & -D'_{cl}(\theta) \\ -D_{cl}(\theta) & \gamma I \end{pmatrix} \tag{13}$$
$$D_{cl}(\theta) = D_{11}(\theta) + D_{12}(\theta)D_K(\theta)D_{21}(\theta)$$

また D_K は $\triangle_{cl} > 0$ を満たすように選ぶものとする．

4.2　制御器の算出

前述の可解条件を満たす $Y(\theta)$，$X(\theta)$，K_B，K_C が与えられれば，一般化 LPV システム (3) 式を内部安定化し，L_2 ゲインを γ 以下にする出力フィードバック制御器の一つは以下に示す式から算出される．B_K，C_K は

$$B_K = Z(\theta)^{-1}K'_B \tag{14}$$
$$C_K = -K_C X(\theta)^{-1} \tag{15}$$

で与えられる．ここで，$Z(\theta) = Y(\theta) - X(\theta)^{-1}$ である．このとき A_K は

$$\begin{aligned} A_K = Z^{-1}(\theta)\Bigg[&\mathcal{A}' + Y(\theta)\mathcal{A}X(\theta) + K'_B C_2(\theta)X(\theta) \\ &+ Y(\theta)B_2(\theta)K_C + v_{max}X^{-1}(\theta)\frac{dX(\theta)}{d\theta} \\ &+ \Big(Y(\theta)\mathcal{B} + K'_B D_{21}(\theta)\ \ \mathcal{C}'\Big)\triangle_{cl}^{-1} \\ &\times \begin{pmatrix} \mathcal{B}' \\ \mathcal{C}X + D_{12}(\theta)K_C \end{pmatrix}\Bigg] X^{-1}(\theta) \end{aligned} \tag{16}$$

で与えられる．出力フィードバック制御器の構成法としては参考文献 [6] の方法が知られているが $\dot{\theta}$ の値が必要である．本構成法の特徴は $\dot{\theta}$ の情報を必要とせずに，その上限 v_{max} を用いて制御器を構成する点にある (導出の詳細は参考文献 [7] 参照)．

しかしながら本構成法では，BMI 制約式 (9) が新たに加わる．この制約式は簡単に解くことができない．そこで，実際の構成では制約式 (9) をゆるめて，次の制約を用いることにする．いま $dX(\theta)/d\theta > 0$ を仮定すれば，(9) 式から $dY(\theta)/d\theta < 0$ がいえる．これら二つの制約式と他の LMI 制約式 (6)，(7)，(8) ともに解の候補を求めた後，それらが BMI 制約式 (9) を満たすかを調べることにする．

5.　制御器設計と実機試験

5.1　一般化 LPV システムと制御器

Fig. 9 に一般化 LPV システム $\Sigma_g(\theta)$ の構成を示す．制御対象 $\Sigma_p(\theta)$ には，入力側に定常偏差を低減する目的で積分器を挿入している．$W_s(s)$ は閉ループ系のバンド幅が 400Hz 以上となるように，$W_t(s)$ は同定モデルとスケジューリングパラメータ依存の 3 次多項式モデルの誤差を考慮するとともに，閉ループ系ステップ応答のオーバシュートが極力小さくなるように選んだ重み関数で，ここでは

$$W_s(s) = \frac{s+1.5}{2(s+0.001)},\quad W_t(s) = \frac{2s}{s+2}$$

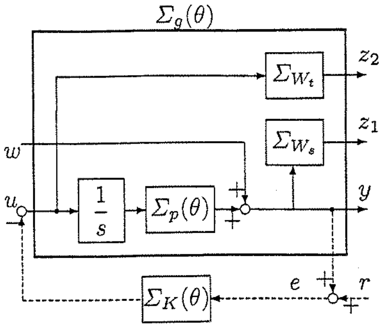

Fig. 9　Generalized LPV system

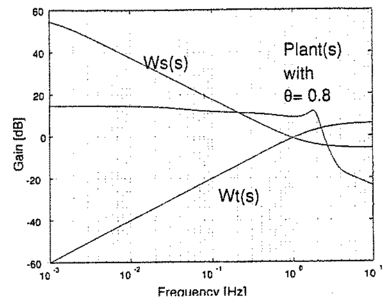

Fig. 10　Bode plots of weighting function

のように決める．また，これらの状態変数表示を

$$\Sigma_{W_s}: \begin{cases} \dot{x}_s(t) = A_s x_s(t) + B_s u_s(t) \\ y_s(t) = C_s x_s(t) + D_s u_s(t) \end{cases}$$

$$\Sigma_{W_t}: \begin{cases} \dot{x}_t(t) = A_t x_t(t) + B_t u_t(t) \\ y_t(t) = C_t x_t(t) + D_t u_t(t) \end{cases}$$

とする．Fig. 10 に $\theta = 0.8$ 時の制御対象と周波数重みのゲイン曲線を示す．また (3) 式に対応する Fig. 9 の一般化 LPV システムの状態変数表示は

$$\Sigma_g(\theta):$$
$$\begin{cases} \begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_i(t) \\ \dot{x}_s(t) \\ \dot{x}_t(t) \\ z_1(t) \\ z_2(t) \\ y(t) \end{bmatrix} = \left(\begin{array}{cccc|c|c} A_p(\theta) & 0 & 0 & 0 & 0 & -B_p \\ C_p(\theta) & 0 & 0 & 0 & 0 & -D_p \\ 0 & B_s & A_s & 0 & B_s & 0 \\ 0 & 0 & 0 & A_t & 0 & -B_t \\ \hline 0 & D_s & C_s & 0 & D_s & 0 \\ 0 & 0 & 0 & C_t & 0 & -D_t \\ \hline 0 & I & 0 & 0 & I & 0 \end{array}\right) \begin{bmatrix} x_p(t) \\ x_i(t) \\ x_s(t) \\ x_t(t) \\ w(t) \\ u(t) \end{bmatrix} \end{cases}$$
$$= \left(\begin{array}{c|c|c} \mathcal{A}(\theta) & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right) \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} \tag{17}$$

のようになる．ただし積分器は

$$\Sigma_{W_i}: \begin{cases} \dot{x}_i(t) = 0 \cdot x_i(t) + u_i(t) \\ y_i(t) = x_i(t) + 0 \cdot u_i(t) \end{cases}$$

とした．

制御器算出では，制御器の保守性を低減する目的で $X(\theta)$ と $Y(\theta)$ も θ に依存する3次多項式とする．v_{max} の値は，つぎのようにして決める．製品への実装時には，高速性の観点から制御器をアナログ素子により構成することを想定し，その操作信号範囲から v_{max} の候補として15を選ぶ．これを基に1.5，15と150の3通りの候補に対する制御器を求めた後，v_{max} が指定した値内に留まるかを調べる．v_{max} が1.5の場合は，条件が満たされない．ここでは，より安定な範囲を得るとの考えから150を採用する．この時 γ の値は6.89を得る．

先に述べたLMI制約は，連続なスケジューリングパラメータ θ に依存しており，解を得るには無限個のLMIを解く必要がある．この問題に対し有限個のLMI制約に帰着させる方法が提案されており，ここでは参考文献[8]による方法を用いている．

このような準備の基で前述のLMIを解き，考慮されなかった制約式(9)を解が満足することを固有値計算から確認し，その後に**4.2**で述べた手順で制御器を算出する．可解条件式(6)，(7)，(8)と(9)を満たす解を用い，θ による制御器の違いを比較するため $\theta = 0.01$, 0.5, 1.0 に対する出力フィードバック制御器を算出し，それらのゲインと位相特性をFigs. 11, 12に示す．ただし，制御器の特徴を示すため積分要素は除いている．

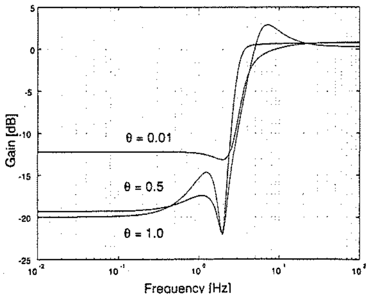

Fig. 11　Controller gain curve

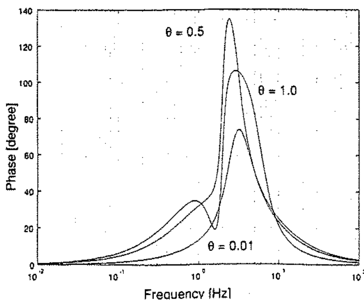

Fig. 12　Controller phase curve

5.2　実装と試験結果

実機における試験条件を示す．

- サンプリング周波数50MHzのDSPと変換速度10μsのAD変換器を持つコントローラを使用．
- 演算時間の問題から，スケジュリングパラメータによりオンラインで制御器のパラメータを算出できないのでTable 2に示すように9通りの制御器を準備し，これらをスケジュリングパラメータで切り換える．

Table 2　Division of θ

range of θ	corresponding θ
0 ≤ 0.015	0.01
0.015 ≤ 0.025	0.02
0.025 ≤ 0.035	0.03
0.035 ≤ 0.055	0.05
0.055 ≤ 0.15	0.1
0.15 ≤ 0.25	0.2
0.25 ≤ 0.45	0.4
0.45 ≤ 0.65	0.6
0.65 ≤ 1.0	0.8

5.2.1　直線性と微小目標信号応答

閉ループ系によって，入出力特性が直線化されている様子をFig. 13に示す．Fig. 14は $0.2\%_{p-p}$ の微小サイン波信号に対する応答である．本制御器により閉ループ系が線形化され，高い分解能を持つことが確認される．

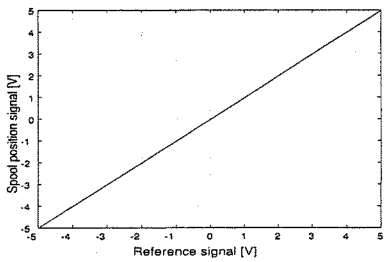

Fig. 13　Closed loop input-output characteristic

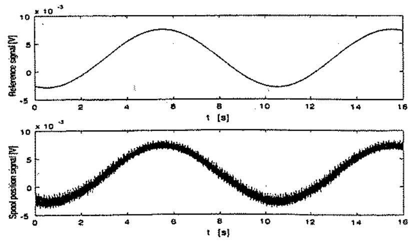

Fig. 14　Sinusoidal response

5.3　ステップ応答と周波数応答

Fig. 15にステップ応答，Fig. 16に周波数応答を示す．いずれも従来の代表的な一つの動作点に対し設計された固定・線形な制御器を用いた時に比べ，5%以下の小振幅ステップ応答の立上がり時間が半分に短縮され，周波数応答ではバンド幅が1.7倍程度改善されている．

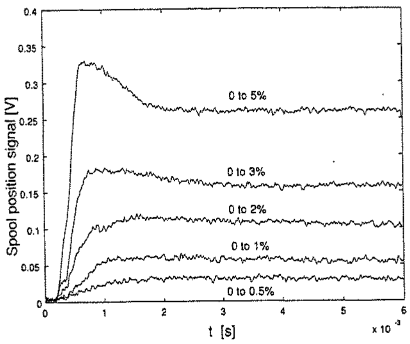

Fig. 15　Closed loop step response

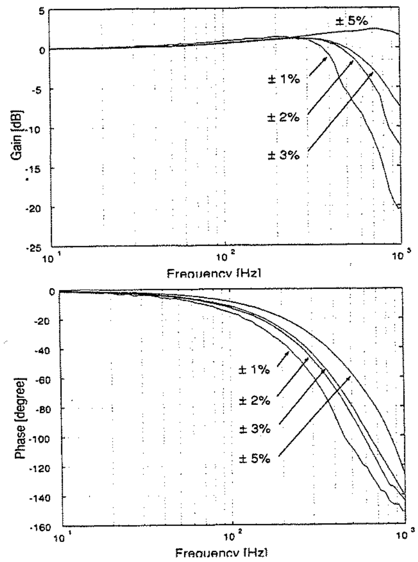

Fig. 16　Closed lood bode plot

6.　おわりに

サーボ弁の重要な特性の一つである小振幅応答の追従特性改善に関し，本手法の有効性が実機による微小信号応答，ステップ応答，周波数応答試験結果から確認できた．また，制御器の実装にあたり問題となるスケジューリングパラメータの変化量情報を必要としない，出力フィードバック制御器の設計法を新たに提案した．なお有限個のLMI条件へ帰着させる過程で生じる制御器の保守性に対する問題とワインドアップ現象問題が今後の課題である．

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