マクロ系及びメソスコピック系の場の量子論

課題番号 08044097

1996 年度～1997 年度科学研究費補助金（国際学術研究）研究成果報告書

1998 年 4 月

研究代表者 中　里　弘　道
（早稲田大学理工学部）

はじめに

本研究課題はマクロ系やメソスコピック系に対する量子論的理解をその最終目的とし，量子力学的観測問題及び関連する諸問題に取り組む中で，micro-macro transition の実現とその物理的，数学的基礎付けを試みてきた．その結果，いくつかの分野では新しい知見を得ることができた．その具体的成果としては，

1. 量子力学的観測過程に対する可解な力学的模型 (modified Coleman-Hepp 模型) を提案し，その弱結合・マクロ極限で出現する確率過程（Wiener 過程）を詳しく調べた．特に，Wiener 過程をもたらす演算子が測定器系の自由 Hamiltonian (Heisenberg 演算子) で与えられること，またその起源は，純粋に系の量子力学的性格に求められことが次第に明らかになってきた．
2. 量子系の時間発展に関して改めて整理し直し，その振る舞い（時間スケールの短い方から順に，ガウス型，指数関数型，べき関数型）と複素エネルギー平面での解析性について一般的な関係を導出した．さらに，簡単な量子系に対しては波束の時間発展を直接追跡し，またより現実的な水素原子の準位間遷移に関しては厳密解を導出した．特に後者では，時間発展初期の特徴的時間スケールを評価するとともに，生存確率に生き残る振動項の存在を指摘した．
3. 量子系特有の時間発展が関連する現象として，またいわゆる量子力学的観測問題との関連で近年注目を集めている量子ゼノン効果を取り上げ，詳細に検討した．特にスペクトル分解過程の果たす役割の重要さを指摘し，その検証実験のひとつとして中性子スピンを利用する実験を提案した．またその実現に向けて，磁場による中性子の反射等，実験でのロスを考慮した検討を行った．
4. いわゆるトンネル時間の問題が取り扱いにくいのは，量子論における時間変数が単なるパラメータであるからである．今回，量子論を確率過程によって再現しようという Nelson の量子力学を用いることによって，トンネル領域を通過する量子力学的粒子の数値シミュレーションを実行した．その結果，いわゆるトンネル時間には，障壁の前で停滞する hesitating time と障壁通過に要する passing time という 2 種類の時間のあることが分かった．さらに，量子系を取り巻く環境からの散逸の効果を考慮した，より現実的なトンネル時間の評価を試みている．
5. 量子力学的観測過程とコヒーレンス消失機構に関して，現時点での総まとめを行い，“Decoherence and Quantum Measurements,” by M. Namiki, S. Pascazio and H. Nakazato (World Sci. Pub., Singapore, 1998) として出版した．

以上の研究成果の多くは既に論文として学術誌に公表されており，代表的なものを掲載することで本研究課題の成果報告書に代える．

本研究課題が目指してきた内容は，量子力学の基礎に関わるという意味において極めて幅広い内容を有しており，今後も引き続き研究を続ける予定である．本研究課題を実行した 2 年間は言うに及ばず，イタリアバリ大学グループとの共同研究は極めて実りの多いものであった．引き続き 1998 年度からも同グループとの共同研究が予定されている．

最後に，共同研究遂行に当たって様々な局面で御支援，御協力いただいた関係諸機関 (早稲田大学理工学部及び教務部，日本学術振興会，Bari University, National Research Council of Italy (CNR), Italian National Institute for Nuclear Physics (INFN)) 及び関係各位に深く感謝いたします.

1998 年 4 月

研究代表者 中　里　弘　道
（早稲田大学理工学部）

研究組織

研究代表者：中里弘道	（早稲田大学・理工学部・助教授）
研究分担者：大場一郎	（早稲田大学・理工学部・教授）
研究分担者：Pascazio, Saverio	（バリ大学・物理・講師）
研究分担者：Cea, Paolo	（バリ大学・物理・講師）
研究分担者 ：Cosmai, Leonardo（1996 年度）	（イタリア国立原子核物理学研究機構・バリ研究所・専任研究員）
研究分担者 ：Rauch, Helmut（1996 年度）	（オーストリア国立原子炉研究所・教授）
研究協力者：並木美喜雄	（早稲田大学・名誉教授）
研究協力者：町田顕	（早稲田大学大学院・理工学研究科・博士後期課程）
研究協力者：Facchi, Paolo	（バリ大学大学院・博士後期課程）

研究経費

1996 年度	1,100 千円
1997 年度	1,600 千円
計	2,700 千円

研究発表

（1）学会誌等

- H. Nakazato, M. Namiki and S. Pascazio, Temporal behavior of quantum mechanical systems, Int. J. Mod. Phys. B **10** No. 3 (1996) 247–295.
- H. Nakazato, M. Namiki, S. Pascazio and H. Rauch, Understanding the quantum Zeno effect, Phys. Lett. A **217** No. 4,5 (1996) 203–208.
- H. Nakazato, M. Namiki, S. Pascazio and Y. Yamanaka, Quantum dephasing by chaos, Phys. Lett. A **222** No. 3 (1996) 130–136.
- K. Machida, H. Nakazato, M. Namiki and S. Pascazio, Effect of fluctuation and dissipation on neutron scattering, J. Phys. Soc. Jpn. **65** Suppl. A (1996) 33–36.

- S. Pascazio, H. Nakazato and M. Namiki, Quantum Zeno effect, J. Phys. Soc. Jpn. **65** Suppl. A (1996) 285–288.
- R. Blasi, H. Nakazato, M. Namiki and S. Pascazio, Dissipative behavior of a quantum system interacting with a macroscopic medium, Phys. Lett. A **223** No. 5 (1996) 320–326.
- K. Watanuki and I. Ohba, Seesaw-type quark mass matrix and neutron electric dipole moment, Mod. Phys. Lett. A **11** No. 22 (1996) 1835–1842.
- I. Ohba, K. Imafuku and Y. Yamanaka, Estimation of tunneling time based on the quantum diffusion process approach and neutron scattering, J. Phys. Soc. Jpn. **65** Suppl. A (1996) 41–44.
- O.I. Zavialov, M. Kanenaga, A.I. Kirillov, V.Y. Mamakin, M. Namiki, I. Ohba and E.V. Polyachenko, On quantization of systems with actions unbounded from below, Theor. Math. Phys. **109** No. 2 (1996) 1379–1387.
- R. Blasi and S. Pascazio, Short-time behavior of the correlation functions for the quantum Langevin equation, Phys. Rev. A **53** No. 4 (1996) 2033–2037.
- N. Kono, K. Machida, M. Namiki and S. Pascazio, Decoherence and dephasing in a quantum measurement process, Phys. Rev. A **54** No. 2 (1996) 1064–1086.
- R. Blasi, H. Nakazato, M. Namiki and S. Pascazio, Emergence of a Wiener process as a result of the quantum mechanical interaction with a macroscopic medium, Physica A **245** No. 1,2 (1997) 189–211.
- H. Nakazato and S. Pascazio, A solvable model for quantum mechanical dissipation, in *New Perspectives in the Physics of Mesoscopic Systems* (S. De Maritno *et al.* eds. World Sci. Pub.) (1997) 216–224.
- H. Nakazato, Time development of a wave packet and the time delay, Found. Phys. **27** No. 12 (1997) 1709–1723.
- K. Imafuku, I. Ohba and Y. Yamanaka, Effects of inelastic scattering on tunneling time based on the generalized diffusion prosess approach, Phys. Rev. A **56** No. 2 (1997) 1142–1153.
- I. Ohba, A novel method to quantize systems of damped motion, Found. Phys. **27** No. 12 (1997) 1725–1738.
- F. Mihokova, S. Pascazio and L.S. Schulman, Hindered decay: quantum Zeno effect through electromagnetic field domination, Phys. Rev. A **56** No. 1 (1997) 25–32.
- S. Pascazio, Dynamical origin of the quantum Zeno effect, Found. Phys. **27** No. 12 (1997) 1655–1670.
- P. Cea and L. Tedesco, Finite-temperature generalized Gaussian effective potential, J. Phys. G **23** No. 2 (1997) 135–141.
- P. Cea, L. Cosmai and A.D. Polosa, The lattice Schroedinger functional and the background field effective action, Phys. Lett. B **392** No. 1,2 (1997) 177–181.
- P. Cea and L. Tedesco, Perturbation theory with a variational basis: the generalized

Gaussian effective potential, Phys. Rev. D **55** No. 8 (1997) 4967–4989.

- P. Cea, L. Cosmai and A.D. Polosa, Finite size analysis of the U(1) background field effective action, Phys. Lett. B **397** No. 3,4 (1997) 229–233.
- A. Agodi, G. Andronico, P. Cea, M. Consoli, L. Cosmai, R. Fiore and P.M. Stevenson, A lattice test of alternative interpretations of "triviality" in $(\lambda\phi^4)_4$ theory, Mod. Phys. Lett. A **12** No. 14 (1997) 1011–1024.
- P. Cea and L. Tedesco, Generalized Gaussian effective potential: second-order thermal corrections, Mod. Phys. Lett. A **12** No. 15 (1997) 1077–1085.
- P. Cea, Vacuum stability for Dirac fermions in three dimensions, Phys. Rev. D **55** No. 12 (1997) 7985–7988.
- Z. Hradil, H. Nakazato, M. Namiki, S. Pascazio and H. Rauch, Infinitely frequent measurements and quantum Zeno effect, to appear in Phys. Lett. A.
- M. Namiki, H. Nakazato and S. Pascazio, Time symmetry and quantum dephasing, to appear in *Proc. Int. Symp. "Symmetries in Science X".*
- H. Nakazato, Time development of a wave packet in potential scattering, to appear in *Proc. Int. Symp. "Symmetries in Science X".*
- S. Pascazio, H. Nakazato and M. Namiki, Temporal behavior of quantum systems and quantum Zeno effect, to appear in *Proc. Int. Symp. "Symmetries in Science X".*

(2) 口頭発表

- H. Nakazato, K. Machida, M. Namiki and S. Pascazio (Poster presentation), Effect of fluctuation and dissipation on neutron scattering, *Int. Symp. on Advances in Neutron Optics and Related Research Facilities* (Osaka), 1996.03.20.
- S. Pascazio, H. Nakazato and M. Namiki, Quantum Zeno Effect, *Int. Symp. on Advances in Neutron Optics and Related Research Facilities* (Osaka), 1996.03.21.
- 町田顕・中里弘道・並木美喜雄，観測過程の数値シミュレーション，日本物理学会第 51 回年会（金沢大学），1996.03.31.
- 中里弘道・堀田浩崇・金長正彦・並木美喜雄，確率過程量子化法にもとづく位相演算子の取り扱い，日本物理学会第 51 回年会（金沢大学），1996.04.01.
- 今福健太郎・大場一郎・山中由也，Nelson の量子力学によるトンネル時間の考察—非弾性散乱の影響—，日本物理学会第 51 回年会（金沢大学），1996.04.01
- 町田顕・中里弘道・並木美喜雄，中性子を用いた量子ゼノン効果，日本物理学会秋の分科会（山口大学），1996.10.02.
- 今福健太郎・大場一郎・山中由也，非線形 Schrödinger 方程式と Nelson 流確率過程，日本物理学会秋の分科会（山口大学），1996.10.04.
- 佐々木貴・中里弘道・並木美喜雄，カオス的少数自由度系におけるデフェイジング II，日本物理学会秋の分科会（佐賀大学），1996.10.09.

- 湯浅一哉・中里弘道，確率過程量子化法による遷移確率振幅の計算，日本物理学会秋の分科会（佐賀大学），1996.10.09.
- 町田顕・並木美喜雄・中里弘道・Saverio Pascazio，中性子を用いた量子ゼノン効果 II，日本物理学会第 52 回年会（名城大学），1997.03.30.
- 楠本将博・今福健太郎・大場一郎，Nelson 流量子力学による中性子のトンネル時間及び量子井戸中の振る舞いの解析，日本物理学会第 52 回年会（名城大学），1997.03.30.
- 荒添直棋・大場一郎・中里弘道，量子力学的コヒーレンスの熱浴との結合による影響，日本物理学会第 52 回年会（名城大学），1997.03.31.
- 中里弘道，ポテンシャル散乱における波束の時間発展と散乱による波束のずれ，日本物理学会第 52 回年会（名城大学），1997.03.31.
- 佐々木貴・中里弘道・並木美喜雄，カオス的少数自由度系におけるデフェイジング III，日本物理学会第 52 回年会（名城大学），1997.03.31.
- 湯浅一哉・中里弘道，確率過程量子化法によるクーロン系の遷移確率振幅の計算，日本物理学会第 52 回年会（名城大学），1997.03.31.
- 大場一郎，散逸系の “正準”量子化と “経路積分”量子化，日本物理学会第 52 回年会（名城大学），1997.03.31.
- 今福健太郎・大場一郎，Nelson 流量子力学の散逸系への拡張，日本物理学会第 52 回年会（名城大学），1997.03.31.
- M. Namiki, H. Nakazato and S. Pascazio, Time symmetry and quantum dephasing, *International symposium “Symmetries in Science X”* (Bregenz, Austria), 1997.07.15
- H. Nakazato, Time development of a wave packet in potential scattering, *International symposium “Symmetries in Science X”* (Bregenz, Austria), 1997.07.15.
- S. Pascazio, H. Nakazato and M. Namiki, Temporal behavior of quantum systems and quantum Zeno effect, *International symposium “Symmetries in Science X”* (Bregenz, Austria), 1997.07.15.
- 湯浅一哉・中里弘道，確率過程量子化法によるクーロン・ポテンシャルに対する遷移確率振幅の計算，日本物理学会秋の分科会（東京都立大学），1997.09.23.
- 今福健太郎・大場一郎・並木美喜雄，超光速トンネリングは存在するか？，日本物理学会秋の分科会（東京都立大学），1997.09.23.
- R. Blasi・並木美喜雄・中里弘道・S. Pascazio，巨視的媒質との相互作用を取り込んだ力学模型における Wiener 過程の出現機構，日本物理学会秋の分科会（神戸大学），1997.10.05.
- 小山哉・荒添直棋・今福健太郎・町田顕・大場一郎・中里弘道，量子力学的位相相関に及ぼす $1/f$ 雑音の影響，日本物理学会秋の分科会（神戸大学），1997.10.05.
- 原光一郎・今福健太郎・楠本将博・大場一郎，散逸を考慮した系におけるトンネル時間，日本物理学会秋の分科会（神戸大学），1997.10.05.
- 楠本将博・今福健太郎・大場一郎，時間振動型ポテンシャルにおけるネルソン流とビュテッカー・ランダウア流トンネル時間の比較，日本物理学会秋の分科会（神戸大学），1997.10.05.

- 井門孝治・今福健太郎・大場一郎，三剛体柱カオスにおける量子軌跡と領域滞在時間 II，日本物理学会第 53 回年会（東邦大学・日本大学），1998.03.31.
- 今福健太郎・中里弘道，トンネル現象におけるタイムスケールについて，日本物理学会第 53 回年会（東邦大学・日本大学），1998.03.31.
- 宮本明啓・大場一郎，Nelson の量子力学による同種粒子散乱における統計性と量子軌跡，日本物理学会第 53 回年会（東邦大学・日本大学），1998.04.02.
- 楠本将博・原光一郎・今福健太郎・大場一郎，時間振動型ポテンシャルにおける Nelson 流と B-L 流，Takagi 流トンネル時間の比較，日本物理学会第 53 回年会（東邦大学・日本大学），1998.04.02.

（3）出版物

- M. Namiki, I. Ohba, K. Maeda and Y. Aizawa, Eds., *Quantum Physics, Chaos Theory, and Cosmology*, (American Institute of Physics, 1996).
- 並木美喜雄・大場一郎,「散乱理論」，(岩波書店，1997).
- M. Namiki, S. Pascazio and H. Nakazato, *Decoherence and Quantum Measurements*, (World Scientific Pub., 1998).

ELSEVIER

15 July 1996

PHYSICS LETTERS A

Physics Letters A 217 (1996) 203–208

Understanding the quantum Zeno effect

Hiromichi Nakazato [a], Mikio Namiki [a], Saverio Pascazio [b], Helmut Rauch [c]

[a] *Department of Physics, Waseda University, Tokyo 169, Japan*

[b] *Dipartimento di Fisica, Università di Bari, and Istituto Nazionale di Fisica Nucleare, Sezione di Bari, I-70126 Bari, Italy*

[c] *Atominstitut der Österreichischen Universitäten, A-1020 Wien, Austria*

Received 3 April 1996; accepted for publication 25 April 1996
Communicated by P.R. Holland

Abstract

The quantum Zeno effect consists in the hindrance of the evolution of a quantum system that is very frequently monitored and *found* to be in its initial state at *every* single measurement. On the basis of the correct formula for the survival probability, i.e. the probability of finding the system in its initial state at *every* single measurement, we critically analyze a recent proposal and experimental test that make use of an oscillating system.

PACS: 03.65.Bz; 42.50.–p

The seminal formulation of the quantum Zeno effect, due to Misra and Sudarshan [1], deals with the probability of observing an unstable system in its initial state *throughout* a time interval $\Delta = [0, t]$. The purpose of this note is to point out that the quantum Zeno effect has not been experimentally observed, yet, in its original formulation. Indeed, we shall argue that the interesting proposal by Cook [2], that makes use of a two-level system undergoing Rabi oscillations, as well as the beautiful experiment performed by Itano et al. [3], investigate the probability of finding the initial state *at* time t, *regardless* of the actual state of the system in the time interval Δ. As we shall see, in general, if the temporal behavior of the system is oscillatory, this probability includes the possibility that transitions of the type: initial state → other states → initial state, actually take place. Of course, this remark does not invalidate the soundness of the analysis in Ref. [2] and of the experiment in Ref. [3].

The temporal behavior of quantum mechanical systems is a long-standing issue of investigation [4] (for a review and a collection of recent developments, see Ref. [5]), and the curious features of the short-time behavior of the so-called "survival" probability of a quantum mechanical state, leading to what was to be named "quantum Zeno paradox" [1], were already known about 30 years ago [6]. However, renewed interest in the above topic was motivated by Cook's idea [2] and its subsequent experimental verification [3]. The experiment by Itano et al. provoked a lively debate [7–9], that has essentially focused on two aspects of the problem. First, it has been shown, and it is now becoming a widespread viewpoint, that the experimental results can be explained by making use of a unitary dynamics [10,8]. Notice that an analogous point was raised by Peres quite a few years ago [11]. Second, it has been argued that the so-called limit of continuous observation is in contradiction with Heisenberg's uncertainty principle and does not take into account unavoidable quantum mechanical losses, and is there-

PII S0375-9601(96)00350-7

fore to be considered unphysical [12].

Nowadays most physicists tend to view this phenomenon as a purely dynamical process, in which von Neumann's projections can be substituted by spectral decompositions [13,14], so that the phase correlation among different branch waves is perfectly kept. For this reason, one often speaks of quantum Zeno *effect* (QZE) [7,8], rather than quantum Zeno *paradox* [1].

However, surprisingly, nobody seems to have realized that, strictly speaking, Cook's proposal and Itano et al.'s experiment are conceptually at variance with the original formulation of the QZE. Misra and Sudarshan, in their seminal paper [1], endeavoured to define "the probability $\mathcal{P}(0,T;\rho_0)$ that no decay is found *throughout the interval* $\Delta = [0,T]$ when the initial state of the system was known to be ρ_0". (Italics in the original. Some symbols have been changed.) The definition given in Ref. [1] is

$$\mathcal{P}(0,T;\rho_0) \equiv \lim_{N\to\infty} P^{(N)}(0,T;\rho_0), \tag{1}$$

where $P^{(N)}(0,T;\rho_0)$ is the probability of observing the initial state ρ_0 in a series of N observations, performed at times $t_n = nT/N$ $(n = 1,\ldots,N)$, in order to ascertain whether the system is still undecayed.

In order to facilitate comprehension of the following analysis, it is worth stressing that the above-mentioned "survival probability" of the initial state ρ_0 is the probability of finding the system under investigation in ρ_0 at *every* measurement, during the interval Δ. This is a subtle point, as we shall see.

For the sake of clarity, we shall first carefully analyze Itano et al.'s derivation of what they interpreted as a realization of the QZE, and then scrutinize Cook's formulae. Consider a three-level atomic system, on which an rf field of frequency ω provokes Rabi oscillations between levels 1 and 2. In the rotating wave approximation and in absence of detuning, the equations of motion for the density matrix ρ_{ij} $(i,j = 1,2)$ read

$$\dot\rho_{11} = \tfrac{1}{2}i\omega(\rho_{21} - \rho_{12}), \quad \dot\rho_{12} = \tfrac{1}{2}i\omega(\rho_{22} - \rho_{11}),$$
$$\dot\rho_{22} = \tfrac{1}{2}i\omega(\rho_{12} - \rho_{21}), \tag{2}$$

where the dot denotes derivative with respect to time.

By applying a technique invented by Feynman, Vernon and Hellwarth [15], one can recast the above equations of motion in a very simple form, in which the use of rotating coordinates, introduced by Block [16] and Rabi, Ramsey and Schwinger [17], turns out to be particularly advantageous. Define

$$R_1 \equiv \rho_{21} + \rho_{12}, \quad R_2 \equiv i(\rho_{12} - \rho_{21}),$$
$$R_3 \equiv \rho_{22} - \rho_{11} \equiv P_2 - P_1, \tag{3}$$

where $P_j \equiv \rho_{jj}$ is the probability that the atom is in level j $(j = 1,2)$. Since $P_1 + P_2 = 1$, one gets

$$P_2 = \tfrac{1}{2}(1 + R_3). \tag{4}$$

In terms of the quantities $\boldsymbol{R} \equiv (R_1, R_2, R_3)$ and $\boldsymbol{\omega} \equiv (\omega, 0, 0)$, Eqs. (2) become

$$\dot{\boldsymbol{R}} = \boldsymbol{\omega} \times \boldsymbol{R}. \tag{5}$$

The solution of the above equation, with initial condition $\boldsymbol{R}(0) \equiv (0,0,-1)$ (only level 1 is initially populated) reads

$$\boldsymbol{R}(t) = (0, \sin\omega t, -\cos\omega t). \tag{6}$$

If the transition between the two levels is driven by an on-resonant π pulse, of duration $T = \pi/\omega$, one gets $\boldsymbol{R}(T) \equiv (0,0,1)$, so that $\rho_{22} = 1, \rho_{11} = 0$, and only level 2 is populated at time T.

The reasoning of Ref. [3] is the following. Assume you perform a measurement at time $\tau = \pi/N\omega = T/N$, by shining on the system a very short "measurement" pulse, that provokes transitions from level 1 to level 3, with subsequent spontaneous emission of a photon[1]. The measurement pulse "projects" the atom into level 1 or 2 ("naive wave function collapse"). Because a measurement "kills" the off-diagonal terms ρ_{12} and ρ_{21} of the density matrix, while leaving unaltered its diagonal terms ρ_{11} and ρ_{22}, one obtains

$$\boldsymbol{R}(\pi/N\omega) = [0, \sin(\pi/N), -\cos(\pi/N)]$$
$$\overset{\text{measurement}}{\longrightarrow} [0, 0, -\cos(\pi/N)] \equiv \boldsymbol{R}^{(1)}. \tag{7}$$

Then the evolution restarts, according to Eq. (5), but with the new initial condition $\boldsymbol{R}^{(1)}$. After N measurements, at time $T = N\tau = \pi/\omega$,

$$\boldsymbol{R}(T) = [0, 0, -\cos^N(\pi/N)] \equiv \boldsymbol{R}^{(N)}. \tag{8}$$

[1] We are not addressing the (delicate) point that a measurement pulse, however short, must have a certain *finite* time duration. As a consequence, one must take into account the inevitable spread $\Delta\omega$ of the measurement pulse, and modify accordingly the following formulae. This problem is a very subtle one and will be properly addressed in a forthcoming paper [18].

The probabilities that the atom is in level 2 or 1 at time T, after the N measurements, are therefore given by (see Eq. (4))

$$P_2^{(N)}(T) = \tfrac{1}{2}(1 + R_3^{(N)}) = \tfrac{1}{2}[1 - \cos^N(\pi/N)], \tag{9}$$

$$P_1^{(N)}(T) = 1 - P_2^{(N)}(T) = \tfrac{1}{2}[1 + \cos^N(\pi/N)], \tag{10}$$

respectively. Since $P_2^{(N)}(T) \to 0$ and $P_1^{(N)}(T) \to 1$ as $N \to \infty$, this is interpreted as quantum Zeno effect [2]. The experimental result are in very good agreement with the above formulae. However, this is *not* the quantum Zeno effect à la Misra and Sudarshan: Eq. (9) ((10)) expresses only the probability that the atom is in level 2 (1) at time T, after N measurements, *independently* of its past history. In particular, Eqs. (9), (10) take into account the *possibility that one level gets repopulated after the atom has made transitions to the other level.* In order to shed light on this very important (and rather subtle) point, let us look explicitly at the first two measurements.

After the first measurement, $R^{(1)}$ is given by Eq. (7) and

$$R_3^{(1)} = -\cos\frac{\pi}{N} = P_2^{(1)} - P_1^{(1)}, \tag{11}$$

where $P_j^{(1)}$ is the occupation probability of level j ($j = 1,2$) at time $\tau = \pi/N\omega$, after the first measurement pulse:

$$P_2^{(1)} = \tfrac{1}{2}(1 + R_3^{(1)}) = \sin^2\frac{\pi}{2N}, \tag{12}$$

$$P_1^{(1)} = 1 - P_2^{(2)} = \cos^2\frac{\pi}{2N}. \tag{13}$$

After the second measurement, one obtains

$$R_3^{(2)} = -\cos^2\frac{\pi}{N} = P_2^{(2)} - P_1^{(2)}, \tag{14}$$

where the occupation probabilities at time $2\tau = 2\pi/N\omega$ read

$$P_2^{(2)} = \tfrac{1}{2}(1 + R_3^{(2)}) = 2\sin^2\frac{\pi}{2N}\cos^2\frac{\pi}{2N}, \tag{15}$$

[2] The $N \to \infty$ limit is in contradiction with the Heisenberg uncertainty principle, and is therefore unphysical. It is possible to set a physical limit on the maximum value that N can attain in a certain experimental situation [12,18].

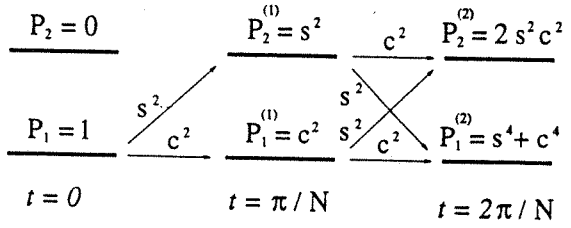

Fig. 1. Transition probabilities after the first two measurements ($s = \sin(\pi/2N)$ and $c = \cos(\pi/2N)$).

$$P_1^{(2)} = 1 - P_2^{(2)} = \cos^4\frac{\pi}{2N} + \sin^4\frac{\pi}{2N}. \tag{16}$$

It is then obvious that $P_1^{(2)}$, in Eq. (16), is *not the survival probability* of level 1, according to the seminal definition (1). It is just the probability that level 1 is populated at time $t = 2\pi/N\omega$, including the possibility that the transition $1 \to 2 \to 1$ took place, with probability $\sin^2(\pi/2N)\sin^2(\pi/2N) = \sin^4(\pi/2N)$. By contrast, the *survival* probability, namely the probability that the atom is found in level 1 *both* in the first and second measurements, is given by $P_1^{(1,2)} = \cos^2(\pi/2N)\cos^2(\pi/2N) = \cos^4(\pi/2N)$. Fig. 1 shows what happens during the first two measurements in the experiment in Ref. [3].

In the general case, after N measurements, the probability that level 1 is populated at time T, independently of its "history", is given by (10), and includes the possibility that transitions to level 2 took place. As a matter of fact, it is not difficult to realize that (9), (10) conceal a binomial distribution,

$$\begin{aligned}
\sum_{n\ \text{even}} \binom{N}{n} s^{2n}c^{2(N-n)} &= c^{2N}\sum_{n\ \text{even}} \binom{N}{n}(s/c)^{2n} \\
&= \tfrac{1}{2}c^{2N}\Bigg[\sum_{n=0}^{N}\binom{N}{n}(s/c)^{2n} \\
&\quad + \sum_{n=0}^{N}\binom{N}{n}(-1)^n(s/c)^{2n}\Bigg] \\
&= \tfrac{1}{2}c^{2N}\{[1 + (s/c)^2]^N + [1 - (s/c)^2]^N\} \\
&= \tfrac{1}{2}[1 + \cos^N(\pi/N)] \\
&= P_1^{(N)}(T) = 1 - P_2^{(N)}(T),
\end{aligned} \tag{17}$$

where $\sum_{n\ \text{even}}$ is a sum over all even values of n between 0 and N, $s = \sin(\pi/2N)$, $c = \cos(\pi/2N)$ [3].

[3] Mensky [9] first noticed the occurrence of a binomial dis-

Therefore

$$P_2^{(N)}(T) = 1 - \sum_{n \text{ even}} \binom{N}{n} \sin^{2n}\frac{\pi}{2N} \cos^{2(N-n)}\frac{\pi}{2N}, \tag{18}$$

$$P_1^{(N)}(T) = \sum_{n \text{ even}} \binom{N}{n} \sin^{2n}\frac{\pi}{2N} \cos^{2(N-n)}\frac{\pi}{2N}, \tag{19}$$

which clearly shows that Eqs. (9), (10) or (18), (19) include all possible transitions between levels 1 and 2, in such a way that at time T the system is, say, in level 1 after having made an even number ($n = 0, 2, \ldots$, etc.) of transitions between levels 1 and 2. It should be clear now that the result (10) is conceptually very different from Misra and Sudarshan's survival probability (1). The correct formula for the survival probability, in the present case, is obtained by considering *only* the $n = 0$ term in (19),

$$\mathcal{P}_1^{(N)}(T) = \cos^{2N}\frac{\pi}{2N}. \tag{20}$$

Eq. (20) is just the "survival probability", namely the probability that level 1 is populated at every measurement, at times $n\tau = nT/N$ ($n = 1, \ldots, N$) [4].

A comparison with the formulae of Ref. [3] is not straightforward, due to the fact that the authors analyzed their results in terms of the quantity $P_2^{(N)}(T)$, rather than $P_1^{(N)}(T)$. At any rate, Eq. (20) implies

$$\mathcal{P}_2^{(N)}(T) = 1 - \cos^{2N}\frac{\pi}{2N}. \tag{21}$$

Eq. (21) can be compared to (9): Even though they tend to the same limiting value 0 (in either case $\sin(\pi/2N) \to 0$ as $N \to \infty$), they give different results, in particular when N is small, as shown in Table 1.

Table 1

N	1[a]	2	4	8	16	32	64
$P_2^{(N)}(T)$	1	0.5	0.3750	0.2346	0.1334	0.0716	0.0371
$\mathcal{P}_2^{(N)}(T)$	1	0.75	0.4692	0.2668	0.1431	0.0742	0.0378

[a] $N = 1$ means that only a final measurement is performed, at time T.

It must be emphasized that we are not criticizing the soundness of the nice experiment in Ref. [3]. Indeed, the experimental results obtained by Itano et al. are in excellent agreement with Eqs. (9) or (18). We only claim that this experiment, although correctly performed, is conceptually at variance with the original idea of the QZE, as defined by Misra and Sudarshan, because the right expression for the survival probability, according to (1), is given by (20) and not by (19).

Let us now look at Cook's derivation of the QZE. For the sake of clarity, we shall present his analysis in a slightly simplified case. Starting from the set of Eqs. (2), Cook obtained the following rate equations,

$$\dot{P}_1 = k(P_2 - P_1), \tag{22}$$

$$\dot{P}_2 = k(P_1 - P_2), \tag{23}$$

where $k = \omega^2\tau/2$, τ being the time interval between measurement pulses. These equations yield, at time $T = \pi/\omega$,

$$P_2(T) = \tfrac{1}{2}[1 - \exp(-\pi^2/2N)]. \tag{24}$$

(A misprint in Ref. [2] has been corrected.) The above formula is interpreted as a quantum Zeno effect. Once again, this is not correct in a strict sense: The above equation expresses the occupation probability of level 2, independently of its history. Clearly, the rate equations (22), (23) take into account the possibility of transitions $1 \to 2 \to 1$, and so on, and therefore cannot be viewed as expressing "survival" probabilities, as in Eq. (1). It should be stressed that the conclusions drawn in this Letter hold true for all those situations in which the temporal behavior of the system under investigation is of the oscillatory type, and no precautions are taken in order to prevent repopulation of the initial state.

Finally, it is worth briefly commenting on the $N \to \infty$ limit (continuous observation). It was shown [12,18] that this limit is unphysical, for it is in contradiction with Heisenberg's uncertainty principle, and set a reasonable physical limit for the maximum value that N can attain in an experimental test of the QZE involving neutron spin. Venugopalan and Ghosh [19]

tribution in connection with the QZE for an oscillating system, without however pointing out the discrepancy with Misra and Sudarshan's definition of survival probability. The result (17) is, to our knowledge, new.

[4] Eq. (20) was first given in Section V of Ref. [10] (see in particular footnote 21).

criticized this result on the basis of an analysis whose starting point was Eq. (24). However, as we have seen, (24) is *not* related to the survival probability, according to the definition (1), so that the calculation of Ref. [19], although mathematically correct, is not physically relevant for our problem. Incidentally, in the light of our analysis, it is not surprising that the authors of Ref. [19], by applying the uncertainty principle, obtained the limiting value $P_2(T) \to 1/2$, in the large-N limit, from Eq. (24). Such a result is to be expected, on the basis of Cook's equations (22), (23), but refers to a physically different situation, not to the QZE.

In conclusion, we would like to put forward a few remarks. The real problem related with Cook's proposal and Itano et al.'s experiment is that the state of the atom is *not* observed at intermediate times. As a matter of fact, its observation would probably raise difficult technical problems, for one should be able to "select", after each measurement pulse, *which* atoms are in level 1 and discard those atoms that are in level 2.

The quantum theory of measurement [20,21] is still full of pitfalls and conceptual difficulties. One has to be extremely careful when applying von Neumann's projection postulate. A quantum measurement implies the occurrence of decoherence, but the inverse is not necessarily true, as we have seen: It may happen that the system is practically incoherent, but one still does not know, in practice, *which* state the atom is in.

Very promising candidates for an experimental observation of a genuine QZE seem to be those experiments involving neutron spin [12] or photon polarization [22]. There is certainly more to come, on this fascinating subject.

We thank C. Presilla for bringing Ref. [9] to our attention. M.N. was partially supported by the Japanese Ministry of Education, Science and Culture, and S.P. by the Japanese Society for the Promotion of Science, under a bilateral exchange program with Italian Consiglio Nazionale delle Ricerche, and by the Administration Council of the University of Bari. S.P. thanks the High Energy Physics Group of Waseda University for their kind hospitality and H.N. acknowledges the kind hospitality at the Department of Physics, University of Bari.

References

[1] B. Misra and E.C.G. Sudarshan, J. Math. Phys. 18 (1977) 756;
C.B. Chiu, B. Misra and E.C.G. Sudarshan, Phys. Rev. D 16 (1977) 520.
[2] R.J. Cook, Phys. Scr. T 21 (1988) 49.
[3] W.H. Itano, D.J. Heinzen, J.J. Bollinger and D.J. Wineland, Phys. Rev. A 41 (1990) 2295.
[4] E.J. Hellund, Phys. Rev. 89 (1953) 919;
M. Namiki and N. Mugibayashi, Prog. Theor. Phys. 10 (1953) 474;
L.A. Khalfin, Dokl. Acad. Nauk USSR 115 (1957) 277 [Sov. Phys. Dokl. 2 (1957) 340]; Zh. Eksp. Teor. Fiz. 33 (1958) 1371 [Sov. Phys. JETP 6 (1958) 1053].
[5] H. Nakazato, M. Namiki and S. Pascazio, Int. J. Mod. Phys. B 10 (1996) 247;
G.-C. Cho, H. Kasari and Y. Yamaguchi, Prog. Theor. Phys. 90 (1993) 803.
[6] A. Beskow and J. Nilsson, Arkiv för Fysik 34 (1967) 561;
L.A. Khalfin, Zh. Eksp. Teor. Fiz. Pis. Red. 8 (1968) 106 [JETP Lett. 8 (1968) 65];
L. Fonda, G.C. Ghirardi, A. Rimini and T. Weber, Nuovo Cimento A 15 (1973) 689; A 18 (1973) 805;
A. DeGasperis, L. Fonda and G.C. Ghirardi, Nuovo Cimento A 21 (1974) 471.
[7] T. Petrosky, S. Tasaki and I. Prigogine, Phys. Lett. A 151 (1990) 109; Physica A 170 (1991) 306;
A. Peres and A. Ron, Phys. Rev. A 42 (1990) 5720;
L.E. Ballentine, Phys. Rev. A 43 (1991) 5165;
W.H. Itano, D.J. Heinzen, J.J. Bollinger and D.J. Wineland, Phys. Rev. A 43 (1991) 5168;
V. Frerichs and A. Schenzle, Phys. Rev. A 44 (1991) 1962;
S. Inagaki, M. Namiki and T. Tajiri, Phys. Lett. A 166 (1992) 5;
D. Home and M.A.B. Whitaker, Phys. Lett. A 173 (1993) 327;
R. Onofrio, C. Presilla and U. Tambini, Phys. Lett. A 183 (1993) 135;
Ph. Blanchard and A. Jadczyk, Phys. Lett. A 183 (1993) 272;
M. Cini, How classical is the quantum Zeno effect?, preprint 1063, Univ. of Rome (1995).
[8] S. Pascazio, M. Namiki, G. Badurek and H. Rauch, Phys. Lett. A 179 (1993) 155.
[9] M.B. Mensky, Continuous quantum measurements and path integrals (Inst. of Phys. Publ., Bristol, 1993).
[10] S. Pascazio and M. Namiki, Phys. Rev. A 50 (1994) 4582.
[11] A. Peres, Am. J. Phys. 48 (1980) 931.
[12] H. Nakazato, M. Namiki, S. Pascazio and H. Rauch, Phys. Lett. A 199 (1995) 27.
[13] E.P. Wigner, Am. J. Phys. 31 (1963) 6.
[14] S. Machida and M. Namiki, Prog. Theor. Phys. 63 (1980) 1457, 1833;
M. Namiki and S. Pascazio, Phys. Rev. A 44 (1991) 39.
[15] R.P. Feynman, F.L. Vernon Jr. and R.W. Hellwarth, J. Appl. Phys. 28 (1957) 49.

[16] F. Block, Phys. Rev. 70 (1946) 460.
[17] I.I. Rabi, N.F. Ramsey and J. Schwinger, Rev. Mod. Phys. 26 (1954) 167.
[18] H. Nakazato, M. Namiki, S. Pascazio and H. Rauch, in preparation.
[19] A. Venugopalan and R. Ghosh, Phys. Lett. A 204 (1995) 11.
[20] J. von Neumann, Die mathematische Grundlagen der Quantenmechanik (Springer, Berlin, 1932).
[21] J.A. Wheeler and W.H. Zurek, eds., Quantum theory and measurement (Princeton Univ. Press, Princeton, NJ, 1983);
B. d'Espagnat, Conceptual foundations of quantum mechanics (Benjamin, New York, 1971);
P. Busch, P.J. Lahti and P. Mittelstaedt, The quantum theory of measurement (Springer, Berlin, 1991);
M. Namiki and S. Pascazio, Phys. Rep. 232 (1993) 301.
[22] P. Kwiat, H. Weinfurter, T. Herzog, A. Zeilinger and M. Kasevich, Ann. Acad. Sci. (New York) 755 (1995) 383.

Short-time behavior of the correlation functions for the quantum Langevin equation

Raffaella Blasi[1] and Saverio Pascazio[1,2]
[1]*Dipartimento di Fisica, Università di Bari, I-70126 Bari, Italy*
[2]*Istituto Nazionale di Fisica Nucleare, Sezione di Bari, I-70126 Bari, Italy*
(Received 6 October 1995)

We analyze the quantum Langevin equation obtained for the Ford-Kac-Mazur and related models. We study an explicit expression for the correlation function of the noise, obtained by making use of the normal-ordered product of operators. Such an expression is divergence-free, does not require any frequency cutoff, and yields the classical (Markoffian) case in the limit of vanishing $\hbar$. We also bring to light and discuss two different regimes for the momentum autocorrelation. The high-temperature and weak-coupling limits are considered, and the latter is shown to be related to van Hove's "$\lambda^2 T$" limit.

PACS number(s): 03.65.Bz, 05.40.+j, 05.70.Ln, 05.30.−d

The derivation of a dissipative equation from an underlying Hamiltonian dynamics is a long-standing problem. Many important contributions on this subject have been given during the last decades. Even though we still lack a thorough comprehension of dissipative and irreversible phenomena, the essential features of a self-consistent physical framework are now becoming more clear.

The analysis of solvable models provides very useful insights into the above-mentioned issues. One of such models was originally proposed by Ford, Kac, and Mazur (FKM) in a pioneering paper [1]. This model consists of an ensemble of coupled oscillators interacting via a quadratic Hamiltonian. The reduced dynamics of one of these oscillators yields, in an appropriate macroscopic limit, a Langevin equation [2,3].

Several other examples of this sort have been proposed. Among others, the "independent oscillator" model [4–7] and a whole class of related models [8–10] have played an important role in clarifying several aspects related to dissipative phenomena [9–11]. The similarities and differences among them have been discussed and clarified in Refs. [5,7]. In all the above-mentioned examples, the influence of a "heat bath," composed of harmonic oscillators, on the equation of motion of the "Brownian particle" is described by a phenomenological equation: The reduced description of the system always turns out to be of the Langevin type.

The aim of the present paper is to discuss some features of the Langevin equation in the quantum case. We shall consider an explicit expression for the correlation function of the noise in the quantum case. This expression, which was first derived by Gardiner [12], is divergence-free and therefore does not require any frequency cutoff. Moreover, by making use of this result, it is possible to discuss some interesting general features of the momentum autocorrelation, and bring to light the presence of two temporal regimes in the evolution of the quantum system. The deviations of the momentum autocorrelation function from a purely exponential behavior were considered by Lee [13], and are very significant in the context of dissipation in quantum mechanics. In the light of a recent remark by Leggett [14], this issue may have interesting spinoffs on the quantum measurement problem [15,16]. Finally, we shall put forward a curious analogy between the short-time behavior of the noise correlation, the momentum autocorrelation and the "survival probability" of quantum-mechanical states (quantum Zeno effect) [17–19].

It should be emphasized that it is possible to adhere to a different standpoint and postulate (rather than derive) an equation of the Langevin type for some quantum-mechanical operators, involving operator-valued random terms [20]. In such a case, the Langevin equation is viewed as a phenomenological starting point. In the white-noise case, this approach can be put on a sound mathematical basis [21] and brings to light other interesting aspects of dissipation, related to van Hove's "$\lambda^2 T$" limit [22]. Interestingly, we shall see that the above limit is related to one of the temporal regimes to be considered in this paper.

In the FKM model [1] an ensemble of $2N+1$ coupled oscillators is described via the Hamiltonian

$$H=\frac{1}{2}\sum_{j=-N}^{N}\frac{p_j^2}{m}+\frac{1}{2}\sum_{j,k=-N}^{N}q_jA_{jk}q_k+V(q_0), \qquad (1)$$

where q_j and p_j are the coordinates and momenta of the oscillators, m their (common) mass, V a potential acting on the zeroth oscillator and the $(2N+1)\times(2N+1)$ interaction matrix A_{ij} is assumed to be symmetric, cyclic, and with nonvanishing eigenvalues (the last condition is eventually relaxed in order to meet the requirement of Markoffianity). The oscillators are taken to be identical (with the only exception that the zeroth one is acted upon by an external force), and one looks for an interaction matrix A_{ij} such that the zeroth oscillator ("Brownian particle"), follows a Langevin equation.

Ford, Lewis, and O'Connell [7] pointed out that the FKM model is related, via a canonical coordinate transformation, to the "independent-oscillator" model

$$H=\frac{P^2}{2m}+V(X)+\frac{1}{2}\sum_j\left[\frac{p_j^2}{m_j}+\omega_j^2(q_j-\gamma_jX)^2\right], \qquad (2)$$

where X and P are the coordinate and momentum of the (Brownian) particle and q_j,p_j the coordinates and momenta of the (bath) oscillators. (The constants γ_j can either be taken equal to one or alternatively proportional to ω_j^{-2}.) For this reason, the conclusions we shall draw in the present

paper hold true also for the Hamiltonian (2). The independent oscillator model frequently appears in the literature, although in somewhat different forms [4–7]. Notice also [5] that a class of similar models [8–10] can be brought into the form (2).

Under certain conditions for the interaction matrix A_{ij}, the Hamiltonian (1) yields the following equation of motion in the $N\to\infty$ limit:

$$\dot{p}_0+\xi\frac{p_0}{m}+V'(q_0)=\eta(t), \tag{3}$$

where ξ is an effective friction constant and $\eta(t)$ an effective "noise" term [2,3]. (For the independent oscillator model, p_0,q_0 are substituted by P,X in the above equation.) The friction constant ξ is closely related to the interaction, and is therefore representative of the coupling between the Brownian particle and the remaining oscillators. In some cases, it is even possible to obtain an explicit expression for ξ. For example, the independent oscillator model, with the ansatz $\gamma_j\equiv\gamma/\omega_j^2$ [4] and the Debye approximation, yields

$$\xi=\frac{9\pi N\gamma^2}{2\omega_T^3}, \tag{4}$$

where N is the total number of degrees of freedom of the system of oscillators and ω_T a frequency cutoff. (A misprint in Ref. [4] has been corrected.) The relation (4) is of more general validity than one might think at first sight, and its general features can also be derived within different schemes than Zwanzig's [6].

The Langevin equation (3) is obtained in both the classical and quantum case. The expectation value of the noise, computed according to the canonical distribution, is zero in both cases. The only important difference concerns the correlation function of the noise. In the classical case

$$\langle\eta(t)\,\eta(t+\tau)\rangle_{\rm cl}=\frac{2\xi}{\beta}\delta(\tau), \tag{5}$$

as we expect from the fluctuation-dissipation theorem [$\beta=(kT)^{-1}$ is the inverse temperature of the bath and $\langle\ \rangle_{\rm cl}$ denotes a statistical (Boltzmann) average]. In the quantum case the result depends on whether one chooses the symmetric or the normal-ordered product of operators. The first case is the one usually considered in the literature: One gets

$$\langle\{\eta(t)\,\eta(t+\tau)\}\rangle\equiv\frac{1}{2}\langle\eta(t)\,\eta(t+\tau)+\eta(t+\tau)\,\eta(t)\rangle$$
$$=\frac{\xi}{\pi}\int_0^\infty d\omega\hbar\omega\,\coth\frac{\beta\hbar\omega}{2}\cos\omega\tau, \tag{6}$$

$\langle\ \rangle$ being the quantum-mechanical expectation value over the thermal state of the oscillators. On the other hand, in the second case one obtains

$$\langle:\eta(t)\,\eta(t+\tau):\rangle=\frac{2\xi}{\pi}\int_0^\infty d\omega\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\cos\omega\tau. \tag{7}$$

Notice that the two definitions differ for the zero-point energy fluctuations of the heat bath. In the first case the energy of each oscillator is taken to be

$$E_\beta(\hbar\omega)=\frac{\hbar\omega}{2}+\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}=\frac{\hbar\omega}{2}\coth\frac{\beta\hbar\omega}{2}, \tag{8}$$

while in the second case

$$E_\beta^{(0)}(\hbar\omega)=E_\beta(\hbar\omega)-\frac{\hbar\omega}{2}=\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}. \tag{9}$$

Both $E_\beta(\hbar\omega)$ and $E_\beta^{(0)}(\hbar\omega)$ yield the classical equipartition energy kT in the high-temperature limit.

As emphasized by FKM, there are many alternative definitions of correlation function, corresponding to different orderings and symmetrizations of the operators. Which of these possible definitions correspond to a real quantum fluctuation phenomenon must, in the last analysis, be determined by experiment [1]. See Gardiner's lucid discussion on this point [12]. The usual definition of correlation function is taken to be (6). The purpose of this paper is to show that the physical results obtained by normal ordering of the operators are very appealing. Let us start by observing that the integral (6) is divergent and requires a frequency cutoff. The normal-ordered product (7) does not suffer from the same drawback. Moreover, the integral (7) is solvable, as first noted in Ref. [12]: By choosing the rectangle $[0,\infty,\infty+2\pi i/\beta\hbar,2\pi i/\beta\hbar]$ as integration contour, one gets after a straightforward, if lengthy, calculation

$$f_\hbar(\beta,\xi;\tau)\equiv\langle:\eta(t)\,\eta(t+\tau):\rangle$$
$$=\frac{\xi\hbar}{\pi}\left[\frac{1}{\tau^2}-\frac{\pi^2}{\beta^2\hbar^2\sinh^2(\pi\tau/\beta\hbar)}\right]. \tag{10}$$

This function is shown in Fig. 1. It must be stressed that the result (10) diverges *only* for $\beta=\tau=0$. Notice that

$$f_\hbar(0^+,\xi;\tau)\sim\frac{\xi\hbar}{\pi\tau^2},\quad f_\hbar(\beta,\xi;0^\pm)\sim\frac{\xi\pi}{3\beta^2\hbar}. \tag{11}$$

It is also worth emphasizing that (for $\beta\neq0$) the correlation function $f_\hbar$ has a vanishing derivative for $\tau=0$. This is an important point to which we shall come back later. It is interesting to look at the very shape of $f_\hbar$ in Fig. 1. Incidentally, observe also that the integrand in Eq. (7) is not analytic in $\beta=0$, so that the naive high-temperature expansion

$$E_\beta(\hbar\omega)\simeq\frac{1}{\beta}+\beta\frac{(\hbar\omega)^2}{12} \tag{12}$$

in (9) and in the integral (7) leads to a different (wrong) result, after integration.

On the other hand, the classical correlation function (5) is obtained by (10) in the $\hbar\to0$ limit: To this end, observe that

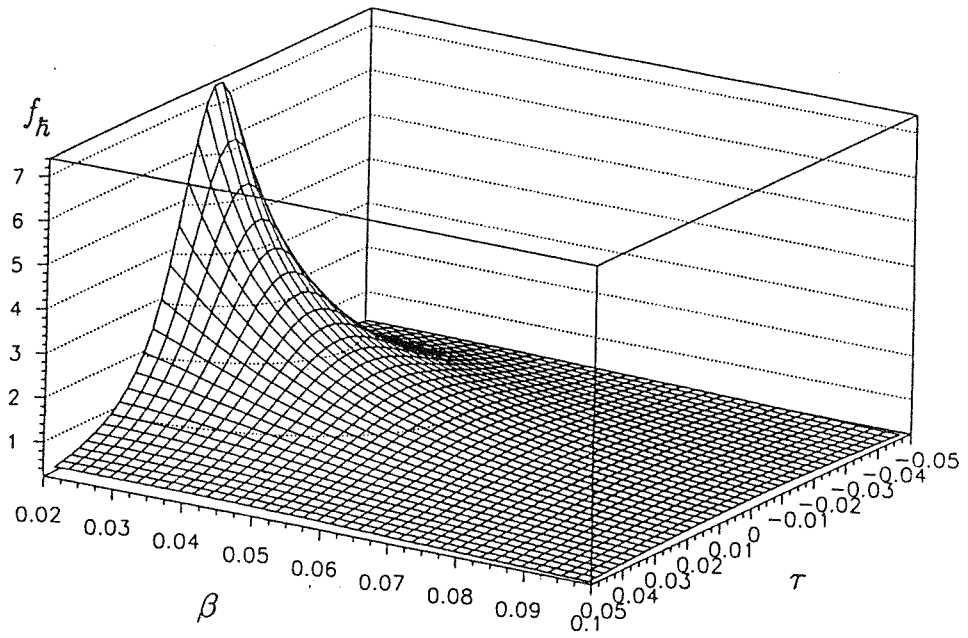

FIG. 1. Normal-ordered correlation function $f_\hbar$ vs τ and β (arbitrary units).

$$\int_{-\infty}^{\infty} d\tau f_\hbar(\beta,\xi;\tau)=\frac{2\xi}{\beta} \quad (\beta\neq 0), \tag{13}$$

a result that is independent of $\hbar$. [Since the function $f_\hbar$ in (10) is continuous for $\beta\neq 0$, the integral (13) is computed in Riemann's sense *without* making use of Cauchy's principal value.] Equation (13) allows us to infer that

$$f_\hbar(\beta,\xi;\tau)\overset{\hbar\to 0}{\longrightarrow}\frac{2\xi}{\beta}\delta(\tau)=\langle\eta(t)\eta(t+\tau)\rangle_{\rm cl}, \tag{14}$$

in agreement with (5). This has interesting spinoffs: The somewhat disturbing divergence of $f_\hbar$ in $\beta=\tau=0$ [see Eq. (11)] turns out to be of classical, rather than of quantum-mechanical origin.

One can also compute the power spectrum of the noise, by observing that (7) is nothing but a Fourier cosine transform. The final result is

$$F(\omega)\equiv\int_{-\infty}^{\infty} d\tau f_\hbar(\beta,\xi;\tau)e^{-i\omega\tau}=\frac{2\xi\hbar|\omega|}{e^{\beta\hbar|\omega|}-1} \tag{15}$$

and vanishes in the $T\to 0$ limit. In this context, it is worth noting that FKM stressed that the normal-ordered product is "physically appropriate because it leads to a noise spectrum that vanishes at absolute zero" [1].

It should be emphasized that also the correlation function (6) yields the classical result (5) in the limit of vanishing $\hbar$. The advantage of (7), however, lies in the explicit expression of all the quantities considered. Moreover, as we shall see, the above analysis will enable us to draw some other general conclusions.

Let us now look at the momentum autocorrelation function. If we normal-order the operators, we obtain (let $\xi'=\xi/m$ and call it ξ again)

$$\begin{aligned} g_\hbar(\beta,\xi;\tau)&\equiv\langle:p_0(t)p_0(t+\tau):\rangle \\ &=\frac{2m}{\pi\beta}\int_0^\infty d\omega\frac{\beta\hbar\omega}{e^{\beta\hbar\omega}-1}\frac{\xi}{\omega^2+\xi^2}\cos\omega\tau \\ &\equiv\frac{2m}{\pi\beta}\int_0^\infty d\omega A(\omega)B(\omega)\cos\omega\tau, \end{aligned} \tag{16}$$

A and B being the first and second factors in the integral, respectively. Let us first observe that, for all nonvanishing (i.e., physical) values of the parameters $\hbar,\beta,\xi$, a short-time expansion of the cosinus yields

$$g_\hbar\simeq a-b\tau^2, \tag{17}$$

where a and b are nonvanishing positive constants that depend on the parameters $\hbar,\beta,\xi$. This result is of *general validity* [13], and shows that $g_\hbar$ [like $f_\hbar$ in Eq. (10)] always has a vanishing derivative for $\tau=0$.

The functions A and B are bell shaped, with their maxima at the origin $\omega=0$. Moreover, $A(\omega)$ has poles in $\omega_A=2\pi ni/\beta\hbar$ (integer n), while $B(\omega)$ has poles in $\omega_B=\pm i\xi$. This suggests the presence of two different regimes, according to the value of the parameter $\xi\beta\hbar/2\pi$: Let us introduce the two relaxation times

$$\tau_{\rm cl}\equiv\xi^{-1}, \quad \tau_q\equiv\beta\hbar/2\pi, \tag{18}$$

which characterize, respectively, a classical and a quantum regime. As previously emphasized, $\tau_{\rm cl}$ is representative of the bath-particle interaction. If

$$\xi\beta\hbar\ll 2\pi\Leftrightarrow\tau_q\ll\tau_{\rm cl}, \tag{19}$$

then the integral (16) can be approximated by the expression

$$g_\hbar \sim \tilde{B}(\tau) = m\beta^{-1} e^{-\xi|\tau|} = m\beta^{-1} e^{-|\tau|/\tau_{\rm cl}} \quad (\tau \gg \tau_q), \tag{20}$$

where the tilde denotes the Fourier transform. Notice that the above expression is only valid for $\tau \gg \tau_q$. This is the *classical* regime [1], in which quantum effects are negligible and the process is Markoffian: FKM remarked how this result, together with (14), could be obtained in the limit of vanishing $\hbar$. Notice that, according to Eq. (19), the exponential behavior (20) can also be obtained in the limits of high temperature ($\beta \to 0$) and/or weak coupling ($\xi \to 0$). However, the latter limits must be considered with great care: For example, the naive high-temperature limit does not yield the correct noise correlation function [see (12)], which can be obtained by letting $\hbar \to 0$ [see (13) and (14)]. On the other hand, if

$$\xi\beta\hbar \gg 2\pi \Leftrightarrow \tau_q \gg \tau_{\rm cl}, \tag{21}$$

then

$$g_\hbar \sim \tilde{A}(\tau) = \frac{\hbar m}{\xi\pi}\left[\frac{1}{\tau^2} - \frac{\pi^2}{\beta^2\hbar^2 \sinh^2(\pi\tau/\beta\hbar)}\right]$$
$$= \frac{\hbar m}{\xi\pi}\left[\frac{1}{\tau^2} - \frac{1}{(2\tau_q)^2 \sinh^2(\tau/2\tau_q)}\right] \quad (\tau \gg \tau_{\rm cl}). \tag{22}$$

In this regime, *quantum coherence effects* are not negligible and the process is not Markoffian. According to Eq. (21), this behavior is obtained in the low-temperature and/or strong-coupling limit.

Remember that the short-time behavior of the function $g_\hbar$ is *always* given by Eq. (17): This yields a vanishing derivative for $t \to 0$, in agreement with (22), but not with (20). As a matter of fact, the exponential behavior is always the result of approximations or limiting procedures of some sort.

This difference in the short-time domain is, in our opinion, of general significance. Indeed, it is well known that the temporal evolution of the so-called "survival probability" of a quantum system, under general conditions, is roughly characterized by three distinct regions [17]: A Gaussian-like behavior at very short times, an (approximately) exponential decay at intermediate times, and a power law at long times. The asymptotic dominance of the exponential behavior is representative of a purely stochastic evolution, in which all quantum-mechanical phase correlations are lost, and this suggests a close connection between dissipation, quantum measurements, and exponential decay [14,18]. On the other hand, the Gaussian short-time behavior [17] is essentially ascribable to the persistence of quantum-mechanical phase correlations, leading to the so-called quantum Zeno effect [19]. The above conclusions are valid under very general conditions for the "survival" probability of any quantum-mechanical system (not necessarily unstable) [23].

Even though the above-mentioned issue and the analysis of the present paper reflect somewhat different aspects of the problem of dissipation, it is peculiar, in our opinion, that in both cases the quantum properties of the system be reflected in a *vanishing* derivative at very short times. This is true both for the quantum Zeno effect and in Eqs. (10), (16), (17), and (22). The contrast with the classical regime (20) is striking, and reminds one of other macroscopic, weak-coupling limits yielding exponential laws [22,18]. It is indeed possible to put forward a curious link between the conditions of approximate validity of the exponential behavior (20) and van Hove's limit. In the latter case [22], one considers the weak-coupling, macroscopic limit by keeping

$$\lambda^2 T = \text{finite}, \tag{23}$$

where T is time and λ the coupling constant, and shows that the temporal evolution of quantum-mechanical systems satisfies a master equation for time scales of order λ^{-2}. Consider now Eq. (4). In the macroscopic limit $N, \omega_T \to \infty$, with $N/\omega_T^3 =$ finite, ξ is simply proportional to the square of the coupling constant

$$\xi \propto \gamma^2. \tag{24}$$

On the other hand, the approximate exponential law (20) is valid in the weak-coupling limit for times of order $\tau \sim \tau_{\rm cl} = \xi^{-1} (\gg \tau_q)$, which, by virtue of (24), implies

$$\gamma^2 \tau = \text{finite}. \tag{25}$$

Equations (23) and (25) reflect the *same physical approximations*. Even though the latter equation is not derived in full generality, we cannot escape the feeling that this result is of broader significance. Incidentally, it is worth stressing that van Hove's limit (23) provokes also the disappearance of the long-time power tail in quantum-mechanical temporal evolutions [17,23]. Such a powerlike long-time behavior appears also in the context of the momentum autocorrelation functions [13,24].

One may wonder whether these curious features of the quantum-mechanical case are of general significance or are rather a "fluke of the special FKM model" [25]. This problem is a very subtle one. The results derived in the present paper are obtained for a specific model of the system-environment interaction, and the (approximate) validity of a quantum Langevin equation in more general cases than a simple array of harmonic oscillators is a very open problem. Observe, for instance, that the product $\xi\beta\hbar$, which plays such an important role in our analysis, is the only dimensionless constant that can be constructed by starting from models leading to Eqs. (3) and (7). Nevertheless, it is difficult to believe that the above-mentioned essential features of the quantum temporal behavior be just casual. Rather, we feel that the vanishing derivative at short times of some quantum-mechanical expectation values reflects a deep, yet unclear, property of "persistence" of the quantum-mechanical phase correlation. Only when such a phase correlation is completely destroyed, for instance, by interacting, in an appropriate limit, with a dissipative environment (a "heat bath") can the quantum system relax towards a classical (Markoffian) behavior, characterized by approximately exponential laws.

The authors thank P. Cea, Y.G. Lu, H. Nakazato, M. Namiki, and in particular E. Sardella for their comments, help, and criticisms.

[1] G.W. Ford, M. Kac, and P. Mazur, J. Math. Phys. **9**, 504 (1965).

[2] *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover, New York, 1954).

[3] C.W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1990).

[4] R. Zwanzig, J. Stat. Phys. **9**, 215 (1973).

[5] V. Hakim and V. Ambegaokar, Phys. Rev. A **32**, 423 (1985).

[6] T. Saito, Studia Humana et Naturalia **20**, 1 (1986); G.W. Ford and M. Kac, J. Stat. Phys. **46**, 803 (1987); H. Grabert, P. Schramm, and G.L. Ingold, Phys. Rev. Lett. **58**, 58 (1987).

[7] G.W. Ford, J.T. Lewis, and R.F. O'Connell, Phys. Rev. A **37**, 4419 (1988).

[8] V.B. Magalinskij, Zh. Éksp. Teor. Fiz. **36** (1959) 1942 [Sov. Phys. JETP **9**, 1381 (1959)]; P. Ullersma, Physica **32**, 27 (1966).

[9] A.O. Caldeira and A.J. Leggett, Phys. Rev. Lett. **46**, 211 (1981); Physica **121A**, 587 (1983); Phys. Rev. A **31**, 1059 (1985).

[10] R.P. Feynman and F.L. Vernon, Ann. Phys. (N.Y.) **149**, 374 (1983).

[11] D. Kumar, Phys. Rev. A **29**, 1571 (1984); A. Venugopalan, *ibid.* **50**, 2742 (1994).

[12] C.W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).

[13] M.H. Lee, Phys. Rev. Lett. **51**, 1227 (1983).

[14] A.J. Leggett, *Proceedings of the 4th International Symposium-Foundations of Quantum Mechanics*, edited by M. Tsukada (Physical Society of Japan, Tokyo, 1993), p. 10.

[15] J. von Neumann, *Die Mathematische Grundlagen der Quantenmechanik* (Springer-Verlag, Berlin, 1932).

[16] For a review, see *Quantum Theory and Measurement*, edited by J.A. Wheeler and W.H. Zurek (Princeton University Press, Princeton, 1983); P. Busch, P.J. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement* (Springer-Verlag, Berlin, 1991); M. Namiki and S. Pascazio, Phys. Rep. **232**, 301 (1993).

[17] E.J. Hellund, Phys. Rev. **89**, 919 (1953); M. Namiki and N. Mugibayashi, Prog. Theor. Phys. **10**, 474 (1953); L.A. Khalfin, Zh. Éksp. Teor. Fiz. **33**, 1371 (1958) [Sov. Phys. JETP **6**, 1053 (1958)]; A. Beskow and J. Nilsson, Ark. Fys. **34**, 561 (1967); L.A. Khalfin, Pis'ma Zh. Éksp. Teor. Fiz. **8**, 106 (1968) [JETP Lett. **8**, 65 (1968)]; L. Fonda, G.C. Ghirardi, A. Rimini, and T. Weber, Nuovo Cimento A **15**, 689 (1973); **18**, 805 (1973); A. DeGasperis, L. Fonda, and G.C. Ghirardi, *ibid.* A **21**, 471 (1974).

[18] H. Nakazato and S. Pascazio, Phys. Rev. Lett. **70**, 1 (1993); H. Nakazato, M. Namiki, and S. Pascazio, Phys. Rev. Lett. **73**, 1063 (1994).

[19] B. Misra and E.C.G. Sudarshan, J. Math. Phys. **18**, 756 (1977). See also A. Peres, Am. J. Phys. **48**, 931 (1980). A dynamical explanation of the quantum Zeno effect is given in S. Pascazio, M. Namiki, G. Badurek, and H. Rauch, Phys. Lett. A **179**, 155 (1993); S. Pascazio and M. Namiki, Phys. Rev. A **50**, 4582 (1994); H. Nakazato, M. Namiki, S. Pascazio, and H. Rauch, Phys. Lett. A **199**, 27 (1995).

[20] M. Mizutani, S. Muroya, and M. Namiki, Phys. Rev. D **37**, 3033 (1988).

[21] L. Accardi, A. Frigerio, and Y.G. Lu, Commun. Math. Phys. **131**, 537 (1990); L. Accardi, Y.G. Lu, and I. Volovich (unpublished).

[22] L. van Hove, Physica **21**, 517 (1955). See also E.B. Davies, Commun. Math. Phys. **39**, 91 (1974); G.L. Sewell, *Quantum Theory of Collective Phenomena* (Clarendon, Oxford, 1986).

[23] H. Nakazato, M. Namiki, and S. Pascazio, Int. J. Mod. Phys. B (to be published).

[24] Y. Pomeau and P. Resibois, Phys. Rep. **19C**, 64 (1975).

[25] R. Benguria and M. Kac, Phys. Rev. Lett. **46**, 1 (1981).

ELSEVIER

Physica A 245 (1997) 189–211

PHYSICA A

Emergence of a Wiener process as a result of the quantum mechanical interaction with a macroscopic medium

Raffaella Blasi[a,*], Hiromichi Nakazato[b], Mikio Namiki[b], Saverio Pascazio[a,c]

[a] *Departimento di Fisica, Università di Bari, I-70126 Bari, Italy*
[b] *Department of Physics, Waseda University, Tokyo 169, Japan*
[c] *Istituto Nazionale di Fisica Nucleare, Sezione di Bari, I-70126 Bari, Italy*

Received 10 December 1996

Abstract

We analyze a modified version of the Coleman–Hepp model, which is able to take into account energy-exchange processes between the incoming particle and the linear array made up of N spin-$\frac{1}{2}$ systems. We bring to light the presence of a Wiener dissipative process in the weak-coupling, macroscopic ($N \to \infty$) limit. In such a limit and a restricted portion of the total Hilbert space, the particle undergoes a sort of Brownian motion, while the free Hamiltonian of the spin array serves as a Wiener process. No partial trace is computed over the states of the spin system (which plays the role of "reservoir"). The mechanism of appearance of the stochastic process is discussed and contrasted to other noteworthy examples in the literature. The links with van Hove's "$\lambda^2 T$" limits are emphasized.

PACS: 03.65.Bz; 05.70.Ln; 05.40.+j; 05.30.-d
Keywords: Wiener process; Quantum statistical mechanics; Theory of quantum measurement

1. Introduction

The derivation of a dissipative dynamics in quantum mechanics and quantum field theory is a long-standing problem. A very important contribution to this issue was given by Van Hove [1], who clarified the main features of a quantum dissipative dynamics and was able to derive a master equation from the Schrödinger equation, in an appropriate limit (his famous "$\lambda^2 T$" limit), via the so-called "diagonal singularity". It is important to stress that Van Hove's ansatz replaced Pauli's random-phase assumption [2].

Dissipation in quantum mechanics can emerge as a result of the interaction between a particle and a macroscopic "environment". Many other important contributions

* Corresponding author.

PII S0378-4371(97)00043-5

have been given to this issue, in the attempt to set up a general physical framework. However, unlike in Van Hove's case, a dissipative dynamics is derived under some assumptions for the energy spectrum of the environment system, by computing the partial trace over the states of the latter. This is true for a whole class of interesting models, in which the environment is schematized with a collection of harmonic oscillators [3,4], and one derives dissipative equations for the "object" particle. It is worth observing that the dissipation constant in the Langevin-type equation is derived from the underlying dynamics under some (reasonable) assumptions on the spectrum of the Hamiltonian of the environment system. Of course, the assumptions on the spectrum are integrating part of the specifications of the model chosen. Yet, it would be somewhat preferable and more consistent to start from a specific microscopic model, solve the dynamics and find an appropriate limit in which the model realizes the desired continuous spectrum. Notice also that a "quantum Langevin equation" has some well-known peculiar properties, such as a non-markoffian random force [5] and colored noise [6,7].

In this paper we shall analyze the so-called "AgBr" model [8,9], which has played an important role in the quantum measurement problem. This model is relatively simple, yet extremely interesting from the physical point of view. We shall base our discussion on a modified version [10] of the above model, that is able to take into account energy-exchange processes. The modified AgBr model provides an interesting nontrivial example of realization of Van Hove's diagonal singularity and displays the occurrence of an exponential regime at all times in the weak-coupling, macroscopic limit [11]. In this way, a door is open to investigate the occurrence of a dissipative dynamics and its link with a quantum measurement process. We shall solve the equations of motion, take a weak-coupling, macroscopic limit, and obtain a Wiener process in a restricted portion of the Hilbert space of the total system. This technique is to be contrasted to the computation of a partial trace over the states of the macroscopic system, and will represent the main difference between the present work and other ones, based on partial tracing.

This work has interesting spinoffs for the quantum measurement problem [12]. It is commonly believed that a quantum measurement occurs via a dephasing (decoherence) process [13–16]. Since "decoherence" is nothing but the disappearance of the off-diagonal elements of the density matrix of the quantum system, and since a system described by a diagonal density matrix exhibits a purely stochastic behavior [17], one is led to expect a connection between dissipation, irreversibility and a quantum measurement process [11,18,19]. In this paper, we shall concentrate on a particular aspect of the above-mentioned issues. Our main purpose is to derive a stochastic process from an underlying Hamiltonian dynamics and to clarify in which sense it is possible to identify a Wiener process for a dynamical variable of the microscopic system under investigation.

The plan of the paper is as follows. We review the main properties of the modified AgBr Hamiltonian in Section 2, focusing on those characteristics that hint at the

presence of a stochastic process. In Section 3 we obtain the operators in the Heisenberg picture. Finally, we bring to light the stochastic process in Section 4, where we show that there is a Gaussian process of the Wiener type. All these results are exact. Section 5 is devoted to conclusions and comments.

2. Review of the AgBr Hamiltonian

The modified AgBr Hamiltonian [10] describes the interaction between an ultrarelativistic particle Q and a one-dimensional array (D-system), made up of N spin-$\frac{1}{2}$ objects. The array can be viewed as a caricature of a linear "photographic emulsion" of AgBr molecules, if one identifies the *down* (ground) state of the spin with the undivided molecule and the *up* state (whose excitation energy is $\hbar\omega$) with the dissociated molecule (Ag and Br atoms). The particle and each molecule interact via a spin-flipping local potential. The total Hamiltonian for the $Q + D$ system reads

$$H = H_0 + H', \qquad H_0 = H_Q + H_D, \tag{2.1}$$

where H_Q and H_D, the free Hamiltonians of the Q particle and H of the "detector" D, respectively, and the interaction Hamiltonian H' are written as

$$H_Q = c\hat{p}, \qquad H_D = \frac{1}{2}\hbar\omega \sum_{n=1}^{N} (1 + \sigma_3^{(n)}),$$

$$H' = \sum_{n=1}^{N} V(\hat{x} - x_n)\left[\sigma_+^{(n)} \exp\left(-i\frac{\omega}{c}\hat{x}\right) + \sigma_-^{(n)} \exp\left(+i\frac{\omega}{c}\hat{x}\right)\right]. \tag{2.2}$$

Here $\hat{p}$ is the momentum of the Q particle, $\hat{x}$ its position, V a real potential, x_n ($n = 1, \ldots, N$) the positions of the scatterers in the array ($x_n > x_{n-1}$) and $\sigma_{i,\pm}^{(n)}$ the Pauli matrices acting on the nth site. The above Hamiltonian has attracted the attention of several researchers [20] due, in particular, to the presence of the free Hamiltonian of the array H_D, which enables one to distinguish energetically the up and down states and makes the energy-exchange processes between Q and D physically meaningful. The original Hamiltonian [8] is reobtained in the $\omega = 0$ limit.

Let us review the main results obtained from this model [10,11]. The evolution operator in the interaction picture

$$U_I(t, t') = e^{iH_0 t/\hbar} e^{-iH(t-t')/\hbar} e^{-iH_0 t'/\hbar} = e^{-i\int_{t'}^{t} H_I'(t')dt'/\hbar}, \tag{2.3}$$

where $H_I'(t)$ is the interaction Hamiltonian in the interaction picture, can be computed exactly as

$$U_I(t) \equiv U_I(t, 0) = e^{iH_0 t/\hbar} e^{-iHt/\hbar}$$

$$= \prod_{n=1}^{N} \exp\left[-\frac{i}{\hbar}\int_0^t dt' V(\hat{x} + ct' - x_n)(\sigma_+^{(n)} e^{-i\omega\hat{x}/c} + \text{h.c.})\right], \tag{2.4}$$

and a straightforward calculation yields the S-matrix

$$S^{[N]} = \lim_{\substack{t\to+\infty \\ t'\to-\infty}} U_I(t,t') = \prod_{n=1}^{N} S_{(n)}, \quad S_{(n)} = \exp\left[-i\frac{V_0\Omega}{\hbar c}\boldsymbol{\sigma}^{(n)}\cdot\boldsymbol{u}\right], \tag{2.5}$$

where $\boldsymbol{u} = [\cos(\omega x/c), \sin(\omega x/c), 0]$ and $V_0\Omega \equiv \int_{-\infty}^{\infty} V(x)\,dx < \infty$. The above expression enables us to define the "spin-flip" probability, i.e. the probability of dissociating one AgBr molecule:

$$q \equiv \sin^2\left[\frac{V_0\Omega}{c\hbar}\right]. \tag{2.6}$$

By defining

$$\alpha_n \equiv \alpha_n(\hat{x}, t) \equiv \int_0^t \frac{dt'}{\hbar} V(\hat{x} + ct' - x_n) \tag{2.7}$$

and

$$\sigma_{\pm}^{(n)}(\hat{x}) \equiv \sigma_{\pm}^{(n)} e^{\mp i\omega\hat{x}/c}, \tag{2.8}$$

which satisfy, together with $\sigma_3^{(n)}$, the SU(2) algebra

$$\begin{aligned} &[\sigma_3^{(n)}, \sigma_{\pm}^{(n)}(\hat{x})] = \pm 2\delta_{mn}\sigma_{\pm}^{(n)}(\hat{x}), \\ &[\sigma_-^{(m)}(\hat{x}), \sigma_+^{(n)}(\hat{x})] = -\delta_{mn}\sigma_3^{(n)}, \end{aligned} \tag{2.9}$$

we can return to the Schrödinger picture by inverting Eq. (2.4). We disentangle the exponential in U_I by making use of (2.9) and obtain

$$e^{-iHt/\hbar} = e^{-iH_0t/\hbar} \prod_{n=1}^{N} \left(e^{-i\tan(\alpha_n)\sigma_+^{(n)}(\hat{x})}\, e^{-\ln\cos(\alpha_n)\sigma_3^{(n)}}\, e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})}\right). \tag{2.10}$$

Let us concentrate our attention on the situation in which the Q particle is initially located at $x' < x_1$, where x_1 is the position of the first scatterer in the linear array, and moves toward the array with speed c. The spin system is initially set in the ground state $|0\rangle_N$ of the free Hamiltonian H_D (all spins down). The propagator is defined by

$$G(x, x', t) \equiv {}_N\langle 0, x|\, e^{-iHt/\hbar}\, |x', 0\rangle_N, \tag{2.11}$$

where $|x, 0\rangle_N \equiv |x\rangle \otimes |0\rangle_N$. If we place the spin array at the far right of the origin ($x_1 > 0$) and consider the case in which the potential V has a compact support and the Q particle is initially located at the origin $x' = 0$, we obtain

$$G(x, 0, t) = \delta(x - ct) \prod_{n=1}^{N} \cos\tilde{\alpha}_n(t), \quad \tilde{\alpha}_n(t) \equiv \int_0^t \frac{dt'}{\hbar} V(ct' - x_n). \tag{2.12}$$

Note that, due to the choice of the free Hamiltonian H_Q, the Q wave packet does not disperse and moves with constant speed c. In this paper we shall exclusively consider the weak-coupling, macroscopic limit

$$N \to \infty \quad \text{and} \quad q \simeq \left(\frac{V_0\Omega}{\hbar c}\right)^2 = O(N^{-1}), \tag{2.13}$$

which is equivalent to the requirement that the total number of spin flips $\bar{n} = qN$ be finite in the macroscopic limit $N \to \infty$. Notice that, if we set

$$x_n = x_1 + (n-1)d, \qquad L = x_N - x_1 = (N-1)d, \tag{2.14}$$

and let $d/L \to 0$ as $N \to \infty$, a summation over n can be replaced by a definite integration according to

$$q \sum_{n=1}^{N} f(x_n) \to \frac{\bar{n}}{L} \int_{x_1}^{x_N} dy\, f(y). \tag{2.15}$$

In this case, by making use of the Fermi–Yang approximation $V(y) = V_0\Omega\delta(y)$, (2.12) becomes

$$G \propto \exp\left(\sum_{n=1}^{N} \ln\left[\cos \int_0^{ct} dx \frac{V_0\Omega}{c\hbar} \delta(x - ct) \right] \right)$$

$$\to \exp\left(-\frac{\bar{n}}{2L}[(ct - x_1)\theta(x_N - ct)\theta(ct - x_1) + L\theta(ct - x_N)] \right), \tag{2.16}$$

where the arrow will henceforth denote the weak-coupling, macroscopic limit (2.13), (2.15). The system attains an exponential regime as soon as the interaction starts: Indeed, if $x_1 < ct < x_N$,

$$G \propto \exp\left(-\bar{n}\frac{(ct - x_1)}{2L} \right). \tag{2.17}$$

Notice the absence of the "Gaussian" regime, characterized by a vanishing derivative at $t = 0$ [21,22], and of the power law at long times [23]. This result is valid for the propagator (2.11), which involves position eigenstates of the Q particle. If these are substituted by (normalizable) wave packets, small deviations from the exponential law appear at short times [19], in agreement with general mathematical theorems [21].

The result (2.17) hints at the presence of a dissipative process of some sort, at least in a restricted portion of the Hilbert space of the total $(Q + D)$ system. Such a dissipative process was brought to light in Ref. [24], where it was shown that a Wiener process

appears in the weak-coupling, macroscopic limit (2.13), (2.15). In the following sections we shall derive all results (including "border effects") in full generality, discuss their meaning and clarify in which sense and under which conditions it is possible to identify the presence of a dissipative process. The important links with Van Hove's $\lambda^2 T$ limit [1] will also be properly emphasized.

3. Heisenberg operators

In order to bring to light the emergence of a dissipative process in the particle-detector interaction, it is essential to study the temporal evolution of all the operators involved in the interaction process. It is therefore convenient to work in the Heisenberg picture. First of all, notice that the total Hamiltonian of the system is constant in time

$$\begin{aligned} H(t) &= c\hat{p}(t) + \frac{\hbar\omega}{2}\sum_n (1 + \sigma_3^{(n)}(t)) \\ &\quad + \sum_n V(\hat{x}(t) - x_n)[\sigma_+^{(n)}(t) e^{-i\omega\hat{x}(t)/c} + \sigma_-^{(n)}(t) e^{i\omega\hat{x}(t)/c}] \\ &= H(0), \end{aligned} \tag{3.1}$$

where $H(0) \equiv H$ is the total Hamiltonian of the system in the Schrödinger picture. Let us focus our attention on the free Hamiltonian of the Q particle. From (3.1) we get

$$\begin{aligned} c\hat{p}(t) &= c\hat{p}(0) + \frac{\hbar\omega}{2}\sum_n (\sigma_3^{(n)} - \sigma_3^{(n)}(t)) \\ &\quad + \sum_n \{V(\hat{x} - x_n)[\sigma_+^{(n)} e^{-i\omega\hat{x}/c} + \text{h.c.}] \\ &\quad - V(\hat{x}(t) - x_n)[\sigma_+^{(n)}(t) e^{i\omega\hat{x}(t)/c} + \text{h.c.}]\}, \end{aligned} \tag{3.2}$$

where $c\hat{p}(0) \equiv c\hat{p}$ and $\sigma_i^{(n)} \equiv \sigma_i^{(n)}(0)$ $(i = 3, \pm)$ are operators in the Schrödinger picture. In order to solve Eq. (3.2), we need the explicit forms of the Heisenberg operators $\hat{x}(t)$, $\sigma_3^{(n)}(t)$ and $\sigma_\pm^{(n)}(t)$. To this end, we shall make use of disentanglement formula (2.10).

The calculation of the operator $\hat{x}(t)$ is straightforward and yields

$$\hat{x}(t) = e^{iHt/\hbar}\hat{x}e^{-iHt/\hbar} = \hat{x} + ct. \tag{3.3}$$

On the other hand, the evaluation of $\sigma_{i,\pm}^{(n)}(t)$ is more involved; let us first show in full how to calculate the operator $\sigma_+^{(n)}(t)$. With the help of (2.10) this operator can be rewritten as

$$\begin{aligned} &\sigma_+^{(n)}(t) = e^{iHt/\hbar}\sigma_+^{(n)} e^{-iH_0 t/\hbar}\prod_m D_m, \\ &D_m \equiv e^{-i\tan(\alpha_m)\sigma_+^{(m)}(\hat{x})} e^{-\ln\cos(\alpha_m)\sigma_3^{(m)}} e^{-i\tan(\alpha_m)\sigma_-^{(m)}(\hat{x})}. \end{aligned} \tag{3.4}$$

By observing that

$$e^{iHt/\hbar}\sigma_+^{(n)}e^{-iHt/\hbar} = e^{iHt/\hbar}e^{i\omega\hat{x}/c}\sigma_+^{(n)}(\hat{x})e^{-iH_0t/\hbar}$$
$$= e^{i\omega(\hat{x}+ct)/c}e^{iHt/\hbar}\sigma_+^{(n)}(\hat{x})e^{-iH_0t/\hbar} = e^{i\omega(\hat{x}+ct)/c}e^{iHt/\hbar}e^{-iH_0t/\hbar}\sigma_+^{(n)}(\hat{x}), \tag{3.5}$$

Eq. (3.4) becomes

$$e^{i\omega(\hat{x}+ct)/c}e^{iHt/\hbar}e^{-iH_0t/\hbar}[\sigma_+^{(n)}(\hat{x})e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})}e^{-\ln\cos(\alpha_n)\sigma_3^{(n)}}e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})}]\prod_{m\neq n}D_m. \tag{3.6}$$

We now evaluate the term in square brackets. By making use of the formulas

$$e^{a\sigma_3^{(n)}}\sigma_+^{(n)}e^{-a\sigma_3^{(n)}} = e^{2a}\sigma_+^{(n)}, \quad e^{b\sigma_-^{(n)}}\sigma_+^{(n)}e^{-b\sigma_-^{(n)}} = \sigma_+^{(n)} - b\sigma_3^{(n)} + \frac{b^2}{2!}(-2\sigma_-^{(n)}), \tag{3.7}$$

we obtain

$$\sigma_+^{(n)}(\hat{x})e^{-i\tan(\alpha_n)\sigma_+^{(n)}(\hat{x})}e^{-\ln\cos(\alpha_n)\sigma_3^{(n)}}e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})}$$
$$= e^{-i\tan(\alpha_n)\sigma_+^{(n)}(\hat{x})}e^{-\ln\cos(\alpha_n)\sigma_3^{(n)}}e^{2\ln\cos(\alpha_n)}\sigma_+^{(n)}(\hat{x})e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})}$$
$$= e^{-i\tan(\alpha_n)\sigma_+^{(n)}(\hat{x})}e^{-\ln\cos(\alpha_n)\sigma_3^{(n)}}e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})}\cos^2\alpha_n$$
$$\times(\sigma_+^{(n)}(\hat{x}) - i\sigma_3^{(n)}\tan\alpha_n - (i\tan\alpha_n)^2\sigma_-(\hat{x})). \tag{3.8}$$

By substituting (3.8) into Eq. (3.6), we get

$$\sigma_+^{(n)}(t) = e^{i\omega(\hat{x}+ct)/c}e^{iHt/\hbar}e^{-iH_0t/\hbar}\prod_m e^{-i\tan(\alpha_m)\sigma_+^{(m)}(\hat{x})}e^{-\ln\cos(\alpha_m)\sigma_3^{(m)}}e^{-i\tan(\alpha_m)\sigma_-^{(m)}(\hat{x})}$$
$$\times\left(\sigma_+^{(n)}(\hat{x})\cos^2\alpha_n - \frac{i}{2}\sigma_3^{(n)}\sin 2\alpha_n + \sigma_-^{(n)}(\hat{x})\sin^2\alpha_n\right)$$
$$= e^{i\omega(\hat{x}+ct)/c}\left(\sigma_+^{(n)}(\hat{x})\cos^2\alpha_n - \frac{i}{2}\sigma_3^{(n)}\sin 2\alpha_n + \sigma_-^{(n)}(\hat{x})\sin^2\alpha_n\right). \tag{3.9}$$

The Hermitian conjugate of (3.9) yields

$$\sigma_-^{(n)}(t) = e^{i\omega(\hat{x}+ct)/c}\left(\sigma_-^{(n)}(\hat{x})\cos^2\alpha_n - \frac{i}{2}\sigma_3^{(n)}\sin 2\alpha_n + \sigma_+^{(n)}(\hat{x})\sin^2\alpha_n\right) \tag{3.10}$$

and by using an analogous procedure, together with the formula

$$e^{b\sigma_+^{(n)}}\sigma_3^{(n)}e^{-b\sigma_+^{(n)}} = (\sigma_3^{(n)} - 2b\sigma_+^{(n)}) \tag{3.11}$$

and its Hermitian conjugate, we get

$$\sigma_3^{(n)}(t) = \sigma_3^{(n)} \cos 2\alpha_n - i(\sigma_+^{(n)}(\hat{x}) - \sigma_-^{(n)}(\hat{x}))\sin 2\alpha_n . \tag{3.12}$$

The results (3.9), (3.10) and (3.12) and the relation

$$\sigma_\pm^{(n)}(t) e^{\mp i\omega(\hat{x}+ct)/c} = \sigma_\pm^{(n)} e^{\mp i\omega\hat{x}/c} = \sigma_\pm^{(n)}(\hat{x}) , \tag{3.13}$$

lead us to the final expression

$$c\hat{p}(t) = c\hat{p}(0) + \hbar\omega \sum_n \sigma_3^{(n)} \sin^2 \alpha_n(\hat{x}, t) + i\frac{\hbar\omega}{2} \sum_n [(\sigma_+^{(n)}(\hat{x}) - \sigma_-^{(n)}(\hat{x})) \sin 2\alpha_n(\hat{x}, t)]$$

$$+ \sum_n [V(\hat{x} - x_n) - V(\hat{x} + ct - x_n)](\sigma_+^{(n)}(\hat{x}) + \sigma_-^{(n)}(\hat{x})) . \tag{3.14}$$

All the above results are *exact*.

4. The stochastic process

Having found the explicit (and exact) expressions for the quantum operators of our system in the Heisenberg picture, we search for the *stochastic process* that is at the origin of the exponential decay (2.17) of the propagator (2.11).

4.1. Expectation value of the Q particle Hamiltonian

By making use of the explicit expression (3.14) for the momentum operator of the Q particle in the Heisenberg picture, we can easily compute its expectation value in the state

$$|\psi, 0\rangle_N \equiv |\psi\rangle \otimes |0\rangle_N = \int dx\, \psi(x)|x\rangle \otimes |0\rangle_N, \qquad \int_{-\infty}^{+\infty} dx |\psi(x)|^2 = 1 . \tag{4.1}$$

The choice of such an uncorrelated initial state is physically consistent: Indeed, we suppose that at initial time $t = 0$ the particle Q is well outside the detector D and moves toward it with constant speed c. It will be clear in the following that the choice of the ground state $|0\rangle_N$ (N spins down) as the initial D state is essential in deriving a stochastic process. For convenience we choose ψ to be symmetrically distributed around the origin and with a compact support. Possible choices for ψ are

$$\psi(x) = (2a)^{-1/2}\theta(a - |x|)e^{ip_0 x/\hbar} , \tag{4.2}$$

or a Gaussian wave packet

$$\psi(x) = \left(\frac{1}{2\pi a^2}\right)^{1/4} e^{-x^2/4a^2 + ip_0 x/\hbar}, \tag{4.3}$$

truncated for, say, $|x| > a$. Notice again that, owing to (3.3), the wave packet does not disperse. We define

$$\langle c\hat{p}(t)\rangle \equiv {}_N\langle 0, \psi | c\hat{p}(t) | \psi, 0\rangle_N \tag{4.4}$$

and, from Eq. (3.14) we obtain

$$\begin{aligned}\langle c\hat{p}(t)\rangle = {}& {}_N\langle 0, \psi | c\hat{p}(0) | \psi, 0\rangle_N + \hbar\omega\, {}_N\langle 0, \psi | \sum_n \sigma_3^{(n)} \sin^2 \alpha_n(\hat{x}, t) | \psi, 0\rangle_N \\ & + i\frac{\hbar\omega}{2}\, {}_N\langle 0, \psi | \sum_n (\sigma_+^{(n)}(\hat{x}) - \sigma_-^{(n)}(\hat{x})) \sin 2\alpha_n(\hat{x}, t) | \psi, 0\rangle_N \\ & + {}_N\langle 0, \psi | \sum_n [V(\hat{x} - x_n) - V(\hat{x} + ct - x_n)] (\sigma_+^{(n)}(\hat{x}) + \sigma_-^{(n)}(\hat{x})) | \psi, 0\rangle_N .\end{aligned} \tag{4.5}$$

It is easy to see that the last two terms in (4.5) give vanishing contributions. Let us now compute the contributions of the first and the second terms. The evaluation of the first term is straightforward: It is nothing but the initial energy of the Q particle

$$\begin{aligned}{}_N\langle 0, \psi | c\hat{p}(0) | \psi, 0\rangle_N = \langle \psi | c\hat{p}(0) | \psi\rangle &= \int dp' \langle \psi | p'\rangle \langle p' | c\hat{p} | \psi\rangle \\ &= \int dp' |\tilde{\psi}(p')|^2 cp' = cp_0 .\end{aligned} \tag{4.6}$$

On the other hand, the calculation of the second term

$$\begin{aligned}\hbar\omega\, {}_N\langle 0, \psi | \sum_n \sigma_3^{(n)} \sin^2 \alpha_n(\hat{x}, t) | \psi, 0\rangle_N &= -\hbar\omega \langle \psi | \sum_n \sin^2 \alpha_n(\hat{x}, t) | \psi\rangle \\ &= -\hbar\omega \int dx |\psi(x)|^2 \sum_n \sin^2 \left[\frac{1}{\hbar}\int_0^t dt' V(x + ct' - x_n)\right]\end{aligned} \tag{4.7}$$

is more involved: We can use, without loss of generality (see the appendix), the Fermi-Yang approximation $V(y) = V_0 \Omega \delta(y)$, and obtain

$$\begin{aligned}& -\hbar\omega \int dx |\psi(x)|^2 \sum_n \sin^2 \left[\frac{V_0\Omega}{c\hbar} \int_{x - x_n}^{x + ct - x_n} dy\, \delta(y)\right] \\ & = -\hbar\omega \int dx |\psi(x)|^2 \sum_n \sin^2 \left[\frac{V_0\Omega}{c\hbar} \theta(x + ct - x_n)\theta(x_n - x)\right].\end{aligned} \tag{4.8}$$

If we require

$$a \ll x_1 \quad \text{and} \quad a \ll L = x_N - x_1 \,, \tag{4.9}$$

we easily get

$$-\hbar\omega \int dx\, |\psi(x)|^2 \sum_n \sin^2\left[\frac{V_0\Omega}{c\hbar}\,\theta(x + ct - x_n)\right] \tag{4.10}$$

since every spin is located at the far right of the initial wave packet $\psi(x)$, whose finite support is $[-a, a]$, so that the inequality $x(<a) \ll x_n$ holds for the integration variable x. In the weak-coupling, macroscopic limit (2.13), the summation over n can be replaced with an integration as in (2.15) and the above quantity is further reduced to

$$\begin{aligned} &-\hbar\omega\frac{\bar{n}}{L}\int dx\,|\psi(x)|^2 \int_{x+ct-x_N}^{x+ct-x_1} dz\,\theta(z) \\ &\quad = -\hbar\omega\frac{\bar{n}}{L}\int dx\,|\psi(x)|^2\,[(x + ct - x_1)\theta(x + ct - x_1)\theta(x_N - ct - x) \\ &\qquad + (x_N - x_1)\theta(x + ct - x_N)]\,. \end{aligned} \tag{4.11}$$

Thus, if we restrict our attention to the situation in which Q is still inside D, i.e. $ct < x_N$, we finally obtain

$$\hbar\omega\, {}_N\langle 0, \psi | \sum_n \sigma_3^{(n)} \sin^2 \alpha_n(\hat{x}, t) | \psi, 0\rangle_N \to -\hbar\omega\frac{\bar{n}}{L}(ct - x_1) + \text{b.e.} \tag{4.12}$$

The shorthand notation "b.e." stands for "border effects", namely terms appearing only when $|ct - x_1|$, $|ct - x_N| \leqslant a$, whose explicit expression is easily computed to be

$$-\hbar\omega\frac{\bar{n}}{L} \times \begin{cases} (ct - x_1 + a)^2/4a & \text{if } x_1 - a \leqslant ct \leqslant x_1 + a\,, \\ L - (x_N + a - ct)^2/4a & \text{if } x_N - a \leqslant ct \leqslant x_N + a\,, \end{cases} \tag{4.13}$$

corresponding, respectively, to the situations in which Q is entering D and Q is going out of D.

Having obtained the explicit expressions (4.6) and (4.12) for the first and the second terms in $\langle c\hat{p}(t)\rangle$ [see (4.5)], we reach the following *exact* expression:

$$\langle c\hat{p}(t)\rangle \equiv {}_N\langle 0, \psi | c\hat{p}(t) | \psi, 0\rangle_N \to cp_0 - \hbar\omega\frac{\bar{n}}{L}(ct - x_1) + \text{b.e}\,. \tag{4.14}$$

In conclusion, the energy of the Q particle (when Q is *inside* D) decreases linearly with respect to t. It is worth mentioning that what is seen here is an energy-dissipative process: If ω were set equal to zero (as in the original AgBr model [8, 9]), we could not have found such a process.

4.2. Correlation functions of the spin-array Hamiltonian

In order to clarify the stochastic nature of the system, let us calculate the correlation functions of the spin-array Hamiltonian H_D.

Consider the operator $\Delta H_D(t) = H_D(t) - H_D$, which represents the energy stored in the detector between time 0 and t. By using Eqs. (3.12), one obtains

$$\Delta H_D(t) = \sum_n \frac{\hbar\omega}{2}(\sigma_3^{(n)}(t) - \sigma_3^{(n)})$$

$$= -\left[\sum_n \hbar\omega\sigma_3^{(n)} \sin^2\alpha_n(\hat{x}, t) + \sum_n i\frac{\hbar\omega}{2}(\sigma_+^{(n)}(\hat{x}) - \sigma_-^{(n)}(\hat{x}))\sin 2\alpha_n(\hat{x}, t)\right]. \tag{4.15}$$

It is easy to see that when the Q particle is inside D, by making use of (4.12), the expectation value of this operator reads

$$\langle \Delta H_D(t)\rangle = {}_N\langle 0, \psi | \Delta H_D(t) | \psi, 0\rangle_N \to \hbar\omega\frac{\bar{n}}{L}(ct - x_1) + \text{b.e.}, \tag{4.16}$$

in agreement with Eq. (4.14): The energy lost by Q is stored in D.

Next we turn our attention to the two-time correlation function, defined by

$$\langle \Delta H_D(t_1)\Delta H_D(t_2)\rangle \equiv {}_N\langle 0, \psi | \Delta H_D(t_1)\Delta H_D(t_2) | \psi, 0\rangle_N . \tag{4.17}$$

Its explicit form, by Eq. (4.15), is

$$\langle \Delta H_D(t_1)\Delta H_D(t_2)\rangle$$

$$= {}_N\langle 0, \psi | \left(\sum_n \hbar\omega\sigma_3^{(n)} \sin^2\alpha_n(\hat{x}, t_1)\right)\left(\sum_m \hbar\omega\sigma_3^{(m)} \sin^2\alpha_m(\hat{x}, t_2)\right) | \psi, 0\rangle_N$$

$$+ \left(i\frac{\hbar\omega}{2}\right)^2 {}_N\langle 0, \psi | \left[\sum_n (\sigma_+^{(n)}(\hat{x}) - \sigma_-^{(n)}(\hat{x}))\sin 2\alpha_n(\hat{x}, t_1)\right]$$

$$\times \left[\sum_m \sigma_+^{(m)}(\hat{x}) - \sigma_-^{(m)}(\hat{x}))\sin 2\alpha_m(\hat{x}, t_2)\right] | \psi, 0\rangle_N + \text{vanishing terms.} \tag{4.18}$$

The first term yields

$$
{}_N\langle 0,\psi|\left(\sum_n \hbar\omega\sigma_3^{(n)}\sin^2\alpha_n(\hat{x},t_1)\right)\left(\sum_m \hbar\omega\sigma_3^{(m)}\sin^2\alpha_m(\hat{x},t_2)\right)|\psi,0\rangle_N
$$

$$
=(\hbar\omega)^2\int dx|\psi(x)|^2\left[\sum_n\sin^2\left(\frac{1}{\hbar}\int_0^{t_1}dt\,V(x+ct-x_n)\right)\right]
$$

$$
\times\left[\sum_m\sin^2\left(\frac{1}{\hbar}\int_0^{t_2}dt'\,V(x+ct'-x_m)\right)\right], \tag{4.19}
$$

which is written, in the Fermi–Yang approximation $V(x)=V_0\Omega\delta(x)$, as

$$
(\hbar\omega)^2\int dx|\psi(x)|^2\left[\sum_n\sin^2\left(\frac{V_0\Omega}{c\hbar}\theta(x+ct_1-x_n)\theta(x_n-x)\right)\right]
$$

$$
\times\left[\sum_m\sin^2\left(\frac{V_0\Omega}{c\hbar}\theta(x+ct_2-x_m)\theta(x_m-x)\right)\right]. \tag{4.20}
$$

The weak-coupling, macroscopic limit (2.13), together with the continuum ansatz (2.15), reduce this quantity to

$$
\left(\hbar\omega\frac{\bar{n}}{L}\right)^2\int dx|\psi(x)|^2\left[(x+ct_1-x_1)\theta(x+ct_1-x_1)\theta(x_N-ct_1-x)\right.
$$

$$
\left.+L\theta(x+ct_1-x_N)\right]
$$

$$
\times\left[(x+ct_2-x_1)\theta(x+ct_2-x_1)\theta(x_N-ct_2-x)+L\theta(x+ct_2-x_N)\right]. \tag{4.21}
$$

Finally, by focusing our attention on the situation in which the Q particle is *inside* D, we obtain

$$
\left(\hbar\omega\frac{\bar{n}}{L}\right)^2\int dx|\psi(x)|^2(x+ct_1-x_1)(x+ct_2-x_1)
$$

$$
=\left(\hbar\omega\frac{\bar{n}}{L}\right)^2(ct_1-x_1)(ct_2-x_1)+\text{b.e.}\quad(\text{for } x_1<ct_{1,2}<x_N). \tag{4.22}
$$

Let us now calculate the second term in Eq. (4.18), the nonvanishing term of which reads

$$
\left(\frac{\hbar\omega}{2}\right)^2\int dx|\psi(x)|^2\sum_n\sin 2\alpha_n(x,t_1)\sin 2\alpha_n(x,t_2). \tag{4.23}
$$

In the usual weak-coupling, macroscopic limit (2.13), we obtain, after a short manipulation,

$$(\hbar\omega)^2 \int dx\,|\psi(x)|^2 \frac{\bar{n}}{L} \int_{x_1}^{x_N} dy\,\theta(x+ct_1-y)\theta(x+ct_2-y)$$

$$= (\hbar\omega)^2 \frac{\bar{n}}{L}[c\min(t_1,t_2)-x_1] + \text{b.e.} \quad (\text{for } x_1 < ct_{1,2} < x_N)\,. \tag{4.24}$$

In conclusion, we are led to the following expression for the two-time correlation function:

$$\langle \Delta H_D(t_1)\Delta H_D(t_2)\rangle \to \left(\hbar\omega\frac{\bar{n}}{L}\right)^2 (ct_1-x_1)(ct_2-x_1)$$

$$+ (\hbar\omega)^2 \frac{\bar{n}}{L}[c\min(t_1,t_2)-x_1] + \text{b.e.}\,, \tag{4.25}$$

which is valid when the Q particle is inside D. The border effects will be discussed in the appendix.

4.3. *The Wiener process*

The results we have obtained so far, (4.16) and (4.25), look quite interesting: Introduce the operator

$$\hat{\Sigma}(t) \equiv \Delta H_D(t) - \langle \Delta H_D(t)\rangle\,, \tag{4.26}$$

where the expectation value is to be evaluated on the state spanned by $|\psi,0\rangle_N \equiv |\psi\rangle \otimes |0\rangle_N$. Then, by Eqs. (4.16) and (4.25), we easily show the following properties

$$\langle \hat{\Sigma}(t)\rangle = 0, \tag{4.27}$$

$$\langle \hat{\Sigma}(t_1)\hat{\Sigma}(t_2)\rangle = \langle \Delta H_D(t_1)\Delta H_D(t_2)\rangle - \langle \Delta H_D(t_1)\rangle\langle \Delta H_D(t_2)\rangle$$

$$\to (\hbar\omega)^2 \frac{\bar{n}}{L}|c\min(t_1,t_2)-x_1] + \text{b.e.}, \tag{4.28}$$

valid in the restricted state space spanned by $|\psi,0\rangle_N$. These properties remind us of the characteristics of a *Wiener stochastic process* [17], i.e., a Gaussian process with a variance proportional to $\min(t,t')$, since the second relation (4.28) can be rewritten as

$$\langle \hat{\Sigma}(t_1)\hat{\Sigma}(t_2)\rangle \to (\hbar\omega)^2 \frac{c\bar{n}}{L}\min(\tau_1,\tau_2) + \text{b.e.}, \tag{4.29}$$

in terms of an "interaction time" $\tau_{1,2} \equiv t_{1,2} - x_1/c$. As a matter of fact, we can prove that the operator $\hat{\Sigma}(t)$ really serves as a Wiener process *in the restricted Hilbert space spanned by* $|\psi, 0\rangle_N$, in the weak-coupling, macroscopic limit. To this end, we must show that the process is Gaussian, namely we must prove that the correlation functions of any order can be written as a sum of products of two-time correlation functions over all possible combinations. This will be done in Section 4.4.

4.4. *Characteristic functional*

In order to demonstrate in full generality the Gaussian property of the process, let us consider the characteristic functional

$$\phi[\beta] \equiv \langle e^{\int dt\, \beta(t)\hat{\Sigma}(t)} \rangle , \tag{4.30}$$

which is subject to the normalization condition

$$\phi[0] = \langle 1 \rangle = {}_N\langle 0, \psi | \psi, 0 \rangle_N = 1 . \tag{4.31}$$

We know that the characteristic functional is the generating functional of correlation functions

$$\langle \hat{\Sigma}(t_1)\hat{\Sigma}(t_2) \cdots \hat{\Sigma}(t_n) \rangle = \left. \frac{\delta^n \phi[\beta]}{\delta\beta(t_1) \cdots \delta\beta(t_n)} \right|_{\beta=0} \tag{4.32}$$

and that Gaussian processes are characterized by Gaussian characteristic functionals.

By making use of (4.26) together with (4.15) and (4.16), we find

$$\phi[\beta] = \int dx |\psi(x)|^2 \left\langle 0 \left| \prod_n e^{-\hbar\omega \int dt\, \beta(t)[\sigma_3^{(n)} \sin^2 \alpha_n(x,t) + \frac{1}{2}(\sigma_+^{(n)}(x) - \sigma_-^{(n)}(x)) \sin 2\alpha_n(x,t)]} \right| 0 \right\rangle_N$$
$$\times e^{-\int dt\, \beta(t) \langle \Delta H_D(t) \rangle} . \tag{4.33}$$

Let us now focus our attention on the factor

$$e^{-\hbar\omega \int dt\, \beta(t)[\sigma_3^{(n)} \sin^2 \alpha_n(x,t) + \frac{1}{2}(\sigma_+^{(n)}(x) - \sigma_-^{(n)}(x)) \sin 2\alpha_n(x,t)]} = e^{a_n [b_n \sigma_3^{(n)} + i(\sigma_+^{(n)}(x) - \sigma_-^{(n)}(x))]}$$
$$\equiv f(a_n, b_n) , \tag{4.34}$$

where we have introduced the quantities a_n, $b_n \in \Re$:

$$a_n \equiv -\frac{\hbar\omega}{2} \int dt\, \beta(t) \sin 2\alpha_n(x,t) , \tag{4.35}$$

$$a_n b_n \equiv -\hbar\omega \int dt\, \beta(t) \sin^2 \alpha_n(x,t) . \tag{4.36}$$

We try a disentanglement of f in the following form:

$$f(a,b) = e^{iX(a,b)\sigma_+(x)} e^{Y(a,b)\sigma_3} e^{-iX(a,b)\sigma_-(x)}, \tag{4.37}$$

where, for the moment, the index n has been suppressed for the sake of simplicity and the functions $X(a,b)$, $Y(a,b) \in \Re$ are to be determined later. The determination of X and Y is straightforward but somewhat involved. Differentiation of f, in (4.34), w.r.t. a yields

$$\frac{\partial f(a,b)}{\partial a} = [b\sigma_3 + i(\sigma_+(x) - \sigma_-(x))] f(a,b), \tag{4.38}$$

while the disentangled form of f in (4.37) implies that the same quantity is to be equated with

$$\left[i\frac{\partial X}{\partial a}\sigma_+(x) + \frac{\partial Y}{\partial a} e^{iX\sigma_+(x)}\sigma_3 e^{-iX\sigma_+(x)} - i\frac{\partial X}{\partial a} e^{iX\sigma_+(x)} e^{Y\sigma_3}\sigma_-(x) e^{-Y\sigma_3} e^{-iX\sigma_+(x)} \right] f(a,b). \tag{4.39}$$

The calculation of the terms in square brackets is simple: The second term is calculated to be

$$e^{iX\sigma_+(x)}\sigma_3 e^{-iX\sigma_+(x)} = \sigma_3 - iX[\sigma_3, \sigma_+(x)] = \sigma_3 - 2iX\sigma_+(x), \tag{4.40}$$

while the third term becomes

$$\begin{aligned} e^{iX\sigma_+(x)} e^{Y\sigma_3}\sigma_-(x) e^{-Y\sigma_3} e^{-iX\sigma_+(x)} &= e^{-2Y} e^{iX\sigma_+(x)}\sigma_-(x) e^{-iX\sigma_+(x)} \\ &= e^{-2Y}(\sigma_-(x) + iX\sigma_3 + X^2\sigma_+(x)). \end{aligned} \tag{4.41}$$

Therefore, we have the equality

$$\begin{aligned} &b\sigma_3 + i(\sigma_+(x) - \sigma_-(x)) \\ &\quad = i\frac{\partial X}{\partial a}\sigma_+(x) + \frac{\partial Y}{\partial a}(\sigma_3 - iX2\sigma_+(x)) - i\frac{\partial X}{\partial a} e^{-2Y}(\sigma_-(x) + iX\sigma_3 + X^2\sigma_+(x)), \end{aligned} \tag{4.42}$$

from which we obtain the following set of differential equations:

$$b = \frac{\partial Y}{\partial a} + X\frac{\partial X}{\partial a} e^{-2Y}, \tag{4.43}$$

$$1 = \frac{\partial Y}{\partial a} - 2X\frac{\partial Y}{\partial a} - X^2\frac{\partial X}{\partial a} e^{-2Y}, \tag{4.44}$$

$$1 = \frac{\partial X}{\partial a} e^{-2Y}. \tag{4.45}$$

We can easily solve these equations since they are equivalent to the following two equations:

$$\frac{\hat{c}X}{\hat{c}a} = 1 + 2bX - X^2 , \tag{4.46}$$

$$\frac{\hat{c}Y}{\hat{c}a} = b - X . \tag{4.47}$$

Under the initial condition $X(0,b) = 0$, the solution of (4.46) is readily obtained

$$X(a,b) = \frac{e^{2a\sqrt{b^2+1}} - 1}{(\sqrt{b^2+1} + b) + (\sqrt{b^2+1} - b)e^{2a\sqrt{b^2+1}}} . \tag{4.48}$$

Then the function Y is calculated, either from (4.47) or (4.45), to be

$$Y(a,b) = a\sqrt{b^2+1} + \ln\left[\frac{2\sqrt{b^2+1}}{(\sqrt{b^2+1} + b) + (\sqrt{b^2+1} - b)e^{2a\sqrt{b^2+1}}}\right] . \tag{4.49}$$

By plugging the solutions (4.48)–(4.49) into disentanglement formula (4.34) we can evaluate the characteristic functional

$$\begin{aligned}\phi(\beta) &= \int dx\,|\psi(x)|^2 \,{}_N\left\langle 0\left|\prod_n e^{iX(a_n,b_n)\sigma_+^{(n)}(x)} e^{Y(a_n,b_n)\sigma_3^{(n)}} e^{-iX(a_n,b_n)\sigma_-^{(n)}(x)}\right|0\right\rangle_N \\ &\quad \times e^{-\int dt\,\beta(t)\langle \Delta H_D(t)\rangle} \\ &= \int dx\,|\psi(x)|^2\, e^{-\sum_n Y(a_n,b_n)} e^{-\int dt\,\beta(t)\langle \Delta H_D(t)\rangle} .\end{aligned} \tag{4.50}$$

Consider now the weak-coupling, macroscopic limit (2.13), together with (2.15). Obviously $a_n, b_n \sim O(1/\sqrt{N}) \to 0$ and keeping only terms up to order $1/N$ in $Y(a_n,b_n)$, we obtain

$$\begin{aligned}&Y(a_n,b_n) \\ &\quad \to a_n\left(1 + \frac{b_n^2}{2}\right) \\ &\qquad + \ln\left[\frac{2(1 + b_n^2/2)}{(1 + b_n + b_n^2/2) + (1 - b_n + b_n^2/2)(1 + 2a_n(1 + b_n^2/2) + (2a_n)^2/2!)}\right] \\ &\quad \simeq -\frac{a_n^2}{2} + a_n b_n .\end{aligned} \tag{4.51}$$

Notice that in this limit, the above qualities are expressed as

$$a_n^2 \to \left[\hbar\omega \int dt\, \beta(t)\alpha_n(x,t)\right]\left[\hbar\omega \int dt'\, \beta(t')\alpha_n(x,t')\right], \tag{4.52}$$

$$a_n b_n \to -\hbar\omega \int dt\, \beta(t)\alpha_n^2(x,t)\,. \tag{4.53}$$

Putting these results together and neglecting all border effects, we finally arrive at the explicit expression of the characteristic functional

$$\begin{aligned}\phi[\beta] \to & \int dx\, |\psi(x)|^2\, e^{((\hbar\omega)^2/2)\sum_n \int dt\, dt'\, \beta(t)\alpha_n(x,t)\alpha_n(x,t')\beta(t')}\, e^{\hbar\omega \sum_n \int dt\, \beta(t)\alpha_n^2(x,t)} \\ & \times e^{-\int dt\, \beta(t)\hbar\omega(\bar{n}/L)(ct - x_1)} \\ \to & \; e^{(1/2)(\hbar\omega)^2(\bar{n}/L)\int dt\, dt'\, \beta(t)[c\min(t,t') - x_1]\beta(t')}\, e^{\int dt\, \beta(t)\hbar\omega(\bar{n}/L)(ct - x_1)} \\ & \times e^{-\int dt\, \beta(t)\hbar\omega(\bar{n}/L)(ct - x_1)} \\ = & \; e^{(1/2)(\hbar\omega)^2(c\bar{n}/L)\int dt\, dt'\, \beta(t)\min(\tau,\tau')\beta(t')}\,,\end{aligned} \tag{4.54}$$

where the interaction time $\tau = t - x_1/c$ has been introduced as before. The characteristic functional turns out to be Gaussian, which proves that the stochastic process under consideration is Gaussian. We understand from the appearance of $\min(\tau,\tau')$ in the exponent, which represents the variance of a Gaussian process, that this process is nothing but a Wiener process. We stress again that this conclusion is only valid in the restricted state space spanned by $|\psi,0\rangle_N$, in the weak-coupling, macroscopic limit.

5. Comments and outlook

We have analyzed the modified Coleman–Hepp model and brought to light a Wiener process in a restricted portion of the total Hilbert space. The operator $\hat{\Sigma}(t)$ becomes a sort of "noise operator" in the $N \to \infty$ limit (2.13), in the sense of Eqs. (4.27) and (4.28). We also proved the Gaussian white noise properties by starting from the characteristic functional (4.30).

Although the appearance of a stochastic process of some sort could probably be expected on the basis of the stochastic behavior of the propagator (2.17), the emergence of the Gaussian white noise is remarkable, for such a nontrivial Hamiltonian like (2.2). We stressed in [24] that the exponential decay form (2.17) of the propagator is *independent* of ω and therefore the presence of an energy-exchange process is essential for the derivation of the Wiener process, through which the energy of the system is

dissipated. We understand that the weak-coupling, macroscopic limit (2.13)–(2.15), that is closely related to Van Hove's limit, plays a crucial role in this respect: It corresponds to a kind of coarse graining and scale-change procedures, some details of which are discussed in the appendix.

It is useful, in this context, to briefly comment on a remark by Leggett [18] that summarizes a widespread opinion among physicists working on these topics. By discussing the role of the environment in connection with the collapse of the wave function, Leggett stressed the central relevance of the problem of dissipation to the quantum measurement theory, and argued that "it is only genuinely dissipative processes, in which the interaction leads to an irreversible exchange of energy between system and environment, which can guarantee that interference is gone beyond the possibility of recovery. Thus, we see that it is not interaction with the environment as such, but specifically dissipation, which is responsible for genuine "decoherence", hence the central relevance of the problem of dissipation to quantum measurement theory". We believe that our analysis contributes to clarify and sharpen the above remark: The behavior just derived, yielding a Wiener process, is certainly related to dephasing ("decoherence") effects of the same kind of those encountered in quantum measurements. The exchange of energy between the particle and the "environment" (our spin system) can be considered practically irreversible. However, the role played by ω is much more subtle, because ω directly contributes to *construct* the stochastic process, as could be seen in Section 4. These remarkable features are manifest in the model here presented and could not be guessed, in our opinion, without an explicit solution. Another important fact, not to be dismissed, is that we have exclusively considered the dynamics within a restricted state space spanned by $|\psi, 0\rangle_N$. We emphasize, once again, that the stochastic process is derived *without* tracing over the states of the macroscopic system D. This is to be contrasted to other work.

We also stress that the link between a dissipative dynamics and a quantum measurement process is not obvious. This point is delicate and somewhat unclear and deserves discussion. In which sense can we talk of "quantum measurement" in the AgBr model? This issue has been analyzed in Refs. [10,11]. One sets up a double-slit experiment, by splitting an incoming Q wave function into two branch waves, only one of which interacts with D. It is possibly to compute exactly several interesting physical quantities, such as the energy "stored" in D after the interaction with Q, as well as the visibility of the interface pattern:

$$\langle H_D \rangle_F = qN\hbar\omega \to \bar{n}\hbar\omega\,, \tag{5.1}$$

$$\mathscr{V} = (1-q)^{N/2} \to e^{-\bar{n}/2} = e^{-\langle H_D\rangle_F/2\hbar\omega}\,, \tag{5.2}$$

where the sandwich is computed over the final state of the $Q + D$ system after the Q particle has gone through D, q is the "spin-flip" probability (2.6) and the arrow denotes, as usual, the weak-coupling, macroscopic limit $N \to \infty$, $qN = \bar{n} =$ finite.

Clearly, the visibility of the interference pattern decreases as $\bar{n}$ increases, so that interference gradually disappears as the average number $\bar{n}$ of spin flips (dissociated molecules) or, equivalently, the energy exchanged between Q and D increases. However, this is *not enough* to state that we are actually facing a genuine dephasing (decoherence) process, leading to a quantum measurement: Technically, "dephasing" consists in the elimination of the off-diagonal elements of the density matrix of the total $(Q + D)$ system. By contrast, here we are simply observing a dynamical process in which the wave functions of the total system do not completely overlap anymore: In order to shed light on the above point, look at the scalar product between the vectors

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi,0\rangle_N \quad \text{and} \quad |\psi_0(t)\rangle = e^{-iH_0t/\hbar}|\psi,0\rangle_N\,, \tag{5.3}$$

obtained by letting $|\psi,0\rangle_N$, in Eq. (4.1), evolve under the action of the total Hamiltonian H and the free Hamiltonian H_0, respectively. We obtain, in the limit of narrow wave packet [19],

$$\langle \Psi_0(t)|\Psi(t)\rangle = {}_N\langle\psi,0|U_I(t)|\psi,0\rangle_N \to \exp\left(-\bar{n}\frac{(ct-x_1)}{2L}\right), \tag{5.4}$$

where $U_I(t)$ is the same operator defined in (2.4). This result is nothing but Eq. (2.17); the above quantity extracts the net effect of the interaction.

One clearly sees that, *in principle*, the quantum coherence can be recovered: Strictly speaking, the underlying dynamics is unitary and there is no irreversible effect. However, the results obtained in the present paper make us understand that an irreversible process of some sort is present, in the AgBr model. How can we reconcile these two apparently contradictory points of view?

In our opinion, what really provokes the appearance of the Wiener process in the present model is the $N \to \infty$ limit, via Van Hove's diagonal singularity. This practically yields a coarse graining procedure over a certain characteristic time, and discards all effects stemming from fine oscillations over small time periods. In such a limit, the AgBr chain of spins makes a transition from the ordinary unitary representation to a unitary-inequivalent one. Such a phenomenon is characteristic of the many-Hillbert-space theory [14] and yields decoherence effects. Admittedly, we are entering a domain of speculation that should be corroborated by more clear-cut arguments. We feel entitled to put forward the above qualitative comments because the mechanisms at the origin of the stochastic process and the very transition to the unitary inequivalent representation in the present model are not completely clear to us.

We would also like to emphasize that it is not entirely trivial to bring to light the dissipative dynamical processes constituting a quantum measurement: In general, a thermal irreversible process is a probabilistic one, described by master equations, that characterize the approach to thermal equilibrium. On the other hand, in a quantum measurement process, the evolution leads to the so-called collapse of the wave function. The final density matrix, that does not contain off-diagonal terms, depends

on the measured observable, on the way one performs the spectral decomposition and on the very measuring apparatus [14,15]. The description of the loss of quantum mechanical coherence in terms of dissipative equations, governing the evolution toward an equilibrium situation of some sort, is therefore a delicate problem, that deserves further investigation.

There are other interesting open problems. For instance, it is well known that the reduced dynamics of a (sub)system in interaction with a larger system (playing the role of reservoir) is well described in terms of quantum dynamical semigroups [25], so that one should be able to derive a master (or Langevin) equal [26] for some dynamical variables of the subsystem (such as the energy-momentum of the Q particle). The derivation of a master or a Langevin equation in the present model would open a door to thoroughly investigate a possible link between a quantum measurement and a genuine dissipative process.

Appendix

In this appendix we shall consider the more realistic situation in which the potential $V(x)$ has a finite width. We shall consider, for simplicity, a square wave packet and potential (actually, the requirement of compact support for ψ and V would suffice)

$$\psi(x) = \frac{1}{\sqrt{2a}}\theta(a-x)\theta(x+a)e^{ip_0 x/\hbar}, \tag{A.1}$$

$$V(x) = V_0\theta\left(\frac{\Omega}{2}-x\right)\theta\left(x+\frac{\Omega}{2}\right). \tag{A.2}$$

In this case the expression (2.7) for α_n becomes

$$\begin{aligned}\alpha_n \equiv \alpha_n(x,t) &= \frac{V_0\Omega}{\hbar c}\int_{x-x_n}^{x+ct-x_n}\frac{dy}{\Omega}\theta\left(\frac{\Omega}{2}-y\right)\theta\left(y+\frac{\Omega}{2}\right)\\ &= \frac{\sqrt{q}}{\Omega}\left[\min\left(\frac{\Omega}{2}, x+ct-x_n\right)-\max\left(x-x_n, -\frac{\Omega}{2}\right)\right].\end{aligned} \tag{A.3}$$

The expectation value of the operator $\Delta H_D(t)$ in (4.15), relative to the initial state $|\psi,0\rangle_N$ reads

$$\begin{aligned}\langle\Delta H_D(t)\rangle &= \hbar\omega\int dx|\psi(x)|^2\sum_n \sin^2\alpha_n(x,t)\\ &\to \hbar\omega\frac{\bar{n}}{L}\int_{-a}^{a}\frac{dx}{2a}\int_{x_1}^{x_N}\frac{dz}{\Omega^2}\left[\min\left(\frac{\Omega}{2}, x+ct-z\right)-\max\left(x-z, -\frac{\Omega}{2}\right)\right]^2,\end{aligned} \tag{A.4}$$

where, in the last step, we have considered the weak-coupling, macroscopic limit (2.13). We consider now the case $ct > \Omega$ and focus our attention on the situation in which the wave packet is fully inside the potential. This means that

$$x_1 + \frac{\Omega}{2} + a < ct < x_N - \frac{\Omega}{2} - a\,. \tag{A.5}$$

By using (A.5) and since $x_1 \gg \Omega, a$, (A.4) simply becomes

$$\hbar\omega\frac{\bar{n}}{L}\int_{-a}^{a}\frac{dx}{2a}\left[\int_{x_1}^{x+ct-\Omega/2}\frac{dz}{\Omega^2}\Omega^2 + \int_{x+ct-\Omega/2}^{x+ct+\Omega/2}\frac{dz}{\Omega^2}\left(x+ct+\frac{\Omega}{2}-z\right)^2\right]$$

$$= \hbar\omega\frac{\bar{n}}{L}\int_{-a}^{a}\frac{dx}{2a}\left(x+ct-\frac{\Omega}{6}-x_1\right) = \hbar\omega\frac{\bar{n}}{L}\left(ct-\frac{\Omega}{6}-x_1\right). \tag{A.6}$$

Notice that there is no effect due to the wave packet width, since it is considered entirely inside the detector, but the finite width Ω of the potential appears in the above formula, in contrast with (4.16). By following a procedure similar to the previous one, we can compute the second-order correlation function of the operator $\Delta H_D(t)$ and, by making use of the definition (4.26) of $\hat{\Sigma}(t)$ we finally obtain

$$\langle\hat{\Sigma}(t)\rangle = 0\,,$$

$$\langle\hat{\Sigma}(t_1)\hat{\Sigma}(t_2)\rangle \to (\hbar\omega)^2\frac{\bar{n}}{L}[\theta(\Delta t-\Omega/c)(ct_1-x_1)$$

$$+\theta(\Delta t)\theta(\Omega/c-\Delta t)\{ct_2-\Omega-x_1-h(-\Delta t,\Omega)\}$$

$$+\theta(-\Delta t)\theta(\Omega/c+\Delta t)\{ct_1-\Omega-x_1-h(-\Delta t,\Omega)\}$$

$$+\theta(-\Delta t-\Omega/c)(ct_2-x_1)]\,, \tag{A.7}$$

where $\Delta t \equiv t_2 - t_1$ and

$$h(t,\Omega) = \frac{1}{6\Omega^2}(ct+\Omega)((ct+\Omega)^2-6\Omega^2)\,. \tag{A.8}$$

From the previous two equations we can observe that, in contrast with (4.27) and (4.28), when the finite width of the potential is taken in account, $\hat{\Sigma}(t)$ is not a Wiener process anymore, unless $\Omega \to 0$ (δ-potential limit). Incidentally, if the time scale is changed like $t = \lambda\bar{t}$ (where t and $\bar{t}$ can be regarded as a microscopic and a macroscopic time respectively) and if we define $\hat{W}(\bar{t}) \equiv \hat{\Sigma}(t)$, then

$$\langle\hat{W}(\bar{t})\rangle = 0 \tag{A.9}$$

$$\langle \hat{W}(\bar{t}_1)\,\hat{W}(\bar{t}_2)\rangle \to (\hbar\omega)^2\frac{\bar{n}}{\bar{L}}[\theta(\Delta\bar{t}-\Omega/\lambda c)(c\bar{t}_1-\bar{x}_1)$$

$$+\theta(\Delta\bar{t})\theta(\Omega/\lambda c-\Delta\bar{t})\{c\bar{t}_2-\Omega/\lambda-\bar{x}_1-h(-\Delta\bar{t},\Omega/\lambda)\}$$

$$+\theta(-\Delta\bar{t})\theta(\Delta\bar{t}+\Omega/\lambda c)\{c\bar{t}_1-\Omega/\lambda-\bar{x}_1-h(\Delta\bar{t},\Omega/\lambda)\}$$

$$+\theta(-\Delta\bar{t}-\Omega/\lambda c)(c\bar{t}_2-\bar{x}_1)]\,, \tag{A.10}$$

where $\bar{x}_1 = x_1/\lambda$ and $\bar{L} = L/\lambda$. In this case, only when $|\bar{t}_2 - \bar{t}_1| \gg \Omega/\lambda c$ (or $\lambda \to \infty$ with $\bar{x}_1, \bar{L}, \bar{t} < \infty$, which is equivalent to a time scale transformation), we reobtain a proper Wiener process. In other words, the δ-potential limit can be *regarded as a realization of the macroscopic time-scale transformation.*

The above considerations bring to light the close link with Van Hove's $\lambda^2 T$ limit, as discussed in [11]. This can be easily evinced by observing that q, in Eq. (2.13), is nothing but the square of a coupling constant (Van Hove's λ), and that $N(\propto L)$ can be considered proportional to the total interaction time T. Notice also that the "lattice spacing" d, the inverse of which corresponds to a density in our one-dimensional model, can be kept finite in the limit. Obviously, in such a case, we have to express everything in terms of scaled variables such as $\bar{t} = t/\lambda$, $\bar{x} = x/\lambda$ and $\zeta \equiv a/L$, where a is the size of the wave packet.

References

[1] L. Van Hove, Physica 21 (1955) 517. See also E.B. Davies, Commun. Math. Phys. 39 (1974) 91; G.L. Sewell, Quantum Theory of Collective Phenomena, Clarendon Press, Oxford, 1986.

[2] W. Pauli, Festschrift zum 60 Geburtstage A. Sommerfelds, Hirzel, Leipzig, 1928, p. 30.

[3] V.B. Magalinskij, Zh. Eksp. Teor. Fiz. 36 (1959) 1942 [Sov. Phys. JEPT 9 (1959) 1381]; G.W. Ford, M. Kac, P. Mazur, J. Math. Phys. 9 (1965) 504; P. Ullersma, Physica 32 (1966) 27; R. Zwanzig, J. Stat. Phys. 9 (1973) 215.

[4] A.O. Caldeira, A.J. Leggett, Phys. Rev. Lett. 46 (1981) 211; Physica A 121 (1983) 587; Phys. Rev. A 31 (1985) 1059; R.P. Feynman, F.L. Vernon, Ann. Phys. (NY) 24 (1963) 118.

[5] G.W. Ford, J.T. Lewis, R.F.O'Connell, Phys. Rev. A 37 (1988) 4419.

[6] M. Mizutani, S. Muroya, M. Namiki, Phys. Rev. D 37 (1988) 3033.

[7] C.W. Gardiner, Quantum Noise, Springer, Berlin, 1991; R. Blasi, S. Pascazio, Phys. Rev. A 53 (1996) 2033.

[8] K. Hepp, Helv. Phys. Acta 45 (1972) 237.

[9] J.S. Bell, Helv. Phys. Acta 48 (1975) 93; S. Machida, M. Namiki, In: S. Kamefuchi et al. (Eds.), Proc. Int. Symp. Foundations of quantum Mechanics, Phys. Soc. Japan, Tokyo, 1984, p. 136; S. Kudaka, S. Matsumoto, K. Kakazu, Prog. Theor. Phys. 82 (1989) 665. See also other references in [10].

[10] H. Nakazato, S. Pascazio, Phys. Rev. Lett. 70 (1993) 1; Phys. Rev. A 48 (1993) 1066.

[11] H. Nakazato, M. Namiki, S. Pascazio, Phys. Rev. Lett. 73 (1994) 1063.

[12] J. von Neumann, Die Mathematische Grundlagen der Quantenmechanik, Springer, Berlin, 1932.

[13] For a review, see in: J.A. Wheeler, W.H. Zurek (Eds.), Quantum Theory and Measurement, Princeton University Press, Princeton, 1983; P. Busch, P.J. Lahti, P. Mittelstaedt, The Quantum Theory of Measurement, Springer, Berlin, 1991; M. Namiki, S. Pascazio, Phys. Rep. 232 (1993) 301.

[14] S. Machida, M. Namini, Prog. Theor. Phys. 63 (1980) 1457; 1833; M. Namiki, Found. Phys. 18 (1988) 29; M. Namiki, S. Pascazio, Phys. Rev. A 44 (1991) 39.

[15] W.H. Zurek, Phys. Rev. D 24 (1981) 1516; D 26 (1982) 1862; J.P. Paz, S. Habib, W.H. Zurek, Phys. Rev. D 47 (1993) 488.
[16] N.G. van Kampen, Physica A 153 (1988) 97.
[17] N. Wax (Ed.), Selected Papers on Noise and Stochastic Processes, Dover, New York, 1954; N.G. van Kampen, Stochastic Processes in Physics and Chemistry, North-Holland, Amsterdam, 1981; C.W. Gardiner, Handbook of Stochastic Methods, Springer, Berlin, 1990.
[18] A.J. Leggett, in: Tsukada et al. (Eds.), Proc. 4th Int. Symp. Foundations of Quantum Mechanics, Phys. Soc. Japan, Tokyo, 1993, p. 10.
[19] H. Nakazato, M. Namiki, S. Pascazio, Int. J. Mod. Phys. B 10 (1996) 247.
[20] C.P. Sun, Phys. Rev. A 48 (1993) 898; K. Hiyama, S. Takagi, Phys. Rev. A 48 (1993) 2568; X.J. Liu, C.P. Sun, Phys. Lett. A 198 (1995) 371; Y.Q. Li, Y.X. Chen, Phys. Lett. A 202 (1995) 325.
[21] A. Beskow, J. Nilsson, Arkiv för Fysik 34 (1967) 561; L.A. Khalfin, Zh. Eksp. Teor. Fiz. Pis. Red. 8 (1968) 106 [JETP Lett. 8 (1968) 65]; L. Fonda, G.C. Ghirardi, A. Rimini, T. Weber, Nuovo Cimento A 15 (1973) 689; A 18 (1973) 805; A. DeGasperis, L. Fonda, G.C. Ghirardi, Nuovo Cimento A 21 (1974) 471; B. Misra, E.C.G. Sundarshan, J. Math. Phys. 19 (1977) 756.
[22] S. Pascazio, M. Namiki, G. Badurek, H. Rauch, Phys. Lett. A 179 (1993) 155; S. Pascazio, M. Namiki, Phys. Rev. A 50 (1994) 4582; H. Nakazato, M. Namiki, S. Pascazio, H. Rauch, Phy. Lett.. A 199 (1995) 27; A 217 (1996) 203.
[23] E.J. Hellund, Phys. Rev. 89 (1953) 919;M. Namiki, N. Mugibayashi, Prog. Theor. Phys. 10 (1953) 474; L.A. Khalfin, Dokl. Acad. Nauk USSR 115 (1957) 277 [Sov. Phys. Dokl. 2, 340 (1957)]; Zh. Eksp. Teor. Fiz. 33 (1958) 1371 [Sov. Phys. JETP 6 (1958) 1053].
[24] R. Blasi, H. Nakazato, M. Namiki, S. Pascazio, Dissipative behavior of a quantum system interacting with a macroscopic medium, Phys. Lett. A 223 (1996) 320.
[25] G. Lindblad, Comm. Math. Phys. 48 (1976) 119; V. Gorini, A. Kossakowski, E.C.G. Sudarshan, J. Math. Phys. 17 (1976) 821; R. Alicki, K. Lendi, Quantum Dynamical Semigroups and Applications, Lecture Notes in Physics, Vol. 286, Springer, Berlin, 1987; A.S. Holevo, Proc. Steklov Inst. Math. 191 (1989) 131.
[26] L. Accardi, A. Frigerio, Y.G. Lu, Comm. Math. Phys. 131 (1990) 537; L. Accardi, Y.G. Lu, I. Volovich, Quantum Probab. Related Topics 9 (1994) 15.

Effects of inelastic scattering on tunneling time based on the generalized diffusion process approach

Kentaro Imafuku,[1,*] Ichiro Ohba,[1,2,3,†] and Yoshiya Yamanaka[4,‡]
[1]*Department of Physics, Waseda University Shinjuku-ku, Tokyo 169, Japan*
[2]*Advanced Research Center for Science and Engineering, Waseda University Shinjuku-ku, Tokyo 169, Japan*
[3]*Kagami Memorial Laboratory for Materials Science and Technology, Waseda University Shinjuku-ku, Tokyo 169, Japan*
[4]*Waseda University Senior High School, Nerima-ku, Tokyo 177, Japan*
(Received 6 November 1996; revised manuscript received 11 April 1997)

We analyze the effects of inelastic scattering on the tunneling time theoretically, using generalized Nelson's quantum mechanics. This generalization enables us to describe a quantum system with optical potential and channel couplings in a real-time stochastic approach, which seems to give us a new insight into quantum mechanics beyond Copenhagen interpretation. [S1050-2947(97)01408-X]

PACS number(s): 03.65.Bz

I. INTRODUCTION

An issue of the tunneling time, i.e., the time associated with the passage of a particle through a tunneling barrier, has been discussed in many theoretical studies [1–17], and is not settled yet. This difficulty arises mainly from the fact that time is not an observable represented by a self-adjoint operator, but is just a parameter in quantum mechanics.

In our previous paper [20], we proposed a method to evaluate the tunneling time, using Nelson's approach of quantum-mechanics [18]. Our aim then was to treat tunneling effects in a detailed time-dependent and fully quantum-mechanical way, as any theoretical expression of the tunneling time must be tested by experiments which are feasible at present and in the near future.

As discussed in Ref. [20], Nelson's approach to quantum mechanics has several advantages to study the tunneling time, a few of which are listed below. First of all, this approach, using the real-time stochastic process, enables us to describe individual experimental runs of a quantum system in terminology of the "analog" of classical mechanics. This is true even in the tunnel region where a classical path is forbidden. From sample paths generated by the stochastic process, we obtain information on the time parameter, in particular, the tunneling time.

As a matter of course, the whole ensemble of sample paths gives us the same results as quantum mechanics in the ordinary approach, e.g., expectation values of the observable, transmission and reflection probabilities in scattering problem, and so on. It is important for us to note that in scattering phenomena (those without bound states) the transmission and reflection ensembles are defined unambiguously, that is, each sample path is classified distinctively into either a transmission ensemble or reflection one.

We need to accumulate a sufficient number of sample paths in numerical simulations. In thick or/and high potential cases the transmission probability is low, and consequently we have the difficulty that a number of sample paths belonging to the transmission ensemble are also low when each sample path is followed in the forward time direction. However, in Nelson's approach there is not only the forward Langevin equation but also the backward Langevin equation [see Eq. (2) below], both being equivalent to each other in physical results. The difficulty above is avoided when the backward Langevin equation is employed.

Taking account of these advantages, we developed a theoretical of time-dependent description of tunneling phenomena based on Nelson's stochastic approach in Ref. [20]. Numerical simulations for a one-dimensional square-well potential barrier model were demonstrated. An important result about the tunneling time then is that there are three characteristic times, i.e., *the passing time and the hesitating time*, and their sum, *the interacting time*. The probability distribution of these three times were calculated numerically.

Our previous study treated only a quantum system of a single particle under a simple potential. But realistic experimental situations are more complicated. Naturally we are tempted to extend our previous formulation to more general scattering phenomena. In this paper we consider cases in which transition processes into other channels or absorptive processes takes place during scattering processes, and look into these effects on the tunneling time.

Processes of transition into other channels and absorption are described by channel coupling and optical potential (complex potential), respectively, in ordinary quantum mechanics using the Schrödinger equation. So far it is known that Nelson's formulation is equivalent to the Schrödinger equation *only for a one-body problem with a single channel and a real potential.* The purpose of this paper is to generalize Nelson's stochastic quantization so that it can deal with multichannel coupling and/or optical potential problems. As will be shown below, one can construct such generalized formulations of Nelson's approach with additional stochastic jumping processes. These theoretical formulations allow us to perform numerical simulations of stochastic processes as before [20]. This way we can investigate the effects of transition into other channels, or absorption on the tunneling time.

*Electronic address: imafuku@mn.waseda.ac.jp
†Electronic address: ohba@mn.waseda.ac.jp
‡Electronic address: yamanaka@mn.waseda.ac.jp

The paper is organized as follows: In Sec. II the original Nelson's quantum mechanics is reviewed briefly for later relevance. We propose a formulation of the Nelson's approach, generalized to a quantum system with channel coupling, in Sec. III. The formulation of Sec. III hints at how to develop a formulation for optical potential, which is shown in Sec. IV. In Sec. V a numerical simulation for a square-well potential model, using the formulations in Secs. III and IV, are demonstrated, and physical implications of these results are analyzed. Section VI is devoted to summary and some comments.

II. BRIEF REVIEW OF NELSON'S QUANTUM MECHANICS

We start with a brief review of the original Nelson's quantum mechanics, which consists of two basic conditions, i.e., the kinematical condition and the dynamical one. The kinematical condition is given by the Ito-type stochastic differential equation: There are two ways to express it, depending on the forward or backward time direction. Explicitly we have, for forward time evolution,

$$dx(t)=b(x(t),t)dt+dw(t), \tag{1}$$

and, for backward time evolution,

$$dx(t)=b_*(x(t),t)dt+dw_*(t). \tag{2}$$

The $dw(t)$ is the Gaussian white noise (representing the quantum fluctuation) with the statistical properties of

$$\langle dw(t)\rangle=0 \quad \text{and} \quad \langle dw(t)dw(t)\rangle=\frac{\hbar}{m}dt, \tag{3}$$

and the same properties for $dw_*(t)$ as in Eq. (3). Here $\langle\ \rangle$ means a sample average. It is easy to show that for these two Langevin equations hold the following Fokker-Planck equations for the distribution function $P(x,t)$ of the random variables $x(t)$,

$$\frac{\partial P(x,t)}{\partial t}=\left[-\frac{\partial}{\partial x}b(x,t)+\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\right]P(x,t) \quad \text{(forward in } t\text{)}, \tag{4}$$

$$-\frac{\partial P(x,t)}{\partial t}=\left[\frac{\partial}{\partial x}b_*(x,t)+\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\right]P(x,t) \quad \text{(backward in } t\text{)}. \tag{5}$$

Thus a pair of equations (1) and (2) is mathematically equivalent to a pair of equations (4) and (5). We obtain an osmotic velocity u from the sum of Eqs. (4) and (5) as

$$u=\frac{b-b_*}{2}=\frac{\hbar}{2m}\frac{1}{P}\frac{\partial P}{\partial x} \tag{6}$$

under the boundary condition of

$$P(x\to\infty,t)\to 0. \tag{7}$$

Subtraction of Eq. (5) from Eq. (4) gives

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}(vP), \tag{8}$$

where v is a current velocity,

$$v=\frac{b+b_*}{2}. \tag{9}$$

The elimination of $P(x,t)$ from Eqs. (6) and (8) leads to an equation called the kinematical equation,

$$\frac{\partial u}{\partial t}=-\frac{\hbar}{2m}\frac{\partial^2 v}{\partial x^2}-\frac{\partial}{\partial x}(uv). \tag{10}$$

The dynamical condition is expressed through the "mean time derivatives" introduced as follows: The "mean forward time derivative" $Df(t)$ is defined as

$$Df(t)\equiv\lim_{\Delta t\to+0}\left\langle\frac{f(t+\Delta t)-f(t)}{\Delta t}\middle| f(s)(s\le t) \text{ fixed}\right\rangle, \tag{11}$$

and the "mean backward time derivative" $D_*f(t)$ is defined as

$$D_*f(t)\equiv\lim_{\Delta t\to+0}\left\langle\frac{f(t)-f(t-\Delta t)}{\Delta t}\middle| f(s)(s\ge t) \text{ fixed}\right\rangle. \tag{12}$$

The "mean balanced acceleration" is introduced through the definitions of Eqs. (11) and (12) as

$$a(x(t),t)\equiv\frac{DD_*+D_*D}{2}x(t). \tag{13}$$

Note that this definition can be rewritten as

$$a(x,t)=-\frac{\hbar}{2m}\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\frac{\partial}{\partial x}(v^2-u^2)+\frac{\partial v}{\partial t} \tag{14}$$

from Eqs. (1) and (2) with Eqs. (6) and (9). The dynamical condition is nothing but the classical Newton equation to this "mean balanced acceleration" $a(x(t),t)$, that is,

$$ma(x,t)=-\frac{\partial V}{\partial x}, \tag{15}$$

from which we derive the "Newton-Nelson equation"

$$\frac{\partial v}{\partial t}=\frac{\hbar}{2m}\frac{\partial^2 u}{\partial x^2}-v\frac{\partial v}{\partial x}+u\frac{\partial u}{\partial x}-\frac{1}{m}\frac{\partial V}{\partial x} \tag{16}$$

because of Eq. (14).

Next we summarize the mathematical structure of Nelson's quantum mechanics. The two basic equations, Eq. (10) from the kinematical condition, and Eq. (16) from the dynamical condition, form a set of simultaneous equations for two unknown functions $u(x,t)$ and $v(x,t)$, or equivalently, $b(x,t)$ and $b_*(x,t)$. Then we can determine the ensemble of sample paths or the distribution function $P(x,t)$. Although it is practically very difficult to solve these equations directly due to their nonlinearity, one can easily show the equiva-

lence between this approach and the ordinary approach of the Schrödinger equation. This fact tells us that one can solve the problem by means of the wave function much more easily. The equation

$$\frac{\partial}{\partial x}\left[i\frac{\hbar}{m}\frac{1}{\psi'}\frac{\partial\psi'}{\partial t}+\frac{1}{2}\left(\frac{\hbar}{m}\right)^2\frac{1}{\psi'}\frac{\partial^2\psi'}{\partial x^2}-\frac{1}{m}V\right]=0 \quad (17)$$

follows from the combination of Eq. (10) and the imaginary version of Eq. (16), where

$$u+iv=\frac{\hbar}{m}\frac{1}{\psi'}\frac{\partial\psi'}{\partial x}. \quad (18)$$

Equation (17) clearly shows the relationship between ψ' and the wave function ψ as the solution of Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V\right)\psi, \quad (19)$$

that is,

$$\psi(x,t)=\psi'(x,t)\exp\left(-\frac{im}{\hbar}\int^t\eta(s)ds\right), \quad (20)$$

with an arbitrary function of t, $\eta(t)$, which has no physical relevance. It is easily seen from this proof of the equivalence that one has expressions for $b(x,t)$, $b_*(x,t)$, and $P(x,t)$ in terms of $\psi(x,t)$,

$$b(x,t)=\frac{\hbar}{m}(\mathrm{Im}+\mathrm{Re})\frac{\partial}{\partial x}\ln\psi(x,t), \quad (21)$$

$$b_*(x,t)=\frac{\hbar}{m}(\mathrm{Im}-\mathrm{Re})\frac{\partial}{\partial x}\ln\psi(x,t), \quad (22)$$

$$P(x,t)=|\psi(x,t)|^2. \quad (23)$$

III. STOCHASTIC FORMULATION FOR QUANTUM SYSTEM WITH CHANNEL COUPLING

We now generalize the above Nelson's approach to a system with a channel coupling. For simplicity, consider the two-channel Schrödinger equations ($\{i,j\}=\{1,2\}$)

$$i\hbar\frac{\partial}{\partial t}\psi_i(x,t)=\left(-\frac{\hbar^2}{2m_i}\frac{\partial^2}{\partial x^2}+V_{ii}(x,t)\right)\psi_i(x,t)+V_{ij}(x,t)\psi_j(x,t), \quad (24)$$

with

$$V_{ij}=V_{ji}^*. \quad (25)$$

Here and below the dummy index does not imply taking a sum. As will be seen, the generalization of the formulation in this section to the N-channel case ($N>2$) is straightforward.

Consider the Fokker-Planck equations in the stochastic formulation, corresponding to Eq. (24). First we require a natural extension of Eq. (23) to the present case,

$$P_i(x,t)=|\psi_i(x,t)|^2. \quad (26)$$

The diagonal parts (the kinetic energy and V_{ii} terms) in Eq. (24) are expected to be dealt with as in Sec. II. The Schrödinger equations (24) and their complex conjugates suggest the following equations for $P_i(x,t)$:

$$\frac{\partial P_i(x,t)}{\partial t}=\left[-\frac{\partial}{\partial x}b_i(x,t)+\frac{\hbar}{2m_i}\frac{\partial^2}{\partial x^2}-W_{(i\to j)}(x,t)\right]P_i(x,t)$$

(forward in time), (27)

$$-\frac{\partial P_i(x,t)}{\partial t}=\left[\frac{\partial}{\partial x}b_{*i}(x,t)+\frac{\hbar}{2m_i}\frac{\partial^2}{\partial x^2}+W_{(i\to j)}(x,t)\right]P_i(x,t)$$

(backward in time) (28)

as $P_i(x,t)$ increases or decreases, due to the potential V_{ij} causing transitions between i and j, at the rate of the absolute value of

$$W_{(i\to j)}P_i=-W_{(j\to i)}P_j=\frac{2}{\hbar}\mathrm{Im}\psi_j^*V_{ji}\psi_i. \quad (29)$$

Although the sum of Eq. (27) and (28) leads to Eq. (6) with the index i,

$$u_i=\frac{b_i-b_{*i}}{2}=\frac{\hbar}{2m_i}\frac{1}{P_i}\frac{\partial P_i}{\partial x}, \quad (30)$$

their difference provides us with

$$\frac{\partial P_i}{\partial t}=-\frac{\partial}{\partial x}(v_iP_i)-W_{(i\to j)}P_i \quad (31)$$

instead of Eq. (8), where

$$v_i=\frac{b_i+b_{*i}}{2}. \quad (32)$$

As a result, eliminating $P_i(x,t)$ from Eqs. (30) and (31), one derives the kinematical equation

$$\frac{\partial u_i}{\partial t}=-\frac{\hbar}{2m_i}\frac{\partial^2 v_i}{\partial x^2}-\frac{\partial}{\partial x}(u_iv_i)-\frac{\hbar}{2m_i}\frac{\partial}{\partial x}W_{(i\to j)} \quad (33)$$

instead of Eq. (10).

Here arises a natural question what are the stochastic differential equations corresponding to the Fokker-Planck equations in Eqs. (27) and (28), just as Eqs. (1) and (2) correspond to Eqs. (4) and (5). Apparently we need two random variables $x_i(t)$ ($i=1$ and 2), which are assumed to be subject to the stochastic differential equations, similar to Eqs. (1) and (2),

$$dx_i(t)=b_i(x_i(t),t)dt+dw_i(t) \quad \text{(forward in time)}, \quad (34)$$

$$dx_i(t)=b_{*i}(x_i(t),t)dt+dw_{*i}(t) \quad \text{(backward in time)}, \quad (35)$$

with the properties for $dw_i(t)$ and $dw_{*i}(t)$,

$$\langle dw_i(t)\rangle = 0, \quad \langle dw_i(t)dw_j(t)\rangle = \frac{\hbar}{m_i}\delta_{ij}dt, \tag{36}$$

$$\langle dw_{*i}(t)\rangle = 0, \quad \langle dw_{*i}(t)dw_{*j}(t)\rangle = \frac{\hbar}{m_i}\delta_{ij}dt.$$

As is easily seen, a naive interpretation of these independent stochastic differential equations leads only to the Fokker-Planck equations in Eqs. (27) and (28) *without the terms proportional to* $W_{(i\to j)}$. An additional mechanism to take account of the quantum jump between i and j represented by the terms involving $W_{(i\to j)}$ is necessary. For this purpose we supplement Eqs. (34) and (35) with a stochastic jumping process between i and j. Thus below we attempt the formulation of two random variables $x_i(t)$, subject to the stochastic differential equations (34) and (35) combined with a stochastic jumping process in the following way.

The "dynamical" rule to determine how each sample path $x_i(t)$ changes its index ($i=1\to 2$, or vice versa) during passage of time is described by the following random jumping process (Fig. 1): At each time a dice is cast, *independently of the stochastic equations* (34) and (35), and each sample path either keeps or changes its index at a certain rate. For the forward time direction, we have the rule in case of $W_{(i\to j)}>0$ $(i\neq j)$,

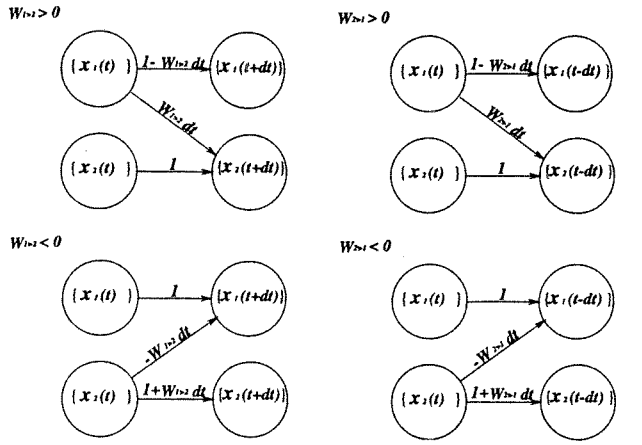

FIG. 1. Schematical illustration of the "dynamical" rule for the stochastic jumping process between two channels.

$$x_i(t)\to\begin{cases} x_j(t+dt) & \text{with the probability } W_{(i\to j)}(x_i(t),t)dt,\\ x_i(t+dt) & \text{with the probability of } 1-W_{(i\to j)}(x_i(t),t)dt,\end{cases}$$
$$x_j(t)\to x_j(t+dt) \quad \text{with the probability 1,} \tag{37}$$

and the rule in case of $W_{(i\to j)}<0$,

$$x_j(t)\to\begin{cases} x_i(t+dt) & \text{with the probability } -W_{(i\to j)}(x_j(t),t)dt,\\ x_j(t+dt) & \text{with the probability } 1+W_{(i\to j)}(x_j(t),t)dt,\end{cases}$$
$$x_i(t)\to x_i(t+dt) \quad \text{with the probability 1.} \tag{38}$$

Likewise, the rules for backward time direction state that, in the case of $W_{(i\to j)}>0$,

$$x_j(t)\to\begin{cases} x_i(t-dt) & \text{with the probability } W_{(i\to j)}(x_j(t),t)dt,\\ x_j(t-dt) & \text{with the probability } 1-W_{(i\to j)}(x_j(t),t)dt,\end{cases}$$
$$x_i(t)\to x_i(t-dt) \quad \text{with the probability 1,} \tag{39}$$

and, in the case of $W_{(i\to j)}<0$,

$$x_i(t)\to\begin{cases} x_j(t-dt) & \text{with the probability } -W_{(i\to j)}(x_i(t),t)dt,\\ x_i(t-dt) & \text{with the probability } 1+W_{(i\to j)}(x_i(t),t)dt,\end{cases}$$
$$x_j(t)\to x_j(t-dt) \quad \text{with the probability 1.} \tag{40}$$

According to the rules of the random jumping process above, the behavior of each sample path is illustrated as follows: For the forward time direction, a sample path starts from $x_i(t_I)$, develops according to Eq. (34) with i for a while, and, when a chance comes, it changes its index from i to j and follows Eq. (34) with j until the next jumping process takes place. The jumping process from x_i to x_j is allowed, and the reverse process is forbidden, where $W_{(i\to j)}>0$, and vice versa where $W_{(i\to j)}<0$. The jumping processes may be repeated or may not occur, depending on the sign and magnitude of $W_{(i\to j)}$. Sample paths show a similar behavior for the backward time direction.

It is remarked that $x_i(t)$ is generally a functional of both $dw_1(s)$ and $dw_2(s)$ $(s<t)$ [or $dw_{*1}(s)$ or $dw_{*2}(s)$

$(s>t)$], as it may repeat jumps between $i=1$ and 2 in the past (in the future). Due to changes in the index for each sample path, there are several types of averages which are distinguished from each other carefully. It is convenient to introduce notations for conditional averages. The simple average $\langle\ \rangle$ should be taken over both of $dw_1(s)$ and $dw_2(s)$ $(s<t)$. To represent a physical average of the i state at t, we introduce a notation of

$$\langle\langle f(x(t))\rangle\rangle_{\{x_i(t)\}} \equiv \langle f(x_i(t))\rangle, \tag{41}$$

where the average on the left-hand side implies a conditional average only over sample paths, labeled by i at t. This average should be expressed in terms of the probability distribution $P_i(x,t)$ as

$$\langle\langle f(x(t))\rangle\rangle_{\{x_i(t)\}} = \int dx\, f(x) P_i(x,t). \tag{42}$$

The notation $\langle\langle f(x(t))\rangle\rangle_{\{x_1(t)\}\cup\{x_2(t)\}}$ has trivial interpretations,

$$\langle\langle f(x(t))\rangle\rangle_{\{x_1(t)\}\cup\{x_2(t)\}} = \langle f(x(t))\rangle. \tag{43}$$

Furthermore, conditional averages with different times such as $\langle\langle f(x(t))\rangle\rangle_{\{x_i(t+dt)\}\cap\{x_j(t)\}}$ can be introduced: This example represents the average only over sample paths which have the index j at t and i at $t+dt$.

Let us now evaluate the time derivative of the physical average $\langle\langle f(x(t))\rangle\rangle_{\{x_i(t)\}}$. For the forward time direction, using appropriate conditional averages, we write

$$\begin{aligned}
&\frac{d}{dt}\langle\langle f(x(t))\rangle\rangle_{\{x_i(t)\}} \\
&= \frac{1}{dt}[\langle\langle f(x(t+dt))\rangle\rangle_{\{x_i(t+dt)\}} - \langle\langle f(x(t))\rangle\rangle_{\{x_i(t)\}}] \\
&= \frac{1}{dt}[\langle\langle f(x(t+dt)) - f(x(t))\rangle\rangle_{\{x_i(t+dt)\}\cap\{x_i(t)\}} \\
&\quad + \langle\langle f(x(t+dt))\rangle\rangle_{\{x_i(t+dt)\}\cap\{x_j(t)\}} \\
&\quad - \langle\langle f(x(t))\rangle\rangle_{\{x_j(t+dt)\}\cap\{x_i(t)\}}].
\end{aligned} \tag{44}$$

The three terms here are manipulated as

$$\begin{aligned}
\langle\langle f(x(t+dt)) - f(x(t))\rangle\rangle_{\{x_i(t+dt)\}\cap\{x_i(t)\}} &= \left\langle\left\langle \left.\frac{df(x)}{dx}\right|_{x=x(t)} dx(t) + \frac{1}{2}\left.\frac{d^2 f(x)}{dx^2}\right|_{x=x(t)} (dx(t))^2 + o(dt^{3/2}) \right\rangle\right\rangle_{\{x_i(t+dt)\}\cap\{x_i(t)\}} \\
&= \left\langle\left\langle \frac{df(x)}{dx} b_i(x(t),t)dt + \frac{d^2 f(x)}{dx^2}\frac{\hbar}{2m_i}dt \right\rangle\right\rangle_{\{x_i(t+dt)\}\cap\{x_i(t)\}} \\
&= \left\langle\left\langle \frac{df(x)}{dx} b_i(x(t),t)dt + \frac{d^2 f(x)}{dx^2}\frac{\hbar}{2m_i}dt \right\rangle\right\rangle_{\{x_i(t)\}} + o(dt^2) \\
&= dt\int dx \left(\frac{df(x)}{dx} b_i(x,t) + \frac{d^2 f(x)}{dx^2}\frac{\hbar}{2m_i}\right) P_i(x,t) + o(dt^2) \\
&= dt\int dx\, f(x)\left(-\frac{\partial}{\partial x} b_i(x,t) + \frac{\hbar}{2m_i}\frac{\partial^2}{\partial x^2}\right) P_i(x,t) + o(dt^2),
\end{aligned} \tag{45}$$

$$\langle\langle f(x(t+dt))\rangle\rangle_{\{x_i(t+dt)\}\cap\{x_j(t)\}} = -dt\int dx\, f(x) W_{(i\to j)}(x,t) P_i(x,t)\,\theta(-W_{(i\to j)}(x,t)) + o(dt^2), \tag{46}$$

and

$$\begin{aligned}
&\langle\langle f(x(t))\rangle\rangle_{\{x_j(t+dt)\}\cap\{x_i(t)\}} \\
&\quad = dt\int dx\, f(x) W_{(i\to j)}(x,t) P_i(x,t)\,\theta(W_{(i\to j)}(x,t)) \\
&\qquad + o(dt^2),
\end{aligned} \tag{47}$$

respectively, from Eqs. (34), (36), (37), (38), and (42). Collecting Eqs. (44)–(47), we obtain the correct time evolution of Eq. (27). This shows the equivalence between Eq. (27) and the stochastic equation (34) supplemented with the stochastic jumping processes (37) and (38). Likewise one can show the equivalence between Eq. (28) and the stochastic equation (35) supplemented with the stochastic jumping processes (39) and (40).

We need some careful treatment on the dynamical condition in the present case. For the equivalence between Nelson's and Schrödinger approaches, the dynamical condition should have the form

$$\frac{\partial v_i}{\partial t} = \frac{\hbar}{2m_i}\frac{\partial^2 u_i}{\partial x^2} - v_i\frac{\partial v_i}{\partial x} + u_i\frac{\partial u_i}{\partial x} - \frac{1}{m_i}\frac{\partial \bar{V}_{ii}}{\partial x}. \tag{48}$$

Here we introduce a "quantum potential" $\bar{V}_{ii}$ which is to include the effect of channel coupling as well as the usual

potential V_{ii}. The simplest way to achieve this equation is to define the "mean balanced acceleration" a_i through the "mean (forward and backward) time derivative" as usual, but for the stochastic process without any jumping process. We simply consider a stochastic process governed by Eq. (34) all the time, and denote X_i instead of x_i to distinguish them from each other. There is no mixing of dw_i and dw_j in X_i, contrary to x_i. For each $X_i(t)$ we define the "mean balanced acceleration" $a_i(X_i(t),t)$, and the "Newton" equations

$$m_i a_i(X_i(t),t) = -\frac{\partial \bar{V}_{ii}}{\partial X_i} \tag{49}$$

becomes Eq. (48).

The combination of equations (33)+i(48) derives

$$\frac{\partial}{\partial x}\left[i\frac{\hbar}{m}\frac{1}{\psi_i'}\frac{\partial \psi_i'}{\partial t} + \frac{1}{2}\left(\frac{\hbar}{m_i}\right)^2 \frac{1}{\psi_i'}\frac{\partial^2 \psi_i'}{\partial x^2} - \frac{1}{m_i}\left\{ \bar{V}_{ii} - \frac{i\hbar}{2} W_{(i\to j)} \right\} \right] = 0, \tag{50}$$

where the relation

$$u_i + iv_j = \frac{\hbar}{m_i}\frac{1}{\psi_i'}\frac{\partial \psi_i'}{\partial x} \tag{51}$$

is used. If we shift the function ψ_i' to

$$\psi_i(x,t) = \psi_i'(x,t)\exp\left(-\frac{im_i}{\hbar}\int^t \eta(s)ds \right), \tag{52}$$

choose the "quantum potential" as

$$\bar{V}_{ii} = V_{ii} + \mathrm{Re}\frac{\psi_i^* V_{ij}\psi_j}{|\psi_i|^2}, \tag{53}$$

and use the relation

$$W_{(i\to j)} = -\frac{2}{\hbar}\mathrm{Im}\frac{\psi_i^* V_{ij}\psi_j}{|\psi_i|^2}, \tag{54}$$

we can reproduce the Schrödinger equations (24). By the use of Eqs. (51) and (52), the relations

$$b_i(x,t) = \frac{\hbar}{m_i}(\mathrm{Im}+\mathrm{Re})\frac{\partial}{\partial x}\ln\psi_i(x,t), \tag{55}$$

$$b_{*i}(x,t) = \frac{\hbar}{m_i}(\mathrm{Im}-\mathrm{Re})\frac{\partial}{\partial x}\ln\psi_i(x,t), \tag{56}$$

and Eq. (26) are established again.

IV. STOCHASTIC FORMULATION FOR QUANTUM SYSTEM OF OPTICAL POTENTIAL

In this section, let us formulate Nelson's stochastic approach to a system of a single degree of freedom described by an optical potential. Then the Schrödinger equation with an imaginary part of the potential, denoted by iU (a physically relevant situation, i.e., an absorptive process corresponds to $U<0$), is written as

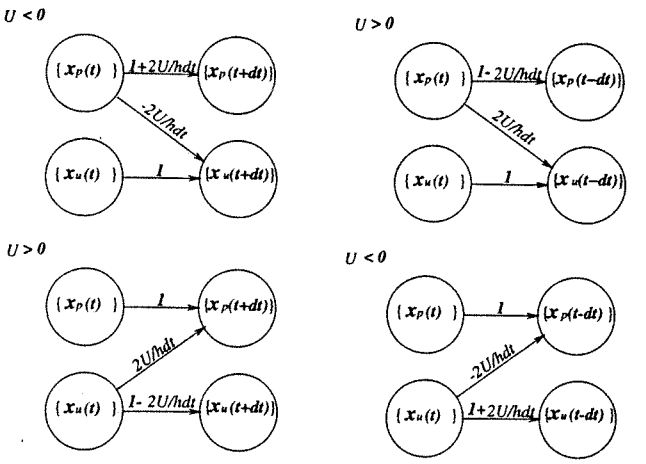

FIG. 2. Schematical illustration of the "dynamical" rule for the stochastic jumping process between physical and unphysical sector.

$$i\hbar\frac{\partial \psi(x,t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t) + iU(x,t) \right)\psi(x,t). \tag{57}$$

The formulation in Sec. III suggests a method to establish a stochastic formulation for this Schrödinger equation. The analogy between the channel-coupling model and the present model becomes apparent when we attempt the Fokker-Planck equation corresponding to Eq. (57) in the forms

$$\frac{\partial P(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x}b + \frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + \frac{2U}{\hbar} \right]P(x,t) \quad \text{(forward in } t\text{)}, \tag{58}$$

$$-\frac{\partial P(x,t)}{\partial t} = \left[\frac{\partial}{\partial x}b_* + \frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} - \frac{2U}{\hbar} \right]P(x,t) \quad \text{(backward in } t\text{)}. \tag{59}$$

Equations (58) and (59) are compared with Eqs. (27) and (28); both are quite similar to each other with the correspondence between $2U/\hbar$ and $-W_{(i\to j)}$.

While the sum of Eqs. (58) and (59) is given by Eq. (6), their difference leads to

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(vP) + \frac{2U}{\hbar}P \tag{60}$$

instead of Eq. (8). From Eqs. (6) and (60) follows the kinematical equation

$$\frac{\partial u}{\partial t} = -\frac{\hbar}{2m}\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}(uv) + \frac{1}{m}\frac{\partial}{\partial x}U, \tag{61}$$

instead of Eq. (10).

The additional term in Eq. (58) simply describes production (absorption) effects for $U>0$ ($U<0$), which one may put in such a way that the production (absorption) process is a transition from an "unphysical" sector to a "physical" one (from a "physical" sector to an "unphysical" one). At this point the analogy between Sec. III and this section is

helpful to find stochastic processes equivalent to the Fokker-Planck equations in Eqs. (58) and (59): We consider the two random variables $x_p(t)$ and $x_u(t)$ for "physical" and "unphysical" sectors, respectively, and stochastic jumping between them occurs according to certain rules, which will be specified below. In contrast with the channel-coupling case with the index i, the stochastic differential equations for both of $x_p(t)$ and $x_u(t)$ can be common. Introducing a notation of a random variable $x(t)$ standing for both of $x_p(t)$ and $x_u(t)$, we require the same form of stochastic differential equations for this $x(t)$ as Eqs. (1) and (2) *all the time*,

$$dx(t)=b(x(t),t)dt+dw(t) \quad \text{forward in time}, \tag{62}$$

$$dx(t)=b_*(x(t),t)dt+dw_*(t) \quad \text{backward in time}, \tag{63}$$

with the same properties for $dw(t)$ as in Eq. (3), and so on. Each sample path is described by $x(t)$ as a whole, but has to be classified into either $x_p(t)$ or $x_u(t)$ at each t. Typically a sample path changes as, for example, $x_p(t_1)\to x_u(t_2)\to\cdots\to x_p(t_n)$ as a result of repeated jumping processes. A sample path is said to be physically relevant at t if the sample is represented by $x_p(t)$, while it is not so if it is represented by $x_u(t)$. In other words, the physical average at t is given by the average over ensemble of not all sample paths but only physically relevant sample paths at t. The notation $\langle\langle f(x(t))\rangle\rangle_{\{x_p(t)\}}$ is introduced to represent this conditional average for $f(x(t))$. Similarly the notations of other conditional averages such as $\langle\langle f(x(t))\rangle\rangle_{\{x_u(t)\}}$ and $\langle\langle f(x(t))\rangle\rangle_{\{x_p(t)\}\cup\{x_u(t)\}}$ are clear, in particular

$$\langle\langle f(x(t))\rangle\rangle_{\{x_p(t)\}\cup\{x_u(t)\}}=\langle f(x(t))\rangle. \tag{64}$$

Again, conditional averages related to many times can be introduced, e.g., $\langle\langle f(x(t))\rangle\rangle_{\{x_p(t+dt)\}\cap\{x_u(t)\}}$ is supposed to represent the average over all the sample paths which are described by x_u at t and x_p at $t+dt$.

Let us summarize the "dynamical" rule for stochastic jumping processes between p and u. The rules are given as follows (Fig. 2): (i) For the forward time direction, in the case of $U<0$,

$$\begin{aligned} x_p(t)&\to\begin{cases} x_u(t+dt) & \text{with the probability } -2U(x_p(t),t)/\hbar\ dt \\ x_p(t+dt) & \text{with the probability } 1+2U(x_p(t),t)/\hbar\ dt, \end{cases} \\ x_u(t)&\to x_u(t+dt) \quad \text{with the probability 1}, \end{aligned} \tag{65}$$

and, in the case of $U>0$,

$$\begin{aligned} x_u(t)&\to\begin{cases} x_p(t+dt) & \text{with the probability } 2U(x_u(t),t)/\hbar\ dt \\ x_u(t+dt) & \text{with the probability } 1-2U(x_u(t),t)/\hbar\ dt, \end{cases} \\ x_p(t)&\to x_p(t+dt) \quad \text{with the probability 1}. \end{aligned} \tag{66}$$

(ii) For the backward time direction, in the case of $U<0$,

$$\begin{aligned} x_u(t)&\to\begin{cases} x_p(t-dt) & \text{with the probability } -2U(x_u(t),t)/\hbar\ dt \\ x_u(t-dt) & \text{with the probability } 1+2U(x_u(t),t)/\hbar\ dt, \end{cases} \\ x_p(t)&\to x_p(t-dt) \quad \text{with the probability 1}, \end{aligned} \tag{67}$$

and, in the case of $U>0$,

$$\begin{aligned} x_p(t)&\to\begin{cases} x_u(t-dt) & \text{with the probability } 2U(x_p(t),t)/\hbar\ dt \\ x_p(t-dt) & \text{with the probability } 1-2U(x_p(t),t)/\hbar\ dt, \end{cases} \\ x_u(t)&\to x_u(t-dt) \quad \text{with the probability 1}. \end{aligned} \tag{68}$$

Note that for the forward time direction a jumping process from x_p to x_u is allowed, and the reverse process is forbidden where $U<0$, and vice versa where $U>0$, and that when U is nonpositive everywhere, the number of sample paths described $x_p(t)$ decreases, and that in $x_u(t)$ increases as t goes, the total number being conserved. Regardless of the indices of p and u, each sample path is a stochastic process described by Eq. (62) [or Eq. (63)].

To prove the equivalence between the Fokker-Planck equation (58) and the stochastic differential equation (62) with the jumping rules (65) and (66), we calculate, for example,

$$\frac{d\langle\langle f(x(t))\rangle\rangle_{\{x_p(t)\}}}{dt}=\frac{1}{dt}[\langle\langle f(x(t+dt))\rangle\rangle_{\{x_p(t+dt)\}}-\langle\langle f(x(t))\rangle\rangle_{\{x_p(t)\}}]$$
$$=\frac{1}{dt}[\langle\langle f(x(t+dt))-f(x(t))\rangle\rangle_{\{x_p(t+dt)\}\cap\{x_p(t)\}}+\langle\langle f(x(t+dt))\rangle\rangle_{\{x_p(t+dt)\}\cap\{x_u(t)\}}-\langle\langle f(x(t))\rangle\rangle_{\{x_u(t+dt)\}\cap\{x_p(t)\}}], \quad (69)$$

with

$$\langle\langle f(x(t+dt))-f(x(t))\rangle\rangle_{\{x_p(t+dt)\}\cap\{x_p(t)\}}=dt\int dx\, f(x)\left(-\frac{\partial}{\partial x}b(x,t)+\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\right)P(x,t)+o(dt^2), \quad (70)$$

$$\langle\langle f(x(t+dt))\rangle\rangle_{\{x_p(t+dt)\}\cap\{x_u(t)\}}=dt\int dx\, f(x)\frac{2U(x,t)}{\hbar}P(x,t)\theta(U(x,t))+o(dt^2), \quad (71)$$

and

$$\langle\langle f(x(t))\rangle\rangle_{\{x_u(t+dt)\}\cap\{x_p(t)\}}=-dt\int dx\, f(x)\frac{2U(x,t)}{\hbar}P(x,t)\theta(-U(x,t))+o(dt^2). \quad (72)$$

These equations (69)–(72) follow Eq. (58). The equivalence between the Fokker-Planck approach and the approach of the stochastic differential equation (62) with the stochastic jumping processes (65) and (66) has been shown for the forward direction. Similarly the equivalence between the two approaches can be proven for the backward time direction.

As for the dynamical condition, we do not modify the original Nelson's formulation. When the mean time derivatives $Df(t)$ and $D_*f(t)$ are concerned, there may be some ambiguity with respect to the taking expectation. Here we will follow the argument given above, Eq. (49). We define the "mean balanced acceleration" through the "mean time derivatives" as usual, but for the stochastic process without any jumping process. We simply consider a stochastic process governed by Eqs. (62) and (63) at all times. This leads to the "Newton-Nelson equation" in Eq. (16) in the present case.

The combination of the equations (61)+i(16) leads to

$$\frac{\partial}{\partial x}\left[i\frac{\hbar}{m}\frac{1}{\psi'}\frac{\partial\psi'}{\partial t}+\frac{1}{2}\left(\frac{\hbar}{m}\right)^2\frac{1}{\psi'}\frac{\partial^2\psi'}{\partial x^2}-\frac{1}{m}(V+iU)\right]=0, \quad (73)$$

where relation (18) is used. Again the relation between ψ' and the solution of Eq. (57) ψ is given as

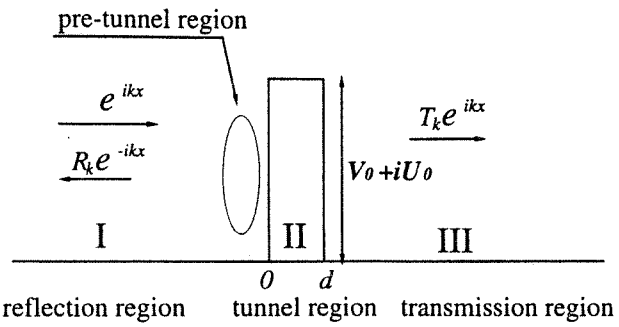

FIG. 3. Schematical illustration of one-dimensional optical barrier tunneling.

$$\psi=\psi'\exp\left(-\frac{im}{\hbar}\int^t\eta(s)ds\right) \quad (74)$$

and

$$b(x,t)=\frac{\hbar}{m}(\mathrm{Im}+\mathrm{Re})\frac{\partial}{\partial x}\ln\psi(x,t), \quad (75)$$

$$b_*(x,t)=\frac{\hbar}{m}(\mathrm{Im}-\mathrm{Re})\frac{\partial}{\partial x}\ln\psi(x,t), \quad (76)$$

$$P(x,t)=|\psi(x,t)|^2 \quad (77)$$

are satisfied.

V. NUMERICAL ANALYSIS

Now we can perform a numerical analysis of the effects of the optical potential and channel coupling on the tunneling time, using above generalized Nelson's approach. First, we discuss one-dimensional system with a static square-well optical potential,

$$V(x)=\begin{cases}0 & \text{in I} \quad (x<0)\\ V_0-iU_0 & \text{in II} \quad (0<x<d)\\ 0 & \text{in III} \quad (d<x)\end{cases} \quad (78)$$

(Fig. 3). We set the solution of the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)=\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\psi(x,t), \quad (79)$$

as

$$\psi(x,t)=\int_{-\infty}^{\infty}A(k)\varphi_k(x)e^{-i(E/\hbar)t}dk, \quad (80)$$

with a coefficient function $A(k)$ and $E=\hbar^2k^2/2m$. It is well known that $\varphi_k(x)$ is written as

$$\varphi_k(x)=\begin{cases}e^{ikx}+R_ke^{-ikx} & \text{in I}\\ C_ke^{\kappa x}+D_ke^{-\kappa x} & \text{in II}\\ T_ke^{ikx} & \text{in III},\end{cases} \quad (81)$$

where

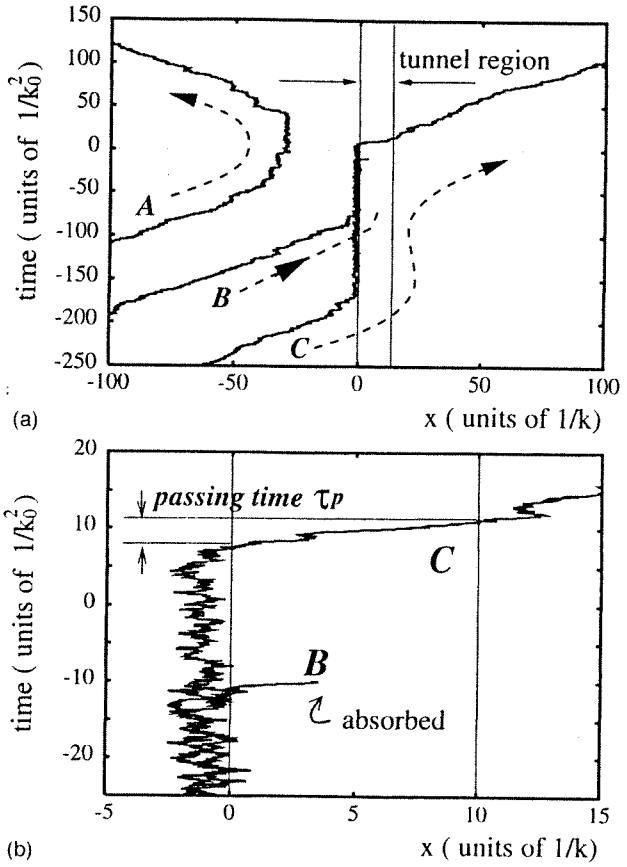

FIG. 4. (a) and (b) Three typical sample paths in the optical potential case. (b) is an enlarged version of (a) ($m=\hbar=1$).

$$\kappa=\frac{\sqrt{2m(V_0-iU_0-E)}}{\hbar}=\kappa_R-i\kappa_I \quad (\kappa_I>0), \tag{82}$$

$$\left(\kappa_0=\frac{\sqrt{2m(V_0-iU_0-E_0)}}{\hbar}=\kappa_{R0}-i\kappa_{I0} \quad (\kappa_{I0}>0)\right), \tag{83}$$

and R_k, T_k, C_k, and D_k are given as

$$\left.\begin{matrix}R_k\\T_k\\C_k\\D_k\end{matrix}\right\}=\mathcal{B}\times\left\{\begin{matrix}-i(\kappa^2+k^2)\sinh\kappa d\\2k\kappa e^{-ikd}\\k(\kappa+ik)e^{-\kappa d}\\k(\kappa-ik)e^{\kappa d}\end{matrix}\right. \tag{84}$$

$$\mathcal{B}=\frac{1}{2k\kappa\cosh\kappa d+i(\kappa^2-k^2)\sinh\kappa d}. \tag{85}$$

We take a Gaussian form with its center at $k=k_0$, or

$$A(k)=A_{k_0}(k)=C\exp\left\{-\frac{(k_0-k)^2}{4\sigma^2}\right\}, \tag{86}$$

with a normalization constant C. Here we put $\sigma=k_0/100$ and $V_0=5E_0=(\hbar k_0/2m)^2$. Using this solution, we numerically calculate Eqs. (62), (63) and (65)–(68) .

Figure 4 shows the three typical sample paths calculated

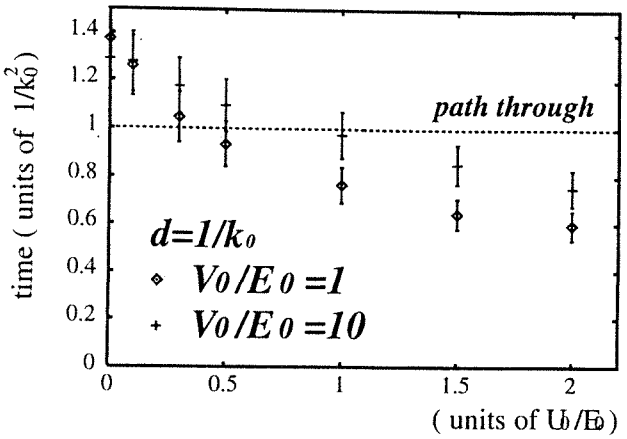

FIG. 5. The mean value of τ_p vs U_0/E_0 (thin potential cases).

by Eqs. (62), (65), and (66). There is a sample path $x(t)$ which changes its property from "physical" to "unphysical" in the tunnel region.

Figures 5 and 6 show the parameter U_0/E_0 vs the average of passing time τ_p, calculated by Eqs. (63), (67), and (68). See the details of this "backward time evolution method" in our previous work [20]. Generally, τ_p decrease as U_0/E_0 become larger. Let us estimate τ_p analytically on the WKB-like approximation. If we can write the wave function in the tunnel region II as

$$\psi(x,t)\sim\psi(x)\sim C'\exp(-\kappa_0 x)=\exp\{-(\kappa_{R0}-i\kappa_{I0})x\}, \tag{87}$$

the drift of Eq. (63) becomes

$$b_*=\frac{\hbar}{m}(\kappa_{I0}+\kappa_{R0})\sim\frac{\hbar\bar{\kappa}_0}{m}\left[1+\frac{\kappa_{I0}}{\kappa_{R0}}+o^2\left(\frac{\kappa_{I0}}{\kappa_{R0}}\right)\right],$$

$$\bar{\kappa}_0=\frac{\sqrt{2m(V_0-E_0)}}{\hbar} \tag{88}$$

from Eq. (76). In these cases, the "backward" time evolution of the distribution function $P_T(x,t)$, which has an "initial" distribution $\delta(x-d)$, is written as

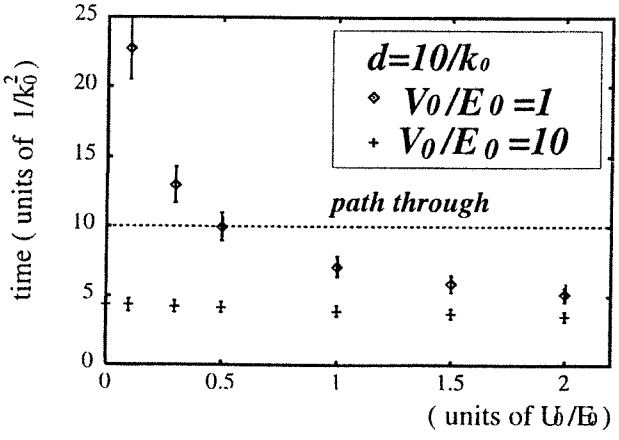

FIG. 6. The mean value of τ_p vs U_0/E_0 (thick potential cases).

$$\frac{\partial P_T(x,t)}{\partial t} = -\left[\frac{\hbar}{m}\bar{\kappa}_0\left(1+\frac{\kappa_{I0}}{\kappa_{R0}}\right)\frac{\partial}{\partial x}+\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} - \frac{2}{\hbar}U_0\right]P_T(x,t), \tag{89}$$

and we can obtain the solution of Eq. (89) easily:

$$P_T(x,t)=\left(\frac{2m\pi}{-\hbar t}\right)^{1/2}\exp\left\{\frac{\left(x-d+\frac{\hbar\bar{\kappa}_0\left(1+\frac{\kappa_{I0}}{\kappa_{R0}}\right)}{m}t\right)^2}{\frac{\hbar t}{2m}} - \frac{2U_0}{\hbar}t\right\} \quad (t<0). \tag{90}$$

There are two characteristic time intervals in this solution. One is the diffusion time $t_d \sim md^2/\hbar$ for which the distribution sizes up to the potential width d. The other is the current time

$$t_c \sim \frac{md}{\hbar\bar{\kappa}_0\left(1+\frac{\kappa_{I0}}{\kappa_{R0}}\right)} \sim \frac{md}{\hbar\bar{\kappa}_0}\left(1-\frac{\kappa_{I0}}{\kappa_{R0}}\right), \tag{91}$$

for which the peak of the distribution moves from $x=d$ to $x=0$. Of course, the approximation of Eq. (87) is justified when $\bar{\kappa}_0 d$ is much larger than 1, and this leads us to the relation

$$t_d \gg t_c, \tag{92}$$

and the time interval t_c becomes the passing time in this extreme case. Note that this t_c has the tendency of decreasing as U_0/E_0 becomes larger.

Second, we discuss a one-dimensional system with a static square-well potential and two-channel coupling, or the case of the Schrödinger equation for this problem written as

$$i\frac{\partial}{\partial t}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = \begin{bmatrix} -\frac{1}{2m}\frac{\partial^2}{\partial x^2}+V & U \\ U & -\frac{1}{2m}\frac{\partial^2}{\partial x^2}+V\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}. \tag{93}$$

V and U are supposed to be

$$V(x)=\begin{cases}0 & \text{in I} \quad (x<0)\\ V_0 & \text{in II} \quad (0<x<d)\\ 0 & \text{in III} \quad (d<x)\end{cases} \tag{94}$$

and

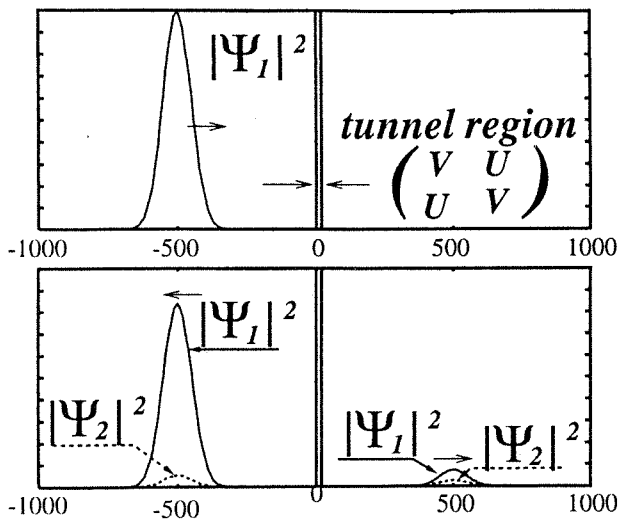

FIG. 7. Schematical illustration of one-dimensional scattering with channel coupling.

$$U(x)=\begin{cases}0 & \text{in I} \quad (x<0)\\ U_0 & \text{in II} \quad (0<x<d)\\ 0 & \text{in III} \quad (d<x).\end{cases} \tag{95}$$

Figure 7 shows the schematical illustration of our simulation. We can diagonalize Eq. (93) as

$$i\frac{\partial}{\partial t}\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix} = \begin{bmatrix} -\frac{1}{2m}\frac{\partial^2}{\partial x^2}+V+U & 0 \\ 0 & -\frac{1}{2m}\frac{\partial^2}{\partial x^2}+V-U\end{bmatrix}\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix}, \tag{96}$$

where

$$\psi_+=\frac{1}{\sqrt{2}}(\psi_1+\psi_2) \quad \text{and} \quad \psi_-=\frac{1}{\sqrt{2}}(\psi_1-\psi_2), \tag{97}$$

and write down the time-dependent solution of ψ_+ and ψ_- easily as the same as Eq. (80), or

$$\psi_\pm(x,t)=\int_{-\infty}^{\infty}A(k)\varphi_{\pm k}(x)e^{-i(E/\hbar)t}dk, \tag{98}$$

with a Gaussian coefficient function $A(k)$ and $E=\hbar^2k^2/2m$. $\varphi_{\pm k}(x)$ is Eq. (81), substituting κ with

$$\kappa_\pm=\frac{\sqrt{2m(V_0\pm U_0-E)}}{\hbar}. \tag{99}$$

Figure 8 shows the same typical sample paths calculated by Eqs. (34), (37), and (38). There is a path which changes its index from 1 to 2 in the passage through the tunneling region, $t_{i=1,2}$. Figures 9 and 10 are the averages of the passing times over the sample paths which belong to $\{x_i(t)\}$ at

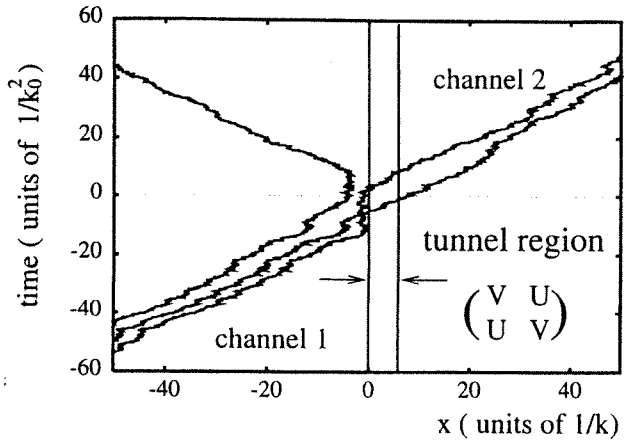

FIG. 8. Three typical sample paths with channel coupling.

$t\to\infty$. We can see in Fig. 10 that there is a critical parameter value of $(V_0-U_0)/E_0=1$ in the behavior of t_1 and t_2. This is understood as following: In the case of $(V_0-U_0)/E_0>1$, the "$-$" channel, which is dominant in the tunnel region II in comparison with "$+$" one, is not the tunneling channel, and it describes a particle which goes over the potential. Regardless of $x_1(t)$ and $x_2(t)$, the time spent in the "potential region" is expected to agree with the one which is expected from classical mechanics or

$$\frac{md}{\hbar k_{-0}} \quad \text{where } k_{-0}=\frac{\sqrt{2m(E_0-V_0+U_0)}}{\hbar}. \tag{100}$$

This is also seen in Fig. 9. On the other hand, in the case of $(V_0-U_0)/E_0<1$, we can approximate the wave functions ψ_+ and ψ_- in the thick tunnel region

$$\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix}\sim\begin{bmatrix}0\\ C_-\exp(-\kappa_{-0}\,x)\end{bmatrix}, \tag{101}$$

and ψ_1 and ψ_2 as

$$\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\sim\frac{1}{\sqrt{2}}\begin{bmatrix}C_-\exp(-\kappa_{-0}\,x)\\ -C_-\exp(-\kappa_{-0}\,x)\end{bmatrix}, \tag{102}$$

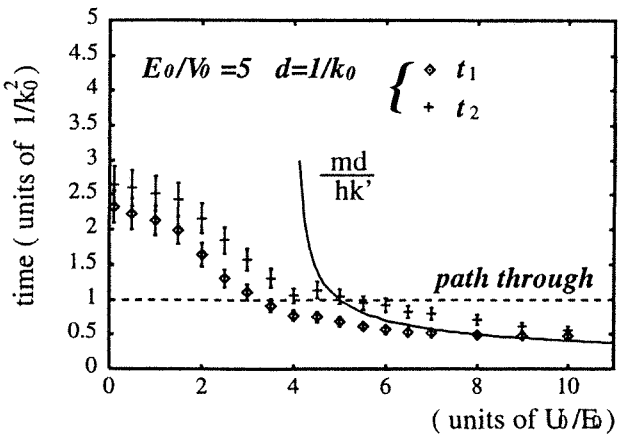

FIG. 9. The mean values of t_1 and t_2 vs U_0/E_0 (thin potential cases).

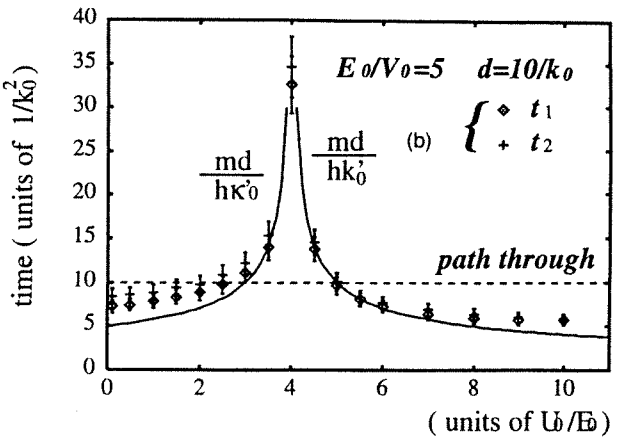

FIG. 10. The mean values of t_1 and t_2 vs U_0/E_0 (thick potential cases).

where

$$\kappa_{-0}=\frac{\sqrt{2m(V_0-U_0-E_0)}}{\hbar}. \tag{103}$$

So we can estimate "passing time" of both channels (1 and 2) at

$$\frac{md}{\hbar\kappa_{-0}}, \tag{104}$$

and likewise Eq. (91).

VI. SUMMARY AND COMMENTS

In this paper, we have analyzed the effects of inelastic scattering on the tunneling time theoretically, using generalized Nelson's quantum mechanics. This generalization enabled us to describe quantum system with optical potential and channel couplings in a real-time stochastic approach. In this formalism, the space-time development of dynamical variable, e.g., the coordinate of the particle, is described by a definite path determined stochastically. Each sample path has a definite form of trajectory in the space-time diagram, while a physical quantity averaged over the ensemble of these sample paths recovers the effect of quantum coherence. This is true even in Young's double slits interference experiment. Nelson's quantum mechanics gives each definite trajectory, and the ensemble of it, but it does not predict which path is selected when one wants to measure the position of a particle. In this sense, this "real-time stochastic process approach" seems to give us a new insight into quantum mechanics beyond the Copenhagen interpretation. On the other hand, the effects of more general cases (many-body systems, environment, temperature, and so on) are subjects for the future, and this work would be the first step to such a study. Recent experimental data of tunneling time using the neutron spin-echo shift through the magnetic films [19] seem to agree with the simulation based on our approach [21], and this study will be reported in near future.

ACKNOWLEDGMENTS

This work was supported by a Grant-in-Aid for Science Research, Ministry of Education, Science and Culture, Japan (Nos. 08240232, and 04244105).

[1] E. P. Wigner, Phys. Rev. **98**, 145 (1955).
[2] M. Büttiker and R. Landauer, Phys. Rev. Lett. **49**, 1739 (1982).
[3] H. W. Lee and M. O. Scully, Found. Phys. **13**, 61 (1983).
[4] M. Büttiker, Phys. Rev. B **27**, 6178 (1983).
[5] D. Sokolvski and L. M. Baskin, Phys. Rev. A **36**, 4604 (1987).
[6] H. A. Fertig, Phys. Rev. Lett. **65**, 2321 (1990).
[7] Ming-Quey Chen and M. S. Wang, Phys. Lett. A **149**, 441 (1990).
[8] D. Sokolovski and J. N. L. Connor, Phys. Rev. A **47**, 4677 (1993).
[9] V. S. Olkhovsky and E. Recami, Phys. Rep. **214**, 339 (1992).
[10] C. R. Leavens and G. C. Aers, in *Scanning Tunneling Microscopy III*, edited by R. Wiesendanger and H.-J. Güntherodt (Springer-Verlag, Berlin, 1993).
[11] T. Martin and R. Landauer, Phys. Rev. A **47**, 2023 (1993).
[12] C. R. Leavens and W. R. McKinnon, Phys. Lett. A **194**, 12 (1994).
[13] S. Brouard, R. Sala, and J. G. Muga, Phys. Rev. A **49**, 4312 (1994).
[14] A. M. Steinberg, Phys. Rev. Lett. **74**, 2405 (1995).
[15] S. Collins, D. Lowe, and J. R. Barker, J. Phys. C **20**, 6213 (1987).
[16] E. H. Hauge and J. A. Støvneng, Rev. Mod. Phys. **61**, 917 (1989).
[17] R. Landauer and Th. Martin, Rev. Mod. Phys. **66**, 217 (1994).
[18] E. Nelson, Phys. Rev. **150**, 1079 (1966).
[19] M. Hino, N. Achiwa, S. Tsaki, T. Ebisawa, T. Akiyoshi, and T. Kawai, J. Phys. Soc. Jpn., Suppl. A, **65**, 203 (1996).
[20] K. Imafuku, I. Ohba, and Y. Yamanaka, Phys. Lett. A **204**, 329 (1995).
[21] I. Ohba, K. Imafuku and Y. Yamanaka, J. Phys. Soc. Jpn., Suppl. A **65**, 41 (1996).