非線形拡散方程式系と関連する
楕円型微分方程式系の研究

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研究代表者　山田　義雄

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はしがき

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の研究成果報告書である．本研究の研究組織および研究経費は次の通りである．

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ただし，上記分担者のうち柴田良弘（平成 15 年度），石渡　通徳 (平成 15・16 年度)，大屋博一（平成 17 年度）は括弧年度の研究分担者である．なお分担者の所属・身分は平成 18 年 4 月 1 日現在のものを採用している．

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研究発表

本研究の成果の一部は Journal of Differential Equations, Journal of Mathematical Analysis and Applications, Differential and Integral Equations, Funkcialaj Ekvacioj, Advances in Mathematical Sciences and Applications, Discrete and Continuous Dynamical Systems などの定評ある専門雑誌に論文の形で出版または出版予定である．また，日本数学会をはじめとする国内の研究集会や国際会議における講演によっても研究成果は公表されている．

なお，発表の詳細は研究代表者による「研究総括」における論文発表・口頭発表の部分，および各研究分担者の成果発表部分を参照していただきたい．

最後に本研究を遂行するにあたり，各方面からいただいた多大な援助に対し深く感謝の意を表わしたい．

平成 18 年 4 月　　　　山田義雄

目次

研究総括

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物理学，化学反応，工学，数理生態学の分野に現れる非線形現象のなかには物質の密度や濃淡の差により，縞模様のパターンとなって現れるものがある．このような現象は非線形の反応拡散方程式系として数理モデル化されることが多い．反応拡散方程式系に対して，コンピュータによるシミュレーションを実行すると，ダイナミクスの激変を伴う分岐，時空的振動，パターンの形成や界面の生成という現象が観測される．これらの現象は非常に興味深いものであるにもかかわらず，発生・展開のメカニズムの理論的解明は十分ではない．したがって，非線形現象がなぜ生まれるか，また時間空間的にどのように展開するのか，これらの疑問を理論的に解き明かすことは数理科学の問題として非常に重要な課題である．

本研究「非線形拡散方程式系と関連する楕円型微分方程式系の研究」においては，反応拡散方程式に見られる，パターンや界面のような空間的な非一様性に着目し，空間的な非均質な状態の発生メカニズムと推移する遷移過程の動きを理論的に明らかにすることを目指した．重点的に扱ったのは

I. 相転移モデル—相転移現象を記述する方程式 (phase transition model)

II. 数理生態学モデル—非線形拡散を伴う Lotka-Voltrra 型方程式
(Shigesada-Kawasaki-Teramoto model)

である．

I. 相転移モデル

パターンや界面の動きを記述するモデルとして相転移モデルと呼ばれる双安定型の反応項を伴う拡散方程式がある．空間 1 次元の場合，$\varepsilon > 0$ を微小な拡散係数として

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + u(1-u)(u-a(x)) & \text{for } (x,t) \in (0,1)\times(0,\infty), \\ u_x(0,t) = u_x(1,t) & \text{for } t \in (0,\infty), \end{cases} \tag{1}$$

を初期条件 $u(x,0) = u_0(x)$　for　$x \in (0,1)$ のもとで考える．ここで $a(x)$ は $C^2(0,1)$ 級の関数で $-1 < a(x) < 1$ をみたす．この問題において $\varepsilon = 0$ のときは $u = 0, 1$ が安定な平衡点であるが，$u = a(x)$ は不安定な平衡点であるため，非線形項は双安定型とも呼ばれる．ここで

$$F(x,u) = -\int_{\phi(x)}^{u} f(x,s)\, ds, \qquad f(x,u) = u(1-u)(u-a(x))$$

とおく．ただし

$$\phi(x) = \begin{cases} 1 & \text{if } a(x) > 1/2, \\ 0 & \text{if } a(x) < 1/2. \end{cases}$$

次にエネルギー関数として

$$E(u) = \frac{\varepsilon^2}{2}\int_0^1 u_x(x)^2\, dx + \int_0^1 F(x, u(x))\, dx$$

を定義し，(1) の解 $u=u(x,t)$ を代入すると

$$\frac{d}{dt}E(u(t)) = -\int_0^1 u_t(x,t)^2\,dx \le 0$$

となる．したがって $E(u)$ は (1) の Lyapunov 関数となり，非定常問題の解 $u(x,t)$ について次の結果が成立する：

(i) $t \longrightarrow E(u(t))$ は単調減少である．

(ii) $\lim_{t\to\infty} u(x,t) = \varphi(x)$ (uniformly in $x\in[0,1]$),

ただし φ は

$$\varepsilon^2\varphi_{xx} + \varphi(1-\phi)(\varphi - a(x)) = 0, \qquad \varphi_x(0)=\varphi_x(1)=0 \tag{2}$$

の解である．

すなわち (1) の解 $u(t)$ は t とともに $E(u(t))$ も小さくするような挙動を示す．これより $F(x,u(x,t))$ を小さくするような挙動をとることが予想される．$u=1$ も $u=-1$ も拡散がない場合は安定な状態であるが，何が $F(x,u)$ の最小値であるかという状況は x とともに変化する．$\phi(x)$ と逆に

$$\phi^*(x) = \begin{cases} 0 & \text{if } a(x) > 1/2, \\ 1 & \text{if } a(x) < 1/2. \end{cases}$$

と定義し，ここでは $u=\phi^*(x)$ を "minimal state", $u=\phi^*(x)$ を "non-minimal state" と呼ぶことにする．定常問題 (2) は安定な自明解 $u=0,1$ をもつが，他にも安定な定常解を持つかどうかが疑問となる．とくに方程式の非均質性を引き起こす $a(x)$ は安定な非自明解をもつことがあるだろうか？

$\varepsilon>0$ が非常に微小な場合には，上の問題 (2) は振動する定常解を持つことが知られている．とくに 2 つの安定な平衡点 $0,1$ を結ぶ，非常にシャープな**内部遷移層** (transition layer) や**スパイク**状の突起を持つ定常解が興味深い．方程式の非均質性を意味する関数 a について次の条件を仮定する：

(A.1) $\Sigma := \{x\in(0,1);\ a(x)=1/2\}$ は空でない有界集合である．

(A.2) $\Lambda := \{x\in(0,1);\ a'(x)=0\}$ は有界集合である．

(A.3) $\Sigma\cap\Lambda=\emptyset$,

単一の内部遷移層を持つ解については，1987 年 Angenent, Mallet-Paret, Peletier らのグループが集合 Σ 内の任意の点 x^* の近傍において $u'(x^*)a'(x^+)<0$ をみたすような内部遷移層を持つ定常解 u で，しかも安定であるものを比較定理を用いて構成することに成功した．その後，Hale-Sakamoto(1988) が逆の不等式, $u'(x^*)a'(x^+)>0$ をみたす不安定な解について研究するなど，(1), (2) は多くの研究者により研究されてきた．

(2) の定常解のなかで，振動する解 φ に着目すると $\varphi(x)-a(x)$ の零点と $\varphi(x)$ の極大点または極小点は交互に現れることに注意する．したがって $\varphi(x)-a(x)$ の零点の個数を利用して，n 個ならば φ を n **モード解**と呼ぶことにする．n モード解について，Ai, Chen, Hatings のグループと我々のグループはそれぞれ独立に研究し，解の形状，安定性などについて非常に詳しい性質を示すことに成功している．例えば定常解 φ と a の交点を x^* とする．拡散係数 ε が非常に小さいならば，$a(x^*)\approx 1/2$ のときには φ は非常にシャープな内部遷移層をも

ち，ここでは $\varphi(x)$ が x^* を含む非常に短い区間で 0 と 1 を結んでいることがわかる．一方，$a(x^*)$ が 1/2 から離れていれば，$a(x^*) > 1/2$ のときには φ は x^* の近傍において 1 をベースとするスパイクを持ち，$a(x^*) < 1/2$ のときには φ は x^* の近傍において 0 をベースとするスパイクを持つことがわかる．これらの結果を空間変数を $\frac{x-x^*}{\varepsilon} = z$ のように引き伸ばすことにより示すことができる．

以下では拡散係数が小さい時の n モード解の性質について，本研究で得られた主な結果を説明しよう．まず遷移層やスパイクの位置に関して次の定理を示すことができる．

定理 1 φ を (2) の n モード解とするとき，拡散係数 ε が十分小さいならば次の性質が成り立つ:

(i) 内部遷移層 (transition layer) は Σ の点の近傍にのみ現れる．

(ii) スパイク (spike) は Λ の点の近傍にのみ現れ，しかも non-minimal state をベースにする．

次に非常に短い区間に複数個の遷移層が重なるように束になって現れるケースを考えよう．このような遷移層の集まり（束）を **multi-layer** と呼ぶことにする．multi-layer を持つ解について，解の形状や遷移層の個数について次の定理が成立する．

定理 2 φ は $(z-\delta, z+\delta)$ $(z \in \Sigma, \delta > 0)$ に multi-layer をもち，ε は十分小さいと仮定する．このとき次が成立する：

(i) multi-layer は奇数個の遷移層から成り立つ．

(ii) multi-layer は non-minimal state と non-minimal state を結ぶ形状に限られる．

定理 2 より φ が $z \in \Sigma$ の近傍において multi-layer をもつとき，$a'(z) > 0$ ならば，x が増加するとき，multi-layer は 0 から 1 を結ぶ形状に限ることがわかる．逆に 1 から 0 を結ぶ遷移層は単一に限られ，multi-layer は現れない．

同様に複数個のスパイクが重なるようになっているケースを考える．このようなスパイクの束を **multi-spike** と呼ぶ．multi-spike を持つ解を考えるために $\Lambda = \{x \in [0,1] |\, a'(x) = 0\}$ の分解

$$\Lambda = \Lambda^+ \cup \Lambda^- \cup \Lambda^0$$

を考える．ただし

$$\begin{aligned}
\Lambda^+ &:= \{x \in \Lambda |\, a(x) < 1/2, a''(x) < 0\},\ (a \text{ の極大点}),\\
\Lambda^- &:= \{x \in \Lambda |\, a(x) > 1/2, a''(x) > 0\},\ (a \text{ の極小点}),\\
\Lambda^0 &:= \Lambda \setminus (\Lambda^+ \cup \Lambda^-)
\end{aligned}$$

である．以上の準備の下，次が成立する．

定理 3 φ は $(z-\delta, z+\delta)$ $(z \in \Sigma, \delta > 0)$ に multi-spike をもち，ε は十分小さいと仮定する．このとき次が成立する：

(i) 0 をベースとする multi-spike は Λ^+ の点の近傍にのみ現れる．

(ii) 1 をベースとする multi-spike は Λ^- の点の近傍にのみ現れる．

定理 1,2,3 の詳しい主張，証明は論文 Urano-Nakashima-Yamada;"Transition layers and spikes for a bistable reaction-diffusion equation", Advances in Mathematical Sciences and Applications Vol.15 (2005), 683-707 において発表している．

さらに n-mode 解の安定性についても結果が得られる．(2) の解 φ の（線形化）安定性は固有値問題

$$\varepsilon^2 w_{xx} + f_u(x,\varphi(x))w = \lambda u, \quad w_x(0) = w_x(1) = 0 \tag{3}$$

の固有値の正負によって判定される．すなわち，第1固有値が負であれば，φ は**漸近安定**，第1固有値が正であれば**不安定**である．とくに (3) の正の固有値の個数に注目し φ の **Morse 指数** を

$$\mathrm{Ind}(\varphi)=(3) \text{ の正の固有値の個数}$$

で定義する．大雑把に述べれば Ind(φ) は定常解の不安定性の度合いを示すものである．単一の内部遷移層については minimal state と minimal state を結べば，安定な遷移層であり，non-minimal state と non-minimal state を結べば，不安定な遷移層となることがわかる．

定理 4 n-mode 解 φ はスパイクを持たないものと仮定し，non-minimal state と non-minimal state を結ぶ m_0 個の単一内部遷移層と ℓ 個の multi-layer を持ち，それぞれの multi-layer は $2m_i - 1$ 個 $(i = 1, 2, \cdots, \ell)$ の遷移層から成り立つと仮定する．このときつぎが成立つ：

$$\mathrm{Ind}(\varphi)= \sum_{i=0}^{\ell} m_i.$$

この結果は論文 Urano-Nakashima-Yamada;"Stability of a solution with transition layers for a bistable reaction-diffusion equation"（プレプリント）として発表予定である．

なお，定理1－4と同様な結果は Ai-Chen-Hastings らによっても我々とは異なる方法で得られており，論文 Ai-Chen-Hastings; "Layers and spikes in non-homogeneous bistable reaction-diffusion equations", Transactions of the American Mathematical Society Vol. 358, No. 7 (2006), 3169-3206, に詳しく述べられている．

II. 数理生態学モデル

同一の生息領域において生存競争をする2種の生物の棲み分けを記述するモデルとして，1979 年 Shigesada-Kawasaki-Teramoto らの数理生態学者のグループにより

$$\begin{cases} u_t = \Delta[(1+\alpha u+\beta v)u] + u(a_1 - b_1 u - c_1 v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta[(1+\gamma u+\delta v)v] + u(a_2 - b_2 u - c_2 v) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0, \quad v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{4}$$

の形の非線形拡散を伴う反応拡散方程式系が提案された（このモデルは SKT モデルと呼ばれることがある）．ここで u, v は生存競争をする生物の個体数密度，$\alpha, \beta, \gamma, \delta, a_i, b_i, c_i$ $(i = 1, 2)$ は非負定数，Ω は滑らかな境界 $\partial\Omega$ をもつ R^N の領域である．上のシステムに現れる非線形拡散は通常のタイプとは少し異なった形をしており，数理生態学的には「人口圧力」によって拡散が大きくなると解釈される．コンピュータによる数値シミュレーションの結果，棲み分け現象が観測され，モデルの重要性が認識されてきている．数学的には

(i) 任意の初期値に対して (4) は時間大域解をもつか？

(ii) (4) に対応する定常問題は空間的に非一様で，安定な定常解をもつか？

などの疑問に答えることが重要である．しかし，これらの疑問に対して十分満足できる解答が得られてないのが実情である．

本研究で主として扱ったのは SKT モデルに関連する次の形の定常問題である：

$$\begin{cases} \Delta[\varphi(u,v)u] + u(a-u-cv) = 0 & \text{in} \quad \Omega, \\ \Delta[\psi(u,v)v] + u(b+du-v) = 0 & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

ここで u, v はそれぞれ prey, predator の個体数密度であり，a, b, c, d は正定数である．問題となるのはこの (5) の正値定常解の存在条件，正値解の個数や形状などである．この問題を理解するために補助的問題

$$\Delta w + w(a-w) = 0 \quad \text{in} \quad \Omega, \quad w = 0 \quad \text{on} \quad \partial\Omega, \qquad w \geq 0 \quad \text{in} \quad \Omega,$$

を考える．同次 Dirichlet 境界条件のもとで $-\Delta$ の第 1 固有値を λ_1 とすると，この問題は $a \leq \lambda_1$ では自明解 $w \equiv 0$ のみを持つが，$a > \lambda_1$ では唯一の正値解 θ_a を持つことが知られている．

Nakashima-Yamada (1996) は (5) において

$$\varphi(u,v) = 1 + \alpha v, \quad \psi(u,v) = 1 + \beta u \tag{6}$$

とおき，θ_a を利用して正値解が存在するための十分条件を導いた．(6) のような拡散は **cross diffusion** と呼ばれている．この形の拡散について $\varphi(u,v) = 1 + \alpha v$ は predator v の存在がより大きな拡散をもたらすということで合理的であるが，$\psi(u,v) = 1 + \beta u$ はどうであろうか？predator v の拡散係数としてこのような状況は不自然に思われるが，自然界では prey となる種が自己防衛のため大きな集団を構成し，この集団が predator を拡散させる圧力となることがある．，$\psi(u,v) = 1 + \beta u$ はこのような状況を考慮したモデルである．(5) の正値解は数理生態学の観点では共存状態として意味のある解であり，数学的にも重要な解である．、

線形拡散の場合 $(\varphi(u,v) = 1, \psi(u,v) = 1)$ には正値解が存在するための必要十分条件が知られている．一方，cross-diffusion 項が存在するケースでは，状況が複雑になることが数値解析からも示唆され，正値解が存在するための必要十分条件を求めることは難しい．正値解の個数はどうなるかなど，解集合の構造を理解するために特別な状況として

cross-diffusion の係数 β が大きく，α が小さい

ような状況を考えよう．このとき $b, d/\beta$ が λ_1 に近いならば，a を分岐パラメータとみなし正値解に関する分岐ダイヤグラムを描くことができ，S 字型の分岐曲線が現れることが示された．このことはパラメータのとり方により 3 組の正値定常解が存在することを意味する．またそれぞれの正値定常解の線形化安定性に関する結果を求めることができる．これらの詳しい結果は論文 Kuto-Yamada; “Multiple coexistence states for a prey-predator system with cross-diffusion”, Journal of Differential Equations Vol. 197, No.2 (2004), 315-348 および Kuto; “Stability of steady state solutions to a prey-pradator system with cross-diffusion”, Journal of Differential Equations Vol. 197, No.2, (2004), 293-314 を参照してほしい．

なお最近は (5) において

$$\varphi(u,v) = 1 + \alpha v, \quad \psi(u,v) = \mu + \frac{1}{1+\beta u}$$

と置き，分岐理論を利用して正値解が存在するための十分条件や正値解集合の構造の解析を行っており，興味深い結果が得られつつある．結果の一部は論文 Kadota-Kuto;"Positive steady states for a prey-predator model with some nonlinear diffusion terms", Journal of Mathematical Analysis and Applications (印刷中) に発表されている．

発表論文

[1] (with Y. S. Choi and Roger Lui) *Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with weak cross-diffusion*, Discrete and Cintinuous Dynamical Systems **9**, No. 5 (2003), 1193- 1200.

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TRANSITION LAYERS AND SPIKES FOR A BISTABLE REACTION-DIFFUSION EQUATION

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Abstract. This paper is concerned with a steady-state problem for

$$u_t = \varepsilon^2 u_{xx} + u(1-u)(u-a(x)), \quad (x,t) \in (0,1) \times (0,\infty),$$

with $u_x(0,t) = u_x(1,t) = 0$, where a is a C^2-function satisfying $0 < a(x) < 1$. When ε is very small, the problem has various solutions. Among them, we are interested in solutions with transition layers and spikes. Our main purpose is to study profiles of such solutions and determine the location of transition layers and spikes. Moreover, we will show some conditions under which one can observe multi-layers and multi-spikes.

1 Introduction

In this paper we consider the following reaction-diffusion equation:

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + f(x,u), & 0 < x < 1, \ t > 0, \\ u_x(0,t) = u_x(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases} \tag{1.1}$$

Here ε is a positive parameter and $f(x,u)$ is given by

$$f(x,u) = u(1-u)(u-a(x)), \tag{1.2}$$

where a is a $C^2[0,1]$-function with the following properties:
(A.1) $0 < a(x) < 1$ in $[0,1]$,
(A.2) if

$$\Sigma := \{x \in (0,1)\,;\, a(x) = 1/2\}, \tag{1.3}$$

then Σ is a finite set and $a'(x) \neq 0$ at any $x \in \Sigma$,
(A.3) if

$$\Lambda := \{x \in (0,1)\,;\, a'(x) = 0\}, \tag{1.4}$$

then Λ is a finite set,
(A.4) $a'(0) = a'(1) = 0$.
The above problem appears as a model which describes a phase transition phenomenon in various fields. See the monograph of Fife [5] and the references therein.

We will mainly discuss the steady state problem associated with (1.1):

$$\begin{cases} \varepsilon^2 u'' + f(x,u) = 0, & 0 < x < 1, \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.5}$$

where ' $'$ ' denotes the derivative with respect to x. Angenent, Mallet-Paret and Peletier [3] proved that, for sufficiently small $\varepsilon > 0$, (1.5) admits a stable solution u_ε which possesses a single transition layer near each $x_0 \in \Sigma$ with $a'(x_0) \neq 0$ and that $u_\varepsilon(x)$ is sufficiently close to 0 (resp. 1) for x in any compact subset of $\{x \in (0,1)\,;\, a(x) > 1/2\}$ (resp. $\{x \in (0,1)\,;\, a(x) < 1/2\}$).

The appearance of such a solution with transition layers is closely related to the bistable property of reaction-term $f(x,u)$. As an energy functional associated with (1.1), one can find

$$E(u) = \int_0^1 \left\{ \frac{1}{2}\varepsilon^2 u_x(x)^2 + W(x,u(x)) \right\} dx,$$

where

$$W(x,u) = -\int_{\phi_0(x)}^{u} f(x,s)ds \qquad \text{with} \qquad \phi_0(x) = \begin{cases} 0 & \text{if } a(x) \leq 1/2, \\ 1 & \text{if } a(x) > 1/2. \end{cases} \tag{1.6}$$

Here W is called a **bistable potential** because W takes its local minimums at $u = 0$ and $u = 1$. It is well known that every solution of (1.1) converges to a solution of (1.5) as $t \to \infty$ and that $E(u(\cdot,t))$ is decreasing with respect to t. Therefore, a minimizer of E will be a stable solution of (1.5). We should note that the minimum of $W(x,\cdot)$ is attained at $u = 1$ (resp. $u = 0$) when $a(x) < 1/2$ (resp. $a(x) > 1/2$). Intuitively, this fact assures that E has a minimizer u_ε with a transition layer near an $x_0 \in \Sigma$ with $u'_\varepsilon(x_0)a'(x_0) < 0$. We also refer to a work of Hale and Sakamoto [6], who proved that (1.5) admits an unstable solution u_ε with a single transition layer near $x_0 \in \Sigma$ and that it satisfies $u'_\varepsilon(x_0)a'(x_0) > 0$. Moreover, Dancer and Yan [4] have shown the existence of a solution u_ε with multi-layers to (1.5). Here a **multi-layer** means a part of u_ε where multiple transition layers appear as a cluster in a neighborhood of a certain point. More precisely, it is proved that there exists a solution which possesses a prescribed number of transition layers near a designated point $x_0 \in \Sigma$. (They have discussed such solutions in a ball of R^N.) See also

Nakashima [7, 8], where a solution with multi-layers is studied in a balanced case with $f(x, u) = A(x)u(1-u)(u-1/2)$. See also the work of Ai and Hastings [2].

Recently, Ai, Chen and Hastings [1] have obtained remarkable results on the structure of solutions u_ε of (1.5) with transition layers and spikes. They give interesting information on complicated patterns of transition layers and spikes. The existence and stability (Morse index) of such solutions are also discussed. In order to discuss patterns, they derived asymptotic results which describes how close $u_\varepsilon(\zeta)$ approaches to 0 or 1 when ε is sufficiently small. Here, ζ denotes a local maximum or minimum point of u_ε. Using these results, they reduce the pattern determination problem to a certain kind of an algebraic system; patterns of transition layers and spikes are determined by solving this algebraic system. This paper is greatly motivated by their work. Our main purpose is to derive more precise results on the profiles of solutions with transition layers and spikes. We will develop more general results on the asymptotic behavior of $u_\varepsilon(x)$ as $\varepsilon \to 0$ (Theorems 3.3 and 3.6). Furthermore, we will discuss patterns by using different approach based on our asymptotic results.

When we concentrate ourselves on a solution u_ε of (1.5) with oscillatory profiles such as transition layers and spikes, it is useful to take account of the number of intersecting points of the graphs of u_ε and a in $(0, 1)$. We introduce the notion of n-mode solution; u_ε is called an **n-mode solution** if the graph of u_ε has n intersecting points with that of a in $(0, 1)$. Roughly speaking, for any n-mode solution u_ε of (1.5), its graph is classified into the following three groups (see Lemmas 2.2 and 2.4):

(i) $u_\varepsilon(x)$ is close to 0 or 1,

(ii) $u_\varepsilon(x)$ forms a transition layer connecting 0 and 1,

(iii) $u_\varepsilon(x)$ forms a spike based on 0 or 1.

Here it should be noted that, if u_ε has a spike, then its peak is distant away from $u = 0$ and $u = 1$. In order to study patterns of solutions with transition layers, we note that $u_\varepsilon(x)$ is very close to 0 or 1 at one of end-points of any transition layer, when ε is sufficiently small. The situation is similar when we discuss a spike; if u_ε has a spike based on 1, then $u_\varepsilon(x)$ is very close to 1 at both end-points of the spike. Therefore, as is stated in the preceeding paragraph, it will be important to study the asymptotic rate of $1 - u_\varepsilon(x)$ (resp. $u_\varepsilon(x)$) as $\varepsilon \to 0$ in a certain interval containing one local maximum point (resp. local minimum point) of u_ε. The analysis to get the asymptotic rate will be carried out by a kind of barrier method.

The content of this paper is as follows. In Section 2, we will give some fundamental properties of n-mode solutions of (1.5). In Section 3, asymptotic rates of $1 - u_\varepsilon(x)$ and $u_\varepsilon(x)$ as $\varepsilon \to 0$ for x in a suitable interval will be discussed. The asymptotic results are given by Theorems 3.3 and 3.6. These results enable us to show that any transition layer (resp. spike) appears only in a neighborhood of a point of Σ (resp. Λ) in Section 4. Finally, Section 5 is devoted to the study of multi-layers and multi-spikes. It will be shown that each multi-layer consists of an odd number of transition layers. Furthermore, we will show that multi-layers (resp. multi-spikes) can appear only in a neighborhood of a point in a suitable subset of Σ (resp. Λ).

2 Transition layers and spikes for n-mode solutions

In this section we will give some basic properties of solutions of (1.5).

Lemma 2.1. *Let u_ε be a solution of* (1.5). *Then*

$$0 \le u_\varepsilon(x) \le 1 \quad \text{for all } x \in (0,1).$$

Furthermore, if $u_\varepsilon \not\equiv 0$ or 1, then

$$0 < u_\varepsilon(x) < 1 \quad \text{for all } x \in (0,1).$$

Proof. Assume that

$$u_\varepsilon(x_0) = \max\{u_\varepsilon(x)\,;\, x \in [0,1]\} > 1 \tag{2.1}$$

for some $x_0 \in [0,1]$. It follows from $u_\varepsilon''(x_0) \le 0$ that $f(x_0, u_\varepsilon(x_0)) \ge 0$. On the other hand, (2.1) together with (1.2) implies $f(x_0, u_\varepsilon(x_0)) < 0$, which is a contradiction. Hence $u_\varepsilon(x) \le 1$. Similarly, it is easy to show $u_\varepsilon(x) \ge 0$.

To give the proof of the last assertion, assume $u_\varepsilon(x_1) = \max\{u_\varepsilon(x)\,;\, x \in [0,1]\} = 1$ at some $x_1 \in [0,1]$. Since $u_\varepsilon'(x_0) = 0$, we immediately get $u_\varepsilon \equiv 1$ by the uniqueness of solutions for the initial value problem of the second order differential equation. Therefore, $u_\varepsilon(x) < 1$ in $[0,1]$ unless $u_\varepsilon \equiv 1$. Similarly, one can see that, if $u \not\equiv 0$, then $u(x) > 0$ in $[0,1]$. This completes the proof. □

Let u_ε be a solution of (1.5). Recall that u_ε is called an n-mode solution of (1.5) if $u_\varepsilon - a$ has exactly n zero-points in $(0,1)$. Denote by $S_{n,\varepsilon}$ the set of all n-mode solutions and we fix arbitrary $n \in \mathbb{N}$. For $u_\varepsilon \in S_{n,\varepsilon}$, define

$$\Xi = \{x \in [0,1]\,;\, u_\varepsilon(x) = a(x)\}. \tag{2.2}$$

In what follows, we sometimes extend u_ε to a function over $\mathbb{R}$ by the standard reflection. This is possible because u_ε satisfies $u_\varepsilon'(0) = u_\varepsilon'(1) = 0$; so that u_ε is regarded as a periodic function with period 2. Similarly, by virtue of (A.4), $f(x,u)$ can be extended for $(x,u) \in \mathbb{R} \times \mathbb{R}$ by the reflection with respect to x-variable. So we may consider that u_ε satisfies (1.5) for all $x \in \mathbb{R}$.

Lemma 2.2. *For $n \in \mathbb{N}$, it holds that*

$$\lim_{\varepsilon \to 0} \sup_{u_\varepsilon \in S_{n,\varepsilon}} \max_{x \in [0,1]} \left| u_\varepsilon(x)(1-u_\varepsilon(x)) \left[\frac{1}{2}\varepsilon^2 u_\varepsilon'(x)^2 - W(x, u_\varepsilon(x)) \right] \right| = 0, \tag{2.3}$$

where $W(x,u)$ is defined by (1.6).

Proof. Although this lemma is given in [1, Lemma 2.1] we will give a proof for the sake of completeness. Suppose that (2.3) is not true, then there exist $\{(\varepsilon_k, u_k, x_k)\}$ such that $u_k \in S_{n,\varepsilon_k}$, $x_k \in [0,1]$ and

$$\left| u_k(x_k)(1-u_k(x_k)) \left[\frac{1}{2}{\varepsilon_k}^2 u_k'(x_k)^2 - W(x_k, u_k(x_k)) \right] \right| \ge \delta \tag{2.4}$$

with some $\delta > 0$.

We use a change of variable $x = x_k + \varepsilon_k t$ and introduce a new function U_k by $U_k(t) = u_k(x_k + \varepsilon_k t)$. Clearly, U_k satisfies

$$\ddot{U}_k + f(x_k + \varepsilon_k t, U_k) = 0 \quad \text{in} \quad \mathbb{R}, \tag{2.5}$$

where '$\dot{}$' denotes the derivative of t.

We first prove the uniform boundedness of $\{U_k\}, \{\dot{U}_k\}$ and $\{\ddot{U}_k\}$. By Lemma 2.1, $\sup\{|U_k(t)|\,;\, t \in \mathbb{R}\} < 1$; so that it follows from (2.5) that $\sup\{|\ddot{U}_k(t)|\,;\, t \in \mathbb{R}\} =: m_1 < \infty$. To study $\dot{U}_k$, we take any $t \in \mathbb{R}$. The mean value theorem assures that there exists a number $t_0 \in (t, t+1)$ such that

$$\dot{U}_k(t_0) = U_k(t+1) - U_k(t)\,;$$

then $-1 < \dot{U}_k(t_0) < 1$ from Lemma 2.1. Hence it holds that

$$|\dot{U}_k(t)| = \left|\dot{U}_k(t_0) + \int_{t_0}^{t} \ddot{U}_k(s)\, ds\right| < 1 + m_1.$$

These estimates implies that $\{U_k\}, \{\dot{U}_k\}, \{\ddot{U}_k\}$ are uniformly bounded in $\mathbb{R}$. Therefore, it is easy to see that $\{U_k\}$ and $\{\dot{U}_k\}$ are equi-continuous. Moreover, it also follows from (2.5) that $\{\ddot{U}_k\}$ is also equi-continuous.

On account of the above results, one can apply Ascoli-Arzelà's theorem and use a diagonal argument to show that $\{U_k\}$ has a subsequence, which is still denoted by $\{U_k\}$, such that

$$\lim_{k\to\infty} U_k = U \qquad \text{in} \quad C^2_{loc}(\mathbb{R})$$

with a suitable function $U \in C^2(\mathbb{R})$. Here we recall that $\{x_k\}$ is bounded. Since one can choose a convergent subsequence from $\{x_k\}$, we may assume

$$\lim_{k\to\infty} x_k = x^* \in [0,1].$$

Then it is seen in the standard manner that U satisfies

$$\ddot{U} + f(x^*, U) = 0 \quad \text{in } \mathbb{R}. \tag{2.6}$$

Multiplying (2.6) by $\dot{U}$ and integrating the resulting expression with respect to t we get

$$\frac{1}{2}\dot{U}(t)^2 - W(x^*, U(t)) = C \quad \text{in } \mathbb{R} \tag{2.7}$$

with some constant C. If $U \equiv 0$ or $U \equiv 1$, then it is easy to derive a contradiction to (2.4) from (2.7).

We will show $C = 0$ in (2.7) in the case that $U \not\equiv 0$ and $U \not\equiv 1$. If $C > 0$, then we see from the phase plane analysis that U is unbounded. This is impossible because $\{U_k\}$ is bounded. If $C < 0$, then the phase plane analysis tells us that U is a periodic function. So the graph of $U(t)$ has infinitely many intersecting points with that of $a(x^*)$ and, therefore, the graph of $U_k(t)$ also has infinitely many intersecting points provided that k is sufficiently large. This fact implies that, if k is sufficiently large, then $u_k(x) - a(x)$ has many zero-points near $x = x^*$. This result contradicts to the definition of n-mode solutions. Thus we have proved $C = 0$ in (2.7).

Hence

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{1}{2}{\varepsilon_k}^2 u_k'(x_k)^2 - W(x_k, u_k(x_k))\right| &= \lim_{k\to\infty}\left|\frac{1}{2}\dot{U}_k(0)^2 - W(x_k, U_k(0))\right| \\ &= \left|\frac{1}{2}\dot{U}(0)^2 - W(x^*, U(0))\right| = 0,\end{aligned}$$

which contradicts to (2.4). Thus the proof is complete. □

Lemma 2.3. *For $u_\varepsilon \in S_{n,\varepsilon}$, set $\Xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$. If ε is sufficiently small, then u_ε' has exactly $(n-1)$ zero points $\{\zeta_k\}_{k=1}^{n-1}$ in $(0,1)$ satisfying*

$$0 < \xi_1 < \zeta_1 < \xi_2 < \zeta_2 < \cdots < \xi_{n-1} < \zeta_{n-1} < \xi_n < 1.$$

Proof. Let $\xi \in \Xi$ and take any small $\eta > 0$. Lemma 2.2 implies that, if ε is sufficiently small, then

$$\begin{aligned}&\left|u_\varepsilon(\xi)(1 - u_\varepsilon(\xi))\left[\frac{1}{2}\varepsilon^2 u_\varepsilon'(\xi)^2 - W(\xi, u_\varepsilon(\xi))\right]\right| \\ &\qquad = \left|a(\xi)(1 - a(\xi))\left[\frac{1}{2}\varepsilon^2 u_\varepsilon'(\xi)^2 - W(\xi, a(\xi))\right]\right| < \eta.\end{aligned}$$

Since $a(\xi)(1 - a(\xi)) > M$ with some ε-independent $M > 0$, we get

$$-\frac{\eta}{M} < \frac{1}{2}\varepsilon^2 u_\varepsilon'(\xi)^2 - W(\xi, a(\xi)) < \frac{\eta}{M}.$$

Observe that $W(\xi, a(\xi)) \geq c_1 > 0$, where c_1 is a positive constant independent of ε. Hence, taking a sufficiently small $\varepsilon > 0$ one can conclude

$$\varepsilon^2 u_\varepsilon'(\xi)^2 \geq c_2^2 > 0$$

with some $c_2 > 0$. Thus

$$|u'(\xi)| > \frac{c_2}{\varepsilon} \tag{2.8}$$

for sufficiently small ε.

We study the case $u_\varepsilon(x) > a(x)$ in (ξ_k, ξ_{k+1}). By (2.8) and the boundedness of $a'(x)$, it is easy to see $u_\varepsilon'(\xi_k) > 0$ and $u_\varepsilon'(\xi_{k+1}) < 0$. On the other hand, since (1.5) implies $u''(x) < 0$ in (ξ_k, ξ_{k+1}), u_ε' has a unique zero point in (ξ_k, ξ_{k+1}), which is denoted by ζ_k. Clearly u_ε attains its local maximum at $x = \zeta_k$.

Since the proof is analogous for the case $u_\varepsilon(x) < a(x)$ in (ξ_k, ξ_{k+1}), it remains to show the nonexistence of zero point of u_ε' in $(0, \xi_1) \cup (\xi_n, 1)$. Assume $u_\varepsilon(x) > a(x)$ in $(0, \xi_1)$. Since $u_\varepsilon'(0) = 0$ and $u_\varepsilon''(x) < 0$ in $(0, \xi_1)$, it is clear that $u_\varepsilon'(x) < 0$ in $(0, \xi_1)$. The other cases can be discussed in the same way. □

Lemma 2.4. *For $u_\varepsilon \in S_{n,\varepsilon}$, let ξ^ε be any point in Ξ and define U_ε by $U_\varepsilon(t) = u_\varepsilon(\xi^\varepsilon + \varepsilon t)$. Then there exists a subsequence $\{\varepsilon_k\} \downarrow 0$ such that $\xi_k = \xi^{\varepsilon_k}$ and $U_k = U_{\varepsilon_k}$ satisfy*

$$\lim_{k\to\infty} \xi_k = \xi^* \quad \textit{and} \quad \lim_{k\to\infty} U_k = \phi \quad \textit{in } C^2_{loc}(\mathbb{R}),$$

where $\phi \in C^2(\mathbb{R})$ is a function satisfying one of the following properties.
(i) *In the case $a(\xi^*) = 1/2$, ϕ is a unique solution of the following problem:*

$$\begin{cases} \ddot{\phi} + f(\xi^*, \phi) = 0 & \textit{in } \mathbb{R}, \\ \phi(-\infty) = 0,\ \phi(+\infty) = 1 & (\textit{resp. } \phi(-\infty) = 1,\ \phi(+\infty) = 0), \\ \phi(0) = 1/2, \end{cases}$$

if $\dot{\phi}(0) > 0$ (resp. $\dot{\phi}(0) < 0$). Moreover, $\dot{\phi}(t) > 0$ for $t \in \mathbb{R}$ if $\dot{\phi}(0) > 0$, while $\dot{\phi}(t) < 0$ for $t \in \mathbb{R}$ if $\dot{\phi}(0) < 0$.
(ii) *In the case $a(\xi^*) < 1/2$, ϕ is a solution of the following problem:*

$$\begin{cases} \ddot{\phi} + f(\xi^*, \phi) = 0 & \textit{in } \mathbb{R}, \\ \phi(0) = a(\xi^*), \phi(\pm\infty) = 0. \end{cases}$$

Moreover, ϕ satisfies $\sup\{\phi(x)\,;\, x \in \mathbb{R}\} > a(\xi^*)$.
(iii) *In the case $a(\xi^*) > 1/2$, ϕ is a solution of the following problem:*

$$\begin{cases} \ddot{\phi} + f(\xi^*, \phi) = 0 & \textit{in } \mathbb{R}, \\ \phi(0) = a(\xi^*), \phi(\pm\infty) = 1. \end{cases}$$

Moreover, ϕ satisfies $\inf\{\phi(x)\,;\, x \in \mathbb{R}\} < a(\xi^*)$.

Proof. Clearly, $U_\varepsilon(t)$ satisfies

$$\ddot{U}_\varepsilon + f(\xi^\varepsilon + \varepsilon t, U_\varepsilon) = 0 \quad \text{and} \quad U_\varepsilon(0) = a(\xi^\varepsilon).$$

As in the proof of Lemma 2.2, one can prove that $\{U_\varepsilon\}$ is bounded in $C^2(\mathbb{R})$; so that there exists a subsequence $\{\varepsilon_k\} \downarrow 0$ such that $U_k = U_{\varepsilon_k}$ is convergent in $C^2_{loc}(\mathbb{R})$; i.e.,

$$\lim_{k\to\infty} U_k = \phi \qquad \text{in } C^2_{loc}(\mathbb{R}) \tag{2.9}$$

with some $\phi \in C^2(\mathbb{R})$. Moreover, since $\{\xi_k\}$ $(\xi_k = \xi^{\varepsilon_k})$ is also bounded, we may assume $\lim_{k\to\infty} \xi_k = \xi^* \in [0,1]$. Therefore, the limiting procedure yields

$$\ddot{\phi}(t) + f(\xi^*, \phi(t)) = 0 \quad \text{with } \phi(0) = a(\xi^*).$$

The same argument as in the proof of (2.7) with $C = 0$ also shows

$$\frac{1}{2}\dot{\phi}(t)^2 - W(\xi^*, \phi(t)) = 0$$

for $t \in \mathbb{R}$. Hence the phase plane analysis enables us to conclude that ϕ satisfies one of (i)-(iii). $\square$

Lemma 2.5. *For $u_\varepsilon \in S_{n,\varepsilon}$, let $\xi_1^\varepsilon, \xi_2^\varepsilon$ be two successive points in Ξ. Then one of the following properties holds true:*
(i) *$(\xi_2^\varepsilon - \xi_1^\varepsilon)/\varepsilon$ is unbounded as $\varepsilon \to 0$,*
(ii) *For sufficiently small $\varepsilon > 0$, it holds that*

$$M_1 < \frac{\xi_2^\varepsilon - \xi_1^\varepsilon}{\varepsilon} < M_2,$$

where M_1 and M_2 are positive constants independent of ε.

Proof. We denotes the derivative with respect to t by ' $\dot{}$ ' and the derivative with respect to x by ' $'$ '. Put $U_\varepsilon(t) = u_\varepsilon(\xi_1^\varepsilon + \varepsilon t)$; then $\dot{U}_\varepsilon(t) = \varepsilon u_\varepsilon'(\xi_1^\varepsilon + \varepsilon t)$. Therefore,

$$\begin{cases} \dot{U}_\varepsilon(0) = \varepsilon u_\varepsilon'(\xi_1^\varepsilon), \\ \dot{U}_\varepsilon((\xi_2^\varepsilon - \xi_1^\varepsilon)/\varepsilon) = \varepsilon u_\varepsilon'(\xi_2^\varepsilon). \end{cases}$$

In view of (2.8) we see

$$\dot{U}_\varepsilon(0)\dot{U}_\varepsilon((\xi_2^\varepsilon - \xi_1^\varepsilon)/\varepsilon) = \varepsilon^2 u_\varepsilon'(\xi_1^\varepsilon) u_\varepsilon'(\xi_2^\varepsilon) < -c_2^2 < 0. \tag{2.10}$$

Suppose that $\{(\xi_2^\varepsilon - \xi_1^\varepsilon)/\varepsilon\}$ is bounded. Then one can choose a subsequence $\{\varepsilon_k\}$ such that

$$0 \le M = \lim_{k\to\infty} \frac{\xi_2^{\varepsilon_k} - \xi_1^{\varepsilon_k}}{\varepsilon_k} < +\infty.$$

Recalling the proof of Lemma 2.4 we may regard $\{U_{\varepsilon_k}\}$ as a convergent sequence satisfying (2.9). Setting $\varepsilon = \varepsilon_k$ in (2.10) and letting $k \to \infty$ we get

$$\dot{\phi}(0)\dot{\phi}(M) \le -c_2^2 < 0.$$

Hence it follows from Lemma 2.4 that M must be positive. Thus we have shown (ii) when (i) does not hold. □

3 Asymptotic profiles of n-mode solutions

In this section we will derive some asymptotic behavior of u_ε or $1 - u_\varepsilon$ as $\varepsilon \to 0$ in a certain interval containing a local minimum or local maximum of u_ε. For this purpose, we first prepare the following lemma.

Lemma 3.1. *Let $g(v) = v(1-v)(v-a_0)$ with $a_0 \in (0,1)$. Then for any $\sigma \in (0,1)$ satisfying $\sigma > \max\{a_0, (a_0+1)/3\}$ and $M > 0$, there exists a unique solution of*

$$\begin{cases} v_{zz} + g(v) = 0 & \text{in } (-M, 0), \\ v(-M) = \sigma,\ v_z(0) = 0, \\ v > \sigma & \text{in } (-M, 0). \end{cases} \tag{3.1}$$

Moreover, there exists a constant $\sigma^* \in ((a_0 + 1 + \sqrt{a_0^2 - a_0 + 1})/3, 1)$ such that, if $\sigma > \sigma^*$, then

$$c_1 \exp(-RM) < 1 - v(0) < c_2 \exp(-rM), \tag{3.2}$$

where $r = \sqrt{-g'(\sigma)}$, $R = \sqrt{-g'(1)}$ and $c_1, c_2\ (0 < c_1 < c_2)$ are positive constants depending only on σ.

Proof. In order to solve (3.1), we employ the time-map method (see, e.g., Smoller and Wasserman [9]). Take $\sigma \in (0,1)$ with $\sigma > \max\{a_0, (a_0+1)/3\}$ and consider the following initial value problem:

$$\begin{cases} v_{zz} + g(v) = 0 & \text{for } z > -M, \\ v(-M) = \sigma,\ v_z(-M) = p, \end{cases} \tag{3.3}$$

where p is a positive parameter. Let $v(z;p)$ the solution of (3.3). Multiplying (3.3) by $v_z(z;p)$ and integrating the resulting expression over $(-M, z)$ we get

$$\frac{1}{2} v_z(z;p)^2 - G(v(z;p)) = \frac{1}{2} p^2, \tag{3.4}$$

where

$$G(v) = -\int_\sigma^v g(s)ds.$$

Since we look for p satisfying $v_z(0;p) = 0$ and $v_z(z;p) > 0$ for $z \in (-M, 0)$, we have to restrict the range of p. By the phase plane analysis, $0 < p < \sqrt{-2G(1)} =: p^*$ (note $-G(1) > 0$ because of $\sigma > a_0$).

For such p, define $\alpha(p) \in (\sigma, 1)$ by $p^2/2 = -G(\alpha(p))$, and let $T(p)$ be a time-map defined by

$$T(p) = \inf\{\, z > -M\,;\, v(z) = \alpha(p)\,\} + M.$$

Then $\alpha(p) = \max\{v(z;p)\,;\, z > -M\}$ and $T(p)$ denotes the distance from $-M$ to the first zero point of v_z. If we can find a number p_M satisfying $T(p_M) = M$, then $v(z;p_M)$ gives a solution of (3.1). Hence the study of $T(p)$ is essential to show the existence of a solution of (3.1).

As a first step, we will show that $T(p)$ is strictly monotone increasing for $0 < p < p^*$. It follows from (3.4) that

$$\frac{1}{\sqrt{G(v) - G(\alpha(p))}} \frac{dv}{dz} = \sqrt{2}$$

Integrating this equation over $(-M, -M + T(p))$ yields

$$\sqrt{2}T(p) = \int_\sigma^{\alpha(p)} \frac{dv}{\sqrt{G(v) - G(\alpha(p))}}. \tag{3.5}$$

From the definition, $\alpha(p)$ is a strictly increasing function of p satisfying $\alpha(p) \to \sigma$ as $p \to 0$ and $\alpha(p) \to 1$ as $p \to p^*$. So it is convenient to treat $T(p)$ in (3.5) as a function of α in place of p. Set

$$S(\alpha) = \int_\sigma^\alpha \frac{dv}{\sqrt{G(v) - G(\alpha)}} = \int_0^1 \frac{\alpha - \sigma}{\sqrt{G(s(\alpha-\sigma)+\sigma) - G(\alpha)}} ds.$$

We will prove that $S(\alpha)$ is strictly monotone increasing for $\alpha \in (\sigma, 1)$. Differentiation of $S(\alpha)$ with respect to α gives

$$\begin{aligned} S'(\alpha) &= \int_0^1 \frac{2(\Delta G) + (\alpha-\sigma)sg(s(\alpha-\sigma)+\sigma) - (\alpha-\sigma)g(\alpha)}{2(\Delta G)^{3/2}} ds \\ &= \frac{1}{\alpha - \sigma} \int_\sigma^\alpha \frac{\theta(v) - \theta(\alpha)}{2(\Delta G)^{3/2}} dv, \end{aligned} \tag{3.6}$$

where

$$\Delta G = G(v) - G(\alpha) \quad \text{and} \quad \theta(v) = 2G(v) + (v-\sigma)g(v).$$

Note $\Delta G > 0$ for $\sigma < v < \alpha$. We will investigate θ to show $S'(\alpha) > 0$ for $\alpha \in (\sigma, 1)$. It is easy to see

$$\theta'(v) = -g(v) + (v-\sigma)g'(v) \text{ and } \theta''(v) = (v-\sigma)g''(v).$$

Observe $\theta'(\sigma) = -g(\sigma) < 0$ for $a_0 < \sigma < 1$. Moreover, $\theta''(v) < 0$ in (σ, α) by the concavity of $g(v)$. Therefore, $\theta'(v) < 0$ in (σ, α). Since θ is monotone decreasing in (σ, α), we see from (3.6) that $S'(\alpha) > 0$ in $(\sigma, 1)$. Therefore, $S(\alpha)$ is monotone increasing in $(\sigma, 1)$ and so is $T(p)$ in $(0, p^*)$.

Furthermore, we will show

$$\lim_{p\to 0} T(p) = 0 \tag{3.7}$$

and

$$\lim_{p\to p^*} T(p) = +\infty. \tag{3.8}$$

We use

$$\begin{aligned} G(v) - G(\alpha) &= \int_v^\alpha g(s)ds \\ &\geq \min\{g(\alpha), g(\sigma)\}(\alpha - v) \quad \text{for } v \in (\sigma, \alpha) \end{aligned}$$

to prove (3.7). Hence it is easy to see $\lim_{\alpha\to\sigma} S(\alpha) = 0$, which implies (3.7). To prove (3.8), we note $\alpha(p) \to 1$ when $p \to p^*$. For $\alpha \to 1$, we see

$$G(v) - G(\alpha) \to -\frac{1}{2}g'(1)(v-1)^2 + o((v-1)^2) \quad \text{as } v \to 1.$$

Therefore, $\lim_{\alpha\to 1} S(\alpha) = +\infty$, whence follows (3.8).

We have shown that $T(p)$ is a strictly increasing function satisfying (3.7) and (3.8). Hence it is easy to see that, for each $M > 0$, there exists a unique $p_M \in (0, p^*)$ such that $T(p_M) = M$. Clearly, p_M is strictly increasing and continuous with respect to M and $\lim_{M\to\infty} p_M = p^*$. Set $v_M = v(0; p_M)$; v_M is also strictly increasing and continuous with respect to M and satisfies $\lim_{M\to\infty} v_M = 1$.

We will prove that v_M satisfies (3.2). Recall

$$\sqrt{2}M = \int_\sigma^{v_M} \frac{dv}{\sqrt{G(v) - G(v_M)}}, \tag{3.9}$$

from (3.5). By the mean value theorem, there exists a constant $\theta_1 \in (\sigma, v_M)$ satisfying

$$\frac{G(v) - G(v_M)}{(1-v)^2 - (1-v_M)^2} = -\frac{g(\theta_1)}{2(\theta_1 - 1)} = -\frac{g(\theta_1) - g(1)}{2(\theta_1 - 1)}. \tag{3.10}$$

Using the mean value theorem again, we see that the right-hand side of (3.10) is equal to $-g'(\theta_2)/2$ with some $\theta_2 \in (\theta_1, 1)$. Take $\sigma^* \in ((a_0 + 1 + \sqrt{a_0^2 - a_0 + 1})/3, 1)$.

It should be noted that $g'(s)$ is decreasing and negative for $s \in (\sigma^*, 1)$. Then for $\sigma \in (\sigma^*, 1)$

$$\frac{r^2}{2} < -\frac{g'(\theta_2)}{2} < \frac{R^2}{2} \tag{3.11}$$

with $r = \sqrt{-g'(\sigma)}$ and $R = \sqrt{-g'(1)}$. With use of (3.10) and (3.11), it follows from (3.9) that

$$\frac{1}{R} B_M < M < \frac{1}{r} B_M, \tag{3.12}$$

where

$$B_M = \int_\sigma^{v_M} \frac{dv}{\sqrt{(1-v)^2 - (1-v_M)^2}} = \log\left(b_M + \sqrt{b_M^2 - 1}\right),$$

with $b_M = (1-\sigma)/(1-v_M)$. Since $B_M \in [\log b_M, \log 2b_M]$, (3.12) yields

$$(1-\sigma)\exp(-RM) < 1 - v_M < 2(1-\sigma)\exp(-rM).$$

Thus the proof is complete. □

Replacing z by $-z$ in the proof of Lemma 3.1, we can show the following lemma.

Lemma 3.2. *Let g be the same function as in Lemma 3.1. Then for any $\sigma \in (0,1)$ satisfying $\sigma > \max\{a_0, (a_0+1)/3\}$ and $M > 0$, there exists a unique solution of*

$$\begin{cases} v_{zz} + g(v) = 0 & \text{in } (0, M), \\ v_z(0) = 0,\ v(M) = \sigma, & \\ v > \sigma & \text{in } (0, M). \end{cases}$$

Furthermore, there exists a constant $\sigma^* \in ((a_0 + 1 + \sqrt{a_0^2 - a_0 + 1})/3, 1)$ such that, if $\sigma > \sigma_*$, then v satisfies (3.2).

In what follows, let ξ_1, ξ_2 be two successive points in Ξ and let (ξ_1, ξ_2) be an interval such that

$$u_\varepsilon(x) - a(x) > 0 \quad \text{in } (\xi_1, \xi_2). \tag{3.13}$$

Let $\zeta \in (\xi_1, \xi_2)$ be a unique point satisfying $u_\varepsilon'(\zeta) = 0$ and $u_\varepsilon'(x) > 0$ in (ξ_1, ζ). The existence of such ζ is assured by Lemma 2.3.

We will study asymptotic behavior of u_ε in (ξ_1, ξ_2) as $\varepsilon \downarrow 0$.

Theorem 3.3. *For $u_\varepsilon \in S_{n,\varepsilon}$, assume (3.13) and let $\zeta \in (\xi_1, \xi_2)$ satisfy $u'(\zeta) = 0$. If $(\zeta - \xi_1)/\varepsilon \to +\infty$ as $\varepsilon \to 0$, then there exist positive constants C_1, C_2, r, R with $C_1 < C_2$ and $r < R$ such that*

$$C_1 \exp\left(-\frac{R(\zeta - \xi_1)}{\varepsilon}\right) < 1 - u_\varepsilon(x) < C_2 \exp\left(-\frac{r(x - \xi_1)}{\varepsilon}\right) \tag{3.14}$$

for $x \in [\xi_1, \zeta]$ and sufficiently small $\varepsilon > 0$.

Proof of Theorem 3.3. We begin with the proof of the right-hand side inequality of (3.14). Let a^* be a constant which satisfies $a^* > \max\{a(x)\,;\, x \in [\xi_1, \zeta]\}$ and take $\delta^* \in (a^*, 1)$ which is close to 1. By assumptions and Lemma 2.4 we can find a point $\tilde{\xi}_1 \in (\xi_1, \zeta)$ such that $u_\varepsilon(\tilde{\xi}_1) = \delta^*$ and $u_\varepsilon(x) > \delta^*$ in $(\tilde{\xi}_1, \zeta)$ provided that ε is sufficiently small. Clearly, $\tilde{\xi}_1 - \xi_1 = O(\varepsilon)$ as $\varepsilon \to 0$; so $\zeta - \tilde{\xi}_1 > \varepsilon$.

Now take any $x^* \in (\tilde{\xi}_1 + \varepsilon, \zeta)$ and apply Lemma 3.1. Let $v(z)$ be a solution of (3.1) with $a_0 = a^*$, $\sigma = \delta^*$ and $M = (x^* - \tilde{\xi}_1 - \varepsilon)/\varepsilon$. We use a change of variable $z = (x - x^*)/\varepsilon$ and define V_1 by $V_1(x) = v((x - x^*)/\varepsilon)$; then

$$\begin{cases} \varepsilon^2 V_1'' + V_1(1 - V_1)(V_1 - a^*) = 0 & \text{in } (\tilde{\xi}_1 + \varepsilon, x^*), \\ V_1(\tilde{\xi}_1 + \varepsilon) = \delta^*,\ V_1'(x^*) = 0, & \\ V_1 > \delta^* & \text{in } (\tilde{\xi}_1 + \varepsilon, x^*). \end{cases} \tag{3.15}$$

By virtue of Lemma 3.1, V_1 satisfies

$$c_1 e^R \exp\left(-\frac{R(x^* - \tilde{\xi}_1)}{\varepsilon}\right) < 1 - V_1(x^*) < c_2 e^r \exp\left(-\frac{r(x^* - \tilde{\xi}_1)}{\varepsilon}\right), \tag{3.16}$$

where c_1, c_2, r and R are positive constants depending only on a^* and δ^*.

We will show

$$V_1(x) \le u_\varepsilon(x) \qquad \text{in} \quad (\tilde{\xi}_1 + \varepsilon, x^*). \tag{3.17}$$

For this purpose, it is convenient to introduce the following auxiliary function

$$h_1(x) = \frac{V_1(x) - a^*}{u_\varepsilon(x) - a^*} \quad \text{in } [\tilde{\xi}_1 + \varepsilon, x^*],$$

and show $h_1(x) \le 1$ in $[\tilde{\xi}_1 + \varepsilon, x^*]$ by contradiction. Suppose that there exists an $x_1 \in [\tilde{\xi}_1 + \varepsilon, x^*]$ such that

$$h_1(x_1) = \max\{h_1(x)\,;\, x \in [\tilde{\xi}_1 + \varepsilon, x^*]\} = \frac{1}{\eta} > 1.$$

Then

$$\begin{cases} V_\eta(x) \le u_\varepsilon(x) & \text{in } [\tilde{\xi}_1 + \varepsilon, x^*], \\ V_\eta(x_1) = u_\varepsilon(x_1), & \end{cases}$$

where

$$V_\eta(x) = \eta(V_1(x) - a^*) + a^* \ (< V_1(x)).$$

We will prove

$$V_\eta''(x_1) \le u_\varepsilon''(x_1). \tag{3.18}$$

Clearly, $h_1(\tilde{\xi}_1 + \varepsilon) < 1$. Moreover, since $u_\varepsilon'(x^*) > 0$ and $V_1'(x^*) = 0$ (by (3.15)), it is easy to see $h_1'(x^*) < 0$. Therefore, x_1 must be an interior point in $(\tilde{\xi}_1 + \varepsilon, x^*)$. So

$$h_1'(x_1) = 0 \quad \text{and} \quad h_1''(x_1) \le 0. \tag{3.19}$$

From the definition of h_1,

$$h_1(x)(u_\varepsilon(x) - a^*) = V_1(x) - a^*.$$

Differentiating the above identity two times with respect to x and setting $x = x_1$ we get

$$u_\varepsilon''(x_1) + 2\eta u_\varepsilon'(x_1)h_1'(x_1) + \eta(u_\varepsilon(x_1) - a^*)h_1''(x_1) = \eta V_1''(x_1) = V_\eta''(x_1). \tag{3.20}$$

Then (3.19) and (3.20) imply (3.18).

We next use $f(x, V_\eta) > \eta V_1(1 - V_1)(V_1 - a^*)$. Indeed, since $V_1(x) > a^* > a(x)$ in $(\tilde{\xi}_1 + \varepsilon, x^*)$, a simple calculation yields

$$\begin{aligned} f(x, V_\eta) &= V_\eta(1 - V_\eta)(V_\eta - a(x)) \\ &= \eta(V_1 - a^*)V_\eta(1 - V_\eta) + (a^* - a(x))V_\eta(1 - V_\eta) \\ &> \eta(V_1 - a^*)V_\eta(1 - V_\eta) \\ &> \eta V_1(1 - V_1)(V_1 - a^*) \end{aligned}$$

provided that δ^* is sufficiently close to 1. Hence it follows from (3.15) that

$$\varepsilon^2 V_\eta'' + f(x, V_\eta) = \eta\varepsilon^2 V_1'' + f(x, V_\eta) > \eta\{\varepsilon^2 V_1'' + V_1(1 - V_1)(V_1 - a^*)\} = 0.$$

Therefore, using (3.18) we have

$$0 = \varepsilon^2 u_\varepsilon''(x_1) + f(x_1, u_\varepsilon(x_1)) \geq \varepsilon^2 V_\eta''(x_1) + f(x_1, V_\eta(x_1)) > 0,$$

which is a contradiction. Thus we have shown (3.17).

Now (3.16) and (3.17) imply

$$1 - u_\varepsilon(x^*) \leq 1 - V_1(x^*) < c_2 e^r \exp\left(-\frac{r(x^* - \tilde{\xi}_1)}{\varepsilon}\right).$$

Here we should note that c_2 and r can be chosen independently of x^*. Recalling that x^* is an arbitrary point in $(\tilde{\xi}_1 + \varepsilon, \zeta)$, one can conclude that

$$1 - u_\varepsilon(x) < c_2 e^r \exp\left(-\frac{r(x - \tilde{\xi}_1)}{\varepsilon}\right) \tag{3.21}$$

is valid for $x \in (\tilde{\xi}_1 + \varepsilon, \zeta)$.

Moreover, since $\tilde{\xi}_1 - \xi_1 < K\varepsilon$ with some $K > 0$, it follows from (3.21) that

$$\begin{aligned} 1 - u_\varepsilon(x) &< c_2 e^r \exp\left(-\frac{r(x - \xi_1)}{\varepsilon}\right) \exp\left(\frac{r(\tilde{\xi}_1 - \xi_1)}{\varepsilon}\right) \\ &< c_2 e^{r(K+1)} \exp\left(-\frac{r(x - \xi_1)}{\varepsilon}\right) \end{aligned} \tag{3.22}$$

for $x \in (\tilde{\xi}_1 + \varepsilon, \zeta)$. On the other hand, we note that

$$\exp(-r(K + 1)) < \exp\left(-\frac{r(x - \xi_1)}{\varepsilon}\right)$$

for $x \in (\xi_1, \tilde{\xi}_1 + \varepsilon)$. Hence, we can choose a sufficiently large constant $L > 0$ such that

$$\begin{aligned} 1 - u_\varepsilon(x) &\leq 1 - u_\varepsilon(\xi_1) = 1 - a(\xi_1) \\ &< L\exp(-r(K+1)) < L\exp\left(-\frac{r(x-\xi_1)}{\varepsilon}\right) \end{aligned} \tag{3.23}$$

for $x \in (\xi_1, \tilde{\xi}_1 + \varepsilon)$. Thus (3.22) and (3.23) enable us to extend (3.21) for all $x \in [\xi_1, \zeta]$ with $\tilde{\xi}_1$ replaced by ξ_1 (for $x = \zeta$, it is sufficient to use the continuity of u_ε with respect to x).

We will prove the left-hand side inequality of (3.14). Let a_* be a constant satisfying $a^* < \min\{a(x)\,;\, x \in [\xi_1, \zeta]\}$ and take $\delta_* \in (a_*, 1)$ which is close to 1. In particular, we assume that $\delta_* > \max\{1/2, \max\{a(x)\,;\, x \in [\xi_1, \zeta]\}\}$. Then there exists a point $\bar{\xi} \in (\xi_1, \zeta)$ such that $u_\varepsilon(\bar{\xi}_1) = \delta_*$ and $\bar{\xi}_1 - \xi_1 = O(\varepsilon)$.

If ε is sufficiently small, then $\zeta - \bar{\xi}_1 > \varepsilon$. We apply Lemma 3.1 by setting $\sigma = \delta_*$, $a_0 = a_*$ and $M = (\zeta - \bar{\xi}_1 + \varepsilon)/\varepsilon$ and define v as the solution of (3.1). With use of the change of variable $z = (x - \zeta)/\varepsilon$, we see that $V_2(x) = v((x-\zeta)/\varepsilon)$ satisfies

$$\begin{cases} \varepsilon^2 V_2'' + V_2(1 - V_2)(V_2 - a_*) = 0 & \text{in } (\bar{\xi}_1 - \varepsilon, \zeta), \\ V_2(\bar{\xi}_1 - \varepsilon) = \delta_*,\ V_2'(\zeta) = 0, & \\ V_2 > \delta_* & \text{in } (\bar{\xi}_1 - \varepsilon, \zeta). \end{cases}$$

Lemma 3.1 gives

$$c_1 e^R \exp\left(-\frac{R(\zeta - \bar{\xi}_1)}{\varepsilon}\right) < 1 - V_2(\zeta). \tag{3.24}$$

We will prove

$$V_2(x) \geq u_\varepsilon(x) \quad \text{in} \quad [\bar{\xi}_1 - \varepsilon, \zeta], \tag{3.25}$$

which, together with (3.24), yields the assertion because $u_\varepsilon(\zeta)$ is the maximum of u_ε in $[\xi_1, \zeta]$ and $\xi_1 < \bar{\xi}_1 < \zeta$. To prove (3.25), we introduce the following function

$$h_2(x) = \frac{u_\varepsilon(x) - a_*}{V_2(x) - a_*} \quad \text{in } [\bar{\xi}_1, \zeta]$$

and will show $h_2(x) \leq 1$ by contradiction. Assume that there exists $x_2 \in [\bar{\xi}_1, \zeta]$ such that

$$h_2(x_2) = \max\{h_2(x)\,;\, x \in [\bar{\xi}_1, \zeta]\} = \eta > 1. \tag{3.26}$$

By (3.26)

$$\begin{cases} u_\varepsilon(x) \leq W_\eta(x) & \text{in } [\bar{\xi}_1, \zeta], \\ u_\varepsilon(x_2) = W_\eta(x_2), & \end{cases}$$

where $W_\eta(x) = \eta(V_2(x) - a_*) + a_*$. Since $h_2(\bar{\xi}_1) < 1$, x_2 must satisfy $\bar{\xi}_1 < x_2 \leq \zeta$. If x_2 lies in $(\bar{\xi}_1, \zeta)$, then it is easy to see

$$u_\varepsilon''(x_2) \leq W_\eta''(x_2). \tag{3.27}$$

For the case $x_2 = \zeta$, note $h_2'(x_2) = h_2'(\zeta) = 0$. Therefore, (3.27) is also valid for $x_2 = \zeta$.

As the next step, we will prove

$$f(x, W_\eta) < \eta V_2(1 - V_2)(V_2 - a_*). \tag{3.28}$$

As a function of η, set $P(\eta) = \eta V_2(1 - V_2)(V_2 - a_*) - f(x, W_\eta)$. Then

$$P'(\eta) = V_2(1 - V_2)(V_2 - a_*) - (V_2 - a_*)f_u(x, W_\eta) = (V_2 - a_*)Q(\eta),$$

where

$$Q(\eta) = V_2(1 - V_2) - f_u(x, W_\eta).$$

Observe that

$$Q'(\eta) = -f_{uu}(x, W_\eta)(V_2 - a_*) = 2(V_2 - a_*)\{(W_\eta - a(x)) + (2W_\eta - 1)\}.$$

Recalling the definition of δ_* and $\eta > 1$, we can see that $W_\eta(x) \geq V_2(x) > \delta_* > \max\{1/2, \max\{a(x)\,;\, x \in [\xi_1, \zeta]\}\}$ in $(\bar{\xi}_1, \zeta)$; this implies $Q'(\eta) > 0$. Therefore,

$$Q(\eta) \geq Q(1) = (V_2 - a(x))(2V_2 - 1) > 0,$$

which leads to $P'(\eta) > 0$ for $\eta \geq 1$. Hence we get

$$P(\eta) \geq P(1) = V_2(1 - V_2)(a(x) - a_*) > 0$$

and (3.28) is proved.

We finally combine (3.27) and (3.28) to get

$$\begin{aligned} 0 = \varepsilon^2 u_\varepsilon''(x_2) + f(x_2, u_\varepsilon(x_2)) &\leq \varepsilon^2 W_\eta''(x_2) + f(x_2, W_\eta(x_2)) \\ &< \eta\{(\varepsilon^2 V_2''(x_2) + V_2(x_2)(1 - V(x_2))(V(x_2) - a_*)\} = 0. \end{aligned}$$

Since this is a contradiction, we have shown (3.25) ; thus the proof is complete. □

Remark 3.4. We should note that (3.14) depends on position x ; for any $x \in [\xi_1, \zeta]$, $1 - u_\varepsilon(x)$ is estimated in terms of the distance between x and ξ_1 when ζ is a local maximum point. Although similar results as Theorem 3.3 have been obtained by Ai, Chen and Hastings [1, Lemma 2.3], their results are only concerned with the order of $1 - u_\varepsilon(\zeta)$. In this point of view, we believe that (3.14) gives us more precise information on the profile of u_ε. Indeed, (3.14) helps us to study the ε-dependence of the width of each transition layer, spike, multi-layer and multi-spike in Sections 4 and 5.

Remark 3.5. In (3.14), we can choose $r = \sqrt{1 - A^*} + O(1)$ and $R = \sqrt{1 - A_*} + O(1)$ where $A_* = \min\{a(x)\,;\, x \in [\xi_1, \zeta]\}$ and $A^* = \max\{a(x)\,;\, x \in [\xi_1, \zeta]\}$. These facts can be shown from the proof of Theorem 3.3 by taking account of the definition of r and R in Lemma 3.1.

Using the same method as the proof of Theorem 3.3 one can prove the following result from Lemma 3.2 :

Theorem 3.6. *For $u \in S_{n,\varepsilon}$, assume* (3.13) *and let $\zeta \in (\xi_1, \xi_2)$ satisfy $u'(\zeta) = 0$. If $(\xi_2 - \zeta)/\varepsilon \to +\infty$, then for sufficiently small $\varepsilon > 0$, there exist positive constants C_1', C_2', r', R' with $C_1' < C_2'$ and $r' < R'$ such that*

$$C_1' \exp\left(-\frac{R'(\xi_2 - \zeta)}{\varepsilon}\right) < 1 - u_\varepsilon(x) < C_2' \exp\left(-\frac{r'(\xi_2 - x)}{\varepsilon}\right) \tag{3.29}$$

for $x \in [\zeta, \xi_2]$.

Remark 3.7. Theorems 3.3 and 3.6 deal with the case that $\zeta \in (\xi_1, \xi_2)$ is a local maximum point of u_ε ; i.e., the case that $u_\varepsilon(\zeta)$ is very close to 1. On the contrary, assume that ζ is a local minimum point of u_ε and $(\zeta - \xi_1)/\varepsilon \to \infty$ or $(\xi_2 - \zeta)/\varepsilon \to \infty$ as $\varepsilon \to 0$. Then we can derive similar estimates as (3.14) and (3.29) with $1 - u_\varepsilon(x)$ replaced by $u_\varepsilon(x)$.

4 Location of transition layers and spikes

We will study the location of transition layers and spikes of n-mode solution u_ε with use of (1.3) and (1.4).

Theorem 4.1. *Let ξ be any point in Ξ. Then ξ lies in a neighborhood of a point in $\Sigma \cup \Lambda$ when ε is sufficiently small. Moreover, if u_ε has a transition layer near a point $x_0 \in \Sigma \cup \Lambda$, then x_0 belongs to Σ, and if u_ε has a spike near a point $x_0 \in \Sigma \cup \Lambda$, then x_0 belongs to Λ.*

Remark 4.2. Theorem 4.1 has been obtained by Ai, Chen and Hastings [1, Theorem 1]. In the proof, they have reduced the location problem to a certain kind of algebraic system. We give a different proof; we will derive a contradiction to the finiteness of Ξ for u_ε by means of asymptotic properties developed in Section 3.

Proof. Define $\{\xi_k\}_{k=1}^{n}$, $\{\zeta_k\}_{k=1}^{n-1}$ as in Lemma 2.3 and set $\zeta_0 = 0$, $\zeta_n = 1$. Let $\Sigma = \{z_1, z_2, \ldots, z_m\}$ with $0 < z_1 < z_2 < \cdots < z_m < 1$. By Lemma 2.4 it can be shown that, if $u_\varepsilon \in S_{n,\varepsilon}$ has a transition layer in a neighborhood of $\xi^\varepsilon \in \Xi$, then ξ^ε must be very close to one of z_j when ε is sufficiently small.

It is sufficient to show that if u_ε has a spike near ξ^ε, then ξ^ε lies in a vicinity of a point in Λ. For this purpose, let $a(x) - 1/2 > 0$ in (z_j, z_{j+1}) and denote all points of $\Lambda \cap (z_j, z_{j+1})$ by $y_1, y_2, \ldots, y_l$ with $z_j < y_1 < y_2 < \cdots < y_l < z_{j+1}$.

We will prove by contradiction that every spike lies near a point in Λ. Take any small $\delta > 0$ and fix it. Assume that u_ε has a spike in an interval $(z_j + \delta, y_1 - \delta)$. Note $a'(x) > 0$ in this interval. By (iii) of Lemma 2.4, then there exist ξ_k and ξ_{k+1} such that

$$z_j + \delta < \xi_k < \xi_{k+1} < y_1 - \delta, \quad u_\varepsilon'(\xi_k) < 0 \quad \text{and} \quad u_\varepsilon'(\xi_{k+1}) > 0,$$

if ε is sufficiently small. By Lemma 2.3 there exist $\zeta_{k-1}, \zeta_k, \zeta_{k+1}$ satisfying $\zeta_{k-1} < \xi_k < \zeta_k < \xi_{k+1} < \zeta_{k+1}$.

We will show

$$1 - u_\varepsilon(\zeta_{k-1}) > \kappa\sqrt{\varepsilon} \tag{4.1}$$

with some $\kappa > 0$, in the case that neither ζ_{k-1} nor ζ_{k+1} belongs to (z_j, y_1). The other cases can be discussed in the same way and the proof is easier.

We rewrite (1.5) as

$$\varepsilon^2 u_\varepsilon'' + f(\zeta_k, u_\varepsilon) = u_\varepsilon(1-u_\varepsilon)(a(x) - a(\zeta_k)). \tag{4.2}$$

Multiplying (4.2) by u_ε' and integrating the resulting expression over $(\zeta_{k-1}, \zeta_{k+1})$ with respect to x we get

$$\begin{aligned}
&W(\zeta_k, u_\varepsilon(\zeta_{k-1})) - W(\zeta_k, u_\varepsilon(\zeta_{k+1})) \\
&= \int_{\zeta_{k-1}}^{\zeta_{k+1}} u_\varepsilon(x)(1-u_\varepsilon(x))(a(x) - a(\zeta_k))u_\varepsilon'(x)dx \\
&= \left(\int_{\zeta_{k-1}}^{z_j} + \int_{z_j}^{y_1} + \int_{y_1}^{\zeta_{k+1}}\right) u_\varepsilon(x)(1-u_\varepsilon(x))(a(x) - a(\zeta_k))u_\varepsilon'(x)dx \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{4.3}$$

We will estimate I, II and III.

We begin with the study of II. Since a is monotone increasing in (z_j, y_1),

$$\begin{aligned}
\mathrm{II} &> \int_{\zeta_k+\varepsilon}^{y_1} u_\varepsilon(x)(1-u_\varepsilon(x))(a(x) - a(\zeta_k))u_\varepsilon'(x)dx \\
&> (a(\zeta_k+\varepsilon) - a(\zeta_k)) \int_{\zeta_k+\varepsilon}^{y_1} u_\varepsilon(x)(1-u_\varepsilon(x))u_\varepsilon'(x)dx \\
&= (a(\zeta_k+\varepsilon) - a(\zeta_k)) \int_{u_\varepsilon(\zeta_k+\varepsilon)}^{u_\varepsilon(y_1)} s(1-s)ds \\
&> K\varepsilon \int_{u_\varepsilon(\zeta_k+\varepsilon)}^{u_\varepsilon(y_1)} s(1-s)ds
\end{aligned}$$

with a positive constant K. Moreover, Theorem 3.3 gives

$$1 - u_\varepsilon(y_1) < C\exp\left(-\frac{r(y_1 - \xi_k)}{\varepsilon}\right) < C\exp\left(-\frac{r\delta}{\varepsilon}\right),$$

and Lemma 2.4 implies $u_\varepsilon(\zeta_k + \varepsilon) < A$ with some $A \in (0,1)$ provided that ε is sufficiently small. Hence

$$\int_{u_\varepsilon(\zeta_k+\varepsilon)}^{u_\varepsilon(y_1)} s(1-s)ds > \int_A^{u_\varepsilon(y_1)} s(1-s)ds > C^* \tag{4.4}$$

with a positive constant C^* independent of ε; so

$$\mathrm{II} > C^* K\varepsilon.$$

We next estimate I;

$$\begin{aligned}
|\mathrm{I}| &\le \int_{\zeta_{k-1}}^{z_j} |u_\varepsilon(1-u_\varepsilon)(a(x) - a(\zeta_k))u_\varepsilon'|dx \\
&\le \int_{\zeta_{k-1}}^{z_j} u_\varepsilon(1-u_\varepsilon)|u_\varepsilon'|dx = \int_{u_\varepsilon(z_j)}^{u_\varepsilon(\zeta_{k-1})} s(1-s)ds \le 1 - u_\varepsilon(z_j).
\end{aligned}$$

Theorem 3.6 implies

$$1 - u_\varepsilon(z_j) \le C_2 \exp\left(-\frac{r(\xi_k - z_j)}{\varepsilon}\right) \le C_2 \exp\left(-\frac{r\delta}{\varepsilon}\right).$$

Therefore, we get $|\mathrm{I}| = O(\exp(-1/\varepsilon)))$. Similarly, one can also derive $|\mathrm{III}| = O(\exp(-1/\varepsilon))$. Thus we get

$$W(\zeta_k, u_\varepsilon(\zeta_{k-1})) - W(\zeta_k, u_\varepsilon(\zeta_{k+1})) = \mathrm{I} + \mathrm{II} + \mathrm{III} > K^*\varepsilon \tag{4.5}$$

with some $K^* > 0$.

On the other hand, we will estimate the left-hand side of (4.3). In the same way as the proof of (3.10), one can see

$$W(\zeta_k, u_\varepsilon(\zeta_{k-1})) - W(\zeta_k, u_\varepsilon(\zeta_{k+1})) = -\frac{1}{2} f_u(\zeta_k, \theta)\{(1 - u_\varepsilon(\zeta_{k-1}))^2 - (1 - u_\varepsilon(\zeta_{k+1}))^2\}$$

with some $\theta \in (u_\varepsilon(\zeta_{k-1}), 1)$. Since θ is very close to 1, there exists a positive constant M, which is independent of ε, such that

$$W(\zeta_k, u_\varepsilon(\zeta_{k-1})) - W(\zeta_k, u_\varepsilon(\zeta_{k+1})) < M(1 - u_\varepsilon(\zeta_{k-1}))^2. \tag{4.6}$$

Hence (4.1) follows from (4.5) and (4.6).

We use (4.1) and Theorem 3.6 with $x = \zeta_{k-1}$ and $\xi_2 = \xi_k$ to get

$$\kappa\sqrt{\varepsilon} < c_2' \exp\left(-\frac{r'(\xi_k - \zeta_{k-1})}{\varepsilon}\right) \tag{4.7}$$

with some $c_2' > 0$ and $r' > 0$. Here recall that u_ε is periodic with period 2. So we see that there exists $\xi_{k-1} \in \Xi$ such that $u_\varepsilon(x) > a(x)$ for $x \in (\xi_{k-1}, \xi_k)$. Therefore, Theorem 3.3 together with (4.1) implies

$$\kappa\sqrt{\varepsilon} < 1 - u_\varepsilon(\zeta_{k-1}) < C \exp\left(-\frac{r(\zeta_{k-1} - \xi_{k-1})}{\varepsilon}\right). \tag{4.8}$$

Hence (4.7) and (4.8) imply

$$\xi_k - \xi_{k-1} < K\varepsilon|\log \varepsilon| \tag{4.9}$$

with some positive constant K. This fact implies that ξ_{k-1} belongs to the interval $(z_j + \delta, y_1 - \delta)$ if ε is sufficiently small.

When ξ_{k-1} lies in $(z_j + \delta, y_1 - \delta)$, Lemma 2.4 tells us that there must be another spike such that $\xi_{k-2}, \xi_{k-1} \in \Xi$ with $z_j + \delta < \xi_{k-2} < \xi_{k-1} < y_1 - \delta$ and $u_\varepsilon'(\xi_{k-2}) < 0$, $u_\varepsilon'(\xi_{k-1}) > 0$. Note that u_ε has a peak at $x = \zeta_{k-1} \in (\xi_{k-2}, \xi_{k-1})$. Repeating this procedure, we see that the number of points of $\Xi \cap (z_j + \delta, y_1 - \delta)$ increases in each process. This is a contradiction to the definition of n-mode solutions; so that u_ε has no spikes in $(z_j + \delta, y_1 - \delta)$.

The same argument is valid to show that u_ε has no spikes in $(y_i + \delta, y_{i+1} - \delta)$ for $i = 1, 2, \ldots, l-1$ and $(y_l + \delta, z_{j+1} - \delta)$. Thus the proof is complete. □

We will discuss the location of each single transition layer more carefully.

Theorem 4.3. *Let $u_\varepsilon \in S_{n,\varepsilon}$ possess a single transition layer near $z \in \Sigma$ for sufficiently small $\varepsilon > 0$. If $\Xi \cap (z-\delta, z+\delta) = \{\xi\}$ with some $\delta > 0$, then $\xi - z = O(\varepsilon)$.*

Proof. We only consider the case where $a'(z) > 0$, $z < \xi$ and $u_\varepsilon'(\xi) > 0$ for the sake of simplicity. The other case can be shown in the same way as follows.

Choose critical points ζ_0 and ζ_1 of u_ε such that $u_\varepsilon'(x) > 0$ for $x \in (\zeta_0, \zeta_1)$. Since u_ε has a single transition layer in (ζ_0, ζ_1), there exists $\xi^* \in \Xi$ such that $\xi^* > \xi + \sigma$ with a positive constant σ independent of ε and $u_\varepsilon(x) > a(x)$ for $x \in (\xi, \xi^*)$. (Regarding u_ε as a function defined for all $x \in \mathbb{R}$ by reflection, we can take such ξ^*.) Since ζ_1 is distant from ξ or ξ^* independently of ε, Theorem 3.3 or 3.6 enables us to get $1 - u_\varepsilon(\zeta_1) = O(\exp(-1/\varepsilon))$. Similarly, we can also show $u_\varepsilon(\zeta_0) = O(\exp(-1/\varepsilon))$.

We introduce

$$\tilde{W}(x,u) := -\int_{\tilde{\phi}_0(x)}^{u} f(x,s)ds \qquad \text{with} \qquad \tilde{\phi}_0(x) := \begin{cases} 0 & \text{in } (\zeta_0, \xi), \\ 1 & \text{in } (\xi, \zeta_1). \end{cases}$$

We use the following identity for $x \in (\zeta_0, \xi) \cup (\xi, \zeta_1)$:

$$\begin{aligned} \frac{d}{dx}\left\{\frac{1}{2}\varepsilon^2 u_\varepsilon'(x)^2 - \tilde{W}(x, u_\varepsilon(x))\right\} &= \{\varepsilon^2 u_\varepsilon''(x) + f(x, u_\varepsilon(x))\}u_\varepsilon'(x) - \tilde{W}_x(x, u_\varepsilon(x)) \\ &= a'(x)G(u(x)), \end{aligned} \tag{4.10}$$

where,

$$G(u(x)) := \begin{cases} -u(x)^2/2 + u(x)^3/3 & \text{in } (\zeta_0, \xi), \\ (1-u(x)^2)/2 - (1-u(x)^3)/3 & \text{in } (\xi, \zeta_1). \end{cases}$$

By Remark 3.7, we see

$$|a'(x)G(u(x))| < C_2 \exp\left(-\frac{r(\xi - x)}{\varepsilon}\right) \quad \text{in } (\zeta_0, \xi)$$

with some positive constants C_2 and r. Therefore, there exists a positive constant K such that

$$\left|\int_{\zeta_0}^{\xi} a'(x)G(u(x))dx\right| < K\varepsilon. \tag{4.11}$$

On the other hand, integrating the left-hand side of (4.10) over (ζ_0, ξ) yields that

$$\int_{\zeta_0}^{\xi} \frac{d}{dx}\left\{\frac{1}{2}\varepsilon^2 u_\varepsilon'(x)^2 - \tilde{W}(x, u_\varepsilon(x))\right\} dx = \frac{1}{2}\varepsilon^2 u_\varepsilon'(\xi)^2 + \int_0^{u(\xi)} f(\xi, s)ds + \tilde{W}(\zeta_0, u_\varepsilon(\zeta_0)).$$

Hence it follows from (4.10) and (4.11) that

$$\frac{1}{2}\varepsilon^2 u_\varepsilon'(\xi)^2 + \int_0^{u(\xi)} f(\xi, s)ds + \tilde{W}(\zeta_0, u_\varepsilon(\zeta_0)) \le K\varepsilon. \tag{4.12}$$

Repeating the same argument as above with (ζ_0, ξ) replaced by (ξ, ζ_1), one can obtain that

$$-\frac{1}{2}\varepsilon^2 u_\varepsilon'(\xi)^2 - \tilde{W}(\zeta_1, u_\varepsilon(\zeta_1)) + \int_{u(\xi)}^{1} f(\xi, s)ds \le K_1\varepsilon. \tag{4.13}$$

with some $K_1 > 0$. Therefore, (4.12) and (4.13) imply that

$$\tilde{W}(\zeta_0, u_\varepsilon(\zeta_0)) - \tilde{W}(\zeta_1, u_\varepsilon(\zeta_1)) + \int_0^1 f(\xi, s)ds = O(\varepsilon).$$

It follows from $\tilde{W}(\zeta_0, u_\varepsilon(\zeta_0)) = O(\exp(-1/\varepsilon))$ and $\tilde{W}(\zeta_1, u_\varepsilon(\zeta_1)) = O(\exp(-1/\varepsilon))$ that

$$\int_0^1 f(\xi, s)ds = O(\varepsilon).$$

Taking account of

$$\int_0^1 f(\xi, s)ds = -\frac{1}{6}a(\xi) + \frac{1}{12} \quad \text{and} \quad a(\xi) = \frac{1}{2} + a'(z)(\xi - z) + O((\xi - z)^2),$$

we can conclude that $\xi - z = O(\varepsilon)$. □

5 Multiplicity of transition layers and spikes

In this section we will discuss a cluster of multiple transition layers and spikes. By Theorem 4.1, such a cluster of multiple transition layers appears in a neighborhood of a point in Σ if it exists, while a cluster of multiple spikes appears in a neighborhood a point in Λ if it exists.

Definition 5.1 (multi-layer). Let u_ε be a solution of (1.5). If u_ε has a cluster of multiple transition layers in a neighborhood of a point in Σ, then such a cluster is called a **multi-layer**.

Definition 5.2 (multi-spike). Let u_ε be a solution of (1.5). If u_ε has a cluster of multiple spikes in a neighborhood of a point in Λ, then such a cluster is called a **multi-spike**.

We introduce some notations to study multi-layers and multi-spikes.

$$\begin{aligned}
\Sigma^+ &= \{x^* \in \Sigma\,;\, a'(x^*) > 0\}, \quad \Sigma^- = \{x^* \in \Sigma\,;\, a'(x^*) < 0\},\\
\Lambda^+ &= \{x^* \in \Lambda\,;\, a(x^*) < 1/2 \text{ and } a \text{ attains its local maximum at } x = x^*\},\\
\Lambda^- &= \{x^* \in \Lambda\,;\, a(x^*) > 1/2 \text{ and } a \text{ attains its local minimum at } x = x^*\}.
\end{aligned}$$

We begin with the study of multi-layers. We only discuss the case where u_ε has a multi-layer in a neighborhood of $z \in \Sigma^+$ because the analysis for the case $z \in \Sigma^-$ is almost the same.

By virtue of Lemma 2.4, there exists a one-to-one correspondence between a transition layer and a point in Ξ defined by (2.2).

Lemma 5.1. *For $z \in \Sigma^+$, let $\xi_1, \xi_2 \in (z-\delta, z+\delta)$ be successive points in Ξ satisfying $u_\varepsilon'(\xi_1) < 0$ and $u_\varepsilon'(\xi_2) > 0$ (resp. $u_\varepsilon'(\xi_1) > 0$ and $u_\varepsilon'(\xi_2) < 0$) with some $\delta > 0$. Then there exitst another $\xi \in \Xi$ such that $z - \delta < \xi < \xi_1$ (resp. $\xi_2 < \xi < z + \delta$) and $u_\varepsilon'(\xi) > 0$ provided that ε is sufficiently small.*

Proof. We give the proof in the case $u'_\varepsilon(\xi_1) < 0$ and $u'_\varepsilon(\xi_2) > 0$. By Lemma 2.3, there exist critical points ζ_0, ζ_1 and ζ_2 of u_ε with $\zeta_0 < \xi_1 < \zeta_1 < \xi_2 < \zeta_2$. Since $a'(x) > 0$ in $(z-\delta, z+\delta)$, the argument used in the proof of (4.1) is valid to show

$$1 - u_\varepsilon(\zeta_0) > \kappa\sqrt{\varepsilon} \tag{5.1}$$

with some $\kappa > 0$ independent of ε. Theorem 3.3 implies the existence of the other successive point $\xi \in \Xi$ to ξ_1 (with $\xi < \xi_1$) satisfying

$$1 - u_\varepsilon(\zeta_0) < C\exp\left(-\frac{r(\zeta_0 - \xi)}{\varepsilon}\right) \tag{5.2}$$

with some $C, r > 0$. As in the proof of (4.9), it follows from (5.1) and (5.2) that $\xi \in \Xi$ satisfies $\xi < \xi_1$, $\xi_1 - \xi < K\varepsilon|\log\varepsilon|$ and $u'_\varepsilon(\xi) > 0$. Hence ξ lies in $(z-\delta, z+\delta)$ if ε is sufficiently small. □

Lemma 5.2. *Let $z \in \Sigma^+$ and assume that u_ε has a multi-layer in $(z-\delta, z+\delta)$ with some $\delta > 0$. If ε is sufficiently small, then $\Xi \cap (z-\delta, z+\delta)$ consists of an odd number of elements. Moreover, if*

$$\Xi \cap (z-\delta, z+\delta) = \{\xi_l, \dots, \xi_m\} \tag{5.3}$$

with some $l, m \in \mathbb{N}$ such that $m-l$ is even, then $u'_\varepsilon(\xi_l) > 0$ and $u'_\varepsilon(\xi_m) > 0$.

Proof. Define ξ_i, $i = l, \cdots, m$, by (5.3). We will show this lemma by contradiction. Assume that $m-l$ is odd. Then one of the following properties holds true:

$$u'_\varepsilon(\xi_l) < 0, u'_\varepsilon(\xi_{l+1}) > 0 \text{ and } u'_\varepsilon(\xi_{m-1}) < 0, u'_\varepsilon(\xi_m) > 0, \tag{5.4}$$

$$u'_\varepsilon(\xi_l) > 0, u'_\varepsilon(\xi_{l+1}) < 0 \text{ and } u'_\varepsilon(\xi_{m-1}) > 0, u'_\varepsilon(\xi_m) < 0. \tag{5.5}$$

Lemma 5.1 implies that there exists $\xi_{l-1} \in \Xi$ (resp. $\xi_{m+1} \in \Xi$) such that $z - \delta < \xi_{l-1} < \xi_l$ (resp. $\xi_m < \xi_{m+1} < z + \delta$) when (5.4) (resp. (5.5)) is satisfied. This is a contradiction to (5.3). Hence $m - l$ is even.

It is clear that either $u'_\varepsilon(\xi_l) > 0$ and $u'_\varepsilon(\xi_m) > 0$, or $u'_\varepsilon(\xi_l) < 0$ and $u'_\varepsilon(\xi_m) < 0$. However, in the latter case, Lemma 5.1 enables us to derive a contradiction in the same way as above. So the proof is complete. □

Let u_ε possess a multi-layer in a neighborhood of $z \in \Sigma^+$. Set $\Xi \cap (z-\delta, z+\delta) = \{\xi_l, \xi_{l+1}, \dots, \xi_m\}$ with some $\delta > 0$. By Lemma 2.3 u_ε has critical points $\zeta_{l-1}, \zeta_l, \dots, \zeta_m$ such that $\zeta_{l-1} < \xi_l < \zeta_l < \cdots < \xi_m < \zeta_m$. Here we should note that $u_\varepsilon(\zeta_{l-1})$ is close to 0 and that $u_\varepsilon(\zeta_m)$ is close to 1. Such a multi-layer is called a multi-layer from 0 to 1. A multi-layer from 1 to 0 is defined in a similar manner.

We can also show that, if there exists a multi-layer in a neighborhood of a point in Σ^-, it must be a multi-layer from 1 to 0.

Summarizing these facts we have the following theorem.

Theorem 5.3. *A multi-layer from 0 to 1 (resp. from 1 to 0) appears only in a neighborhood of a point in Σ^+ (resp. Σ^-).*

Next we will study multi-spikes. Note that for each spike there exist exactly two points in Ξ. So if u_ε has a multi-spike in a neighborhood of some $y \in \Lambda$, we can denote $\Xi \cap (y-\delta, y+\delta) = \{\xi_l, \xi_{l+1}, \cdots, \xi_m\}$ with some $\delta > 0$ and some $l, m \in \mathbb{N}$ such that $m-l$ is odd. Moreover, by Lemmas 2.3 and 2.4, there exist critical points of u_ε denoted by $\{\zeta_k\}_{k=l-1}^m$ such that $\zeta_{l-1} < \xi_l < \zeta_l, \cdots, \xi_m < \zeta_m$ and both $u_\varepsilon(\zeta_{l-1})$ and $u_\varepsilon(\zeta_m)$ are sufficiently close to 0 or 1. If $u_\varepsilon(\zeta_{l-1})$ and $u_\varepsilon(\zeta_m)$ are close to 1 (resp. 0), then such a multi-spike is called a multi-spike based on 1 (resp. 0).

Theorem 5.4. *A multi-spike based on* 1 *(resp.* 0*) appears only in a neighborhood of a point in* Λ^- *(resp.* Λ^+*).*

Proof. We only show that a multi-spike based on 1 appears in a neighborhood of a point of Λ^-. Since any spike based on 1 appears only in a neighborhood of a critical point y of a with $a(y) > 1/2$, it sufficeis to show that, if a takes its local maximum at y, then any multi-spike based on 1 can not appear in a neighnorhood of such y in order to complete the proof.

We take a contradiction method. Let y be a local maximum point of a satisfying $a(y) > 1/2$. Assume that u_ε has a multi-spike based on 1 in $(y-\delta, y+\delta)$ with some $\delta > 0$. Observe that, if there is a multi-spike in $(y-\delta, y+\delta)$, then

$$\Xi \cap (y-\delta, y+\delta) = \{\xi_l, \xi_{l+1}, \cdots, \xi_m\}$$

with some $l, m \in \mathbb{N}$ such that $m-l$ is odd. By Lemma 2.3, we can choose $\zeta_{l-1}, \zeta_l, \ldots, \zeta_m$ such that $u'(\zeta_k) = 0$ for $k = l-1, l, \ldots, m$ and $\zeta_{l-1} < \xi_l < \zeta_l < \cdots < \xi_m < \zeta_m$. Moreover, Lemma 2.4 implies that $\xi_{k+1} - \xi_k = O(\varepsilon)$ for $k = l, l+2, l+4, \ldots, m-2$; so that at least two points in Ξ belong to either $(y-\delta, y)$ or $(y, y+\delta)$.

We will consider the case when $\xi_l, \xi_{l+1} \in (y-\delta, y)$. Note that $a'(x) > 0$ in $(y-\delta, y)$. For the sake of simplicity, we assume that ζ_{l-1} lies in $(y-\delta, y)$. (If not, see the argument developed in the proof of Theorem 4.1.) Similarly to the proof of (4.2) and (4.3) we have

$$W(\zeta_l, u_\varepsilon(\zeta_{l-1})) - W(\zeta_l, u_\varepsilon(\zeta_{l+1})) = \int_{\zeta_{l-1}}^{\zeta_{l+1}} u_\varepsilon(x)(1-u_\varepsilon(x))(a(x)-a(\zeta_l))u'_\varepsilon(x)dx. \tag{5.6}$$

For the left-hand side of (5.6), observe that (4.6) is valid with k replaced by l. So it is sufficient to consider the right-hand side of (5.6). Since $a''(x) < 0$ in $(y-\delta, y)$ by (A.3), the right-hand side of (5.6) is bounded from below by

$$\begin{aligned}
&\int_{\zeta_{l-1}}^{\zeta_l-\varepsilon} u_\varepsilon(x)(1-u_\varepsilon(x))(a(\zeta_l)-a(x))(-u'_\varepsilon(x))dx \\
>&\int_{\zeta_{l-1}}^{\zeta_l-\varepsilon} u_\varepsilon(x)(1-u_\varepsilon(x))(a(\zeta_l)-a(\zeta_l-\varepsilon))(-u'_\varepsilon(x))dx \\
=&(a(\zeta_l)-a(\zeta_l-\varepsilon))\int_{u_\varepsilon(\zeta_l-\varepsilon)}^{u_\varepsilon(\zeta_{l-1})} s(1-s)ds.
\end{aligned}$$

when ε is sufficiently small. By the Taylor expansion, we see that

$$a(\zeta_l) - a(\zeta_l-\varepsilon) = -\frac{a''(z)}{2}\varepsilon\{(y-\zeta_l)+(y-\zeta_l+\varepsilon)\} + h.o.t.$$

We should note that Lemma 2.4 implies $y - \zeta_l > \xi_{l+1} - \zeta_l > C\varepsilon$ with some positive constant C independent of ε. Thus there exists a positive constant C' such that

$$a(\zeta_l) - a(\zeta_l - \varepsilon) > C'\varepsilon^2.$$

Moreover, the same argument as in the proof of (4.4) leads to

$$\int_{u_\varepsilon(\zeta_l-\varepsilon)}^{u_\varepsilon(\zeta_{l-1})} s(1-s)ds > C^*$$

with some positive constant C^*. Thus one can deduce

$$1 - u_\varepsilon(\zeta_{l-1}) > \kappa\varepsilon$$

with some $\kappa > 0$ (cf. (4.1)). We repeat the argument developed in the proof of Theorem 4.1 with use of Theorems 3.3 and 3.6. It is seen that there exists another spike in $(y-\delta, y)$ when ε is sufficiently small. This is a contradiction to the definition of ξ_l. Thus we complete the proof. □

Finally, we will discuss ε-dependence of the width and the location of multi-layer and multi-spike. For this purpose, we will collect important properties of multi-layers and multi-spikes.

By Lemma 5.2 any multi-layer consists of an odd number of transition layers. If u_ε has a multi-layer in δ-neighborhood of $z \in \Sigma = \Sigma^+ \cup \Sigma^-$ with small $\delta > 0$, then there exist $m \in \mathbb{N} \setminus \{1\}$ and $\{\xi_k\}_{k=1}^{2m-1} \subset \Xi$ satisfying

$$(z-\delta, z+\delta) \cap \Xi = \{\xi_k\}_{k=1}^{2m-1} \tag{5.7}$$

when ε is sufficiently small. Then from Lemma 2.3 we can choose a set of critical points of u_ε, which is denoted by $\{\zeta_k\}_{k=0}^{2m-1}$, satisfying $\zeta_0 < \xi_1 < \zeta_1 < \cdots < \xi_{2m-1} < \zeta_{2m-1}$. We should note that $\xi_k - \xi_{k-1} = O(\varepsilon|\log\varepsilon|)$ for any $k = 1, 2, \ldots, 2m-1$ by (4.9). It also should be noted that $u(\zeta_0) = O(\exp(-1/\varepsilon))$ and $1 - u(\zeta_{2m-1}) = O(\exp(-1/\varepsilon))$ if $z \in \Sigma^+$, while $1 - u(\zeta_0) = O(\exp(-1/\varepsilon))$ and $u(\zeta_{2m-1}) = O(\exp(-1/\varepsilon))$ if $z \in \Sigma^-$ by the same reasoning as in the proof of Theorem 4.3.

Similarly, if u_ε has a multi-spike in a neighborhood of $y \in \Lambda^+ \cup \Lambda^- \subset \Lambda$, then there exist $l \in \mathbb{N} \setminus \{1\}$, $\{\xi_k\}_{k=1}^{2l} \subset \Xi$ and critical points $\{\zeta_k\}_{k=0}^{2l}$ of u_ε which satisfy

$$(y-\delta, y+\delta) \cap \Xi = \{\xi_k\}_{k=1}^{2l} \tag{5.8}$$

and $\zeta_0 < \xi_1 < \zeta_1 < \cdots < \xi_{2l} < \zeta_{2l}$. Observe that Lemma 2.4 implies that $\xi_{2k} - \xi_{2k-1} = O(\varepsilon)$ for any $k = 1, 2, \ldots, l$. Furthermore, by the same argument as in the proof of Theorem 5.4, we obtain that $\xi_{2k+1} - \xi_{2k} = O(\varepsilon|\log\varepsilon|)$ for any $k = 1, 2, \ldots, l-1$. We also note that, if $y \in \Lambda^+$, then $u_\varepsilon(\zeta_0) = O(\exp(-1/\varepsilon))$ and $u_\varepsilon(\zeta_{2l}) = O(\exp(-1/\varepsilon))$, while if $y \in \Lambda^-$, then $1 - u_\varepsilon(\zeta_0) = O(\exp(-1/\varepsilon))$ and $1 - u_\varepsilon(\zeta_{2l}) = O(\exp(-1/\varepsilon))$.

Theorem 5.5. *Let $u_\varepsilon \in S_{n,\varepsilon}$ possess a multi-layer satisfying* (5.7) *for sufficiently small $\varepsilon > 0$. Then $\xi_k - z = O(\varepsilon|\log\varepsilon|)$ for $k = 1, 2, \ldots, 2m-1$.*

Proof. For the sake of simplicity, we only consider the case that $m = 2$ and $z \in \Sigma^+$. In this case, there exist a set of critical points $\{\zeta_k\}_{k=0}^3$ of u_ε satisfying $\zeta_0 < \xi_1 < \zeta_1 < \xi_2 < \zeta_2 < \xi_3 < \zeta_3$, and a constant $C > 0$ such that $\xi_3 - \xi_1 < C\varepsilon|\log \varepsilon|$. Therefore, it suffices to consider the case $z < \xi_1$ or $\xi_3 < z$ in order to complete the proof.

We will give the proof in the case $\xi_3 < z$. It also should be noted that a multi-layer near $z \in \Sigma^+$ must be a multi-layer from 0 to 1. Rewrite (1.5) as

$$\varepsilon^2 u_\varepsilon'' + f(z, u_\varepsilon) = u_\varepsilon(1 - u_\varepsilon)(a(x) - 1/2). \tag{5.9}$$

Multiplying (5.9) by u_ε' and integrating the resulting expression over (ζ_2, z) we get

$$\frac{1}{2}\varepsilon^2 u_\varepsilon'(z)^2 - W(z, u_\varepsilon(z)) + W(z, u_\varepsilon(\zeta_2)) = \int_{\zeta_2}^{z} u_\varepsilon(x)(1 - u_\varepsilon(x))(a(x) - 1/2)u_\varepsilon'(x)dx. \tag{5.10}$$

We should note that both a and u_ε are monotone increasing in (ζ_2, z). Hence the right-hand side of (5.10) is negative; so that $W(z, u_\varepsilon(z)) > W(z, u_\varepsilon(\zeta_2))$. Taking account of the profile of the graph of $W(z, u)$, we get

$$u_\varepsilon(\zeta_2) < 1 - u_\varepsilon(z). \tag{5.11}$$

Applying Theorems 3.3, 3.6 and Remark 3.7 to (5.11), we can obtain

$$C_1 \exp\left(-\frac{R(\xi_3 - \zeta_2)}{\varepsilon}\right) < C_2 \exp\left(-\frac{r(z - \xi_3)}{\varepsilon}\right)$$

with some positive constants C_1, C_2, r and R. This implies that there is a constant $K > 0$ such that

$$0 < z - \xi_3 < K(\xi_3 - \zeta_2) < K(\xi_3 - \xi_1) < KC\varepsilon|\log \varepsilon|$$

when ε is sufficiently small. Thus the proof is complete. □

Theorem 5.6. *Let $u_\varepsilon \in S_{n,\varepsilon}$ possess a multi-spike satisfying* (5.8) *for sufficiently small $\varepsilon > 0$. Then $\xi_k - y = O(\varepsilon|\log \varepsilon|)$ for $k = 1, 2, \ldots, 2l$.*

Proof. For the sake of simplicity, we only consider the case $m = 2$ and $y \in \Lambda^-$. Then there exists a set of critical points $\{\zeta_k\}_{k=0}^4$ of u_ε satisfying $\zeta_0 < \xi_1 < \zeta_1 < \cdots < \xi_4 < \zeta_4$. We should note that this multi-spike is based on 1. Then $\xi_2 - \xi_1 = O(\varepsilon), \xi_4 - \xi_3 = O(\varepsilon)$ and $\xi_3 - \xi_2 = O(\varepsilon|\log \varepsilon|)$; so $\xi_4 - \xi_1 = O(\varepsilon|\log \varepsilon|)$. Therefore, it is sufficient to discuss the case $y < \xi_1$ or $\xi_4 < y$ in order to complete the proof. We only consider the latter case.

We rewrite (1.5) as

$$\varepsilon^2 u_\varepsilon'' + f(\zeta_3, u_\varepsilon) = u_\varepsilon(1 - u_\varepsilon)(a(x) - a(\zeta_3)). \tag{5.12}$$

Multiplying (5.12) by u_ε' and integrating the resulting expression over (ζ_2, y) with respect to x, we obtain

$$\frac{1}{2}\varepsilon^2 u_\varepsilon'(y)^2 - W(\zeta_3, u_\varepsilon(y)) + W(\zeta_3, u_\varepsilon(\zeta_2)) = \int_{\zeta_2}^{y} u_\varepsilon(x)(1 - u_\varepsilon(x))(a(x) - a(\zeta_3))u_\varepsilon'(x)dx. \tag{5.13}$$

Since $a'(x) < 0$ in (ζ_2, y), $u_\varepsilon'(x) > 0$ in (ζ_3, y) and $u_\varepsilon'(x) < 0$ in (ζ_2, ζ_3), the right-hand side of (5.13) is negative. This fact implies $W(\zeta_3, u_\varepsilon(\zeta_2)) < W(\zeta_3, u_\varepsilon(y))$. Therefore, $1 - u_\varepsilon(\zeta_2) < 1 - u_\varepsilon(y)$. Applying Theorems 3.3 and 3.6 we can obtain

$$C_1 \exp\left(-\frac{R(\xi_3 - \zeta_2)}{\varepsilon}\right) < C_2 \exp\left(-\frac{r(y - \xi_4)}{\varepsilon}\right)$$

with some positive constants C_1, C_2, r and R. Thus we can conclude that $y - \xi_4 < \xi_3 - \zeta_2 < \xi_4 - \xi_1 = O(\varepsilon|\log\varepsilon|)$. □

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Stability of steady-state solutions with transition layers for a bistable reaction-diffusion equation

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1 Introduction

In this paper, we will consider the following reaction-diffusion problem:

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + f(x,u), & 0 < x < 1, \ t > 0, \\ u_x(0,t) = u_x(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases} \tag{1.1}$$

Here ε is a positive parameter and

$$f(x,u) = u(1-u)(u-a(x)),$$

where a is a $C^2[0,1]$-function with the following properties:
(A1) $0 < a(x) < 1$ in $[0,1]$,
(A2) if Σ is defined by

$$\Sigma := \{x \in (0,1)\,;\, a(x) = 1/2\}, \tag{1.2}$$

then Σ is a finite set and $a'(x) \neq 0$ at any $x \in \Sigma$,
(A3) $a'(0) = a'(1) = 0$.

It is well known that (1.1) describes phase transition phenomena in various fields, such as physics, chemistry and mathematical biology. This problem is a gradient system with the following energy functional:

$$E(u) := \int_0^1 \left\{ \frac{1}{2}\varepsilon^2 |u_x|^2 + W(x,u) \right\} dx,$$

where

$$W(x,u) := -\int_0^u f(x,s)ds.$$

For every solution of (1.1), $E(u(\cdot,t))$ is decreasing with respect to t and it is well known that $u(x,t)$ is convergent to a solution of the corresponding steady-state problem as $t \to \infty$. The graph of W has two local minimums at $u = 0$ and $u = 1$ so that we can regard both $u = 0$ and $u = 1$ as stable states when ε is sufficiently small. Furthermore, the minimal energy state depends on whether $a(x)$ is greater than $1/2$ or not, that is, if $a(x) < 1/2$, then W attains its minimum at $u = 1$, while if $a(x) > 1/2$, then the minimum of W is attained at $u = 0$. The interaction of the bistability and the spatial inhomogeneity yields a complicated structure of solutions to (1.1).

In this point of view, one of the most important problems for (1.1) is to know the structure of steady state solutions. So we will mainly consider the following steady state problem associated with (1.1):

$$\begin{cases} \varepsilon^2 u'' + f(x,u) = 0 \quad \text{in } (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.3}$$

where ' $'$ ' denotes the derivative with respect to x.

Above all solutions of (1.3), we are interested in a solution with transition layers, especially, it is interesting to know the locations of transition layers. Here **transition layer** is a part of a solution u where $u(x)$ drastically changes from 0 to 1 or 1 to 0 when x varies in a very small interval. For (1.3), we can observe a cluster of transition layers. This is called a **multi-layer**, while a single transition layer is called a **single-layer**. It is known that any single- or multi-layer appears only in a vicinity of a point in Σ. These results are proved by Ai, Chen and Hastings [1] (see also Urano, Nakashima and Yamada [7], whose method of the proof is different from that of in [1]), and they are given in Theorems 2.6 and 2.7. It should be noted that the existence of such solutions is also discussed in [1] by shooting method. Furthermore, they have also discussed the stability problem of such solutions with use of Sturm's comparison theorem (Proposition 3.1). The study of stability properties of such solutions is also a great important problem.

For (1.3), Angenent, Mallet-Paret and Peletier [3] proved that there exist solutions with single-layers from minimal energy state to minimal energy state when ε is sufficiently small. They also showed all solutions with such transition layers are stable. See also Hale and Sakamoto [4], who proved that solutions with single-layers from nonminimal energy state to nonminimal energy state; all of their solutions are unstable. In a special case that $\int_0^1 f(x,u)du = 0$, which is called a balanced case, Nakashima [5, 6] has shown the existence of solutions with transition layers, especially, in [6], she showed the existence of a solution with multi-layers and obtained its stability property.

The main purpose of this paper is to study stability properties of a solution u_ε of (1.3) which possesses transition layers. For this purpose, we consider the following linearized problem:

$$\begin{cases} -\varepsilon^2\phi'' - f_u(x,u_\varepsilon)\phi = \lambda\phi & \text{in } (0,1), \\ \phi'(0) = \phi'(1) = 0. \end{cases} \tag{1.4}$$

We will show that all solutions with transition layers are non-degenerate. We also study the stability property of u_ε in terms of Morse index. The notion of non-degeneracy and Morse index are defined as follows:

Definition 1.1 (Non-degeneracy). Let u_ε be a solution of (1.3). If (1.4) does not admit zero eigenvalue, then u_ε is said to be **non-degenerate.**

Definition 1.2 (Morse index). Let u_ε be a solution of (1.3). The Morse index of u_ε is defined by the number of negative eigenvalues of (1.4).

In general, the stability property of a solution has a close relationship to its profiles. In particular, the results of Angenent, Mallet-Paret and Peletier [3] and Hale and Sakamoto [4] (Proposition 4.1) tell us that the stability properties of solutions with single-layers are greatly influenced by the direction of each transition layer. Therefore we can expect that such facts are valid for solutions with multi-layers. Indeed, we can show that the Morse index of a solution with multi-layers is equal to the number of transition layers from nonminimal energy state to nonminimal energy state (Theorem 4.2). Our method of proof is based on the Courant min-max principle and is different from that of Ai, Chen and Hastings [1].

The content of this paper is as follows: In Section 2, we will collect some information on profiles of solutions with transition layers . In Section 3 we will recall the theory of Sturm-Liouville type eigenvalue problem. Finally, Section 4 is devoted to the stability analysis for solutions with transition layers.

2 Profiles of steady-state solutions with transition layers

In this section, we will give some important properties concerning to the profiles of solutions with transition layers. Such oscillating solutions have at most a finite number of intersecting points with a in $(0, 1)$. So, we take account of the number of these points. Let u_ε be a solution of (1.3) and set

$$\Xi := \{x \in (0,1)\,;\, u_\varepsilon(x) = a(x)\}. \tag{2.1}$$

We now introduce the notion of n-mode solutions.

Definition 2.1. Let u_ε be a solution of (1.3) and set Ξ by (2.1). If $\#\Xi = n$, then u_ε is called an **n-mode solution**.

In what follows, we denote the set of all of n-mode solutions by $S_{n,\varepsilon}$. We collect some properties of solutions in $S_{n,\varepsilon}$. By the maximum principle, one can easily see that any $u_\varepsilon \in S_{n,\varepsilon}$ satisfies $0 < u_\varepsilon(x) < 1$ in $(0, 1)$.

Lemma 2.2. *For $u_\varepsilon \in S_{n,\varepsilon}$, assume $\Xi = \{\xi_k\}_{k=1}^n$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$. Then there exist exactly $n-1$ critical points $\{\zeta_k\}_{k=1}^{n-1}$ of u_ε satisfying*

$$0 < \xi_1 < \zeta_1 < \xi_2 < \cdots < \zeta_{n-1} < \xi_n < 1,$$

provided that ε is sufficiently small.

Lemma 2.3. *For $u_\varepsilon \in S_{n,\varepsilon}$, let ξ^ε be any point in Ξ and define U_ε by $U_\varepsilon(t) = u_\varepsilon(\xi^\varepsilon + \varepsilon t)$. Then there exists a subsequence $\{\varepsilon_k\} \downarrow 0$ such that $\xi_k = \xi^{\varepsilon_k}$ and $U_k = U_{\varepsilon_k}$ satisfy*

$$\lim_{k\to\infty} \xi_k = \xi^* \quad \text{and} \quad \lim_{k\to\infty} U_k = U \quad \text{in} \quad C^2_{loc}(\mathbb{R}),$$

with some $\xi^* \in [0, 1]$ and $U \in C^2(\mathbb{R})$. Furthermore, if $\xi^* \in \Sigma$ and $\dot{U}(\xi^*) > 0$ (resp. $\dot{U}(\xi^*) < 0$), then U is a unique solution of the following problem:

$$\begin{cases} \ddot{U} + U(1-U)(U-1/2) = 0 & \text{in } \mathbb{R}, \\ \dot{U} > 0 \quad (\text{resp. } \dot{U} < 0) & \text{in } \mathbb{R}, \\ U(-\infty) = 0,\ U(\infty) = 1 \quad (\text{resp. } U(-\infty) = 1,\ U(\infty) = 0), & \\ U(0) = 1/2, & \end{cases}$$

where '$\cdot$' denotes the derivative with respect to t.

Theorem 2.4. *For $u_\varepsilon \in S_{n,\varepsilon}$, let ξ_1, ξ_2 be successive points in Ξ satisfying $\xi_1 < \xi_2$ and $(\xi_2 - \xi_1)/\varepsilon \to \infty$ as $\varepsilon \to 0$ and let $\zeta \in (\xi_1, \xi_2)$ be a critical point of u_ε. Furthermore, set*

$$d(x) = \begin{cases} x - \xi_1 & \text{if } \xi_1 \leq x \leq \zeta, \\ \xi_2 - x & \text{if } \zeta \leq x \leq \xi_2. \end{cases}$$

Then one of the following assertions holds true:
(i) If u_ε attains its local maximum at ζ, then there exist positive constants C_1, C_2, r, R with $C_1 < C_2$ and $r < R$ such that

$$C_1 \exp\left(-\frac{Rd(\zeta)}{\varepsilon}\right) < 1 - u_\varepsilon(x) < C_2 \exp\left(-\frac{rd(x)}{\varepsilon}\right) \quad \text{in } [\xi_1, \xi_2]. \tag{2.2}$$

(ii) If u_ε attains its local minimum at ζ, then there exist positive constants C_1', C_2', r', R' with $C_1' < C_2'$ and $r' < R'$ such that

$$C_1' \exp\left(-\frac{R'd(\zeta)}{\varepsilon}\right) < u_\varepsilon(x) < C_2' \exp\left(-\frac{r'd(x)}{\varepsilon}\right) \quad \text{in } [\xi_1, \xi_2]. \tag{2.3}$$

Remark 2.5. Theorem 2.4 tells us that $u_\varepsilon(x)$ and $1 - u_\varepsilon(x)$ are very small when x does not lie in an $O(\varepsilon)$-neighborhood of a point in Ξ. On the contrary, one can see that u_ε has a sharp transition in a small neighborhood of a point in Ξ.

Theorem 2.6. *For $u_\varepsilon \in S_{n,\varepsilon}$, define Ξ by (2.1) and assume that u_ε forms a transition layer near $\xi \in \Xi$. Then there exists a positive number ε_0 such that, for any $\varepsilon \in (0, \varepsilon_0)$, $\xi - z = O(\varepsilon|\log\varepsilon|)$ with some $z \in \Sigma$.*

We also give a result on multi-layers. For this purpose, we decompose Σ into the following subsets:

$$\Sigma^+ = \{x \in \Sigma\,;\, a'(x) > 0\}, \quad \Sigma^- = \{x \in \Sigma\,;\, a'(x) < 0\}.$$

Theorem 2.7. *For $u_\varepsilon \in S_{n,\varepsilon}$, assume that u_ε has a multi-layer near $z \in \Sigma$ when ε is sufficiently small. Then there exists a positive number K such that $\#(\Xi \cap (z - K\varepsilon|\log\varepsilon|, z + K\varepsilon|\log\varepsilon|)) = 2m - 1$ with some $m \in \mathbb{N}$. Furthermore, if the multi-layer is a multi-layer from 0 to 1 (resp. from 1 to 0), then $z \in \Sigma^+$ (resp. $z \in \Sigma^-$).*

Remark 2.8. Theorem 2.7 gives us more precise information on the profile of u_ε. Set $\Xi \cap (z - K\varepsilon|\log\varepsilon|, z + K\varepsilon|\log\varepsilon|) = \{\xi_k\}_{k=1}^{2m-1}$ with $\xi_1 < \xi_2 < \cdots < \xi_{2m-1}$ and let $\{\zeta_k\}_{k=0}^{2m-1}$ be a set of critical points of u_ε satisfying $\zeta_0 < \xi_1 < \zeta_1 < \cdots < \xi_{2m-1} < \zeta_{2m-1}$. Then, by Theorem 2.7, there exists a positive constant M such that $\zeta_{k+1} - \zeta_k < M\varepsilon|\log\varepsilon|$ for each $k = 1, 2, \ldots, 2m-3$.

The proofs of Lemmas and Theorems in this section can be found in [7].

3 Basic theory for Sturm-Liouville eigenvalue problem

In this section, we recall the Sturm-Liouville theory for (1.4).

Proposition 3.1. *There exist infinitely number of eigenvalues of* (1.4) *and all of them are real and simple. Furthermore, if λ_j denotes the j-th eigenvalue of* (1.4)*, then it holds that*

$$-\infty < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \to \infty \quad \text{as } j \to \infty$$

and the eigenfunction corresponding to λ_j has exactly $j-1$ zeros in $(0,1)$.

The following results is well known as the Courant min-max principle:

Proposition 3.2. *Let λ_j be the j-th eigenvalue of* (1.4). *Then λ_j is characterized by*

$$\lambda_1 = \inf_{\phi \in H^1(0,1)\setminus\{0\}} \frac{\mathscr{H}(\phi)}{\|\phi\|^2_{L^2(0,1)}},$$

$$\lambda_j = \sup_{\psi_1,\ldots,\psi_{j-1} \in L^2(0,1)} \inf_{\phi \in X[\psi_1,\ldots,\psi_{j-1}]} \frac{\mathscr{H}(\phi)}{\|\phi\|^2_{L^2(0,1)}} \quad \text{for } j = 2, 3, \ldots, \tag{3.1}$$

where

$$\mathscr{H}(\phi) := \int_0^1 \left\{ \varepsilon^2 |\phi'(x)|^2 - f_u(x, u_\varepsilon(x)) |\phi(x)|^2 \right\} dx$$

and

$$X[\psi_1, \ldots, \psi_{j-1}] := \{\phi \in H^1(0,1)\setminus\{0\}\,;\, (\phi, \psi_i)_{L^2(0,1)} = 0\,(i = 1, 2, \cdots, j-1)\}.$$

Remark 3.3. If ψ_i is the eigenfunction corresponding to the i-th eigenvalue λ_i of (1.4) for every $i = 1, 2, \ldots, j-1$ in (3.1), then λ_j is characterized by

$$\lambda_j = \inf_{\phi \in X[\psi_1, \ldots, \psi_{j-1}]} \frac{\mathscr{H}(\phi)}{\|\phi\|^2_{L^2(0,1)}}.$$

It is possible to prove the following result from Proposition 3.2:

Proposition 3.4. *Let λ_j be the j-th eigenvalue of* (1.4) *and let $\widetilde{\lambda}_j$ be the j-th eigenvalue of the following eigenvalue problem:*

$$\begin{cases} -\varepsilon^2 \phi'' - f_u(x, u_\varepsilon)\phi + p(x)\phi = \lambda\phi & \text{in } (0,1), \\ \phi'(0) = \phi'(1) = 0, \end{cases}$$

where $p \in C([0,1])$. If $p(x) \geq 0$ (resp. $p(x) \leq 0$) and $p(x) \not\equiv 0$ in $(0,1)$, then $\widetilde{\lambda}_j > \lambda_j$ (resp. $\widetilde{\lambda}_j < \lambda_j$).

4 Stability of solutions with transition layers

We will study stability properties of solutions with transition layers. In order to study a solution with transition layers, assume that a solution u_ε of (1.3) does not have any oscilation in $(0,1)$. For such u_ε, we can choose a positive constant M and a subset $\{z_i\}_{i=1}^l$ of Σ satisfying

$$\Xi \cap (z_i - M\varepsilon|\log\varepsilon|, z_i + M\varepsilon|\log\varepsilon|) \neq \emptyset \tag{4.1}$$

and

$$\#(\Xi \cap (z_i - M\varepsilon|\log\varepsilon|, z_i + M\varepsilon|\log\varepsilon|)) = 2m_i - 1 \tag{4.2}$$

with some $m_i \in \mathbb{N}$ for each $i = 1, 2, \ldots, l$, and

$$\Xi = \Xi \cap \bigcup_{i=1}^{l} (z_i - M\varepsilon|\log\varepsilon|, z_i + M\varepsilon|\log\varepsilon|), \tag{4.3}$$

provided that ε is sufficiently small. We should note that, if $m_i = 1$, then u_ε forms a single-layer near z_i, while, if $m_i \geq 2$, then u_ε forms a multi-layer near z_i.

In the case that $m_i = 1$ for each $i = 1, 2, \ldots, l$, the stability or instability of u_ε has been established by Angenent, Mallet-Paret and Peletier [3] and Hale and Sakamoto [4].

Proposition 4.1 ([3], [4]). *Let u_ε be a solution of* (1.3) *satisfying* (4.1), (4.2) *and* (4.3) *with $m_i = 1$ for every $i = 1, 2, \dots l$. Then the following statements hold true:*
(*i*) *If $u'_\varepsilon(z_i)a'(z_i) < 0$ for all i, then u_ε is stable.*
(*ii*) *If $u'_\varepsilon(z_i)a'(z_i) > 0$ for all i, then u_ε is unstable. Furthermore,*

$$\text{the Morse index of } u_\varepsilon = l.$$

We will discuss stability properties of a solution u_ε in the case where $m_i \geq 1$. The stability property of such u_ε is described as follows:

Theorem 4.2. *Let u_ε be a solution of* (1.3). *Assume that there exist a positive constant M and a subset $\{z_i\}_{i=1}^{l}$ of Σ, which satisfy* (4.1), (4.2) *and* (4.3). *Then the following assertions hold true:*
(*i*) *If $m_i = 1$ and $u'_\varepsilon(z_i)a'(z_i) < 0$ for all $i = 1, 2, \dots, l$, then u_ε is stable.*
(*ii*) *If there exists an $i \in \{1, 2, \dots, l\}$ which satisfies either $m_i \geq 2$ or $m_i = 1$ with $u'_\varepsilon(z_i)a'(z_i) > 0$, then u_ε is unstable. Furthermore, u_ε is non-degenerated and*

$$\text{the Morse index of } u_\varepsilon = \sum_{i \in \{1,2,\dots,l\} \setminus \mathscr{I}} m_i,$$

where

$$\mathscr{I} := \left\{ i \in \{1, 2, \dots, l\}\,;\, m_i = 1 \text{ and } u'_\varepsilon(z_i)a'(z_i) < 0 \right\}.$$

Remark 4.3. Proposition 4.1 is a special case of Theorem 4.2; so Theorem 4.2 is generalization of Proposition 4.1.

Remark 4.4. The same result as Theorem 4.2 has been obtained by Ai, Chen and Hastings [1] with use of Sturm's comparison theorem (Proposition 3.1). In this paper, we will show a different approach based on the Courant min-max principle (Proposition 3.2).

We will discuss the simplest case, $l = 1$, in Theorem 4.2. We should note that $m_1 = 1$ implies that u_ε has only one single-layer, while $m_1 \geq 2$ implies that u_ε has only one multi-layer in $(0, 1)$. We will prove the following theorem in place of Theorem 4.2:

Theorem 4.5. *Under the same assumptions as in Theorem 4.2 with $l = 1$ and $m_1 = m \geq 2$, u_ε is non-degenerate and unstable. Furthermore, the Morse index of u_ε is exactly m.*

In what follows, we denote the j-th eigenvalue of (1.4) by λ_j. By virtue of Proposition 3.1, it is sufficient to show the following two lemmas to prove Theorem 4.5 :

Lemma 4.6. *Under the same assumptions as in Theorem 4.5, it holds that*

$$\lambda_m < 0.$$

Lemma 4.7. *Under the same assumptions as in Theorem 4.5, it holds that*

$$\lambda_{m+1} > 0.$$

We will give the essential idea of proofs of Lemmas 4.6 and 4.7. For details, see [9].

Proof of Lemma 4.6. We will consider the case that $a'(z_1) > 0$. It follows from Theorem 2.7 that u_ε forms a multi-layer from 0 to 1 near z_1. Since u_ε and a have $2m-1$ intersecting points in $(z_1 - M\varepsilon|\log\varepsilon|, z_1 + M\varepsilon|\log\varepsilon|)$, we can denote these points by $\{\xi_k\}_{k=1}^{2m-1}$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{2m-1} < 1$. In this case, there exist critical points $\{\zeta_k\}_{k=0}^{2m-1}$ of u_ε satisfying

$$0 = \zeta_0 < \xi_1 < \zeta_1 < \cdots < \xi_{2m-1} < \zeta_{2m-1} = 1.$$

Define $\{w_k\}_{k=1}^m$ by

$$w_k(x) := \begin{cases} u_\varepsilon'(x) & \text{in } (\zeta_{2k-2}, \zeta_{2k-1}), \\ 0 & \text{in } (0,1) \setminus (\zeta_{2k-2}, \zeta_{2k-1}). \end{cases}$$

Then $\{w_k\}_{k=1}^m$ is a family of linearly independent functions in $H^1(0,1)$ and $(w_j, w_k)_{L^2(0,1)} = 0$ for $j \neq k$. Note that w_k satisfy

$$\varepsilon^2 w_k'' + f_u(x, u_\varepsilon) w_k + f_x(x, u_\varepsilon) = 0 \qquad \text{in } (\zeta_{2k-2}, \zeta_{2k-1}). \tag{4.4}$$

Taking $L^2(\zeta_{2k-2}, \zeta_{2k-1})$-inner product of (4.4) with w_k, we get

$$\mathscr{H}(w_k) = -\int_{\zeta_{2k-2}}^{\zeta_{2k-1}} a'(x) u_\varepsilon(x)(1 - u_\varepsilon(x)) u_\varepsilon'(x) dx.$$

Since a is monotone increasing in $(z_1 - M\varepsilon|\log\varepsilon|, z_1 + M\varepsilon|\log\varepsilon|)$, it is easy to see

$$\mathscr{H}(w_k) < 0 \tag{4.5}$$

for $k = 2, \ldots, m-1$.

It should be noted that $a'(x)$ is not necessarily positive in (ζ_0, ζ_1) and $(\zeta_{2m-2}, \zeta_{2m-1})$. However, we can show that both $\mathscr{H}(w_1)$ and $\mathscr{H}(w_m)$ are negative without the monotonicity condition of a. For the proofs, see [9].

Thus $\mathscr{H}(w_k) < 0$ for every $k = 1, 2, \ldots, m$. This fact together with Proposition 3.2 implies $\lambda_m < 0$. □

We now show Lemma 4.7. For this purpose, we will introduce auxiliary eigenvalue problems as follows:

$$\begin{cases} -\varepsilon^2\phi'' - f_u(x, u_\varepsilon)\phi = \lambda\phi & \text{in } J_k^+ := (\zeta_{2k-2}, \zeta_{2k-1}), \\ \phi'(\zeta_{2k-1}) = \phi'(\zeta_{2k}) = 0, & k = 1, 2, \ldots, m, \end{cases} \tag{4.6}$$

$$\begin{cases} -\varepsilon^2\phi'' - f_u(x, u_\varepsilon)\phi = \lambda\phi & \text{in } J_k^- := (\zeta_{2k-1}, \zeta_{2k}), \\ \phi'(\zeta_{2k-1}) = \phi'(\zeta_{2k}) = 0 & k = 1, 2, \ldots, m-1. \end{cases} \tag{4.7}$$

It should be noted that u_ε' is positive in J_k^+, while u_ε' is negative in J_k^-. We denote the j-th eigenvalue of (4.6) (resp. (4.7)) by $\lambda_j(J_k^+)$ for $k = 1, 2, \ldots, m$ (resp. $\lambda_j(J_k^-)$ for $k = 1, 2, \ldots, m-1$).

For (4.6) and (4.7), we can show the following two lemmas:

Lemma 4.8. *For each $k = 1, 2, \ldots, m$, it holds that*

$$\lambda_1(J_k^+) < 0 < \lambda_2(J_k^+).$$

Lemma 4.9. *For each $k = 1, 2, \ldots, m-1$, it holds that*

$$\lambda_1(J_k^-) > 0.$$

Before giving proofs of Lemmas 4.8 and 4.9, we will prove Lemma 4.7, which is essential in our analysis.

Proof of Lemma 4.7. Let $\phi_{1,k}^+$ be the first eigenfunction of (4.6) and set

$$\mathscr{H}_k^{\pm}(\phi) := \int_{J_k^{\pm}} \left\{ \varepsilon^2|\phi'(x)|^2 - f_u(x, u_\varepsilon(x))|\phi(x)|^2 \right\} dx.$$

For each $k = 1, 2, \ldots, m$, take any $w_k \in H^1(J_k^+) \setminus \{0\}$ satisfying

$$\int_{J_k^+} w_k(x)\phi_{1,k}^+(x)dx = 0.$$

Then, it follows from Lemma 4.8 that

$$\lambda_2(J_k^+)\int_{J_k^+}|w_k(x)|^2dx \le \mathscr{H}_k^+(w_k).$$

We extend $\phi_{1,k}^+$ to $\psi_k \in L^2(0,1)$ by

$$\psi_k(x) := \begin{cases} \phi_{1,k}^+ & \text{in } J_k^+, \\ 0 & \text{in } (0,1)\setminus J_k^+. \end{cases} \tag{4.8}$$

For any $w \in X[\psi_1, \psi_2, \dots, \psi_m]$, it follows from (4.8) that

$$(w, \psi_k)_{L^2(0,1)} = \int_{J_k^+} w(x)\phi_{1,k}^+(x)dx = 0.$$

Hence we have

$$\mathscr{H}_k^+(w) \ge \lambda_2(J_k^+)\int_{J_k^+}|w_k(x)|^2dx > 0.$$

On the other hand, Lemma 4.9 yields

$$0 < \lambda_1(J_k^-)\int_{J_k^-}|w(x)|^2dx \le \mathscr{H}_k^-(w),$$

for $k = 1, 2, \dots, m-1$. Therefore, one can see that

$$\begin{aligned}
\mathscr{H}(w) &= \sum_{k=1}^{m}\mathscr{H}_k^+(w) + \sum_{k=1}^{m-1}\mathscr{H}_k^-(w) \\
&\ge \sum_{k=1}^{m}\lambda_2(J_k^+)\int_{J_k^+}|w(x)|^2dx + \sum_{k=1}^{m-1}\lambda_1(J_k^-)\int_{J_k^-}|w(x)|^2dx \\
&\ge \lambda^*\int_0^1|w(x)|^2dx,
\end{aligned}$$

where

$$\lambda^* := \min\left\{\min_{k=1,2,\dots,m}\lambda_2(J_k^+), \min_{k=1,2,\dots,m-1}\lambda_1(J_k^-)\right\} > 0.$$

Thus we can conclude by Proposition 3.2 that

$$\lambda_{m+1} = \sup_{\psi_1,\dots,\psi_m}\ \inf_{w\in X[\psi_1,\dots,\psi_m]}\frac{\mathscr{H}(w)}{\|w\|_{L^2(0,1)}} \ge \lambda^* > 0.$$

□

We next discuss Lemmas 4.8 and 4.9. However, their proofs require quite lengthly argument. So we will only give the outline of proofs. For the complete proofs, see [9].

Outline of the proof of Lemma 4.8. By virtue of Propositions 3.1, 3.2 and 3.4, it suffices to show the existence of a pair of functions $A \in C(J_k^+)$ and $w \in C^2(J_k^+)$ with the following properties:
(i) A and w satisfy the following equation:

$$\begin{cases} -\varepsilon^2 w'' + A(x)w = 0 & \text{in } (\zeta_{2k-2}, \zeta_{2k-1}), \\ w'(\zeta_{2k-2}) = w'(\zeta_{2k-1}) = 0, & \\ -f_u(x, u_\varepsilon) \geq A(x) & \text{in } (\zeta_{2k-2}, \zeta_{2k-1}), \end{cases} \tag{4.9}$$

(ii) w has only one zero point in $(\zeta_{2k-2}, \zeta_{2k-1})$.

Take a small number $\delta > 0$ and let g be a smooth function satisfying

$$g(x) = \begin{cases} 1 & \text{for } |x| \leq \delta, \\ 0 & \text{for } |x| \geq 2\delta, \end{cases}$$

and $|g(x)| \leq 1$ for any $x \in \mathbb{R}$. We introduce a cut-off function ρ by

$$\rho(x) := g\left(\frac{x - z_{2k-1}}{\varepsilon}\right) \qquad \text{in } J_k^+.$$

Furthermore, let φ be a C^2-function which satisfying

$$\begin{cases} -\varepsilon^3 \varphi'' - (1/2 - a(x) + 2a(x)u_\varepsilon - u_\varepsilon^2)\varphi \\ \qquad + (u_\varepsilon^2 - u_\varepsilon + 1/2)(1/2 - a(x)) = 0 \quad \text{in } (z_{2k-1} - 2\varepsilon\delta, z_{2k-1} + 2\varepsilon\delta), \\ \varphi(z_{2k-1} - 2\varepsilon\delta) = \varphi(z_{2k-1} + 2\varepsilon\delta) = 0, \\ \sup\{|\varphi(x)|\,;\, x \in (z_{2k-1} - 2\varepsilon\delta, z_{2k-1} + 2\varepsilon\delta)\} = O(|\log \varepsilon|). \end{cases} \tag{4.10}$$

We should note that such φ can be constructed by super and subsolution method.

We are ready to define w and A by

$$w(x) := u_\varepsilon(x) - \frac{1}{2} + \varepsilon\rho(x)\varphi(x)$$

and

$$A(x) := -\frac{\varepsilon^2 w''(x)}{w(x)}.$$

Then one can prove by direct calculations that A and w fulfill properties (i) and (ii). $\square$

Outline of the proof of Lemma 4.9. For each $k = 1, 2, \ldots, m-1$, we consider the following eigenvalue problem.

$$\begin{cases} -\varepsilon^2\phi'' - f_u(x, u_\varepsilon)\phi + \dfrac{e^{-1/\varepsilon}}{\psi}\phi = \mu\phi & \text{in } J_k^-, \\ \phi'(\zeta_{2k-1}) = \phi'(\zeta_{2k}) = 0, \end{cases} \tag{4.11}$$

where ψ is a C^2-function satisfying

$$\begin{cases} \varepsilon^2\psi'' + f_u(x, u_\varepsilon)\psi - e^{-1/\varepsilon} = 0 & \text{in } J_k^-, \\ \psi'(\zeta_{2k-1}) = \psi'(\zeta_{2k}) = 0, \\ \psi < 0 & \text{in } J_k^-. \end{cases} \tag{4.12}$$

The existence of such ψ is not trivial. However, if (4.12) has a solution ψ, then ψ is an eigenfunction corresponding to zero eigenvalue of (4.11). Clearly, 0 is the first eigenvalue of (4.11) because ψ does not change its sign in J_k^-. Furthermore, the third term of the first equation of (4.12) is negative. Hence, Proposition 3.4 enables us to derive $\lambda_1(J_k^-) > 0$. Therefore, we have only to show the existence of a solution of (4.12).

We will take a super and subsolution method to solve (4.12). Set

$$\overline{\psi}(x) := 0 \qquad \text{in } J_k^- ;$$

clearly $\overline{\psi}$ is a supersolution of (4.12).

We will construct a subsolution of (4.12). We only discuss for $x \geq \xi_{2k}$ because the argument for $x \leq \xi_{2k}$ is essentially the same. It should be noted that there exists a positive constants κ and P such that

$$f_u(x, u_\varepsilon(x)) \leq -P \quad \text{in } (\xi_{2k} + \kappa\varepsilon, \zeta_{2k}) \tag{4.13}$$

when ε is sufficiently small. We set $\theta(z) = q(z)e^z$ with $q(z) = z^2/(z^2+1)$ and introduce

$$\eta(x) = \begin{cases} 0 & \text{in } (\xi_{2k}, \xi_{2k} + \kappa\varepsilon), \\ \varepsilon^{K_1}\theta\left(\dfrac{K_2(x - \xi_{2k} - \kappa\varepsilon)}{\varepsilon}\right) & \text{in } (\xi_{2k} + \kappa\varepsilon, \zeta_{2k}]. \end{cases} \tag{4.14}$$

Here, K_1 is a sufficiently large positive number and K_2 is a positive constant satisfying $(1+\gamma)K_2^2 < P$ with small $\gamma > 0$. We define

$$\underline{\psi}(x) := u_\varepsilon'(x) - \eta(x) \quad \text{in } [\xi_{2k}, \zeta_{2k}]$$

and

$$z^* := \inf\{x \in [\xi_{2k}, \zeta_{2k}]\,;\, \underline{\psi}'(x) = 0\}.$$

If $z^* \le \zeta_{2k}$, then it is easy to show that $\underline{\psi}$ is a subsolution of (4.12) by direct calculation. On the other hand, if $z^* > \zeta_{2k}$, the argument is somewhat complicated. For details, see [9]

Finally, it is obvious that

$$\underline{\psi} < \overline{\psi} \qquad \text{in } J_k^-.$$

Thus there exists a solution ψ of (4.12) satisfying $\underline{\psi} < \psi < \overline{\psi}$ in J_k^-. □

We are ready to show Theorem 4.2.

Proof of Theorem 4.2. From the proof of Theorem 4.5, it is sufficient to sum up the number of layers at each multi-layer. Thus the proof is complete. □

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COEXISTENCE STATES FOR A PREY-PREDATOR MODEL WITH CROSS-DIFFUSION

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Abstract. This paper discusses a prey-predator system with cross-diffusion. We can prove that the set of coexistence steady-states of this system contains an S or ⊃-shaped branch with respect to a bifurcation parameter in a large cross-diffusion case. We give also some criteria on the stability of these positive steady-states. Furthermore, we find the Hopf bifurcation point on the steady-state solution branch in a certain case.

1. **Introduction.** This paper is concerned with the following Lotka-Volterra prey-predator interaction model with cross-diffusion;

$$\text{(P)}\quad \begin{cases} u_t = \Delta u + u(a - u - cv) & \text{in } \Omega \times (0,\infty), \\ \sigma v_t = \Delta[(1+\beta u)v] + v(b + du - v) & \text{in } \Omega \times (0,\infty), \\ u = v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\,\cdot\,,0) = u_0 \geq 0, \quad v(\,\cdot\,,0) = v_0 \geq 0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\boldsymbol{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$; σ, a, b, c, d are positive constants and $\beta \geq 0$ is the *cross-diffusion* coefficient. In (P), unknown functions u and v represent the population densities of prey and predator species, respectively, which are interacting and migrating in the same habitat Ω. This system is concerned with an ecological situation such that the population pressure due to the high density of prey induces the diffusion of the form $\beta\Delta(uv)$ in the second equation. See also the monograph of Okubo and Levin [16] for the ecological background. The time local solvability of (P) has been established by Amann [1], where a wide class of quasilinear parabolic systems is discussed. According to his result, (P) has a unique local solution (u, v) provided $(u_0, v_0) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ for $p > N$. Recently, Le Dung [5] has found the global attractor for a class of triangular cross diffusion systems involving (P).

System (P) originates from the competition population model with cross-diffusion proposed by Shigesada, Kawasaki and Teramoto [19]. Since their pioneer work, many mathematicians have discussed such cross-diffusion systems. We refer to [3],[5],[6] and references therein for a recent progress on the global solvability of time-depending solutions. See e.g., [7],[12],[13],[14],[15],[18] about steady-state problems.

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Despite all their works concerning cross-diffusion systems, many problems still remain open. In particular, it is very difficult to know the complete structure of the steady-state solution set (e.g., the number, the stability or the shape of steady-states) to cross-diffusion systems such as (P).

We are interested in the global bifurcation structure of positive steady-state solutions to (P). Regarding a as a bifurcation parameter, we set

$$\mathcal{S} := \{(u, v, a) \,:\, (u, v) \text{ is a positive steady-state solution of (P)}\}.$$

Among other things, we will prove that when (β, b, c, d) belongs to a certain range, $\mathcal{S}$ contains a bounded S or $\supset$-shaped curve with respect to a. So (P) admits two or three positive steady-state solutions if a belongs to suitable ranges. This result implies a great contrast to the linear diffusion case ($\beta = 0$), where the uniqueness of positive steady-states is obtained by López-Gómez and Pardo [11] if the spatial dimension is one. Our method of analysis uses the idea developed by Du and Lou [4] and is based on bifurcation theory and the Lyapunov-Schmidt reduction procedure. If β is large and both of $b-\lambda_1$ and $\lambda_1 - d/\beta$ are small positive numbers, this reduction enables us to find an approximate limiting problem in a suitable finite dimensional space. Further, we can get the exact solution set of the limiting problem. Making use of the perturbation theory developed in [4], we will depict an S or $\supset$-shaped curve of $\mathcal{S}$ near the limiting solution set.

In Section 2. we will discuss such multiple existence of steady-state solutions. In Section 3, we will give some criteria on the stability of the positive steady-states. Furthermore, we will find a Hopf bifurcation point on the S or $\supset$-shaped solution set if σ is sufficiently large. Throughout the paper, the usual norms of the spaces $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\overline{\Omega})$ are defined by

$$\|u\|_p := \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p} \quad \text{and} \quad \|u\|_\infty := \max_{x \in \overline{\Omega}} |u(x)|.$$

In particular, we simply write $\|u\|$ instead of $\|u\|_2$. Furthermore, we will denote by Φ a unique positive solution of

$$-\Delta \Phi = \lambda_1 \Phi \text{ in } \Omega, \quad \Phi = 0 \text{ on } \partial\Omega, \quad \|\Phi\| = 1,$$

where λ_1 is the least eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition on $\partial\Omega$.

2. Bifurcation branch of positive steady-states.

2.1. Main Result. It is well known that the following elliptic boundary value problem

$$\Delta u + u(a - u) = 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

has a unique positive solution θ_a if $a > \lambda_1$; moreover, $a \in [\lambda_1, \infty) \to \theta_a \in C(\overline{\Omega})$ is continuous and strictly increasing function. It is easily verified that (P) has two semitrivial steady-state solutions

$$(u, v) = (\theta_a, 0) \text{ for } a > \lambda_1 \quad \text{and} \quad (u, v) = (0, \theta_b) \text{ for } b > \lambda_1$$

in addition to the trivial solution $(u, v) = (0, 0)$.

Theorem 2.1. *Suppose that $\beta b > \beta\lambda_1 > d$. For any $c > 0$, there exist a large number M and an open set*

$$O = O(c) \subset \{(\beta, b, d) \,:\, \beta \geq M,\, 0 < \lambda_1 - d/\beta, b - \lambda_1 \leq M^{-1}\}$$

such that if $(\beta, b, d) \in O$, *then* $\mathcal{S}$ *contains a bounded smooth curve*

$$\Gamma = \{(u(r), v(r), a(r)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times (\lambda_1, \infty), r \in (0, C)\},$$

which possesses the following properties,

(i) $(u(0), v(0)) = (0, \theta_b)$, $a(0) > \lambda_1$, $a'(0) > 0$;
(ii) $(u(C), v(C)) = (\theta_{a(C)}, 0)$, $a(C) > \lambda_1$;
(iii) $a(r)$ *attains a strict local maximum in* $(0, C)$. *Additionally, there exists an open set* $O' \subset O$ *such that, if* $(\beta, b, d) \in O'$, *then* $a(r)$ *attains a strict local minimum in* $(0, C)$.

Our result asserts that $\mathcal{S}$ contains a bounded S or $\supset$-shaped branch, which connects the above two semitrivial solutions, in a certain case. We can also find an *unbounded* S-shaped branch of $\mathcal{S}$, under another coefficient assumption [10, Theorem 1.2].

2.2. **Outline of the proof of Theorem 2.1.** In (P), we employ the following change of variables;

$$a = \lambda_1 + \varepsilon a_1, \quad b = \lambda_1 + \varepsilon b_1, \quad d/\beta = \lambda_1 - \varepsilon\tau, \quad \beta = \gamma/\varepsilon, \quad u = \varepsilon w, \quad (1+\beta u)v = \varepsilon z. \tag{2.1}$$

Here a_1, b_1, τ are positive constants. Furthermore, ε is a small positive constant, thus γ is also a positive constant. In what follows, we will mainly discuss the case when β is large and both of $b - \lambda_1$ and $\lambda_1 - d/\beta$ are small positives. We note that a_1 plays a role of a bifurcation parameter. By (2.1), a pair of new unknown functions (w, z) satisfies

$$\text{(PP)} \begin{cases} w_t = \Delta w + \lambda_1 w + \varepsilon f(w, z, a_1) & \text{in } \Omega \times (0, \infty), \\ \sigma \left[-\dfrac{\gamma z}{(1+\gamma w)^2} w_t + \dfrac{z_t}{1+\gamma w} \right] = \Delta z + \lambda_1 z + \varepsilon g(w, z) & \text{in } \Omega \times (0, \infty), \\ w = z = 0 & \text{on } \partial\Omega \times (0, \infty), \\ w(\,\cdot\,, 0) = u_0/\varepsilon, \quad z(\,\cdot\,, 0) = (1 + \beta u_0) v_0/\varepsilon & \text{in } \Omega, \end{cases}$$

where

$$f(w, z, a_1) := w \left(a_1 - w - \frac{cz}{1+\gamma w} \right), \quad g(w, z) := \frac{z}{1+\gamma w} \left(b_1 - \tau\gamma w - \frac{z}{1+\gamma w} \right).$$

The steady-state problem associated with (PP) is reduced to the following semilinear elliptic equations;

$$\begin{cases} \Delta w + \lambda_1 w + \varepsilon f(w, z, a_1) = 0 & \text{in } \Omega, \\ \Delta z + \lambda_1 z + \varepsilon g(w, z) = 0 & \text{in } \Omega, \\ w = z = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

By virtue of (2.1), it is easy to see that (2.2) has two semitrivial solutions

$$(w, z) = (\varepsilon^{-1}\theta_{\lambda_1 + \varepsilon a_1}, 0), \quad (w, z) = (0, \varepsilon^{-1}\theta_{\lambda_1 + \varepsilon b_1})$$

in addition to the trivial solution. For the Lyapunov-Schmidt reduction, we will give a similar framework to that of Du and Lou [4]. For $p > N$, we prepare two Banach spaces

$$X := [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)] \times [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)], \quad Y := L^p(\Omega) \times L^p(\Omega).$$

We note that $X \subset C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ by the Sobolev embedding theorem. Define mappings $H : X \to Y$ and $B : X \times \boldsymbol{R} \to Y$ by

$$H(w, z) := (\Delta w + \lambda_1 w, \Delta z + \lambda_1 z), \quad B(w, z, a_1) := (f(w, z, a_1), g(w, z)). \tag{2.3}$$

Then (2.2) is equivalent to the following equation

$$H(w,z)+\varepsilon B(w,z,a_1)=0. \tag{2.4}$$

Let X_1 and Y_1 be the L^2-orthogonal complements of span $\{(\Phi,0),(0,\Phi)\}$ in X and Y, respectively. Let $P: X\to X_1$ and $Q: Y\to Y_1$ represent L^2-orthogonal projections. Thus a pair of unknown functions $(w,z)\in X$ is decomposed as

$$(w,z)=(r,s)\,\Phi+\boldsymbol{u},\quad \boldsymbol{u}=P(w,z).$$

Since $H((r,s)\Phi)=0$ and $(I-Q)H(X_1)=0$, (2.4) is consequently reduced to

$$QH(\boldsymbol{u})+\varepsilon QB((r,s)\,\Phi+\boldsymbol{u},a_1)=0 \tag{2.5}$$

and

$$(I-Q)B((r,s)\,\Phi+\boldsymbol{u},a_1)=0.$$

The Lyapunov-Schmidt reduction procedure leads us to the next lemma:

Lemma 2.1. *For any $C>0$, there exist a neighborhood N_0 of the set*

$$\{(w,z,a_1,\varepsilon)=(r\Phi,s\Phi,a_1,0)\in X\times \boldsymbol{R}^2 : |r|,|s|,|a_1|\le C\}$$

and a positive constant ε_0 such that all solutions of (2.5) in N_0 are given by

$$\{((r,s)\,\Phi+\varepsilon\boldsymbol{U}(r,s,a_1,\varepsilon),a_1,\varepsilon) : |r|,|s|,|a_1|\le C+\varepsilon_0, |\varepsilon|\le\varepsilon_0\}.$$

with a smooth X_1-valued function $\boldsymbol{U}$. Then

$$(w,z,a_1,\varepsilon)=((r,s)\Phi+\varepsilon\boldsymbol{U}(r,s,a_1,\varepsilon),a_1,\varepsilon)$$

becomes a solution of (2.4), or equivalently (2.2), in N_0 if and only if

$$F^{\varepsilon}(r,s,a_1)\Phi := (I-Q)B((r,s)\Phi+\varepsilon\boldsymbol{U}(r,s,a_1,\varepsilon),a_1)=0.$$

See [10] for the proof of Lemma 2.1. Since $(I-Q)(u,v)=(\int_\Omega u\Phi dx,\ \int_\Omega v\Phi dx)\Phi$, it follows from (2.3) that

$$\begin{aligned} F^0(r,s,a_1) &= \left(\int_\Omega f(r\Phi,s\Phi,a_1)\Phi,\ \int_\Omega g(r\Phi,s\Phi)\Phi\right) \\ &= {}^t\!\begin{pmatrix} r\left(a_1-r\|\Phi\|_3^3-cs\displaystyle\int_\Omega\frac{\Phi^3}{1+\gamma r\Phi}\right) \\ s\left\{b_1-(b_1+\tau)\gamma r\displaystyle\int_\Omega\frac{\Phi^3}{1+\gamma r\Phi}-s\int_\Omega\frac{\Phi^3}{(1+\gamma r\Phi)^2}\right\}\end{pmatrix}. \end{aligned} \tag{2.6}$$

Thus $\operatorname{Ker} F^0$ is a union of the following four sets;

$$\mathcal{L}_0=\{(0,0,a_1) : a_1\in\boldsymbol{R}\},\quad \mathcal{L}_1=\{(a_1/\|\phi_1\|_3^3,0,a_1) : a_1\in\boldsymbol{R}\},$$
$$\mathcal{L}_2=\{(0,b_1/\|\phi_1\|_3^3,a_1) : a_1\in\boldsymbol{R}\},\quad \mathcal{L}_p=\{(r,\varphi(\gamma r),\psi(r)) : r\in\boldsymbol{R}\},$$

where

$$\begin{cases} \varphi(r)=\left[b_1-(b_1+\tau)r\displaystyle\int_\Omega\frac{\Phi^3}{1+r\Phi}\right]\left(\displaystyle\int_\Omega\frac{\Phi^3}{(1+r\Phi)^2}\right)^{-1}, \\ \psi(r)=r\|\Phi\|_3^3+c\varphi(\gamma r)\displaystyle\int_\Omega\frac{\Phi^3}{1+\gamma r\Phi}. \end{cases} \tag{2.7}$$

We note that $\mathcal{L}_p\cap\overline{\boldsymbol{R}_+^3}$ is identical with the limiting set of positive solutions of (2.2) as $\varepsilon\to 0$. Indeed the following proposition holds true:

Proposition 2.1. *For a sufficiently large $A_1 > 0$, there exist $\varepsilon_0 > 0$ and a family of smooth curves*

$$\{(r(\xi,\varepsilon), s(\xi,\varepsilon), a_1(\xi,\varepsilon)) \in \boldsymbol{R}_+^3 : (\xi,\varepsilon) \in (0, C_\varepsilon) \times (0, \varepsilon_0)\}$$

such that for each fixed $\varepsilon \in (0, \varepsilon_0]$, all positive solutions of (2.2) *with $a_1 \in (0, A_1]$ can be parameterized as*

$$\Gamma^\varepsilon = \{(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) = ((r,s)\Phi + \varepsilon \boldsymbol{U}(r,s,a_1,\varepsilon), a_1) : (r,s,a_1) = (r(\xi,\varepsilon), s(\xi,\varepsilon), a_1(\xi,\varepsilon)) \ \ for \ \ \xi \in (0, C_\varepsilon)\}$$

and $(r(\xi,0), s(\xi,0), a_1(\xi,0)) = (\xi, \varphi(\gamma\xi), \psi(\xi))$, $r(0,\varepsilon) = 0$. Here $C_\varepsilon > 0$ depends continuously on $\varepsilon \in [0, \varepsilon_0]$. Furthermore, $w(C_\varepsilon, \varepsilon) > 0$ in Ω and $z(C_\varepsilon, \varepsilon) \equiv 0$.

The above proposition implies that if $\varepsilon > 0$ is sufficiently small, then Γ^ε forms a positive solution branch near the curve $\{(r\Phi, \varphi(\gamma r)\Phi, \psi(r)) : 0 < r < C\}$. So it is important to study the profile of $\mathcal{L}_p$. By virtue of (2.7), $(0, \varphi(0), \psi(0)) = (0, b_1/\|\Phi\|_3^3, cb_1) \in \mathcal{L}_2$. It is easy to find a positive constant $r_0 = r_0(\tau/b_1)$ such that $\varphi(r) > 0$ for $r \in [0, r_0)$ and $\varphi(r) < 0$ for $r \in (r_0, \infty)$. Thus it follows that

$$(r_0/\gamma, \varphi(r_0), \psi(r_0/\gamma)) = (r_0/\gamma, 0, r_0\|\Phi\|_3^3/\gamma) \in \mathcal{L}_1.$$

We note that C_ε stated in Proposition 2.1 satisfies $C_0 = r_0/\gamma$. Additionally the next lemma gives profiles of $\psi(r)$ in the interval of $\{r > 0 : \varphi(\gamma r) > 0\}$ if τ is sufficiently small and γ is sufficiently large.

Lemma 2.2. *There exist positive constants $\tilde{\tau} = \tilde{\tau}(c, b_1)$ and $\tilde{\gamma} = \tilde{\gamma}(c, b_1)$ such that if $(\tau, \gamma) \in (0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$, then $\psi'(0) > 0$ and $\psi(r)$ achieves a strict local maximum in $(0, r_0/\gamma)$. Furthermore, there exists a continuous function $\hat{\gamma}(\tau)$ in $(0, \tilde{\tau}]$ satisfying*

$$\tilde{\gamma} < \hat{\gamma}(\tau) \quad for\ all \quad \tau \in (0, \tilde{\tau}] \quad and \quad \lim_{\tau \downarrow 0} \hat{\gamma}(\tau) = \infty$$

and that, if $\gamma \in [\tilde{\gamma}, \hat{\gamma}(\tau))$ for $\tau \in (0, \tilde{\tau}]$, then $\psi(r)$ attains a strict local minimum in $(0, r_0/\gamma)$.

From Proposition 2.1 and Lemma 2.2, one can see the following proposition.

Proposition 2.2. *Suppose that $(\tau, \gamma) \in (0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$ and that $\varepsilon > 0$ is small enough. Then the positive solution set of* (2.2) *contains a bounded smooth curve*

$$\Gamma^\varepsilon = \{(w(\xi), z(\xi), a_1(\xi)) \in X \times \boldsymbol{R} : \xi \in (0, C_\varepsilon)\},$$

which possesses the following properties;

(i) $(w(0), z(0)) = (0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1})$, $a_1(0) > 0$, $a_1'(0) > 0$;
(ii) $(w(C_\varepsilon), z(C_\varepsilon)) = (\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_{1*}}, 0)$, $a_{1*} := a_1(C_\varepsilon) > 0$;
(iii) *$a_1(\xi)$ attains a strict local maximum in $(0, C_\varepsilon)$. In particular, if $\gamma \in [\tilde{\gamma}, \hat{\gamma}(\tau))$ for $\tau \in (0, \tilde{\tau}]$, then $a_1(\xi)$ attains a strict local minimum in $(0, C_\varepsilon)$.*

With use of (2.1), Theorem 2.1 immediately follows from Proposition 2.2. Actually, for small $\varepsilon > 0$, open sets stated in Theorem 2.1 are given as

$$O = \{(\beta, b, d) = (\gamma/\varepsilon, \lambda_1 + \varepsilon b_1, (\lambda_1 + \varepsilon\tau)\gamma/\varepsilon) : (\tau, \gamma) \in (0, \tilde{\tau}) \times (\tilde{\gamma}, \infty)\},$$

$$O' = \{(\beta, b, d) = (\gamma/\varepsilon, \lambda_1 + \varepsilon b_1, (\lambda_1 + \varepsilon\tau)\gamma/\varepsilon) : (\tau, \gamma) \in (0, \tilde{\tau}) \times (\tilde{\gamma}, \hat{\gamma}(\tau))\}.$$

We refer to [10] for the complete proofs.

3. Stability analysis.

3.1. Main results. In this section, we will discuss the stability of steady-state solutions on Γ obtained in Theorem 2.1. Before stating our stability results, we need to divide Γ at every turning point with respect to a. In case $(\beta, b, d) \in O$, let

$$0 < r_1 < r_2 < \cdots < r_{k-1} < C$$

be all strict local maximum or minimum points of $a(r)$. Because of $a'(0) > 0$ (see Theorem 2.1), $r_{2j-1}\,(j = 1, 2, \ldots, [k/2])$ are strict local maximum points, and $r_{2j}\,(j = 1, 2, \ldots, [(k-1)/2])$ are strict local minimum points. For each $1 \leq i \leq k$, we set

$$\Gamma_i := \{(u(r), v(r), a(r)) \in \Gamma : r \in (r_{i-1}, r_i)\},$$

where $r_0 := 0$ and $r_k := C$.

We are ready to state stability results. In a case when σ is sufficiently small, we can deduce that the stability of steady-states on Γ changes only at the *turning points*, and moreover, we can know whether each solution on Γ_i is asymptotically stable or not:

Theorem 3.1. *For almost every $(\beta, b, d) \in O$, there exists a small positive constant δ such that if $\sigma \leq \delta$, then all steady-state solutions on $\Gamma_{2j-1}\,(j = 1, 2, \ldots, [(k+1)/2])$ are asymptotically stable in the topology of X, while all steady-state solutions on $\Gamma_{2j}\,(j = 1, 2, \ldots, [k/2])$ are unstable.*

In the above case, we remark that $(u(0), v(0)) = (0, \theta_b)$ and $(u(C), v(C)) = (\theta_{a(C)}, 0)$ by Theorem 2.1. So Theorem 3.1 implies that stable positive steady-states bifurcate from the semitrivial solution $(0, \theta_b)$, the stability on Γ changes at every turning point with respect to a, and moreover Γ connects the other semitrivial solution $(\theta_{a(C)}, 0)$. On the other hand, when σ becomes large enough, we can find a Hopf bifurcation point on Γ_1; so that, time-periodic solutions of (P) appear from the point:

Theorem 3.2. *For any $(\beta, b, d) \in O$, there exists a large positive D such that if $\sigma \geq D$, then the Hopf bifurcation occurs at some point $(u(r^*), v(r^*), a(r^*)) \in \Gamma_1$. In this case, there exists a periodic solution of* (P) *if a lies in a neighborhood of $a(r^*)$ with $a > a(r^*)$.*

3.2. Outline of the proofs of Theorems 3.1 and 3.2. By virtue of the regularity of (2.1), the stability of a steady-state (u^*, v^*) of (P) coincides with that of the steady-state $(w^*, z^*) = (u^*/\varepsilon, (1 + \beta u^*)z^*/\varepsilon)$ of (PP). So we will concentrate on the stability analysis for the steady-states on Γ^ε given in Proposition 2.2. By virtue of Proposition 2.1, all positive steady-states of (PP) with $a_1 \in (0, A_1)$ can be parameterized as $\Gamma^\varepsilon = \{(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) : \xi \in (0, C_\varepsilon)\}$ when $\varepsilon > 0$ is sufficiently small. For each $(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) \in \Gamma^\varepsilon$, we define a linear operator $L(\xi, \varepsilon) : X \to Y$ by

$$L(\xi,\varepsilon)\begin{pmatrix} h \\ k \end{pmatrix} := -H\begin{pmatrix} h \\ k \end{pmatrix} - \varepsilon B_{(w,z)}(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon))\begin{pmatrix} h \\ k \end{pmatrix},$$

where H, B are mappings defined by (2.3) and $B_{(w,z)}$ denotes the Fréchet derivative of B with respect to (w, z). Furthermore, in view of the left hand side of (PP), we set

$$J(\xi,\varepsilon) := \begin{bmatrix} 1 & 0 \\ -\dfrac{\sigma\gamma z(\xi,\varepsilon)}{(1+\gamma w(\xi,\varepsilon))^2} & \dfrac{\sigma}{1+\gamma w(\xi,\varepsilon)} \end{bmatrix}.$$

Then the linearized eigenvalue problem associated with $(w(\xi,\varepsilon), z(\xi,\varepsilon))$ is given by

$$L(\xi,\varepsilon)\begin{pmatrix} h \\ k \end{pmatrix} = \mu J(\xi,\varepsilon)\begin{pmatrix} h \\ k \end{pmatrix}. \tag{3.1}$$

In this subsection, we study the linearized stability of steady-states on Γ^{ε} by the spectral analysis for (3.1). Put

$$\rho(\xi,\varepsilon) := \{\mu \in \boldsymbol{C} \ : \ (3.1) \text{ has no solution except for } h = k = 0\}.$$

We begin with the following lemma.

Lemma 3.1. *Suppose that $\varepsilon > 0$ is sufficiently small. Then there exist positive constants κ_1, ω independent of (ξ,ε) such that $-\rho(\xi,\varepsilon) \supset \{z \in \boldsymbol{C} : |z| \geq \kappa_1$ and $|\arg z| \leq \pi/2 + \omega\}$. On the other hand, all eigenvalues $\{\mu_i(\xi,\varepsilon)\}_{i=1}^{\infty}$ (counting multiplicity) of (3.1) satisfy*

$$\lim_{\varepsilon \downarrow 0} \mu_1(\xi,\varepsilon) = \lim_{\varepsilon \downarrow 0} \mu_2(\xi,\varepsilon) = 0 \tag{3.2}$$

and $\mathrm{Re}\,\mu_i(\xi,\varepsilon) > \kappa_2$ for all $i \geq 3$ and $\xi \in (0, C_\varepsilon)$ for some positive constant κ_2 independent of (ξ,ε).

The proof of Lemma 3.1 can be established by employing a limiting eigenvalue problem as $\varepsilon \downarrow 0$ in (3.1), and making use of the perturbation theory by T. Kato [8, Chapter 8]. See [9] for details.

We note that all eigenvalues $\{\mu_i(\xi,\varepsilon)\}$ form a symmetric set with respect to the real axis in the complex space $\boldsymbol{C}$. Then $\mu_1(\xi,\varepsilon)$ and $\mu_2(\xi,\varepsilon)$ (with (3.2)) satisfy the following properties (i) or (ii);

(i) both of $\mu_1(\xi,\varepsilon)$ and $\mu_2(\xi,\varepsilon)$ are real numbers;
(ii) $\mu_1(\xi,\varepsilon)$ is a complex conjugate of $\mu_2(\xi,\varepsilon)$.

In what follows, we assume that $\mu_1(\xi,\varepsilon) \leq \mu_2(\xi,\varepsilon)$ in case (i), and $\mathrm{Im}\mu_1(\xi,\varepsilon) \geq \mathrm{Im}\mu_2(\xi,\varepsilon)$ in case (ii).

Definition 3.1. A steady-state $(w(\xi,\varepsilon), z(\xi,\varepsilon))$ of (PP) is called *linearly stable* if $\mathrm{Re}\,\mu_1(\xi,\varepsilon) > 0$. If $\mathrm{Re}\,\mu_1(\xi,\varepsilon) < 0$, then it is called *linearly unstable.*

We define matrices $K(r)$ and $M(r)$ by

$$\begin{aligned} K(r) &= \begin{bmatrix} 1 & 0 \\ -\sigma\gamma\varphi(\gamma r)\displaystyle\int_\Omega \frac{\Phi^3}{(1+\gamma r\Phi)^2} & \sigma\displaystyle\int_\Omega \frac{\Phi^2}{1+\gamma r\Phi} \end{bmatrix}, \\ M(r) &= -K(r)^{-1}F^0_{(r,s)}(r,\varphi(\gamma r),\psi(r)) \end{aligned} \tag{3.3}$$

for the mapping F^0 defined by (2.6). To determine the sign of $\mathrm{Re}\,\mu_1(\xi,\varepsilon)$, the following lemma plays an important role.

Lemma 3.2. *Let $\nu_1(r)$ and $\nu_2(r)$ be eigenvalues of $M(r)$ and satisfy $\mathrm{Re}\,\nu_1(r) \leq \mathrm{Re}\,\nu_2(r)$, $\mathrm{Im}\,\nu_1(r) \geq \mathrm{Im}\,\nu_2(r)$. Then for any $r \in (0, C_0)$, it holds true that*

$$\lim_{(\xi,\varepsilon)\to(r,0)} \frac{\mu_i(\xi,\varepsilon)}{\varepsilon} = \nu_i(r) \quad \text{for} \quad i = 1, 2. \tag{3.4}$$

Lemma 3.2 can be proved by taking L^2-inner product of (3.1) with Φ and letting $\varepsilon \to 0$. See [9] for details.

Lemma 3.3. *Suppose that $\varepsilon > 0$ is sufficiently small. Suppose further that $\xi \in (0, C_\varepsilon)$. Thus all zeros of $\mu_1(\xi,\varepsilon)$ coincide with all zeros of $\partial_\xi a_1(\xi,\varepsilon)$.*

The above lemma asserts that the degeneracy of steady-states on Γ^ε is equivalent to the criticality of $a_1(\xi,\varepsilon)$ with respect to ξ. We refer the proof of Lemma 3.3 to the perturbation theory for the Fredholm operator developed by Du and Lou [4, Theorem 3.13 and Appendix]

Since ψ is analytic, ψ' possesses at most a finite number of zeros in $(0,C_0)$. Furthermore, by virtue of (2.7), any zero of ψ' must be a strictly critical point of ψ for almost every $(\tau,\gamma)\in(0,\tilde{\tau}\,]\times[\,\tilde{\gamma},\infty)$. For such $(\tau,\gamma)\in(0,\tilde{\tau}\,]\times[\,\tilde{\gamma},\infty)$ and sufficiently small $\varepsilon>0$, all zeros of $\partial_\xi a_1(\xi,\varepsilon)$ are denoted by

$$0<\xi_1(\varepsilon)<\xi_2(\varepsilon)<\cdots<\xi_{k-1}(\varepsilon)<C_\varepsilon.$$

That is,

$$(w_i,z_i,a_1^i):=(w(\xi_i(\varepsilon),\varepsilon),z(\xi_i(\varepsilon),\varepsilon),a_1(\xi_i(\varepsilon),\varepsilon))\in\Gamma^\varepsilon\quad(i=1,2,\ldots,k-1)$$

are all turning points on Γ^ε with respect to a_1. Here we remark that $\lim_{\varepsilon\downarrow 0}a_1(\,\cdot\,,\varepsilon)=\psi$ in $C^2([\,0,C_0\,])$ by Proposition 2.1 (see also the proof of [10, Lemma 5.3]). Additionally, for each $1\le i\le k$ we set

$$\Gamma_i^\varepsilon:=\{((w(\xi,\varepsilon),z(\xi,\varepsilon),a_1(\xi,\varepsilon))\;:\;\xi\in(\xi_{i-1}(\varepsilon),\xi_i(\varepsilon))\},$$

where $\xi_0(\varepsilon):=0$ and $\xi_k(\varepsilon)=C_\varepsilon$. This implies $\bigcup_{i=1}^{k}\Gamma_i^\varepsilon=\Gamma^\varepsilon\setminus\bigcup_{i=1}^{k-1}\{(w_i,z_i,a_1^i)\}$.

Lemma 3.4. *For almost every $(\tau,\gamma)\in(0,\tilde{\tau}\,]\times[\,\tilde{\gamma},\infty)$, there exist small positive constants δ,ε_0 such that if $\sigma\le\delta$ and $\varepsilon\le\varepsilon_0$, then all steady-state solutions on $\Gamma_{2j-1}^\varepsilon$ $(j=1,2,\ldots,[(k+1)/2])$ are linearly stable, while all steady-state solutions on Γ_{2j}^ε $(j=1,2,\ldots,[k/2])$ are linearly unstable.*

Proof. Taking the trace of $M(r)$, one can see

$$\begin{aligned}\nu_1(r)+\nu_2(r)=\frac{\varphi(\gamma r)}{\sigma}&\left[\int_\Omega\frac{\Phi^3}{(1+\gamma r\Phi)^2}\left(\int_\Omega\frac{\Phi^2}{1+\gamma r\Phi}\right)^{-1}-\sigma c\gamma r\int_\Omega\frac{\Phi^4}{(1+\gamma r\Phi)^2}\right]\\&+r\|\Phi\|_3^3+c\gamma r\varphi(\gamma r)\int_\Omega\frac{\Phi^3}{1+\gamma r\Phi}\int_\Omega\frac{\Phi^3}{(1+\gamma r\Phi)^2}\left(\int_\Omega\frac{\Phi^2}{1+\gamma r\Phi}\right)^{-1}.\end{aligned}\tag{3.5}$$

We set $y_1(r):=\int_\Omega r\Phi^4/(1+r\Phi)^2$. Since $y_1(0)=0$ and $y_1(r)=O(r^{-1})\,(r\to\infty)$, $y_1(\hat{r})=\sup_{r>0}y_1(r)$ for some $\hat{r}>0$. Then by (3.5),

$$\nu_1(r)+\nu_2(r)>\frac{\varphi(\gamma r)}{\sigma}\left[\int_\Omega\frac{\Phi^3}{(1+\gamma C_0\Phi)^2}-\sigma c y_1(\hat{r})\right]+r\|\Phi\|_3^3$$

for all $r\in[\,0,C_0\,]$. Therefore, it follows from $\varphi(\gamma r)>0$ $(r\in[\,0,C_0))$ that, if

$$\sigma<\frac{1}{2cy_1(\hat{r})}\int_\Omega\frac{\Phi^3}{(1+\gamma C_0\Phi)^2},$$

then $\nu_1(r)+\nu_2(r)>0$ for all $r\in[\,0,C_0\,]$. Thus we can see by Lemma 3.2 that for sufficiently small $\varepsilon>0$,

$$\mu_1(\xi,\varepsilon)+\mu_2(\xi,\varepsilon)>0\quad\text{for all}\quad\xi\in[\,0,C_\varepsilon\,].\tag{3.6}$$

Hence (3.6) also implies $\operatorname{Re}\mu_2(\xi,\varepsilon)>0$ for all $\xi\in[\,0,C_\varepsilon\,]$. On the other hand, in view of (3.3), (2.6) and (2.7), direct calculations enable us to obtain

$$\nu_1(r)\nu_2(r)=\det M(r)=\frac{r\varphi(\gamma r)\psi'(r)}{\sigma}\int_\Omega\frac{\Phi^3}{(1+\gamma r\Phi)^2}\left(\int_\Omega\frac{\Phi^2}{1+\gamma r\Phi}\right)^{-1}.\tag{3.7}$$

So it holds that $\operatorname{sign}\nu_1(r)\nu_2(r) = \operatorname{sign}\psi'(r)$ for all $r \in (0, C_0)$. Let $r_0 \in (0, C_0)$ be any fixed point. If $\psi'(r_0) > 0$, then Lemma 3.2 implies $\mu_1(\xi,\varepsilon)\mu_2(\xi,\varepsilon) > 0$ if (ξ,ε) is sufficiently near $(r_0, 0)$. Furthermore, together with (3.6), we obtain $\operatorname{Re}\mu_1(\xi,\varepsilon) > 0$. Similarly if $\psi'(r_0) < 0$ and (ξ,ε) is close to $(r_0,0)$, then $\operatorname{Re}\mu_1(\xi,\varepsilon) < 0$. Additionally it follows from Lemma 3.3 that $\mu_1(\xi,\varepsilon) = 0$ if and only if $\xi = \xi_i(\varepsilon)$ for some $1 \le i \le k-1$ provided that $\varepsilon > 0$ is sufficiently small. Since $\operatorname{Re}\mu_2(\xi,\varepsilon) > 0$ for all $\xi \in [0, C_\varepsilon]$, consequently $\operatorname{Re}\mu_1(\xi,\varepsilon) = 0$ holds if and only if $\xi = \xi_i(\varepsilon)$ for some $1 \le i \le k-1$. We now remark $\psi'(0) > 0$ if $(\tau,\gamma) \in (0,\tilde{\tau}\,] \times [\,\tilde{\gamma},\infty)$ (see [10, Lemma 4.1]). Therefore we obtain

$$\begin{cases} \operatorname{Re}\mu_1(\xi,\varepsilon) > 0 & \text{if } \ (w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) \in \Gamma^\varepsilon_{2j-1}, \\ \operatorname{Re}\mu_1(\xi,\varepsilon) < 0 & \text{if } \ (w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) \in \Gamma^\varepsilon_{2j}. \end{cases}$$

Thus the proof of Lemma 3.4 is complete. □

By virtue of (2.1), we can complete the proof of Theorem 3.1 from Lemma 3.4. It should be noted that we use the linearized stability theory developed by Potier-Ferry [17]. See [9] for details.

Proposition 3.1. *For any $(\tau,\gamma) \in (0,\tilde{\tau}\,] \times [\,\tilde{\gamma},\infty)$, there exist a large $D > 0$ and a small $\varepsilon_0 > 0$ such that if $\sigma \ge D$ and $\varepsilon \le \varepsilon_0$, then the Hopf bifurcation occurs at a certain point on Γ^ε_1.*

Proof. To accomplish the proof, it suffices to find small positive numbers ξ^* and ε such that $\mu_1(\xi^*,\varepsilon)$, $\mu_2(\xi^*,\varepsilon)$ form a pure imaginary pair and satisfy $\partial_\xi \operatorname{Re}\mu_i(\xi^*,\varepsilon) < 0$ for $i = 1,2$. We refer to Amann [2] for the abstract Hopf bifurcation theorem for strongly coupled parabolic equations.

Take $(\tau,\gamma) \in (0,\tilde{\tau}\,] \times [\,\tilde{\gamma},\infty)$. Let $\nu_1(r)$ and $\nu_2(r)$ be eigenvalues of $M(r)$ defined by (3.3). We first remark that by (3.7) and $\psi'(0) > 0$,

$$\nu_1(r)\nu_2(r) > 0 \quad \text{for all} \quad r \in (0, r_1) \tag{3.8}$$

with some $r_1 > 0$. If we set

$$y_2(r) := \int_\Omega \frac{\Phi^4}{(1+\gamma r\Phi)^2} - \int_\Omega \frac{\Phi^3}{1+\gamma r\Phi} \int_\Omega \frac{\Phi^3}{(1+\gamma r\Phi)^2} \left(\int_\Omega \frac{\Phi^2}{1+\gamma r\Phi} \right)^{-1} - \frac{\|\Phi\|_3^3}{c\gamma\varphi(\gamma r)}$$

then, (3.5) is rewritten as

$$\nu_1(r) + \nu_2(r) = \frac{\varphi(\gamma r)}{\sigma} \left[\int_\Omega \frac{\Phi^3}{(1+\gamma r\Phi)^2} \left(\int_\Omega \frac{\Phi^2}{1+\gamma r\Phi} \right)^{-1} - \sigma c\gamma r y_2(r) \right].$$

Thus direct calculations imply

$$\nu_1(0) + \nu_2(0) = \frac{b_1}{\sigma}, \quad \nu_1'(0) + \nu_2'(0) = \frac{1}{\sigma}\left(\tilde{C} - \sigma c\gamma y_2(0)\right) \tag{3.9}$$

with some constant $\tilde{C}$ independent of σ. By virtue of Schwarz' inequality and $\|\Phi\| = 1$, we see $\|\Phi\|_4^4 > \|\Phi\|_3^6$. Thus it turns out that $y_2(0) = \|\Phi\|_4^4 - \|\Phi\|_3^6 - \|\Phi\|_3^3(cb_1\gamma)^{-1} > 0$ if γ is large enough. It follows from (3.9) that if σ is sufficiently large, we can find a small positive number $r_0 \in (0, r_1)$ such that

$$\nu_1(r) + \nu_2(r) > 0 \ \text{ in } \ (0, r_0), \quad \nu_1(r_0) + \nu_2(r_0) = 0 \quad \text{and} \quad \nu_1'(r_0) + \nu_2'(r_0) < 0. \tag{3.10}$$

We can find a certain (ξ^*,ε) near $(r_0, 0)$, such that eigenvalues $\mu_1(\xi^*,\varepsilon)$, $\mu_2(\xi^*,\varepsilon)$ are pure imaginary pair and satisfy $\partial_\xi \operatorname{Re}\mu_i(\xi^*,\varepsilon) < 0$ $(i = 1,2)$. In this part of the proof, we make use of Lemma 3.4 and Lyapunov-Schmidt reduction technique (see

[9]). Therefore the Hopf bifurcation occurs at $(w(\xi^*,\varepsilon), z(\xi^*,\varepsilon),\ a_1(\xi^*,\varepsilon))$, which belongs to Γ_1^ε because ξ^* is sufficiently small. □

By virtue of (2.1), Proposition 3.1 immediately yields Theorem 3.2.

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研究成果報告

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研究成果の概要

1) 調和平均曲率流の時間発展の挙動を，対応する非線形拡散方程式の爆発解の振舞いによって分類し，それを数値解析によって確認した．(論文 [1])

2) ロバチェフスキー平面に値をとるギルバート減衰を伴うランダウ・リフシッツ方程式に対する初期値問題の大域的適切性を高次放物型近似を用いて示した．ガレルキン法とエネルギー評価を用いている．(論文 [2])

3) 境界条件として，完全壁条件を持つ，静電磁場のレゾルベントの L^p 評価を確立した．これを用いて放物型超伝導方程式に対する初期値問題の L^p 局所適切性を示した．(論文 [3])

4) 高次元空間における S^2 に値を取るランダウ・リフシッツ方程式の初期値問題の局所適切性を差分近似解法と非線形半群理論を用いて示した．(論文 [4])

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研究成果報告

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研究成果の概要

1) 従来の方法では得られなかった，準線形放物型方程式の解の高い微分可能性を保証する「L^∞-エネルギー法」を開発した．この方法により，まず，充分一般的な二重非線形放物型方程式のリプシッツ連続な時間局所解の存在が示され (1996, 2002)，さらには，1950 年代以来，未解決であった「Porous Medium 方程式は C^∞-級の時間局所解を許すか？」という問題が肯定的に解決されるという重要な知見が得られた (2001).「L^∞-エネルギー法」は，これらの成果のみならず，いろいろな局面で応用可能な極めて有用な解析手段を与えていることを，現在進行中の研究が示唆している．

2) 「劣微分作用素の非単調摂動理論」が，バナッハ空間上の枠組みへ拡大された．これにより，従来 ガレルキン法で構成されていた退化放物型方程式の解の存在と正則性がより自然な枠組みで，より一般的な条件のもとで議論できるようになり，いくつかの具体的な方程式に対して，従来の方法では解決できなかった未解決問題が解決された．

3) 部分対称性を有する Concentration Compactness 理論を構築した．コンパクト性の欠如した問題を解析する有力な方法として，Concentration Compactness 理論が知られているが，一方で球対称性などの高い対称性がある場合には，コンパクト性が回復することが知られている．コンパクト性が回復しない程度の部分的対称性が存在する場合に，Concentration Compactness 理論がどのように，その部分対称性を反映するかを解明した．この応用として，無限柱状領域において，臨界指数を越える非線形性をもつ楕円型方程式の非自明解の存在が示された．

4) 「ある条件のもとでは，対称性をもつ部分空間での臨界点が，全体での臨界点を与える」という R.Palais による対称臨界性原理は，本来　変分構造をもつ楕円型方程式に限られた理論であった．この理論が，必ずしも変分構造をもたない楕円型方程式や時間発展を含むの発展方程式へ適用可能な一般的な理論に拡張された．これにより，従来の理論では不可能であった，放物型方程式や波動方程式への応用の道が開かれた．

5) 劣微分作用素を含む多価写像に対する写像度の理論が構築された．これにより，従来ではカバーできなかった，種々の多価性をもつ非線形偏微分方程式への写像度の理論が適用可能になった．

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研究成果報告

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研究成果の概要

(i) 非線型楕円型方程式について非線型楕円型方程式に対する特異摂動問題を非線型 Schrödinger 方程式等の方程式を対象として peak をもつ解の構成を行った．非線型 Schrödinger では今まで扱われていなかった漸近的に線型となる非線型項を含む広いクラスの非線型項に対して，ポテンシャルの極小値に集中してゆく peak をもつ解の構成に成功した（論文 [9, 10, 12]）．また Bartsch-Pankov-Wang の設定の下で複数の bump をもつ解の構成 (論文 [11]), 非常に一般的な非線型項をもつ Schrödinger 方程式の全域解の構成 (論文 [7]) を行った．空間次元が 1 の場合, broken geodesic argument により, Allen-Cahn 方程式の遷移層をもつ解の構成を elementary な方法により行った (論文 [13]). また broken geodesic argument を用いて Allen-Cahn, Fisher, 非線型 Schrödinger 方程式に対して高振動解の特徴付けおよび構成を行った (論文 [2, 3, 4, 6]) また disrupted environment における生物モデルに対して遷移層をもつ安定解を見いだした (論文 [5]).

(ii) ハミルトン系については, 2 体問題をモデルとした特異性をもつハミルトン系に対して周期軌道の存在を議論した. 論文 [8] においては特異集合が体積をもつ場合に weak force であっても周期軌道が存在することを示した．このような状況は従来 strong force の下でのみ議論されている．また論文 [1] においては strong force をもつ 2 体問題をモデルとした 1 階のハミルトン系に対して prescribed energy 問題を考えその存在を Hofer-Viterbo の方法を援用することに示した.

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研究成果報告

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研究成果の概要

aperture(通路) により連結された上・下半空間を占める非圧縮粘性流体の運動を考える. 空間次元 $n \geq 3$ とし, この領域 –aperture domain– における Navier-Stokes 方程式の初期値問題を考察する. aperture domain は, 問題が線型であっても, 通常の付帯条件だけのもとでは解の一意性が成立しない場合が起こりうる興味深い領域である. その場合, 一意解を得るには, 付加境界条件として aperture を通る流量あるいは上・下半空間の無限遠での圧力差を指定しなくてはならない. 論文 [1] では, 流量がゼロの場合に, L_n-ノルムが小さい初期関数に対する強解の時間大域的存在と漸近挙動を調べた.

次に, 回転する 3 次元物体の外部領域を占める流体の運動を考察する. ただし, 回転角速度 ω は定数ベクトルとする. この問題の解析においては, 偏微分作用素 $L = -\Delta - (\omega \wedge x) \cdot \nabla + \omega \wedge$ をよく調べることが大切である. 作用素 L の基本解 $\Gamma(x, y)$ は, 各点評価 $|\Gamma(x, y)| \leq c/|x - y|$ を許さず, ラプラス作用素からの摂動として扱えない. 論文 [2] では, 全空間 $\mathbb{R}^3$ における方程式 $Lu = f$ に対する L_q 評価 $\|\nabla^2 u\|_q \leq C\|f\|_q$ を示した $(1 < q < \infty)$. 圧力勾配 ∇p と div u がある場合も扱える.

論文 [3] では, 同じ作用素 L に対して, 全空間 $\mathbb{R}^3$ における方程式 $Lu + \nabla p = f, \mathrm{div}\, u = 0$ の弱解の L_q 評価 $\|\nabla u\|_q + \|p\|_q \leq C\|f\|_{-1,q}$ を示した $(1 < q < \infty)$. 適当な関数 g に対して, div $u = g$ を伴う場合も扱える. また, この結果を用いて, 外部領域における線型定常問題の弱解の一意存在と L_q 評価を, $3/2 < q < 3$ に対して証明した. 指数 q に対するこの条件は避けられず, 特に非線型問題を解く際に必要な $q = 3/2$ の場合は成り立たない.

そこで, 論文 [5] では, 同じ外部問題に対する弱 $L_{3/2}$ 評価

$$\|\nabla u\|_{3/2,\infty} + \|p\|_{3/2,\infty} \leq C\|f\|_{-1,3/2,\infty}$$

をそのようなクラスでの弱解の一意存在とともに証明した. また, これを用いて, 外部領域での Navier-Stokes 方程式の境界値問題のクラス $(\nabla u, p) \in L_{3/2,\infty}$ における一意解の存在を, 回転角速度 ω および外力が小さい場合に示した. このクラスの定常解は, その安定性が期待される.

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研究成果報告

研究分担者　廣瀬　宗光（明治大学・理工学部・講師）

研究成果の概要

Xuefeng Wang は論文 " On the Cauchy problem for reaction-diffusion equations " において，反応拡散方程式の非線形項に重みをつけた方程式

$$u_t = \Delta\ u + |x|^{\ell} u^p$$

の初期値問題に対して，大域解の存在や時間無限大での解の振る舞いを調べたが，未解決な部分も残されていた．これに完全な答えを与えるために，Haraux-Weissler 型方程式の解構造を調べた際に用いた方法が適用できると判断し，この問題に取り組むことにした．すなわち，上記の方程式の自己相似解を考え，これが満たす半線形楕円型方程式について，無限遠方で代数関数オーダーでゆっくりと減衰する正値解の存在を示す事が目標となる．前述の論文では，初期値が非常に小さく押さえられているとき，$l > -2$, $p \geqq (n+2+2\ell)/(n-2)$ のもとで，初期値問題の大域解の存在が示され，また，時間無限大での解のオーダーが与えられている．これに対し，私の研究により $-2 < \ell < 0$, $1+(2+\ell)/n < p < (n+2+2\ell)/(n-2)$ のもとで同様の結果が得られることがわかった．さらに ℓ が正の場合についても考察を進めており，例えば $n=3$ のとき，$0 \leqq \ell < 2/3$ まで望むべき結果に到達している．

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研究成果報告

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研究成果の概要

I. 双安定型方程式にあらわれる定常遷移層の位置と安定性について

双安定型方程式の１次元の定常問題を考える．拡散係数を微小とすると定常解は遷移層を形成する．遷移層は折り重なって現れることもある．本研究ではこのような折り重なった遷移層はどこに現れるかを調べ，またそれらの定常解のモース指数は遷移層の数と位置により完全に決定されることを示す（浦野氏、山田氏 (早稲田大) との共同研究）．

II. 競争系の遷移層の形成と運動について

競争系は数理生態学にあらわれるモデルで，同じ領域内に生息する２種の生物の個体数密度を記述するものである．２種の生物の競争が比較的激しい場合に系は双安定となる．この系においてある係数を０あるいは無限大とした特異極限をみつけ、解の運動がこの特異極限より近似されることを数学的に厳密に示すのが本研究の目的である．

１．拡散係数微小の場合

拡散係数を小さくした場合には解はきわめて短時間のあいだに遷移層を形成する．いったん形成された遷移層は次の運動をはじめるが，この運動は, 界面の平均曲率と移流の項の和で表される，ある界面方程式に支配されることを厳密に示す. (カラリ氏 (トロント大)，ヒルホスト氏 (パリ南大)，俣野氏 (東大) らとの共同研究).

２．競争係数無限大の場合

競争の非常に激しい状況下における２種の生物の境界面の形成とその挙動についても研究を進めている．競争系において競争係数を非常に大きくすると，２種の生物のすみわけの境界が現れる．拡散係数を微小にしたときとは異なり，この境界の近傍において解は限りなく角に近い形状をもつ．本研究では，この角の形成とその後の挙動を扱う．角は非常に短時間内に形成された後，その挙動はある自由境界問題に支配されるが，この一連の挙動を数学的に厳密に証明する (若狭氏 (早稲田大) との共同研究, 飯田氏 (岩手大)，カラリ氏 (トロント大)，三村氏 (明治大)，柳田氏 (東北大) との共同研究).

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研究成果報告

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研究成果の概要

大きく分けて次の二つに関する研究を行った．

（1）退化拡散項を伴う複素ギンツブルグ・ランダウ方程式淺川氏（岐阜大）と横田氏（東京理科大）との共同研究である．横田氏と岡沢氏（東京理科大）によって既に得られていた条件よりも弱い条件において大域解の存在や大域アトラクターの存在が証明された．特に，解の一意性が保証されない場合は解作用素が多価作用素となるが、この場合にも（拡張された）大域アトラクターの存在を証明することが出来た．大域解と大域アトラクターの存在を保証するために複素係数に課した条件は最良と思われ，この条件が成り立たない場合には爆発解が存在すると予想されるが，証明には至っていない．

（2）非一様な環境収容力を持つ退化拡散ロジスティック方程式 p ラプラシアンを拡散項として持つロジスティック方程式の定常問題において，環境収容力が空間非一様である場合を考えた．環境収容力が空間一様である場合は，拡散項の係数がある程度小さいならば解が環境収容力と一致し，さらに小さくするにつれてその一致集合は領域全体に広がっていくことが知られている．本研究では，非一様な場合でも，拡散項の係数が小さくなるにつれて解は環境収容力にコンパクト一様収束していくこと，および環境収容力が一定であるような領域があればそこでは一致集合が存在することが示された．この問題は退化放物型方程式の定常解集合の位相的な性質を知る上で大きな手がかりとなる．

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研究成果報告

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研究成果の概要

「相互拡散 (cross-diffusion)」と呼ばれる非線形項をもつ連立反応拡散方程式は，微分方程式論の従来的技法が直接的には通用しないケースも多く，更なる理論研究が待たれる状況にある．その中で平成 15 年度においては，早稲田大学・山田義雄教授とともに「相互拡散項をもつ数理生態学モデル（ロトカ・ボルテラ系）」の研究に従事した．具体的成果として，相互拡散効果によって，方程式系の正値定常解が複数個存在することを数学的に証明した．このような解の多重性は，定常解集合のなす大域分岐枝が「S 字型」等に変形することによるもので，係数パラメーターの値に応じて正値定常解は 3 個存在することを意味する．正値定常解の多重構造は，同じ反応拡散方程式系で相互拡散項を外した場合では起こらことから，相互拡散による非線型メカニズムの多様な一面が抽出されたことになる．さらに，線形拡散係数の比率に応じて，上記の「S 字型大域分岐枝」上でホップ分岐が起こることも明らかにした．

平成 16 年度以降においては，被食生物系の数理生態学モデルにおいて「被食生物の多い場所では捕食生物の拡散が鈍化する状況」をモデル化した分数型の非線形拡散系を解析した．成果として，正値定常解のなす集合の大域分岐構造が，拡散の非線形効果が非常に大きいとき，ふたつの「極限系」の解からの摂動で特徴付けられることを証明した．この結果により，正値定常解は，拡散の非線形性の増大に伴い，ふたつのタイプに分類されることが判明した．

最終年度においては，宇部高専・大崎浩一助教授とともに，走化性粘菌の空間分布を記述する反応拡散移流系（三村・辻川モデル）の定常問題の研究に従事した．成果として，移流項が増大すると，ストライプ状や正六角形状のパターン解が定数解から分岐することを証明した．この結果により，大阪大学・八木厚志教授グループによる数値実験結果が、理論的に裏付けられた．さらに，パターン解集合のなす分岐枝の安定性解析を行った．

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研究成果報告

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研究成果の概要

非有界領域における非線形楕円型方程式の解構造を解明する事を究極的な目標として変分法的アプローチを中心に解析を進めてきた．特に対象とする方程式には無限遠方で発散するような係数関数を含んでおり，このような方程式を通常のソボレフ空間を用いて解析を行おうとすると，その係数関数の非有界性から既存の議論，解析方法が破綻してしまう，と言う欠点がある．ここで方程式を解析するにあたり，まず方程式のエネルギー汎関数から解析を行うのに自然な関数空間を準備し，その性質を詳しく調べる事を研究対象とした．ここで扱う関数空間は指数増大する係数を持つ重み付きソボレフ空間である．このような関数空間をもとに非線形楕円型方程式の解の存在を述べるため，この関数空間の埋め込みの性質と係数関数の無限遠方での増大度との関係を調べた．このようなタイプの関数空間に対する埋め込みの議論は，先行する研究としては Escobedo-Kavian の研究があげられるが，論文 [1] においてより単純な議論を用いて彼らの結果の拡張を行うことに成功した．

また，上記の楕円型方程式において，臨界指数を持つ方程式に対する解の存在についても議論を進めてきた．臨界指数を含む楕円型方程式の解析においては，Lions による "Concentration-Compactness Argument" が最も有効的であり，このような考え方と Talenti により導出された特殊解を用いた汎関数のエネルギー解析を組み合わせる事により解の存在を導出することが一般的である．論文 [2] において，非有界な係数を含む場合における Concentration-Compactness Argument を新しく構築することに成功した．これを用いる事により，非有界な係数を持ち，かつ臨界指数を含むような楕円型方程式の解の存在に関する議論を進めることが出来た．

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