無限語について

Infinitary Words

Applications to Fundamental Groups

江田 勝哉
早稲田大学理工学術院

INFINITARY WORDS

KATSUYA EDA

1. Words

In this section we define infinitary words. In combinatorial group theory we adopt generators as letters. On the other hand, when we treat with free products, we adopt elements of groups as letters. Therefore, there are two different treatments about free groups. Here we define the both notions and make some distinctions.

First we define words over a set D. For each $d \in D$ we formally define d^+ and d^- and call d^+ and d^- *letters*. Here we identify d^+ and $(d^-)^-$ with d and so usually use d. A *word* $W \in \mathcal{W}(D)$ is a map from a linearly ordered set $\overline{W}$ to the set of letters $\{d^+ : d \in D\} \cup \{d^- : d \in D\}$ such that $\{\alpha \in \overline{W} : W(\alpha) = d^\pm\}$ is finite for each $d \in D$. We define W^- by: $\overline{W^-}$ is the reversed ordering of $\overline{W}$ and $W^-(\alpha) = W(\alpha)^-$.

Next let G_i $(i \in I)$ be groups. We assume $G_i \cap G_j = \{e\}$ for distinct $i, j \in I$. An element of $\bigcup\{G_i \setminus \{e\} : i \in I\}$ is called a *letter*. A *word* $W \in \mathcal{W}(G_i : i \in I)$ is a map from a linearly ordered set $\overline{W}$ to the set letters $\bigcup\{G_i \setminus \{e\} : i \in I\}$ such that $\{\alpha \in \overline{W} : W(\alpha) \in G_i\}$ is finite for each $i \in I$.

To make a distinction well we say a word in $\mathcal{W}(D)$ or a word in $\mathcal{W}(G_i : i \in I)$, when a confusion possibly occurs. In the both case the empty map is the empty word. The identification of words is given by the following, For words U and V, $U \equiv V$ means the existence of an order isomorphism $\varphi : \overline{U} \to \overline{V}$ such that $U(\alpha) = V(\varphi(\alpha))$ for each $\alpha \in \overline{U}$.

For two words U and V the concatenation UV is the word such that $\overline{UV}$ is the disjoint union of $\overline{U}$ and $\overline{V}$ and $UV(\alpha) = U(\alpha)$ if $\alpha \in \overline{U}$ and $UV(\alpha) = V(\alpha)$ otherwise. We remark that we may always suppose that $\overline{U}$ and $\overline{V}$ are disjoint according to the identification $\equiv$. Since $(UV)W \equiv U(VW)$, we may write UVW. Moreover we can define an infinite concatenation. Let U_α $(\alpha \in L)$ be words in $\mathcal{W}(D)$ for a linearly ordered set L such that for each $d \in D$, $\bigcup\{\alpha : U_\alpha(\beta) = d^\pm$ for some $\beta \in$

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$\overline{U_\alpha}\}$ is finite. Then the infinite concatenation $\Pi_{\alpha\in L}U_\alpha$ is also a word in $\mathcal{W}(D)$, i.e.

(1) $\overline{\Pi_{\alpha\in L}U_\alpha} = \{(\alpha,\beta) : \beta \in \overline{U_\alpha}\}$ where $(\alpha_0,\beta_0) \leq (\alpha_1,\beta_1)$, if $\alpha_0 < \alpha_1$, or $\alpha_0 = \alpha_1$ and $\beta_0 \leq \beta_1$;
(2) $\Pi_{\alpha\in L}U_\alpha(\alpha_0,\beta_0) = U_{\alpha_0}(\beta_0)$ for $(\alpha_0,\beta_0) \in \overline{\Pi_{\alpha\in L}U_\alpha}$.

In case U_α $(\alpha \in L)$ are words in $\mathcal{W}(G_i : i \in I)$ such that for each $i \in I$, $\bigcup\{\alpha : U_\alpha(\beta) \in G_i$ for some $\beta \in \overline{U_\alpha}\}$ is finite, the word $\Pi_{\alpha\in L}U_\alpha$ is defined by the same formula. We remark that in each case the word $\Pi_{\alpha\in L}U_\alpha$ is in $\mathcal{W}(D)$ or $\mathcal{W}^\sigma(G_i : i \in I)$ respectively.

We call a word U a *subword* of a word V, if there exist words such that $V \equiv XVY$.

The set of words whose cardinality is countable is expressed as $\mathcal{W}^\sigma(D)$ or $\mathcal{W}^\sigma(G_i : i \in I)$ respectively, which is related to fundamental groups. The set of words of finite length is expressed as $\mathcal{W}^f(D)$ or $\mathcal{W}^f(G_i : i \in I)$ respectively, which is more usual and familiar.

2. Reduced words and basic results

We define reduced words and shall show that every word corresponds to a unique reduced word. First we define an equivalence relation $\sim$ for finitary words in the usual way, then we regard infinitary words an expression of certain elements of the inverse limit of finitely generated free groups or free products of finitely many factors.

For a definition precise definition of a reduction of a word of finite length, we refer the reader to standard textbooks as [2, 4, 3].

We just explain a reduction process of a word of finite length and state the definition of free groups and free products.

For a word $W \in \mathcal{W}^f(D)$, if there appears a subword of W the form dd^- or d^-d, we delete it. For a word $W \in \mathcal{W}^f(G_i : i \in I)$, if there appears a subword of W of the form uv such that u and v belong to the same group G_i, we replace uv by the element w of G_i with $w = uv$ if $uv \neq e$ and we delete uv otherwise, i.e. $uv = e$. Then in each case, in finite steps we get a word for which we cannot reduce it anymore. That is a reduced word. We remark that for each word there exists a unique reduced word which can be obtained by reduction procedures, but there may be different procedures to obatain it.

Now two words U and V are said to be equivalent, written as $U \sim V$, if the reduced words of U and V are the same. The free group over a set D is defined as the quotient set $\mathcal{W}^f(D)/\sim$ with the concatenation as the multiplication. The free product $*_{i\in I}G_i$ is similarly defined as the quotient set $\mathcal{W}^f(G_i : i \in I)/\sim$ with the concatenation as the multiplication. For a word W, $[W]$ denotes the equivalence class containing W.

To introduce an equivalence to the set of infinitary words, we introduce the unrestricted free product due to Higman [1]. For two finite subsets E, F of an index set I with $E \subseteq F$, let $p_{EF} : *_{i\in F}G_i \to *_{i\in E}G_i$ be the projection. (We write $E \Subset I$, when E is a finite subset of I.) The unrestricted free product is defined as: $\varprojlim(*_{i\in E}G_i, p_{EF} : E \subset F \Subset I)$ [1]. For a given word $W \in \mathcal{W}(G_i : i \in I)$ and $F \Subset I$, let W_F be a word obtained by the restriction of W to $\alpha \in \overline{W}$ such that $W(\alpha) \in \bigcup_{i\in F} G_i$. Then, W_F is a word of finite length and so expresses an element of the free product $*_{i\in F}G_i$. We extend the equivalence $\sim$ on $\mathcal{W}(G_i : i \in I)$ by: $V_F \sim W_F$ for every $F \Subset I$. Since $p_{EF}([W_F]) = [W_E]$ for $E \subseteq F \Subset I$,

Hence $V, W \in \mathcal{W}(G_i : i \in I)$ express elements of $\varprojlim(*_{i\in E}G_i, p_{EF} : E \subset F \Subset I)$ and $V \sim W$ is equivalent to the fact that the expressed element of V and that of W are the same.

In case $\mathcal{W}(D)$, let $\mathbb{Z}_d$ be a copy of $\mathbb{Z}$ for each $d \in D$. Then, we can regard a word in $\mathcal{W}(D)$ as a word in $\mathcal{W}(\mathbb{Z}_d : d \in D)$ and extend the equivalence $\sim$.

Definition 2.1. A word $W \in \mathcal{W}(D)$ is reduced, if any $W \equiv UXV$ implies $[X] \neq e$ for any non-empty word X. A word $W \in \mathcal{W}(G_i : i \in I)$ is reduced, if any $W \equiv UXV$ implies $[X] \neq e$ for any non-empty word X and for any neighboring elements α and β of $\overline{W}$, $W(\alpha)$ and $W(\beta)$ do not belong to the same G_i. A word $W \in \mathcal{W}(G_i : i \in I)$ is quasi-reduced, if $W \simeq UXV$ with $[X] = e$ implies $Im(X) \subset G_i$ and the existence of $e \neq g \in G_i$ for some i such that g is the right most letter of U or the left most letter of V. In other words, W is quasi-reduced, if the reduced word of W is obtained just multiplying contiguous elements belonging to the same group.

The existence of reduced words is very important even in the case of infinitary words. As we can see in the following examples, infinitary words are complicated sometimes. But if we start from a concatenation of finitely many reduced words, we have finite steps of reduction procedure due to [5, Corollary 1.7], which helps to investigate group theoretic properties of the free σ-product, for which we have introduced infinitary words.

The subgroup of the unrestricted free product $\varprojlim(*_{i\in E}G_i, p_{EF} : E \subset F \Subset I)$ consisting all elements expressed by words in $\mathcal{W}(G_i : i \in I)$ is called the free complete product and the countable version, i.e. that consisting of all elements expressed by words in $\mathcal{W}^\sigma(G_i : i \in I)$ is called the free σ-product. Of course, the one obtained by $\mathcal{W}^f(G_i : i \in I)$ is the free product.

Proposition 2.2. [7, Lemma 2.4] *Any non-empty word $W \in \mathcal{W}(D)$ is equivalent to a reduced word $V^- XV$ such that XX is a non-empty reduced word. Any non-empty word $W \in \mathcal{W}(G_i : i \in I)$ is equivalent to a quasi-reduced word $V^- XV$ such that*

(1) *XV is reduced;*
(2) *X is a single letter or XX is reduced.*

For other results we refer the reader to [5, 7].

The first example shows that we have no reduction procedure as in the case of words of finite length, even if we allow infinite steps of reductions.

Example 2.3. A word W which is equivalent to the empty word but does not contain a subword of the form $V^- V$.

Let $Seq(2)$ be the set of all finite sequences $s = \langle s_0, \cdots, s_n \rangle$ of $0, 1$ with the lexicographical ordering and a_n be letters. The length of $s \in Seq(2)$ is denoted by $lh(s)$. Define $\overline{W} = Seq(2) \setminus \{\langle\ \rangle\}$ and $W(s) = a_n$ if $s_{lh(s)} = 0$ and $W(s) = a_n^-$ if $s_{lh(s)} = 1$.

To show that W is equivalent to the empty word, let $F_n = \{a_i : 0 \leq i \leq n\}$. Then, in W_{F_n} a_n and a_n^- are contigous and after deleting such pairs we have $W_{F_{n-1}}$ and so on we conclude that W_{F_n} is equivalent to the empty word.

The reason of the non-existence of a subword of W of the form $V^- V$ follows from the fact that in W every letter has an immediate successor, but has no immediate predecessor.

The next example shows that we cannot reduce the notion of the reducedness of W to those of W_F s for $F \Subset I$ or $F \Subset D$.

Example 2.4. A reduced word W such that W_F is not reduced for any finite set $F \Subset \{a_n : n < \omega\}$.

Define $\overline{W} = Seq(2) \setminus \{\langle\ \rangle\}$ and $W(s) = a_n a_n$ if $s_{lh(s)} = 0$ and $W(s) = a_n^{-1}$ if $s_{lh(s)} = 1$.

The fact that W_F is not reduced can be seen, if we consider the largest n such that a_n appears in F. Then, there is a subword $a_n a_n a_n^-$ of W_F and hence W_F is not reduced. To see the reducedness of W by contradiction suppose that a subword X of W stisfies $X = e$. Choose n such that a_n or a_n^- appears in X. Since $X = e$, there exists a subword of X of the form $a_n Y a_n^-$ or $a_n^- Y a_n$ such that $Y = e$. In each case the number of appearances of a_n in Y is 2 but that of a_n^- is 1, which contradicts $Y = e$.

I attach three papers [5], [7] and [17]. The first ones are published ones and contain basic results, but also contain many misprints and so here I offer fixed ones. The third one has been submitted some years ago.

3. History of infinitary words

I introduced infinitary words for a presentation of the fundamental group of the Hawaiian earring and related spaces in [5]. Actually I started to study about this topic early in 1980's and results of this paper was obtained around 1985, but I submitted it in 1990. At that time I thought no other person was interested in the fundamental group of the Hawaiian earring, but Morgan and Morrison published a paper of this subject, detecting an error of H. B. Griffiths's paper and correcting it. It seems that their interests to the Hawaiian earring was related to Gromov's research of limit spaces of hyperbolic spaces and this stream started early in 1980's. But I did not know such a stream until 1998. Around the year some people who were interested in geometric group theory, in the above stream, started writing papers about the fundamental group of the Hawaiian earring. (See the references of [16].) Then, Cannon and Conner [10] introduced infinitary words from the point of view of geometric group theory. Particularly, free complete products introduced in [5] was realized as big fundamental groups [11].

My interest to the Hawaiian earring started from its relationship to the Specker phenomenon and so motivations are very different. But after all the both interests are concentrated to fundamental groups of wild spaces and so infinitary words and paths in one dimentional spaces as its generalization are good weapons to attack this topic.

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School of Science and Engineering, Waseda University, Tokyo 169-8555, JAPAN
E-mail address: eda@logic.info.waseda.ac.jp

FREE σ-PRODUCTS AND NONCOMMUTATIVELY SLENDER GROUPS

KATSUYA EDA

An infinitary version of the notion of free products has been introduced and investigated by G. Higman [11]. Let $G_i (i \in I)$ be groups and $*_{i\in X} G_i$ the free product of $G_i (i \in X)$ for $X \subset I$ and $p_{XY} : *_{i\in Y} G_i \to *_{i\in X} G_i$ the canonical homomorphism for $X \subset Y \subset I$. Then, the unrestricted free product is the inverse limit $\lim_{\leftarrow}(*_{i\in X} G_i, p_{XY} : X \subset Y \subset\subset I)$, where $Y \subset\subset I$ means that Y is a finite subset of I. In the present paper we introduce a similar one to the unrestricted free product, which is a subgroup of the unrestricted free product and equal to the subgroup P in [11, Section 6] if $G_i \simeq \mathbb{Z}$ and I is countable. There were also related investigations due to H. B. Griffiths [8, 9]. Free products are defined using words of finite length. Our infinitary version of free products will be defined using words of infinite length instead of finite one. The group $\circledast_{i\in I} G_i$ is called a free complete product and is isomorphic to a subgroup of the unrestricted free product, that is, $\bigcap_{F\subset\subset I}\{*_{i\in F} G_i * \lim_{\leftarrow}(*_{i\in X} G_i, p_{XY} : X \subset Y \subset\subset I \backslash F)\}$. Our interest will be concentrated to free σ-products, which are defined using words of countable length and a subgroup of the free complete product. One reason to do so is that free σ-products are naturally related to fundamental groups of certain spaces [9], as we shall explain and state applications in the appendix. Another reason is that these behave well concerning noncommutatively slender groups, which will be defined later, but we have not found a slender property concerning free complete products.

In Section 1 we shall define free complete products and free σ-products and state some preliminary results. In Section 2 we shall prove a noncommutative version of Chase's lemma, that is, a theorem about homomorphisms from free σ-products to free products of infinite components. In Section 3 we shall introduce a new notion "noncommutatively slender groups" and investigate it. We remark that this notion is strictly stronger than that of slender groups in the sense of [7]. In Section 4 we shall investigate the abelianizations of free σ-products and

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related ones. In the appendix we shall explain the relationship with algebraic topology.

First, we state basic notations. For a subset X of a group G, $\langle X\rangle$ is the subgroup generated by X. The direct product $\Pi_{i\in I}G_i$ is the group consisting of all functions x from the index set I such that $x(i) \in G_i (i \in I)$. The restricted direct product $\Pi^r_{i\in I}G_i$ is the subgroup of $\Pi_{i\in I}G_i$ consisting of all x such that $\{i : x(i) \neq e\}$ is finite. (The symbol "e" is always used for the identity of a group in question. We use "0" instead of "e" for an abelian group as usual.) The σ-product $\Pi^\sigma_{i\in I}G_i$ is the subgroup of $\Pi_{i\in I}G_i$ consisting of all x such that $\{i : x(i) \neq e\}$ is countable. In case $G_i (i \in I)$ are isomorphic to a group G, $\Pi_{i\in I}G_i$ is denoted by $\Pi_I G$. The group of rational integers is denoted by $\mathbb{Z}$ and the set of natural numbers is denoted by N.

1. Free complete products and free σ-products

First we introduce words of infinite length.

Definition 1.1. Let $G_i (i \in I)$ be groups. We assume $G_i \cap G_j = \{e\}$ for distinct $i, j \in I$. Elements of $\bigcup_{i\in I} G_i$ are called letters. W is a word, if W is a function from a linearly ordered set $\overline{W}$ to $\bigcup_{i\in I} G_i$ such that $W^{-1}(G_i)$ is finite for each i. In case the cardinality of $\overline{W}$ is countable, we say that W is a σ-word. The class of all words is denoted by $\mathcal{W}(G_i : i \in I)$ (abbreviated by $\mathcal{W}$ and the class of all σ-words is denoted by $\mathcal{W}^\sigma(G_i : i \in I)$ (abbreviated by $\mathcal{W}^\sigma$).

If there exists an isomorphism $i : \overline{U} \to \overline{V}$ as linearly ordered sets and $U(\alpha) = V(i(\alpha))$ for all $\alpha \in \overline{U}$, we say that U and V are isomorphic and denote it by $U \simeq V$. In this case we identify U and V. Since the cardinality of $\overline{W}$ is less than or equal to Max$\{| I |, \aleph_0\}$ for a word W, $\mathcal{W}$ becomes a set under this identification. For words of finite length, this is the same as the usual definition. For the definition of free products we refer the reader to [10] or [13]. For a word $W \in \mathcal{W}(G_i : i \in I)$ and a subset $X \subset I$, W_X is the word obtained by eliminating letters not in $\bigcup_{i\in X} G_i$, that is, $W_X \in \mathcal{W}(G_i : i \in X)$, $\overline{W_X} = \{\alpha \in \overline{W} : W(\alpha) \in \bigcup_{i\in X} G_i\}$ and $W_X(\alpha) = W(\alpha)$ for $\alpha \in \overline{W_X}$. For words U and V, we say that $U \sim V$ holds if $U_F = V_F$ for every $F \subset\subset I$, where we regard U_F, V_F as elements of the free product $*_{i\in F}G_i$. Then, $\sim$ is an equivalence relation on $\mathcal{W}$ clearly. Denote the equivalence class containing U by $[U]$. For $U, V \in \mathcal{W}$, let UV be the composition of U and V, that is, $\overline{UV} = \{(0,\alpha), (1,\beta) : \alpha \in \overline{U}, \beta \in \overline{V}\}$ where $(0,\alpha) < (1,\beta)$ for $\alpha \in \overline{U}$ and $\beta \in \overline{V}$ and $(i,\alpha) < (i,\beta)$ for $\alpha < \beta$ and $i = 0, 1$; $UV((0,\alpha)) = U(\alpha)$ and $UV((1,\beta)) = V(\beta)$. Let U^{-1} be the word such that $\overline{U^{-1}} = \{(0,\alpha) : \alpha \in \overline{U}\}$ where $(0,\alpha) < (0,\beta)$ if $\alpha > \beta$ and $U^{-1}((0,\alpha)) = U(\alpha)^{-1}$. Then, $\mathcal{W}/\sim = \{[W] : W \in \mathcal{W}\}$ clearly becomes a group with its operation $[U][V] = [UV]$. We define U^0 as the empty word, $U^{n+1} = U^n U$ and $U^{-n-1} = U^{-n}U^{-1}$ for $n \in N$.

Definition 1.2. The free complete product $\times_{i\in I} G_i$ is the group $\mathcal{W}(G_i : i \in I)/\sim$. The free σ-product $\times^{\sigma}_{i\in I} G_i$ is the group $\mathcal{W}^{\sigma}(G_i : i \in I)/\sim$, which is a subgroup of $\times_{i\in I} G_i$. In case every G_i is isomorphic to G, we abbreviate $\times_{i\in I} G_i$ by $\times_I G$ and similarly for free σ-products.

Restricting the length of words to be finite, we get the free product $*_{i\in I} G_i$. Obviously, $\times_{i\in I} G_i$ and $\times^{\sigma}_{i\in I} G_i$ are isomorphic to $*_{i\in I} G_i$, if I is finite. We define reduced words and shall show that every word corresponds to a unique reduced word.

Definition 1.3. A word W is reduced, if $W \simeq UXV$ implies $[X] \neq e$ for any non-empty word X, where e is the identity, and for any neighboring elements α and β of $\overline{W}$ it never occurs that $W(\alpha)$ and $W(\beta)$ belong to the same G_i. A word W is quasi-reduced, if $W \simeq UXV$ with $[X] = e$ implies $Im(X) \subset G_i$ and the existence of $e \neq g \in G_i$ for some i such that g is the right most letter of U or the left most letter of V.

In other words, W is quasi-reduced if a reduced word is obtained by multiplying all neighboring letters which belong to the same G_i.

Theorem 1.4. *For any word W, there exists a reduced word V such that $[W] = [V]$ and V is unique up to isomorphism.*

Proof. We define words W_μ for ordinals μ by induction. Let W_0 be W. If there exists a non-empty word X such that W_μ is isomorphic to UXV and $[X] = e$, let $\overline{W_{\mu+1}} = \{\alpha \in \overline{W_\mu} : i(\alpha) \in \overline{U} \text{ or } i(\alpha) \in \overline{V}\} \subset \overline{W}$ and $W_{\mu+1}(\alpha) = W(\alpha)$ for $\alpha \in \overline{W_{\mu+1}}$, where the ordering is the restriction of that of $\overline{W}$ and $i : \overline{W_\mu} \to \overline{UXV}$ is the order isomorphism. Otherwise, the procedure is completed. For a limit ordinal μ, let $\overline{W_\mu} = \bigcap_{\nu<\mu} \overline{W_\nu}$ and $W_\mu(\alpha) = W(\alpha)$ for $\alpha \in \overline{W_\mu}$. This procedure must stop at some ordinal whose cardinality is at most $\mathrm{Max}\{\mid I \mid, \aleph_0\}$ because the cardinality of $\overline{W}$ is equal to or less than $\mathrm{Max}\{\mid I \mid, \aleph_0\}$. Let W_∞ be the obtained word. By induction we can see that $[W_\mu] = [W]$ and hence $[W_\infty] = [W]$. There may be neighboring $\alpha, \beta \in \overline{W_\infty}$ such that $W_\infty(\alpha)$ and $W_\infty(\beta)$ belong the same G_i. Since such occasions happen only finitely many times for each i, performing the calculation in each G_i we get the desired reduced word of W. Next, suppose that $[U] = [V]$ for reduced words U and V. We define $\varphi : \overline{U} \to \overline{V}$ in the following manner. For $\alpha \in \overline{U}$ there exists a unique $i \in I$ such that $U(\alpha) \in G_i$. Then, there exist $i \in E \subset\subset I$, letters $g_1, \cdots, g_m \in G_i$ and $X_1, \cdots, X_{m+1} \in \mathcal{W}(G_j : i \neq j \in I)$ such that $U \simeq X_1 g_1 X_2 \cdots X_m g_m X_{m+1}$, α corresponds to g_k, $U_E \simeq (X_1)_E g_1 (X_2)_E \cdots (X_m)_E g_m (X_{m+1})_E$ and $[(X_2)_E], \cdots, [(X_m)_E] \neq e$. On the other hand there exist $E \subset F \subset\subset I$, letters $g'_1, \cdots, g'_n \in G_i$ and $Y_1, \cdots, Y_{n+1} \in \mathcal{W}(G_j : i \neq j \in I)$ such that $V \simeq Y_1 g'_1 Y_2 \cdots Y_n g'_n Y_{n+1}$, $V_F \simeq (Y_1)_F g'_1 (Y_2)_F \cdots (Y_n)_F g'_n (Y_{n+1})_F$ and $[(Y_2)_F], \cdots, [(Y_n)_F] \neq e$.

Since $[U_F] = [V_F]$ by definition, $m = n$ and $g(l) = g'(l)$ for $1 \leq l \leq m$. Let $\varphi(\alpha) \in \overline{V}$ be the member corresponding to g'_k in V. Clearly φ is a 1-1 onto map and $U(\alpha) = V(\varphi(\alpha))$. Taking large enough $F \subset\subset I$ as the above, we can see that φ preserves the order. Therefore, U and V are isomorphic. □

From now on we regard a word as an element of $\times_{i \in I} G_i$ in case no confusion will occur. Hence, $U = V$ means $[U] = [V]$ for words U and V.

Corollary 1.5. *Let U and V be reduced words. If $UV = e$, then V is isomorphic to U^{-1}.*

Corollary 1.6. *Let U be a reduced word. There exists no nonempty reduced word X such that $U = UX$ or $U = XU$. If U is nonempty and $U = U^{-1}$, then there exist a reduced word X and a letter g such that U is isomorphic to $X^{-1}gX$ and $g^2 = e$.*

Proof. The first proposition is clear. Since U^{-1} is also reduced, $U = U^{-1}$ implies $U \simeq U^{-1}$ and hence let $i : \overline{U} \to \overline{U^{-1}}$ be the order isomorphism. Under the notation before Definition 1.2, let $\overline{X}$ be the maximal subset of $\overline{U}$ such that $\alpha > \beta \in \overline{X}$ implies $\alpha \in \overline{X}$ and $i^{-1}(0, \alpha) \notin \overline{X}$ for any $\alpha \in \overline{X}$, and let $X(\alpha) = U(\alpha)$ for $\alpha \in \overline{X}$. If $\overline{X} \cup i^{-1}\{(0, \alpha) : \alpha \in \overline{X}\} = \overline{U}$ then $U = e$. Hence $U \simeq X^{-1}gX$ for some letter $g \neq e$ by the maximality of $\overline{X}$. Then, $g^2 = e$ by $U \simeq U^{-1}$. □

Considering the reduction in the proof of Theorem 1.4, we get,

Corollary 1.7. *Let U and V be reduced words. Then, there exist reduced words X, Y, Z such that $U \simeq XY$, $V \simeq Y^{-1}Z$ and XZ is quasi-reduced.*

Next we show another presentation of $\times_{i \in I} G_i$ as a subgroup of an inverse limit.

Proposition 1.8. *The free complete product $\times_{i \in I} G_i$ is isomorphic to $\bigcap_{F \subset\subset I} *_{i \in F} G_i * \lim_{\leftarrow}(*_{i \in X} G_i, p_{XY} : X \subset Y \subset\subset I \backslash F)$, which is a subgroup of $\lim_{\leftarrow}(*_{i \in X} G_i, p_{XY} : X \subset Y \subset\subset I)$.*

Proof. Define $\varphi_X : \times_{i\in I} G_i \to *_{i\in X} G_i$ for $X \subset\subset I$ as $\varphi_X(W) = W_X$ for a word W. Then, φ_X is a homomorphism by definition and $p_{XY} \cdot \varphi_Y = \varphi_X$ for $X \subset Y \subset\subset I$. Let $\varphi : \times_{i\in I} G_i \to \lim_{\leftarrow}(*_{i\in X} G_i, p_{XY} : X \subset Y \subset\subset I)$ be the homomorphism induced by φ_X $(X \subset\subset I)$. Then, φ is clearly injective . Let $x \in \bigcap_{F \subset\subset I} *_{i\in F} G_i * \lim_{\leftarrow}(*_{i\in X} G_i, p_{XY} : X \subset Y \subset\subset I \backslash F)$. For each $i \in I$, let W_i be the reduced word corresponding to x as a member of $G_i * \lim_{\leftarrow}(*_{j\in X} G_j, p_{XY} : X \subset Y \subset\subset I \backslash \{i\})$. Let $g(i,1), \cdots, g(i,k_i)$ be the sequence of letters in G_i appearing in W_i in this order. Now let $\overline{W} = \{(i,1), \cdots, (i,k_i) : i \in I\}$. Consider the reduced word $W_{i,j}$ corresponding to x as a member of $G_i * G_j * \lim_{\leftarrow}(*_{k\in X} G_k, p_{XY} : X \subset Y \subset\subset I \backslash \{i,j\})$, then we can see that $g(i,1), \cdots, g(i,k_i)$ and $g(j,1), \cdots, g(i,k_j)$ are appearing in $W_{i,j}$. Define $(i,p) < (j,q)$ if $g(i,p)$ is left of $g(j,q)$ in the word $W_{i,j}$. Then, it is easy to see that this is a linear ordering on $\overline{W}$. Let $W(i,p) = g(i,p)$, then $W \in \mathcal{W}(G_i : i \in I)$ and $\varphi(W) = x$.

For $x \in \times_{i\in I} G_i$, the i-length of x (say $l_i(x)$) is the cardinality of $\{\alpha \in \overline{W} : W(\alpha) \in G_i\}$, where W is the reduced word of x. $\square$

$\times_{i\in I} G_i$ and $\times^{\sigma}_{i\in I} G_i$ naturally admits infinite operations for certain sequences as $\Pi_{i\in I} G_i$. Namely,

Proposition 1.9. *Let $g_\lambda (\lambda \in \Lambda)$ be elements of $\times_{i\in I} G_i$ such that $\{\lambda \in \Lambda : l_i(g_\lambda) \neq e\}$ are finite for all $i \in I$ and denote the element corresponding 1 of the λ-th component of $\times_\Lambda \mathbb{Z}$ by δ_λ. Then, there exists a natural homomorphism $\varphi : \times_\Lambda \mathbb{Z} \to \times_{i\in I} G_i$ such that $\varphi(\delta_\lambda) = g_\lambda (\lambda \in \Lambda)$. Consequently, in case $\Lambda = N$ and $g_n \in \times^{\sigma}_{i\in I} G_i (n \in N)$, we get $\varphi : \times_N \mathbb{Z} \to \times^{\sigma}_{i\in I} G_i$ so that $\varphi(\delta_n) = g_n (n \in N)$.*

Proof. Let W_λ be the reduced word of g_λ for $\lambda \in \Lambda$. For $W \in \mathcal{W}(\mathbb{Z}, \lambda \in \Lambda)$, let $\overline{W^*} = \{(\alpha, \beta) : \alpha \in \overline{W}, \beta \in \overline{W^a_\lambda}$ where $W(\alpha) = a\delta_\lambda$ for $a \in \mathbb{Z}\}$ and $(\alpha, \beta) < (\alpha', \beta')$ if and only if $\alpha < \alpha'$, or $\alpha = \alpha'$ and $\beta < \beta'$. And let $W^*((\alpha,\beta)) = W^a_\lambda(\beta)$, where $W(\alpha) = a\delta$. Finally let $\varphi(W) = W^*$. It is easy to check by Corollary 1.5 that φ is the desired homomorphism. $\square$

2. A NONCOMMUTATIVE VERSION OF CHASE'S LEMMA

Roughly speaking, Chase's lemma [1] says that any homomorphism from an infinite direct product to an infinite direct sum maps a large part to a small part. More precisely, let $h : \Pi_{n\in N} A_n \to \bigoplus_{j\in J} B_j$ be a homomorphism for abelian groups $A_n (n \in N)$ and $B_j (j \in J)$. Then, there exist $k, m \in N$ and a finite subset F of J such that

$h(m \cdot \Pi_{n\geq k} A_n) \leq \bigoplus_{j\in F} B_j + U(\bigoplus_{j\in J} B_j)$, where $U(X)$ is the Ulm subgroup of X, that is, $\bigcap_{n\in N} nX$. We prove the following,

Theorem 2.1. *(A noncommutative version of Chase's lemma)*
Let $h : \times^{\sigma}_{i\in I} G_i \to *_{j\in J} H_j$ *be a homomorphism for groups* $G_i (i \in I)$ *and* $H_j (j \in J)$. *Then, there exist* $E \subset\subset I$ *and* $F \subset\subset J$ *such that* $h(\times^{\sigma}_{i\in I\setminus E} G_i) \leq *_{j\in F} H_j$.

To show this some notion and lemmas are necessary. For $g \in *_{j\in J} H_j$, $l(g)$ denotes the length of the reduced word corresponding to g. Let $W \simeq Xg$, where W, X are words and g is a letter. We say that g is stable in W, if the reduced word corresponding to Xg is of form Ug. Similarly for $W \simeq gX$.

Lemma 2.2. *Let* $H_j (j \in J)$ *be groups and* U *and* X *be reduced words. If the left most letter* g *or the right most one* g^{-1} *in* XUX^{-1} *is not stable in* XUX^{-1}, *then* $l(XUX^{-1}) \leq l(U) + 1$.

Proof. It is enough to deal with the case that g^{-1} is not stable. Let V be the reduced word of XU. Then, $l(V) \leq l(X) + l(U)$. Since the right most letter g^{-1} in VX^{-1} is not stable, $l(VX^{-1}) \leq l(V) - l(X^{-1}) + 1$. Therefore, $l(XUX^{-1}) \leq l(U) + 1$. □

Lemma 2.3. *Let* $H_j (j \in J)$ *be groups. Let* $m+n+2 \leq k$ *for* $m, n, k \in N$ *and* $u, x_i, z \in *_{j\in J} H_j (1 \leq i \leq M)$. *If* $l(u) \leq m, u = x_1 z^k \cdots x_M z^k$ *and* $l(x_i) \leq n$ *for all* $1 \leq i \leq M$, *then* z *is a conjugate of a member of some* H_j *or* $z = x^{-1} f x y^{-1} g y$ *for some* $f \in H_j$ *and* $g \in H_{j'}$ *with* $f^2 = g^2 = e$.

Proof. It is easy to see that there exist reduced words U and W such that $z = W^{-1}UW$ and the both words UU and $W^{-1}UW$ are quasi-reduced or $UU = e$. If $l(U) \geq 2$, we can take the above U and W so that UU is reduced. If $l(U) \leq 1$, the proof is done. Hence, we assume $l(U) \geq 2$ and so also assume that UU is reduced. Let X_i be the reduced word of x_i for each $1 \leq i \leq M$. Then, $x_1 z_1 \cdots x_M z_M = X_1 W^{-1} U^k W X_2 W^{-1} \cdots W X_M W^{-1} U^k W$. Suppose that the left most letter g and the right most one g^{-1} of WX_iW^{-1} are stable in WX_iW^{-1}. Then, the reduced word of $U^2 W X_i W^{-1} U^2$ is of form UY_iU. On the other hand, if at least one of g and g^{-1} is not stable, then $l(WX_iW^{-1}) \leq l(X_i) + 1$ by Lemma 2.2. Let Z_i be the reduced word of WX_iW^{-1}. Let p be the least number so that $2p \geq n + 1$. Then, the reduced word of $Z_i U^{p+1}$ is of form $Y_i U$, where $Y_i \simeq Y'_i V_i$ and $l(Y'_i) \leq l(Z_i)$ and $U^p \simeq W_i V_i$ for some W_i. Hence, the reduced word of $U^{p+2} Y_i$ is of form UZ'_i. Suppose that the reduced word of $UZ'_i U^2$ is of form $UZ''_i U$. Since $l(U^k) \geq l(U) + 2(k-1)$, the reduced form of $X_1 W^{-1} U^k W X_2 W^{-1} U^k \cdots X_M W^{-1} U^k W$ is of form $P_1 U P_2 U \cdots P_M U^m V$, where $V = UW$. This contradicts $l(u) \leq m$. Therefore, the reduced word of $UZ_i'U^2$ is not of form $UZ_i''U$. Since UU is reduced, not only the right most letter of Y_i is not stable in $U^{p+2}Y_i$, but also Z_i' must be of form U^dS_i for some d and $U \simeq S_iT_i$ for some T_i. By the assumption, S_i must disappear in the reduction of $UU^dS_iU^2$ and hence $S_i \simeq S_i^{-1}$. By Corollary 1.6, S_i is empty or $S_i = x^{-1}fx$ for some $f \in H_j$ with $f^2 = e$. If S_i is empty, then UZ_iU^2 itself is reduced, which is a contradiction. Hence, the latter holds. Since UU is reduced and $l(U) \geq 2$, T_i is not empty. Apply the same reasoning for S_i to T_i, then we get that $T_i = y^{-1}gy$ for some $g \in H_{j'}$ with $g^2 = e$.

Lemma 2.4. *Let $h : \times_N\mathbb{Z} \to *_{j\in J}H_j$ be a homomorphism. Then, there exists $F \subset\subset J$ such that $h(\times_N\mathbb{Z}) \leq *_{j\in F}H_j$.*

Proof. By Kuroś's theorem [13, Sec. 34 or 10, Ch.17], $h(\times_N\mathbb{Z})$ is a free product of copies of $\mathbb{Z}$ and conjugate groups of subgroups of some H_j. If the number of components of this free product is finite, then we get the conclusion. Hence, it suffices to deduce a contradiction from $h(\times_N\mathbb{Z}) = *_{j\in J}H_j$ for infinite J. Let $p_j : *_{j\in J}H_j \to H_j$ be the projection. First, we inductively define $k_n \in N$, $j_n \in J$, $x_n \in \times_{N\setminus\{1,\cdots,n\}}\mathbb{Z}$ and finite subsets F_n of J for $n = 0, 1, \cdots$. Let $k_0 = 1$ and take x_0 and finite $F_0 \subset J$ so that $h(x_0) \in *_{j\in F_0}H_j$, $p_{j_\alpha} \cdot h(x_0) \neq e$ for $0 \leq \alpha \leq 4$ where the j_α's are distinct. Suppose that we have defined the $(n-1)$-step. Since $h(\times_{\{1,\cdots,n\}}\mathbb{Z}) \leq *_{j\in E}H_j$ for some finite E and h is surjective, there exist $x_n \in \times_{N\setminus\{1,\cdots,n\}}\mathbb{Z}$ and distinct $j_{5n+\alpha} \in J\setminus F_n$ such that $p_{j_{5n+\alpha}} \cdot h(x_n) \neq e$ for $0 \leq \alpha \leq 4$. Let $k_n = n + 2 + \max\{l(h(x_k \cdots x_{n-1})) : 0 \leq k \leq n-1\}$ and F_n be a finite subset of J such that $F_{n-1} \subset F_n$ and $h(x_n) \in *_{j\in F_n}H_j$. Then, let Seq be the set of all finite sequences of natural numbers and denote the length of $s \in Seq$ by $lh(s)$. The empty sequence is denoted by $<>$ and generally $s \in Seq$ is denoted by $\langle s_1, \cdots, s_n\rangle$ where $s_k \in N$ $(1 \leq k \leq n)$. For $s, t \in Seq$, $s \prec t$ if $s(i) < t(i)$ for the minimal i with $s(i) \neq t(i)$ or t extends s. Let $D_n = \{s \in Seq : 0 \leq lh(s) \leq n, 1 \leq s(i) \leq k_i \text{ for } 1 \leq i \leq n\}$ and $\overline{W_n} = D_n$ with the ordering $\prec$ and $W_n(s) = x_n$ where $n = lh(s)$. Similarly, let $D_n^{(m)} = \{s \in Seq : 0 \leq lh(s) \leq n, 1 \leq s(i) \leq k_{m+i} \text{ for } 1 \leq i \leq n\}$ and $\overline{W_n^{(m)}} = D_n^{(m)}$ with the ordering $\prec$ and $W_n^{(m)}(s) = x_{m+n}$ for $n = lh(s)$. Then, there exist σ-words W and $W^{(m)} (m \in N)$ such that $W_{\{1,\cdots,n\}} = (W_{n-1})_{\{1,\cdots,n\}}$ and $(W^{(m)})_{\{1,\cdots,n\}} = (W_{n-1}^{(m)})_{\{1,\cdots,n\}}$ for $n \in N$. There exists $E \subset\subset J$ such that $h(W) \in *_{j\in E}H_j$. Let m be a number such that $E \cap F_{m-1} = E \cap \bigcup_{n\in N} F_n$ and $l(h(W)) \leq m$. Then, $h(x_k) \in *_{j\in F_{m-1}}H_j$ for $0 \leq k \leq m-1$. Since $h(W) = y_1 \cdot h(W^{(m)})^{k_m} \cdot y_2 \cdot h(W^{(m)})^{k_m} \cdots y_M \cdot h(W^{(m)})^{k_m}$ for some y_k with $l(y_k) \leq k_m - (m+2)$, $p_{j_{5m+\alpha}} \cdot h(W^{(m)}) = e$ for at least three $\alpha \in \{0, 1, 2, 3, 4\}$ by Lemma 2.3. A similar argument for $h(W^{(m+1)})^{k_{m+1}}$

and the fact $W^{(m)} = x_m \cdot (W^{(m+1)})^{k_{m+1}}$ imply that $p_{j_{5m+\alpha}} h(x_m) = e$ for at least one $\alpha \in \{0,1,2,3,4\}$, which is a contradiction. □

Proof of Theorem 2.1. Suppose the negation of the conclusion. Then, we get $x_n \in \times^{\sigma}_{i \in I \setminus E_n} G_i$, $E_n \subset\subset I$ and $F_n \subset\subset J$ such that $E_n \subset E_{n+1}$, $F_n \subset F_{n+1}$, $x_n \in \times_{i \in \cup_n E_n} G_i$, $h(x_n) \notin *_{j \in F_n} H_j$ and $h(x_n) \in *_{j \in F_{n+1}} H_j$ $(n \in N)$. Finally, we get a contradiction by Lemma 2.4 and Proposition 1.9. □

Corollary 2.5. *Let* $h : \times^{\sigma}_{i \in I} G_i \to *_{j \in J} H_j$ *be a homomorphism for groups* $G_i (i \in I)$ *and* $H_j (j \in J)$. *If every* G_i *is finitely generated, then there exist* $F \subset\subset J$ *such that* $h(\times^{\sigma}_{i \in I} G_i) \leq *_{j \in F} H_j$.

We remark that Theorem 2.1 for the unrestricted free product can be shown similarly, if an index set I is countable.

3. Noncommutatively slender groups

We introduce a new notion "noncommutatively slender groups", which is a noncommutative version of slender abelian groups [6, Sec.94]. Recall that an abelian group A is slender if and only if for any homomorphism $h : \Pi_N \mathbb{Z} \to A$ there exists $n \in N$ such that $h(\Pi_{N \setminus \{1,\cdots,n\}} \mathbb{Z}) = \{0\}$.

Definition 3.1. A group G is noncommutatively slender, if for any homomorphism $h : \times_N \mathbb{Z} \to G$ there exists an $n \in N$ such that $h(\times_{N \setminus \{1,\cdots,n\}} \mathbb{Z}) = \{e\}$. We say "n-slender" instead of "noncommutatively slender" for short.

This notion is equivalent to a seemingly weaker condition as in case of slender abelian groups, which we show now.

Proposition 3.2. *If for any homomorphism* $h : \times_N \mathbb{Z} \to G$ *the set* $\{n \in N : h(\delta_n) \neq e\}$ *is finite, then* G *is n-slender.*

Proof. Let $h : \times_N \mathbb{Z} \to G$ be a homomorphism. Then, there exists n such that $h(\delta_k) = e$ for $k > n$. Suppose that $h(\times_{N \setminus \{1,\cdots n\}} \mathbb{Z}) \neq \{e\}$, and take $x \in \times_{N \setminus \{1,\cdots n\}} \mathbb{Z}$ so that $h(x) \neq e$. Let W be a word corresponding to x and $x_k = W_{N \setminus \{1,\cdots,k\}} (k \in N)$. Then, there exists a homomorphism $\varphi : \times_N \mathbb{Z} \to \times_N \mathbb{Z}$ such that $\varphi(\delta_k) = x_k (k \in N)$ by Proposition 1.9. Now, $h \cdot \varphi(\delta_k) = h(x_k) = h(x) \neq e$ for every k, which is a contradiction.

Clearly, $\times_N \mathbb{Z}$ is not n-slender. However, $\times_N \mathbb{Z}$ is slender in the sense of [7], which is a straightforward generalization of slenderness of abelian groups. To see this, let A be an abelian subgroup of $\times_N \mathbb{Z}$. Then, A is isomorphic to $\mathbb{Z}$ or trivial by [11, Theorem 6]. Hence, $\times_N \mathbb{Z}$ is slender in the sense of [7] by Specker's theorem [**16**, or **6**]. On the other hand, it easy to see that every n-slender group is slender in the sense of [7]. □

Theorem 3.3. *An abelian group A is n-slender, if and only if A is slender.*

Proof. Let $\sigma : \times_N \mathbb{Z} \to \Pi_N \mathbb{Z}$ be the canonical homomorphism. Let $h : \Pi_N \mathbb{Z} \to A$ be a homomorphism for an n-slender group A. Then, there exists $n \in N$ such that $h \cdot \sigma(\times_{N \setminus \{1,\cdots n\}} \mathbb{Z}) = \{0\}$. Since $\sigma(\times_{N \setminus \{1,\cdots n\}} \mathbb{Z}) = \Pi_{N \setminus \{1,\cdots n\}} \mathbb{Z}$, $h(\Pi_{N \setminus \{1,\cdots n\}} \mathbb{Z}) = \{0\}$. Next, let $h : \times_N \mathbb{Z} \to A$ be a homomorphism for a slender abelian group A. Then, $h((\times_N \mathbb{Z})') = \{0\}$, where G' denotes the commutator subgroup of G. By Corollary 4.8, which we shall show in the next section, there exists no nonzero homomorphism from $\mathrm{Ker}(\sigma)/(\times_N \mathbb{Z})'$ to any slender abelian group. Hence, there exists a homomorphism $\overline{h} : \Pi_N \mathbb{Z} \to A$ such that $\overline{h} = h \cdot \sigma$. Take n so that $\overline{h}(\Pi_{N \setminus \{1,\cdots n\}} \mathbb{Z}) = \{0\}$, then we get $h(\times_{N \setminus \{1,\cdots n\}} \mathbb{Z}) = \{0\}$. □

Corollary 3.4. *An n-slender group is torsion-free.*

Proof. Suppose that G is not torsion-free. Then, there exists a nontrivial finite cyclic subgroup C. Since C is abelian but not slender, C is not n-slender. Hence, G is not n-slender. □

&

Proposition 3.5. *Let S be an n-slender group and $h : \times^\sigma_{i \in I} G_i \to S$ be a homomorphism. Then, there exist a finite subset F and a homomorphism $\overline{h} : *_{i \in F} G_i \to S$ such that $h = \overline{h} \cdot p_F$, where $p_F(W) = W_F$.*

Proof. First we show that $h(G_i) \neq \{e\}$ for almost all i. Suppose the contrary hols. Then there exist $i_n \in I (n \in N)$ such that $h(G_{i_n}) \neq \{e\}$ for $n \in N$ and $i_m \neq i_n$ for $m \neq n$. Let $g_n \in G_{i_n}$ so that $h(g_n) \neq e$. We can naturally define a homomorphism $\varphi : \times_N \mathbb{Z} \to \times^\sigma_{i \in I} G_i$ such that $\varphi(\delta_n) = g_n (n \in N)$ by Proposition 1.9. Then, $h \cdot \varphi(\delta_n) \neq e$ for every $n \in N$, which is a contradiction. Let $F = \{i \in I : h(G_i) \neq \{e\}\}$. Similarly as the proof of Proposition 3.2, we can conclude $h(\times_{i \in I \setminus F} G_i) = \{e\}$. Since $\times_{i \in I} G_i \simeq *_{i \in F} G_i * (\times_{i \in I \setminus F} G_i)$ naturally, we get the conclusion. □

Theorem 3.6. *Let $G_j (j \in J)$ be n-slender groups. Then, both the restricted direct product $\Pi^r_{j \in J} G_j$ and the free product $*_{j \in J} G_j$ are n-slender.*

The next corollary due to G. Higman [11, Theorem 1 with a remark on p.80], is a fundamental result about n-slenderness.

Corollary 3.7. *(Higman [10, Corollary 3.7]) Every free group is n-slender.*

Proof of Theorem 3.6. Let $h : \times_N \mathbb{Z} \to \Pi^r_{j\in J} G_j$ be a homomorphism and $p_F : \Pi^r_{j\in J} G_j \to \Pi_{j\in F} G_j$ the projection for $F \subset\subset J$. Suppose the negation of the conclusion. By the n-slenderness of of G_j, we get $i_n \in N$ and finite subsets J_n of J such that $i_n < i_{n+1}$, $J_n \subset J_{n+1}, J_n \neq J_{n+1}, h(\delta_{i_n}) \neq e, h(\delta_k) \in \Pi^r_{j\in J_n} G_j$ for $1 \leq k \leq i_n$ and $p_{J_n} \cdot h(\times_{N\setminus\{1,\cdots,i_n\}}\mathbb{Z}) = \{e\}$. Let $\overline{W_k} = N$ with the usual ordering and $W_n(k) = \delta_{i_{n+k}}$ for $k \in N$ and $n \in N \cup \{0\}$. Let $n \in N$ be a number such that $p_j \cdot h(W_0) = e$ for $j \in \bigcup_{k\in N} J_k \setminus J_n$. By definition $p_{J_n} \cdot h(W_n) = e$ and $p_j \cdot h(\delta_{i_1} \cdots \delta_{i_n}) = e$ for $j \notin J_n$. Therefore, $p_j \cdot h(W_n) = p_j((\delta_{i_1} \cdots \delta_{i_n})^{-1} W_0) = e$ for $j \in \bigcup_{k\in N} J_k \setminus J_n$ and consequently $p_j \cdot h(W_n) = e$ for $j \in \bigcup_{k\in N} J_k$. By the same reasoning, $p_j \cdot h(W_{n+1}) = e$ for $j \in \bigcup_{k\in N} J_k$. Then, $p_j \cdot h(\delta_{i_{n+1}}) = p_j \cdot h(W_n \cdot (W_{n+1})^{-1}) = e$ for $j \in \bigcup_{k\in N} J_k$, which is a contradiction.

Next, to show the n-slenderness of $*_{j\in J} G_j$, let $h : \times_N \mathbb{Z} \to *_{j\in J} G_j$ be a homomorphism and $\sigma : *_{j\in J} G_j \to \Pi^r_{j\in J} G_j$ be the canonical homomorphism. To the contrary, suppose that $h(\delta_k) \neq e$ for infinitely many k. By n-slenderness of $\Pi^r_{j\in J} G_j$, there exists $n \in N$ such that $\sigma \cdot h(\times_{N\setminus\{1,\cdots n\}}\mathbb{Z}) = \{e\}$. Let $\varphi : \times_N \mathbb{Z} \to \times_N \mathbb{Z}$ be a natural homomorphism such that $\varphi(\delta_k) = \delta_{n+k}$ according to Proposition 1.9. Then, $\sigma \cdot h \cdot \varphi(x) = e$, for $x \in \times_N \mathbb{Z}$. We claim $h \cdot \varphi(\delta_k) = e$ for almost all k, which implies the conclusion. By modifying φ, we may assume $h \cdot \varphi(\delta_k) \neq e$ for all k and $\sigma \cdot h \cdot \varphi$ is trivial. Though we can deduce a contradiction from these assumptions using Kuroš's theorem and Higman's theorem (Corollary 3.7), we present a proof which is similar to the proof of Theorem 2.1 for completeness. Remark that $l(u) \geq 4$ if $u \neq e$ and $\sigma(u) = e$. Let $k_1 = 1$ and $k_{n+1} = \Sigma^n_{i=1} l(h(\delta_i)) + k_n + 2$. Then, $k_j < k_{j+1}$ clearly. Let $D_j = \{s \in Seq : 0 \leq lh(s) \leq j, 1 \leq s(i) \leq k_i \text{ for } 1 \leq i \leq j\}$ and $\overline{W_j} = D_j$ with the ordering $\prec$ and $W_j(s) = \delta_{lh(s)+1}$ for $s \in D_j$. Then, there exists a unique σ-word W such that $W_{\{1,\cdots,j\}} = W_j$ for $j \in N$. Take m so that $l(h \cdot \varphi(W)) \leq k_m$. As in the proof of Theorem 2.1, we get U_s and V_s for $s \in D_m$ with $lh(s) = m$ so that $U_s = \delta_{i_1} \cdots \delta_{i_k}$ where $0 \leq i_1 < \cdots < i_k \leq m$ and $V_s \simeq Z^{k_{m+1}}$, where $\overline{Z} = \{s \in Seq : 1 = s(1), lh(s) \geq 1, 1 \leq s(i) \leq k_{m+i} \text{ for } i \geq 2\}$ with $\prec$ and $Z(s) = \delta_{m+lh(s)}$. Now, $h \cdot \varphi(W) = \cdots h \cdot \varphi(u_s) \cdot (h \cdot \varphi(Z))^{k_{m+1}} \cdots$. Since $G_j (j \in J)$ are torsion-free, $h \cdot \varphi(Z)$ is a conjugate of a member of some G_m by Lemma 2.3. Since $\sigma \cdot h \cdot \varphi$ is trivial, $h \cdot \varphi(Z) = e$. By the same argument for k_{m+2} we can conclude $h \cdot \varphi(Y) = e$, where $\overline{Y} = \{s \in Seq : 1 = s(1), lh(s) \geq 1, 1 \leq s(i) \leq k_{m+1+i} \text{ for } i \geq 2\}$ with $\prec$ and $Y(s) = \delta_{m+1+lh(s)}$. Then, $Z = \delta_{m+1} \cdot Y^{k_{m+2}}$ and hence $h(\delta_{m+1}) = e$, which is a contradiction. □

We close this section by stating a question.

Question 3.8. Let $h : \times_{\omega_1}\mathbb{Z} \to \mathbb{Z}$ be a homomorphism such that $h(\delta_\alpha) = 0$ for $0 \leq \alpha < \omega_1$, where ω_1 is the least uncountable ordinal. Is h trivial?

It is equivalent to ask whether each homomorphism $h : C\omega_1/(\times_{\omega_1}\mathbb{Z})^{\sigma\prime} \to \mathbb{Z}$ is trivial, according to the notation in Section 4.

4. Commutator subgroups, abelianizations and σ-abelianizations

Let C_I be the subgroup of $\times_I\mathbb{Z}$ consisting of x such that $p_i(x) = 0$ for all $i \in I$, where p_i is the canonical projection to the i-th component. The commutator subgroup G' is the subgroup generated by all commutators $x^{-1}y^{-1}xy$ (= [x,y]), that is, $\langle [x,y] : x, y \in G\rangle$. Then, $G' = \langle h(C_2) : h \in \mathrm{Hom}(\mathbb{Z} * \mathbb{Z}, G)\rangle = \langle h(C_F) : h \in \mathrm{Hom}(\times_F\mathbb{Z}, G), F \subset\subset N\rangle$. Generalizing this in our scope, let $G^{\sigma\prime} = \langle h(C_N) : h \in \mathrm{Hom}(\times_N\mathbb{Z}, G)\rangle$ and $G^{\infty\prime} = \langle h(C_I) : h \in \mathrm{Hom}(\times_I\mathbb{Z}, G)$ for some $I\rangle$. Clearly, $G^{\sigma\prime}$ and $G^{\infty\prime}$ are normal subgroups of G. Though $G^{\infty\prime}$ also seems to be a natural subgroup of G, we have not found any interesting phenomenon about it. Therefore, we deal only with $G^{\sigma\prime}$. The abelianization of G, that is, G/G', is denoted by $Ab(G)$. Similarly we define Ab^σ as $G/G^{\sigma'}$ and call $Ab^\sigma(G)$ the σ-abelianization of G. $Ab^\sigma(G)$ is a homomorphic image of $Ab(G)$. To investigate $Ab(G)$ and $Ab^\sigma(G)$, we recall some notions for abelian groups.

An abelian group A is called complete modulo the Ulm subgroup (abbreviated by "complete mod-U"), if for any $x_n \in A (n \in N)$ with $n! \mid x_{n+1} - x_n$ there exists $x \in A$ such that $n! \mid x - x_n$ for all $n \in N$. It is known that A is algebraically compact, if and only if $UU(A) = U(A)$ and A is complete mod-U [2]. A is cotorsion-free if A does not contain nonzero cotorsion subgroup, that is, A is torsion-free, reduced and contains no copy of the p-adic integer group $\mathbb{J}_p$ for any prime p. It is known that A is slender if and only if A is cotorsion-free and contains no copy of $\mathbb{Z}^N$. First we state some preliminary facts about this notion. Since the proofs are straightforward, we omit them.

Proposition 4.1. *Any homomorphic image of a group which is complete mod-U is also complete mod-U. A direct product of groups which are complete mod-U is also complete mod-U.*

Proposition 4.2. *Let A be an abelian group and H its pure subgroup. If both H and A/H are complete mod-U, then A itself is complete mod-U.*

Proposition 4.3. *If an abelian group A is complete mod-U, then* $\mathrm{Hom}(A, B) = \{0\}$ *for any cotorsion-free abelian group B.*

Proof. Let $h \in \mathrm{Hom}(A, B)$. Since $\mathrm{Im}(h)$ becomes torsion-free, $U^2(\mathrm{Im}(h)) = U(\mathrm{Im}(h))$. Hence, $\mathrm{Im}(h)$ is algebraically compact by [2, Theorem 2.5]. The cotorsion-freeness of B implies $\mathrm{Im}(h) = \{0\}$. □

Now, we investigate G', $G^{\sigma\prime}$, $Ab(G)$, $Ab^{\sigma}(G)$ and so on.

Lemma 4.4. *If G is an n-slender group, then $G^{\sigma\prime} = G'$.*

Proof. Let $h : \times_N \mathbb{Z} \to G$ be a homomorphism. Then, there exist $F \subset\subset N$ and $\overline{h} : *_F \mathbb{Z} \to G$ such that $h = \overline{h} \cdot p_F$ by Proposition 2.4. Hence, $h(C_N) = \overline{h}(C_F) \leq G'$ and consequently $G^{\sigma\prime} = G'$. □

Theorem 4.5. *Let $G_i (i \in I)$ be n-slender groups. Then,*
$(\times^{\sigma}_{i\in I} G_i)^{\sigma\prime} = \{x \in \times^{\sigma}_{i\in I} G_i : p_i(x) \in G'_i \text{ for all } i\}$ and hence
$Ab^{\sigma}(\times^{\sigma}_{i\in I} G_i) \simeq \Pi^{\sigma}_{i\in I} Ab(G_i)$ naturally. In case of σ-products the analogous facts hold, that is, $(\Pi^{\sigma}_{i\in I} G_i)^{\sigma\prime} = \{x \in \Pi^{\sigma}_{i\in I} G_i : x(i) \in G'_i \text{ for all } i\}$, and also $Ab^{\sigma}(\Pi^{\sigma}_{i\in I} G_i) \simeq \Pi^{\sigma}_{i\in I} Ab(G_i)$ naturally.

Proof. Since $p_j((\times^{\sigma}_{i\in I} G_i)^{\sigma\prime}) \leq G_j^{\sigma\prime} = G'_j$ for each j by Lemma 3.4, the one inclusion is obvious. Let $p_i(g) \in G'_i (i \in I)$ for $g \in \times^{\sigma}_{i\in I} G_i$ and W be a word corresponding to g. Let $g_{i1}, \cdots, g_{ik_i} \in G_i$ so that the word $g_{i1} \cdots g_{ik_i}$ is $W_{\{i\}}$ for each $i \in I$. Then, $g_{i1} \cdots g_{ik_i} \in G'_i$. There exist $m_i \in N$, $h_i : *^{m_i}_{k=1} \mathbb{Z} \to G_i$ and $x_{ij} (1 \leq j \leq k_i)$ such that $h_i(x_{ij}) = g_{ij}$ and $x_{i1} \cdots x_{ik_i} \in (*^{m_i}_{k=1} \mathbb{Z})'$. Let $L = \{(i, j) : i \in I, \mathrm{Im}(W) \cap (G_i \backslash \{e\}) \neq \phi$ and $1 \leq j \leq m_i\}$. By $\mathbb{Z}(i, j)$, we mean the (i, j)-th component of $\times_L \mathbb{Z}$. Define $h : \times_L \mathbb{Z} \to \times^{\sigma}_{i\in I} G_i$ naturally so that $h(\delta_{ij}) = h_i(\delta_j)$ for $(i, j) \in L$ according to Proposition 1.9, where δ_{ij} corresponds to 1 of $\mathbb{Z}(i, j)$. Joining all words corresponding to x_{ij}'s under the corresponding ordering of g_{ij}'s in $\overline{W}$, we get a word $X \in \mathcal{W}(\mathbb{Z}(i, j) : (i, j) \in L)$ so that $h(X) = g$. Since $X_{\{(i,j):1\leq j\leq k_i\}} \in (*^{m_i}_{k=1} \mathbb{Z})'$ for each i, $X \in C_L$ and hence $g \in (\times^{\sigma}_{i\in I} G_i)^{\sigma\prime}$. The second proposition follows immediately and the case of σ-products is proved analogously. □

Corollary 4.6. $(\times_N \mathbb{Z})^{\sigma\prime} = C_N$.

Theorem 4.7. *Let $G_i (i \in I)$ be n-slender groups. Then,*
$(\times^{\sigma}_{i\in I} G_i)^{\sigma\prime} / (\times^{\sigma}_{i\in I} G_i)'$ is complete modulo the Ulm subgroup.

Proof. Let $E = \{x \in \times^{\sigma}_{i\in I} G_i : p_i(x) \in G'_i$ for all $i\}$. By Theorem 3.5 it suffices to show that $E/(\times^{\sigma}_{i\in I} G_i)'$ is complete mod-U. Since the property in question depends on countably many members only and each member is related to a σ-word, we may assume $I = N$. Let $\sigma : E \to E/(\times_{n\in N} G_n)'$ be the canonical homomorphism and $n! \mid \sigma(x_{n+1}) - \sigma(x_n) (n \in N)$. We can take σ-words $V_n (n \in N)$ so that $x_{n+1} \cdot x_n^{-1} \in V_n^{n!} (\times_{n\in N} G_n)'$ and $\mathrm{Im}(V_n) \cap \bigcup^n_{k=1} G_k = \phi$. Let $B = \{s \in Seq : s \neq <>, 1 \leq s_i \leq i \text{ for } 1 \leq i \leq lh(s)\}$. Let $\overline{V_\infty} =$

$\{(s,\alpha) : s \in B$ and $\alpha \in \overline{V_{lh(s)}}\}$ where $(s,\alpha) < (t,\beta)$ if $s \prec t$ or $s = t$ and $\alpha < \beta$ as members of $\overline{V_{lh(s)}}$ and $V_\infty(s,\alpha) = V_{lh(s)}(\alpha)$. Since $V_n \in E(n \in N)$, $V_\infty \in E$. Let $\overline{U_n} = \{(s,\alpha) \in \overline{V_\infty} : lh(s) \geq n, s(i) = 1$ for $1 \leq i \leq n\}$ and $U_n(s,\alpha) = V_\infty(s,\alpha)$ for $(s,\alpha) \in \overline{U_n}$. Then, $V_\infty \in {U_n}^{n!} \cdot {V_{n-1}}^{(n-1)!} \cdots V_1 \cdot (\times_{n\in N} G_n)'$. Hence, $V_\infty \cdot x_1 \cdot x_n^{-1} \in U_n^{n!} \cdot x_n \cdot x_{n-1}^{-1} \cdot x_{n-1} \cdots x_2 \cdot x_1^{-1} \cdot x_1 \cdot x_n^{-1}(\times_{n\in N} G_n)' = U_n^{n!}(\times_{n\in N} G_n)'$ and consequently $n! \mid \sigma(V_\infty x_1) - \sigma(x_n)$ for all $n \in N$. □

By Theorem 4.7 and Proposition 4.3, we get,

Corollary 4.8. *Let A be a cotorsion-free abelian group. Then,* $\mathrm{Hom}(C_N, A) = \{0\}$.

Corollary 4.9. *Let $G_i (i \in I)$ be n-slender groups. Then,* $(\Pi^\sigma_{i\in I} G_i)^{\sigma\prime}/(\Pi^\sigma_{i\in I} G_i)'$ *is complete modulo the Ulm subgroup.*

Proof. Let $\varphi : \times^\sigma_{i\in I} G_i \to \Pi^\sigma_{i\in I} G_i$ and $\psi : \Pi^\sigma_{i\in I} G_i \to Ab(\Pi^\sigma_{i\in I} G_i)$ be the canonical homomorphisms. Then, $\varphi((\times^\sigma_{i\in I} G_i)^{\sigma\prime}) = (\Pi^\sigma_{i\in I} G_i)^{\sigma\prime}$ by Theorem 4.5. For $x \in (\times^\sigma_{i\in I} G_i)^{\sigma\prime}$, $\psi \cdot \varphi(x) = e$ if and only if $\rho(\varphi(x)(i))(i \in I)$ are bounded. Hence, $(\Pi^\sigma_{i\in I} G_i)^{\sigma\prime}/(\Pi^\sigma_{i\in I} G_i)'$ is a homomorphic image of $(\times^\sigma_{i\in I} G_i)^{\sigma\prime}/(\times^\sigma_{i\in I} G_i)'$. Now, the conclusion follows from Theorem 4.7 and Proposition 4.1. □

To get further information about Ab^σ, we need some definitions about words.

Definition 4.10. A finite sequence of words $U_1, \cdots, U_\mu$ is of n-form, if $U_i (1 \leq i \leq \mu)$ are reduced and there exist a partition $A_1, \cdots, A_M, B$ of $\{1, \cdots, \mu\}$ and $i_k, j_k (1 \leq k \leq m)$ such that $\{i_k, j_k : 1 \leq k \leq m\} = B$, U_{i_k} is $U_{j_k}^{-1}$ as words for each k, $U_\alpha = U_\beta$ for any $\alpha, \beta \in A_\gamma$ and $|A_\gamma| = n$ $(1 \leq \gamma \leq M)$. In addition if the word $U_1 \cdots U_\mu$ is quasi-reduced, we say that $U_1, \cdots, U_\mu$ is of canonical n-form. In case $n = 0$, we say that it is of commutator form and canonical commutator form respectively.

Sometimes we shall confuse a sequence of words $U_1, \cdots, U_k$ with a word $U_1 \cdots U_k$ for simplicity of expression.

Lemma 4.11. *Let $\varphi : \times_N \mathbb{Z} \to Ab(\times_N \mathbb{Z})$ be the canonical homomorphism. Suppose that $\varphi(x)$ is divided by $n \in \mathbb{N}$ in $Ab(\times_N \mathbb{Z})$ for $x \in \times_N \mathbb{Z}$. Then, there exists a canonical n-form $U_1, \cdots, U_k$ such that $x = U_1 \cdots U_k$.*

Proof. First we describe a transformation of commutator forms corresponding to $c \in (\times_N \mathbb{Z})'$. There exists a sequence of reduced words $W_1, \cdots, W_l$ of commutator form with $c = W_1 \cdots W_l$. Let $U_1, \cdots, U_{2m}$ is of commutator form and $U_{i+1} \cdots U_{2m}$ is quasi-reduced. If $U_i U_{i+1} \cdots U_{2m}$ is not quasi-reduced, there exist reduced words X, V, W such that

$U_i \simeq VX$, $U_{i+1} \cdots U_{2m} = X^{-1}W$ and $X^{-1}W$ is quasi-reduced. Cancelling XX^{-1} and arranging pairings, we get a sequence of commutator forms of length equal to or less than $2(m+1)$. Observe that the occasion "U_iU_{i+1} is not quasi-reduced" happens in the process, only when $W_j = XU_i$, $W_{j+1} \cdots W_l = U_{i+1}Y$ where both XU_i and $U_{i+1}Y$ are quasi-reduced for some X and Y. Therefore, this transformation stops in finite times and we get a canonical commutator form which is equal to c.

Under the given condition, there exist y and $c \in (\times_N \mathbb{Z})'$ such that $x = y^n c$ and hence reduced words U and W such that $x = U^{-1}W^nUc$ and $U^{-1}W^nU$ is quasi-reduced. By a similar argument as above we get the conclusion. □

Lemma 4.12. *There exists a pure subgroup of* $Ab(\times_N \mathbb{Z})$ *which is also contained in* $C_N/(\times_N \mathbb{Z})'$ *and isomorphic to* $\mathbb{Z}$.

Proof. Let $a = e_1 \cdots e_k \cdots e_1^{-1} \cdots e_k^{-1} \cdots \in \times_N \mathbb{Z}$, where e_k is the generator of the k-component. Then, $\varphi(a) \in \varphi(C_N)$. Suppose that $\varphi(a^m)$ $(m > 0)$ is divided by n in $Ab(\times_N \mathbb{Z})$. Then, a^m is equal to a word $U_1 \cdots U_\mu$ of canonical n-form in Definition 4.10 by Lemma 4.11. Since the reduced word of a^m is well ordered from left to right, U_α is a finite word for every $\alpha \in B$. A word of form $e_k \cdots$ for large enough k must be a part of some U_α where $\alpha \in A_\gamma$. Hence, n divides m. Now, we have shown that $< \varphi(a) >$ is isomorphic to $\mathbb{Z}$ and a pure subgroup of $Ab(\times_N \mathbb{Z})$. □

Theorem 4.13. *For a group* G, $Ab^\sigma(G) = G$ *if and only if* G *is a cotorsion-free abelian group.*

Proof. If G is a cotorsion-free abelian group, $h(C_N) = 0$ for any $h \in \mathrm{Hom}(\times_N \mathbb{Z}, G)$ by Proposition 4.3 and hence $Ab^\sigma(G) = G$. Now suppose that $Ab^\sigma(G) = G$. Then, G is abelian. Let $\psi : Ab(\times_N \mathbb{Z}) \to Ab(\times_N \mathbb{Z})/U(Ab(\times_N \mathbb{Z}))$ be the canonical homomorphism. Then, Lemma 4.12 implies that there exists a pure subgroup of $Ab(\times_N \mathbb{Z})/U(Ab(\times_N \mathbb{Z}))$ which is isomorphic to $\mathbb{Z}$ and contained in $\psi\varphi(C_N)$. Since $\psi\varphi(C_N)$ is complete mod-U by Proposition 4.1, $\psi\varphi(C_N)$ contains $\hat{\mathbb{Z}}$, that is, the $\mathbb{Z}$-adic completion of $\mathbb{Z}$, as a subgroup. The subgroup $\hat{\mathbb{Z}}$ is pure in $Ab(\times_N \mathbb{Z})/U(Ab(\times_N \mathbb{Z}))$ by purity of $\mathbb{Z}$ in $Ab(\times_N \mathbb{Z})$. Hence, $\hat{\mathbb{Z}}$ is a summand of $Ab(\times_N \mathbb{Z})/U(Ab(\times_N \mathbb{Z}))$. Now, $Ab^\sigma(G) = G$ implies that $h(\hat{\mathbb{Z}}) = 0$ for any $h \in \mathrm{Hom}(\hat{\mathbb{Z}}, G)$. Hence, G is cotorsion-free. □

Theorem 4.14. *Let* $G_i (i \in I)$ *be groups where infinitely many of them are nontrivial. Then,* $(\times^\sigma_{i\in I} G_i)^{\sigma\prime}/(\times^\sigma_{i\in I} G_i)'$ *and hence* $Ab(\times^\sigma_{i\in I} G_i)$ *includes a subgroup isomorphic to the direct sum of* $2^{\aleph_0}$*-many copies of the rational group* $\mathbb{Q}$.

To prove Theorem 4.14, some notions and lemmas are necessary.

Definition 4.15. For $c \in G'$, let $\rho(c) = \min\{n : c = [x_1, y_1] \cdots [x_n, y_n]$ for $x_i, y_i \in G\}$. For $c \in \times^{\sigma}_{i\in I} G_i$, $\rho^*(c)$ is the minimal number m such that there exist $U_1, \cdots, U_{2m}$ of canonical commutator form with $c = U_1 \cdots U_{2m}$.

If we consider the case $G = \times_{\{0\}} G$ in the definition of ρ^* a sequence of words $U_1, \cdots, U_{2m}$ is a sequence of members of G. Therefore, $\rho^*(c)$ for $c \in G'$ depends on representations of G, which is different from the case of ρ. However, the following hold, where some G_i may be trivial.

Lemma 4.16. *Let $G = \times^{\sigma}_{i\in I} G_i$. Then, $\rho(c) \leq \rho^*(c) - 1$ and $\rho^*(c) \leq 6\rho(c) - 1$ for $c \in G'$.*

Proof. Observing the role of commutators, that is, $xz^{-1}yz[yz, z^{-1}] = xy$, we can see $\rho(c) \leq \rho^*(c) - 1$. The second equation follows from the proof of Lemma 4.11. □

Lemma 4.17. *Suppose that $x \in G$ and $y \in H$ satisfy $x^2 \neq e$ and $y^2 \neq e$. Then, $\rho([x, y]^n) > n/12$ for $n \in N$ in $G * H$.*

Proof. Suppose that $\rho([x, y]^n) \leq n/12$. Then, $\rho^*([x, y]^n) \leq n/2 - 1$. There exists a sequence of words $U_1, \cdots, U_{2p}$ of quasi-reduced commutator form such that $p \leq n/2 - 1$ and $[x, y]^n = U_1 \cdots U_{2p}$. Then, one of $U_i's$ is of form $Vy^{-1}xW$ so that V and W are nonempty and hence another one of $U_i's$ is of form $W^{-1}x^{-1}yV^{-1}$. On the other hand,
$x^{-1}y^{-1}xyx^{-1}y^{-1} \cdots xy = U_1 \cdots U_{2p}$ and $U_1 \cdots U_{2p}$ is quasi-reduced and each U_i is reduced. Since $x \neq x^{-1}$ and $y \neq y^{-1}$, it never occurs that U_i is of form $W^{-1}x^{-1}yV^{-1}$ with nonempty V and W. □

Proof of} Theorem 4.14. It suffices to deal with the case that $I = N$ and $G_n (n \in N)$ are nontrivial groups. First we construct a subgroup which is isomorphic to $\mathbb{Q}$. Since $\times_{n\in N} G_n \simeq \times_{n\in N}(G_{2n-1} * G_{2n})$, we may assume the existence of $g_n \in G_n$ such that $g_n \neq g_n^{-1}$. Let V_n be the word $g_{2n-1}^{-1} g_{2n}^{-1} g_{2n-1} g_{2n}$ and next V_∞ and $U_n (n \in N)$ the σ-words defined from $V_n (n \in N)$ just in the same way as in the proof of Theorem 4.7. Then, $V_\infty \in {U_n}^{n!} \cdot {V_{n-1}}^{(n-1)!} \cdots V_1 (\times_{n\in N} G_n)' \subset (\times_{n\in N} G_n)^{\sigma\prime}$. We claim $\{\varphi(V_\infty), \varphi({U_n}^a) : a \in \mathbb{Z}, n \in N\}$ $(= H)$ is isomorphic to $\mathbb{Q}$, where $\varphi : \times_{n\in N} G_n \to Ab(\times_{n\in N} G_n)$ is the canonical homomorphism. Since ${U_n}^n = U_{n-1} \cdot {V_{n-1}}^{-1} \in U_{n-1}(\times_{n\in N} G_n)'$, H is divisible and of rank 1. It suffices to show that H is torsion-free and nonzero. Suppose that $V_\infty \in (\times_{n\in N} G_n)'$. Then, $V_\infty = [x_1, y_1] \cdots [x_m, y_m]$ for some $x_i, y_i \in \times_{n\in N} G_n$, which implies $\rho([g_{2n-1}, g_{2n}]^{n!}) = \rho(p_{\{2n-1,2n\}}(V_\infty)) \leq m$ for every n. This contradicts Lemma 4.17 for large enough n. By *a similar argument we can see that H is torsion-free. To get the conclusion of the theorem, we modify the above construction. There exist $X_\alpha \subset N(\alpha < 2^{\aleph_0})$ such that each X_α is infinite and $X_\alpha \cap X_\beta$ is finite for distinct α, β. Let $k_{n\alpha}(n \in N)$ be an enumeration of X_α without repetition. Let $V_{\infty\alpha}$ and $U_{n\alpha}$ be the σ-words obtained by replacing n by $k_{n\alpha}$ in the above construction. Then, we get subgroups $H_\alpha(\alpha < 2^{\aleph_0})$ of $(\times_{n\in N} G_n)^{\sigma\prime}/(\times_{n\in N} G_n)'$ which are linearly independent and isomorphic to $\mathbb{Q}$.* □

Since a theorem analogous to Theorems 4.14 holds for σ-products of free products, we prove it in the remaining part of this section. We need a lemma which is a version of Lemma 4.17. As is well-known, the commutator subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2$, that is, the infinite dihedral group, is consisting of all commutators, where $\mathbb{Z}_2$ is the group of order 2. Except this case we get the following,

Lemma 4.18. *Let G and H be nontrivial groups at least one of which is not isomorphic to $\mathbb{Z}_2$. Then, there exists $c \in (G * H)'$ such that $\rho^*(c^m) > (m-1)/2$ and consequently $\rho(c^m) > (m+1)/12$ for $m \in N$.*

Proof. We assume that G is not isomorphic to $\mathbb{Z}_2$.
Case 1. There exists $g \in G$ such that $g \neq g^{-1}$.

Take $h \in H$ with $h \neq e$ and let
$c = g^{-1}hghghghg^{-1}h^{-1}g^{-1}h^{-1}gh^{-1}g^{-1}h^{-1}$. Then,
$c^{-1} = hghg^{-1}hghgh^{-1}g^{-1}h^{-1}g^{-1}h^{-1}g^{-1}h^{-1}g$. We only remark the ordering of g and g^{-1} and the fact $h, h^{-1} \neq e$. Then, we can conclude $\rho^*(c^m) > (m-1)/2$ by a similar reasoning to the proof of Lemma 4.17. Hence $\rho(c^m) > (m+1)/12$ by Lemma 4.16.
Case 2. Otherwise.

Then, $g^2 = e$ for every $g \in G$. Take distinct $g_1, g_2 \in G$ with $g_1, g_2 \neq e$ and $h \in H$ with $h \neq e$ and let $c = hg_1g_2hg_1hg_2h^{-1}g_1g_2h^{-1}g_1h^{-1}g_2$. Since $g_1g_2 \neq g_1$ and $g_1g_2 \neq g_2$, we conclude that $\rho^*(c^m) > (m-1)/2$ and $\rho(c^m) > (m+1)/12$ as before. □

Theorem 4.19. *Let G_i and H_i be nontrivial groups at least one of which is not isomorphic to $\mathbb{Z}_2$ and I an infinite index set, then $(\Pi^\sigma_{i\in I} G_i * H_i)^{\sigma\prime}/(\Pi^\sigma_{i\in I} G_i * H_i)'$ and consequently $Ab(\Pi^\sigma_{i\in I} G_i * H_i)$ include a subgroup isomorphic to the direct sum of $2^{\aleph_0}$-many copies of the rational group $\mathbb{Q}$.*

Proof. It is enough to prove this in case $I = N$. Let $\varphi : \Pi_{n\in N} G_n * H_n \to Ab(\Pi_{n\in N} G_n * H_n)$ be the canonical homomorphism. Take $c_n \in (G_n * H_n)'$ $(n \in N)$ so that $\rho({c_n}^m) > (m+1)/12$ for $m \in N$ by Lemma 4.18. Define $x_m \in \Pi_{n\in N} G_n * H_n(m \in N)$ by: $x_m(n) = e$ for $n < m$ and $x_m(n) = {c_n}^{n!/m!}$ for $n \geq m$. Let $H = \{\varphi({x_m}^a) : m \in N, a \in \mathbb{Z}\}$.

Then, H is isomorphic to $\mathbb{Q}$. The rest of the proof is similar to that of Theorem 4.14. □

We remark that $(\Pi_N(\mathbb{Z}_2*\mathbb{Z}_2))' = \Pi_N(\mathbb{Z}_2*\mathbb{Z}_2)'$ *and hence* $Ab(\Pi_N(\mathbb{Z}_2*\mathbb{Z}_2)) \simeq \Pi_N(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ *canonically.*

Question 4.20. Are $Ab(\times_N\mathbb{Z})$ and $Ab(\Pi_N(\mathbb{Z}*\mathbb{Z}))$ torsion-free?

It is equivalent to ask whether $(\times_N\mathbb{Z})^{\sigma\prime}/(\times_N\mathbb{Z})'$ *and* $(\Pi_N(\mathbb{Z}*\mathbb{Z}))^{\sigma\prime}/(\Pi_N(\mathbb{Z}*\mathbb{Z}))'$ *are torsion-free or not. Especially,* $(\Pi_N(\mathbb{Z}*\mathbb{Z}))^{\sigma\prime}/(\Pi_N(\mathbb{Z}*\mathbb{Z}))'$ *is not torsion-free, if and only if there exist* $m, M \in N$ *such that* $\sup\{\rho(c) : c \in (\mathbb{Z}*\mathbb{Z})', \rho(c^m) \leq M\} = \infty$. *If the answers to these questions are affirmative,* $Ab(\times_N\mathbb{Z})$ *and* $Ab(\Pi_N(\mathbb{Z}*\mathbb{Z}))$ *have summands isomorphic to* $\mathbb{Z}^N$ *and* $(\mathbb{Z}\oplus\mathbb{Z})^N$ *respectively by Theorem 4.7.*

5. Appendix

Here, we state applications to algebraic topology, which are back ground of the context. Topological spaces in this appendix are always Hausdorff. Undefined notions about algebraic topology are standard and can be found in [12, 15]. *Let* (X_i, x_i) *be pointed spaces such that* $X_i \cap X_j = \phi$ *for* $i \neq j$. *There are two typical way of attaching spaces* (X_i, x_i) *under the identification of all* $x_i(= x^*)$. *The underlying set of the two spaces* $\bigvee_{i\in I}(X_i, x_i)$ *and* $\widetilde{\bigvee}_{i\in I}(X_i, x_i)$ *are* $\{x^*\} \cup \bigcup_{i\in I} X_i\backslash\{x_i\}$. *It suffices to define open neighborhood of* x^*. *Let* U *be a subset of* $\{x^*\} \cup \bigcup_{i\in I} X_i\backslash\{x_i\}$ *containing* x^*. U *is open in* $\bigvee_{i\in I}(X_i, x_i)$ *if* $U \cap X_i$ *is open in each* X_i, *while* U *is open in* $\widetilde{\bigvee}_{i\in I}(X_i, x_i)$ *if* U *is open in* $\bigvee_{i\in I}(X_i, x_i)$ *and* $(U\backslash\{x^*\}) \cap X_i = X_i\backslash\{x_i\}$ *for almost all* i. *If each* X_i *is locally simply connected at* x_i *and also first countable at* x_i, *then* $\pi_1(\bigvee_{i\in I}(X_i, x_i)) \simeq *_{i\in I}\pi_1(X_i)$. *Here, the first countability is essential even if* I *is finite* [3, 4]. *On the other hand we have the following, which was a theorem of H. B. Griffiths* [9] *and J. W. Morgan and I. Morrison* [14] *have completed its proof.*

Theorem 5.1. *(H.B.Griffiths, J.W.Morgan and I.Morrison) Let* X_i *be locally simply connected at* x_i *and also have countable basic neighborhoods of* x_i *for each* $i \in I$. *Then,* $\pi_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*) \simeq \times^{\sigma}_{i\in I}\pi_1(X_i, x_i)$.

Originally this was proved in case I *is countable, but it is not hard to see that this also holds for arbitrary* I, *which we shall explain in the sequel. In the introduction of* [14], *they stated that the proof contains a noneffective construction of a homotopy. Though we do not insist that the following proof is effective, it seems that it is more direct. Since* *the proof will clarify the meaning of equivalence of infinitary words, we outline the proof and present a direct construction of a homotopy.*

For a pointed space (X, x) a loop f in (X, x) is a continuous map from a closed interval $[a, b]$ (where $a < b$) to a space with $f(a) = f(b) = x$. Two loops f and g in (X, x) with their domain $[a, b]$ are briefly said to be homotopic, if there exists a homotopy from f to g which is constant with respect to a and b. When we do not mention the domain of a loop, the domain is always $[0, 1]$. For an interval I, $\dot{I}$ is the set of end points of I. For a loop f in $(\widetilde{\bigvee}_{i\in I}(X_i, x_i, x^*))$ there exist at most countable pairwise disjoint open subintervals $(a_n, b_n)(n \in M)$ of $[0, 1]$ such that $\bigcup_{n\in M}(a_n, b_n) = f^{-1}(\widetilde{\bigvee}_{i\in I}(X_i, x_i)\backslash\{x^*\})$. Each loop $f \mid [a_n, b_n]$ lies in some X_i and for each i almost all loops $f \mid [a_n, b_n]$ in (X_i, x_i) are homotopic to the constant map, since X_i is locally simply connected at x_i. Hence, we can get a σ-word $W^f \in \mathcal{W}(\pi_1(X_i, x_i) : i \in I)$ naturally using the ordering of (a_n, b_n)'s. If f is homotopic to the constant map, it is easy to see that $(W^f)_F = e$ for any $F \subset\subset J$. Hence, we can define a natural homomorphism $\psi : \pi_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*) \to \circledast^{\sigma}_{i\in I}\pi_1(X_i, x_i)$ by: $\pi_F \cdot \psi([f]) = (W^f)_F$ for $F \subset\subset J$, where $[f]$ is the member of $\pi_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i))$ corresponding to f. It is also easy to see that ψ is surjective. To see the injectivity of ψ, some notion is necessary. A loop f in (X, x) is proper, if f satisfies the following: Let $(a_n, b_n)(n \in M)$ be pairwise disjoint open intervals such that $\bigcup_{n\in M}(a_n, b_n) = f^{-1}(X\backslash\{f(0)\})$. Then, if $f \mid [a_n, b_n]$ is homotopic to the constant loop, $f \mid [a_n, b_n]$ itself is constant. The next lemma is the only part where we use the first countability.

Lemma 5.2. *(Essentially in [8, 1.2]) Let X be locally simply connected at x which has countable neighborhood bases. Let f be a loop in $((X, x) \vee (Y, y), x^*)$ such that $f(a_n) = f(b_n) = x^*$ for $n \in N$, $f([a_n, b_n]) \subset X$ and $f \mid [a_n, b_n]$ is homotopic to the constant loop, where $(a_n, b_n)(n \in N)$ are pairwise disjoint open subintervals of $[0, 1]$. Then, there exists a continuous map $H : [0, 1] \times [0, 1] \to (X, x) \vee (Y, y)$ with the following:*
(1) $H(1, t) = f(t)$ for $t \in [0, 1]$;
(2) $H(s, 0) = H(s, 1) = H(s, a_n) = H(s, b_n) = x$ for $s \in [0, 1]$ and $n \in N$;
(3) $H(s, t) \in X$ for $s \in [0, 1]$ and $t \in \bigcup_{n\in N}[a_n, b_n]$;
(4) $H(0, t) = x$ for $t \in \bigcup_{n\in N}[a_n, b_n]$.

Since this is not so hard to prove if we use the two given local properties, we omit the proof. Since the image of any loop f in $(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$ is included by $\widetilde{\bigvee}_{i\in C}(X_i, x_i)$ for some countable $C \subset I$, by iterating use of this lemma we get,

Lemma 5.3. ([14, Lemma 4.2]) *Under the same conditions as in Theorem 5.1, any loop f in $(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$ is homotopic to some proper loop.*

Proof of} Theorem 5.1. Let f be a loop in $(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*)$ with $W^f = e$. Since there exists a countable subset C of I such that $Im(f) \subset \widetilde{\bigvee}_{i\in C}(X_i, x_i)$, it suffices to deal the case $I = N$. By Lemma A.3, we may assume that f is a proper loop. Now, we construct a homotopy H from f to the constant loop. In the k-th step, we define H on subrectangles of $[0,1]\times[0,1]$ which makes loops in (X_k, x^*) homotopic to the constant loop expecting loops in $\widetilde{\bigvee}_{n>k}X_n$ will be made homotopic to the constant loop in a suitable way in future.
(Step 1) Let $H(t,1) = f(t)$ and $H(t,0) = x^*$ for $0 \leq t \leq 1$. Let $W^f = W_1 \cdots W_{n_1}$ where $W_i \in \mathcal{W}(G_1)$ or $W_i \in \mathcal{W}(G_n : n \geq 2)$ for $1 \leq i \leq n_1$ and $W_i \in \mathcal{W}(G_1)$ if and only if $W_{i+1} \in \mathcal{W}(G_j : j \geq 2)$ for $1 \leq i \leq n_1 - 1$.
(Substep 1) We can correspond a closed interval I_i to each W_i so that $W_i = W^{f|I_i}$ for $1 \leq i \leq n_1$, $\bigcup_{i=1}^{n_1} I_i = [0,1]$ and the right end of I_i is the left end of I_{i+1} for $1 \leq i \leq n_1 - 1$. We claim that $W_i = e$ for some $1 \leq i \leq n_1$. Suppose not. There exists $F \subset\subset N$ such that $p_F(W_i) \neq e$ for every $1 \leq i \leq n_1$. Then, $p_F(W^f) \neq e$, which is a contradiction. We choose one W_i with $W_i = e$. Let $H(s,t) = f(s)$ for $(s,t) \in \bigcup_{j\neq 1} I_j \times [1/2, 1]$.

In case $W_i \in \mathcal{W}(G_1)$, $f|I_i$ is homotopic to the constant loop in X_1. Let $H|I_i \times [1/2,1]$ be a continuous map such that $H(s,1/2) = x^*$ for $s \in I_i$ and $H(s,t) = x^*$ for $s \in \dot{I}_i$ and $t \in [1/2,1]$. In case $W_i \in \mathcal{W}(G_n : n \geq 2)$ we do not define H on $(I_i \backslash \dot{I}_i) \times (1/2, 1)$ in this step, but we let $H(s,1/2) = x*$ for $s \in I_i$. Next, we reform the word W^f to $W_1 \cdots V \cdots W_{n_1}$ by eliminating W_i, where $V = W_{i-1}W_{i+1}$. Then, $W_1 \cdots V \cdots W_{n_1} = e$ and members of $\mathcal{W}(G_1)$ and $\mathcal{W}(G_n : n \geq 2)$ are neighboring in $W_1, \cdots, V, \cdots, W_{n_1}$.
(Substep $k+1$) In the substep k, $H(s, 1/2^k)(s \in [0,1])$ have been defined and there is a corresponding word reformed from W^f. By the same reasoning as in the substep 1, one of the words equal e as member of the group, of course. We perform the work as in the substep 1. The substeps would finish in at most n_1-steps. If they finish in the k-step, then $H(t, 1/2^k)(0 \leq t \leq 1)$ have been defined and equal to x^*. Let $H([0,1] \times [0, 1/2^k]) = x^*$.
(Step k) After the $(k-1)$-step, there possibly exist finitely many subrectangles of $[0,1]\times[0,1]$ on which H has not been defined. Their forms are $[a,b] \times (\Sigma_{i=1}^{m-1} s_i/2^i + 1/2^m, \Sigma_{i=1}^{m-1} s_i/2^i + 1/2^{m-1})$, where $s_i = 0$ or 1

and $m \leq \Sigma_{i=1}^{k} n_i$. *H has been defined on the upper side of a rectangle and it is corresponding to a word in* $\mathcal{W}(\pi_n(X_n, x_n) : n \geq k)$. *H maps the lower side to* x^*. *In each rectangle, we work as in the step 1 as if the rectangle were* $[0,1] \times [0,1]$. *Note that the values of H which we define in this step are in* $\widetilde{\bigvee}_{n \geq k}(X_n, x_n)$, *because the loops in question are in* $\widetilde{\bigvee}_{n \geq k}(X_n, x_n)$.

Let $H(t,u) = x^*$, *if* $H(t,u)$ *has not been defined in any step. Now, the continuity of H is clear and the proof of Theorem 5.1 is complete.* □

Next we state a characterization of n-slender groups using π_1*-groups. Some preliminaries and definitions are necessary to state it.*

A continuous map $f : X \to Y$ *with* $f(x) = y$ *naturally induces a homomorphism* $f_* : \pi_1(X,x) \to \pi_1(Y,y)$. *A homomorphism* $h : \pi_1(X,x) \to \pi_1(Y,y)$ *is spatial with respect to pointed spaces* (X,x) *and* (Y,y), *if there exists a continuous map* $f : X \to Y$ *with* $f(x) = y$ *such that* $f_* = h$. *Denote the circle with a base point by* $(\mathbb{S}^1, b)$ *and let* $(\mathbb{S}^1_n, b_n)$ $(n \in N)$ *be copies of it. Then,* $\widetilde{\bigvee}_{n \in N}(\mathbb{S}^1_n, b_n)$ $(= (\mathbb{H}, b^*))$ *is the so-called the Hawaiian earring.*

Theorem 5.4. *For a group G the following are equivalent.*
(1) *G is n-slender;*
(2) *Let* $X_i (i \in I)$ *be 2-simplicial complexes with* $x_i \in X_i$. *If* $\pi_1(Y,y) \simeq G$ *for an arbitrary pointed space* (Y,y), *then any homomorphism* $h : \pi_1(\widetilde{\bigvee}_{i \in I}(X_i, x_i), x^*) \to \pi_1(Y,y)$ *is spatial with respect to* $(\widetilde{\bigvee}_{i \in I}(X_i, x_i), x^*)$ *and* (Y,y) *;*
(3) *If* $\pi_1(Y,y) \simeq G$ *for a pointed space* (Y,y), *then any homomorphism* $h : \pi_1(\mathbb{H}, b^*) \to \pi_1(Y,y)$ *is spatial with respect to* $(\mathbb{H}, b^*)$ *and* (Y,y).

Proof. It suffices to show the implications (1)→(2) and (3)→(1).
(1)→(2): By Theorem 5.1, $\pi_1(\widetilde{\bigvee}_{i \in I}(X_i, x_i), x^*) \simeq \times^{\sigma}_{i \in I} \pi_1(X_i, x_i)$ naturally. Therefore, there exist $E \subset\subset I$ and $\overline{h} : \pi_1(X_i, x_i) \to \pi_1(Y,y)$ such that $h = \overline{h} \cdot p_E$ by Proposition 3.5. Since $X_i (i \in I)$ are 2-simplicial complexes, a standard method shows that any homomorphism from $\pi_1(X_i, x_i)$ to $\pi_1(Y,y)$ is spatial. Hence, there exist continuous maps $f_i : X_i \to Y (i \in E)$ such that $f_i(x_i) = y$ and $(f_i)_* = h|\pi_1(X_i, x_i)$. Define a continuous map $f : \widetilde{\bigvee}_{i \in I}(X_i, x_i) \to Y$ by: $f|X_i = f_i$ for $i \in E$ and $f(\widetilde{\bigvee}_{i \in I \setminus F}(X_i, x_i)) = \{y\}$. Then, $f_* = h$.
(3)→(1): Let $h : \times_N \mathbb{Z} \to G$ be a homomorphism. There exists a simplicial complex Y with y such that $\pi_1(Y,y) \simeq G$, for example, the Eilenberg-Maclane complex $K(1,G)$. Identify $\times_N \mathbb{Z}$ with $\pi_1(\mathbb{H}, b^*)$, then there exists a continuous map $f : \mathbb{H} \to Y$ such that $f(b^*) =$

y and $f_* = \underset{\sim}{h}$. Since Y is locally contractible, there exists $n \in N$ such that $f(\widetilde{\bigvee}_{m\geq n}(\mathbb{S}^1_m, b_m))$ contained in some contractible neighborhood of y. On the other hand $\mathbb{H} = \bigvee_{m<n}(\mathbb{S}^1_m, b_m) \vee \widetilde{\bigvee}_{m\geq n}(\mathbb{S}^1_m, b_m)$ and $\pi_1(\mathbb{H}, b^*) \simeq *_{m<n}\pi_1(\mathbb{S}^1_m, b_m) * \times_{m\geq n}\pi_1(\mathbb{S}^1_m, b_m)$ naturally. Therefore, $h(\times_{N\setminus\{1,\cdots,m\}}\mathbb{Z}) = f_*(\times_{m\geq n}\pi_1(\mathbb{S}^1_m, b_m)) = \{e\}$. $\square$

As is well-known, the first integral singular homology group $H_1(X)$ is isomorphic to $Ab(\pi_1(X,x))$ for a path-connected pointed space (X,x) [14]. For certain spaces we can interpret $Ab^\sigma(G)$ naturally. Let $X_i (i \in I)$ be simplicial complexes, or ANR's more generally, such that $\pi_1(X_i, x_i)(i \in I)$ are n-slender. Then, $\pi_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*) \simeq \times^\sigma_{i\in I}\pi_1(X_i, x_i)$. As we we have shown in Section 3, $H_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i))$ becomes a rather complicated group for an infinite I, even if $X_i (i \in I)$ are copies of $\mathbb{S}^1$. On the other hand the factor $H^T_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i))$ of $H_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i))$, introduced in [5]*, is naturally isomorphic to $\Pi^\sigma_{i\in I}H^T_1(X_i)$ for path-connected spaces (X_i, x_i) by* [5, Theorem 4.6]*. Therefore, $H^T_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i))$ is isomorphic to $Ab^\sigma(\pi_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*))$. We explain the situation a little.*
$H_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i))$ consists of loops modulo the image of the boundary map, $Im(\partial_2)$. A loop f with base point x^* represents an element of $\pi_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*)'$ if and only if f belongs to $Im(\partial_2)$. H^T_1 is defined as H_1 replacing $Im(\partial_2)$ by $\overline{Im(\partial_2)}$, where the topological closure is taken under the topology of a free topological abelian group. (See [5] *for precise definition.) Now, f represents an element of $\pi_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*)^{\sigma\prime}$ if and only if f belongs to $\overline{Im(\partial_2)}$. Hence, $\overline{Im(\partial_2)}/Im(\partial_2)$ is complete modulo the Ulm subgroup by Theorem 4.7 in such a case.*

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, JAPAN
E-mail address: `eda@sakura.tsukuba.ac.jp`

FREE σ-PRODUCTS AND FUNDAMENTAL GROUPS OF SUBSPACES OF THE PLANE

KATSUYA EDA

ABSTRACT. Let $\mathbb{H}$ be the so-called Hawaiian earring, i.e. $\mathbb{H} = \{(x,y) : (x - 1/n)^2 + y^2 = 1/n^2, 1 \leq n < \omega\}$ and $o = (0,0)$. We prove:
(1) If Y is a subspace of a line in the Euclidean plane $\mathbb{R}^2$ and X its complement $\mathbb{R}^2 \setminus Y$ with $x \in X$, then the fundamental group $\pi_1(X,x)$ is isomorphic to a subgroup of $\pi_1(\mathbb{H},o)$.
(2) Let Y be a subspace of a line in the Euclidean plane $\mathbb{R}^2$. Then, $\pi_1(\mathbb{R}^2 \setminus Y, x)$ for $x \in \mathbb{R}^2 \setminus Y$ is isomorphic to $\pi_1(\mathbb{H},o)$, if and only if there exists infinitely many connected components of Y which converge to a point outside of Y.
(3) Every homomorphism from $\pi_1(\mathbb{H},o)$ to itself is conjugate to a homomorphism induced from a continuous map.

1. INTRODUCTION AND SUMMARY

Let X be a subspace of the Euclidean plane $\mathbb{R}^2$. If X has topologically good properties, the fundamental group $\pi_1(X,x)$ becomes a free group whose generators correspond to the holes of X in $\mathbb{R}^2$, i.e. the bounded connected components of $\mathbb{R}^2 \setminus X$. On the other hand, without such conditions the fundamental groups become complicated in general. For instance, consider the so-called Hawaiian earring, i.e. $\mathbb{H} = \{(x,y) : (x - 1/n)^2 + y^2 = 1/n^2, 1 \leq n < \omega\}$. The fundamental group $\pi_1(\mathbb{H},o)$ is uncountable and not free [4, 5, 6, 2]. We introduced 'free σ-products' in [2] and a modified notion 'free π-products' in [1]. They are fitted to present fundamental groups of spaces which are not locally simply connected. In the first section of the present paper, we investigate homomorphisms from free σ-products to free σ-products of n-slender groups, which are introduced in [2]. We shall show that any such homomorphism is conjugate to a standard homomorphism, defined in Definition 2.2. The notion 'standard homomorphism' turns out to be a spatial homomorphism under certain settings, where a homomorphism induced by a continuous map called *spatial* [3, 2]. In Section 3, we investigate fundamental groups of subspaces of the Euclidean plane. One of theorems in the section is the following: If Y is a subspace of a line in the Euclidean plane $\mathbb{R}^2$ and X its complement $\mathbb{R}^2 \setminus Y$ with $x \in X$, then the fundamental group $\pi_1(X,x)$ is isomorphic to a subgroup of $\pi_1(\mathbb{H},o)$.

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Since the Hawaiian earring $\mathbb{H}$ is homotopic to the space $\mathbb{R}^2 \setminus \{(1/n, 0) : 1 \leq n < \omega\}$ (Remark 3.12), we may consider $\mathbb{H}$ itself as one of the spaces X in the statement. In Section 4, using a result in Section 3, we characterize a subspace Y of a line in $\mathbb{R}^2$ for which $\pi_1(\mathbb{R}^2 \setminus Y, x)$ with $x \in \mathbb{R}^2 \setminus Y$ is isomorphic to $\pi_1(\mathbb{H}, o)$. We also discuss the relation between a factor of singular homology [3] and the σ-abelianization [2].

2. Free σ-products and standard homomorphisms

In this section we show that any homomorphism from a free σ-product to a free σ-product of n-slender groups is a natural one. (See Theorem 2.3 below.) We use notions of words of countable length (called σ-words), free σ-products and n-slenderness. For the definitions and basic facts, we refer the reader to [2]. The merit of using σ-words instead of inverse systems is the existence of reduced σ-words. According to it, we can investigate more precise properties like the case of usual words of finite length.

We simply say 'word' instead of 'σ-word'. A word U is called a *subword* of V, if $U \subseteq V$ and $V \simeq XUY$ for some X, Y. The next notion was defined in [2, p.247] for elements of a free σ-product, under the identification of an element and a reduced word. Here, we define it also for words.

Definition 2.1. Let G_i $(i \in I)$ be groups. For a word $W \in \mathcal{W}^\sigma(G_i : i \in I)$, the i-length $l_i(W)$ is the number of elements of G_i which appear in W. For an element x in the free σ-product $\times^\sigma_{i\in I} G_i$, $l_i(x)$ is $l_i(W)$ for the reduced word W of x [2, p.247].

A sequence $(x_j : j \in J)$ of elements of $\times^\sigma_{i\in I} G_i$ is *proper*, if $\{j \in J : l_i(x_j) \neq 0\}$ is finite for each $i \in I$.

For a proper sequence $(x_j : j \in J)$, we can naturally define the infinite multiplication depending on an ordering of J, which is defined in [2, Proposition 1.9]. We shall not distinguish a word W and an element $[W]$ of a free σ-product, since no confusion will occur except the definition of l_i. More precisely, when we are using the notions $\simeq$ or $\overline{W}$, the domain of W, we are interested in words. In other cases, we are interested in elements of groups.

Definition 2.2. Let G_i $(i \in I)$ and H_j $(j \in J)$ be groups. A homomorphism $h : \times^\sigma_{i\in I} G_i \to \times^\sigma_{j\in J} H_j$ is *standard*, if $(h(g_i) : i \in I)$ is proper for any $g_i \in G_i$ $(i \in I)$ and $h(W) = V$ for a word $W \in \mathcal{W}^\sigma(G_i : i \in I)$, where V is the word in $\mathcal{W}^\sigma(H_j : j \in J)$ defined as follows:

(1) $\overline{V} = \{(\alpha, \beta) : \alpha \in \overline{W}, \beta \in \overline{V_\alpha}\}$, where V_α is the reduced word of $h(W(\alpha))$;
(2) The order $(\alpha, \beta) < (\alpha', \beta')$ is lexicographical, i.e. $\alpha < \alpha'$, or $\alpha = \alpha'$ and $\beta < \beta'$;
(3) $V(\alpha, \beta) = V_\alpha(\beta)$ for $(\alpha, \beta) \in \overline{V}$.

The following is the main theorem of this section, which is a free σ-product version of a trivial fact about usual free products, i.e. every homomorphism $h : *_{i\in I} G_i \to *_{j\in J} H_j$ is determined by all restrictions $h \restriction G_i$. We cannot assert the same result for free σ-products. See Remark 2.12.

Theorem 2.3. *Let G_i $(i \in I)$ and H_j $(j \in J)$ be groups. If each H_j is n-slender, every homomorphism $h : \times^\sigma_{i\in I} G_i \to \times^\sigma_{j\in J} H_j$ is conjugate to a standard homomorphism $\overline{h}$, that is, there exist $u \in \times^\sigma_{j\in J} H_j$ such that $h(x) = u^{-1}\overline{h}(x)u$ for $x \in \times^\sigma_{i\in I} G_i$. In addition if the set $\{i \in I : h(g) \neq e$ for some $g \in G_i\}$ is infinite, such u is unique.*

To prove this theorem, some lemmas are necessary. First we prove a lemma about a presentation of an element.

Lemma 2.4. *For $a \in \times^\sigma_{i\in I} G_i$, there exist reduced words V and W in $\mathcal{W}^\sigma(G_i : i \in I)$ such that*

(1) *$a = W^{-1}VW$ and $W^{-1}VW$ is quasi-reduced;*
(2) *VW is reduced;*
(3) *If V is a single letter, $W^{-1}VW$ is reduced; otherwise VV is reduced.*

Proof. Let U be a reduced word for a and take a maximal subword W' of U such that $W'^{-1}V'W' \simeq U$ for some V'. Then, $V'W'$ is reduced. If V' is a single letter or $V'V'$ is reduced, we have done. Otherwise, $V' \simeq g'V''g$ for some $g, g' \in G_i \setminus \{e\}$ and some non-empty word V''. Then, neither the left side letter of W' nor the both side letters of V'' belong to G_i. Let $W \simeq gW'$ and $V \simeq g''V''$, where $g'' = gg'$. Then, $g'' \neq e$ and consequently V and W have the required properties. □

Lemma 2.5. *Let H_j $(j \in J)$ be n-slender groups and $h, h' : \times^\sigma_{i\in I} G_i \to \times^\sigma_{j\in J} H_j$ be homomorphisms. Then, $h \restriction *_{i\in I} G_i = h' \restriction *_{i\in I} G_i$ implies $h = h'$.*

Proof. Suppose that $h(x) \neq h'(x)$, that is, there exists a finite subset E of J such that $p_E h(x) \neq p_E h'(x)$, where $p_E : \times^\sigma_{j\in J} H_j \to *_{j\in E} H_j$ is the projection. Since $*_{j\in E} H_j$ is n-slender by [2, Theorem 3.6], there exists a finite subset F of I such that $p_E h(\times^\sigma_{i\in I\setminus F} G_i) = \{e\}$, see the proof of [2, Proposition 2]. Since $\times^\sigma_{i\in I} G_i = (*_{i\in F} G_i) * (\times^\sigma_{i\in I\setminus F} G_i)$ and $h \restriction *_{i\in I} G_i = h' \restriction *_{i\in I} G_i$, we have $p_E h(x) = p_E h'(x)$, which is a contradiction. □

In the above lemma, assume that $(h(g_i) : i \in I)$ is proper for any $g_i \in G_i$ $(i \in I)$ and let h' be a standard homomorphism defined by the restrictions $h \restriction G_i$ (cf. Definition 2.2). Then, we have $h = h'$, namely h is a standard homomorphism.

Lemma 2.6. *Let H_j $(j \in J)$ be n-slender groups and $h : \times_\omega \mathbb{Z} \to \times^\sigma_{j\in J} H_j$ be a homomorphism. If $x_n \in \times_{\omega\setminus n} \mathbb{Z}$ for $n < \omega$, $\sup\{l_j(h(x_n)) : n < \omega\} < \infty$ for each $j \in J$.*

Proof. To the contrary, suppose $\sup\{l_{j^*}(h(x_n)) : n < \omega\} = \infty$ for some $j^* \in J$. Since we can take a countable subset J' of J so that $j^* \in J'$ and $h(x_n) \in \times^\sigma_{j\in J'} H_j$ and there is a projection from $\times^\sigma_{j\in J} H_j$ to $\times^\sigma_{j\in J'} H_j$, we may assume $J = \omega$. As in the proof of Lemma 2.5, for each projection $p_m : \times^\sigma_{i<\omega} H_i \to *_{i<m} H_i$ there exists k_m such that $p_m h \restriction \times_{\omega\setminus k_m} \mathbb{Z}$ is trivial.

We define $n_0 = 0$ and n_i, m_i by induction as follows. For given n_j $(j \leq i)$ and m_j $(j < i)$, choose $m_i > m_{i-1}$ such that $l_{j^*}(h(x_{n_0} \cdots x_{n_i})) = l_{j^*}(p_{m_i} h(x_{n_0} \cdots x_{n_i}))$. Take $n_{i+1} \geq k_{m_i}$ so that $2l_{j^*}(h(x_{n_0} \cdots x_{n_i})) < l_{j^*}(h(x_{n_{i+1}}))$.

Let W_i be a reduced word for x_{n_i}. Since $(x_{n_i} : i < \omega)$ is a proper sequence, $W_0 \cdots W_i \cdots = W$ becomes a word naturally. Choose m_i so that $l_{j^*}(h(W)) = l_{j^*}(p_{m_i} h(W))$. Since $W_{i+1} \cdots \in \times_{\omega \setminus n_{i+1}} \mathbb{Z} \subseteq \times_{\omega \setminus k_{m_i}} \mathbb{Z}$,

$$\begin{aligned} l_{j^*}(h(W)) = l_{j^*}(p_{m_i} h(W_0 \cdots W_i)) &= l_{j^*}(h(W_0 \cdots W_i)) \\ &< l_{j^*}(h(W_0 \cdots W_i W_{i+1})) = l_{j^*}(p_{m_{i+1}} h(W_0 \cdots W_i W_{i+1})) \\ &= l_{j^*}(p_{m_{i+1}} h(W)) = l_{j^*}(h(W)), \end{aligned}$$

which is a contradiction. □

To get an element $u \in \times^{\sigma}_{j \in J} H_j$ for the conjugate form in Theorem 2.3, we define some notion.

Definition 2.7. For a sequence $(W_n : n < \omega)$ of words, W is a *tail-limit* of $(W_n : n < \omega)$, if the following hold:

(1) For each $\alpha \in \overline{W}$ and for all but finite n there exists a word X_n such that $W_n \simeq X_n W^{\alpha}$, where W^{α} is defined by $\overline{W^{\alpha}} = \{\beta \in \overline{W} : \beta \geq \alpha\}$ and $W^{\alpha}(\beta) = W(\beta)$ for $\beta \geq \alpha$;

(2) W is maximal among words satisfying (1), i.e. if V satisfies (1), $W \simeq XV$ for some word X.

Lemma 2.8. *Let $W_n \in \mathcal{W}(G_i : i \in I)$ $(n < \omega)$ be reduced words. If* $\sup\{l_i(W_n) : n < \omega\} < \infty$ *for each $i \in I$, there exists a unique tail-limit of $(W_n : n < \omega)$.*

Proof. Since each W_n is reduced, a tail-limit is also reduced if it exists. The uniqueness follows from the maximality. To see the existence, we consider tails of W_n. First, take the maximal word V_0 such that for any n there exists X_{n0} with $X_{n0} V_0 \simeq W_n$. Inductively, we extend V_{k-1} to the maximal word V_k such that for any $n \geq k$ there exists X_{nk} with $X_{nk} V_k \simeq W_n$.

Finally, let $\overline{W} = \varinjlim\{\overline{V_n} : n < \omega\}$ and $W(i_n(\alpha)) = V_n(\alpha)$ for $\alpha \in \overline{V_n}$, where $i_n : \overline{V_n} \to \overline{W}$ is the canonical map. If we can see W is a word, we easily see W is the desired word. It suffices to see $l_i(W) \leq \max\{l_i(V_n) : n < \omega\} (\leq \sup\{l_i(W_n) : n < \omega\})$ for each $i \in I$. To the contrary, suppose that there exists $i \in I$ such that $|\{\alpha \in \overline{W} : W(\alpha) \in G_i\}| > \max\{l_i(V_n) : n < \omega\}$ and consequently there exists $\alpha^* \in \overline{W}$ such that $W(\alpha^*) \in G_i$ and $|\{\alpha \in \overline{W} : \alpha^* < \alpha, W(\alpha) \in G_i\}| \geq \max\{l_i(V_n) : n < \omega\}$. By the construction of W, there exists V_n such that $\alpha^* = i_n(\beta)$ for some $\beta \in \overline{V_n}$. Then, $l_i(V_n) > |\{i_n(\gamma) : \beta < \gamma, V_n(\gamma) \in G_i\}| = |\{\alpha \in \overline{W} : \alpha^* < \alpha, W(\alpha) \in G_i\}|$, which is a contradiction. □

Let $\delta_n \in \times_{\omega} \mathbb{Z}$ be the element corresponding 1 in the n-th copy of $\mathbb{Z}$.

Lemma 2.9. *Let H_j $(j \in J)$ be n-slender groups. Then, every homomorphism $h : \times_\omega \mathbb{Z} \to \times^\sigma_{j\in J} H_j$ is conjugate to a standard homomorphism, that is, there exist $u \in \times^\sigma_{j\in J} H_j$ and a standard homomorphism $\overline{h} : \times_\omega \mathbb{Z} \to \times^\sigma_{j\in J} H_j$ such that $h(x) = u^{-1}\overline{h}(x)u$ for $x \in \times_\omega \mathbb{Z}$. In addition if the set $\{n < \omega : h(\delta_n) \neq e\}$ is infinite, such u is unique.*

Proof. By Lemma 2.4, there is a presentation $h(\delta_n) = W_n^{-1}V_nW_n$ for each $n < \omega$, where V_n and W_n satisfy the conditions in the lemma.
(Claim 1) $(V_n : n < \omega)$ is proper.

We can divide our argument into two cases when V_n are single words for all $n < \omega$ and when V_nV_n are reduced for all $n < \omega$. In the former case, let $p_j : \times^\sigma_{j\in J} H_j \to H_j$ be the projection. Then, $p_j(W_n)^{-1}p_j(V_n)p_j(W_n) = p_jh(\delta_n) = e$ for almost all n by the n-slenderness of H_j, which implies $p_j(V_n) = e$ and hence $l_j(V_n) = 0$ for almost all n. To show the properness in the latter case by contradiction, suppose that $\{n < \omega : l_j(V_n) \neq 0\}$ is infinite for some $j \in J$. Since $h(\delta_n{}^n) = W_n^{-1}V_n^nW_n$ and $l_j(h(\delta_n{}^n)) \geq l_j(V_n^n) = nl_j(V_n)$, we get a sequence $(\delta_n{}^n : n < \omega)$ forbidden by Lemma 2.6. Now, we have shown Claim 1.

Let $I = \{n < \omega : h(\delta_n) \neq e\}$. In case I is finite, h itself is a standard homomorphism by Lemma 2.5. Hence, we deal with the case I is infinite. Since $l_j(W_n) \leq l_j([W_n{}^{-1}V_nW_n])$ for $n < \omega$ and $\sup\{l_j([W_n^{-1}V_nW_n]) : n < \omega\} < \infty$ by Lemma 2.6, $\sup\{l_j(W_n) : n < \omega\} < \infty$ for each $j \in J$. By Lemma 2.8, there exists a tail-limit W of $(W_n : n \in I)$.
(Claim 2) $(W_nW^{-1} : n \in I)$ is proper.

Suppose the contrary. There is $j \in J$ such that $l_j([W_nW^{-1}]) \neq 0$ for infinitely many n. (Note that W_nW^{-1} is not reduced in general.) Let $\alpha \in \overline{W}$ be the leftmost element such that $W(\alpha) \in H_j$. (If there is no $\alpha \in \overline{W}$ satisfying $W(\alpha) \in H_j$, we let W^α to be an empty word in the following.) Take n_0 so that $l_j(V_n) = 0$ for any $n \geq n_0$ and there are reduced words X_n $(n \geq n_0, n \in I)$ satisfying $W_n \simeq X_nW^\alpha$. By the hypothesis, there are infinitely many $n \geq n_0$ such that $l_j(X_n) \neq 0$. By the maximality of W, for such n there exists $m \in I$ such that $m > n$ and $l_j([W_nW_m^{-1}]) \neq 0$. Let $c = h(\delta_n)h(\delta_m)$. We shall show that $\sup\{l_j(c^k) : k < \omega\} = \infty$. Remark that some letter $g \in H_j$ in X_n remains in the reduced word of $W_nW_m^{-1}$, i.e. the head part of W_n remains. Hence, if the tail of W_m^{-1} remains in the reduced word of $W_nW_m^{-1}$, it is easy to see $\sup\{l_j(c^k) : k < \omega\} = \infty$ in the following argument. In case W_m^{-1} is cancelled, we may let $W_n = UW_m$, where U is reduced and UW_m is quasi-reduced. Now,

$$\begin{aligned} c^2 &= W_m^{-1}U^{-1}V_nUW_mW_m^{-1}V_mW_mW_m^{-1}U^{-1}V_nUW_mW_m^{-1}V_mW_m \\ &= W_m^{-1}U^{-1}V_nUV_mU^{-1}V_nUV_mW_m. \end{aligned}$$

Since V_m is a single letter or V_mV_m is reduced, at least one of UV_m and V_mU^{-1} is quasi-reduced. Since $l_j(V_m) = 0$ and V_m is non-empty, the right most appearance of members of H_j in U and the left most one in U^{-1} remain in a reduced word of UV_mU^{-1} and hence the tail and head of UV_mU^{-1} remain in its reduced word and $l_j([UV_mU^{-1}]) \geq 2$. Hence, $\sup\{l_j(c^k) : k < \omega\} = \infty$. Now, using members of form $(\delta_n\delta_m)^k$, we can easily get a sequence $x_n \in \times_{\omega\setminus n}\mathbb{Z}$ $(n < \omega)$ with $\sup\{l_j(h(x_n)) : n < \omega\} = \infty$, which contradicts Lemma 2.6. Now, we have proved Claim 2.

By these claims, $(WW_n^{-1}V_nW_nW^{-1} : n < \omega)$ is proper. Let $\overline{h}(x) = Wh(x)W^{-1}$ for $x \in \times_\omega\mathbb{Z}$. Then, $\overline{h}$ is a standard homomorphism and we get the conclusion.

To show the uniqueness, it suffices to show $u^{-1}h(x)u$ is not standard for a standard homomorphism h, if $u \neq e$ and $\{n < \omega : h(\delta_n) \neq e\}$ is infinite. Choose j so that $l_j(u) \neq 0$. There is m such that $l_j(h(\delta_n)) = 0$ for any $n \geq m$ by the standardness of h. If $l_j(h(\delta_n)) = 0$ and $h(\delta_n) \neq e$, $l_j(u^{-1}h(\delta_n)u) \neq 0$. Therefore, a homomorphism $u^{-1}h(x)u$ is not standard. □

Proof of Theorem 2.3. In case $\{i \in I : h(g) \neq e \text{ for some } g \in G_i\} = F$ is finite, $h(\times^\sigma_{i\in I\setminus F}G_i) = \{e\}$ by Lemma 2.5. Hence, h itself is standard.

In the other case, we have $g_n \in G_{i_n}$ $(n < \omega)$ such that $h(g_n) \neq e$ and $i_m \neq i_n$ if $m \neq n$. Take a standard homomorphim $\varphi : \times_\omega\mathbb{Z} \to \times^\sigma_{i\in I}G_i$ such that $\varphi(\delta_n) = g_n$ $(n < \omega)$. By Lemma 2.9, there exists $u \in \times^\sigma_{j\in J}H_j$ such that $u^{-1}h\varphi(x)u$ is standard. We claim $(u^{-1}h(g_i')u : i \in I)$ is proper for any $g_i' \in G_i$. Otherwise, there exist $k_n \in I$ $(n < \omega)$ such that $k_m \neq k_n$ for $m \neq n$ and $(u^{-1}h(g'_{k_n})u : n < \omega)$ is not proper. Take a standard homomorphism $\psi : \times_\omega\mathbb{Z} \to \times^\sigma_{i\in I}G_i$ such that $\psi(\delta_{2n}) = g'_{k_n}$ and $\psi(\delta_{2n+1}) = g_n$. By Lemma 2.9, there exists $v \in \times^\sigma_{j\in J}H_j$ such that $v^{-1}h\psi(x)v$ is standard. Let $\xi : \times_\omega\mathbb{Z} \to \times_\omega\mathbb{Z}$ be a standard homomorphism such that $\xi(\delta_n) = \delta_{2n+1}$. Then, $\varphi = \psi\xi$ holds and both $v^{-1}h\psi\xi(x)v$ and $u^{-1}h\varphi(x)u$ are standard. Since $h\varphi(\delta_n) = h\psi(\delta_{2n+1}) \neq e$, $u = v$ by the uniqueness. Hence, $(u^{-1}h(g'_{k_n})u : n < \omega)$ is proper, which is a contradiction. □

The next notion 'spatial homomorphism' has been defined in [3] and [2], but we restate it for reader's convenience. Let (X, x) and (Y, y) be pointed spaces. A homomorphism $h : \pi_1(X, x) \to \pi_1(Y, y)$ is *spatial*, if there exists a continuous map $f : X \to Y$ with $f(x) = y$ such that $f_* = h$, where $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ is a homomorphism naturally induced from f.

Let (X_i, x_i) be pointed spaces such that $X_i \cap X_j = \phi$ for $i \neq j$. We identify all $x_i(= x^*)$ and define a topology on $\bigcup_{i\in I} X_i$ in the following way: the topology on $\bigcup_{i\in I} X_i\setminus\{x_i\}$ is induced from the topologies of the components; for an open neighborhood U of x^*, each $U \cap X_i$ is not only open in X_i but also $X_i \subseteq U$ for almost all i. We denote this space by $\widetilde{\bigvee}_{i\in I}(X_i, x_i)$.

Proposition 2.10. *Let (X_i, x_i) and (Y_j, y_j) be path-connected pointed spaces such that X_i and Y_j are locally simply connected at x_i and y_j and also first countable at x_i and y_j respectively. For a continuous map $\varphi : \widetilde{\bigvee}_{i\in I}(X_i, x_i) \to \widetilde{\bigvee}_{j\in J}(Y_j, y_j)$, the induced homomorphism $\varphi_* : \pi_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*) \to \pi_1(\widetilde{\bigvee}_{j\in J}(Y_j, y_j), y^*)$ is a standard* *homomorphism under the natural identification* $\pi_1(\widetilde{\bigvee}_{i\in I}(X_i,x_i),x^*) = \times^\sigma_{i\in I}\pi_1(X_i,x_i)$ *and* $\pi_1(\widetilde{\bigvee}_{j\in J}(Y_j,y_j),y^*) = \times^\sigma_{j\in J}\pi_1(Y_j,y_j)$.

In case X_i*'s are copies of the circle, the converse also holds, i.e. any standard homomorphism from* $\times^\sigma_{i\in I}\pi_1(X_i,x_i)$ *to* $\times^\sigma_{j\in J}\pi_1(Y_j,y_j)$ *is a spatial homomorphism.*

Proof. Under the given condition, we can identify $\pi_1(\widetilde{\bigvee}_{i\in I}(X_i,x_i),x^*) = \times^\sigma_{i\in I}\pi_1(X_i,x_i)$ and $\pi_1(\widetilde{\bigvee}_{j\in J}(Y_j,y_j),y^*) = \times^\sigma_{j\in J}\pi_1(Y_j,y_j)$ by [2, Theorem A.1]. Let $u \in \pi_1(\widetilde{\bigvee}_{i\in I}(X_i,x_i),x$ and $j_0 \in J$. Then, u is expressed by a word $W \in \mathcal{W}(\pi_1(X_i,x_i) : i \in I)$. Take a neighborhood U of y_{j_0} in Y_{j_0} so that any loop in U is homotopic to the constant relative to the end points. For almost all $i \in I$, $\varphi(X_i) \subseteq \{y^*\} \cup \bigcup_{j\neq j_0} Y_j \cup U$. So $\{\alpha \in \overline{W} : l_{j_0}(\varphi_*(W(\alpha)) \neq 0\}$ is finite. Now, the standardness of φ_* follows from the definition of the word $W^{\varphi f} \in \mathcal{W}^\sigma(\pi_1(Y_j,y_j) : j \in J)$ in the proof of [2, Theorem A.1] for a loop f in $\widetilde{\bigvee}_{i\in I}(X_i,x_i)$.

Suppose that X_i's are copies of the circle and $h : \times^\sigma_{i\in I}\pi_1(X_i,x_i) \to \times^\sigma_{j\in J}\pi_1(Y_j,y_j)$ is a standard homomorphism. Then each $h(\delta_i)$ is presented by a loop $f_i : [0,1] \to \widetilde{\bigvee}_{j\in J}(Y_j,y_j)$ such that $f_i(0) = f_i(1) = y^*$. Define $\varphi : \widetilde{\bigvee}_{i\in I}(X_i,x_i) \to \widetilde{\bigvee}_{j\in J}(Y_j,y_j)$ by $\varphi \restriction X_i = f_i$ for all $i \in I$. Then, $\varphi_* = h$. □

By Theorem 2.3 and this proposition, we get the following corollary.

Corollary 2.11. *Every homomorphism from the fundamental group of the Hawaiian earring* $\mathbb{H}$ *to itself is conjugate to a spatial homomorphism.*

Remark 2.12. (1) We note that n-slenderness and conjugacy in Theorem 2.3 are necessary. If H is not n-slender, there is a homomorphism $h : \times_\omega\mathbb{Z} \to H$ such that $h(\delta_n) \neq e$ for all n. Let $H_{j_0} = H$. For any $e \neq u \in \times^\sigma_{j\in J}H_j$, $l_{j_0}(u^{-1}h(\delta_n)u) \neq 0$. This implies that h cannot be conjugate to a standard homomorphism. Next, take $u \in \times_\omega\mathbb{Z}$ with $l_n(u) \neq 0$ for infinitely many n. Then, $h(x) = u^{-1}xu$ is not a standard homomorphism.

(2) In Proposition 2.10, the condition 'local simple connectivity and first countability' can be weakened to a certain notion (†) in [1, Proposition 3.5]. The second statement of the proposition holds for 2-simplicial complexes X_i as in [2, Theorem A.4]. Consequently, in addition if $\pi_1(X_i,x_i)$ are n-slender, Corollary 2.11 holds for such 2-simplicial complexes X_i.

3. Fundamental groups of subsets of the plane

In this section we represent fundamental groups of certain subsets of the Euclidean plane. To do this, we introduce some notion for words.

For a subset Y of $\mathbb{R}$, $D \subseteq Y$ is *quasi-dense* in Y, if $(u,v) \cap Y \neq \emptyset$ implies $(u,v) \cap D \neq \emptyset$ for $u,v \in \mathbb{R} \setminus Y$ with $u < v$. Let $Y \subseteq \mathbb{R}$ and D be a countable quasi-dense subset of Y. Any subset of $\mathbb{R}$ is endowed with the linear ordering of

$\mathbb{R}$. For a linearly ordered set L, L^{-1} is the inversely ordered set of L consisting of $\{-u : u \in L\}$.

Let $\mathcal{W}(D) = \mathcal{W}(\mathbb{Z}_d : d \in D)$ where $\mathbb{Z}_d$ is a copy of $\mathbb{Z}$. By $d \in D$ and $-d \in D^{-1}$, we denote 1 in $\mathbb{Z}_d = \mathbb{Z}$ and -1 respectively. We define $|d| = |-d| = d$ for $d \in D$.

Let $W \in \mathcal{W}(D)$ such that each $W(\alpha)$ is d or $-d$ for some $d \in D$. A word $V \in \mathcal{W}(D)$ is a *component* of W, if V is a maximal subword of W which satisfies the following:

(+) There exist $u, v \in \mathbb{R} \setminus Y$ such that $u < v$ and $V : \overline{V} \to (u, v) \cap D$ is the order isomorphism;

or

(−) There exist $u, v \in \mathbb{R} \setminus Y$ such that $u < v$ and $V : \overline{V} \to ((u, v) \cap D)^{-1}$ is the order isomorphism.

For any non-empty word W, the word W^2 is not contained in one component of any word. This observation is important in the argument in the proof of Theorem 4.1.

Definition 3.1. $\mathcal{U}(D, Y)$ consists of $W \in \mathcal{W}(D)$ which satisfies the following:

(1) Each $W(\alpha)$ is d or $-d$ for some $d \in D$;
(2) For any $\alpha \in \overline{W}$, there exists a component V of W such that $\alpha \in \overline{V}$;
(3) Let $\alpha_n \in \overline{W}$ $(n < \omega)$ be an increasing or decreasing sequence. If α_n's belong to different components V_n's, i.e. $\alpha_n \in \overline{V_n}$ with $V_m \neq V_n$ $(m \neq n)$, there exists $x \in \mathbb{R} \setminus Y$ such that $\lim_{n \to \infty} |W(\alpha_n)| = x$.

Now, we can state our main theorem of this section.

Theorem 3.2. *Let Y be a subset of $\mathbb{R}$, D a countable quasi-dense subset of Y and $X = \mathbb{R}^2 \setminus Y \times \{0\}$ with $x_0 \in X$ the base point. Then, the fundamental group $\pi_1(X, x_0)$ is isomorphic to the subgroup $\{[W] : W \in \mathcal{U}(D, Y)\}$ of $\times_D \mathbb{Z}$.*

The basic idea of the proof is the same as those in Theorem A.1 in [2] and Theorem 3.9 in [1], but it's more complicated. To prove this theorem we introduce some auxiliary notions. A *path* f in X is a continuous map $f : [a, b] \to X$ for some $a < b$. For two paths $f : [a, b] \to X$ and $g : [c, d] \to X$ we define $f \equiv g$ as $f(t) = g(c + \frac{t-a}{b-a}(d-c))$ for $a \leq t \leq b$. For a map $f : Y \to \mathbb{R}^2$, define $f_1, f_2 : Y \to \mathbb{R}$ by $(f_1(y), f_2(y)) = f(y)$ for $y \in Y$. For distinct real numbers a and b, we denote the open interval $(\min\{a, b\}, \max\{a, b\})$ by $\langle a, b \rangle$.

For a path $f : [0, 1] \to X$ with $f(0), f(1) \in \mathbb{R} \times \{0\}$, we define a word $W^f \in \mathcal{U}(D, Y) \subseteq \mathcal{W}(D)$ as follows:

Let $\bigcup_{\alpha \in L} (a_\alpha, b_\alpha) = f^{-1}(\mathbb{R} \times (-\infty, 0))$, where $(a_\alpha, b_\alpha) \cap (a_\beta, b_\beta) = \emptyset$ for $\alpha \neq \beta$. Let $\overline{W^f} = \{(\alpha, u) : \alpha \in L, u \in D \cap (f(a_\alpha), f(b_\alpha))$ if $f(a_\alpha) < f(b_\alpha), u \in (D \cap (f(b_\alpha), f(a_\alpha)))^{-1}$ otherwise$\}$ and let $(\alpha, u) < (\beta, v)$, if $a_\alpha < a_\beta$, or $\alpha = \beta$ and $u < v$. Finally, let $W^f(\alpha, u) = u$.

To see $W^f \in \mathcal{U}(D, Y)$, it suffices to check the third property of Definition 3.1, which follows from the continuity of the path.

Definition 3.3. For $u, v \in \mathbb{R}$, we define $S^+(u,v), S^-(u,v) : [0,1] \to \mathbb{R}^2$ as follows:
(1) $S^+(u,v)_1(t) = S^-(u,v)_1(t) = u + t(v-u)$ for $0 \leq t \leq 1$;
(2) $S^+(u,v)_2(t) = -S^-(u,v)_2(t) = \begin{cases} t \cdot |u-v| & 0 \leq t \leq 1/2; \\ (1-t)|u-v| & 1/2 \leq t \leq 1. \end{cases}$

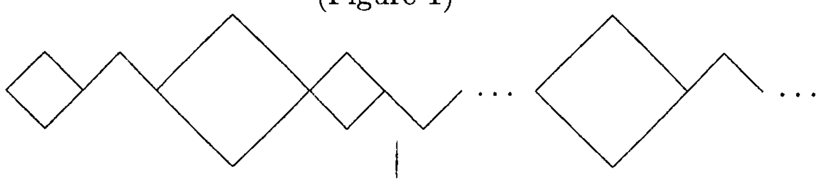
According to the definition, $S^\pm(u,u)(t) = (u,0)$ and $W^{S^+(u,v)}$ is an empty word.

Definition 3.4. A path $g : [a,b] \to \mathbb{R}^2$ is *canonical*, if $g(a), g(b) \in \mathbb{R} \times \{0\}$ and the following hold: Let $\bigcup_{n<\nu}(a_n, b_n) = g^{-1}(\mathbb{R}^2 \setminus \mathbb{R} \times \{0\})$, where $\nu \leq \omega$ and $(a_m, b_m) \cap (a_n, b_n) = \emptyset$ for $m \neq n$, and let $C = \{x : (x,0) \in \mathrm{Im}(g)\}$. Then

(1) $g(a_n) \neq g(b_n)$ and $g \restriction [a_n, b_n] \equiv \begin{cases} S^+(g_1(a_n), g_1(b_n)) \text{ or} \\ S^-(g_1(a_n), g_1(b_n)); \end{cases}$
(2) if $g \restriction [a_n, b_n] \equiv S^-(g_1(a_n), g_1(b_n))$, then $\langle g_1(a_n), g_1(b_n)\rangle \cap Y \neq \emptyset$;
(3) C is nowhere dense and $\mathrm{Im}(g) \cap C \times \mathbb{R} = C \times \{0\}$.

In short, a canonical path consists of paths similar to $S^\pm(u,v)$ and, for instance, is a path in Figure 1. (In the figure, the path may go through lines more than twice.)

(Figure 1)



Lemma 3.5. *Let Y be a subset of $\mathbb{R}$ and $X = \mathbb{R}^2 \setminus Y \times \{0\}$. Then, any path $f : [0,1] \to X$ with $f(0), f(1) \in \mathbb{R} \times \{0\}$ is homotopic to a canonical path g relative to $\{0,1\}$ which satisfies $W^g \simeq W^f$.*

Proof. Let $\bigcup_{n<\mu}(a_n, b_n) = f^{-1}(\mathbb{R}^2 \setminus \mathbb{R} \times \{0\})$, where $\mu \leq \omega$ and $(a_m, b_m) \cap (a_n, b_n) = \emptyset$ for $m < n < \mu$.

Define $H : [0,1] \times [0,1] \to \mathbb{R}^2$ as follows:

$$H(s,t) = \begin{cases} (1-t)f(s) + tS^+(f_1(a_n), f_1(b_n))(\dfrac{s-a_n}{b_n-a_n}) & \text{if } f_2(s) > 0 \text{ and } a_n < s < b_n; \\ (1-t)f(s) + tS^-(f_1(a_n), f_1(b_n))(\dfrac{s-a_n}{b_n-a_n}) & \text{if } f_2(s) < 0 \text{ and } a_n < s < b_n; \\ f(s) & \text{otherwise.} \end{cases}$$

It is easy to see that the image of H is in X. We check the continuity of H especially in case that s is an accumulation point of a_n's. Since the diameters of the sets $\{f(s) : a_n \leq s \leq b_n\}$, $\mathrm{Im}(S^+(f_1(a_n), f_1(b_n)))$ and $\mathrm{Im}(S^-(f_1(a_n), f_1(b_n)))$ converge to 0, the diameters of $\{H(s,t) : a_n \leq s \leq b_n, 0 \leq t \leq 1\}$ also converge to 0. Hence, the continuity of H for such an (s,t) follows from that of f. Let $g(s) = H(s,1)$. Then, $W^g \simeq W^f$ holds, because the values do not move through $\{(x,0) : x \in \mathbb{R}\}$. In case $f_1(a_n) = f_1(b_n)$, $g_1(s) = f_1(a_n) = f_1(b_n)$, and otherwise $g_1(s) \in \langle f_1(a_n), f_1(b_n)\rangle$, for $a_n < s < b_n$.

Let $B = \{g_1(s) : g_2(s) = 0, 0 \leq s \leq 1\}$, $C = (B \setminus \mathrm{int}(B)) \cup \{g_1(0), g_1(1)\}$ and $\bigcup_{n<\mu'}(a'_n, b'_n) = g_1{}^{-1}(\mathbb{R} \setminus C)$, where $\mu' \leq \omega$ and $(a'_m, b'_m) \cap (a'_n, b'_n) = \emptyset$ for $m < n < \mu'$. We observe the following:

(1) C is nowhere dense;

(2) C is equal to the closure of the set $\{g_1(a'_n), g_1(b'_n) : n < \mu'\}$;

(3) if $g_1((a'_n, b'_n)) \cap B \neq \emptyset$, then $g_1((a'_n, b'_n)) \subseteq \mathrm{int}(B)$ and hence $g_1([a'_n, b'_n]) \cap Y = \emptyset$.

The first statement (1) is obvious. To see (2), it suffices to show that C is contained in $\overline{\{g_1(a'_n), g_1(b'_n) : n < \mu'\}}$. For $c \in C$, there is $s \in [0,1]$ with $g_1(s) = c$. If $s \in \overline{\bigcup_{n<\mu'}(a'_n, b'_n)\}}$, then $s \in \overline{\{a'_n, b'_n : n < \mu'\}}$ and hence $c \in \overline{\{g_1(a'_n), g_1(b'_n) : n < \mu'\}}$. Otherwise, there are $a^*, b^* \in \overline{\bigcup_{n<\mu'}(a'_n, b'_n)}$ such that $a^* < s < b^*$ and $(a^*, b^*) \cap \bigcup_{n<\mu'}(a'_n, b'_n) = \emptyset$. Since $g_1([a^*, b^*]) \subseteq C$ and C is nowhere dense, $g_1(a^*) = g_1(s) = g_1(b^*)$ and hence $c \in \overline{\{g_1(a'_n), g_1(b'_n) : n < \mu'\}}$. To see (3), let $g_1(t) \in B$ for some $t \in (a'_n, b'_n)$. Then, $g_1(t) \in \mathrm{int}\, B$ holds. Take a^* and b^* so that $a^* < t < b^*$, $g_1((a^*, b^*)) \subseteq B$ and (a^*, b^*) is such a maximal interval. Then, $g_1(a^*)$ and $g_1(b^*)$ are still in B. By the maximality, the both $g_1(a^*)$ and $g_1(b^*)$ belong to C and hence $a^* = a'_n$ and $b^* = b'_n$, which imply $g_1((a'_n, b'_n)) \subseteq \mathrm{int}(B)$.

Let $I = \{n < \mu' : g_1((a'_n, b'_n)) \cap B = \emptyset\}$, and define $H' : [0,1] \times [0,1] \to X$ as follows:

$$H'(s,t) = \begin{cases} tS^+(g_1(a'_n), g_1(b'_n))(\dfrac{s - a'_n}{b'_n - a'_n}) + (1-t)g(s) & \text{if } n \notin I \text{ and } a'_n \leq s \leq b'_n, \\ & \text{or } n \in I, g_2(s) \geq 0 \text{ and} \\ & a'_n \leq s \leq b'_n; \\ tS^-(g_1(a'_n), g_1(b'_n))(\dfrac{s - a'_n}{b'_n - a'_n}) + (1-t)g(s) & \text{if } n \in I, g_2(s) < 0 \text{ and} \\ & a'_n \leq s \leq b'_n; \\ t(g_1(s), 0) + (1-t)g(s) & \text{otherwise.} \end{cases}$$

In case $n \in I$, $g_2(s) > 0$ for all $s \in (a'_n, b'_n)$ or $g_2(s) < 0$ for all $s \in (a'_n, b'_n)$. Otherwise, $g_1([a'_n, b'_n]) \cap Y = \emptyset$. Therefore, the image of H' is in X. Observe that $H_1(a'_n, t) = g_1(a'_n), H_1(b'_n, t) = g_1(b'_n), H_2(a'_n, t) = (1-t)g_2(a'_n)$, and $H_2(b'_n, t) = (1-t)g_1(b'_n)$. Then, we can check the continuity of H' as in case of H. Let $g'(s) = H'(s,1)$. Then, $C = \{x : (x,0) \in \mathrm{Im}(g')\}$ holds and $g'^{-1}_1(\mathbb{R}^2 \setminus \mathbb{R} \times \{0\}) = \bigcup_{n \in I'}(a'_n, b'_n)$, where $I' = \{n < \mu' : n \notin I \text{ and } g_1(a'_n) \neq g(b'_n)\}$. Now, we can see that g' satisfies the first and third properties of a canonical path and $W^{g'} \simeq W^g \simeq W^f$ holds.

Finally, define $H'' : [0,1] \times [0,1] \longrightarrow \mathbb{R}^2$ as follows:
$H''_1(s,t) = g'_1(s)$;

$$H''_2(s,t) = \begin{cases} (1-2t)g'_2(s) & \text{if } g'_2(s) < 0, a'_n < s < b'_n \text{ and } \langle g'_1(a'_n), g'_1(b'_n)\rangle \cap Y = \emptyset, \\ g'_2(s) & \text{otherwise.} \end{cases}$$

When $H''(s,t)$ $(0 \le t \le 1)$ are not constant, $H''_1(s,t) \in \mathbb{R} \setminus Y$. Therefore, the image of H'' is in X. Let $g''(s) = H''(s,1)$. Then, $W^{g''} \simeq W^{g'} \simeq W^f$ holds and g'' is the desired path. □

Lemma 3.6. *Let $f : [a,b] \to X$ $(a < b)$ be a canonical path and $\{f_1(a), f_1(b)\} \subseteq \{u,v\} = \{x : (x,0) \in \mathrm{Im}(f)\}$ with $u < v$. If $W^f = e$, there is a homotopy $H(s,t)$ in X such that*

(1) $H(s,1) = f(s), H(s,0) = S^+(f_1(a), f_1(b))(\frac{s-a}{b-a})$;
(2) $H(a,t) = f(a), H(b,t) = f(b)$; *and*
(3) $\mathrm{Im}(H) \subseteq \mathrm{Im}(S^+(u,v)) \cup \mathrm{Im}((S^-(u,v))$.

Proof. If $D \cap (u,v) = \emptyset$, $\mathrm{Im}(f) \subseteq \mathrm{Im}(S^+(u,v))$ by the definition of a canonical path. Therefore, f consists of similar maps to $S^+(u,v), S^+(v,u)$ and constant maps and the conclusion is obvious. Otherwise, f consists of similar maps to $S^+(u,v), S^+(v,u), S^-(u,v), S^-(v,u)$ and constant maps, the conclusion is also obvious from $W^f = e$. □

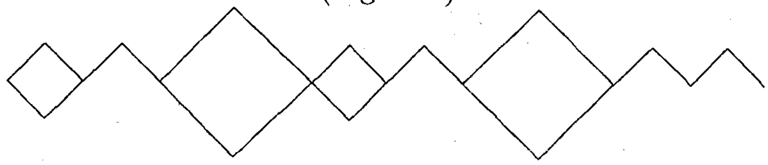
Lemma 3.7. *Let Y be a subset of $\mathbb{R}$, $X = \mathbb{R}^2 \setminus Y \times \{0\}$ and $f : [0,1] \longrightarrow X$ be a path such that $f(0), f(1) \in \mathbb{R} \times \{0\}$ and f is homotopic to $S^+(f_1(0), f_1(1))$ relative to $\{0,1\}$. Then, $W^f = e$ holds.*

Proof. Let $H : [0,1] \times [0,1] \longrightarrow X$ be a homotopy from f to $S^+(f_1(0), f_1(1))$, i.e. $H(0,t) = f(t), H(1,t) = S^+(f_1(0), f_1(1))(t)$ and $H(s,0) = f(0) = (f_1(0),0), H(s,1) = f(1) = (f_1(1),0)$ for $0 \le s,t \le 1$. To see $W^f = e$, let F be a finite subset of D. There exist $a_0 < \cdots < a_n$ in $\mathbb{R} \setminus Y$ satisfying the following:

(1) If there exists an $x \in \mathrm{Im}(H_1)$ such that $d < x < d'$ for $d, d' \in F$, then $d < a_i < d'$ for some a_i.
(2) For any $d \in F$, $a_0 < d < a_n$ holds.
(3) $f_1(0)$ and $f_1(1)$ belong to $\{a_0, \cdots, a_n\}$.

$$\text{Let } X_0 = \bigcup\{\mathrm{Im}(S^+(a_i, a_{i+1})) : 0 \le i \le n-1\} \cup \bigcup\{\mathrm{Im}(S^-(a_i, a_{i+1})) : a_i < d < a_{i+1} \text{ for some } d \in F\}.$$

(Figure 2)



Since $\mathrm{Im}(H)$ is compact, by the properties (1) and (2) we have a subspace Y of X containing $\mathrm{Im}(H)$ and a retraction $r : Y \to X_0$ with the following: $r_1((x,y)) = x$ holds for $a_i < x < a_{i+1}$ where $(a_i, a_{i+1}) \cap F = \emptyset$; $r((x,y)) = (a_0, 0)$ for $x \leq a_0$ and $r((x,y)) = (a_n, 0)$ for $x \geq a_n$; and $r_2((x,y)) \geq 0$ or $r_2((x,y)) \leq 0$ according to $y \geq 0$ or $y \leq 0$ for a member (x,y) in a bounded connected component of $\mathbb{R}^2 \setminus X_0$. We recall W_F is a word for a free product $*_{d\in F}\mathbb{Z}_d$ for a word $W \in \mathcal{W}(D)$. Then, by the properties (1),(2) and (3) $(W^f)_F = (W^{r \cdot f})_F$ holds and moreover $r \cdot S^+(f_1(0), f_1(1))$ is a path in the upper half plane. Therefore, $(W^f)_F = e$ in the free product $*_{d\in F}\mathbb{Z}_d$, which implies $W^f = e$. □

Definition 3.8. For a word $W \in \mathcal{U}(D,Y)$, UV is a *regular decomposition* of W, if $UV \simeq W$ and any component of W is contained in U or V.

If UV is a regular decomposition of $W \in \mathcal{U}(D,Y)$, U and V also belong to $\mathcal{U}(D,Y)$. Especially, any component of W also belongs to $\mathcal{U}(D,Y)$. By the third property of Definition 3.1, it is straightforward to get the following:

Lemma 3.9. *Suppose that UV is a regular decomposition of $W \in \mathcal{U}(D,Y)$ and $\overline{U}$ has no largest element. Then, there exists a unique $u \in \mathbb{R} \setminus Y$ such that $\lim_{n\to\infty} |W(\gamma_n)| = u$ for any unbounded increasing sequence $\gamma_n \in \overline{U}$ $(n < \omega)$. The dual statement for V also holds.*

Lemma 3.10. *For any $W \in \mathcal{U}(D,Y)$, there exists a path $f : [0,1] \to X$ with $f(0), f(1) \in \mathbb{R} \times \{0\}$ such that $W^f \simeq W$.*

Proof. First we decompose W to components. There exist a countable linearly ordered set L and components V_α $(\alpha \in L)$ of W such that $\overline{W}$ is isomorphic to $\{(\alpha, \beta) : \alpha \in L, \beta \in \overline{V_\alpha}\}$ under the lexicographical ordering and $W(\alpha,\beta) = V_\alpha(\beta)$ under the identification through this isomorphism. By the definition of components, there exist $u_\alpha, v_\alpha \in \mathbb{R} \setminus Y$ such that $u_\alpha < v_\alpha$ and $V_\alpha : \overline{V_\alpha} \to (u_\alpha, v_\alpha) \cap D$ or $V_\alpha : \overline{V_\alpha} \to ((u_\alpha, v_\alpha) \cap D)^{-1}$ is an order isomorphism. We choose such u_α, v_α so that $u_\alpha = \inf(D \cap (u_\alpha, v_\alpha))$ if $\inf(D \cap (u_\alpha, v_\alpha)) \notin Y$ and $v_\alpha = \sup(D \cap (u_\alpha, v_\alpha))$ if $\sup(D \cap (u_\alpha, v_\alpha)) \notin Y$.

Shrinking the complement of the Cantor ternary set, we can choose $0 < a_\alpha < b_\alpha < 1$ $(\alpha \in L)$ so that $b_\alpha < a_\beta$ for $\alpha < \beta$ and $\{[a_\alpha, b_\alpha] : \alpha \in L\}$ is discrete in $[0,1]$. Define $f : [0,1] \to X$ as follows: Let $f \restriction [a_\alpha, b_\alpha] \equiv S^-(u_\alpha, v_\alpha)$ if $V_\alpha : \overline{V_\alpha} \to (u_\alpha, v_\alpha) \cap D$ and $f \restriction [a_\alpha, b_\alpha] \equiv S^-(v_\alpha, u_\alpha)$ if $V_\alpha : \overline{V_\alpha} \to ((u_\alpha, v_\alpha) \cap D)^{-1}$. Note that $f \restriction \bigcup\{[a_\alpha, b_\alpha] : \alpha \in L\}$ is continuous by the choice of a_α, b_α. For each $a \in [0,1] \setminus \bigcup\{[a_\alpha, b_\alpha] : \alpha \in L\}$, let $a^- = \sup\{b_\alpha : b_\alpha < a\}$ and $a^+ = \inf\{a_\alpha : a < a_\alpha\}$. We shall define $f(a^-)$ and $f(a^+)$. In case $a^- = \max\{b_\alpha : b_\alpha < a\}$ or $a^+ = \min\{a_\alpha : a < a_\alpha\}$, the value $f(a^-)$ or $f(a^+)$ has been defined respectively. In the following, we treat the case $f(a^-)$ is not defined. Since a induces a regular decomposition UV of W and $\overline{U}$ has no largest element, we get $u \in \mathbb{R} \setminus Y$ in Lemma 3.9. Observe that $\overline{U} = \{(\alpha, \beta) : \alpha \in L, b_\alpha < a, \beta \in \overline{V_\alpha}\}$.

(Claim) $\lim_{n\to\infty} u_{\alpha_n} = \lim_{n\to\infty} v_{\alpha_n} = u$ for any unbounded increasing sequence $(\alpha_n : n \in \omega)$ in $\{\alpha \in L : b_\alpha < a\}$.

To the contrary, suppose that there exist an unbounded sequence $(\alpha_n : n \in \omega)$ and $\varepsilon > 0$ such that $u_{\alpha_n} < u - \varepsilon$ for all $n < \omega$ or $v_{\alpha_n} > u + \varepsilon$ for all $n < \omega$. We only deal with the former case. We observe the following fact:

For $\beta_n \in \overline{V_{\alpha_n}}$ $(n < \omega)$, $|V_\alpha(\beta_n)| \in (u_{\alpha_n}, v_{\alpha_n}) \cap D$ and $\lim_{n\to\infty} |V_{\alpha_n}(\beta_n)| = \lim_{n\to\infty} |W(\alpha_n, \beta_n)| = u$.

(Case 1): $(u - \varepsilon, u) \cap D = \emptyset$. By the fact, $(u_{\alpha_n}, u) \cap D = \emptyset$ for almost all n. Take n so that $(u_{\alpha_n}, u) \cap D = \emptyset$, then $(u_{\alpha_n}, v_{\alpha_n}) \cap D = (u, v_{\alpha_n}) \cap D$. Again by the fact, $u = \inf(D \cap (u_{\alpha_n}, v_{\alpha_n}))$ holds. Therefore, $u_{\alpha_n} = u$ for such u_{α_n} by the choice of u_{α_n}, which contradicts $u_{\alpha_n} < u - \varepsilon$.

(Case 2): $(u - \varepsilon, u) \cap D \neq \emptyset$. Take $d \in (u - \varepsilon, u) \cap D$. Again by the fact, $d < v_{\alpha_n}$ for almost all n, which implies $l_d(W) = \infty$. But, this contradicts $W \in \mathcal{W}(D)$.

According to this claim and its dual claim for V, there exist unique u and v such that when we set $f(a^-) = u$ and $f(a^+) = v$, $f \restriction \{a^-, a^+\} \cup \bigcup_{\alpha \in L}[a_\alpha, b_\alpha]$ becomes continuous at a^- and a^+. Finally, let $f \restriction [a^-, a^+] \equiv S^+(u, v)$ for each $a \in [0,1] \setminus \bigcup_{\alpha \in L}[a_\alpha, b_\alpha]$. Then, f is continuous and $W^f \simeq W$. □

Proof of Theorem 3.2. We may assume that the base point x_0 of a loop $g : [0,1] \to X$ is $(0,1)$. Let $a = \min\{t : g_2(t) = 0\}$ and $b = \max\{t : g_2(t) = 0\}$. According to Lemma 3.10, any word in $\mathcal{U}(D, Y)$ is realized by $W^{g \restriction [a,b]}$ for some loop g in X. Therefore, by Lemma 3.7 the map corresponding f to W^f induces a surjective homomorphism from $\pi_1(X, x_0)$ to $\{[W] : W \in \mathcal{U}(D, Y)\}$. It suffices to show that a path $g \restriction [a, b]$ is homotopic relative to $\{a, b\}$ to a path which lies in the upper half plane $\{(x, y) : y \geq 0\}$ in case $W^{g \restriction [a,b]} = e$. By Lemma 3.5, it also suffices to show that $W^f = e$ implies that f is homotopic to a path $S^+(u^*, v^*)$ relative to $\{0, 1\}$ for a canonical path f with $f(0) = (u^*, 0)$, $f(1) = (v^*, 0)$.

Now, let f be a canonical path with $f(0) = (u^*, 0)$, $f(1) = (v^*, 0)$ and $W^f = e$. Let $C = \{x : (x, 0) \in \mathrm{Im}(f)\}$ and $\bigcup_{n<\mu}(u_n, v_n) = (\min \mathrm{Im}(f_1), \max \mathrm{Im}(f_1)) \setminus C$, where $(u_m, v_m) \cap (u_n, v_n) = \emptyset$ for $m \neq n$ and $\mu \leq \omega$. We define parts of a homotopy $H : [0,1] \times [0,1] \to X$ by induction. First, let $H(s, 1) = f(s)$, $H(0, t) = (u^*, 0)$, $H(1, t) = (v^*, 0)$ and $H(s, 0) = S^+(u^*, v^*)(s)$.

(Step 0) We are given canonical paths f and $S^+(u^*, v^*)$. There exist a division $0 = a_0 < a_1 < \cdots < a_{n_0} < a_{n_0+1} = 1$ such that

($\star$) (1) $f_1(a_i) \in \{u_0, v_0\}$ for $1 \leq i \leq u_0$;

(2) $\mathrm{Im}(f_1 \restriction [a_i, a_{i+1}])$ is contained in the only one of $(-\infty, u_0], [u_0, v_0]$ and $[v_0, \infty)$;
(3) $\mathrm{Im}(f_1 \restriction [a_i, a_{i+1}])$ and $\mathrm{Im}(f_1 \restriction [a_{i+1}, a_{i+2}])$ are contained in different ones of $(-\infty, u_0], [u_0, v_0]$ and $[v_0, \infty)$;
(4) $f_1 \restriction [a_i, a_{i+1}]$ is not constant for $0 \leq i \leq n_0$.

Let $f^i = f \restriction [a_i, a_{i+1}]$ for $0 \leq i \leq n_0$ to simplify the notation. We observe the following:

(1) $\mathrm{Im}(f_1^i) \subseteq [u_0, v_0]$ for an alternate i.
(2) If $\mathrm{Im}(f_1^{i-1}) \cup \mathrm{Im}(f_1^{i+1}) \subseteq [u_0, v_0]$, $f_1(a_i) = f_1(a_{i+1}) = u_0$ or $f_1(a_i) = f_1(a_{i+1}) = v_0$ holds according to $\mathrm{Im}(f_1^i) \subseteq (-\infty, u_0]$ or $\mathrm{Im}(f_1^i) \subseteq [v_0, \infty)$.
(3) If $\mathrm{Im}(f_1^{i-1}) \cup \mathrm{Im}(f_1^{i+1}) \subseteq (-\infty, u_0]$, then $f_1(a_i) = f_1(a_{i+1}) = u_0$. If $\mathrm{Im}(f_1^{i-1}) \cup \mathrm{Im}(f_1^{i+1}) \subseteq [v_0, \infty)$, then $f_1(a_i) = f_1(a_{i+1}) = v_0$.

(Substep 0) Since $W^{f^0} \cdots W^{f^{n_0}} \simeq W^f$ belongs to the free product $\times_{D\cap(-\infty,u_0)}\mathbb{Z} * \times_{D\cap(u_0,v_0)}\mathbb{Z} * \times_{D\cap(v_0,\infty)}\mathbb{Z}$, we have $W^{f^i} = e$ for some i. If there exists f^i such that $f^i(a_i) = f^i(a_{i+1})$ (equivalently $f_1(a_i) = f_1(a_{i+1})$) in addition to $W^{f^i} = e$, we pick such an f^i. Otherwise, $n_0 \leq 2$ holds and $W^{f^i} = e$ for each $0 \leq i \leq n_0$ and we pick f_i arbitrarily.

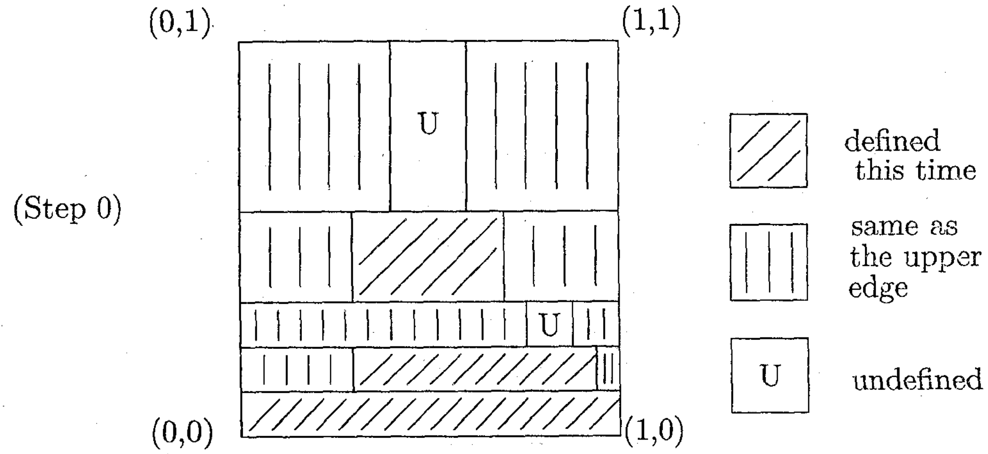
In case $W^{f^i} \in \mathcal{W}(D\cap(u_0, v_0))$, we define $H(s,t)$ for $a_i \leq s \leq a_{i+1}, 1/2 \leq t \leq 1$ by using Lemma 3.6 so that $H(s, 1/2) = S^+(f_1(a_i), f_1(a_{i+1}))((s-a_i)/(a_{i+1}-a_i))$ and let $H(s,t) = f(t)$ for $s \leq a_i$ or $a_{i+1} \leq s$ and $1/2 \leq t \leq 1$. In case $W^{f_i} \notin \mathcal{W}(D\cap(u_0, v_0))$, we just let $H(s, 1/2) = S^+(f_1(a_i), f_1(a_{i+1}))((s-a_i)/(a_{i+1}-a_i))$ and also $H(s,t) = f(t)$ for $s \leq a_i$ or $a_{i+1} \leq s$ and $1/2 \leq t \leq 1$, but do not define for $(a_i, a_{i+1}) \times (1/2, 1)$. Let $g(s) = H(s, 1/2)$. Then, $W^g \simeq W^{f^0} \cdots W^{f^{i-1}} W^{f^{i+1}} \cdots W^{f^{n_0}} = e$. In case $f(a_i) = f(a_{i+1})$ for $2 \leq i \leq n_0 - 2$, $g \restriction [a_i, a_{i+1}]$ is constant and moreover $\mathrm{Im}(g_1 \restriction [a_{i-1}, a_{i+1}])$ is contained in one of $(-\infty, u_0], [u_0, v_0]$ and $[v_0, \infty)$. Unless $\mathrm{Im}(f_1^i) \subseteq [u_0, v_0]$, $\mathrm{Im}(f_1^i) \subseteq (-\infty, u_0]$ or $\mathrm{Im}(f_1^i) \subseteq [v_0, \infty)$.

(Substep k) We have defined $H(s, 1/2^k)(= g(s))$ so that g is a canonical path and $W^g = e$. There exist a division $0 = b_0 < b_1 < \cdots < b_{n_k} < b_{n_k+1} = 1$ which is an amalgamation of $0 = a_0 < a_1 < \cdots < a_{n_0} < a_{n_0+1} = 1$ and has the property $(\star)$ for g. We work in $[0,1] \times [1/2^{k+1}, 1/2^k]$ as in Substep 0 using g instead of f and $b_0, b_1, \cdots, b_{n_k+1}$ instead of $a_0, a_1, \cdots, a_{n_0+1}$.

The substeps will finish in at most n_0-step, say k step. Then, we have defined $H(s, 1/2^{k+1}) = g(s)$ and W^g is an empty word. Since the image of g is in the upper half plane, i.e. $\{(x,y) : y \geq 0\}$, we define

$$H(s,t) = 2^{k+1} t g(s) + 2^{k+1}(1/2^{k+1} - t) S^+(u^*, v^*)$$

for $0 \leq s \leq 1, 0 \leq t \leq 1/2^{k+1}$.



(Step k) After the $(k-1)$-step, there possibly exist finitely many subrectangles of $[0,1] \times [0,1]$ on which H has not been defined. Their forms are $[c,d] \times (\Sigma_{i=1}^{m-1} s_i/2^i + 1/2^m, \Sigma_{i=1}^{m-1} s_i/2^i + 1/2^{m-1})$, where $s_i = 0$ or 1. H has been defined on all the sides of these rectangles. In this step we work in each rectangle as in Step 0 replacing (u_0, v_0) by (u_k, v_k), as if the rectangle is $[0,1] \times [0,1]$.

If the above procedure finishes in finite steps, we have done. Otherwise, after all the steps $H(s_0, t_0)$ may not be defined. We consider such (s_0, t_0). In each k-step, there exists an rectangle R_k which contains (s_0, t_0). On their sides, i.e. ∂R_k, $H(s,t)$ is defined and $\mathrm{Im}(H \restriction \partial R_k)$ is contained in one of closed components of $[\min \mathrm{Im}(f_1), \max \mathrm{Im}(f_1)] \setminus \bigcup_{i=0}^{k-1} (u_i v_i)$, say $[w_0, w_1]$. Then, $\mathrm{Im}(H \restriction \partial R_k)$ is contained in $\{(x,y) : w_0 \leq x \leq w_1, |y| \leq (w_1 - w_0)/2\}$. Since $\bigcup_{n<\mu}(u_n, v_n)$ is dense in $[\min \mathrm{Im}(f_1), \max \mathrm{Im}(f_1)]$, $\mathrm{Im}(H_1 \restriction \partial R_k)$ converge to some $c \in C$ and consequently $\mathrm{Im}(H \restriction \partial R_k)$ converge to $(c, 0)$. Let $H(s_0, t_0) = (c, 0)$. Now, the continuity of H is clear by definition and the proof has been finished. □

After Theorem 3.2 the following natural question occurs, which we cannot answer so far.

Question 3.11. Is the fundamental group of any subspace of the plane isomorphic to a subgroup of that of the Hawaiian earring? More specifically, is the fundamental group of the Sierpinski gasket (or carpet) isomorphic to a subgroup of that of the Hawaiian earring?

Remark 3.12. (1) First, we remark that the Hawaiian earring $\mathbb{H}$ is homotopic to the space $\mathbb{R}^2 \setminus \{p_n : n < \omega\}$ where p_n $(n < \omega)$ converge to $c \notin \{p_n : n < \omega\}$. Since $\mathbb{R}^2 \setminus \{p_n : n < \omega\}$ is homeomorphic to $\mathbb{R}^2 \setminus \{(3/2^n, 0) : 1 \leq n < \omega\}$, it suffices to show that $\mathbb{H}$ is homotopic to $X = \bigcup_{1 \leq n < \omega} \{(x,y) : (x - 3/2^n)^2 + y^2 = 1/2^{2n}\} \cup \{(0,0)\}$. Making $\{(x,y) : y \geq 0\} \cap X$ as one point, we get a space homeomorphic to $\mathbb{H}$. Let $\varphi : X \to \mathbb{H}$ be the map through this quotient. Next, let $\psi : \mathbb{H} \to X$ be a continuous map satisfying the following: Let $\psi(0,0) = (0,0)$, $\psi(1/n, 1/n) = \psi(1/n, -1/n) = (1/2^{n-1}, 0)$, $\psi(\frac{1}{n}(\cos\theta + 1), \frac{1}{n}\sin\theta) \in \mathbb{R} \times [0, \infty)$ $(\pi/2 \leq \theta \leq 3\pi/2)$

and $\psi(\frac{1}{n}(\cos\theta+1), \frac{1}{n}\sin\theta) = (\frac{3}{2^n}+\frac{1}{2^n}\cos 2\theta, \frac{1}{2^n}\sin 2\theta)$ $(-\pi/2 \leq \theta \leq \pi/2)$. Then, it is easy to see that both $\varphi\psi$ and $\psi\varphi$ are homotopic to the identities on $\mathbb{H}$ and X respectively.

(2) As mentioned in the introduction of this paper, the fundamental groups of subspaces of the plane are related to the numbers of holes of the subspaces in the plane. However, in the cases treated in Theorem 3.2 this correspondence is not clear. Here, we offer some examples which clarify this correspondence.

Let Y be a subset of $\mathbb{R}$ and $X = \mathbb{R}^2 \setminus Y \times [0,1]$. Take $a_0 \in \mathbb{R}\setminus Y$. Then, $\pi_1(X,x)$ becomes a free group whose generator is the set $\{Y \cap (a,a_0), Y \cap (a_0,b) : a,b \in \mathbb{R}\setminus Y, a < a_0 < b\}$. This holds, since any loop in X cannot go and back between $\{(x,y) : y \leq 1\}$ and $\{(x,y) : y \leq 0\}$ infinitely many times, which is very different from the case of Theorem 3.2, i.e. $\{(x,y) : y > 0\}$ and $\{(x,y) : y < 0\}$ instead. Consider the two cases when Y are the set of rationals and that of irrationals. The generators of the free groups are uncountable and countable respectively according to $Y = \mathbb{Q}$ and $Y = \mathbb{R}\setminus\mathbb{Q}$. (Not the converse!) Hence, the correspondence is not between generators and holes, but between generators and paths modulo homotopy by definition after all.

4. Homomorphisms from $\pi_1(\mathbb{H},(1,1))$ and σ-abelianization

In this section we investigate homomorphisms from $\pi_1(\mathbb{H},(1,1))$ to the fundamental groups appearing in Theorem 3.2. Using this we shall discuss relationship between the 'σ-abelianization' [2] and a 'canonical factor' of singular homology [3]. The next theorem generalizes Corollary 2.11.

Theorem 4.1. *Let X and Y be as in Theorem 3.2 and $x_0 = (0,1)$. Then, every homomorphism $h : \pi_1(\mathbb{H},(1,1)) \to \pi_1(X,x_0)$ is conjugate to a spatial homomorphism, i.e. there exist a continuous map $f : \mathbb{H} \to X$ with $f((1,1)) = x_0$ and $u \in \pi_1(X,x_0)$ such that $h(x) = u^{-1}f_*(x)u$.*

To prove this theorem, we need some lemmas. In the following, let X, Y and D be as in Theorem 3.2.

Lemma 4.2. *Let $h : \times_\omega\mathbb{Z} \to \{[W] : W \in \mathcal{U}(D,Y)\} \hookrightarrow \times_D\mathbb{Z}$ be a standard homomorphism. If $h(\delta_n) \neq e$ for infinitely many $n < \omega$, there exists a unique $c \in \mathbb{R}\setminus Y$ such that $\lim_{n\to\infty} \operatorname{supp} h(\delta_n) = c$, where $\operatorname{supp} u = \{d \in D : l_d(u) \neq 0\}$ for $u \in \times_D\mathbb{Z}$ and the limit is taken over for n with $h(\delta_n) \neq e$.*

Proof. Let $u_n = h(\delta_n)$. First, we show the uniqueness of an accumulation point. To the contrary, suppose that O and P are open subsets of $\mathbb{R}$ such that $\overline{O}\cap\overline{P} = \emptyset$ and both $\{n : \operatorname{supp} u_n \cap O \neq \emptyset\}$ and $\{n : \operatorname{supp} u_n \cap P \neq \emptyset\}$ are infinite. Since $(u_n : n < \omega)$ is a proper sequence by Definition 2.2, we can take a subsequence $(u_{n_k} : k < \omega)$ and $d_k \in D$ $(k < \omega)$ so that $d_{2k} \in \operatorname{supp} u_{n_{2k}} \cap O$, $d_{2k+1} \in \operatorname{supp} u_{n_{2k+1}} \cap P$ and $d_k \notin \operatorname{supp} u_{n_j}$ for any $j > k$. Then, inductively choose m_k so large that $l_{d_k}(u_{n_0}^{m_0}\cdots u_{n_k}^{m_k}) \neq 0$. A reduced word W for $u_{n_0}^{m_0}\cdots u_{n_k}^{m_k}\cdots$ becomes of form $W_0\cdots W_k\cdots$ such that

$d_k \in \operatorname{supp} W_k$, but $d_k \notin \operatorname{supp} W_j$ for $j > k$, since the right most appearance d_k in the reduced word for $u_{n_k}^{m_k}$ remains in W. There exists $W' \in \mathcal{U}(D, Y)$ such that $W' = W$. Then, $l_{d_k}(W') \neq 0$ for all k. By Lemma 3.10, there exists a path $f : [0,1] \to X$ with $f(0), f(1) \in \mathbb{R} \times \{0\}$ such that $W^f \simeq W'$. Since the values of f_1 cannot be taken alternately in O and P infinitely many times, we get a contradiction.

Next, we show that the sequence $\operatorname{supp} u_n$ $(n < \omega)$ is bounded in $\mathbb{R}$. Suppose the contrary. We take an increasing sequence $(n_k : k < \omega)$ and an unbounded sequence $(d_k : k < \omega)$ in D such that $d_k \in \operatorname{supp}\ u_{n_k}$, $d_k \notin \operatorname{supp}\ u_{n_j}$ for any $j < k$ and $d_k \notin \operatorname{supp}(u_{k_j})$ for any $j \geq k+1$. Then, a reduced word W for $h(\delta_{n_0}\delta_{n_1}\cdots\delta_{n_k}\cdots)$ contains all letters d_k $(k < \omega)$, since d_k in the reduced word for u_{n_k} remains in W. On the other hand, by the properties (2) and (3) of Definition 3.1, $\operatorname{supp}[W]$ is bounded for any $W \in \mathcal{U}(D, Y)$. Now, a contradiction occurs.

Finally, we show $c = \lim_{n\to\infty} \operatorname{supp} u_n$ does not belong to Y. Suppose the contrary. We get a subsequence u_{n_k} and $d_k \in \operatorname{supp} u_{n_k}$ such that $\lim_{k\to\infty} d_k = c$ and $d_k \notin \operatorname{supp} u_{n_j}$ for any $j > k$. Then, similarly as in the proof of uniqueness we inductively choose m_k so large that one of d_k and $-d_k$ appears at least twice in the reduced word for $u_{n_k}^{m_k}$ remains in the reduced word for $u_{n_0}^{m_0}\cdots u_{n_k}^{m_k}$. Since $d_k \notin \operatorname{supp}\ u_{n_j}$ for any $j > k$, these $\pm d_k$ appear in the reduced word W for $u_{n_0}^{m_0}\cdots u_{n_k}^{m_k}\cdots$. Then, W is of form $W_0\cdots W_k\cdots$ where $\pm d_k$ appears at least twice in W_k. Note that two appearances of a letter in W belong to different components of W. Then, for $k \neq k'$ there are appearances of $\pm d_k$ and $\pm d'_k$ which belong to different components of W. Take a path f so that $W^f \simeq W$ by By Lemma 3.10. Then, $\operatorname{Im}(f) \cap \langle d_k, d_{k'}\rangle \times \{0\} \neq \emptyset$. This implies $(c, 0) \in \operatorname{Im} f$, which is a contradiction. □

To investigate $\mathcal{U}(D, Y)$ more, we need a notion 'g-reduced', an abbreviation of 'generator-wise reduced.' Since arguments involving the difference between the reducedness and g-reducedness are tedious, we explain the intention of introducing the g-reducedness here. Let $W^{-1}VW$ be a presentation according to Lemma 2.4 with $l_d(V) \neq 0$ for $d \in D$. If the length of V is greater than or equal to 2, $l_d([(W^{-1}VW)^m]) = l_d([W^{-1}V^mW])$ increases according to the increase of m. However, $l_d([(W^{-1}dW)^m]) = l_d([W^{-1}d^mW]) = l_d([W^{-1}dW])$ for any m. In the proof of Theorem 4.1, we need to make a distinction between $W^{-1}d^mW$ and $W^{-1}dW$ as forms of words. Hence, we do not want to treat d^m as a single letter, but want to treat as the m-letters $d\cdots d$.

Definition 4.3. A word $W \in \mathcal{W}(D)$ is *g-reduced*, if $W(\alpha) = d$ or $-d$ $(d \in D)$ for each $\alpha \in \overline{W}$ and $U \neq e$ for any nonempty U with $XUY \simeq W$.

For instance, the word ddd is g-reduced and quasi-reduced but not reduced, and $d(-d)d$ is quasi-reduced but not g-reduced. For any word $W \in \mathcal{W}(D)$, there exists a unique g-reduced word of W, as in case of a reduced word [2, Theorem 1.4]. The proof is its easy corollary and so we omit it. The difference of the reduced word W' of W and the g-reduced word W'' of W is such that a letter d^n or $(-d)^n$ in W' corresponds to $d\cdots d$ or $(-d)\cdots(-d)$ in W'' respectively. Therefore, the g-reduced word of a word in $\mathcal{U}(D,Y)$ also belongs to $\mathcal{U}(D,Y)$ and a reduced word in $\mathcal{U}(D,Y)$ itself is g-reduced.

Lemma 4.4. *Let $U \in \mathcal{U}(D,Y)$ be a g-reduced word. Then, there exist unique g-reduced words $V, W \in \mathcal{U}(D,Y)$ such that $W^{-1}VW \simeq U$ and VV is g-reduced.*

Proof. As in the proof of Lemma 2.4, just looking at U from the right hand side, we get a maximal W so that $U \simeq W^{-1}VW$. Then, V, W and VV are g-reduced. Since VV is not g-reduced unless W is maximal, the uniqueness is clear. We show that $W \in \mathcal{U}(D,Y)$, which implies $W^{-1} \in \mathcal{U}(D,Y)$. Since a similar proof can be done for V, we omit the proof for V.

Each $\alpha \in \overline{W}$ belongs to $\overline{U'}$ for some component U' of U. Let U' be an order-isomorphism from $\overline{U'}$ to $(u,v) \cap D$ or $((u,v) \cap D)^{-1}$ with $u, v \in \mathbb{R} \setminus Y$. We only deal with the case of $(u,v) \cap D$. Take the maximal $W' \subseteq W$ such that $U' \simeq SW'$ for some S. If we can take $w \in \mathbb{R} \setminus Y$ so that $W' : \overline{W'} \to (w,v) \cap D$ is an order-isomorphism, W' becomes a component of W and we have done. Otherwise, there exists a unique $w \in Y$ such that $W' : \overline{W'} \to (w,v) \cap D$ is an order-isomorphism, $w = \inf((w,v) \cap D)$ and $(w - \varepsilon, w] \cap D \neq \emptyset$ for any $\varepsilon > 0$, because D is quasi-dense in Y. Hence, the tail of S contains a subword corresponding to $(w - \varepsilon, w] \cap D$ for some $\varepsilon > 0$. Then, W'^{-1} in $W^{-1}V$ is also not a component of U and there is a part $((w - \varepsilon', w] \cap D)^{-1}$ neighboring to the right of W'^{-1} for some $\varepsilon' > 0$, which contradicts the maximality of W. □

The next lemma is easy to see and we omit its proof.

Lemma 4.5. *Suppose that $W^{-1}VW = W'^{-1}V'W'$ holds, where the left side and the right side of the equation are the presentations according to Lemma 2.4 and 4.4 respectively and $V, W, V', W' \in \mathcal{W}(D)$. Let W'' be a reduced word of W'. Then, $W'' \simeq W$ holds, or there exist $d \in D$ and non-zero integer n such that $d^n W'' \simeq W$.*

Lemma 4.6. *Let $h : \times_\omega \mathbb{Z} \to \times_D \mathbb{Z}$ be a homomorphism and $h(\delta_n) = {W_n}^{-1} V_n W_n = {W'_n}^{-1} V'_n W'_n$, where the second term and third one are presentations according to Lemmas 2.4 and 4.4 respectively. Let W and W' be the tail-limits of $(W_n : n < \omega)$ and $(W'_n : n < \omega)$ respectively. Then, $W = W'$ holds.*

Proof. In this proof, we denote the g-reduced word of U by U^* for $U \in \mathcal{W}(D)$. As in the proof of Lemma 2.8, the tail-limit W is taken as a direct limit of $U_0 \subset U_1 \subset \cdots$, where each U_n is the maximal word such that there exist Y_{nk} $(k \geq n)$ with $W_k \simeq Y_{nk} U_n$. If $(U_n : n < \omega)$ is not eventually constant, the tail-limit of $(U'_n : n < \omega)$ is equal to W', where U'_n is a g-reduced word of U_n, and consequently $W = W^* \simeq W'$ holds. Next, we deal with the case that $(U_n : n < \omega)$ is eventually constant, i.e. $U_n \simeq W$ for $n \geq n_0$. Let $Y_n = Y_{nn}$. Since $V_n W_n = V_n Y_n W$ is reduced, $V_n^* Y_n^* W^*$ is g-reduced. Therefore, the tail-limit W' of $(W'_n : n < \omega)$ is a subword of the tail-limit of $(Y_n^* W^* : n < \omega)$. We claim that the tail-limit of $(Y_n^* W^* : n < \omega)$ is W^*. To the contrary, suppose the existence of a non-empty word X such that XW^* is a subword of $Y_n^* W^*$ for almost all $n < \omega$. This contradicts the fact $(Y_n^{-1} V_n Y_n : n < \omega)$ is a proper sequence (cf. the proof of Lemma 2.9). Therefore, W' is a subword of W^*. To see $W' \simeq W^*$ by contradiction, suppose that $W^* \simeq XW'$ for some non-empty word X. If Y_n is non-empty, W_n' contains $W^* \simeq XW'$ in tail by Lemma 4.5. Therefore, in case almost all $Y_n W$ are non-empty, almost all W_n' contain XW' in tail, which contradicts the maximality of W'. In case infinitely many Y_n are empty, infinitely many $X^{-1}V_n^*$ are not g-reduced, otherwise almost all W_n' contain $W^* \simeq XW'$ in tail. This contradicts that $(V_n : n < \omega)$ is proper (cf. the proof of Lemma 2.9). Now, $W^* \simeq W'$ and hence $W = W'$ holds. □

Lemma 4.7. *Let U_n $(n < \omega)$ be g-reduced words such that U_n is a subword of U_{n+1} with $U_{n+1} \simeq X_n U_n$ for some X_n. Then, $\bigcup_{n<\omega} U_n$ is a g-reduced word.*

Proof. Suppose not. There exists a non-empty subword U of $\bigcup_{n<\omega} U_n$ such that $U = e$. U is not contained in any U_n. Choose d so that $l_d(U) \neq 0$. Then, U is of form $U'V'$ where $l_d(U') = 0$, U' and V' are non-empty and the left most letter of V' belongs to $\mathbb{Z}_d$. Consider U as a word of the free product $\mathbb{Z}_d * (\times_{D\setminus\{d\}}\mathbb{Z})$. Then, V' contains a non-empty subword V'' with $V'' = e$. Since V' is contained in some U_n, a contradiction occurs. □

Proof of Theorem 4.1. By Theorem 3.2, we can identify $\pi_1(X, x_0) = \{[W] : W \in \mathcal{U}(D,Y)\} \subset \times_D\mathbb{Z}$. We also identify $\pi_1(\mathbb{H}, (1,1)) = \times_\omega\mathbb{Z}$ [2, Theorem A.1]. In case $h(\delta_n) = e$ for almost all $n < \omega$, it is straightforward to see that h itself is spatial. Hence, we assume that the set $I = \{n < \omega : h(\delta_n) \neq e\}$ is infinite. By Theorem 2.3, there exist $w \in \times_D\mathbb{Z}$ and a standard homomorphism $\overline{h} : \pi_1(\mathbb{H}, (1,1)) \to \times_D\mathbb{Z}$ such that $h(x) = w^{-1}\overline{h}(x)w$.

(Claim) $w \in \{[W] : W \in \mathcal{U}(D,Y)\}$.

First we prove the theorem by assuming this claim. By the claim, the image of $\overline{h}$ is a subgroup of $\{[W] : W \in \mathcal{U}(D,Y)\}$. By Lemma 4.2 there exists a unique $c \in \mathbb{R} \setminus Y$ such that $\lim_{n\to\infty} \operatorname{supp} \overline{h}(\delta_n) = c$. For each $1 \leq n < \omega$ we choose a loop $f_n : [0,1] \to X$ so that $f_n(0) = f_n(1) = (c, 0)$, $W^{f_n} = \overline{h}(\delta_n)$ and $\lim_{n\to\infty} \operatorname{Im}(f_n) = (c, 0)$. Finally, we define $\varphi : \mathbb{H} \to X$ as follows:

(1) $\varphi(\frac{1}{n}(\cos\theta + 1), \frac{1}{n}\sin\theta) = f_n(\frac{1}{2\pi}(\theta + \pi))$ for $2 \leq n < \omega, -\pi \leq \theta \leq \pi$,

(2) $\varphi(\cos\theta + 1, \sin\theta) = f_1(\frac{1}{\pi}(\theta + \pi))$ for $-\pi \leq \theta \leq 0$, $\varphi(\cos\theta + 1, \sin\theta) = ((1 - 2\theta/\pi)c, 2\theta/\pi)$ for $0 \leq \theta \leq \pi/2$, and $\varphi(\cos\theta + 1, \sin\theta) = ((2\theta/\pi - 1)c, 2 - 2\theta/\pi)$ for $\pi/2 \leq \theta \leq \pi$.

Then, φ is continuous, $\varphi((1,1)) = x_0 (= (0,1))$ and $\varphi_* = \overline{h}$.

In the remaining part, we prove the claim. Recall that w was given by the tail-limit of $(W_n : n \in I)$, where $h(\delta_n) = W_n^{-1}V_nW_n$ $(n \in I)$ are of form in Lemma 2.4. By lemma 4.6, we may assume $W_n^{-1}V_nW_n$ $(n \in I)$ are of form in Lemma 4.4. Hence, $W = \bigcup_{n<\omega} U_{i_n}$ holds, where $I = \{i_n : n < \omega\}$, $i_n < i_{n+1}$, $U_{i_0} \subset U_{i_1} \subset \cdots$ and each U_{i_k} is the maximal word such that there exist Y_{nk} $(n \geq k)$ with $W_n \simeq Y_{nk}U_k$. Using the maximality of U_{i_k} and reasoning as in the proof of Lemma 4.4, we can see $U_{i_k} \in \mathcal{U}(D,Y)$. Therefore, in case $W \simeq U_{i_n}$ for some n, $W \in \mathcal{U}(D,Y)$ holds. In case $W \not\simeq U_{i_n}$ for any n, let $(j_n : n < \omega)$ be the subsequence of $(i_n : n < \omega)$ such that $j_m < i_n \leq j_{m+1}$ implies $U_{i_n} = U_{j_{m+1}}$. Then, $U_{j_m} \subsetneq U_{j_{m+1}}$ for $m < \omega$ and U_{j_m} is a maximal subword of $U_{j_{m+1}}$ such that a g-reduced word for $h(\delta_{j_m})$ is of form $U_{j_m}^{-1} X U_{j_m}$ for some X. For the simplicity of notation, let n denote j_n. Then, $W = \bigcup_{n<\omega} U_n$, where W is g-reduced by Lemma 4.7. LetX_{nm} and X_n are g-reduced words such that $X_{nm} = U_n U_m^{-1}$ for $m \geq n$ and $X_n = U_n W^{-1}$. Then, $X_n X_m^{-1} = X_{nm}$ holds. By induction we choose $n_0 \leq n_k < n_{k+1} < \omega, d_k \in D$ so that $l_{d_k}(V_{n_k}) \neq 0$ and $l_{d_k}(\overline{h}(x)) = 0$ for any $x \in \times_{n \geq n_{k+1}} \mathbb{Z}$. Next, we choose large enough m_k by induction so that $m_0 > 2l_{d_0}(W) + 2$ and $m_k > l_{d_k}(T_{k-1}) + 2l_{d_k}(W) + 2$ for $k \geq 1$, where T_{k-1} is the g-reduced word for $\overline{h}(\delta_{n_{k-1}})$. Let U be a g-reduced word for $h(\delta_{n_0}^{m_0} \cdots \delta_{n_k}^{m_k} \cdots)$and denote $Y_{nn} = Y_n$ for simplicity. Now, it holds that

$$\begin{aligned} \overline{h}(\delta_n^{m_n}) &= W W_n^{-1} V_n^{m_n} W_n W^{-1} \\ &= W U_n^{-1} Y_n^{-1} V_n^{m_n} Y_n U_n W^{-1} \\ &= X_n^{-1} Y_n^{-1} V_n^{m_n} Y_n X_n. \end{aligned}$$

Observe the following:

(1) $\overline{h}$ is a standard homomorphism;
(2) $(V_n^{m_n} : n < \omega)$ and $(Y_n X_n : n < \omega)$ are proper sequences;
(3) $V_n^{m_n}$ and $Y_n X_n$ are g-reduced.

Then, we get

$$\begin{aligned} \overline{h}(\delta_{n_0}^{m_0} \cdots \delta_{n_k}^{m_k} \cdots) &= X_{n_0}^{-1} Y_{n_0}^{-1} V_{n_0}^{m_0} Y_{n_0} X_{n_0} \cdots X_{n_k}^{-1} Y_{n_k}^{-1} V_{n_k}^{m_k} Y_{n_k} X_{n_k} \cdots \\ &= X_{n_0}^{-1} Y_{n_0}^{-1} V_{n_0}^{m_0} Y_{n_0} X_{n_0 n_1} Y_{n_1}^{-1} \cdots X_{n_{k-1} n_k}^{-1} Y_{n_k}^{-1} V_{n_k}^{m_k} Y_{n_k} X_{n_k n_{k+1}} \cdots . \end{aligned}$$

Since Y_{n_k} or $Y_{n_{k+1}}^{-1}$ may be empty, the word $V_{n_k}^{m_k} Y_{n_k} X_{n_k n_{k+1}} Y_{n_{k+1}}^{-1} V_{n_{k+1}}^{m_{k+1}}$ may not be g-reduced. We observe the appearances of εd_{n_k} and $\varepsilon d_{n_{k+1}}$ for $\varepsilon = \pm 1$, i.e. $l_{d_k}(V_k^{m_k}) \geq m_k$, $l_{d_{k+1}}(Y_{n_k}) \leq l_{d_{k+1}}(T_k)$, $l_{d_k}(Y_{n_{k+1}}^{-1} V_{n_{k+1}}^{m_{k+1}}) = 0$, $l_{d_k}(X_{n_k n_{k+1}}) \leq l_{d_k}(W)$ and $l_{d_{k+1}}(X_{n_k n_{k+1}}) \leq l_{d_{k+1}}(W)$. Then, we can see a g-reduced word of this word is of form

$$V_{n_k}^{m'_k} Z_{k+1} V_{n_{k+1}}^{m'_{k+1}}$$

for some g-reduced word Z_{k+1}, where

$$m'_k > l_{d_k}(T_{k-1}) + l_{d_k}(W) + 2 \quad \text{and} \quad m'_{k+1} > l_{d_{k+1}}(W) + 2 \quad \cdots \quad (\star\star).$$

Now,

$$\begin{aligned} U &= h(\delta_{n_0}^{m_0} \cdots \delta_{n_k}^{m_k} \cdots) \\ &= W^{-1} \overline{h}(\delta_{n_0}^{m_0} \cdots \delta_{n_k}^{m_k} \cdots) W \\ &= W^{-1} X_{n_0}^{-1} Y_{n_0}^{-1} V_{n_0}^{m_0} Y_{n_0} X_{n_0 n_1} Y_{n_1}^{-1} \cdots X_{n_{k-1} n_k}^{-1} Y_{n_k}^{-1} V_{n_k}^{m_k} Y_{n_k} X_{n_k n_{k+1}} \cdots W. \end{aligned}$$

According to ($\star\star$), we can see each $V_{n_k}^2$ remains in the g-reduced word U and consequently W remains in its tail. Therefore, U is of form

$$Z_0 V_{n_0}^{m''_0} Z_1 V_{n_1}^{m''_1} \cdots V_{n_k}^{m''_k} Z_{k+1} V_{n_{k+1}}^{m''_{k+1}} \cdots W,$$

where $m''_k \geq 2$. Since $V_{n_k}^{m''_k}$ does not belong to the same component, the decomposition of U into $Z_0 V_{n_0}^{m''_0} Z_1 V_{n_1}^{m''_1} \cdots V_{n_k}^{m''_k} Z_{k+1} V_{n_{k+1}}^{m''_{k+1}} \cdots$ and W is a regular decomposition. Hence, $W \in \mathcal{U}(D, Y)$ holds and we have shown the claim. □

Corollary 4.8. *Let X be a subset of $\mathbb{R}^2$ such that $X = \mathbb{R}^2 \setminus Y \times \{0\}$ for some $Y \subset \mathbb{R}$ and $x_0 = (0,1)$. Then, $\pi_1(X, x_0) \simeq \times_\omega \mathbb{Z}$ if and only if Y has countably infinite connected components which converge to a point in $\mathbb{R} \setminus Y$.*

Proof. We remark that a bounded connected component of Y is an interval and a connected component of Y may be unbounded. First we show the necessity of the condition. An isomorphism from $\pi_1(\mathbb{H}, (1,1))$ to $\pi_1(X, x_0)$ is conjugate to a spatial homomorphism by Theorem 4.1. Then, the spatial homomorphism itself is an isomorphism from $\pi_1(\mathbb{H}, (1,1))$ to $\pi_1(X, x_0)$. Let $\varphi : \mathbb{H} \to X$ be a continuous map which induces the isomorphism. Note that Y has infinitely many connected components, because, otherwise, $\pi_1(X, x_0)$ is a finitely generated free group. Therefore, there exist accumulation points of the family of connected components or there exists a family of connected components which diverges to ∞ or $-\infty$. Suppose that $y \in Y$ is an accumulation point of the family of connected components of Y. Since $\mathbb{H}$ is compact, there exists a neighborhood U of $(y, 0)$ such that $\mathrm{Im}(\varphi) \cap U = \emptyset$. There is a circle in $U \cap X$ which divides $U \cap (Y \times \{0\})$ into two parts. Take a loop in X corresponding to a winding of this circle. Then, the homotopy class of the loop does not belong to the image of φ_*, which contradicts that φ_* is an isomorphism. Similarly, we can see no family of connected components diverges to ∞ nor $-\infty$. Hence, the accumulation points are in $\mathbb{R} \setminus Y$. Suppose that there are two accumulation points. At least one of them is not equal to $\varphi(o)_1$, say $p \neq \varphi(o)_1$. We only deal with the case $p < \varphi(o)_1$, but the other case is proved similarly. There exists $x_0 \in \mathbb{R}$ such that $(x_0, 0) \in X$, $p \leq x_0 < \varphi(o)_1$ and there are infinitely many connected components of Y in $(-\infty, x_0)$. Let $X_0 = X \cap (-\infty, x_0] \times \mathbb{R}$ and $r : X \to X_0$ be the retraction. Then, $(r \cdot \varphi)_* : \pi_1(\mathbb{H}, o) \to \pi_1(X_0, (x_0, 0))$ is surjective. On the other hand, since $\{\varphi(x,y) : (x - 1/n)^2 + y^2 = 1/n^2\} \cap X_0 = \emptyset$ for almost all $1 \leq n < \omega$, the image of $(r \cdot \varphi)_*$ is finitely generated, which contradicts that there infinitely many holes in X_0. Since the existence of uncountable many connected components implies the existence of infinitely many accumulation points, these show the necessity.

Conversely, suppose that Y has countably infinite connected components which converge to a point in $\mathbb{R} \setminus Y$. Then, X is homotopic to the space $Z = \mathbb{R}^2 \setminus \bigcup_{n<\omega} D_n$, where $\{D_n : n < \omega\}$ is a pair-wise disjoint family of open disks whose centers are on the line $\mathbb{R}$ and which converge to a point. As we have shown in Remark 3.12, Z is homotopic to $\mathbb{H}$ and we obtain the conclusion. □

We recall the σ-abelianization from [2, p.252] and a canonical factor of singular homology from [3]. We refer a reader to [2] and [3] for undefined notion. As we mentioned in the last remark in [2, p.262], under the condition in Proposition 2.10 a canonical factor of singular homology group $H_1^T(\widetilde{\bigvee}_{i\in I}(X_i, x_i))$ is isomorphic to the σ-abelianization $Ab^\sigma(\pi_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i), x^*))$. We state it more precisely. There is a canonical surjection from $H_1(X)$ to $H_1^T(X)$. Let $\Phi_X : \pi_1(X, x) \to H_1^T(X)$ be a homomorphism obtained by the composition of the Hurewicz homomorphism from $\pi_1(X, x)$ to $H_1(X)$ and this surjection. In case X is path-connected, Φ_X is surjective. In the case of Corollary 4.8 with $\pi_1(X, x) \simeq \times_\omega \mathbb{Z}$, $\operatorname{Ker} \Phi_X = \pi_1(X, x)^{\sigma\prime}$ and consequently $H_1^T(X) \simeq Ab^\sigma(\pi_1(X, x))$. Here, we show that this correspondence does not hold for spaces in Theorem 3.2 in general.

Proposition 4.9. *There exists $X \subseteq \mathbb{R}^2$ with $x \in X$ such that* $\operatorname{Ker} \Phi_X$ *is not equal to $\pi_1(X, x)^{\sigma\prime}$, where $X = \mathbb{R}^2 \setminus Y \times \{0\}$ for some $Y \subset \mathbb{R}$.*

Proof. Let $Y \subseteq [0,1]$ be the complement of the Cantor ternary set, i.e. $Y = \bigcup\{(\Sigma_{i=1}^{n-1}\varepsilon_i/3^i + 1/3^n, \Sigma_{1=1}^{n-1}\varepsilon_i/3^i + 2/3^n) : \varepsilon_i = 0 \text{ or } 2,\ n < \omega\}$, and D be a quasi-dense subset $\{\Sigma_{i=1}^{n-1}\varepsilon_i/3^i + 1/(2\cdot 3^{n-1}) : \varepsilon_i = 0 \text{ or } 2,\ n < \omega\}$ of Y. First, we claim the existence of a loop f_0 in X such that $W^{f_0} \simeq \cdots - u \cdots - v \cdots u \cdots v \cdots$ where $u < v$ and $u, v \in D$. Since the loop $S^-(a, b) \cdot S^+(b, a)$ corresponds to a word $\cdots u \cdots v \cdots$, it suffices to construct a path $g : [0,1] \to X$ such that $g(0) = (0,0)$, $g(1) = (1,0)$ and $W^g \simeq \cdots - u \cdots - v \cdots$ where $u < v$. Put $a = \Sigma_{i=1}^{n-1}\varepsilon_i/3^i + 1/3^n$, $b = \Sigma_{1=1}^{n-1}\varepsilon_i/3^i + 2/3^n$ and let $g \restriction [a, b] \equiv S^+(a, b) \cdot S^-(b, a) \cdot S^+(a, b)$ and $g(t) = (t, 0)$ for $t \in [0,1] \setminus \bigcup\{[\Sigma_{i=1}^{n-1}\varepsilon_i/3^i + 1/3^n, \Sigma_{1=1}^{n-1}\varepsilon_i/3^i + 2/3^n] : \varepsilon_i = 0 \text{ or } 2,\ n < \omega\}$. Then, it is easy to see that g is continuous and $W^g \simeq \cdots - u \cdots - v \cdots$ where $u < v$. Next, we show that $[f_0] \in \operatorname{Ker} \Phi_X$ but $[f_0] \notin \pi_1(X, x)^{\sigma\prime}$, where $[f_0]$ is the homotopy class of f_0.

By [3, Theorem 5.1], $H_1^T(X) \simeq C(Y, \mathbb{Z})$ and the isomorphism is given in the following way: For a loop f in X, the winding numbers of f at $y \in Y$ form a continuous map from Y to $\mathbb{Z}$. Therefore, $\Phi_X([f_0]) = 0$. Toward the contradiction, suppose $[f_0] \in \pi_1(X, x)^{\sigma\prime}$. According to Theorem 4.1, there exist spatial homomorphisms $h_i : \pi_1(\mathbb{H}, (1,1)) \to \pi_1(X, (0,1))$, $u_i \in \pi_1(X, (0,1))$ and σ-commutators c_i, (i.e. $c_i \in C_\omega \leq \times_\omega \mathbb{Z} \simeq \pi_1(\mathbb{H}, (1,1))$ [2, p. 252]), such that $[f_0] = u_0^{-1}h_0(c_0)u_0 \cdots u_n^{-1}h_n(c_n)u_n$. Since each h_i is spatial, there is a continuous map $\varphi_i : \mathbb{H} \to X$ such that $\varphi_{i*} = h_i$. Choose an accumulation point $x^* \in X$ of $D \times \{0\}$ and a closed disk C so that x^* is an interior point of C, the boundary ∂C is in X and $\varphi_i(o) \notin C$ for any $0 \leq i \leq n$ similarly as in the proof of Corollary 4.8. Let $r : X \to C \setminus Y \times \{0\}$ be a retraction and $D' = \{d \in D : (d, 0) \in C\}$ and $y_0 \in C \setminus Y \times \{0\}$. Remark the following fact:

Let $h : \times_\omega \mathbb{Z} \to \times_\omega \mathbb{Z}$. If $h(\delta_n) = e$ for almost all $n < \omega$, then $h(c)$ belongs the commutator subgroup of $\times_\omega \mathbb{Z}$ for $c \in C_\omega$.

Then, since each $r_* \cdot h_i$ is a standard homomorphism and $h_i(\delta_n) = e$ for almost all n, $r_*(u_0^{-1}h_0(c_0)u_0 \cdots u_n h_n(c_n)u_n)$ belongs to the commutator subgroup of $\pi_1(C \setminus Y \times$

$\{0\}, y_0) \hookrightarrow \times_{D'}\mathbb{Z}$. A reduced word for $r_*([f_0])$ is of form $\cdots - u \cdots - v \cdots u \cdots v \cdots$ where $u < v$ and $u, v \in D'$. According to the fact in the proof of [2, Lemma 4.11], it should be of canonical commutator form [2, Definition 4.10], that is, $r_*([f_0]) = B_1 \cdots B_{2n}$ such that each B_i is reduced, $B_1 \cdots B_{2n}$ is quasi-reduced and there are $\{i_k, j_k : 1 \leq k \leq n\} = \{1, \cdots, 2n\}$ with $B_{i_k} \simeq (B_{j_k})^{-1}$. Since D' is infinite, there appears a word $u \cdots v$ for $u < v$ in some B_i and consequently $(-v) \cdots (-u)$ in some other B_j. However, this cannot be occur from the form of a reduced word for $r_*([f_0])$. □

Remark 4.10. Even if $Y = D$ in the proof of Proposition 4.9, we get the same conclusion. Therefore, the difference between $\operatorname{Ker} \Phi_X$ and $\pi_1(X, x)^{\sigma'}$ for $X = \mathbb{R}^2 \setminus Y \times \{0\}$ occurs even when Y is discrete.

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School of Science and Engineering, Waseda University, Tokyo 169, JAPAN
E-mail address: eda@logic.info.waseda.ac.jp

ATOMIC PROPERTY OF THE FUNDAMENTAL GROUPS OF THE HAWAIIAN EARRING AND WILD PEANO CONTINUA

KATSUYA EDA

ABSTRACT. We strengthen previous results on the fundamental groups of the Hawaiian earring and the wild Peano continua. If the fundamental group of a wild Peano continuum, i.e. a Peano continuum which is not locally semi-simplyconnected at any point, is a subgroup of a free product $*_{j \in J} H_j$, it is contained in a conjugate subgroup to some H_j.

1. INTRODUCTION

Until recently the Hawaiian earring had been only a typical example of a non-locally simply connected space [10], [15, p. 59], but the fundamental group of the Hawaiian earring has gotten attentions of several authurs now [1, 8, 9, 16]. We call the fundamental group of the Hawaiian earring as the Hawaiian earring group for short, following [1]. Particularly the Hawaiian earring group played a central role in [8], where we can detect a point from the endomorphic images of the Hawaiian earring group in the fundamental group of one-dimensional wild Peano continua. There we see the reflection of the following result about quasi-atomness.

We call a group G *quasi-atomic*, if for each homomorphism $h : G \rightarrow A * B$ there exists a finitely generated subgroup A' of A or B' of B such that $\mathrm{Im}(h)$ is contained in $A' * B$ or $A * B'$.

By definition, finitely generated groups and abelian groups are quasi-atomic, but free products of infinitely generated groups are not quasi-atomic. Every homomorphic image of a quasi-atomic group is also quasi-atomic.

Theorem 1.1. [3, Theorem 4.1] *(See also* [2].*)*
Let G_i be a finitely generated group for each $i \in I$. Then $\times^{\sigma}_{i \in I} G_i$ is quasi-atomic. Consequently the Hawaiian earring group is quasi-atomic.

A similar result is

Theorem 1.2. [4, Theorem 1.5] *Let X be a Peano continuum which is not semi-locally simply connected at any point. For an injective homomorphism $h : \pi_1(X, x_0) \to A * B$, there exists a finitely generated subgroup A_0 of A such that $\mathrm{Im}(h) \leq A_0 * B$ or there exists a finitely generated subgroup B_0 of B such that $\mathrm{Im}(h) \leq A * B_0$.*

In the present paper we strengthen the above results as follows:

Theorem 1.3. *Let G_i $(i \in I)$ and H_j $(j \in J)$ be groups and h : $\times^{\sigma}_{i \in I} G_i \to *_{j \in J} H_j$ be a homomorphism to the free product of groups H_j's. Then there exists a finite subset F of I such that $h(\times^{\sigma}_{i \in I \setminus F} G_i)$ is contained in a conjugate subgroup to some H_j.*

Theorem 1.4. *Let X be a path-connected, locally path-connected, first countable space which is not semi-locally simply connected at any point and $h : \pi_1(X, x_0) \to *_{j \in J} H_j$ be an injective homomorphism. Then the image of h is contained in a conjugate subgroup to some H_j.*

Corollary 1.5. *Let X be a Peano continuum which is not semi-locally simply connected at any point and $h : \pi_1(X, x_0) \to *_{j \in J} H_j$ be an injective homomorphism. Then the image of h is contained in a conjugate subgroup to some H_j.*

We do not define the atomic property of a group in the title, because it does not seem to be suitable to call some particular property so far. However the consequences in Theorems 1.1 and 1.2 play impotant roles in [8] and [3], where groups in question are hard to be taken apart into free products in essential ways. We sum up these properties as atomic property.

A weaker form of Theorem 1.3 for the fundamental group of the Hawaiian earring, i.e. Theorem 3.1, will be applied in our forth coming paper [5].

2. Word theoretic arguments

Since our argument requires of results in [7] in detail, we review and reprove some parts and lemmmas of [7] and [2] for the reader's covenience.

For given groups G_i $(i \in I)$ with $G_i \cap G_j = \{e\}$ for distinct i and j, a letter is a non-identity element of $\bigcup_{i \in I} G_i$. Two letters are *of the same kind*, if they belong to the same G_i.

For a word W let W^- be a word defined by the following: $\overline{W^-}$ consists α^-'s, where α^- is a formal symbol related to each $\alpha \in \overline{W}$; and $\alpha^- \leq \beta^-$ if and only if $\beta \leq \alpha$; and $W^-(\alpha^-) = W(\alpha)^{-1}$. For words we use the notation "$\cong$", when two words are the same as words, i.e. there exists an order preserving isomorphism $\varphi : \overline{U} \to \overline{V}$ with $U(\alpha) =$

$V(\varphi(\alpha))$, and use the notation "=", when two words present the same element in the inverse limit $\text{proj}\lim(*_{i\in F}G_i, p_{FF'} : F \subseteq F' \Subset I)$. Here $F' \Subset I$ implies F' is a finite subset of I and $p_{FF'} : *_{i\in F'}G_i \to *_{i\in F}G_i$ is the projection.

The notion "reduced" and "quasi-reduced" for words was defined in [7]. The reducedness of words is defined so that it coincideds with the usual one [12, 14] for words of finite length. A word W is *quasi-reduced*, if the reduced word of W is obtained by multiplying contiguous elements belonging to the same groups without cancellation. A word W is *cyclically reduced*, if W is empty, a single letter, or WW is reduced.

We define W^n for an interger n as follows: W^0 is an empty word and $W^{n+1} \cong W^n W$ and $W^{-(n+1)} \cong W^{-n}W^-$ for a nonnegative integer n;

For a word W and a letter g, let $a_W(g)$ be the number of appearances of g in W, i.e. the cardinality of the set $\{\alpha \in \overline{W} : W(\alpha) = g\}$.

For $a \in *_{j\in J}H_j$, $l(a)$ denotes the length of the reduced word which expresses a. Hence $l(W)$ means the length of the reduced word of W, but not the length of W itself in general.

Let $W \cong gX$, where W, X are words and g is a letter. The *head* of W is g denoted by gX. Similarly the *tail* of W is g denoted by $Xg \cong W$. We say that an appearence of a letter g is *stable* in XgY, if the reduced word of XgY is of the form $X'gY'$ where X' and Y' are the reduced words of X and Y respectively. We simply say that the head of W is stable, instead of "stable in W", and similarly for the tail. (In the present paper we use the notions "head" and "tail" only for non-empty words of finite length and so they always exist.)

The following lemma was stated for infinitary words, but here we only use them for words of finite length.

Lemma 2.1. [7, Corollary 1.6] *Let U be a non-empty reduced word such that $U \cong U^-$. Then there exist a letter u and a word W such that $u^2 = e$ and $U \cong W^-uW$.*

Lemma 2.2. *Let U be a non-empty word such that $UU \cong XU^-Y$ for some words X and Y. Then there exist U_0 and U_1 such that $U \cong U_0U_1$, $U_0 \cong U_0^-$ and $U_1 \cong U_1^-$.*

Proof. We have U_0 and U_1 such that $U_1U_0 \cong U^-$ by the assumption. This implies $U_0 \cong U_0^-$ and $U_1 \cong U_1^-$. □

Lemma 2.3. *Let $H_j (j \in J)$ be groups and U and X be reduced words. If the head g or the tail g^{-1} in XUX^- is not stable, then $l(XUX^-) \leq l(U) + 1$.*

Proof. It suffices to deal with the case that the tail g^{-1} is not stable. Let V be the reduced word of XU. Then, $l(V) \leq l(X) + l(U)$. Since the tail g^{-1} in VX^- is not stable, $l(VX^-) \leq l(V) - l(X^-) + 1$. Therefore, $l(XUX^-) \leq l(U) + 1$. □

Lemma 2.4. [7, Lemma 2.3] *Let H_j ($j \in J$) be groups. Let $m + n + 2 \leq k$ for $m, n, k \in \mathbb{N}$ and $u, y_i, z \in *_{j\in J} H_j$ ($1 \leq i \leq M$). If $l(u) \leq m, u = y_1 z^k \cdots y_M z^k$ and $l(y_i) \leq n$ for all $1 \leq i \leq M$, then one of the following holds:*

(1) *z is a conjugate to an element of some H_j;*
(2) *$z = x^{-1}fxy^{-1}gy$ for some $f \in H_j$ and $g \in H_{j'}$ with $f^2 = g^2 = e$, and some $x, y \in *_{j\in J} H_j$ such that $y_i = z^r x^{-1} f x$ or $y_i = y^{-1} g y z^r$ for some i and r.*

Proof. It is easy to see that there exist reduced words U and W such that $z = W^- UW$ and the both words UU and $W^- UW$ are quasi-reduced or $UU = e$. If $l(U) \geq 2$, we can take the above U and W so that UU is reduced. If $l(U) \leq 1$, the proof is done. Hence, we assume $l(U) \geq 2$ and so also assume that UU is reduced. Let Y_i be the reduced word for y_i for each $1 \leq i \leq M$. Then,

$$u = Y_1 W^- U^k W Y_2 W^- \cdots W Y_M W^- U^k W.$$

Since $M = 1$ implies the contradiction, we assume $M \geq 2$. Suppose that the head and tail of WY_iW^- are stable for all $i \geq 2$. Then, the head and tail of UWY_iW^-U are also stable, which implie $l(u) > 2(k - 2) \geq 2(m + n)$. The last inequality contradicts $l(u) \leq m$. Therefore at least one of the head and the tail of WY_iW^- is not stable. It implies $l(WY_iW^-) \leq l(Y_i) + 1 \leq n + 1$ by Lemma 2.3. Let Z_i be the reduced word of WY_iW^- for $2 \leq i \leq M$. Let p be the least number so that $2p \geq n + 1$. (Since this p is used again later, it is taken a little larger than it is necessary here.) We remark that $W^- U^k$ is reduced and $l(Y_1) \leq n$. Whether the tail of $Y_1 W^-$ are stable or not, the reduced word of $Y_1 W^- U^k$ is of the form $Z_1 U^{p+2}$. Now we have

$$u = Z_1 U^{p+2} Z_2 U^k \cdots U^k Z_M U^k W.$$

We are concerned with the reduced word of $U^{p+2} Z_i U^{p+2}$. The reduced word of $Z_i U^{p+2}$ is of form $Z_i' X_i' U^{q+2}$, where $q \geq 0$, $X_i X_i' \simeq U$ and $l(Z'_i) \leq l(Z_i)$. If the head and the tail of $U^{p+2} Z_i' X_i' U^{q+2}$ are stable, then $l(u) \geq 2k - 2p > n$, which is a contradiction. Hence, at least one of the head and the tail of $U^{p+2} Z_i' X_i' U^{q+2}$ is not stable.

(Case 1):The tail of $U^{p+2} Z_i' X_i' U^{q+2}$ is not stable.

We observe the cancellations in the rightmost U^2. Then there are S_i, T_i such that $S_i T_i \cong U$ and $S_i \simeq S_i^-$ and $T_i \cong T_i^-$. Since UU is reduced, neither S_i nor T_i is empty. By Lemma 2.1, $S_i = x_0^{-1} f x_0$ and $T_i = y_0^{-1} g y_0$ for some $f \in H_j$ and $g \in H_{j'}$ with $f^2 = g^2 = e$

and $x_0, y_0 \in *_{j\in J}H_j$. Let $x = W^- x_0 W$ and $y = W^- y_0 W$. Then $z = x^{-1}fxy^{-1}gy$. Moreover $WY_iW^-U^{p-q} = Z_i'X_i' = U^{-j}T_i$ for some j. Hence

$$y_i = Y_i = W^-U^{-j}T_iU^{-(p-q)}W = W^-(T_iS_i)^jT_i(T_iS_i)^{p-q}W.$$

Now the last term is equal to $z^{p-q-j-1}x^{-1}fx$ if $p-q > j$ and $y_i = y^{-1}gyz^{j-p+q}$ otherwise, i.e. $p - q \leq j$.

(Case 2): The head of $U^{p+2}Z_i'X_i'U^{q+2}$ is not stable.

We observe the cancellations in the left most U^2. Since $l(Z_i') \leq 2p \leq l(U^p)$, we conclude the existence of S_i, T_i similarly as the above and get the conclusion. □

Let A and B be groups, and let C_1 and C_2 be subsets of $A*B$ defined by: $C_1 = \{x^{-1}ux : u \in A \cup B, x \in A * B\}$ and $C_2 = \{xy : x, y \in C_1\}$.

Since C_2 is closed under conjugacy, that is, $u^{-1}xu \in C_2$ iff $x \in C_2$, we consider cyclically reduced forms of words of elments in C_2. A word U is *cyclically equivalent* to a word V, if U is equivalent to X^-VX for some X, i.e. U presents a conjugate to the element presented by V. Now we easily have:

Lemma 2.5. *A word W for an element of $C_2(\subseteq A * B)$ is cyclically equivalent to a word of one of the following forms:*

(1) *empty;*
(2) *u_0 where $u_0 \in A \cup B$;*
(3) *$V_0^-u_0V_0v_0$ where $u_0, v_0 \in A \cup B$ and V_0 is reduced words.*

We remark the following: if $W \in C_2 \setminus C_1$ for a cyclically reduced word W, then W is of the form $W_0^-w_0W_0W_1^-w_1W_1$. In the remaining part of this section A and B are groups. We prove lemmas which have word theoretic characters and will be used in the proof of Theorem 3.1 in the next section.

Lemma 2.6. *Suppose that every element of A has its order 2. Let $a_1, a_2 \in A$ $(a_1 \neq a_2)$ and $b \in B$ be nontrivial elements be a nontrivial element. Then $u^{-1}a_1uv^{-1}bvu^{-1}a_2uv^{-1}bvu^{-1}(a_1a_2)uv^{-1}bv$ does not belong to C_2 for any $u, v \in A * B$.*

Proof. Let $a_3 = a_1a_2$. Since

$$\begin{aligned} &vu^{-1}a_1uv^{-1}bvu^{-1}a_2uv^{-1}bvu^{-1}(a_1a_2)uv^{-1}bvv^{-1} \\ = \ &w^{-1}a_1wbw^{-1}a_2wbw^{-1}a_2wbw^{-1}a_3wb \end{aligned}$$

where $w = uv^{-1}$, we may assume $v = e$ and moreover that $U^-a_1UbU^-a_2UbU^-(a_1a_2)Ub$ is cyclically reduced for the reduced word U for u. Let $V \cong U^-a_1UbU^-a_2UbU^-a_3Ub$. We remark $a_3 \neq a_1$ and $a_3 \neq a_2$. Then $a_V(a_1)$, $a_V(a_2)$ and $a_V(a_3)$ are odd, because each letter

a_i appearing in U again appears in U^-. That is, we have three distinct letters g for which $a_V(g)$ is odd and $g^2 = e$

Since V is cyclically reduced, V does not belong to C_1 and consequently V is of the form $W_0^- w_0 W_0 W_1^- w_1 W_1$ if V belongs to C_2. But in such a case we have at most two distinct letters g for which $a_V(g)$ is odd and $g^2 = e$. Hence we conclude $V \notin C_2$. □

Lemma 2.7. *Let $a \in A$ be an element satisfying $a^2 \neq e$ and $b \in B$ be a non-trivial element. Then $u^{-1}auv^{-1}bvu^{-1}auv^{-1}bvu^{-1}auv^{-1}bv$ does not belong to C_2 for every $u, v \in A * B$.*

Proof. As in the preceding proof, we may assume $v = e$ and that $U^-aUbU^-aUbU^-aUb$ is cyclically reduced for the reduced word U for u. Let $V = U^-aUbU^-aUbU^-aUb$, then V is cyclically reduced and does not belong to C_1. We see $a_V(a) - a_V(a^{-1}) = 3$, but if V is of the form $W_0^- w_0 W_0 W_1^- w_1 W_1$, $a_V(g) - a_V(g^{-1}) \leq 2$ for every letter g. Hence we conclude $V \notin C_2$. □

Lemma 2.8. *Let H be a subgroup of $A * B$ containing $\langle W^-aW\rangle * \langle V\rangle$, where $a \in A \cup B$, W^-aW is a reduced word and V is a cyclically reduced word with $l(V) \geq 2$. Then there exist $u \in H$ such that $u \notin C_2$.*

Proof. Since V is cyclically reduced and $l(V) \geq 2$, one of W^-aWV and VW^-aW is reduced. Since the argument will go symmetrically, we assume that VW^-aW is reduced. Choose k so that $k \cdot l(V) > l(W^-aW)$. Then the head of VW^-aW is stable in VW^-aWV^{k+1}. To show this by contradiction, suppose the negation of the conclusion. Considering the cancellation of the left most V in VW^-aWV^{k+1}, we have V_0 and V_1 such that $V_0V_1 \cong V$, $V_0^- \cong V_0$, $V_1^- \cong V_1$ and $W^-aW \cong V_0^-(V^-)^l$ for some $l \geq 0$. This implies $(W^-aWV^{l+1})^2 = (V_0V)^2 = (V_1)^2 = e$ by Lemma 2.1, which contradicts that $\langle W^-aW\rangle$ and $\langle V\rangle$ have no relation.

By the preceding argument we conclude that for the reduced word W_0 of VW^-aWV^{k+1} the word VW_0V is reduced. Since we worked in $A * B$, the reduction of VW^-aWV^{k+1} stops when the mutiplication in A or B occurs. We have a non-negative integer l, letters u_0, u of the same kind and words U_0, U_1, X such that $U_0u_0U_1 \cong V$, $W_0 \cong XuU_1V^l$, and $u \neq u_0$.

Let U be the reduced word of

$$VW^-aWV^{k+1}V^{2k+2}VW^-aWV^{k+1}V^{6k+8}VW^-aWV^{k+1}V^{14k+18},$$

i.e. $W_0V^{2k+2}W_0V^{6k+8}W_0V^{14k+18}$. To show $U \notin C_2$ by contradiction, suppose the negation. Since U is cyclically reduced, U does not belong to C_1 and hence is of the form $X_0^- x_0 X_0 X_1^- x_1 X_1$. Remark that $l(W_0) <$

$(2k+2)l(V)$. Then x_1 is located in V^{14k+18}. If the right most W_0V is located in $X_1^- x_1 X_1$, it is located in X_1^- because the length of V^{14k+17} is larger than that of $W_0V^{2k+2}W_0V^{6k+8}W_0V$. Since X_1 is a subword of V^{14k+18}, we have V_0 and V_1 such that $V_0V_1 \cong V$, $V_0^- \cong V_0$, $V_1^- \cong V_1$ and $W_0V \cong XuU_1V^lV \cong YV_0V_1V_0V^{-l}V_1$ for some word Y. This implies $XuU_1 \cong YV_0V_1 \cong YV$, which contradicts $u \neq u_0$.

Otherwise, i.e. the right most W_0V is not located in $X_1^- x_1 X_1$, the leftmost W_0V and the middle W_0V are located in X_0^-, because the length of V^{6k+7} is larger than $W_0V^{2k+2}W_0V$. Since $l(V^{2k+2}) > l(W_0)$, the length of the word between the leftmost W_0 and the middle W_0, is greater than the length of the rightmost W_0, we can argue similarly for at least one of the leftmost W_0V and the middle W_0V as in the case when the right most W_0V is located in X_1^-. Thus we deduce a contradiction.

□

Lemma 2.9. *Let H be a subgroup of $A * B$ such that*

(1) *H contains a non-trivial element which is conjugate to an element in A or B; and*
(2) *H is not contained in any conjugate subgroup to A nor B; and*
(3) *H is not contained in any subgroup of the form $\langle u_0 \rangle * \langle u_1 \rangle$ with $u_0^2 = u_1^2 = e$.*

Then, H contains an element $u \notin C_2$.

Proof. By the Kurosh subgroup theorem [13], H is of the form $*_{i\in I} u_i^{-1} H_i u_i * *_{j\in J} v_j^{-1} \langle V_j \rangle v_j$, where H_i's are subgroups of A or B and V_j's are cyclically reduced words and $l(V_j) \geq 2$. Under the given condition H contains

(a) a subgroup $u^{-1}\langle a \rangle u * v^{-1} \langle b \rangle v$ for some non-trivial elements $a, b \in A \cup B$ with $a^2 \neq e$; or
(b) a subgroup $\langle w^{-1}aw \rangle * \langle v^{-1}Vv \rangle$, where $a \in A \cup B$ and V is a non-empty cyclically reduced word with $l(V) \geq 2$.

When (a) holds, Lemma 2.7 implies the conclusion. When (b) holds, vHv^{-1} contains a subgroup $\langle (wv^{-1})^{-1}awv^{-1} \rangle * \langle V \rangle$. Let W be the reduced word for wv^{-1}. If $W^{-1}aW$ is a reduced word, we can apply Lemma 2.8 to vHv^{-1}. Otherwise, $W \cong a_0W_0$ and a and a_0 belong to the same group A or B. Let $a_1 = a_0^{-1}aa_0$. Then, ${a_1}^2 = e$ and $W_0^- a_1 W_0$ is a reduced word. Hence we can apply Lemma 2.8 to vHv^{-1}. Therefore in the both cases we have $u \in vHv^{-1}$ satisfying $u \notin C_2$. We have $v^{-1}uv \in H$ and $v^{-1}uv \notin C_2$. □

Lemma 2.10. *Let $x \notin C_2$. Then $x^m \notin C_2$ for every integer $m \geq 4$. For given $x_1, \cdots, x_n$, there exists a positive integer $m \geq 4$ such that $x_i x^m \notin C_1$ for every $1 \leq i \leq n$.*

Proof. Let V be the reduced word for x. First we assume that V is cyclically reduced and $l(V) \geq 2$. To show the first proposition by contradiction, suppose that $V^m \in C_2$. Then we have letters w_0, w_1 and words W_0, W_1 such that $V^m \cong W_0^- w_0 W_0 W_1^- w_1 W_1$. Since $l(W_0) \leq l(V)$ or $l(W_1) \leq l(V)$, V is a subword of W_0^- or W_1. By Lemma 2.2 we have V_0, V_1 such that $V_0 V_1 \cong V$, $V_0 \cong V_0^-$ and $V_1 \cong V_1^-$ and consequently $V \in C_2$, which is a contradiction.

Next we show the second proposition. Let m_0 be a natural number such that $l(x_i) < m_0 l(V)$ for every $1 \leq i \leq n$ and $m = m_0 + 3$. To show $x_i V^m \notin C_1$ by contradiction, suppose that $x_i V^m \notin C_2$. Then the reduced word for $x_i V^m$ is of the form XV^{k+3} where $l(X) \leq l(V^k)$. By a similar argument as above we conclude $V \in C_2$, which is a contradiction.

When V is not cyclically reduced, we have u such that the reduced word for $u^{-1}xu$ is cyclically reduced. Since C_2 is closed under conjugacy, we have the first proposition for x from the corresponding statement for $u^{-1}xu$. To show the second proposition, we choose m for $u^{-1}x_i u$ $(1 \leq i \leq n)$ so that $u^{-1}x_i u(u^{-1}xu)^m \notin C_1$ for every $1 \leq i \leq n$, which implies $x_i x^m \notin C_1$. □

Lemma 2.11. *Let H be a non-trivial subgroup of $\langle a \rangle * \langle b \rangle$ where $a^2 = b^2 = e$.*

If H is not conjugate to $\langle a \rangle$ nor $\langle b \rangle$, then H contains an element of the form $(ab)^k$ for some $k > 0$.

Proof. Every non-empty reduced word of even length is of the form $(ab)^k$ or $(ba)^k$ for some $k > 0$ and every reduced word of odd length is of the form $W^- aW$ or $W^- bW$ for some word W. The reduced word of the concatenation of two words of odd length is of even length and $(ba)^k$ is the inverse of $(ab)^k$. Hence, if $(ab)^k$ does not belong to H for any $k > 0$ and H is trivial, then H is a conjugate to $\langle a \rangle$ or $\langle b \rangle$. □

Lemma 2.12. *Let $u, w_0 \in *_{j \in J} H_j$. If $e \neq u^{-1} w_0^{-1} h w_0 u \in w_0^{-1} H_{j_0} w_0$, then $u \in w_0^{-1} H_{j_0} w_0$.*

Proof. Under the assumption $(w_0 u w_0^{-1})^{-1} h w_0 u w_0^{-1} \in H_{j_0}$ and hence $w_0 u w_0^{-1} \in H_{j_0}$, that is, $u \in w_0^{-1} H_{j_0} w_0$. □

3. Proofs of Theorems 1.3 and 1.4

The following theorem strengthens a part of [3, Theorem 4.1] (see also [2]) and is a special case of Theorem 1.3.

Theorem 3.1. *Let A, B be arbitrary groups and $h : \times_{n<\omega}\mathbb{Z}_n \to A * B$ be a homomorphism. Then there exist $m < \omega$ and $u \in A * B$ such that $h(\times_{n\geq m}\mathbb{Z}_n) \leq u^{-1}Au$ or $h(\times_{n\geq m}\mathbb{Z}_n) \leq u^{-1}Bu$.*

This section is devoted to the proof of this theorem and those of Theorems 1.3 and 1.4. The first lemma is a very restricted case of Theorem 3.1,

Lemma 3.2. *Theorem 3.1 holds, if $A = B = \mathbb{Z}/2\mathbb{Z}$.*

Proof. Let a and b be a generators of A and B respectively. To show the conclusion by contradiction we suppose the negation of the conculsion. We construct $x_m \in \times_{n\geq m}\mathbb{Z}_n$ and positive integers k_m by induction. The subgroup $h(\times_{n\geq 0}\mathbb{Z}_n)$ contains an element of the form $h(x_0) = (ab)^{k_0}$ with $k_0 > 0$ by Lemma 2.11. For m, we choose x_m and even k_m so that $h(x_m) = (ab)^{k_m}$ and $k_m > \Sigma_{i=0}^{m-1}k_i$.

The following construction of a certain element in $\times_{n<\omega}\mathbb{Z}_n$ is similar to that in the proof of Theorem 3.1 and is a modification of that in the proofs of [7, Theorem 2.1 and etc.], [6, Theorem 1.1], [3, Theorem 4.1] and [4, Thereom 1.5]. For this purpose we recall the notions for the construction.

Let Seq be the set of all finite sequences of natural numbers and denote the length of $s \in Seq$ by $lh(s)$. An element $s \in Seq$ is denoted by $\langle s_0, \cdots, s_{n-1}\rangle$ where $s_k \in \mathbb{N}$ $(0 \leq k < n)$. For $s, t \in Seq$, $s \prec t$ if $s(i) < t(i)$ for the minimal i with $s(i) \neq t(i)$ or t extends s.

Let $W_m \in \mathcal{W}(\mathbb{Z}_n : n < \omega)$ be the reduced word for x_m, i.e. $W_m = x_m$. Let $\overline{V} = \{(s,p) : s \in Seq, 0 \leq s(i) < k_i \text{ for } 0 \leq i < lh(s), p \in \overline{W_i}\}$ with the lexicographical ordering and $V(s,p) = W_{lh(s)}(p)$. (We remark $h(W_{lh(s)}) = h(x_{lh(s)}) = (ab)^{k_{lh(s)}}$.) Then V is a word in $\mathcal{W}(\mathbb{Z}_n : n < \omega)$. Let $\overline{V_m} = \overline{V} \cap \{s : lh(s) \geq m, s_i = 0 \text{ for } 0 \leq i < m\}$ and $V_m = V \restriction \overline{V_m}$. We remark that $V \cong V_0 \cong (W_0V_1)^{k_0}$ and $V_m \cong (W_mV_{m+1})^{k_m}$ generally.

We consider $h(V) \in \langle a\rangle * \langle b\rangle$ and take $m > 0$ so that the length of the reduced word for $h(V)$ is less than m. If the length of the reduced word for $h(W_{m+1}V_{m+2})$ is odd, then $h(W_{m+1}V_{m+2})^2 = e$ and hence $h(V_{m+1}) = h((W_{m+1}V_{m+2})^{k_{m+1}}) = e$. Therefore $h(V) = (ab)^k$, where $k = \Sigma_{i=0}^{m}k_i\Pi_{j=0}^{i}k_j \geq k_m$, which contradicts $m > l(h(V))$.

Otherwise, $h(W_{m+1}V_{m+2}) = (ab)^p$ for some $p \geq 0$ or $(ba)^p$ for some $p > 0$. The former case is similar to the preceding case and we deduce a contradiction similarly. In the latter case, we have $h(V) = (ba)^k$, where

$k = pk_{m+1}\Pi_{j=0}^{m}k_j - \Sigma_{i=0}^{m}k_i\Pi_{j=0}^{i}k_j \geq (pk_{m+1} - \Sigma_{i=0}^{m}k_i)\Pi_{j=0}^{m}k_j > m$, which is a contradiction. Now we have shown the lemma. □

Proof of Theorem 3.1. Let $h : \times_{n<\omega}\mathbb{Z}_n \to A * B$ be a homomorphism. We consider the subgroups $h(\times_{n\geq m}\mathbb{Z}_n)$ for $m < \omega$ and recall the Kurosh subgroup theorem, i.e a subgroup $A * B$ is of the form $*_{i\in I}u_i^{-1}H_iu_i * *_{j\in J}v_j^{-1}\langle V_j\rangle v_j$, where H_i's are subgroups of A or B and V_j's are cyclically reduced and $l(V_j) \geq 2$. We remark that $v_j^{-1}\langle V_j\rangle v_j$'s are freesubgroups.

If there exists $m < \omega$ such that $h(\times_{n\geq m}\mathbb{Z}_n)$ is contained in a free subgroup, we have $m_0 \geq m$ such that $h(\times_{n\geq m_0}\mathbb{Z}_n)$ is trivial by the Higman theorem [11] (see also [7, Corollary 3.7]). Therefore, every $h(\times_{n\geq m_0}\mathbb{Z}_n)$ has a free factor $u^{-1}Hu$ for some non-trivial subgroup H of A or B. On the other hand, if there exists $m < \omega$, $u_0, u_1 \in A \cup B$ and $w_0, w_1 \in A * B$ such that $u_0^2 = u_1^2 = e$, $h(\times_{n\geq m}\mathbb{Z}_n) \leq w_0^{-1}\langle u_0\rangle w_0 * w_1^{-1}\langle u_1\rangle w_1$, then we get the conclusion for $w_0^{-1}\langle u_0\rangle w_0 * w_1^{-1}\langle u_1\rangle w_1$ by Lemma 3.2, which implies the conlusion for $A * B$. Therefore, in the following argument we assume that $h(\times_{n\geq m}\mathbb{Z}_n)$ is not a subgroup of $w_0^{-1}\langle u_0\rangle w_0 * w_1^{-1}\langle u_1\rangle w_1$ for any $m < \omega$, $u_0, u_1 \in A \cup B$ with $u_0^2 = u_1^2 = e$ and $w_0, w_1 \in A * B$.

We prove the theorem similarly as the proof of Lemma 3.2. Hence we suppose the negation of the conculsion. Then by the above assumption and Lemma 2.9 we have $x_m \in \times_{n\geq m_0}\mathbb{Z}_n$ such that $h(x_m) \notin C_2$. Then we choose natural numbers k_m's by induction. Let $k_0 = 1$ and k_m be a natural number which satisfies the following two requirements:

(1) $k_m \geq 4$ and $\max\{l(h(x_i^{k_i}\cdots x_{m-1}^{k_{m-1}})) : 0 \leq i \leq m-1\} + m + 2 \leq k_m$;
(2) $h(x_i^{k_i}\cdots x_{m-1}^{k_{m-1}})h(x_m)^{k_m} \notin C_1$ for every $0 \leq i \leq m-1$.

The existence of k_m and also $h(x_m^{k_m}) \notin C_2$ are assured by Lemma 2.10.

Now we modify the proof of Lemma 3.2. Let $W_m \in \mathcal{W}(\mathbb{Z}_n : n < \omega)$ be a reduced word such that $W_m = x_m^{k_m}$ for each m. Let

$$\overline{V} = \{(s,p) : s \in Seq, 0 \leq s(i) < k_i \text{ for } 0 \leq i < lh(s), p \in \overline{W_i}\}$$

with the lexicographical ordering and $V(s,p) = W_{lh(s)}(p)$. (We remark $h(W_{lh(s)}) = h(x_{lh(s)}^{k_{lh(s)}})$.) And also let

$$\overline{V_m} = \{(s,p) : \quad s \in Seq,\ lh(s) \geq m, s(i) = 0 \text{ for } 0 \leq i < m,\ 0 \leq s(i) < k_i \text{ for } m \leq i < lh(s), p \in \overline{W_i}\}$$

and V_m is the restriction of V to $\overline{V_m}$. We remark $V_m = x_m^{k_m}V_{m+1}^{k_{m+1}}$.

Finally choose m so that $l(h(V)) \leq m$ and apply Lemma 2.4 for $n = \max\{l(h(x_i^{k_i} \cdots x_{m-1}^{k_{m-1}})) : 0 \leq i \leq m-1\}$, $z = V_m$, $k = k_m$ and each y_i is $h(x_j^{k_j} \cdots x_{m-1}^{k_{m-1}})$ for some $0 \leq j < m$.

When (2) of Lemma 2.4 holds, $y_j \in C_1$ for some j, but according to our construction $h(x_j^{k_j} \cdots x_{m-1}^{k_{m-1}})$ does not belong to C_1 for any $0 \leq j < m$. Therefore, (1) of Lemma 2.4 holds, i.e. $V_m \in C_1$.

We apply the above argument to $m+1$, then we have $V_{m+1} \in C_1$ and consequently $h(x_m^{k_m}) = V_m V_{m+1}^{-k_{m+1}} \in C_2$, which contradicts our construction by Lemma 2.10. □

Now we recall

Lemma 3.3. [7, Theorem 2.1] *Let G_i $(i \in I)$ and H_j $(j \in J)$ be groups and $h : \times_{i\in I}^{\sigma} G_i \to *_{j\in J} H_j$ be a homomorphism to the free product of groups H_j's. Then there exist finite subsets F of I and E of J such that $h(\times_{i\in I\setminus F}^{\sigma} G_i)$ is contained in $*_{j\in E} H_j$.*

Apparently Theorem 1.3 strengthens this result.

Proof of Theorem 1.3. The proof is an application of Theorem 3.1, which is a special case of Theorem 1.3. First we show Theorem 1.3 in case $J = \{0, 1\}$. Let $H_0 = A$ and $H_1 = B$.

To show by contradiction suppose that the conclusion does not hold. We claim that for each finite subset F of I there exists $x \in \times_{i\in I\setminus F}^{\sigma} G_i$ such that $h(x) = w^{-1}aw$ or $w^{-1}bw$ for some $a \in A$, $b \in B$ and $w \in A * B$. This follows from the Kuroś subgroup Theorem and [7, Proposition 3.5] by the same argument in the first half of the proof of Theorem 3.1. Now we construct $x_m \in \times_{i\in I}^{\sigma} G_i, w_m \in A * B$, $u_m \in A \cup B$ and finite subsets F_m of I by induction so that the following hold:

(1) $x_m \in \times_{i\in I^*\setminus F_m}^{\sigma} G_i$ where $I^* = \bigcup_{m<\omega} F_m$ and $F_m \subseteq F_{m+1}$;
(2) $h(x_m) = w_m^{-1} u_m w_m$ with $u_m \neq e$;
(3) if the both u_m and u_{m+1} belong to A or B, then $w_m \neq w_{m+1}$, in the other words, if $w_m = w_{m+1}$, then $u_m \in A$ and $u_{m+1} \in B$ are equivalent.

Apart from I^* and F_m's we can choose x_m, u_m and w_m by the assumptions. For the care for I^*, before choosing x_{m+1} we let K_m to be a countable subset of I such that $x_m \in \times_{i\in K_m}^{\sigma} G_i$ and enumerate K_m so that $\{p(m,n) : n < \omega\}$. Then apply the standard book-keeping method to $\{p(k,n) : n < \omega, k \leq m\}$ to define $F_{m+1} \subseteq \bigcup_{k\leq m} K_k$ and we can get the desired ones.

Now we have a homomorphism $\varphi : \times_{n<\omega} \mathbb{Z}_n \to \times_{i\in I}^{\sigma} G_i$ such that $h(\delta_m) = x_m$ for $m < \omega$ by [7, Propostion 1.9], where δ_n is the generator of $\mathbb{Z}_n$. By Theorem 3.1 we have $n_0 < \omega$ such that $h \circ \varphi(\times_{n\geq n_0} \mathbb{Z}_n)$ is contained in a conjugate subgroup to A or B, which contradicts the construction of x_m's. Now we have shown the case that $J = \{0, 1\}$.

For a general case let $h : \times^{\sigma}_{i\in I} G_i \to *_{j\in J} H_j$ be a homomorphism. By Lemma 3.3 there exist finite subsets F of I and E of J such that $h(\times^{\sigma}_{i\in I\setminus F} G_i)$ is contained in $*_{j\in E} H_j$.

Now the restriction of h to $\times^{\sigma}_{i\in I\setminus F} G_i$ maps into $*_{j\in E} H_j$. Since $u^{-1}(*_{j\in E'} H_j)u = *_{j\in E'}\langle u^{-1} H_j u\rangle$ for $E' \subseteq I$, by succesive use of the case of $J = \{0, 1\}$ we have the conclusion. □

Next we prove Theorem 1.4. We recall some notions about loops. For a path $f : [0, 1] \to X$, f^- denotes the path defined by: $f^-(s) = f(1-s)$ for $0 \leq s \leq 1$. For paths $f : [0, 1] \to X$ and $g : [0, 1] \to X$ with $f(1) = g(0)$ we denote the concatenation of the paths f and g by fg. The Hawaiian earring is the plane compactum $\mathbb{H} = \{(x, y) : (x+1)^2 + y^2 = 1/n^2, 1 \leq n < \omega\}$ and each simple closed curve of the Hawaiian earring $\mathbb{H}$ is parametrized as follows: $\mathbf{e}_n(t) = ((\cos 2\pi t - 1)/n, \sin 2\pi t/n)$ for $1 \leq n < \omega, 0 \leq t \leq 1$. (Here, $\mathbf{e}_n$ refers to the n-th earring, that is the n-th simple closed curve.)

Lemma 3.4. *Let X be a path-connected, locally path-connected space which is not semi-locally simply connected at any point and $h : \pi_1(X, x_0) \to *_{j\in J} H_j$ be an injective homomorphism. For each point $x \in X$ and a path p_x from x to x_0 there exists a path-connected open neighborhood U of x satisfying: there exist $w_x \in *_{j\in J} H_j$ and $j(x) \in I$ such that for every loop l in U with base point x, $h([p_x^- l p_x]) \in w_x^{-1} H_{j(x)} w_x$. Moreover $j(x)$ does not depend on the choice of a path p_x.*

Proof. To show this by contradiction suppose that such a neighborhood does not exist for a point x and a path p_x. Let $\{U_n : n < \omega\}$ be a neighrhood base of x consisting of path-connected open sets. We construct loops l_n in U_n with base point x as follows. If there exists an essential loop l in U_n with base point x such that $h([p_x^- l p_x])$ does not belong to a conjugate subgroup to some H_j, then we let l_n be such a loop. Otherwise, but if $n = 0$ or $h(p_x^- l_{n-1} p_x)$ does not belong to a conjugate subgroup to some H_j, we let l_n be an arbitrary essential loop in U_n with base point x. Otherwise, and if $h(p_x^- l_{n-1} p_x)$ belongs to a conjugate subgroup to some H_j, we let $w_{n-1}^{-1} u_{n-1} w_{n-1} = h(p_x^- l_{n-1} p_x)$ where u_{n-1} belong to some H_j. By our construction, for each essential loop l in U_n with base point x, $h([p_x^- l p_x])$ belongs to a conjugate subgroup to some H_j. By the assumption we can choose an essential loop l_n in U_n with base point x so that $h(p_x^- l_n p_x) = w_n^{-1} u_n w_n$, $u_n \in H_j$ for some j, but $w_n \neq w_{n-1}$ or u_n does not belong to the same H_j to which u_{n-1} belongs.

We define a continous map $f : \mathbb{H} \to X$ so that $f((0,0)) = x$ and $f \circ \mathbf{e}_n = l_n$. Then f_* is a forbidden homomorphism by Theorem 1.3 and the contradiction occurs.

To see the additional property, let q_x be another path from x to x_0. Then the conclusion follows from an equation $[q_x^- l q_x] = [q_x^- p_x p_x^- l p_x p_x^- q_x p_x^- q_x] = [p_x^- q_x]^{-1}[p_x^- l p_x][p_x^- q_x]$. □

Proof of Theorem 1.4. By Lemma 3.4 for each point $x \in X$ we have a path-connected open neighborhood $U(x)$ of x and $j(x) \in J$ such that for any path p from x to x_0 and loop l in U_x with base point x $h([p^- l p])$ is contained in a conjugate subgroup to $H_{j(x)}$. We have $w_0 \in *_{j \in J} H_j$ such that for any loop l in $U(x_0)$ with base point x_0 $h([l])$ is contained in $w_0^{-1} H_{j(x_0)} w_0$. We'll show $j(x) = j(x_0)$ for all x and a conjugator w_0 does not change under taking different paths to the base point.

Let p be an arbitrary path from x to x_0. Considering open intervals contained in $p^{-1}(U(y))$'s, we have $0 = t_n < \cdots < t_0 = 1$ and $x_1, \cdots, x_{n-1} \in X$ such that $p(0) = x_{n-1} = x$, $p(1) = x_0$, and $p([t_{i+1}, t_i]) \subseteq U(y_i)$. Let p_i and p'_i be the restricted paths of p to $[t_i, 1]$ and $[t_i, t_{i-1}]$ for $0 \le i \le n-1$ and $1 \le i \le n-1$ respectively. Since each $U(x_i)$ is path-connected, we choose a path q_i in $U(x_i)$ from x_i to $p(t_i)$ for $1 \le i \le n-2$ and we let q_0 and q_{n-1} to be the constant paths.

We show that for any loop l in $U(x_i)$ with base point x_i $h([(q_i p_i)^- l q_i p_i])$ is contained in $w_0^{-1} H_{j(x_0)} w_0$ by induction on i. The case $i = 0$ is our assumption. Suppose that this holds for $i-1$. We have an essential loop l_i in $U(x_i) \cap U(x_{i-1})$ with base point $p(t_i)$. Now

$$
\begin{aligned}
[p_i^- l p_i] &= [p_{i-1}^- q_{i-1}^- q_{i-1} p_i'^- l p_i' q_{i-1}^- q_{i-1} p_{i-1}] \\
&= [(q_{i-1} p_{i-1})^- (p_i' q_{i-1}^-)^- l (p_i' q_{i-1}^-)(q_{i-1} p_{i-1})] \text{ and} \\
[p_i^- l p_i] &= [p_i^- q_i^- q_i l q_i^- q_i p_i] = [(q_i p_i)^- q_i l q_i^- (q_i p_i)].
\end{aligned}
$$

We remark that $(p_i' q_{i-1}^-)^- l (p_i' q_{i-1}^-)$ and $q_i l q_i^-$ are loops in $U(x_{i-1})$ and $U(x_i)$ with base points x_{i-1} and x_i respectively. By induction hypothesis we have $h([p_i^- l p_i]) \in w_o^{-1} H_{j(x_0)} w_0$ and by the choice of $U(x_i)$ we have the desired property for i. At the final step, i.e. the $n-1$-th step, we have $[p_{n-1}^- l p_{n-1}] = [p_n^- p_n' l p_n'^- p_n] = [p^- (p_n' l p_n'^-) p]$ and $p_n' l p_n'^- p_n$ is a loop in $U(x)$ with base point x. We conclude that for any loop l in $U(x)$ with base point x $h([p^- l p]) \in w_0^{-1} H_{j(x_0) w_0}$ holds.

Now we apply this fact to tha case $x = x_0$ and l be an essential loop in $U(x)$ with base point x_0 and consequently p is an arbitrary loop. Then we have $h([p])^{-1} h([l]) h([p]) = h([p^- l p]) \in w_0^{-1} H_{j(x_0)} w_0$ and also $h([l]) \in w_0^{-1} H_{j(x_0)} w_0$ by the assumption. Now Lemma 2.12 implies $h([p]) \in w_0^{-1} H_{j(x_0)} w_0$. □

Remark 3.5. *(1) The injectivity of a homomorphism in Theorem 1.4 is essential, which is shown in* [4, Remark 4.6]. *On the other hand the injectivity can be weakened as follows. For a non-empty open set* U *and a path from a point* $p(0)$ *in* U *to* x_0, *let* H_U^p *be a subgroup of* $\pi_1(X, x_0)$ *consisting all* $[p^- l p]$*'s for loops* l *in* U *with base point* $p(0)$. *The injectivity can be weakened to:* $h(H_U^p)$ *is non-trivial for an arbitrary non-empty open set* U *and a path from a point* $p(0)$ *in* U *to* x_0.

(2) The free σ*-product* $\times^\sigma_{i\in I} G_i$ *is realized as the fundamental group of the one point union of spaces whose fundamental groups are isomorphic to* G_i*'s. More precisely, let* X_i *be a space locally strongly contractible at* x_i *with* $pi_1(X_i, x_i) = G_i$ *and identify all* x_i*'s as one point* $*$. *Let* $\widetilde{\bigvee}_{i\in I}(X_i, x_i)$ *be the space with this base set. The topology of each* $X_i \setminus \{x_i\}$ *is the same as in* X_i. *A neighborhood of* $*$ *is of the form* $\bigcup_{i\in I} O_i$ *where each* O_i *is a neiborhood of* x_i *and all but finite many* O_i*'s are the whole spaces* X_i. *Then* $\pi_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i))$ *is isomorphic to* $\times^\sigma_{i\in I} G_i$ [7, Theorem A.1].

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School of Science and Engineering, Waseda University, Tokyo 169-8555, JAPAN
E-mail address: eda@logic.info.waseda.ac.jp