非線形拡散方程式系および関連する
界面問題の解析

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はしがき

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なお，発表の詳細は研究代表者による「総括」における論文発表・口頭発表の部分，および各研究分担者の成果発表部分を参照していただきたい。

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総括

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「非線形拡散方程式系および関連する界面問題の解析」

について、研究分担者と協力しながら、次のような準線形拡散項を含む反応拡散方程式系

$$\begin{cases} u_t = \mu\Delta[(1+\alpha v+\gamma u)u] + uf(u,v), \\ v_t = \nu\Delta[(1+\beta u+\delta v)v] + vg(u,v), \end{cases}$$

に対する定常問題、および非定常問題を集中的に研究した。このシステムは同一領域で生存競争する2種類の生物の棲み分け現象を記述する数理モデルとして有名であり、1979年 Shigesada-Kawasaki-Teramoto らの数理生態学者のグループによって提起されたものである。未知関数 u, v は個体数密度を表わし、f, g は u, v 間の相互作用を表す。Lotka - Volterra 型の競合モデルのケースでは

$$f(u,v) = a - u - cv, \quad g(u,v) = b - du - v,$$

prey-predator モデルのケースでは

$$f(u,v) = a - u - cv, \quad g(u,v) = b + du - v,$$

(u: prey, v: predator) の形となる。ただし、上の例において a, b, c, d は正定数である。

研究成果を大別すると、以下のテーマのようになる。

- (I) cross-diffusion 項を伴う反応拡散方程式系の時間大域解の存在
- (II) Lotka-Volterra 競合型の反応拡散方程式系の正値定常解の多重性
- (III) cross-diffusion 項を伴う反応拡散方程式系に関する正値定常解の構造

それぞれのテーマについてもう少し詳しく述べる。

(I) 非線形拡散 (cross-diffusion) を伴う上記モデルに対する非定常問題

$$\begin{cases} u_t = \mu\Delta[(1+\alpha v+\gamma u)u] + u(a-u-cv) & \text{in} \quad \Omega\times(0,\infty), \\ v_t = \nu\Delta[(1+\beta u+\delta v)v] + v(b-du-v) & \text{in} \quad \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega\times(0,\infty), \\ u(\cdot,0) = u_0(\geq 0), v(\cdot,0) = v_0(\geq 0) & \text{in} \quad \Omega, \end{cases} \tag{1}$$

を考える。ここで Ω は滑らかな境界 $\partial\Omega$ をもつ、R^N における有界領域、μ,ν は正定数、$\alpha,\beta,\gamma,\delta$ は非負定数である。初期値境界値問題 (1) に対して、時間大域解の存在に関する既存の結果は N が 2 以下のケースに限られていた。本研究では、$\alpha,\gamma>0$ の場合、もう一方の方程式の拡散項が線形（$\beta=\delta=0$）ならば、空間次元 N や初期データ (u_0,v_0) の大きさと無関係に大域解 (u,v) が一意的に存在することを示すことができた（論文 [7]）。うまくいった理由は、システムを準線形放物型方程式と半線形放物型方程式に分解し、各方程式について別々に評価を導いた点にある。まず

$$u_t = \mu\Delta[(1+\alpha v+\gamma u)u] + u(a-u-cv) \tag{2}$$

に対して Moser の技法を用いてエネルギー評価を導くと, u についての $L^p(\Omega\times(0,T))$ 評価が求められる。一方、最大値原理を

$$v_t = \nu\Delta v + v(b-du-v) \tag{3}$$

に適用すると、v の一様有界性が導かれるだけではなく、さらに v の $W_p^{2,1}(\Omega\times(0,T))$ ノルムの有界性も得られる。u を変数係数の線形方程式の弱解とみなすと、最大値原理が適用できて u の一様有界性も得られる。この後 (3), (2) を交互に利用して正則性を挙げていけば、Hölder ノルムに関するアプリオリ評価が求められる。

解の大域的存在を示すアイデアは上に述べたとおりである。さらにこのアイデアは (1) について $\beta=0,\delta>0$ のケースについても有効であり、N が 5 以下の場合にも大域解の一意的存在を示すことができた（論文 [8]）。

(II) Lotka-Volterra 型数理生態学モデルに対する定常問題について、正値定常解は共存解として大きな意味があり、定常解の個数を知ることは重要な問題である。しかし線形拡散のケースですら、正値定常解集合の構造については完全な解明から程遠い状態である。それ故、正値定常解が複数個存在するのはいかなる状況か？を調べることは価値がある。線形拡散を伴う競合モデル

$$\begin{cases} \Delta u + u(a-u-cv) = 0 & \text{in} \quad \Omega, \\ \Delta v + v(b-du-v) = 0 & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \\ u\geq 0, \quad v\geq 0 & \text{in} \quad \Omega, \end{cases} \tag{4}$$

を考える。u, v 間の相互作用の係数 c, d を無限大とすると、(4) はある種の極限問題と密接な関係があることが知られている（Dancer-Du (1994))。実際、c, d を $d/c \to m$ をみたしながら、無限に大きくすると

$$\begin{cases} \Delta w = w^+(a - w^-/m) + w^-(b + w^-) & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

が極限問題として得られる。w_0 を孤立した、(5) の符号変化解とするとき、粗く述べると c, d が十分大ならば、(4) は $(w_0^+/m, -w_0^-)$ の近くに正値解 (u, v) を持つことが知られている（Dancer-Du)。我々の研究では $N = 1$ の場合に (5) の解集合の構造を完全に解明できることを示した。とくに、c, d が大きくなるにつれて解集合はより複雑になり、解の個数も無限に大きくなっていく。これに応じて、(4) の正値解集合の構造に関する重要な知見が得られた（論文 [4],[6])。

(III) Lotka-Volterra 型の数理生態学モデルに対する定常問題について、cross-difusion 項を伴う場合を考える。近年我々が扱っているモデルは

$$\begin{cases} \Delta[(1 + \alpha v)u] + u(a - u - cv) = 0 & \text{in} \quad \Omega, \\ \Delta[(1 + \beta u)v] + v(b + du - v) = 0 & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \\ u \geq 0, \quad v \geq 0 & \text{in} \quad \Omega, \end{cases} \tag{6}$$

の形の prey-predator 型である。これについて複数個の正値定常解はいかなる条件で存在するか調べるために、分岐理論を用いる。出発点は (6) の半自明解

$$(\theta_a, 0), \quad (0, \theta_b)$$

である。ただし、θ_a は

$$\Delta w + w(a - w) = 0 \ \text{ in } \ \Omega, \quad w = 0 \ \text{ on } \ \partial\Omega$$

をみたす唯一の正値解であり、$a > \lambda_1$ ($\lambda_1 = -\Delta$ の最小固有値) のときに限り存在する。β が非常に大きく、α がゼロに近い場合は、a を分岐パラメータとみなすと $(0, \theta_b)$ から S 字型の分岐ブランチが現れることなど、興味深い事実が明らかになりつつある。この結果は本報告書における論文
"Multiple coexistence states for a prey-predator system with cross-diffusion" by Kousuke Kuto and Yoshio Yamada
にまとめられている。また、定常解の個数のみならず、各解の安定性についても情報が得られつつある。

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Cross-Diffusion　項を伴う反応拡散方程式の定常問題と関連する話題

Positive Solutions for Reaction-Diffusion Systems with Cross-Diffusion and Related Topics

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A part of this talk is a joint work with Dr. Kousuke KUTO (Waseda University).

1　Cross-Diffusion Model

The cross-diffusion model was first introduced by Shigesada, Kawasaki and Teramoto [25] in 1979 to describe the habitat segregation phenomena of two competing species. Our system consists of two reaction diffusion equations with nonlinear diffusion terms and prey-predator interaction terms:

$$
\text{(P)} \quad \begin{cases} u_t = \Delta[(1+\alpha v)u] + u(a-u-cv) & \text{in } \Omega\times(0,\infty), \\ v_t = \Delta[(1+\beta u)v] + v(b+du-v) & \text{in } \Omega\times(0,\infty), \\ u = v = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(\cdot,0)=u_0\ge 0, \quad v(\cdot,0)=v_0\ge 0, & \text{in } \Omega, \end{cases}
$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, α, β are nonnegative constants, constants a, b, c, d are positive except for b and u_0, v_0 are nonnegative functions.

In (P) u and v denote the population densities of prey and predator species, respectively. The presence of cross-dissusion terms means that the diffusion also depend on the population density due to the population pressure from other species. The mathematical derivation of cross-diffusion can be found in the monograph of Okubo [22]. In reaction terms, a, b correspond to the carrying capacity of the environments and c, d denote the prey-predator interaction between two species.

2 Nonstationary Problem

Mathematically, one of the most important problems is to show the existence of global solutions for any initial nonnegative initial functions and for any space dimension. However, this is generally an open problem. The global existence can be proved in a very restricted case.

Theorem 1 *Let $N = 1, 2$ and assume $\alpha = 0$ and $\beta > 0$. If (u_0, v_0) satisfies $u_0, v_0 \in W_0^{1,p}(\Omega)$ with $p > N$, then* (P) *has a unique global solution $u, v \in C([0,\infty); W_0^{1,p}(\Omega)) \cap C((0,\infty); W^{2,p}(\Omega) \cap C^1((0,\infty); L^p(\Omega))$.*

Remark. A similar result was obtained by Lou, Ni and Wu [18] for the competion model with the same nonlinear diffusion as (P) with $N = 2$. We can follow their arguments to get some a priori estimates of $\| u(t) \|_{W^{2,p}}$ and $\| v(t) \|_{W^{2,p}}$ by the energy method.

I will give some comments on the global existence of solutions for reaction-diffusion systems with cross-diffusion terms such as (P). The global existence in the case $N = 1$ was first established by Kim [12]. In higeher dimensional case, it will be standard to carry out the folowing procedure to show the global existence.

1. The first step is tp prepare the local existence theorem.
2. The second step is to derive necessary a priori estimates of solutions.
3. Finally it is sufficient to combine the local existence theorem and suitable estimates to show the globa existence.

As to the local theory for cross-diffusion systems, there are two results. The first one is given in the framework of $L^p(\Omega)$ theory by Amann [1]. For (P), the local existence result can be stated as follows: if $u_0, v_0 \in W_0^{1,p}$ with $p > N$, then there exists a unique local solution of (P).

The second one is proved by Yagi [26]. His result is given by the following: let $N = 1, 2, 3$ and assume $u_0, v_0 \in H_0^{N/2+\epsilon}$ with some $\epsilon > 0$; then there exists a unique local solution of (P).

The global results are established for competition systems by several authors, Deuring[6], Ichikawa and Yamada [9], Lou, Ni and Wu [18], Pozio [23], Redlinger [24], Yagi[26], [27] and Yamada[30]. However, all works have some restrictionss on space dimension, nonlinear diffusion terms or reactoin terms. See also the recent work of Choi, Lui and Yamada [3].

3 Stationary Problem; Existence of Positive Solutions

The stationary problem associated wuth (P) is given by

$$\text{(SP)} \qquad \begin{cases} \Delta[(1+\alpha v)u] + u(a - u - cv) = 0 & \text{in } \Omega, \\ \Delta[(1+\beta u)v] + v(b + du - v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u \geq 0, \quad v \geq 0 & \text{in } \Omega. \end{cases}$$

Here we are interested with nonnegative functions and, especially, possitive solutions of (SP), that is a pair (u, v) such taht $u > 0$ and $v > 0$ in Ω. Important subjects are to study the existence, non-existence multiplicity, bifurcation and stality of postive solutions.

In the linear diffusion case ($\alpha = \beta = 0$), there are lots of works concerning Lotka-Volterra diffusion systems, see, e.g., Blat-Brown[2], Dancer[4],[5], Eilbeck, Furter López-Gómez [8]. Especilly, a necessary and sufficient condition for the existence of positive solutions is well known (see Li [15]). The uniqueness is shown by Løpez-Gømez and Parod [16] in case $N = 1$. For higher dimensional case, there is a conjecture that (SP) has a unique solution. However, it is not proved yet.

In the nonlinear diffusion, a sufficient condition for the existence of positive solutions is established by Nakashima and myself [21]. However, understnading of the structure of positive solutions for (SP) is far from complete.

We prepare some notation and give some basic results for positive solutions of (SP). For $q \in C(\overline{\Omega})$, denote by $\lambda_1(q)$ the least eigenvalue of

$$(3.1) \qquad \begin{cases} -\Delta w + q(x)w &= \lambda w \qquad \text{in } \Omega, \\ w &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

If $q \equiv 0$, then we simply write $\lambda_1 = \lambda_1(0)$ and denote the corresponding eigenfunction by φ. Since λ_1 is the principal eigenvalue, φ can be normalized so that it satisfies

$$\varphi > 0 \text{ in } \Omega \text{ and } \int_\Omega \varphi^2 dx = 1.$$

The following logisitic equation holps us to understand the cross-diffusion model:

$$\left\{ \begin{array}{ll} \Delta w + w(a - w) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{3.2}$$

It is well known that (3.2) has a unique positive solution θ_a if and only if $a > \lambda_1$. Moreover, this positive solution θ_a possesses the following prperties.

Lemma 3.1 (i) *The mapping $a \to \theta_a$ is of class C^1 in $C(\overline{\Omega})$ and $\partial\theta_a/\partial a > 0$ in Ω.*
(ii) $\theta_a = \dfrac{a - \lambda_1}{\sigma}\varphi + o(a - \lambda_1)$ *as* $a \to \lambda_1$,

$$\text{where } \sigma = \int_\Omega \varphi^3 dx.$$

In what follows, we assume $\alpha = 0$ for the sake of simplicity. We introduce a new unknown function

$$V = (1 + \beta u)v.$$

The original problem (SP) is reduced to the following semilinear elliptic system:

$$\text{(SE)} \quad \left\{ \begin{array}{ll} \Delta u + u\left(a - u - \dfrac{cV}{1 + \beta u}\right) = 0 & \text{in } \Omega, \\ \Delta V + \dfrac{V}{1 + \beta u}\left(b + du - \dfrac{V}{1 + \beta u}\right) = 0 & \text{in } \Omega, \\ u = V = 0 & \text{on } \partial\Omega, \\ u \geq 0, \quad V \geq 0, & \text{in } \Omega. \end{array} \right.$$

Clearly, (SE) admits the following trivial and semi-trivial solutions:

$$(u, V) = \left\{ \begin{array}{ll} (0, 0), & \\ (\theta_a, 0) & \text{if } a > \lambda_1, \\ (0, \theta_b) & \text{if } b > \lambda_1. \end{array} \right.$$

We can construct positive solutions of (SE) from these semi-trivial solutions by the method developed by Nakashima and the author [21].

Theorem 2 *Define*

$$\Sigma^+ = \{(a,b) \in R^2; a > \lambda_1, \lambda_1\left(-\frac{b+d\theta_a}{1+\beta\theta_a}\right) < 0, \lambda_1(c\theta_b - a) < 0\},$$
$$\Sigma^- = \{(a,b) \in R^2; a > \lambda_1, \lambda_1\left(-\frac{b+d\theta_a}{1+\beta\theta_a}\right) > 0, \lambda_1(c\theta_b - a) > 0\}$$

and set $\Sigma = \Sigma^+ \cup \Sigma^-$, where $\lambda_1(q)$ is defined the least eigenvalue of (3.1). If $(a,b) \in \Sigma$, then (SE) has at least one positive solution $u, v \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$.

Remark 3.1. Define two curves Γ_1 and Γ_2 by

$$\Gamma_1 = \left\{(a,b) \in R^2; \lambda_1\left(-\tfrac{b+d\theta_a}{1+\beta\theta_a}\right) = 0\right\},$$
$$\Gamma_2 = \{(a,b) \in R^2; \lambda_1(c\theta_b - a) = 0\}.$$

Then Σ in Theorem 2 a region surrouded by Γ_1 and Γ_2 in ab-plane. One can show that Γ_1, Γ_2 are smooth curves. Indeed, Γ_1 is represented as $b = \Phi(a)$ with $a > \lambda_1$, where Φ is a C^1-function such that

(i) $\Phi' < 0$ if $d > \beta\lambda_1$ and $\Phi' > 0$ if $d < \beta\lambda_1$,

(ii) $\lim_{a\to\lambda_1} \Phi(a) = \lambda_1$ and $\lim_{a\to\lambda_1} \Phi'(a) = \beta\lambda_1 - d$.

Therefore, Γ_1 is strictly decreasing curve if β is small ($d > \beta\Lambda_1$) and strictly increasiing if β is large ($d < \beta\Lambda_1$).

Silimarly, Γ_2 is represented as $a = \Psi(b)$ with $b > \lambda_1$, where Ψ is also a C^1-function satisfying

$$\Psi' > 0, \quad \lim_{b\to\lambda_1} \Psi(b) = \lambda_1 \text{and} \lim_{b\to\lambda_1} \Psi'(a) = c.$$

Remark 3.2. I will give some biological meaning of Theorem 2. In (SE) β is a cross-diffusion coefficient and the presence of β makes predator disperse due to the population pressure from prey-species. Therefore, assording as β becomes larger and larger, it brings about disadvantage to predator species. It will becoome more difficult for predator to survive as β becomes large.

I will explain the meaning of Theorem 2 from the view-point of bifurcation theory. Let b be fixed and regard a as a bifurcation paramater.

Suppose $b > \lambda_1$. We first consider the case $\beta\lambda_1 < d$. Then one can find that $a^* = \Psi(b)$ is a bifurcation point. The local bifurcation theory implies that positive solutions of (SE) bifurcate from semi-trivial solution $(0, \theta_b)$ at $(a_*, b) \in \Gamma_2$. The global bifucation theory assures that this branch of positive solutions can be extended over (a_*, ∞) as an unbounded set of $R \times X \times X$ with $X = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$.

We next consider the case $\beta\lambda_1 > d, 1 + c(d - \beta\lambda_1) > 0$ with $b > \lambda_1$. In this case curve Γ_1 is located in the right-hand side of $Gamma_2$ and one can find a^* such that $b = \Phi(a^*)$ and $a^* > a_*$. So there exists a branch of positive solutions of (SE) such that the branch connects two semi- trivial solutions; $(0, \theta_b)$ at $(a_*, b) \in \Gamma_2$ and $(0, \theta_{a^*})$ at $(a^*, b) \in \Gamma_1$.

Finally, in case $1 + c(d - \beta\lambda_1) < 0$ the situation is quite similar to the above. We should note the location of Γ_1 and Γ_2 is exchanged differently from the above case.

Remark 3.3. Theorem 2 does not give us precise information about the structure of positive solutions of (SE). We need some device to know the number of positive solutions and the optimal coexistence region in ab plane.

4 Branch of Positive Solutions

We will get better understanding of the structure of the set of positive solutions of (SE) in a special case:

$$(a, b, d/\beta): \text{ is very close to } (\lambda_1, \lambda_1, \lambda_1)$$
$$\beta: \text{ is sufficiently large}$$

Regarding a as a bifurcation parameter we have the following theorem.

Theorem 3 *For every $c > 0$, there exists an open set $\mathcal{O} \subset \{(b, d, \beta) \in R^3 : b > \lambda_1\}$ such that, if $(b, d, \beta) \in \mathcal{O}$, then $\mathcal{S} := \{(a, u, V, a) \in R \times \{W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)\}^2; (u, V)$ is a positive solution of (SE) with $a > \lambda_1\}$ contains a smooth curve $\mathcal{S}^+$ which bifurcates from $(0, \theta_b)$ at $a = a^*(:= \Psi(b))$ and has at least two turning points with respect to a.*

Remark 4.1. It follows from Theorem 3 that the branch of positive solutions looks like an 'S' shaped curve when $a, b, d/\beta$ are close to $lambda_1$ and β is sufficiently large. Since the branch has two turning points at $a = a_1, a_2$ $(a_1 > a_2)$, (SE) admits at least three positive solutions when a lies in (a_2, a_1).

An analogous result has been obatained by Du and Lou [7]. They have shown the S-shaped global bifurcation diagram for Holling-Tanner predator-prey model with linear diffusion.

The proof of Theorem 3 is very long; so we will give only the idea of its proof. For details, see Kuto and the author [13].

1. Introduction of new variables

Let $\epsilon > 0$ be a small parameter and set

$$a = \lambda_1 + a_1\epsilon, \quad b = \lambda_1 + b_1\epsilon, \quad u = \epsilon w,$$
$$V = \epsilon z, \quad \frac{d}{\beta} = \lambda_1 + \tau\epsilon, \quad \beta = \frac{\gamma}{\epsilon}.$$

With introduction of new variables a_1, b_1, τ, γ, original system (SE) is rewritten as follows for new unknown funtions $\mathbb{U} = (w, z)$:

$$\text{(RP)} \qquad \begin{cases} \mathbb{L}\mathbb{U} + \epsilon\mathbb{F}(\mathbb{U}, a_1) = 0 & \text{in } \Omega, \\ \mathbb{U} = 0 & \text{on } \partial\Omega, \\ \mathbb{U} \geq 0 & \text{in } \Omega, \end{cases}$$

where $\mathbb{L}$ is a principal linear term defined by

$$\mathbb{L}\mathbb{U} := \begin{pmatrix} \Delta w + \lambda_1 w \\ \Delta z + \lambda_1 z \end{pmatrix},$$

and $\mathbb{F}$ is a nonlinear term given by

$$\begin{aligned} \mathbb{F}(\mathbb{U}, a_1) : &= \begin{pmatrix} w\left(a_1 - w - \frac{cz}{1+\gamma w}\right) \\ \frac{z}{1+\gamma w}\left(b_1 + \gamma\tau w - \frac{z}{1+\gamma w}\right) \end{pmatrix} \\ &= \begin{pmatrix} F_1(\mathbb{U}, a_1) \\ F_2(\mathbb{U}, a_1) \end{pmatrix}. \end{aligned}$$

2. Lyapunov-Schmidt Reduction

We will use the Lyapunov-Schmidt procedure to reduce (RP) to a suitable system in a finite dimensional space. Set

$$X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \qquad \text{and} \qquad Y = L^p(\Omega)$$

with $p > N$ and define a projection from X or Y into infinite dimensional subspace by

$$Pw = w - \langle w, \varphi\rangle\varphi \qquad \text{for} \quad w \in X(orY) \quad \text{with} \quad \langle w, \varphi\rangle = \int_\Omega w\varphi dx,$$

where φ is the eigenfunctions corresponding to λ_1 (see (3.1)). Therefore, X (resp. Y) is expressed as follows:

$$X = \{\varphi\} \oplus \widetilde{X} \quad (\text{resp.} \quad Y = \{\varphi\} \oplus \widetilde{Y}).$$

Every $\mathbb{U} = (w, z) \in X \times X$ can be decomposed as

$$\mathbb{U} = \begin{pmatrix} s\varphi_1 + \widetilde{w} \\ t\varphi_1 + \widetilde{z} \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix} \varphi + \widetilde{\mathbb{U}} \quad \text{with} \quad \widetilde{w}, \widetilde{z} \in \widetilde{X},$$

where $s = \langle w, \varphi \rangle$ and $t = \langle z, \varphi \rangle$. Every element in $Y \times Y$ can be also decomposed in the same manner. With use of this decomposition, (RP) is rewritten in the equivalent form:

$$\mathbb{L}\widetilde{\mathbb{U}} + \epsilon \mathbb{P}\mathbb{F}((s,t)\varphi + \widetilde{\mathbb{U}}, a_1) = 0 \tag{4.1}$$

$$\begin{cases} \displaystyle\int_\Omega F_1((s,t)\varphi + \widetilde{\mathbb{U}}, a_1)\varphi dx = 0, \\ \displaystyle\int_\Omega F_2((s,t)\varphi + \widetilde{\mathbb{U}}, a_1)\varphi dx = 0, \end{cases} \tag{4.2}$$

where $\mathbb{P}\mathbb{U} = (Pw, Pz)$ for $\mathbb{U} = (w, z) \in Y \times Y$. It should be noted that $\mathbb{L}$ is an isomorphism from $\widetilde{\mathbb{X}} := \mathbb{P}(X \times X)$ to $\widetilde{\mathbb{Y}} := \mathbb{P}(Y \times Y)$. Hence one can apply the implicit function theorem to (4.1) and solve $\widetilde{\mathbb{U}}$ as a function of s, t, a_1 and ϵ. As a result we have

Lemma 4.1 *For every (s, t, a_1) with $|s|, |t|, |a_1| \leq C$, there exists ϵ_0 such that, if $|\epsilon| < \epsilon_0$, then (4.1) has a solution*

$$\widetilde{\mathbb{U}}(s, t, a_1; \epsilon) \in C^1([-C, C]^3 \times [-\epsilon_0, \epsilon_0]; \widetilde{\mathbb{X}})$$

such that $\widetilde{\mathbb{U}}(s, t, a_1; 0) = 0$.

Since $\widetilde{\mathbb{U}}(s, t, a_1; 0) = 0$, we may define $(\widetilde{u}, \widetilde{v})(s, t, a_1; \epsilon) \in C^1([-C, C]^3 \times [-\epsilon_0, \epsilon_0]; \widetilde{\mathbb{X}})$ by

$$\widetilde{\mathbb{U}}(s, t, a_1; \epsilon) = (\epsilon\widetilde{u}, \epsilon\widetilde{v})(s, t, a_1; \epsilon).$$

Setting $\widetilde{\mathbb{U}} = (\epsilon\widetilde{u}, \epsilon\widetilde{v})$ in (4.2) we get the following finite dimensional problem to solve (s, t, a_1):

$$\mathbb{G}_\epsilon(s,t,a_1) := \begin{pmatrix} \displaystyle\int_\Omega (s\varphi + \epsilon\widetilde{u}) \{a_1 - (s\varphi + \epsilon\widetilde{u}) \\ \qquad\qquad - \frac{c(t\varphi+\epsilon\widetilde{v})}{1+\gamma(s\varphi+\epsilon\widetilde{u})}\Big\} \varphi dx \\ \displaystyle\int_\Omega \frac{t\varphi + \epsilon\widetilde{v}}{1+\gamma(s\varphi+\epsilon\widetilde{u})} \{b_1 + \tau\gamma(s\varphi + \epsilon\widetilde{u}) \\ \qquad\qquad - \frac{t\varphi+\epsilon\widetilde{v}}{1+\gamma(s\varphi+\epsilon\widetilde{u})}\Big\} \varphi dx \end{pmatrix}. \tag{4.3}$$

Here we should note that $\widetilde{u}, \widetilde{v}$ are smooth functions of s, t, a_1, ϵ. Thus we arrive at the equivalence between (RP) and

$$\mathbb{G}_\epsilon(s,t,a_1) = 0. \tag{4.4}$$

So what we should do is to find (s, t, a_1) satisfying (4.4).

3. Analysis of Limit problem

Our strategy to solve (4.4) is to study the corresponding limit problem. Letting $\epsilon \to 0$ in (4.3) leads to

$$\mathbb{G}_0(s,t,a_1) :=$$

$$\begin{pmatrix} s\left(a_1 - s\sigma - ct\displaystyle\int_\Omega \frac{\varphi^3}{1+\gamma s\varphi}dx\right) \\ t\left(b_1 + (\tau - b_1)\displaystyle\int_\Omega \frac{\gamma s\varphi^3}{1+\gamma s\varphi}dx - t\int_\Omega \frac{\varphi^3}{(1+\gamma s\varphi)^2}dx\right) \end{pmatrix}. \tag{4.5}$$

Here the expression of $\mathbb{G}_0(s, t, a_1)$ becomes simpler; so that will be easier to find (s, t, a_1) satisfying

$$\mathbb{G}_0(s,t,a_1) = 0. \tag{4.6}$$

We may expect that solutions of $G_\epsilon(s, t, a_1) = 0$ are close to solutions of $\mathbb{G}_0(s, t, a_1) = 0$ when ϵ is very small. This idea was used by Du and Lou [7] in the study of Holling-Tanner system for predator-prey interaction.

In view of (4.5), positive solutions of (4.6) are given by

$$(s,t,a_1) = (s, f(\gamma s), g(s)) \tag{4.7}$$

where

$$f(s) = \frac{b_1 + (\tau - b_1)\int_\Omega \frac{s\varphi^3}{1+s\varphi}dx}{\int_\Omega \frac{\varphi^3}{(1+s\varphi)^2}dx},$$

$$g(s) = s\sigma + cf(\gamma s)\int_\Omega \frac{\varphi^3}{1+\gamma s\varphi}dx$$

with $\sigma = \displaystyle\int_\Omega \varphi^3 dx$.

We follow the arguments of Du and Lou as in [7]; then we can positive solutions of (RP) in a neighborhood of

$$(z, w, a_1) = (s\varphi, f(\gamma s)\varphi, g(s))$$

provided ϵ is sufficiently small.
So our task becomes to study the profile of g in order to accomplish the proof of Theorem 3. Recall that g is expressed as

$$g(s) = s\sigma + h(\gamma s)$$

with

$$h(s) = cf(s)\int_\Omega \frac{\varphi^3}{1+s\varphi}dx.$$

When $\tau = 0$, one can show that h has a unique local maximum. Hence, if $|\tau|$ is sufficiently small, then h has also at least one local maximum as in the case $\tau = 0$. Therefore, if $|\tau|$ is small and γ is sufficiently large, then one can prove that $g(s)$ has one local maximum and one local minimum for for s close to zero. This fact implies the the branch of positive solutions looks like 'S' shaped.

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EXISTENCE OF GLOBAL SOLUTIONS FOR THE SHIGESADA-KAWASAKI-TERAMOTO MODEL WITH STRONGLY COUPLED CROSS-DIFFUSION

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Abstract. This paper is a continuation of [3] by the same authors to study the problem of global existence of strong solutions for the Shigesada-Kawasaki-Teramoto model. We shall prove global existence of strong solutions assuming that there are self- and cross-diffusions in the first species and there is no cross-diffusion in the second species. If self-diffusion is also present in the second species, then our result requires that the space dimension be less than 6.

1. Introduction. In this paper, we shall prove the global existence of solutions to the following strongly-coupled time-dependent system

$$\begin{cases} u_t = d_1\Delta[(1+\alpha v+\gamma u)u] + au(1-u-cv) & \text{in } \Omega\times[0,\infty)\,, \\ v_t = d_2\Delta[(1+\delta v)v] + bv(1-du-v) & \text{in } \Omega\times[0,\infty)\,, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,\infty)\,, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega\,, \end{cases} \tag{1.1}$$

when $\alpha > 0, \gamma > 0$ and $\delta \geq 0$. Here, Ω is a bounded region in R^n with smooth boundary $\partial\Omega$, $\partial/\partial\nu$ denotes the directional derivative along the outward normal on $\partial\Omega$, u_0 and v_0 are non-negative initial conditions, d_1, d_2, a, b, c, d are positive

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constants, α is the coefficient of cross-diffusion of the first species and γ, δ are the coefficients of self-diffusion for the first and second species, respectively. Since c and d are positive, u, v denote the population densities of competing species. The system is strongly-coupled because of the coupling in the highest derivatives in the first equation. Strongly-coupled systems occur frequently in biological and chemical models and they are notoriously difficult to analyze.

System (1.1) is a special case of a model proposed by Shigesada, Kawasaki and Teramoto in 1979 ([12]). Over the past ten years, many mathematicians have attempted to prove the global existence of solutions for this model. See [11] for a list of references for the time-dependent case and [9], [10] for the steady-state case. In [3], the authors have shown that global solutions exist for (1.1) if $\gamma = 0, \delta = 0$ and $\alpha \leq \alpha^*$ where $\alpha^* > 0$ is a constant depending on $\| v_0 \|_{L^\infty(\Omega)}$. The purpose of this paper is to remove this smallness assumption and show the global existence result for (1.1). The main result is stated as follows.

Theorem 1.1. *Let $\gamma > 0$ and suppose that $u_0 \geq 0, v_0 \geq 0$ satisfy zero Neumann boundary conditions and belong to $C^{2+\lambda}(\overline{\Omega})$ for some $\lambda > 0$. Then (1.1) possesses a unique nonnegative solution $u, v \in C^{2+\lambda,(2+\lambda)/2}(\overline{\Omega} \times [0,\infty))$ if either* (i) *$\delta = 0$ or* (ii) *$\delta > 0$ and $n < 6$.*

Let $Q_T = \Omega \times [0, T)$. Standard notations similar to those in [7] are adopted throughout this paper. In particular, $u \in W_q^{2,1}(Q_T)$ means that u, u_{x_i}, $u_{x_i x_j}$, $i, j = 1, \dots, n$ and u_t are in $L^q(Q_T)$, $\|u\|_{L^{p,q}(Q_T)} = [\int_0^T (\int_\Omega |u(x,t)|^p \, dx)^{q/p} \, dt]^{1/q}$ and $\|u\|_{V_2(Q_T)} = \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)}$. In addition, $u \in W_q^1(\Omega)$ means that u and ∇u are in $L^q(\Omega)$.

The organization of this paper is as follows. In §2, we collect some well known results and prove two new lemmas that are needed in §3. The main part of the proof is to show that there exists a constant $C_1(q,T) > 0$ such that $\|u\|_{L^q(Q_T)} \leq C_1(q,T)$ for some $q > (n+2)/2$. This is done in §3. In §4, we show that this estimate implies that u is also bounded in Q_T and hence smooth. Our theorem follows easily from this and Amann's theorem. For blow-up of solutions of nonlinear parabolic equations, see the survey paper by Galaktionov and Vázquez [5]. Throughout this paper, we shall assume that $n \geq 3$ since the cases $n = 2$ have been solved by Lou, Ni and Wu in [11] by a completely different method. This paper also gives a correct proof of [13].

2. **Preliminaries.** In [1], Amann proved that there exists a unique nonnegative local smooth solution for the Shigesada-Kawasaki-Teramoto model if cross-diffusion only appears in one of the species. Throughout this paper, (u, v) shall denote such a smooth solution and T the maximal time of existence. The following result, also from [1], gives a criterion for the existence of global solutions to the Shigesada-Kawasaki-Teramoto model and in particular to (1.1).

Theorem 2.1. *Suppose the local solution (u, v) satisfies the conditions*

$$\sup_{0 \leq t < T} \|u(\cdot, t)\|_{W_p^1(\Omega)} < \infty \quad \text{and} \quad \sup_{0 \leq t < T} \|v(\cdot, t)\|_{W_p^1(\Omega)} < \infty \tag{2.1}$$

for some $p > n$, then $T = \infty$.

The following result is valid as long as there is no cross-diffusion for v.

Lemma 2.1. *There exist positive constants* $m = \max\{1, \|v_0\|_{L^\infty(\Omega)}\}$ *and* $C_1(T)$ *such that*

$$0 \le u, \qquad 0 \le v \le m \quad \text{in} \;\; Q_T$$

and

$$\sup_{0\le t\le T} \|u(\cdot,t)\|_{L^1(\Omega)} \le C_1(T), \qquad \|u\|_{L^2(Q_T)} \le C_1(T).$$

PROOF. The first three inequalities follow from the maximum principle. The last two inequalities are derived by integrating the equation for u. The details are given in [11, Lemma 2.1 and Lemma 2.2]. □

Lemma 2.2. *Let* $w_2 = (1+\delta v)v$. *Then there exists a constant* $C_2(T)$, *depending on* $\|v_0\|_{W_2^1(\Omega)}$ *and* $\|v_0\|_{L^\infty(\Omega)}$ *such that*

$$\|w_2\|_{W_2^{2,1}(Q_T)} \le C_2(T). \tag{2.2}$$

Furthermore, $\nabla w_2 \in V_2(Q_T)$.

PROOF. The proof may be found in [11, Lemma 2.4], but for completeness we repeat it here. First, w_2 satisfies the equation

$$w_{2t} = d_2(1+2\delta v)\Delta w_2 + c_1 + c_2\, u, \tag{2.3}$$

where c_1, c_2 depend on v and are bounded functions because of Lemma 2.1. Multiplying the above equation by $-\Delta w_2$ and integrating by parts over Q_t, we have

$$\frac{1}{2}\int_\Omega |\nabla w_2|^2(x,t)dx - \frac{1}{2}\int_\Omega |\nabla w_2|^2(x,0)dx + d_2\int_{Q_t} |\Delta w_2|^2 dxdt \le \int_{Q_T} |\Delta w_2||c_1 + c_2 u|\, dxdt$$

Here the right-hand side of the above inequality is bounded from above by

$$\|\Delta w_2\|_{L^2(Q_T)}\|c_1 + c_2 u\|_{L^2(Q_T)} \le m_1\|\Delta w_2\|_{L^2(Q_T)}(1 + \|u\|_{L^2(Q_T)}) \le \frac{d_2}{2}\|\Delta w_2\|^2_{L^2(Q_T)} + \frac{m_1^2(1+C_1(T))^2}{2d_2}$$

with some $m_1 > 0$. Rearrranging,

$$\sup_{0\le t\le T}\int_\Omega |\nabla w_2|^2(x,t)dx + d_2\int_{Q_T} |\Delta w_2|^2 dxdt \le m_2$$

where m_2 depends on $\|v_0\|_{W_2^1(\Omega)}$ and $\|v_0\|_{L^\infty(\Omega)}$. Since $w_2 \in L^2(Q_T)$, we have from the elliptic regularity estimate [11, Lemma 2.3]

$$\int_{Q_T} |(w_2)_{x_i x_j}|^2 dxdt \le m_3 \quad \text{for } i,j = 1,\cdots,n.$$

From (2.3), since c_1, c_2 and v are bounded and $u \in L^2(Q_T)$, we have $w_{2t} \in L^2(Q_T)$. Hence, $w_2 \in W_2^{2,1}(Q_T)$. The proof of the lemma is complete. □

For the rest of this paper, we shall use the inequality $(a+b)^p \le 2^p(a^p + b^p)$ freely without mentioning it.

Lemma 2.3. *Let $q > 1$ and $\tilde{q} = 2 + 4q/n(q+1)$. Then there exists a positive constant M_1 such that for any w satisfying $\sup_{0\le t\le T} \|w\|_{L^{2q/(q+1)}(\Omega)} + \|\nabla w\|_{L^2(Q_T)} < \infty$, we have*

$$\|w\|_{L^{\tilde{q}}(Q_T)} \le M_1 \{ (\sup_{0\le t\le T} \|w(t)\|_{L^{2q/(q+1)}(\Omega)})^{4q/n(q+1)\tilde{q}} \|\nabla w\|_{L^2(Q_T)}^{2/\tilde{q}} + \|w\|_{L^{1,\tilde{q}}(Q_T)}\} . \tag{2.4}$$

Proof. Since $\tilde{q} \in (2q/(q+1), 2n/(n-2))$, the embedding theorem [7, Theorem 2.2, p.62] implies that for any smooth function w satisfying the zero-mean condition $\int_\Omega w\, dx = 0$, we have

$$\|w\|_{L^{\tilde{q}}(\Omega)} \le C_3 \|\nabla w\|_{L^2(\Omega)}^{\tilde{\alpha}} \|w\|_{L^{2q/(q+1)}(\Omega)}^{1-\tilde{\alpha}}, \tag{2.5}$$

where

$$\tilde{\alpha} = \frac{\frac{q+1}{2q} - \frac{1}{\tilde{q}}}{\frac{1}{n} + \frac{1}{2q}} \in (0,1).$$

If the zero-mean condition is not satisfied, the corresponding result is

$$\|w\|_{L^{\tilde{q}}(\Omega)} \le C_3 (\|\nabla w\|_{L^2(\Omega)}^{\tilde{\alpha}} \|w\|_{L^{2q/(q+1)}(\Omega)}^{1-\tilde{\alpha}} + \|w\|_{L^1(\Omega)}) \tag{2.6}$$

with the same $\tilde{\alpha}$ and is well known (see for example [2, Lemma 3.4]). Such result can be derived by putting the zero-mean function $w - \int_\Omega w\, dx$ in place of w in (2.5) and (2.6) leads us to conclude that

$$\|w(\cdot,t)\|_{L^{\tilde{q}}(\Omega)}^{\tilde{q}} \le 2^{\tilde{q}} C_3^{\tilde{q}} \{ \|\nabla w(\cdot,t)\|_{L^2(\Omega)}^{\tilde{\alpha}\tilde{q}} \|w(\cdot,t)\|_{L^{2q/(q+1)}(\Omega)}^{(1-\tilde{\alpha})\tilde{q}} + \|w(\cdot,t)\|_{L^1(\Omega)}^{\tilde{q}} \}$$

Since $\tilde{\alpha}\tilde{q} = 2$, the above inequality is simplified to

$$\begin{aligned} \|w(\cdot,t)\|_{L^{\tilde{q}}(\Omega)}^{\tilde{q}} &\le 2^{\tilde{q}} C_3^{\tilde{q}} \{ \|\nabla w(\cdot,t)\|_{L^2(\Omega)}^{2} \|w(\cdot,t)\|_{L^{2q/(q+1)}(\Omega)}^{\tilde{q}-2} + \|w(\cdot,t)\|_{L^1(\Omega)}^{\tilde{q}} \} \\ &\le 2^{\tilde{q}} C_3^{\tilde{q}} \{ \|\nabla w(\cdot,t)\|_{L^2(\Omega)}^{2} \|w(\cdot,t)\|_{L^{2q/(q+1)}(\Omega)}^{4q/n(q+1)} + \|w(\cdot,t)\|_{L^1(\Omega)}^{\tilde{q}} \} . \end{aligned}$$

We now integrate the above over $(0,T)$ and obtain (2.4) with $M_1 = 2C_3$. □

Lemma 2.4. *Let $q > 1$, $\tilde{q} = 2 + 4q/n(q+1)$ and assume that there exists a $0 < \tilde{\beta} < 1$ such that $(\int_\Omega |w(\cdot,t)|^{\tilde{\beta}} dx)^{1/\tilde{\beta}} \le C_T$ for all $t \in [0,T]$. Then there exists a positive constant M_2 independent of w but may depend on $\tilde{\beta}$ and C_T such that*

$$\|w\|_{L^{\tilde{q}}(Q_T)} \le M_2 \{1 + (\sup_{0\le t\le T} \|w(t)\|_{L^{2q/(q+1)}(\Omega)})^{4q/n(q+1)\tilde{q}} \|\nabla w\|_{L^2(Q_T)}^{2/\tilde{q}} \} . \tag{2.7}$$

Proof. Although $(\int_\Omega |u|^{\tilde{\beta}} dx)^{1/\tilde{\beta}}$ is not a norm for $0 < \tilde{\beta} < 1$, we will still use $\|u\|_{L^{\tilde{\beta}}(\Omega)}$ to denote it. The standard interpolation theorem for L^p spaces [6, (7.9) on p.146] yields

$$\|w(\cdot,t)\|_{L^1(\Omega)} \le \|w(\cdot,t)\|_{L^{\tilde{q}}(\Omega)}^{1-\lambda} \|w(\cdot,t)\|_{L^{\tilde{\beta}}(\Omega)}^{\lambda} \le C_T^{\lambda} \|w(\cdot,t)\|_{L^{\tilde{q}}(\Omega)}^{1-\lambda}$$

for $\lambda = \tilde{\beta}(\tilde{q}-1)/(\tilde{q}-\tilde{\beta}) \in (0,1)$. Raising the above inequality to power $\tilde{q}$ and integrating the resulting expression with respect to t, we obtain

$$\|w\|_{L^{1,\tilde{q}}(Q_T)}^{\tilde{q}} \le C_T^{\lambda\tilde{q}} \int_0^T (\int_\Omega |w(x,t)|^{\tilde{q}} dx)^{1-\lambda} dt \le C_T^{\lambda\tilde{q}} T^{\lambda} (\int_{Q_T} |w|^{\tilde{q}} dx dt)^{1-\lambda} .$$

Thus

$$\|w\|_{L^{1,\tilde{q}}(Q_T)} \le C_T^{\lambda} T^{\lambda/\tilde{q}} \|w\|_{L^{\tilde{q}}(Q_T)}^{1-\lambda} \le C_T^{\lambda} T^{\lambda/\tilde{q}} (\delta \|w\|_{L^{\tilde{q}}(Q_T)} + C_{\delta,\lambda}) \tag{2.8}$$

for any $\delta > 0$ with some $C_{\delta,\lambda} > 0$. Substituting (2.8) into (2.4), we obtain

$$\begin{aligned}\|w\|_{L^{\bar q}(Q_T)} &\le M_1 \{(\sup_{0\le t\le T}\|w(t)\|_{L^{2q/(q+1)}(\Omega)})^{4q/n(q+1)\bar q}\,\|\nabla w\|^{2/\bar q}_{L^2(Q_T)}\\ &+ C_T^{\lambda}T^{\lambda/\bar q}\,(\delta\|w\|_{L^{\bar q}(Q_T)} + C_{\delta,\lambda})\}\,.\end{aligned}$$

Finally we choose δ such that $\delta M_1 C_T^{\lambda}T^{\lambda/\bar q} = 1/2$ and (2.7) follows easily. The proof of the lemma is complete. □

3. L^q estimates for u. Let $t \in (0,T]$ and define $Q_t \equiv \Omega \times (0,t)$. In this section, $C_1, C_2, \ldots$ and $M_1, M_2, \ldots$ shall denote positive constants that may depend on Ω, q, n, the constants in the equations, T (but not on t), and the following norms $\|v_0\|_{W_q^{2-2/q}(\Omega)}$, $\|v_0\|_{L^\infty(\Omega)}$ and $\|u_0\|_{L^q(\Omega)}$.

Proposition 3.1. *Let $\gamma > 0, q > 1$ and $T > 0$, then if either (i) $\delta = 0$ or (ii) $\delta > 0$ and $q < 2(n+1)/(n-2)$, then there exist positive constants $C_{q,T}$ and C_T such that $\| u \|_{L^q(Q_T)} \le C_{q,T}$ and $\| u \|_{V_2(Q_T)} \le C_T$.*

Proof. The first part of the proof is valid irrespective of whether δ is positive or not. We begin by multiplying (1.1a) by $qu^{q-1}, q > 1$, and then integrate by parts to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega u^q\,dx &= q\int_\Omega u^{q-1}\{d_1\Delta[(1+\alpha v+\gamma u)u] + au(1-u-cv)\}\,dx\\ &= q\int_\Omega d_1\,u^{q-1}\{\nabla\cdot[(1+\alpha v+2\gamma u)\nabla u] + \alpha\nabla\cdot[u\nabla v]\}\\ &+ au^q(1-u-cv)\,dx \qquad (3.1)\\ &= -q(q-1)d_1\int_\Omega u^{q-2}(1+\alpha v+2\gamma u)|\nabla u|^2\,dx\\ &- (q-1)d_1\alpha\int_\Omega \nabla(u^q)\cdot\nabla v dx + qa\int_\Omega u^q(1-u-cv)\,dx.\end{aligned}$$

Dropping the αv term in the last expression of (3.1) and integrating from 0 to t, we have

$$\begin{aligned}&\int_\Omega u^q(x,t)\,dx - \int_\Omega u^q(x,0)\,dx + 4d_1\frac{q-1}{q}\int_{Q_t}|\nabla(u^{q/2})|^2\,dxdt\\ &+8\gamma d_1\frac{q(q-1)}{(q+1)^2}\int_{Q_t}|\nabla(u^{(q+1)/2})|^2\,dxdt \qquad (3.2)\\ \le\ & -(q-1)\,d_1\alpha\int_{Q_t}\nabla(u^q)\cdot\nabla v\,dxdt + qa\int_{Q_t}u^q(1-u-cv)\,dxdt\,.\end{aligned}$$

The last term in (3.2) may be estimated by

$$qa\int_{Q_t} u^q(1-u-cv)\,dxdt \;\le\; -qa \parallel u \parallel^{q+1}_{L^{q+1}(Q_t)} + qa \parallel u \parallel^{q}_{L^q(Q_t)}$$
$$\le\; -qa \parallel u \parallel^{q+1}_{L^{q+1}(Q_t)} + qaQ_T^{1/(q+1)} \parallel u \parallel^{q}_{L^{q+1}(Q_t)}$$
$$\le\; C_4$$

for $t \in [0,T]$.

At this point, it becomes important whether $\delta = 0$ or $\delta > 0$ in estimating the term $\int_{Q_T} \nabla(u^q)\cdot\nabla v\,dx\,dt$ in (3.2). If $\delta = 0$, we can integrate by parts once to obtain from Lemma 2.1 and an analogue of [7, Theorem 9.1, p. 341-342] for Neumann boundary conditions (see [7, p. 351]),

$$\left|\int_{Q_t} \nabla(u^q)\cdot\nabla v\,dx\,dt\right| = \left|\int_{Q_t} u^q\Delta v\,dxdt\right|$$
$$\le \left(\int_{Q_T} u^{q+1}\,dxdt\right)^{\frac{q}{q+1}} \left(\int_{Q_T} |\Delta v|^{q+1}\,dxdt\right)^{\frac{1}{q+1}}$$
$$\le C_5 \parallel u \parallel^{q}_{L^{q+1}(Q_T)} \left(\parallel bv(1-du-v) \parallel_{L^{q+1}(Q_T)} + \parallel v_0 \parallel_{W_q^{2-2/q}(\Omega)}\right)$$
$$\le C_6(\parallel u \parallel^{q+1}_{L^{q+1}(Q_T)} + 1)\,.$$

We have used the fact that $\alpha^q\beta \le q\alpha^{q+1}/(q+1) + \beta^{q+1}/(q+1)$ for any $\alpha, \beta > 0$ in the last inequality. Therefore, it follows from (3.2) that

$$\int_\Omega u^q(x,t)\,dx + 8\gamma d_1\frac{q(q-1)}{(q+1)^2}\int_{Q_t} |\nabla(u^{(q+1)/2})|^2\,dxdt$$
$$+\, 4d_1\frac{q-1}{q}\int_{Q_t} |\nabla(u^{q/2})|^2\,dxdt$$
$$\le C_7\,(\parallel u \parallel^{q+1}_{L^{q+1}(Q_T)} + 1) + \int_\Omega u^q(x,0)\,dx\,. \tag{3.3}$$

Since (3.3) holds for all $t \in (0,T]$ with the right hand side independent of t, we have

$$\sup_{0\le t\le T}\int_\Omega u^q(x,t)\,dx + \int_{Q_T} |\nabla(u^{(q+1)/2})|^2\,dxdt \le C_8\,(\parallel u \parallel^{q+1}_{L^{q+1}(Q_T)} + 1) \tag{3.4}$$

where C_8 depends on $\|u_0\|_{L^q(\Omega)}$.

Set $w = u^{(q+1)/2}$ so that $u^q = w^{2q/(q+1)}$ and $u^{q+1} = w^2$. Then (3.4) becomes

$$\sup_{0\le t\le T}\int_\Omega w^{2q/(q+1)}(x,t)dx + \int_{Q_T} |\nabla w|^2\,dxdt \le C_8\,(1 + \parallel w \parallel^{2}_{L^2(Q_T)})\,. \tag{3.5}$$

Recall from Lemma 2.3 that $\bar{q} = 2 + 4q/n(q+1)$ so that $4/(q+1) < 2 \le \bar{q}$. Hölder's inequality then gives

$$\|w\|_{L^2(Q_T)} \le \|w\|^{1-\lambda}_{L^{\bar{q}}(Q_T)}\,\|w\|^{\lambda}_{L^{4/(q+1)}(Q_T)} \tag{3.6}$$

with

$$\lambda = \frac{\frac{1}{2} - \frac{1}{\tilde{q}}}{\frac{q+1}{4} - \frac{1}{\tilde{q}}} \in (0,1) \tag{3.7}$$

(see [6, (7.9)]). From the definition of w and Lemma 2.1, we have $\|w\|_{L^{4/(q+1)}(Q_T)} = \|u\|_{L^2(Q_T)}^{(q+1)/2} \le C_1(T)^{(q+1)/2}$. Hence (3.5) together with (3.6) yields

$$E \equiv \sup_{0\le t\le T} \int_\Omega w^{2q/(q+1)}(x,t)dx + \int_{Q_T} |\nabla w|^2\, dxdt \le C_9 \left(1 + \| w \|_{L^{\tilde{q}}(Q_T)}^{2(1-\lambda)}\right). \tag{3.8}$$

Setting $\tilde{\beta} = 2/(q+1) \in (0,1)$, we have

$$\|w(t)\|_{L^{\tilde{\beta}}(\Omega)} = \|u(t)\|_{L^1(\Omega)}^{1/\tilde{\beta}} \le C_1(T)^{1/\tilde{\beta}}$$

for all $t \in [0,T]$ because of Lemma 2.1. Applying (2.7) to (3.8), we have

$$\begin{aligned} E &\le C_9 \left\{ 1 + \left(M_2 [1 + (\sup_{0\le t\le T} \|w(t)\|_{L^{2q/(q+1)}(\Omega)})^{\frac{4q}{n(q+1)\tilde{q}}} \, \|\nabla w\|_{L^2(Q_T)}^{2/\tilde{q}}] \right)^{2(1-\lambda)} \right\} \\ &\le C_{10} \left\{ 1 + \left(\sup_{0\le t\le T} \|w(t)\|_{L^{2q/(q+1)}(\Omega)}^{2q/(q+1)} \right)^{\frac{4(1-\lambda)}{n\tilde{q}}} \left(\|\nabla w\|_{L^2(Q_T)}^2 \right)^{\frac{2(1-\lambda)}{\tilde{q}}} \right\} \\ &\le C_{10} \left\{ 1 + E^{4(1-\lambda)/n\tilde{q}} \, E^{2(1-\lambda)/\tilde{q}} \right\}. \end{aligned} \tag{3.9}$$

In other words,

$$E \le C_{10} \left(1 + E^\theta\right) \tag{3.10}$$

where

$$\theta = \frac{2(1-\lambda)}{\tilde{q}} \left(\frac{2}{n} + 1 \right).$$

We want to show that $0 < \theta < 1$. Obviously, $\theta > 0$. From (3.7) $\theta < 1$ is equivalent to

$$\frac{4}{n} - \frac{8}{n(q+1)} + 2 < \tilde{q}$$

which is also satisfied because $\tilde{q} = 2 + 4q/n(q+1)$ and $q > 1$. It then follows from (3.10) that there exists a positive constant C_{11} such that

$$E \le C_{11}. \tag{3.11}$$

Since $\tilde{q} > 2$, from (2.7) and the definition of w, we have for any $q > 1$,

$$\|u\|_{L^{q+1}(Q_T)} = \|w\|_{L^2(Q_T)}^{2/(q+1)} \le C_{12} \|w\|_{L^{\tilde{q}}(Q_T)}^{2/(q+1)} \le C_{13}. \tag{3.12}$$

It remains to show the boundedness of $\|u\|_{V_2(Q_T)}$. Letting $q = 2$ in (3.3) and use (3.12), we have

$$\sup_{0\le t\le T} \|u(t)\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(Q_T)}^2 \le C_{14}$$

for some $C_{14} > 0$. This completes the proof of Proposition 3.1 when $\delta = 0$.

We next consider the case $\delta > 0$. From Lemma 2.2 and the Sobolev inequality (see [8, Theorem 6.9]), we have $\nabla w_2 = (1 + 2\delta v)\nabla v \in L^{2(n+2)/n}(Q_T)$. Therefore,

$$
\begin{aligned}
&\left|\int_{Q_T} \nabla(u^q)\cdot\nabla v\,dxdt\right| \\
&= q\left|\int_{Q_T} u^{q-1}\nabla u\cdot\nabla v\,dxdt\right| \\
&= \frac{2q}{q+1}\left|\int_{Q_T} u^{\frac{q-1}{2}}\,\nabla\left(u^{\frac{q+1}{2}}\right)\cdot\nabla v\,dxdt\right| \\
&\le \frac{2q}{q+1}\left\|u\right\|^{\frac{q-1}{2}}_{L^{\frac{(q-1)(n+2)}{2}}(Q_T)}\left\|\nabla(u^{\frac{q+1}{2}})\right\|_{L^2(Q_T)}\left\|\nabla v\right\|_{L^{\frac{2(n+2)}{n}}(Q_T)} \\
&\le C_{15}\left\|u\right\|^{\frac{q-1}{2}}_{L^{\frac{(q-1)(n+2)}{2}}(Q_T)}\left\|\nabla(u^{\frac{q+1}{2}})\right\|_{L^2(Q_T)} \\
&\le \epsilon\, C_{15}\left\|\nabla(u^{\frac{q+1}{2}})\right\|^2_{L^2(Q_T)} + \frac{C_{15}}{4\epsilon}\left\|u\right\|^{q-1}_{L^{\frac{(q-1)(n+2)}{2}}(Q_T)} \\
&= \epsilon\, C_{15}\left\|\nabla w\right\|^2_{L^2(Q_T)} + \frac{C_{15}}{4\epsilon}\left(\int_{Q_T} w^{\frac{(q-1)(n+2)}{q+1}}\,dxdt\right)^{\frac{2}{n+2}} \\
&= \epsilon\, C_{15}\left\|\nabla w\right\|^2_{L^2(Q_T)} + C_{16}\left\|w\right\|^{\frac{2(q-1)}{q+1}}_{L^{\frac{(q-1)(n+2)}{q+1}}(Q_T)}
\end{aligned}
\tag{3.13}
$$

where ϵ is arbitrary and $w = u^{(q+1)/2}$ as defined earlier. Choose ϵ such that

$$\epsilon\alpha C_{15} < \frac{4\gamma q}{(q+1)^2}\,,$$

then substitution of (3.13) into (3.2) leads to

$$
\begin{aligned}
\sup_{0\le t\le T}\int_\Omega w^{\frac{2q}{q+1}}(x,t)\,dx \quad &+ \quad \int_{Q_T}|\nabla w|^2\,dxdt \\
&\le \quad C_{17}\left(1+\|w\|^{\frac{2(q-1)}{q+1}}_{L^{\frac{(q-1)(n+2)}{q+1}}(Q_T)}\right)
\end{aligned}
\tag{3.14}
$$

which is analogous to (3.5). It is easily checked that for $1 < q < n(n+4)/(n^2-4)$ we have $(q-1)(n+2)/(q+1) < \tilde{q}$ so that E, defined by (3.8), satisfies the inequality

$$E \le C_{17}\left(1+\|w\|^{2(q-1)/(q+1)}_{L^{\tilde{q}}(Q_T)}\right).$$

It follows from Lemma 2.4 and the definition of E that

$$\|w\|_{L^{\tilde{q}}(Q_T)} \le M_2\left(1+E^{2/n\tilde{q}}\,E^{1/\tilde{q}}\right).$$

Combining these two inequalities, we have

$$E \le C_{17}\left[1+\left(M_2\left(1+E^{2/n\tilde{q}}\,E^{1/\tilde{q}}\right)\right)^{2(q-1)/(q+1)}\right].$$

Therefore,

$$E \le C_{18}\,(1+E^{\mu}E^{\nu}) \tag{3.15}$$

with

$$\mu = \frac{4(q-1)}{n\tilde{q}(q+1)} \qquad \text{and} \qquad \nu = \frac{2(q-1)}{\tilde{q}(q+1)}.$$

Since

$$\begin{aligned}\mu+\nu &= \frac{2(q-1)}{\tilde{q}(q+1)}\left[\frac{2}{n}+1\right] \\ &< \frac{1}{\tilde{q}}\left[\frac{4q}{n(q+1)}+2\right] \\ &= 1,\end{aligned}$$

it is easy to see from (3.15) that E is bounded. Therefore, $w \in L^{\tilde{q}}(Q_T)$ which in turn implies that $u \in L^r(Q_T)$ with $r = \tilde{q}(q+1)/2 = q+1+2q/n$. Since $q < n(n+4)/(n^2-4)$, we get $u \in L^r(Q_T)$ with $r < 2(n+1)/(n-2)$.

Finally, observe that one can take $q = 2 < n(n+4)/(n^2-4)$ in (3.2) for $n = 2,3,4,5$ so that $\|u\|_{V_2(Q_T)} \le C_T$. The proof of the proposition is complete. □

4. **Proof of Theorem 1.1.** The first step of the proof is to show that $u \in L^\infty(Q_T)$.

Lemma 4.1. *Suppose* (i) $\delta = 0$ *or* (ii) $\delta > 0$ *and* $n = 2,3,4,5$. *Then there exists* $M_1 > 0$ *such that* $\| u \|_{L^\infty(Q_T)} \le M_1$.

PROOF. The proof of the case $\delta = 0$ is almost identical to [3, Lemma 3.1] so we shall be brief here. The idea is to write equation (1.1a) as a linear parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(a_i u) - \tilde{a}u \tag{4.1}$$

where

$$a_{ij}(x,t) = d_1(1+\alpha v+2\gamma u)\delta_{ij}, \qquad a_i(x,t) = d_1\,\alpha\,\frac{\partial v}{\partial x_i} \quad \text{and} \quad \tilde{a}(x,t) = -a(1-u-cv)$$

with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \ne j$. Since $u \in L^q(Q_T)$ for some $q > (n+2)/2$, application of the parabolic regularity result (see [7, Theorem 9.1, p. 341-342]) to the equation for v gives

$$v \in W_q^{2,1}(Q_T) \quad \text{for some} \quad q > \frac{n+2}{2},$$

which, together with [7, Lemma 3.3, p. 80], implies that $\nabla v \in L^{(n+2)q/(n+2-q)}(Q_T)$. Since $\|u\|_{V_2(Q_T)}$ is finite by Proposition 3.1, one can apply the maximum principle as stated in [7, Theorem 7.1, p. 181] to conclude that u is bounded in $\overline{Q_T}$.

We next consider the case $\delta > 0$ and $n = 2,3,4,5$. Proposition 3.1 implies $u \in L^q(Q_T)$ for $q < 2(n+1)/(n-2)$. For $n = 2,3,4,5$, we have $(n+2)/2 < 2(n+1)/(n-2)$ so we assume that q lies in the interval $((n+2)/2, 2(n+1)/(n-2))$. The equation for v can be written in the divergence form as

$$\frac{\partial v}{\partial t} = \nabla\cdot((1+2\delta v)\nabla v) + bv(1-du-v) \tag{4.2}$$

where $1+2\delta v$ is bounded in $\overline{Q_T}$ by Lemma 2.1 and $bv(1-du-v) \in L^q(Q_T)$ for $q > (n+2)/2$ from the above consideration. Application of the Hölder continuity result (see [4, Theorem 1.3, p. 43] or [7, Theorem 10.1, p. 204]) [1] to (4.2) yields

$$v \in C^{\beta,\beta/2}(\overline{Q_T}) \quad \text{with some} \quad \beta > 0. \tag{4.3}$$

Now let $w_2 = (1+\delta v)v$ as in Lemma 2.2 and recall that w_2 satisfies (2.3). In (2.3), $c_1 + c_2 u \in L^q(Q_T)$ with $q < 2(n+1)/(n-2)$ by Proposition 3.1. Since $1+2\delta v \in C^{\beta,\beta/2}(\overline{Q_T})$ by (4.3), the parabolic regularity theorem ([7, Theorem 9.1, p. 341-342]) can be applied to (2.3) so that

$$\|w_2\|_{W_q^{2,1}(Q_T)} \le M_4 \quad \text{for} \quad \frac{n+2}{2} < q < \frac{2(n+1)}{n-2}\,. \tag{4.4}$$

Hence it follows from Lemma 3.3 in [7, p. 80] that

$$\nabla w_2 \in L^{(n+2)q/(n+2-q)}(Q_T)\,. \tag{4.5}$$

Since $\nabla v = \nabla w_2/(1+2\delta v)$, we have

$$\nabla v \in L^{(n+2)q/(n+2-q)}(Q_T) \quad \text{for} \quad \frac{n+2}{2} < q < \frac{2(n+1)}{n-2}\,.$$

The rest of the proof is the same as in the case $\delta = 0$. Hence u is bounded in $\overline{Q_T}$ and the proof of the lemma is complete. □

Proof of Theorem 1.1.

We give the proof only in case $\delta > 0$ because the proof for $\delta = 0$ is essentially the same.

Let $[0,T)$ be a maximal existence interval of the solution (u,v) to (1.1). From Lemma 4.1, u is bounded in $\overline{Q_T}$. Since c_1 and c_2 in (2.3) are also bounded, we have $c_1 + c_2 u \in L^q(Q_T)$ for any $q > 1$ and [7, Theorem 9.1, p. 341-342] implies that w_2 satisfies (4.4) actually for any $q > 1$. Hence it follows from [7, Lemma 3.3, p. 80] that

$$w_2 \in C^{1+\beta^*,(1+\beta^*)/2}(\overline{Q_T}) \quad \text{for any} \quad 0 < \beta^* < 1. \tag{4.6}$$

Since $v = (-1+\sqrt{1+4\delta w_2})/2\delta$, (4.6) implies that

$$v \in C^{1+\beta^*,(1+\beta^*)/2}(\overline{Q_T}) \quad \text{for any} \quad 0 < \beta^* < 1. \tag{4.7}$$

We next rewrite the equation for u in divergence form as

$$\frac{\partial u}{\partial t} = \nabla \cdot \{d_1(1+\alpha v + 2\gamma u)\nabla u + d_1 \alpha u \nabla v\} + f(x,t) \tag{4.8}$$

where $f(x,t) = au(1-u-cv) \in L^\infty(Q_T)$. In (4.8), u, v and ∇v are bounded functions because of Lemma 2.1, Lemma 4.1 and (4.7) so that [4, Theorem 1.3 p. 43] implies that

$$u \in C^{\sigma,\sigma/2}(\overline{Q_T}) \quad \text{with some} \quad 0 < \sigma < 1. \tag{4.9}$$

We then return to the equation for v and write it as

$$\frac{\partial v}{\partial t} = d_2(1+2\delta v)\Delta v + g(x,t) \tag{4.10}$$

[1] Theorem 10.1 in [7] is stated for the Dirichlet boundary condition and as such the Hölder norm estimates is for an interior domain with positive distance from the boundary. For zero Neumann boundary condition, since the test function in the proof of the theorem is not required to vanish at the boundary, we can modify the proof in [7] so that the Hölder estimate holds for Q_T.

where $g(x,t) = 2d_2\delta|\nabla v|^2 + bv(1-du-v) \in C^{\sigma,\sigma/2}(\overline{Q_T})$ by (4.7) and (4.9). Then Schauder estimate ([7, Theorem 5.3, p. 320-321]) applied to (4.10) yields

$$v \in C^{2+\sigma^*,(2+\sigma^*)/2}(\overline{Q_T}), \quad \sigma^* = \min\{\sigma, \lambda\}. \tag{4.11}$$

In order to derive the regularity of u, it is convenient to introduce the function

$$w_1 = (1+\alpha v + \gamma u)u \tag{4.12}$$

which satisfies the equation

$$\frac{\partial w_1}{\partial t} = d_1(1+\alpha v + 2\gamma u)\Delta w_1 + h(x,t). \tag{4.13}$$

Since $d_1(1+\alpha v+2\gamma u) \in C^{\sigma,\sigma/2}(\overline{Q_T})$ and

$$h = au(1+\alpha v+2\gamma u)(1-u-cv) + \alpha u\partial v/\partial t \in C^{\sigma^*,\sigma^*/2}(\overline{Q_T})$$

(see (4.9) and (4.11)), applying Schauder estimate to (4.13) gives

$$w_1 \in C^{2+\sigma^*,(2+\sigma^*)/2}(\overline{Q_T}). \tag{4.14}$$

By solving the quadratic equation (4.12) for u, we have

$$u \in C^{2+\sigma^*,(2+\sigma^*)/2}(\overline{Q_T}). \tag{4.15}$$

We can also show that (4.11) and (4.15) are valid with σ^* replaced by λ, which is the exponent concerned with the Hölder continuity of (u_0, v_0). For this result, we have only to repeat the above procedure by making use of (4.11) and (4.15) in place of (4.7) and (4.9)

Finally, the estimates (4.11) and (4.15) imply that the hypotheses of Theorem 2.1 are satisfied so that (u,v) exists globally in time. The proof of our theorem is complete.

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GLOBAL SOLUTIONS FOR LOTKA-VOLTERRA COMPETITION SYSTEMS WITH CROSS-DIFFUSION

YOSHIO YAMADA

1. Problem and Result

This article is concerned with the initial boundary value problem for the following system of nonlinear parabolic equations

$$\text{(P)} \qquad \begin{cases} u_t = \mu\Delta[(1+\alpha v)u] + au(1-u-cv) & \text{in} \quad \Omega \times (0,\infty), \\ v_t = \nu\Delta v + bv(1-du-v) & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0 & \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded domain in R^N ($N \geq 2$) with smooth boundary $\partial\Omega$, Δ is the Laplacian, $\mu, \nu, \alpha, a, b, c, d$ are positive constants, $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega$ and u_0, v_0 are nonnegative functions which are not identically zero. The above system was first introduced by Shigesada-Kawasaki-Teramoto [15] as a population model to describe the habitat segregation phenomena between two competing species; u and v represent the population densities. In (P), one of two species is migrating under cross-diffusion effect in addition to the usual linear diffusion.

Mathematically, (P) and related problems have been discussed by many authors. After Kim [9] showed the existence of global solutions for quasilinear parabolic systems in case $N = 1$, the global solvability for (P) has been a very important subject. In particular, Lou-Ni-Wu [11] have shown that there exists a unique global solution of (P) for every initial function $(u_0, v_0) \in W_p^1(\Omega) \times W_p^1(\Omega)$ with $p > 2$. (See also Yagi[17], where the global existence is proved for similar quasilinear parabolic systems with self-diffusion terms.) For $N = 3$, the global solvability of (P) has been shown by Yang [19]. However, his proof contains a serious error. So the existence of global solutions for (P) is still an open problem for $N \geq 3$.

The purpose of the present paper is to establsih a sufficient condition for the existence of global solutions for (P) without any restrictions on space dimension N and the amplitude of initial functions u_0, v_0. We will study this parabolic system in the framework of $L_r(\Omega) \times L_r(\Omega)$ where r is a real number satisfying

$$r > N. \tag{1.1}$$

We assume

(A.1) $$(u_0, v_0) \in W_r^1(\Omega) \times W_r^1(\Omega).$$

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Our main result is the following.

Theorem 1.1. *Assume that* (u_0, v_0) *satisfies* (A.1). *Morover, assume that any solution of* (P) *satisfies*

$$\| u \|_{L_q(Q_T)} \leq C(q,T) \qquad \textit{with some} \qquad q \geq \frac{N+2}{2},$$

where $Q_T = \Omega \times (0,T)$ *and* T *is an arbitary positive number. Then* (P) *admits a unique global solution* u, v *in* $C([0,\infty); W_r^1(\Omega)) \cap C((0,\infty); W_r^2(\Omega)) \cap C^1((0,\infty); L_r(\Omega))$ *satisfying*

$$u \geq 0 \quad \textit{and} \quad v \geq 0 \quad \textit{in} \quad \Omega \times [0,\infty).$$

It should be noted that analogous global existence results have been obtained by Pozio-Tesei[13], Redlinger[14] and Yamada [18]. These authors have discussed quasilinear parabolic systems of the form

$$\begin{cases} u_t = \mu\Delta[(1+\alpha v)u] + au(1 - f(u) - cv) & \text{in} \quad \Omega \times (0,\infty), \\ v_t = \nu\Delta v + bv(1 - du - v) & \text{in} \quad \Omega \times (0,\infty), \end{cases} \tag{1.2}$$

where $f(u)$ satisfies

$$\liminf_{u\to\infty} \frac{f(u)}{u^\gamma} = \infty,$$

with $\gamma > 1$ if $N \geq 2$ and $\gamma > 0$ if $N = 1$. In [14] and [18], the existence of global solutions for (1.2) with Neumann or Dirichlet boundary conditions has been established for every N, but no information is given for (P).

The content of the present paper is as follows. In §2, we give some preliminary results and a brief survey of procedures to accomplish the proof of Theorem 1.1. Our basic idea is to decouple the system and study each single equation for u or v separately. In §3 we give some remarks on the duality method to get an $L_p(Q_T)$ estimate of u. This estimate plays a very important role in §4 to derive a maximal regularity result for v by considering the second equation of (P) as a linear parabolic equation with inhomogeneous term. In §5 we employ the energy method to derive $L^p(\Omega)$ estimate from $\| u \|_{L_q(Q_T)}$ estimate with some $q \geq (N+2)/2$. In §6 we take an analytic semigroup approach to study the Hölder continuity of v with respect to t variable as well as $W_r^1(\Omega)$ estimate of v. The abstract theory of evolution operators is developed to get $W_r^1(\Omega)$ estimate of u in §7.

Finally we should remark that Theorem 1.1 still holds true if we replace Neumann boundary conditions in (P) by homogeneous Dirichlet boundary conditions. The procedure of the proof is essentially the same as in the Neumann case.

2. Preliminaries

In this section we will collect some preliminary results on (P) and explain our strategy to prove Theorem 1.1.

We begin with a local existence result for (P) due to Amann [4, Theorem] or [3, Theorem 1].

Theorem 2.1. *Assume that $u_0, v_0 \in W_r^1(\Omega)$ for $r > N$. Then* (P) *possesses a unique local solution (u, v) in $C([0,T); W_r^1(\Omega)) \cap C((0,T); W_r^2(\Omega)) \cap C^1((0,T); L_r(\Omega))$, where T is a maximal existence time for the solution.*

Moreover, it is also possible to show the following result.

Lemma 2.1. *Under the same assumption as Theorem 2.1, let (u, v) be the solution of* (P) *in $[0, T)$. Then*

$$u(x,t) \geq 0 \quad \text{and} \quad m \geq v(x,t) \geq 0 \quad \text{for} \quad (x,t) \in \Omega \times [0,T),$$

where $m = \max\{\| v_0 \|_\infty, 1\}$ and $\| \cdot \|_\infty$ denotes the supremum norm in Ω.

Proof. Since u_0 and v_0 are nonnegative functions, the nonnegativity of u and v comes from the maximum principle (for details, see, e.g., Redlinger[14, Proposition 3.1]). Moreover, it follows from $u \geq 0$ that

$$v_t \leq \nu \Delta v + bv(1-v) \qquad \text{in} \quad \Omega \times (0, \infty).$$

Hence it is easy to see from the comparison principle that

$$v(x,t) \leq \max\{\| v_0 \|_\infty, 1\}.$$

Here note $\| v_0 \|_\infty < \infty$ because Sobolev's embedding theorem implies $W_r^1(\Omega) \subset C(\Omega)$ for $r > N$. □

By virtue of Theorem 2.1, it suffices to derive some a priori estimates of $\| u(t) \|_{W_r^1(\Omega)}$ and $\| v(t) \|_{W_r^1(\Omega)}$ to show the existence of global solutions of (P). Indeed, the following result has been established by Amann[3], [4].

Theorem 2.2. *If the solution (u, v) of* (P) *satisfies*

$$\| u(t) \|_{W_r^1(\Omega)} \leq C(T) \quad \text{and} \quad \| v(t) \|_{W_r^1(\Omega)} \leq C(T) \quad \text{for all} \quad t \in (0,T) \tag{2.1}$$

with a positive constant $C(T)$ depending on T, then $T = +\infty$.

On account of Theorem 2.2, our task for the proof of Theorem 1.1 is to derive (2.1). However, its proof is very long; so we will roughly explain the procedure to get (2.1).

For any $T > 0$, let (u, v) be the solution of (P) such that

$$u, v \in C([0,T]; W_r^1(\Omega)) \cap C((0,T]; W_r^2(\Omega)) \cap C^1((0,T]; L_r(\Omega)). \tag{2.2}$$

We decouple the parabolic system in (P) and study the first equation for u and the second one for v separately.

We first assume

(i) $\qquad L_q(Q_T)$ boundedness of u with some $q \geq (N+2)/2$.

The next step is to study the second equation for v. It is regarded as a linear parabolic equation with an inhomogeneous term:

$$\begin{cases} v_t = \nu \Delta v + g & \text{in} \quad \Omega \times (0, \infty), \\ \partial v / \partial n = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \end{cases} \tag{2.3}$$

with $g(x,t) = bv(x,t)(1-du(x,t)-v(x,t))$. Denote by $W_p^s(\Omega)$ for $s \geq 0$ and $p > 1$ the fractional order Sobolev spaces (see [1, Chapt. VII]). We also define

$$W_p^{2,1}(Q_T) := \{w : Q_T \to \mathbf{R}; w, w_t, \nabla w, \nabla^2 w \in L_p(Q_T)\} \quad \text{for} \quad p > 1$$

with norm

$$\| w \|_{W_p^{2,1}(Q_T)} = \| w \|_{L_p(Q_T)} + \| w_t \|_{L_p(Q_T)} + \| \nabla w \|_{L_p(Q_T)} + \| \nabla^2 w \|_{L_p(Q_T)} . \tag{2.4}$$

Since g belongs to $L_q(Q_T)$ with $q \geq (N+2)/2$ by Lemma 2.1 and property (i), the maximal regularity result holds true; one can show

(ii) $\qquad W_q^{2,1}(Q_T)$ boundedness of v

if $v_0 \in W_q^{2-2/q}(\Omega)$.

In the third step we employ the standard energy method. We multiply the equation for u by u^{p-1} $(p > 1)$ and integrate the resulting expression. Using Lemma 2.1 and property (ii) one can get

(iii) $\qquad$ boundedness of $\sup\{\| u(t) \|_{L_p(\Omega)}; 0 \leq t \leq T\}$

for every $p > 1$.

In the fourth step, we study (2.3) again by analytic semi-group approach. Let B be the $L_r(\Omega)$ realization of $-\nu\Delta + I$ with homogeneous Neumann boundary condition. Then (2.3) can be rewritten in the integral form in $L_r(\Omega)$:

$$v(t) = e^{-tB}v_0 + \int_0^t e^{-(t-s)B}(g(s) + v(s))ds. \tag{2.5}$$

Applying $B^\rho, \rho \in (0,1)$, to the both sides of (2.5) we will prove

(iv). $\qquad$ boundedness of $\sup\{\| \nabla v(t) \|_{L_r(\Omega)}; 0 \leq t \leq T\}$.

Here Lemma 2.1 and property (iii) will be used. Moreover, it is also possible to show the Hölder continuity of $t \to B^\rho v(t)$ with respect to $L_r(\Omega)$ norm if v_0 satisfies an additional regularity condition. By Sobolev's embedding theorem, this fact implies

(v) $\quad$ Hölder continuity of $v(t)$ and $\nabla v(t)$ with respect to $t \in [0,T]$ in $C(\Omega)$ norm.

Finally, we focus on the first equation for u and regard v as a given function with properties (iv) and (v). We set

$$\begin{aligned} A(t)u &= -\mu(1+\alpha v(t))\Delta u - 2\mu\alpha\nabla v(t)\cdot\nabla u + \omega u \\ D(A(t)) &= \{u \in W_r^2(\Omega); \partial u/\partial n = 0 \text{ on } \partial\Omega)\}, \end{aligned}$$

where ω is a suitable positive number. By virtue of (v), one can see that $\{A(t)\}_{0\leq t\leq T}$ generates a system of evolution operators $\{U(t,s)\}_{0\leq s\leq t\leq T}$. The first equation for u is written as

$$u_t = -A(t)u + h(t)$$

with

$$h(t) = u(t)[\mu\alpha\Delta v(t) - \omega + a(1-u(t)-cv(t)] \in L_r(Q_T)$$

provided v_0 satisfies a regularity condition. Hence the abstract integral form associated with the above equation is given by

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)h(s)ds. \tag{2.6}$$

Application of ∇ to (2.6) yields

(vi) boundedness of $\sup\{\| \nabla u(t) \|_{L_r(\Omega)}; 0 \le t \le T\}$.

Summarizing (i)-(vi) we will get

$$\| u(t) \|_{W^1_r(\Omega)} \le C(T) \quad \text{and} \quad \| v(t) \|_{W^1_r(\Omega)} \le C(T) \quad \text{for} \quad 0 \le t \le T$$

with a positive constant $C(T)$ when (u_0, v_0) satisfies (A.1) and some additional regularity conditions. These additional assumptions will be removed by smoothing effect of parabolic equations.

3. L^p- ESTIMATE FOR u

In what follows, let (u,v) be the solution of (P) with properties (2.2). So Lemma 2.1, in particular, implies that v is bounded and continuous in $(x,t) \in \Omega \times [0,T]$. The important step is to derive

$$\| u \|_{L_p(Q_T)} \le C(p,T) \qquad \text{for } p > 1 \text{ and } T \in (0,\infty), \tag{3.1}$$

with some $C(p,T) > 0$. However, its derivation is very difficult. When the first equation for u includes a self-diffusion term, Choi, Lui and the author [6] have succeeded in getting (3.1) under some conditions.

Another possibility to get (3.1) is the use of the duality method. We will briefly explain the idea (see [19]). Let v be a continuous function in $\overline{\Omega} \times [0,T]$. For any nonnegative function $\theta \in L_q(Q_T)$ with $q \in (1,\infty)$ satisfying $1/p + 1/q = 1$, consider the following backward parabolic equation

$$\begin{cases} \psi_t = -\mu(1+\alpha v)\Delta\psi - \theta & \text{in} \quad \Omega \times (0,T), \\ \dfrac{\partial \psi}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ \psi(\cdot, T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{3.2}$$

Let $\psi : \overline{\Omega} \times [0,T] \to R$ be a nonnegative solution of (3.2). Then

$$\begin{aligned} \frac{d}{dt}\int_\Omega u\psi dx = & \int_\Omega u_t \psi dx + \int_\Omega u\psi_t dx \\ & = \int_\Omega \{\mu\Delta[(1+\alpha v)u] + au(1-u-cv)\}\, \psi dx \\ & \quad - \int_\Omega \{\mu u(1+\alpha v)\Delta\psi + u\theta\} dx. \end{aligned}$$

Integrating by parts and making use of Neumann boundary conditions for u, v and ψ we see that

$$\frac{d}{dt}\int_\Omega u\psi dx = \int_\Omega au(1-u-cv)\psi dx - \int_\Omega u\theta dx. \tag{3.3}$$

Integration of (3.3) with respect to t over $(0, T)$ yields

$$\int_{Q_T} u\theta dxdt \leq \int_{\Omega} u_0 \psi(0) dx + \frac{a}{4} \int_{\Omega} \psi dxdt, \tag{3.4}$$

where we have used the nonnegativity of u, v and ψ. On account of Hölder's inequality, the right-hand side of (3.4) is bounded from above by

$$\| u_0 \|_{L_p(\Omega)} \| \psi(0) \|_{L_q(\Omega)} + \frac{1}{4} a \| \psi \|_{L_q(Q_T)} |Q_T|^{1/p}. \tag{3.5}$$

If $\| \psi(0) \|_{L_q(\Omega)}$ and $\| \psi \|_{L_q(Q_T)}$ are bounded from above by $C \| \theta \|_{L_q(Q_T)}$, i.e.,

$$\| \psi \|_{W_q^{2,1}(Q_T)} \leq C_1 \| \theta \|_{L_q(Q_T)}, \qquad \text{and } \| \psi(0) \|_{L_q(\Omega)} \leq C_1 \| \theta \|_{L_q(Q_T)}, \tag{3.6}$$

then we can get an apriori estimate for $\| u \|_{L_p(Q_T)}$. In Yang's paper [19, Lemma 3.1], he asserted that (3.6) is a consequence of a parabolic estimate by Ladyzenskaja, Solonnikov and Ural'ceva [10, Theorem 9.1, Chapter IV]. However, their result cannnot be applied because the modulus of the continuity of $\mu(1 + \alpha v(x,t))$, which is the coefficient of the highest-order term in (3.2), is not established yet. In our situation, Yang's duality argument is not correct.

4. Maximal regularity for v

In this section we assume (3.1) with $p > 1$. If we write the boundary value problem for v in the form of (2.3) with

$$g(x,t) = bv(x,t)(1 - du(x,t) - v(x,t)),$$

Lemma 2.1 implies

$$\| g \|_{L_p(Q_T)} \leq bm(1+m)|Q_T|^{1/p} + bdm \| u \|_{L_p(Q_T)}$$

for $p > 1$. Hence (3.1) gives

$$\| g \|_{L_p(Q_T)} \leq K_1(p, T) \quad \text{for } p > 1 \tag{4.1}$$

with a positive number $K_1(p,T)$ depending on p, m, T and $\| u \|_{L_p(Q_T)}$.

We now put an additional assumption on v_0;

$$v_0 \in \{v \in W_p^{2-2/p}(\Omega); \partial v/\partial n = 0 \text{ on } \partial\Omega\} \qquad \text{with some } \quad p > 1. \tag{4.2}$$

Recall the maximal regularity result for (2.3) with initial condition $v(0) = v_0$ (see, e.g., [5, §4.10 in Chap. 3]or [10, Theorem 9.1 in Chapt. IV]):

$$\| v \|_{W_p^{2,1}(Q_T)} \leq C_2(\| v_0 \|_{W_p^{2-2/p}(\Omega)} + \| g \|_{L_p(Q_T)}), \tag{4.3}$$

with some C_2 depending on p and T. Here $\| \cdot \|_{W_p^{2,1}(Q_T)}$ is defined by (2.4).

From (4.1) and (4.3) we can show the following lemma.

Proposition 4.1. *In addition to* (A.1), *assume* (3.1) *and* (4.2) *for* $p > 1$. *Then*

$$\| v \|_{W_p^{2,1}(Q_T)} \leq K_2(p, T)$$

with a positive number $K_2(p,T)$ *depending on* $p, m, T, \| u \|_{L_p(Q_T)}$ *and* $\| v_0 \|_{W_p^{2-2/p}(\Omega)}$.

5. Estimates for u by energy method

In this section we will show that $L_q(Q_T)$ estimtate of u with some $q \geq (N+2)/2$ yields $L_p(Q_T)$ estimtate of u for every $p > 1$. More precisely we have

Proposition 5.1. *Assume* (A.1) *and*

$$v_0 \in W_q^{2-2/q}(\Omega) \qquad \textit{with some} \quad q \geq \frac{N+2}{2} \quad \textit{and} \quad \partial v_0/\partial n = 0 \ \textit{on}\ \partial\Omega. \tag{5.1}$$

Additionally, if

$$\| u \|_{L_q(Q_T)} \leq C(q,T) \qquad \textit{for}\ T \in (0,\infty), \tag{5.2}$$

then it holds that, for every $p > 1$,

$$\sup_{0 \leq t \leq T} \| u(t) \|^p_{L_p(\Omega)} \leq K_3(p,T) \tag{5.3}$$

with a positive constant $K_3(p,T)$.

Proof. We first note that Proposition 4.1 together with (5.1) and (5.2) implies

$$\| \Delta v \|_{L_q(Q_T)} \leq K_2(q,T) \tag{5.4}$$

As a next step we employ Moser's technique. Multiplying the first equation of (P) by u^{p-1} and integrating by parts we get

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx &= \int_\Omega u^{p-1} u_t dx \\
&= \int_\Omega u^{p-1}[\mu\nabla\{(1+\alpha v)\nabla u + \alpha u \nabla v\} + au(1-u-cv)]dx \\
&= -\mu(p-1)\int_\Omega (1+\alpha v)u^{p-2}|\nabla u|^2 dx - \mu\alpha(p-1)\int_\Omega u^{p-1}\nabla v \cdot \nabla u dx \\
&\qquad + a\int_\Omega u^p(1-u-cv)dx.
\end{aligned} \tag{5.5}$$

In (5.5), u and v are nonnegative; so that

$$\int_\Omega (1+\alpha v)u^{p-2}|\nabla u|^2 dx \geq \frac{4}{p^2}\int_\Omega |\nabla(u^{p/2})|^2 dx, \tag{5.6}$$

$$a\int_\Omega u^p(1-u-cv)dx \leq a\int_\Omega u^p dx. \tag{5.7}$$

Moreover, Grees's formula gives

$$\int_\Omega u^{p-1}\nabla v \cdot \nabla u dx = \frac{1}{p}\int_\Omega \nabla(u^p)\cdot\nabla v dx = -\frac{1}{p}\int_\Omega u^p \Delta v dx.$$

Hence it follows from Hölder's inequality that

$$\left|\int_\Omega u^{p-1}\nabla v \cdot \nabla u dx\right| \leq \frac{1}{p} \| u^p \|_{L_{q}^*(\Omega)} \| \Delta v \|_{L_q(\Omega)}, \tag{5.8}$$

where q^* is a number satisfying $\frac{1}{q} + \frac{1}{q^*} = 1$.

We now set $w = u^{p/2}$. Making use of (5.6), (5.7) and (5.8), we see from (5.5) that

$$\begin{aligned}\frac{d}{dt} \| w(t) \|^2 \; &\leq -\frac{4\mu(p-1)}{p} \| \nabla w(t) \|^2 + ap \| w(t) \|^2 \\ &\qquad + \mu\alpha(p-1) \| w(t) \|^2_{L_{2q^*}(\Omega)} \| \Delta v(t) \|_{L_q(\Omega)},\end{aligned} \tag{5.9}$$

where $\| \cdot \|$ denotes $L_2(\Omega)$ norm. By Gagliardo-Nirenberg inequality ([7])

$$\begin{aligned}\| w \|^2_{L_{2q^*}(\Omega)} &\leq C(\| \nabla w \| + \| w \|)^{2\kappa} \| w \|^{2(1-\kappa)} \\ &\leq 2^{2\kappa} C\{\| \nabla w \|^{2\kappa} \| w \|^{2(1-\kappa)} + \| w \|^2\},\end{aligned} \tag{5.10}$$

with $\kappa = \dfrac{N}{2q} \in (0,1)$. We apply Young's inequality to $\| \nabla w \|^{2\kappa} \| w \|^{2(1-\kappa)} \| \Delta v \|_{L_q(\Omega)}$; then for any $\epsilon > 0$

$$\| \nabla w \|^{2\kappa} \| w \|^{2(1-\kappa)} \| \Delta v \|_{L_q(\Omega)} \leq \epsilon \| \nabla w \|^2 + C_\epsilon \| w \|^2 \| \Delta v \|^{1/(1-\kappa)}_{L_q(\Omega)} \tag{5.11}$$

with a positive constant $C_\epsilon > 0$ depending on ϵ. Taking $\epsilon = \dfrac{2}{p\alpha}$ in (5.11) one can deduce from (5.9), (5.10) and (5.11)

$$\frac{d}{dt} \| w(t) \|^2 \leq -\frac{2\mu(p-1)}{p} \| \nabla w(t) \|^2 + C^* h(t) \| w(t) \|^2 \tag{5.12}$$

with

$$h(t) = 1 + \| \Delta v \|_{L_q(\Omega)} + \| \Delta v \|^{1/(1-\kappa)}_{L_q(\Omega)} .$$

Integration of (5.12) with respect to t yields

$$\| w(t) \|^2 + \frac{2\mu(p-1)}{p} \int_0^t \| \nabla w(s) \|^2 \, ds \leq \| w(0) \|^2 + C^* \int_0^t h(s) \| w(s) \|^2 \, ds. \tag{5.13}$$

Since $2q \geq N + 2$, we see

$$\frac{1}{1-\kappa} = \frac{2q}{2q-N} \leq q;$$

so that (5.4) implies $h \in L_q(0,T)$. This fact enables us to apply Gronwall's inequality to (5.13):

$$\| w(t) \|^2 + \frac{2\mu(p-1)}{p} \int_0^t \| \nabla w(s) \|^2 \, ds \leq \| w(0) \|^2 \exp\left(C^* \int_0^T h(s) ds \right) \tag{5.14}$$

for all $0 \leq t \leq T$. Recalling $w = u^{p/2}$ we can obtain (5.3) from (5.14).

□

6. Various estimates for v by semigroup approach

In this section we assume (5.1) and (5.2). We will reconsider (2.3) by analytic semigroup approach. Let r satisfy (1.1) and define a closed linear operator B in $L_r(\Omega)$ by

$$Bv = -\mu \Delta v + v \qquad \text{for} \quad v \in D(B), \tag{6.1}$$

where $D(B) = \{v \in W_r^2(\Omega); \partial v/\partial n = 0 \text{ on } \partial\Omega\}$ is a dense domain of B. It is well known that $-B$ generates an analytic semigroup $\{e^{-tB}\}_{t\geq 0}$ in $L_r(\Omega)$, which satisfies

$$\| B^\rho e^{-tB} \|_r \leq M(r,\rho)t^{-\rho} \qquad \text{for} \quad \rho > 0 \quad \text{and} \quad t > 0, \tag{6.2}$$

where $\| \cdot \|_r$ denotes the operator norm in $L^r(\Omega)$ and B^ρ is the fractional power of B (see [8, Theorem 1.4.3] or [12, Theorem 6.1.3]). Note that $D(B^\rho)$ becomes a Banach space with graph norm $\| w \|_{D(B^\rho)} := \| B^\rho w \|_{L_r(\Omega)}$.

We write the initial boundary value problem for (2.3) as

$$\frac{dv}{dt} + Bv = g_1, \qquad v(0) = v_0 \tag{6.3}$$

with $g_1(x,t) = g(x,t) + v(x,t)$. Note that, on account of Lemma 2.1 and Proposition 5.1,

$$\sup_{0\leq t\leq T} \| g_1(t) \|_{L_r(\Omega)} \leq K_4(r,T) \tag{6.4}$$

with $K_4(r,T) = m\{b(1+m)+1\}|\Omega|^{1/r} + bdmK_3(r,T)$. We will study (6.3) in the integral form

$$v(t) = e^{-tB}v_0 + \int_0^t e^{-(t-s)B} g_1(s)ds, \quad 0 \leq t \leq T, \tag{6.5}$$

to show the following lemma.

Lemma 6.1. *In addition to* (A.1),(5.1) *and* (5.2), *assume*

$$v_0 \in D(B^\sigma) \quad \textit{with some} \quad 0 < \sigma < 1.$$

(i) *For any* $\rho \geq 0$ *satisfying* $\rho \leq \sigma$

$$\| B^\rho v(t) \|_{L_r(\Omega)} \leq K_5, \qquad 0 \leq t \leq T,$$

where K_5 *is a positive constant depending on* $r, m, T, \rho, K_4(r,T)$ *and* $\| B^\sigma v_0 \|_{L_r(\Omega)}$.
(ii) *For any* $\rho \geq 0$ *satisfying* $\rho < \sigma$

$$\| B^\rho(v(t+h) - v(t)) \|_{L_r(\Omega)} \leq K_6 h^\theta, \quad 0 \leq t \leq t+h \leq T,$$

where θ *is any positive number satisfying*

$$\theta \leq \sigma - \rho \quad \text{and} \quad \theta < 1 - \rho$$

and K_6 *is a positive constant depending on* $r, m, T, \rho, \theta, K_4(r,T)$ *and* $\| B^\sigma v_0 \|_{L_r(\Omega)}$.

Proof. (i) It follows from (6.5) that

$$B^\rho v(t) = B^\rho e^{-tB} v_0 + \int_0^t B^\rho e^{-(t-s)B} g_1(s)ds, \quad 0 \leq t \leq T. \tag{6.6}$$

Since B^{-1} is bounded in $L_r(\Omega)$, $v_0 \in D(B^\sigma)$ with $\rho \leq \sigma$ implies

$$\| B^\rho e^{-tB} v_0 \|_{L_r(\Omega)} = \| B^{\rho-\sigma} e^{-tB} B^\sigma v_0 \|_{L_r(\Omega)} \leq M^*(r) \| B^\sigma v_0 \|_{L_r(\Omega)} \tag{6.7}$$

with some $M^*(r) > 0$. Moreover, (6.2) gives

$$\int_0^t \| B^\rho e^{-(t-s)B} g_1(s) \|_{L_r(\Omega)} ds \leq M(r,\rho) \int_0^t (t-s)^{-\rho} \| g_1(s) \|_{L_r(\Omega)} ds.$$

By (6.4)

$$\int_0^t (t-s)^{-\rho} \parallel g_1(s) \parallel_{L_r(\Omega)} ds \leq \frac{K_4(r,T)t^{1-\rho}}{1-\rho}.$$

Therefore,

$$\int_0^t \parallel B^\rho e^{-(t-s)B} g_1(s) \parallel_{L_r(\Omega)} ds \leq \frac{M(r,\rho)K_4(r,T)T^{1-\rho}}{1-\rho}. \tag{6.8}$$

Thus we have shown (i) from from (6.7) and (6.8).

(ii) In order to prove the Hölder continuity we note from (6.2) that for $t \geq s > 0$ and $\rho > 0$

$$\begin{aligned} \parallel B^\rho(e^{-tB} - e^{-sB}) \parallel_r &= \parallel B^\rho \int_s^t \frac{d}{d\tau} e^{-\tau B} d\tau \parallel_r = \parallel \int_s^t B^{\rho+1} e^{-\tau B} d\tau \parallel_r \\ &\leq \int_s^t M(r,\rho+1)\tau^{-\rho-1} d\tau \leq \frac{M(r,\rho+1)(t-s)}{s^{\rho+1}}. \end{aligned} \tag{6.9}$$

On the other hand, (6,2) also implies

$$\parallel B^\rho(e^{-tB} - e^{-sB}) \parallel_r \leq \parallel B^\rho e^{-tB} \parallel_r + \parallel B^\rho e^{-sB} \parallel_r \leq \frac{2M(r,\rho)}{s^\rho} \tag{6.10}$$

for $t \geq s > 0$ and $\rho > 0$. Combining (6.9) and (6.10) we get

$$\parallel B^\rho(e^{-tB} - e^{-sB}) \parallel_r \leq \frac{M_1(\theta)(t-s)^\theta}{s^{\rho+\theta}} \quad \text{for any} \quad 0 \leq \theta \leq 1, \tag{6.11}$$

where $M_1(\theta) = 2^{1-\theta} M(r,\rho+1)^\theta M(r,\rho)^{1-\theta}$.

We are ready to prove the Hölder continuity of $t \to B^\rho v(t)$. From (6.6)

$$\begin{aligned} B^\rho(v(t+h) - v(t)) &= B^\rho(e^{-(t+h)B} - e^{-tB})v_0 + \int_t^{t+h} B^\rho e^{-(t+h-s)B} g_1(s) ds \\ &\quad + \int_0^t B^\rho \{e^{-(t+h-s)B} - e^{-(t-s)B}\} g_1(s) ds \end{aligned} \tag{6.12}$$

for $0 \leq t \leq t+h \leq T$.

Since v_0 belongs to $D(B^\sigma)$, the first term in the right-hand side of (6.12) is estimated as follows:

$$\begin{aligned} \parallel B^\rho(e^{-(t+h)B} - e^{-tB})v_0 \parallel_{L_r(\Omega)} = & \parallel \int_t^{t+h} B^{\rho+1} e^{-\tau B} v_0 d\tau \parallel_{L_r(\Omega)} \\ & \leq \int_t^{t+h} \parallel B^{\rho+1-\sigma} e^{-\tau B} B^\sigma v_0 \parallel_{L_r(\Omega)} d\tau \\ & \leq M_2 h^{\sigma-\rho}, \end{aligned} \tag{6.13}$$

where $M_2 = M(r, 1+\rho-\sigma) \parallel B^\sigma v_0 \parallel_{L_r(\Omega)} /(\sigma-\rho)$.

Application of (6.2) to the first integral in (6.12) also yields

$$\begin{aligned}\| \int_t^{t+h} B^\rho e^{-(t+h-s)B} g_1(s) ds \|_{L_r(\Omega)} &\leq M(r,\rho) \int_t^{t+h} (t+h-s)^{-\rho} \| g_1(s) \|_{L_r(\Omega)} ds \\ &\leq \frac{M(r,\rho) K_4(r,T) h^{1-\rho}}{1-\rho}.\end{aligned} \tag{6.14}$$

Finally, we make use of (6.11) to estimate the last integral in (6.12). Take $\theta > 0$ such that $\rho + \theta < 1$. Then we get

$$\begin{aligned}&\| \int_0^t B^\rho \{ e^{-(t+h-s)B} - e^{-(t-s)B} \} g_1(s) ds \|_{L_r(\Omega)} \\ &\leq \int_0^t M_1(\theta) h^\theta (t-s)^{-\rho-\theta} \| g_1(s) \|_{L_r(\Omega)} ds \\ &\leq \frac{M_1(\theta) K_4(r,T) T^{1-\rho-\theta} h^\theta}{1-\rho-\theta}.\end{aligned} \tag{6.15}$$

Thus we can obtain the conclusion of (ii) from (6.13), (6.14) and (6.15). □

Since we have shown Lemma 6.1, we invoke the embedding result for $D(B^\rho)$ (see [8, Theorem 1.6.1] or [12, Theorem 4.3 in Chap.8]):

$$D(B^\rho) \subset C^1(\overline{\Omega}) \qquad \text{if} \quad \rho > \frac{1}{2} + \frac{N}{2r}. \tag{6.16}$$

Proposition 6.1. *In addition to* (A.1), (5.1) *and* (5.2), *assume*

$$v_0 \in D(B^\sigma) \qquad \textit{with} \quad 1 > \sigma > \frac{1}{2} + \frac{N}{2r}. \tag{6.17}$$

Then there exists a positive number K_7 determined by $r, m, T, K_4(r,T)$ and $\| B^\sigma v_0 \|_{L_r(\Omega)}$, such that

$$\| \nabla v(t) \|_\infty \leq K_7$$

for $0 \leq t \leq T$.

Proof. In view of (6.17) it is possible to choose ρ satisfying

$$\frac{1}{2} + \frac{N}{2r} < \rho \leq \sigma.$$

For such ρ, (i) of Lemma 6.1 together with (6.16) gives the assertion. □

Similarly to Proposition 6.1, one can also show the Hölder continuity of $t \to v(t)$ and $t \to \nabla v(t)$ with respect to the supremum norm in Ω from (ii) of Lemma 6.1.

Proposition 6.2. *Under the same assumptions as Proposition* 6.1 *there exist positive constants θ and K_8 such that*

$$\| v(t) - v(s) \|_\infty \leq K_8 |t-s|^\theta \quad \textit{and} \quad \| \nabla v(t) - \nabla v(s) \|_\infty \leq K_8 |t-s|^\theta$$

for $0 \leq s \leq t \leq T$.

7. Evolution equation for u

In this section we assume (5.1), (5.2), (6.17) and concentrate on the study of the first equation of (P). We regard v as a known function possessing several properties given by Propositions 5.1, 6.1 and 6.2. The initial boundary value problem for u is written as

$$\begin{cases} u_t = \mu(1+\alpha v)\Delta u + 2\alpha\mu\nabla v\cdot\nabla u \\ \qquad\qquad +\alpha\mu u\Delta v + au(1-u-cv) & \text{in} \quad \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega\times(0,\infty), \\ u(\cdot,0) = u_0, & \text{in} \quad \Omega. \end{cases} \tag{7.1}$$

We will treat (7.1) as the Cauchy problem for an abstract evolution equation in $L_r(\Omega)$. Define a closed linear operator $A(t)$ in $L_r(\Omega)$ by

$$A(t)u = -\mu(1+\alpha v(t))\Delta u - 2\alpha\mu\nabla v(t)\cdot\nabla u + \omega u$$

with dense domain

$$D(A(t)) = D := \{u \in W_r^2(\Omega); \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\},$$

where ω is a sufficiently large number satisfying

$$\omega \geq \mu\alpha^2 \sup_{0\leq t\leq T} \| \nabla v(t) \|_\infty^2 /(r-1)$$

It is possible to choose such ω by Proposition 6.1. Repeating the method in the author's previous paper [18, p.1404] one can show

$$\| (\lambda + A(t))^{-1} \|_r \leq \frac{M_3}{|\lambda|}, \qquad t\in[0,T], \quad \lambda \in \{\zeta\in\mathbf{C}; |\arg\zeta| \leq \frac{\pi}{2}+\gamma\} \tag{7.2}$$

and

$$\| A(t)^{-1} \|_r \leq M_4, \qquad t \in [0,T], \tag{7.3}$$

where M_3, M_4 are positive constants and γ is a suitable constant satisfying $0<\gamma<\pi/2$.

Proposition 6.2 gives

$$\| (A(t)-A(s))u \|_{L_r(\Omega)} \leq M_5|t-s|^\theta\{\| \Delta u \|_{L_r(\Omega)} + \| \nabla u \|_{L_r(\Omega)}\} \tag{7.4}$$

for all $u \in D$, where M_5 is a positive constant depending on α, μ and K_8. Here we should recall the following regularity result for elliptic operators

$$\| u \|_{W_r^2(\Omega)} \leq C_3\left(\| A(t)u \|_{L_r(\Omega)} + \| u \|_{L_r(\Omega)}\right) \tag{7.5}$$

for $u \in D$, where C_3 is a positive constant independent of u and t (see, e.g., [2] or [7, Part 1, Theorem 19.1]). Hence it follows from (7.3), (7.4) and (7.5) that

$$\| (A(t)-A(s))A(\tau)^{-1} \|_r \leq M_6|t-s|^\theta \qquad \text{for} \quad 0\leq t,s,\tau\leq T, \tag{7.6}$$

with some $M_6 > 0$.

Since the family $\{A(t)\}_{0\leq t\leq T}$ satisfies (7.2) and (7.6), we can apply the theory of abstract evolution equations. According to Tanabe's result [16] (see also [12, Theorem

6.1 in Chapt. 5]), there exists a unique family of evolution operators $\{U(t,s)\}_{0\le s\le t\le T}$. It is also possible to show

$$\| \nabla U(t,s) \|_r \le M_7 |t-s|^{-1/2} \quad \text{for} \quad 0 \le s \le t \le T \tag{7.7}$$

in the same way as the proof of Lemma 5.2 in [18]. Indeed, it is sufficient to employ Gagliardo-Nirenberg's inequality (see [7, Part 1, Theorem 10.1]

$$\| \nabla u \|_{L_r(\Omega)} \le C_4 \| u \|^{1/2}_{W^2_r(\Omega)} \| u \|^{1/2}_{L_r(\Omega)} \le C_5 \| A(t)u \|^{1/2}_{L_r(\Omega)} \| u \|^{1/2}_{L_r(\Omega)} \quad \text{for} \quad u \in D$$

in place of (A.12) in [18], where the last inequality follows from (7.5).

We are ready to get the following estimate.

Proposition 7.1. *In addition to* (A.1), *assume* (5.1),(5.2) *and* (6.17). *Then there exists a positive constant* K_9 *such that*

$$\| \nabla u(t) \|_{L_r(\Omega)} \le K_9$$

for $0 \le t \le T$.

Proof. . In terms of evolution operators $\{U(t,s)\}_{0\le s\le t\le T}$, (7.1) is written in the integral form

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)\{h_1(s) + h_2(s)\}ds \tag{7.8}$$

with

$$h_1(t) = \alpha\mu u(t)\Delta v(t) \quad \text{and} \quad h_2(t) = au(t)(1 - u(t) - cv(t)).$$

Proposition 5.1 assures $u \in L_p(Q_T)$ for every $p > 1$. Hence repeating the arguments in §4 we see from Proposition 4.1 that $v \in W^{2,1}_p(Q_T)$ for every $p > 1$. By Hölder's inequality

$$\int_0^T \| h_1(t) \|^r_{L_r(\Omega)} dt \le \alpha\mu \| \Delta v \|^r_{L_{2r}(Q_T)} \| u \|^r_{L^{2r}(Q_T)} .$$

Since the right-hand side of this inequality is bounded, it is easy to see $h_1 \in L_r(Q_T)$. Similarly, $h_2 \in L_r(Q_T)$. Therefore,

$$h := h_1 + h_2 \in L_r(Q_T). \tag{7.9}$$

Define $r^* > 1$ by $1/r + 1/r^* = 1$. Since $r > 2$ by (1.1), note $r^* < 2$. Making use of (7.7) and (7.8) we obtain

$$\begin{aligned} \| \nabla u(t) \|_{L_r(\Omega)} &\le \| \nabla U(t,0)u_0 \|_{L_r(\Omega)} + \int_0^t \| \nabla U(t,s)h(s) \|_{L_r(\Omega)} ds \\ &\le M_7 \left\{ t^{-1/2} \| u_0 \|_{L_r(\Omega)} + \int_0^t (t-s)^{-1/2} \| h(s) \|_{L_r(\Omega)} ds \right\} \qquad (7.10) \\ &\le M_7 \left\{ t^{-1/2} \| u_0 \|_{L_r(\Omega)} + \frac{\| h \|_{L_r(Q_T)} t^{1/r^*-1/2}}{(1 - r^*/2)^{1/r^*}} \right\} \end{aligned}$$

for $0 \le t \le T$. On the other hand, $\nabla u(t)$ is continuous in $t \in [0,T]$ with respect to $L_r(\Omega)$ norm; so that

$$\| \nabla u(t) \|_{L_r(\Omega)} \le 2 \| \nabla u_0 \|_{L_r(\Omega)}, \quad 0 \le t \le t_0,$$

with some $t_0 \in (0,T)$. It follows from (7.9) and (7.10) that

$$\| \nabla u(t) \|_{L_r(\Omega)} \leq M_6 \left\{ t_0^{-1/2} \| u_0 \|_{L_r(\Omega)} + \frac{\| h \|_{L_r(Q_T)} T^{1/r^*-1/2}}{(1-r^*/2)^{1/r^*}} \right\}$$

for $t_0 \leq t \leq T$. Thus we complete the proof.

□

Proof of Theorem 1.1.

We first assume that (u_0, v_0) satisfies (5.2) and (6.17) as well as (A.1). In this case Propositions 6.1 and 7.1 enable us to verify (2.1); so that Theorem 1.1 follows from Theorem 2.2.

We next drop conditions (5.2),(6.17) and accomplish the proof. We recall Theorem 2.1 to get

$$u(t), v(t) \in D = \left\{ w \in W_r^2(\Omega); \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \right\}$$

for every $t \in (0,T]$ with some $T > 0$. Take any $t_0 \in (0,T)$ and fix it; then

$$v(t_0) \in D \subset D(B^\sigma) \quad \text{for any} \quad \sigma \leq 1. \tag{7.11}$$

Since r satisfies (1.1), it is easy to see

$$v(t_0) \in D(B^\sigma) \quad \text{for} \quad 1 \geq \sigma > \frac{1}{2} + \frac{N}{2r} \tag{7.12}$$

from (7.11). Moreover, Sobolev-Besov's embedding theorem ([1, Theorem 7.58]) yields

$$W_r^2(\Omega) \subset W_q^\rho(\Omega) \quad \text{if} \quad q > r \quad \text{and} \quad \rho = 2 - \frac{N}{r} + \frac{N}{q}. \tag{7.13}$$

If $q \leq r$, then (7.13) is valid for every $\rho \leq 2$. Therefore, we see from (7.11) and (7.13)

$$v(t_0) \in W_q^{2-2/q}(\Omega) \quad \text{and} \quad \frac{\partial v}{\partial n}(t_0) = 0 \quad \partial\Omega. \tag{7.14}$$

These facts (7.12) and (7.14) mean that the pair $(u(t_0), v(t_0))$ satisfies (5.2) and (6.17) in addition to (A.1). Regarding $(u(t_0), v(t_0))$ as initial data we repeat the preceding arguments for (P) with (u_0, v_0) replaced by $(u(t_0), v(t_0))$. Then we can get a priori estimates for $\| \nabla u(t) \|_{L_r(\Omega)}$ and $\| \nabla v(t) \|_{L_r(\Omega)}$ and complete the proof.

Remark. In (5.5), set $p = 1$; then

$$\frac{d}{dt} \| u(t) \|_{L_1(\Omega)} + a \| u(t) \|^2 \leq a \| u(t) \|_{L_1(\Omega)} .$$

Therefore, Gronwall's inequality yields

$$\| u(t) \|_{L_1(\Omega)} + a \| u(t) \|^2_{L_2(Qt)} \leq \| u_0 \|_{L_1(\Omega)} e^{aT}$$

for $0 \leq t \leq T$. In case $N = 2$, the above inequality assures the assumption of Theorem 1.1. Hence the global solvability in case $N = 2$ easily follows.

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Multiple Coexistence States for a Prey-Predator System with Cross-Diffusion

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1 Introduction

In this paper we study nonnegative steady-state solutions of the following strongly-coupled parabolic system

$$\begin{cases} u_t = \Delta[(d_1 + \rho_{12}v)u] + u(a_1 - b_1 u - c_1 v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta[(d_2 + \rho_{21}u)v] + v(a_2 + b_2 u - c_2 v) & \text{in } \Omega \times (0,\infty), \\ u = v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\,\cdot\,,0) = u_0 \geq 0, \quad v(\,\cdot\,,0) = v_0 \geq 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\boldsymbol{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$; ρ_{12}, ρ_{21} are nonnegative constants; d_i, a_i, b_i, c_i $(i = 1, 2)$ are also constants, and they are all positive except for a_2 which may be nonpositive. The system (1.1) is known as the Lotka-Volterra prey-predator system with *cross-diffusion* effects. In (1.1), u and v, respectively, represent the population densities of prey and predator species which are interacting and migrating in the same habitat Ω. Such a density-dependent population model was first proposed by Shigesada, Kawasaki and Teramoto [20] to investigate the habitat segregation phenomena.

In diffusion terms, d_i represents natural dispersive force of movement of an individual, while ρ_{ij} describes mutual interferences between individuals; ρ_{12} and ρ_{21} are usually referred as cross-diffusion pressures. The above model means that, in addition to dispersive force, diffusion also depends on population pressure from other species. For details, see the monograph of Okubo and Levin [16]. First cross-diffusion pressure ρ_{12} means the tendency that the prey keeps away from the predator. In a certain kind of prey-predator relationships, a great number of prey species form a huge group to protect themselves from the attack of predator. So we assume that the population pressure due to the high density of prey induces the diffusion of the form $\rho_{21}\Delta(uv)$ in the second equation. This kind of prey-predator models has also been discussed in [8, 15, 19]. The boundary condition means that the habitat Ω is surrounded by a hostile environment.

The system with the aggregation term $\nabla(d_2\nabla v - \rho_{21}v\nabla u)$ (instead of $\Delta[(d_2+\rho_{21}u)v]$ in (1.1)) is also an interesting model. We will discuss such a problem elsewhere. See also [16] for the biological background.

The purpose of the present paper is to investigate nonnegative steady-state solutions of (1.1). Thus we will concentrate on the following strongly-coupled elliptic system

$$
\text{(SP)}\quad \begin{cases} \Delta[(1+\alpha v)u] + u(a-u-cv) = 0 & \text{in } \Omega, \\ \Delta[(1+\beta u)v] + v(b+du-v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}
$$

which is obtained from (1.1) by employing the rescaling

$$
\alpha = \frac{d_2\rho_{12}}{c_2 d_1},\ \beta = \frac{d_1\rho_{21}}{b_1 d_2},\ a = \frac{a_1}{d_1},\ b = \frac{a_2}{d_2},\ c = \frac{c_1 d_2}{c_2 d_1},\ d = \frac{b_2 d_1}{b_1 d_2},\ \tilde{u} = \frac{b_1 u}{d_1},\ \tilde{v} = \frac{c_2 v}{d_2}. \tag{1.2}
$$

For simplicity, we have dropped the '$\,\tilde{}\,$' sign in (SP).

We are mainly interested in positive solutions of (SP). It is said that (u,v) is a positive solution of (SP) if $u>0$ and $v>0$ in Ω. Among other things, we will prove that when (α,β,b,c,d) belongs to a certain range, the positive solution set $\{(u,v,a)\}$ of (SP) contains an unbounded S-shaped curve with respect to a, while when (α,β,b,c,d) falls into another range, the positive solution set $\{(u,v,a)\}$ contains a bounded S or $\supset$-shaped curve. These results not only confirm multiple existence of positive solutions for (SP) but also suggest that the dynamical behavior of (1.1) is quite complicate. The stability analysis for the above coexistence steady-states will be treated in a forthcoming paper.

When there are no cross-diffusion effect ($\alpha=\beta=0$), (SP) is reduced to the classical Lotka-Volterra prey-predator system. This system has been discussed extensively by many authors (e.g., [1, 4, 5, 6, 10, 11, 12, 13, 14, 17, 21]). In particular, we know the exact range of parameter (a,b,c,d) for the existence of a positive solution of (SP) (see Li [10, Theorem 1.A] or López-Gómez and Pardo [13, Theorem 3.1]). So it is possible to determine completely the coexistence region in a parameter space (a,b) (see [13, Figure 1]). Furthermore, López-Gómez and Pardo [14] have proved the uniqueness of positive solutions for the special case when the spatial dimension is one ($N=1$, $\alpha=\beta=0$).

To discuss the case $(\alpha,\beta)\neq(0,0)$, we need some notation. Let $\lambda_1(q)$ be the least eigenvalue for the following eigenvalue problem

$$
-\Delta u + q(x)u = \lambda u \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega, \tag{1.3}
$$

where $q(x)$ is a continuous function in $\overline{\Omega}$. We simply write λ_1 instead of $\lambda_1(0)$. It is well known that the problem

$$
\Delta u + u(a-u) = 0 \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega \tag{1.4}
$$

has a unique positive solution θ_a if $a > \lambda_1$ (see, e.g. [1]); moreover, $a \to \theta_a : [\lambda_1, \infty) \to C(\overline{\Omega})$ and $q \to \lambda_1(q) : C(\overline{\Omega}) \to \boldsymbol{R}$ are continuous and strictly increasing functions. Here $C(\overline{\Omega})$ is equipped with the uniform convergence topology in $\overline{\Omega}$. It is possible to show that (SP) has two semitrivial solutions

$$(\theta_a, 0) \text{ for } a > \lambda_1 \quad \text{and} \quad (0, \theta_b) \text{ for } b > \lambda_1$$

in addition to the trivial solution $(0,0)$.

Concerning the problem (SP), Nakashima and Yamada [15] have obtained a sufficient condition of parameter $(\alpha, \beta, a, b, c, d)$ for the existence of a positive solution with use of the index theory.

Theorem 1.1 (Nakashima and Yamada [15]). *For $a > \lambda_1$, there exists a positive solution for* (SP) *if one of the following conditions is satisfied:*

$$\lambda_1\left(\frac{c\theta_b - a}{1 + \alpha\theta_b}\right) < 0 \quad \text{and} \quad \lambda_1\left(\frac{-b - d\theta_a}{1 + \beta\theta_a}\right) < 0, \tag{1.5}$$

$$\lambda_1\left(\frac{c\theta_b - a}{1 + \alpha\theta_b}\right) > 0 \quad \text{and} \quad \lambda_1\left(\frac{-b - d\theta_a}{1 + \beta\theta_a}\right) > 0, \tag{1.6}$$

where it is understood that $\theta_b \equiv 0$ for $b \leq \lambda_1$.

For $a \leq \lambda_1$, (SP) *has no positive solution.*

Before stating our results, we will explain the meaning of Theorem 1.1. Regarding a and b as parameters, we define

$$S_1 := \left\{(a,b) \in \boldsymbol{R}^2 \,:\, \lambda_1\left(\frac{-b - d\theta_a}{1 + \beta\theta_a}\right) = 0 \text{ for } a \geq \lambda_1\right\}, \tag{1.7}$$

$$S_2 := \left\{(a,b) \in \boldsymbol{R}^2 \,:\, \lambda_1\left(\frac{c\theta_b - a}{1 + \alpha\theta_b}\right) = 0 \text{ for } b \geq \lambda_1\right\}. \tag{1.8}$$

Lemma A.1 in Appendix implies that if $\beta\lambda_1 < d$ (resp. $\beta\lambda_1 > d$), then S_1 forms a monotone decreasing (resp. increasing) curve starting from (λ_1, λ_1). Lemma A.3 asserts that S_2 is a monotone increasing curve which starts from (λ_1, λ_1). See Fig. 1. Combining these properties, one can deduce from Theorem 1.1 that if (a,b) lies in a region R surrounded by S_1 and S_2, then (SP) has a positive solution. This R, in case $\alpha = \beta = 0$, corresponds to the exact coexistence region shown by López-Gómez and Pardo [13]. Furthermore, Lemma A.5 implies that S_1 curve is located below (resp. above) S_2 curve near (λ_1, λ_1) if $(\alpha\lambda_1 + c)(\beta\lambda_1 - d) < 1$ (resp. $(\alpha\lambda_1 + c)(\beta\lambda_1 - d) > 1$). From the viewpoint of the bifurcation theory, we can see that positive solutions bifurcate from $(\theta_a, 0)$ when (a,b) crosses S_1 curve. Similarly positive solutions also bifurcate from $(0, \theta_b)$ when (a,b) moves across S_2. We will give proofs of these bifurcation properties in Lemma 2.4. In this sense, Theorem 1.1 suggests that the structure to the positive solution set changes at $d/\beta = \lambda_1$.

Once Theorem 1.1 is obtained, we are led to the next interesting problems; whether or not (SP) has *multiple* positive solutions and whether (SP) admits a positive solution even if (a,b) lies in the outside of R. Regarding a as a bifurcation parameter, we set

$$\mathcal{S} := \{(u,v,a) \,:\, (u,v) \text{ is a positive solution of (SP)}, \ a > \lambda_1\}.$$

Our main results are concerned with the global structure of $\mathcal{S}$. The first result asserts that for some (α,β,b,c,d) with $\min\{\beta b, d\} > \beta\lambda_1$, $\mathcal{S}$ contains an unbounded S-shaped curve (with respect to a) which bifurcates from the semitrivial solution curve $\{(0,\theta_b,a) \,:\, a > 0\}$:

Theorem 1.2. *Assume* $\min\{\beta b, d\} > \beta\lambda_1$. *For any* $c > 0$, *there exist a large number* M *and an open set*

$$O_1 = O_1(c) \subset \{(\alpha,\beta,b,d) \,:\, \beta \geq M,\, 0 < \alpha, d/\beta - \lambda_1, b - \lambda_1 \leq M^{-1}\}$$

such that $\partial O_1 \cap \{(\alpha,\beta,b,d) \,:\, d/\beta = \lambda_1\}$ *is not empty and, if* $(\alpha,\beta,b,d) \in O_1$, *then* $\mathcal{S}$ *contains an unbounded smooth curve*

$$\Gamma_1 = \{(u(s),v(s),a(s)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times (\lambda_1,\infty) \,:\, s \in (0,\infty)\},$$

which possesses the following properties:

(i) $(u(0),v(0)) = (0,\theta_b)$, $a(0) = a^* > \lambda_1$, $a'(0) > 0$, *where* a^* *is a unique number satisfying* $(a^*,b) \in S_2$.

(ii) $a(s) > a(0)$ *for all* $s \in (0,\infty)$ *and* $\lim_{s\to\infty} a(s) = \infty$;

(iii) $a(s)$ *attains a strict local maximum and a strict local minimum at some* $s = \overline{s}$ *and* $s = \underline{s}$ $(0 < \overline{s} < \underline{s})$, *respectively.*

Let $\overline{a} := a(\overline{s})$ and $\underline{a} := a(\underline{s})$. Theorem 1.2 implies that (SP) has at least one positive solution if $a \in (a^*,\underline{a}) \cup (\overline{a},\infty)$; at least two positive solutions if $a = \underline{a}$ or $a = \overline{a}$ and at least three positive solutions if $a \in (\underline{a},\overline{a})$. Furthermore, we will show the nonexistence of positive solutions in $a \in (0,a^*]$. We remark that a^*, $\underline{a}$, $\overline{a}$ depend continuously on (α,β,b,c,d) and, moreover, (a^*,b) lies on S_2. Since Theorem 1.2 implies that (SP) possesses multiple positive solutions for any (a,b) such that $(\alpha,\beta,b,d) \in O_1$ and $a \in [\underline{a},\overline{a}]$, a multiple coexistence region can be constructed in (a,b) space. Furthermore, this region is contained in R, because $a^* < \underline{a} < \overline{a}$ and $(a^*,b) \in S_2$, and S_2 is the left side boundary of R.

For some (α,β,a,b,c,d) with $\beta b > \beta\lambda_1 > d$, $\mathcal{S}$ contains a bounded S or $\supset$-shaped curve, which bifurcates from the semitrivial solution curve $\{(0,\theta_b,a) \,:\, a > 0\}$ and connects the other semitrivial solution curve $\{(\theta_a,0,a)\,;\, a > \lambda_1\}$:

Theorem 1.3. *Assume* $\beta b > \beta\lambda_1 > d$. *For any* $c > 0$, *there exist a large number* M *and an open set*

$$O_2 = O_2(c) \subset \{(\alpha,\beta,b,d) \,:\, \beta \geq M,\, 0 < \alpha, \lambda_1 - d/\beta, b - \lambda_1 \leq M^{-1}\}$$

such that if $(\alpha, \beta, b, d) \in O_2$, *then* $\mathcal{S}$ *contains a bounded smooth curve*

$$\Gamma_2 = \{(u(s), v(s), a(s)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times (\lambda_1, \infty) \,:\, s \in (0, C)\},$$

which possesses the following properties:

(i) $(u(0), v(0)) = (0, \theta_b)$, $a(0) = a^* > \lambda_1$, $a'(0) > 0$, *where* a^* *is a unique number satisfying* $(a^*, b) \in S_2$.

(ii) $(u(C), v(C)) = (\theta_{a(C)}, 0)$, $a(C) = a_* > \lambda_1$, *where* a_* *is a unique number satisfying* $(a_*, b) \in S_1$.

(iii) Γ_2 *has at least one turning point with respect to* a. *Furthermore, there exists an open set* $O_2' \subset O_2$ *such that* $\partial O_2' \cap \{(\alpha, \beta, b, d) : d/\beta = \lambda_1\}$ *is not empty and if* $(\alpha, \beta, b, d) \in O_2'$, *then* Γ_2 *has at least two turning points with respect to* a.

The above result asserts that if $(\alpha, \beta, b, d) \in O_2'$, then Γ_2 forms a bounded S-shaped branch of positive solutions. Furthermore, it can be shown that $\hat{a} := \max_{s \in [0,C]} a(s) > \max\{a^*, a_*\}$ if $(\alpha, \beta, b, d) \in O_2 \setminus O_2'$. This fact means not only that (SP) has multiple positive solutions for any $a \in (\max\{a_*, a^*\}, \hat{a})$ but also even in the right-hand outside of R, there exists a region where (SP) admits multiple positive solutions, because $(\max\{a_*, a^*\}, b)$ lies on the right side boundary of R. In particular, for the one dimensional case ($N = 1$), the above multiple coexistence results in Theorems 1.2 and 1.3 are very different from the uniqueness result in the linear diffusion case $\alpha = \beta = 0$ (see [14]). By virtue of (1.2), Theorems 1.2 and 1.3 assert that the original model (1.1) possibly possesses multiple coexistence steady-states in cases when $\beta = d_1\rho_{21}/(b_1 d_2)$ is large and $\alpha = d_2\rho_{12}/(c_2 d_1), |d/\beta - \lambda_1| = |b_2/\rho_{21} - \lambda_1|$, $b - \lambda_1 = a_2/d_2 - \lambda_1$ are small positives.

A crucial point of proofs for Theorems 1.2 and 1.3 is to construct a positive solution curve of (SP) in the extreme case $\alpha = 0$. The analysis is based on the bifurcation theory and the Lyapunov-Schmidt reduction procedure. If β is large and $b - \lambda_1 > 0, |d/\beta - \lambda_1|$ are small, then this reduction enables us to find a relationship to a suitable limit problem. Making use of the perturbation theory developed by Du and Lou [7], we will depict precise solution curves Γ_i of (SP) near limit solution sets. In [7], they have obtained an S-shaped positive solution curve of a prey-predator system with the Holling-Tanner interaction terms.

The contents of the present paper are as follows: In Section 2, we first reduce (SP) to a related semilinear problem (EP) by employing new unknown functions $U = (1 + \alpha v)u$ and $V = (1 + \beta u)v$. Next we give preliminary results about a priori estimates and bifurcation properties of positive solutions to (EP). In Section 3, we will introduce a perturbed problem (PP) for the Lyapunov-Schmidt reduction scheme. This problem (PP) can be reduced to (EP) with $\alpha = 0$ through some changing of variables and will play an important role in proofs of Theorems 1.2 and 1.3. The solution set of (PP) will be investigated by way of a finite dimensional limit problem in Sections 4 and 5. In Section 6, we will accomplish the proofs of our main results. Some basic properties of S_1 and S_2 defined by (1.7) and (1.8), which are needed in Sections 2-6, are presented in Appendix.

Throughout the paper, the usual norms of the spaces $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\overline{\Omega})$ are defined by

$$\|u\|_p := \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p} \quad \text{and} \quad \|u\|_\infty := \max_{x \in \overline{\Omega}} |u(x)|.$$

In particular, we simply write $\|u\|$ instead of $\|u\|_2$. Furthermore, we will denote by Φ a unique positive solution of

$$-\Delta \Phi = \lambda_1 \Phi \ \text{ in } \ \Omega, \quad \Phi = 0 \ \text{ on } \ \partial\Omega, \quad \|\Phi\| = 1.$$

2 Preliminaries

In this section, we first introduce a semilinear elliptic system equivalent to (SP). Next we give some a priori estimates and local bifurcation properties of positive solutions to the semilinear system.

2.1 Reduction to the Semilinear Problem

Suppose $(\alpha, \beta) \neq (0,0)$ in (SP). Since we are interested in nonnegative solutions, it is convenient to introduce two unknown functions U and V by

$$U = (1 + \alpha v)u \quad \text{and} \quad V = (1 + \beta u)v. \tag{2.1}$$

There is a one-to-one correspondence between $(u, v) \geq 0$ and $(U, V) \geq 0$. It is possible to describe their relations by

$$u = u(U,V) = \frac{1}{2\beta}\left[\{(1 - \beta U + \alpha V)^2 + 4\beta U\}^{1/2} + \beta U - \alpha V - 1 \right], \tag{2.2}$$

$$v = v(U,V) = \frac{1}{2\alpha}\left[\{(1 - \alpha V + \beta U)^2 + 4\alpha V\}^{1/2} + \alpha V - \beta U - 1 \right]. \tag{2.3}$$

As far as we are concerned with nonnegative solutions, (SP) is rewritten in the following equivalent form

$$\text{(EP)} \begin{cases} \Delta U + u(a - u - cv) = 0 & \text{in } \ \Omega, \\ \Delta V + v(b + du - v) = 0 & \text{in } \ \Omega, \\ U = V = 0 & \text{on } \ \partial\Omega, \end{cases}$$

where $u = u(U, V)$ and $v = v(U, V)$ are understood to be functions of (U, V) defined by (2.2) and (2.3). It is easy to show that (EP) has two semitrivial solutions

$$(U, V) = (\theta_a, 0) \ \text{ for } \ a > \lambda_1 \quad \text{and} \quad (U, V) = (0, \theta_b) \ \text{ for } \ b > \lambda_1,$$

in addition to the trivial solution $(0, 0)$.

2.2 A Priori Estimates

We will derive some a priori estimates for positive solutions of (EP).

Lemma 2.1. *If $a \leq \lambda_1$ or $\max\{\beta b, d\} \leq \beta\lambda_1$, then* (EP) *(or equivalently,* (SP)*) has no positive solution.*

Proof. Suppose for contradiction that (U, V) is a positive solution of (EP) for the case $\max\{\beta b, d\} \leq \beta\lambda_1$. Observe that, if $d \leq \beta b \leq \beta\lambda_1$, then

$$\frac{V}{1+\beta u}(b + du - v) \leq \frac{V}{1+\beta u}\{b(1+\beta u) - v\} < bV \quad \text{in} \quad \Omega,$$

while, if $\beta b < d \leq \beta\lambda_1$, then

$$\frac{V}{1+\beta u}(b + du - v) \leq \frac{dV}{\beta(1+\beta u)}\left(1 + \beta u - \frac{\beta}{d}v\right) < \frac{d}{\beta}V \quad \text{in} \quad \Omega.$$

Then multiplying by V the second equation of (EP) and integrating the resulting expression in Ω, we obtain

$$\begin{cases} ||\nabla V||^2 < b\,||V||^2 & \text{if} \;\; d \leq \beta b \leq \beta\lambda_1, \\ ||\nabla V||^2 < \dfrac{d}{\beta}\,||V||^2 & \text{if} \;\; \beta b < d \leq \beta\lambda_1. \end{cases} \tag{2.4}$$

Since $||\nabla V||^2 \geq \lambda_1 ||V||^2$ by Poincaré's inequality, (2.4) obviously yields a contradiction. By virtue of $U(a - u - cv)/(1 + \alpha v) < aU$ in Ω, one can derive the assertion for the case $a \leq \lambda_1$ in a similar manner. □

We will give a priori estimates for positive solutions in the case $a > \lambda_1$ and $\max\{\beta b, d\} > \beta\lambda_1$.

Lemma 2.2. *Let (U, V) be a positive solution of* (EP)*. Then*

$$0 \leq u(x) \leq U(x) \leq M(a) := \begin{cases} a & \text{if} \;\; \alpha a \leq c, \\ \dfrac{(c + \alpha a)^2}{4\alpha c} & \text{if} \;\; \alpha a \geq c, \end{cases}$$
$$0 \leq v(x) \leq V(x) \leq (1 + \beta M(a))(b + dM(a))$$

for all $x \in \Omega$.

The following lemma gives other a priori estimates in the special cases.

Lemma 2.3. *Let (U, V) be a positive solution of* (EP)*. If $\alpha a \leq c$, then*

$$\theta_a \geq U \geq u \quad \text{in} \quad \Omega. \tag{2.5}$$

If $\beta b \leq d$, then

$$V \geq \theta_b \quad \text{in} \quad \Omega.$$

For the proofs of Lemmas 2.2 and 2.3, see Lemmas 2 and 3 in [15].

2.3 Bifurcations from Semitrivial Solutions

In this subsection, we regard a as a bifurcation parameter with b fixed. We will derive local bifurcation properties for positive solutions of (EP) from the semitrivial solution curves

$$\{(U,V,a)\,:\,(U,V)=(\theta_a,0),\ a>\lambda_1\}\ \text{ and }\ \{(U,V,a)\,:\,(U,V)=(0,\theta_b),\ a>\lambda_1\}.$$

Corollary A.2 in Appendix implies that, if $\beta b>\beta\lambda_1>d$ or $d>\beta\lambda_1>\beta b$, then there exists a unique constant $a_*\in(\lambda_1,\infty)$ such that

$$\lambda_1\left(\frac{-b-d\theta_{a_*}}{1+\beta\theta_{a_*}}\right)=0. \tag{2.6}$$

Corollary A.4 asserts that, if $b>\lambda_1$, then there exists a unique $a^*\in(\lambda_1,\infty)$ such that

$$\lambda_1\left(\frac{c\theta_b-a^*}{1+\alpha\theta_b}\right)=0. \tag{2.7}$$

Let ϕ_* and ϕ^* denote positive functions such that

$$-\Delta\phi_*-\frac{b+d\theta_{a_*}}{1+\beta\theta_{a_*}}\phi_*=0\ \text{ in }\ \Omega,\quad \phi_*=0\ \text{ on }\ \partial\Omega,\quad \|\phi_*\|=1$$

and

$$-\Delta\phi^*+\frac{c\theta_b-a^*}{1+\alpha\theta_b}\phi^*=0\ \text{ in }\ \Omega,\quad \phi^*=0\ \text{ on }\ \partial\Omega,\quad \|\phi^*\|=1.$$

By the definition of a_* and a^*, such ϕ_* and ϕ^* are uniquely determined from the above eigenvalue problems, respectively. Furthermore, for $p>N$, we define

$$\begin{cases} X:=\left[W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\right]\times\left[W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\right],\\ Y:=L^p(\Omega)\times L^p(\Omega). \end{cases} \tag{2.8}$$

Lemma 2.4. *Suppose $a>\lambda_1$. Then the following local bifurcation properties* (i) *and* (ii) *hold true:*

(i) *Let $\beta b>\beta\lambda_1>d$ or $d>\beta\lambda_1>\beta b$. Then positive solutions of* (EP) *bifurcate from the semitrivial solution curve $\{(\theta_a,0,a)\,:\,a>\lambda_1\}$ if and only if $a=a_*$. To be precise, all positive solutions of* (EP) *near $(\theta_{a_*},0,a_*)\in X\times\boldsymbol{R}$ can be expressed as*

$$\{(\theta_{a_*}+s\psi+s\hat{U}(s),\ s\phi_*+s\hat{V}(s),\ a(s))\,:\,0<s\le\delta\}$$

for some $\psi\in X$ and $\delta>0$. Here $(\hat{U}(s),\hat{V}(s),a(s))$ is a smooth function with respect to s and satisfies $(\hat{U}(0),\hat{V}(0),a(0))=(0,0,a_*)$ and $\int_\Omega\hat{V}(s)\phi_*=0$.

(ii) *Let $b>\lambda_1$. Then positive solutions of* (EP) *bifurcate from the semitrivial solution curve $\{(0,\theta_b,a)\,:\,a>\lambda_1\}$ if and only if $a=a^*$. More precisely, all positive solutions of* (EP) *near $(0,\theta_b,a^*)\in X\times\boldsymbol{R}$ are given by*

$$\{(s\phi^*+s\tilde{U}(s),\ \theta_b+s\chi+s\tilde{V}(s),\ a(s))\,:\,0<s\le\delta\} \tag{2.9}$$

for some $\chi\in X$ and $\delta>0$. Here $(\tilde{U}(s),\tilde{V}(s),a(s))$ is a smooth function with respect to s and satisfies $(\tilde{U}(0),\tilde{V}(0),a(0))=(0,0,a^*)$ and $\int_\Omega\tilde{U}(s)\phi^*=0$.

Proof. For $a > \lambda_1$, set

$$f(u,v) = u(a-u-cv), \qquad g(u,v) = v(b+du-v),$$

where u, v are functions of U, V (see (2.2) and (2.3)). By Taylor's expansion at (U^*, V^*), we reduce differential equations of (EP) to the form

$$\begin{pmatrix} \Delta U \\ \Delta V \end{pmatrix} + \begin{pmatrix} f(u(U^*,V^*), v(U^*,V^*)) \\ g(u(U^*,V^*), v(U^*,V^*)) \end{pmatrix}$$
$$+ \begin{pmatrix} f_u^* & f_v^* \\ g_u^* & g_v^* \end{pmatrix} \begin{pmatrix} u_U^* & u_V^* \\ v_U^* & v_V^* \end{pmatrix} \begin{pmatrix} U-U^* \\ V-V^* \end{pmatrix} + \begin{pmatrix} \rho^1(U-U^*, V-V^*) \\ \rho^2(U-U^*, V-V^*) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.10}$$

where $f_u^* := f_u(u(U^*,V^*), v(U^*,V^*))$, $u_U^* := u_U(U^*,V^*)$ and other notations are defined by similar rules. Here $\rho^i(U-U^*, V-V^*)$ $(i = 1, 2)$ are smooth functions such that $\rho^i(0,0) = \rho^i_{(U,V)}(0,0) = 0$. Differentiation of (2.1) yields

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\alpha v & \alpha u \\ \beta v & 1+\beta u \end{pmatrix} \begin{pmatrix} u_U & u_V \\ v_U & v_V \end{pmatrix}.$$

Since u and v are both nonnegative, we have

$$\begin{pmatrix} u_U & u_V \\ v_U & v_V \end{pmatrix} = \frac{1}{1+\alpha v+\beta u} \begin{pmatrix} 1+\beta u & -\alpha u \\ -\beta v & 1+\alpha v \end{pmatrix}. \tag{2.11}$$

Let $\beta b > \beta\lambda_1 > d$ or $d > \beta\lambda_1 > \beta b$. We note that

$$f(\theta_a, 0) = \theta_a(a-\theta_a) = -\Delta\theta_a, \qquad g(\theta_a, 0) = 0.$$

So by virtue of (2.3), setting $(U^*, V^*) = (\theta_a, 0)$ and $\overline{U} := U - \theta_a$ in (2.10) yields

$$\begin{pmatrix} \Delta\overline{U} \\ \Delta V \end{pmatrix} + \frac{1}{1+\beta\theta_a} \begin{pmatrix} a-2\theta_a & -c\theta_a \\ 0 & b+d\theta_a \end{pmatrix} \begin{pmatrix} 1+\beta\theta_a & -\alpha\theta_a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{U} \\ V \end{pmatrix}$$
$$+ \begin{pmatrix} \rho^1(\overline{U}, V; a) \\ \rho^2(\overline{U}, V; a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.12}$$

where $\rho^i(\overline{U}, V; a)$ $(i = 1, 2)$ are smooth functions satisfying

$$\rho^1_{(\overline{U},V)}(0,0;a) = \rho^2_{(\overline{U},V)}(0,0;a) = 0 \quad \text{for all} \quad a > \lambda_1. \tag{2.13}$$

Define a mapping $F : X \times \boldsymbol{R} \to Y$ by the left hand side of (2.12):

$$F(\overline{U}, V, a) = \begin{pmatrix} \Delta\overline{U} + (a-2\theta_a)\overline{U} - \dfrac{(\alpha a + c - 2\alpha\theta_a)\theta_a}{1+\beta\theta_a} V + \rho^1(\overline{U}, V, a) \\ \Delta V + \dfrac{b+d\theta_a}{1+\beta\theta_a} V + \rho^2(\overline{U}, V; a) \end{pmatrix}. \tag{2.14}$$

Since $(U,V)=(\theta_a,0)$ is a semitrivial solution of (EP), it turns out $F(0,0,a)=0$ for $a>\lambda_1$. It follows from (2.13) and (2.14) that the Fréchet derivative of F at $(\overline{U},V,a)=(0,0,a)$ is given by

$$F_{(\overline{U},V)}(0,0,a)\begin{pmatrix} h \\ k \end{pmatrix}=\begin{pmatrix} \Delta h+(a-2\theta_a)h-\dfrac{(\alpha a+c-2\alpha\theta_a)\theta_a}{1+\beta\theta_a}k \\ \Delta k+\dfrac{b+d\theta_a}{1+\beta\theta_a}k \end{pmatrix}. \tag{2.15}$$

By virtue of (2.6), we see that $\operatorname{Ker} F_{(\overline{U},V)}(0,0,a)$ is nontrivial for $a=a_*$ and that

$$\operatorname{Ker} F_{(\overline{U},V)}(0,0,a_*)=\operatorname{span}\{\psi,\phi_*\},$$

with

$$\psi=-(-\Delta-a_*+2\theta_{a_*})^{-1}\left[\frac{(\alpha a_*+c-2\alpha\theta_{a_*})\theta_{a_*}}{1+\beta\theta_{a_*}}\phi_*\right],$$

where $(-\Delta-a_*+2\theta_{a_*})^{-1}$ is the inverse operator of $-\Delta-a_*+2\theta_{a_*}$ with zero Dirichlet boundary condition on $\partial\Omega$. (Recall that $-\Delta-a_*+2\theta_{a_*}$ is invertible; see, e.g., [4].) If ${}^t(\tilde{h},\tilde{k})\in \operatorname{Range} F_{(\overline{U},V)}(0,0,a^*)$, there must exist $(h,k)\in X$ such that

$$\begin{cases} \Delta h+(a_*-2\theta_{a_*})h-\dfrac{(\alpha a_*+c-2\alpha\theta_{a_*})\theta_{a_*}}{1+\beta\theta_{a_*}}k=\tilde{h} & \text{in } \Omega, \\ \Delta k+\dfrac{b+d\theta_{a_*}}{1+\beta\theta_{a_*}}k=\tilde{k} & \text{in } \Omega, \\ h=k=0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that the second equation has a solution k if and only if $\int_\Omega \tilde{k}\phi_*=0$. For such a solution k, the first equation has a unique solution h because of the invertibility of $-\Delta-a_*+2\theta_{a_*}$. Then, it holds that $\operatorname{codimRange} F_{(\overline{U},V)}(0,0,a^*)=1$. In order to use the local bifurcation theory by Crandall and Rabinowitz [2] at $(\overline{U},V,a)=(0,0,a_*)$, we need to verify

$$F_{(\overline{U},V),a}(0,0,a_*)\begin{pmatrix} \psi \\ \phi_* \end{pmatrix}\notin \operatorname{Range} F_{(\overline{U},V)}(0,0,a_*).$$

Since $\rho^i_{(\overline{U},V),a}(0,0,a_*)=0$ by (2.13), it follows from (2.14) that

$$F_{(\overline{U},V),a}(0,0,a_*)\begin{pmatrix} \psi \\ \phi_* \end{pmatrix} =\begin{pmatrix} \left(1-2\left.\dfrac{d\theta_a}{da}\right|_{a=a_*}\right)\psi-\dfrac{\partial}{\partial a}\left.\left(\dfrac{(\alpha a+c-2\alpha\theta_a)\theta_a}{1+\beta\theta_a}\right)\right|_{a=a_*}\phi_* \\ \dfrac{d-\beta b}{(1+\beta\theta_{a_*})^2}\left.\dfrac{d\theta_a}{da}\right|_{a=a_*}\phi_* \end{pmatrix}.$$

Suppose for contradiction that there exists $k \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\Delta k + \frac{b + d\theta_{a_*}}{1 + \beta\theta_{a_*}} k = \frac{d - \beta b}{(1 + \beta\theta_{a_*})^2} \left. \frac{d\theta_a}{da} \right|_{a=a_*} \phi_*.$$

Multiplying the above equality by ϕ_* and integrating, we have

$$(d - \beta b) \int_\Omega \frac{1}{(1 + \beta\theta_{a_*})^2} \left. \frac{d\theta_a}{da} \right|_{a=a_*} \phi_*^2 = 0.$$

Thus it follows from the strict increasing property of θ_a that $d = \beta b$, which is impossible.

Recall that $\overline{U} = U - \theta_a$, one can immediately obtain the assertion of (i) by applying the local bifurcation theorem ([2]). We note that the possibility of other bifurcation points except $a = a_*$ is excluded by virtue of the Krein-Rutman theorem.

Next assume $b > \lambda_1$. Setting $(U^*, V^*) = (0, \theta_b)$ and $\overline{V} = V - \theta_b$ in (2.10) implies

$$\begin{pmatrix} \Delta U - \dfrac{c\theta_b - a}{1 + \alpha\theta_b} U + \rho^1(U, \overline{V}; a) \\ \Delta \overline{V} - \dfrac{(\beta b - d - 2\beta\theta_b)\theta_b}{1 + \alpha\theta_b} U + (b - 2\theta_b)\overline{V} + \rho^2(u, \overline{V}; a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.16}$$

where $\rho^i(U, \overline{V}; a)$ $(i = 1, 2)$ are smooth functions with $\rho^i_{(U,\overline{V})}(0, 0; a) = 0$ for all $a > \lambda_1$. Define a mapping $G(U, \overline{V}, a)$ from $X \times \boldsymbol{R}$ to Y by the left hand side of (2.16). So it turns out that $G(0, 0, a) = 0$ for $a > \lambda_1$ and

$$G_{(U,\overline{V})}(0, 0, a) \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \Delta h - \dfrac{c\theta_b - a}{1 + \alpha\theta_b} h \\ \Delta k - \dfrac{(\beta b - d - 2\beta\theta_b)\theta_b}{1 + \alpha\theta_b} h + (b - 2\theta_b) k \end{pmatrix}.$$

Thus it follows from (2.7) that $\operatorname{Ker} G_{(U,\overline{V})}(0, 0, a)$ is nontrivial if $a = a^*$ and

$$\operatorname{Ker} G_{(U,\overline{V})}(0, 0, a^*) = \operatorname{span}\{\phi^*, \chi\},$$

where

$$\chi = -(-\Delta - b + 2\theta_b)^{-1} \left[\frac{(\beta b - d - 2\beta\theta_b)\theta_b}{1 + \alpha\theta_b} \phi^* \right]. \tag{2.17}$$

Furthermore, a similar procedure to the proof of (i) yields

$$G_{(U,\overline{V}),a}(0, 0, a^*) \begin{pmatrix} \phi^* \\ \chi \end{pmatrix} \notin \operatorname{Range} G_{(U,\overline{V})}(0, 0, a^*).$$

Hence the local bifurcation theory ensures the assertion of (ii). □

Remark 2.1. Corollary A.2 in Appendix also asserts that if $b > \lambda_1$ and $d \geq \beta\lambda_1$, then

$$\lambda_1 \left(\frac{-b - d\theta_a}{1 + \beta\theta_a} \right) < 0 \quad \text{for all} \quad a \in (\lambda_1, \infty).$$

So it follows from (2.15) that $F_{(\overline{U},V)}(0, 0, a)$ is invertible for any $a \in (\lambda_1, \infty)$. By the implicit function theorem, we see that, if $b > \lambda_1$ and $d \geq \beta\lambda_1$, then no positive solution bifurcates from the semitrivial solution curve $\{(\theta_a, 0, a) : a > \lambda_1\}$.

3 Lyapunov-Schmidt Reduction Scheme

We will carry out the Lyapunov-Schmidt reduction procedure in case $\alpha = 0$: Observe that, in $\alpha = 0$, (EP) is reduced to the problem

$$(\mathrm{EP})_0 \begin{cases} \Delta U + U\left(a - U - \dfrac{cV}{1+\beta U}\right) = 0 & \text{in } \Omega, \\ \Delta V + \dfrac{V}{1+\beta U}\left(b + dU - \dfrac{V}{1+\beta U}\right) = 0 & \text{in } \Omega, \\ U = V = 0 & \text{on } \partial\Omega. \end{cases}$$

We introduce the following change of variables in $(\mathrm{EP})_0$;

$$a = \lambda_1 + \varepsilon a_1,\ \ b = \lambda_1 + \varepsilon b_1,\ \ d/\beta = \lambda_1 + \varepsilon\tau,\ \ \beta = \gamma/\varepsilon,\ \ U = \varepsilon w,\ \ V = \varepsilon z, \tag{3.1}$$

where ε is a small positive parameter and τ is a constant which may be nonpositive. In what follows, we will mainly discuss the case that d/β and $b\,(> \lambda_1)$ are close to λ_1 and β is large. Through (3.1), $(\mathrm{EP})_0$ is rewritten in the form

$$(\mathrm{PP}) \begin{cases} \Delta w + \lambda_1 w + \varepsilon w\left(a_1 - w - \dfrac{cz}{1+\gamma w}\right) = 0 & \text{in } \Omega, \\ \Delta z + \lambda_1 z + \dfrac{\varepsilon z}{1+\gamma w}\left(b_1 + \tau\gamma w - \dfrac{z}{1+\gamma w}\right) = 0 & \text{in } \Omega, \\ w = z = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that (3.1) maps semitrivial solutions

$$(U,V) = (\theta_a, 0)\ (a > \lambda_1) \quad \text{and} \quad (U,V) = (0, \theta_b)\ (b > \lambda_1)$$

of $(\mathrm{EP})_0$ to semitrivial ones

$$(w,z) = \left(\frac{1}{\varepsilon}\theta_{\lambda_1+\varepsilon a_1}, 0\right) \quad \text{and} \quad (w,z) = \left(0, \frac{1}{\varepsilon}\theta_{\lambda_1+\varepsilon b_1}\right)$$

of (PP), respectively. Further, it follows from Lemma 2.4 and (3.1) that in case $\tau < 0$, positive solutions of (PP) bifurcate from semitrivial solution curve $\{(\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_1}, 0, a_1) : a_1 > 0\}$ if and only if

$$a_1 = a_{1*}(\varepsilon) := \frac{1}{\varepsilon}(a_* - \lambda_1). \tag{3.2}$$

Similarly, positive solutions of (PP) bifurcate from other semitrivial solution curve $\{(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1) : a_1 > 0\}$ if and only if

$$a_1 = a_1^*(\varepsilon) := \frac{1}{\varepsilon}(\lambda_1(c\theta_{\lambda_1+\varepsilon b_1}) - \lambda_1). \tag{3.3}$$

In order to apply the Lyapunov-Schmidt reduction method, we will give a similar framework to that of Du and Lou [7]. For X and Y defined by (2.8), we introduce mappings $H : X \to Y$ and $B : X \times \boldsymbol{R} \to Y$ by

$$
\begin{aligned}
&H(w, z) = (\Delta w + \lambda_1 w, \Delta z + \lambda_1 z), \\
&B(w, z, a_1) = \left(w\left(a_1 - w - \frac{cz}{1+\gamma w}\right), \frac{z}{1+\gamma w}\left(b_1 + \tau\gamma w - \frac{z}{1+\gamma w}\right)\right).
\end{aligned}
\tag{3.4}
$$

Then (PP) is equivalent to

$$
H(w, z) + \varepsilon B(w, z, a_1) = 0. \tag{3.5}
$$

Denote by X_1 and Y_1 the L^2-orthogonal complements of span $\{(\Phi, 0), (0, \Phi)\}$ in X and Y, respectively. Furthermore, let $P : X \to X_1$ and $Q : Y \to Y_1$ be the L^2-orthogonal projections. Hence for each $(w, z) \in X$, there exists a unique $(s, t) \in \boldsymbol{R}^2$ such that

$$
(w, z) = (s, t)\Phi + \boldsymbol{u}, \quad \text{where } \boldsymbol{u} = P(w, z). \tag{3.6}
$$

Additionally, (3.5) is decomposed as

$$
\begin{cases}
QH((s, t)\Phi + \boldsymbol{u}) + \varepsilon QB((s, t)\Phi + \boldsymbol{u}, a_1) = 0, \\
(I - Q)H((s, t)\Phi + \boldsymbol{u}) + \varepsilon(I - Q)B((s, t)\Phi + \boldsymbol{u}, a_1) = 0.
\end{cases}
$$

By virtue of $H((s, t)\Phi) = 0$ and $(I - Q)H(X_1) = 0$, (3.5) (that is (PP)) is equivalent to

$$
QH(\boldsymbol{u}) + \varepsilon QB((s, t)\Phi + \boldsymbol{u}, a_1) = 0 \tag{3.7}
$$

and

$$
(I - Q)B((s, t)\Phi + \boldsymbol{u}, a_1) = 0.
$$

In view of (3.7), we define a mapping $G : \boldsymbol{R}^4 \times X_1 \to Y_1$ by

$$
G(s, t, a_1, \varepsilon, \boldsymbol{u}) = QH(\boldsymbol{u}) + \varepsilon QB((s, t)\Phi + \boldsymbol{u}, a_1).
$$

Then it follows that

$$
G(s, t, a_1, 0, 0) = 0 \quad \text{for any} \quad (s, t, a_1) \in \boldsymbol{R}^3.
$$

Furthermore, it is possible to verify that

$$
G_{\boldsymbol{u}}(s, t, a_1, 0, 0) = QH \quad \text{for any} \quad (s, t, a_1) \in \boldsymbol{R}^3;
$$

so that $G_{\boldsymbol{u}}(s, t, a_1, 0, 0)$ is an isomorphism from X_1 onto Y_1. Therefore, the implicit function theorem implies that for any $(s', t', a_1') \in \boldsymbol{R}^3$ there exist a positive constant $\varepsilon' = \varepsilon'(s', t', a_1')$ and a neighborhood N' of $(w, z, a_1, \varepsilon) = (s'\Phi, t'\Phi, a_1', 0)$ in $X \times \boldsymbol{R}^2$ such that all solutions of (3.7) in N' are expressed as

$$
\{((s, t)\Phi + \boldsymbol{u}(s, t, a_1, \varepsilon), a_1, \varepsilon) \, : \, |s - s'|, \, |t - t'|, \, |a_1 - a_1'|, \, |\varepsilon| \le \varepsilon'\}.
$$

Taking account for the compactness of $\{(s,t,a_1) : |s|, |t|, |a_1| \le C\}$, one can find a positive $\varepsilon_0 = \varepsilon_0(C)$ and a neighborhood N_0 of $\{(s\Phi, t\Phi, a_1, 0) : |s|, |t|, |a_1| \le C\}$ such that all solutions of (3.7) in N_0 are given by

$$\{((s,t)\Phi + \boldsymbol{u}(s,t,a_1,\varepsilon), a_1, \varepsilon) : |s|, |t|, |a_1| \le C + \varepsilon_0, |\varepsilon| \le \varepsilon_0\}. \tag{3.8}$$

Here we note that $\boldsymbol{u}(s,t,a_1,\varepsilon)$ is an X_1-valued smooth function with $\boldsymbol{u}(s,t,a_1,0) = 0$. Hence if we put

$$\varepsilon \boldsymbol{U}(s,t,a_1,\varepsilon) = \boldsymbol{u}(s,t,a_1,\varepsilon), \tag{3.9}$$

then $\boldsymbol{U}(s,t,a_1,\varepsilon)$ is also smooth for $|s|, |t|, |a_1| \le C + \varepsilon_0$ and $|\varepsilon| \le \varepsilon_0$. The above consideration gives the following lemma:

Lemma 3.1. *Suppose that* $|s|, |t|, |a_1| \le C + \varepsilon_0$ *and* $|\varepsilon| \le \varepsilon_0$. *Then any element of the set defined by* (3.8);

$$(w, z, a_1, \varepsilon) = ((s,t)\Phi + \varepsilon \boldsymbol{U}(s,t,a_1,\varepsilon), a_1, \varepsilon)$$

becomes a solution of (3.5) (*or equivalently* (PP)) *in* N_0 *if and only if*

$$(I - Q)B((s,t)\Phi + \varepsilon \boldsymbol{U}(s,t,a_1,\varepsilon), a_1) = 0.$$

Let $M = \{(s,t,a_1) : |s|, |t|, |a_1| \le C + \varepsilon_0\}$. For each $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, define a mapping $F^\varepsilon : M \to \boldsymbol{R}^2$ by

$$F^\varepsilon(s,t,a_1)\Phi = (I - Q)B((s,t)\Phi + \varepsilon \boldsymbol{U}(s,t,a_1,\varepsilon), a_1).$$

It follows from (3.4) that, if we put $\boldsymbol{U}(s,t,a_1,\varepsilon) = (W(s,t,a_1,\varepsilon), Z(s,t,a_1,\varepsilon))$, then

$$F^\varepsilon(s,t,a_1) = \begin{pmatrix} \displaystyle\int_\Omega (s\Phi + \varepsilon W)\left[a_1 - (s\Phi + \varepsilon W) - \frac{c(t\Phi + \varepsilon Z)}{1 + \gamma(s\Phi + \varepsilon W)}\right]\Phi \\ \displaystyle\int_\Omega \frac{t\Phi + \varepsilon Z}{1 + \gamma(s\Phi + \varepsilon W)}\left[b_1 + \tau\gamma(s\Phi + \varepsilon W) - \frac{t\Phi + \varepsilon Z}{1 + \gamma(s\Phi + \varepsilon W)}\right]\Phi \end{pmatrix}. \tag{3.10}$$

Lemma 3.1 asserts that for each $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, the solution set of (3.5) in N_0 is identical to $\operatorname{Ker} F^\varepsilon$.

4 Analysis of Limiting Solution Set

In this section we investigate the structure of $\operatorname{Ker} F^0(s,t,a_1)$. It will give a lot of important information on a set of positive solutions of (PP) when $\varepsilon > 0$ is very small. It follows from (3.10) that

$$F^0(s,t,a_1) = \begin{pmatrix} \displaystyle s\left(a_1 - s\|\phi\|_3^3 - ct\int_\Omega \frac{\Phi^3}{1 + \gamma s\Phi}\right) \\ \displaystyle t\left[b_1 - (b_1 - \tau)\gamma s\int_\Omega \frac{\Phi^3}{1 + \gamma s\Phi} - t\int_\Omega \frac{\Phi^3}{(1 + \gamma s\Phi)^2}\right] \end{pmatrix}. \tag{4.1}$$

Therefore, $\operatorname{Ker} F^0(s,t,a_1)$ is a union of the following four sets:

$$\begin{aligned}
\mathcal{L}_0 &= \{(0,0,a_1) \;:\; a_1 \in \boldsymbol{R}\},\\
\mathcal{L}_1 &= \{(a_1/\|\Phi\|_3^3, 0, a_1) \;:\; a_1 \in \boldsymbol{R}\},\\
\mathcal{L}_2 &= \{(0, b_1/\|\Phi\|_3^3, a_1) \;:\; a_1 \in \boldsymbol{R}\},\\
\mathcal{L}_p &= \{(s, \varphi(\gamma s), \psi(s)) \;:\; s \in \boldsymbol{R}\},
\end{aligned}$$

where

$$\begin{cases}
\varphi(s) = \left[b_1 - (b_1 - \tau)s\displaystyle\int_\Omega \frac{\Phi^3}{1+s\Phi}\right]\left(\displaystyle\int_\Omega \frac{\Phi^3}{(1+s\Phi)^2}\right)^{-1},\\
\psi(s) = s\|\Phi\|_3^3 + c\varphi(\gamma s)\displaystyle\int_\Omega \frac{\Phi^3}{1+\gamma s\Phi}.
\end{cases} \tag{4.2}$$

By the identification $(s,t)\Phi$ with (s,t), $\mathcal{L}_1 \cap \overline{\boldsymbol{R}_+}^3$, $\mathcal{L}_2 \cap \overline{\boldsymbol{R}_+}^3$ and $\mathcal{L}_p \cap \overline{\boldsymbol{R}_+}^3$ can be regarded as the limiting sets of semitrivial solution curves $\{(\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_1}, 0, a_1) \;:\; a_1 > 0\}$, $\{(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1) \;:\; a_1 > 0\}$ and the positive solution set of (PP) as $\varepsilon \to 0$, respectively. By virtue of (4.2),

$$(0, \varphi(0), \psi(0)) = (0, b_1/\|\phi\|_3^3, cb_1) \in \mathcal{L}_2. \tag{4.3}$$

It is easily verified that in case $\tau \geq 0$,

$$\varphi(s) > 0 \;\text{ for all }\; s \in [0,\infty).$$

On the other hand, if $\tau < 0$, we can find a positive constant $s_0 = s_0(\tau/b_1)$ such that

$$\begin{cases}
\varphi(s) > 0 \;\text{ for }\; s \in [0, s_0),\\
\varphi(s) < 0 \;\text{ for }\; s \in (s_0, \infty).
\end{cases} \tag{4.4}$$

Thus it follows that

$$(s_0/\gamma, \varphi(s_0), \psi(s_0/\gamma)) = (s_0/\gamma, 0, s_0\|\Phi\|_3^3/\gamma) \in \mathcal{L}_1 \tag{4.5}$$

provided $\tau < 0$.

We will study profiles of ψ:

Lemma 4.1. *The following properties of $\psi(s)$ hold true:*
(a) *If $\tau \geq 0$, then $\psi(s) > \psi(0) = cb_1$ for all $s \in (0,\infty)$ and $\lim_{s\to\infty} \psi(s) = \infty$,*
(b) *there exist positive constants $\tilde{\tau} = \tilde{\tau}(c, b_1)$ and $\tilde{\gamma} = \tilde{\gamma}(c, b_1)$ such that*

(i) *if $(\tau,\gamma) \in [0,\tilde{\tau}] \times [\tilde{\gamma}, \infty)$, then $\psi(s)$ attains a strict local maximum and a strict local minimum at some $s = \overline{s}$ and $s = \underline{s}$ $(0 < \overline{s} < \underline{s})$, respectively, and $\psi(\overline{s}) > \psi(\underline{s})$;*

(ii) *if $(\tau,\gamma) \in [-\tilde{\tau}, 0) \times [\tilde{\gamma}, \infty)$, then $\psi(s)$ achieves a strict local maximum in $(0, s_0/\gamma)$. Furthermore, there exists a continuous function $\hat{\gamma}(\tau)$ in $[-\tilde{\tau}, 0)$ with*

$$\tilde{\gamma} < \hat{\gamma}(\tau) \;\; \textit{for all} \;\; \tau \in [-\tilde{\tau}, 0) \;\; \textit{and} \;\; \lim_{\tau\uparrow 0} \hat{\gamma}(\tau) = \infty \tag{4.6}$$

such that, if $\gamma \in [\tilde{\gamma}, \hat{\gamma}(\tau)]$ *for* $\tau \in [-\tilde{\tau}, 0)$, *then* $\psi(s)$ *attains a strict local minimum in* $(0, s_0/\gamma)$ *and, moreover, if* $\gamma \in [\hat{\gamma}(\tau), \infty)$ *for* $\tau \in [-\tilde{\tau}, 0)$, *then* $\max_{s\in[0,s_0/\gamma]} \psi(s) = \psi(\hat{s})$ *for some* $\hat{s} \in (0, s_0/\gamma)$.

Proof. In view of (4.2), if we define

$$\begin{aligned} h(s;\tau) &:= \varphi(s) \int_\Omega \frac{\Phi^3}{1+s\Phi} \\ &= \left[b_1 - (b_1-\tau)s\int_\Omega \frac{\Phi^3}{1+s\Phi}\right] \int_\Omega \frac{\Phi^3}{1+s\Phi} \left(\int_\Omega \frac{\Phi^3}{(1+s\Phi)^2}\right)^{-1}, \end{aligned} \tag{4.7}$$

then

$$\psi(s) = s\|\Phi\|_3^3 + ch(\gamma s;\tau). \tag{4.8}$$

Recalling $\|\Phi\| = 1$, one can see

$$\lim_{s\to\infty} \frac{h(s;\tau)}{s} = \frac{\tau}{\|\Phi\|_1},$$

which immediately yields

$$\lim_{s\to\infty} \frac{\psi(s)}{s} = \|\Phi\|_3^3 + \frac{c\gamma\tau}{\|\Phi\|_1} \quad \text{for any } \tau \in \boldsymbol{R}.$$

In particular, from Schwarz' inequality

$$\left(\int_\Omega \frac{\Phi^3}{1+s\Phi}\right)^2 \le \int_\Omega \frac{\Phi^4}{(1+s\Phi)^2} \int_\Omega \Phi^2 = \int_\Omega \frac{\Phi^4}{(1+s\Phi)^2};$$

so that

$$\begin{aligned} h(s;0) &= b_1\left(1 - s\int_\Omega \frac{\Phi^3}{1+s\Phi}\right) \int_\Omega \frac{\Phi^3}{1+s\Phi} \left(\int_\Omega \frac{\Phi^3}{(1+s\Phi)^2}\right)^{-1} \\ &\ge b_1\left(\int_\Omega \frac{\Phi^3}{1+s\Phi} - s\int_\Omega \frac{\Phi^4}{(1+s\Phi)^2}\right) \left(\int_\Omega \frac{\Phi^3}{(1+s\Phi)^2}\right)^{-1} \\ &= b_1 = h(0;0) \quad \text{for all } s \in [0,\infty). \end{aligned} \tag{4.9}$$

It obviously follows from (4.7)-(4.9) that $\psi(s) > \psi(0) = cb_1$ for all $s \in (0,\infty)$ provided $\tau \ge 0$. Furthermore, note

$$1 - s\int_\Omega \frac{\Phi^3}{1+s\Phi} = \int_\Omega \frac{(1+s\Phi)\Phi^2}{1+s\Phi} - s\int_\Omega \frac{\Phi^3}{1+s\Phi} = \int_\Omega \frac{\Phi^2}{1+s\Phi}.$$

Hence

$$\begin{aligned} \lim_{s\to\infty} h(s;0) =& b_1 \lim_{s\to\infty} \int_\Omega \frac{\Phi^2}{1+s\Phi} \int_\Omega \frac{\Phi^3}{1+s\Phi} \left(\int_\Omega \frac{\Phi^3}{(1+s\Phi)^2}\right)^{-1} \\ =& b_1 = h(0;0). \end{aligned} \tag{4.10}$$

Thus (4.9) and (4.10) imply that $h(s;0)$ attains a global maximum at some point in $(0,\infty)$. Therefore, in view of (4.8), one can see that, if $\tau = 0$ and γ is large enough, then $\psi(s)$ forms a '$\sim$'-shaped curve in the sense of the assertion (i). Hence this property of $\psi(s)$ is invariant for small $\tau > 0$ and the proof of (i) is complete.

In case $\tau < 0$, (4.4) and (4.7) yield

$$\begin{cases} h(s;\tau) > 0 & \text{for} \quad s \in [0, s_0), \\ h(s;\tau) < 0 & \text{for} \quad s \in (s_0, \infty). \end{cases}$$

Hence, if $|\tau|$ is sufficiently small, then $h(s;\tau)$ achieves a global maximum at some point contained in $(0, s_0)$ because of (4.9). Thus by (4.8), we may assume that if $(\tau,\gamma) \in [-\tilde{\tau}, 0) \times [\tilde{\gamma}, \infty)$, then $\psi(s)$ possesses at least one strict local maximum in $(0, s_0/\gamma)$. Observe that ψ depends continuously on (τ,γ); so we get a continuous function $\hat{\gamma}(\tau)$ in $[-\tilde{\tau}, 0)$ with (4.6) such that, if $\gamma \in [\tilde{\gamma}, \hat{\gamma}(\tau))$ for $\tau \in [-\tilde{\tau}, 0)$, then $\psi(s)$ forms a '$\sim$'-shaped curve in $(0, s_0/\gamma)$ and, if $\gamma \in [\hat{\gamma}(\tau), \infty)$ for $\tau \in [-\tilde{\tau}, 0)$, then $\max_{s\in[0,s_0/\gamma]} \psi(s) = \psi(\hat{s})$ for some $\hat{s} \in (0, s_0/\gamma)$. Thus the proof of Lemma 4.1 is accomplished. □

5 Perturbed Solution Set of (PP)

5.1 Case $\tau \geq 0$

Let $\tau \geq 0$. By Lemma 4.1, there exist sufficient large numbers A_1 and C such that

$$A_1 = \psi(C) = \max_{s\in[0,C]} \psi(s). \tag{5.1}$$

In this subsection, we will prove that if $\varepsilon > 0$ is small enough, then all positive solutions of (PP) in the range of $a_1 \in [0, A_1]$ form a one-dimensional submanifold near

$$\{(w, z, a_1) = (s\Phi, \varphi(\gamma s)\Phi, \psi(s)) \ : \ 0 < s \leq C\}.$$

More precisely, we will prove the following proposition:

Proposition 5.1. *Let $\tau \geq 0$. Then there exist a positive constant $\varepsilon_0 = \varepsilon_0(A_1)$ and a family of bounded smooth curves*

$$\{S(\xi,\varepsilon) = (s(\xi,\varepsilon), t(\xi,\varepsilon), a_1(\xi,\varepsilon)) \in \boldsymbol{R}^3 \ : \ (\xi,\varepsilon) \in [0, C(\varepsilon)] \times [0, \varepsilon_0]\}$$

such that for each fixed $\varepsilon \in (0, \varepsilon_0]$, all positive solutions of (PP) *with $a_1 \in (0, A_1]$ can be parameterized as*

$$\begin{aligned} \Gamma^\varepsilon = \{(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) &= ((s,t)\Phi + \varepsilon U(s,t,a_1,\varepsilon), a_1) \ : \\ (s,t,a_1) &= (s(\xi,\varepsilon), t(\xi,\varepsilon), a_1(\xi,\varepsilon)) \ \textit{for} \ \ \xi \in (0, C(\varepsilon)]\}, \end{aligned} \tag{5.2}$$

and

$$S(\xi, 0) = (\xi, \varphi(\gamma\xi), \psi(\xi)), \quad S(0,\varepsilon) = (0, t(\varepsilon), a_1^*(\varepsilon)).$$

Here $C(\varepsilon)$ is a smooth positive function in $[0,\varepsilon_0]$ with $C(0)=C$ and $a_1(C(\varepsilon),\varepsilon)=A_1$, $t(\varepsilon) := \varepsilon^{-1}\int_\Omega \theta_{\lambda_1+\varepsilon b_1}\Phi$, $a_1^*(\varepsilon)$ is the positive number defined by (3.3) *and $\boldsymbol{U}$ is the X_1-valued function defined by* (3.9). *Furthermore, Γ^ε can be extended to the range $a_1 \in [A_1,\infty)$ as a positive solution curve of* (PP).

As the first step to the proof of Proposition 5.1, we will express the nonnegative solution set of (3.5) (or equivalently (PP)) near the intersection point of $\mathcal{L}_p$ and $\mathcal{L}_2$; $(s,t,a_1)=(0,b_1/\|\Phi\|_3^3, cb_1)$.

Lemma 5.2. *Let F^ε be the mapping defined by* (3.10). *Then there exist a neighborhood U_0 of $(0,b_1/\|\Phi\|_3^3, cb_1)$ and a positive constant δ_0 such that for any $\varepsilon\in[0,\delta_0]$,*

$$\begin{aligned}&\operatorname{Ker} F^\varepsilon \cap U_0 \cap \overline{\boldsymbol{R}_+}^3\\ =&\{(s(\xi,\varepsilon),t(\xi,\varepsilon),a_1(\xi,\varepsilon))\ :\ \xi\in[0,\delta_0)\}\cup\{(0,t(\varepsilon),a_1)\in U_0\}\end{aligned} \tag{5.3}$$

with some smooth function $(s(\xi,\varepsilon),t(\xi,\varepsilon),a_1(\xi,\varepsilon))$ in $[0,\delta_0]\times[0,\delta_0]$ satisfying

$$\begin{aligned}(s(\xi,0),t(\xi,0),a_1(\xi,0)) &= (\xi,\varphi(\gamma\xi),\psi(\xi)),\\ (s(0,\varepsilon),t(0,\varepsilon),a_1(0,\varepsilon)) &= (0,t(\varepsilon),a_1^*(\varepsilon)).\end{aligned}$$

Proof. By Lemma 2.4 and (3.3), we recall that for any $\varepsilon>0$, there exist a positive number $\delta=\delta(\varepsilon)$ and a neighborhood V_ε of the bifurcation point $(w,z,a_1)=(0,\varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1},a_1^*(\varepsilon))$ such that all positive solutions of (PP) in V_ε are given by

$$\begin{aligned}&(w,z,a_1)=(w(\xi,\varepsilon),z(\xi,\varepsilon),a_1(\xi,\varepsilon))\\ =&(\xi\phi^*+\xi W(\xi,\varepsilon),\ \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}+\xi\chi+\xi Z(\xi,\varepsilon),\ a_1(\xi,\varepsilon))\end{aligned}$$

for $\xi\in(0,\delta]$. Here χ is the function defined by (2.17), $(W(\xi,\varepsilon),Z(\xi,\varepsilon),a_1(\xi,\varepsilon))$ is a certain smooth function such that $a_1(0,\varepsilon)=a_1^*(\varepsilon)$ and $\int_\Omega W(\xi,\varepsilon)\phi^*=0$. We define an open set U_ε of $\boldsymbol{R}^3$ by

$$U_\varepsilon := \left\{(s,t,a_1)\ :\ s=\int_\Omega w\Phi,\ t=\int_\Omega z\Phi,\ (w,z,a_1)\in V_\varepsilon\right\}.$$

and put

$$s(\xi,\varepsilon) := \int_\Omega w(\xi,\varepsilon)\Phi, \quad t(\xi,\varepsilon) := \int_\Omega z(\xi,\varepsilon)\Phi.$$

By virtue of the equivalence of (PP) and (3.5), we can verify that, if $\varepsilon\in[0,\varepsilon_0]$, then

$$\begin{aligned}&\operatorname{Ker} F^\varepsilon \cap U_\varepsilon \cap \overline{\boldsymbol{R}_+}^3\\ =&\{(s(\xi,\varepsilon),t(\xi,\varepsilon),a_1(\xi,\varepsilon))\ :\ \xi\in[0,\delta]\}\cup\{(0,t(\varepsilon),a_1)\in U_\varepsilon\}.\end{aligned}$$

Since $(0,t(\varepsilon),a_1^*(\varepsilon))$ is a bifurcation point for any $\varepsilon\in[0,\varepsilon_0]$, it is possible to show that U_ε contains a neighborhood U_0 of $(0,b_1/\|\Phi\|_3^3, cb_1)$ if $\varepsilon>0$ is sufficiently small. Thus the proof of Lemma 5.2 is complete. □

Lemma 5.3. *Assume $\tau \geq 0$ and let A_1, C be positive constants obtained in* (5.1). *There exist $\varepsilon_0 = \varepsilon(A_1) > 0$ and a neighborhood U of $\{(s\Phi, \varphi(\gamma s)\Phi, \psi(s)) : 0 \leq s \leq C\}$ such that for each fixed $\varepsilon \in (0, \varepsilon_0]$, all positive solutions of* (PP) *contained in $U \cap (X \times (0, A_1])$ can be expressed as* (5.2).

Proof. We will prove this lemma along the perturbation theory by Du and Lou [7, Appendix]. Define $\mathcal{L}_p([\delta_0/2, C]) = \{(s, \varphi(\gamma s), \psi(s)) : s \in [\delta_0/2, C]\}$ for the positive constant δ_0 obtained in Lemma 5.2. By (4.1) and (4.2), direct calculations lead to

$$\det F^0_{(s,t)}(s, \varphi(\gamma s), \psi(s)) = s\varphi(\gamma s)\psi'(s) \int_\Omega \frac{\Phi^3}{(1+\gamma s\Phi)^2}. \tag{5.4}$$

Let $(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})) \in \mathcal{L}_p([\delta_0/2, C])$ be any fixed point. Note that $\varphi(\gamma\overline{s}) > 0$ for $\tau \geq 0$. Thus (5.4) implies that, if $\psi'(\overline{s}) \neq 0$, then $F^0_{(s,t)}(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s}))$ is invertible. In this case, the implicit function theorem gives a positive number $\delta = \delta(\overline{s})$ and neighborhood $W_{\overline{s}}$ of $(\overline{s}, \varphi(\gamma\overline{s}))$ such that for all $\varepsilon \in [0, \delta]$,

$$\operatorname{Ker} F^\varepsilon \cap U_{\overline{s}} = \{(s(a_1, \varepsilon), t(a_1, \varepsilon), a_1) : a_1 \in (\psi(\overline{s}) - \delta, \psi(\overline{s}) + \delta)\}, \tag{5.5}$$

where $U_{\overline{s}} = W_{\overline{s}} \times (\psi(\overline{s}) - \delta, \psi(\overline{s}) + \delta)$ and $(s(a_1, \varepsilon), t(a_1, \varepsilon))$ is a smooth function satisfying $(s(\psi(\overline{s}), 0), t(\psi(\overline{s}), 0)) = (\overline{s}, \varphi(\gamma\overline{s}))$.

On the other hand, if $\psi'(\overline{s}) = 0$, then (5.4) leads to $\operatorname{rank} F^0_{(s,t)}(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})) = 1$; so that

$$\dim \operatorname{Ker} F^0_{(s,t)}(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})) = \operatorname{codim} \operatorname{Range} F^0_{(s,t)}(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})) = 1. \tag{5.6}$$

After some calculations, one can see

$$F^0_{a_1}(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})) = \begin{pmatrix} \overline{s} \\ 0 \end{pmatrix} \notin \operatorname{Range} F^0_{(s,t)}(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})). \tag{5.7}$$

According to the spontaneous bifurcation theory by Crandall and Rabinowitz [3, Theorem 3.2 and Remark 3.3], (5.6) and (5.7) enable us to get a positive number $\delta = \delta(\overline{s})$ and a neighborhood $U_{\overline{s}}$ of $(\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s}))$ such that for each $\varepsilon \in [0, \delta]$,

$$\operatorname{Ker} F^\varepsilon \cap U_{\overline{s}} = \{(s(\xi, \varepsilon), t(\xi, \varepsilon), a_1(\xi, \varepsilon)) : \xi \in (-\delta, \delta)\} \tag{5.8}$$

with a suitable smooth function $(s(\xi, \varepsilon), t(\xi, \varepsilon), a_1(\xi, \varepsilon))$ in $[-\delta, \delta] \times [0, \delta]$ with

$$(s(0,0), t(0,0), a_1(0,0)) = (\overline{s}, \varphi(\gamma\overline{s}), \psi(\overline{s})).$$

For each $U_{\overline{s}}$ satisfying (5.5) or (5.8), it clearly follows that

$$\mathcal{L}_p([\delta_0/2, C]) \subset \bigcup \{U_{\overline{s}} : \overline{s} \in [\delta_0/2, C]\}.$$

Since $\mathcal{L}_p([\delta_0/2, C])$ is compact, there are a finite number of points $\{s_j\}_{j=1}^k$ such that

$$\begin{cases} (s_j, \varphi(s_j), \psi(s_j)) \in \mathcal{L}_p([\delta_0/2, C]) \quad \text{for} \quad 1 \le j \le k, \\ \mathcal{L}_p([\delta_0/2, C]) \subset \bigcup_{j=1}^{k} U_j, \quad \text{where} \quad U_j := U_{s_j}. \end{cases}$$

We may assume $U_j \cap U_{j+1}$ are not empty for all $0 \le j \le k-1$. Here U_0 is an open set obtained in Lemma 5.2. Thus by (5.5) and (5.8), if we put $\delta_j = \delta(s_j)$, then for any $\varepsilon \in [0, \delta_j]$ $(1 \le j \le k)$,

$$\operatorname{Ker} F^\varepsilon \cap U_j = \{(s^j(\xi,\varepsilon), t^j(\xi,\varepsilon), a_1^j(\xi,\varepsilon)) \;:\; \xi \in (-\delta_j, \delta_j)\} =: J_j^\varepsilon$$

with some smooth functions $s^j(\xi,\varepsilon)$, $t^j(\xi,\varepsilon)$ and $a_1^j(\xi,\varepsilon)$ which satisfy

$$(s^j(0,0), t^j(0,0), a_1^j(0,0)) = (s_j, \varphi(\gamma s_j), \psi(s_j)).$$

Additionally in view of Lemma 5.2, if we set

$$J_0^\varepsilon = \{(s(\xi,\varepsilon), t(\xi,\varepsilon), a_1(\xi,\varepsilon)) \;:\; \xi \in (0, \delta_0]\}$$

and $U = \bigcup_{j=0}^{k} U_j$, then

$$\operatorname{Ker} F^\varepsilon \cap U \cap \boldsymbol{R}_+^3 = \bigcup_{j=0}^{k} J_j^\varepsilon \quad \text{for any} \quad \varepsilon \in [0, \min_{0 \le j \le k} \delta_j]. \tag{5.9}$$

Clearly (5.9) implies that $\operatorname{Ker} F^\varepsilon \cap U \cap \boldsymbol{R}_+^3$ forms a one-dimensional submanifold. Indeed, with the aid of the procedure by Du and Lou [7, Proposition A3], it is possible to construct a smooth curve $S(\xi,\varepsilon) = (s(\xi,\varepsilon), t(\xi,\varepsilon), a_1(\xi,\varepsilon))$ such that

$$\begin{cases} \bigcup_{j=0}^{k} J_j^\varepsilon = S((0, C(\varepsilon)], \varepsilon), \\ (s(\xi,0), t(\xi,0), a_1(\xi,0)) = (\xi, \varphi(\gamma\xi), \psi(\xi)), \\ (s(0,\varepsilon), t(0,\varepsilon), a_1(0,\varepsilon)) = (0, t(\varepsilon), a_1^*(\varepsilon)) \end{cases} \tag{5.10}$$

for sufficiently small $\varepsilon > 0$ and $\xi \in [0, C(\varepsilon)]$ with some smooth function $C(\varepsilon)$. In view of Lemma 3.1, one can get the conclusion from (5.10). □

The next lemma means that if $a_1 \in (0, A_1]$ and $\varepsilon > 0$ is small enough, then (PP) has no positive solution outside of U.

Lemma 5.4. *Assume $\tau \ge 0$ and V is any neighborhood of $\{(s\Phi, \varphi(\gamma s)\Phi, \psi(s)) : 0 \le s \le C\}$. Then there exists a positive constant ε_1 such that for each $\varepsilon \in (0, \varepsilon_1]$, any solution of* (PP) *with $a_1 \in (0, A_1]$ is given by*

$$(w, z) = (s,t)\Phi + \varepsilon \boldsymbol{U}(s,t,a_1,\varepsilon) \quad \textit{for some} \quad (s\Phi, t\Phi, a_1) \in V.$$

Proof. We will prove this lemma by a contradiction argument. Suppose that for a certain sequence $\{(a_1^n, \varepsilon_n)\}$ satisfying $a_1^n \in (0, A_1]$ and $\lim_{n\to 0} \varepsilon_n = 0$, (PP) with $(a_1, \varepsilon) = (a_1^n, \varepsilon_n)$ have positive solutions (w_n, z_n) such that $(w_n, z_n, a_1^n) \notin V$ for all $n \in \boldsymbol{N}$. To derive a contradiction, it suffices to find a subsequence $\{(w_{n(k)}, z_{n(k)}, a_1^{n(k)}, \varepsilon_{n(k)})\}$ and a sequence $\{(s_k, t_k)\}$ such that

$$\begin{cases} (w_{n(k)}, z_{n(k)}) = (s_k, t_k)\Phi + \varepsilon_{n(k)} \boldsymbol{U}(s_k, t_k, a_1^{n(k)}, \varepsilon_{n(k)}) \ \text{ for all } \ k \in \boldsymbol{N}, \\ \lim_{k\to\infty} (s_k, t_k, a_1^{n(k)}) = (s, \varphi(\gamma s), \psi(s)) \ \text{ for some } \ s \in [0, C]. \end{cases} \tag{5.11}$$

We begin with a priori bounds for $\{w_n\}$ and $\{z_n\}$. It follows from (2.5) and (3.1) that

$$w_n \le \frac{1}{\varepsilon_n} \theta_{\lambda_1 + \varepsilon_n A_1} \ \text{ in } \ \Omega$$

for all $n \in \boldsymbol{N}$. Recall that

$$\lim_{\lambda \downarrow \lambda_1} \frac{\theta_\lambda}{\lambda - \lambda_1} = \frac{\Phi}{\|\Phi\|_3^3} \ \text{ uniformly in } \ \Omega, \tag{5.12}$$

(see, e.g., Gui and Lou [9, Proposition 6.4]) one can see that

$$w_n \le 1 + A_1 \frac{\|\Phi\|_\infty}{\|\Phi\|_3^3} =: M \ \text{ in } \ \Omega$$

for sufficiently large n. Therefore, $\{w_n\}$ is a bounded sequence in $C(\overline{\Omega})$. From the second equation of (PP), we see

$$\begin{aligned} -\Delta z_n &= \lambda_1 z_n + \frac{\varepsilon_n z_n}{1 + \gamma w_n} \left(b_1 + \tau \gamma w_n - \frac{z_n}{1 + \gamma w_n} \right) \\ &\le \lambda_1 z_n + \varepsilon_n z_n \left[b_1 + \tau - \frac{z_n}{(1 + \gamma w_n)^2} \right] \\ &= z_n \left[\lambda_1 + \varepsilon_n (b_1 + \tau) - \frac{\varepsilon_n z_n}{(1 + \gamma M)^2} \right] \ \text{ in } \ \Omega \end{aligned}$$

for sufficiently large n. This fact implies that $\varepsilon_n z_n/(1 + \gamma M)^2$ is a subsolution of (1.4) with a replaced by $\lambda_1 + \varepsilon_n(b_1 + \tau)$. Thus by the well-known comparison result, one can obtain $\varepsilon_n z_n/(1 + \gamma M)^2 \le \theta_{\lambda_1 + \varepsilon_n(b_1 + \tau)}$ in Ω; so that

$$z_n \le (1 + \gamma M)^2 \frac{\theta_{\lambda_1 + \varepsilon_n(b_1 + \tau)}}{\varepsilon_n} \ \text{ in } \ \Omega.$$

Owing to (5.12), we get

$$z_n \le 1 + \frac{(1 + \gamma M)^2 (b_1 + \tau)}{\|\Phi\|_3^3} \Phi \ \text{ in } \ \Omega \tag{5.13}$$

for sufficiently large n. Therefore $\{z_n\}$ is uniformly bounded in $C(\overline{\Omega})$.

Let $\overline{w}_n = w_n/\|w_n\|_\infty$ and $\overline{z}_n = z_n/\|z_n\|_\infty$. Thus it follows from (PP) that $\overline{w}_n$ and $\overline{z}_n$ satisfy

$$\begin{cases} -\Delta \overline{w}_n = \lambda_1 \overline{w}_n + \varepsilon_n \overline{w}_n \left(a_1^n - w_n - \dfrac{c z_n}{1+\gamma w_n} \right) & \text{in } \Omega, \\ -\Delta \overline{z}_n = \lambda_1 \overline{z}_n + \dfrac{\varepsilon_n \overline{z}_n}{1+\gamma w_n} \left(b_1 + \tau\gamma w_n - \dfrac{z_n}{1+\gamma w_n} \right) & \text{in } \Omega, \\ \overline{w}_n = \overline{z}_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.14}$$

Since $\{(w_n, z_n, a_1^n)\}$ is uniformly bounded in $C(\overline{\Omega})^2 \times \boldsymbol{R}$, $\{a_1^n - w_n - cz_n/(1+\gamma w_n)\}$ and $\{b_1 + \tau\gamma w_n - z_n/(1+\gamma w_n)\}$ are also bounded in $C(\overline{\Omega})$. With the aid of the standard elliptic regularity theory, $\{w_n\}$, $\{z_n\}$ are uniformly bounded in $C^2(\overline{\Omega})$. So one can choose a subsequence $\{(w_{n(k)}, z_{n(k)}, a_1^{n(k)})\}$ such that

$$\lim_{k\to\infty} (\overline{w}_{n(k)}, \overline{z}_{n(k)}, a_1^{n(k)}) = (\overline{w}, \overline{z}, a_1^\infty) \quad \text{in} \quad C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times \boldsymbol{R}$$

with some $(\overline{w}, \overline{z}, a_1^\infty)$. For simplicity, we rewrite this subsequence by $\{(w_n, z_n, a_1^n)\}$. Since $\lim_{n\to\infty} \varepsilon_n = 0$, letting $n \to \infty$ in (5.14) implies that $\overline{w}$ and $\overline{z}$ satisfy

$$-\Delta \overline{w} = \lambda_1 \overline{w}, \quad -\Delta \overline{z} = \lambda_1 \overline{z} \ \text{ in } \ \Omega, \qquad \overline{w} = \overline{z} = 0 \ \text{ on } \ \partial\Omega.$$

Together with $\|\overline{w}\|_\infty = \|\overline{z}\|_\infty = 1$, we can deduce $\overline{w} = \overline{z} = \Phi/\|\Phi\|_\infty$. So the boundness of $\{(w_n, z_n)\}$ in $C^2(\overline{\Omega})^2$ yields

$$\lim_{n\to\infty} (w_n, z_n) = (s\Phi, t\Phi) \quad \text{in} \quad C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \tag{5.15}$$

for some $s \geq 0$ and $t \geq 0$. By virtue of (3.8) and (5.15), for sufficiently large n, (w_n, z_n) must be given by

$$(w_n, z_n) = (s_n, t_n)\Phi + \varepsilon_n U(s_n, t_n, a_1^n, \varepsilon_n)$$

with some sequence $\{(s_n, t_n)\}$ such that $\lim_{n\to\infty} (s_n, t_n) = (s, t)$.

To prove $t = \varphi(\gamma s)$, we multiply by Φ the second equation of (5.14) and integrate the resulting expression;

$$\int_\Omega \frac{\overline{z}_n \Phi}{1+\gamma w_n} \left(b_1 + \tau\gamma w_n - \frac{z_n}{1+\gamma w_n} \right) = 0.$$

By (5.15), letting $n \to \infty$ in the above equality yields

$$b_1 - (b_1 - \tau)\gamma s \int_\Omega \frac{\Phi^3}{1+\gamma s\Phi} = t \int_\Omega \frac{\Phi^3}{(1+\gamma s\Phi)^2},$$

which, together with (4.2), implies $t = \varphi(\gamma s)$.

Finally we will prove $a_1^\infty = \psi(s)$. Multiply the first equation of (5.14) by Φ and integrate it;

$$\int_\Omega \overline{w}_n \Phi \left(a_1^n - w_n - \frac{cz_n}{1+\gamma w_n} \right) = 0.$$

Letting $n \to \infty$ in the above equality, we have

$$a_1^\infty - s\|\Phi\|_3^3 - ct \int_\Omega \frac{\Phi^3}{1+\gamma s \Phi} = 0,$$

which immediately leads to $a_1^\infty = \psi(s)$ by (4.2). Then we obtain (5.11), which completes the proof of Lemma 5.4. □

Proof of Proposition 5.1. We have already shown (5.2) by Lemmas 5.3 and 5.4. To accomplish the proof of Proposition 5.1, it remains to show that Γ^ε can be extended to the range $a_1 \in [A_1, \infty)$ as a positive solution curve of (PP). Let $\hat{\Gamma}^\varepsilon$ be a maximum extension of Γ^ε in the direction $a_1 \geq A_1$ as a solution curve of (PP). According to the global bifurcation theorem by Rabinowitz [18], the following (i) or (ii) must hold true;

(i) $\hat{\Gamma}^\varepsilon$ is unbounded in $X \times \boldsymbol{R}$;

(ii) $\hat{\Gamma}^\varepsilon$ meets the trivial or a semitrivial solution curve at some point except for $(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1^*)$.

We introduce the following positive cone

$$P = \left\{ (w,z) \ : \ w > 0,\, z > 0 \text{ in } \Omega \text{ and } \frac{\partial w}{\partial \nu} < 0,\, \frac{\partial z}{\partial \nu} < 0 \text{ on } \partial\Omega \right\}.$$

Suppose that $(\hat{w}, \hat{z}, \hat{a}_1) \in \hat{\Gamma}^\varepsilon$ satisfies $(\hat{w}, \hat{z}) \in \partial P$ at $\hat{a}_1 \in (A_1, \infty)$. Thus it follows that $\hat{w} \geq 0, \hat{z} \geq 0$ for all $x \in \Omega$ and

$$\hat{w}(x_0)\hat{z}(x_0) = 0 \ \text{ at some } \ x_0 \in \Omega \tag{5.16}$$

or

$$\frac{\partial \hat{w}}{\partial \nu}(x_1)\frac{\partial \hat{z}}{\partial \nu}(x_1) = 0 \ \text{ at some } \ x_1 \in \partial\Omega. \tag{5.17}$$

By applying the strong maximum principle to (PP), it is possible to prove that both (5.16) and (5.17) imply $\hat{w} \equiv 0$ or $\hat{z} \equiv 0$.

We now recall that positive solutions of (PP) bifurcate from the semitrivial solution curve $\{(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1) \ : \ a_1 > 0\}$ if and only if $a_1 = a_1^*$ and no positive solution bifurcates from other semitrivial solution curve $\{(\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_1}, 0, a_1) \ : \ a_1 > 0\}$ if $\tau \geq 0$ (see Remark 2.1). In addition, it is easily verified that the trivial solution is non-degenerate. Therefore, we can deduce that $(\hat{w}, \hat{z}, \hat{a}_1) = (0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1^*)$, which contradicts (ii). Thus (ii) is excluded and (i) must be satisfied. Lemma 2.2 and (3.1) imply the boundness of w and z

$$\begin{cases} w(x) \leq \dfrac{1}{\varepsilon}(\lambda_1 + \varepsilon a_1), \\ z(x) \leq \dfrac{1}{\varepsilon}\{1 + \beta(\lambda_1 + \varepsilon a_1)\}\{\lambda_1 + \varepsilon b_1 + \beta(\lambda_1 + \varepsilon\tau)(\lambda_1 + \varepsilon a_1)\} \end{cases}$$

for all $x \in \Omega$. Therefore, Γ^ε must be extended with respect to $a_1 \in [A_1, \infty)$ as a positive solution curve of (PP). Thus the proof of Proposition 5.1 is complete. □

Proposition 5.1 in combination with Lemma 4.1 implies that Γ^ε forms an unbounded S-shaped curve with respect to a_1 for the special case when $(\tau, \gamma) \in [0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$ and $\varepsilon > 0$ is small enough:

Corollary 5.5. *Suppose that $(\tau, \gamma) \in [0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$ and $\varepsilon > 0$ is sufficiently small. Then the positive solution set of* (PP) *contains an unbounded S-shaped curve Γ^ε which bifurcates from the semitrivial solution curve $\{(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1) : a_1 > 0\}$ at $a_1 = a_1^*(\varepsilon)$. Furthermore, there exist two positive numbers $\overline{a}_1(\varepsilon) > \underline{a}_1(\varepsilon)$ $(> a_1^*(\varepsilon))$ such that*

(i) *if $a_1 \in (0, a_1^*(\varepsilon)]$, then* (PP) *has no positive solution;*
(ii) *if $a_1 \in (a_1^*(\varepsilon), \underline{a}_1(\varepsilon)) \cup (\overline{a}_1(\varepsilon), \infty)$, then* (PP) *has at least one positive solution;*
(iii) *if $a_1 = \underline{a}_1(\varepsilon)$ or $a_1 = \overline{a}_1(\varepsilon)$, then* (PP) *has at least two positive solutions;*
(iv) *if $a_1 \in (\underline{a}_1(\varepsilon), \overline{a}_1(\varepsilon))$, then* (PP) *has at least three positive solutions.*

Proof. Let $S(\xi, \varepsilon) = (s(\xi, \varepsilon), t(\xi, \varepsilon), a_1(\xi, \varepsilon))$ be the smooth curve obtained in Proposition 5.1. We recall that $S(\xi, 0) = (\xi, \varphi(\xi), \psi(\xi))$. Additionally it is possible to verify that $\psi'(0) > 0$ if $\tau \geq 0$ and

$$\lim_{\varepsilon \to 0} (t(\xi, \varepsilon), a_1(\xi, \varepsilon)) = (\varphi(\xi), \psi(\xi)) \text{ in } C^1([0, C]) \times C^1([0, C]),$$

where C is the positive constant defined in (5.1). Thus it follows from Lemma 4.1 that if $(\tau, \gamma) \in [0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$ and $\varepsilon > 0$ is small enough, then $\psi_\varepsilon(\xi) := a_1(\xi, \varepsilon)$ $(0 \leq \xi \leq C(\varepsilon))$ satisfies $\psi_\varepsilon'(0) > 0$, $\psi_\varepsilon(\xi) > \psi_\varepsilon(0) = a_1^*(\varepsilon)$ for all $\xi \in (0, C(\varepsilon)]$ and achieves a local minimum and a local maximum at some $\overline{\xi}(\varepsilon)$ and $\underline{\xi}(\varepsilon)$, respectively, which satisfy $\lim_{\varepsilon \to 0} \overline{\xi}(\varepsilon) = \overline{s}$ and $\lim_{\varepsilon \to 0} \underline{\xi}(\varepsilon) = \underline{s}$. Here, $\overline{s}$ and $\underline{s}$ are critical points of ψ obtained in Lemma 4.1. Define $\overline{a}_1(\varepsilon) := \psi_\varepsilon(\overline{\xi}(\varepsilon))$, $\underline{a}_1(\varepsilon) := \psi_\varepsilon(\underline{\xi}(\varepsilon))$ and

$$K_\varepsilon(a_1) := \{\xi \in (0, \infty) : \psi_\varepsilon(\xi) = a_1\}.$$

Obviously if $\varepsilon > 0$ is small enough, then $K_\varepsilon(a_1)$ has no element for $a_1 \in (0, a_1^*(\varepsilon)]$; at least one element for $a_1 \in (a_1^*(\varepsilon), \underline{a}_1(\varepsilon)) \cup (\overline{a}_1(\varepsilon), A_1]$; at least two elements for $a_1 = \underline{a}_1(\varepsilon)$ or $\overline{a}_1(\varepsilon)$; at least three elements for $a_1 \in (\underline{a}_1(\varepsilon), \overline{a}_1(\varepsilon))$. We observe that (5.2) implies that the number of elements of $K_\varepsilon(a_1)$ is equal to the number of positive solutions of (PP) provided $\varepsilon \in (0, \varepsilon_0]$ and $a_1 \in (0, A_1]$. Since the extension of Γ^ε implies that (PP) has at least one positive solution for $a_1 \in [A_1, \infty)$, we obtain the assertion. □

5.2 Case $\tau < 0$

For the case $\tau < 0$, let A_1 be a sufficiently large number. In this subsection, we will prove that all positive solutions (PP) with $a_1 \in [0, A_1]$ lie on a bounded curve near

$$\{(s\Phi, \varphi(\gamma s)\Phi, \psi(s)) : 0 < s < s_0/\gamma\}$$

if $\varepsilon >$ is sufficiently small:

Proposition 5.6. *Let $\tau < 0$. Then there exists a positive constant $\varepsilon_0 = \varepsilon_0(A_1)$ such that for each $\varepsilon \in (0, \varepsilon_0]$, all positive solutions of* (PP) *with $a_1 \in (0, A_1]$ are given by*

$$\begin{aligned} \Gamma^\varepsilon = \{(w, z, a_1) = ((s,t)\Phi + \varepsilon \boldsymbol{U}(s,t,a_1,\varepsilon), a_1) \ : \\ (s,t,a_1) \in \{S(\xi,\varepsilon) : 0 < \xi < C(\varepsilon)\}\}, \end{aligned} \tag{5.18}$$

where $S(\xi,\varepsilon) \in \boldsymbol{R}^3$ is a suitable smooth curve for $(\xi,\varepsilon) \in [0, C(\varepsilon)] \times [0, \varepsilon_0]$ satisfying

$$S(\xi, 0) = (\xi, \varphi(\xi), \psi(\xi)), S(0,\varepsilon) = (0, t(\varepsilon), a_1^*(\varepsilon)), S(C(\varepsilon),\varepsilon) = (s(\varepsilon), 0, a_{1*}(\varepsilon)).$$

Here, $t(\varepsilon) := \varepsilon^{-1} \int_\Omega \theta_{\lambda_1+\varepsilon b_1}\Phi$, $s(\varepsilon) := \varepsilon^{-1} \int_\Omega \theta_{\lambda_1+\varepsilon a_{1*}(\varepsilon)}\Phi$ and $C(\varepsilon)$ is a certain smooth function in $[0, \varepsilon_0]$ such that $C(0) = s_0/\gamma$.

It follows from (4.3) and (4.5) that if $\tau < 0$, then $\mathcal{L}_p$ intersects $\mathcal{L}_1$ and $\mathcal{L}_2$ at $(s_0/\gamma, 0, s_0\|\Phi\|_3^3/\gamma)$ and $(0, b_1/\|\Phi\|_3^3, cb_1)$, respectively. Even if $\tau < 0$, Lemma 5.2 remains valid; so that one can obtain the expression (5.3) for all nonnegative solutions of (3.5) near $(0, b_1/\|\Phi\|_3^3, cb_1)$. We can also express all nonnegative solutions of (3.5) near $(s_0/\gamma, 0, s_0\|\Phi\|_3^3/\gamma)$:

Lemma 5.7. *Let $\tau < 0$. Then there exist a positive number δ_e and a neighborhood U_e of $(s_0/\gamma, 0, s_0\|\Phi\|_3^3/\gamma)$ such that for each $\varepsilon \in [0, \delta_e]$*

$$\operatorname{Ker} F^\varepsilon \cap U_e \cap \overline{\boldsymbol{R}_+}^3 = \{\tilde{S}(\xi,\varepsilon) : \xi \in [0, \delta_e)\} \cup \left\{ \left(\frac{1}{\varepsilon} \int_\Omega \theta_{\lambda_1+\varepsilon a_1}\Phi, 0, a_1 \right) \in U_e \right\}$$

with a smooth curve $\tilde{S}(\xi,\varepsilon) \in \boldsymbol{R}^3$ $(0 \le \xi \le \delta_e)$ which satisfies

$$\tilde{S}(\xi, 0) = (s_0 - \xi/\gamma, \varphi(s_0 - \xi/\gamma), \psi(s_0/\gamma - \xi)) \quad \text{and} \quad \tilde{S}(0,\varepsilon) = (s(\varepsilon), 0, a_{1*}(\varepsilon)).$$

Recall that $(w, z, a_1) = (\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_{1*}(\varepsilon)}, 0, a_{1*}(\varepsilon))$ is a bifurcation point of positive solutions of (PP) on the semitrivial solution curve $\{(\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_1}, 0, a_1) : a_1 > 0\}$. So the proof of Lemma 5.7 can be carried out by the same argument as in the proof of Lemma 5.2.

Lemma 5.8. *Let $\tau < 0$. So there exists a neighborhood U of $\{(s\Phi, \varphi(\gamma s)\Phi, \psi(s)) : s \in [0, s_0/\gamma]\}$ such that, if $\varepsilon > 0$ is sufficiently small, then all positive solutions of* (PP) *contained in U are given by* (5.18).

Proof. Let δ_0 and δ_e be positive numbers in Lemmas 5.2 and 5.7, respectively. Put

$$\mathcal{L}_p([\delta_0/2, s_0/\gamma - \delta_e/2]) := \{(s, \varphi(\gamma s), \psi(s)) : s \in [\delta_0/2, s_0/\gamma - \delta_e/2]\}.$$

Hence $\mathcal{L}_p([\delta_0/2, s_0/\gamma - \delta_e/2])$ is a compact set and both $\mathcal{L}_p([\delta_0/2, s_0/\gamma - \delta_e/2]) \cap U_0$ and $\mathcal{L}_p([\delta_0/2, s_0/\gamma - \delta_e/2]) \cap U_e$ are not empty. Here U_0 and U_e are open sets obtained in Lemmas 5.2 and 5.7. Therefore, when $\varepsilon > 0$ is small enough, a similar procedure to the proof of Lemma 5.3 enables us to construct the solution curve of (3.5) in a neighborhood U' of $\mathcal{L}_p([\delta_0/2, s_0/\gamma - \delta_e/2])$. We note that both $U' \cap U_0$ and $U' \cap U_e$ are not empty. Therefore, together with Lemmas 5.2 and 5.7, we obtain the assertion. □

Lemma 5.9. *Let $\tau < 0$ and assume that $A_1 > 0$ is sufficient large. Let V be any neighborhood of $\{(s\Phi, \varphi(\gamma s)\Phi, \psi(s)) : s \in [0, s_0/\gamma]\}$. If $\varepsilon > 0$ is sufficiently small, then any positive solution of* (PP) *with $a_1 \in (0, A_1]$ can be expressed by*

$$(w, z, a_1) = (s, t)\Phi + \varepsilon \boldsymbol{U}(s, t, a_1, \varepsilon) \quad \textit{for some} \quad (s\Phi, t\Phi, a_1) \in V.$$

Proof. Let $\{(w_n, z_n)\}$ be any sequence of positive solutions of (PP) with $\varepsilon = \varepsilon_n \downarrow 0$ and $a_1 = a_1^n \in (0, A_1]$. It suffices to get a subsequence $\{(w_{n(k)}, z_{n(k)}, a_1^{n(k)}, \varepsilon_{n(k)})\}$ and a sequence $\{(s_k, t_k)\}$ satisfying (5.11) with C replaced by s_0/γ. The proof of this assertion is almost the same as that of Lemma 5.4. We have only to note that in case $\tau < 0$, (5.13) is replaced by $z_n \le 1 + (1 + \gamma M)^2 b_1 \Phi / \|\Phi\|_3^3$ in Ω. □

Proposition 5.6 follows from Lemmas 5.8 and 5.9. Furthermore, we can employ Lemma 4.1 to obtain the following corollary about the positive solution set of (PP) for the case when $(\tau, \gamma) \in [-\tilde{\tau}, 0) \times [\tilde{\gamma}, \infty)$ and $\varepsilon > 0$ is sufficiently small.

Corollary 5.10. *Suppose that $(\tau, \gamma) \in [-\tilde{\tau}, 0) \times [\tilde{\gamma}, \infty)$ and that $\varepsilon > 0$ is sufficiently small. Then the positive solution set of* (PP) *contains a bounded smooth curve*

$$\Gamma^{\varepsilon} = \{(w(\xi), z(\xi), a_1(\xi)) : \xi \in (0, C(\varepsilon))\},$$

which possesses the following properties;
(i) $(w(0), z(0), a_1(0)) = (0, \varepsilon^{-1}\theta_{\lambda_1 + \varepsilon b_1}, a_1^*(\varepsilon)), \quad a_1'(0) > 0$;
(ii) $(w(C(\varepsilon)), z(C(\varepsilon)), a_1(C(\varepsilon))) = (\varepsilon^{-1}\theta_{\lambda_1 + \varepsilon a_{1*}(\varepsilon)}, 0, a_{1*}(\varepsilon))$;
(iii) *$a_1(\xi)$ attains a strict local maximum in $(0, C(\varepsilon))$. In particular, if $(\tau, \gamma) \in [-\tilde{\tau}, 0) \times [\tilde{\gamma}, \hat{\gamma}(\tau))$, then $a_1(\xi)$ attains a strict local minimum in $(0, C(\varepsilon))$.*

The proof of Corollary 5.10 is essentially the same as that of Corollary 5.5.

6 Proofs of Main Results

Proof of Theorem 1.2. We begin with the case $\alpha = 0$. The bifurcation point of Γ^{ε} obtained in Proposition 5.1; $(w, z, a_1) = (0, \varepsilon^{-1}\theta_{\lambda_1 + \varepsilon b_1}, a_1^*(\varepsilon))$ is mapped by (3.1) to the bifurcation point $(U, V, a) = (0, \theta_b, \lambda_1(c\theta_b))$ on the semitrivial solution curve $\{(0, \theta_b, a) : a > 0\}$ of $(\mathrm{EP})_0$. In view of (3.1), we define

$$O_1^0 := \{(\beta, b, d) = (\gamma/\varepsilon, \lambda_1 + \varepsilon b_1, (\lambda_1 + \varepsilon\tau)\gamma/\varepsilon) : (\tau, \gamma) \in [0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)\}$$

for sufficiently small $\varepsilon > 0$. It follows from Corollary 5.5 that if $(\beta, b, d) \in O_1^0$ and $\varepsilon > 0$ is small enough, then the positive solution set of $(\mathrm{EP})_0$ contains an unbounded S-shaped curve $\Gamma_{(\mathrm{EP})_0}$ which bifurcates from the semitrivial solution curve $\{(0, \theta_b, a) : a > 0\}$ at $a = \lambda_1(c\theta_b)$. To be precise, if we let

$$\overline{a} = \lambda_1 + \varepsilon\overline{a}_1(\varepsilon) \quad \text{and} \quad \underline{a} = \lambda_1 + \varepsilon\underline{a}_1(\varepsilon),$$

then $(\mathrm{EP})_0$ has no positive solution for $a \in (0, \lambda_1(c\theta_b)]$; at least one positive solution for $a \in (\lambda_1(c\theta_b), \underline{a}) \cup (\overline{a}, \infty)$; at least two positive solutions for $a = \underline{a}$ or $\overline{a}$; at least three positive solutions for $a \in (\underline{a}, \overline{a})$. By virtue of the one-to-one correspondence between $(u, v) \geq 0$ and $(U, V) \geq 0$ through (2.1), define $\Gamma_1 = \{(u, v, a) : (U, V, a) \in \Gamma_{(\mathrm{EP})_0}\}$ for $\alpha = 0$ and $(\beta, b, d) \in O_1^0$. Hence Γ_1 is contained in the positive solution set of (SP) and possesses at least two turning points with respect to a. Therefore, Theorem 1.2 is proved for the special case $\alpha = 0$.

Next we will justify that the above S-shaped property of Γ_1 allows a small perturbation with respect to α. Define $F : X \times \boldsymbol{R}^3 \to Y$ by

$$F(U, V, a, \alpha, \beta) = (\Delta U + u(a - u - cv), \Delta V + v(b + dv - v))$$

to study (EP). Here $u = u(U, V, \alpha, \beta)$ and $v = v(U, V, \alpha, \beta)$ are given by (2.2) and (2.3), respectively. For any $(\beta, b, d) \in O_1^0$, let (U_0, V_0) be any positive solution of $(\mathrm{EP})_0$. We note that all positive solutions of $(\mathrm{EP})_0$ near $(0, \theta_b, a^*)$ are given by (2.9). By following the procedure by Du and Lou [7, Lemma 3.14], we can prove that, if $\|U_0\|_{W^{2,p}} \geq \delta/2$ for the positive number δ in (2.9), then $F_{(U,V)}(U_0, V_0, a, 0, \beta)$ is a Fredholm operator with index 0; so that the following (i) or (ii) holds true alternatively.

(i) $F_{(U,V)}(U_0, V_0, a, 0, \beta) : X \to Y$ is an isomorphism;

$$\text{(ii)} \quad \begin{cases} \dim \operatorname{Ker} F_{(U,V)}(U_0, V_0, a, 0, \beta) = \operatorname{codim} \operatorname{Range} F_{(U,V)}(U_0, V_0, a, 0, \beta) = 1, \\ F_a(U_0, V_0, a, 0, \beta) \notin \operatorname{Range} F_{(U,V)}(U_0, V_0, a, 0, \beta). \end{cases}$$

For sufficiently large A, denote by $\Gamma_{(\mathrm{EP})_0}|_{0<a\leq A}$ the restriction of $\Gamma_{(\mathrm{EP})_0}$ in the range $0 < a \leq A$. Thus, in the same way as the proof of Lemma 5.3, we can construct a positive solution curve $\Gamma_{(\mathrm{EP})}|_{0<a\leq A}$ of (EP) in a neighborhood W of $\Gamma_{(\mathrm{EP})_0}|_{0<a\leq A}$ if $\alpha > 0$ is sufficiently small. By taking account for the continuity of positive solutions of (EP) with respect to α, it can be verified that $\Gamma_{(\mathrm{EP})}|_{0<a\leq A}$ converges to $\Gamma_{(\mathrm{EP})_0}|_{0<a\leq A}$ in $C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times [0, A]$ as $\alpha \downarrow 0$. Furthermore, it is also possible to prove that if $0 < a \leq A$ and $\alpha > 0$ is small enough, then there is no positive solution of (EP) outside of W. With the aid of Lemma 2.2, we can extend $\Gamma_{(\mathrm{EP})}|_{0<a\leq A}$ to the range $a \in [A, \infty)$ as a positive solution curve of (EP) by applying the global bifurcation theorem ([18]). By virtue of (2.1), we can get an S-shaped positive solution curve of (SP) when $(\beta, b, d) \in O_1^0$ and $\alpha > 0$ is small. Thus the proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3. In view of Corollary 5.10, the proof of Theorem 1.3 can be carried out in a similar way. □

A Appendix

In this section, we will give some properties of S_1 and S_2 defined by (1.7) and (1.8).

Lemma A.1. *If $\beta\lambda_1 < d$ (resp. $\beta\lambda_1 > d$), then S_1 can be expressed as*

$$S_1 = \{(a, b) : a = f(b) \;\; \textit{for} \;\; b \leq \lambda_1 \; (\textit{resp.}\, b \geq \lambda_1)\},$$

where $f(\cdot)$ is a continuous function with respect to b in $(-\infty, \lambda_1]$ (resp. $[\lambda_1, \infty)$) and possesses the following properties:

(i) $f(\cdot)$ *is* $\begin{cases} \textit{strictly monotone decreasing if } \beta\lambda_1 < d, \\ \textit{strictly monotone increasing if } \beta\lambda_1 > d; \end{cases}$

(ii) $f(\lambda_1) = \lambda_1$;

(iii) $\lim_{b\to-\infty} f(b) = \infty$ *if* $\beta\lambda_1 < d$ *and* $\lim_{b\to\infty} f(b) = \infty$ *if* $\beta\lambda_1 > d$.

Proof. If we put

$$S(a,b) := \lambda_1\left(\frac{-b - d\theta_a}{1+\beta\theta_a}\right),$$

then

$$S(\lambda_1, b) = \lambda_1(-b) = \lambda_1 - b. \tag{A.1}$$

We note that for each compact set K in Ω

$$\lim_{a\to\infty} \frac{-b - d\theta_a}{1+\beta\theta_a} = -\frac{d}{\beta} \quad \text{uniformly in } K$$

(see, e.g., Dancer [4, Lemma 1]). Therefore, it can be seen that for any $b \in \boldsymbol{R}$

$$\lim_{a\to\infty} S(a,b) = \lambda_1\left(-\frac{d}{\beta}\right) = \lambda_1 - \frac{d}{\beta}. \tag{A.2}$$

Recall that both of the mappings $q \to \lambda_1(q) : C(\overline{\Omega}) \to \boldsymbol{R}$ and $a \to \theta_a : [\lambda_1, \infty) \to C(\overline{\Omega})$ are strictly increasing. Therefore, by virtue of

$$\frac{\partial}{\partial a}\left(\frac{-b - d\theta_a}{1+\beta\theta_a}\right) = \frac{\beta b - d}{(1+\beta\theta_a)^2}\frac{d\theta_a}{da} \quad \text{in } \Omega,$$

we know that

$$S(a,b) \text{ is } \begin{cases} \text{strictly decreasing with respect to } a \in (\lambda_1, \infty) \text{ if } \beta b < d, \\ \text{strictly increasing with respect to } a \in (\lambda_1, \infty) \text{ if } \beta b > d. \end{cases} \tag{A.3}$$

Suppose $\beta\lambda_1 < d$. By virtue of (A.1) and (A.2), $S(\lambda_1, b_0) \le 0$ (resp. $S(\lambda_1, b_0) > 0$) if $b_0 \ge \lambda_1$ (resp. $b_0 < \lambda_1$) and $\lim_{a\to\infty} S(a, b_0) < 0$. Since $S(a, b_0)$ is a monotone function with respect to a by (A.3), we see that, if $b_0 \ge \lambda_1$, then $S(a, b_0) < 0$ for all $a \in (\lambda_1, \infty)$. On the other hand, if $b_0 < \lambda_1$, then the intermediate theorem gives a unique $a_0 \in (\lambda_1, \infty)$ such that $S(a_0, b_0) = 0$. Furthermore, it follows from (A.3) that $S_a(a_0, b_0) < 0$. Therefore, by the implicit function theorem, there exists a smooth function $a = f(b)$ such that

$$\begin{cases} f(b_0) = a_0, \\ S(f(b), b) = 0 \quad \text{for all} \quad b \in [b_0 - \delta, b_0 + \delta] \end{cases}$$

with some $\delta > 0$. Since $b_0 \in (-\infty, \lambda_1)$ can be taken arbitrarily, we deduce that there exists a unique smooth function $a = f(b)$ such that

$$S(f(b), b) = 0 \quad \text{for} \quad b \in (-\infty, \lambda_1). \tag{A.4}$$

Differentiation of (A.4) with respect to b implies

$$S_a(f(b), b) f'(b) + S_b(f(b), b) = 0.$$

Note that $S_b(a, b) = -\lambda_1'((-b - d\theta_a)/(1 + \beta\theta_a)) < 0$. Thus from (A.3) one can see $f'(b) < 0$ for $b \in (-\infty, \lambda_1)$. It is easy to see $f(\lambda_1) = \lambda_1$. Furthermore, we can show $\lim_{b \to -\infty} f(b) = \infty$. Indeed, if $\lim_{b \to -\infty} f(b) < \infty$,

$$\lim_{b \to -\infty} S(f(b), b) = \lim_{b \to -\infty} \lambda_1 \left(\frac{-b - d\theta_{f(b)}}{1 + \beta\theta_{f(b)}} \right) = +\infty,$$

which obviously contradicts (A.4). Thus the proof for $\beta b < \beta\lambda_1 < d$ is complete. For the case $\beta b > \beta\lambda_1 > d$, a similar argument is valid to get the conclusion. □

The following corollary immediately follows from Lemma A.1

Corollary A.2. *For each b satisfying $\beta b < \beta\lambda_1 < d$ or $\beta b > \beta\lambda_1 > d$, there exists a unique $a = a_* \in (\lambda_1, \infty)$ such that*

$$\lambda_1 \left(\frac{-b - d\theta_{a_*}}{1 + \beta\theta_{a_*}} \right) = 0.$$

If $b > \lambda_1$ and $d \geq \beta\lambda_1$, then

$$\lambda_1 \left(\frac{-b - d\theta_a}{1 + \beta\theta_a} \right) < 0 \quad \textit{for all} \quad a \in (\lambda_1, \infty).$$

We can also give an analogous property for S_2:

Lemma A.3. *If S_2 is defined by (1.8), it can be expressed as*

$$S_2 = \{(a, b) \, : \, b = g(a) \quad \text{for} \quad a \geq \lambda_1\},$$

where $g(\cdot)$ is a continuous function for $a \in [\lambda_1, \infty)$ and has the following properties:
(i) *$g(\cdot)$ is strictly monotone increasing;*
(ii) $g(\lambda_1) = \lambda_1$;
(iii) $\lim_{a \to \infty} g(a) = \infty$.

The proof of Lemma A.3 is essentially the same as Lemma A.1; so we omit it. The following corollary immediately follows from Lemma A.3.

Corollary A.4. *For each $b > \lambda_1$, then there exists a unique $a = a^* \in (\lambda_1, \infty)$ such that*

$$\lambda_1\left(\frac{c\theta_b - a^*}{1+\alpha\theta_b}\right) = 0.$$

We can give more information about S_1 and S_2 near (λ_1, λ_1).

Lemma A.5. (i) *The function $f(\cdot)$ defined in Lemma A.1 satisfies*

$$f(b) = \lambda_1 + \frac{1}{\beta\lambda_1 - d}(b-\lambda_1) + o(b-\lambda_1) \quad \text{near} \quad \lambda_1.$$

(ii) *The function $g(\cdot)$ defined in Lemma A.3 satisfies*

$$g(a) = \lambda_1 + \frac{1}{\alpha\lambda_1 + c}(a-\lambda_1) + o(a-\lambda_1) \quad \text{near} \quad \lambda_1.$$

The proof is based on the local bifurcation analysis and is accomplished in the same manner as [21, Lemmas 3.4 and 3.6].

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研究分担者　大谷光春の報告

研究成果の概要

1) 従来の方法では得られなかった、準線形放物型方程式の解の高い微分可能性を保証する「L^∞-エネルギー法」を開発した。この方法により、まず、充分一般的な二重非線形放物型方程式のリプシッツ連続な時間局所解の存在が示され (1996, 2002)、さらには、1950 年代以来、未解決であった「Porous Medium 方程式は C^∞-級の時間局所解を許すか？」という問題が肯定的に解決されるという重要な知見が得られた (2001)。「L^∞-エネルギー法」は、これらの成果のみならず、いろいろな局面で応用可能な極めて有用な解析手段を与えていることを、現在進行中の研究が示唆している。

2) 「劣微分作用素の非単調摂動理論」が、バナッハ空間上の枠組みへ拡大された。これにより、従来 ガレルキン法で構成されていた退化放物型方程式の解の存在と正則性がより自然な枠組みで、より一般的な条件のもとで、議論できるようになり、いくつかの具体的な方程式に対して、従来の方法では解決できなかった未解決問題が解決された。

3) 部分対称性を有する Concentration Compactness 理論を構築した。コンパクト性の欠如した問題を解析する有力な方法として、Concentration Compactness 理論が知られているが、一方で球対称性などの高い対称性がある場合には、コンパクト性が回復することが知られている。コンパクト性が回復しない程度の部分的対称性が存在する場合に、Concentration Compactness 理論がどのように、その部分対称性を反映するかを解明した。この応用として、無限柱状領域において、臨界指数を越える非線形性をもつ楕円型方程式の非自明解の存在が示された。

4) 「ある条件のもとでは、対称性をもつ部分空間での臨界点が、全体での臨界点を与える」という R.Palais による対称臨界性原理は、本来変分構造をもつ楕円型方程式に限られた理論であった。この理論が、必ずしも変分構造をもたない楕円型方程式や時間発展を含む発展方程式へ適用可能な一般的な理論に拡張された。これにより、従来の理論では不可能であった放物型方程式や波動方程式への応用の道が開かれた。

5) 劣微分作用素を含む多価写像に対する写像度の理論が構築された。これにより、従来ではカバーできなかった、種々の多価性をもつ非線形偏微分方程式への写像度の理論が適用可能になった。

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研究分担者　田中和永の報告

研究成果の概要

変分的手法により非線型微分方程式の解の存在問題に関する研究を行い，主として非線型楕円型方程式，ハミルトン系を扱った．ここでは非線型楕円型方程式に対する特異摂動問題に限定して述べる．

方程式が空間変数 x に依存する場合，空間に関する非一様性の解構造への影響を研究した．まず1次元では中島, del Pino, Felmer, Jeanjean, Ding らと共に非常に複雑なプロファイルをもつ解の構成を Allen-Cahn 方程式, スカラーフィールド方程式に対して行った．より高次元では, “$\mathbf{R}^N$ における least energy solution がいつ Mountain Pass Theorem により特徴づけられるか?” という疑問から研究を始め, Jeanjean 氏らとの共同研究により, 非常に一般的な仮定の下で, 特徴付けが可能であることが示された．そしてその応用として空間変数 x に依存性をもつ非線型楕円型方程式の正値解の存在および特異摂動問題の仮定の下での解の存在結果を得ることが可能となった．

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研究分担者　四ツ谷晶二の報告

研究成果の概要

さまざまな非線形楕円型方程式の解の存在・一意性などに関する理論的な研究を行ってきた．具体的には、以下のような研究である。

＊　流体現象を説明するのに大変有効な、Oseen の螺旋流を分岐理論の立場から完全に解明した。

＊　数理生態学において，1979 年に Shigesada - Kawasaki - Teramoto によって提案された cross-diffusion 方程式の定常極限方程式の解構造の全容を明らかにした。

＊　半線形楕円型方程式の正値球対称解に関しては，いろんな領域，境界条件のもとで，個別にいろいろな結果が得られているが，一つの標準形に帰着でき，一見複雑にみえる様々な境界条件も，Dirichlet, Neumann, Robin 境界条件と解釈できることを発見した。　この結果の応用として、Brezis-Nirenberg 型の半線形楕円型方程式、スカラー曲率方程式、 ball の内部・外部領域における Dirichlet 問題、をはじめとするこれまで未解明であった解の全体構造を明らかにした。

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各論文の概要

[1] の概要

スカラー曲率方程式において，スカラー曲率にわずかな摂動を加えても解の構造が激変すること

が 1985年に Ding-Ni により示されている．しかしながら，その理由についてはよくわからなかった．本論文において，非線形項の指数 p をもパラメータとみることにより解の構造は連続的に変化していることを示した．指数を $p=(n+2)/(n-2)$ に固定してみると解の構造が不連続に激変して見える訳である．

[2] の概要
半線形楕円型方程式の正値球対称解に関しては，いろんな領域，境界条件のもとで，個別にいろいろな結果が得られているが，一つの標準形に帰着でき，一見複雑にみえる様々な境界条件も，Dirichlet, Neumann, Robin 境界条件と解釈できることを紹介した．この結果を Yanagida-Yotsutani の全空間での解の構造定理と結びつけることにより，従来のほとんど未解明であった，半線形楕円型方程式の（原点と無限遠方で特異性をもつ）正値球対称特異解の解の構造を明らかにできる．その基本となるアイデアを例を用いて解説した．

[3] の概要
半線形楕円型方程式の正値球対称解に関しては，ここ 20 年間いろんな領域，境界条件のもとで，個別にいろいろな結果が得られている．方程式や境界条件がわずかにかわるだけで，全く別の計算を必要とすることがしばしばである．本論文では，球対称半線形楕円型方程式の一つの標準形に帰着でき，一見複雑にみえる様々な境界条件も，Dirichlet, Neumann, Robin 境界条件と解釈できることを示した．この結果を Yanagida-Yotsutani の全空間での解の構造定理と結びつけることにより，従来の未知であった解の構造を明らかにできると同時に，解の構造の統一的な理解が可能となった．

[4] の概要
従来，Brezis-Nirenberg 型の半線形楕円型方程式のなかで，最も基本的な境界条件である Dirichlet の場合でさえ正値解の構造はよく分かっていなかった．さらに，境界条件を Dirichlet から Neumann, Robin と連続的に動かしていくとき解の構造がどのように変化していくかという大域的な構造の問題は全く未解明であった．この論文において，解の構造が連続的に変化していくことと，構造の詳細な変化の様子を数値計算により明らかにし，それらの数学的証明を与えた．証明には，論文 [3] で提案した標準形を考え方が本質的な役割をはたしている．

[5] の概要
論文 [3] で提案した標準形を実際に応用しやすくするため，具体的に，標準形の方程式に対する解の構造定理を示した．この応用例のひとつとして，従来全く解明であった，全空間での共形スカラー曲率方程式の，原点と無限遠方で特異性をもつ解の全体構造を明らかにした．

[6] の概要
論文 [3] で提案した標準形を用いることにより，従来未解明であった，半線形楕円型方程式の ball の内部領域および外部領域における Dirichlet 問題の解の全体構造を明らかにすることができることを示した．

[7] の概要
数理生態学において，1979年に Shigesada-Kawasaki-Teramoto によって提案された cross-diffusion 方程式に関する研究である．1996年に Lou-Ni は，cross-diffusion をあらわすパラメータを無限大にしてたときの極限方程式を導出した．これは，未知定数を含む積分制約条件付きの半線形楕円型方程式である．従来，この方程式の解の構造はほとんど分かっていなかった．未知定数と積分制約条件をいかに処理すればよいか分からなかったからである．本論文では，積分制約条件を無視し，候補となる解をすべて間接的ながらできるだけ詳細に表示し，その中から解を求めるという，全く新しい方法を提案し，それを実行した．その結果，空間 1 次元の場合には，ほぼ完全な解答を与えた．様々なパラメータを動かしたとき，解の構造がどのようであり，どのように変化していくかという大域的な構造を明らかにした．なお，ここで発見された方法は，流体力学等さまざまな分野に現れる方程式

に有効な手段であるほとが分かってきている.

[8] の概要

この論文は，従来解決が困難と思われていた流体力学の基本的かつ古典的な問題に対して，全く画期的な方法により完全解決を与えたものである．1927年，流体力学者 Oseen (オセーン) は流体の運動を記述するナビエ・ストークス方程式の具体的な解を求めることを提案し，現在，Oseen の螺旋流と呼ばれている，流体現象を説明するのに大変有効なさまざまな解を発見した．1998年，岡本久教授（京大数理研）が Oseen の螺旋流の統一的な理解の重要性を認識し，そのための方程式を提唱し，数値計算により全体像に対する示唆を与えた．多くの研究者が，その数学的な証明を試みたが果たせずにきていた．しかし，この論文おいて，古典的な楕円関数論と計算機の数式処理能力を有効に融合し，この問題に完全解決を与えた.

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研究分担者　菱田俊明の報告

研究成果の概要

(1) aperture domain において, Navier-Stokes 方程式を考察した。aperture における flux の総量を指定する問題に対して, Stokes 半群の L^p-L^q 評価を証明し, それを応用して, 初期関数の L^n ノルムが小さい場合の Navier-Stokes 方程式の時間大域解の存在と漸近挙動を明らかにした ([4],[5])。

(2) 回転する物体の外部において, Navier-Stokes 方程式を考察した。回転角速度が時間により変化する問題に対して, 附随する発展作用素の平滑化効果を用いて, 時間局所解を構成した ([3])。また, 角速度一定な問題において現れる線型偏微分作用素の L^q 評価を示した (Farwig-Hishida-Müller, preprint)。

(3) Navier-Stokes 方程式の弱解が $L^\infty(0,T;L^n)$ に属するときの正則性を考察した。正則性を得るための付加条件を従来より弱いものに改良し, 全く新しい見方による簡単な証明を与えた ([2])。

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研究分担者　廣瀬宗光の報告

研究成果の概要

ボース・アインシュタイン凝縮と呼ばれる物理現象を記述する非線形シュレディンガー方程式について、その定常波解を考えることにより「調和ポテンシャル項を含む半線形楕円型方程式」が得られる。定常波解の安定性を議論するために、この楕円型方程式のあるクラスにおける正値解の存在と一意性を調べることは極めて重要である。変分法を用いて、正値解の存在を示すことは比較的容易であるが、一意性は微妙な問題である。この研究では、変分法により得られる正値解が原点について球対称であることを示した上で、この楕円型方程式の「正値球対称解集合の構造」を調べることを試みた。いわゆる解構造分類定理を用いることにより、無限遠方で急速に減衰する正値球対称解の一意性を示すことに成功し、その結果、この楕円型方程式の正値解の一意性が結論された。さらに無限遠方での解の減衰オーダーも得られ、正値解の大域的な振る舞いについても明らかにすることが出来た。

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研究分担者　中島主恵の報告

研究成果の概要

本研究では数理生態学に現れる競争系とよばれる連立系について考える．この方程式系は同じ領域内で相争って生息する２種の生物の個体数密度を記述したものである．２種の生物の競争が比較的激しい場合には，競争系はいわゆる双安定型のシステムになる．双安定型の反応拡散方程式系において，拡散係数を非常に小さくすると，“内部遷移層”をもつ解が現れる．内部遷移層とは，空間内のある曲面を境に，解の値がほとんど不連続にみえるほど急激に変化している部分のことである．拡散係数を 0 に近づけた特異極限下では，この内部遷移層は厚さが 0 の曲面“界面”に収束し，もとの非線型拡散方程式系に対する解析はこの界面の挙動を記述する方程式の解析に帰着される．本研究では競争系の界面方程式を導出し，解が形成した遷移層の動きが導出された界面方程式に支配されることを数学的に厳密に証明することができた．

特異極限における界面方程式を厳密に導出する作業は，通例，優解劣解を構成する手法を用いる．本研究もその手法に従っている．ただ，この方法の難点は優解と劣解の間隔が拡散係数を０に近づけると共に急速に狭くなっていき，そのため優解劣解ではさまれる初期値のクラスが非常に制限されてしまう点にある．本研究では，非常に広いクラスの初期値から出発した解が，きわめて短時間の間に遷移層を形成することを証明し（generation of interface），その結果，解が優解劣解に挟まれる狭い領域に入ることを示すことで上記の困難を克服する．

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研究分担者　竹内慎吾の報告

研究成果の概要

退化放物型方程式のうち，とりわけ p ーラプラス作用素を拡散項としてもったロジスティック方程式（以下，ｐＬとよぶ）の漸近挙動や定常状態を研究してきた．$p=2$ の場合，数理生態学，超伝導理論，界面ダイナミクスの理論などにおいて重要な半線型方程式となり，このときの解集合の構造の研究は，従来の常微分方程式に対する（有限次元）力学系の理論が無限次元にも適用できることを明確に示した例として純粋数学的な対象としても頻繁に引用されている．

ｐＬについて漸近挙動の研究では，解のプロファイルの時間的変化を詳しく考察し，定常状態に落ち着く過程や定常解の安定性について有益な情報を得ることができた．具体的には，解の値が永遠に変化しないような領域の発見がある．このことは定常解のある部分が平坦になっていることを示唆しており，半線型（$p=2$）には見られない興味深い現象である．また定常状態の解析では，そのような平坦な部分は実際に存在し，拡散係数が小さいほど現れやすいこと，またその結果としてある定常解のいくらでも近くに別の定常解が存在する（定常解の孤立性の崩れ）ことが確認された．ｐＬの研究を通じて退化方程式独特の現象が浮き彫りになり，これまでの半線型方程式の研究にはない新しい研究対象が見出されたことになる．そしてこの研究は半線型のそれとは無関係ではなく，現在わが国で活発に行われている反応拡散系の自由境界問題や遷移層に関する研究とも深く関わっている．

さらにロジスティック方程式として自然な結果である大域的アトラクターの存在を示した．定常解の孤立性の崩れからこの次元は無限大であることが予想されるがまだ未解決である．また最近では，ｐＬの係数を複素数にした（退化）複素ギンツブルグ・ランダウ方程式の研究も行っている．この方程式は見た目はロジスティック方程式と同じ形だが，係数の如何によっては爆発解の存在も否定できない．それゆえ私が行ってきた実係数の場合の研究結果と比較することは有益であると思う．現在論文を準備中である．

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研究分担者　久藤衡介の報告

研究成果の概要

反応拡散方程式に対する時間発展問題、および関連する時間定常問題の解構造の究明を、反応項が滑らかでないケースを中心に取り組んだ。「滑らかさを欠く反応項を伴う拡散方程式」は、数学的に「滑らかな反応項を伴う拡散方程式」と比べて解析が著しく困難になるケースが多く、未解決問題を多く残している。滑らかでない反応項の典型例として、反応項が劣線型項と優線型項の和で表されるケースを研究し、具体的成果として、次の(1)～(3)を得た。

(1) 空間1次元のケースにおける、定常解構造を解明。

(2) 領域が多次元における球のときの、正値定常解の構造を解明。

(3) 領域が上記(1)、(2)と同様のときの、正値非定常解の時間的挙動の解析。

上記(2)に関しては、楕円型境界値問題の正値解の個数を球領域の半径の大きさに応じて特定することに成功している。さらに符号変化を伴う球対称解の構造についても言及している。一方で(3)に関しては、放物型初期値境界値問題の正値解の時間的挙動（解の爆発、あるいは大域的存在）が、楕円型境界値問題の正値解の解の有無に密接に関係することを証明している。さらに正値定常解の安定性の解析も併せて進めた。

また、被食生物(プレイ)と捕食生物(プレデター)の個体数密度のダイナミクスを記述する反応拡散方程式系の研究にも従事した。この反応拡散方程式系は、プレイおよびプレデターに対応するそれぞれの生物の個体数に対する空間的拡散が、自種のみならず多種の個体数にも依存するケース（相互拡散）をモデル化したものである。この研究の具体的成果としては、次の(4)～(6)が挙げられる。

(4) 正値定常解（関連する非線型楕円型方程式の正値解）が複数個存在すること証明した。

(5) ホップ分岐現象による時間周期解の存在を証明した。

(6) 正値定常解の漸近安定性の判定に成功した。

(4)と(5)は、相互拡散のケースでのみ起こりうる数理現象を示したことに意義がある。この結果は、解のもつ時空的なダイナミクス（被食生物と捕食生物の個体数密度のダイナミクス）が、相互拡散効果の有無によって本質的に異なることを示唆する。(6)については、局所安定性の判定にとどまるが、相互拡散系に対する定常解の安定性解析は目新しいものである。

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