Classification of polarized manifolds admitting a low degree cover of projective space among their hyperplane sections

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Introduction

By a *polarized manifold* we mean a pair (X,L) consisting of a smooth complex projective variety X and an ample line bundle L on it. If L is very ample, then a member of the complete linear series $|L|$ is nothing but a hyperplane section of X embedded by the morphism associated to $|L|$.

In studies on the structures of polarized manifolds (X,L), it has been considered that the nature of an ample divisor $A \in |L|$ strongly influences that of the ambient space X. Based on this philosophy, several kind of classification problems of polarized manifolds whose ample divisors have special properties have been considered by several authors (e.g. [Se], [SV], [Fa], [LPS 3], [BFS]).

In this thesis, we first study the case where a hyperplane section has a finite map of low degree onto $\mathbb{P}^n$, as indicated in the thesis subject. Next we investigate the case where an ample divisor has a maximal sectional genus with respect to its Δ-genus and degree; A variety with this property is said to be a Castelnuovo variety though explained later.

This thesis consists of three chapters. In Chapters 1 and 2, we shall deal with the former case, and classify the polarized manifolds (X,L) with very ample line bundles L in the cases where the degrees of the finite map are 4 and 5. In Chapter 3, we shall treat the latter case. There we raise a classification problem of (X,L) whose ample divisors A are Castelnuovo manifolds, and provide a classification of those polarized manifolds in the case where the degree of A is smaller than its dimension.

In Chapters 1 and 2, we consider the following

Problem 1 (A. Lanteri–M. Palleschi–A. J. Sommese [LPS 1]) *Let (X,L) be a polarized manifold such that its line bundle L is very ample. Fix an integer $d \geq 2$. Assume that there exists a smooth member $A \in |L|$ such that there exists a finite morphism $\pi\colon A \to \mathbb{P}^n$ of degree d. Then classify (X,L).*

The beginning of this problem goes back to the studies on the structures of projective surfaces by the Italian school in the late 19th century. In fact, this problem originated in G. Castelnuovo's work [Ca2] on the classification problem of projective surfaces admitting a hyperelliptic curve among their hyperplane sections. In 1980s, the revisions of Castelnuovo's result were made by F. Serrano [Se] and Sommese–A. Van de Ven [SV].

In 1994, Lanteri–Palleschi–Sommese (LPS, for short) were inspired by the revisions, and they raised and solved Problem 1 in the cases where $d = 2$ and $n > d = 3$ ([LPS 1], [LPS 2]). For the cases of $n > d = 4$ and 5, there was an attempt by Lanteri [Lan] to classify (X,L). After my paper [A2] was submitted, I was informed about Lanteri's paper by the referee. But his results, obtained by using freely the method of LPS, give an only partial answer to the classification problems for $d = 4$ and 5 cases because they contain doubtful cases.

Here the method of LPS is the way to determine the structure of (X,L) by using the Δ-genus theory, i.e. T. Fujita's classification theory of polarized manifolds (e.g. [Fu 5, Chapter I]), for $(X,\mathscr{H})$ after investigating the possible values of the following three invariants: the degree $d(X,\mathscr{H})$, the Δ-genus $\Delta(X,\mathscr{H})$, and the sectional genus $g(X,\mathscr{H})$, where $\mathscr{H}$ is a line bundle on X such that $\mathscr{H}|_A \cong \pi^*\mathscr{O}_{\mathbb{P}}(1)$. For the cases of $d \leq 3$, the Δ-genus theory applies well in classifying (X,L) since the possible values of the three invariants turn out to be small. However, in the cases of $d \geq 4$, the situation is rather complicated because the range of the possible values of their invariants go beyond the applicable one of the Δ-genus theory (e.g. $\Delta(X,\mathscr{H}) = d(X,\mathscr{H}) = 1$ and $g(X,\mathscr{H}) \geq 3$). The parts that the Δ-genus theory does not apply have remained unsettled in Lanteri's results [Lan, Theorems 3.4 & 3.5] for $d = 4$ and 5 cases.

In Chapter 1, we discuss the case of $n > d = 5$, and provide a complete classification of (X,L) by resolving the unsettled parts positively.

Theorem 1.1.1 *Let X be a smooth complex projective variety with* $\dim X = n+1 \geq 7$. *Then the following* (I) *and* (II) *are equivalent.*

(I) *There exists a very ample line bundle on X, L, such that $|L|$ contains a smooth member A endowed with a finite morphism $\pi\colon A \to \mathbb{P}^n$ of degree* 5.

(II) (X,L) *is one of the following*:

(i) $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}}(5))$;

(ii) $(H_5^{n+1}, \mathscr{O}_{H_5}(1))$, *where* H_5^{n+1} *is a hypersurface of degree* 5 *in* $\mathbb{P}^{n+2}$;

(iii) $(Y_1, 5\mathscr{L})$, *where* $(Y_1, \mathscr{L})$ *is a del Pezzo manifold of degree one*;

(iv) $(W_{10}, \mathscr{O}_{W_{10}}(5))$, *where* W_{10} *is a weighted hypersurface of degree* 10 *in the weighted projective space* $\mathbb{P}(5,2,1^{n+1})$; *or*

(v) $(W_{20}, \mathscr{O}_{W_{20}}(5))$, *where* W_{20} *is a weighted hypersurface of degree* 20 *in* $\mathbb{P}(5,4,1^{n+1})$.

It turns out that the pairs (iv) and (v) newly show up by comparing our result for $d=5$ to the results for $d \leq 3$ by LPS. Moreover the existences for those new pairs are verified (see Section 1.3).

One of the key ingredients of our proof in the degree 5 case is to describe the structure of the polarized manifold in question, $(X, \mathscr{H})$, which is of $\Delta(X, \mathscr{H}) = d(X, \mathscr{H}) = 1$ and $g(X, \mathscr{H}) = 6$ (Theorem 1.6.2). Although the polarized manifolds with these invariants are yet to be classified, we can successfully determine the structure of $(X, \mathscr{H})$ in question by focusing attention on certain ring-theoretic properties of the graded ring

$$R(X, \mathscr{H}) := \bigoplus_{i=0}^{\infty} H^0(X, i\mathscr{H}).$$

We use the ladder method to find the generators of $R(X, \mathscr{H})$ and the relations among them. By using the Riemann–Roch theorem for a smooth curve $X_1 \subset X$ cut out by $|\mathscr{H}|$, we first describe the structure of $R(X_1, \mathscr{H}|_{X_1})$ in terms of generators and relations. After that, we prove that $R(X_2, \mathscr{H}|_{X_2})$ is a Cohen–Macaulay ring, where X_2 is a smooth surface cut out by $|\mathscr{H}|$. It enables us to use a vanishing theorem for $H^1(l\mathscr{H}|_{X_2})$. Consequently it is lead that (X, L) coincides with a weighted hypersurface of type (v).

In Chapter 2, we classify (X, L) completely in the case of $n > d = 4$. Indeed, we obtain the following

Theorem 2.1.1 *Let* X *be a smooth projective variety with* $\dim X = n+1 \geq 6$. *Then the following* (I) *and* (II) *are equivalent.*

(I) *There exists a very ample line bundle on* X, L, *such that* $|L|$ *contains a smooth member* A *endowed with a finite morphism* $\pi\colon A \to \mathbb{P}^n$ *of degree* 4.

(II) (X, L) *is one of the following*:

(i) $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}}(4))$;

(ii) $(H_4^{n+1}, \mathscr{O}_{H_4}(1))$;

(iii) $(Y_1, 4\mathscr{L})$;

(iv) $(W_{12}, \mathscr{O}_{W_{12}}(4))$, *where* W_{12} *is a weighted hypersurface of degree* 12 *in* $\mathbb{P}(4,3,1^{n+1})$;

(v) $(\mathbb{Q}^{n+1}, \mathscr{O}_{\mathbb{Q}}(2))$, *where* $\mathbb{Q}^{n+1}$ *is a hyperquadric in* $\mathbb{P}^{n+2}$;

(vi) $(V_{2,2}^{n+1}, \mathscr{O}_{V_{2,2}}(1))$, *where* $V_{2,2}^{n+1}$ *is a complete intersection of two hyperquadrics in* $\mathbb{P}^{n+3}$; *or*

(vii) $(Y_2, 2\mathfrak{L})$, *where* $(Y_2, \mathfrak{L})$ *is a del Pezzo manifold of degree* 2.

Compared to Theorem 1.1.1, Theorem 2.1.1 includes new pairs (v)–(vii). The existences of those new pairs are immediately verified.

Note that a complexity is caused by the compositeness of the degree d of the finite map π. In the case where d is prime, the birationality of A and $q(A)$ plays a key role to classify (X, L), where q is the morphism associated to $|\pi^*\mathscr{O}_{\mathbb{P}^n}(1)|$. On the other hand, in the case where d is composite, by the diagram below ($t := \dim|\pi^*\mathscr{O}_{\mathbb{P}^n}(1)| - n$), we immediately see that the birationality of those varieties does *not* always hold, which complicates the analyses of (X, L).

$$\begin{array}{ccl} A & \xrightarrow{\;q\;} & q(A) \subset \mathbb{P}^{n+t} \\ & \searrow^{\pi} & \Big\downarrow p\colon \text{ The projection from a } \mathbb{P}^{t-1} \text{ in } \mathbb{P}^{n+t} \text{ with } q(A) \cap \mathbb{P}^{t-1} = \emptyset. \\ & & \mathbb{P}^n \end{array}$$

In Theorem 2.1.1 (vii), the case where A is not birational to $q(A)$ really occurs.

We lead Theorem 2.1.1 by using a technique different from that in Theorem 1.1.1 although we use the Δ-genus theory in some parts. In fact, the key ingredient of our proof in the degree 4 case is to show the nonexistence of the polarized manifolds $(X, \mathscr{H})$ with $\Delta(X, \mathscr{H}) = d(X, \mathscr{H}) = 2, g(X, \mathscr{H}) = 3$ and with $L \cong 2\mathscr{H}$ (Proposition 2.3.2). Although the polarized manifolds with these invariants are yet to be classified, we can lead a consequence that contradicts the very ampleness of L by using the double point formula for a surface cut out by $|\mathscr{H}|$, therefore we successfully show the nonexistence of those $(X, \mathscr{H})$.

In Chapter 3, we treat the classification problem of polarized manifolds $(X, \mathscr{L})$ such that $|\mathscr{L}|$ contains a Castelnuovo manifold A. In 1990, for a polarized variety

$(A, \mathscr{H})$ with a birationally very ample line bundle $\mathscr{H}$, Fujita [Fu 5, (16.3)] gave an upper bound for the sectional genus $g(A, \mathscr{H})$ in terms of its Δ-genus and degree. Moreover he called a polarized variety $(A, \mathscr{H})$ whose sectional genus attains the maximum a *Castelnuovo variety* after Castelnuovo's study [Ca1] on curves of maximal genus.

We first pose the following

Problem 2 *Let $(X, \mathscr{L})$ be a polarized manifold of dimension $n+1$. Assume that $|\mathscr{L}|$ contains a member A such that $(A, \mathscr{H})$ is a Castelnuovo manifold with some line bundle $\mathscr{H} \in \mathrm{Pic}(A)$. Then classify $(X, \mathscr{L})$.*

The structures of Castelnuovo manifolds themselves have been studied by Fujita [Fu 5, §16] and S. Mukai [Mu]. However, to the best of my knowledge, it seems that the problem is raised for the first time.

The main result of this chapter is to give an answer to this problem in the case of $d(A, \mathscr{H}) < n$. Precisely speaking we have the following

Theorem 3.1.1 *Let X be a smooth complex projective variety of dimension $n+1$. Assume that $0 < d < n$. Then the following* (I)–(III) *are equivalent*:

(I) *There exists an ample line bundle on X, $\mathscr{L}$, such that $|\mathscr{L}|$ contains a member A such that $(A, \mathscr{H})$ is a Castelnuovo manifold of degree d with some $\mathscr{H} \in \mathrm{Pic}(A)$.*

(II) *There exists an very ample line bundle on X, $\mathscr{L}$, such that $|\mathscr{L}|$ contains a member A such that $(A, \mathscr{H})$ is a Castelnuovo manifold of degree d with some $\mathscr{H} \in \mathrm{Pic}(A)$.*

(III) $(X, \mathscr{L})$ *is one of the following*:

  (i) $(W_d, \mathscr{O}_W(l))$ *with some positive integer l dividing d, where W_d is a weighted hypersurface of degree d in $\mathbb{P}(l, 1^{n+2})$;*

  (ii) $(W_{2,d/2}, \mathscr{O}_W(l))$ *with $l = 1, 2$ or l dividing $d/2$, where the given d is an even number ≥ 4, and $W_{2,d/2}$ is a weighted complete intersection of type $(2, d/2)$ in $\mathbb{P}(l, 1^{n+3})$; or*

  (iii) $(W_{2,2,2}, \mathscr{O}_W(l))$ *with $l = 1$ or 2, where $W_{2,2,2}$ is a weighted complete intersection of type $(2,2,2)$ in $\mathbb{P}(l, 1^{n+4})$.*

Moreover, for each of the list (i)–(iii), $\mathscr{L}|_A \cong \mathscr{H}$ *holds if and only if* $l = 1$.

Note that if $(X, \mathscr{L})$ is a Castelnuovo manifold then so is $(A, \mathscr{L}|_A)$. In this case, $\mathscr{L}|_A \cong \mathscr{H}$ holds. Meanwhile, Theorem 3.1.1 indicates that polarized manifolds $(X, \mathscr{L})$ which are non-Castelnuovo manifolds do appear. In fact, the existences of (i)–(iii) with $l \neq 1$ are verified.

Besides, by this theorem, it turns out that a polarized manifold which contains a Castelnuovo manifold of small degree as an ample divisor is confined to be a weighted complete intersection, which is what I would like to stress. The consequence would be interesting in the point of view of characterization of weighted complete intersections (e.g. [Laz 2, 3.2.B], [L'v]).

Finally, in the proof of Theorem 3.1.1, we successfully classify the Castelnuovo manifolds $(A, \mathscr{H})$ of the first kind, i.e. $(A, \mathscr{H})$ with $d(A, \mathscr{H}) > 2\Delta(A, \mathscr{H})$, under $n > d(A, \mathscr{H})$ although the structures of Castelnuovo manifolds of the first kind are still unrevealed in general (cf. [Fu 5, (16.7)]). By using Barth's theorem (see [Laz 2, Corollary 3.2.3]), which is valid under $n > d(A, \mathscr{H})$, we prove that those Castelnuovo manifolds are confined to be Fano manifolds of coindex at most 2. Therefore we can describe the structures of those Castelnuovo manifolds explicitly under $n > d(A, \mathscr{H})$ by utilizing classification results (e.g. [Fu 5, (8.11)]) of those Fano manifolds. Here is our result.

Proposition 3.4.1 *Let* $(A, \mathscr{H})$ *be an* n*-dimensional Castelnuovo manifold with* $n > d(A, \mathscr{H})$. *Then the following* (I) *and* (II) *are equivalent.*

(I) $(A, \mathscr{H})$ *is of the first kind.*

(II) $(A, \mathscr{H})$ *is one of the following*:

- (i) $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}}(1))$;
- (ii) $(\mathbb{Q}^n, \mathscr{O}_{\mathbb{Q}}(1))$;
- (iii) $(H_3^n, \mathscr{O}_{H_3}(1))$;
- (iv) $(V_{2,2}^n, \mathscr{O}_{V_{2,2}}(1))$; *or*
- (v) $(\mathrm{Gr}(5,2), \mathscr{O}_{\mathrm{Gr}}(1))$, *where* $\mathrm{Gr}(5,2)$ *is a Grassmann variety parametrizing the* 2*-dimensional linear subspace in* $\mathbb{C}^5$.

Conventions Throughout this thesis, we work over the complex number field $\mathbb{C}$. We adopt the standard notation from algebraic geometry as in [Hart]. By a *manifold* we mean a smooth projective variety. The words "Cartier divisors", "line bundles" and "invertible sheaves" are used interchangeably, and "vector bundles" and "locally free sheaves", too. The tensor products of line bundles are denoted additively, while we use multiplicative notation for intersection products in Chow rings.

Chapter 1

Polarized manifolds admitting a five-sheeted cover of $\mathbb{P}^n$ among their hyperplane sections

1.1 Introduction

Let X be an $(n+1)$-dimensional smooth complex projective variety and L a very ample line bundle on X. Consider the following condition:

$(*)_d$ There exists a smooth member $A \in |L|$ such that there exists a branched covering $\pi\colon A \to \mathbb{P}^n$ of degree d.

Needless to say, the following "obvious" pairs (X,L) satisfy $(*)_d$: $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(d))$ and $(H_d^{n+1}, \mathscr{O}_{H_d^{n+1}}(1))$, where H_d^{n+1} is a smooth hypersurface of degree d in $\mathbb{P}^{n+2}$.

The study of (X,L) satisfying $(*)_d$ is a natural generalization of a classical problem of Castelnuovo [Ca2]. The classical problem is to classify the pairs (X,L) satisfying $(*)_d$ when $n=1$ and $d=2$, and was solved by F. Serrano [Se], Sommese–A. Van de Ven [SV], independently. When $n=1$ and $d=3$, M. L. Fania [Fa] studied the pairs (X,L). In cases $n \geq d = 2$ [LPS 1], $n > d = 3$ [LPS 2], Lanteri–Palleschi–Sommese (LPS, for short) classified the pairs.

Surprisingly, in case $n > d \in \{2,3\}$, it turns out that the results of the classifications are simple; this relies on topological restrictions imposed X by A. In fact, in case $d=2$, the "non-obvious" pairs never arise in the classification. In case $d=3$, the "non-obvious" pair is only $(Y_1, 3\mathscr{L})$, where $(Y_1, \mathscr{L})$ is a del Pezzo manifold of degree 1, i.e., a polarized manifold satisfying $-K_{Y_1} \cong n\mathscr{L}$ and $\mathscr{L}^{n+1} = 1$.

So, what kind of "non-obvious" pairs (X,L) arise in case $n > d \geq 4$? We shall deal with the case of $d = 4$ in Chapter 2.

The purpose of this chapter is to give a complete classification of the pairs (X,L) that satisfy $(*)_5$ under $n \geq 6$. Our result here is as follows:

Theorem 1.1.1 *Let X be a smooth projective variety with $\dim X = n+1 \geq 7$. Then there exists a very ample line bundle L on X that satisfies the condition $(*)_5$ if and only if (X,L) is one of the following:*

(i) $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(5))$;

(ii) $(H_5^{n+1}, \mathscr{O}_{H_5^{n+1}}(1))$;

(iii) $(Y_1, 5\mathscr{L})$;

(iv) $(W_{10}, \mathscr{O}_{W_{10}}(5))$*, where W_{10} is a weighted hypersurface of degree* 10 *in the weighted projective space* $\mathbb{P}(5,2,1^{n+1})$; *or*

(v) $(W_{20}, \mathscr{O}_{W_{20}}(5))$*, where W_{20} is a weighted hypersurface of degree* 20 *in* $\mathbb{P}(5,4,1^{n+1})$.

Two "non-obvious" pairs (iv) and (v) newly show up. Moreover the existences of those new pairs are verified (see Section 1.3).

LPS ([LPS 1], [LPS 2]), in cases $n > d \in \{2,3\}$, use the classification theory of polarized manifolds by means of sectional genera.

The difficulty in our study is that a polarized manifold $(X,\mathscr{H})$ with $\Delta(X,\mathscr{H}) = d(X,\mathscr{H}) = 1$ and sectional genus ≥ 3 arises; the classification problem of polarized manifolds with these invariants is yet to be solved completely (cf. [Fu 5, (6.18)]).

Our study involves a new strategy although the starting point of the proof is inspired by the ideas of LPS. The key ingredients of the proof are twofold:

(I) To show the very ampleness of $\mathscr{O}_{W_{20}}(5)$ (Proposition 1.3.3).

(II) To characterize $(X,\mathscr{H})$ with invariants $\Delta(X,\mathscr{H}) = \mathscr{H}^{n+1} = 1, g(X,\mathscr{H}) = 6$ and with $L \cong 5\mathscr{H}$ (Theorem 1.6.2).

For (I), after finding a basis of $H^0(\mathscr{O}_{W_{20}}(5))$, we check that the freeness, the separation of points and the separation of tangent vectors for $|\mathscr{O}_{W_{20}}(5)|$.
For (II), our strategy is to find the generators of the graded ring of $(X,\mathscr{H})$

$$R(X,\mathscr{H}) := \bigoplus_{i=0}^{\infty} H^0(X, i\mathscr{H}),$$

and the relations among them. Using the ladder method, we reduce this to describing the structure of $R(X_1, \mathscr{H}|_{X_1})$ in terms of generators and relations, where X_1 is a smooth curve section of X that is an intersection of n-general members of $|\mathscr{H}|$. By the Riemann–Roch theorem and some ring-theoretic arguments, we can describe the structure of $R(X, \mathscr{H})$ successfully.

This chapter is organized as follows: In Section 1.2, we give some notation, definitions and general facts. In Section 1.3, we prove (I), consequently the 'if' part in Theorem 1.1.1 is proved. From Section 1.4 to 1.6, we concentrate on proving the 'only if' part. In Section 1.4, we prove a basic result on $h^0(A, \pi^*\mathscr{O}_{\mathbb{P}^n}(1))$. Section 1.5 is devoted to the cases (i) and (ii) in Theorem 1.1.1. Section 1.6 is devoted to the proof of (II) (Theorem 1.6.2), as a consequence we see that the polarized manifolds (iii)–(v) actually show up.

After Theorem1.1.1 had been obtained, I found Lanteri's result on a classification of the pairs (X, L) in question [Lan, Theorem 3.5]. However, his classification result contains one doubtful case: In fact, his result says that the cases (i)-(iv) in our Theorem 1.1.1 arise. But he gave only a numerical characterization and invariants for the case (v). In contrast, I determine the structure of a unique polarized manifold appearing in that case, completely.

1.2 Preliminaries

A *branched covering of degree d* means a finite surjective morphism of degree d. A *manifold* means a smooth variety. A line bundle on a variety is said to be *spanned* if it is generated by global sections.

A *polarized variety* means a pair $(V, \mathscr{L})$ where V is a projective variety and $\mathscr{L}$ is an ample line bundle on V. Set $m = \dim V$.

A member of $|\mathscr{L}|$ is called a *rung* of $(V, \mathscr{L})$ if it is an irreducible and reduced subscheme of V. A rung D of $(V, \mathscr{L})$ is said to be *regular* if the restriction map $H^0(V, \mathscr{L}) \to H^0(D, \mathscr{L}|_D)$ is surjective. A sequence $V = V_m \supset V_{m-1} \supset \cdots \supset V_1$ of subvarieties of V is called a *ladder* of $(V, \mathscr{L})$ if each V_j is a rung of $(V_{j+1}, \mathscr{L}_{j+1})$ for $j \geq 1$, where $\mathscr{L}_j$ is the restriction of $\mathscr{L}$ to V_j.

The Δ*-genus* of $(V, \mathscr{L})$ is defined by $\Delta(V, \mathscr{L}) = m + d(V, \mathscr{L}) - h^0(V, \mathscr{L})$, where $d(V, \mathscr{L}) := \mathscr{L}^m$ is the *degree* of $(V, \mathscr{L})$. For a manifold V, the *sectional genus* of

$(V, \mathscr{L})$, denoted by $g(V, \mathscr{L})$, is defined by the formula

$$2g(V, \mathscr{L}) - 2 = (K_V + (m-1)\mathscr{L}) \cdot \mathscr{L}^{m-1}.$$

A polarized variety $(V, \mathscr{L})$ is called a *scroll over* a smooth curve C if it is of the form $(\mathbb{P}(\mathscr{E}), H(\mathscr{E}))$ for some locally free sheaf $\mathscr{E}$ on C, where $H(\mathscr{E})$ denotes the tautological line bundle of $\mathbb{P}(\mathscr{E})$.

For an integer $r \geq 1$, a line bundle $\mathscr{L}$ on V is said to be *r-generated* if the graded ring $R(V, \mathscr{L}) = \bigoplus_{i=0}^{\infty} H^0(V, i\mathscr{L})$ is generated by the global sections of $\mathscr{L}, \ldots, r\mathscr{L}$. In particular $\mathscr{L}$ is said to be *simply generated* if it is one-generated.

The following is used in the study of polarized manifolds with small Δ-genera.

Proposition 1.2.1 (Fujita) *Let $(M, \mathscr{L})$ be an m-dimensional polarized manifold having a ladder. Assume that $g := g(M, \mathscr{L}) \geq \Delta(M, \mathscr{L}) =: \Delta$ and $\mathscr{L}^m \geq 2\Delta + 1$. Then $\mathscr{L}$ is simply generated, $g = \Delta$ and $H^q(M, t\mathscr{L}) = 0$ for any integers t, q with $0 < q < m$.*

For the proof, we refer to [Fu 5, Chapter I (3.5)].

The following lemma is trivial but useful in studying the structures of graded rings.

Lemma 1.2.2 *Let $(V, \mathscr{L})$ be a polarized variety, D a rung of $(V, \mathscr{L})$ defined by $\delta \in H^0(V, \mathscr{L})$, and $\rho_t \colon H^0(V, t\mathscr{L}) \to H^0(D, t\mathscr{L}|_D)$ the restriction map. Then* $\mathrm{Ker}(\rho_t) = \delta H^0(V, (t-1)\mathscr{L})$.

A *weighted projective space* $\mathbb{P}(e_0, \ldots, e_N)$ is defined to be $\mathrm{Proj}(\mathbb{C}[s_0, \ldots, s_N])$, where $\mathrm{wt}(s_0, \ldots, s_N) = (e_0, \ldots, e_N) \in \mathbb{N}^{\oplus(N+1)}$. A projective variety W is called a *weighted complete intersection* of type $(a_1, \ldots, a_c)$ in $\mathbb{P}(e_0, \ldots, e_N)$(w.c.i., for short) if the following two conditions are satisfied:

(1) $W \cong \mathrm{Proj}(\mathbb{C}[s_0, \ldots, s_N]/(F_1, \ldots, F_c))$, where $(F_1, \ldots, F_c)$ is a regular sequence and each F_i is a homogeneous polynomial of degree $a_i > 0$;

(2) $V_+(F_1, \ldots, F_c) \cap (\bigcup_{1<k}(s_j = 0 \mid k \nmid e_j)) = \emptyset$ in $\mathbb{P}(e_0, \ldots, e_N)$.

We put $S(e_0, \ldots, e_N) := \bigcup_{1<k}(s_j = 0 \mid k \nmid e_j)$.

Proposition 1.2.3 (S. Mori) *Let D be an effective ample divisor of an m-dimensional projective manifold M. Assume D is a w.c.i. of type $(a_1, \ldots, a_c)$ in $\mathbb{P}(e_0, \ldots, e_N)$. Then the following hold.*

(1) *If* $m \geq 4$, *M is a w.c.i. of type* $(a_1,\dots,a_c)$ *in* $\mathbb{P}(e_0,\dots,e_N,a)$, *where* $a > 0$ *is an integer such that* $\mathscr{O}_M(D)\otimes\mathscr{O}_D \cong \mathscr{O}_D(a)$.

(2) *If* $m = 3$ *and there exists a positive integer a such that* $\mathscr{O}_M(D)\otimes\mathscr{O}_D \cong \mathscr{O}_D(a)$, *then M satisfies the same conclusion of* (1).

For the proof, see [Mo, Corollary 3.8 & Proposition 3.10].

1.3 Polarized manifolds of $\Delta = d = 1$ and special examples: the 'if' part

In this section we consider the three special classes (iii)–(v) of polarized manifolds appearing in Theorem 1.1.1. These classes are constructed from polarized manifolds $(M,\mathscr{L})$ of $\Delta(M,\mathscr{L}) = d(M,\mathscr{L}) = 1$.

We begin with the following fact:

Fact 1.3.1 *Let* $(M,\mathscr{L})$ *be an m-dimensional polarized manifold of* $\Delta(M,\mathscr{L}) = \mathscr{L}^m = 1$, *and let* $H_1,\dots,H_{m-1}$ *be general members of* $|\mathscr{L}|$. *For each integer* $1 \leq k \leq m-1$, *we put* $X_k := \bigcap_{k\leq i\leq m-1} H_i$. *Then the following hold.*

(1) *The base locus* $\mathrm{Bs}|\mathscr{L}|$ *consists of a single point.*

(2) *The linear system* $|b^*\mathscr{L} - E|$ *defines a flat surjective morphism* $f\colon \tilde{M} \to \mathbb{P}^{m-1}$, *where* $b\colon \tilde{M} \to M$ *is the blowing up at* $\mathrm{Bs}|\mathscr{L}|$ *and E is the exceptional divisor lying over* $\mathrm{Bs}|\mathscr{L}|$. *The set E is a section of f, and every fiber of f is an integral curve of arithmetic genus* $g(M,\mathscr{L}) \geq 1$.

(3) X_k *is a k-dimensional submanifold of M, and* $X_1 \subset \cdots \subset X_{m-1} \subset M$ *is a regular ladder of* $(M,\mathscr{L})$.

For the proof, we refer to [Fu 4, §13].

Proposition 1.3.2 *Let* $(M,\mathscr{L})$ *be as in Fact* 1.3.1, *and let* $d \geq 2$ *be an integer such that* $L := d\mathscr{L}$ *is spanned. Then there exists a smooth member A of* $|L|$ *with a finite surjective morphism of degree d,*

$$\pi\colon A \longrightarrow \mathbb{P}^{m-1}.$$

Proof. From Fact 1.3.1 (2), we obtain the flat surjective morphism $f\colon \tilde{M} \to \mathbb{P}^{m-1}$. Since L is spanned, there exists a smooth member A of $|L|$ not passing through $\mathrm{Bs}|\mathscr{L}|$. Since $H^i(M,(1-d)\mathscr{L}) = 0$ for $i = 0,1$ by the Kodaira vanishing theorem, we see that $h^0(A,\mathscr{L}|_A) = m$, especially $|\mathscr{L}|_A| = |\mathscr{L}|_A$. Therefore, combining these and $\mathscr{L}|_A^{m-1} = d$, we see that $|\mathscr{L}|_A|$ gives a branched covering of degree d from A to $\mathbb{P}^{m-1}$. ■

Example 1 Let $(X,L) = (Y_1, 5\mathscr{L})$, where $(Y_1,\mathscr{L})$ is an $(n+1)$-dimensional del Pezzo manifold of degree 1, i.e., $-K_{Y_1} \cong n\mathscr{L}$ with $\mathscr{L}^{n+1} = 1$. We see $\Delta(Y_1,\mathscr{L}) = 1$. The very ampleness of $5\mathscr{L}$ follows from the fact that $2\mathscr{L}$ is spanned [Fu 4, §14] and $3\mathscr{L}$ is very ample [LPS 2, (1.2)]. Hence, by Proposition 1.3.2, there exists a smooth five-sheeted cover of $\mathbb{P}^n$ that is a member of $|5\mathscr{L}|$.

Example 2 Let $(X,L) = (W_{10}, \mathscr{O}_{W_{10}}(5))$, where W_{10} is an $(n+1)$-dimensional smooth weighted hypersurface of degree 10 in $\mathbb{P}(5,2,1^{n+1})$. We see that $\Delta(W_{10},\mathscr{O}_{W_{10}}(1)) = \mathscr{O}_{W_{10}}(1)^{n+1} = 1$. Moreover, it follows from $g(W_{10},\mathscr{O}_{W_{10}}(1)) = 2$ that $(W_{10},\mathscr{O}_{W_{10}}(1))$ is a sectionally hyperelliptic polarized manifold of type $(-)$ [Fu 4, §15 and 16]. Therefore $\mathscr{O}_{W_{10}}(5)$ is very ample due to [Laf, Theorem 3.3]. Consequently we obtain a smooth five-sheeted cover of $\mathbb{P}^n$ in $|\mathscr{O}_{W_{10}}(5)|$.

Example 3 Let $(X,L) = (W_{20}, \mathscr{O}_{W_{20}}(5))$, where W_{20} is an $(n+1)$-dimensional smooth weighted hypersurface of degree 20 in $\mathbb{P}(5,4,1^{n+1})$. Since we have $\Delta(W_{20},\mathscr{O}_{W_{20}}(1)) = \mathscr{O}_{W_{20}}(1)^{n+1} = 1$, we get a five-sheeted cover of $\mathbb{P}^n$ in $|\mathscr{O}_{W_{20}}(5)|$ from the following:

Proposition 1.3.3 *The line bundle $\mathscr{O}_{W_{20}}(5)$ is very ample.*

Proof. We prove the conclusion with the following steps:

(a) $\mathrm{Bs}|\mathscr{O}_{W_{20}}(5)| = \emptyset$;

(b) the morphism φ associated with $|\mathscr{O}_{W_{20}}(5)|$ is injective;

(c) the linear system $|\mathscr{O}_{W_{20}}(5)|$ separates the tangent vectors.

By combining 5-generatedness of $\mathscr{O}_{W_{20}}(1)$ and [Laf, Theorem 2.2], the rational map φ is an embedding outside the single point $p := \mathrm{Bs}|\mathscr{O}_{W_{20}}(1)|$.

Let $x, y, z_0, \ldots, z_n$ generate the graded ring $R(W_{20},\mathscr{O}_{W_{20}}(1))$, where $\deg(x,y,z_0,\ldots,z_n) = (5,4,1,\ldots,1)$.

(a) We see that $H^0(\mathscr{O}_{W_{20}}(5))$ is generated by the sections

$$x, yz_0, \ldots, yz_n, z_{j_1}\cdots z_{j_5}, \text{ with } 0 \le j_1 \le \cdots \le j_5 \le n.$$

Therefore it follows that

$$\mathrm{Bs}|\mathscr{O}_{W_{20}}(5)| = (x=0) \cap \left(\bigcap_{0\le i\le n} (z_i = 0) \right),$$

which is empty since W_{20} does not meet the locus $S(5,4,1^{n+1})$.

(b) Suppose that $\varphi(p) = \varphi(q)$ for some $q \in W_{20}$. Then we see that $z_i(q) = 0$ for any $0 \le i \le n$, which indicates $q \in \mathrm{Bs}|\mathscr{O}_{W_{20}}(1)|$. Therefore $p = q$.

(c) Let τ be a non-zero tangent vector in $T_p(W_{20})$. We need to show that there exists a section $\sigma \in H^0(\mathscr{O}_{W_{20}}(5))$ satisfying the following conditions:

$$\sigma(p) = 0 \text{ and } d\sigma(\tau) \neq 0.$$

We claim that $\sigma_i := yz_i$ satisfies the above conditions for some $0 \le i \le n$. The former condition is satisfied for all σ_i since $z_i(p) = 0$. We prove that the latter holds. Suppose that there exists non-zero $\tau \in T_p(W_{20})$ such that $d\sigma_i(\tau) = 0$ for all i. Since $d\sigma_i(\tau) = y(p)dz_i(\tau)$ and $y(p) \neq 0$, we see that $dz_i(\tau) = 0$ for all i. Hence it follows that

$$\tau \in T_p(\Gamma), \text{ where } \Gamma := \bigcap_{1\le i\le n} (z_i = 0).$$

From $dz_0(\tau) = 0$, we have $\Gamma \cdot \mathscr{O}_{W_{20}}(1) \ge 2$, which contradicts $\mathscr{O}_{W_{20}}(1)^{n+1} = 1$. This concludes the proof. ∎

1.4 The 'only if' part

We are now going to classify the polarized manifolds in question.

Suppose that (X, L) satsifies $(*)_5$ and $n > d = 5$. Let $\pi\colon A \to \mathbb{P}^n$ denote the finite morphism of degree 5. Then a Barth-type theorem of R. Lazarsfeld [Laz 1, Theorem1] implies that $H^2(A, \mathbb{Z}) \cong H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$ and $H^1(A, \mathscr{O}_A) = 0$. Therefore $\mathrm{Pic}(A) \cong \mathbb{Z}$, generated by $\pi^*\mathscr{O}_{\mathbb{P}^n}(1)$. The Lefschetz hyperplane section theorem implies $\mathrm{Pic}(X) \cong \mathbb{Z}$. We denote by $\mathscr{H}$ the ample generator of $\mathrm{Pic}(X)$; we have $\mathscr{H}|_A \cong \pi^*\mathscr{O}_{\mathbb{P}^n}(1)$. Combining the ampleness of $\mathscr{H}|_A$ and the fact that Δ-genus is non-negative for every polarized manifold [Fu 5, Chapter I (4.2)], we see

$$n+1 \le h^0(A, \mathscr{H}|_A) \le n+5.$$

In fact, we have the following

Proposition 1.4.1 *$h^0(A, \mathscr{H}|_A) = n+1$ or $n+2$.*

Proof. At first, suppose $h^0(A, \mathscr{H}|_A) = n+5$. then we have $\Delta(A, \mathscr{H}|_A) = 0$. Therefore, by [Fu 5, Chapter I (5.10)], $(A, \mathscr{H}|_A)$ is either (a) $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$, (b) $(\mathbb{Q}^n, \mathscr{O}_{\mathbb{Q}^n}(1))$ or (c) a scroll over $\mathbb{P}^1$. Cases (a) and (b) cannot occur by $\mathscr{H}|_A^n = 5$. Case (c) also cannot occur because of $\mathrm{Pic}(A) \cong \mathbb{Z}$.

Secondly, suppose $h^0(A, \mathscr{H}|_A) = n+4$. Then we obtain $\Delta(A, \mathscr{H}|_A) = 1$. By Proposition 1.2.1, we have $g(A, \mathscr{H}|_A) = 1$. Therefore it follows from [Fu 5, (12.3)] that $(A, \mathscr{H}|_A)$ is either a del Pezzo manifold or a scroll over an elliptic curve. The latter case is ruled out because of $\mathrm{Pic}(A) \cong \mathbb{Z}$. The former case is also ruled out by the following reason: if $(A, \mathscr{H}|_A)$ is a del Pezzo manifold of degree 5, then we see that A is the Grassmann variety parametrizing lines in $\mathbb{P}^4$, $\mathrm{Gr}(5,2)$, by combining the result of [Fu 5, (8.11)] and our assumption $n > 5$. But $\mathrm{Gr}(5,2)$ cannot be ample divisors on X by virtue of [Fu 2, (5.2)].

Lastly, we suppose $h^0(A, \mathscr{H}_A) = n+3$. By Proposition 1.2.1, we see that $g(A, \mathscr{H}_A) = \Delta(A, \mathscr{H}|_A) = 2$ and $\mathscr{H}|_A$ is simply generated, hence very ample. According to [I], we have $\dim A \leq 4$, which contradicts our assumption. ■

From now on, we will discuss the case $h^0(A, \mathscr{H}|_A) = n+2$ in Section 1.5 and the case $h^0(A, \mathscr{H}|_A) = n+1$ in Section 1.6.

1.5 The case of $h^0(A, \mathscr{H}|_A) = n+2$

In this section we treat the case $h^0(A, \mathscr{H}|_A) = n+2$. The aim of this section is to prove the following

Proposition 1.5.1 *If $h^0(A, \mathscr{H}|_A) = n+2$, then (X, L) is either $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(5))$ or $(H_5^{n+1}, \mathscr{O}_{H_5^{n+1}}(1))$.*

The following lemma is a special case of [LPS 1, (1.3)]:

Lemma 1.5.2 (Lanteri–Palleschi–Sommese) *If $h^0(A, \mathscr{H}|_A) = n+2$, then the morphism $q \colon A \to \mathbb{P}^{n+1}$ associated to $|\mathscr{H}|_A|$ is birational and its image $q(A)$ is a hypersurface (possibly singular) of degree 5 in $\mathbb{P}^{n+1}$.*

Remark 1.5.3 By virtue of the Bertini theorem, we obtain a smooth k-dimensional rung A_k of $(A_{k+1}, \mathscr{H}|_{A_{k+1}})$ inductively, with $A_n := A$. Put $C := A_1$. Then one can easily obtain an inequality

$$g(C, \mathscr{H}|_C) \geq \Delta(C, \mathscr{H}|_C). \qquad (\star)$$

Lemma 1.5.4 *The ladder $C \subset A_2 \subset \cdots \subset A_n$ is regular.*

Proof. It suffices to prove $H^1(A_k, \mathscr{O}_{A_k}) = 0$ for all $k \geq 2$. By the Lefschetz hyperplane section theorem [Fu 5, (7.1.4)], we have $H^1(A_k, \mathscr{O}_{A_k}) \cong H^1(A_{k-1}, \mathscr{O}_{A_{k-1}})$ for all $k \geq 3$. Combining these and $H^1(A, \mathscr{O}_A) = 0$, we obtain the assertion. ■

By Lemma 1.5.2, the smooth curve C is the normalization of $q(C)$, which is a plane quintic curve of arithmetic genus 6. Since $h^0(A_{k+1}, \mathscr{H}|_{A_{k+1}}) = k+3$ for all k by virtue of Lemma 1.5.4, we have $\Delta(C, \mathscr{H}|_C) = 3$.

Lemma 1.5.5 *The line bundle $\mathscr{H}|_C$ is simply generated.*

Proof. We prove that $g(C, \mathscr{H}|_C) = 6$ as follows: We have inequalities

$$3 \leq g(C, \mathscr{H}|_C) \leq 6.$$

Indeed, the right inequality is obvious and the left is obtained by combining $(\star)$ and $\Delta(C, \mathscr{H}|_C) = 3$. We have $K_A \cong r\mathscr{H}|_A$ for some integer r due to $\mathrm{Pic}(A) \cong \mathbb{Z}$. By the sectional genus formula

$$2g(A, \mathscr{H}|_A) - 2 = (K_A + (n-1)\mathscr{H}|_A) \cdot \mathscr{H}|_A^{n-1} = 5(r+n-1),$$

we see that $g(A, \mathscr{H}|_A) - 1$ is divisible by 5. Combining this and the above inequlities, we obtain $g(C, \mathscr{H}|_C) = 6$.

It follows from $g(C, \mathscr{H}|_C) = 6 = p_a(q(C))$ that $\mathscr{H}|_C$ is very ample, i.e., $C \cong q(C)$. Moreover $q(C)$ is a smooth plane curve. Therefore $\mathscr{H}|_C$ is simply generated. ■

Proof of Proposition 1.5.1. By combining Lemma 1.5.4, 1.5.5 and [Fu 5, Chapter I (2.5)], we see that $\mathscr{H}|_A$ is very ample. Thus

$$(A, \mathscr{H}|_A) \cong (H_5^n, \mathscr{O}_{H_5^n}(1)).$$

We can write $L = l\mathscr{H}$ with some integer $l \geq 1$. It follows from $5 = \mathscr{H}|_A^n = l\mathscr{H}^{n+1}$ that $(l, \mathscr{H}^{n+1})$ is either $(1,5)$ or $(5,1)$.

The case of $(l, \mathscr{H}^{n+1}) = (1,5)$ The ladder $C \subset \cdots \subset A \subset X$ is regular, hence $\Delta(X,L) = 3$. Therefore, from $h^0(X,L) = n+3$, it follows $(X,L) \cong (H_5^{n+1}, \mathscr{O}_{H_5^{n+1}}(1))$.

The case of $(l, \mathscr{H}^{n+1}) = (5,1)$ Since $H^i(X, -4\mathscr{H}) = 0$ for $i = 0,1$ due to the Kodaira vanishing theorem, we see that $h^0(X, \mathscr{H}) = n+2$, hence we have $\Delta(X, \mathscr{H}) = 0$. Since $\mathscr{H}^{n+1} = 1$, we obtain $(X,L) \cong (\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(5))$. ■

1.6 The case of $h^0(A, \mathscr{H}|_A) = n+1$

In this section, we deal with the case $h^0(A, \mathscr{H}|_A) = n+1$. The heart of this section is to prove Theorem 1.6.2.

Lemma 1.6.1 *If* $h^0(A, \mathscr{H}|_A) = n+1$*, then we have* $L \cong 5\mathscr{H}$*,* $\mathscr{H}^{n+1} = 1$ *and* $\Delta(X, \mathscr{H}) = 1$.

Proof. We see that $L = l\mathscr{H}$ for $l \neq 1$ as follows. Suppose $l = 1$. Then $|\mathscr{H}|_A|$ gives an embedding of A into $\mathbb{P}^n$, which contradicts $\deg \pi = 5$. From this, we see $l \neq 1$.

Therefore $(l, \mathscr{H}^{n+1}) = (5,1)$. Furthermore, from the Kodaira vanishing theorem, it follows $h^0(X, \mathscr{H}) = h^0(A, \mathscr{H}|_A) = n+1$. Hence we obtain $\Delta(X, \mathscr{H}) = 1$. ■

Let $H_1, \ldots, H_n$ be general members of $|\mathscr{H}|$, and put $X_k := \bigcap_{k \leq i \leq n} H_i$ for all $1 \leq k \leq n$. Recalling Fact 1.3.1 (3), we see that X_k is a k-dimensional manifold. We put $p := \mathrm{Bs}|\mathscr{H}|$.

We now consider the morphism associated to $|L|$

$$\varphi_L : X \longrightarrow \mathbb{P}(|L|),$$

which is an embedding of X, and $\varphi_L(X_1)$ is a smooth curve of degree 5. Then we obtain $g(X, \mathscr{H}) = g(\varphi_L(X_1)) = 0, 1, 2$ or 6 (see [Hart, p.354]).

The case of $g(X, \mathscr{H}) = 0$ From [Fu 5, (12.1)], we see $\Delta(X, \mathscr{H}) = 0$, which is absurd.

The case of $g(X, \mathscr{H}) = 1$ By virtue of a result of Fujita [Fu 5, (6.5)], We see that $(X, \mathscr{H})$ is a del Pezzo manifold of degree 1, hence we are in the case of (iii) in Theorem 1.1.1.

The case of $g(X, \mathscr{H}) = 2$ From [Fu 4, §15 & Appendix 1] and $n \geq 6$, $(X, \mathscr{H})$ is a sectionally hyperelliptic polarized manifold of type $(-)$, which is also classified by Fujita. We are in the case (iv).

The case of $g(X,\mathscr{H})=6$ Then we see that X_1 is isomorphic to a smooth plane quintic curve. What we are going to prove is the following

Theorem 1.6.2 *If* $h^0(A,\mathscr{H}|_A)=n+1$ *and* $g(X,\mathscr{H})=6$*, then*

$$(X,\mathscr{H})\cong(W_{20},\mathscr{O}_{W_{20}}(1)).$$

We will use the ladder method to prove this, where the key is to describe the structure of $R(X_2,\mathscr{H}|_{X_2})$ explicitly. In fact, in order to get the conclusion, we need the description of the structure of $R(X_1,\mathscr{H}|_{X_1})$ and the surjectivity of the restriction map

$$\rho\colon R(X_2,\mathscr{H}|_{X_2})\longrightarrow R(X_1,\mathscr{H}|_{X_1}).$$

We first describe the structure of $R(X_1,\mathscr{H}|_{X_1})$:

Proposition 1.6.3 *Under the assumption of Theorem* 1.6.2*, there exists an isomorphism*

$$R(X_1,\mathscr{H}|_{X_1})\cong\mathbb{C}[x,y,z]/(F_{20}),$$

where $\mathrm{wt}(x,y,z)=(5,4,1)$ *and* F_{20} *is an irreducible weighted homogeneous polynomial of degree* 20*.*

Proof. Using the Riemann–Roch theorem for X_1, we find the generators of $R(X_1,\mathscr{H}|_{X_1})$ and the relations among them. We proceed in three steps.

Step 1 We show that the dimension of $H^0(l\mathscr{H}|_{X_1})$ for $l\geq 1$ is as follows:

l	$h^0(l\mathscr{H}\|_{X_1})$	l	$h^0(l\mathscr{H}\|_{X_1})$
1	1	6	3
2	1	7	3
3	1	8	4
4	2	9	5
5	3	10	6

and $h^0(l\mathscr{H}|_{X_1})=l-5$ for all $l\geq 11$. Indeed, by the Riemann–Roch theorem, we obtain

$$h^0(l\mathscr{H}|_{X_1})=h^0((10-l)\mathscr{H}|_{X_1})+l-5,$$

which implies the latter assertion. We prove the former. Note that $h^0(5\mathscr{H}|_{X_1})=3$ since $|L|_{X_1}|$ gives an embedding of X_1 into $\mathbb{P}^2$. By Fact 1.3.1 (3), we see $h^0(\mathscr{H}|_{X_k})=k$ in particular $h^0(\mathscr{H}|_{X_1}) = 1$, thus $h^0(9\mathscr{H}|_{X_1}) = 5$. From the well-known fact that a smooth plane quintic curve has neither g^1_2 nor g^1_3, we have $h^0(2\mathscr{H}|_{X_1}) = h^0(3\mathscr{H}|_{X_1}) = 1$, thus $h^0(8\mathscr{H}|_{X_2}) = 4, h^0(7\mathscr{H}|_{X_1}) = 3$. Then we see $h^0(6\mathscr{H}|_{X_1}) = 3$ and $h^0(4\mathscr{H}|_{X_1}) = 2$. Therefore the former assertion is proved.

Let z be a basis of $H^0(\mathscr{H}|_{X_1})$. Choose $y \in H^0(4\mathscr{H}|_{X_1})$ such that $H^0(4\mathscr{H}|_{X_1}) = \langle y, z^4\rangle$. Moreover, choose $x \in H^0(5\mathscr{H}|_{X_1})$ such that $H^0(5\mathscr{H}|_{X_1}) = \langle x, yz, z^5\rangle$.

Step 2 We claim that the graded ring $R(X_1, \mathscr{H}|_{X_1})$ is generated by x,y,z. Indeed, it suffices to prove that there exist some monomials in x,y,z which form a basis of $H^0(l\mathscr{H}|_{X_1})$ for each l. Note that

$$h^0(l\mathscr{H}|_{X_1}) - h^0((l-1)\mathscr{H}|_{X_1}) = \delta \in \{0,1\}.$$

The cases of $6 \le l \le 11$ We may assume $\delta = 1$: otherwise, we have $H^0(l\mathscr{H}|_{X_1}) = zH^0((l-1)\mathscr{H}|_{X_1})$. Therefore we only consider the cases $l = 8, 9, 10$. Each monomial in x,y contained in $H^0(l\mathscr{H}|_{X_1})$ has a pole of order exactly l at p. Comparing their orders of poles, we see from Step 1 that the following monomials are linearly independent for each $8 \le l \le 10$, hence form a basis for $H^0(l\mathscr{H}|_{X_1})$:

l	monomials in $H^0(l\mathscr{H}|_{X_1})$
8	y^2, xz^3, yz^4, z^8
9	$xy, y^2z, xz^4, yz^5, z^9$
10	$x^2, xyz, y^2z^2, xz^5, yz^6, z^{10}$.

Therefore the assertion holds in these cases.

The cases of $l \ge 12$ We see $\delta = 1$ from Step 1. We prove the assertion by induction. When $l = 12$, it is easy to see that the following monomials are linearly independent as before, hence form a basis of $H^0(12\mathscr{H}|_{X_1})$:

$$y^3, x^2z^2, xyz^3, y^2z^4, xz^7, yz^8, z^{12}.$$

Suppose $l > 12$ and that the assertion holds for $l-1$. It is easily shown that

> for two coprime positive integers a,b and an integer l with $l \ge (a-1)(b-1)$, the equation $ai + bj = l$ has at least one solution (i,j) of non-negative integers.

Set $(a,b) = (5,4)$. Then, since $l > 12$, there exists at least one section written as x^iy^j $(i,j \ge 0)$ in $H^0(l\mathscr{H}|_{X_1})$, not contained in $zH^0((l-1)\mathscr{H}|_{X_1})$. Hence $H^0(l\mathscr{H}|_{X_1}) =$

$\mathbb{C}x^iy^j \bigoplus zH^0((l-1)\mathscr{H}|_{X_1})$. From the assumption of induction, the assertion holds. This proves our claim.

By Step 2, there exists a surjective homomorphism of graded rings

$$\Phi\colon \mathbb{C}[x,y,z] \longrightarrow R(X_1, \mathscr{H}|_{X_1}).$$

Step 3 We show that there exists an irreducible homogeneous polynomial F_{20} of degree 20 in $\mathbb{C}[x,y,z]$ such that $\mathrm{Ker}(\Phi) = (F_{20})$. Indeed, there exist no relations of degree $l < 20$ because the equation $5i+4j = l$ has at most one solution (i, j) of non-negative integers. For $l = 20$, there are exactly 16 monomials of x,y,z in $H^0(20\mathscr{H}|_{X_1})$. On the other hand, $h^0(20\mathscr{H}|_{X_1}) = 15$. Hence there exists one relation F_{20} of degree 20, which is written as

$$F_{20} = x^4 + y^5 + z\psi_{19}(x,y,z)$$

after we replace x and y by suitable scalar multiples, where ψ_{19} is a homogeneous polynomial of x,y,z of degree 19. The irreducibility of F_{20} is proved as follows: One can easily show that $x^4 + y^5$ is irreducible. Write $F_{20}(x,y,z) = P_1(x,y,z)P_2(x,y,z)$ with some $P_1, P_2 \in \mathbb{C}[x,y,z]$. Then we may assume $P_1(x,y,0) = 1$ without loss of generality. Hence $P_1(x,y,z) = 1 + z\xi_1$ and $P_2 = x^4 + y^5 + z\xi_2$, where ξ_1, ξ_2 are polynomials in x,y,z. We obtain that

$$\psi_{19}(x,y,z) = \xi_1(x^4 + y^5 + z\xi_2) + \xi_2.$$

It follows that $\xi_1 = 0$. Indeed, otherwise, the highest term of the right-hand side has degree ≥ 20, which is absurd. Therefore F_{20} is irreducible. Furthermore, combining this and the fact that $\mathrm{ht}(\mathrm{Ker}(\Phi)) \leq \dim \mathbb{C}[x,y,z] - \dim R(X_1, \mathscr{H}|_{X_1}) = 1$, we see $\mathrm{Ker}(\Phi) = (F_{20})$. ■

Next we will show the surjectivity of the restriction map ρ. Let $\mathbf{s} = \{s_0, \ldots, s_N\}$ be a minimal set of generators of $R(X_2, \mathscr{H}|_{X_2})$. Then there exists an isomorphism

$$R(X_2, \mathscr{H}|_{X_2}) \cong \mathbb{C}[s_0, \ldots, s_N]/(F_1, \ldots, F_h),$$

where $F_1, \ldots, F_h$ are homogeneous polynomials in $\mathbb{C}[s_0, \ldots, s_N]$. Put $I_{\mathbf{s}} := (F_1, \ldots, F_h)$.

It follows from Fact 1.3.1 (3) that the vector space $H^0(\mathscr{H}|_{X_2})$ is of dimension 2, hence has a basis $\{s,t\}$ such that $\rho(s) = z$ and $(t)_0 = X_1$. We may assume that $\mathbf{s}$ contains these two elements.

Lemma 1.6.4 *The sequence t, s contained in $\mathfrak{m} := R(X_2, \mathscr{H}|_{X_2})_+$ is regular.*

Proof. Let m be a homogeneous element of degree a in $R(X_2, \mathscr{H}|_{X_2})$ such that $tm = 0$. We see that $R(X_2, \mathscr{H}|_{X_2})_+$ has no zero-divisors since $X_2 \cong \mathrm{Proj}(R(X_2, \mathscr{H}|_{X_2}))$ is integral. Hence, if $a > 0$, then we obtain $m = 0$. If $a = 0$, then the minimality of $\mathbf{s}$ implies that $I_{\mathbf{s}}$ has no generators of degree one. Thus we have $m = 0$. Therefore t is $R(X_2, \mathscr{H}|_{X_2})$-regular. By the same argument, we see that s is $R(X_2, \mathscr{H}|_{X_2})/(t)$-regular since $X_1 \cong \mathrm{Proj}(R(X_2, \mathscr{H}|_{X_2})/(t))$ is integral. Consequently the assertion holds. ■

In order to prove Proposition 1.6.6, we need some information about generators of $I_{\mathbf{s}}$. Let

$$\rho_l \colon H^0(l\mathscr{H}|_{X_2}) \twoheadrightarrow H^0(l\mathscr{H}|_{X_2})/\langle t \rangle \hookrightarrow H^0(l\mathscr{H}|_{X_1})$$

denote the restriction map. Here we show the following

Lemma 1.6.5 *The ideal $I_{\mathbf{s}}$ has no generators in degrees ≤ 5.*

Proof. We first prove that

$$\mathrm{Im}(\rho_5) = H^0(5\mathscr{H}|_{X_1}). \qquad (\dagger)$$

It follows that $\mathrm{rank}(\rho_5) \geq 3$. Indeed, the morphism $\varphi_L|_{X_1} \colon X_1 \to \mathbb{P}(\mathrm{Im}(\rho_5))$ is an embedding of a curve of genus 6. Consequently $(\dagger)$ follows by virtue of Step 1 in the proof of Proposition 1.6.3.

Subsequently, we find a basis of $H^0(l\mathscr{H}|_{X_2})$ for $1 \leq l \leq 5$ by using Lemma 1.2.2.

For $l = 1$, there exist no relations in $H^0(\mathscr{H}|_{X_2})$ because of the minimality of $\mathbf{s}$.

For $l = 2$, there exist no relations. In fact, it follows $H^0(2\mathscr{H}|_{X_2}) = \langle s^2, st, t^2 \rangle$. Indeed, let $\eta \in H^0(2\mathscr{H}|_{X_2})$. We can write $\rho_2(\eta) = cz^2$ with some $c \in \mathbb{C}$. Then, from Lemma 1.2.2, it follows that η is a linear combination of s^2, st, t^2. These three monomials are linearly independent because each order of pole along X_1 differs from that of the others.

For $l = 3$, there are no relations: we see that $H^0(3\mathscr{H}|_{X_2}) = \langle s^3, s^2t, st^2, t^3 \rangle$ by the same argument as in the case $l = 2$.

As for $l = 4$, we note that $1 \leq \mathrm{rank}(\rho_4) \leq h^0(4\mathscr{H}|_{X_1}) = 2$. We first suppose $\mathrm{rank}(\rho_4) = 1$. Then $H^0(4\mathscr{H}|_{X_2}) = \langle s^4, s^3t, s^2t^2, st^3, t^4 \rangle$ holds, which implies that there exist no relations. By $(\dagger)$, there exist sections $u, v \in H^0(5\mathscr{H}|_{X_2})$ such that $\rho_5(u) = x, \rho_5(v) = yz$. Since it follows from Lemma 1.2.2 that

$$H^0(5\mathscr{H}|_{X_2}) = \langle u, v, s^5, s^4t, s^3t^2, s^2t^3, st^4, t^5 \rangle,$$

there exist no relations in $H^0(5\mathscr{H}|_{X_2})$.

Next we suppose $\mathrm{rank}(\rho_4) = 2$. Let w denote a section such that $\rho_4(w) = y$. Then we see

$$H^0(4\mathscr{H}|_{X_2}) = \langle w, s^4, s^3t, s^2t^2, st^3, t^4\rangle,$$
$$H^0(5\mathscr{H}|_{X_2}) = \langle u, sw, tw, s^5, s^4t, s^3t^2, s^2t^3, st^4, t^5\rangle,$$

where u is a section such that $\rho_5(u) = x$. Therefore there exist no relations. ■

Proposition 1.6.6 *The restriction map*

$$\rho\colon R(X_2, \mathscr{H}|_{X_2}) \longrightarrow R(X_1, \mathscr{H}|_{X_1})$$

is surjective.

Proof. It suffices to prove that $H^1(l\mathscr{H}|_{X_2}) = 0$ for every $l \geq 0$, which is equivalent to showing that $R(X_2, \mathscr{H}|_{X_2})$ is a Cohen–Macaulay ring (see [W, (2.4)]).

We find a regular sequence of length 3 contained in $\mathfrak{m}$. The sequence t, s is regular by Lemma 1.6.4. Let $u \in H^0(5\mathscr{H}|_{X_2})$ denote a unique section such that $\rho_5(u) = x$. We assert that u is $R(X_2, \mathscr{H}|_{X_2})/(t,s)$-regular. Indeed, $\mathrm{Proj}(R(X_2, \mathscr{H}|_{X_2})/(t,s))$ is an integral scheme p because of $\mathscr{H}|_{X_2}^2 = 1$. Thus we see that $(R(X_2, \mathscr{H}|_{X_2})/(t,s))_+$ has no zero-divisors. Let m be a homogeneous element of degree a in $R(X_2, \mathscr{H}|_{X_2})/(t,s)$ such that $um = 0$. If $a > 0$, then we have $m = 0$ obviously. If $a = 0$, then we have $m = 0$ by Lemma 1.6.5. Therefore t, s, u form a regular sequence. ■

At last, we can prove Theorem 1.6.2 as follows:

Proof of Theorem 1.6.2. Combining Proposition 1.6.3 and 1.6.6, we see that X_2 is a weighted hypersurface of degree 20 in $\mathbb{P}(5,4,1^2)$. Furthermore, the assertion follows from Proposition 1.2.3. ■

Chapter 2

Polarized manifolds admitting a four-sheeted cover of $\mathbb{P}^n$ among their hyperplane sections

2.1 Introduction

Let X be an $(n+1)$-dimensional smooth complex projective variety and L a very ample line bundle on X. Consider the following condition:

$(*)_d$ There exists a smooth member $A \in |L|$ such that there exists a finite surjective morphism $\pi\colon A \to \mathbb{P}^n$ of degree d.

Needless to say, the following "obvious" pairs (X,L) satisfy $(*)_d$: $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(d))$ and $(H_d^{n+1}, \mathscr{O}_{H_d^{n+1}}(1))$, where H_d^{n+1} is a smooth hypersurface of degree d in $\mathbb{P}^{n+2}$.

It is an interesting subject to investigate, for a fixed d, what kind of the "non-obvious" pairs show up. In fact, for small prime numbers d, the pairs (X,L) satisfying $(*)_d$ and $n > d$ have been classified completely: For $d = 2$ and 3, Lanteri–Palleschi–Sommese ([LPS 1], [LPS 2]) classified the pairs. For $d = 5$, we classified the pairs in Chapter 1.

Let q be the morphism associated to $|\pi^*\mathscr{O}_{\mathbb{P}^n}(1)|$, and assume $t := h^0(A, \pi^*\mathscr{O}_{\mathbb{P}^n}(1)) - n - 1 > 0$. Then we have a factorization of π as follows:

$$\begin{array}{ccl} A & \xrightarrow{\ q\ } & q(A) \subset \mathbb{P}^{n+t} \\ & {\scriptstyle \pi} \searrow & \downarrow p\colon \text{The projection from a } \mathbb{P}^{t-1} \text{ in } \mathbb{P}^{n+t} \text{ with } q(A) \cap \mathbb{P}^{t-1} = \emptyset. \\ & & \mathbb{P}^n \end{array}$$

In the case where d is a prime, it immediately follows that q is birational onto its image $q(A)$, which is a variety of degree d. This plays a key role in the classification problem for a small d.

Now then, for a composite number d, there may exist pairs (X,L) with a non-birational morphism q. Therefore it is natural to study the structures of these pairs.

The purpose of this chapter is to provide a complete classification of the pairs (X,L) in case $n > d = 4$. Our result is

Theorem 2.1.1 *Let X be a smooth projective variety with* $\dim X = n+1 \geq 6$. *Then there exists a very ample line bundle L on X that satisfies the condition $(*)_4$ if and only if (X,L) is one of the following*:

(i) $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(4))$;

(ii) $(H_4^{n+1}, \mathscr{O}_{H_4^{n+1}}(1))$;

(iii) $(Y_1, 4\mathscr{L})$, *where* $(Y_1, \mathscr{L})$ *is a del Pezzo manifold of degree one*;

(iv) $(W_{12}, \mathscr{O}_{W_{12}}(4))$, *where W_{12} is a weighted hypersurface of degree* 12 *in the weighed projective space* $\mathbb{P}(4,3,1^{n+1})$ *with its ample invertible sheaf* $\mathscr{O}_{W_{12}}(1)$;

(v) $(\mathbb{Q}^{n+1}, \mathscr{O}_{\mathbb{Q}^{n+1}}(2))$, *where* $\mathbb{Q}^{n+1}$ *is a hyperquadric in* $\mathbb{P}^{n+2}$;

(vi) $(V_{2,2}^{n+1}, \mathscr{O}_{V_{2,2}^{n+1}}(1))$, *where* $V_{2,2}^{n+1}$ *is a complete intersection of two hyperquadrics in* $\mathbb{P}^{n+3}$; *or*

(vii) $(Z, 2\mathfrak{L})$, *where* $(Z, \mathfrak{L})$ *is a del Pezzo manifold of degree* 2.

There are five “non-obvious” pairs. By comparing this theorem to Theorem 1.1.1, it turns out that no fewer than three pairs (v)–(vii) newly show up. In particular, the pair (vii) is a unique one with a non-birational morphism q. In fact, we see that $q(A)$ is a smooth hyperquadric in this case.

Our basic strategy is to reduce to Fujita’s classification theory of polarized manifolds, which leads us to study the structure of (X,L) with a non-birational morphism q.

The strategy is roughly summarized as follows: As we will see in the section 3, it follows that $\mathrm{Pic}(X) = \mathbb{Z}[\mathscr{H}]$, where $\mathscr{H}$ is the ample generator. And we can show that invariants of $(X,\mathscr{H})$ are small. Therefore the classification theory is applicable except certain polarized manifolds with sectional genera $g(X,\mathscr{H})=3$, Δ-genera and degrees (I) $\Delta(X,\mathscr{H})=\mathscr{H}^{n+1}=1$ or (II) 2. The classification problem of polarized manifolds with these invariants, in general, are yet to be solved completely (cf. [Fu 5, (6.18)&(10.10)]).

As for (I), it turns out that $(X,\mathscr{H})$ is not sectionally hyperelliptic. Furthermore, we find that a curve which is an intersection of n-general members of $|\mathscr{H}|$ is a smooth plane quartic. In this case, we can determine the structure of $(X,\mathscr{H})$ by using a new method provided in Section 1.6.

As for (II), we can prove that this case is ruled out by using the Riemann–Roch theorem for curves and the double point formula for surfaces, successfully (see Proposition 2.3.2).

After Theorem 2.1.1 had been obtained, I found Lanteri's result [Lan, Theorem 3.4], which is similar to ours. But his result contains one doubtful case: In fact, for the case (iv) in Theorem 2.1.1, his result has given only some invariants. In contrast, our theorem reveals the structure of a unique polarized manifold appearing in the case. So our classification result is complete.

2.2 Three special examples: the 'if' part

In this section the 'if' part of Theorem 2.1.1 is proved. We only consider the three special classes (iii), (iv) and (vii) of polarized manifolds appearing in Theorem because one can easily check that the other classes (i), (ii), (v) and (vi) satisfy the assertion.

Example 1 Let $(X,L)=(Y_1,4\mathscr{L})$, where $(Y_1,\mathscr{L})$ is an $(n+1)$-dimensional del Pezzo manifold of degree 1. We have $\Delta(Y_1,\mathscr{L})=1$. As in the proof of [LPS 2, (1.2)], we see that $4\mathscr{L}$ is very ample. Therefore it follows from Proposition 1.3.2 that there exists a four-sheeted cover of $\mathbb{P}^n$ that is a member of $|4\mathscr{L}|$.

Example 2 Let $(X,L)=(Y_2,2\mathcal{L})$, where $(Y_2,\mathcal{L})$ is an $(n+1)$-dimensional del Pezzo manifold of degree 2, i.e., $-K_{Y_2}\cong n\mathcal{L}$ with $\mathcal{L}^{n+1}=2$. Then, from [Fu 5, (8.11)], $(Y_2,\mathcal{L})$ is a double covering of $\mathbb{P}^{n+1}$ branched along a smooth hypersurface of degree 4 and $\mathcal{L}$ is the pull-back of $\mathscr{O}_{\mathbb{P}^{n+1}}(1)$. The graded ring $R(Y_2,\mathcal{L})$ is 2-generated since $(Y_2,\mathcal{L})$ is a smooth weighted hypersurface of degree 4 in $\mathbb{P}(2,1^{n+2})$. We obtain that $2\mathcal{L}$ is very ample by combining the spannedness of $\mathcal{L}$ and [Laf, Corollary 2.3]. Therefore there exists a smooth member $A \in |2\mathcal{L}|$ that is a double covering of $\mathbb{Q}^n$. By projecting $\mathbb{Q}^n$ from a point of $\mathbb{P}^{n+1} \setminus \mathbb{Q}^n$ to $\mathbb{P}^n$, we see that A is a four-sheeted cover of $\mathbb{P}^n$.

Example 3 Let $(X,L) = (W_{12}, \mathscr{O}_{W_{12}}(4))$, where W_{12} is a smooth weighted hypersurface of degree 12 in $\mathbb{P}(4,3,1^{n+1})$. By easy calculations, we obtain that $\Delta(W_{12}, \mathscr{O}_{W_{12}}(1)) = \mathscr{O}_{W_{12}}(1)^{n+1} = 1$. From [Fu 4, § 13], we see that $\mathrm{Bs}|\mathscr{O}_{W_{12}}(1)|$ consists of a single point, which is denoted by p. We obtain a smooth four-sheeted cover of $\mathbb{P}^n$ that is contained in $|\mathscr{O}_{W_{12}}(4)|$ by combining Proposition 1.3.2 and the following

Lemma 2.2.1 *The line bundle $\mathscr{O}_{W_{12}}(4)$ is very ample.*

Proof. We obtain the conclusion with the following steps:

(a) $\mathrm{Bs}|\mathscr{O}_{W_{12}}(4)| = \emptyset$;

(b) The morphism $\varphi := \varphi_{\mathscr{O}_{W_{12}}(4)}$ associated to $\mathscr{O}_{W_{12}}(4)$ is injective;

(c) The linear system $|\mathscr{O}_{W_{12}}(4)|$ separates the tangent vectors.

From the 4-generatedness of $R(W_{12}, \mathscr{O}_{W_{12}}(1))$ and [Laf, Theorem 2.2], φ is an embedding outside the single point p. Let x, y, z_j $(0 \le j \le n)$ generate the graded ring $R(W_{12}, \mathscr{O}_{W_{12}}(1))$, where $\mathrm{wt}(x,y,z_j) = (4,3,1)$ for all j.

(a) It follows that $H^0(\mathscr{O}_{W_{12}}(4))$ is generated by the sections

$$x, yz_0, \ldots, yz_n, z_{j_1} \cdots z_{j_4}, \text{ with } 0 \le j_1 \le \cdots \le j_4 \le n.$$

Therefore we see that

$$\mathrm{Bs}|\mathscr{O}_{W_{12}}(4)| = (x=0) \cap \left(\bigcap_{0 \le j \le n} (z_j = 0) \right),$$

which is empty since W_{12} does not meet the singular points of $\mathbb{P}(4,3,1^{n+1})$.

(b) If we assume $\varphi(p) = \varphi(q)$ for some $q \in W_{12}$, then we find that $z_j = 0$ for any $0 \le j \le n$, which implies $q \in \mathrm{Bs}|\mathscr{O}_{W_{12}}(1)|$. Thus $p = q$.

(c) For a non-zero tangent vector $\tau \in T_p(W_{12})$, we need to show that there exists a section $\sigma \in H^0(\mathscr{O}_{W_{12}}(4))$ satisfying the following conditions:

$$\sigma(p) = 0 \text{ and } d\sigma(\tau) \neq 0.$$

We show that $\sigma_j := yz_j$ satisfies the above conditions for some $0 \leq j \leq n$. The former holds because $z_j(p) = 0$ for all j. We prove that the latter holds by contradiction. Assume that there exists a non-zero $\tau \in T_p(W_{12})$ with $d\sigma_j(\tau) = 0$ for all j. Since $d\sigma_j(\tau) = y(p)dz_j(\tau)$ and $y(p) \neq 0$, we see that $dz_j(\tau) = 0$ for all j. Thus we have

$$\tau \in T_p(\Gamma), \text{ where } \Gamma := \bigcap_{1 \leq j \leq n} (z_j = 0).$$

It follows from $dz_0(\tau) = 0$ that $\Gamma \cdot \mathscr{O}_{W_{12}}(1) \geq 2$, which contradicts $\mathscr{O}_{W_{12}}(1)^{n+1} = 1$. This completes the proof. ■

2.3 The 'only if' part

Let (X, L) satisfy $n \geq 5$ and $(*)_4$. And let $\pi \colon A \to \mathbb{P}^n$ denote the finite morphism of degree 4. Then a Barth-type theorem of Lazarsfeld [Laz 1, Theorem 1] implies that $H^2(A, \mathbb{Z}) \cong H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$ and $H^1(A, \mathscr{O}_A) = 0$. Therefore we have $\mathrm{Pic}(A) \cong \mathbb{Z}$, generated by $\pi^*\mathscr{O}_{\mathbb{P}^n}(1)$. The Lefschetz hyperplane section theorem implies $\mathrm{Pic}(X) \cong \mathbb{Z}$. We denote by $\mathscr{H}$ the ample generator of $\mathrm{Pic}(X)$; we have $\mathscr{H}|_A \cong \pi^*\mathscr{O}_{\mathbb{P}^n}(1)$. Thus we can write $L = l\mathscr{H}$ with some $l > 0$. Since $l\mathscr{H}^{n+1} = \mathscr{H}|_A^n = 4$, we see that

$$\mathscr{H}^{n+1} = 1, 2 \text{ or } 4.$$

Combining the ampleness of $\mathscr{H}|_A$ and the fact that Δ-genus is non-negative for every polarized manifold [Fu 5, Chapter I (4.2)], we see

$$n + 1 \leq h^0(A, \mathscr{H}|_A) \leq n + 4.$$

In this section, we investigate the polarized manifolds in question case by case.

The case of $h^0(A, \mathscr{H}|_A) = n + 4$ Since $\Delta(A, \mathscr{H}|_A) = 0$ and $\mathrm{Pic}(A) \cong \mathbb{Z}$, it follows from [Fu 5, Chapter I (5.10)] that $(A, \mathscr{H}|_A)$ is either $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$ or $(\mathbb{Q}^n, \mathscr{O}_{\mathbb{Q}^n}(1))$. Moreover, since $\mathscr{H}|_A^n = 4$, we get a contradiction. Hence this case does not occur.

The case of $h^0(A, \mathscr{H}|_A) = n + 3$ We see that $(A, \mathscr{H}|_A)$ has a regular ladder by the argument as in the proof of Lemma 1.5.4. Then we obtain that $g(A, \mathscr{H}|_A) \geq \Delta(A, \mathscr{H}|_A) = 1$ by the Riemann–Roch theorem. Therefore we see $g(A, \mathscr{H}|_A) = 1$ by combining $4 = \mathscr{H}|_A^n > 2\Delta(A, \mathscr{H}|_A) = 2$ and [Fu 5, Chapter I (3.5.3)]. This implies that $(A, \mathscr{H}|_A)$ is a del Pezzo manifold of degree 4, which is $(V_{2,2}^n, \mathscr{O}_{V_{2,2}^n}(1))$ due to [Fu 5, (8.11)].

For $(l, \mathscr{H}^{n+1}) = (1,4)$, $L = \mathscr{H}$ gives an embedding of X into $\mathbb{P}^{n+3}$. Hence it follows from [Mo, Corollary 3.8] that $(X,L) \cong (V_{2,2}^{n+1}, \mathscr{O}_{V_{2,2}^{n+1}}(1))$. We are in the case (vi) in Theorem 2.1.1.

For $(l, \mathscr{H}^{n+1}) = (2,2)$, we see that $h^0(X, \mathscr{H}) = n+3$ from the Kodaira vanishing theorem. Since $\Delta(X, \mathscr{H}) = 0$ and $\mathscr{H}^{n+1} = 2$, we have $(X,L) \cong (\mathbb{Q}^{n+1}, \mathscr{O}_{\mathbb{Q}^{n+1}}(2))$. Hence we are in the case (v).

For $(l, \mathscr{H}^{n+1}) = (4,1)$, we see that this case does not occur as follows: Since $h^0(X, \mathscr{H}) = n+3$, we obtain that $\Delta(X, \mathscr{H}) = -1$, which is absurd.

The case of $h^0(A, \mathscr{H}|_A) = n+2$ For $(l, \mathscr{H}^{n+1}) = (1,4)$, we have $h^0(X, \mathscr{H}) = n+3$ by the Kodaira vanishing theorem. Hence we obtain that $\Delta(X, \mathscr{H}) = 2$. Combining $\dim X > 5$ and [Fu 5, (10.8.1)], we see that $(X,L) \cong (H_4^{n+1}, \mathscr{O}_{H_4^{n+1}}(1))$. Thus we are in the case (ii) in the Theorem.

For $(l, \mathscr{H}^{n+1}) = (2,2)$, we have $h^0(X, \mathscr{H}) = n+2$, hence $\Delta(X, \mathscr{H}) = 1$. It follows from [Fu 5, (6.13)] that $(X,L) \cong (Y_2, 2\mathscr{L})$. Thus we are in the case (vii).

For $(l, \mathscr{H}^{n+1}) = (4,1)$, we have $\Delta(X, \mathscr{H}) = 0$. Therefore $(X,L) \cong (\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(4))$, which is the case (i).

The case of $h^0(A, \mathscr{H}|_A) = n+1$ Since $\mathscr{H}|_A^n = 4$, we have $l \neq 1$, hence

(I) $\Delta(X, \mathscr{H}) = \mathscr{H}^{n+1} = 1$;

(II) $\Delta(X, \mathscr{H}) = \mathscr{H}^{n+1} = 2$.

Let $H_1, \ldots, H_n \in |\mathscr{H}|$ be general members, and put $X_k := \bigcap_{k \le i \le n} H_i$ for every $1 \le k \le n$. Then each X_k is a k-dimensional submanifold of X due to [Fu 4, (13.1)] and [Fu 3, (4.1)]. Moreover, by combining $H^1(X, \mathscr{O}_X) = 0$ and the Lefschetz-type theorem [Fu 5, (7.1.4)], we see that the ladder $\{X_k\}_{1 \le k \le n+1}$ is regular, where we put $X_{n+1} := X$. Therefore we have $h^0(X_k, \mathscr{H}|_{X_k}) = k$ for all $1 \le k \le n+1$. Since $L|_{X_1}$ is very ample and has degree 4, we have $g(X, \mathscr{H}) = g(X_1) = 1$ or 3. Then we argue case by case.

For the case $g(X, \mathscr{H}) = 1$, we are in the case (I) by [Fu 5, (12.3)] and $\mathrm{Pic}(X) \cong \mathbb{Z}$. Hence $(X, \mathscr{H})$ is a del Pezzo manifold of degree one, which is the case (iii) in Theorem 2.1.1.

For the case $g(X, \mathscr{H}) = 3$ and (I), it turns out that (iv) shows up. In fact, we prove the following

Proposition 2.3.1 *Assume that* $g(X,\mathscr{H}) = 3$ *and* (I). *Then* $(X,\mathscr{H}) \cong (W_{12}, \mathscr{O}_{W_{12}}(1))$, *where* $W_{12} \subset \mathbb{P}(4,3,1^{n+1})$ *is a smooth weighted hypersurface of degree* 12.

Proof. We first note that X_1 is isomorphic to a plane quartic curve because of $g(X_1) = 3$. Next, we will show that

(1) $R(X_1, \mathscr{H}|_{X_1}) \cong \mathbb{C}[x,y,z]/(F_{12})$, where $\mathrm{wt}(x,y,z) = (4,3,1)$ and $F_{12} = x^3 + y^4 + z\psi_{11}$ for some homogeneous polynomial $\psi_{11} \in \mathbb{C}[x,y,z]$ of degree 11; and

(2) The restriction map $\rho\colon R(X_2, \mathscr{H}|_{X_2}) \to R(X_1, \mathscr{H}|_{X_1})$ is surjective.

It suffices to prove the above: In fact, from (1) and (2), we see that X_2 is a weighted hypersurface of degree 12 in $\mathbb{P}(4,3,1^2)$, and therefore the assertion follows from [Mo, Proposition 3.10].

(1) We find the generators of $R(X_1, \mathscr{H}|_{X_1})$ and the relations among them by using the Riemann–Roch theorem for X_1. By the sectional genus formula, we obtain $K_{X_1} = 4\mathscr{H}|_{X_1}$. Therefore we have

$$h^0(l\mathscr{H}|_{X_1}) = h^0((4-l)\mathscr{H}|_{X_1}) + l - 2.$$

For all $l \geq 5$, we see $h^0(l\mathscr{H}|_{X_1}) = l - 2$. For $l \leq 4$, we get the following table because of the well-known fact that a smooth plane quartic has no g^1_2:

l	$h^0(l\mathscr{H}_{X_1})$	l	$h^0(l\mathscr{H}_{X_1})$
1	1	3	2
2	1	4	3

Let z be a basis of the vector space $H^0(\mathscr{H}|_{X_1})$. Choose $y \in H^0(3\mathscr{H}|_{X_1})$ such that $H^0(3\mathscr{H}|_{X_1}) = \langle y, z^3\rangle$. Similarly, choose $x \in H^0(4\mathscr{H}|_{X_1})$ such that $H^0(4\mathscr{H}|_{X_1}) = \langle x, yz, z^4\rangle$. From now on, we proceed in two steps.

Step 1 We claim that the graded ring $R(X_1, \mathscr{H}|_{X_1})$ is generated by three elements x,y,z. Indeed, it suffices to show that there exist some monomials in x,y,z which form a basis of $H^0(l\mathscr{H}|_{X_1})$ for each $l \geq 5$.

We use induction on l. By the assumption (I), we see that $\mathrm{Bs}|\mathscr{H}|$ is a single point p. Note that each monomial in x,y contained in $H^0(l\mathscr{H}|_{X_1})$ has a pole of order exactly l at p. When $l = 5$, we see that the monomials xz, yz^2, z^5 are linearly independent by comparing their orders of poles at p, hence form a basis of $H^0(5\mathscr{H}|_{X_1})$.

Suppose that the assertion holds for $l - 1 \geq 5$. Note that $h^0(l\mathscr{H}|_{X_1}) = h^0((l-1)\mathscr{H}|_{X_1}) + 1$. It is easily shown that for two coprime positive integers a,b and an integer l with $l \geq (a-1)(b-1)$, the equation $ai+bj=l$ has at least one solution (i,j) of non-negative integers.

Set $(a,b)=(4,3)$. Then, due to $l \geq 6$, there exists at least one section written as $x^i y^j$ $(i,j \geq 0)$ in $H^0(l\mathscr{H}|_{X_1})$, not contained in $zH^0((l-1)\mathscr{H}|_{X_1})$. Hence $H^0(l\mathscr{H}|_{X_1}) = \mathbb{C}x^i y^j \bigoplus zH^0((l-1)\mathscr{H}|_{X_1})$. From the induction hypothesis, the assertion holds for l. This proves our claim.

By Step 1, there exists a surjective homomorphism of graded rings

$$\Phi\colon \mathbb{C}[x,y,z] \to R(X_1, \mathscr{H}|_{X_1}).$$

Step 2 We show that there exists an irreducible homogeneous polynomial F_{12} of degree 12 in $\mathbb{C}[x,y,z]$ such that $\operatorname{Ker}(\Phi) = (F_{12})$. Indeed, there exist no relations of degree $l < 12$ since the equation $4i+3j=l$ has at most one solution (i,j) of non-negative integers. For $l=12$, there are exactly 11 monomials in x,y,z of degree 12. On the other hand, $h^0(12\mathscr{H}|_{X_1}) = 10$. Therefore there exists one relation F_{12} of degree 12, which is written as

$$F_{12} = x^3 + y^4 + z\psi_{11}(x,y,z)$$

after we replace x and y by suitable scalar multiples, where ψ_{11} is a homogeneous polynomial in x,y,z of degree 11.

It turns out that F_{12} is irreducible as follows: We can show that x^3+y^4 is irreducible, immediately. Write $F_{12} = P_1(x,y,z)P_2(x,y,z)$ with some $P_1, P_2 \in \mathbb{C}[x,y,z]$. Without loss of generality, we may assume $P_1(x,y,0)=1$. Hence $P_1(x,y,z) = 1+z\xi_1$ and $P_2 = x^3+y^4+z\xi_2$, where ξ_1, ξ_2 are polynomials in x,y,z. We have

$$\psi_{11}(x,y,z) = \xi_1(x^3+y^4+z\xi_2) + \xi_2.$$

It follows that $\xi_1 = 0$. Indeed, otherwise, the highest term of the right-hand side has degree ≥ 12, which is absurd. Therefore F_{12} is irreducible.

Moreover, combining this and the fact that

$$\operatorname{ht}(\operatorname{Ker}(\Phi)) \leq \dim \mathbb{C}[x,y,z] - \dim R(X_1, \mathscr{H}|_{X_1}) = 1,$$

we obtain $\operatorname{Ker}(\Phi) = (F_{12})$. Thus (1) is proved.

(2) It suffices to prove that $R(X_2, \mathscr{H}|_{X_2})$ is Cohen–Macaulay, which is equivalent to finding a regular sequence of length $\dim R(X_2, \mathscr{H}|_2) = 3$ contained in $R(X_2, \mathscr{H}|_{X_2})_+ := \bigoplus_{l>0} H^0(X_2, l\mathscr{H}|_{X_2})$.

Before proving this, we fix our notation: Let $\mathbf{s} = \{s_0, \ldots, s_N\}$ be a minimal set of generators of $R(X_2, \mathscr{H}|_{X_2})$. Then there exists an isomorphism

$$R(X_2, \mathscr{H}|_{X_2}) \cong \mathbb{C}[s_0, \ldots, s_N]/I_{\mathbf{s}},$$

where $I_{\mathbf{s}}$ is the homogeneous ideal defining X_2.

First we find a regular sequence of length 2 contained in $R(X_2, \mathscr{H}|_{X_2})_+$ as follows: Since $h^0(X_2, \mathscr{H}|_{X_2}) = 2$, we see that $H^0(\mathscr{H}|_{X_2})$ has a basis $\{s, t\}$ satisfying

$$\rho(s) = z \text{ and } (t)_0 = X_1.$$

We may assume that $\mathbf{s}$ contains these two elements. It is easy to check that $t, s \in R(X_2, \mathscr{H}|_{X_2})_+$ form a regular sequence of length 2.

Next, we find an $R(X_2, \mathscr{H}|_{X_2})/(t, s)$-regular element. One needs some information about generators of $I_{\mathbf{s}}$. For each $l \geq 0$, let

$$\rho_l \colon H^0(l\mathscr{H}|_{X_2}) \twoheadrightarrow H^0(l\mathscr{H}|_{X_2})/\langle t \rangle \hookrightarrow H^0(l\mathscr{H}|_{X_1}).$$

denote the restriction map. We proceed in two steps.

Step 1 We show that the ideal $I_{\mathbf{s}}$ has no generators in degrees ≤ 4 as follows: Firstly, we see that

$$\mathrm{Im}(\rho_4) = H^0(4\mathscr{H}|_{X_2}) \tag{†}$$

combining $h^0(4\mathscr{H}|_{X_1}) = 3$, the very ampleness of $L = 4\mathscr{H}$ and the irrationality of X_1.

Subsequently, we find a basis of $H^0(l\mathscr{H}|_{X_2})$ for each $1 \leq l \leq 4$.

For $l = 1$, there exist no relations in $H^0(\mathscr{H}|_{X_2})$ by virtue of the minimality of $\mathbf{s}$.

For $l = 2$, there are no relations: In fact, it follows that $H^0(2\mathscr{H}|_{X_2}) = \langle s^2, st, t^2 \rangle$. Indeed, for any $\eta \in H^0(2\mathscr{H}|_{X_2})$, we can write $\rho_2(\eta) = cz^2$ with some $c \in \mathbb{C}$. Therefore we see that η is a linear combination of s^2, st, t^2. These three monomials are linearly independent because each order of pole along X_1 differs from that of the others.

For $l = 3$, we note that $1 \leq \mathrm{rank}(\rho_3) \leq h^0(3\mathscr{H}|_{X_1}) = 2$. We argue whether there are relations or not, case by case. We first suppose $\mathrm{rank}(\rho_3) = 1$. Then, by the same argument as in the case $l = 2$, we see $H^0(3\mathscr{H}|_{X_2}) = \langle s^3, s^2t, st^2, t^3 \rangle$, which asserts that there are no relations. By (†), there exist sections $u, v \in H^0(4\mathscr{H}|_{X_2})$ such that

$\rho_4(u) = x, \rho_4(v) = yz$. It is easy to see that $H^0(4\mathscr{H}|_{X_2}) = \langle u, v, s^4, s^3t, s^2t^2, st^3, t^4 \rangle$, therefore there are no relations in $H^0(4\mathscr{H}|_{X_2})$.

Next, suppose that $\mathrm{rank}(\rho_3) = 2$. Let w denote a section such that $\rho_3(w) = y$. Then we see that

$$H^0(3\mathscr{H}|_{X_2}) = \langle w, s^3, s^2t, st^2, t^3 \rangle,$$
$$H^0(4\mathscr{H}|_{X_2}) = \langle u, sw, tw, s^4, s^3t, s^2t^2, st^3, t^4 \rangle,$$

where u is a section such that $\rho_4(u) = x$. Therefore there exist no relations. In this way, it turns out that $I_{\mathbf{s}}$ has no generators in degrees ≤ 4.

Step 2 We claim that there exists an $R(X_2, \mathscr{H}|_{X_2})/(t,s)$-regular element. Let u denote a section of $H^0(4\mathscr{H}|_{X_2})$ such that $\rho_4(u) = x$. We assert that u is $R(X_2, \mathscr{H}|_{X_2})/(t,s)$-regular. Indeed, $\mathrm{Proj}(R(X_2, \mathscr{H}|_{X_2})/(t,s))$ is an integral scheme p because of $\mathscr{H}|_{X_2}^2 = 1$. Thus we see that $(R(X_2, \mathscr{H}|_{X_2})/(t,s))_+$ has no zero-divisors. Let m be a homogeneous element of degree a in $R(X_2, \mathscr{H}|_{X_2})/(t,s)$ such that $um = 0$. If $a > 0$, we have $m = 0$ obviously. If $a = 0$, then we obtain $m = 0$ by Step 1. Therefore our claim is proved.

Consequently, due to (1) and (2), the proposition is proved. ■

For the case $g(X, \mathscr{H}) = 3$ and (II), we have $K_X \cong (2-n)\mathscr{H}$. Hence it follows that $H^1(X_3, m\mathscr{H}|_{X_3}) = 0$ for all $m \geq 0$. We also see that the restriction map

$$\rho : H^0(X_2, m\mathscr{H}|_{X_2}) \to H^0(X_1, m\mathscr{H}|_{X_1}) \qquad (\ddagger)$$

is surjective for all $m \geq 0$.

Proposition 2.3.2 *Assume that* $g(X, \mathscr{H}) = 3$ *and* (II). *Then* $L = 2\mathscr{H}$ *is not very ample.*

Proof. Using (‡), we obtain that $h^0(X_2, 2\mathscr{H}|_{X_2}) = h^0(X_1, 2\mathscr{H}|_{X_1}) + 2 = 5$. Suppose that L is very ample. Then we see that $L|_{X_2}$ gives an embedding of X_2 into $\mathbb{P}^4$. But the double point formula for surfaces (see [BS, Lemma 8.2.1]) $L|_{X_2}^2(L|_{X_2}^2 - 5) - 10(g(X_2, L|_{X_2}) - 1) + 12\chi(\mathscr{O}_{X_2}) - 2K_{X_2}^2 = 0$ implies that $-7 + 3p_g(X_2) = 0$, which is absurd. ■

Therefore we see that this case cannot occur, which completes the proof of Theorem 2.1.1.

Chapter 3

Polarized manifolds admitting a Castelnuovo manifold among their ample divisors

3.1 Introduction

Let $\mathscr{L}$ be an ample line bundle on a smooth complex projective $(n+1)$-fold X. To determine the structure of X such that the complete linear series $|\mathscr{L}|$ contains a "special" variety has been an interesting subject in adjunction theory and, therefore, investigated by several authors (e.g. [LPS 3]; [Se], [SV], [Fa], [LPS 2], [LPS 1], [Lan], [BFS] in a more strict setting that $\mathscr{L}$ is very ample).

In this chapter, we shall study the case where $|\mathscr{L}|$ contains a Castelnuovo manifold, precisely speaking, the structures of the polarized manifolds $(X,\mathscr{L})$ with the following condition:

($\star$) There exists a member A of $|\mathscr{L}|$ such that $(A,\mathscr{H})$ is a Castelnuovo manifold with some ample and spanned line bundle $\mathscr{H} \in \mathrm{Pic}(A)$.

By a *Castelnuovo manifold* we here mean a polarized manifold $(A,\mathscr{H})$ such that $|\mathscr{H}|$ defines the birational morphism onto its image and that the sectional genus $g(A,\mathscr{H})$ attains the maximum $\gamma(A,\mathscr{H})$ given in terms of both the Δ-genus $\Delta(A,\mathscr{H})$ and the degree $\mathscr{H}^n$ (for the definition of γ, see Section 3.2). This is a generalization of a curve of maximal genus studied originally by Castelnuovo ([Ca1]; cf. [GH, pp. 527–533], [ACGH, Chapter III, §2]), to the higher dimensional cases, due to Fujita [Fu 5, (16.7)](for other generalizations, see, e.g. [Harr], [Ci]).

Note that if $(X,\mathscr{L})$ is a Castelnuovo manifold then so is $(A,\mathscr{L}|_A)$ (see [Fu 5, (16.6)]). Therefore we have $\mathscr{L}|_A \cong \mathscr{H}$ in this case. Now, in the case of $\mathscr{L}|_A \ncong \mathscr{H}$, what kind of the pairs $(X,\mathscr{L})$ show up?

The purpose of this chapter is to provide a complete classification of the pairs $(X,\mathscr{L})$ with $(\star)$ under the assumption $n > \mathscr{H}^n$. It turns out that those pairs $(X,\mathscr{L})$ fall into only three simple series and, moreover, that those X must be weighted complete intersections of codimension ≤ 3. To be more precise, our main result is

Theorem 3.1.1 *Let X be a smooth complex projective variety of dimension $n+1$. Assume that $0 < d < n$. Then the following* (I)–(III) *are equivalent*:

(I) *There exists an ample line bundle $\mathscr{L}$ on X satisfying the condition $(\star)$ and $d = \mathscr{H}^n$.*

(II) *There exists a very ample line bundle $\mathscr{L}$ on X satisfying $(\star)$ and $d = \mathscr{H}^n$.*

(III) *$(X,\mathscr{L})$ is one of the following*:

- (i) *$(W_d, \mathscr{O}_W(l))$ with some positive integer l dividing d, where W_d is a smooth weighted hypersurface of degree d in the weighted projective space $\mathbb{P}(l,1^{n+2})$;*
- (ii) *$(W_{2,d/2}, \mathscr{O}_W(l))$ with $l = 1,2$ or l dividing $d/2$, where the given d is an even number ≥ 4, and $W_{2,d/2}$ is a smooth weighted complete intersection of type $(2,d/2)$ in $\mathbb{P}(l,1^{n+3})$; or*
- (iii) *$(W_{2,2,2}, \mathscr{O}_W(l))$ with $l = 1$ or 2, where $W_{2,2,2}$ is a smooth weighted complete intersection of type $(2,2,2)$ in $\mathbb{P}(l,1^{n+4})$.*

Moreover, for each of the list (i)–(iii), *$\mathscr{L}|_A \cong \mathscr{H}$ holds if and only if $l = 1$.*

Our proof consists of two parts: (I) $\Rightarrow$ (III) and (III) $\Rightarrow$ (II) ((II) $\Rightarrow$ (I) is trivial). The main part is to prove the former. We utilize Fujita's basic structure theorem of Castelnuovo manifolds (see [Fu 5, (16.7)–(16.14)] or Theorem 3.2.6 in this chapter). Specifically, we shall describe the structure of $(X,\mathscr{L})$ by classifying $(A,\mathscr{H})$ in terms of d and $\Delta(A,\mathscr{H})$: (A) $d > 2\Delta(A,\mathscr{H})$, (B) $d = 2\Delta(A,\mathscr{H})$ and (C) $d < 2\Delta(A,\mathscr{H})$. Castelnuovo varieties of type (A), (B) and (C) are called of the first kind, the second kind and the third kind in [Fu 5, §16], respectively.

Basically the study of the case (B) reduces to M. C. Beltrametti-Fania-Sommese's result [BFS, Proposition 3.1] and to a classification result of Mukai manifolds [Mu].

In the case (C), Fujita's basic structure theorem gives explicit descriptions of the possible types of a Castelnuovo variety $(A, \mathscr{H})$. However the theorem does *not* tell whether or not A does become an ample divisor on X, whence some detailed arguments are needed to exclude certain pairs $(A, \mathscr{H})$. In fact, by showing that the intersection $\mathscr{W}_A$ of hyperquadrics containing A in $\mathbb{P}(|\mathscr{H}|)$ is neither a generalized cone over the Veronese surface nor one over a smooth rational normal scroll under $n > d$ (see Lemma 3.4.5), we complete the proof in the case (C).

The difficulty in dealing with the case (A) is as follows: The structures of Castelnuovo manifolds $(A, \mathscr{H})$ still remain unrevealed in general (cf. [Fu 5, (16.7) below & Chapter I (3.5.3)]). By using Lemma 3.2.7, we can successfully describe $(A, \mathscr{H})$ explicitly under $n > d$, reducing to classification results of Fano manifolds of coindex ≤ 2 (see Proposition 3.4.1).

This chapter is organized as follows: In Section 3.2, we first state two lemmas needed in Sections 3.3 and 3.4. After that, we introduce and summarize several fundamental results on Castelnuovo varieties. Also we prove Lemma 3.2.7 that plays a key role in the proof of (I) $\Rightarrow$ (III). In Section 3.3, for each of the list (i)–(iii) in Theorem 3.1.1, we show the very ampleness of its line bundle by using Lemma 3.2.1 and prove that $(\star)$ is satisfied by taking an appropriate $\mathscr{H} \in \mathrm{Pic}(A)$. The latter part of the theorem is immediately verified from a result that $l\mathscr{H} \cong \mathscr{L}_A$ with some $l \geq 1$. Section 3.4 is devoted to proving (I) $\Rightarrow$ (III).

Notation

In this chapter, we adopt the following notation.

- $H^n_{d_1,\ldots,d_r}$: an n-dimensional smooth complete intersection of type $(d_1, \ldots, d_r)$ in $\mathbb{P}^{n+r}$.
- $\mathbb{P}(\mathscr{E})$: the $\mathbb{P}^{s-1}$-bundle associated to a locally free sheaf $\mathscr{E}$ of rank s over $\mathbb{P}^1$.
- $W^n_{d_1,\ldots,d_r}$: a smooth weighted complete intersection of type $(d_1, \ldots, d_r)$ in the weighted projective space $\mathbb{P}(l, 1^{n+r}) := \mathbb{P}(l, \underbrace{1, \ldots, 1}_{n+r})$ of dimension n.
- $S * T$: the closure of the union of all the lines passing through $s \in S$ and $t \in T$ in $\mathbb{P}^n$, where S and T are subsets of $\mathbb{P}^n$.
- $\mathrm{Ridge}(X) := \{x \in X \mid x * X = X\}$ for a projective variety $X \subset \mathbb{P}^n$.

- $[q]$: the integer part of a rational number $q \geq 0$.

3.2 Preliminaries: Castelnuovo varieties

We begin with two lemmas needed in later. Here we simply state the results, referring to [Laf, Corollary 2.3] for a proof of the former and to the proof of [Fu 1, (3.8)] for a proof of the latter.

Lemma 3.2.1 (A. Laface) *Let (M,L) be a polarized manifold. Suppose that L is spanned and that the graded ring $R(M,L) := \bigoplus_{i=0}^{\infty} H^0(M,iL)$ is generated in degrees $\leq r$. Then the line bundle rL is very ample.*

Lemma 3.2.2 (Fujita) *Let (M,L) be a polarized manifold of dimension $m \geq 4$ and A a member of $|lL|$ for some $l \geq 1$. Assume that $(A, L|_A)$ is a smooth complete intersection of type $(d_1,\ldots,d_r)$ in $\mathbb{P}^{m+r}$. Then (M,L) is a weighted complete intersection of type $(d_1,\ldots,d_r)$ in $\mathbb{P}(l,1^{m+r+1})$. Furthermore l divides one of $d_1,\ldots,d_r$.*

In what follows, we give a brief summary of fundamental results on Castelnuovo varieties, referring to [Fu 5, §16]. Let (V,L) be a pair consisting of a projective variety V and a spanned line bundle L such that $|L|$ defines the birational morphism φ onto its image. Fujita proved that the following inequality holds for arbitrary (V,L):

$$g(V,L) \leq \Delta(V,L)F - \big(d - \Delta(V,L) - 1\big)\binom{F}{2},$$

where $d := L^{\dim V}$ and $F := \left[\dfrac{d-1}{d-\Delta(V,L)-1}\right]$. Define $\gamma(V,L)$ as the right-hand side.

Definition 3.2.3 A *Castelnuovo variety* is a polarized variety (V,L) with its spanned line bundle L such that φ is birational and that $g(V,L) = \gamma(V,L)$.

Castelnuovo varieties have distinguished properties as below. We only state the result, referring to [Fu 5, (16.6) & (16.9)] for a proof.

Proposition 3.2.4 (Fujita) *Let (V,L) be an m-dimensional Castelnuovo variety. Then L is simply generated, hence very ample. Furthermore, let $\mathscr{W}_V$ be the intersection of all the hyperquadrics containing V in $\mathbb{P}(|L|)$, and assume that $L^m < 2\Delta(V,L)$. Then it follows that $\dim \mathscr{W}_V = m+1$ and $\Delta(\mathscr{W}_V, \mathscr{O}_{\mathscr{W}_V}(1)) = 0$.*

From now on, we deal with only the case where a Castelnuovo variety (M,L) is smooth. Especially, in the case where $L^{\dim M} < 2\Delta(M,L)$, according to Proposition 3.2.4, one can describe $(\mathscr{W}_M, \mathscr{O}_{\mathscr{W}_M}(1))$ by using Fujita's classification result of polarized varieties of Δ-genus zero [Fu 5, Chapter I, (5.10) & (5.15)], where a generalized cone emerges.

Here we define a generalized cone (cf. [Fu 5, (5.13)]) and fix some notation to introduce Theorem 3.2.6.

Definition 3.2.5 Let (X,L) be a polarized k-fold with its very ample line bundle L. Then (X,L) is said to be a *generalized cone* over a polarized s-fold $(S,\mathscr{L})$ if the following conditions are satisfied:

(1) $S = \bigcap_{j=1}^{k-s} V_j \subset X \subset \mathbb{P}(|L|)$ and $\mathscr{L} \cong L|_S$, where each V_j is some general member of $|L|$;

(2) $\mathrm{Ridge}(X) \neq \emptyset$ and $\mathrm{Ridge}(X) \cap S = \emptyset$; and

(3) $X = S * \mathrm{Ridge}(X) \subset \mathbb{P}(|L|)$.

Notation and Remark. If $(\mathscr{W}_M, \mathscr{O}_{\mathscr{W}_M}(1))$ is a generalized cone over $(\mathbb{P}(\mathscr{E}), H(\mathscr{E}))$, where $\mathscr{E} := \bigoplus_{i=1}^{n-r} \mathscr{O}_{\mathbb{P}^1}(\delta_i)$ with some $\delta_i > 0$ and $r := \dim \mathrm{Ridge}(\mathscr{W}_M)$, then it follows from [Fu 5, Chapter I, (5.15)] that $\mathrm{Ridge}(\mathscr{W}_M)$, which is a linear space of $\mathscr{W}_M$, coincides with the singular locus $\mathrm{Sing}(\mathscr{W}_M)$. Set $(\tilde{\mathscr{W}}_M, \mathscr{O}_{\tilde{\mathscr{W}}_M}(1)) := (\mathbb{P}(\mathscr{F}), H(\mathscr{F}))$, where $\mathscr{F} := \mathscr{E} \bigoplus \mathscr{O}_{\mathbb{P}^1}^{\oplus(r+1)}$. Then $|H(\mathscr{F})|$ defines a birational morphism $\tilde{\mathscr{W}}_M \to \mathscr{W}_M$. Let $\tilde{M}$ be the strict transform of M on $\tilde{\mathscr{W}}_M$ and $p : \tilde{\mathscr{W}}_M \to \mathbb{P}^1$ the bundle projection.

The following theorem, obtained by Fujita, gives a rough classification of Castelnuovo manifolds. For a proof, we refer to [Fu 5, (16.7)–(16.14)].

Theorem 3.2.6 (Fujita's basic structure theorem of Castelnuovo manifolds) *Let (M,L) be a Castelnuovo manifold of dimension $m \geq 1$. Then one of* (A)–(C) *holds.*

(A) $d := L^m > 2\Delta(M,L)$;

(B) $d = 2\Delta(M,L)$, *then (M,L) is a Mukai manifold, i.e. a polarized manifold with $-K_M \cong (m-2)L$, with its simply generated line bundle L; or*

(C) $d < 2\Delta(M,L)$, *then one of the following holds.*

(a) $\mathscr{W}_M \cong \mathbb{P}^{m+1}$ *and* $(M,L) \cong (H_d, \mathscr{O}_H(1))$.

(b) $\mathscr{W}_M \cong \mathbb{Q}^{m+1}$ *and* $(M,L) \cong (H_{2,d/2}, \mathscr{O}_H(1))$.

(c) $\mathscr{W}_M$ *is a generalized cone over* $(\mathbb{P}(\mathscr{E}), H(\mathscr{E}))$, $M \cong \tilde{M} \in |aH(\mathscr{F}) + bp^*\mathscr{O}_{\mathbb{P}^1}(1)|$, $L \cong H(\mathscr{F})|_M$ *and* $r = -1, 0, 1$. *Furthermore we have*

  (i) $a > 0$ *and* $1 - \sum_{i=1}^{n+1} \delta_i \le b \le 1$ *if* $r = -1$, *i.e.,* $\mathscr{W}_M$ *is smooth.*

  (ii) $a > 0$ *and* $0 \le b \le 1$ *if* $r = 0$.

  (iii) $a > 0$ *and* $b = 1$ *if* $r = 1$.

(d) $\mathscr{W}_M$ *is a generalized cone over* $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}}(2))$, $M \in |\mathscr{O}_{\mathscr{W}_M}(a)|$ *for some* $a > 0$, $L \cong \mathscr{O}_{\mathscr{W}_M}(1)|_M$ *and* $r \le 0$, *therefore* $\dim \mathscr{W}_M \le 3$.

We conclude this section with the following

Lemma 3.2.7 *Let* $(X, \mathscr{L})$ *be a polarized manifold of dimension* $n+1$ *with* $(\star)$ *and* $d = \mathscr{H}^n < n$. *Then* $\mathrm{Pic}(A) = \mathbb{Z}[\mathscr{H}]$ *and* $\mathrm{Pic}(X) \cong \mathbb{Z}$.

Proof. We first prove the former assertion. Since $\mathscr{H}$ is very ample, A is embedded into $\mathbb{P}(|\mathscr{H}|)$. As $\dim A - \mathrm{codim}_{\mathbb{P}} A \ge n - (d-1) \ge 2$ by virtue of our assumption $n > d$, Barth's theorem [Laz 2, Corollary 3.2.3] applies, therefore $\mathrm{Pic}(A) \cong \mathbb{Z}$. Moreover $\mathscr{H}$ turns out to be the ample generator. Indeed, if not so, then we can write $\mathscr{H} = tH$ with some $t \ge 2$ and the ample generator $H \in \mathrm{Pic}(A)$. Taking the both self-intersection numbers, we have $d = t^n H^n > t^d \ge 2^d$, which is absurd. Hence the former is obtained.

Next we show the latter. If $n > d \ge 2$, then the assertion follows from the Lefschetz theorem [Fu 5, (7.1.5)]. When $n > d = 1$, we see that $A \cong \mathbb{P}^n$. It follows from [Fu 5, (7.18)] that $X \cong \mathbb{P}^{n+1}$, hence $\mathrm{Pic}(X) \cong \mathbb{Z}$. Thus the latter is proved. ■

3.3 Proof of (III) ⇒ (II)

Let $(W, \mathscr{O}_W(l))$ be one of the list (i)–(iii) in Theorem 3.1.1. The aim of this section is to prove that $(W, \mathscr{O}_W(l))$ satisfies the condition $(\star)$. We obtain the conclusion with the following steps:

(a) The line bundle $\mathscr{O}_W(l)$ is very ample;

(b) There exists a pair $(A, \mathscr{H})$ consisting of a smooth member $A \in |\mathscr{O}_W(l)|$ and an ample and spanned line bundle $\mathscr{H}$ with $\mathscr{H}^n = d$ such that $|\mathscr{H}|$ defines a birational morphism; and

(c) For a pair $(A,\mathscr{H})$ as in (b), the equality $g(A,\mathscr{H}) = \gamma(A,\mathscr{H})$ holds.

We first prove (a). Thanks to Lemma 3.2.1, it suffices to show that $R(W,\mathscr{O}_W(1))$ is generated in degrees $\leq l$ and that $\mathscr{O}_W(1)$ is spanned. We show the former: Let s be the number of defining equations of W (e.g. $s = 2$ when $W = W_{2,d/2}$). We may assume that $l \neq 1$. Since W is a weighted complete intersection of $\dim W = n+1 > d+1 \geq 2$, we infer that $\mathrm{Pic}(W) = \mathbb{Z}[\mathscr{O}_W(1)]$ and that the restriction map of graded rings

$$r\colon \mathbb{C}[x,y_0,\dots,y_{n+s}] \to R(W,\mathscr{O}_W(1))$$

is surjective, where $\mathrm{wt}(x,y_i) = (l,1)$ for each $0 \leq i \leq n+s$. Therefore $H^0(W,\mathscr{O}_W(1)) = \langle y_0,\dots,y_{n+s}\rangle$, and $R(W,\mathscr{O}_W(1))$ is generated in degrees $\leq l$. It only remains to show that $\mathscr{O}_W(1)$ is spanned. Indeed, it is verified as follows: The base locus is

$$\mathrm{Bs}|\mathscr{O}_W(1)| = \bigcap_{i=0}^{n+s}(y_i = 0) \subset W.$$

Since W does not meet the locus $\bigcup_{1<k} S_k = \bigcap_{i=0}^{n+s}(y_i = 0) \subset \mathrm{Proj}(\mathbb{C}[x,y_0,\dots,y_{n+s}])$, we see the spannedness of $\mathscr{O}_W(1)$, hence (a) is proved.

Next we show (b). At least one of the given defining equations of $W \subset \mathbb{P}(l,1^{n+s})$ is monic in the variable x because W does not meet $\bigcup_{1<k} S_k$. Now define $A := (x + f_l(y_0,\dots,y_{n+s}) = 0)$ in W, where f_l is a homogeneous polynomial of degree l. Furthermore Bertini's theorem assures that $A \in |\mathscr{O}_W(l)|$ is smooth if f_l is chosen to be general. Also define $\mathscr{H} := \mathscr{O}_W(1)|_A$. The amplitude and spannedness of $\mathscr{H}$ follow from that of $\mathscr{O}_W(1)$. We easily check that $\mathscr{H}^n = d$ by using [Mo, Proposition 3.2]. We can verify the birationality of the map φ associated to $|\mathscr{H}|$. Indeed, we see that $\varphi(A) \subset \mathbb{P}^{n+s}$ is an n-dimensional complete intersection of type similar to that of W. Taking the Stein factorization of φ, we infer that A is birational to a finite covering T of $\varphi(A)$. The degree of this covering $\varpi\colon T \to \varphi(A)$ must be one because $d = \mathscr{H}^n = \deg(\varpi)\deg_{\mathbb{P}^{n+s}}\varphi(A) = d\deg(\varpi)$. Thus φ is birational, so (b) is proved.

Finally, we verify (c) with a case-by-case analysis on (i)–(iii) in Theorem 3.1.1. For the case (1), we obtain $\Delta(A,\mathscr{H}) = d-2$ due to $h^0(A,\mathscr{H}) = n+2$. Also we easily check that $K_A \cong (d-n-2)\mathscr{H}$, hence we have

$$g(A,\mathscr{H}) = 1 + \frac{1}{2}(d-3)d = (d-2)(d-1) - \binom{d-1}{2} = \gamma(A,\mathscr{H}).$$

Consider the case (2). Write $d = 2k$ with some integer $k \geq 2$. Using $\Delta(A,\mathscr{H}) = 2k-3$ and $K_A \cong (k-n-1)\mathscr{H}$, we obtain

$$g(A,\mathscr{H}) = 1 + (k-2)k = (2k-3)(k-1) - \binom{k-1}{2} = \gamma(A,\mathscr{H}).$$

We treat the remaining case (3). By easy calculations, we see that $d = 8, \Delta(A,\mathscr{H}) = 4$ and $K_A \cong (2-n)\mathscr{H}$. Thus we have $g(A,\mathscr{H}) = \gamma(A,\mathscr{H}) = 5$.

To conclude, the implication (III) $\Rightarrow$ (I) is proved.

3.4 Proof of (I) $\Rightarrow$ (III)

For a given $d < n$, let $(X,\mathscr{L})$ be a smooth polarized $(n+1)$-fold satisfying $(\star)$ and $d = \mathscr{H}^n$, where $(A,\mathscr{H})$ is a Castelnuovo manifold with $A \in |\mathscr{L}|$. In this section, we classify the pairs $(X,\mathscr{L})$. Our proof is divided into the three parts as in Theorem 3.2.6: (A) $d > 2\Delta(A,\mathscr{H})$; (B) $d = 2\Delta(A,\mathscr{H})$ or (C) $d < 2\Delta(A,\mathscr{H})$.

The case (A)

We first prove the following

Proposition 3.4.1 *Let $(A,\mathscr{H})$ be an n-dimensional Castelnuovo manifold of degree $d < n$. Then the following* (1) *and* (2) *are equivalent.*

(1) *$(A,\mathscr{H})$ is of the first kind.*

(2) *$(A,\mathscr{H})$ is one of the following*:

- (a) $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$;
- (b) $(\mathbb{Q}^n, \mathscr{O}_{\mathbb{Q}^n}(1))$;
- (c) $(H_3, \mathscr{O}_{H_3}(1))$;
- (d) $(H_{2,2}, \mathscr{O}_{H_{2,2}}(1))$; *or*
- (e) $(\mathrm{Gr}(5,2), \mathscr{O}_{\mathrm{Gr}}(1))$, *where* $\mathrm{Gr}(5,2)$ *is a Grassmann variety parametrizing the* 2*-dimensional linear subspace in* $\mathbb{C}^5$.

Proof. Since the implication (2) $\Rightarrow$ (1) is immediately proved, we here prove the converse. Since $\mathscr{H}$ is spanned, Bertini's theorem yields that $(A,\mathscr{H})$ has a ladder consisting of smooth rungs. Hence it follows from [Fu 5, Chapter 1, (3.5.3)] that $g(A,\mathscr{H}) = \Delta(A,\mathscr{H})$. Combining this and the assumption of the case (A), we obtain that

$$d-2 > 2g(A,\mathscr{H})-2 = \bigl(K_A+(n-1)\mathscr{H}\bigr)\cdot\mathscr{H}^{n-1}.$$

Furthermore, we can write $K_A = r\mathscr{H}$ with some integer r thanks to Lemma 3.2.7. Hence we have $(r+n-2)d < -2$, therefore

$$-(n+1) \leq r \leq -(n-1).$$

Now proceed with a case-by-case analysis on the value of r. If $r = -(n+1)$, then it follows from [Fu 5, (11.2)] that $(A,\mathscr{H}) \cong (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}}(1))$, hence we are in (a).

If $r = -n$, then it follows from both [Fu 5, (11.7)] and Lemma 3.2.7 that $(A,\mathscr{H}) \cong (\mathbb{Q}^n, \mathscr{O}_{\mathbb{Q}}(1))$, thus we are in (b).

If $r = -(n-1)$, then $(A,\mathscr{H})$ is a del Pezzo manifold of degree d, which satisfies $\Delta(A,\mathscr{H}) = 1$. By the assumption of (A), we infer that $n > d \geq 3$. Combining Lemma 3.2.7, the very ampleness of $\mathscr{H}$ and a classification result of del Pezzo manifolds by Fujita [Fu 5, (8.11)], we see that $(A,\mathscr{H})$ is one of the following: (c) $(H_3, \mathscr{O}_H(1))$; (d) $(H_{2,2}, \mathscr{O}_H(1))$; or (e) $(\mathrm{Gr}(5,2), \mathscr{O}_{\mathrm{Gr}}(1))$. ∎

Claim 3.4.2 *In the case* (A), *the pair* $(X,\mathscr{L})$ *is isomorphic to one of the following*: (i) $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}}(1))$; (ii) $(W_2, \mathscr{O}_W(l))$ *with* $l = 1,2$; (iii) $(W_3, \mathscr{O}_W(l))$ *with* $l = 1,3$; *or* (iv) $(W_{2,2}, \mathscr{O}_W(l))$ *with* $l = 1,2$.

Proof. For (a) in Proposition 3.4.1, combining $n > d = 1$ and [Fu 5, (7.18)], we see $(X,\mathscr{L}) \cong (\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}}(1))$. Thus we are in (i) of our claim. As for (b), since $n > d = 2$, by using Lemma 3.2.2, we see that $(X,\mathscr{L}) \cong (W_2, \mathscr{O}_W(l))$ with $l = 1$ or 2, which is (ii).

For (c), it follows from Lemma 3.2.2 that $(X,\mathscr{L}) \cong (W_3, \mathscr{O}_W(l))$ with $l = 1$ or 3. Hence we are in (iii). For (d), we similarly see that $(X,\mathscr{L}) \cong (W_{2,2}, \mathscr{O}_W(l))$ with $l = 1$ or 2, which is (iv).

The case (e) cannot occur. Indeed, A must be $\mathrm{Gr}(5,2)$ since $n > d = 5$. On the other hand, it is impossible that $\mathrm{Gr}(5,2)$ is contained as an ample divisor on a smooth projective variety X because of [Fu 2, (5.2)]. Thus we are done. ∎

The case (B)

Claim 3.4.3 *In the case* (B), *the pair* $(X,\mathscr{L})$ *is isomorphic to one of the following*: (i) $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}}(4))$; (ii) $(\mathbb{Q}^{n+1}, \mathscr{O}_{\mathbb{Q}}(3))$; (iii) $(H_3, \mathscr{O}_H(2))$; (iv) $(W_4, \mathscr{O}_W(2))$, *where* $W_4 \subset \mathbb{P}(2,1^{n+2})$; (v) $(H_{2,2}, \mathscr{O}_H(2))$; (vi) $(H_4, \mathscr{O}_H(1))$; (vii) $(H_{2,3}, \mathscr{O}_H(1))$; *or* (viii) $(H_{2,2,2}, \mathscr{O}_H(1))$.

Proof. Due to Theorem 3.2.6, $(A, \mathscr{H})$ is a Mukai manifold with its simply generated line bundle $\mathscr{H}$. Note that $n > d \geq 2$. Combining Beltrametti-Fania-Sommese's result [BFS, Proposition 3.1] and Lemma 3.2.7, we see that $(X, \mathscr{L})$ is one of the following:

(a) a Mukai manifold $(\mathfrak{M}, \mathfrak{L})$;

(b) $(\mathscr{M}, 2\mathscr{L})$, where $(\mathscr{M}, \mathscr{L})$ is a del Pezzo manifold;

(c) $(\mathbb{Q}^{n+1}, \mathscr{O}_{\mathbb{Q}}(3))$, which is (ii); or

(d) $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}}(4))$, which is (i).

We first use a classification of Mukai manifolds [Mu] to describe the structure of $(X, \mathscr{L})$ explicitly in the case (a). Due to $n > d \geq 2$, we see that $d = 2, 4, 6$ or 8. Also the very ampleness of $\mathfrak{L}$ implies that $(\mathfrak{M}, \mathfrak{L})$ is one of the following:

(vi) $\mathfrak{M} \cong H_4$ and $\mathfrak{L} \cong \mathscr{O}_H(1)$;

(vii) $\mathfrak{M} \cong H_{2,3}$ and $\mathfrak{L} \cong \mathscr{O}_H(1)$; or

(viii) $\mathfrak{M} \cong H_{2,2,2}$ and $\mathfrak{L} \cong \mathscr{O}_H(1)$.

Next we treat the case (b). Now we utilize Fujita's classification result of del Pezzo manifolds. Since $\mathrm{Pic}(\mathscr{M}) \cong \mathbb{Z}$, we see that $(\mathscr{M}, 2\mathscr{L})$ is one of the following possible cases:

(b1) $\mathscr{M}$ is a weighted hypersurface of degree 6 in $\mathbb{P}(3,2,1^{n+1})$, and $2\mathscr{L} \cong \mathscr{O}_{\mathscr{M}}(2)$;

(iv) $\mathscr{M} \cong W_4 \subset \mathbb{P}(2, 1^{n+2})$ and $2\mathscr{L} \cong \mathscr{O}_W(2)$;

(iii) $\mathscr{M} \cong H_3$ and $2\mathscr{L} \cong \mathscr{O}_H(2)$;

(v) $\mathscr{M} \cong H_{2,2}$ and $2\mathscr{L} \cong \mathscr{O}_H(2)$;

(b2) $\mathscr{M} \cong G$, a linear section of $\mathrm{Gr}(5,2) \subset \mathbb{P}^9$, and $2\mathscr{L} \cong \mathscr{O}_G(2)$; or

(b3) $\mathscr{M} \cong \mathbb{P}^3$ and $2\mathscr{L} \cong \mathscr{O}_{\mathbb{P}}(4)$.

It is proved that the cases (b1), (b2) and (b3) cannot occur. In fact, as for (b1), note that $\mathscr{H}^n = 2$ and $\Delta(A, \mathscr{H}) = 1$. Since $|\mathscr{H}|$ gives a double covering of $\mathbb{P}^n$ (see [Fu 5, (6.10)]), it turns out that $\mathscr{H}$ is not simply generated, which is absurd. As to (b2) (resp. (b3)), we have that $\dim \mathscr{M} = n+1 \leq 6$ (resp. $= 3$) and $d = 5$ (resp. 8), which contradicts the assumption $n > d$. Consequently the claim is proved. ■

The case (C)

Claim 3.4.4 *In the case* (C), $(X,\mathscr{L})$ *is either* (i) $(W_d,\mathscr{O}_W(l))$ *with* l *dividing* $d\geq 5$ *or* (ii) $(W_{2,d/2},\mathscr{O}_W(l))$ *with* $l=1,2$ *or* l *dividing* $d/2$*, where the given* $d\geq 8$ *is even.*

Proof. Let $\mathscr{W}_A$ be the intersection of hyperquadrics containing A in $\mathbb{P}(|\mathscr{H}|)$. Due to Proposition 3.2.4, we see that $\Delta(\mathscr{W}_A,\mathscr{O}_{\mathscr{W}_A}(1))=0$. Therefore, by using a classification of polarized varieties of Δ-genus zero [Fu 5, (5.10)–(5.15)], $(\mathscr{W}_A,\mathscr{O}_{\mathscr{W}_A}(1))$ is isomorphic to either (a) $(\mathbb{P}^{n+1},\mathscr{O}_{\mathbb{P}}(1))$; (b) $(\mathbb{Q}^{n+1},\mathscr{O}_{\mathbb{Q}}(1))$; (c) $(\mathbb{P}(\mathscr{E}),H(\mathscr{E}))$, or a generalized cone over it, where $\mathscr{E}$ is an ample vector bundle over $\mathbb{P}^1$; or (d) a Veronese surface $(\mathbb{P}^2,\mathscr{O}_{\mathbb{P}^2}(2))$, or a generalized cone over it.

In the case of (a), we see $A\cong H_d$ of degree $d\geq 5$ since $d<2\Delta(H_d,\mathscr{O}_H(1))=2(d-2)$. Hence, using Lemma 3.2.2, we have $(X,\mathscr{L})\cong(W_d,\mathscr{O}_W(l))$ with some positive l dividing $d\geq 5$, which is (i).

In the case of (b), due to $d<2\Delta(H_{2,d/2},\mathscr{O}_H(1))=2(d-3)$, we have $A\cong H_{2,d/2}$ with even $d\geq 8$. Therefore we see that $(X,\mathscr{L})\cong(W_{2,d/2},\mathscr{O}_W(l))$ with some positive l dividing either 2 or $d/2\geq 4$, which is (ii).

Consequently, by the following Lemma 3.4.5, the claim is proved. ∎

Lemma 3.4.5 *The cases* (c) *and* (d) *cannot occur.*

Proof. The proof is divided into two cases as follows:

The case of (c) First, assume that $\mathscr{W}_A$ is smooth, which is isomorphic to $\mathbb{P}(\mathscr{E})$ for some ample vector bundle $\mathscr{E}$ over $\mathbb{P}^1$. The linear-normality of $\mathscr{W}_A\subset\mathbb{P}(|\mathscr{O}_{\mathscr{W}_A}(1)|)$ (see [EH, §3 (1)]) yields that $h^0(\mathscr{W}_A,\mathscr{O}_{\mathscr{W}_A}(1))=h^0(A,\mathscr{H})$. Therefore, since

$$\dim\mathscr{W}_A-\operatorname{codim}_{\mathbb{P}(|\mathscr{H}|)}\mathscr{W}_A=(n+1)-(d-2-\Delta(A,\mathscr{H}))=n-d+3+\Delta(A,\mathscr{H})>2,$$

the Barth theorem [Laz 2, Corollary 3.2.3] implies that $\operatorname{Pic}(\mathscr{W}_A)\cong\mathbb{Z}$, which is a contradiction.

Next we assume that $\mathscr{W}_A$ is singular. Remind that $\operatorname{Sing}(\mathscr{W}_A)=\operatorname{Ridge}(\mathscr{W}_A)$. It follows from Theorem 3.2.6 that $r:=\dim(\operatorname{Ridge}(\mathscr{W}_A))$ is either 0 or 1. First, suppose that $r=0$. Set $R:=\operatorname{Ridge}(\mathscr{W}_A)$, which is the vertex of the cone $\mathscr{W}_A$. Then we have two possibilities: $R\notin A$; or $R\in A$. For the former, A is a smooth member of $|\mathscr{O}_{\mathscr{W}_A}(a)|$ for some $a>0$. Furthermore, $\mathscr{W}_A$ is projectively normal in $\mathbb{P}(|\mathscr{H}|)$ since $\mathscr{O}_{\mathscr{W}_A}(1)$ is simply generated. Therefore, putting $\mathscr{E}:=\bigoplus_{i=1}^n\mathscr{O}_{\mathbb{P}^1}(\delta_i)$ with some $\delta_i>0$ for each $1\leq i\leq n$, we obtain

$$d = \deg A = a \deg \mathscr{W}_A = a(\sum_{i=1}^{n} \delta_i) \geq n,$$

which contradicts $n > d$. For the latter $R \in A$, using notation in §2, we have $\tilde{A} \in |aH(\mathscr{F}) + p^* \mathscr{O}_{\mathbb{P}^1}(1)|$, which is ample by virtue of [BS, Lemma 3.2.4]. Hence, by the Lefschetz theorem, we get a contradiction:

$$\mathbb{Z} \cong \mathrm{Pic}(A) \cong \mathrm{Pic}(\tilde{A}) \cong \mathrm{Pic}(\tilde{\mathscr{W}}_A) \cong \mathbb{Z}^{\oplus 2}.$$

Next we treat the case $r = 1$. Then we obtain that $A \in |aH(\mathscr{F}) + p^* \mathscr{O}_{\mathbb{P}^1}(1)|$ with some $a > 0$, which is an ample Cartier divisor. Again, due to the argument similar to that in the case $r = 0$, we get a contradiction. Therefore the case (c) does not occur.

The case of (d) Since $A \in |\mathscr{O}_{\mathscr{W}_A}(a)|$ for some $a > 0$ according to Theorem 3.2.6, we similarly have

$$2 \geq n > d = a \deg \mathscr{W}_A = 4a,$$

which is absurd. Thus we are done. ■

We sum up the above case-by-case arguments as follows:

Proof of (I) ⇒ (III) *in Theorem* 3.1.1. We will show that any pair $(X, \mathscr{L})$ in the above three Claims falls into (i)–(iii) in Theorem 3.1.1. Firstly, each of cases (i)–(iii) in Claim 3.4.2 is $(W_d, \mathscr{O}_W(l))$, where l divides $1 \leq d \leq 3$. Each of cases (i), (iv) and (vi) in Claim 3.4.3 can be viewed as $(W_4, \mathscr{O}_W(l))$ with $l = 4, 2, 1$, respectively. Consequently, we see that the above cases and (i) in Claim 3.4.4 fall into (i) in Theorem 3.1.1.

Secondly, each of cases (ii), (iii) and (vii) in Claim 3.4.3 can be regarded as $(W_{2,3}, \mathscr{O}_W(l))$, where $l = 3, 2, 1$ respectively. Therefore each of the following falls into (ii) in our theorem: (iv) in Claim 3.4.2, the above three cases in Claim 3.4.3 and (ii) in Claim 3.4.4.

Finally, it is easy to see that each of (v) and (viii) in Claim 3.4.3 can be viewed as $(W_{2,2,2}, \mathscr{O}_W(l))$ with $l = 2$ and 1, respectively. Hence our theorem is proved. ■

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