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Abstract

Internet advertisements that are displayed along with the search results for a keyword or a combination of keywords are sold through keyword auctions. In this study, we explore a slot supply strategy of a search engine. Based on the revenue prediction from a game theoretic analysis, we show that restricting the number of advertisements in a search result page and highlighting the top ads are consistent with the revenue maximization of the sellers. We also conduct a comparative statics analysis of the optimal number of the advertising slots to the change in the exogenous conditions.

JEL classification: C72, C91, D44. *Keywords:* Keyword auction, Generalized second price auction, Slot restriction, Optimal number of advertising slots.

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1 Introduction

Internet advertisements that are displayed along with the search results for a keyword or a combination of keywords are sold through keyword auctions. The generalized second price auction (GSP) is the most widely used auction mechanism for selling advertisements on Internet search engines. Each time a user enters a search keyword into a search engine, a GSP-type auction allocates the advertising slots within that user's search results. The Internet advertisement revenue via keyword auctions has increased constantly in the last few years and is a principal source of revenue of search engines.¹

The advertisements related to a searched keyword are usually displayed on the right-hand side of the search result pages. The number of the ads on each search result page is limited to less than or equal to a fixed number, and this holds true for major search engines such as Google and Yahoo!.² This seems to be a puzzle because a search engine can add a new advertising slot with no cost burden and the new advertisement creates an additional advertising revenue. A naive answer for the limit of the number of the advertisements is that it serves the interests of search engine users; too many advertisements on a search result page reduces the users' benefit and the search engine loses endorsement from them.

In this study, we explain why a search engine restricts the number of the advertisements from the perspective of the search engine's slot supply strategy. There are no existing studies investigating the reason for the limited number of advertis-

¹For example, Google's revenue from these auctions increased more than 2.5-fold from \$6 billion in 2005 to \$16.4 billion in 2007. Since 2003, the revenue generated from keyword auctions has accounted for approximately 47% of the total Internet advertisement revenue in the U.S., which increased from \$2.7 billion in 2003 to \$7.2 billion in 2009 (from annual reports by the Interactive Advertising Bureau (http://www.iab.net/insights_research/947883/adrevenuereport)).

²Interestingly, even if you, as a user of Google search or Yahoo! search, change the setting on the number of search results on each page, this ceiling of the number of advertisements on a page remains unchanged.

ing slots on a search result page. We show that limiting the number of advertising slots is a simple but powerful method of increasing the search engine's revenue.

Many studies that consider how a search engine sells advertising slots follow Myerson (1981) and Riley and Samuelson (1981) on the optimal design of an auction mechanism for a single object. Iyengar and Kumar (2006a,b) use a direct mechanism for a keyword auction and considers the computation problem of the optimal allocation of advertising slots. Assuming the separable click through-rates (CTRs), Edelman and Schwarz (2010) analyze the optimal reserve price and show that the GSP with the optimal reserve price is the optimal auction. In Ostrovsky and Schwarz (2009), the impact of the reserve price is measured in a field experiment on the actual keyword auction conducted by Yahoo!. Although in reality, the auctioneer does not have the exact information on the distributions of the bidders' values and cannot set the optimal reserve price, these results suggest that a reserve price is an effective method of increasing the seller's revenue. In contrast, in this study, we explore another method, limiting the number of advertising slots, to improve the revenue.

Our analysis adopts a simplified model of keyword auctions. We assume that each advertiser knows his value (expected revenue) per click. The CTR of an advertising slot depends only on its position, and we assume that the CTRs of advertising slots are common knowledge. All advertisers maximize expected profit (defined as total value of clicks received minus total payment in the auction). We describe a keyword auction as a one-shot incomplete information game where each advertiser simultaneously announces his bid to a search engine.

From the game theoretic analysis, we obtain an explicit representation of a search engine's revenue. Then, we show that for a given keyword, there exists an optimal number of advertising slots that maximizes the search engine's expected revenue and this optimal number is independent of the values of CTRs of all advertising slots. This result is obtained from a suggestive theorem of *decreasing marginal return across slots*, i.e., the marginal return on CTR at a *higher* advertis-

ing slot is greater than that at a *lower* advertising slot.

The decreasing marginal return across slots gives us a useful insight to understand a slot supply strategy of a search engine. It implies that a search engine can increase its revenue by ramping the CTRs across slots. In other words, to increase the expected revenue, the search engine designs the placement of the advertisements in a manner that the advertisements in higher positions receive several clicks and the advertisements in lower positions receive few. This insight seems to be consistent with the current form of keyword auctions where a few advertisements in the top positions are highlighted and the advertisements in lower positions are displayed only in the second or third page of search results.

We also analyze a more general question of how a seller of divisible goods such as land, forest and water, divides the goods into a finite number of items with different sizes when the items are sold through position auctions. We show that our result from the analysis of keyword auctions can be applied to this general framework, and the optimal number defined in a keyword auction becomes the upper-bound of the optimal number of items in this context.

Finally, we analyze the comparative statics on the optimal number of advertising slots in keyword auction. We provide a sufficient condition for the optimal number of slots for a certain keyword being greater than or equal to that for another keyword. We also show that the optimal number of slots is non-decreasing as the number of potential advertisers increases.

The remainder of the paper is organized as follows. In Section 2, we explain the basic setup of our model and provide the result on the seller's expected revenue. In Section 3, we propose the slot supply strategy of a search engine, and show that a restriction on the number of advertising slots on a page and an accentuation of the advertisement in the higher position are effective methods of increasing the seller's revenue. We also apply our result to the more general question of how a seller of divisible goods divides the goods into a finite number of items in Section 4. In Section 5, we conduct a comparative statics analysis on the optimal number of

slots. We conclude in Section 6.

2 Model

2.1 Basic setup

A keyword auction is defined by the following components. There are $n, n \geq 2$, advertisers (bidders) participating in a keyword auction and each advertiser i has a value or expected revenue v_i for a click on the advertisement. There are K advertising slots with click-through rates (CTRs) $\alpha_1, \alpha_2, \dots, \alpha_K$, where α_k is the estimated probability of being clicked or the estimated number of clicks per given period, for an advertiser in the k -th advertising slot (the slot k). We assume $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_K$ and set $\alpha_k = 0$ for all $k > K$ for notational convenience. We assume $n \geq K$.³ Each advertiser submits a bid to the auction. The bid submitted by i is denoted by b_i . We denote the bid profile of n advertisers by $\mathbf{b} = (b_1, \dots, b_n)$.

In the generalized second price auction (GSP), advertisers are allocated advertising slots in the descending order of the bids b_1, b_2, \dots, b_n . Let $d(k)$ denote the name of the bidder who submits k -th highest bid among \mathbf{b} . In the GSP, bidder $d(k)$ acquires the slot k . The advertiser obtaining the slot k pays the bid of the advertiser obtaining one lower advertising slot (the slot $k + 1$) for each click. Hence, the payment $p_k^G(\mathbf{b})$ is $\alpha_k b_{d(k+1)}$. To complete the definition of the payments, we assume that $b_{d(k)} = 0$ if $k > n$. From this, when $K = n$, the payments of $d(K)$ is assumed to be zero, and for $k > K$, bidder $d(k)$ pays $\alpha_k b_{d(k+1)} = 0$ by the definition of α_k . The payoff of bidder $d(k)$ is given by $\alpha_k v_{d(k)} - \alpha_k b_{d(k+1)}$.

2.2 Revenue of a search engine

We model a keyword auction in the GSP as a normal form game of incomplete information. Thus, we assume that v_i , the valuation of i to obtain the advertising slot, is private information.

³In reality, this is not a restriction because when $n < K$, it is sufficient to redefine K by $K = n$.

Let $F_i : [0, \bar{v}] \rightarrow [0, 1]$ be a distribution function of the valuation of bidder i and $f_i(v_i) \equiv F'_i(v_i)$ is its density function. We assume that the value of each bidder is independent and identically distributed and thus $F_i(\cdot) = F(\cdot)$ and $f_i(\cdot) = f(\cdot)$ for all i . We call $\{v \in [0, \bar{v}] | 0 < F(v) < 1\}$ the effective domain of F . We further assume that F is increasing in its effective domain. This implies that $F^{-1}(u)$ is well defined and increasing for $u \in (0, 1)$. We define $F^{-1}(0)$ by $\lim_{u \rightarrow 0^+} F^{-1}(u)$ and $F^{-1}(1)$ by $\lim_{u \rightarrow 1^-} F^{-1}(u)$.

To analyze the equilibrium behavior of the bidders in the keyword auction, we focus on the symmetric equilibrium where every bidder follows the same strategy that determines the bid depending on his true valuation. Let $\beta : [0, \bar{v}] \rightarrow \mathbb{R}_+$ be such a strategy and $\beta(v)$ be the bid of the advertiser whose value is v . We assume that $\beta(\cdot)$ is increasing and differentiable and $\beta(0) = 0$.

Given a bidding profile $\mathbf{b} = (b_1, \dots, b_n)$, the revenue of the auctioneer is defined by

$$m(\mathbf{b}) = \sum_{k=1}^K \alpha_k b_{d(k+1)}.$$

For any vector of values, $\mathbf{v} = (v_1, \dots, v_n)$, let $\beta(\mathbf{v}) = (\beta(v_1), \dots, \beta(v_n))$.

Let V_i denote a random variable of bidder i 's valuation and $\mathbf{V} = (V_1, \dots, V_n)$ be a random vector. Note that $\beta(\mathbf{V})$ is also a random vector. The expected revenue of the auctioneer is

$$E[m(\beta(\mathbf{V}))],$$

where $E[\cdot]$ is a expectation operator with respect to a distribution function F .

In this section, we explore the explicit representation of the expected revenue in order to analyze the revenue maximizing behavior of the auctioneer. To find the explicit formula of the expected revenue, we rely on the *revenue equivalence theorem* (see, for example, Lahaie, Pennock, Saberi, and Vohra 2007). From the revenue equivalence theorem, we know that the expected revenue in the GSP auction is the same as the one in the VCG (Vickrey 1961, Clarke 1971, Groves 1973) mechanism, in which submitting bidder's true value is his dominant strategy and thus the symmetric equilibrium is constructed by $\beta(v_i) = v_i$.

The VCG mechanism uses the following allocation and payments rules. The allocation rule is the same as the one in the GSP. For the payment rule, advertiser $i = d(k)$ pays the negative externality that i imposes on other advertisers. The payment of advertiser i who acquires the slot k is

$$p_k^V(\mathbf{b}) = \left[\sum_{h=1}^{k-1} \alpha_h b_{d(h)} + \sum_{h=k}^K \alpha_h b_{d(h+1)} \right] - \left[\sum_{h=1}^{k-1} \alpha_h b_{d(h)} + \sum_{h=k+1}^K \alpha_h b_{d(h)} \right], \quad (1)$$

where, assuming that each advertiser submits the bid of its true value for a click, the expression in the first square bracket is the sum of the revenue of advertisers other than i when i leaves the auction, and the expression in the second square bracket is the sum of their revenue when i participates in the auction.

By simple calculation, Eq. (1) is reduced to

$$p_k^V(\mathbf{b}) = \sum_{h=k+1}^{K+1} (\alpha_{h-1} - \alpha_h) b_{d(h)} = \sum_{h=k}^K (\alpha_h - \alpha_{h+1}) b_{d(h+1)}.$$

From this, we obtain a convenient recursive formula for the payment of advertisers:

$$p_K^V(\mathbf{b}) = \alpha_K b_{d(K+1)}, \text{ and for each } k = 1, 2, \dots, K-1, p_k^V(\mathbf{b}) = (\alpha_k - \alpha_{k+1}) b_{d(k+1)} + p_{k+1}^V(\mathbf{b}).$$

For any value vector $\mathbf{v} = (v_1, \dots, v_n)$, let $(v_{(1)}, \dots, v_{(n)})$ be the rearrangement of v_i by the non-increasing order, i.e., $v_{(k)} \geq v_{(h)}$ for any k and h with $k \geq h$. For notational convenience, we set $v_{(n+1)} = 0$. For a given \mathbf{v} , the revenue of the auctioneer in the VCG is

$$m^V(\mathbf{v}) = \sum_{k=1}^K p_k^V(\mathbf{v}) = \sum_{k=1}^K \sum_{h=k}^K (\alpha_h - \alpha_{h+1}) v_{(h+1)} = \sum_{k=1}^K k(\alpha_k - \alpha_{k+1}) v_{(k+1)} \quad (2)$$

where note that $\alpha_{K+1} = 0$.

Proposition 1. *The expected revenue of the auctioneer in the GSP auction with incomplete information, M , is*

$$\sum_{k=1}^K k(\alpha_k - \alpha_{k+1}) E[V_{(k+1)}]$$

where $V_{(k+1)}$ is the $(k+1)$ -th highest order statistic generated from distribution function F and we set $E[V_{(n+1)}] = 0$.

Proof. From the revenue equivalence theorem and Eq. (2),

$$M = E[m^V(\mathbf{V})] = E\left[\sum_{k=1}^K k(\alpha_k - \alpha_{k+1})V_{(k+1)}\right] = \sum_{k=1}^K k(\alpha_k - \alpha_{k+1})E[V_{(k+1)}].$$

□

Thus, the expected revenue of the auctioneer is the weighted average of the expectation of the $(k+1)$ -th order statistic of advertisers' valuation when we normalize $\sum_{k=1}^K \alpha_k = 1$ because $\sum_{k=1}^K k(\alpha_k - \alpha_{k+1}) = \sum_{k=1}^K \alpha_k$.

When there is only one advertising slot and $\alpha_1 = 1$, the expected revenue is $E[V_{(2)}]$ and this is similar to the expected revenue in a single-object auction without reserve price (see, for example, Krishna 2002). Thus, this result is a natural extension to the case of selling multiple items.

The expected revenue of a search engine defined in the proposition is supported from other frameworks. In a complete information setting, Varian (2007) and Edelman et al. (2007) introduce an equilibrium concept, known as a locally envy-free equilibrium, and show that the revenue defined in Proposition 1 is equal to the expectation of the revenue predicted from the bidder-optimal locally envy-free equilibrium.⁴ In the recent work of Edelman and Schwarz (2010), the bidder-optimal locally envy-free equilibrium is uniquely justified from the criterion obtained by considering the upper bound of the revenue of a finitely repeated keyword auction of incomplete information when the reserve price is optimally selected. Several works focus on the dynamic aspect of keyword auctions. Cary, Das, Edelman, Giotis, Heimerl, Karlin, Mathieu, and Schwarz (2007) examine the dynamic process of bidding behavior where, in each period, one bidder changes the bid to the one that produces the most favorable outcome for the bidder, taking other bidders' bids in the previous period as given. They show that this bidding behavior converges to the bidder-optimal locally envy-free equilibrium. Kamiyo (2010) considers the bidding behavior in an environment where each bidder changes his bid without

⁴The bidder-optimal locally envy-free equilibrium gives the lowest revenue among the set of all locally envy-free equilibria.

realizing others' current and past bids. Kamijo (2010) shows that the same result as Cary et al. (2007) holds even in this setting. In a laboratory experiment, Fukuda, Kamijo, Takeuchi, Masui, and Funaki (2010) found that there is no statistical difference between the average revenue of a seller in a repeatedly played keyword auction in the lab and the revenue in the bidder-optimal locally envy-free equilibrium.

3 Slot supply strategy of a search engine

In this section, we consider how a search engine can increase its revenue by modifying the placement of advertising slots and the number of the advertising slots displayed on a search result page.

3.1 Motivating Example

Let us consider a situation where there are five bidders and five advertising slots for a keyword. The values of the bidders are distributed according to a uniform distribution between 0 and 100. The CTRs are given by $(\alpha_1, \dots, \alpha_5) = (100, 80, 60, 40, 20)$.

In the case of uniform distribution, the expectation of the k -th highest order statistic is given by

$$E[V_{(k)}] = \bar{v} \frac{n+1-k}{n+1}, \quad (3)$$

where v is distributed between 0 and \bar{v} and the number of the bidders is n . From Proposition 1 and the above equation, we calculate the expected revenue of the auctioneer and obtain $M \cong 6667$.

Next, let us consider a situation where the CTRs of advertising slots diminish. This is not a difficult task for the search engine because it determines the place where the advertisements will appear or the number of the advertisements on the first search result page. For example, we consider a situation where α_5 becomes 0 (i.e., deleting slot 5). In this new situation, the expected revenue of the auctioneer becomes $M = 8000$. This implies that the auctioneer's revenue is improved by

reducing CTRs of advertising slots.⁵

As shown in the example above, the revenue of the auctioneer can be improved by manipulating the CTRs, especially by reducing the CTRs. This is an easier method of increasing revenue than manipulating the auction mechanism.

To obtain greater insight on the auctioneer's slot supply strategy of the auctioneer, we consider the case where the CTR of one slot is improved by one point. Let M^k , $k = 1, 2, 3, 4, 5$, denote the expected revenue when the CTR of slot k is slightly improved and is changed from α_k to $\alpha_k + 1$. By easy calculation, we have

$$M^1 - M \cong 67,$$

$$M^2 - M \cong 33,$$

$$M^3 - M = 0,$$

$$M^4 - M \cong -33,$$

$$M^5 - M \cong -67.$$

From these equations, we find that on one hand, an increase in CTR for slots 1 or 2 enhances the auctioneer's revenue, and on the other hand, that for slots 4 or 5 decreases the revenue. We also find that $M^k - M$ is decreasing in k with $M^1 - M > 0$ and $M^5 - M < 0$. This implies that there exist L such that $M^L - M \geq 0$ and $M^{L+1} - M < 0$ (for this example, $L = 3$). Since the expected revenue is linearly varied to the change in the CTRs, it is indicated that supplying the top L advertising slots is the optimal strategy for the search engine.

Why does the revenue decrease even though the CTR of an advertising slot is improved? This counter-intuitive observation is explained as follows.⁶ When the

⁵This result does not rely on our simplified setting that the price of the lowest advertising slot (slot 5) is zero. Even if there exists a minimal price $\varepsilon > 0$ which the bidder in the last position pays for each click and ε is sufficiently small, the search engine can increase its revenue by deleting the slot 5.

⁶Because we do not obtain the explicit representation of equilibrium strategy for a keyword auction with incomplete information, the following explanation is based on the equilibrium bid at the bidder-optimal locally envy-free equilibrium for a keyword auction with complete information. As

CTR of an advertising slot, for example slot k , is improved, the bidder $d(k)$ who currently occupies slot k is satisfied with the current slot and thus, loses the incentive to obtain higher advertising slots, thereby decreasing his bid. This decrease occurs even though bidder $d(k+1)$ increases his bid in order to obtain slot k . The decrease in the bid of $d(k)$ then motivates the decrease in the bid of $d(k-1)$, and this induces the further decrease in the bid of $d(k-2)$, and so on. Therefore, while the search engine's revenue obtained from slot k increases due to the improvement of CTR of slot k combined with an increase of $b_{d(k+1)}$ (i.e., the positive *direct* effect of the improvement of CTR of slot k), the revenues obtained from higher advertising slots will decrease (i.e., the negative *indirect* effect of the improvement of CTR of slot k). If slot k is in a lower position, the former effect (the positive direct effect) is dominated by the latter (the negative indirect effect), and thus, the revenue of the search engine will decrease. On the other hand, if slot k is in a higher position, the former effect dominates the latter, and thus, the search engine's revenue is improved.

3.2 Optimal slot restriction and accentuating certain slots

In this subsection, we examine the observation made in the previous subsection by a general framework. To illustrate the results in this subsection, we assume a situation where the seller initially provides a sufficient number of advertising slots to the advertisers. Thus, we assume $K = n$.

We define the *marginal return of slot k* , denoted by μ_k , as the derivative of M by α_k . Thus, μ_k is defined as follows: for $k = 1, 2, \dots, K$,

$$\mu_k \equiv \frac{\partial M}{\partial \alpha_k} = -(k-1)E[V_{(k)}] + kE[V_{(k+1)}].$$

From this equation, we find that the marginal expected return of slot k is a constant and independent of the values of CTRs of advertising slots. If we compare this with an example in the previous subsection, μ^k corresponds to $M^k - M$. Thus, our aim is to show that $\mu_1, \mu_2, \dots, \mu_K$ are in the decreasing order.

explained in subsection 2.2, this prediction is consistent with the expected revenue in Proposition 1.

Next we explain our assumption on distribution function F . We define the virtual valuation of the bidder whose value is v by $\phi_F(v) = v - \frac{1-F(v)}{f(v)}$. We say that F is *regular* (or F satisfies regularity) if a virtual valuation is increasing in the effective domain of F .⁷ The regularity is a weaker condition than the increasing hazard rate (the hazard rate defined by $\frac{f(v)}{1-F(v)}$ is increasing in v). F is *weakly regular* (or F satisfies weak regularity) if a virtual valuation is non-decreasing but not a constant.

The next theorem is a generalization of the observation in the previous subsection and one of our main results.

Theorem 1 (Decreasing marginal return across slots). *Assume that F satisfies regularity. The following three statements hold:*

- (i) $\mu_1 > \mu_2 > \dots > \mu_K$,
- (ii) $\mu_1 > 0$, and
- (iii) $\mu_K < 0$.

The proof is in the appendix. It is not difficult to show that the statements of this theorem hold for the case where F is weakly regular.

This theorem gives us a useful insight to understand a search engine's slot-supply strategy. It implies that a search engine can increase its revenue not by uniformly improving CTRs across slots but by ramping the CTRs across advertising slots. In fact, in the example given in the previous subsection, the expected revenue of the auctioneer is unchanged even if the CTRs of advertising slots are uniformly improved and then become $(\alpha_1, \dots, \alpha_5) = (100 + a, 80 + a, 60 + a, 40 + a, 20 + a)$ for any $a > 0$. On the other hand, if the CTRs of advertising slots are inclined and become $(\alpha_1, \dots, \alpha_5) = (100 + a, 80 + a, 60, 40 - a, 20 - a)$ for any $a \in (0, 20)$, the expected revenue increases by $200a$. Thus, to increase the expected revenue, the search engine designs the placement of the advertisements in a manner that the

⁷This assumption is the same as one in Myerson (1981) on the optimal design of auction.

ads in higher positions receive several clicks and the ads in lower positions receive fewer clicks.

In the rest of this subsection, we explain the consequences of Theorem 1 for two types of search engine's slot supply strategies. One strategy is to restrict the number of the advertising slots displayed on the search result page. In the real world, this strategy is executed in a keyword auction.⁸ The other is accentuating certain advertising slots to improve their CTRs.

For $L \leq K$, an *L-slot restriction* is the slot supply strategy such that a search engine sells only top L advertising slots. In other words, the search engine set $\alpha_k = 0$ for any $k > L$ with keeping the CTRs of top L advertising slots unchanged. From Theorem 1, the optimal slot restriction is obtained.

Proposition 2. *Assume that F is regular. Let L^* satisfy*

$$\mu_{L^*} \geq 0 > \mu_{L^*+1}.$$

Then, the L^ -slot restriction maximizes the expected revenue of the seller among any L -slot restrictions.*

Proof. This is clear from Theorem 1. □

There are three remarks on this proposition. First, L^* is determined only by the distribution function F (correctly speaking, the order statistics of F). Thus, a change in CTRs of advertising slots does not influence the optimal number L^* . This enables a search engine to execute the different types of slot supply strategy separately. Second, even if we consider the *slot restriction on the first page*, L^* is still the optimal number. An *L-slot restriction on the first page* is the slot supply strategy such that a search engine makes the CTR of slot $k, k > L$, $r\alpha_k$, where $r \in (0, 1)$. Third, in the case of uniform distribution on $[0, \bar{v}]$, μ_k is directly calculated

⁸In 2007, the number of ads on a page was restricted to less than or equal to eight (Varian 2007).

as follows.

$$\begin{aligned}\mu_k &= -(k-1)E[V_{(k)}] + kE[V_{(k+1)}] \\ &= -(k-1)\bar{v} \frac{n+1-k}{n+1} + k\bar{v} \frac{n-k}{n+1} \\ &= \bar{v} \frac{n+1-2k}{n+1}.\end{aligned}$$

Thus L^* is $\frac{n+1}{2}$ if n is odd and $\frac{n}{2}$ if n is even.

Since irrespective of the values of CTRs, the optimal number L^* of slots is determined, another search engine's slot-supply strategy is to improve CTR of certain advertising slots. Assume that $\alpha_1 > \alpha_2 > \dots > \alpha_K > 0$. For k , a *slot k -accentuation* is the slot supply strategy such that the CTR of the slot k is slightly improved and that of the other slots remains unchanged. Thus, the search engine makes the CTR of slot k $\alpha_k + \eta$, where η is a small positive number satisfying $\alpha_k + \eta < \alpha_{k-1}$ for any k . The slot k -accentuation is a simplified form of the real world slot supply strategy, in which search engines highlight specific advertising slots.

Proposition 3. *Assume that F is regular. The slot 1-accentuation maximizes the expected revenue of the auctioneer among any slot k -accentuation.*

Proof. From Theorem 1, the marginal return of slot 1 is the highest. \square

The statement of this proposition can hold in a more general framework. First, we consider a more realistic setting such that after slot k -accentuation, the CTRs of the other slots decrease. Thus, while the new CTR of slot k becomes $\alpha_k + \eta$, the new CTR of slot $h, h \neq k$, is $\alpha_h - \delta$, where $(K-1)\delta \leq \eta$, $\delta > 0$ and $\alpha_k + \eta < \alpha_{k-1} - \delta$ for any $k > 1$. Even in this general setting, it is easily shown from Theorem 1 that the slot 1-accentuation gives the highest revenue.

Second, we consider the situation where a seller can freely select the values of CTRs of slots with a restriction such that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_K \geq 0$ and the sum of the CTRs is a constant number $C, C > 0$. What is the solution for this allocation problem? Applying Theorem 1 again, the answer is to set $\alpha_1 = C$ and $\alpha_k = 0$ for

any $k > 1$. Therefore, if the sum of the CTRs is restricted to a constant value, there is no reason for the seller to divide them into several slots.

4 Other applications

In this section, we consider a slightly different situation from the keyword auction. A divisible good that amounts to C , for example, a land, is sold to several producers (i.e., farmers). The landowner can freely partition the lands into K blocks, and the sizes of blocks are $\alpha_1, \alpha_2, \dots, \alpha_K$, where $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_K$. Farmers have different technologies and a farmer with technology v , $v \in \mathbb{R}_+$, obtains a return $v\alpha_k$ if he uses a land whose size is α_k . Assume that technology v is independent and identically distributed according to F . Now consider what happens if the owner sells his land by a generalized second price auction. In this context, the utility of a farmer i who acquires block k is $v_i\alpha_k - b_{d(k+1)}$, where $b_{d(k+1)}$ is defined in a similar manner as in a keyword auction. The only difference between this model and a keyword auction is the payment rule. Then, applying the revenue equivalence theorem, the expected revenue of the landowner is that defined in Proposition 1. Therefore, Theorem 1 also holds for this model, and thus, from the last paragraph in the previous subsection, this landowner maximizes his expected revenue by setting $\alpha_1 = C$ and $\alpha_k = 0$ for other $k > 1$.

The result in the previous paragraph indicates that there is no room for the landowner to partition the land if he wants to sell it.⁹ The reason for this unintuitive result is that in the model above, there is no *benefit from partitioning*. Thus, if we consider the model of diminishing marginal return from land, the result is completely different. We now consider the same problem of the landowner as

⁹Wilson (1979) observed a similar result. He compared the revenue when a single item is sold in a first price auction (unit auction) with the revenue when the share of the item is sold in a share auction in which each bidder submits a schedule which specifies the number of shares requested for each possible price per share. Wilson (1979) observed that in some cases, the revenue in the share auction is only half of that in the unit auction.

that in the previous paragraph, but we now incorporate the benefit of partition by modifying the payoff of the farmers. Let $s : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function satisfying $s(\alpha) > 0, s'(\alpha) > 0$ and $s''(\alpha) < 0$ for any $\alpha > 0$ and $s(0) = 0$. A farmer with technology $v, v \in \mathbb{R}_+$, obtains a return $vs(\alpha_k)$ if he uses a land whose size is α_k . Let $M(\alpha_1, \alpha_2, \dots, \alpha_K)$ be the expected revenue defined in Proposition 1, i.e.,

$$M(\alpha_1, \alpha_2, \dots, \alpha_K) := \sum_{k=1}^K k(\alpha_k - \alpha_{k+1})E[V_{(k+1)}].$$

Then, the problem of the landowner is to maximize $M(s(\alpha_1), s(\alpha_2), \dots, s(\alpha_K))$ under the condition that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_K \geq 0$ and $\sum_{k=1}^K \alpha_k = C$.

This maximization problem can be solved by the Lagrange multiplier method. We have the following necessity conditions for the optimum:

$$\alpha_k(s'(\alpha_k)\mu_k - \lambda) = 0 \text{ for any } k = 1, 2, \dots, K, \quad (4)$$

and

$$s'(\alpha_k)\mu_k - \lambda \leq 0 \text{ for any } k = 1, 2, \dots, K,$$

where λ is a Lagrange multiplier and satisfies $\lambda \geq 0$.

Let L^* be that defined in Proposition 2 and be the optimal number of advertising slots for a keyword auction. Interestingly, we will show that L^* is also an important number in this context. Let k be an integer greater than L^* . By the definition of L^* , $\mu_k < 0$ holds. This implies that for any non-negative λ , $s'(\alpha_k)\mu_k - \lambda < 0$ for any $\alpha_k > 0$. Then, Condition (4) implies that in the optimum, α_k must be zero for any $k > L^*$. Therefore, L^* is the upper bound for the number of land partition. It is readily shown that when $\lim_{\alpha \rightarrow 0^+} s'(\alpha) = \infty$, partitioning the land into L^* blocks is the optimum for the landowner and $\alpha_1, \dots, \alpha_{L^*}$ are determined to satisfy $s'(\alpha_1)\mu_1 = \dots = s'(\alpha_{L^*})\mu_{L^*}$ and $\sum_{k=1}^{L^*} \alpha_k = C$.

The models considered in this section are applied to many other examples such as selling government bonds to securities companies and selling leases of tract for oil and gas exploitation.

5 Comparative statics analysis on the optimal number of slots

In this section, we explore how the optimal number of advertising slots varies with changes in exogenous conditions. Specifically, we consider the comparative statics of a change in a distribution function for the bidders' valuation and a change in the number of the potential bidders in the market on the optimal number of slots.

5.1 Comparative statics of the change in distribution functions

Let F and G be different distribution functions for the bidders' values. We first prepare dominance relations on the set of random variables or the set of distribution functions. In the following analysis, we use notations X and Y to denote random variables distributed according to F and G , respectively.

The first one describes a situation where one random variable X is *larger* than another random variable Y . We say that F *stochastic-dominates* (s-dominates) G and write $F \geq_s G$ if for any $u \in (0, 1)$, $F^{-1}(u) > G^{-1}(u)$. If $F \geq_s G$, we say that X s-dominates Y .

We say that F is the right parallel shift of G if for some $c \in \mathbb{R}$, $F(v) = G(v - c)$ for $v \in [c, \bar{v}]$.

The next is a variability comparison between two random variables. We say that F *dispersive-dominates* (d-dominates) G and write $F \geq_d G$ if $F^{-1}(u) - G^{-1}(u)$ is non-decreasing in $u \in (0, 1)$. Then, if F is the right parallel shift of G , both $F \geq_d G$ and $G \geq_d F$ hold.¹⁰

Assume F and G satisfy regularity. Let $\phi_F(\cdot)$ and $\phi_G(\cdot)$ denote the virtual valuations under distribution function F and G , respectively. Thus, $\phi_F(v) = v - \frac{1-F(v)}{f(v)}$ and $\phi_G(v) = v - \frac{1-G(v)}{g(v)}$. We say that F *virtual valuation-dominates* (v-dominates) G and write $F \geq_v G$ if a random variable $\phi_F(X)$ s-dominates another $\phi_G(Y)$. Thus,

¹⁰For basic results on s-domination and d-domination, refer to Shaked and Shanthikumar (1994) and Boland, Shaked, and Shanthikumar (1998).

if $F \geq_v G$, the virtual valuation calculated from distribution function F is stochastic larger than the virtual valuation calculated from distribution function G .

An important consequence of v -dominance is that $F \geq_v G$ holds if and only if $(1-u)(G^{-1}(u) - F^{-1}(u))$ is increasing in $u \in (0, 1)$. To confirm this, let $\Phi_F(\cdot)$ and $\Phi_G(\cdot)$ be distribution functions of random variables $\phi_F(X)$ and $\phi_G(Y)$, respectively. Then, by the assumption on the regularity, the inverse function of $\Phi_F(\cdot)$ is determined by

$$\Phi_F^{-1}(u) = \phi_F(F^{-1}(u)) = F^{-1}(u) - \frac{1-u}{f(F^{-1}(u))}$$

for any $u \in (0, 1)$. Differentiating $(1-u)(G^{-1}(u) - F^{-1}(u))$ by $u \in (0, 1)$, we have

$$\begin{aligned} & -(G^{-1}(u) - F^{-1}(u)) + (1-u)\left(\frac{1}{g(G^{-1}(u))} - \frac{1}{f(F^{-1}(u))}\right) \\ &= -\left(G^{-1}(u) - \frac{1-u}{g(G^{-1}(u))}\right) + \left(F^{-1}(u) - \frac{1-u}{f(F^{-1}(u))}\right) = -\Phi_G^{-1}(u) + \Phi_F^{-1}(u). \end{aligned}$$

If $F \geq_v G$, this must be positive because $F \geq_v G$ implies $\Phi_F^{-1}(u) > \Phi_G^{-1}(u)$. On the other hand, if this is positive for any $u \in (0, 1)$, $F \geq_v G$ holds.¹¹

To avoid confusion, in this section we write μ_k^F (resp. μ_k^G) instead of μ_k when the values of bidders are i.i.d. F (resp. G). Similarly, we denote the optimal number of slots defined in Proposition 2 by L^{*F} (resp. L^{*G}) when the values of bidders are i.i.d. F (resp. G).

The following theorem shows that the v -dominance is a sufficient condition for the inequality relationship between μ_k^F and μ_k^G .

Theorem 2. *Let F and G be two different distribution functions for the bidders' valuations and satisfy regularity. If F v -dominates G , then*

$$\mu_k^F > \mu_k^G$$

holds for any $k \leq K$.

¹¹Following Bulow and Roberts (1989), for $q \in (0, 1)$, the virtual valuation $\phi_F(F^{-1}(1-q))$ is interpreted as the marginal revenue of the discriminating monopolist when he sells q unit of goods. Then, since $F \geq_v G$ is equivalent to $\Phi_F^{-1}(u) > \Phi_G^{-1}(u)$ for any $u \in (0, 1)$, it is equivalent to the condition that the marginal revenue under F is greater than that under G .

The proof is in the appendix.

From this theorem, we know that if $F \succeq_v G$, for any k , the marginal expected return of slot k when the bidder's values are distributed according to F is greater than that when the bidder's values are distributed according to G .

Combining Theorems 1 and 2, we obtain the result on the comparative statics of the optimal number of advertising slots.

Corollary 1. *Let F and G be two different distribution functions for the bidders' valuations. If both F and G are regular and F v -dominates G , $L^{*F} \geq L^{*G}$.*

Thus, the v -dominance is a sufficient condition for the inequality relationship between the optimal number of advertising slots under different distribution functions. As applications of this corollary, we list the following two propositions.

Proposition 4. *Assume that F and G are regular. If F is the right parallel shift of G , then $L^{*F} \geq L^{*G}$.*

Proof. From the definition of the right parallel shift, there exists $c > 0$ such that $F(v) = G(v - c)$ for any $v \in [c, \bar{v}]$. This implies that $f(v) = g(v - c)$ for any $v \in [c, \bar{v}]$ and $F^{-1}(u) - c = G^{-1}(u)$ hold for any $u \in (0, 1)$. Then,

$$-\left(G^{-1}(u) - \frac{1-u}{g(G^{-1}(u))}\right) + \left(F^{-1}(u) - \frac{1-u}{f(F^{-1}(u))}\right) = c > 0.$$

Thus $F \succeq_v G$, and from Corollary 1, $L^{*F} \geq L^{*G}$. \square

This proposition implies that if the values of the bidders are uniformly increased by some positive constant, the optimal number of slots is increasing or unchanged.

Proposition 5. *Assume that F and G are regular. If $F \succeq_s G$ and $G \succeq_d F$, then $L^{*F} \geq L^{*G}$.*

Proof. By the definitions of the s -dominance and d -dominance, $(G^{-1}(u) - F^{-1}(u))$ is a negative and increasing function in $u \in (0, 1)$. Thus, $(1-u)(G^{-1}(u) - F^{-1}(u))$ is an increasing function; therefore, $F \succeq_v G$. From Corollary 1, $L^{*F} \geq L^{*G}$. \square

This proposition states that if F is larger (in a stochastic sense) than G and is less scattered (in a stochastic sense) than G , the optimal number of slots under F is greater than or equal to that under G .

How can we interpret these propositions on the optimal number of slots? The answer seems to come from the following two intuitive statements: keeping other conditions constant,

- (1) when large bidder's values are more realized, the seller can increase its revenue by supplying more advertising slots, and
- (2) when values of bidders are less scattered, the seller can increase its revenue by supplying more advertising slots.

We explain why these two statements hold. For the first statement, there is a simple reason. When the higher values of the bidders are likely to be realized, the cost of deleting advertising slots becomes expensive. Thus, if the values of the bidders are expected to be high, the search engine can increase the revenue by providing more slots.

Next, we explain the second statement. When the valuations of the advertisers are less scattered, each bidder has a chance to obtain a higher advertising slot by slightly increasing his bid, and this leads to higher bids by advertisers in equilibrium. This results in higher revenue for the auctioneer. In this case, the auctioneer has a weak incentive to decrease the number of slots. On the other hand, when the valuations of the advertisers are more scattered, each bidder cannot obtain a higher advertising slot by a small increase in his bid, and this leads to lower bids by advertisers in equilibrium. This results in less revenue for the auctioneer. In such a case, the auctioneer has a strong incentive to decrease the number of slots in order to make the bids of advertisers higher.

To check these statements, we consider a general uniform distribution on $[a, b]$ where $a \geq 0$ and $b > a$. The expectation of k -th highest order statistic is given by

$$E[V_{(k)}] = (b - a) \frac{n + 1 - k}{n + 1} + a.$$

Thus, by a simple calculation, the marginal return of slot k is

$$\mu_k = \frac{2(b-a)}{n+1} \left(\frac{b(n+1)}{2(b-a)} - k \right).$$

From this, the optimal number of slots is $\lfloor \frac{b}{2(b-a)}(n+1) \rfloor$ if $\lfloor \frac{b}{2(b-a)}(n+1) \rfloor \leq n-1$ and the optimal number is $n-1$ otherwise.¹² Statement (1) holds in this case because if b increases with keeping $(b-a)$ a constant value, the optimal number increases or is unchanged. In addition, Statement (2) holds because if $(b-a)$ decreases with keeping b a constant value, the optimal number increases or is unchanged.

From a general uniform distribution, we can provide examples of why both s-dominance and d-dominance are needed in Proposition 5. Let F and G be uniform distributions on $[1, 10]$ and $[1, 3]$, respectively. Then, $F \geq_s G$ holds but $G \geq_d F$ does not. From the result in the previous paragraph, L^{*F} is $\lfloor \frac{5}{9}(n+1) \rfloor$ and L^{*G} is $\lfloor \frac{3}{4}(n+1) \rfloor$. Thus, we have $L^{*F} < L^{*G}$. Next, let F and G be uniform distributions on $[1, 3]$ and $[3, 7]$, respectively. Then, $G \geq_d F$ holds but $F \geq_s G$ does not. Then, L^{*F} is $\lfloor \frac{3}{4}(n+1) \rfloor$ and L^{*G} is $\lfloor \frac{7}{8}(n+1) \rfloor$. Thus, we have $L^{*F} < L^{*G}$.

5.2 Comparative statics analysis of the change in the number of the bidders

In this subsection, we explore how the optimal number of slots changes due to a change in the number of the bidders in the market. We denote by $\mu_{k:n}^F$ the marginal return of slot k and by L_n^{F*} the optimal number of slots defined in Proposition 2 when there are n bidders whose values are i.i.d. F . We obtain the following result.

Theorem 3. *Assume that F is regular. Then,*

$$\mu_{k:n+1}^F > \mu_{k:n}^F$$

holds for any $k \leq K$.

¹²For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the maximal integer that is less than or equal to x .

The proof is in the appendix.

From this theorem, we obtain a natural result on the optimal numbers of slots for the markets with different depths.

Corollary 2. *Assume that F satisfies regularity. Then, $L_{n+1}^{F*} \geq L_n^{F*}$.*

From this corollary, we find that the optimal number of slots increases as the number of potential advertisers increases.

6 Concluding remarks

In this study, we explored the optimal number of the advertising slots that maximizes the auctioneer's revenue. Our result showed that the auctioneer indeed has an incentive to delete certain lower advertising slots. In reality, the search engines like Google and Yahoo! restrict the number of the advertising slots for each search result page, and this is well explained by our analysis. In the real world keyword auction, a reserve price is a familiar and convenient way to increase the seller's revenue. However, in the recent change in the rule of Google (AdWords) keyword auctions, AdWords removed the reserve price, called a "minimum bids." Instead, it began to present the information on "first page bid estimates," which is an estimated value of a bid needed for the ad to be displayed on the first search result page.¹³ This may be the evidence for the importance of a slot-restriction in the first result page as a revenue improving mechanism of a search engine. The analysis should be expanded to explore the difference between setting the reserve price and restricting the number of slots as slot supply strategy of a search engine.

¹³See, <http://adwords.blogspot.com/2008/09/quality-score-improvements-to-go-live.html>, last checked May 29, 2010.

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Appendix

From the theory of order statistic, it is known that

$$E[V_{(k)}] = \int_0^{\bar{v}} x \frac{n!}{(n-k)!(k-1)!} f(x) F(x)^{n-k} (1-F(x))^{k-1} dx.$$

Put $u = F(x)$. Then, $x = F^{-1}(u)$ and $du = f(x)dx$. Thus, we obtain

$$E[V_{(k)}] = \int_0^1 F^{-1}(u) \frac{n!}{(n-k)!(k-1)!} u^{n-k} (1-u)^{k-1} du. \quad (5)$$

From this, we can calculate μ_k as follows. For all $k = 1, 2, \dots, n-1$,

$$\begin{aligned} \mu_k &= -(k-1)E[V_{(k)}] + kE[V_{(k+1)}] \\ &= -(k-1) \int_0^1 F^{-1}(u) \frac{n!}{(n-k)!(k-1)!} u^{n-k} (1-u)^{k-1} du \\ &\quad + k \int_0^1 F^{-1}(u) \frac{n!}{(n-k-1)!k!} u^{n-k-1} (1-u)^k du \\ &= \frac{n!}{(n-k)!(k-1)!} \int_0^1 F^{-1}(u) \left[(n-k)u^{n-k-1} (1-u)^k - (k-1)u^{n-k} (1-u)^{k-1} \right] du \end{aligned} \quad (6)$$

For $k = n$,

$$\mu_n = -(n-1)E[V_{(n)}] = -n(n-1) \int_0^1 F^{-1}(u) (1-u)^{n-1} du \quad (7)$$

From (6) and (7), for all $k = 1, 2, \dots, n$,

$$\mu_k = \frac{n!}{(n-k)!(k-1)!} \int_0^1 F^{-1}(u) \left[(n-k)u^{n-k-1} (1-u)^k - (k-1)u^{n-k} (1-u)^{k-1} \right] du \quad (8)$$

Proof of Theorem 1

(i). From (8), for $k = 1, 2, \dots, n-1$,

$$\begin{aligned} \mu_k - \mu_{k+1} &= \frac{n!}{(n-k)!(k-1)!} \int_0^1 F^{-1}(u) \left[(n-k)u^{n-k-1} (1-u)^k - (k-1)u^{n-k} (1-u)^{k-1} \right] du \\ &\quad - \frac{n!}{(n-k-1)!k!} \int_0^1 F^{-1}(u) \left[(n-k-1)u^{n-k-2} (1-u)^{k+1} - ku^{n-k-1} (1-u)^k \right] du \end{aligned}$$

$$\begin{aligned}
 &= \frac{n!}{(n-k)!k!} \int_0^1 F^{-1}(u) u^{n-k-2} (1-u)^{k-2} \left[k(n-k)u(1-u) - k(k-1)u^2 \right. \\
 &\quad \left. - (n-k)(n-k-1)(1-u)^2 + k(n-k)u(1-u) \right] du \\
 &= \frac{n!}{(n-k)!k!} \int_0^1 F^{-1}(u) u^{n-k-2} (1-u)^{k-2} \left[(n-k-1)(ku - (n-k)(1-u))(1-u) \right. \\
 &\quad \left. - (k-1)u(ku - (n-k)(1-u)) + nu(1-u) \right] du \\
 &= \frac{n!}{(n-k)!k!} \int_0^1 (1-u) F^{-1}(u) \frac{d}{du} \left[u^{n-k-1} (1-u)^{k-1} (ku - (n-k)(1-u)) \right] du \\
 &= \frac{n!}{(n-k)!k!} \left\{ \left[(1-u) F^{-1}(u) u^{n-k-1} (1-u)^{k-1} (ku - (n-k)(1-u)) \right]_0^1 \right. \\
 &\quad \left. + \int_0^1 \left(F^{-1}(u) - \frac{(1-u)}{f^{-1}(u)} \right) u^{n-k-1} (1-u)^{k-1} (ku - (n-k)(1-u)) du \right\}. \tag{9}
 \end{aligned}$$

Note that

$$\left[(1-u) F^{-1}(u) u^{n-k-1} (1-u)^{k-1} (ku - (n-k)(1-u)) \right]_0^1 = \begin{cases} 0 & \text{if } k < n-1, \\ F^{-1}(0) \geq 0 & \text{if } k = n-1. \end{cases} \tag{10}$$

From (9) and (10),

$$\mu_k - \mu_{k+1} \geq \frac{n!}{(n-k)!k!} \int_0^1 \left(F^{-1}(u) - \frac{(1-u)}{f^{-1}(u)} \right) u^{n-k-1} (1-u)^{k-1} (ku - (n-k)(1-u)) du. \tag{11}$$

Note that

$$\int_0^1 u^{n-k-1} (1-u)^{k-1} (ku - (n-k)(1-u)) du = \left[-u^{n-k} (1-u)^k \right]_0^1 = 0. \tag{12}$$

From (11) and (12),

$$\mu_k - \mu_{k+1} \geq \frac{n!}{(n-k)!k!} \int_0^1 \left[\left(F^{-1}(u) - \frac{(1-u)}{f^{-1}(u)} \right) - \left(F^{-1}(r) - \frac{(1-r)}{f^{-1}(r)} \right) \right] u^{n-k-1} (1-u)^{k-1} (ku - (n-k)(1-u)) du \tag{13}$$

where $r = \frac{n-k}{n}$. Note that

$$u^{n-k-1} (1-u)^{k-1} (ku - (n-k)(1-u)) \begin{cases} < 0 & \text{if } 0 < u < r, \\ > 0 & \text{if } r < u < 1. \end{cases} \tag{14}$$

Furthermore, since by regularity, $F^{-1}(u) - \frac{(1-u)}{f^{-1}(u)}$ is increasing in u ,¹⁴

$$\left(F^{-1}(u) - \frac{(1-u)}{f^{-1}(u)} \right) - \left(F^{-1}(r) - \frac{(1-r)}{f^{-1}(r)} \right) \begin{cases} < 0 & \text{if } 0 < u < r, \\ > 0 & \text{if } r < u < 1. \end{cases} \quad (15)$$

From (13), (14), and (15), it follows that $\mu_k - \mu_{k+1} > 0$.

(ii). From (8),

$$\mu_1 = n \int_0^1 F^{-1}(u) [(n-1)u^n(1-u)] du > 0.$$

(iii). It is obvious from (7).

Proof of Theorem 2

From (8),

$$\begin{aligned} \mu_k^F - \mu_k^G &= \frac{n!}{(n-k)!(k-1)!} \int_0^1 F^{-1}(u) \left[(n-k)u^{n-k-1}(1-u)^k - (k-1)u^{n-k}(1-u)^{k-1} \right] du \\ &\quad - \frac{n!}{(n-k)!(k-1)!} \int_0^1 G^{-1}(u) \left[(n-k)u^{n-k-1}(1-u)^k - (k-1)u^{n-k}(1-u)^{k-1} \right] du \\ &= \frac{n!}{(n-k)!k!} \int_0^1 (1-u) (F^{-1}(u) - G^{-1}(u)) \\ &\quad \left[(n-k)u^{n-k-1}(1-u)^{k-1} - (k-1)u^{n-k}(1-u)^{k-2} \right] du \end{aligned} \quad (16)$$

Note that

$$\int_0^1 (n-k)u^{n-k-1}(1-u)^{k-1} - (k-1)u^{n-k}(1-u)^{k-2} du = \left[u^{n-k}(1-u)^{k-1} \right]_0^1 = 0. \quad (17)$$

¹⁴This is checked as follows. Let $\varphi(u) = \phi_F(F^{-1}(u))$ for $u \in (0, 1)$. Then,

$$\frac{d\varphi}{du} = \frac{d\phi_F}{dv}(F^{-1}(u)) \frac{dF^{-1}}{du}(u) = \frac{d\phi_F}{dv}(F^{-1}(u)) \frac{1}{f(F^{-1}(u))}.$$

Since the regularity implies $\frac{d\phi_F}{dv}(F^{-1}(u)) > 0$, we have $\frac{d\varphi}{du} > 0$.

From (16) and (17),

$$\mu_k^F - \mu_k^G = \frac{n!}{(n-k)!k!} \int_0^1 [(1-u)(F^{-1}(u) - G^{-1}(u)) - (1-r)(F^{-1}(r) - G^{-1}(r))] \left[(n-k)u^{n-k-1}(1-u)^{k-1} - (k-1)u^{n-k}(1-u)^{k-2} \right] du \quad (18)$$

where $r = \frac{n-k}{n-1}$. Note that

$$(n-k)u^{n-k-1}(1-u)^{k-1} - (k-1)u^{n-k}(1-u)^{k-2} \begin{cases} > 0 & \text{if } 0 < u < r, \\ < 0 & \text{if } r < u < 1. \end{cases} \quad (19)$$

Since F v -dominates G , $(1-u)(F^{-1}(u) - G^{-1}(u))$ is decreasing in u . Thus,

$$(1-u)(F^{-1}(u) - G^{-1}(u)) - (1-r)(F^{-1}(r) - G^{-1}(r)) \begin{cases} > 0 & \text{if } 0 < u < r \\ < 0 & \text{if } r < u < 1. \end{cases} \quad (20)$$

From (18), (19), and (20), it follows that $\mu_k^F - \mu_k^G > 0$.

Proof of Theorem 3

From (8),

$$\begin{aligned}
 & \mu_{k:n+1}^F - \mu_{k:n}^F \\
 &= \frac{(n+1)!}{(n-k+1)!(k-1)!} \int_0^1 F^{-1}(u) \left[(n+1-k)u^{n-k}(1-u)^k - (k-1)u^{n+1-k}(1-u)^{k-1} \right] du \\
 &\quad - \frac{n!}{(n-k)!(k-1)!} \int_0^1 F^{-1}(u) \left[(n-k)u^{n-k-1}(1-u)^k - (k-1)u^{n-k}(1-u)^{k-1} \right] du \\
 &= \frac{n!}{(n-k+1)!(k-1)!} \int_0^1 F^{-1}(u) u^{n-k-1} (1-u)^{k-1} \left[(n+1)(n-k+1)u(1-u) \right. \\
 &\quad \left. - (n+1)(k-1)u^2 - (n-k+1)(n-k)(1-u) + (n-k+1)(k-1)u \right] du \\
 &= \frac{n!}{(n-k+1)!(k-1)!} \int_0^1 F^{-1}(u) u^{n-k-1} (1-u)^{k-1} \left[(n-k)(1-u)(ku - (n-k+1)(1-u)) \right. \\
 &\quad \left. - (k-1)u(ku - (n-k+1)(1-u)) + (n+1)u(1-u) \right] du \\
 &= \frac{n!}{(n+1-k)!(k-1)!} \int_0^1 (1-u) F^{-1}(u) \frac{d}{du} \left[u^{n-k}(1-u)^{k-1} (ku - (n-k+1)(1-u)) \right] du. \\
 &= \frac{n!}{(n-k+1)!(k-1)!} \left\{ \left[(1-u) F^{-1}(u) u^{n-k} (1-u)^{k-1} (ku - (n-k+1)(1-u)) \right]_0^1 \right. \\
 &\quad \left. + \int_0^1 \left(F^{-1}(u) - \frac{(1-u)}{f^{-1}(u)} \right) u^{n-k} (1-u)^{k-1} (ku - (n-k+1)(1-u)) du \right\}. \tag{21}
 \end{aligned}$$

Note that

$$\left[(1-u) F^{-1}(u) u^{n-k} (1-u)^{k-1} (ku - (n-k+1)(1-u)) \right]_0^1 = \begin{cases} 0 & \text{if } k < n, \\ F^{-1}(0) \geq 0 & \text{if } k = n. \end{cases} \tag{22}$$

From (21) and (22),

$$\mu_{k:n+1}^F - \mu_{k:n}^F \geq \frac{n!}{(n-k+1)!(k-1)!} \int_0^1 \left(F^{-1}(u) - \frac{(1-u)}{f^{-1}(u)} \right) u^{n-k} (1-u)^{k-1} (ku - (n-k+1)(1-u)) du. \tag{23}$$

Note that

$$\int_0^1 u^{n-k} (1-u)^{k-1} (ku - (n-k+1)(1-u)) du = \left[-u^{n-k+1} (1-u)^k \right]_0^1 = 0. \tag{24}$$

From (23) and (24),

$$\mu_{k:n+1}^F - \mu_{k:n}^F \geq \frac{n!}{(n-k+1)!(k-1)!} \int_0^1 \left[\left(F^{-1}(u) - \frac{(1-u)}{f^{-1}(u)} \right) - \left(F^{-1}(r) - \frac{(1-r)}{f^{-1}(r)} \right) \right] u^{n-k}(1-u)^{k-1}(ku - (n-k+1)(1-u)) du \quad (25)$$

where $r = \frac{n-k+1}{n+1}$. Note that

$$u^{n-k}(1-u)^{k-1}(ku - (n-k+1)(1-u)) \begin{cases} < 0 & \text{if } 0 < u < r, \\ > 0 & \text{if } r < u < 1. \end{cases} \quad (26)$$

Furthermore, since by regularity, $F^{-1}(u) - \frac{(1-u)}{f^{-1}(u)}$ is increasing in u ,

$$\left(F^{-1}(u) - \frac{(1-u)}{f^{-1}(u)} \right) - \left(F^{-1}(r) - \frac{(1-r)}{f^{-1}(r)} \right) \begin{cases} < 0 & \text{if } 0 < u < r, \\ > 0 & \text{if } r < u < 1. \end{cases} \quad (27)$$

From (25), (26), and (27), it follows that $\mu_{k:n+1}^F - \mu_{k:n}^F > 0$.