WIAS Discussion Paper No.2011-004

Characteristics finite difference schemes of second order in time for convection-diffusion problems

February 24, 2012

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Abstract

Two new finite difference schemes based on the method of characteristics are presented for convection-diffusion problems. Both of the schemes are of second order in time, and the matrices of the derived systems of linear equations are symmetric. No numerical integration is required for them. The one is of first order in space and the other is of second order. For the former scheme, an optimal error estimate is proved in the framework of discrete L^2-theory. Numerical results are shown to recognize the convergence rates of the schemes.

JEL classification: C0, C6
Keywords: The method of characteristics, Finite Difference Method, Second order in time

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1 Introduction

Convection-diffusion equation describes phenomena including both convection and diffusion effects, and appears in various fields of natural sciences, e.g., heat transfer, weather prediction and atmospheric radioactivity propagation. It may also be treated as a simplified model of the system of the Navier-Stokes equations, which are representative equations in fluid dynamics. Although the convection-diffusion equation is linear, numerical difficulty caused by convection effect is still remained. Nowadays, to deal with convection-dominant problems several upwind type ideas have been developed for flow problems, e.g., upwind methods [2, 7, 8, 15], characteristics (-based) methods [1, 4, 5, 6, 9, 10, 11, 12, 13, 14] and so on. We focus on the approximation based on the method of characteristics. The idea of the method is to consider the trajectory of the fluid particle and discretize the material derivative term along the trajectory. The method has such a common advantage that the resulting matrix is symmetric, which is especially useful when we employ implicit schemes for the benefit of a good stability.

The characteristics finite element method of first order in time has been well studied in [4, 9, 10]. As for the scheme of second order in time, a multi-step scheme has been considered in [5] while a single-step scheme has been developed in [13], where they have pointed out that the conventional Crank-Nicolson method is not sufficient and that an additional correction term is indispensable in order to obtain a real second order scheme. In this paper we extend their idea to the finite difference method, and present two new characteristics schemes with a proper additional correction term for convection-diffusion problems in 2D.

In general, the finite difference method has less flexibility in the shape of domains to be applied and is less familiar in L^2-analysis than the finite element method. The reason why we consider the finite difference method nevertheless is that it requires no numerical integration in the execution. Every characteristics scheme includes a composite function term. When we employ the finite element method, some numerical integration procedure is often required to compute the integration of the composite function, since it is not a polynomial in each element. In the papers [16] and [17], they have remarked that much attention should be paid to the numerical integration, because a rough numerical integration formula may yield oscillating results caused by the non-smoothness of the composite function. In order to overcome such a problem a characteristics finite element scheme without numerical integration has been presented in [12], where a mass-lumping technique is used to $P1$ element and L^∞-theory is applied to establish the convergence. For the application to flow problems and higher order elements, L^2-analysis is preferable. Our two finite difference schemes require no numerical integration, and they are analyzed by a discrete L^2-theory.

Both of the schemes have such advantages that these are of second order in time and the resulting matrices are symmetric and positive definite. The extension to 3D problems is straightforward with the expense of a little complicated notation. The difference of the two schemes is accuracy in space. The one is of first order in space, and the stability and convergence theorems are proved in the framework of a discrete L^2-theory. The other is of second order in space by the use of a quadratic interpolation in dealing with the composition of functions. The convergence orders of both schemes are observed by numerical results.

Let m be a non-negative integer. We use the Sobolev spaces $W^{1,\infty}(\Omega)$ and $H^m(\Omega)$ as well as $C^m(\overline{\Omega})$. For any normed space X with norm $\|\cdot\|_X$, we define the function space $C^m([0,T];X)$ consisting of X-valued functions in $C^m([0,T])$. We often omit $[0,T]$ if there is no confusion, e.g., we write $C^j(C^m(\overline{\Omega}))$ in place of $C^j([0,T];C^m(\overline{\Omega}))$. We introduce function spaces Z^m and Z^m_C,

$$Z^m \equiv \left\{\phi \in H^j(H^{m-j}(\Omega));\ j=0,\cdots,m,\ ||\phi||_{Z^m} < +\infty\right\},$$
$$Z^m_C \equiv \left\{\phi \in C^j(C^{m-j}(\overline{\Omega}));\ j=0,\cdots,m,\ ||\phi||_{Z^m_C} < +\infty\right\},$$

where the norms $||\cdot||_{Z^m}$ and $||\cdot||_{Z_C^m}$ are defined by

$$||\phi||_{Z^m} \equiv \max_{j=0,\cdots,m} \|\phi\|_{H^j(H^{m-j}(\Omega))}, \quad \|\phi\|_{Z_C^m} \equiv \max_{j=0,\cdots,m} \|\phi\|_{C^j(C^{m-j}(\overline{\Omega}))}.$$

The partial derivative $\partial\phi/\partial x_i$ of a function ϕ is simply denoted by $D^i\phi$. We often consider a continuous function in $\overline{\Omega}$ as a function defined on lattice points in $\overline{\Omega}$. δ_{ij} $(i, j = 1, 2)$ is Kronecker's delta, and $\mathbb{Z}^\alpha \equiv \{\mathbb{Z} + \alpha\}$ for $\alpha \in [0, 1)$. The abbreviations LHS and RHS mean left- and right-hand sides, respectively.

2 A characteristics finite difference scheme of second order in time

In this section we present a characteristics finite difference scheme of second order in time and of first order in space.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $\Gamma \equiv \partial\Omega$ be the boundary of Ω and T be a positive constant. We consider an initial boundary value problem; find $\phi : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\frac{\partial\phi}{\partial t} + u\cdot\nabla\phi - \nu\Delta\phi = f \quad \text{in } \Omega \times (0, T), \tag{1a}$$

$$\phi = 0 \quad \text{on } \Gamma \times (0, T), \tag{1b}$$

$$\phi = \phi^0 \quad \text{in } \Omega, \text{ at } t = 0, \tag{1c}$$

where ν is a positive constant, and $u : \Omega \times (0, T) \to \mathbb{R}^2$, $f : \Omega \times (0, T) \to \mathbb{R}$ and $\phi^0 : \Omega \to \mathbb{R}$ are given functions.

To begin with, we summarize conditions to be used in the paper for the functions u, ϕ^0, f and ϕ. Each condition is referred to simply by, e.g., $[\mathrm{H}_{0,1}(u)]$ in place of Hypothesis 1 $[\mathrm{H}_{0,1}(u)]$.

Hypothesis 1 (u).

$[\mathrm{H}_{0,1}(u)]$ $u \in C^0(C^1(\overline{\Omega}))$, $[\mathrm{H}_{0,2}(u)]$ $u \in C^0(C^2(\overline{\Omega}))$, $[\mathrm{H}_{0,3}(u)]$ $u \in C^0(C^3(\overline{\Omega}))$,

$[\mathrm{H}_{1C}(u)]$ $u \in Z_C^1$, $[\mathrm{H}_{2C}(u)]$ $u \in Z_C^2$, $[\mathrm{H}_\Gamma(u)]$ $u = 0$ *on* $\Gamma \times [0, T]$.

Hypothesis 2 (ϕ^0).

$[\mathrm{H}_{0,\Gamma}(\phi^0)]$ $\phi^0 \in C^0(\overline{\Omega})$ *and* $\phi^0 = 0$ *on* Γ.

Hypothesis 3 (f).

$[\mathrm{H}_{0,0}(f)]$ $f \in C^0(C^0(\overline{\Omega}))$, $[\mathrm{H}_{2C}(f)]$ $f \in Z_C^2$.

Hypothesis 4 (ϕ).

$[\mathrm{H}_{0,1}(\phi)]$ $\phi \in C^0(C^1(\overline{\Omega}))$, $[\mathrm{H}_{0,2}(\phi)]$ $\phi \in C^0(C^2(\overline{\Omega}))$, $[\mathrm{H}_{0,3}(\phi)]$ $\phi \in C^0(C^3(\overline{\Omega}))$,

$[\mathrm{H}_{0,4}(\phi)]$ $\phi \in C^0(C^4(\overline{\Omega}))$, $[\mathrm{H}_{1,0}(\phi)]$ $\phi \in C^1(C^0(\overline{\Omega}))$, $[\mathrm{H}_{3C}(\phi)]$ $\phi \in Z_C^3$,

$[\mathrm{H}_{1C}(\nabla\phi)]$ $\nabla\phi \in Z_C^1$, $[\mathrm{H}_{2C}(\Delta\phi)]$ $\Delta\phi \in Z_C^2$.

For the sake of simplicity we consider a rectangle domain $\Omega = (0, L_1) \times (0, L_2)$ for positive numbers L_1 and L_2. For $i = 1$ and 2 let N_i be a positive integer and $h_i \equiv L_i/N_i$ be the mesh size of x_i-direction. We set lattice points $x_{i,j} \equiv (ih_1, jh_2)^T$ for i and $j \in \mathbb{Z} \cup \mathbb{Z}^{1/2}$, and the minimum and maximum mesh sizes $h_{\min} \equiv \min\{h_1, h_2\}$ and $h \equiv \max\{h_1, h_2\}$, respectively, where the superscript "T" means the transposition. The following hypothesis is for a family of meshes.

Hypothesis 5 (h_1, h_2)**.** *There exist positive constants h_0, γ_1 and γ_2 such that*

$$h_1, h_2 \in (0, h_0] \quad \textit{and} \quad \gamma_1 \le \frac{h_2}{h_1} \le \gamma_2.$$

We assume that Hypothesis 5 holds for the family of meshes to be considered in the paper.

Remark 1. *For a positive constant $\gamma_0 \equiv \max\{1/\gamma_1,\ \gamma_2\}$ it holds that*

$$h_{\min} \le h_i \le \gamma_0 h_{\min} \quad (i = 1, 2). \tag{2}$$

Let Δt be a time increment and $N_T \equiv [T/\Delta t]$ be a total step number. We set $t^n \equiv n\Delta t$ for $n \in \mathbb{Z} \cup \mathbb{Z}^{1/2}$, and $\phi^n \equiv \phi(\cdot, t^n)$ for any function ϕ defined in $\Omega \times (0, T)$. Let U_0^∞ and U_1^∞ be constants defined by

$$\begin{aligned}
U_0^\infty &\equiv \max\big\{|u(x,t)|_\infty;\ x \in \overline{\Omega},\ t \in [0,T]\big\},\\
U_1^\infty &\equiv \max\big\{|\nabla u_j(x,t)|_1;\ x \in \overline{\Omega},\ t \in [0,T],\ j = 1,2\big\},
\end{aligned}$$

where, for a vector $a \in \mathbb{R}^2$, $|a|_\infty \equiv \max\big\{|a_i|;\ i = 1,2\big\}$ and $|a|_1 \equiv \sum_{i=1}^2 |a_i|$. Before the presentation of the scheme we summarize conditions on Δt.

Hypothesis 6 (Δt)**.** *Let C_1 be any positive constant independent of h and Δt.*

$[\mathrm{H}_u(\Delta t)]$ $\quad \Delta t < 1/\|u\|_{C^0(W^{1,\infty}(\Omega))}$, $\quad [\mathrm{H}_{wCFL}(\Delta t)]$ $\quad \Delta t \le C_1 h_{\min}/U_0^\infty$,

$[\mathrm{H}_{CFL}(\Delta t)]$ $\quad \Delta t \le h_{\min}/U_0^\infty$.

Remark 2. *(i)* $[\mathrm{H}_u(\Delta t)]$ *guarantees that all upwind points to be used in our schemes are in $\overline{\Omega}$ (cf.* Proposition 1*). (ii)* $[\mathrm{H}_{wCFL}(\Delta t)]$ *with $C_1 = 1$ is the same as* $[\mathrm{H}_{CFL}(\Delta t)]$*, which is so-called the CFL condition (cf. [10]). Since $C_1 > 1$ can be chosen, we call* $[\mathrm{H}_{wCFL}(\Delta t)]$ *"weak-CFL condition", whose abbreviation is put in the subscript.*

Let $X : (0,T) \to \mathbb{R}^2$ be a solution of the ordinary differential equation

$$\frac{dX}{dt} = u(X,t). \tag{4}$$

Then, for a smooth function ϕ we can write

$$\Big(\frac{\partial}{\partial t} + u \cdot \nabla\Big)\phi(X(t),t) = \frac{d}{dt}\phi(X(t),t),$$

which is a basic idea of the method of characteristics. Let $X(t; x, t^n)$ be the solution of (4) subject to an initial condition $X(t^n) = x$. Approximating the value $X(t^{n-1}; x, t^n)$ by the Euler method and the second order Runge-Kutta method, we obtain

$$X_1^n(x) \equiv x - u^n(x)\Delta t, \qquad X_2^n(x) \equiv x - u^{n-1/2}\Big(x - u^n(x)\frac{\Delta t}{2}\Big)\Delta t.$$

Remark 3. *Instead of the second order Runge-Kutta method we can also use the Heun method,*

$$X_2^n(x) = x - \Big\{u^n(x) + u^{n-1}\big(x - u^n(x)\Delta t\big)\Big\}\frac{\Delta t}{2}.$$

The following result has been proved in [13, Proposition 1] for any bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$.

Proposition 1. *Suppose* $[\mathrm{H}_{0,1}(u)]$, $[\mathrm{H}_\Gamma(u)]$ *and* $[\mathrm{H}_u(\Delta t)]$. *Then, it holds that*

$$X_1^n(\Omega) = X_2^n(\Omega) = \Omega.$$

For a pair $(\alpha, \beta) \in \{(0, 0),\ (\frac{1}{2}, 0),\ (0, \frac{1}{2})\}$ we define sets of lattice points (cf. Fig. 1),

$$\Omega_h^{(\alpha,\beta)} \equiv \Big\{x_{i,j} \in \Omega;\ (i,\, j) \in \{\mathbb{Z}+\alpha\} \times \{\mathbb{Z}+\beta\}\Big\}, \quad \Omega_h \equiv \Omega_h^{(0,0)}, \tag{5a}$$

$$\overline{\Omega}_h^{(\alpha,\beta)} \equiv \Big\{x_{i,j} \in \overline{\Omega};\ (i,\, j) \in \{\mathbb{Z}+\alpha\} \times \{\mathbb{Z}+\beta\}\Big\}, \quad \overline{\Omega}_h \equiv \overline{\Omega}_h^{(0,0)}, \tag{5b}$$

$$\Gamma_h^{(\alpha,\beta)} \equiv \overline{\Omega}_h^{(\alpha,\beta)} \setminus \Omega_h^{(\alpha,\beta)}, \quad \Gamma_h \equiv \Gamma_h^{(0,0)}, \tag{5c}$$

$$\widetilde{\Omega}_h^{(\frac{1}{2},0)} \equiv \overline{\Omega}_h^{(\frac{1}{2},0)} \cup \big\{x_{i,j};\ (i,\, j) \in \{-1/2, N_1+1/2\} \times \{0, \cdots, N_2\}\big\}, \tag{5d}$$

$$\widetilde{\Omega}_h^{(0,\frac{1}{2})} \equiv \overline{\Omega}_h^{(0,\frac{1}{2})} \cup \big\{x_{i,j};\ (i,\, j) \in \{0, \cdots, N_1\} \times \{-1/2, N_2+1/2\}\big\}, \tag{5e}$$

and function spaces,

$$V_h^{(\alpha,\beta)} \equiv \big\{v_h : \overline{\Omega}_h^{(\alpha,\beta)} \to \mathbb{R}\big\}, \qquad V_h \equiv V_h^{(0,0)}, \tag{6a}$$

$$V_{h0}^{(\alpha,\beta)} \equiv \big\{v_h \in V_h^{(\alpha,\beta)};\ v_h|_{\Gamma_h^{(\alpha,\beta)}} = 0\big\}, \quad V_{h0} \equiv V_{h0}^{(0,0)}, \tag{6b}$$

$$V_{0h}^{(\alpha,\beta)} \equiv \big\{v_h : \Omega_h^{(\alpha,\beta)} \to \mathbb{R}\big\}, \qquad V_{0h} \equiv V_{0h}^{(0,0)}. \tag{6c}$$

The space V_{h0} includes the essential boundary condition (1b).

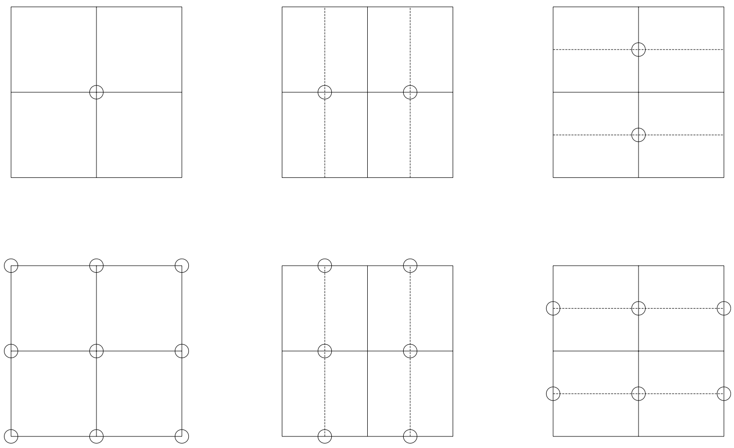

Figure 1: The sets of lattice points, Ω_h, $\Omega_h^{(\frac{1}{2},0)}$, $\Omega_h^{(0,\frac{1}{2})}$, $\overline{\Omega}_h$, $\overline{\Omega}_h^{(\frac{1}{2},0)}$ and $\overline{\Omega}_h^{(0,\frac{1}{2})}$ (left to right, top to bottom). The bottom three figures also exhibit locations where function values are used in the interpolation operators, $\Pi_h^{(1)}$, $\Pi_h^{(\frac{1}{2},0),(1)}$ and $\Pi_h^{(0,\frac{1}{2}),(1)}$.

Let $\eta(\cdot; i, h) : \mathbb{R} \to \mathbb{R}$ $(i \in \mathbb{Z} \cup \mathbb{Z}^{1/2},\ h > 0)$ be a function (cf. Fig. 2),

$$\eta(\xi; i, h) \equiv \begin{cases} 1 + \dfrac{\xi - ih}{h} & (\xi \in [(i-1)h, ih]), \\ 1 - \dfrac{\xi - ih}{h} & (\xi \in [ih, (i+1)h]), \\ 0 & (\text{otherwise}), \end{cases}$$

and $K_{i,j}$ $((i,j) \in \mathbb{Z} \cup \mathbb{Z}^{1/2})$ be a closed rectangle,

$$K_{i,j} \equiv \Big[\Big(i-\frac{1}{2}\Big)h_1,\ \Big(i+\frac{1}{2}\Big)h_1\Big] \times \Big[\Big(j-\frac{1}{2}\Big)h_2,\ \Big(j+\frac{1}{2}\Big)h_2\Big].$$

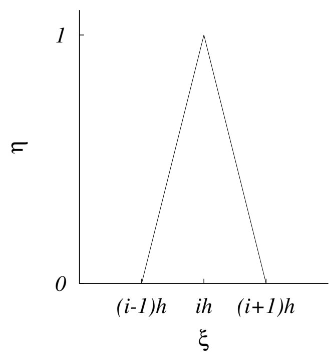

Figure 2: The graph of the function $\eta(\xi;\, i, h)$.

For each (i,j) we define a function $\phi_{i,j}(x)$,

$$\phi_{i,j}(x_1,x_2) \equiv \eta(x_1;i,h_1)\eta(x_2;j,h_2),$$

whose support is $\cup_{\alpha=i\pm 1/2,\beta=j\pm 1/2} K_{\alpha,\beta}$. We define a bilinear interpolation operator $\Pi_h^{(1)}$: $V_h \to C^0(\overline{\Omega})$ by

$$\Pi_h^{(1)} v_h \equiv \sum_{x_{i,j}\in\overline{\Omega}_h} v_h(x_{i,j})\phi_{i,j}.$$

We also define bilinear interpolation operators $\Pi_h^{(\alpha,\beta),(1)} : V_h^{(\alpha,\beta)} \to C^0(\overline{\Omega})$ by

$$\Pi_h^{(\alpha,\beta),(1)} v_h \equiv \sum_{x_{i,j}\in\widetilde{\Omega}_h^{(\alpha,\beta)}} \tilde{v}_h(x_{i,j})\phi_{i,j}$$

for $(\alpha,\beta) = (1/2,0)$ and $(0,1/2)$, where

$$\tilde{v}_h(x_{i,j}) \equiv \begin{cases} v_h(x_{i,j}) & (x_{i,j} \in \overline{\Omega}_h^{(\frac{1}{2},0)} \cup \overline{\Omega}_h^{(0,\frac{1}{2})}), \\ 2v_h(x_{1/2,j}) - v_h(x_{3/2,j}) & (x_{i,j} \in \widetilde{\Omega}_h^{(\frac{1}{2},0)},\ i=-1/2), \\ 2v_h(x_{N_1-1/2,j}) - v_h(x_{N_1-3/2,j}) & (x_{i,j} \in \widetilde{\Omega}_h^{(\frac{1}{2},0)},\ i=N_1+1/2), \\ 2v_h(x_{i,1/2}) - v_h(x_{i,3/2}) & (x_{i,j} \in \widetilde{\Omega}_h^{(0,\frac{1}{2})},\ j=-1/2), \\ 2v_h(x_{i,N_2-1/2}) - v_h(x_{i,N_2-3/2}) & (x_{i,j} \in \widetilde{\Omega}_h^{(0,\frac{1}{2})},\ j=N_2+1/2). \end{cases}$$

For $(\alpha,\beta) \in \{(0,0),\ (\frac{1}{2},0),\ (0,\frac{1}{2})\}$ and $x \in \overline{\Omega}$ we set $\Lambda^{(\alpha,\beta)}(x)$ by

$$\Lambda^{(\alpha,\beta)}(x) \equiv \big\{(i,j),(i+1,j),(i,j+1),(i+1,j+1) \in \mathbb{Z}^\alpha \times \mathbb{Z}^\beta;\\ x_1 \in [ih_1,(i+1)h_1),\ x_2 \in [jh_2,(j+1)h_2)\big\}.$$

Remark 4. *The functions $\{\phi_{i,j}\}_{i,j}$ are called bilinear basis functions in the finite element method.*

Remark 5. *The superscript "(1)" of the operators $\Pi_h^{(1)}$, $\Pi_h^{(\frac{1}{2},0),(1)}$ and $\Pi_h^{(0,\frac{1}{2}),(1)}$ means first order approximation in space, i.e., bilinear basis functions are used in the interpolation operators, and we often omit the superscript "(1)" if there is no confusion. A biquadratic interpolation operator $\Pi_h^{(2)}$ and modified bilinear interpolation operators $\check{\Pi}_h^{(\frac{1}{2},0),(1)}$ and $\check{\Pi}_h^{(0,\frac{1}{2}),(1)}$ appear in section 6.*

We use the symbol $\circ$ to represent the composition of functions, e.g.,

$$(\phi \circ X_1^n)(x) \equiv \phi\big(X_1^n(x)\big).$$

Let $e_i \equiv (\delta_{i1}, \delta_{i2})^T$ $(i = 1,2)$ be unit vectors and $T_a^{h,i}$ be a translation operator,

$$(T_a^{h,i} v)(x) \equiv v(x + ahe_i).$$

For a discrete function v_h and an integer n $(= 1, \cdots, N_T)$ we set finite difference operators,

$$\big(\tilde{\nabla}_{h1}^{(n)}(\Delta t)\, v_h\big)(x) \equiv \Big\{\Pi_h^{(\frac{1}{2},0),(1)}\Big(\frac{T_{1/2}^{h_1,1} - T_{-1/2}^{h_1,1}}{h_1}\, v_h\Big)\Big\}\big(x - u^n(x)\Delta t\big), \tag{7a}$$

$$\big(\tilde{\nabla}_{h2}^{(n)}(\Delta t)\, v_h\big)(x) \equiv \Big\{\Pi_h^{(0,\frac{1}{2}),(1)}\Big(\frac{T_{1/2}^{h_2,2} - T_{-1/2}^{h_2,2}}{h_2}\, v_h\Big)\Big\}\big(x - u^n(x)\Delta t\big), \tag{7b}$$

$$\tilde{\nabla}_h^{(n)}(\Delta t) \equiv \big(\tilde{\nabla}_{h1}^{(n)}(\Delta t), \tilde{\nabla}_{h2}^{(n)}(\Delta t)\big)^T, \tag{7c}$$

$$\nabla_{hi} \equiv \tilde{\nabla}_{hi}^{(n)}(0)\ (i = 1, 2), \qquad \nabla_h \equiv \big(\nabla_{h1},\, \nabla_{h2}\big)^T, \tag{7d}$$

$$\tilde{\Delta}_{h,i}^{(n)}(\Delta t) \equiv \nabla_{hi}\tilde{\nabla}_{hi}^{(n)}(\Delta t)\ (i = 1,2), \qquad \tilde{\Delta}_h^{(n)}(\Delta t) \equiv \sum_{i=1}^{2} \tilde{\Delta}_{h,i}^{(n)}(\Delta t), \tag{7e}$$

$$\Delta_{h,i} \equiv \nabla_{hi}^2 \quad (i = 1,2), \qquad \Delta_h \equiv \sum_{i=1}^{2} \Delta_{h,i}. \tag{7f}$$

Δt is often omitted from above operators, e.g., $\tilde{\nabla}_h^{(n)} = \tilde{\nabla}_h^{(n)}(\Delta t)$. For $\{\phi_h^n\}_{n=0}^{N_T} \subset V_h$ we define finite difference operators,

$$\mathscr{M}_h^{n-1/2,(1)}\phi_h \equiv \frac{\phi_h^n - (\Pi_h^{(1)}\phi_h^{n-1}) \circ X_2^n}{\Delta t}, \tag{8a}$$

$$\mathscr{L}_{h,0}^{n-1/2,(1)}\phi_h \equiv -\frac{\nu}{2}(\Delta_h \phi_h^n + \tilde{\Delta}_h^{(n)}\phi_h^{n-1}), \tag{8b}$$

$$\mathscr{L}_{h,1}^{n-1/2}\phi_h \equiv -\frac{\nu\Delta t}{2}\Big\{\sum_{i=1}^{2}(D^i u_i^n)\Delta_{h,i} + (D^2 u_1^n + D^1 u_2^n)\nabla_{(2h)1}\nabla_{(2h)2}\Big\}\phi_h^{n-1}, \tag{8c}$$

$$\mathscr{L}_h^{n-1/2,(1)}\phi_h \equiv (\mathscr{L}_{h,0}^{n-1/2,(1)} + \mathscr{L}_{h,1}^{n-1/2})\phi_h, \tag{8d}$$

$$\mathscr{A}_h^{n-1/2,(1)} \equiv \mathscr{M}_h^{n-1/2,(1)} + \mathscr{L}_h^{n-1/2,(1)}. \tag{8e}$$

Remark 6. *(i) For $i = 1$ and 2 we can write*

$$\nabla_{hi} = \frac{T_{1/2}^{h_i,i} - T_{-1/2}^{h_i,i}}{h_i}, \text{ i.e., } \big(\nabla_{hi} v_h\big)(x) = \frac{1}{h_i}\Big\{v_h\big(x + \frac{h_i}{2}e_i\big) - v_h\big(x - \frac{h_i}{2}e_i\big)\Big\},$$

$$\tilde{\nabla}_{h1}^{(n)} v_h = \left(\Pi_h^{(\frac{1}{2},0),(1)} \nabla_{h1} v_h\right) \circ X_1^n, \quad \tilde{\nabla}_{h2}^{(n)} v_h \equiv \left(\Pi_h^{(0,\frac{1}{2}),(1)} \nabla_{h2} v_h\right) \circ X_1^n.$$

(ii) We note that, for $v_h \in V_h$, $\nabla_{h1} v_h \in V_h^{(\frac{1}{2},0)} \cup V_{0h}^{(\frac{1}{2},0)}$, $\nabla_{h2} v_h \in V_h^{(0,\frac{1}{2})} \cup V_{0h}^{(0,\frac{1}{2})}$ and $\nabla_{(2h)1}\nabla_{(2h)2} v_h$, $\Delta_h v_h$ and $\tilde{\Delta}_h^{(n)} v_h \in V_{0h}$.

Remark 7. *$\mathscr{L}_{h,1}^{n-1/2}\phi_h$, i.e.,*

$$-\frac{\nu\Delta t}{2}\left\{\sum_{i=1}^{2}(D^i u_i^n)\Delta_{h,i} + (D^2 u_1^n + D^1 u_2^n)\nabla_{(2h)1}\nabla_{(2h)2}\right\}\phi_h^{n-1},$$

is an additional correction term in order to obtain a real second order scheme in time, which will be shown in section 5.

A characteristics finite difference scheme of second order in time for (1) is to find $\{\phi_h^n\}_{n=0}^{N_T} \subset V_{h0}$ such that, for $n = 1, \cdots, N_T$,

$$\mathscr{A}_h^{n-1/2}\phi_h = \frac{1}{2}(f^n + f^{n-1}\circ X_1^n) \quad \text{on } \Omega_h, \tag{9a}$$

$$\phi_h^0 = \phi^0 \quad \text{on } \overline{\Omega}_h. \tag{9b}$$

The equation (9a) is equivalent to

$$\begin{aligned}
&\frac{\phi_h^n - \left(\Pi_h^{(1)}\phi_h^{n-1}\right)\circ X_2^n}{\Delta t}(x) - \frac{\nu}{2}\left(\Delta_h\phi_h^n + \tilde{\Delta}_h^{(n)}\phi_h^{n-1}\right)(x)\\
&-\frac{\nu\Delta t}{2}\left\{\sum_{i=1}^{2}(D^i u_i^n)\Delta_{h,i} + (D^2 u_1^n + D^1 u_2^n)\nabla_{(2h)1}\nabla_{(2h)2}\right\}\phi_h^{n-1}(x)\\
&= \frac{1}{2}(f^n + f^{n-1}\circ X_1^n)(x), \quad x \in \Omega_h.
\end{aligned}$$

We also consider a scheme corresponding to (9) for general initial values and right-hand sides. Let $a_h \in V_{h0}$ and $\{\mathscr{F}_h^{n-1/2}\}_{n=1}^{N_T} \subset V_{0h}$ be given. A general scheme is to find $\{\phi_h^n\}_{n=0}^{N_T} \subset V_{h0}$ such that, for $n = 1, \cdots, N_T$,

$$\mathscr{A}_h^{n-1/2}\phi_h = \mathscr{F}_h^{n-1/2} \quad \text{on } \Omega_h, \tag{10a}$$

$$\phi_h^0 = a_h \quad \text{on } \overline{\Omega}_h. \tag{10b}$$

Then, $\phi_h = \{\phi_h^n\}_{n=0}^{N_T}$ is called the solution of scheme (10) with $(a_h, \mathscr{F}_h^{n-1/2})$. Obviously, the solution $\phi_h = \{\phi_h^n\}_{n=0}^{N_T}$ of scheme (9) is the solution of scheme (10) with $(\phi^0, \frac{1}{2}(f^n + f^{n-1}\circ X_1^n))$.

3 Main results

In this section we give a stability theorem for scheme (10) and an error estimate for scheme (9), whose proofs are shown in sections 4 and 5, respectively.

For a set S_h of lattice points and functions v_h and w_h in a function space $\{v_h : S_h \to \mathbb{R}\}$, we define an inner product by

$$(v_h,\, w_h)_{S_h} \equiv h_1 h_2 \sum_{x\in S_h} v_h(x) w_h(x).$$

Let $(\alpha,\beta)\in\{(0,0),(\frac{1}{2},0),(0,\frac{1}{2})\}$ be a pair of numbers. We define norms and seminorms,

$$\begin{aligned}
&\|v_h\|_{l^2(\overline{\Omega}_h^{(\alpha,\beta)})} \equiv \left\{(v_h,v_h)_{\overline{\Omega}_h^{(\alpha,\beta)}}\right\}^{1/2} \quad (v_h\in V_h^{(\alpha,\beta)}),\\
&\|v_h\|_{l^2(\Omega_h^{(\alpha,\beta)})} \equiv \left\{(v_h,v_h)_{\Omega_h^{(\alpha,\beta)}}\right\}^{1/2} \quad (v_h\in V_{0h}^{(\alpha,\beta)}\cup V_{h0}^{(\alpha,\beta)}),\\
&\|\cdot\|_{l^2(\overline{\Omega}_h)} \equiv \|\cdot\|_{l^2(\overline{\Omega}_h^{(0,0)})}, \quad \|\cdot\|_{l^2(\Omega_h)} \equiv \|\cdot\|_{l^2(\Omega_h^{(0,0)})},\\
&\|w_h\|_{l^2(\overline{\Omega}_h^{(\frac{1}{2},0)})\times l^2(\overline{\Omega}_h^{(0,\frac{1}{2})})} \equiv \left\{\|w_{h1}\|^2_{l^2(\overline{\Omega}_h^{(\frac{1}{2},0)})}+\|w_{h2}\|^2_{l^2(\overline{\Omega}_h^{(0,\frac{1}{2})})}\right\}^{1/2}\\
&\qquad (w_h=(w_{h1},w_{h2})^T\in V_h^{(\frac{1}{2},0)}\times V_h^{(0,\frac{1}{2})}),\\
&\|w_h\|_{l^2(\Omega_h^{(\frac{1}{2},0)})\times l^2(\Omega_h^{(0,\frac{1}{2})})} \equiv \left\{\|w_{h1}\|^2_{l^2(\Omega_h^{(\frac{1}{2},0)})}+\|w_{h2}\|^2_{l^2(\Omega_h^{(0,\frac{1}{2})})}\right\}^{1/2}\\
&\qquad (w_h=(w_{h1},w_{h2})^T\in (V_{0h}^{(\frac{1}{2},0)}\cup V_{h0}^{(\frac{1}{2},0)})\times(V_{0h}^{(0,\frac{1}{2})}\cup V_{h0}^{(0,\frac{1}{2})}),\\
&|v_h|_{h^1(\overline{\Omega}_h)} \equiv \|\nabla_h v_h\|_{l^2(\overline{\Omega}_h^{(\frac{1}{2},0)})\times l^2(\overline{\Omega}_h^{(0,\frac{1}{2})})} \quad (v_h\in V_h),\\
&|v_h|_{h^1(\Omega_h)} \equiv \|\nabla_h v_h\|_{l^2(\Omega_h^{(\frac{1}{2},0)})\times l^2(\Omega_h^{(0,\frac{1}{2})})} \quad (v_h\in V_{h0}),\\
&\|\phi_h\|_{l^\infty(l^2)} \equiv \max_{n=0,\cdots,N_T}\|\phi_h^n\|_{l^2(\Omega_h)} \quad (\phi_h=\{\phi_h^n\}_{n=0}^{N_T}\subset V_{0h}\cup V_{h0}),\\
&|\phi_h|_{l^\infty(h^1)} \equiv \max_{n=0,\cdots,N_T}|\phi_h^n|_{h^1(\Omega_h)} \quad (\phi_h=\{\phi_h^n\}_{n=0}^{N_T}\subset V_{h0}),\\
&\|\phi_h\|_{l^2(l^2)} \equiv \left\{\Delta t\sum_{n=1}^{N_T}\|\phi_h^n\|^2_{l^2(\Omega_h)}\right\}^{1/2} \quad (\phi_h=\{\phi_h^n\}_{n=1}^{N_T}\subset V_{0h}\cup V_{h0}),\\
&\|\phi_h\|_{\overline{l}^2(\overline{l}^2)} \equiv \left\{\Delta t\sum_{n=0}^{N_T}\|\phi_h^n\|^2_{l^2(\overline{\Omega}_h)}\right\}^{1/2} \quad (\phi_h=\{\phi_h^n\}_{n=0}^{N_T}\subset V_h),\\
&|\phi_h|_{l^2(h^{1\prime})} \equiv \left\{\Delta t\sum_{n=1}^{N_T}\left\|\frac{\nabla_h\phi_h^n+\tilde{\nabla}_h^{(n)}\phi_h^{n-1}}{2}\right\|^2_{l^2(\Omega_h^{(\frac{1}{2},0)})\times l^2(\Omega_h^{(0,\frac{1}{2})})}\right\}^{1/2}\\
&\qquad (\phi_h=\{\phi_h^n\}_{n=0}^{N_T}\subset V_{h0}).
\end{aligned}$$

We use the same notation $\|\cdot\|_{l^2(l^2)}$ for $\phi_h=\{\phi_h^{n-1/2}\}_{n=1}^{N_T}\subset V_{0h}\cup V_{h0}$, which represents

$$\|\phi_h\|_{l^2(l^2)}=\left\{\Delta t\sum_{n=1}^{N_T}\|\phi_h^{n-1/2}\|^2_{l^2(\Omega_h)}\right\}^{1/2}.$$

Remark 8. *We note that*

$$\|v_h\|_{l^2(\Omega_h^{(\alpha,\beta)})}=\|v_h\|_{l^2(\overline{\Omega}_h^{(\alpha,\beta)})} \quad (v_h\in V_{h0}^{(\alpha,\beta)}),$$
$$\|w_h\|_{l^2(\Omega_h^{(\frac{1}{2},0)})\times l^2(\Omega_h^{(0,\frac{1}{2})})}=\|w_h\|_{l^2(\overline{\Omega}_h^{(\frac{1}{2},0)})\times l^2(\overline{\Omega}_h^{(0,\frac{1}{2})})} \quad (w_h\in V_{h0}^{(\frac{1}{2},0)}\times V_{h0}^{(0,\frac{1}{2})}),$$

and, especially,

$$|\nabla_h v_h|_{h^1(\Omega_h)}=|\nabla_h v_h|_{h^1(\overline{\Omega}_h)} \quad (v_h\in V_{h0}).$$

Theorem 1 (stability)**.** *Suppose* $[\mathrm{H}_{0,1}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_u(\Delta t)]$ *and* $[\mathrm{H}_{wCFL}(\Delta t)]$. *Let* $a_h \in V_{h0}$ *and* $\{\mathscr{F}_h^{n-1/2}\}_{n=1}^{N_T} \subset V_{0h}$ *be given. Let* $\phi_h = \{\phi_h^n\}_{n=0}^{N_T} \subset V_{h0}$ *be the solution of* (10). *Then, there exists a positive constant* $c = c(\|u\|_{C^0(C^1(\overline{\Omega}))})$, *independent of* h *and* Δt, *such that*

$$\begin{aligned} &\|\phi_h\|_{l^\infty(l^2)} + \sqrt{\nu\Delta t}|\phi_h|_{l^\infty(h^1)} + \sqrt{\nu}|\phi_h|_{l^2(h^{1\prime})} \\ &\quad \le c\,(\|a_h\|_{l^2(\Omega_h)} + \sqrt{\nu\Delta t}|a_h|_{h^1(\Omega_h)} + \|\mathscr{F}_h\|_{l^2(l^2)}). \end{aligned} \tag{11}$$

Corollary 1. *Suppose* $[\mathrm{H}_{0,1}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_{0,0}(f)]$, $[\mathrm{H}_{0,\Gamma}(\phi^0)]$, $[\mathrm{H}_u(\Delta t)]$ *and* $[\mathrm{H}_{wCFL}(\Delta t)]$. *Let* $\phi_h = \{\phi_h^n\}_{n=0}^{N_T} \subset V_{h0}$ *be the solution of scheme* (9). *Then, there exists a positive constant* $c = c(\|u\|_{C^0(C^1(\overline{\Omega}))})$, *independent of* h *and* Δt, *such that*

$$\begin{aligned} &\|\phi_h\|_{l^\infty(l^2)} + \sqrt{\nu\Delta t}|\phi_h|_{l^\infty(h^1)} + \sqrt{\nu}|\phi_h|_{l^2(h^{1\prime})} \\ &\quad \le c\,(\|\phi^0\|_{l^2(\Omega_h)} + \sqrt{\nu\Delta t}|\phi^0|_{h^1(\Omega_h)} + \|f\|_{\bar{l}^2(\bar{l}^2)}). \end{aligned} \tag{12}$$

Theorem 2 (error estimate)**.** *Suppose* $[\mathrm{H}_{2C}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_{3C}(\phi)]$, $[\mathrm{H}_{2C}(\Delta\phi)]$, $[\mathrm{H}_u(\Delta t)]$ *and* $[\mathrm{H}_{wCFL}(\Delta t)]$. *Let* $\phi_h = \{\phi_h^n\}_{n=0}^{N_T} \subset V_{h0}$ *be the solution of scheme* (9), *and* ϕ *be the solution of* (1). *Then, there exists a positive constant* $c = c(\|u\|_{Z_C^2})$, *independent of* h *and* Δt, *such that*

$$\begin{aligned} &\|\phi - \phi_h\|_{l^\infty(l^2)} + \sqrt{\nu\Delta t}|\phi - \phi_h|_{l^\infty(h^1)} + \sqrt{\nu}|\phi - \phi_h|_{l^2(h^{1\prime})} \\ &\quad \le c\,(\Delta t^2 + h)(\|\phi\|_{Z_C^3} + \|\Delta\phi\|_{Z_C^2}). \end{aligned} \tag{13}$$

Corollary 2. *Suppose* $[\mathrm{H}_{1,0}(\phi)]$ *and* $[\mathrm{H}_{0,2}(\phi)]$ *instead of* $[\mathrm{H}_{3C}(\phi)]$ *and* $[\mathrm{H}_{2C}(\Delta\phi)]$ *in the assumptions of Theorem 2. Then, it holds that*

$$\|\phi - \phi_h\|_{l^\infty(l^2)} + \sqrt{\nu}|\phi - \phi_h|_{l^2(h^{1\prime})} \to 0 \quad (h \downarrow 0). \tag{14}$$

Corollary 3. *RHS of* (13) *can be replaced by*

$$c\,(\Delta t^2 + h)(\|\phi\|_{Z^3} + \|\Delta\phi\|_{Z^2}). \tag{15}$$

Remark 9. *Since the relation* $[\mathrm{H}_{wCFL}(\Delta t)]$ *is assumed, RHS of* (13) *can be written as*

$$c\,h\,(\|\phi\|_{Z_C^3} + \|\Delta\phi\|_{Z_C^2}),$$

and $h \downarrow 0$ *in* (14) *is equivalent to the condition that* h *and* $\Delta t \downarrow 0$ *under that relation.*

Throughout the paper, we use c with or without subscript to denote the generic positive constant independent of h and Δt, which may take different values at different places, e.g., $c(A)$ means a constant depending on A. We prepare positive constants,

$$\begin{aligned} &c_0 = c_0(\|u\|_{C^0(C^0(\overline{\Omega}))}), \qquad c_1 = c_1(\|u\|_{C^0(C^1(\overline{\Omega}))}), \qquad c_2 = c_2(\|u\|_{C^0(C^2(\overline{\Omega}))}), \\ &c_3 = c_3(\|u\|_{C^0(C^3(\overline{\Omega}))}), \qquad c_4 = c_4(\|u\|_{Z_C^1}), \qquad c_5 = c_5(\|u\|_{Z_C^2}), \\ &c_6 = c_6(\|u\|_{C^0(C^3(\overline{\Omega}))\cap Z_C^2}), \end{aligned}$$

and sometimes add "$\prime$ (prime)" to the constants, e.g., c_0'.

4 Proof of Theorem 1

In this section we prove Theorem 1 and Corollary 1 after preparing three lemmas. A key of the proof is Lemma 1, which describes a property of the bilinear interpolation operator $\Pi_h^{(1)}$.

For a vector $w \in \mathbb{R}^2$, mesh sizes h_1 and h_2 and a time increment Δt, we define a "*proportional weight*" of the w-upwind point of a lattice point $x_{i,j}$ with respect to a lattice point $x_{l,m}$ by

$$c_{i,j}^{l,m}(w;\ \Delta t, h_1, h_2) \equiv \phi_{l,m}\big(x_{i,j} - w\Delta t\big), \tag{16}$$

whose properties are summarized in Lemma A.1 of Appendix A.1.

Lemma 1. *Suppose* $[\mathrm{H}_{0,1}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_u(\Delta t)]$ *and* $[\mathrm{H}_{wCFL}(\Delta t)]$. *Then, for any function* $v_h \in V_h$, $n = 1, \cdots, N_T$ *and* $k = 1$ *and* 2, *it holds that*

$$\|(\Pi_h v_h) \circ X_k^n\|_{l^2(\overline{\Omega}_h)} \le (1 + c_1 \Delta t)\|v_h\|_{l^2(\overline{\Omega}_h)}. \tag{17}$$

Proof. Let C_1 be the constant in $[\mathrm{H}_{wCFL}(\Delta t)]$. We consider the case $k = 1$, as the other case is treated similarly. Let n $(\le N_T)$ be a positive integer and $x_{i,j} \in \overline{\Omega}_h$ be a lattice point. Since we have $X_1^n(x_{i,j}) \in \overline{\Omega}$ by $[\mathrm{H}_u(\Delta t)]$, it holds that, from Lemma A.1 (iv) with $w = u^n(x_{i,j})$,

$$(\Pi_h v_h) \circ X_k^n(x_{i,j}) = \sum_{x_{l,m} \in \overline{\Omega}_h} c_{i,j}^{l,m}\big(u^n(x_{i,j})\big)\, v_h(x_{l,m}), \tag{18}$$

where $c_{i,j}^{l,m} = c_{i,j}^{l,m}\big(\cdot;\ \Delta t, h_1, h_2\big)$. Using the properties of $\{c_{i,j}^{l,m}(w)\}_{i,j,l,m}$ in Lemma A.1 and the Schwarz inequality, we have

$$
\begin{aligned}
\big(\text{LHS of (17)}\big)^2 &= h_1 h_2 \sum_{x_{i,j} \in \overline{\Omega}_h} \Big\{ \sum_{x_{l,m} \in \overline{\Omega}_h} c_{i,j}^{l,m}\big(u^n(x_{i,j})\big)\, v_h(x_{l,m}) \Big\}^2 \\
&\le h_1 h_2 \sum_{x_{i,j} \in \overline{\Omega}_h} \Big\{ \sum_{x_{l,m} \in \overline{\Omega}_h} c_{i,j}^{l,m}\big(u^n(x_{i,j})\big) \sum_{x_{l,m} \in \overline{\Omega}_h} c_{i,j}^{l,m}\big(u^n(x_{i,j})\big) v_h(x_{l,m})^2 \Big\} \\
&\qquad \text{(by Lemma A.1 (i))} \\
&\le h_1 h_2 \sum_{x_{i,j} \in \overline{\Omega}_h} \sum_{x_{l,m} \in \overline{\Omega}_h} c_{i,j}^{l,m}\big(u^n(x_{i,j})\big)\, v_h(x_{l,m})^2 \qquad \text{(by Lemma A.1 (ii))} \\
&= h_1 h_2 \sum_{x_{l,m} \in \overline{\Omega}_h} v_h(x_{l,m})^2 \sum_{x_{i,j} \in \overline{\Omega}_h} c_{i,j}^{l,m}\big(u^n(x_{i,j})\big) \\
&= h_1 h_2 \sum_{x_{l,m} \in \overline{\Omega}_h} v_h(x_{l,m})^2 + h_1 h_2 \sum_{x_{l,m} \in \overline{\Omega}_h} v_h(x_{l,m})^2 \Big\{ \sum_{x_{i,j} \in \overline{\Omega}_h} c_{i,j}^{l,m}\big(u^n(x_{i,j})\big) - 1 \Big\} \\
&\le h_1 h_2 \sum_{x_{l,m} \in \overline{\Omega}_h} v_h(x_{l,m})^2 \\
&\quad + h_1 h_2 \sum_{x_{l,m} \in \overline{\Omega}_h} v_h(x_{l,m})^2 \Big\{ \sum_{x_{i,j} \in \overline{\Omega}_h} c_{i,j}^{l,m}\big(u^n(x_{i,j})\big) - \sum_{x_{i,j} \in \overline{\Omega}_h} c_{i,j}^{l,m}\big(u^n(x_{l,m})\big) \Big\} \\
&\qquad \text{(by Lemma A.1 (iii))} \\
&\le h_1 h_2 \sum_{x_{l,m} \in \overline{\Omega}_h} v_h(x_{l,m})^2 + h_1 h_2 \sum_{x_{l,m} \in \overline{\Omega}_h} v_h(x_{l,m})^2 \sum_{x_{i,j} \in \overline{\Omega}_h} \xi_{i,j}^{l,m},
\end{aligned} \tag{19}
$$

where $\xi_{i,j}^{l,m} \equiv |c_{i,j}^{l,m}(u^n(x_{i,j})) - c_{i,j}^{l,m}(u^n(x_{l,m}))|$. Let $\Xi^{l,m}$, $\Xi_0^{l,m}$ and $\Xi_1^{l,m}$ be sets of lattice points,

$$\Xi^{l,m} \equiv \Big\{x_{i,j} \in \overline{\Omega}_h;\ \xi_{i,j}^{l,m} \ne 0\Big\}, \quad \Xi_0^{l,m} \equiv \Big\{x_{i,j} \in \overline{\Omega}_h;\ c_{i,j}^{l,m}\big(u^n(x_{i,j})\big) \ne 0\Big\},$$

$$\Xi_1^{l,m} \equiv \Big\{x_{i,j} \in \overline{\Omega}_h;\ c_{i,j}^{l,m}\big(u^n(x_{l,m})\big) \neq 0\Big\},$$

and $\tilde{C}_1$ and N_Ξ be integers,

$$\tilde{C}_1 \equiv [C_1\gamma_0]+1, \qquad N_\Xi \equiv 2(2\tilde{C}_1+1)^2. \tag{20}$$

We note that, from an inequality

$$\sharp\Xi_k^{l,m} \leq (2\tilde{C}_1+1)^2 \quad (k=0,\ 1),$$

it holds that

$$\sharp\Xi^{l,m} \leq \sharp\Xi_0^{l,m} + \sharp\Xi_1^{l,m} \leq N_\Xi. \tag{21}$$

Therefore, from Lemma A.2 and (21) the sum $\sum_{x_{i,j}\in\overline{\Omega}_h} \xi_{i,j}^{l,m}$ is estimated as

$$\begin{aligned}
\sum_{x_{i,j}\in\overline{\Omega}_h} \xi_{i,j}^{l,m} = \sum_{x_{i,j}\in\Xi^{l,m}} \xi_{i,j}^{l,m} &\leq \sum_{x_{i,j}\in\Xi_0^{l,m}\cup\Xi_1^{l,m}} \xi_{i,j}^{l,m} \leq \sum_{x_{i,j}\in\Xi_0^{l,m}} \xi_{i,j}^{l,m} + \sum_{x_{i,j}\in\Xi_1^{l,m}} \xi_{i,j}^{l,m}\\
&\leq (\sharp\Xi_0^{l,m} + \sharp\Xi_1^{l,m})\, 2U_1^\infty(C_1+\gamma_0)\Delta t \qquad \text{(by Lemma A.2)}\\
&\leq 2N_\Xi U_1^\infty(C_1+\gamma_0)\Delta t \qquad \text{(by (21))}.
\end{aligned} \tag{22}$$

Combining (22) with (19), we get (17) for $c_1 = N_\Xi U_1^\infty(C_1+\gamma_0)$. □

Applying Lemma 1, we have an estimate on $\nabla_h v_h$.

Lemma 2. *Suppose* $[\mathrm{H}_{0,1}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_u(\Delta t)]$ *and* $[\mathrm{H}_{wCFL}(\Delta t)]$. *Then, for any function* $v_h \in V_h$ *and* $n = 1, \cdots, N_T$, *it holds that*

$$\begin{aligned}
\bigg\{\Big\|(\Pi_h^{(\frac12,0)}\nabla_{h1}v_h)\circ X_1^n\Big\|^2_{l^2(\overline{\Omega}_h^{(\frac12,0)})} + \Big\|(\Pi_h^{(0,\frac12)}\nabla_{h2}v_h)\circ X_1^n\Big\|^2_{l^2(\overline{\Omega}_h^{(0,\frac12)})}\bigg\}^{1/2}&\\
\leq (1+c_1\Delta t)|v_h|_{h^1(\overline{\Omega}_h)}&.
\end{aligned} \tag{23}$$

Proof. Regarding $\Pi_h^{(\frac12,0)}$ and $\nabla_{h1}v_h$ as Π_h and v_h in Lemma 1, respectively, we have

$$\Big\|(\Pi_h^{(\frac12,0)}\nabla_{h1}v_h)\circ X_1^n\Big\|_{l^2(\overline{\Omega}_h^{(\frac12,0)})} \leq (1+c_1\Delta t)\|\nabla_{h1}v_h\|_{l^2(\overline{\Omega}_h^{(\frac12,0)})}, \tag{24}$$

which implies the result. □

Remark 10. *If* $v_h \in V_{h0}$ *in Lemmas 1 and 2, the inequalities* (17) *and* (23) *become*

$$\|(\Pi_h v_h)\circ X_k^n\|_{l^2(\Omega_h)} \leq (1+c_1\Delta t)\|v_h\|_{l^2(\Omega_h)},$$

$$\begin{aligned}
\bigg\{\Big\|(\Pi_h^{(\frac12,0)}\nabla_{h1}v_h)\circ X_1^n\Big\|^2_{l^2(\Omega_h^{(\frac12,0)})} + \Big\|(\Pi_h^{(0,\frac12)}\nabla_{h2}v_h)\circ X_1^n\Big\|^2_{l^2(\Omega_h^{(0,\frac12)})}\bigg\}^{1/2}&\\
\leq (1+c_1\Delta t)|v_h|_{h^1(\Omega_h)}&.
\end{aligned}$$

In the next lemma we present discrete formulas of integration by parts, whose proofs are given in Appendix A.2.

Lemma 3 (summation by parts)**.** *For* v_h *and* $w_h \in V_{h0}$ *we have*

$$-(\tilde{\Delta}_{h,1}^{(n)}v_h, w_h)_{\Omega_h} = (\tilde{\nabla}_{h1}^{(n)}v_h, \nabla_{h1}w_h)_{\Omega_h^{(\frac12,0)}},\ -(\tilde{\Delta}_{h,2}^{(n)}v_h, w_h)_{\Omega_h} = \Big(\tilde{\nabla}_{h2}^{(n)}v_h, \nabla_{h2}w_h\Big)_{\Omega_h^{(0,\frac12)}}, \tag{25a}$$

$$-(\nabla_{(2h)2}\nabla_{(2h)1}v_h, w_h)_{\Omega_h} = (\nabla_{(2h)1}v_h, \nabla_{(2h)2}w_h)_{\Omega_h} = (\nabla_{(2h)2}v_h, \nabla_{(2h)1}w_h)_{\Omega_h}. \tag{25b}$$

Now we prove the stability theorem and its corollary.

Proof of Theorem 1. Multiplying both sides of (10a) by $h_1h_2\phi_h^n$ and summing up for all $x\in\Omega_h$, we have

$$\big(\mathscr{A}_h^{n-1/2}\phi_h,\ \phi_h^n\big)_{\Omega_h} = \big(\mathscr{F}_h^{n-1/2},\ \phi_h^n\big)_{\Omega_h}. \tag{26}$$

The definition of $\mathscr{A}_h^{n-1/2}$ leads to

$$\begin{aligned}
\text{LHS of (26)} &= \left(\frac{\phi_h^n-\big(\Pi_h\phi_h^{n-1}\big)\circ X_2^n}{\Delta t},\ \phi_h^n\right)_{\Omega_h} - \frac{\nu}{2}\Big(\Delta_h\phi_h^n+\tilde{\Delta}_h^{(n)}\phi_h^{n-1},\ \phi_h^n\Big)_{\Omega_h}\\
&\quad -\frac{\nu\Delta t}{2}\left(\Big\{\sum_{i=1}^2 (D^iu_i^n)\Delta_{h,i}+(D^2u_1^n+D^1u_2^n)\nabla_{(2h)1}\nabla_{(2h)2}\Big\}\phi_h^{n-1},\ \phi_h^n\right)_{\Omega_h}\\
&\equiv I_1+I_2+I_3.
\end{aligned}$$

Let $\overline{D}_{\Delta t}$ be the backward difference operator

$$\overline{D}_{\Delta t}\phi^n \equiv \frac{\phi^n-\phi^{n-1}}{\Delta t}.$$

Lemmas 1, 2 and 3 imply the estimates,

$$I_1 \geq \overline{D}_{\Delta t}\Big(\frac{1}{2}\|\phi_h^n\|_{l^2(\Omega_h)}^2\Big) - c_1\|\phi_h^{n-1}\|_{l^2(\Omega_h)}^2 + \frac{1}{2\Delta t}\|\phi_h^n-(\Pi_h\phi_h^{n-1})\circ X_2^n\|_{l^2(\Omega_h)}^2, \tag{27a}$$

$$\begin{aligned}
I_2 &\geq \overline{D}_{\Delta t}\Big(\frac{\nu\Delta t}{4}|\phi_h^n|_{h^1(\Omega_h)}^2\Big) - c_1\nu\Delta t|\phi_h^{n-1}|_{h^1(\Omega_h)}^2\\
&\quad +\nu\Big\|\frac{\nabla_h\phi_h^n+\tilde{\nabla}_h^{(n)}\phi_h^{n-1}}{2}\Big\|_{l^2(\Omega_h^{(\frac12,0)})\times l^2(\Omega_h^{(0,\frac12)})}^2,
\end{aligned} \tag{27b}$$

$$I_3 \leq c_1\nu\Delta t\Big(\frac{1}{\delta_0}|\phi_h^{n-1}|_{h^1(\Omega_h)}^2+\delta_0|\phi_h^n|_{h^1(\Omega_h)}^2\Big), \tag{27c}$$

for any positive number δ_0. Here we have used the following inequalities to obtain (27c), for $v_h\in V_{h0}$,

$$\begin{aligned}
\big(\nabla_{(2h)1}v_h,\nabla_{(2h)1}v_h\big)_{\Omega_h} &\leq \|\nabla_{h1}v_h\|_{l^2(\Omega_h^{(\frac12,0)})}^2,\\
\big(\nabla_{(2h)2}v_h,\nabla_{(2h)2}v_h\big)_{\Omega_h} &\leq \|\nabla_{h2}v_h\|_{l^2(\Omega_h^{(0,\frac12)})}^2.
\end{aligned}$$

It is obvious that

$$\text{RHS of (26)} = (\mathscr{F}_h^{n-1/2},\phi_h^n)_{\Omega_h} \leq \frac{\delta_0}{2}\|\phi_h^n\|_{l^2(\Omega_h)}^2+\frac{1}{2\delta_0}\|\mathscr{F}_h^{n-1/2}\|_{l^2(\Omega_h)}^2. \tag{28}$$

Combining the inequalities (27) and (28) with (26), we have

$$\begin{aligned}
&\overline{D}_{\Delta t}\Big(\frac{1}{2}\|\phi_h^n\|_{l^2(\Omega_h)}^2+\frac{\nu\Delta t}{4}|\phi_h^n|_{h^1(\Omega_h)}^2\Big)\\
&+\frac{1}{2\Delta t}\|\phi_h^n-(\Pi_h\phi_h^{n-1})\circ X_2^n\|_{l^2(\Omega_h)}^2+\nu\Big\|\frac{\nabla_h\phi_h^n+\tilde{\nabla}_h^{(n)}\phi_h^{n-1}}{2}\Big\|_{l^2(\Omega_h^{(\frac12,0)})\times l^2(\Omega_h^{(0,\frac12)})}^2\\
&\leq c_1\Big\{\delta_0(\|\phi_h^n\|_{l^2(\Omega_h)}^2+\nu\Delta t|\phi_h^n|_{h^1(\Omega_h)}^2)+\|\phi_h^{n-1}\|_{l^2(\Omega_h)}^2+\nu\Delta t\big(1+\frac{1}{\delta_0}\big)|\phi_h^{n-1}|_{h^1(\Omega_h)}^2\Big\}
\end{aligned}$$

$$+\frac{1}{2\delta_0}\|\mathscr{F}_h^{n-1/2}\|^2_{l^2(\Omega_h)}. \tag{29}$$

Applying the discrete Gronwall inequality (cf. [18]) to (29) with a proper δ_0, we get (11). □

Proof of Corollary 1. Since ϕ_h is nothing but the solution of (10) with $(\phi^0, \frac{1}{2}(f^n + f^{n-1}\circ X_1^n))$, it holds that

$$\text{LHS of (12)} \leq c\,(\|\phi^0\|_{l^2(\Omega_h)} + \sqrt{\nu\Delta t}|\phi^0|_{h^1(\Omega_h)} + \|\mathscr{F}_h\|_{l^2(l^2)}).$$

From Lemma 1 we have

$$\begin{aligned}\|\mathscr{F}_h^{n-1/2}\|_{l^2(\Omega_h)} &= \Big\|\frac{1}{2}(f^n + f^{n-1}\circ X_1^n)\Big\|_{l^2(\Omega_h)}\\ &\leq \frac{1}{2}\big\{\|f^n\|_{l^2(\Omega_h)} + (1+c_1\Delta t)\|f^{n-1}\|_{l^2(\overline{\Omega}_h)}\big\},\end{aligned}$$

which implies

$$\|\mathscr{F}_h\|_{l^2(l^2)} \leq c_1\|f\|_{\bar{l}^2(\bar{l}^2)}.$$

□

5 Proof of Theorem 2

In this section we prove Theorem 2 and Corollaries 2 and 3. The choice of a proper evaluation point of the scheme plays a key role.

For functions $u \in C^0(C^0(\overline{\Omega}))$ and $\phi \in C^1(C^0(\overline{\Omega})) \cap C^0(C^2(\overline{\Omega}))$ we define an operator $\mathscr{A}^{n-1/2}$ and a function $Y_1^n(x)$ by

$$\begin{aligned}\mathscr{A}^{n-1/2}\phi &\equiv \frac{\partial\phi}{\partial t}^{n-1/2} + u^{n-1/2}\cdot\nabla\phi^{n-1/2} - \nu\Delta\phi^{n-1/2},\\ Y_1^n(x) &\equiv \frac{x + X_1^n(x)}{2}.\end{aligned}$$

We evaluate scheme (9) at a point $P^{n-1/2}(x) \equiv (Y_1^n(x), t^{n-1/2})$ (cf. Fig. 3).

Remark 11. *$Y_1^n(x)$ approximates $X(t^{n-1/2}; x, t^n)$ in $O(\Delta t^2)$,*

$$Y_1^n(x) = X(t^{n-1/2}; x, t^n) - \frac{\Delta t^2}{4}\int_0^1 ds_1 \int_{s_1}^1 X''(t^{n-1/2} + s_2\frac{\Delta t}{2}; x, t^n)ds_2, \tag{30}$$

since both sides are equal to

$$x - u^n(x)\frac{\Delta t}{2} = X(t^n) - X'(t^n)\frac{\Delta t}{2}.$$

Let ϕ be the solution of (1), $\phi_h = \{\phi_h^n\}_{n=0}^{N_T} \subset V_{h0}$ be the solution of (9) and $e_h = \{e_h^n\}_{n=0}^{N_T} \subset V_{h0}$ be a function set defined by

$$e_h^n(x) \equiv \phi_h^n(x) - \phi^n(x) \quad (x \in \overline{\Omega}_h). \tag{31}$$

From (9) and the fact that

$$\mathscr{A}^{n-1/2}\phi = f^{n-1/2} \quad \text{in } \Omega,$$

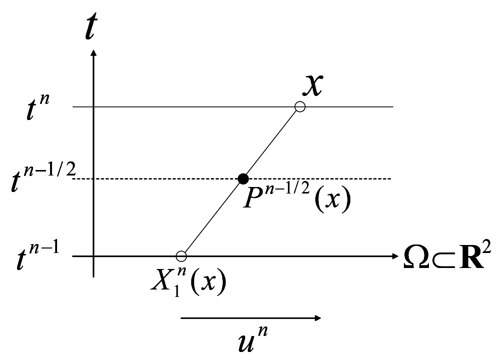

Figure 3: The evaluation point used in the proof of Theorem 2

we have, for $n = 1, \cdots, N_T$,

$$\mathscr{A}_h^{n-1/2} e_h = R_f^n + R_{\mathscr{A}}^n, \tag{32}$$

where

$$R_f^n \equiv \mathscr{F}_h^{n-1/2} - f^{n-1/2} \circ Y_1^n = \frac{1}{2}\big(f^n + f^{n-1} \circ X_1^n\big) - f^{n-1/2} \circ Y_1^n, \tag{33a}$$

$$R_{\mathscr{A}}^n \equiv (\mathscr{A}^{n-1/2}\phi) \circ Y_1^n - \mathscr{A}_h^{n-1/2}\phi \equiv \sum_{i=1}^{4} R_i^n + \nu \sum_{i=5}^{8} R_i^n, \tag{33b}$$

$$R_1^n \equiv \frac{D\phi}{Dt}^{n-1/2} \circ Y_1^n - \frac{D\phi}{Dt}^{n-1/2}\big(X(t^{n-1/2};\cdot,t^n)\big), \tag{33c}$$

$$R_2^n \equiv \frac{D\phi}{Dt}^{n-1/2}\big(X(t^{n-1/2};\cdot,t^n)\big) - \frac{\phi^n - \phi^{n-1}\big(X(t^{n-1};\cdot,t^n)\big)}{\Delta t}, \tag{33d}$$

$$R_3^n \equiv \frac{\phi^{n-1} \circ X_2^n - \phi^{n-1}\big(X(t^{n-1};\cdot,t^n)\big)}{\Delta t}, \tag{33e}$$

$$R_4^n \equiv \frac{(\Pi_h \phi^{n-1}) \circ X_2^n - \phi^{n-1} \circ X_2^n}{\Delta t}, \tag{33f}$$

$$R_5^n \equiv \frac{1}{2}(\Delta_h - \Delta)\phi^n, \tag{33g}$$

$$R_6^n \equiv \frac{1}{2}\big\{\tilde{\Delta}_h^{(n)} \phi^{n-1} - \nabla \cdot \big((\nabla\phi^{n-1}) \circ X_1^n\big)\big\}, \tag{33h}$$

$$\begin{aligned} R_7^n \equiv \frac{1}{2}\big\{&\nabla \cdot \big((\nabla\phi^{n-1}) \circ X_1^n\big) + \Delta t \sum_{i=1}^{2} (D^i u_i^n)\Delta_{h,i}\phi^{n-1} \\ &+ \Delta t (D^2 u_1^n + D^1 u_2^n)\nabla_{(2h)1}\nabla_{(2h)2}\phi^{n-1} - \Delta\phi^{n-1} \circ X_1^n\big\}, \end{aligned} \tag{33i}$$

$$R_8^n \equiv \frac{1}{2}\big(\Delta\phi^n + \Delta\phi^{n-1} \circ X_1^n\big) - \Delta\phi^{n-1/2} \circ Y_1^n. \tag{33j}$$

In order to prove Theorem 2 we prepare two lemmas, which give estimates of $\|R_f\|_{l^2(l^2)}$ and $\|R_{\mathscr{A}}\|_{l^2(l^2)}$.

Lemma 4. *Suppose* $[\mathrm{H}_{0,1}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_{2C}(f)]$ *and* $[\mathrm{H}_u(\Delta t)]$. *Then, there exists a positive constant* M_f *such that*

$$\|R_f\|_{l^2(l^2)} \le c\Delta t^2 M_f, \tag{34a}$$

where M_f satisfies

$$M_f \le c_1 \|f\|_{Z_C^2},\ c_1' \|f\|_{Z^2}. \tag{34b}$$

Proof. Let g_0 and F be functions defined by

$$\begin{aligned} g_0(x,t) &\equiv f\big(x-(t^n-t)u^n(x),t\big), \qquad (x,t)\in\overline{\Omega}\times(t^{n-1},t^n],\\ F(s) = F(s;x,t^n) &\equiv g_0(x,t^{n-1/2}+s). \end{aligned}$$

Then, it holds that

$$R_f^n(x) = \Gamma_1(F(\cdot;x,t^n);\Delta t),$$

where Γ_1 is given by (A.5a). From (A.6a) and the relation

$$\int_{-1}^1 F''(\frac{\Delta t}{2}s;x,t^n)^2\,ds = \int_{-1}^1 \frac{\partial^2 g_0}{\partial t^2}\big(x,t^{n-1/2}+\frac{\Delta t}{2}s\big)^2\,ds = \frac{2}{\Delta t}\int_{t^{n-1}}^{t^n} \frac{\partial^2 g_0}{\partial t^2}(x,t)^2\,dt,$$

we have

$$\begin{aligned} \|R_f\|_{l^2(l^2)} &\le \frac{\Delta t^2}{8}\Big\|\Big\{\frac{2}{\Delta t}\int_{t^{n-1}}^{t^n}\frac{\partial^2 g_0}{\partial t^2}(\cdot,t)^2\,dt\Big\}^{1/2}\Big\|_{l^2(l^2)}\\ &= \frac{\Delta t^2}{8}\Big[\Delta t\sum_{n=1}^{N_T} h_1h_2\sum_{x\in\Omega_h}\frac{2}{\Delta t}\int_{t^{n-1}}^{t^n}\frac{\partial^2 g_0}{\partial t^2}(x,t)^2\,dt\Big]^{1/2}\\ &= \frac{\sqrt{2}\Delta t^2}{8}\Big[\int_0^T \Big\|\frac{\partial^2 g_0}{\partial t^2}(\cdot,t)\Big\|_{l^2(\Omega_h)}^2\,dt\Big]^{1/2}\\ &= \frac{\sqrt{2}\Delta t^2}{8}\Big\|\frac{\partial^2 g_0}{\partial t^2}\Big\|_{L^2(0,T;l^2(\Omega_h))}, \end{aligned}$$

which leads to (34a) for

$$M_f \equiv \Big\|\frac{\partial^2 g_0}{\partial t^2}\Big\|_{L^2(0,T;l^2(\Omega_h))}.$$

The first inequality of (34b) follows from $f\in Z_C^2$ and an identity

$$\frac{\partial^2 g_0}{\partial t^2}(x,t) = \Big\{\big(\frac{\partial}{\partial t}+u^n(x)\cdot\nabla\big)^2 f\Big\}\big(x-(t^n-t)u^n(x),t\big).$$

Since any sequence of Riemann sums $\{\|\partial^2 g_0/\partial t^2(\cdot,t)\|_{l^2(\Omega_h)}\}_{h\downarrow 0}$ converges to $\|\partial^2 g_0/\partial t^2(\cdot,t)\|_{L^2(\Omega)}$, there exists a constant $h_* = h_*(g_0) > 0$ such that, for any $h\le h_*$,

$$\Big\|\frac{\partial^2 g_0}{\partial t^2}\Big\|_{L^2(0,T;l^2(\Omega_h))} \le 2\Big\|\frac{\partial^2 g_0}{\partial t^2}\Big\|_{L^2(0,T;L^2(\Omega))}.$$

Transforming the variable x into $y = x-(t^n-t)u^n(x)$ and evaluating the Jacobian by $1+c_1\Delta t$, we have

$$\begin{aligned} \Big\|\frac{\partial^2 g_0}{\partial t^2}\Big\|_{L^2(0,T;L^2(\Omega))} &\le c_0\Big[\sum_{n=1}^{N_T}\int_{t^{n-1}}^{t^n} dt\\ &\times\int_\Omega\Big\{\Big(\frac{\partial^2 f}{\partial t^2}\Big)^2+\sum_{i=1}^2\Big(\frac{\partial^2 f}{\partial t\partial x_i}\Big)^2+\sum_{i,j=1}^2\Big(\frac{\partial^2 f}{\partial x_i\partial x_j}\Big)^2\Big\}(x-(t^n-t)u^n(x),t)\,dx\Big]^{1/2} \end{aligned}$$

$$
\leq c_0(1+c_1\Delta t)\Big[\int_0^T dt \int_\Omega \Big\{\Big(\frac{\partial^2 f}{\partial t^2}\Big)^2 + \sum_{i=1}^2 \Big(\frac{\partial^2 f}{\partial t \partial x_i}\Big)^2 + \sum_{i,j=1}^2 \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big)^2\Big\}(y,t)\,dy\Big]^{1/2}
$$

$$
\leq c_1 \|f\|_{Z^2},
$$

which implies the second inequality of (34b). □

Remark 12. *In the following Lemmas A.7–A.14 we omit similar discussions to prove inequalities corresponding to the second one of* (34b).

Lemma 5 (truncation error of $\mathscr{A}_h^{n-1/2}$)**.** *Suppose* $[\mathrm{H}_{2C}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_{3C}(\phi)]$, $[\mathrm{H}_{2C}(\Delta\phi)]$, $[\mathrm{H}_u(\Delta t)]$ *and* $[\mathrm{H}_{wCFL}(\Delta t)]$. *Then, there exists a positive constant* $M_\mathscr{A}$ *such that*

$$
\|R_\mathscr{A}\|_{l^2(l^2)} \leq c_1(\Delta t^2 + h)M_\mathscr{A}, \tag{35a}
$$

where $M_\mathscr{A}$ *satisfies*

$$
M_\mathscr{A} \leq c_5(\|\phi\|_{Z_C^3} + \|\Delta\phi\|_{Z_C^2}),\ \ c_5'(\|\phi\|_{Z^3} + \|\Delta\phi\|_{Z^2}). \tag{35b}
$$

Proof. Let $M_\mathscr{A} \equiv \sum_{i=1}^4 M_i + \nu \sum_{i=5}^8 M_i$ (cf. Lemmas A.7–A.14 for M_i). From (33b) and Lemmas A.7–A.14 we have

$$
\begin{aligned}
\|R_\mathscr{A}\|_{l^2(l^2)} &\leq \sum_{i=1}^4 \|R_i\|_{l^2(l^2)} + \nu \sum_{i=5}^8 \|R_i\|_{l^2(l^2)} \\
&\leq \Delta t^2 \big\{c(M_1+M_2+M_3+\nu M_8) + c_1 \nu M_7\big\} \\
&\quad + h\big\{c\nu M_5 + c_0(M_4+\nu M_6) + c_1 \nu h M_7\big\} \\
&\leq c_1(\Delta t^2 + h)M_\mathscr{A},
\end{aligned}
$$

which leads to (35a). From Lemmas A.7–A.14, (35b) follows. □

Now we prove the error estimate.

Proof of Theorem 2. Let $e_h^n \in V_{h0}$, R_f^n and $R_\mathscr{A}^n$ be functions defined by (31), (33a) and (33b), respectively. Then, (32) implies that $e_h = \{e_h^n\}_{n=0}^{N_T} \subset V_{h0}$ is the solution of scheme (10) with $(0, R_f^n + R_\mathscr{A}^n)$. Applying Theorem 1 for e_h, we have, from Lemmas 4 and 5,

$$
\begin{aligned}
\text{LHS of (13)} &\leq c_1\|R_f + R_\mathscr{A}\|_{l^2(l^2)} \leq c_1(\|R_f\|_{l^2(l^2)} + \|R_\mathscr{A}\|_{l^2(l^2)}) \\
&\leq c_1(\Delta t^2 + h)M,
\end{aligned}
$$

where

$$
M \equiv M_f + M_\mathscr{A} \leq c_5(\|\phi\|_{Z_C^3} + \|\Delta\phi\|_{Z_C^2}). \tag{36}
$$

Therefore the inequality (13) holds for a constant $c = c_5$ independent of h and Δt. □

Corollaries 2 and 3 are proved as follows.

Proof of Corollary 2. Let $\varepsilon > 0$ be any fixed number. It holds that

$$
\|\phi - \phi_h\|_X \leq \|\phi - \phi^\delta\|_X + \|\phi^\delta - \phi_h^\delta\|_X + \|\phi_h^\delta - \phi_h\|_X, \tag{37}
$$

where $\|\cdot\|_X \equiv \|\cdot\|_{l^\infty(l^2)} + \sqrt{\nu}|\cdot|_{l^2(h^{1\prime})}$, $\delta > 0$ is any (small) number, ϕ^δ is a mollification of ϕ [3], $\phi_h^\delta = \{\phi_h^{\delta,n}\}_{n=0}^{N_T}$ is the solution of scheme (10) with $(\phi^{\delta,0}, \frac{1}{2}(f^{\delta,n} + f^{\delta,n-1}\circ X_1^n))$,

$\phi^{\delta,0} \equiv \phi^{\delta}(\cdot,0) \in C^0(\overline{\Omega})$, $f^{\delta,n} \equiv f^{\delta}(\cdot,n\Delta t)$ and $f^{\delta} \equiv D\phi^{\delta}/Dt - \nu\Delta\phi^{\delta} \in C^0(C^0(\overline{\Omega}))$. There exists $\delta_1 > 0$, independent of h, such that, for $\delta \le \delta_1$,

$$\|\phi - \phi^{\delta}\|_X \le c\|\phi - \phi^{\delta}\|_{C^0(C^1(\overline{\Omega}))} < \frac{\varepsilon}{3}. \tag{38a}$$

Let us consider $\|\phi_h^{\delta} - \phi_h\|_X$. Since ϕ_h is the solution of scheme (10) with $(\phi^0, \frac{1}{2}(f^n + f^{n-1} \circ X_1^n))$, there exists $\delta_2 > 0$, independent of h, such that, for $\delta \le \delta_2$,

$$\begin{aligned}
\|\phi_h^{\delta} - \phi_h\|_X &\le c(\|\phi^{\delta,0} - \phi^0\|_{l^2(\Omega_h)} + \sqrt{\nu\Delta t}|\phi^{\delta,0} - \phi^0|_{h^1(\Omega_h)} + \|f^{\delta} - f\|_{\bar{l}^2(\bar{l}^2)}) \\
&\le c(\|\phi^{\delta,0} - \phi^0\|_{C^0(\overline{\Omega})} + \sqrt{\nu\Delta t}|\phi^{\delta,0} - \phi^0|_{C^1(\overline{\Omega})} + \|f^{\delta} - f\|_{C^0(C^0(\overline{\Omega}))}) \\
&< \frac{\varepsilon}{3},
\end{aligned} \tag{38b}$$

from Theorem 1 (stability), $[\mathrm{H}_{1,0}(\phi)]$ and $[\mathrm{H}_{0,2}(\phi)]$. Now we fix $\delta = \min\{\delta_1, \delta_2\}$. Then, there exists a constant $h_* = h_*(\phi^{\delta}) > 0$ such that, for $h \le h_*$,

$$\|\phi^{\delta} - \phi_h^{\delta}\|_X \le c(\Delta t^2 + h)(\|\phi^{\delta}\|_{Z_C^3} + \|\Delta\phi^{\delta}\|_{Z_C^2}) < \frac{\varepsilon}{3}, \tag{38c}$$

from Theorem 2 (error estimate) and $[\mathrm{H}_{wCFL}(\Delta t)]$. Combining (38) with (37), we obtain

$$\|\phi - \phi_h\|_X < \varepsilon,$$

which implies (14). □

Proof of Corollary 3. Since (36) can be replaced by

$$M \le c_5(\|\phi\|_{Z^3} + \|\Delta\phi\|_{Z^2})$$

in virtue of Lemmas 4 and 5 in the proof of Theorem 2, we obtain the result. □

6 A characteristics finite difference scheme of second order in both time and space

Theorem 2 shows that the convergence order of scheme (9) is $O(\Delta t^2 + h)$. In this section we improve the accuracy in space by introducing a biquadratic interpolation operator.

Let N_1 and N_2 be a pair of even numbers. Let $\eta_k(\cdot; i, h) : \mathbb{R} \to \mathbb{R}$ $(i \in \mathbb{Z},\ h > 0,\ k = 0, 1)$ be functions defined by

$$\eta_0(\xi; i, h) \equiv \begin{cases} \Big(1 + \dfrac{\xi - ih}{h}\Big)\Big(1 + \dfrac{\xi - ih}{2h}\Big) & (\xi \in [(i-2)h, ih]), \\ \Big(1 - \dfrac{\xi - ih}{h}\Big)\Big(1 - \dfrac{\xi - ih}{2h}\Big) & (\xi \in [ih, (i+2)h]), \\ 0 & (\text{otherwise}), \end{cases}$$

$$\eta_1(\xi; i, h) \equiv \begin{cases} \Big(1 - \dfrac{\xi - ih}{h}\Big)\Big(1 + \dfrac{\xi - ih}{h}\Big) & (\xi \in [(i-1)h, (i+1)h]), \\ 0 & (\text{otherwise}). \end{cases}$$

For each (i, j) we define a function $\phi_{i,j}^{(2)}(x)$,

$$\phi_{i,j}^{(2)}(x_1, x_2) \equiv \eta_{k(i)}(x_1; i, h_1)\eta_{k(j)}(x_2; j, h_2), \quad k(l) \equiv \begin{cases} 0 & (l \in 2\mathbb{Z}), \\ 1 & (l \in \{2\mathbb{Z} + 1\}). \end{cases}$$

We define a biquadratic interpolation operator $\Pi_h^{(2)}:\ V_h \to C^0(\overline{\Omega})$ by

$$\Pi_h^{(2)} v_h \equiv \sum_{x_{i,j}\in\overline{\Omega}_h} v_h(x_{i,j})\phi_{i,j}^{(2)}.$$

We also define bilinear interpolation operators $\check{\Pi}_h^{(\alpha,\beta),(1)} : V_h^{(\alpha,\beta)} \to C^0(\overline{\Omega})$ by

$$\check{\Pi}_h^{(\alpha,\beta),(1)} v_h \equiv \sum_{x_{i,j}\in\widetilde{\Omega}_h^{(\alpha,\beta)}} \tilde{v}_h(x_{i,j})\phi_{i,j}$$

for $(\alpha,\beta)=(1/2,0)$ and $(0,1/2)$, where

$$\tilde{v}_h(x_{i,j}) \equiv$$
$$\begin{cases} v_h(x_{i,j}) & (x_{i,j}\in\overline{\Omega}_h^{(\frac{1}{2},0)}\cup\overline{\Omega}_h^{(0,\frac{1}{2})}),\\ 3v_h(x_{1/2,j})-3v_h(x_{3/2,j})+v_h(x_{5/2,j}) & (x_{i,j}\in\widetilde{\Omega}_h^{(\frac{1}{2},0)},\ i=-1/2),\\ 3v_h(x_{N_1-1/2,j})-3v_h(x_{N_1-3/2,j})+v_h(x_{N_1-5/2,j}) & (x_{i,j}\in\widetilde{\Omega}_h^{(\frac{1}{2},0)},\ i=N_1+1/2),\\ 3v_h(x_{i,1/2})-3v_h(x_{i,3/2})+v_h(x_{i,5/2}) & (x_{i,j}\in\widetilde{\Omega}_h^{(0,\frac{1}{2})},\ j=-1/2),\\ 3v_h(x_{i,N_2-1/2})-3v_h(x_{i,N_2-3/2})+v_h(x_{i,N_2-5/2}) & (x_{i,j}\in\widetilde{\Omega}_h^{(0,\frac{1}{2})},\ j=N_2+1/2). \end{cases}$$

A characteristics finite difference scheme of second order in both time and space for problem (1) is to find $\{\phi_h^n\}_{n=0}^{N_T}\subset V_{h0}$ such that, for $n=1,\cdots,N_T$,

$$\mathscr{A}_h^{n-1/2,(2)}\phi_h = \frac{1}{2}(f^n + f^{n-1}\circ X_1^n) \quad \text{on } \Omega_h, \tag{39a}$$
$$\phi_h^0 = \phi^0 \quad \text{on } \overline{\Omega}_h, \tag{39b}$$

where

$$\mathscr{A}_h^{n-1/2,(2)} \equiv \mathscr{M}_h^{n-1/2,(2)} + \check{\mathscr{L}}_h^{n-1/2,(1)},\ \ \mathscr{M}_h^{n-1/2,(2)}\phi_h \equiv \frac{\phi_h^n - \big(\Pi_h^{(2)}\phi_h^{n-1}\big)\circ X_2^n}{\Delta t},$$

and $\check{\mathscr{L}}_h^{n-1/2,(1)}$ is a modified operator of $\mathscr{L}_h^{n-1/2,(1)}$ obtained by replacing $\Pi_h^{(\frac{1}{2},0),(1)}$ in (7a) and $\Pi_h^{(0,\frac{1}{2}),(1)}$ in (7b) by $\check{\Pi}_h^{(\frac{1}{2},0),(1)}$ and $\check{\Pi}_h^{(0,\frac{1}{2}),(1)}$, respectively.

The interpolation operators $\Pi_h^{(2)}$, $\check{\Pi}_h^{(\frac{1}{2},0),(1)}$ and $\check{\Pi}_h^{(0,\frac{1}{2}),(1)}$ derive higher-order estimates, which are described in Lemmas A.10 (ii) and A.12 (ii). By Lemmas A.7–A.14 under [$\mathrm{H}_{CFL}(\Delta t)$], we get the next proposition.

Proposition 2 (truncation error of $\mathscr{A}_h^{n-1/2,(2)}$)**.** *Suppose* [$\mathrm{H}_{0,3}(u)$], [$\mathrm{H}_{2C}(u)$], [$\mathrm{H}_{\Gamma}(u)$], [$\mathrm{H}_{3C}(\phi)$], [$\mathrm{H}_{2C}(\Delta\phi)$], [$\mathrm{H}_u(\Delta t)$] *and* [$\mathrm{H}_{CFL}(\Delta t)$]. *Let $R_{\mathscr{A}}^{n,(2)}$ be a function defined by*

$$R_{\mathscr{A}}^{n,(2)} \equiv (\mathscr{A}^{n-1/2}\phi)\circ Y_1^n - \mathscr{A}_h^{n-1/2,(2)}\phi. \tag{40}$$

Then, there exists a positive constant $M_{\mathscr{A}}^{(2)}$ such that

$$\|R_{\mathscr{A}}^{(2)}\|_{l^2(l^2)} \le c_1(\Delta t^2 + h^2)M_{\mathscr{A}}^{(2)}, \tag{41a}$$

where $M_{\mathscr{A}}^{(2)}$ satisfies

$$M_{\mathscr{A}}^{(2)} \le c_6(\|\phi\|_{Z_C^3} + \|\Delta\phi\|_{Z_C^2}),\ c_6'(\|\phi\|_{Z^3} + \|\Delta\phi\|_{Z^2}). \tag{41b}$$

Proof. In the proof of Lemma 5 we can replace M_4, M_5 and M_6 by $M_4^{(2)}$, $M_5^{(2)}$ and $M_6^{(2)}$, which are evaluated by $O(h^2)$ in virtue of Lemmas A.10–A.12. Thus we get (41a). The condition $[\mathrm{H}_{CFL}(\Delta t)]$ is required in Lemma A.12 (ii). □

Proposition 2 implies that scheme (39) has higher accuracy in space than scheme (9). Stability and convergence theorems for scheme (39) will be obtained, if we can prove the estimate

$$\|(\Pi_h^{(2)} v_h) \circ X_2^n\|_{l^2(\overline{\Omega}_h)} \leq (1 + c\Delta t)\|v_h\|_{l^2(\overline{\Omega}_h)},$$

which is corresponding to (17) in Lemma 1. To prove the above estimate is a future work.

7 Numerical results

In this section we show numerical results for the following problem.

Example 1 (rotating Gaussian hill)**.** *In the problem* (1) *we set*

$$\Omega = (0,1)^2, \quad T = 2\pi, \quad u = \big(-(x_2 - 0.5), x_1 - 0.5\big)^T, \quad f = 0,$$

and three values of

$$\nu = 5 \times 10^{-4},\ 10^{-3},\ 2 \times 10^{-3}.$$

The initial function ϕ^0 is given so that the exact solution is

$$\phi(x_1, x_2, t) = \frac{\sigma}{\sigma + 4\nu t} \exp\left\{ -\frac{(\bar{x}_1(t) - x_{1,c})^2 + (\bar{x}_2(t) - x_{2,c})^2}{\sigma + 4\nu t} \right\},$$

where

$$\begin{aligned}
&(\bar{x}_1(t),\ \bar{x}_2(t))^T \\
&\equiv \big((x_1 - 0.5)\cos t + (x_2 - 0.5)\sin t,\ -(x_1 - 0.5)\sin t + (x_2 - 0.5)\cos t\big)^T, \\
&(x_{1,c}, x_{2,c}) \equiv (0.25, 0), \quad \sigma \equiv 0.01.
\end{aligned}$$

We take division numbers $N_1 = N_2 = 16,\ 32,\ 64,\ 128$ and 256, and the relations $h_1 = h_2 = h_{\min} = h$ and $\gamma_0 = 1$ hold for such meshes. In the example $U_0^\infty = 1/2$ and $U_1^\infty = 1$. We choose $\Delta t = 4h$ and h for schemes (9) and (39), respectively, where the relations satisfy $[\mathrm{H}_{wCFL}(\Delta t)]$ with $C_1 = 2$ for (9) and $[\mathrm{H}_{CFL}(\Delta t)]$ for (39). We calculate Err defined by

$$Err \equiv \frac{\|\phi - \phi_h\|_{l^\infty(l^2)}}{\|\phi\|_{l^\infty(l^2)}}$$

as an error between the finite difference and the exact solutions. Fig. 4 shows the graphs of Err versus h by schemes (9) (left) and (39) (right) in logarithmic scale for all ν. As mentioned in Remark 9, the theoretical convergence order of scheme (9) under $[\mathrm{H}_{wCFL}(\Delta t)]$ is $O(\Delta t^2 + h) = O(h)$. In the left graph of Fig. 4 we can observe Err is almost of first order in h for all ν. These results are consistent with Theorem 2. For scheme (39) with $\Delta t = h$ the accuracy is $O(\Delta t^2 + h^2) = O(h^2)$ by Proposition 2. The right graph of Fig. 4 exhibits that Err is almost of second order in h for all ν, which is the advantage of scheme (39), though it remains to prove stability and convergence theorems for the scheme.

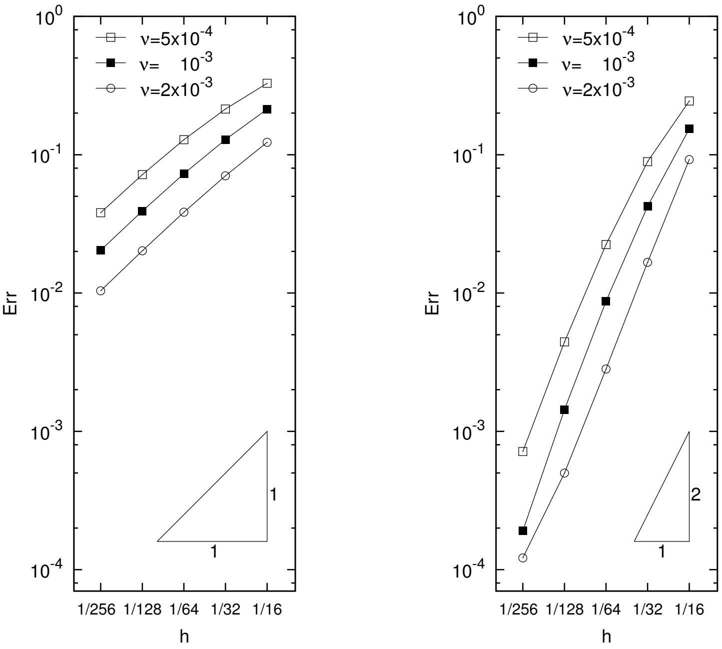

Figure 4: Err versus h by schemes (9) (left) and (39) (right) for $\nu = 5 \times 10^{-4}$, 10^{-3} and 2×10^{-3}.

Remark 13. *For the computation of* $\Pi_h^{(1)} \phi_h^{n-1} \circ X_1^n(x)$ *in scheme* (9), *we have to find a pair* $(i,\, j) \in \mathbb{Z}^{1/2} \times \mathbb{Z}^{1/2}$ *such that* $X_1^n(x) \in K_{i,j}$. *For* $y = X_1^n(x)$ *it is written as*

$$(i,\, j) = \Big(\Big[\frac{y_1}{h_1}\Big] + \frac{1}{2},\ \Big[\frac{y_2}{h_2}\Big] + \frac{1}{2} \Big),$$

while it costs much more to find an element where $X_1^n(x)$ *belong in unstructured meshes.*

8 Conclusions

We have presented two new characteristics finite difference schemes for convection-diffusion problems, which are of second order in Δt and symmetric. These finite difference schemes are extensions of the characteristics finite element scheme of second order in time in [13]. In the case of characteristics finite element methods we need to pay attention to numerical integration of composite functions. However, in the case of characteristics finite difference methods we do not need it. For scheme (9) we have proved the stability and convergence theorems under some conditions including $U_0^\infty \Delta t \leq c\,h$, and the convergence order is $O(\Delta t^2 + h)$. For scheme (39) we have shown that the accuracy is of second order in both time and space. To prove stability and convergence theorems for scheme (39) is a future work. We have also given numerical results to observe the convergence orders of the schemes. For scheme (9) the convergence order proved in Theorem 2 has been recognized in the numerical results. For scheme (39) the numerical results have been correspondent to the accuracy given in Proposition 2, and have shown the advantage of the scheme.

Acknowledgements

The first author was supported by MEXT (the Ministry of Education, Culture, Sports, Science and Technology) under the Special Coordination Funds for Promoting Science and Technology (the Tenure Track Program at Waseda Institute for Advanced Study), and by JSPS (the Japan Society for the Promotion of Science) under Grant-in-Aid for Young Scientists (B), No. 23740090 and the Japanese-German Graduate Externship (Mathematical Fluid Dynamics). The second author was supported by the National Natural Science Foundation of China, No.11171190, and by the National Basic Research Program of China, No.2007CB814906. The third author was supported by JSPS under Grants-in-Aid for Scientific Research (C), No. 22540143 and (S), No. 20224001, and by Waseda University under Specific-Research Subsidiary, No. 2010A-849.

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Appendix

Here we omit '(1)' from $\Pi_h^{(1)}$, $\Pi_h^{(\frac{1}{2},0),(1)}$ and $\Pi_h^{(0,\frac{1}{2}),(1)}$ except in Lemma A.10.

A.1 Tools for the proof of Lemma 1

Lemma A.1 (Properties of the "*proportional weight*")**.** *Let $w \in \mathbb{R}^2$ be a constant vector, h_1 and h_2 be mesh sizes and Δt be a time increment. The proportional weights $\{c_{i,j}^{l,m}(w)\}_{i,j,l,m\in\mathbb{Z}} = \{c_{i,j}^{l,m}(w;\, \Delta t, h_1, h_2)\}_{i,j,l,m\in\mathbb{Z}}$ defined by* (16) *have the following properties.*

(i) $c_{i,j}^{l,m}(w) \geq 0 \quad (i,\, j,\, l,\, m \in \mathbb{Z})$.

(ii) For any fixed integers i and j there are at most four non-zero values in $\{c_{i,j}^{l,m}(w)\}_{l,m\in\mathbb{Z}}$, and it holds that

$$\sum_{x_{l,m}\in\Omega_h} c_{i,j}^{l,m}(w) \leq \sum_{x_{l,m}\in\overline{\Omega}_h} c_{i,j}^{l,m}(w) \leq \sum_{l,m\in\mathbb{Z}} c_{i,j}^{l,m}(w) = 1.$$

(iii) For any fixed integers l and m there are at most four non-zero values in $\{c_{i,j}^{l,m}(w)\}_{i,j\in\mathbb{Z}}$, and it holds that

$$\sum_{x_{i,j}\in\Omega_h} c_{i,j}^{l,m}(w) \leq \sum_{x_{i,j}\in\overline{\Omega}_h} c_{i,j}^{l,m}(w) \leq \sum_{i,j\in\mathbb{Z}} c_{i,j}^{l,m}(w) = 1.$$

(iv) Assume $v_h \in V_h$, i and $j \in \mathbb{Z}$, and $x_{i,j} - w\Delta t \in \overline{\Omega}$. Then, it holds that

$$(\Pi_h v_h)(x_{i,j} - w\Delta t) = \sum_{x_{l,m} \in \overline{\Omega}_h} c_{i,j}^{l,m}(w)\, v_h(x_{l,m}) \Big(= \sum_{(l,m) \in \Lambda^{(0,0)}(x_{i,j} - w\Delta t)} c_{i,j}^{l,m}(w)\, v_h(x_{l,m}) \Big).$$

Proof. Since the support of $\phi_{l,m}$ is equal to $\bigcup_{\alpha = l \pm 1/2, \beta = m \pm 1/2} K_{\alpha,\beta}$, the above results follow immediately from the definition (16). □

Lemma A.2. *Let $w \in C^1(\overline{\Omega})$ be a velocity satisfying $w|_\Gamma = 0$ and W_0 and W_1 be positive constants defined by*

$$W_0 \equiv \max\Big\{ |w(x)|_\infty;\ x \in \overline{\Omega} \Big\}, \quad W_1 \equiv \max\Big\{ |\nabla w_j(x)|_1;\ x \in \overline{\Omega},\ j = 1, 2 \Big\}.$$

Let C_1 be any positive constant independent of h and Δt. Assume Δt satisfies inequalities $\Delta t < 1/\|w\|_{W^{1,\infty}(\Omega)}$ and $\Delta t \le C_1 h_{\min}/W_0$. Suppose $x_{i,j}$ and $x_{l,m} \in \overline{\Omega}_h$ and $x_{i,j} - w(x_{i,j})\Delta t \in \mathrm{supp}(\phi_{l,m})$. Then, it holds that

$$\Big| c_{i,j}^{l,m}\big(w(x_{i,j})\big) - c_{i,j}^{l,m}\big(w(x_{l,m})\big) \Big| \le 2W_1 \Delta t (C_1 + \gamma_0), \tag{A.1}$$

where $c_{i,j}^{l,m} = c_{i,j}^{l,m}(\cdot;\ \Delta t, h_1, h_2)$.

Proof. From the Taylor formula we have

$$\begin{aligned}
\text{LHS of (A.1)} &= \big| \phi_{l,m}(x_{i,j} - w(x_{i,j})\Delta t) - \phi_{l,m}(x_{i,j} - w(x_{l,m})\Delta t) \big| \\
&= \Big| \int_0^1 \nabla \phi_{l,m}\Big(s\big(x_{i,j} - w(x_{i,j})\Delta t\big) + (1-s)\big(x_{i,j} - w(x_{l,m})\Delta t\big) \Big) \\
&\qquad\quad \cdot \big(-w(x_{i,j})\Delta t + w(x_{l,m})\Delta t \big)\ ds \Big| \\
&\le \frac{1}{h_1} |w_1(x_{i,j})\Delta t - w_1(x_{l,m})\Delta t| + \frac{1}{h_2} |w_2(x_{i,j})\Delta t - w_2(x_{l,m})\Delta t| \\
&\le \frac{\Delta t}{h_{\min}} \big\{ |w_1(x_{i,j}) - w_1(x_{l,m})| + |w_2(x_{i,j}) - w_2(x_{l,m})| \big\} \\
&\le 2W_1 \frac{\Delta t}{h_{\min}} |x_{i,j} - x_{l,m}|_\infty \\
&\le 2W_1 \frac{\Delta t}{h_{\min}} (W_0 \Delta t + h) \quad (\text{by } x_{i,j} - w(x_{i,j})\Delta t \in \mathrm{supp}(\phi_{l,m})) \\
&\le 2W_1 \Delta t (C_1 + \gamma_0) \quad (\text{by } \Delta t \le C_1 h_{\min}/W_0,\ (2)).
\end{aligned}$$

□

A.2 Proof of Lemma 3

At first, we prove the first equation of (25a). We have, from the definition of $\tilde{\Delta}_{h,1}^{(n)}$ (cf. (7e)) and $v_h|_{\Gamma_h} = w_h|_{\Gamma_h} = 0$,

$$\begin{aligned}
&\text{LHS of the first equation of (25a)} \\
&= -h_2 \sum_{\substack{0<i<N_1 \\ 0<j<N_2}} (\tilde{\nabla}_{h1}^{(n)} v_h)(x_{i+\frac{1}{2},j}) w_h(x_{i,j}) + h_2 \sum_{\substack{0<i<N_1 \\ 0<j<N_2}} (\tilde{\nabla}_{h1}^{(n)} v_h)(x_{i-\frac{1}{2},j}) w_h(x_{i,j})
\end{aligned}$$

$$
\begin{aligned}
&= -h_2 \sum_{\substack{0\le i<N_1\\0<j<N_2}} (\tilde{\nabla}_{h1}^{(n)} v_h)(x_{i+\frac{1}{2},j}) w_h(x_{i,j}) + h_2 \sum_{\substack{0\le i<N_1\\0<j<N_2}} (\tilde{\nabla}_{h1}^{(n)} v_h)(x_{i+\frac{1}{2},j}) w_h(x_{i+1,j})\\
&= h_1 h_2 \sum_{\substack{0\le i<N_1\\0<j<N_2}} (\tilde{\nabla}_{h1}^{(n)} v_h)(x_{i+\frac{1}{2},j}) (\nabla_{h1} w_h)(x_{i+\frac{1}{2},j})\\
&= h_1 h_2 \sum_{x\in\overline{\Omega}_h^{(\frac{1}{2},0)}} (\tilde{\nabla}_{h1}^{(n)} v_h)(x) (\nabla_{h1} w_h)(x),
\end{aligned}
$$

which implies the first equation of (25a). Since the other equation of (25a) similarly holds, we get (25a). For the first equality of (25b) we also have

$$
\begin{aligned}
&\text{LHS of the first equation of (25b)}\\
&= -\frac{h_1}{2} \sum_{\substack{0<i<N_1\\0<j<N_2}} (\nabla_{(2h)1} v_h)(x_{i,j+1}) w_h(x_{i,j}) + \frac{h_1}{2} \sum_{\substack{0<i<N_1\\0<j<N_2}} (\nabla_{(2h)1} v_h)(x_{i,j-1}) w_h(x_{i,j})\\
&= -\frac{h_1}{2} \sum_{\substack{0<i<N_1\\0<j<N_2}} (\nabla_{(2h)1} v_h)(x_{i,j}) w_h(x_{i,j-1}) + \frac{h_1}{2} \sum_{\substack{0<i<N_1\\0<j<N_2}} (\nabla_{(2h)1} v_h)(x_{i,j}) w_h(x_{i,j+1})\\
&= h_1 h_2 \sum_{\substack{0<i<N_1\\0<j<N_2}} (\nabla_{(2h)1} v_h)(x_{i,j}) (\nabla_{(2h)2} w_h)(x_{i,j})\\
&= h_1 h_2 \sum_{x\in\Omega_h} (\nabla_{(2h)1} v_h)(x) (\nabla_{(2h)2} w_h)(x),
\end{aligned}
$$

which guarantees the first equality of (25b). The proof of the other equation is similar. □

A.3 Tools for the estimate of truncation errors

We prepare four lemmas used for the estimate of the truncation error of $\mathscr{A}_h$. We often use the notation $X = X(\cdot; x, t^n)$ if there is no confusion.

At first we show a lemma on the bilinear interpolation and its corollary. Let $\hat{I}$ be the identity operator, $\hat{D}^k \equiv \partial/\partial \hat{x}_k$, $\hat{\nabla} \equiv (\hat{D}^1, \hat{D}^2)^T$, $\hat{e}_k \equiv (\delta_{k1}, \delta_{k2})^T$ $(k = 1, 2)$, $\hat{\Lambda} \equiv \{(0,0), (1,0), (0,1), (1,1)\}$, $\hat{x}_{i,j} \equiv i\hat{e}_1 + j\hat{e}_2$,

$$
\begin{aligned}
&\hat{\phi}_{0,0}(\hat{x}) \equiv (1-\hat{x}_1)(1-\hat{x}_2), \quad &&\hat{\phi}_{1,0}(\hat{x}) \equiv \hat{x}_1(1-\hat{x}_2),\\
&\hat{\phi}_{0,1}(\hat{x}) \equiv (1-\hat{x}_1)\hat{x}_2, &&\hat{\phi}_{1,1}(\hat{x}) \equiv \hat{x}_1\hat{x}_2,
\end{aligned}
$$

and

$$
(\hat{\Pi}\hat{f})(\hat{x}) \equiv \sum_{(i,j)\in\hat{\Lambda}} \hat{f}(\hat{x}_{i,j}) \hat{\phi}_{i,j}(\hat{x}) \quad (\hat{f} \in C^0([0,1]^2)).
$$

Lemma A.3. *(i) Let $\hat{f} \in C^2([0,1]^2)$ and $\hat{x} \in [0,1]^2$ be any point. Then, it holds that*

$$
(\hat{\Pi} - \hat{I})\hat{f}(\hat{x}) = \sum_{(i,j)\in\hat{\Lambda}} \hat{T}_1(\hat{x}; i, j) \hat{\phi}_{i,j}(\hat{x}),
$$

where

$$
\hat{T}_1(\hat{x}; i, j) \equiv \int_0^1 d\hat{s}_1 \int_0^{\hat{s}_1} \big\{ \big(\hat{a}(\hat{x}; i, j) \cdot \hat{\nabla}\big)^2 \hat{f} \big\} \big(\hat{A}(\hat{x}; i, j, \hat{s}_2)\big) d\hat{s}_2,
$$

$$\hat{a}(\hat{x};i,j) \equiv \hat{x}_{i,j} - \hat{x}, \quad \hat{A}(\hat{x};i,j,\hat{s}) \equiv \hat{x} + \hat{s}\,\hat{a}(\hat{x};i,j).$$

(ii) If $\hat{f} \in C^3([0,1]^2)$, $(\hat{\Pi} - \hat{I})\hat{f}$ can be also written as

$$(\hat{\Pi} - \hat{I})\hat{f}(\hat{x}) = \frac{1}{2}\sum_{k=1}^{2}\big\{\hat{x}_k(1-\hat{x}_k)\hat{D}^{kk}\hat{f}\big\}(\hat{x}) + \hat{S}_2(\hat{x}), \tag{A.2}$$

where

$$\hat{S}_2(\hat{x}) \equiv \sum_{(i,j)\in\hat{\Lambda}} \hat{T}_2(\hat{x};i,j)\hat{\phi}_{i,j}(\hat{x}),$$
$$\hat{T}_2(\hat{x};i,j) \equiv \int_0^1 d\hat{s}_1 \int_0^{\hat{s}_1} d\hat{s}_2 \int_0^{\hat{s}_2} \big\{\big(\hat{a}(\hat{x};i,j)\cdot\hat{\nabla}\big)^3\hat{f}\big\}\big(\hat{A}(\hat{x};i,j,\hat{s}_3)\big)d\hat{s}_3.$$

Proof. We show only (ii), because the proof of (i) is easier than one of (ii). From the following identities,

$$\hat{f}(\hat{x}) = \sum_{(i,j)\in\hat{\Lambda}} \hat{f}(\hat{x})\hat{\phi}_{i,j}(\hat{x}), \tag{A.3a}$$
$$\sum_{(i,j)\in\hat{\Lambda}} \big\{\big(\hat{a}(\hat{x};i,j)\cdot\hat{\nabla}\big)\hat{f}\big\}(\hat{x})\hat{\phi}_{i,j}(\hat{x}) = 0, \tag{A.3b}$$
$$\sum_{(i,j)\in\hat{\Lambda}} \hat{a}_1(\hat{x};i,j)\,\hat{a}_2(\hat{x};i,j)\hat{\phi}_{i,j}(\hat{x}) = 0, \tag{A.3c}$$
$$\sum_{(i,j)\in\hat{\Lambda}} \hat{a}_k(\hat{x};i,j)^2\hat{\phi}_{i,j}(\hat{x}) = \hat{x}_k(1-\hat{x}_k) \quad (k=1,2), \tag{A.3d}$$

we have

$$\begin{aligned}
\text{LHS of (A.2)} &= \sum_{(i,j)\in\hat{\Lambda}} \{\hat{f}(\hat{x}_{i,j}) - \hat{f}(\hat{x})\}\phi_{i,j}(\hat{x}) \quad \text{(by (A.3a))} \\
&= \sum_{(i,j)\in\hat{\Lambda}} \int_0^1 \big\{\big(\hat{a}(\hat{x};i,j)\cdot\hat{\nabla}\big)\hat{f}\big\}\big(\hat{A}(\hat{x};i,j,\hat{s}_1)\big)d\hat{s}_1\,\phi_{i,j}(\hat{x}) \\
&= \sum_{(i,j)\in\hat{\Lambda}} \int_0^1 d\hat{s}_1 \int_0^{\hat{s}_1} \big\{\big(\hat{a}(\hat{x};i,j)\cdot\hat{\nabla}\big)^2\hat{f}\big\}\big(\hat{A}(\hat{x};i,j,\hat{s}_2)\big)d\hat{s}_2\,\phi_{i,j}(\hat{x}) \\
&\qquad \text{(by (A.3b))} \\
&= \frac{1}{2}\sum_{(i,j)\in\hat{\Lambda}} \big\{\big(\hat{a}(\hat{x};i,j)\cdot\hat{\nabla}\big)^2\hat{f}\big\}(\hat{x})d\hat{s}_2\,\phi_{i,j}(\hat{x}) + \hat{S}_2(\hat{x}) \\
&= \frac{1}{2}\sum_{k=1}^{2}\big\{\hat{x}_k(1-\hat{x}_k)\hat{D}^{kk}\hat{f}\big\}(\hat{x}) + \hat{S}_2(\hat{x}) \quad \text{(by (A.3c), (A.3d))},
\end{aligned}$$

which implies (A.2). □

Corollary A.1. *(i) Let $(\alpha,\beta) \in \{(0,0),(\frac{1}{2},0),(0,\frac{1}{2})\}$ be a fixed pair, $(l,m) \in \mathbb{Z}^{\alpha+1/2} \times \mathbb{Z}^{\beta+1/2}$ be another pair, $v \in C^2(K_{l,m})$ be a function and $x \in K_{l,m}$ be any point. Then, it holds that*

$$(\Pi_h^{(\alpha,\beta)} - I)v(x) = \sum_{(i,j)\in\Lambda^{(\alpha,\beta)}(x)} T_1(x;i,j)\,\phi_{i,j}(x) \equiv S_1(x),$$

where

$$T_1(x;i,j) \equiv \int_0^1 ds_1 \int_0^{s_1} \left\{ \left(a(x;i,j)\cdot\nabla\right)^2 v \right\} \left(A(x;i,j,s_2)\right) ds_2,$$
$$a(x;i,j) \equiv x_{i,j} - x, \quad A(x;i,j,s) \equiv x + s\,a(x;i,j).$$

Moreover, it holds that

$$|S_1(x)| \le ch^2 \|v\|_{C^2(K_{l,m})}.$$

(ii) If $v \in C^3(K_{l,m})$, then $(\Pi_h^{(\alpha,\beta)} - I)v$ can be also written as

$$(\Pi_h^{(\alpha,\beta)} - I)v(x) = \frac{1}{2}\left\{(p\tilde{p}D^{11} + q\tilde{q}D^{22})v\right\}(x) + S_2(x),$$

where

$$p = p(x) \equiv x_1 - (l-1/2)h_1, \quad \tilde{p} = \tilde{p}(x) \equiv (l+1/2)h_1 - x_1,$$
$$q = q(x) \equiv x_2 - (m-1/2)h_2, \quad \tilde{q} = \tilde{q}(x) \equiv (m+1/2)h_2 - x_2,$$
$$S_2(x) \equiv \sum_{(i,j)\in\Lambda^{(\alpha,\beta)}(x)} T_2(x;i,j)\,\phi_{i,j}(x),$$
$$T_2(x;i,j) \equiv \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} \left\{ \left(a(x;i,j)\cdot\nabla\right)^3 v \right\} \left(A(x;i,j,s_3)\right) ds_3.$$

Moreover, it holds that

$$|S_2(x)| \le ch^3 \|v\|_{C^3(K_{l,m})}.$$

Proof. Considering a function defined by

$$\hat{f}(\hat{x}) = v\left(x_{l-1/2,m-1/2} + (h_1\hat{x}_1, h_2\hat{x}_2)^T\right),$$

and applying Lemma A.3 to above $\hat{f}$, we obtain the result. □

Next, we present a basic lemma on finite difference formulae and its corollary. The proof of the lemma is omitted, as it is easy.

Lemma A.4. *Let $f : [-1,1] \to \mathbb{R}$ be a function. Then, it holds that*

$$\frac{1}{2}\{f(1)+f(-1)\} - f(0) = \frac{1}{2}\int_0^1 ds_1 \int_{-s_1}^{s_1} f''(s_2)\,ds_2 \qquad (f \in C^2[-1,1]), \tag{A.4a}$$

$$\frac{1}{2}\{f(1)-f(-1)\} - f'(0) = \frac{1}{2}\int_0^1 ds_1 \int_0^{s_1} ds_2 \int_{-s_2}^{s_2} f'''(s_3)\,ds_3 \qquad (f \in C^3[-1,1]), \tag{A.4b}$$

$$\{f(1)-2f(0)+f(-1)\} - f''(0) = \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_{-s_3}^{s_3} f''''(s_4)\,ds_4 \qquad (f \in C^4[-1,1]). \tag{A.4c}$$

Corollary A.2. *Let δ be a positive number and $F : [-\delta/2, \delta/2] \to \mathbb{R}$ be a function. Then, it holds that*

$$\begin{aligned}
\Gamma_1(F;\delta) &\equiv \frac{1}{2}\Big\{F\big(\frac{\delta}{2}\big)+F\big(-\frac{\delta}{2}\big)\Big\}-F(0)\\
&= \frac{\delta^2}{8}\int_0^1 ds_1 \int_{-s_1}^{s_1} F''\big(\frac{\delta}{2}s_2\big)\, ds_2 \quad \big(F \in C^2\big[-\frac{\delta}{2},\frac{\delta}{2}\big]\big), && \text{(A.5a)}\\
\Gamma_2(F;\delta) &\equiv \frac{F\big(\frac{\delta}{2}\big)-F\big(-\frac{\delta}{2}\big)}{\delta}-F'(0)\\
&= \frac{\delta^2}{8}\int_0^1 ds_1 \int_0^{s_1} ds_2 \int_{-s_2}^{s_2} F'''\big(\frac{\delta}{2}s_3\big)\, ds_3 \quad \big(F \in C^3\big[-\frac{\delta}{2},\frac{\delta}{2}\big]\big), && \text{(A.5b)}\\
\Gamma_3(F;\delta) &\equiv \frac{F\big(\frac{\delta}{2}\big)-2F(0)+F\big(-\frac{\delta}{2}\big)}{\big(\frac{\delta}{2}\big)^2}-F''(0)\\
&= \frac{\delta^2}{4}\int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_{-s_3}^{s_3} F''''\big(\frac{\delta}{2}s_4\big) ds_4 \quad \big(F \in C^4\big[-\frac{\delta}{2},\frac{\delta}{2}\big]\big). && \text{(A.5c)}
\end{aligned}$$

Proof. Setting $f(s) \equiv F(s\delta/2)$ and applying Lemma A.4, we immediately get the results. □

The following two lemmas are useful for our analysis.

Lemma A.5. *Let δ be a positive number, $F = F(\cdot;x,t^n) : [-\delta/2, \delta/2] \to \mathbb{R}$ be a function for $x \in \overline{\Omega}_h$ and $n = 1, \cdots, N_T$. Let $r_i^n : \overline{\Omega}_h \to \mathbb{R}$ $(i = 1,2,3)$ be functions defined by*

$$r_i^n(x) \equiv \Gamma_i(F(\cdot;x,t^n);\delta) \quad (i = 1,2,3).$$

Then, it holds that

$$\begin{aligned}
\|r_1\|_{l^2(l^2)} &\le \frac{\delta^2}{8}\Big\|\Big\{\int_{-1}^1 F''\big(\frac{\delta}{2}s;\cdot,\cdot\big)^2\, ds\Big\}^{1/2}\Big\|_{l^2(l^2)} && \big(F \in C^2\big[-\frac{\delta}{2},\frac{\delta}{2}\big]\big), && \text{(A.6a)}\\
\|r_2\|_{l^2(l^2)} &\le \frac{\delta^2}{8\sqrt{6}}\Big\|\Big\{\int_{-1}^1 F'''\big(\frac{\delta}{2}s;\cdot,\cdot\big)^2\, ds\Big\}^{1/2}\Big\|_{l^2(l^2)} && \big(F \in C^3\big[-\frac{\delta}{2},\frac{\delta}{2}\big]\big), && \text{(A.6b)}\\
\|r_3\|_{l^2(l^2)} &\le \frac{\delta^2}{24\sqrt{2}}\Big\|\Big\{\int_{-1}^1 F''''\big(\frac{\delta}{2}s;\cdot,\cdot\big)^2\, ds\Big\}^{1/2}\Big\|_{l^2(l^2)} && \big(F \in C^4\big[-\frac{\delta}{2},\frac{\delta}{2}\big]\big). && \text{(A.6c)}
\end{aligned}$$

Proof. We prove (A.6b). From (A.5b) and the Schwarz inequality we have

$$\begin{aligned}
r_2^n(x)^2 &\le \big(\frac{\delta^2}{8}\big)^2\Big\{\int_0^1 ds_1 \int_0^{s_1} ds_2 \int_{-s_2}^{s_2} ds_3\Big\}\Big\{\int_0^1 ds_1 \int_0^{s_1} ds_2 \int_{-1}^1 F'''\big(\frac{\delta}{2}s_3\big)^2\, ds_3\Big\}\\
&= \big(\frac{\delta^2}{8\sqrt{6}}\big)^2 \int_{-1}^1 F'''\big(\frac{\delta}{2}s\big)^2\, ds,
\end{aligned}$$

which implies (A.6b). The proofs of (A.6a) and (A.6c) are similar. □

Lemma A.6. *Let $f \in C^1(\overline{\Omega};\mathbb{R})$ and $a,b \in C^0(\overline{\Omega};\overline{\Omega})$ be given functions. Let $r \in C^0(\overline{\Omega};\mathbb{R})$ be a function defined by*

$$r \equiv f \circ b - f \circ a.$$

Then, it holds that

$$r = \int_0^1 (b-a)\cdot\nabla f(sb+(1-s)a)\, ds, \qquad \|r\|_{l^2(\Omega_h)} \le \|g\|_{l^2(\Omega_h)}, \qquad \text{(A.7)}$$

where

$$g \equiv \Big[\int_0^1 \big\{ (b-a) \cdot \nabla f(sb + (1-s)a) \big\}^2 ds \Big]^{1/2} \in C^0(\overline{\Omega}; \mathbb{R}).$$

Proof. The first equation (A.7) follows from

$$r = \Big[f(sb + (1-s)a) \Big]_{s=0}^{1} = \int_0^1 (b-a) \cdot \nabla f(sb + (1-s)a) \; ds.$$

The Schwarz inequality yields

$$r^2 \le \int_0^1 \big\{ (b-a) \cdot \nabla f(sb + (1-s)a) \big\}^2 \; ds = g^2,$$

which implies the rest of (A.7). □

A.4 Estimates of the truncation error

Here we evaluate each term R_i $(i = 1, \cdots, 8)$ of the truncation error $R_{\mathscr{A}}$ in (33c)–(33j).

Lemma A.7. *Suppose* $[\mathrm{H}_{1C}(u)]$, $[\mathrm{H}_{\Gamma}(u)]$, $[\mathrm{H}_{1C}(\nabla\phi)]$ *and* $[\mathrm{H}_u(\Delta t)]$. *Then, there exists a positive constant* M_1 *such that*

$$\|R_1\|_{l^2(l^2)} \le c \Delta t^2 M_1, \tag{A.8a}$$

where M_1 *satisfies*

$$M_1 \le c_4 \|\nabla \phi\|_{Z^1_C}, \;\; c_4' \|\nabla \phi\|_{Z^1}. \tag{A.8b}$$

Proof. Substituting $(D\phi/Dt)^{n-1/2}$, $X(t^{n-1/2}; \cdot, t^n)$ and Y_1^n into f, a and b in Lemma A.6, respectively, we have

$$\|R_1\|_{l^2(l^2)} \le \|\tilde{g}_1\|_{l^2(l^2)}, \tag{A.9}$$

where

$$\tilde{g}_1^n(x) \equiv \Big[\int_0^1 \; \Big\{ \big(Y_1^n(x) - X(t^{n-1/2}; x, t^n) \big) \cdot \nabla \frac{D\phi}{Dt}^{n-1/2} \big(s_0 Y_1^n(x) + (1-s_0) X(t^{n-1/2}; x, t^n) \big) \Big\}^2 \; ds_0 \Big]^{1/2}.$$

We evaluate $\|\tilde{g}_1\|_{l^2(l^2)}$. Let g_1 be a function defined by

$$g_1(x,t) \equiv \Big[\int_0^1 \Big\{ X''(t; x, t^n) \cdot \nabla \frac{D\phi}{Dt}^{n-1/2} \big(s Y_1^n(x) + (1-s) X(t^{n-1/2}; x, t^n) \big) \Big\}^2 ds \Big]^{1/2}, \quad (x,t) \in \overline{\Omega} \times (t^{n-1}, t^n].$$

From (30) we have

$$\tilde{g}_1^n(x)^2 = \frac{\Delta t^4}{16} \int_0^1 \; \Big\{ \int_0^1 ds_1 \int_{s_1}^1 X''(t^{n-1/2} + s_2 \frac{\Delta t}{2}; x, t^n) ds_2 \cdot \nabla \frac{D\phi}{Dt}^{n-1/2} \big(s_0 Y_1^n(x) + (1-s_0) X(t^{n-1/2}; x, t^n) \big) \Big\}^2 ds_0$$

$$\le \frac{\Delta t^4}{32}\int_0^1 ds_0 \int_0^1 \Big\{X''(t^{n-1/2}+s_2\frac{\Delta t}{2};x,t^n)$$
$$\cdot\nabla\frac{D\phi}{Dt}^{n-1/2}\big(s_0Y_1^n(x)+(1-s_0)X(t^{n-1/2};x,t^n)\big)\Big\}^2 ds_2$$
$$\le \frac{\Delta t^3}{16}\int_0^1 ds_0 \int_{t^{n-1/2}}^{t^n} \Big\{X''(t;x,t^n)$$
$$\cdot\nabla\frac{D\phi}{Dt}^{n-1/2}\big(s_0Y_1^n(x)+(1-s_0)X(t^{n-1/2};x,t^n)\big)\Big\}^2 dt$$
$$(\text{by } t=t^{n-1/2}+s_2\Delta t/2)$$
$$\le \frac{\Delta t^3}{16}\int_{t^{n-1}}^{t^n} dt \int_0^1 \Big\{X''(t;x,t^n)\cdot\nabla\frac{D\phi}{Dt}^{n-1/2}\big(s_0Y_1^n(x)+(1-s_0)X(t^{n-1/2};x,t^n)\big)\Big\}^2 ds_0$$
$$=\frac{\Delta t^3}{16}\int_{t^{n-1}}^{t^n} g_1(x,t)^2 dt,$$

which leads to

$$\|\tilde{g}_1\|_{l^2(l^2)} \le \frac{\Delta t^2}{4}\|g_1\|_{L^2(0,T;l^2(\Omega_h))}. \tag{A.10}$$

From (A.9) and (A.10) the inequality (A.8a) is obtained for

$$M_1 \equiv \|g_1\|_{L^2(0,T;l^2(\Omega_h))}.$$

□

Lemma A.8. *Suppose* $[\mathrm{H}_{2C}(u)]$, $[\mathrm{H}_\Gamma(u)]$ *and* $[\mathrm{H}_{3C}(\phi)]$. *Then, there exists a positive constant* M_2 *such that*

$$\|R_2\|_{l^2(l^2)} \le c\Delta t^2 M_2, \tag{A.11a}$$

where M_2 *satisfies*

$$M_2 \le c_5\|\phi\|_{Z_C^3},\ \ c_5'\|\phi\|_{Z^3}. \tag{A.11b}$$

Proof. Using Γ_2 in (A.5b), we can write

$$R_2^n(x) = \Gamma_2(F(\cdot;x,t^n);\Delta t),$$

where

$$F(s;x,t^n) \equiv -\phi(X(t^{n-1/2}+s;x,t^n),t^{n-1/2}+s).$$

Let g_2 be a function defined by

$$g_2(x,t) \equiv \frac{D^3\phi}{Dt^3}\big(X(t;x,t^n),t\big), \quad (x,t)\in\overline{\Omega}\times(t^{n-1},t^n].$$

Then, from (A.6b) and the relation,

$$\int_{-1}^1 F'''(\frac{\Delta t}{2}s;x,t^n)^2 ds = \int_{-1}^1 \frac{D^3\phi}{Dt^3}\big((X(t^{n-1/2}+\frac{\Delta t}{2}s;x,t^n),t^{n-1/2}+\frac{\Delta t}{2}s)\big)^2 ds$$
$$=\frac{2}{\Delta t}\int_{t^{n-1}}^{t^n}\frac{D^3\phi}{Dt^3}\big(X(t;x,t^n),t\big)^2 dt$$

$$= \frac{2}{\Delta t} \int_{t^{n-1}}^{t^n} g_2(x,t)^2 dt,$$

we obtain (A.11a) for

$$M_2 \equiv \|g_2\|_{L^2(0,T;l^2(\Omega_h))}.$$

Inequalities (A.11b) follow from $D^3\phi/Dt^3 = (\partial/\partial t + u \cdot \nabla)^3 \phi$. $\square$

Lemma A.9. *Suppose* $[\mathrm{H}_{2C}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_{0,1}(\phi)]$ *and* $[\mathrm{H}_u(\Delta t)]$. *Then, there exists a positive constant* M_3 *such that*

$$\|R_3\|_{l^2(l^2)} \le c\Delta t^2 M_3, \tag{A.12a}$$

where M_3 *satisfies*

$$M_3 \le c_5 \|\phi\|_{C^0(C^1(\overline{\Omega}))}, \ \ c_5' \|\phi\|_{L^2(H^1(\Omega))}. \tag{A.12b}$$

Proof. Substituting ϕ^{n-1}, $X(t^{n-1};\cdot,t^n)$ and X_2^n into f, a and b in Lemma A.6, respectively, we have

$$\|R_3\|_{l^2(l^2)} \le \|\tilde{g}_3\|_{l^2(l^2)}, \tag{A.13}$$

where

$$\tilde{g}_3^n(x) \equiv \frac{1}{\Delta t} \Big[\int_0^1 \Big\{ \big(X_2^n(x) - X(t^{n-1};x,t^n) \big) \\ \cdot \nabla \phi^{n-1} \big(s_0 X_2^n(x) + (1-s_0) X(t^{n-1}) \big) \Big\}^2 ds_0 \Big]^{1/2}.$$

We evaluate $\|\tilde{g}_3\|_{l^2(l^2)}$. It holds that

$$\begin{aligned} X_2^n(x) - X(t^{n-1};x,t^n) &= \{X(t^n;x,t^n) - X'(t^{n-1/2};x,t^n)\Delta t - X(t^{n-1};x,t^n)\} \\ &\quad + \{u^{n-1/2}(X(t^{n-1/2};x,t^n)) - u^{n-1/2}(Y_1^n(x))\}\Delta t \\ &\equiv I_1^n(x) + I_2^n(x). \end{aligned} \tag{A.14}$$

For $F(s) = F(s;x,t^n) \equiv X(t^{n-1/2}+s)$ we have, from (A.5b),

$$\begin{aligned} I_1^n(x) &= \Delta t\, \Gamma_2(F(\cdot;x,t^n);\Delta t) \\ &= \frac{\Delta t^3}{8} \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_{-s_2}^{s_2} X'''(t^{n-1/2} + \frac{\Delta t}{2} s_3;x,t^n) ds_3. \end{aligned} \tag{A.15a}$$

Substituting $u^{n-1/2}$, Y_1^n and $X(t^{n-1/2};\cdot,t^n)$ into r, a and b in Lemma A.6, respectively, and using (30), we have

$$\begin{aligned} I_2^n(x) &= \Delta t \int_0^1 \{X(t^{n-1/2};x,t^n) - Y_1^n(x)\} \\ &\qquad \cdot \nabla u^{n-1/2} \big(s_1 X(t^{n-1/2};x,t^n) + (1-s_1) Y_1^n(x) \big) ds_1 \\ &= \frac{\Delta t^3}{4} \int_0^1 \Big\{ \int_0^1 ds_2 \int_{s_2}^1 X''(t^{n-1/2} + \frac{\Delta t}{2} s_3) ds_3 \Big\} \\ &\qquad \cdot \nabla u^{n-1/2} \big(s_1 X(t^{n-1/2};x,t^n) + (1-s_1) Y_1^n(x) \big) ds_1. \end{aligned} \tag{A.15b}$$

Then, the equations (A.14) and (A.15) yield

$$
\begin{aligned}
\tilde{g}_3^n(x)^2 &= \frac{1}{\Delta t^2}\int_0^1 \Big\{ (I_1^n + I_2^n)(x) \cdot \nabla\phi^{n-1}\big(s_0 X_2^n(x) + (1-s_0)X(t^{n-1})\big) \Big\}^2 ds_0 \\
&\le c\Delta t^4 \int_0^1 \Big\{ \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_{-s_2}^{s_2} g_{31}\big(x, t^{n-1/2} + \frac{\Delta t}{2}s_3, s_0\big) ds_3 \\
&\qquad + \int_0^1 ds_1 \int_0^1 ds_2 \int_{s_2}^1 g_{32}\big(x, t^{n-1/2} + \frac{\Delta t}{2}s_3, s_0, s_1\big) ds_3 \Big\}^2 ds_0 \\
&\le c\Delta t^4 \Big\{ \int_0^1 ds_0 \int_{-1}^1 g_{31}\big(x, t^{n-1/2} + \frac{\Delta t}{2}s_3, s_0\big)^2 ds_3 \\
&\qquad + \int_0^1 ds_0 \int_0^1 ds_1 \int_0^1 g_{32}\big(x, t^{n-1/2} + \frac{\Delta t}{2}s_3, s_0, s_1\big)^2 ds_3 \Big\} \\
&\le c\Delta t^3 \Big\{ \int_0^1 ds_0 \int_{t^{n-1}}^{t^n} g_{31}(x,t,s_0)^2 dt + \int_0^1 ds_0 \int_0^1 ds_1 \int_{t^{n-1/2}}^{t^n} g_{32}(x,t,s_0,s_1)^2 dt \Big\} \\
&\le c\Delta t^3 \int_{t^{n-1}}^{t^n} g_3(x,t)^2\, dt,
\end{aligned}
$$

where

$$
\begin{aligned}
g_{31}(x,t,s_0) &\equiv X'''(t;x,t^n) \cdot \nabla\phi^{n-1}\big(s_0 X_2^n(x) + (1-s_0)X(t^{n-1};x,t^n)\big), \\
&\qquad (x,t) \in \overline{\Omega} \times (t^{n-1}, t^n], \\
g_{32}(x,t,s_0,s_1) &\equiv \big\{ X''(t;x,t^n)\nabla u^{n-1/2}\big(s_1 X(t^{n-1/2};x,t^n) + (1-s_1)Y_1^n(x)\big) \big\} \\
&\qquad \cdot \nabla\phi^{n-1}\big(s_0 X_2^n(x) + (1-s_0)X(t^{n-1};x,t^n)\big), \\
&\qquad (x,t) \in \overline{\Omega} \times (t^{n-1}, t^n], \\
g_3(x,t) &\equiv \Big\{ \int_0^1 g_{31}(x,t,s_0)^2 ds_0 + \int_0^1 ds_0 \int_0^1 g_{32}(x,t,s_0,s_1)^2\, ds_1 \Big\}^{1/2}, \\
&\qquad (x,t) \in \overline{\Omega} \times (t^{n-1}, t^n].
\end{aligned}
$$

Therefore, we obtain

$$
\|\tilde{g}_3\|_{l^2(l^2)} \le c\Delta t^2 \|g_3\|_{L^2(0,T;l^2(\Omega_h))},
$$

which implies that, from (A.13), the inequality (A.12a) holds for

$$
M_3 \equiv \|g_3\|_{L^2(0,T;l^2(\Omega_h))}.
$$

Inequalities (A.12b) follow from $X'''(t) = (d/dt)^2 u(X(t),t)$ appearing in g_{31}. $\square$

Lemma A.10. *(i) Suppose* $[\mathrm{H}_{0,1}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_{0,2}(\phi)]$ *and* $[\mathrm{H}_u(\Delta t)]$. *Then, there exists a positive constant* M_4 *such that*

$$
\|R_4\|_{l^2(l^2)} \le c_0 h M_4, \tag{A.16a}
$$

where M_4 *satisfies*

$$
M_4 \le c\|\phi\|_{C^0(C^2(\overline{\Omega}))},\ \ c'\|\phi\|_{L^2(H^2(\Omega))}. \tag{A.16b}
$$

(ii) Suppose $[\mathrm{H}_{0,1}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_{0,3}(\phi)]$ *and* $[\mathrm{H}_u(\Delta t)]$. *Let* $R_4^{n,(2)}$ *be a function defined by*

$$
R_4^{n,(2)} \equiv \frac{(\Pi_h^{(2)}\phi^{n-1})\circ X_2^n - \phi^{n-1}\circ X_2^n}{\Delta t}. \tag{A.17}
$$

Then, there exists a positive constant $M_4^{(2)}$ such that

$$\|R_4^{(2)}\|_{l^2(l^2)} \le c_0 h^2 M_4^{(2)}, \tag{A.18a}$$

where $M_4^{(2)}$ satisfies

$$M_4^{(2)} \le c\|\phi\|_{C^0(C^3(\overline{\Omega}))}, \;\; c'\|\phi\|_{L^2(H^3(\Omega))}. \tag{A.18b}$$

Proof. Let $n(=1,\cdots,N_T)$ be any fixed integer and $x = x_{\alpha,\beta} \in \Omega_h$ be any fixed lattice point, and assume $X_2^n(x) \in K_{l,m}$ $(\subset \overline{\Omega})$, where l and $m \in \mathbb{Z}^{1/2}$. We set

$$\begin{aligned}
y &\equiv X_2^n(x), \quad (\xi,\eta)^T \equiv u^{n-1/2}(x - u^n(x)\Delta t/2),\\
(p,q)^T &\equiv y - x_{l-1/2,m-1/2} = \big((\alpha - l + 1/2)h_1 - \xi\Delta t,\; (\beta - m + 1/2)h_2 - \eta\Delta t\big)^T,\\
(\tilde{p},\tilde{q})^T &\equiv x_{l+1/2,m+1/2} - y.
\end{aligned}$$

Without loss of generality we can assume $\xi \ge 0$, $\eta \ge 0$, $l < \alpha$ and $m < \beta$ (cf. Fig. A.1).

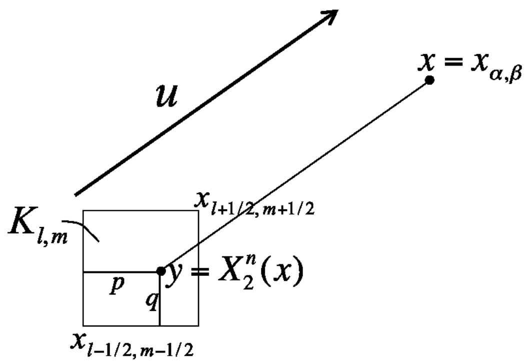

Figure A.1: Notation for the proof of Lemma A.10.

From Corollary A.1 (i) it holds that

$$\big\{(\Pi_h^{(1)} - I)\phi^{n-1}\big\}(y) = \sum_{(i,j)\in\Lambda^{(0,0)}(y)} T_1(y;i,j)\phi_{i,j}(y),$$

which implies

$$R_4^n(x)^2 \le \frac{4}{\Delta t^2} \sum_{(i,j)\in\Lambda^{(0,0)}(y)} \Big\{T_1(y;i,j)\phi_{i,j}(y)\Big\}^2.$$

When, e.g., $(i,j) = (l-1/2, m-1/2)$, we have

$$\begin{aligned}
&\Big\{T_1(y;i,j)\phi_{i,j}(y)\Big\}^2\\
&\le \Big[\int_0^1 ds_1 \int_0^{s_1} \Big\{(pD^1 + qD^2)^2\phi^{n-1}\Big\}\big(A(y;i,j,s_2)\big)\, ds_2\, \frac{\tilde{p}\tilde{q}}{h_1h_2}\Big]^2\\
&\le c\Big(\frac{\tilde{p}\tilde{q}}{h_1h_2}\Big)^2 (p+q)^4 \|\phi^{n-1}\|^2_{C^2(K_{l,m})}
\end{aligned}$$

$$\le c\{(p\tilde{p})^2+(q\tilde{q})^2\}\|\phi^{n-1}\|^2_{C^2(K_{l,m})} \quad (\text{by } p,\, \tilde{p},\, q, \tilde{q} \le h, \text{ Hypothesis 5}),$$

and, consequently, it holds that

$$\Big\{T_1(y;i,j)\phi_{i,j}(y)\Big\}^2 \le c\{(p\tilde{p})^2+(q\tilde{q})^2\}\|\phi^{n-1}\|^2_{C^2(K_{l,m})} \quad ((i,j)\in\Lambda^{(0,0)}(y)),$$

which implies

$$R_4^n(x)^2 \le c\frac{(p\tilde{p})^2+(q\tilde{q})^2}{\Delta t^2}\|\phi^{n-1}\|^2_{C^2(K_{l,m})}. \tag{A.19}$$

Now we evaluate $p\tilde{p}$. From $(0\le)p\le h_1$ it holds that

$$(\alpha-l-1/2)h_1 \le \xi\Delta t.$$

In the case of $\alpha-l-1/2\in\mathbb{N}$, from $h_1\le \frac{\xi}{\alpha-l-1/2}\Delta t \le U_0^\infty \Delta t$ we have

$$p\tilde{p}\le h_1^2 \le U_0^\infty \Delta t h_1.$$

Otherwise, from $\alpha-l-1/2=0$ we have $\xi\Delta t\in(0,h_1]$, $p=h_1-\xi\Delta t$ and $\tilde{p}=\xi\Delta t$, which imply

$$p\tilde{p}\le h_1\xi\Delta t \le U_0^\infty \Delta t h_1.$$

Thus, in any case, it holds that

$$p\tilde{p}\le U_0^\infty \Delta t h_1. \tag{A.20a}$$

Similarly, it holds that

$$q\tilde{q}\le U_0^\infty \Delta t h_2. \tag{A.20b}$$

Combining (A.20) with (A.19), we have

$$R_4^n(x)^2 \le c_0 h^2 \|\phi^{n-1}\|^2_{C^2(K_{l,m})}.$$

By the inequality $\sharp\{(l,m)\in\mathbb{Z}^{1/2}\times\mathbb{Z}^{1/2};\, X_2^n(x)\in K_{l,m},\, x\in\overline{\Omega}_h\}\le N_\Xi$ (cf. (20)), it holds that

$$\begin{aligned}\|R_4^n\|^2_{l^2(\Omega_h)} &\le c_0 h_1 h_2 \sum_{x\in\Omega_h} h^2\|\phi^{n-1}\|^2_{C^2(K_{l,m})}\\ &\le c_0 N_\Xi h_1 h_2 \sum_{x_{l,m}\in\Omega_h^{(1/2,1/2)}} h^2\|\phi^{n-1}\|^2_{C^2(K_{l,m})},\end{aligned}$$

which implies

$$\begin{aligned}||R_4||_{l^2(l^2)} &\le \Big\{\Delta t\sum_{n=1}^{N_T} c_0 N_\Xi h_1 h_2 \sum_{x_{l,m}\in\Omega_h^{(1/2,1/2)}} h^2\|\phi^{n-1}\|^2_{C^2(K_{l,m})}\Big\}^{1/2}\\ &\le c_0 h||\phi||_{C^0(C^2(\overline{\Omega}))}.\end{aligned}$$

Thus we obtain (A.16).

By a similar proof after replacing $\Pi_h^{(1)}$ with $\Pi_h^{(2)}$ we get (A.18). □

Lemma A.11. *(i) Suppose* $[\mathrm{H}_{0,3}(\phi)]$*. Then, there exists a positive constant* M_5 *such that*

$$\|R_5\|_{l^2(l^2)} \le chM_5, \tag{A.21a}$$

where M_5 *satisfies*

$$M_5 \le c\|\phi\|_{C^0(C^3(\overline{\Omega}))},\ \ c'\|\phi\|_{L^2(H^3(\Omega))}. \tag{A.21b}$$

(ii) Suppose $[\mathrm{H}_{0,4}(\phi)]$*. Then, there exists a positive constant* $M_5^{(2)}$ *such that*

$$\|R_5\|_{l^2(l^2)} \le ch^2M_5^{(2)}, \tag{A.22a}$$

where $M_5^{(2)}$ *satisfies*

$$M_5^{(2)} \le c\|\phi\|_{C^0(C^4(\overline{\Omega}))},\ \ c'\|\phi\|_{L^2(H^4(\Omega))}. \tag{A.22b}$$

In the proof of above lemma we use the following norms, for a function $v \in C^0(\overline{\Omega})$ and a function set $\{\phi^n\}_{n=0}^{N_T} \subset C^0(\overline{\Omega})$,

$$\|v\|_{(L^2,l^2)} \equiv \Big\{h_2 \sum_{0<j<N_2} \|v(\cdot, jh_2)\|^2_{L^2(0,L_1)}\Big\}^{1/2},$$

$$\|v\|_{(l^2,L^2)} \equiv \Big\{h_1 \sum_{0<i<N_1} \|v(ih_1,\cdot)\|^2_{L^2(0,L_2)}\Big\}^{1/2},$$

$$\|\phi\|_{l^2(L^2,l^2)} \equiv \Big\{\Delta t \sum_{n=1}^{N_T} \|\phi^n\|^2_{(L^2,l^2)}\Big\}^{1/2}, \quad \|\phi\|_{l^2(l^2,L^2)} \equiv \Big\{\Delta t \sum_{n=1}^{N_T} \|\phi^n\|^2_{(l^2,L^2)}\Big\}^{1/2}.$$

Proof of Lemma A.11. We prove only (ii), as the other proof is similar. Let $x = x_{i,j} \in \Omega_h$ be a lattice point, and R_{5k}^n and $F_k(\cdot;x,t^n)$ $(k = 1,2)$ be functions defined by

$$R_{5k}^n(x) \equiv \frac{1}{2}\big(\Delta_{h,k} - D^{kk}\big)\phi(x,t^n), \quad F_k(s;x,t^n) \equiv \frac{1}{2}\phi(x+se_k,t^n) \quad (k = 1,2).$$

Then, it holds that

$$R_{5k}^n(x) = \Gamma_3(F_k(\cdot;x,t^n), 2h_k).$$

From the relations

$$\begin{aligned}
\int_{-1}^{1} F_1''''(h_1 s;x,t^n)^2 ds &= \frac{1}{2}\int_{-1}^{1} D^{1111}\phi^n(x+sh_1e_1)^2 ds \\
&= \frac{1}{2h_1}\int_{(i-1)h_1}^{(i+1)h_1} D^{1111}\phi^n(\eta, jh_2)^2 d\eta \\
&= \frac{1}{2h_1}\|D^{1111}\phi^n(\cdot, jh_2)\|^2_{L^2((i-1)h_1,(i+1)h_1)}, \\
\int_{-1}^{1} F_2''''(h_2 s;x,t^n)^2 ds &= \frac{1}{2h_2}\|D^{2222}\phi^n(ih_1,\cdot)\|^2_{L^2((j-1)h_2,(j+1)h_2)},
\end{aligned}$$

we have, from (A.6c),

$$\|R_5^n\|^2_{l^2(\Omega_h)} = \|\sum_{k=1}^{2} R_{5k}^n\|^2_{l^2(\Omega_h)} \le 2\sum_{k=1}^{2} \|R_{5k}^n\|^2_{l^2(\Omega_h)}$$

$$
\begin{aligned}
&\le \frac{h_1h_2}{72}\sum_{x_{i,j}\in\Omega_h}\Big\{h_1^3\big\|D^{1111}\phi^n(\cdot,jh_2)\big\|^2_{L^2((i-1)h_1,(i+1)h_1)}\\
&\qquad\qquad + h_2^3\big\|D^{2222}\phi^n(ih_1,\cdot)\big\|^2_{L^2((j-1)h_2,(j+1)h_2)}\Big\}\\
&\le \frac{1}{36}\Big\{h_1^4\big\|D^{1111}\phi^n\big\|^2_{(L^2,l^2)} + h_2^4\big\|D^{2222}\phi^n\big\|^2_{(l^2,L^2)}\Big\},
\end{aligned}
$$

which implies (A.22a) for

$$
M_5^{(2)} \equiv \|D^{1111}\phi\|_{l^2(L^2,l^2)} + \|D^{2222}\phi\|_{l^2(l^2,L^2)}.
$$

□

Lemma A.12. *(i) Suppose* $[\mathrm{H}_{0,2}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_{0,3}(\phi)]$, $[\mathrm{H}_u(\Delta t)]$ *and* $[\mathrm{H}_{wCFL}(\Delta t)]$. *Then, there exists a positive constant* M_6 *such that*

$$
\|R_6\|_{l^2(l^2)} \le c_0 h M_6, \tag{A.23a}
$$

where M_6 *satisfies*

$$
M_6 \le c_2\|\phi\|_{C^0(C^3(\overline{\Omega}))},\ \ c_2'\|\phi\|_{L^2(H^3(\Omega))}. \tag{A.23b}
$$

(ii) Suppose $[\mathrm{H}_{0,3}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_{0,4}(\phi)]$, $[\mathrm{H}_u(\Delta t)]$ *and* $[\mathrm{H}_{CFL}(\Delta t)]$. *Replace* $\Pi_h^{(\frac{1}{2},0),(1)}$ *in* (7a) *and* $\Pi_h^{(0,\frac{1}{2}),(1)}$ *in* (7b) *by* $\check{\Pi}_h^{(\frac{1}{2},0),(1)}$ *and* $\check{\Pi}_h^{(0,\frac{1}{2}),(1)}$, *respectively. Then, there exists a positive constant* $M_6^{(2)}$ *such that*

$$
\|R_6\|_{l^2(l^2)} \le c_1 h^2 M_6^{(2)}, \tag{A.24a}
$$

where $M_6^{(2)}$ *satisfies*

$$
M_6^{(2)} \le c_3\|\phi\|_{C^0(C^4(\overline{\Omega}))},\ \ c_3'\|\phi\|_{L^2(H^4(\Omega))}. \tag{A.24b}
$$

Proof. We prove only (ii), as the other is similar. Let R_{6k}^n $(k=1,2)$ be functions defined by

$$
R_{6k}^n \equiv \frac{1}{2}\Big\{\nabla_{hk}\tilde{\nabla}_{hk}^{(n)}\phi^{n-1} - D^k\big((D^k\phi^{n-1})\circ X_1^n\big)\Big\} \quad (k=1,2).
$$

Then, we have

$$
R_6^n = \sum_{k=1}^{2} R_{6k}^n.
$$

It is sufficient for the proof of (A.24a) to show that there exist positive constants $M_{6k}^{(2)}$ $(k=1,2)$ such that

$$
\|R_{6k}\|_{l^2(l^2)} \le c_3 h^2 M_{6k}^{(2)} \quad (k=1,\ 2). \tag{A.25}
$$

We prove only the case $k=1$ of (A.25), as the other proof is similar. For $x=x_{\alpha,\beta}$ $(\alpha,\beta\in\mathbb{Z})$ R_{61} can be written as

$$
R_{61}^n(x) = \frac{1}{2}\Big[\nabla_{h1}\big\{\check{\Pi}_h^{(\frac{1}{2},0)}\nabla_{h1}\phi^{n-1}\circ X_1^n\big\}(x) - \nabla_{h1}\big\{\check{\Pi}_h^{(\frac{1}{2},0)}D^1\phi^{n-1}\circ X_1^n\big\}(x)\Big]
$$

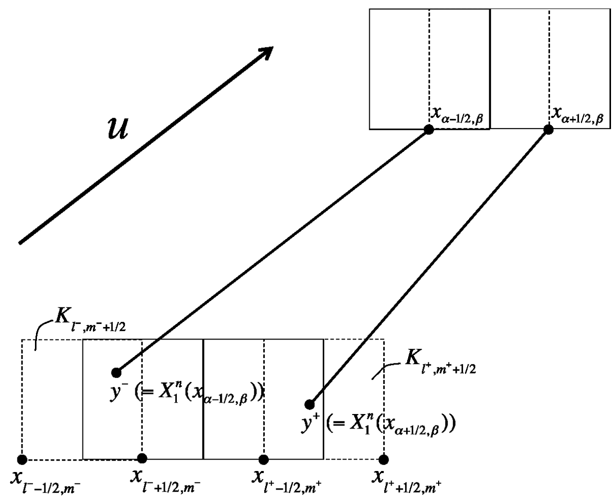

Figure A.2: Notation for the proof of Lemma A.12

$$
\begin{aligned}
&+\frac{1}{2}\Big[\nabla_{h1}\big\{\check{\Pi}_h^{(\frac{1}{2},0)}D^1\phi^{n-1}\circ X_1^n\big\}(x)-\nabla_{h1}\big\{D^1\phi^{n-1}\circ X_1^n\big\}(x)\Big]\\
&+\frac{1}{2}\Big[\nabla_{h1}\big\{D^1\phi^{n-1}\circ X_1^n\big\}(x)-D^1\big((D^1\phi^{n-1})\circ X_1^n\big)(x)\Big]\\
\equiv\ & R_{611}^n(x)+R_{612}^n(x)+R_{613}^n(x).
\end{aligned}
$$

At first we evaluate R_{611}. Let ω_1, ω_2 and ω be sets defined by

$$\omega_1\equiv\{x\in\Omega;\ 0<x_1<h_1/2\},\ \ \omega_2\equiv\{x\in\Omega;\ L_1-h_1/2<x_1<L_1\},\ \ \omega\equiv\omega_1\cup\omega_2.$$

For $y\in\overline{\Omega}\setminus\overline{\omega}$ we have, from (A.5b),

$$
\begin{aligned}
&\check{\Pi}_h^{(\frac{1}{2},0)}\nabla_{h1}\phi^{n-1}(y)-\check{\Pi}_h^{(\frac{1}{2},0)}D^1\phi^{n-1}(y)\\
&=\sum_{(i,j)\in\Lambda^{(\frac{1}{2},0)}(y)}\big(\nabla_{h1}\phi^{n-1}-D^1\phi^{n-1}\big)(x_{i,j})\phi_{i,j}(y)\\
&=\sum_{(i,j)\in\Lambda^{(\frac{1}{2},0)}(y)}\frac{h_1^2}{8}\int_0^1ds_1\int_0^{s_1}ds_2\int_{-s_2}^{s_2}D^{111}\phi^{n-1}(x_{i,j}+s_3\frac{h_1}{2}e_1)ds_3\ \phi_{i,j}(y)\\
&=\sum_{(i,j)\in\Lambda^{(\frac{1}{2},0)}(y)}\Big\{\frac{h_1^2}{24}D^{111}\phi^{n-1}(y)+J_1^n(y;i,j)\Big\}\phi_{i,j}(y)\\
&=\frac{h_1^2}{24}D^{111}\phi^{n-1}(y)+I_1^n(y),
\end{aligned}
\tag{A.26}
$$

where

$$
\begin{aligned}
I_1^n(y)&\equiv\sum_{(i,j)\in\Lambda^{(\frac{1}{2},0)}(y)}J_1^n(y;i,j)\ \phi_{i,j}(y),\\
J_1^n(y;i,j)&\equiv\frac{h_1^2}{8}\int_0^1ds_1\int_0^{s_1}ds_2\int_{-s_2}^{s_2}ds_3
\end{aligned}
$$

$$\times \int_0^1 \Big\{\big((a(y;i,j)+s_3\frac{h_1}{2}e_1)\cdot\nabla\big)D^{111}\phi^{n-1}\Big\}\big(y+s_4(a(y;i,j)+s_3\frac{h_1}{2}e_1)\big)ds_4,$$

$$a(y;i,j)\equiv x_{i,j}-y.$$

Set $y^{\pm}\equiv X_1^n(x_{\alpha\pm1/2,\beta})$. In the case of $y^{\pm}\in\overline{\Omega}\setminus\overline{\omega}$, from (A.26) we have

$$\begin{aligned}
&R^n_{611}(x_{\alpha,\beta})\\
&=\frac{1}{2}\nabla_{h1}\Big[\big\{\check{\Pi}_h^{(\frac12,0)}\nabla_{h1}\phi^{n-1}-\check{\Pi}_h^{(\frac12,0)}D^1\phi^{n-1}\big\}\circ X_1^n\Big](x_{\alpha,\beta})\\
&=\frac{h_1}{48}\big\{D^{111}\phi^{n-1}(y^+)-D^{111}\phi^{n-1}(y^-)\big\}+\frac{1}{h_1}\big\{I_1^n(y^+)-I_1^n(y^-)\big\}\\
&=\frac{h_1}{48}\int_0^1\Big[\big\{(y^+-y^-)\cdot\nabla\big\}D^{111}\phi^{n-1}\Big](y^-+s(y^+-y^-))ds+\frac{1}{h_1}\big\{I_1^n(y^+)-I_1^n(y^-)\big\},
\end{aligned}$$

which implies

$$|R^n_{611}(x)|\le c_0(\Delta t^2+h^2)\|\phi^{n-1}\|_{C^4(\overline{\Omega})} \tag{A.27}$$

in virtue of $|y^+-y^-|\le c_0\Delta t$ and $|I_1^n(y^{\pm})|\le ch^3\|\phi^{n-1}\|_{C^4(\overline{\Omega})}$. Now we consider the case of $(y^+,y^-)\in(\overline{\Omega}\setminus\overline{\omega})\times\overline{\omega}_1$. Since it holds that, for $y\in\overline{\omega}_1$,

$$\begin{aligned}
&\check{\Pi}_h^{(\frac12,0)}\nabla_{h1}\phi^{n-1}(y)-\check{\Pi}_h^{(\frac12,0)}D^1\phi^{n-1}(y)\\
&=\sum_{(i,j)\in\Lambda^{(\frac12,0)}(y),\ i=1/2}(\nabla_{h1}-D^1)\phi^{n-1}(x_{i,j})\phi_{i,j}(y)\\
&\quad+\sum_{(i,j)\in\Lambda^{(\frac12,0)}(y),\ i=-1/2}\big\{3(\nabla_{h1}-D^1)\phi^{n-1}(x_{i+1,j})-3(\nabla_{h1}-D^1)\phi^{n-1}(x_{i+2,j})\\
&\qquad\qquad+(\nabla_{h1}-D^1)\phi^{n-1}(x_{i+3,j})\big\}\phi_{i,j}(y),
\end{aligned}$$

we have

$$\begin{aligned}
&R^n_{611}(x)\\
&=\frac{1}{2h_1}\Big\{\sum_{(i,j)\in\Lambda^{(\frac12,0)}(y^+)}(\nabla_{h1}-D^1)\phi^{n-1}(x_{i,j})\phi_{i,j}(y^+)\\
&\qquad-\sum_{(i,j)\in\Lambda^{(\frac12,0)}(y^-)}(\nabla_{h1}-D^1)\phi^{n-1}(x_{i,j})\phi_{i,j}(y^-)\Big\}\\
&\quad+\frac{1}{2h_1}\sum_{(i,j)\in\Lambda^{(\frac12,0)}(y^-),\ i=-1/2}\Big[(\nabla_{h1}-D^1)\phi^{n-1}(x_{i,j})-\big\{3(\nabla_{h1}-D^1)\phi^{n-1}(x_{i+1,j})\\
&\qquad-3(\nabla_{h1}-D^1)\phi^{n-1}(x_{i+2,j})+(\nabla_{h1}-D^1)\phi^{n-1}(x_{i+3,j})\big\}\Big]\phi_{i,j}(y^-)\\
&\equiv r_{61}(x)+r_{62}(x).
\end{aligned}$$

Similarly to the previous case of $y^{\pm}\in\overline{\Omega}\setminus\overline{\omega}$ we have

$$|r_{61}(x)|\le c_0(\Delta^2+h^2)||\phi||_{C^0(C^4(\overline{\Omega}))}.$$

As for the evaluation of $r_{62}(x)$ we use the identity, for $a\in\mathbb{R}$ and $f\in C^3[a,a+h]$,

$$f(a)-\big\{3f(a+h)-3f(a+2h)+f(a+3h)\big\}$$

$$= h^3 \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} \big\{-3f'''(a+s_3h) + 24f'''(a+2s_3h) - 27f'''(a+3s_3h)\big\} ds_3, \tag{A.28}$$

which implies

$$|r_{62}(x)| \le c_0 h^2 \|\phi\|_{C^0(C^4(\overline{\Omega}))}.$$

Thus we get (A.27). The proof of (A.27) in the case of $(y^+, y^-) \in (\overline{\Omega} \setminus \overline{\varpi}) \times \overline{\varpi}_2$ or $(y^+, y^-) \in \overline{\varpi} \times (\overline{\Omega} \setminus \overline{\varpi})$ is similar. In the other case, i.e., $y^\pm \in \overline{\varpi}$, the inequality (A.27) holds by a similar argument to the case of $y^\pm \in \overline{\Omega} \setminus \overline{\varpi}$.

Next we evaluate R_{612}. Assume $y \in \overline{\Omega} \setminus \overline{\varpi}$ and $y \in K_{l,m}$ for $(l,m) \in \mathbb{Z} \times \mathbb{Z}^{1/2}$. Then, from Corollary A.1 (ii) we have

$$(\check{\Pi}_h^{(\frac{1}{2},0)} - I) D^1 \phi^{n-1}(y) = \frac{1}{2} \big\{ \big(p\tilde{p} D^{111} + q\tilde{q} D^{122} \big) \phi^{n-1} \big\}(y) + I_2^n(y), \tag{A.29}$$

where

$$\begin{aligned}
(p,\ q)^T &= (p(y),\ q(y))^T \equiv y - x_{l-1/2,m-1/2},\\
(\tilde{p},\ \tilde{q})^T &= (\tilde{p}(y),\ \tilde{q}(y))^T \equiv x_{l+1/2,m+1/2} - y,\\
I_2^n(y) &\equiv \sum_{(i,j)\in\Lambda^{(\frac{1}{2},0)}(y)} J_2^n(y;i,j)\phi_{i,j}(y),\\
J_2^n(y;i,j) &\equiv \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} \big\{ \big(a(y;i,j)\cdot\nabla\big)^3 D^1 \phi^{n-1} \big\} \big(A(y;i,j,s_3)\big) ds_3,\\
a(y;i,j) &\equiv x_{i,j} - y, \quad A(y;i,j,s) \equiv y + s\,a(y;i,j).
\end{aligned}$$

Suppose $y^\pm \equiv X_1^n(x_{\alpha\pm1/2,\beta}) \in \overline{\Omega} \setminus \overline{\varpi}$. Then (A.29) yields

$$\begin{aligned}
&R_{612}^n(x)\\
&= \frac{1}{2} \nabla_{h1} \Big[\big\{ \big(\check{\Pi}_h^{(\frac{1}{2},0)} - I\big) D^1 \phi^{n-1} \big\} \circ X_1^n \Big](x)\\
&= \frac{1}{2h_1} \Big[\big\{ p(y^+)\tilde{p}(y^+) D^{111} + q(y^+)\tilde{q}(y^+) D^{122} \big\} \phi^{n-1}(y^+)\\
&\qquad - \big\{ p(y^-)\tilde{p}(y^-) D^{111} + q(y^-)\tilde{q}(y^-) D^{122} \big\} \phi^{n-1}(y^-) \Big] + \frac{1}{h_1} \big\{ I_2^n(y^+) - I_2^n(y^-) \big\}\\
&= \frac{1}{2h_1} \Big[\big\{ p(y^+)\tilde{p}(y^+) - p(y^-)\tilde{p}(y^-) \big\} D^{111}\\
&\qquad\qquad + \big\{ q(y^+)\tilde{q}(y^+) - q(y^-)\tilde{q}(y^-) \big\} D^{122} \Big] \phi^{n-1}(y^-)\\
&\quad + \frac{1}{2h_1} p(y^+)\tilde{p}(y^+) \int_0^1 \big\{ (y^+ - y^-)\cdot\nabla \big\} D^{111} \phi^{n-1}(y^- + s(y^+ - y^-))\, ds\\
&\quad + \frac{1}{2h_1} q(y^+)\tilde{q}(y^+) \int_0^1 \big\{ (y^+ - y^-)\cdot\nabla \big\} D^{122} \phi^{n-1}(y^- + s(y^+ - y^-))\, ds\\
&\quad + \frac{1}{h_1} \big\{ I_2^n(y^+) - I_2^n(y^-) \big\}.
\end{aligned} \tag{A.30}$$

Here we show

$$E \equiv \big| p(y^+)\tilde{p}(y^+) - p(y^-)\tilde{p}(y^-) \big| \le c_1 h(\Delta t^2 + h^2). \tag{A.31}$$

In the case of $u_1(x_{\alpha\pm1/2,\beta}) \geq 0$, from [H$_{CFL}(\Delta t)$] we have

$$p(y^\pm) = h_1 - u_1(x_{\alpha\pm1/2,\beta})\Delta t, \quad \tilde{p}(y^\pm) = u_1(x_{\alpha\pm1/2,\beta})\Delta t,$$

and

$$\begin{aligned}
E &= \Big| \big\{h_1 - u_1(x_{\alpha+1/2,\beta})\Delta t\big\} u_1(x_{\alpha+1/2,\beta})\Delta t \\
&\qquad - \big\{h_1 - u_1(x_{\alpha-1/2,\beta})\Delta t\big\} u_1(x_{\alpha-1/2,\beta})\Delta t \Big| \\
&= \Big| h_1 \Delta t \big\{u_1(x_{\alpha+1/2,\beta}) - u_1(x_{\alpha-1/2,\beta})\big\} \\
&\qquad - \Delta t^2 \big\{u_1(x_{\alpha+1/2,\beta}) + u_1(x_{\alpha-1/2,\beta})\big\}\big\{u_1(x_{\alpha+1/2,\beta}) - u_1(x_{\alpha-1/2,\beta})\big\} \Big| \\
&\leq c_1 h(\Delta t^2 + h^2).
\end{aligned}$$

In the case of $u_1(x_{\alpha+1/2,\beta}) \leq 0$ and $u_1(x_{\alpha-1/2,\beta}) \geq 0$ there exists a point x^* between $x_{\alpha+1/2,\beta}$ and $x_{\alpha-1/2,\beta}$ such that $u_1(x^*) = 0$ to have

$$\big|u_1(x_{\alpha\pm1/2,\beta})\big| \leq c_1 h_1,$$

which implies (A.31). Since proofs for the other cases are similar, we obtain the inequality (A.31) for all cases, and similarly it holds that

$$\big|q(y^+)\tilde{q}(y^+) - q(y^-)\tilde{q}(y^-)\big| \leq c_1 h(\Delta t^2 + h^2). \tag{A.32}$$

Combining the inequalities (A.31) and (A.32) with (A.30) and using the estimates $|y^+ - y^-| \leq c_0\Delta t$ and $|I_2^n(y^\pm)| \leq ch^3\|\phi^{n-1}\|_{C^4(\overline{\Omega})}$, we have

$$|R_{612}^n(x)| \leq c_0(\Delta t^2 + h^2)\|\phi^{n-1}\|_{C^4(\overline{\Omega})}. \tag{A.33}$$

In the case of y^+ or $y^- \in \overline{\varpi}$ the inequality (A.33) holds similarly by using (A.28).

It is obvious that, from (A.5b),

$$|R_{613}^n(x)| \leq c_0 h^2 \|\phi^{n-1}\|_{C^4(\overline{\Omega})}. \tag{A.34}$$

Combining (A.27), (A.33) and (A.34), we obtain the desired result. □

Lemma A.13. *Suppose* [H$_{0,1}(u)$], [H$_\Gamma(u)$], [H$_{0,3}(\phi)$] *and* [H$_u(\Delta t)$]. *Then, there exists a positive constant M_7 such that*

$$\|R_7\|_{l^2(l^2)} \leq c_1(\Delta t^2 + h^2)M_7, \tag{A.35a}$$

where M_7 satisfies

$$M_7 \leq c_1\|\phi\|_{C^0(C^3(\overline{\Omega}))}, \;\; c_1'\|\phi\|_{L^2(H^3(\Omega))}. \tag{A.35b}$$

Proof. At first we prepare three identities (A.36)–(A.38). For $x \in \Omega$ it holds that

$$\begin{aligned}
&\nabla \cdot \big((\nabla\phi^{n-1}) \circ X_1^n\big)(x) \\
&= \sum_{i=1}^2 D^i\big((D^i\phi^{n-1}) \circ X_1^n\big)(x) = \sum_{i,j=1}^2 (D^{ij}\phi^{n-1}) \circ X_1^n(x)\big(\delta_{ji} - D^i u_j^n(x)\Delta t\big) \\
&= (\Delta\phi^{n-1}) \circ X_1^n(x) - \Delta t \sum_{i,j=1}^2 D^i u_j^n(x)(D^{ij}\phi^{n-1}) \circ X_1^n(x)
\end{aligned}$$

$$= (\Delta\phi^{n-1}) \circ X_1^n(x) - \Delta t \sum_{i,j=1}^{2} D^i u_j^n(x) D^{ij}\phi^{n-1}(x) + \Delta t^2 \rho_1^n(x), \tag{A.36}$$

where

$$\rho_1^n(x) \equiv \sum_{i,j,k=1}^{2} D^i u_j^n(x) u_k^n(x) \rho_{1(i,j,k)}^n(x), \;\; \rho_{1(i,j,k)}^n(x) \equiv \int_0^1 (D^{ijk}\phi^{n-1})(x - su^n(x)\Delta t) ds.$$

In the last equality we have used the identity

$$(D^{ij}\phi^{n-1}) \circ X_1^n(x) = D^{ij}\phi^{n-1}(x) - \int_0^1 u^n(x) \cdot \nabla D^{ij}\phi^{n-1}(x - su^n(x)\Delta t) ds \;\; (i, j = 1, 2)$$

obtained from Lemma A.6 with $f = D^{ij}\phi^{n-1}$, $a(x) = x$ and $b(x) = X_1^n(x)$. For $x \in \Omega_h$ it holds that

$$\Delta_{h,i}\phi^{n-1}(x) = D^{ii}\phi^{n-1}(x) + h_i \rho_{2i}^n(x) \quad (i = 1, 2), \tag{A.37}$$

$$\nabla_{(2h)1}\nabla_{(2h)2}\phi^{n-1}(x) = D^{12}\phi^{n-1}(x) + 2\sum_{i=1}^{2} h_i \rho_{3i}^n(x), \tag{A.38}$$

where, for $i = 1, 2$,

$$\rho_{2i}^n(x) \equiv \int_0^1 ds_1 \int_{-s_1}^{s_1} ds_2 \int_0^{s_2} D^{iii}\phi^{n-1}(x + s_3 h_i e_i)\, ds_3,$$
$$\rho_{3i}^n(x) \equiv \int_{-1/2}^{1/2} ds_1 \int_{-1/2}^{1/2} ds_2 \int_0^1 s_i (D^{12i}\phi^{n-1})\big(x + s_3(2h_1 s_1, 2h_2 s_2)^T\big) ds_3.$$

(A.37) and (A.38) are proved similarly to (A.4c). We set $\rho_k^n \equiv \sum_{i=1}^2 \rho_{ki}^n$ $(k = 2, 3)$.

Now we evaluate R_7^n. Let $x \in \Omega_h$ be any lattice point. From the identities (A.36), (A.37) and (A.38) we have

$$\begin{aligned}
R_7^n(x) &= \frac{1}{2}\big\{\nabla \cdot \big((\nabla\phi^{n-1}) \circ X_1^n\big) - \Delta\phi^{n-1} \circ X_1^n + \Delta t \sum_{i=1}^{2} (D^i u_i^n)\Delta_{h,i}\phi^{n-1} \\
&\qquad + \Delta t (D^2 u_1^n + D^1 u_2^n)\nabla_{(2h)1}\nabla_{(2h)2}\phi^{n-1}\big\}(x) \\
&= \frac{1}{2}\big\{-\Delta t \sum_{i,j=1}^{2} D^i u_j^n(x) D^{ij}\phi^{n-1}(x) + \Delta t^2 \rho_1^n(x) \\
&\qquad + \Delta t \sum_{i=1}^{2} D^i u_i^n(x)(D^{ii}\phi^{n-1} + h_i \rho_{2i}^n)(x) \\
&\qquad + \Delta t (D^2 u_1^n + D^1 u_2^n)(x)(D^{12}\phi^{n-1} + 2\sum_{i=1}^{2} h_i \rho_{3i}^n)(x)\big\} \\
&= \frac{\Delta t^2}{2}\rho_1^n(x) + \frac{\Delta t}{2}\sum_{i=1}^{2} h_i D^i u_i^n(x)\rho_{2i}^n(x) + \Delta t (D^2 u_1^n + D^1 u_2^n)(x)\sum_{i=1}^{2} h_i \rho_{3i}^n(x) \\
&\equiv \sum_{i=1}^{3} R_{7i}^n(x).
\end{aligned} \tag{A.39}$$

Let g_{7i}^n $(i = 1, 2)$ be functions defined by

$$g_{71}^n \equiv \sum_{i,j,k=1}^{2} \rho_{1(i,j,k)}^n, \qquad g_{72}^n \equiv \sum_{i=1}^{2} D^{iii}\phi^{n-1}.$$

Then, from $[\mathrm{H}_{0,1}(u)]$ it holds that

$$\|R_{71}\|_{l^2(l^2)} = \frac{\Delta t^2}{2}\|\rho_1\|_{l^2(l^2)} \le c_1\Delta t^2\|g_{71}\|_{l^2(l^2)} \le c_1\Delta t^2\|\phi\|_{C^0(C^3(\overline{\Omega}))}, \tag{A.40}$$

$$\begin{aligned}\|R_{72}\|_{l^2(l^2)} &\le c_1(\Delta t^2+h^2)\|\rho_2\|_{l^2(l^2)} \le c_1(\Delta t^2+h^2)\|g_{72}\|_{l^2(l^2)} \\ &\le c_1(\Delta t^2+h^2)\|\phi\|_{C^0(C^3(\overline{\Omega}))}.\end{aligned} \tag{A.41}$$

From the Schwarz inequality we have, for $x_{\alpha,\beta} \in \Omega_h$,

$$\begin{aligned}&\rho_{3i}^n(x_{\alpha,\beta})^2 \\ &\le \frac{1}{12}\int_{-1/2}^{1/2} ds_1\int_{-1/2}^{1/2} ds_2\int_0^1 D^{12i}\phi^{n-1}\big(x_{\alpha,\beta}+s_3(2h_1s_1,2h_2s_2)^T\big)^2 ds_3 \\ &\le \frac{c}{h_1h_2}\int_0^1 \frac{ds_3}{s_3^2}\int_{h_1(\alpha-s_3)}^{h_1(\alpha+s_3)} dy_1\int_{h_2(\beta-s_3)}^{h_2(\beta+s_3)} D^{12i}\phi^{n-1}(y)^2 dy_2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\text{by } y = x_{\alpha,\beta}+s_3(2h_1s_1,2h_2s_2)^T) \\ &\le \frac{c}{h_1h_2}\int_0^1 \frac{1}{s_3^2}\max\{D^{12i}\phi^{n-1}(y)^2;\ |(y-x_{\alpha,\beta})_k| \le h_ks_3,\ k=1,2\}(4h_1h_2s_3^2)\, ds_3 \\ &\le c\max\{D^{12i}\phi^{n-1}(y)^2;\ |(y-x_{\alpha,\beta})_l| \le h_k,\ k=1,2\}.\end{aligned}$$

Hence it follows

$$\begin{aligned}&\|R_{73}^n\|_{l^2(\Omega_h)}^2 \\ &\le c_1(\Delta t^2+h^2)^2\sum_{x_{\alpha,\beta}\in\Omega_h}\rho_3^n(x_{\alpha,\beta})^2 \le c_1(\Delta t^2+h^2)^2 h_1h_2\sum_{x_{\alpha,\beta}\in\Omega_h}\sum_{i=1}^2\rho_{3i}^n(x_{\alpha,\beta})^2 \\ &\le c_1(\Delta t^2+h^2)^2 h_1h_2\sum_{x_{\alpha,\beta}\in\Omega_h}\sum_{i=1}^2\max\{D^{12i}\phi^{n-1}(y)^2;\ |(y-x_{\alpha,\beta})_k| \le h_k,\ k=1,2\} \\ &\le c_1(\Delta t^2+h^2)^2\|\phi^{n-1}\|_{C^3(\overline{\Omega})}^2,\end{aligned} \tag{A.42}$$

which implies

$$\|R_{73}\|_{l^2(l^2)} \le c_1(\Delta t^2+h^2)\|\phi\|_{C^0(C^3(\overline{\Omega}))}. \tag{A.43}$$

Combining the inequalities (A.40), (A.41) and (A.43) with (A.39), we obtain (A.35a) with the first inequality of (A.35b). From (A.42) there exists a constant $h_* = h_*(\phi) > 0$ such that, for any $h \le h_*$,

$$\|R_{73}^n\|_{l^2(\overline{\Omega}_h)}^2 \le 2c_1(\Delta t^2+h^2)^2\|\phi^{n-1}\|_{H^3(\Omega)}^2, \tag{A.44}$$

which implies the second inequality of (A.35b) with similar estimates for R_{7i} $(i=1,2)$. $\square$

Lemma A.14. *Suppose* $[\mathrm{H}_{0,1}(u)]$, $[\mathrm{H}_\Gamma(u)]$, $[\mathrm{H}_{2C}(\Delta\phi)]$ *and* $[\mathrm{H}_u(\Delta t)]$. *Then, there exists a positive constant* M_8 *such that*

$$\|R_8\|_{l^2(l^2)} \le c\Delta t^2 M_8, \tag{A.45a}$$

where M_8 *satisfies*

$$M_8 \le c_1\|\Delta\phi\|_{Z_C^2},\ \ c_1'\|\Delta\phi\|_{Z^2}. \tag{A.45b}$$

Proof. Regarding $\Delta\phi$ as f in Lemma 4, we get the result for

$$M_8 \equiv \left\| \frac{\partial^2 g_8}{\partial t^2} \right\|_{L^2(0,T;l^2(\Omega_h))},$$
$$g_8(x,t) \equiv \Delta\phi\big(x-(t^n-t)u^n(x),t\big), \quad (x,t)\in\overline{\Omega}\times(t^{n-1},t^n].$$

□