Numerical approaches to local risk-minimization for exponential Lévy models

幾何レヴィ過程に対する局所リスク最小化戦略とその数値解析的研究

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Chapter 1

Introduction

How should we price contingent claims in incomplete markets? What is the optimal hedging structure in incomplete markets? These two questions are most important topics for finance theory. Markets are complete if any claim in the market is attainable. In general there is no friction like transaction costs and so on. The prices of contingent claims in complete markets are given as the initial cost of its self-financing strategies. Complete markets are characterized in terms of the martingale measure: *The market model is complete if, and only if, the martingale measure is unique.* In this models any contingent claim can be priced by no-arbitrage considerations. It is true our real markets are *not* complete. Therefore we face the problem of an incomplete market. In such markets a contingent claim cannot be perfectly hedged by choosing a unique self-financing trading strategy. There are infinitely many martingale measures, each of which produces a no-arbitrage price. To treat this problem we consider an 'optimal' hedging strategy and regard its initial cost as price. In this paper we choose Local risk minimization (LRM) strategies as such 'optimal' hedging strategies. LRM strategies for incomplete market models whose asset price process is described by a stochastic differential equation (SDE) driven by a Lévy process, are typical framework of incomplete market models.

Local risk minimization, which has a history of more than twenty years, is a very famous hedging method for contingent claims in incomplete markets. Although its theoretical aspects have been very well studied, corresponding computational methods have yet to be thoroughly developed. This paper aims to illustrate how to numerically calculate LRM for call options in exponential Lévy models.

Our aims, in this paper, are two points: The first point is how to and how fast to compute local risk minimization (LRM) of call options for exponential Lévy models. Here, LRM is a popular hedging method through a quadratic criterion for contingent claims in incomplete markets. [Arai & Suzuki(2015.1)] have previously obtained a representation of LRM for call options; here we transform it into a form that allows use of the fast Fourier transform (FFT) method suggested by [Carr & Madan(1999)]. FFT is a very forceful algorithm to compute the Discrete Fourier Transform (DFT). Using FFT, we can reduce computational complexity $O(N^2)$ to $O(N \log N)$, where N is the size of data. Considering Merton jump-diffusion models and variance gamma models as typical examples of exponential Lévy models, we provide the forms for the FFT explicitly; and compute the values of LRM numerically for given parameter sets. We show that our FFT method can reduce computation time to calculate LRM dramatically. When Monte Carlo methods, in general, need hours or days to calculate, our FFT method needs only one–tenth seconds. Considering Merton jump-diffusion models and variance gamma models as typical examples of exponential Lévy models, we provide the forms for the FFT explicitly; and compute the values of LRM numerically for given parameter sets. Furthermore, we illustrate numerical results for a variance gamma model with estimated parameters from the Nikkei 225 index.

In response to this, the second is comparing delta hedging strategies and LRM strategies. We discuss the differences of LRM strategies and delta hedging strategies, in exponential Lévy models, where delta hedging strategies in this paper (Δ^*) are defined under the minimal martingale measures (MMM). We give inequality estimations for the differences of LRM and delta hedging strategies, and then show numerical examples for the two typical exponential Lévy models, Merton models and variance gamma models. Furthermore we show FFT can calculate Δ^* in a one–tenth seconds as an application of the first point.

In order to calculate LRM_t numerically, we have to calculate conditional expectations of functionals of S_T under $\mathbb{P}^*$. However, there does not appear to be any straightforward way to specify the probability density function of S_T (or equivalently L_T) under $\mathbb{P}^*$. Instead, since L is a Lévy process, it may be comparatively easy to specify its characteristic function under $\mathbb{P}^*$. Hence, a numerical method based on the Fourier transform is appropriate for computing LRM. Moreover, [Carr & Madan(1999)] introduced a numerical method for valuing options based on the FFT. We take advantage of this to develop a numerical method for LRM. In this paper, we consider two concrete exponential Lévy processes for L. The first is a jump-diffusion process as introduced by [Merton(1976)]. Note that he also suggested a hedging method for these models, but this is different from LRM. For additional details, see Section 10.1 of [Cont & Tankov(2004)]. This jump-diffusion process consists of a Brownian motion and compound Poisson jumps with normally distributed jump sizes. The second is a variance gamma process, which is a Lévy process with infinitely many jumps in any finite time interval and no Brownian component. This was introduced by [Madan & Seneta(1987)] and can be defined as a time-changed Brownian motion. Many papers (e.g., [Carr & Madan (1999)], [Madan et al. (1998)]) have studied it in the context of asset prices. [Schoutens(2003)] provides more details on these two Lévy processes and more examples of exponential Lévy models.

There is great deal of literature on numerical experiments related to LRM (e.g., [Bonetti et al. (2015)], [Ewald, Nawar & Siu (2013)], [Kang & Lee (2014)], [Lee & Song (2007)], [Leoni et al. (2014)] and [Yang et al. (2010)]), but to our knowledge, ours is the first attempt to develop an FFT-based numerical LRM scheme for exponential Lévy models.
[Kélani & Quittard-Pinon(2014)] studied an optimal hedging strategy that they call *θ-hedging*, which is similar to but different from LRM, for exponential Lévy models, and adopted a Fourier transform approach separate from [Carr & Madan(1999)]'s method. As an important difference, they assumed that S is a martingale under the underlying probability measure. In contrast, we do not make this assumption. We therefore need to treat S under $\mathbb{P}^*$, that is, calculate conditional expectations of functionals of S under $\mathbb{P}^*$. However, the structure of S is no longer preserved under a change of measure. For example, when L is a variance gamma process under $\mathbb{P}$, it is not so under $\mathbb{P}^*$. Thus, our setting is more challenging but also more natural.

Delta hedging strategies, which are also well-known and often used by practitioners, are given by differentiating the option price under a certain martingale measure with respect to the underlying asset price. Due to the relationship between LRM and the MMM, we consider delta hedging strategies under the MMM.

This paper is organized as follows: Chapter 2 gives a short introduction of a LRM and its representations.
In Chapter 3, we illustrate how to compute LRM of call options for exponential Lévy models. [Arai & Suzuki(2015.1)] have previously obtained a representation of LRM for call options; here we transform it into a form that allows use of the FFT method suggested by [Carr & Madan(1999)]. Consid-

ering Merton jump-diffusion models and variance gamma models as typical examples of exponential Lévy models, we provide the forms for the FFT explicitly; and compute the values of LRM numerically for given parameter sets. Furthermore, we illustrate numerical results for a variance gamma model with estimated parameters from the Nikkei 225 index.
In chapter 4, we discuss the differences of LRM and delta hedging strategies, in exponential Lévy models, where delta hedging strategies in this paper are defined under the MMM. First of all we give inequality estimations for the differences of LRM and delta hedging strategies, and then show numerical examples for the two typical exponential Lévy models, Merton models and variance gamma models.

Chapter 2

Local Risk-Minimization and Its Representations

We will give a short survey of LRM here. More precise definitions or examples are shown in [Schweizer(2001)], [Schweizer(2008)], [Arai & Suzuki(2015.1)], and [Arai & Suzuki(2015.0)]. We consider a financial market which is composed of one risk-free asset and one risky asset with maturity T. We may assume that the interest rate of the market is given by 0. To put it plainly, the price of the risk-free asset is 1 at all time. The fluctuation of the risky asset is assumed to be given by a solution to the following stochastic differential equation:

$$dS_t = S_{t-}\left[\alpha_t dt + \beta_t dW_t + \int_{\mathbb{R}_0} \gamma_{t,z}\widetilde{N}(dt, dz)\right], \quad S_0 > 0\,, \tag{2.1}$$

where α, β and γ are predictable processes. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be the canonical filtration completed for $\mathbb{P}$. W_t is 1-dim. standard Brownian motion, $N(dt, dz)$ is Poisson random measure, and $\widetilde{N}(dt, dz)$ is its composed random measure. In other words, using Lévy measure ν we can write $\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$. Moreover γ is a stochastic process measurable with respect to the σ-algebra generated by $A \times (s, u] \times B$, $A \in \mathcal{F}_s$, $0 \leq s < u \leq T$, $B \in \mathcal{B}(\mathbb{R}_0)$. Now, we specurate the following:

Assumption 2.0.1. *1. (2.1) has a solution S satisfying the so-called structure condition. More precisely, S is a special semimartingale with the canonical decomposition $S = S_0 + M + A$ such that*

(a)

$$\left\| [M]_T^{1/2} + \int_0^T |dA_t| \right\|_{L^2(\mathbb{P})} < \infty.$$

where $dM_t = S_{t-}(\beta_t dW_t + \int_{\mathbb{R}_0} \gamma_{t,z}\widetilde{N}(dt,dz))$ *and* $dA_t = S_{t-}\alpha_t dt$.

(b) *Defyning a process* $\lambda_t := \dfrac{\alpha_t}{S_{t-}(\beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz))}$, *we have* $A = \int \lambda d\langle M \rangle$.

(c) *The mean-variance trade-off process* $K_t := \int_0^t \lambda_s^2 d\langle M \rangle_s$ *is finite. Id est* K_T *is finite* $\mathbb{P}$*-a.s.*

These conditions (a)-(c) are called structure condition (see [Schweizer(2001)], [Schweizer(2008)]).

2. $\gamma_{t,z} > -1$, (t,z,ω)*-a.e. In other words,* $\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \mathbf{1}_{\{\gamma_{t,z} \leq -1\}} \nu(dz)dt\right] = 0$. *Remark that this condition guarantees* $S_t > 0$ *for arbitrary* $t \in [0,T]$.

We define LRM for a contingent claim $F \in L^2(\mathbb{P})$ based on Theorem 1.6 of [Schweizer(2008)].

Definition 2.0.2. 1. Θ_S *denotes the space of all* $\mathbb{R}$*-valued predictable process* ξ *satisfying* $\mathbb{E}\left[\int_0^T \xi_t^2 d\langle M \rangle_t + (\int_0^T |\xi_t dA_t|)^2\right] < \infty$.

2. *An* L^2*-strategy is given by a pair* $\varphi = (\xi, \eta)$, *where* $\xi \in \Theta_S$ *and* η *is an adopted process such that* $V(\varphi) := \xi S + \eta$ *is a right continuous process with* $\mathbb{E}[V_t^2(\varphi)] < \infty$ *for every* $t \in [0,T]$. *Note that* ξ_t *(resp.* η_t*) represents the amount of units of the risky asset (resp. risk-free asset) an investor holds at time* t.

3. *For* $F \in L^2(\mathbb{P})$, *the process* $C^F(\varphi)$ *defined by* $C_t^F(\varphi) := F1_{\{t=T\}} + V_t(\varphi) - \int_0^t \xi_s dS_s$ *is called the cost process of* $\varphi = (\xi, \eta)$ *for* F.

4. *An* L^2*-strategy* φ *is called locally risk-minimizing for* F *if* $V_T(\varphi) = 0$ *and* $C^F(\varphi)$ *is a martingale orthogonal to* M, *that is,* $[C^F(\varphi), M]$ *is a uniformly integrable martingale.*

Now we discuss a representation of LRM here. First of all, we recall Föllmer-Schweizer decomposition here.

Definition 2.0.3. *An $F \in L^2(\mathbb{P})$ admits Föllmer-Schweizer decomposition if it can be described*

$$F = F_0 + \int_0^T \xi_t^F dS_t + L_T^F \tag{2.2}$$

where $F_0 \in \mathbb{R}$, $\xi^F \in \Theta_S$ and L^F is a square-integrable martingale orthogonal to M with $L_0^F = 0$.

Proposition 5.2 of [Schweizer(2008)] shows the following:

Proposition 2.0.4. *(Proposition 5.2 of [Schweizer(2008)])*
Under Assumption 2.0.1, an LRM $\varphi = (\xi, \eta)$ for F exists if and only if F admits a Föllmer-Schweizer decomposition. Its relationship is given by

$$\xi_t = \xi_t^F, \quad \eta_t = F_0 + \int_0^t \xi_s^F dS_s + L_t^F - F1_{\{t=T\}} - \xi_t^F S_t.$$

As a result, it suffices to obtain a representation of ξ^F in 2.2 in order to obtain LRM. Throughout of this paper we identify ξ^F with LRM. We consider the process $Z := \mathcal{E}(-\int \lambda dM)$, where $\mathcal{E}(Y)$ represents the stochastic exponential of Y, that is, Z is a solution to the SDE $dZ_t = -\lambda_t Z_{t-} dM_t$. In addition to Assumption 2.0.1, we suppose the following:

Assumption 2.0.5. *Z is a positive square-integrable martingale; and $Z_T F \in L^2(\mathbb{P})$.*

A martingale measure $\mathbb{P}^* \sim \mathbb{P}$ is called 'minimal' if any square-integrable $\mathbb{P}$-martingale orthogonal to M remains a martingale under $\mathbb{P}^*$. We can see the following:

Lemma 2.0.6. *Under the Assumption 2.0.1, if Z is a positive square-integrable martingale, then a minimal martingale measure $\mathbb{P}^*$ exists with $d\mathbb{P}^* = Z_T d\mathbb{P}$.*

Example 2.0.7. *We provide a framework here. The postulates are that Assumption 2.0.1 is satisfied, and Z is a positive square integrable martingale. We consider the following three conditions:*

1.

$$\gamma_{t,z} > -1\,, \qquad (t, z, \omega)\text{-}a.e.$$

2. $\sup_{t\in[0,T]}(|\alpha_t| + \beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)) < C$. *for some* $C > 0$

3. *There exist an* $\varepsilon > 0$ *such that*

$$\frac{\alpha_t \gamma_{t,z}}{\beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)} < 1 - \varepsilon \quad \text{and} \quad \beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz) > \varepsilon, \quad (t, z, \omega)\text{-a.e.}$$

The above condition 2 ensures the existence of a unique solution S *to (2.1) satisfying* $\sup_{t\in[0,T]} |S_t| \in L^2(\mathbb{P})$ *by Theorem 117 of [Situ(2005)]. Hence an MMM exists by Lemma 2.0.6*

Next, we concentrate on representations of LRM ξ^F for contingent claim F. As a first step, we study the representation through the martingale representation theorem.

We assume Assumptions 2.0.1 and 2.0.5. Let $\mathbb{P}^*$ be a minimal martingale measure, that is, $d\mathbb{P}^* = Z_T d\mathbb{P}$ holds. The martingale representation theorem (see, e.g. Proposition 9.4 of [Cooley & Tukey (1965)]) provides

$$Z_T F = \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T g_t^0 dW_t + \int_0^T \int_{\mathbb{R}_0} g_{t,z}^1 \widetilde{N}(dt, dz)$$

for some predictable processes g_t^0 and $g_{t,z}^1$. From Ito's lemma, we have

$$\begin{aligned} F =& \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T \frac{g_t^0 + \mathbb{E}[Z_T F|\mathcal{F}_{t-}]u_t}{Z_{t-}} dW_t^{\mathbb{P}^*} \\ &+ \int_0^T \int_{\mathbb{R}_0} \frac{g_{t,z}^1 + \mathbb{E}[Z_T F|\mathcal{F}_{t-}]\theta_{t,z}}{Z_{t-}(1-\theta_{t,z})} \widetilde{N}^{\mathbb{P}^*}(dt, dz) \\ =:& \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T h_t^0 dW_t^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} h_{t,z}^1 \widetilde{N}^{\mathbb{P}^*}(dt, dz) \end{aligned}$$

where $u_t := \lambda_t S_{t-} \beta_t$, $\theta_{t,z} := \lambda_t S_{t-} \gamma_{t,z}$, $dW_t^{\mathbb{P}^*} := dW_t + u_t dt$, and $\widetilde{N}^{\mathbb{P}^*}(dt, dz) := \widetilde{N}(dt, dz) + \theta_{t,z}\nu(dz)dt$. Girsanov's theorem implies that the compensated Poisson random measure of N under $\mathbb{P}^*$ and $W^{\mathbb{P}^*}$ and $\widetilde{N}^{\mathbb{P}^*}$ are a Brownian motion, respectively. Addition to that, we assume that

$$\mathbb{E}\left[\int_0^T \left\{(h_t^0)^2 + \int_{\mathbb{R}_0} (h_{t,z}^1)^2 \nu(dz)\right\} dt\right] < \infty \,. \tag{2.3}$$

Denoting $i_t^0 := h_t^0 - \xi_t S_{t-}\beta_t$, $i_{t,z}^1 := h_{t,z}^1 - \xi_t S_{t-}\gamma_{t,z}$, and

$$\xi_t := \frac{\lambda_t}{\alpha_t}\{h_t^0\beta_t + \int_{\mathbb{R}_0} h_{t,z}^1\gamma_{t,z}\nu(dz)\}\,, \tag{2.4}$$

we can see $i_t^0\beta_t + \int_{\mathbb{R}_0} i_{t,z}^1\gamma_{t,z}\nu(dz) = 0$ for any $t \in [0,T]$. This implies $i_t^0 u_t + \int_{\mathbb{R}_0} i_{t,z}^1\theta_{t,z}\nu(dz) = 0$. We have then

$$\begin{aligned} F - \mathbb{E}_{\mathbb{P}^*}[F] - \int_0^T \xi_t dS_t &= \int_0^T i_t^0 dW_t^{\mathbb{P}^*} + \int_0^T\int_{\mathbb{R}_0} i_{t,z}^1 \widetilde{N}^{\mathbb{P}^*}(dt,dz) \\ &= \int_0^T i_t^0 dW_t + \int_0^T\int_{\mathbb{R}_0} i_{t,z}^1 \widetilde{N}(dt,dz)\,. \end{aligned}$$

The following lemma implies that $L_t^F := \mathbb{E}[F - \mathbb{E}_{\mathbb{P}^*}[F] - \int_0^T \xi_s dS_s|\mathcal{F}_t]$ is a square-integrable martingale orthogonal to M with $L_0^F = 0$.

Lemma 2.0.8. *Under Assumption 2.0.1, 2.0.5, and (2.3), we have*

$$\mathbb{E}\left[\int_0^T (i_t^0)^2 dt + \int_0^T\int_{\mathbb{R}_0} (i_{t,z}^1)^2\nu(dz)dt\right] < \infty\,.$$

Consequently, we can conclude the following:

Proposition 2.0.9. *Assume that Assumption 2.0.1, 2.0.5, and equation (2.3). We have then $\xi^F = \xi$ defined equation (2.4).*

In the above proposition, a representation of LRM ξ^F is obtained under a soft setting. The processes h^0 and h^1 appeared in equation (2.4) are induced by the martingale representation theorem so that it is almost impossible to calculate them explicitly, and confirm wheather equation (2.3) holds. In the paragraph, we introduce concrete expressions for h^0 and h^1 by use of Malliavin calculus.

In this part, we prepare some definitions and terminologies with respect to Malliavin calculus. We treat a Clark-Ocone type formula under change of measure (under $\mathbb{P}^*$) particurarly, see [Solé *et al.*(2007)] and [Delong & Imkeller (2010)].

We adopt the canonical Lévy space framework treated by [Solé *et al.*(2007)]. Remark that Malliavin calculus is discussed based on the underlying Lévy process X. We put $X_t := W_t + \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz)$ here. In the first place, we define measures q and Q on $[0, T] \times \mathbb{R}$ as

$$q(E) := \int_E \delta_0(dz)dt + \int_E z^2\nu(dz)dt\,,$$
$$Q(E) := \int_E \delta_0(dz)dW_t + \int_E z\widetilde{N}(dt, dz)$$

where δ_0 is the Dirac measure at 0 and $E \in \mathcal{B}([0, T] \times \mathbb{R})$. Deterministic functions $h : ([0, T] \times \mathbb{R})^n \to \mathbb{R}$ satisfy

$$\|h\|^2_{L^2_{T,q,n}} := \int_{([0,T]\times\mathbb{R})^n} |h((t_1, z_1), \cdots, (t_n, z_n))|^2 q(dt_1, dz_1) \cdots q(t_n, z_n) < \infty\,,$$

where we denote that $L^2_{T,q,n}$ is the set of product measurable. For $h \in L^2_{T,q,n}$ and $n \in \mathbb{N}$, we define

$$I_n(h) := \int_{([0,T]\times\mathbb{R})^n} h((t_1, z_1), \cdots, (t_n, z_n))Q(dt_1, dz_1) \cdots Q(dt_n, dz_n)\,.$$

To make it formal we denote $I_0(h) := h$ for $h \in \mathbb{R}$ and $L^2_{T,q,0} := \mathbb{R}$. Under this preparations, any $F \in L^2(\mathbb{P})$ has the unique representation $F = \sum_{n=0}^{\infty} I_n(h_n)$ with functions $h_n \in L^2_{T,q,n}$ that are symmetric in the n pairs $(t_i, z_i), 1 \leq i \leq n$, and we have $\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n!\|h_n\|^2_{L^2_{T,q,n}}$. This is called chaos expansion. Note that chaos expansion is unique expansion. Then we define Malliavin derivative.

Definition 2.0.10. *1. Set Sobolev space $\mathbb{D}^{1,2}$ as follows:*

$$\mathbb{D}^{1,2} := \left\{ F \in L^2(\mathbb{P}) \| F = \sum_{n=0}^{\infty} I_n(h_n)\,, \sum_{n=1}^{\infty} nn!\|h_n\|^2_{L^2_{T,q,n}} < \infty \right\}\,.$$

2. For any $F \in \mathbb{D}^{1,2}$, we define $DF : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ as

$$D_{t,z}F := \sum_{n=1}^{\infty} nI_{n-1}(h_n((t, z), \bullet))\,.$$

Then we call this DF as Malliavin derivative of F.

We can obtain Clark-Ocone type formula under MMM $\mathbb{P}^*$ with this Malliavin derivation and some additional assumptions. Here we omit the precise introduction of Clark-Ocone type formula. Under the above preparations, we obtain the representations of h^0 and h^1 as follows:

Proposition 2.0.11. *If Clark-Ocone type formula under MMM, Assumption 2.0.1, and 2.0.5 hold, h^0 and h^1 are described as*

$$h_t^0 = \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - F\left[\int_0^T D_{t,0}u_s dW_s^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}} \widetilde{N}^{\mathbb{P}^*}(ds,dx)\right] \Big| \mathcal{F}_{t-}\right], \tag{2.5}$$

$$h_{t,z}^1 = \mathbb{E}_{\mathbb{P}^*}[F(H_{t,z}^* - 1) + zH_{t,z}^* D_{t,z}F|\mathcal{F}_{t-}]\,. \tag{2.6}$$

Moreover, LRM ξ^F is given by substituting equations (2.5) and (2.6) for h^0 and h^1 in equation (2.4) respectively, if equation (2.3) holds.

In a very real sense, the condition 'if Clark-Ocone type formula under MMM holds' is most important. To check the condition whether this Clark-Ocone type formula holds or not is very complicated. Whereas SDE (2.1) are deterministic function, we need not to check this condition. We propose a framework which satisfies all the above Assumptions here.

Corollary 2.0.12. *We consider the case where α, β, and γ in SDE (2.1) are deterministic functions satisfying the three conditions 2.0.7. Additionally, we assume that*

1. $Z_T F \in L^2(\mathbb{P})$,
2. $F \in \mathbb{D}^{1,2}$,
3. $Z_T D_{t,z}F + FD_{t,z}Z_T + zD_{t,z}F \cdot D_{t,z}Z_T \in L^2(q \times \mathbb{P})$

Then all conditions in Proposition 2.0.11 are satisfied and LRM ξ^F is given by

$$\xi_t^F = \frac{\beta_t \mathbb{E}_{\mathbb{P}^*}[D_{t,0}F|\mathcal{F}_{t-}] + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[zD_{t,z}F|\mathcal{F}_{t-}]\gamma_{t,z}\nu(dz)}{S_{t-}\left(\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)\right)}\,.$$

Now, we discuss the representations of LRM for call options as the last part of this chapter. The pay-off of the call option is expressed $(S_T - K)^+$, where $K > 0$ is strike price, $T > 0$ is maturity, and $X^+ := \max(X, 0)$. We regard $(S_T - K)^+$ as a function of F which is continuous but not smooth. Because of that we can not use the chain rule, we use the mollifier approximation. As a preparation for a representation of LRM for call options, we show the following without proof.

Proposition 2.0.13. *For any $F \in \mathbb{D}^{1,2}$, $K \in \mathbb{R}$ and q-a.e. $(t, z) \in [0, T] \times \mathbb{R}$, we have $(F - K)^+ \in \mathbb{D}^{1,2}$ and*

$$D_{t,z}(F-K)^+ = \mathbf{1}_{\{F>K\}} D_{t,0}F \cdot \mathbf{1}_{\{0\}}(z) + \frac{(F + zD_{t,z}F - K)^+ - (F-K)^+}{z} \mathbf{1}_{\mathbb{R}_0}(z) \, .$$

Lemma 2.0.14. *For any $F \in \mathbb{D}^{1,2}$, we have $\mathbf{1}_{\{F=0\}} D_{t,0}F = 0$, (t, ω)-a.e.*

We consider the case where α, β, and γ in 2.0.7 and assume the next condition:

$$\int_{\mathbb{R}_0} \{\gamma_{t,z}^4 + |\log(1+\gamma_{t,z})|^2\} \nu(dz) < C \text{ for some } C > 0. \tag{2.7}$$

When this condition 2.7 and there conditions on Example 2.0.7 are satisfied, then all conditions on Corollary 2.0.12 are automatically satisfied. By using the preparation, which is the above proposition and lemma, we obtain an explicit representation of LRM for call options.

Proposition 2.0.15. *For any $K > 0$ and $t \in [0, T]$, we have*

$$\begin{aligned}\xi_t^{(S_T-K)^+} = & \frac{1}{S_{t-}\left(\beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)\right)} \Bigg\{ \beta_t^2 \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T > K\}} S_T | \mathcal{F}_{t-}] \\ & + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[(S_T(1+\gamma_{t,z}) - K)^+ - (S_T - K)^+ | \mathcal{F}_{t-}] \gamma_{t,z} \nu(dz) \Bigg\} .\end{aligned}$$

Chapter 3

Numerical local risk minimization for exponential Lévy models

3.1 Preliminaries

We introduced a general representation of LRM for call options by using Malliavin calculus for Lévy processes based on the canonical Lévy space on Chapter 1. One of our main purpose is to transform that result into a form that allows the fast Fourier transform method suggested by [Carr & Madan(1999)] to be applied. In particular, Merton jump-diffusion and variance gamma models, being common classes of exponential Lévy models, are discussed as concrete applications of our approach.

The fluctuation of the risky asset (e.g. liquidity, transaction costs, portfolio constraints, non-continuous trading, and so on) is assumed to be described by an exponential Lévy process S on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, described by

$$S_t := S_0 \exp\left\{\mu t + \sigma W_t + \int_{\mathbb{R}_0} x\widetilde{N}([0,t], dx)\right\} \qquad \text{for } t \in [0,T],$$

where $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, and $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Here W is a one-dimensional Brownian motion and $\widetilde{N}$ is the compensated version of a Poisson random measure N. Denoting the Lévy measure of N by ν, we have $\widetilde{N}([0,t], A) = N([0,t], A) - t\nu(A)$ for any $t \in [0,T]$ and $A \in \mathcal{B}(\mathbb{R}_0)$. Now, $(\Omega, \mathcal{F}, \mathbb{P})$ is taken as the product of a one-dimensional Wiener space and the canonical Lévy space for N. In addition, we take $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ as the completed canonical filtration for $\mathbb{P}$. For more details on the canonical Lévy space, see [Solé *et al.*(2007)] and [Arai & Suzuki(2015.1)]. Moreover, S is also a solution to the stochastic differential equation

$$dS_t = S_{t-}\left[\mu^S\,dt + \sigma\,dW_t + \int_{\mathbb{R}_0}(e^x - 1)\widetilde{N}(dt,dx)\right],$$

where

$$\mu^S := \mu + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}_0}(e^x - 1 - x)\nu(dx).$$

Without loss of generality, we may assume that $S_0 = 1$ for simplicity. Now, defining $L_t := \log S_t$ for all $t \in [0,T]$, we obtain a Lévy process L. Moreover,

$$dM_t := S_{t-}\big[\sigma\,dW_t + \int_{\mathbb{R}_0}(e^x - 1)\widetilde{N}(dt,dx)\big]$$

is the martingale part of S.

Our focus is the development of a computational method for LRM with respect to a call option $(S_T - K)^+$ with strike price $K > 0$. We do not review the definition of LRM in this paper; for details, see [Schweizer(2001)] and [Schweizer(2008)].We first briefly introduce the explicit LRM representation of such options in exponential Lévy models given in [Arai & Suzuki(2015.1)].

Define the minimal martingale measure (MMM) $\mathbb{P}^*$ as an equivalent martingale measure under which any square-integrable $\mathbb{P}$-martingale orthogonal to M remains a martingale. Its density is then given by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left\{-\xi W_T - \frac{\xi^2}{2}T + \int_{\mathbb{R}_0}\log(1-\theta_x)N([0,T],dx) + T\int_{\mathbb{R}_0}\theta_x\nu(dx)\right\},$$

where

$$\xi := \frac{\mu^S\sigma}{\sigma^2 + \int_{\mathbb{R}_0}(e^y-1)^2\nu(dy)} \quad \text{and} \quad \theta_x := \frac{\mu^S(e^x-1)}{\sigma^2 + \int_{\mathbb{R}_0}(e^y-1)^2\nu(dy)}$$

for $x \in \mathbb{R}_0$. In the development of our approach, we rely on the following:

Assumption 3.1.1. *1.* $\int_{\mathbb{R}_0}(|x| \vee x^2)\nu(dx) < \infty$, *and* $\int_{\mathbb{R}_0}(e^x - 1)^n\nu(dx) < \infty$ *for* $n = 4$.

2. $0 \geq \mu^S > -\sigma^2 - \int_{\mathbb{R}_0}(e^x-1)^2\nu(dx)$.

The first condition ensures that μ^S, ξ, and θ_x are well defined, the square integrability of L, and the finiteness of $\int_{\mathbb{R}_0}(e^x-1)^n\nu(dx)$ for $n=1,3$. The second guarantees that $\theta_x < 1$ for any $x \in \mathbb{R}_0$. Moreover, by the Girsanov theorem,

$$W_t^{\mathbb{P}^*} := W_t + \xi t$$

and

$$\widetilde{N}^{\mathbb{P}^*}([0,t],dx) := \theta_x\nu(dx)t + \widetilde{N}([0,t],dx)$$

are a $\mathbb{P}^*$-Brownian motion and the compensated Poisson random measure of N under $\mathbb{P}^*$, respectively. We can then rewrite L_t as

$$L_t = \mu^* t + \sigma W_t^{\mathbb{P}^*} + \int_{\mathbb{R}_0} x\widetilde{N}^{\mathbb{P}^*}([0,t],dx),$$

where

$$\mu^* := -\frac{1}{2}\sigma^2 + \int_{\mathbb{R}_0}(x-e^x+1)(1-\theta_x)\nu(dx).$$

Note that L is a Lévy process even under $\mathbb{P}^*$, with Lévy measure given by

$$\nu^{\mathbb{P}^*}(dx) := (1-\theta_x)\nu(dx).$$

The LRM will be given as a predictable process LRM_t, which represents the number of units of the risky asset the investor holds at time t. First, we define

$$I_1 := \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T>K\}}S_T \mid \mathcal{F}_{t-}]\,, \tag{3.1}$$

$$I_2 := \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[(S_Te^x-K)^+ - (S_T-K)^+ \mid \mathcal{F}_{t-}](e^x-1)\nu(dx)\,. \tag{3.2}$$

Our explicit representation of LRM for call option $(S_T-K)^+$ is then as follows:

Proposition 3.1.2 (Arai & Suzuki (2015)). *For any $K>0$ and $t\in[0,T]$,*

$$LRM_t = \frac{\sigma^2 I_1 + I_2}{S_{t-}\big(\sigma^2 + \int_{\mathbb{R}_0}(e^x-1)^2\nu(dx)\big)}. \tag{3.3}$$

Remark 3.1.3. *1. The assumption* $\int_{\mathbb{R}_0}(e^x-1)^4\nu(dx)<\infty$ *is imposed in Proposition 4.6 of [Arai & Suzuki(2015.1)].*

2. If the interest rate of our market is instead $r>0$*, then equation (3.3) becomes*

$$LRM_t = e^{-r(T-t)}\frac{\sigma^2 I_1 + I_2}{S_{t-}\left(\sigma^2+\int_{\mathbb{R}_0}(e^x-1)^2\nu(dx)\right)},$$

and $\mathbb{P}^*$ *is rewritten with* ξ *and* θ_x *becoming*

$$\frac{(\mu^S-r)\sigma}{\sigma^2+\int_{\mathbb{R}_0}(e^y-1)^2\nu(dy)} \quad \text{and} \quad \frac{(\mu^S-r)(e^x-1)}{\sigma^2+\int_{\mathbb{R}_0}(e^y-1)^2\nu(dy)},$$

respectively. Moreover, the second condition in Assumption 3.1.1 would be revised to

$$0\ge \mu^S - r > -\sigma^2 - \int_{\mathbb{R}_0}(e^x-1)^2\nu(dx).$$

That is, a nonzero r *requires only that we replace* μ *with* $\mu-r$ *and multiply the the expression for* LRM_t *by* $e^{-r(T-t)}$*, which means that we can easily generalize results for the* $r=0$ *case to those for* $r>0$*. For simplicity, in this paper we treat only the case* $r=0$*.*

From the point of view of Proposition 3.1.2, we have to calculate conditional expectations of functionals of S_T under $\mathbb{P}^*$ in order to calculate LRM_t numerically. However, there does not appear to be any straightforward way to specify the probability density function of S_T (or equivalently L_T) under $\mathbb{P}^*$. Instead, since L is a Lévy process, it may be comparatively easy to specify its characteristic function under $\mathbb{P}^*$. Hence, a numerical method based on the Fourier transform is appropriate for computing LRM. Moreover, [Carr & Madan(1999)] introduced a numerical method for valuing options based on the fast Fourier transform (FFT). We take advantage of this to develop a numerical method for LRM. To this end, we induce integral expressions for I_1 and I_2 in terms of the characteristic function of L_{T-t} under $\mathbb{P}^*$ and recast them into a form that allows the Carr–Madan approach to be applied. In particular, I_2 will be given as a linear combination of Fourier transforms.

The rest of this section is organized as follows: An introductory review of the Carr–Madan approach is given in Subsection 2.1.1, and the integral representations of I_1 and I_2 are presented in Subsection 2.1.2. Merton jump-diffusion models are examined in Section 2.2, which starts with mathematical preliminaries and proceeds to numerical results. Section 2.3 is similarly devoted to variance gamma models.

3.1.1 Numerical method

We briefly review the Carr–Madan approach, which is an FFT-based numerical approach for option pricing. The FFT, introduced by [Cooley & Tukey (1965)], is a numerical method for computing a discrete Fourier transform given by

$$F(l) := \sum_{j=0}^{N-1} e^{-i(2\pi/N)jl} x_j \tag{3.4}$$

for $l = 0, \dots, N-1$, where $\{x_j\}_{j=0,\dots,N-1}$ is a sequence on $\mathbb{R}$ and where N is typically a power of 2. The FFT requires only $O(N \log_2 N)$ arithmetic operations, as compared with the usual Fourier transform method's $O(N^2)$.

The aim of the Carr–Madan approach is efficient calculation of $\mathbb{E}[(S_T - K)^+]$ when S is a $\mathbb{P}$-martingale. Recall that we are considering only the case in which the interest rate is zero. Denoting $k := \log K$ and $C(k) := \mathbb{E}[(S_T - e^k)^+]$, we have

$$C(k) = \frac{1}{\pi} \int_0^\infty e^{-i(v-i\alpha)k} \frac{\phi(v - i\alpha - i)}{i(v - i\alpha)[i(v - i\alpha) + 1]} \, dv \tag{3.5}$$

for $\alpha > 0$ with $\mathbb{E}[S_T^{\alpha+1}] < \infty$, where ϕ is the characteristic function of L_T. Note that the right-hand side of equation (3.5) is independent of the choice of α. Now, we denote

$$\psi(z) := \frac{\phi(z - i)}{iz(iz + 1)}$$

for $z \in \mathbb{C}$. Using the trapezoidal rule, we can therefore approximate $C(k)$ as

$$C(k) \approx \frac{1}{\pi} \sum_{j=0}^{N-1} e^{-i(\eta j - i\alpha)k} \psi(\eta j - i\alpha)\eta \,, \tag{3.6}$$

where N represents the number of grid points and $\eta > 0$ is the distance between adjacent grid points. The right-hand side of equation (3.6) corresponds to the integral in equation (3.5) over the interval $[0, N\eta]$, so we need to specify N and η such that

$$\left| \frac{1}{\pi} \int_{N\eta}^{\infty} e^{-i(v-i\alpha)k} \psi(v - i\alpha) dv \right| < \varepsilon \tag{3.7}$$

for a sufficiently small value $\varepsilon > 0$, which represents the allowable error. By incorporating Simpson's rule weightings, we may rewrite equation (3.6) as

$$C(k) \approx \frac{1}{\pi} \sum_{j=0}^{N-1} e^{-i(\eta j - i\alpha)k} \psi(\eta j - i\alpha) \frac{\eta}{3} (3 + (-1)^{j+1} - \delta_j),$$

where δ_j is the Kronecker delta function. We define

$$F(l) := \frac{e^{-\alpha k}}{\pi} \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N} jl} e^{i\pi j} \psi(\eta j - i\alpha) \frac{\eta}{3} (3 + (-1)^{j+1} - \delta_j)$$

for $l = 0, \ldots, N-1$, which is a discrete Fourier transform as given in equation (3.4). This yields

$$C(k) \approx F\left(\left(k + \frac{\pi}{\eta} \right) \frac{N\eta}{2\pi} \right).$$

So long as we take η so that $|k| < \pi/\eta$, we can employ the FFT to compute $C(k)$.

3.1.2 Integral representations

We next induce integral expressions for I_1 and I_2, defined in equations (3.1) and (3.2), and evolve them so that the Carr–Madan approach is available. Recall that Assumption 3.1.1 applies throughout. As can be seen from Subsection 2.1, if I_1 and I_2 are represented in the same form as equation (3.5) we can compute them by means of the Carr–Madan approach. Because the conditional expectations appearing in I_1 and I_2 are under $\mathbb{P}^*$, the functions corresponding to ψ in equation (3.5) should include the characteristic function of L_{T-t} under $\mathbb{P}^*$, denoted by

$$\phi_{T-t}(z) := \mathbb{E}_{\mathbb{P}^*}[e^{izL_{T-t}}]$$

for $z \in \mathbb{C}$.

First, we induce an integral representation for

$$I_1 (= \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T>K\}} S_T \mid \mathcal{F}_{t-}])$$

with ϕ_{T-t} by using Proposition 2 from [Tankov(2010)]:

Proposition 3.1.4. *For $K > 0$,*

$$\mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T>K\}} \cdot S_T \mid \mathcal{F}_{t-}] = \frac{1}{\pi} \int_0^\infty \frac{K^{-iv-\alpha+1}}{\alpha - 1 + iv} \phi_{T-t}(v - i\alpha) S_{t-}^{\alpha+iv} \, dv \tag{3.8}$$

for all $t \in [0, T]$ and $\alpha \in (1, 2]$. Note that the right-hand side is independent of the choice of α.

Proof. Define $G(x) := \mathbf{1}_{\{x>K\}} \cdot x$, $g(x) := G(e^x)$ for any $x \in \mathbb{R}$, and $\hat{g}(z) := \int_{\mathbb{R}} e^{izx} g(x) dx$ for any $z \in \mathbb{C}$. We employ one lemma:

Lemma 3.1.5. *Let L' be an independent copy of L. Then,*

$$L'_{T-t} + L_{t-} \overset{\mathbb{P}^*\text{-}d}{=} L_T$$

for all $t \in [0, T]$, where

$$A \overset{\mathbb{P}^*\text{-}d}{=} B$$

means that $A = B$ in law for $\mathbb{P}^*$.

Proof of Lemma 3.1.5. Proposition I.7 of [Bertoin(1998)] implies that $\mathbb{P}^*(L_{t-} = L_t) = 1$. Therefore,

$$L_t \overset{\mathbb{P}^*\text{-}d}{=} L_{t-}.$$

Because Lévy processes have independent and stationary increments, we have

$$L_T = L_T - L_t + L_t \overset{\mathbb{P}^*\text{-}d}{=} L'_{T-t} + L_t.$$

□

Returning to the proof of Proposition 3.1.4, from Lemma 3.1.5 we have

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T>K\}} \cdot S_T \mid \mathcal{F}_{t-}] &= \mathbb{E}_{\mathbb{P}^*}[G(S_T) \mid \mathcal{F}_{t-}] = \mathbb{E}_{\mathbb{P}^*}[g(L'_{T-t} + L_{t-}) \mid \mathcal{F}_{t-}] \\ &= \int_{\mathbb{R}} g(x + L_{t-}) p(dx),\end{aligned}$$

where $p(A) := \mathbb{P}^*(L'_{T-t} \in A)$ for any $A \in \mathcal{B}(\mathbb{R})$. By (22)–(25) in the proof of Proposition 2 of [Tankov(2010)], if any $\alpha \in (1, 2]$ satisfies the conditions that

(a) $g(x)e^{-\alpha x}$ has finite variation on $\mathbb{R}$,

(b) $g(x)e^{-\alpha x} \in L^1(\mathbb{R})$,

(c) $\mathbb{E}_{\mathbb{P}^*}[e^{\alpha L_{T-t}}] < \infty$, and

(d) $\displaystyle\int_{\mathbb{R}} \frac{|\phi_{T-t}(v - i\alpha)|}{1 + |v|} dv < \infty$,

then

$$\int_{\mathbb{R}} g(x + L_{t-})p(dx) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(v + i\alpha)\phi_{T-t}(-v - i\alpha)S_{t-}^{\alpha - iv}\, dv$$

for $\alpha \in (1, 2]$, which is independent of the choice of α. As a result, under conditions (a)–(d), we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T > K\}} \cdot S_T \mid \mathcal{F}_{t-}] &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(v + i\alpha)\phi_{T-t}(-v - i\alpha)S_{t-}^{\alpha - iv}\, dv \\
&= \frac{1}{\pi} \int_0^\infty \hat{g}(-v + i\alpha)\phi_{T-t}(v - i\alpha)S_{t-}^{\alpha + iv}\, dv \\
&= \frac{1}{\pi} \int_0^\infty \frac{K^{-iv-\alpha+1}}{\alpha - 1 + iv}\phi_{T-t}(v - i\alpha)S_{t-}^{\alpha + iv}\, dv.
\end{aligned}$$

We need only to confirm that conditions (a)–(d) hold. Conditions (a) and (b) are obvious. To demonstrate condition (c), it suffices to show $S_{T-t} \in L^2(\mathbb{P}^*)$ for any $t \in [0, T]$. Note that we have

$$\begin{aligned}
&\int_{\mathbb{R}_0} (e^x - 1)^2 \nu^{\mathbb{P}^*}(dx) \\
&\quad = \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx) + \frac{|\mu^S|}{\sigma^2 + \int_{\mathbb{R}_0}(e^x - 1)^2\nu(dx)} \int_{\mathbb{R}_0} (e^x - 1)^3 \nu(dx) < \infty\,.
\end{aligned}$$

Because S is a solution to

$$dS_t = S_{t-}(\sigma\, dW_t^{\mathbb{P}^*} + \int_{\mathbb{R}} (e^x - 1)\widetilde{N}^{\mathbb{P}^*}(dt, dx)),$$

Theorem 117 of [Situ(2005)] implies that

$$\sup_{t\in[0,T]} |S_t| \in L^2(\mathbb{P}^*).$$

Next, we show condition (d). Note that

$$\begin{aligned}
&\phi_{T-t}(v-i\alpha) \\
&\quad = \mathbb{E}_{\mathbb{P}^*}\left[\exp\left\{(iv+\alpha)\left[\mu^*(T-t)+\sigma W^{\mathbb{P}^*}_{T-t}+\int_{\mathbb{R}_0} x\widetilde{N}^{\mathbb{P}^*}([0,T-t],dx)\right]\right\}\right].
\end{aligned} \tag{3.9}$$

For the right-hand side, we have

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbb{P}^*}\left[\exp\left\{(iv+\alpha)\int_{\mathbb{R}_0} x\widetilde{N}^{\mathbb{P}^*}([0,T-t],dx)\right\}\right]\right| \\
&\qquad \le \mathbb{E}_{\mathbb{P}^*}\left[\exp\left\{\alpha\int_{\mathbb{R}_0} x\widetilde{N}^{\mathbb{P}^*}([0,T-t],dx)\right\}\right] < \infty\,,
\end{aligned}$$

because

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^*}\left[e^{\alpha L_{T-t}}\right] &= \mathbb{E}_{\mathbb{P}^*}\left[\exp\left\{\alpha\left[\mu^*(T-t)+\sigma W^{\mathbb{P}^*}_{T-t}+\int_{\mathbb{R}_0} x\widetilde{N}^{\mathbb{P}^*}([0,T-t],dx)\right]\right\}\right] \\
&= e^{\mu^*(T-t)}\mathbb{E}_{\mathbb{P}^*}\left[e^{\alpha\sigma W^{\mathbb{P}^*}_{T-t}}\right]\mathbb{E}_{\mathbb{P}^*}\left[e^{\alpha\int_{\mathbb{R}_0} x\widetilde{N}^{\mathbb{P}^*}([0,T-t],dx)}\right], \\
\mathbb{E}_{\mathbb{P}^*}\left[e^{\alpha\sigma W^{\mathbb{P}^*}_{T-t}}\right] &= \exp\left\{\frac{1}{2}\alpha^2\sigma^2(T-t)\right\},
\end{aligned}$$

and

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{\alpha L_{T-t}}\right] < \infty.$$

In addition, we obtain

$$\left|\mathbb{E}_{\mathbb{P}^*}[\exp\{(iv+\alpha)\sigma W^{\mathbb{P}^*}_{T-t}\}]\right| = \exp\left\{\frac{(\alpha^2-v^2)\sigma^2(T-t)}{2}\right\}. \tag{3.10}$$

As a result, we have from equations (3.9)–(3.10)

$$\int_{\mathbb{R}} \frac{|\phi_{T-t}(v-i\alpha)|}{1+|v|}dv < C\int_{\mathbb{R}} \frac{1}{1+|v|}\exp\left\{-\frac{\sigma^2(T-t)}{2}v^2\right\}dv < \infty$$

for some $C>0$. This completes the proof of Proposition 3.1.4. □

We evolve (3.8) into the same form as (3.5) as follows:

$$I_1 = \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T>K\}} \cdot S_T \mid \mathcal{F}_{t-}] = \frac{1}{\pi}\int_0^\infty \frac{K^{-iv-\alpha+1}}{\alpha-1+iv}\phi_{T-t}(v-i\alpha)S_{t-}^{\alpha+iv}dv$$
$$= \frac{e^k}{\pi}\int_0^\infty e^{-i(v-i\alpha)k}\psi_1(v-i\alpha)dv \qquad (3.11)$$

where $k := \log K$ and

$$\psi_1(z) := \frac{\phi_{T-t}(z)S_{t-}^{iz}}{iz-1}$$

for $z \in \mathbb{C}$. Thus, we can compute I_1 with the FFT based on Subsection 2.1.

We turn next to

$$I_2\left(= \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[(S_T e^x - K)^+ - (S_T-K)^+ \mid \mathcal{F}_{t-}](e^x-1)\nu(dx)\right).$$

First, we have the following integral representation:

Proposition 3.1.6. *For any $K > 0$,*

$$\mathbb{E}_{\mathbb{P}^*}[(S_T-K)^+ \mid \mathcal{F}_{t-}] = \frac{1}{\pi}\int_0^\infty K^{-iv-\alpha+1}\frac{\phi_{T-t}(v-i\alpha)S_{t-}^{\alpha+iv}}{(\alpha-1+iv)(\alpha+iv)}dv \qquad (3.12)$$

for any $t \in [0,T]$ and any $\alpha \in (1,2]$. Note that the right-hand side is independent of the choice of α.

Proof. We can see this in the same manner as Proposition 3.1.4 but with $G(x) = (x-K)^+$. □

Note that (3.12) coincides with (3.5), where $\alpha - 1$ in (3.12) corresponds to α in (3.5). Denoting

$$\psi_2(z) := \frac{\phi_{T-t}(z)S_{t-}^{iz}}{(iz-1)iz}$$

for $z \in \mathbb{C}$ and $\zeta := v - i\alpha$, we have

$$\mathbb{E}_{\mathbb{P}^*}[(S_T-K)^+ \mid \mathcal{F}_{t-}] = \frac{1}{\pi}\int_0^\infty K^{-iv-\alpha+1}\frac{\phi_{T-t}(v-i\alpha)S_{t-}^{\alpha+iv}}{(\alpha-1+iv)(\alpha+iv)}dv$$

$$= \frac{1}{\pi}\int_0^\infty K^{-i\zeta+1}\frac{\phi_{T-t}(\zeta)S_{t-}^{i\zeta}}{(i\zeta-1)i\zeta}dv$$
$$= \frac{1}{\pi}\int_0^\infty K^{-i\zeta+1}\psi_2(\zeta)dv =: f(K). \tag{3.13}$$

Note that $f(K)$ is computed with the FFT. Moreover, Fubini's theorem implies

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[(S_T e^x - K)^+ - (S_T - K)^+ \mid \mathcal{F}_{t-}](e^x-1)\nu(dx) \\
&= \int_{\mathbb{R}_0} \left\{ e^x f(e^{-x}K) - f(K) \right\} (e^x-1)\nu(dx) \\
&= \int_{\mathbb{R}_0} \left\{ \frac{e^x}{\pi}\int_0^\infty (Ke^{-x})^{-i\zeta+1}\psi_2(\zeta)dv - \frac{1}{\pi}\int_0^\infty K^{-i\zeta+1}\psi_2(\zeta)dv \right\} (e^x-1)\nu(dx) \\
&= \int_{\mathbb{R}_0} \left\{ \frac{1}{\pi}\int_0^\infty (e^{i\zeta x}-1)K^{-i\zeta+1}\psi_2(\zeta)dv \right\} (e^x-1)\nu(dx) \\
&= \frac{1}{\pi}\int_0^\infty K^{-i\zeta+1}\int_{\mathbb{R}_0}(e^{i\zeta x}-1)(e^x-1)\nu(dx)\psi_2(\zeta)dv\,,
\end{aligned} \tag{3.14}$$

which is the same form as (3.5), because the integrand of (3.14) is a function of ζ. However, we cannot compute (3.14) numerically as it stands, because it is not possible to compute the integral $\int_{\mathbb{R}_0}(e^{i\zeta x}-1)(e^x-1)\nu(dx)$ directly. Thus, we need to make further model-dependent calculations. In Sections 3 and 4, respectively, we evolve (3.14) into a linear combination of Fourier transforms for Merton jump-diffusion models and variance gamma models.

Remark 3.1.7. *Regarding LRM_t, I_1, and I_2 as functions of S_{t-} and K, we have $I_i(S_{t-},K)/S_{t-} = I_i(1, K/S_{t-})$ for $i = 1, 2$ by (3.8) and (3.14), and*

$$LRM_t(S_{t-},K) = \frac{\sigma^2 I_1(S_{t-},K) + I_2(S_{t-},K)}{S_{t-}\big(\sigma^2 + \int_{\mathbb{R}_0}(e^x-1)^2\nu(dx)\big)} = \frac{\sigma^2 I_1(1,K/S_{t-}) + I_2(1,K/S_{t-})}{\sigma^2 + \int_{\mathbb{R}_0}(e^x-1)^2\nu(dx)}$$

by (3.3). As a result, LRM_t is given as a function of $K/S_{t-} =: m_{t-}$, where m_{t-} is called moneyness. Thus, we denote LRM_t by $LRM_t(m_{t-})$. As a by-product of this, we can analyze jump impacts on LRM. If the process L has a jump with size $y \in \mathbb{R}_0$ at time t, then the moneyness m_{t-} changes into $m_{t-}e^{-y}$ at the moment when the jump occurs. Thus, LRM also changes from $LRM_t(m_{t-})$ to $LRM_t(m_{t-}e^{-y})$. We can regard the difference $LRM_t(m_{t-}e^{-y}) - LRM_t(m_{t-})$ as a jump impact. In particular, $LRM_t(e^{-y}) - LRM_t(1)$ represents a jump impact when the option is at the money.

Remark 3.1.8. *Hereafter, we fix* $\alpha \in (1, 2]$ *arbitrarily. Moreover, we denote* $\zeta := v - i\alpha$ *for* $v \in \mathbb{R}$*, so we may regard* ζ *as a function of* v*.*

3.2 The Merton Jump-Diffusion Model

We consider the case in which L is given as a Merton jump-diffusion process, which consists of a diffusion component with volatility $\sigma > 0$ and compound Poisson jumps with three parameters, $m \in \mathbb{R}$, $\delta > 0$, and $\gamma > 0$. Note that γ represents the jump intensity and that the sizes of the jumps are distributed normally with mean m and variance δ^2. Thus, its Lévy measure ν is given by

$$\nu(dx) = \frac{\gamma}{\sqrt{2\pi}\delta} \exp\left\{-\frac{(x-m)^2}{2\delta^2}\right\} dx.$$

When it desirable to emphasize the parameters, we write ν as $\nu[\gamma, m, \delta]$. Note that the first condition of Assumption 3.1.1 is satisfied for any $m \in \mathbb{R}$, $\delta > 0$, and $\gamma > 0$. In addition, the second condition is equivalent to

$$0 \geq \mu + \frac{\sigma^2}{2} + \gamma \left\{ \exp\left(m + \frac{\delta^2}{2}\right) - 1 - m \right\}$$

and

$$\mu + \frac{3\sigma^2}{2} + \gamma \left\{ \exp(2m + 2\delta^2) - \exp\left(m + \frac{\delta^2}{2}\right) - m \right\} > 0.$$

We consider only the case in which the parameters satisfy Assumption 3.1.1.

3.2.1 Mathematical preliminaries

Our aim here is threefold: (1) to give an analytic form for

$$\phi_{T-t}(z)(:= \mathbb{E}_{\mathbb{P}^*}[e^{izL_{T-t}}]);$$

(2) to evolve (3.14) into a linear combination of three Fourier transforms; and (3) to give sufficient conditions for $N\eta$ under which (3.7) holds for a given $\varepsilon > 0$.

First, we provide an analytic form of ϕ_{T-t}. To this end, we begin by calculating $\nu^{\mathbb{P}^*}$.

Proposition 3.2.1. *We have*

$$\nu^{\mathbb{P}^*}(dx) = \nu[(1+h)\gamma, m, \delta^2](dx) + \nu\left[-h\gamma\exp\left\{\frac{2m+\delta^2}{2}\right\}, m+\delta^2, \delta^2\right](dx), \tag{3.15}$$

where

$$h := \frac{\mu^S}{\sigma^2 + \int_{\mathbb{R}_0}(e^x-1)^2\nu(dx)}.$$

Proof. By Assumption 3.1.1, $0 \geq h > -1$. Hence,

$$\nu^{\mathbb{P}^*}(dx) = (1-\theta_x)\nu(dx) = (1-h(e^x-1))\nu(dx) = (1+h)\nu(dx) - he^x\nu(dx).$$

Moreover,

$$\begin{aligned} e^x\nu(dx) &= \frac{\gamma}{\sqrt{2\pi}\delta}\exp\left\{x - \frac{(x-m)^2}{2\delta^2}\right\}dx \\ &= \frac{\gamma}{\sqrt{2\pi}\delta}\exp\left\{-\frac{[x-(m+\delta^2)]^2}{2\delta^2} + \frac{2m+\delta^2}{2}\right\}dx \\ &= \nu\left[\gamma\exp\left\{\frac{2m+\delta^2}{2}\right\}, m+\delta^2, \delta^2\right](dx), \end{aligned}$$

from which (3.15) follows. □

Next, we calculate $\phi_{T-t}(\zeta)$ for $t \in [0,T]$.

Proposition 3.2.2. *For any $t \in [0,T]$ and $v \in \mathbb{R}$, with $\zeta := v - i\alpha$,*

$$\begin{aligned} \phi_{T-t}(\zeta) &= \exp\left\{(T-t)\left[i\zeta\mu^* - \frac{\sigma^2\zeta^2}{2} + \int_{\mathbb{R}_0}(e^{i\zeta x} - 1 - i\zeta x)\nu^{\mathbb{P}^*}(dx)\right]\right\} \\ &= \exp\Bigg\{(T-t)\Bigg[i\zeta\mu^* - \frac{\sigma^2\zeta^2}{2} + (1+h)\gamma(e^{im\zeta - \frac{\zeta^2\delta^2}{2}} - 1 - im\zeta) \\ &\qquad - h\gamma e^{\frac{2m+\delta^2}{2}}[e^{i(m+\delta^2)\zeta - \frac{\zeta^2\delta^2}{2}} - 1 - i\zeta(m+\delta^2)]\Bigg]\Bigg\}. \end{aligned}$$

Proof. We only have to show the first equality:

$$\phi_{T-t}(\zeta) = \mathbb{E}_{\mathbb{P}^*}\left[\exp\left\{i\zeta\left[\mu^*(T-t) + \sigma W^{\mathbb{P}^*}_{T-t} + \int_{\mathbb{R}_0} x\widetilde{N}^{\mathbb{P}^*}([0,T-t],dx)\right]\right\}\right]$$

$$= \exp\{(T-t)i\zeta\mu^*\}\,\mathbb{E}_{\mathbb{P}^*}[e^{i\zeta\sigma W^{\mathbb{P}^*}_{T-t}}]\mathbb{E}_{\mathbb{P}^*}\left[\exp\left\{i\zeta\int_{\mathbb{R}_0} x\widetilde{N}^{\mathbb{P}^*}([0,T-t],dx)\right\}\right]$$
$$= \exp\left\{(T-t)\left[i\zeta\mu^* - \frac{\sigma^2\zeta^2}{2} + \int_{\mathbb{R}_0}(e^{i\zeta x}-1-i\zeta x)\nu^{\mathbb{P}^*}(dx)\right]\right\}.$$

□

Second, we evolve (3.14). We define

$$\tilde{\psi}(z) := \psi_2(z)\exp\left\{-\frac{1}{2}\delta^2 z^2\right\}$$

for $z \in \mathbb{C}$ and

$$\tilde{f}(K) := \frac{1}{\pi}\int_0^\infty K^{-i\zeta+1}\tilde{\psi}(\zeta)dv.$$

Remark that $\tilde{f}$ is computed with the FFT as well as f defined in (3.13). The following proposition demonstrates (3.14), namely, I_2 is given by a linear combination of three Fourier transforms.

Proposition 3.2.3. *We have*

$$\int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{P}^*}[(S_T e^x - K)^+ - (S_T-K)^+ \mid \mathcal{F}_{t-}](e^x-1)\nu(dx)$$
$$= \gamma e^{2m+\frac{3}{2}\delta^2}\tilde{f}(Ke^{-m-\delta^2}) - \gamma e^m \tilde{f}(Ke^{-m}) + \gamma(1-e^{m+\frac{\delta^2}{2}})f(K) \quad (3.16)$$

for any $t \in [0,T]$.

Proof. We calculate

$$\int_{\mathbb{R}_0}(e^{i\zeta x}-1)(e^x-1)\nu(dx)$$
$$= \int_{\mathbb{R}_0}(e^{(i\zeta+1)x} - e^{i\zeta x} + 1 - e^x)\nu(dx)$$
$$= \gamma\exp\left\{(i\zeta+1)m + \frac{\delta^2}{2}(i\zeta+1)^2\right\} - \gamma\exp\left\{i\zeta m - \frac{\delta^2}{2}\zeta^2\right\} + \gamma(1-e^{m+\frac{\delta^2}{2}}).$$

Hence, we obtain

$$\begin{aligned}(3.14) &= \frac{\gamma}{\pi} e^{m+\frac{\delta^2}{2}} \int_0^\infty e^{i\zeta(m+\delta^2)} K^{-i\zeta+1} e^{-\frac{\delta^2}{2}\zeta^2} \psi_2(\zeta) dv \\ &\quad - \frac{\gamma}{\pi} \int_0^\infty (Ke^{-m})^{-i\zeta+1} e^m e^{-\frac{\delta^2}{2}\zeta^2} \psi_2(\zeta) dv + \gamma(1 - e^{m+\frac{\delta^2}{2}}) f(K) \\ &= \gamma e^{2m+\frac{3}{2}\delta^2} \tilde{f}(Ke^{-m-\delta^2}) - \gamma e^m \tilde{f}(Ke^{-m}) + \gamma(1 - e^{m+\frac{\delta^2}{2}}) f(K).\end{aligned}$$

□

Third, we provide sufficient conditions for the product $N\eta$ under which (3.7) holds for a given allowable error $\varepsilon > 0$. First of all, we determine an upper estimate for ϕ_{T-t}.

Proposition 3.2.4. *We have*

$$|\phi_{T-t}(v - i\alpha)| \leq C_1 \exp\left\{-\frac{\sigma^2 v^2 (T-t)}{2}\right\}$$

for any $v \in \mathbb{R}$, *where*

$$\begin{aligned}C_1 &= \exp\left\{(T-t)\left[\alpha\mu^* + \frac{\sigma^2\alpha^2}{2} + \int_{\mathbb{R}_0} (e^{\alpha x} - 1 - \alpha x)\nu^{\mathbb{P}^*}(dx)\right]\right\} \\ &= \exp\left\{(T-t)\left[\alpha\mu^* + \frac{\sigma^2\alpha^2}{2} + (1+h)\gamma(e^{m\alpha + \frac{\alpha^2\delta^2}{2}} - 1 - \alpha m)\right.\right. \\ &\qquad \left.\left. - h\gamma e^{\frac{2m+\delta^2}{2}}\left[e^{(m+\delta^2)\alpha + \frac{\alpha^2\delta^2}{2}} - 1 - \alpha(m+\delta^2)\right]\right]\right\}.\end{aligned}$$

Proof. Proposition 3.2.2 implies that

$$\begin{aligned}&\phi_{T-t}(v - i\alpha) \\ &= \exp\left\{(T-t)\left[i(v-i\alpha)\mu^* - \frac{\sigma^2(v-i\alpha)^2}{2} + \int_{\mathbb{R}_0} (e^{i(v-i\alpha)x} - 1 - i(v-i\alpha)x)\nu^{\mathbb{P}^*}(dx)\right]\right\} \\ &= \exp\left\{(T-t)\left[(iv+\alpha)\mu^* - \frac{\sigma^2(v^2 - 2i\alpha v - \alpha^2)}{2} + \int_{\mathbb{R}_0} (e^{(iv+\alpha)x} - 1 - (iv+\alpha)x)\nu^{\mathbb{P}^*}(dx)\right]\right\} \\ &= \exp\left\{(T-t)iv\left[\mu^* + \sigma^2\alpha - \int_{\mathbb{R}_0} x\nu^{\mathbb{P}^*}(dx)\right]\right\} \exp\left\{(T-t)\int_{\mathbb{R}_0} e^{(iv+\alpha)x}\nu^{\mathbb{P}^*}(dx)\right\}\end{aligned}$$

$$\times \exp\left\{(T-t)\left[\alpha\mu^* - \frac{\sigma^2(v^2-\alpha^2)}{2} + \int_{\mathbb{R}_0}(-1-\alpha x)\nu^{\mathbb{P}^*}(dx)\right]\right\}.$$

Noting that

$$\left|\exp\left\{(T-t)\int_{\mathbb{R}_0} e^{(iv+\alpha)x}\nu^{\mathbb{P}^*}(dx)\right\}\right| \le \exp\left\{(T-t)\int_{\mathbb{R}_0} e^{\alpha x}\nu^{\mathbb{P}^*}(dx)\right\},$$

we have

$$|\phi_{T-t}(v-i\alpha)| \le \exp\left\{(T-t)\left[\alpha\mu^* - \frac{\sigma^2(v^2-\alpha^2)}{2} + \int_{\mathbb{R}_0}(e^{\alpha x}-1-\alpha x)\nu^{\mathbb{P}^*}(dx)\right]\right\}.$$

□

Propositions 3.2.5 and 3.2.6 below give sufficient conditions for $N\eta$ under which I_1 and I_2 satisfy (3.7) for a given allowable error $\varepsilon > 0$, respectively.

Proposition 3.2.5. *Let $\varepsilon > 0$ and $t \in [0, T)$. When $a > 0$ satisfies*

$$\left(\frac{K}{\pi}\left(\frac{K}{S_{t-}}\right)^{-\alpha} C_1\right)^{1/4} \frac{1}{\sigma\sqrt{T-t}\varepsilon^{1/4}} \le a, \tag{3.17}$$

we have

$$\left|\frac{1}{\pi}\int_a^\infty \frac{K^{-iv-\alpha+1}}{\alpha-1+iv}\phi_{T-t}(v-i\alpha)S_{t-}^{\alpha+iv}dv\right| \le \varepsilon.$$

Proof. Noting that $e^{-x} \le x^{-2}$ for any $x > 0$, we have, by Proposition 3.2.4,

$$\begin{aligned}
&\left|\frac{1}{\pi}\int_a^\infty \frac{K^{-iv-\alpha+1}}{\alpha-1+iv}\phi_{T-t}(v-i\alpha)S_{t-}^{\alpha+iv}dv\right| \\
&\quad \le \frac{1}{\pi}\int_a^\infty \frac{K^{-\alpha+1}}{|\alpha-1+iv|}|\phi_{T-t}(v-i\alpha)|S_{t-}^{\alpha}dv \\
&\quad \le \frac{K}{\pi}\left(\frac{K}{S_{t-}}\right)^{-\alpha}\int_a^\infty \frac{1}{|\alpha-1+iv|}C_1 e^{-\frac{\sigma^2v^2}{2}(T-t)}dv \\
&\quad \le \frac{K}{\pi}\left(\frac{K}{S_{t-}}\right)^{-\alpha} C_1\int_a^\infty \frac{1}{v}\left\{\frac{\sigma^2v^2}{2}(T-t)\right\}^{-2}dv \\
&\quad = \frac{K}{\pi}\left(\frac{K}{S_{t-}}\right)^{-\alpha} C_1\int_a^\infty \frac{4v^{-5}}{\sigma^4(T-t)^2}dv
\end{aligned}$$

$$= \frac{K}{\pi}\left(\frac{K}{S_{t-}}\right)^{-\alpha}\frac{C_1}{\sigma^4(T-t)^2a^4}$$
$$\leq \varepsilon\,.$$

□

Proposition 3.2.6. *Let $\varepsilon > 0$ and $t \in [0, T)$. If $a > 0$ satisfies*

$$\frac{4C_1\gamma K}{5\pi\sigma^4(T-t)^2\varepsilon}\left(\frac{K}{S_{t-}}\right)^{-\alpha}\left\{e^{(\alpha+1)m+(\frac{\alpha^2}{2}+\alpha+\frac{1}{2})\delta^2}+e^{m\alpha+\frac{\delta^2\alpha^2}{2}}+|1-e^{m+\frac{\delta^2}{2}}|\right\}\leq a^5, \tag{3.18}$$

then

$$\left|\frac{1}{\pi}\int_a^\infty K^{-i\zeta+1}\int_{\mathbb{R}_0}(e^{i\zeta x}-1)(e^x-1)\nu(dx)\psi_2(\zeta)dv\right| < \varepsilon\,. \tag{3.19}$$

Proof. First, we estimate $\int_a^\infty |\psi_2(\zeta)|dv$. Noting that

$$\left|\frac{1}{(i\zeta-1)i\zeta}\right| = \left|\frac{1}{(iv+\alpha-1)(iv+\alpha)}\right| \leq \frac{1}{v^2}\,,$$

Proposition 3.2.4 implies

$$\begin{aligned}\int_a^\infty |\psi_2(\zeta)|dv &= \int_a^\infty\left|\frac{\phi_{T-t}(v-i\alpha)S_{t-}^{i(v-i\alpha)}}{(i\zeta-1)i\zeta}\right|dv\\ &\leq C_1S_{t-}^\alpha\int_a^\infty\frac{e^{-\frac{\sigma^2v^2}{2}(T-t)}}{v^2}\,dv\\ &\leq \frac{4C_1S_{t-}^\alpha}{\sigma^4(T-t)^2}\int_a^\infty v^{-6}\,dv\\ &= \frac{4C_1S_{t-}^\alpha}{5\sigma^4(T-t)^2a^5}\,.\end{aligned}$$

Hence, Proposition 3.2.3 implies that

L.H.S. of (3.19)

$$= \left|\frac{\gamma e^{2m+\frac{3}{2}\delta^2}}{\pi}\int_a^\infty(Ke^{-m-\delta^2})^{-i\zeta+1}\tilde{\psi}(\zeta)dv - \frac{\gamma e^m}{\pi}\int_a^\infty(Ke^{-m})^{-i\zeta+1}\tilde{\psi}(\zeta)dv\right.$$

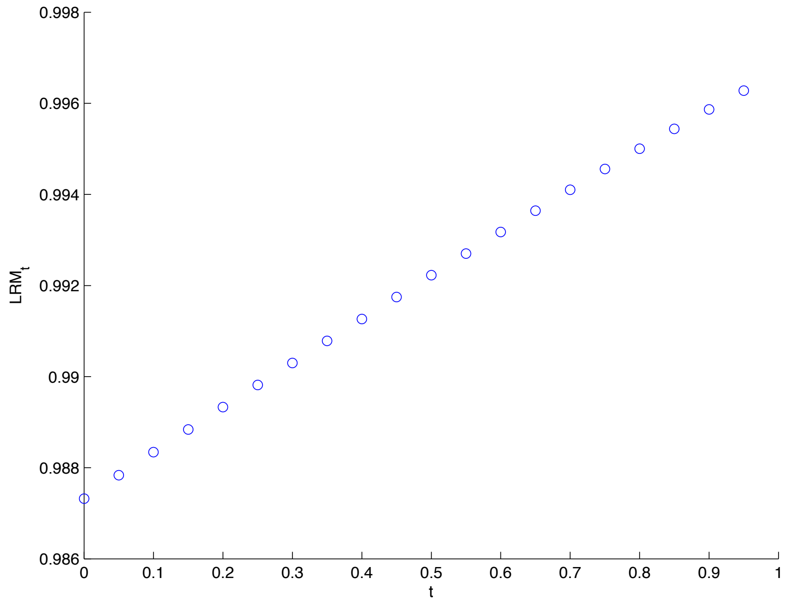
$$
\begin{aligned}
&\quad + \frac{\gamma(1-e^{m+\frac{\delta^2}{2}})}{\pi}\int_a^\infty K^{-i\zeta+1}\psi_2(\zeta)dv\Bigg| \\
&\le \frac{\gamma}{\pi}\int_a^\infty \left|e^{2m+\frac{3}{2}\delta^2}(Ke^{-m-\delta^2})^{-iv-\alpha+1} - e^m(Ke^{-m})^{-iv-\alpha+1}\right| |\psi_2(\zeta)| \left|e^{-\frac{\delta^2\zeta^2}{2}}\right| dv \\
&\quad + \frac{\gamma|1-e^{m+\frac{\delta^2}{2}}|}{\pi}\int_a^\infty |K^{-iv-\alpha+1}|\,|\psi_2(\zeta)|dv \\
&\le \frac{\gamma}{\pi}\int_a^\infty \left\{e^{2m+\frac{3}{2}\delta^2}(Ke^{-m-\delta^2})^{-\alpha+1} + e^m(Ke^{-m})^{-\alpha+1}\right\} |\psi_2(\zeta)| e^{-\frac{\delta^2(v^2-\alpha^2)}{2}} dv \\
&\quad + \frac{\gamma|1-e^{m+\frac{\delta^2}{2}}|}{\pi}\int_a^\infty K^{-\alpha+1}|\psi_2(\zeta)|dv \\
&\le \frac{\gamma K^{-\alpha+1}}{\pi}\left\{e^{(\alpha+1)m+(\frac{\alpha^2}{2}+\alpha+\frac{1}{2})\delta^2} + e^{m\alpha+\frac{\delta^2\alpha^2}{2}} + |1-e^{m+\frac{\delta^2}{2}}|\right\}\int_a^\infty |\psi_2(\zeta)|dv \\
&\le \frac{4C_1\gamma K}{5\pi\sigma^4(T-t)^2a^5}\left(\frac{K}{S_{t-}}\right)^{-\alpha}\left\{e^{(\alpha+1)m+(\frac{\alpha^2}{2}+\alpha+\frac{1}{2})\delta^2} + e^{m\alpha+\frac{\delta^2\alpha^2}{2}} + |1-e^{m+\frac{\delta^2}{2}}|\right\} \\
&\le \varepsilon\,.
\end{aligned}
$$

□

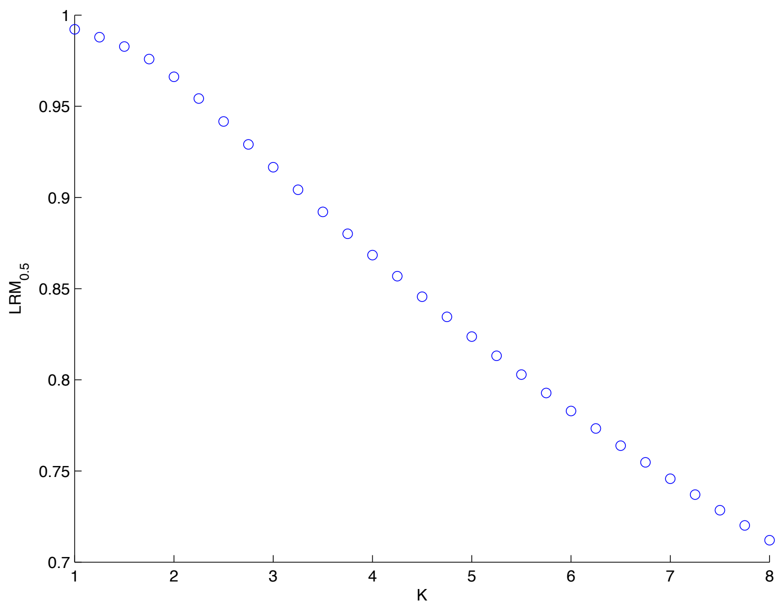
3.2.2 Numerical results

As seen in the previous subsection, substituting (3.11) and (3.16) for I_1 and I_2 respectively, we can compute LRM_t given in (3.3) with the FFT. Note that we need Proposition 3.2.2 in order to calculate ψ_1, ψ_2, and $\widetilde{\psi}$. In this subsection, we provide numerical results for a Merton jump-diffusion model with parameters $T = 1$, $\mu = -0.7$, $\sigma = 0.2$, $\gamma = 1$, $m = 0$, and $\delta = 1$. Note that μ^S is given by -0.03, which satisfies the second condition of Assumption 3.1.1. In particular, we consider the following two cases: First, fixing the strike price K to 1, we compute LRM_t for times $t = 0, 0.05, \dots, 0.95$. Second, t is fixed to 0.5 and we instead vary K from 1 to 8 at steps of 0.25 and compute $LRM_{0.5}$. Note that we take $L_{t-} = 1$ whatever the value of t is taken. Moreover, we choose $N = 2^{14}$, $\eta = 0.025$, and $\alpha = 1.75$ as parameters related to the FFT. We have then $N\eta = 409.6$. For any parameter set mentioned above, both (3.17) and (3.18) are satisfied for $\epsilon = 10^{-2}$. Figure 3.1 shows the results for these two cases. The computation time to obtain

Fig. 3.1(b) was 0.59 s. Note that all numerical experiments in this paper were carried out using MATLAB (8.1.0.604 R2013a) on an Intel Core i7 3.4 GHz CPU with 16 GB 1333 MHz DDR3 memory.



(a) Values of LRM_t of a call option with strike price $K = 1$ and maturity $T = 1$ vs. times $t = 0, 0.05, \ldots, 0.95$ for a Merton jump-diffusion model with parameters $\mu = -0.7$, $\sigma = 0.2$, $\gamma = 1$, $m = 0$, and $\delta = 1$. These parameters satisfy the second condition of Assumption 3.1.1. Moreover, the FFT parameters are chosen as $N = 2^{14}$, $\eta = 0.025$, and $\alpha = 1.75$.



(b) Values of $LRM_{0.5}$ of call options at a fixed time 0.5 vs. strike price K from 1 to 8 at steps of 0.25 for the same Merton jump-diffusion model as (a) with $S_{0.5} = 1$.

Figure 3.1: Merton jump-diffusion model

3.3 The Variance Gamma Model

We now consider the case in which L is given as a variance gamma process. Note that L does not have a diffusion component. This means that $\sigma = 0$, that is, I_1 vanishes. A variance gamma process, which has three parameters $\kappa > 0$, $m \in \mathbb{R}$, and $\delta > 0$, is defined as a time-changed Brownian motion with volatility δ, drift m, and subordinator G_t, where G_t is a gamma process with parameters $(1/\kappa, 1/\kappa)$. In summary, L is represented as

$$L_t = mG_t + \delta B_{G_t} \quad \text{for } t \in [0, T],$$

where B is a one-dimensional standard Brownian motion. Moreover, the Lévy measure of L is given by

$$\nu(dx) = C(\mathbf{1}_{\{x<0\}} e^{-G|x|} + \mathbf{1}_{\{x>0\}} e^{-M|x|}) \frac{dx}{|x|} = C(\mathbf{1}_{\{x<0\}} e^{Gx} + \mathbf{1}_{\{x>0\}} e^{-Mx}) \frac{dx}{|x|},$$

where

$$C := \frac{1}{\kappa}, \qquad G := \frac{1}{\delta^2}\sqrt{m^2 + \frac{2\delta^2}{\kappa}} + \frac{m}{\delta^2}, \qquad M := \frac{1}{\delta^2}\sqrt{m^2 + \frac{2\delta^2}{\kappa}} - \frac{m}{\delta^2}.$$

Note that C, G, and M are positive. To emphasize the parameters, we write ν with parameters κ, m, and δ as $\nu(dx) = \nu[\kappa, m, \delta](dx)$. Moreover, by regarding C, G, and M as parameters, we may express ν as $\nu(dx) = \nu_{C,G,M}(dx)$. In addition, we assume $M > 4$ in this section, which ensures that the first condition of Assumption 3.1.1 holds, by the following lemma:

Lemma 3.3.1. *When $M > 4$, $\int_{\mathbb{R}_0} (e^x - 1)^n \nu(dx) < \infty$ for $n = 2, 4$.*

Proof. For $n = 2, 4$, we have

$$\begin{aligned}
\int_1^\infty (e^x - 1)^n \nu(dx) &\le C \int_1^\infty e^{(n-M)x} dx < \infty, \\
\int_0^1 (e^x - 1)^n \nu(dx) &\le \int_0^1 x^n (e - 1)^n \nu(dx) \le C(e-1)^n < \infty, \\
\int_{-1}^0 (e^x - 1)^n \nu(dx) &\le \int_{-1}^0 (-x)^n \nu(dx) \le C \int_{-1}^0 (-x)^{n-1} dx < \infty, \\
\int_{-\infty}^{-1} (e^x - 1)^n \nu(dx) &\le \int_{-\infty}^{-1} \nu(dx) \le C \int_1^\infty e^{-Gx} dx < \infty,
\end{aligned}$$

because $n - M < 0$, $0 \le e^x - 1 \le x(e-1)$ whenever $x \in [0, 1]$, $1 + x \le e^x$ for any $x \in \mathbb{R}$, and $e^x \le 1$ if $x \le 0$. ∎

Remark 3.3.2. *We can generalize this lemma to* $\int_{\mathbb{R}_0} |e^x - 1|^a \nu(dx) < \infty$ *for any* $a \in [1, M)$.

Because $\mu = \int_{\mathbb{R}_0} x\nu(dx)$, (3.21) below implies that the second condition of Assumption 3.1.1 can be rewritten as

$$\log\left(\frac{(M-1)(G+1)}{(M-2)(G+2)}\right) > 0 \geq \log\left(\frac{MG}{(M-1)(G+1)}\right),$$

which is equivalent to $-3 < G - M \leq -1$.

3.3.1 Mathematical preliminaries

The approach to variance gamma models is similar to that in Subsection 3.1. We begin by calculating of $\nu^{\mathbb{P}^*}$.

Proposition 3.3.3.

$$\nu^{\mathbb{P}^*}(dx) = \nu_{(1+h)C,G,M}(dx) + \nu_{-hC,G+1,M-1}(dx),$$

where

$$h = \frac{\mu^S}{\int_{\mathbb{R}_0}(e^x - 1)^2 \nu(dx)}.$$

Proof. By the same argument as Proposition 3.2.1,

$$\nu^{\mathbb{P}^*}(dx) = (1+h)\nu(dx) - he^x\nu(dx).$$

We have $\lambda\nu_{C,G,M}(dx) = \nu_{\lambda C,G,M}(dx)$ for any $\lambda > 0$, and

$$\begin{aligned} e^x\nu_{C,G,M}(dx) &= e^x C(\mathbf{1}_{\{x<0\}}e^{Gx} + \mathbf{1}_{\{x>0\}}e^{-Mx})\frac{dx}{|x|} \\ &= C(\mathbf{1}_{\{x<0\}}e^{(G+1)x} + \mathbf{1}_{\{x>0\}}e^{-(M-1)x})\frac{dx}{|x|} \\ &= \nu_{C,G+1,M-1}(dx) \end{aligned}$$

because $M - 1 > 0$. □

Remark 3.3.4. *For any* $\lambda > 0$, $\lambda\nu[\kappa, m, \delta](dx)$ *is a Lévy measure corresponding to the variance gamma process with parameters* κ/λ, λm, *and* $\delta\sqrt{\lambda}$. *However,* $\nu_{C,G+1,M-1}(dx)$ *is not necessarily a Lévy measure corresponding to a variance gamma process.*

Next we calculate the characteristic function ϕ_{T-t} of L under $\mathbb{P}^*$:

Proposition 3.3.5. *For any $t \in [0,T]$ and $v \in \mathbb{R}$, with $\zeta := v - i\alpha$, we have*

$$\phi_{T-t}(\zeta) = \left[\left(1+\frac{i\zeta}{G}\right)\left(1-\frac{i\zeta}{M}\right)\right]^{-(1+h)(T-t)C}\left[\left(1+\frac{i\zeta}{G+1}\right)\left(1-\frac{i\zeta}{M-1}\right)\right]^{h(T-t)C}$$
$$\times \exp\left\{(T-t)i\zeta\left[\mu^* + (1+h)C\frac{M-G}{GM} - hC\frac{M-G-2}{(G+1)(M-1)}\right]\right\},$$

where

$$\mu^* = \int_{\mathbb{R}_0}(x - e^x + 1)\nu^{\mathbb{P}^*}(dx).$$

Proof. First of all, we have

$$\begin{aligned}
\int_0^\infty (e^{i\zeta x}-1)\frac{e^{-Mx}}{x}\,dx &= \int_0^\infty \frac{e^{-(M-\alpha-iv)x} - e^{-Mx}}{x}\,dx \\
&= \int_0^\infty \frac{e^{-(M-\alpha-iv)x} - e^{-(M-\alpha)x} + e^{-(M-\alpha)x} - e^{-Mx}}{x}\,dx \\
&= i\int_0^\infty e^{-(M-\alpha)x}\int_0^v e^{itx}\,dt\,dx + \int_0^\infty\int_{M-\alpha}^M e^{-tx}\,dt\,dx \\
&= i\int_0^v\int_0^\infty e^{-(M-\alpha-it)x}\,dx\,dt + \int_{M-\alpha}^M\int_0^\infty e^{-tx}\,dx\,dt \\
&= \log\left(\frac{M-\alpha}{M-\alpha-iv}\right) + \log\left(\frac{M}{M-\alpha}\right) \\
&= -\log\left(1-\frac{i\zeta}{M}\right), \qquad (3.20)
\end{aligned}$$

which provides

$$\begin{aligned}
\int_{\mathbb{R}_0}(e^{i\zeta x}-1)\nu_{C,G,M}(dx) &= C\int_{-\infty}^0 (e^{i\zeta x}-1)\frac{e^{Gx}}{-x}\,dx + C\int_0^\infty (e^{i\zeta x}-1)\frac{e^{-Mx}}{x}\,dx \\
&= -C\left(\log\left(1+\frac{i\zeta}{G}\right) + \log\left(1-\frac{i\zeta}{M}\right)\right).
\end{aligned}$$

In addition, we have

$$\int_{\mathbb{R}_0} x\nu_{C,G,M}(dx) = -C\int_{-\infty}^0 e^{Gx}\,dx + C\int_0^\infty e^{-Mx}\,dx = -C\frac{M-G}{GM}.$$

Together with Proposition 3.3.3, we obtain

$$\begin{aligned}\int_{\mathbb{R}_0}(e^{i\zeta x}-1-i\zeta x)\nu^{\mathbb{P}^*}(dx) &= \log\left(1+\frac{i\zeta}{G}\right)^{-(1+h)C}+\log\left(1-\frac{i\zeta}{M}\right)^{-(1+h)C}\\ &\quad+\log\left(1+\frac{i\zeta}{G+1}\right)^{hC}+\log\left(1-\frac{i\zeta}{M-1}\right)^{hC}\\ &\quad+i(1+h)C\zeta\frac{M-G}{GM}-ihC\zeta\frac{M-G-2}{(G+1)(M-1)},\end{aligned}$$

from which Proposition 3.3.5 follows. □

Now, we reformulate (3.14) into a linear combination of two Fourier transforms in order to allow use of the FFT. As preparation, we show the following:

Lemma 3.3.6.

$$\int_{\mathbb{R}_0} e^{i\zeta x}(e^x-1)\nu(dx) = C\log\left(\frac{M-i\zeta}{M-1-i\zeta}\frac{G+i\zeta}{G+1+i\zeta}\right). \tag{3.21}$$

Proof. First of all, we have

$$\begin{aligned}&\int_{\mathbb{R}_0} e^{i\zeta x}(e^x-1)\nu(dx)\\ &\qquad=\int_{\mathbb{R}_0} e^{(iv+\alpha)x}(e^x-1)\nu(dx)\\ &\qquad=C\left\{\int_0^\infty\frac{1-e^x}{x}e^{-(G+\alpha+1+iv)x}\,dx+\int_0^\infty\frac{e^x-1}{x}e^{-(M-\alpha-iv)x}\,dx\right\}.\end{aligned} \tag{3.22}$$

To calculate (3.22), we compute

$$\int_0^\infty\frac{e^x-1}{x}e^{-ax}\cos bx\,dx \qquad \text{and} \qquad \int_0^\infty\frac{e^x-1}{x}e^{-ax}\sin bx\,dx$$

for $a>1$ and $b\in\mathbb{R}$. First, we have

$$\int_0^\infty\frac{e^x-1}{x}e^{-ax}\cos bx\,dx$$

$$
\begin{aligned}
&= \int_0^\infty \frac{\cos bx}{x} \int_{a-1}^a xe^{-tx}\, dt\, dx \\
&= \int_{a-1}^a \int_0^\infty \cos bx \cdot e^{-tx}\, dx\, dt \\
&= \int_{a-1}^a \frac{t}{t^2+b^2} dt \\
&= \frac{1}{2} \log\left(\frac{a^2+b^2}{(a-1)^2+b^2}\right). \qquad (3.23)
\end{aligned}
$$

A similar calculation implies that

$$
\begin{aligned}
&\int_0^\infty \frac{e^x-1}{x} e^{-ax} \sin bx\, dx \\
&= \int_{a-1}^a \frac{b}{t^2+b^2}\, dt \\
&= \tan^{-1}\frac{a}{b} - \tan^{-1}\frac{a-1}{b}. \qquad (3.24)
\end{aligned}
$$

Noting that $M-\alpha > 2$ and

$$
\tan^{-1} x = \frac{i}{2} \log \frac{i+x}{i-x}
$$

for $x \in \mathbb{R}$, we have, by (3.23) and (3.24),

$$
\begin{aligned}
&\int_0^\infty \frac{e^x-1}{x} e^{-(M-\alpha-iv)x}\, dx \\
&\quad = \int_0^\infty \frac{e^x-1}{x} e^{-(M-\alpha)x} \cos vx\, dx + i \int_0^\infty \frac{e^x-1}{x} e^{-(M-\alpha)x} \sin vx\, dx \\
&\quad = \frac{1}{2} \log\left(\frac{(M-\alpha)^2+v^2}{(M-\alpha-1)^2+v^2}\right) + i\left(\tan^{-1}\frac{M-\alpha}{v} - \tan^{-1}\frac{M-\alpha-1}{v}\right) \\
&\quad = \log\left(\frac{M-\alpha-iv}{M-\alpha-1-iv}\right). \qquad (3.25)
\end{aligned}
$$

Calculating the first term of the right-hand side of (3.22) in the same way as the above, we obtain

$$
\int_0^\infty \frac{1-e^x}{x} e^{-(G+\alpha+1+iv)x}\, dx = \log\left(\frac{G+\alpha+iv}{G+\alpha+1+iv}\right). \qquad (3.26)
$$

Substituting (3.25) and (3.26) for (3.22), we arrive at (3.21). □

From the above lemma, I_2 is given as follows:

$$
\begin{aligned}
&\int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[(S_T e^x - K)^+ - (S_T - K)^+ \mid \mathcal{F}_{t-}](e^x - 1)\nu(dx) \\
&\quad = \frac{1}{\pi}\int_0^\infty K^{-i\zeta+1}\int_{\mathbb{R}_0}(e^{i\zeta x} - 1)(e^x - 1)\nu(dx)\psi_2(\zeta)dv \\
&\quad = \frac{1}{\pi}\int_0^\infty K^{-i\zeta+1}\widetilde{\psi}_{VG}(\zeta)dv - \frac{1}{\pi}\int_0^\infty C\log\left(\frac{MG}{(M-1)(G+1)}\right)K^{-i\zeta+1}\psi_2(\zeta)dv.
\end{aligned}
\tag{3.27}
$$

where

$$
\widetilde{\psi}_{VG}(\zeta) := C\log\left(\frac{M - i\zeta}{M - 1 - i\zeta}\frac{G + i\zeta}{G + 1 + i\zeta}\right)\psi_2(\zeta).
$$

Recall that

$$
\psi_2(\zeta) = \frac{\phi_{T-t}(\zeta)S_{t-}^{i\zeta}}{(i\zeta - 1)i\zeta}.
$$

As a result, we need only use the FFT twice for computing I_2.

As the final item of this subsection, we estimate a sufficient length for the integration interval of (3.27) for a given allowable error $\varepsilon > 0$ in the sense of (3.7). We first provide an upper estimate of ϕ_{T-t} as follows:

Proposition 3.3.7. *For any $v \in \mathbb{R}$,*

$$
|\phi_{T-t}(v - i\alpha)| \le C_2|v|^{-2C(T-t)},
$$

where

$$
\begin{aligned}
C_2 = {}& (GM)^{(1+h)(T-t)C}[(G+1)(M-1)]^{-h(T-t)C} \\
&\times \exp\left\{(T-t)\alpha\left[\mu^* + (1+h)C\frac{M-G}{GM} - hC\frac{M-G-2}{(G+1)(M-1)}\right]\right\}.
\end{aligned}
\tag{3.28}
$$

Proof. This can be seen because

$$
\left|\left(1 + \frac{iv + \alpha}{G}\right)^{-a}\right| \le \frac{G^a}{|v|^a}
$$

for any $a > 0$. □

We need to prepare one more lemma:

Lemma 3.3.8.

$$\left|\int_{\mathbb{R}_0} e^{i\zeta x}(e^x-1)\nu(dx)\right| \le C\left\{\frac{1}{G+\alpha}+\frac{1}{M-\alpha-1}\right\}. \tag{3.29}$$

Proof. The same sort of calculations as in (3.20) imply

$$\begin{aligned}
&\left|\int_{\mathbb{R}_0} e^{i\zeta x}(e^x-1)\nu(dx)\right| \\
&\quad\le C\left\{\left|\int_0^\infty \frac{1-e^x}{x}e^{-(G+\alpha+1+iv)x}\,dx\right| + \left|\int_0^\infty \frac{e^x-1}{x}e^{-(M-\alpha-iv)x}\,dx\right|\right\} \\
&\quad\le C\left\{\int_0^\infty \frac{e^x-1}{x}e^{-(G+\alpha+1)x}\,dx + \int_0^\infty \frac{e^x-1}{x}e^{-(M-\alpha)x}\,dx\right\} \\
&\quad= C\left\{\log\left(1+\frac{1}{G+\alpha}\right)+\log\left(1+\frac{1}{M-\alpha-1}\right)\right\} \\
&\quad\le C\left\{\frac{1}{G+\alpha}+\frac{1}{M-\alpha-1}\right\}.
\end{aligned}$$

□

When we calculate (3.27), N and η should be taken so that $N\eta$ satisfies (3.30) below for a given allowable error $\varepsilon > 0$.

Proposition 3.3.9. *Let $\varepsilon > 0$. When $a > 0$ satisfies*

$$\frac{CC_2K^{-\alpha+1}S_{t-}^{\alpha}}{\pi\varepsilon(2C(T-t)+1)}\left[\frac{1}{G+\alpha}+\frac{1}{M-\alpha-1}+\left|\log\left(\frac{MG}{(M-1)(G+1)}\right)\right|\right] \le a^{2C(T-t)+1}, \tag{3.30}$$

we have

$$\left|\frac{1}{\pi}\int_a^\infty K^{-i\zeta+1}\int_{\mathbb{R}_0}(e^{i\zeta x}-1)(e^x-1)\nu(dx)\psi_2(\zeta)dv\right| < \varepsilon\,, \tag{3.31}$$

where C_2 is defined in (3.28).

Proof. By (3.29), we have

$$
\begin{aligned}
&\left|\frac{1}{\pi}\int_a^\infty K^{-i\zeta+1}\int_{\mathbb{R}_0}(e^{i\zeta x}-1)(e^x-1)\nu(dx)\psi_2(\zeta)dv\right| \\
&\le \frac{1}{\pi}\Bigg\{\left|\int_a^\infty K^{-i\zeta+1}\int_{\mathbb{R}_0}e^{i\zeta x}(e^x-1)\nu(dx)\psi_2(\zeta)dv\right| \\
&+\left|\int_a^\infty K^{-i\zeta+1}\int_{\mathbb{R}_0}(e^x-1)\nu(dx)\psi_2(\zeta)dv\right|\Bigg\} \\
&\le \frac{1}{\pi}\left\{\int_a^\infty \left|K^{-i\zeta+1}\right|\left[\left|\int_{\mathbb{R}_0}e^{i\zeta x}(e^x-1)\nu(dx)\right|+\left|\int_{\mathbb{R}_0}(e^x-1)\nu(dx)\right|\right]\left|\psi_2(\zeta)\right|dv\right\} \\
&\le \frac{1}{\pi}\left\{K^{-\alpha+1}C\left(\frac{1}{G+\alpha}+\frac{1}{M-\alpha-1}+\left|\log\left(\frac{MG}{(M-1)(G+1)}\right)\right|\right)\int_a^\infty\left|\psi_2(\zeta)\right|dv\right\}.
\end{aligned}
\tag{3.32}
$$

Because Proposition 3.3.7 implies

$$
\begin{aligned}
|\psi_2(\zeta)| &= \left|\frac{\phi_{T-t}(\zeta)S_{t-}^{i\zeta}}{(i\zeta-1)i\zeta}\right| \\
&\le \frac{1}{v^2}C_2|v|^{-2C(T-t)}S_{t-}^{\alpha} \\
&= C_2 S_{t-}^{\alpha}|v|^{-2C(T-t)-2},
\end{aligned}
$$

we have, together with (3.32),

$$
\begin{aligned}
\text{R.H.S. of (3.31)} &\le \frac{1}{\pi}CC_2K^{-\alpha+1}S_{t-}^{\alpha}\left[\frac{1}{G+\alpha}+\frac{1}{M-\alpha-1}+\left|\log\left(\frac{MG}{(M-1)(G+1)}\right)\right|\right] \\
&\times\int_a^\infty |v|^{-2C(T-t)-2}dv \\
&= \frac{1}{\pi}CC_2K^{-\alpha+1}S_{t-}^{\alpha}\left[\frac{1}{G+\alpha}+\frac{1}{M-\alpha-1}+\left|\log\left(\frac{MG}{(M-1)(G+1)}\right)\right|\right] \\
&\times\frac{a^{-2C(T-t)-1}}{2C(T-t)+1}.
\end{aligned}
$$

□

3.3.2 Numerical results

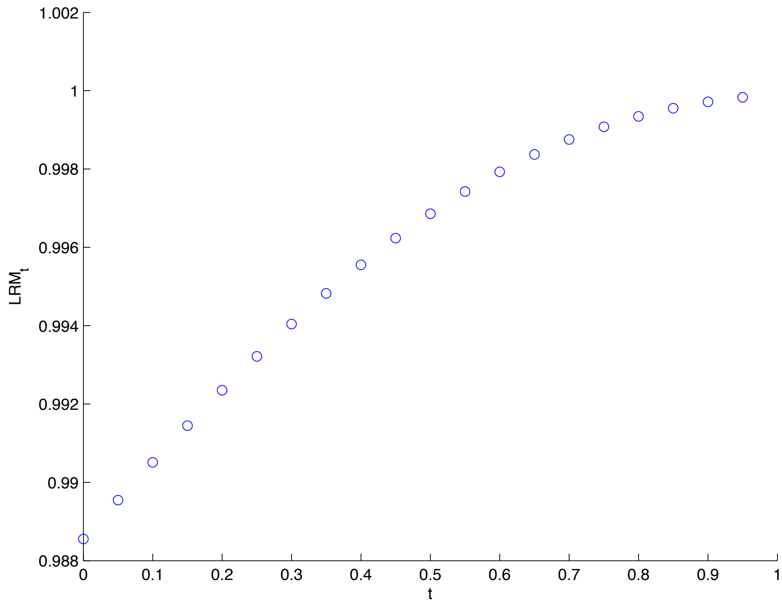
We illustrate our numerical results for a variance gamma model. Choosing the model parameters as $\kappa = 0.15$, $m = -0.2$, and $\delta = 0.45$, which meet the second condition of Assumption 3.1.1, we compute LRM_t for the same numerical experiments as in Subsection 3.2. Note that $M > 4$ is satisfied. Moreover, we also take the same parameters related to the FFT as in Subsection 3.2. $N\eta$ satisfies (3.30) for any parameter set. The results are shown in Fig. 3.2. The computation time to obtain Fig. 3.2(b) was 0.19 s.

In addition, we implemented the same type of numerical experiments as the above based on market data. We used the Nikkei 225 index for March 2014. We need to set the log price $L_t := \log(S_t/S_0)$, where S_0 is the price on 28 February 2014, which was 14841.07. We estimate the parameters C, G, and M in Table 3.1 from the mean, variance, and skewness of the log price by using the generalized method of moments and the Levenberg–Marquardt method.

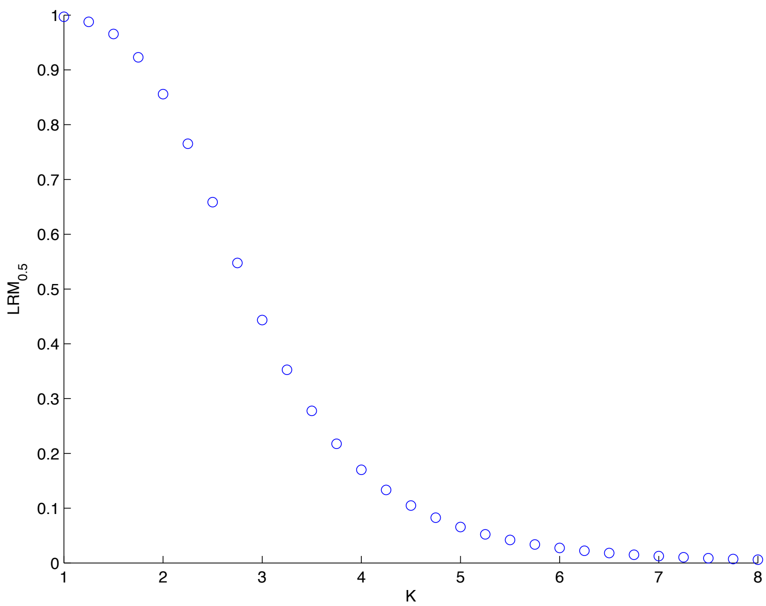
Table 3.1: Estimated parameters

C	2.469395026815120
G	23.743109051760964
M	24.903251787154687

Because $G - M \approx -1.16$, this parameter set satisfies Assumption 3.1.1. We take $T = 1$ and $S_{t-} = 14841.07$, that is, $L_{t-} = 0$. First, fixing the strike price $K = 14000$, we compute LRM_t for $t = 0, 0.05, \ldots, 0.95$. Next, fixing t to 0.5, the values of $LRM_{0.5}$ are computed for $K = 10000, 11000, \ldots, 20000$. Note that $N\eta$ satisfies (3.30). The results of the computation are illustrated in Fig. 3.3.

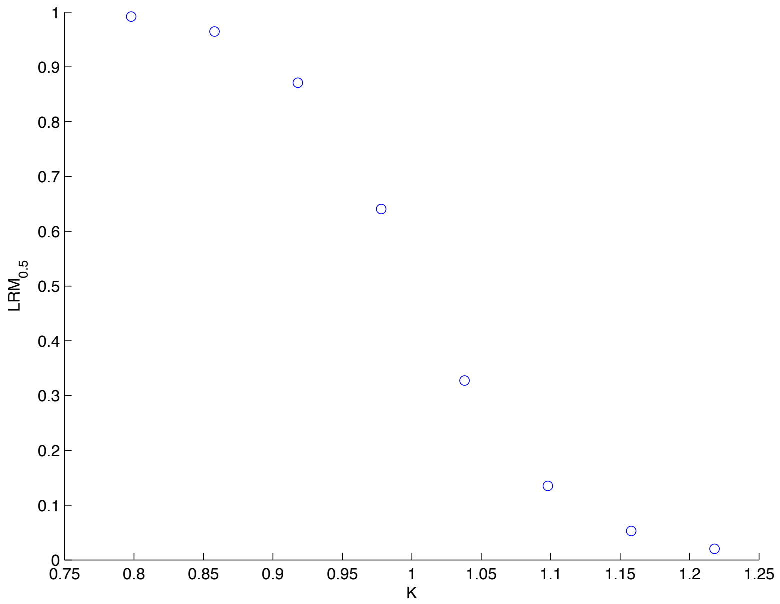


(a) Values of LRM_t of a call option with strike price $K = 1$ and maturity $T = 1$ vs. times $t = 0, 0.05, \ldots, 0.95$ for a variance gamma model with parameters $\kappa = 0.15$, $m = -0.2$, and $\delta = 0.45$. These parameters meet the second condition of Assumption 3.1.1. Moreover, the same FFT parameters as Figure 3.1 are taken.

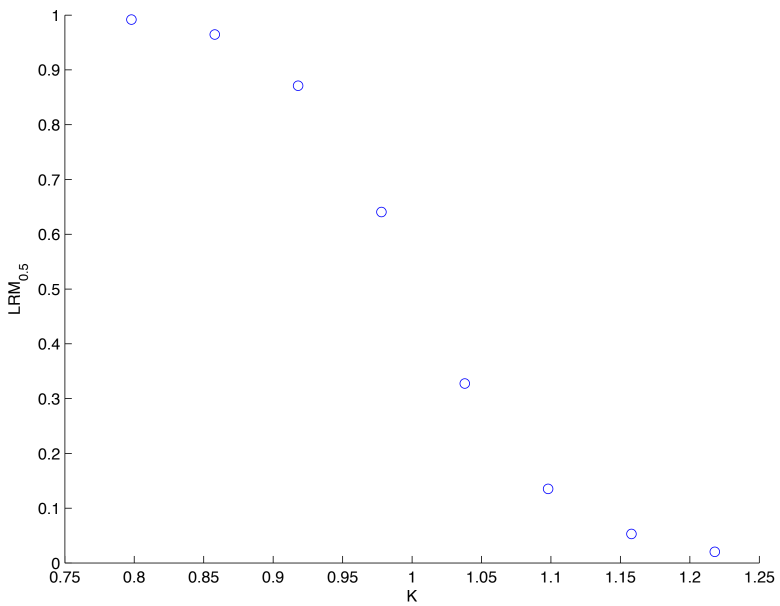


(b) Values of $LRM_{0.5}$ of call options at a fixed time 0.5 vs. strike price K from 1 to 8 at steps of 0.25 for the same variance gamma model as (a) with $S_{0.5} = 1$.

Figure 3.2: Variance gamma model with parameters $\kappa = 0.15$, $m = -0.2$, $\delta = 0.45$



(a) Values of LRM_t for a variance gamma model with strike price $K = 14000$ and $S_{t-} = 14841.07$ vs. $t = 0, 0.05, \dots, 0.95$. The three parameters C, G, and M, given in Table 3.1, are estimated from the Nikkei 225 index for March 2014. This parameter set satisfies Assumption 3.1.1.



(b) Values of $LRM_{0.5}$ at a fixed time 0.5 vs. strike price $K = 10000, 11000, \dots, 20000$ for the same variance gamma model as (a) with $S_{0.5} = 14841.07$.

Figure 3.3: Variance gamma model based on the Nikkei 225 index for March 2014

Chapter 4

Comparison of Local Risk Minimization and Delta Hedging for Exponential Lévy Models

Delta hedging strategies, which are also well-known and often used by practitioners, are given by differentiating the option price under a certain martingale measure with respect to the underlying asset price. Due to the relationship between LRM and the MMM, we consider delta hedging strategies under the MMM. Its precise definition will be introduced in Section 3.1.

[Arai & Suzuki(2015.1)] showed explicit representations of LRM for call options by using Malliavin calculus for Lévy processes based on the canonical Lévy space. Carr and Madan introduced a numerical method for valuing options based on the FFT, see [Carr & Madan(1999)]. In Chapter 2, we adopted Carr and Madan's method to compute LRM of call options for exponential Lévy models. In particular, the authors discussed Merton models and variance Gamma (VG) models as typical examples of exponential Lévy models.

This chapter aims to illustrate, based on [Arai & Suzuki(2015.1)], how different is LRM from delta hedging strategies for call options in exponential Lévy models. Furthermore, we show that delta hedging strategies are easily calculated by using the numerical scheme developed in Chapter 2. We give inequality estimations of the differences of LRM and delta hedging strategies for the typical exponential Lévy models, known as Merton models and VG models. Merton models are composed of a Brownian motion and compound Poisson jumps with normally distributed jump sizes. VG models, which are exponential Lévy processes with infinitely many jumps in any finite time interval and no Brownian component, are the second example. We show that the difference of LRM and delta hedging strategies converges to zero when moneyness tends to zero or infinity. In addition to this, we give numerical results of the difference of LRM and delta hedging strategies since there are mathematical difficulties to follow the behaviours of the option prices around at the money.

4.1 Preliminaries

We consider a financial market composed of one risk-free asset and one risky asset with finite maturity $T > 0$. For simplicity, we assume that market's interest rate is zero, that is, the price of the risk-free asset is 1 at all times. The fluctuation of the risky asset is assumed to be described by an exponential Lévy process S on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, described by

$$S_t := S_0 \exp\left\{\mu t + \sigma W_t + \int_{\mathbb{R}_0} x\widetilde{N}([0,t], dx)\right\}$$

for any $t \in [0,T]$, where $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, and $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Here W is a one-dimensional Brownian motion and $\widetilde{N}$ is the compensated version of a Poisson random measure N. Denoting the Lévy measure of N by ν, we have

$$\widetilde{N}([0,t], A) = N([0,t], A) - t\nu(A)$$

for any $t \in [0,T]$ and $A \in \mathcal{B}(\mathbb{R}_0)$. Moreover, S is also a solution of the stochastic differential equation

$$dS_t = S_{t-}\left[\mu^S\, dt + \sigma\, dW_t + \int_{\mathbb{R}_0} (e^x - 1)\widetilde{N}(dt, dx)\right],$$

where

$$\mu^S := \mu + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}_0} (e^x - 1 - x)\nu(dx)\,.$$

Without loss of generality, we may assume that $S_0 = 1$ for simplicity. Now, defining $L_t := \log S_t$ for all $t \in [0, T]$, we obtain a Lévy process L. Moreover,

$$dM_t := S_{t-}\big[\sigma\, dW_t + \int_{\mathbb{R}_0} (e^x - 1)\widetilde{N}(dt, dx)\big]$$

is the martingale part of S.

Our focus is to compare LRM to delta hedging strategies with respect to a call option $(S_T - K)^+$ with strike price $K > 0$. We first give some preparations and assumptions to introduce an explicit LRM representation of such options in exponential Lévy models. Define the MMM $\mathbb{P}^*$ as an equivalent martingale measure under which any square-integrable $\mathbb{P}$-martingale orthogonal to M remains a martingale. Its density is given by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\big\{ -\xi W_T - \frac{\xi^2}{2}T + \int_{\mathbb{R}_0} \log(1 - \theta_x)N([0, T], dx) + T\int_{\mathbb{R}_0} \theta_x \nu(dx)\big\},$$

where

$$\xi := \frac{\mu^S \sigma}{\sigma^2 + \int_{\mathbb{R}_0}(e^y - 1)^2\nu(dy)} \quad \text{and} \quad \theta_x := \frac{\mu^S(e^x - 1)}{\sigma^2 + \int_{\mathbb{R}_0}(e^y - 1)^2\nu(dy)}$$

for $x \in \mathbb{R}_0$. In the development of our approach, we rely on the following assumption.

Assumption 4.1.1. *1.* $\int_{\mathbb{R}_0}(|x| \vee x^2)\nu(dx) < \infty$, *and* $\int_{\mathbb{R}_0}(e^x - 1)^n\nu(dx) < \infty$ *for* $n = 4$.

2. $0 \geq \mu^S > -\sigma^2 - \int_{\mathbb{R}_0}(e^x - 1)^2\nu(dx)$.

The first condition ensures that μ^S, ξ, and θ_x are well defined, the square integrability of L, and the finiteness of $\int_{\mathbb{R}_0}(e^x - 1)^n\nu(dx)$ for $n = 1, 3$. The second condition guarantees that $\theta_x < 1$ for any $x \in \mathbb{R}_0$. Moreover, by the Girsanov theorem,

$$W_t^{\mathbb{P}^*} := W_t + \xi t \quad \text{and} \quad \widetilde{N}^{\mathbb{P}^*}([0, t], dx) := \theta_x \nu(dx)t + \widetilde{N}([0, t], dx)$$

are a $\mathbb{P}^*$-Brownian motion and the compensated Poisson random measure of N under $\mathbb{P}^*$, respectively. We can then rewrite L_t as

$$L_t = \mu^* t + \sigma W_t^{\mathbb{P}^*} + \int_{\mathbb{R}_0} x\widetilde{N}^{\mathbb{P}^*}([0, t], dx)\,,$$

where

$$\mu^* := -\frac{1}{2}\sigma^2 + \int_{\mathbb{R}_0} (x - e^x + 1)(1 - \theta_x)\nu(dx)\,.$$

Note that L is a Lévy process even under $\mathbb{P}^*$, with Lévy measure given by $\nu^{\mathbb{P}^*}(dx) := (1-\theta_x)\nu(dx)$. LRM will be given as a predictable process LRM_t, which represents the number of units of the risky asset the investor holds at time t. We introduce a representation of LRM for call option. We define

$$\begin{aligned} I_1 &:= \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T>K\}} S_T \mid \mathcal{F}_{t-}]\,, \\ I_2 &:= \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[(S_T e^x - K)^+ - (S_T - K)^+ \mid \mathcal{F}_{t-}] \times (e^x - 1)\nu(dx)\,, \end{aligned}$$

where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is the $\mathbb{P}$-completed filtration generated by W and N. By using these symbols, we can write an explicit representation of LRM for call option $(S_T - K)^+$ as follows:

Proposition 4.1.2 (Proposition 4.6 of [Arai & Suzuki(2015.1)])**.** *For any $K > 0$ and $t \in [0,T]$,*

$$LRM_t = \frac{\sigma^2 I_1 + I_2}{S_{t-}\big(\sigma^2 + \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx)\big)}. \tag{4.1}$$

Next, we introduce integral representations of I_1 and I_2 given in [Arai & Suzuki(2015.1)] in order to show we can adopt Carr and Madan's method. The characteristic function of L_{T-t} under $\mathbb{P}^*$ is denoted by

$$\phi_{T-t}(z) := \mathbb{E}_{\mathbb{P}^*}[e^{izL_{T-t}}] \quad \text{for} \quad z \in \mathbb{C}\,.$$

We induce an integral representation for I_1 with ϕ_{T-t} firstly.

$$\begin{aligned} I_1 &= \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T>K\}} \cdot S_T \mid \mathcal{F}_{t-}] \\ &= \frac{1}{\pi}\int_0^\infty \frac{K^{-iv-\alpha+1}}{\alpha - 1 + iv} \phi_{T-t}(v - i\alpha) S_{t-}^{\alpha+iv} dv \\ &= \frac{e^k}{\pi}\int_0^\infty e^{-i(v-i\alpha)k} \psi_1(v - i\alpha) dv \end{aligned}$$

where $k := \log K$ and

$$\psi_1(z) := \frac{\phi_{T-t}(z) S_{t-}^{iz}}{iz - 1}$$

and $\alpha \in (1,2]$. Note that the right-hand side is independent of the choice of α. We turn next to I_2. Denoting

$$\psi_2(z) := \frac{\phi_{T-t}(z)S_{t-}^{iz}}{(iz-1)iz}$$

and $\zeta := v - i\alpha$, we have

$$\begin{aligned} I_2 &= \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[(S_T e^x - K)^+ - (S_T - K)^+ \mid \mathcal{F}_{t-}](e^x - 1)\nu(dx) \\ &= \frac{1}{\pi}\int_0^\infty K^{-i\zeta+1} \int_{\mathbb{R}_0} (e^{i\zeta x} - 1)(e^x - 1)\nu(dx)\psi_2(\zeta) dv\,. \end{aligned} \tag{4.2}$$

Note that we can not calculate (4.2) numerically as it stands, because it is not possible to compute the integral $\int_{\mathbb{R}_0}(e^{i\zeta x}-1)(e^x-1)\nu(dx)$ directly. Thus, we introduce model-dependent calculations for Merton models in Secs. 3.2 and for VG models in Secs. 3.3, respectively. Regarding LRM_t, I_1, and I_2 as functions of S_{t-} and K, we have $I_i(S_{t-}, K)/S_{t-} = I_i(1, K/S_{t-})$ for $i = 1, 2$. We obtain

$$LRM_t(S_{t-}, K) = \frac{\sigma^2 I_1(1, K/S_{t-}) + I_2(1, K/S_{t-})}{\sigma^2 + \int_{\mathbb{R}_0}(e^x - 1)^2\nu(dx)}$$

from (4.1). As a result, LRM_t is given as a function of $K/S_{t-} =: \chi_{t-}$, where χ_{t-} is called *moneyness*. Thus, we denote LRM_t by $LRM_t(\chi_{t-})$. Moreover, we regard $I_1(A, B) := \frac{1}{\pi}\int_0^\infty \frac{A^{-iv-\alpha+1}}{\alpha-1+iv}\phi_{T-t}(v - i\alpha)B^{\alpha+iv}dv$ and the same thing is valid for I_2. Hereinafter we fix $\alpha \in (1, 2]$ arbitrarily. Moreover, we denote $\zeta := v - i\alpha$ for $v \in \mathbb{R}$, so we may regard ζ as a function of v.

Next, we define delta hedging strategies.

Definition 4.1.3. *For any $K > 0$ and $s > 0$, a delta hedging strategy under the minimal martingale measure is defined as*

$$\Delta_t^{\mathbb{P}^*}(\chi_{t-}) := \frac{\partial \mathbb{E}_{\mathbb{P}^*}[(S_T - K)^+ \mid S_{t-} = s]}{\partial s}\,.$$

Remark that the above definition of delta hedging strategies coincide with the usual delta hedging strategies in the case of Black–Scholes. The next theorem follows from the direct calculation.

Theorem 4.1.4.

$$\Delta_t^{\mathbb{P}^*}(\chi_{t-}) = \frac{I_1}{S_{t-}} \ .$$

Remark 4.1.5. *Using the numerical scheme developed in Chapter 2, we can calculate $\Delta_t^{\mathbb{P}^*}(\chi_{t-})$ easily from Theorem 4.1.4.*

Remark 4.1.6. *[Denkl, et al. (2013)] introduced the definition of Δ-strategies which are generalized delta hedging strategies. The authors derived semi-explicit formulas for the mean-squared hedging error of a European-style contingent claim in terms of Δ-strategies. This has been done for delta hedging strategies including Black-Scholes hedging strategies. They also showed two numerical examples. First, they compared the performance of Black-Scholes strategies and variance-optimal strategies in the normal Gaussian Lévy model. Second, they assessed the hedging errors of Black-Scholes strategies, the delta hedge and the variance-optimal strategy in a diffusion-extended CGMY Lévy model. As in Example 3.2, they discussed the delta hedge by computing the derivatives of a price process with respect to the underlying exponential Lévy models. This delta hedge is equivalent to our $\Delta_t^{\mathbb{P}^*}$.*

We see behaviours of $LRM_t(\chi_{t-})$ and $\Delta_t^{\mathbb{P}^*}(\chi_{t-})$, when moneyness χ_{t-} sufficiently small. Taking strike price $K \to 0$ then S_{t-} goes to relatively and sufficiently large. Under such a condition, we write $\chi_{t-} \to 0$ as one representation of sufficiently small moneyness.

Theorem 4.1.7. *When moneyness χ_{t-} goes to zero relatively, LRM_t coincides with $\Delta_t^{\mathbb{P}^*}$;*

$$\lim_{\chi_{t-}\to 0} |LRM_t(\chi_{t-}) - \Delta_t^{\mathbb{P}^*}(\chi_{t-})| = 0 \ . \tag{4.3}$$

Proof. From monotone convergence theorem,

$$\begin{aligned}
\frac{I_1}{S_{t-}} &= \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\frac{S_T}{S_{t-}} > \chi_{t-}} \frac{S_T}{S_{t-}} | \mathcal{F}_{t-}] \\
&= \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{e^{L_{T-t}} > \chi_{t-}} e^{L_{T-t}}] \\
&= \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{L_{T-t} > \log \chi_{t-}} e^{L_{T-t}}] \\
&= \int_{\log \chi_{t-}}^{\infty} e^x p^*(dx)
\end{aligned}$$

$$
\begin{aligned}
&\to_{\chi_{t-}\to 0} \mathbb{E}_{\mathbb{P}^*}[e^{L_{T-t}}] \\
&= 1\,.
\end{aligned}
$$

Another term is little complicated.

$$
\begin{aligned}
\frac{I_2}{S_{t-}} &= \frac{1}{S_{t-}} \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[(S_T e^x - K)^+ - (S_T - K)^+ | \mathcal{F}_{t-}](e^x - 1)\nu(dx) \\
&= \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[(e^{L_{T-t}+x} - \chi_{t-})^+ - (e^{L_{T-t}} - \chi_{t-})^+](e^x - 1)\nu(dx) \qquad (4.4)
\end{aligned}
$$

To make it easy to see, we put $e^{L_{T-t}}$ as e^y and separate (4.4) into four parts:

$$
\begin{aligned}
J_1 &:= \int_{-\infty}^{\infty} \int_{\mathbb{R}_0} (e^{y+x} - e^y)\mathbf{1}_{y+x \geq \log \chi_{t-}} \mathbf{1}_{y \geq \log \chi_{t-}} p^*(dy)(e^x - 1)\nu(dx) \\
J_2 &:= \int_{-\infty}^{\infty} \int_{\mathbb{R}_0} (e^{y+x} - \chi_{t-})\mathbf{1}_{y+x \geq \log \chi_{t-}} \mathbf{1}_{y < \log \chi_{t-}} p^*(dy)(e^x - 1)\nu(dx) \\
J_3 &:= -\int_{-\infty}^{\infty} \int_{\mathbb{R}_0} (e^{y} - \chi_{t-})\mathbf{1}_{y+x < \log \chi_{t-}} \mathbf{1}_{y \geq \log \chi_{t-}} p^*(dy)(e^x - 1)\nu(dx) \\
J_4 &:= \int_{-\infty}^{\infty} \int_{\mathbb{R}_0} (-\chi_{t-})^+ (-\chi_{t-})^+ \mathbf{1}_{y+x < \log \chi_{t-}} \mathbf{1}_{y < \log \chi_{t-}} p^*(dy)(e^x - 1)\nu(dx) \\
&= 0\ .
\end{aligned}
$$

First part is J_1. Adopting Lebesgue's dominated convergence theorem, we obtain

$$
\begin{aligned}
\lim_{\chi_{t-}\to 0} J_1 &= \lim_{\chi_{t-}\to 0} \int_{-\infty}^{\infty} \int_{\mathbb{R}_0} (e^{y+x} - e^y)\mathbf{1}_{y+x \geq \log \chi_{t-}} \mathbf{1}_{y \geq \log \chi_{t-}} p^*(dy)(e^x - 1)\nu(dx) \\
&= \int_{-\infty}^{\infty} \int_{\mathbb{R}_0} (e^{y+x} - e^y) \lim_{\chi_{t-}\to 0} \mathbf{1}_{y+x \geq \log \chi_{t-}} \mathbf{1}_{y \geq \log \chi_{t-}} p^*(dy)(e^x - 1)\nu(dx) \\
&= \int_{-\infty}^{\infty} \int_{\mathbb{R}_0} e^y p^*(dy)(e^x - 1)^2 \nu(dx) \\
&= \mathbb{E}_{\mathbb{P}^*}[e^{L_{T-t}}] \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx) \\
&= \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx) \\
&= \mathcal{C} < \infty
\end{aligned}
$$

The next part is J_2. We can adopt Lebesgue's dominated convergence theorem also, and we obtain

$$\begin{aligned}
\lim_{\chi_{t-}\to 0} J_2 &= \lim_{\chi_{t-}\to 0} \int_{-\infty}^{\infty}\int_{\mathbb{R}_0} (e^{y+x} - \chi_{t-})\mathbf{1}_{y+x\geq \log\chi_{t-}}\mathbf{1}_{y<\log\chi_{t-}} p^*(dy)(e^x-1)\nu(dx) \\
&= \int_{-\infty}^{\infty}\int_{\mathbb{R}_0} (e^{y+x} - \chi_{t-}) \lim_{\chi_{t-}\to 0}\mathbf{1}_{y+x\geq \log\chi_{t-}}\mathbf{1}_{y<\log\chi_{t-}} p^*(dy)(e^x-1)\nu(dx) \\
&= 0
\end{aligned}$$

The last part is J_3. This part is the same as the former parts so Adopting Lebesgue's dominated convergence theorem that we obtain

$$\begin{aligned}
\lim_{\chi_{t-}\to 0} J_3 &= -\lim_{\chi_{t-}\to 0} \int_{-\infty}^{\infty}\int_{\mathbb{R}_0} (e^{y} - \chi_{t-})\mathbf{1}_{y+x< -\log\chi_{t-}}\mathbf{1}_{y\geq -\log_{t-}} p^*(dy)(e^x-1)\nu(dx) \\
&= -\int_{-\infty}^{\infty}\int_{\mathbb{R}_0} \lim_{\chi_{t-}\to 0}(e^{y} - \chi_{t-})\mathbf{1}_{y+x< -\log\chi_{t-}}\mathbf{1}_{y\geq -\log\chi_{t-}} p^*(dy)(e^x-1)\nu(dx) \\
&= 0
\end{aligned}$$

To summarize the above, we conclude

$$\begin{aligned}
LRM_t - \Delta_t^{\mathbb{P}^*} &= \frac{I_2}{S_{t-}}\frac{1}{\sigma^2 + \int_{\mathbb{R}_0}(e^x-1)^2\nu(dx)} - \frac{I_1}{S_{t-}}\frac{\int_{\mathbb{R}_0}(e^x-1)^2\nu(dx)}{\sigma^2 + \int_{\mathbb{R}_0}(e^x-1)^2\nu(dx)} \\
&= \frac{I_2}{S_{t-}}\frac{1}{\sigma^2+\mathcal{C}} - \frac{I_1}{S_{t-}}\frac{\mathcal{C}}{\sigma^2+\mathcal{C}} \\
\lim_{\chi_{t-}\to 0}(LRM_t - \Delta_t^{\mathbb{P}^*}) &= \frac{\mathcal{C}}{\sigma^2+\mathcal{C}} - \frac{\mathcal{C}}{\sigma^2+\mathcal{C}} \\
&= 0
\end{aligned}$$

□

4.2 The Merton Jump-Diffusion Model

We consider the case where L is given as a Merton jump-diffusion process, which consists of a diffusion component with volatility $\sigma > 0$ and compound Poisson jumps with three parameters, $m \in \mathbb{R}$, $\delta > 0$, and $\gamma > 0$. Note that γ represents the jump intensity, and that the sizes of the jumps are distributed normally with mean m and variance δ^2. Thus, its Lévy measure ν is given by

$$\nu(dx) = \frac{\gamma}{\sqrt{2\pi}\delta} \exp\left\{ -\frac{(x-m)^2}{2\delta^2} \right\} dx .$$

Note that the first condition of Assumption 4.1.1 is satisfied for any $m \in \mathbb{R}$, $\delta > 0$, and $\gamma > 0$. We consider only parameter sets satisfying the second condition of Assumption 4.1.1.

4.2.1 Mathematical preliminaries

Our aim here is to give an inequality estimation of $|LRM_t - \Delta_t^{\mathbb{P}^*}|$. An analytic form of ϕ_{T-t} was given in Proposition 3.2.1 and of $\nu^{\mathbb{P}^*}$ can be seen in Proposition 3.2.2 also. The letter $\mathcal{C}$ and others denote generic constants and the values of constants $\mathcal{C}$ may change from line to line.

Theorem 4.2.1. *There exists a positive constant* $\mathcal{C}$ *such that*

$$|LRM_t(\chi_{t-}) - \Delta_t^{\mathbb{P}^*}(\chi_{t-})| \leq \mathcal{C}\chi_{t-}^{1-\alpha} . \tag{4.5}$$

Proof. First of all, we show the inequality estimation (4.5).

$$\begin{aligned} & I_2 - I_1 \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx) \\ &= \frac{1}{\pi} \int_0^\infty K^{-(\alpha+iv)+1} \int_{\mathbb{R}_0} (e^{(\alpha+iv)x} - 1)(e^x - 1)\nu(dx) \frac{\phi_{T-t}(v - i\alpha) S_{t-}^{\alpha+iv}}{(\alpha + iv - 1)(\alpha + iv)} dv \\ &- \frac{1}{\pi} \int_0^\infty K^{-(\alpha+iv)+1} \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx) \frac{\phi_{T-t}(v - i\alpha) S_{t-}^{\alpha+iv}}{\alpha + iv - 1} dv \end{aligned}$$

Noting that

$$F(y) := \frac{1}{y} \int_{\mathbb{R}_0} (e^x - 1)(e^{yx} - 1)\nu(dx) ,$$

we have

$$\begin{aligned} & I_2 - I_1 \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx) \\ &= \frac{1}{\pi} \int_0^\infty K^{-(\alpha+iv)+1} (F(\alpha + iv) - F(1)) \frac{\phi_{T-t}(v - i\alpha) S_{t-}^{\alpha+iv}}{\alpha + iv - 1} dv . \end{aligned}$$

Hence

$$\begin{aligned}
&|I_2 - I_1 \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx)| \\
&\leq \frac{1}{\pi} \int_0^\infty |K^{-(\alpha+iv)+1}||F(\alpha+iv) - F(1)| \frac{|\phi_{T-t}(v - i\alpha)||S_{t-}^{\alpha+iv}|}{|\alpha + iv - 1|} dv \\
&= \frac{1}{\pi} S_{t-} \chi_{t-}^{1-\alpha} \int_0^\infty |F(\alpha+iv) - F(1)| \frac{|\phi_{T-t}(v - i\alpha)|}{|\alpha + iv - 1|} dv \\
&=: I \ .
\end{aligned}$$

Now we take

$$y(t) = (\alpha + iv - 1)t + 1 \ ,$$

then

$$\begin{aligned}
|F(\alpha+iv) - F(1)| &= \left| \int_1^{\alpha+iv} F'(y) dy \right| \\
&= \left| \int_0^1 F'(y(t)) y'(t) dt \right| \\
&\leq | \sup_{0 \leq t \leq 1} F'(y(t))||(\alpha + iv - 1)| \\
&\leq \mathcal{C}_{(m,\delta,\alpha)} |(\alpha + iv - 1)| \ ,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{C}_{(m,\delta,\alpha)} =& \gamma\{e^{m(\alpha+1) + \frac{\delta^2}{2}(\alpha+1)^2} + e^{m + \frac{\delta^2}{2}} + e^{m\alpha + \frac{\delta^2}{2}\alpha^2} + 1\} \\
&+ \gamma(m + \delta^2 \sqrt{2 + 2\alpha}) e^{m(\alpha+1) + \frac{\delta^2}{2}(\alpha+1)^2} \\
&+ \gamma(m + \delta^2) e^{m\alpha + \frac{\delta^2}{2}\alpha^2} \ .
\end{aligned}$$

Using lemma 3.2.4, we have

$$\begin{aligned}
I &\leq \frac{1}{\pi} \chi_{t-}^{1-\alpha} S_{t-} \mathcal{C}_{(m,\delta,\alpha)} \int_0^\infty |\phi_{T-t}(v - i\alpha)| dv \\
&\leq \frac{\mathcal{C}_1 \mathcal{C}_{(m,\delta,\alpha)}}{\pi} \chi_{t-}^{1-\alpha} S_{t-} \int_0^\infty e^{-\frac{\sigma^2(T-t)}{2} v^2} dv \ .
\end{aligned}$$

Its integral part is easy to calculate and then we have

$$I \leq \frac{\mathcal{C}_1 \mathcal{C}_{(m,\delta,\alpha)}}{\sigma} \frac{1}{\sqrt{2\pi(T-t)}} K^{1-\alpha} S_{t-}^{\alpha} \ .$$

Finally taking a constant $\mathcal{C}$ as

$$\int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx) =: \mathcal{C} < \infty \ ,$$

we obtain the following estimate:

$$\begin{aligned} |LRM_t - \Delta_t^{\mathbb{P}^*}| &\leq \frac{\mathcal{C}_1 \mathcal{C}_{(m,\delta,\alpha)}}{\sigma\sqrt{2\pi(T-t)}} \frac{\chi_{t-}^{1-\alpha}}{\sigma^2 + \int_{\mathbb{R}_0}(e^x-1)^2\nu(dx)} \\ &= \frac{\mathcal{C}_1 \mathcal{C}_{(m,\delta,\alpha)}}{\sigma(\sigma^2 + \mathcal{C})\sqrt{2\pi(T-t)}} \ \chi_{t-}^{1-\alpha} \ . \end{aligned}$$

From

□

4.2.2 Numerical results

We compute $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ with the FFT. In this subsection, we provide a numerical result for a Merton jump-diffusion model with parameters $T = 0.5$, $L_t = 0$, $\mu = -0.7$, $\sigma = 0.2$, $\gamma = 1$, $m = 0$, and $\delta = 1$. Note that μ^S is given by -0.03, which satisfies the second condition of Assumption 4.1.1. We compute and plot the data of $|LRM_{0.5} - \Delta_t^{\mathbb{P}^*}|$ shown as Figure 4.1. FFT parameters are chosen as $N = 2^{14}$, $\eta = 0.025$ and $\alpha = 1.75$.

4.3 The Variance Gamma Model

We now consider the case where L is given as a variance Gamma process, which has three parameters $\kappa > 0$, $m \in \mathbb{R}$, and $\delta > 0$. This is defined as a time-changed Brownian motion with volatility δ, drift m, and subordinator G_t, where G_t is a Gamma process with parameters $(1/\kappa, 1/\kappa)$. In summary, L is represented as

$$L_t = mG_t + \delta B_{G_t} \quad \text{for } t \in [0, T] \ ,$$

where B is a one-dimensional standard Brownian motion. Moreover, the Lévy measure of L is given by

$$\nu(dx) = C(\mathbf{1}_{\{x<0\}} e^{-G|x|} + \mathbf{1}_{\{x>0\}} e^{-M|x|}) \frac{dx}{|x|}$$

where

$$C := \frac{1}{\kappa} > 0, \quad G := \frac{1}{\delta^2}\sqrt{m^2 + \frac{2\delta^2}{\kappa}} + \frac{m}{\delta^2} > 0,$$
$$M := \frac{1}{\sqrt{m^2 + \frac{2\delta^2}{\kappa}}}\delta^2 - \frac{m}{\delta^2} > 0.$$

In addition, we assume $M > 4$, which ensures that the first condition of Assumption 4.1.1 holds. An analytic form of ϕ_{T-t} was given in Proposition 3.3.5, and that of $\nu^{\mathbb{P}^*}$ can be seen in Proposition 3.3.3 also. The letter $\mathcal{C}$ and others denote generic constants and the values of constants $\mathcal{C}$ may change from line to line.

Theorem 4.3.1. *There exists a positive constant $\mathcal{C}$ such that*

$$|LRM_t(\chi_{t-}) - \Delta_t^{\mathbb{P}^*}(\chi_{t-})| \leq \mathcal{C}\chi_{t-}^{1-\alpha} .$$

Proof.

$$\begin{aligned}
&I_2 - I_1 \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx) \\
&= \frac{1}{\pi}\int_0^\infty K^{-(\alpha+iv)+1} \int_{\mathbb{R}_0} (e^{(\alpha+iv)x} - 1)(e^x - 1)\nu(dx) \frac{\phi_{T-t}(v - i\alpha)S_{t-}^{\alpha+iv}}{(\alpha + iv - 1)(\alpha + iv)} dv \\
&- \frac{1}{\pi}\int_0^\infty K^{-(\alpha+iv)+1} \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx) \frac{\phi_{T-t}(v - i\alpha)S_{t-}^{\alpha+iv}}{\alpha + iv - 1} dv
\end{aligned}$$

Noting that

$$F(y) := \frac{1}{y}\int_{\mathbb{R}_0} (e^x - 1)(e^{yx} - 1)\nu(dx) ,$$

we have

$$\begin{aligned}
&I_2 - I_1 \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx) \\
&= \frac{1}{\pi}\int_0^\infty K^{-(\alpha+iv)+1}(F(\alpha + iv) - F(1)) \frac{\phi_{T-t}(v - i\alpha)S_{t-}^{\alpha+iv}}{\alpha + iv - 1} dv .
\end{aligned}$$

Hence

$$|I_2 - I_1 \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx)|$$

$$\leq \frac{1}{\pi}\int_0^\infty |K^{-(\alpha+iv)+1}||F(\alpha+iv)-F(1)|\frac{|\phi_{T-t}(v-i\alpha)||S_{t-}^{\alpha+iv}|}{|\alpha+iv-1|}dv$$
$$= \frac{1}{\pi}S_{t-}\chi_{t-}^{1-\alpha}\int_0^\infty |F(\alpha+iv)-F(1)|\frac{|\phi_{T-t}(v-i\alpha)|}{|\alpha+iv-1|}dv$$
$$=: I \ .$$

Now we take

$$y(t) = (\alpha+iv-1)t+1 \ ,$$

then

$$\begin{aligned}|F(\alpha+iv)-F(1)| &= \left|\int_1^{\alpha+iv} F'(y)dy\right| \\ &= \left|\int_0^1 F'(y(t))y'(t)dt\right| \\ &\leq |\sup_{0\leq t\leq 1} F'(y(t))||(\alpha+iv-1)| \\ &\leq \mathcal{C}_{(C,G,M,\alpha)}|(\alpha+iv-1)| \ .\end{aligned}$$

From the characteristic function of VG

$$\begin{aligned}\phi_{T-t}(v-i\alpha) = &\left[\left(1+\frac{iv+\alpha}{G}\right)\left(1-\frac{iv+\alpha}{M}\right)\right]^{-(1+h)(T-t)C}\left[\left(1+\frac{iv+\alpha}{G+1}\right)\left(1-\frac{iv+\alpha}{M-1}\right)\right]^{h(T-t)C} \\ &\times \exp\left\{(T-t)(iv+\alpha)\left[\mu^* + (1+h)C\frac{M-G}{GM} - hC\frac{M-G-2}{(G+1)(M-1)}\right]\right\},\end{aligned}$$

we put

$$\left[\left(1+\frac{iv+\alpha}{G}\right)\left(1-\frac{iv+\alpha}{M}\right)\right]^{-(1+h)(T-t)C}\left[\left(1+\frac{iv+\alpha}{G+1}\right)\left(1-\frac{iv+\alpha}{M-1}\right)\right]^{h(T-t)C}$$
$$= \left[\left(1+\frac{iv+\alpha}{G}\right)\left(1-\frac{iv+\alpha}{M}\right)\right]^{-a_1}\left[\left(1+\frac{iv+\alpha}{G+1}\right)\left(1-\frac{iv+\alpha}{M-1}\right)\right]^{a_2}$$
$$=: I_{VG} \ .$$

We estimate I_{VG} here.

First of all we estimate $|\left(1+\frac{iv+\alpha}{G}\right)\left(1-\frac{iv+\alpha}{M}\right)|^{-a_1}$.

$$\left|1+\frac{iv+\alpha}{G}\right|^{-a_1} = \left|(1+\frac{\alpha}{G})+i\frac{v}{G}\right|^{-a_1}$$

$$= \left|(1+\frac{\alpha}{G})^2 + (\frac{v}{G})^2\right|^{-\frac{a_1}{2}}$$
$$= \left|1+\frac{2\alpha}{G} + (\frac{\alpha}{G})^2 + (\frac{v}{G})^2\right|^{-\frac{a_1}{2}}$$
$$= \left|\mathcal{C}_0 + (\frac{v}{G})^2\right|^{-\frac{a_1}{2}} .$$

where $\mathcal{C}_0 := 1 + \frac{2\alpha}{G} + (\frac{\alpha}{G})^2 > 1$.

$$\left|1 - \frac{iv+\alpha}{M}\right|^{-a_1} = \left|(1-\frac{\alpha}{M}) - i\frac{v}{M}\right|^{-a_1}$$
$$= \left|(1-\frac{\alpha}{M})^2 + (\frac{v}{M})^2\right|^{-\frac{a_1}{2}}$$

Let $\epsilon := (1 - \frac{\alpha}{M})^2$, then

$$\left|\left(1+\frac{iv+\alpha}{G}\right)\left(1-\frac{iv+\alpha}{M}\right)\right|^{-a_1} = \left|\left(\mathcal{C}_0 + (\frac{v}{G})^2\right)\left(\epsilon + (\frac{v}{M})^2\right)\right|^{-\frac{a_1}{2}}$$
$$= \left|\epsilon\mathcal{C}_0 + \mathcal{C}_0(\frac{v}{M})^2 + \epsilon(\frac{v}{G})^2 + \frac{1}{M^2G^2}v^4\right|^{-\frac{a_1}{2}}$$
$$= |\epsilon\mathcal{C}_0 + \mathcal{C}_1 v^2 + \mathcal{C}_2 v^4|^{-\frac{a_1}{2}} , \qquad (4.6)$$

where we put $\mathcal{C}_1 := \frac{\mathcal{C}_0}{M^2} + \frac{\epsilon}{G^2}$ and $\mathcal{C}_2 := \frac{1}{M^2G^2}$.
The next is $|\left(1+\frac{iv+\alpha}{G+1}\right)\left(1-\frac{iv+\alpha}{M-1}\right)|^{a_2}$.

$$\left|\left(1+\frac{iv+\alpha}{G+1}\right)\left(1-\frac{iv+\alpha}{M-1}\right)\right|^{a_2}$$
$$= \left|\left(1+\frac{\alpha}{G+1}\right)^2 + \left(\frac{v}{G+1}\right)^2\right|^{\frac{a_2}{2}} \left|\left(1-\frac{\alpha}{M-1}\right)^2 + \left(\frac{v}{M-1}\right)^2\right|^{\frac{a_2}{2}}$$
$$= \Bigg| \left(1+\frac{\alpha}{G+1}\right)^2 \left(1-\frac{\alpha}{M-1}\right)^2$$
$$+ \left[\left(1+\frac{\alpha}{G+1}\right)^2 \frac{1}{(M-1)^2} + \left(1-\frac{\alpha}{M-1}\right)^2 \frac{1}{(G+1)^2}\right] v^2 + \frac{1}{(G+1)^2(M-1)^2} v^4 \Bigg|$$

$$= |\tilde{K} + \mathcal{C}_3 v^2 + \mathcal{C}_4 v^4|^{\frac{a_2}{2}} \ , \tag{4.7}$$

where

$$\tilde{K} := \left(1 + \frac{\alpha}{G+1}\right)^2 \left(1 - \frac{\alpha}{M-1}\right)^2$$

$$\mathcal{C}_3 := \left(1 + \frac{\alpha}{G+1}\right)^2 \frac{1}{(M-1)^2} + \left(1 - \frac{\alpha}{M-1}\right)^2 \frac{1}{(G+1)^2}$$

$$\mathcal{C}_4 := \frac{1}{(G+1)^2(M-1)^2} \ .$$

From (4.6) and (4.7),

$$\begin{aligned} I &= |\epsilon\mathcal{C}_0 + \mathcal{C}_1 v^2 + \mathcal{C}_2 v^4|^{-\frac{a_1}{2}} |\tilde{K} + \mathcal{C}_3 v^2 + \mathcal{C}_4 v^4|^{\frac{a_2}{2}} \\ &= |\epsilon\mathcal{C}_0 + \mathcal{C}_1 v^2 + \mathcal{C}_2 v^4|^{-\frac{1}{2}(1+h)(T-t)C} |\tilde{K} + \mathcal{C}_3 v^2 + \mathcal{C}_4 v^4|^{\frac{1}{2}h(T-t)C} \\ &\le (\epsilon\mathcal{C}_0)^{-\frac{1}{2}(1+h)(T-t)C} |\tilde{K} + \mathcal{C}_3 v^2 + \mathcal{C}_4 v^4|^{\frac{1}{2}h(T-t)C} \ . \end{aligned}$$

$$\begin{aligned} \int_0^\infty |I| &\le (\epsilon\mathcal{C}_0)^{-\frac{1}{2}(1+h)(T-t)C} \int_0^\infty |\tilde{K} + \mathcal{C}_3 v^2 + \mathcal{C}_4 v^4|^{\frac{1}{2}h(T-t)C} dv \\ &= (\epsilon\mathcal{C}_0)^{-\frac{1}{2}(1+h)(T-t)C} \left(\int_0^a + \int_a^\infty\right) |\tilde{K} + \mathcal{C}_3 v^2 + \mathcal{C}_4 v^4|^{\frac{1}{2}h(T-t)C} dv \\ &\le (\epsilon\mathcal{C}_0)^{-\frac{1}{2}(1+h)(T-t)C} \left[\int_0^a \tilde{K}^{\frac{1}{2}h(T-t)C} dv + \int_a^\infty (\mathcal{C}_4 v^4)^{\frac{1}{2}h(T-t)C} dv\right] \\ &= (\epsilon\mathcal{C}_0)^{-\frac{1}{2}(1+h)(T-t)C} \left[\tilde{K}^{\frac{1}{2}h(T-t)C} a - \frac{\mathcal{C}_4^{\frac{1}{2}h(T-t)C}}{2h(T-t)C+1} a^{2h(T-t)C+1}\right] \ . \end{aligned}$$

We have

$$\begin{aligned} I &\le \frac{1}{\pi} \chi_{t-}^{1-\alpha} S_{t-} \mathcal{C}_{(C,G,M,\alpha)} \int_0^\infty |\phi_{T-t}(v - i\alpha)| dv \\ &\le \frac{\mathcal{C}_1 \mathcal{C}_{(C,G,M,\alpha)}}{\pi} \chi_{t-}^{1-\alpha} S_{t-} (\epsilon\mathcal{C}_0)^{-\frac{1}{2}(1+h)(T-t)C} \left[\tilde{K}^{\frac{1}{2}h(T-t)C} a - \frac{\mathcal{C}_4^{\frac{1}{2}h(T-t)C}}{2h(T-t)C+1} a^{2h(T-t)C+1}\right] \ . \end{aligned}$$

Finally we obtain the following estimate:

$$|LRM_t - \Delta_t^{\mathbb{P}^*}| \le \frac{1}{\pi} \frac{\mathcal{C}_{(C,G,M,\alpha)}}{C} (\epsilon\mathcal{C}_0)^{-\frac{1}{2}(1+h)(T-t)C} \left[\tilde{K}^{\frac{1}{2}h(T-t)C} a - \frac{\mathcal{C}_4^{\frac{1}{2}h(T-t)C}}{2h(T-t)C+1} a^{2h(T-t)C+1}\right] \chi_{t-}^{1-\alpha} \ .$$

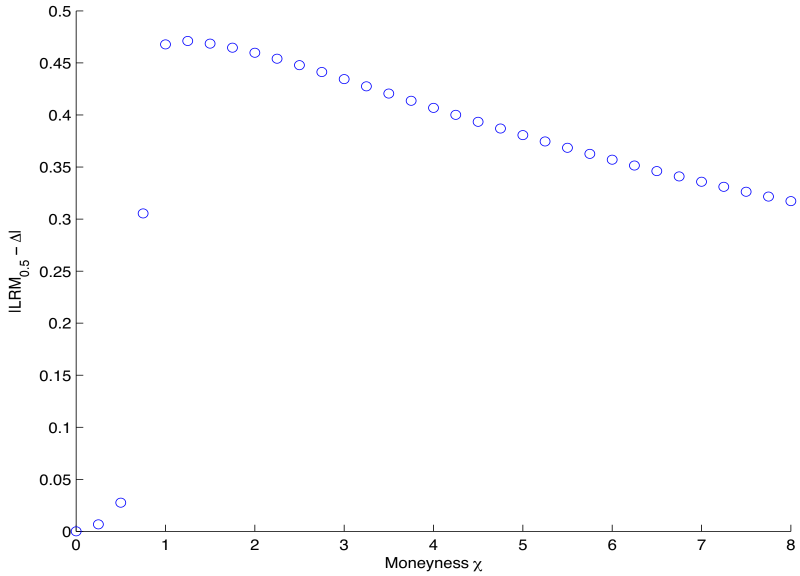
The latter part is the same as the proof of Theorem 4.2.1 □

4.3.1 Numerical results

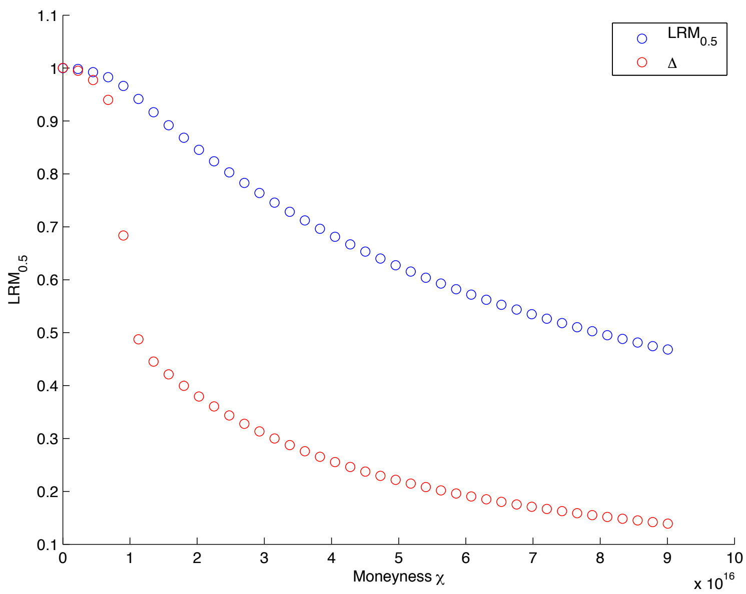
In this subsection, we compute $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ for a VG model with a parameter set based on market data. We use the Nikkei 225 index for March 2014, as in numerical part of VG models in Section 2. We need to set the log price $L_t := \log(S_t/S_0)$, where S_0 is the price on 28 February 2014, which is 14841.07. The parameters C, G, and M are estimated from the mean, variance, and skewness of the log price by using the generalized method of moments and the Levenberg–Marquardt method. The values of C, G and M are $C = 2.469395026815120$, $G = 23.743109051760964$ and $M = 24.903251787154687$. For $G - M \approx -1.16$, this parameter set satisfies Assumption 4.1.1. We take $T = 1$ and $S_{t-} = 14841.07$, that is, $L_{t-} = 0$. We fix t to 0.5, the values of $LRM_{0.5}$ and $\Delta_{0.5}^{\mathbb{P}^*}$ are computed for $K = 10000, 10250, \ldots, 20000$. The computational results are given as Figure 4.2.

4.4 Conclusion

For Merton models and VG models, we have derived inequality estimations for the differences of LRM_t and $\Delta_t^{\mathbb{P}^*}$. Moreover the difference converges to zero when moneyness tends to zero or infinity. We have computed the behaviours of $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ for two cases. The first case is a Merton model with an artificial parameter set. The other is a VG model with a parameter set based on market data. Numerical examples have shown that the behaviours of $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ are different between the two cases. We have deduced four points from the numerical experiments: (i) the differences in VG models have converged faster than the Merton models when moneyness tends to zero or infinity. (ii) Under the given conditions, the values of $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ for the Merton models are larger than that for the VG models. (iii) For the Merton model, $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ has the maximum value around at the money. (iv) For the VG model, the behaviours of $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ are unstable around at the money.

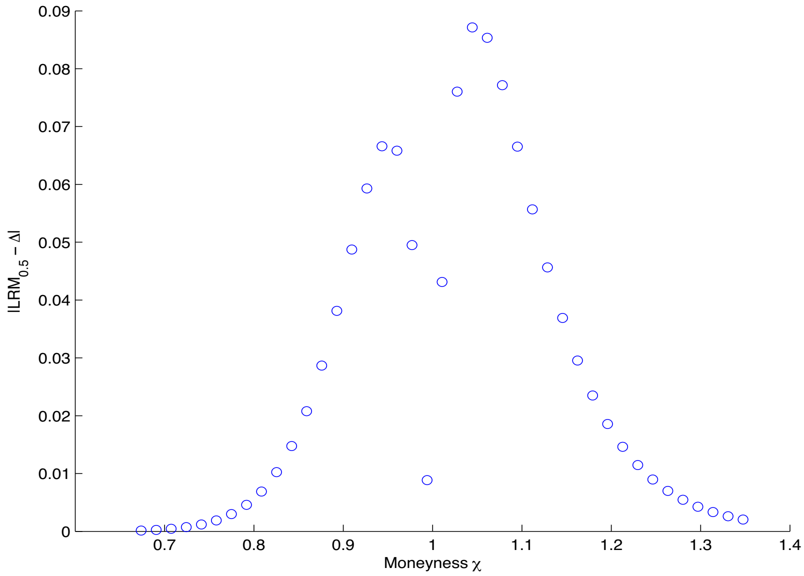


(a) Merton models, plotted $|LRM_{0.5} - \Delta^{\mathbb{P}^*}_{0.5}|$

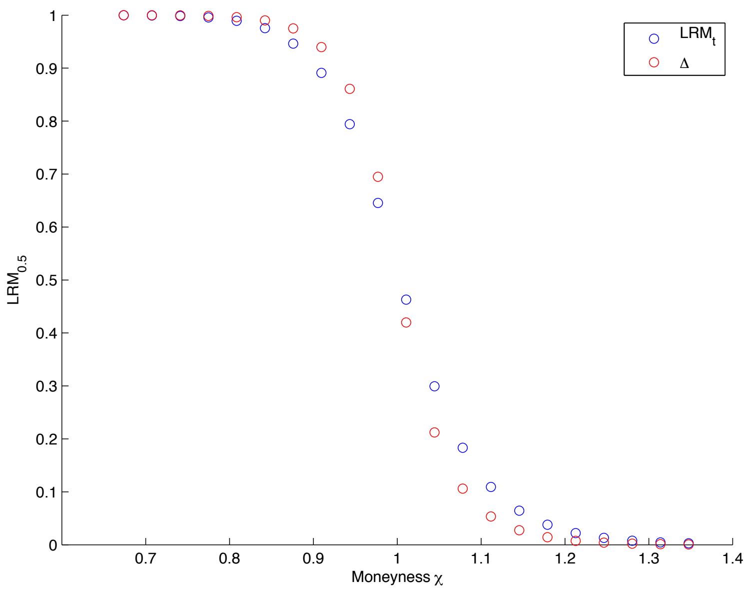


(b) Merton models, plotted $LRM_{0.5}$ and $\Delta^{\mathbb{P}^*}_{0.5}$ separetely

Figure 4.1: $LRM_{0.5}$ and $\Delta^{\mathbb{P}^*}_{0.5}$ for Merton models



(a) VG models, plotted $|LRM_{0.5} - \Delta^{\mathbb{P}^*}_{0.5}|$



(b) VG models, plotted $LRM_{0.5}$ and $\Delta^{\mathbb{P}^*}_{0.5}$ separetely

Figure 4.2: $LRM_{0.5}$ and $\Delta^{\mathbb{P}^*}_{0.5}$ for VG models

Chapter 5

Bibliography

Bibliography

[Arai & Suzuki(2015.0)] T. Arai & R. Suzuki (2015) Local risk-minimization for Lévy markets. Research Report, KSTS/RR–15/002, Keio University.

[Arai & Suzuki(2015.1)] T. Arai & R. Suzuki (2015) Local risk-minimization for Lévy markets. *International Journal of Financial Engineering* **2**, 1550015.

[Arai, Imai & Suzuki (2015)] T. Arai, Y. Imai and R. Suzuki (2015), Numerical local risk minimization for exponential Lévy models. *International Journal of Theoretical and Applied Finance* TBP.

[Bertoin(1998)] G. Bertoin (1998) *Lévy processes*. Cambridge: Cambridge University Press.

[Bonetti *et al.*(2015)] D. Bonetti, D. Leão, A. Ohashi & V. Siqueira (2015) A general multidimensional Monte Carlo approach for dynamic hedging under stochastic volatility, *International Journal of Stochastic Analysis* **2015**, Article ID 863165.

[Carr & Madan(1999)] P. Carr & D. Madan (1999) Option valuation using the fast Fourier transform, *Journal of Computational Finance* **2** (4), 61–73.

[Cont & Tankov(2004)] R. Cont & P. Tankov (2004) *Financial Modelling with Jump Processes*. Chapman & Hall, London.

[Cooley & Tukey (1965)] J. W. Cooley & J. W. Tukey (1965) An algorithm for the machine calculation of complex Fourier series, *Mathematics of Computation* **19**, 297–30.

[Delong & Imkeller (2010)] L. Delong & P. Imkeller (2010) On Malliavin's differentiability of BSDEs with time delayed generators driven by Brownian motions and Poisson random measures, Stochastic Processes and their Applications, **120**, 1748–1775.

[Denkl, et al. (2013)] Stephan Denkl, Martina Goy, Jan Kallsen, Johannes Muhle-Karbe and Arnd Pauwels (2013) On the performance of delta hedging strategies in exponential Lévy models, *Quantitative Finance*, **13**, 1173–1184.

[Ewald *et al.*(2013)] C.O. Ewald, R. Nawar & T. K. Siu (2013) Minimal variance hedging of natural gas derivatives in exponential Lévy models: theory and empirical performance, *Energy Economics* **36**, 97–107.

[Imai & Arai (2015)] Comparison of Local Risk Minimization and Delta Hedging for Exponential Lévy Models, Yuto Imai and Takuji Arai, JSIAM Letters TBP.

[Kang & Lee(2014)] W. Kang & K. Lee (2014) Information on jump sizes and hedging, *Stochastics* **86**, 889–905.

[Kélani & Quittard-Pinon(2014)] A. Kélani & F. Quittard-Pinon (2014) Pricing, hedging and assessing risk in a general Lévy context, *Bankers, Markets & Investors* **131**, 30–42.

[Lee & Song(2007)] K. Lee & S. Song (2007) Insiders' hedging in a jump diffusion model, *Quantitative Finance* **7**, 537–545.

[Leoni *et al.*(2014)] P. Leoni, N. Vandaele & M. Vanmaele (2014) Hedging strategies for energy derivatives, *Quantitative Finance* **14**, 1725–1737.

[Madan & Seneta(1987)] D. B. Madan & E. Seneta (1987) Chebyshev Polynomial Approximations for Characteristic Function Estimation, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **49** (2), 163–169.

[Madan *et al.*(1998)] D. B. Madan, P. Carr & E. Chung (1998) The variance gamma process and option pricing, *European Finance Review* **2**, 79–105.

[Merton(1976)] R. Merton (1976) Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics* **3**, 125–144.

[Schoutens(2003)] W. Schoutens (2003) *Lévy Processes in Finance: Pricing Financial Derivatives*, John Wiley & Sons, Hoboken.

[Schweizer(2001)] M. Schweizer (2001) A Guided Tour through Quadratic Hedging Approaches. In: *Handbooks in Mathematical Finance: Option Pricing, Interest Rates and Risk Management* (E. Jouini, J. Cvitanic & M. Musiela eds.), 538–574. Cambridge: Cambridge University Press.

[Schweizer(2008)] M. Schweizer (2008) Local Risk-Minimization for Multidimensional Assets and Payment Streams, *Banach Center Publications* **83**, 213–229.

[Situ(2005)] R. Situ (2005) *Theory of Stochastic Differential Equations with Jumps and Applications*, Mathematical and Analytical Techniques with Applications to Engineering, Berlin: Springer.

[Solé *et al.*(2007)] J. L. Solé, F. Utzet, J. Vives (2007) Canonical Lévy process and Malliavin calculus, *Stochastic Processes and their Applications* **117**, 165–187.

[Tankov(2010)] P. Tankov (2010) Pricing and hedging in exponential Lévy models: Review of recent results. In: *Paris-Princeton Lectures on Mathematical Finance 2010* (R.A. Carmona, E. Cinlar, I. Ekeland, E. Jouini, J.A. Scheinkman & A.N. Touzi Eds.), 319–359, Berlin: Springer.

[Yang *et al.*(2010)] Z. Yang, C.O. Ewald & K.R. Schenk-Hoppé (2010) An explicit expression to the locally R-minimizing hedge of a European call in the Hull and White model, *Quantitative and Qualitative Analysis in Social Sciences* **4**, 1–18.

Chapter 6

研究業績一覧

論文:

1. Numerical analysis on local risk-minimization for exponential Lévy models, Takuji Arai, Yuto Imai and Ryoichi Suzuki, *International Journal of Theoretical and Applied Finance* (掲載決定済)
2. Comparison of Local Risk Minimization and Delta Hedging for Exponential Lévy Models, Yuto Imai and Takuji Arai, *JSIAM Letters*, Vol. 7 (2015), pp.77–80
3. Lie algebra extensions of current algebras on S^3, Tosiaki Kori and Yuto Imai, *International Journal of Geometric Methods in Modern Physics*, Vol. 12 (2015)
4. Quaternifications and Extensions of Current Algebras on S^3, Tosiaki Kori and Yuto Imai, *Symmetry*, **2015**, 7, 2150-2180.

国際会議

1. Quaternifications and extensions of current algebras on S^3, 9th Japanese German International Workshop on Mathematical Fluid Dynamics, Waseda University, 2013
2. Comparison of local risk minimization and delta hedging strategy for exponential Lévy models, The Fourth Asian Quantitative Finance Conference, Osaka, 2016

国内会議 (査読付き)

1. Numerical analysis on local risk-minimization for exponential Lévy models, 第 42 回ジャフィー大会, 筑波大学, 2015

研究会・全国大会等

1. split 型 Affine Lie 環の拡張としての新しい無限次元 Lie 環の定義とその構造, 2011.11, 量子化の幾何学 2011(早稲田大学)
2. 四元数化した Lie 環の中心拡大, 2012.9, 非可換幾何学と数理物理学 2012(慶応大学)
3. The quarternification of the Lie algebra $Map(S^3, \mathfrak{g})$ and its central extension (for non-professional), 2012.11, TU Darmstadt
4. On the quarternification of the Lie algebra $Map(S^3, \mathfrak{g})$ and its extensions (for professional), 2012.12, TU Darmstadt
5. Lie algebra – Finite and Infinite Lie algebra extensions of current algebras on S^3, Technischen Universit'at Darmstadt (Germany), 2013
6. On the quarternification of the Lie algebra $Map(S^3, \mathfrak{g})$ and its extensions, 日本数学会秋季総合分科会, 愛媛大学, 2013
7. Numerical analysis on local risk-minimization for exponential Lévy models, JAFEE 学会主催第 4 回数理ファイナンス合宿型セミナー, 慶應義塾大学, 2014
8. 幾何 Lévy モデルに対する Local Risk Minimization の数値解析, 日本応用数理学会 2015 年度年会, 金沢大学, 2015

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