Left-right sequences and positive-negative sequences of knot diagrams
(結び目図式の左右列と正負列)

Kuniyuki Takaoka

Contents

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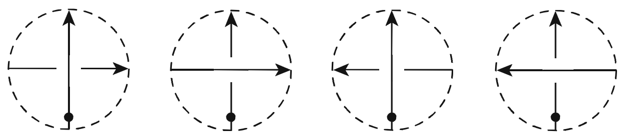
Introduction

In knot theory, knot diagrams play an important role to study and classify knots. We have three points of view for the crossing information on an oriented knot diagram: over/under crossing, left/right crossing and positive/negative crossing. See Fig. 1. We have the following relationship among these three crossing information: if we give a crossing point two of these information, then another is obtained immediately. For instance, if a crossing point is over crossing and left crossing, then it is negative crossing.



O, L, N $\quad$ U, L, P $\quad$ O, R, P $\quad$ U, R, N

Figure 1 : Each dot indicates a base point. Symbols O/U, L/R and P/N mean over/under crossng, left/right crossing and positive/negative crossing, respectively.

In 2012, Higa, Nakanishi, Satoh and Yamamoto defined an OU sequence for a knot diagram, where the OU sequence is obtained from crossing information by reading a sequence of over- and under- crossing points along the orientation of a knot diagram [1]. They studied about sequences which are realized by diagrams of the trefoil knot and characterized the warping polynomials for diagrams of trefoil knot, where the warping polynomials for knot diagrams is introduced by A. Shimizu in [2]. After their study, K. Taniyama suggested the author to study LR sequences and PN sequences for knot diagrams. The LR sequence (respectively PN sequence) for a knot diagram is obtained from crossing information by reading a sequence of left- and right-crossing points (respectively positive- and negative- crossing points) along the orientation. The precise definitions are given later respectively.

The author divides this paper into two parts: Chapter 1 and Chapter 2.

In Chapter 1, we introduce LR sequences for knot diagrams. Since an LR sequence does not reflect any over/under crossing information, we define an LR sequence for an oriented closed curve on the 2-sphere with only finitely many transversal double points. For a given oriented spherical closed curve with n transversal double points, we assign a cyclic word, namely LR sequence, of length $2n$ on two letters L standing left and R standing right by reading the crossing sign so that each crossing point is read once L and once R. The LR number of the curve is the number of appearance of subsequences LR in the LR sequence. We note that there is a related study [3]. We completely determine oriented spherical closed curves whose LR numbers are less than or equal to three.

In Chapter 2, we introduce PN sequences for knot diagrams. For an oriented knot diagram, we define a cyclic word, namely PN sequence, in letters P and N corresponding to positive and negative crossings along the diagram, respectively. We give a necessary and sufficient condition for a PN sequence to be obtained from some knot diagram. Also we prove that any PN sequence of a diagram of a non-trivial knot contains at least four subsequences PP and NN.

Chapter 1

LR number of spherical closed curves

1.1 Introduction

Let L and R be symbols. An *LR pre-sequence* of length $l \in \mathbb{Z}_{\geq 1}$ is a map $\varphi : \{1, 2, \ldots, l\} \longrightarrow \{L, R\}$. Such a pre-sequence is encoded by $\varphi(1)\,\varphi(2) \cdots \varphi(l)$. We define a *cyclic permutation* $\rho : \{1, 2, \ldots, l\} \longrightarrow \{1, 2, \ldots, l\}$ by $\rho(i) = i + 1$ $(i = 1, 2, \ldots, l-1)$ and $\rho(l) = 1$.

Let φ_1 and φ_2 be LR pre-sequences with the same length. If there exists $i \in \mathbb{Z}$ such that $\varphi_2 = \varphi_1 \circ \rho^i$, then we say that φ_1 and φ_2 are *equivalent* and denote it by $\varphi_1 \sim \varphi_2$. It is clear that this is an equivalence relation. We do not distinguish an LR pre-sequence and its equivalence class so long as no confusion occurs. Let S_{2l} be the set of LR pre-sequences of length $2l$ with $\sharp\varphi^{-1}(L) = \sharp\varphi^{-1}(R) = l$. The elements of the quotient set $S_{2l}/\sim$ are called *LR sequences*. For convenience, a consecutive sequence of m $(m \in \mathbb{Z}_{\geq 1})$ copies of L (resp. R) is denoted by L^m (resp. R^m). Then an LR sequence w of length $2l$ can be written as $w = L^{\alpha_1} R^{\beta_1} L^{\alpha_2} R^{\beta_2} \cdots L^{\alpha_n} R^{\beta_n}$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ are positive integers with $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \beta_1 + \beta_2 + \cdots + \beta_n = l$. Note that

$$\begin{aligned} w &= L^{\alpha_1} R^{\beta_1} L^{\alpha_2} R^{\beta_2} \cdots L^{\alpha_n} R^{\beta_n} \\ &= L^{\alpha_1 - 1} R^{\beta_1} L^{\alpha_2} R^{\beta_2} \cdots L^{\alpha_n} R^{\beta_n} L \\ &= \cdots \\ &= R^{\beta_1} L^{\alpha_2} R^{\beta_2} \cdots L^{\alpha_n} R^{\beta_n} L^{\alpha_1} \\ &= R^{\beta_1 - 1} L^{\alpha_2} R^{\beta_2} \cdots L^{\alpha_n} R^{\beta_n} L^{\alpha_1} R \\ &= \cdots . \end{aligned}$$

Then the *LR number* of w, denoted by $lr(w)$, is defined to be n.

We consider an oriented closed curve with finitely many transversal double points without any other singularities in the two-dimensional sphere S^2. In this chapter, by an *oriented spherical closed curve*, we mean such a curve. We consider it up to orientation preserving auto-homeomorphisms on S^2. A transversal double point is called a *crossing point*.

We take a base point except for the crossing points on an oriented spherical closed curve P and trace P along the orientation direction. When we pass a crossing point, the crossing is read to be *left* (resp. *right*) if the curve that one crosses travel from left to right (resp. from right to left). Then we record the symbol L (resp. R) if the crossing is left (resp. right) at each crossing point (see Fig. 1.1). We continue this recording until we return to the base point.

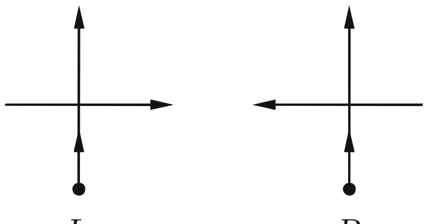


Figure 1.1

Note that we pass each crossing point twice, and once it is read L and once R. Therefore we obtain an LR sequence from the crossing information on P by reading a sequence of left- and right- crossing along the orientation direction of P. It is called an *LR sequence* for P and denoted by w_P. We define the *LR number* of P, denoted by $lr(P)$, to be the LR number of w_P. Namely $lr(P) = lr(w_p)$. See for example Fig. 1.2.

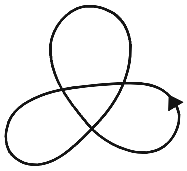


Figure 1.2 : $w_P = LRLRLR$, $lr(P) = 3$.

We define the LR sequence for a simple closed curve in S^2 to be the empty sequence $\emptyset$ and $lr(\emptyset)$ to be zero.

For any P, we obtain an LR sequence for P. Conversely, given an LR sequence w, we can construct a spherical closed curve whose LR sequence coincides with w (Proposition 1.2.2). Therefore, for any positive integer n, there exists an oriented spherical closed curve whose LR number is n.

A spherical closed curve P is said to be *prime* if for any simple closed curve C in S^2 which intersects P transversally in two points, exactly one of subcurves of P cut by C is a simple arc. A spherical closed curve P is said to be *reduced* if P does not have subcurves illustrated in Fig. 1.3.

The main results of this chapter are Theorems 1.2.4, 1.3.2, 1.3.5 and 1.3.8. Theorem 1.2.4 is proved in Section 1.2. In Section 1.3, by Theorem 1.2.4, we describe all prime oriented spherical closed curves whose LR

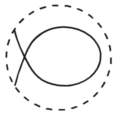


Figure 1.3

numbers are less than or equal to three (Theorems 1.3.2, 1.3.5 and 1.3.8). Then we describe all non-prime oriented spherical closed curves whose LR numbers are less than or equal to three (Theorem 1.3.2, Corollaries 1.3.10 and 1.3.11).

1.2 A concentric circular curve

Let w be an LR sequence. Assume that $w = u_1LRu_2$ or $w = u_1RLu_2$, where u_1 and u_2 are LR subsequences. Then we say that an LR sequence $w' = u_1u_2$ is obtained from w by a *contraction* (cf. [1]).

Lemma 1.2.1. *Let w be an LR sequence, and w' an LR sequence obtained from w by a contraction. If there exists an oriented spherical closed curve P' with $w_{P'} = w'$, then there exists an oriented spherical closed curve P with $w_P = w$.*

Proof. Following the similar lines as [1], we make a local change illustrated in Fig. 1.4 to the arc of P' corresponding to the LR subsequence LR or RL contracted. Then we obtain a new oriented spherical closed curve P with $w_P = w$. □

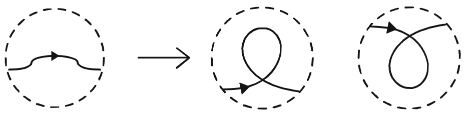


Figure 1.4

Proposition 1.2.2. *For any LR sequence w, there exists an oriented spherical closed curve P such that $w_P = w$.*

Proof. For any LR sequence w, there exists an LR subsequence LR or RL in w. By the induction on the length of w, we obtain an empty sequence $\emptyset$ from w by a finite sequence of contractions. Since $\emptyset$ corresponds to a simple closed curve in S^2, we obtain an oriented spherical closed curve whose LR sequence coincides with w inductively by Lemma 2.2.1. □

Let P be an oriented spherical closed curve. In a small neighborhood of each crossing point of P, we make the following local change to P: delete the crossing point and connect the ends in the only way compatible with orientation as in Fig. 1.5. We call this local change a *smoothing*.

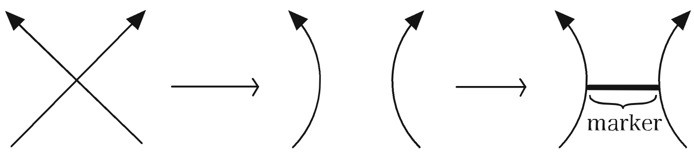


Figure 1.5 : A smoothing, then giving a marker to a crossing point.

When the smoothing has been done at all crossing points, P becomes a set of disjoint simple closed curves in S^2. We call these closed curves *Seifert circles*. Applying smoothings to all crossing points of P and giving line segments called *markers* on the sites of crossing points, we obtain a figure called a *Seifert diagram* for P, which is denoted by $S_d(P)$. The Seifert diagram for P is constructed by the Seifert circles of P and the markers corresponding to the crossing points of P.

In the same way as Section 1.1, we will obtain the LR sequence for $S_d(P)$. First we determine how to circulate $S_d(P)$ in the following: tracing a Seifert circle γ of $S_d(P)$, if we encounter a marker on the left- (resp. right-) hand side for the direction of travelling, then we leave γ, move along the marker and go into a Seifert circle on the left- (resp. right-) hand side of γ along the marker. A marker is said to be *left* (resp. *right*) for γ if we pass the marker and take to a Seifert circle on left- (resp. right-) hand side of γ. A left marker is labeled L and a right marker R (see Fig. 1.6).

Then, circulating $S_d(P)$, we obtain the LR sequence from marker information of $S_d(P)$. Notice that since we can restore $S_d(P)$ to P, the LR sequence for $S_d(P)$ coincides with w_P. Therefore we may treat $S_d(P)$ instead of P in some cases.

Figure 1.6 : A left (resp. right) crossing point corresponds to the left (resp. right) marker.

Let $\mathcal{S}(P)$ be the set of Seifert circles of P in S^2. Define

$$\begin{aligned} I_+(\mathcal{S}(P)) &:= \{\gamma \in \mathcal{S}(P) \mid \gamma \text{ is a clockwise innermost circle in } \mathcal{S}(P)\}, \\ I_-(\mathcal{S}(P)) &:= \{\gamma \in \mathcal{S}(P) \mid \gamma \text{ is a counterclockwise innermost circle in } \mathcal{S}(P)\}, \\ I(\mathcal{S}(P)) &:= \{\gamma \in \mathcal{S}(P) \mid \gamma \text{ is an innermost circle in } \mathcal{S}(P)\}. \end{aligned}$$

Then we have $I(\mathcal{S}(P)) = I_+(\mathcal{S}(P)) \sqcup I_-(\mathcal{S}(P))$.

We say that P is a *concentric circular curve* if $\sharp I_+(\mathcal{S}(P)) = \sharp I_-(\mathcal{S}(P)) = 1$.

Let $c(\gamma)$ be the number of markers which the innermost circle γ touches. We define

$$i_{c_+}(P) := \sum_{\gamma \in I_+(\mathcal{S}(P))} c(\gamma), \quad i_{c_-}(P) := \sum_{\gamma \in I_-(\mathcal{S}(P))} c(\gamma).$$

Lemma 1.2.3. *Let P be an oriented spherical closed curve. Then*
(1) $lr(P) \geq i_{c_+}(P)$,
(2) $lr(P) \geq i_{c_-}(P)$ *and*
(3) $lr(P) \geq$ *max* $\{i_{c_+}(P),\ i_{c_-}(P)\}$.

Proof. (1) Let $S_d(P)$ be the Seifert diagram for P, and $m_1, m_2, \dots, m_n$ the markers which a clockwise innermost circle γ touches in $S_d(P)$ (see Fig. 1.7). Pass through m_i, trace the arc of γ and pass again through m_{i+1} (m_1 when $i = n$), then we have an LR subsequence RL obtained from the marker information.

Hence, if a clockwise innermost circle touches n markers, then the LR sequence for $S_d(P)$ contains at least n LR subsequences RL. Note that no two clockwise innermost circles touch a common marker. Therefore $lr(P) \geq i_{c_+}(P)$.

By a similar argument, we have (2). Then (3) follows immediately. □

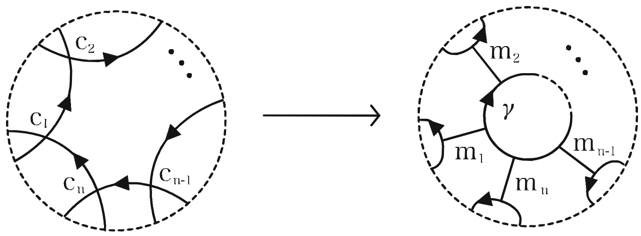


Figure 1.7 : To each crossing point c_i $(1 \leq i \leq n)$, we apply smoothing and giving a maker.

Theorem 1.2.4. *Let P be a reduced oriented closed spherical curve. If the LR number of w_P is less than or equal to three, then P is a concentric circular curve.*

Proof. If P is not a concentric circular curve, then $\sharp I(\mathcal{S}(P)) \geq 3$. Namely, $\sharp I_+(\mathcal{S}(P)) \geq 2$ or $\sharp I_-(\mathcal{S}(P)) \geq 2$. On the other hand, since P is reduced, we have $c(\gamma) \geq 2$ for all $\gamma \in I(\mathcal{S}(P))$. Therefore $i_{c_+}(P) \geq 4$ or $i_{c_-}(P) \geq 4$. Thus $lr(P) \geq 4$ by Lemma 1.2.3. □

1.3 Spherical closed curves with LR number 1, 2 or 3

In this section we determine prime oriented spherical closed curves whose LR numbers are less than or equal to three. We need some lemmas in order to construct such curves.

Let P be a concentric circular curve in S^2, and $S_d(P)$ the Seifert diagram for P. There are four patterns of the parts of the circulation of $S_d(P)$ as illustrated in Fig. 1.8. They are between two or three adjacent Seifert circles of $S_d(P)$.

A pattern on the left-hand side of Fig. 1.8 is called *type A*, one on the second from the left *type B*, and the others *type C*. We obtain an LR subsequence LR or RL from marker information of type A or B, and LL or RR from that of type C. Then the LR number of $S_d(P)$ is n if and only if $S_d(P)$ has exactly n type A patterns, and then $S_d(P)$ has exactly n type B patterns. Therefore we may count the number of patterns of type A or B in $S_d(P)$ to calculate the LR number of P.

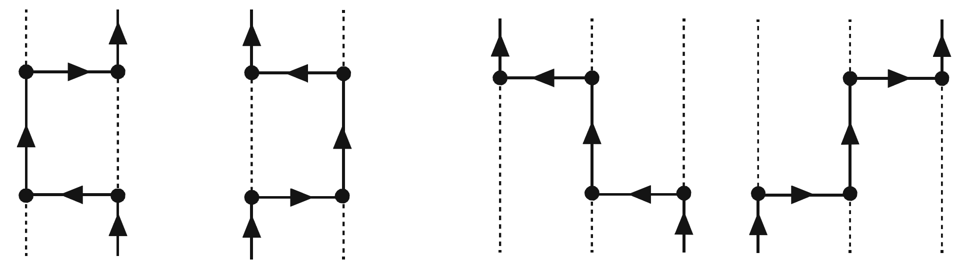


Figure 1.8

Suppose that $S_d(P)$ has k Seifert circles $\gamma_1, \ldots, \gamma_k$ where $k \geq 2$. Let γ_1 be the counterclockwise innermost circle, and γ_k the clockwise innermost circle. Assume that γ_i and γ_{i+1} are adjacent $(1 \leq i \leq k-1)$. Such Seifert circles are called the *Seifert circles of $S_d(P)$ arranged in order.*

Lemma 1.3.1. *Let P be a concentric circular curve, and $S_d(P)$ the Seifert diagram for P. Let $\gamma_1, \ldots, \gamma_k$ be the Seifert circles of $S_d(P)$ arranged in order where $k \geq 2$. If there exists γ_i $(1 \leq i \leq k-1)$ such that the number of the markers between γ_i and γ_{i+1} is n, then the LR number of P is greater than or equal to n.*

Proof. Let $m_1, \ldots, m_n$ be n markers between γ_i and γ_{i+1} (see Fig. 1.9).

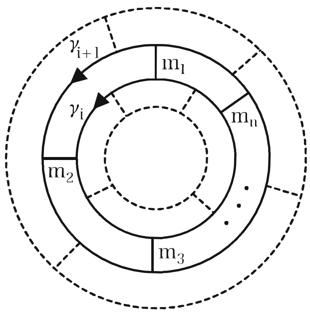


Figure 1.9 : In the neighborhood of γ_i and γ_{i+1}.

Assume that we leave γ_i, pass m_1 and take to γ_{i+1} when circulating the $S_d(P)$. Since m_1 is a right marker for γ_i, we obtain an LR subsequence R. In order to turn back to γ_i again, we need to pass m_h $(1 \leq h \leq n)$ in direction of γ_i once. At this time, m_h is a left marker for γ_{i+1}. Then we obtain an LR subsequence φ of length l ($l \geq 2$), which contains at least one R and one L in this process. Therefore there exists j ($j < l$) such that $\varphi(j) = R$ and $\varphi(j+1) = L$. This shows that we have a type B pattern in the area between γ_i and γ_k.

Applying a similar argument for each marker, we have at least n type B patterns in the area between γ_i and γ_k. Therefore $lr(P) \geq n$. □

Theorem 1.3.2. *Let P be an oriented spherical closed curve. If the LR number of P is one, then P is one of the following curves illustrated in Fig.* 1.10*. In particular a spherical closed curve in the far left-hand side of Fig.* 1.10 *is prime and the others are not prime.*

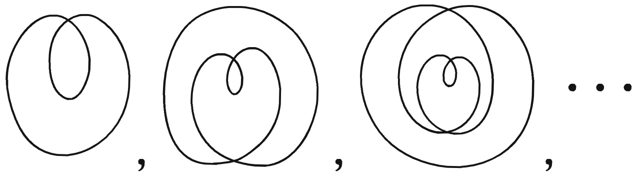


Figure 1.10

Proof. By Lemma 1.2.3, $i_{c_+}(P) = i_{c_-}(P) = 1$. Thus $\sharp I_+(\mathcal{S}(P)) = \sharp I_-(\mathcal{S}(P)) = 1$, that is, P is a concentric circular curve. Let $\gamma_1, \ldots, \gamma_k$ be the Seifert circles of $S_d(P)$ arranged in order where $k \geq 2$. By Lemma 1.3.1, we see that for any γ_i ($1 \leq i \leq k-1$), the number of markers between γ_i and γ_{i+1} is one. Therefore we obtain the following Seifert diagrams for P illustrated in Fig. 1.11. □

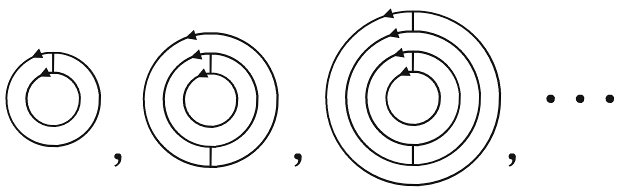


Figure 1.11

Lemma 1.3.3. *Let P be a concentric circular curve, and $S_d(P)$ a Seifert diagram for P. Let $\gamma_1, \ldots, \gamma_k$ be the Seifert circles of $S_d(P)$ arranged in* *order where $k \geq 3$. If there exists γ_i $(1 \leq i \leq k-1)$ such that the number of the markers between γ_i and γ_{i+1} is one, then P is not prime.*

Proof. The marker between γ_i and γ_{i+1} represents a crossing point c in P such that $P \setminus \{c\}$ is disconnected. Therefore P is not prime. □

Let $\gamma_1, \ldots, \gamma_k$ be the Seifert circles of $S_d(P)$ arranged in order where $k \geq 3$. Let $m_1^i, \ldots, m_p^i$ be p markers between γ_i and γ_{i+1}, and $m_1^{i-1}, \ldots, m_q^{i-1}$ q markers between γ_{i-1} and γ_i where $2 \leq i \leq k-1$ and p and q are positive integers. We define the endpoints of the marker

$$e_h^i := m_h^i \cap \gamma_i \ (1 \leq h \leq p), \quad e_j^{i-1} := m_j^{i-1} \cap \gamma_i \ (1 \leq j \leq q).$$

Tracing γ_i, we obtain a cyclic sequence of endpoints. We denote it by $E(\gamma_i)$.

Lemma 1.3.4. *Let P be a concentric circular curve, and $S_d(P)$ the Seifert diagram for P. Let $\gamma_1, \ldots, \gamma_k$ be the Seifert circles of $S_d(P)$ arranged in order where $k \geq 3$. If there exists a Seifert circle γ_i $(2 \leq i \leq k-1)$ such that $E(\gamma_i)$ coincides $e_1^i e_2^i \cdots e_p^i e_1^{i-1} e_2^{i-1} \cdots e_q^{i-1}$ $(p, q \in \mathbb{N})$, then P is not prime.*

Proof. There exists a simple closed curve α that intersects $S_d(P)$ transversally in two points as illustrated in Fig. 1.12. Therefore P restored from $S_d(P)$ is not prime. □

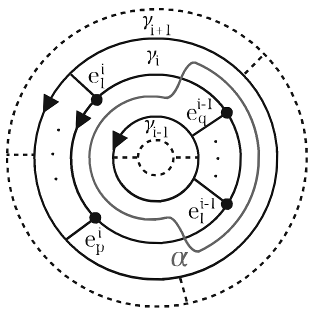


Figure 1.12

Now we construct prime oriented spherical closed curves whose LR numbers are two.

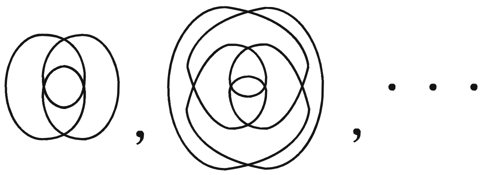


Figure 1.13

Theorem 1.3.5. *Let P be a prime oriented spherical closed curve. If the LR number of P is two, then P is one of the following curves illustrated in Fig.* 1.13.

Proof. It is clear that prime spherical closed curves with two or more crossing points are reduced. Thus P is a concentric circular curve by Theorem 1.2.4.

We deal with $S_d(P)$ instead of P. By Lemmas 1.3.1 and 1.3.3, any Seifert circles of $S_d(P)$ have just two markers. Furthermore, $S_d(P)$ does not contain the portions illustrated in Fig. 1.12 by Lemma 1.3.4. Therefore we obtain the following Seifert diagram for P illustrated in Fig. 1.14. $\square$

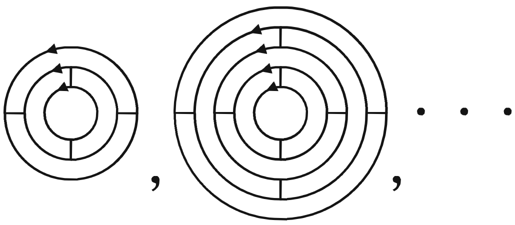


Figure 1.14

Lastly we construct prime oriented spherical closed curves whose LR numbers are three.

Lemma 1.3.6. *Let P be a concentric circular curve, $S_d(P)$ the Seifert diagram for P, and $\gamma_1, \ldots, \gamma_k$ the Seifert circles of $S_d(P)$ arranged in order where $k \geq 3$. If there exists γ_i $(2 \leq i \leq k-1)$ such that $S_d(P)$ has p markers between γ_i and γ_{i+1} and q markers between γ_{i-1} and γ_i, and $E(\gamma_i)$ contains j subsequences $e_h^{i-1} e_{h+1}^{i-1}$ $(h,\ j \leq q-1)$, then the LR number of P is greater than or equal to $p + j$.*

Proof. We will count the number of type B patterns in $S_d(P)$ to calculate the LR number of P.

Since the number of the markers between γ_i and γ_{i+1} is p, $S_d(P)$ has at least p type B patterns in the area between γ_i and γ_k. Then $lr(P) \geq p$. In addition, since $E(\gamma_i)$ contains j subsequences $e_h^{i-1} e_{h+1}^{i-1}$, $S_d(P)$ has j type B patterns in the area between γ_{i-1} and γ_i. Therefore $lr(P) \geq p + j$. □

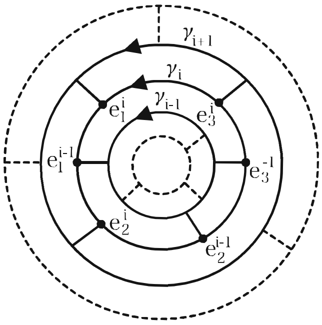


Figure 1.15 : The case of $p = q = 3$ and $E(\gamma_i)$ contains a sequence $e_2^{i-1} e_3^{i-1}$.

Lemma 1.3.7. *Let P be a concentric circular curve, $S_d(P)$ the Seifert diagram for P, and $\gamma_1, \ldots, \gamma_k$ the Seifert circles of $S_d(P)$ arranged in order where $k \geq 3$. If there exists γ_{i_1} and γ_{i_2} $(2 \leq i_2 < i_1 \leq k-1)$ such that $S_d(P)$ has p_1 markers between γ_{i_1} and γ_{i_1+1}, p_2 markers between γ_{i_2} and γ_{i_2+1}, and p_3 markers between γ_{i_2-1} and γ_{i_2} $(p_1 > p_2,\ p_3 > p_2)$, then the LR number of P is greater than or equal to $p_1 - p_2 + p_3$.*

Proof. We will count the number of type B patterns in $S_d(P)$ to calculate the LR number of P. Since the number of the markers between γ_{i_1} and γ_{i_1+1} is p_1, $S_d(P)$ has at least p_1 type B patterns in the area between γ_{i_1} and γ_k. Then $lr(P) \geq p_1$. In addition, $S_d(P)$ has $p_3 - p_2$ type B patterns in the area between γ_{i_2-1} and γ_{i_2}. Therefore $lr(P) \geq p_1 - p_2 + p_3$. □

Theorem 1.3.8. *Let P be a prime oriented spherical closed curve. If the LR number of P is three, then P is obtained from $T(3, n)$ by replacing one of the areas A, B and C with $S(2, l)$, and by replacing the area D with $S(2, m)$ as in Fig.* 1.16 *where $n \geq 2$ and $l, m \geq 0$.*

Proof. Since P has more than two crossing points, P is reduced. Thus, by Theorem 1.2.4, P is a concentric circular curve.

We deal with $S_d(P)$ instead of P. By Lemmas 1.3.1 and 1.3.3, any Seifert circles of $S_d(P)$ have two or three markers. In addition, by Lemma 1.3.4,

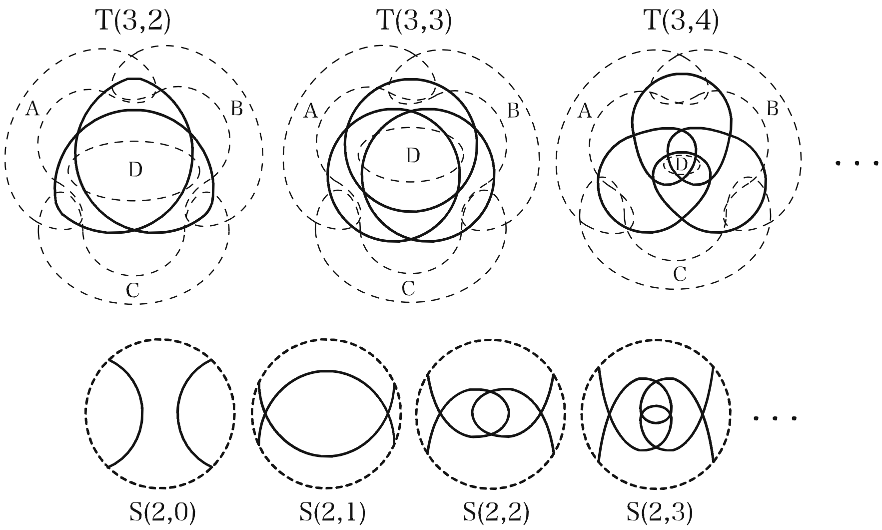


Figure 1.16

$S_d(P)$ does not contain the portions illustrated in Fig. 1.12. Furthermore, $S_d(P)$ does not satisfy the conditions of Lemmas 1.3.6 and 1.3.7. By these facts, we have the desired conclusion. □

Some examples of prime oriented spherical closed curves with LR number three are illustrated in Fig. 1.17. The curve (1) is $T(3,2)$, (2) is $T(3,4)$, (3) is $T(3,5)$, (4) is obtained from $T(3,4)$ by replacing D with $S(2,2)$, and (5) is obtained from $T(3,4)$ by replacing both C and D with $S(2,2)$.

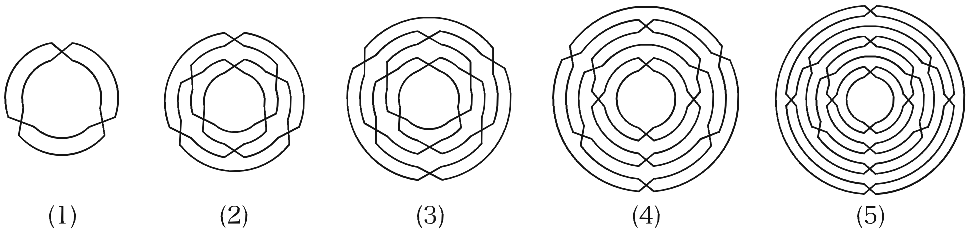


Figure 1.17

Let P_1 and P_2 be spherical closed curves. Suppose that P_1 and P_2 are disjoint. We make the following local change to P_1 and P_2. Find a disk

$D \subset S^2$ such that $D \cap P_i = \partial D \cap P_i$ is a simple arc for $i = 1, 2$.

Let $P = (P_1 \cup P_2 \cup \partial D) \backslash \mathrm{int}(\partial D \cap (P_1 \cup P_2))$. Then P is said to be a *composition* of P_1 and P_2.

Lemma 1.3.9. *Let P_1 and P_2 be oriented spherical closed curves. If the LR number of P_1 is n_1 and that of P_2 is n_2, then the LR number of a composition of P_1 and P_2 is $n_1 + n_2 + 1$, $n_1 + n_2$ or $n_1 + n_2 - 1$.*

Proof. Let w_{P_1} be the LR sequence for P_1, and w_{P_2} the LR sequence for P_2. The proof immediately follows by considering to put w_{P_2} between the letters of w_{P_1}. □

By Theorems 1.3.5 and 1.3.8, and Lemma 1.3.9, we describe how to construct non-prime oriented spherical closed curves whose LR numbers are two or three.

Corollary 1.3.10. *Let P be a non-prime oriented spherical closed curve whose LR number is two. Then P is one of the following curves.*

(1) *The curve P is a composition of two oriented spherical closed curves whose LR numbers are one.*

(2) *The curve P is a composition of a prime oriented spherical closed curve whose LR number is two and at most four oriented spherical closed curves whose LR numbers are one.*

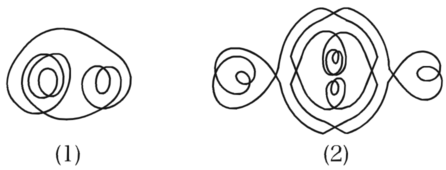


Figure 1.18 : Examples of cases (1) and (2) of Corollary 1.3.10.

Corollary 1.3.11. *Let P be a non-prime oriented spherical closed curve whose LR number is three. Then P is one of the following curves.*

(1) *The curve P is a composition of at most three oriented spherical closed curves whose LR numbers are one.*

(2) *The curve P is a composition of an oriented spherical closed curve whose LR number is two and an oriented spherical closed curve whose LR number is one.*

(3) *The curve P is a composition of two oriented spherical closed curves whose LR numbers are two.*

(4) *The curve P is a composition of a prime oriented spherical closed curve whose LR number is three and at most six oriented spherical closed curves whose LR numbers are one.*

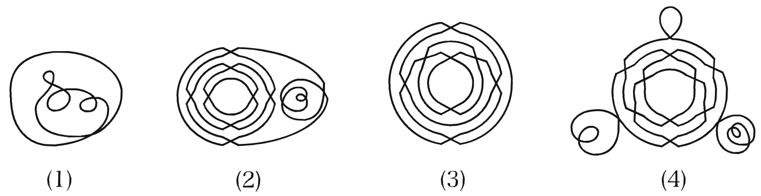


Figure 1.19 : Examples of cases (1), (2), (3) and (4) of Corollary 1.3.11.

Chapter 2

PN sequences obtained from signs of crossings of knot diagrams

2.1 Introduction

A *PN sequence* is a cyclic word $w = X_1X_2 \cdots X_n$ ($n \in \mathbb{N}$) in letters P and N such that both of the numbers of P's and N's in w are even. We denote by $|w|$ the length n of w, which is even by definition. Let n_1 and n_2 be the number of P's in $\{X_1, X_3, \ldots, X_{n-1}\}$ and $\{X_2, X_4, \ldots, X_n\}$, respectively. We denote by $\Delta(w)$ the difference $|n_1 - n_2|$. For convenience, the empty sequence $\emptyset$ is regarded as a PN sequence with $|\emptyset| = 0$ and $\Delta(\emptyset) = 0$.

We use the following notations for subsequences of w:

(i) $P^m = \underbrace{PP \cdots P}_{m}$ and $N^m = \underbrace{NN \cdots N}_{m}$.

(ii) $(PN)^m = \underbrace{PNPN \cdots PN}_{2m}$ and $(NP)^m = \underbrace{NPNP \cdots NP}_{2m}$.

We say that a PN sequence w' is obtained from w by a *contraction* if w' is obtained by deleting a subsequence PP or NN in w. An *interval number* of a PN sequence w is defined to be the number of the subsequences PP and NN in w, and denoted by $I(w)$. For example, we have $I(P^2) = 2$, $I(N^6) = 6$, and $I(P^2N^2P^2N^2) = 4$. We remark that $I(w)$ is always even (Lemma 2.3.1).

Let D be an oriented knot diagram in the 2-sphere. We take a base point on D except for the crossings and trace D from the base point with respect to the orientation of D. When we meet a positive or negative crossing (see Fig. 2.1), we record the letter P or N, respectively, so that we obtain a word $w(D)$ in letters P and N. Since we pass each crossing twice, the word $w(D)$ is a PN sequence.

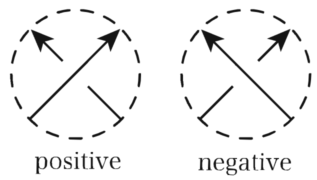


Figure 2.1 : A positive crossing and a negative crossing.

We remark that $w(D)$ is a cyclic word and it is independent of a particular choice of base points. If a pair of letters in $w(D)$ correspond to the same crossing, it is called a *realized pair* for D. The interval number $I(D)$ of D is defined by $I(D) = I(w(D))$. In particular, if $I(D) = 0$, then D has no crossing, and hence, it is a diagram of the trivial knot (Lemma 2.3.2).

The main results of this chapter are Theorems 2.1.1 and 2.1.2.

Theorem 2.1.1. *For a PN sequence w, the following are equivalent.*

(1) *There exists an oriented diagram D of the trivial knot such that $w(D) = w$.*
(2) *There exists a diagram D of some oriented knot such that $w(D) = w$.*
(3) $\Delta(w) = 0$.
(4) *There is a finite sequence of contractions from w to $\emptyset$.*

Theorem 2.1.2. *Let D be an oriented knot diagram. If $I(D) = 2$, then D is a diagram of the trivial knot.*

For an oriented knot K in the 3-sphere, we define the interval number of K by

$$I(K) := \min\{I(D) \mid D \text{ is an oriented diagram of } K\}.$$

By Theorem 2.1.2, we obtain the following immediately.

Corollary 2.1.3. *If K is a non-trivial knot, then $I(K) \geq 4$ holds.*

For instance, let D be a diagram of the figure-eight knot 4_1 with four crossings. Since $w(D) = P^2N^2P^2N^2$ holds, we have $I(4_1) = 4$ by Corollary 2.1.3. We do not know whether there exists an oriented knot K with $I(K) = 6$.

In Section 2.2, we prepare two lemmas on contractions (Lemmas 2.2.1 and 2.2.2) and prove Theorem 2.1.1. In Section 2.3, we characterize a PN sequence $w(D)$ with $I(w(D)) = 2$ (Lemma 2.3.3) and prove Theorem 2.1.2.

2.2 Proof of Theorem 2.1.1

To prove Theorem 2.1.1, we prepare the following lemmas.

Lemma 2.2.1. *Let w and w' be PN sequences. Suppose that w' is obtained from w by a contraction. Then we have the following.*

(i) $\Delta(w') = \Delta(w)$.
(ii) *If there exists an oriented diagram D' of a knot K with $w(D') = w'$, then there exists another oriented diagram D of K with $w(D) = w$.*

Proof. (i) This follows by definition immediately.

(ii) Following the similar lines as [1], we perform a Reidemeister move I as shown in Fig. 2.2 for the arc of D' corresponding to the contraction of PP or NN so that we obtain a diagram D of the same knot K with $w(D) = w$. □

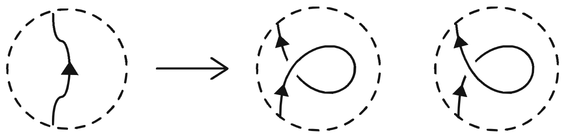


Figure 2.2

Lemma 2.2.2. *For any PN sequence w, there is a PN sequence w' which satisfies the following.*

(i) *w' is obtained from w by a finite sequence of contradictions.*

(ii) *$w' = (PN)^k$ with $k = \Delta(w)$.*

Proof. For $w = \emptyset$, we have $w' = \emptyset$. Assume that $|w| > 0$. If $I(w) > 0$, then w contains PP or NN by definition so that we can perform a contraction for w. By repeating this process, we obtain a finite sequence of PN sequences:

$$w = w_0, w_1, w_2, \ldots, w_s = w',$$

with $|w_{i+1}| = |w_i| - 2$ and $I(w') = 0$, where w_{i+1} is obtained from w_i by a contraction. Since w' contains neither PP nor NN, we have $w' = (PN)^k$ for some $k \geq 0$. Therefore, it follows by Lemma 2.2.1 (i) that $\Delta(w) = \Delta(w') = |k - 0| = k$. □

We are ready to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. (1) $\Longrightarrow$ (2). This is trivial.

(2) $\Longrightarrow$ (3). Suppose that $w = w(D)$ holds for a diagram D of some oriented knot. Since the number of letters between any realized pair X_i and X_j of $w(D) = X_1 X_2 \cdots X_n$ is even, i and j have opposite parities (Lemma 2.3.4). Thus we have $\Delta(w) = 0$ by definition.

(3) $\Longrightarrow$ (4). By Lemma 2.2.2, there is a PN sequence $w' = (PN)^k$ with $k = \Delta(w)$ such that w' is obtained from w by a finite sequence of contractions. Since $\Delta(w) = 0$, we have $w' = \emptyset$.

(4) $\Longrightarrow$ (1). The empty sequence $\emptyset$ defines the diagram with no crossing which presents the trivial knot. By Lemma 2.2.1(ii), w is realized by some oriented diagram of the trivial knot. □

2.3 Proof of Theorem 2.1.2

Lemma 2.3.1. *For any PN sequence w, the interval number $I(w)$ is even.*

Proof. Let m_1 and m_2 be the numbers of the subsequences PN and NP in w, respectively. Then it holds by definition that $I(w) = |w| - (m_1 + m_2)$. Since w is cyclic, we have $m_1 = m_2$. Since $|w|$ is even, so is $I(w)$. $\square$

Lemma 2.3.2. *Let D be an oriented knot diagram. If $I(D) = 0$, then D is a diagram of the trivial knot.*

Proof. By the definition of $I(D)$, we have $w(D) = (PN)^k$ for some $k \geq 0$. Since $k = \Delta(w(D)) = 0$ by Theorem 2.1.1, we have $k = 0$ and $w(D) = \emptyset$ so that D has no crossing. $\square$

Lemma 2.3.3. *Let D be an oriented knot diagram. If $I(D) = 2$, then $w(D)$ is coincident with one of the following.*
(i) $(PN)^m P^2 (NP)^m$ $(m \geq 0)$.
(ii) $(NP)^m N^2 (PN)^m$ $(m \geq 0)$.
(iii) $(PN)^m (NP)^m$ $(m \geq 1)$.

Proof. Since $w(D)$ has two subsequences PP and/or NN, it is coincident with P^2, N^2, or one of the following for some $m, \ell \geq 1$ with $m \equiv \ell \pmod 2$.
(i) $P^2 \underbrace{NPN\cdots PN}_{2m-1} P^2 \underbrace{NPN\cdots PN}_{2\ell-1}$.
(ii) $N^2 \underbrace{PNP\cdots NP}_{2m-1} N^2 \underbrace{PNP\cdots NP}_{2\ell-1}$.
(iii) $P^2 \underbrace{NPN\cdots NP}_{2m-2} N^2 \underbrace{PNP\cdots PN}_{2\ell-2}$.
(iv) $P^3 \underbrace{NPN\cdots PN}_{4m-1}$.
(v) $N^3 \underbrace{PNP\cdots NP}_{4m-1}$.
We remark that the number of P's and N's are even, respectively. Since we have $\Delta(w(D)) = |m - \ell|$ for (i)$-$(iii) and $2m$ for (iv) and (v), it follows by Theorem 2.1.1 that $m = \ell$ holds for (i)$-$(iii) and that the cases (iv) and (v) do not happen. $\square$

Let $c(D)$ be the number of crossings of D. Then we have $c(D) = 2m+1$ for (i) and (ii), and $2m$ for (iii). The following is a well-known fact.

Lemma 2.3.4. *Let $w(D) = X_1 X_2 \cdots X_n$ be the PN sequence of an oriented knot diagram D, where $X_1, X_2, \ldots, X_n \in \{P, N\}$. If X_i and X_j are a realized pair for D, then i and j have opposite parities.*

We are ready to prove Theorem 2.1.2.

Proof of Theorem 2.1.2. We prove the theorem by induction on $c(D)$. We remark that any diagram D with $c(D) \leq 2$ presents the trivial knot.

Assume that $c(D) > 2$. We divide the PN sequence $w(D)$ in Lemma 2.3.3 into halves as follows.

(i) $(\underbrace{PNP \cdots NP}_{2m+1})(\underbrace{PNP \cdots NP}_{2m+1})$.

(ii) $(\underbrace{NPN \cdots PN}_{2m+1})(\underbrace{NPN \cdots PN}_{2m+1})$.

(iii) $(\underbrace{PN \cdots PN}_{2m})(\underbrace{NP \cdots NP}_{2m})$.

By Lemma 2.3.4, there is no realized pair in any half subsequence.

Now we divide the diagram D into two arcs A_1 and A_2 which provide the first and latter half subsequences of $w(D)$, respectively. Then each A_i has no self-crossing ($i = 1, 2$). We may assume that D is located in a 2-sphere, which can be divided into two disks E_1 and E_2 such that $D \cap E_1$ consists of A_1 and short arcs transverse to A_1. See the left of Fig. 2.3. Let p and q be the endpoints of A_1.

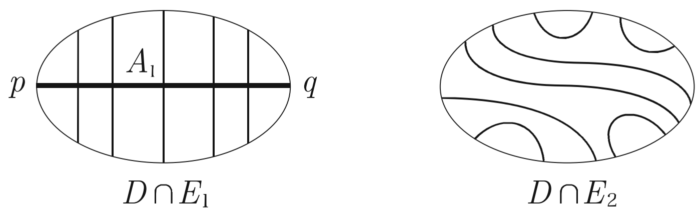


Figure 2.3

Since $D \cap E_2$ consists of disjoint union of embedded arcs as in the right of the figure, there is an innermost arc of $D \cap E_2$, say α, such that one of the disk components of $E_2 \setminus \alpha$ misses any arcs of $D \cap E_2$.

If one of the endpoints of α is p or q, then we can perform a Reidemeister move I containing α to remove a crossing from A_1. The obtained diagram D' of K satisfies $c(D') = c(D) - 1$ and $I(D') = 2$. If the endpoints of α are neither p nor q, then we can perform a Reidemeister move II containing α to cancel a pair of crossings with opposite signs form A_1. See Fig. 2.4. The obtained diagram D'' of K also satisfies $c(D'') = c(D) - 2$ and $I(D'') = 2$. In any case, D presents the trivial knot by the assumption. $\square$

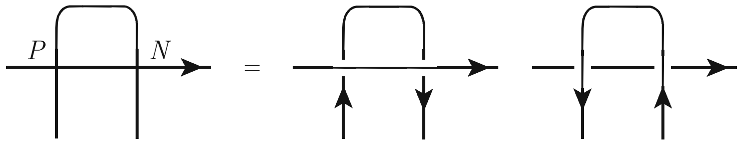


Figure 2.4

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[3] R. Shinjo and K. Tanaka, On the inclusive relation of three properties of knot diagrams, Titles and Short Summaries of the Talks, MSJ 2012 autumn meeting at Kyushu University.

A list of papers by Kuniyuki Takaoka

1. K. Takaoka, LR number of spherical closed curves, *Tokyo J. Math* **38** (2015) no. 2, 491-503.

2. K. Takaoka, PN sequences obtained from signs of crossings of knot diagrams, to appear in *J. Knot Theory Ramifications*.