Verified numerical computation for elliptic partial differential equations and related problems

楕円型偏微分方程式に対する精度保証付き数値計算と関連する問題

February 2017

Waseda University
Graduate School of Fundamental Science and Engineering
Department of Pure and Applied Mathematics,
Research on Numerical Analysis

Kazuaki TANAKA
田中 一成

Acknowledgements

First and foremost, I wish to express my deepest gratitude to Professor Oishi, who has guided me in authoring this work. In a career spanning more than twenty years, Professor Oishi has made numerous achievements in the field of verified numerical computation and continues to be active. In spring of 2012, he was awarded a Purple Ribbon Medal by the Emperor. Additionally, he has produced a great number of knowledgeable researchers in the same field, and in this respect as well, he has contributed greatly to its development. I am certain that I have been able to carry out my present research as a result of the favourable environment within which I have received such excellent guidance.

Next, I would like to extend my thanks to Professor Kashiwagi and Professor Tanaka who took on the positions of vice-examiners on my thesis. My research laboratory has been close to Professor Kashiwagi, and since around the time my lab was assigned in my fourth year of study, in addition to an immeasurable degree of advice on my research, I have received a great deal of support with my student life from Professor Kashiwagi. I must mention that in my recent numerical calculation program, I have relied upon fundamental parts of the KV-Library [8] developed by Professor Kashiwagi with regards to interval arithmetic among other things. Moreover, I remain truly grateful for the wealth of advice I have received on its use. Professor Tanaka's fundamental descriptions of elliptic differential equations in particular have contributed greatly to my methods for verifying the positivity of solutions. The advice I have received from the perspective of an analyst of differential equations has not only benefitted the quality of this paper, but will also doubtlessly play a key part in my future research.

I also thank all the teachers, seniors, and others who provided me with comments and advice on my research as a part of Laboratory Oishi. In particular, I want to thank Assistant Professor Sekine and Professor Kashiwagi who day in and day out took the time to debate and discuss with me one on one. Assistant Professor Morikura supported me in various aspects of my research life with regards to the necessary administrative procedures involved in conducting research, etc. Professor Rump, who has come to Japan as a visiting professor at Laboratory Oishi in recent years, also gave me helpful advice on my research. INTLAB – the Matlab/Octave toolbox for reliable computing [21] – produced by Professor Rump is used by technicians and researchers the world over (including myself) for numerical computations. In addition, Professor Nakao of Kyushu University, Professor Plum of Karlsruhe Institute of Technology, and Associate Professor Watanabe of Kyushu University have been engaged in pioneering research on computer-assisted proofs, and I was able to receive various advice from them that served as hints about my own research during the academic year at places where we were able to have exchanges.

Finally, I would like to express my gratitude to all those who supported me in my research and my student life. Mr. Uchiumi, Mr. Sonoda, and Mr. Yatabe, as friends going through the same process, allowed me the pleasure of having company that I could confide in and speak honestly with, which provided major support for my research life. I hope that in the future we can continue to communicate with one another as researchers. I would also like to express my sincere thanks to all the patrons of the restaurant in Takadanobaba, Shinjuku that became my regular haunt and helped me survive because I was able to find comforting support on the personal side of my student life. In particular, I cannot count the times that “Enji”, “Jizo”, “Takadanobaba-Kenkyujo”, “TontoriQ”, “Narikura”, “Hachiman”, “BABA Picasso”, and “Hitoritabi” helped me out. I would also like to take this chance to thank my family in Nagasaki who cheered me on and provided me with economic and moral support while I was in Tokyo and struggled with my economic situation. I would like to further express my heartfelt gratitude to my wife, Yumiko, who decided to marry me, even as I worked towards a degree under less than favourable circum-

stances. Let me conclude by stating that with this thesis as a point on my journey, I pledge to all those I have mentioned here to further develop my research moving forward.

Kazuaki TANAKA

February 2017

謝辞

謝辞に関しては筆者の母国語である日本語でも併記させて頂きたい．

第一に，筆者の指導教授である大石進一教授にこの上ない感謝を申し上げる．大石教授は 20 年以上に渡って精度保証付き数値計算の分野で活躍され続け，同分野に多大なる功績を残されてこられた．2012 年春には紫綬褒章を天皇陛下より授与されている．また，同分野に精通する多くの研究者を輩出され，この点でも同分野の発展に大きく貢献されてこられた．私が現在の研究を遂行できたのは，このような恵まれた環境の中で指導を受けることができたためであると確信している．

次に私の学位審査の副査をお引き受け頂いた柏木雅英教授と田中和永教授に感謝を申し上げる．柏木教授とは研究室が近いということもあり，私が研究室配属された学部 4 年生の頃から，研究面で数え切れないほどのアドバイスを頂いただけではなく，学生生活の面でも多くのサポートをして頂いた．最近の私の数値計算プログラムの殆どは，柏木教授の開発されている KV ライブラリ [8] に区間演算等の基礎的な部分の多くを頼っているということは特筆すべきであろう．また，その使用方法についても多くのアドバイスを頂いたことは誠に有難く思っている．田中教授には楕円型微分方程式に関する基本的記述や，とりわけ解の正値性の検証法について多大なご貢献を頂いた．微分方程式の解析者の視点からアドバイスを頂いたことは，この論文の質の向上に繋がっただけでなく，私自身の今後の研究の大きな糧となったことに疑いの余地はない．

次に，大石研究室に所属する全ての先生方，先輩方，そして私の研究に対してコメントやアドバイスを下さった全ての方々に感謝を申し上げる．特に，関根晃太助教とは柏木教授と共に連日のように膝を突き合わせて議論を交わさせて頂いたことは記憶に新しい．森倉悠介助教には研究遂行に必要な事務手続き等，研究生活の様々な面でサポートをして頂いた．近年大石研究室に客員教授として来日されている Rump 教授にも研究に関して有用なアドバイスを頂いた．Rump 教授が制作されている精度保証付き数値計算のための Matlab・Octave 用ツールボックス INTLAB[21] は私を含めた世界中の多くの数値計算の研究者，技術者に使用されている．また，九州大学の中尾充宏教授，Karlsruher 工科大学の Plum 教授，九州大学の渡部善隆准教授は偏微分方程式の解の存在性を示す計算機援用証明の先駆的研究をされており，学会期間中をはじめとする交流の場で研究のヒントとなる様々な助言を頂いた．

最後に，私の研究および学生生活を支えて下さった全ての方々に対して感謝の気持ちを示させて頂きたい．内海晋弥氏 (早稲田大学基幹理工学研究科)，園田翔氏 (早稲田大学先進理工学研究科)，矢田部浩平氏（早稲田大学基幹理工学研究科)，は同期の友人として親密に交流させて頂き，それが研究生活の大きな支えとなっていた．今後も研究者としての将来を語り合える存在でありたいと願っている．また，いわゆる「いきつけ」として私の最も心地よい居場所を造り，存続させ，学生生活を私生活の面で支えてくれた新宿区高田馬場の飲食店の皆様にも感謝を申し上げたい．とりわけ『えんじ』,『寺蔵』,『高田馬場研究所』,『とんとり Q』,『成蔵』,『はちまん』,『BABA Picasso』,『ひとり旅』(五十音順) には何度お世話になったことか分からない．決して経済状況が良くない中で，一人上京している筆者を経済面，精神面で支え，温かく応援してくれた長崎の家族にもこの場を借りて改めて感謝の気持ちを申し上げたい．また，私が学位申請中でまだ不安定な身分であったにも関わらず，私との結婚を決断してくれた妻の友美子にも同様に感謝を申し上げる．

この学位論文を一つの通過点として，今後更に研究を発展させていくことを先に挙げた全ての人に誓い，結びとさせて頂く．

田中一成

2017 年 2 月

Contents

Chapter 1

Introduction

We are concerned with verified numerical computation methods for solutions to the following elliptic problem:

$$\begin{cases} -\Delta u(x) = f(u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded polygonal domain (i.e., an open connected bounded set with polygonal shape) in $\mathbb{R}^2$, Δ is the usual Laplace operator, and $f : \mathbb{R} \to \mathbb{R}$ is a given nonlinear function. Our objective includes the problem of finding a solution to

$$\begin{cases} -\Delta u(x) = f(u(x)), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Therefore, we also discuss the positivity of the verified solution to (1.1) if necessary. If a solution u to (1.1) is also a solution to (1.2), then u is called positive solution to (1.1). In this thesis we employ, as typical choices of f, $f(t) = |t|^{p-1}t$ $(p > 1)$ and $f(t) = \varepsilon^{-2}(t - t^3)$ with a small parameter $\varepsilon > 0$ related with the so called singular perturbation phenomenon. In some places of this thesis, the convexity of Ω will be assumed; the main reason is that we require the H^2-regularity of solution u for deriving an L^∞-error estimation (see Section 2.2).

1.1 Background

Numerical analysis plays an important role in a wide range of scientific and engineering fields to understand various phenomena, especially derived from biology and physics. However, the usual numerical analysis generally accompanies several kinds of errors (e.g., rounding errors, truncated errors, and discretization errors), which may cause serious fault in final results. On the other hand, numerical computation with its quantitative error estimation (including rounding errors, truncated errors, discretization errors, and so on) is called "*verified numerical computation*". Our interest is in verified numerical computation methods for the elliptic problems (1.1) and (1.2), and their related problems. Since problem (1.1) (including (1.2)) arises from various models, especially derived from biology and physics, this problem has been widely investigated both analytically and numerically. For example, we can find some analytical results in [10, 6].

Verified numerical computation methods for elliptic problems originate from [15, 17], and have been further developed by many researchers. These methods enable us to obtain an explicit ball containing the exact solution to a target equation, and therefore have the additional advantage that quantitative information of the exact solution is provided accurately in a strict mathematical sense. In the verification procedure of these methods, tight estimations of several constants are required. For example, a norm bound K for the inverse of a linearized operators, which will be defined in (2.7), has to be estimated explicitly. Moreover, the norm bound $C_p(\Omega)$ for the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is also important. More precisely, $C_p(\Omega)$ is a positive number that satisfies

$$\|u\|_{L^p(\Omega)} \leq C_p(\Omega) \|u\|_{H_0^1(\Omega)} \quad \text{for all } u \in H_0^1(\Omega). \tag{1.3}$$

For simple notation, we denote $C_p(\Omega)$ by C_p if no confusion arises. The precision in evaluating these bounds directly affects the precision of the verification results for target equations. Occasionally, rough estimations of the bounds lead to failure in the verification. Therefore, accurately estimating such bounds is essential.

1.2 Organization

The remainder of thesis is organized as follows: In Chapter 2, we prepare the notation used throughout this thesis, and introduce the verification theory based on [18, 19]. In Chapter 3, we discuss a method of estimating the norm bound of the inverse of a linearized operator. In Chapter 4, we propose a method of evaluating the best constant $C_p(\Omega)$ for the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ with $\Omega = (0,1)^2$. In Chapter 5, we apply the present method to the verified numerical computation for some concrete problems. Chapter 6 concludes this thesis with mentioning future work.

Chapter 2

Preparation and verification theory

In this chapter, we present verification methods (including existing theory) for (1.1). We apply the methods summarized in [18, 19] to obtaining a rigorous numerical inclusion of solutions to (1.1).

Throughout this thesis, we use the following notation:

- $L^p(\Omega)$ $(1 \le p < \infty)$ is the functional space of pth power Lebesgue integrable functions over Ω;
- $L^\infty(\Omega)$ is the functional space of Lebesgue measurable functions over Ω;
- $H^k(\Omega)$ $(k > 0)$ is the kth order L^2-Sobolev space on Ω;
- $H_0^1(\Omega) := \{u \in H^1(\Omega) \ : \ u = 0 \text{ on } \partial\Omega \text{ in the trace sense}\}$;
- We denote $V = H_0^1(\Omega)$;
- We denote $V^* = H^{-1}(\Omega)(:=(\text{dual of } V))$ with the usual sup-norm;
- The L^2-inner product and the L^2-norm are simply denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively, if no confusion arises;
- $B(x, r; \|\cdot\|)$ denotes the closed ball whose center is x and whose radius is $r \ge 0$ in the sense of the norm $\|\cdot\|$;
- For function u, we define

  $$u_+ = \max\{u, 0\} \quad \text{and} \quad u_- = \max\{-u, 0\},$$

  respectively. If $u \in H^1(\Omega)$, then u_+, $u_- \in H^1(\Omega)$, and

  $$\nabla u_+ = \begin{cases} \nabla u, & u > 0 \\ 0, & u \le 0 \end{cases}, \qquad \nabla u_- = \begin{cases} -\nabla u, & u \le 0 \\ 0, & u > 0 \end{cases};$$

  a proof can be found, e.g., in [6, Lemma 7.6].

We assume that $f(u(\cdot)) \in V^*$ for each $u \in V$, and denote

$$F : \begin{cases} V & \to & V^*, \\ u & \mapsto & f(u(\cdot)). \end{cases} \tag{2.1}$$

We need F defined in (2.1) is Fréchet differentiable (the Fréchet derivative of F at $\omega \in V$ is denoted by F'_ω). For this purpose, we require the nonlinearity f to satisfy

$$f(0) = 0, \tag{2.2}$$

$$f \in C^1(\overline{\Omega}), \tag{2.3}$$

$$f'(x) - f'(0) \leq C|x|^p \ (x \in \Omega), \tag{2.4}$$

with $C > 0$ and $1 < p < \infty$; recall that $\Omega \subset \mathbb{R}^2$. Our objective $f(t) = |t|^{p-1}t$ $(p > 1)$ and $f(t) = \varepsilon^{-2}(t - t^3)$ $(\varepsilon > 0)$ satisfy these conditions.

Let $\hat{u} \in V$ denote some approximate solution to (1.1) constructed numerically, e.g., by a finite element basis, a Fourier-Galerkin basis, or a Legendre polynomial basis (in this thesis, we will employ a Legendre polynomial basis; therefore $\hat{u} \in C^\infty(\overline{\Omega})$). We define the operator $\mathcal{F} : V \to V^*$ as $\mathcal{F}(u) := -\Delta u - F(u)$ $(u \in V)$, more precisely, which is given by

$$\langle \mathcal{F}(u), v \rangle := (\nabla u, \nabla v) - (F(u), v) \quad \text{for } u, v \in V.$$

Note that, the Fréchet differentiability of F leads to that of $\mathcal{F}$, and the Fréchet derivative $\mathcal{F}'_\omega : V \to V^*$ of $\mathcal{F}$ at $\omega \in V$ is given by $\mathcal{F}'_\omega u = -\Delta u - F'_\omega u$ $(u \in V)$. We first rewrite (1.1) into

$$\mathcal{F}(u) = 0, \tag{2.5}$$

and discuss a rigorous inclusion of a solution to (2.5). In other words, we first consider the existence of a weak solution to (1.1) (a solution to (2.5) in V), and then we discuss its H^2-regularity if necessary.

2.1 Inclusion in $H_0^1(\Omega)$

We use the following theorem, which is similar to the Newton-Kantrovich theorem, for deriving a rigorous inclusion of a solution to (2.5).

Theorem 2.1 ([19])**.** *Let $\mathcal{F} : V \to V^*$ be a Fréchet differentiable operator, and let $\hat{u} \in V$ be some numerical approximation of a solution to $\mathcal{F}(u) = 0$. Suppose that there exist $\delta > 0$, $K > 0$, and a non-decreasing function g satisfying*

$$\|\mathcal{F}(\hat{u})\|_{V^*} \leq \delta, \tag{2.6}$$

$$\|u\|_V \leq K \|\mathcal{F}'_{\hat{u}} u\|_{V^*} \quad \text{for all } u \in V, \tag{2.7}$$

$$\left\|\mathcal{F}'_{\hat{u}+u} - \mathcal{F}'_{\hat{u}}\right\|_{\mathcal{B}(V,V^*)} \leq g\left(\|u\|_V\right) \quad \text{for all } u \in V, \tag{2.8}$$

and

$$g(t) \to 0 \quad \text{as} \quad t \to 0, \tag{2.9}$$

where $\mathcal{F}'_{\hat{u}}$ is the Fréchet derivative of $\mathcal{F}$ at $\hat{u} \in V$. Moreover, suppose that some $\alpha > 0$ exists such that

$$\delta \leq \frac{\alpha}{K} - G(\alpha) \quad \textit{and} \quad Kg(\alpha) < 1, \tag{2.10}$$

where $G(t) := \displaystyle\int_0^t g(s)ds$. Then, there exists a solution $u \in V$ to the equation $\mathcal{F}(u) = 0$ satisfying

$$\|u - \hat{u}\|_V \leq \alpha. \tag{2.11}$$

Furthermore, the solution is unique under the side condition (2.11).

Remark 2.2. *For $\hat{u} \in V$ that satisfies $\Delta\hat{u} + F(\hat{u}) \in L^2(\Omega)$, the norm of the residual*

$\|\mathcal{F}(\hat{u})\|_{V^*}$ *is bounded by*

$$C_2 \|\Delta\hat{u} + F(\hat{u})\|_{L^2(\Omega)}.$$

Here, C_2 *is the norm bound for the embedding* $V^* \hookrightarrow L^2(\Omega)$ *(the same embedding constant for* $L^2(\Omega) \hookrightarrow V$*), and the* L^2*-norm (in the above formula) can be computed by a numerical integration method with verification.*

We will discuss a method for computing the bound K *for the operator for the operator norm of* $\mathcal{F}_{\hat{u}}'^{-1}$ *in Chapter 3.*

An explicit construction of the function g *satisfying* (2.8) *and* (2.9) *is determined for each* f*. For* $f(t) = |t|^{p-1}t$ $(p \geq 2)$ *and* $f(t) = \varepsilon^{-2}(t - t^3)$ $(\varepsilon > 0)$*, one can employ*

$$g(t) = p(p-1)C_{p+1}^3 t \left(\|\hat{u}\|_{L^{p+1}(\Omega)} + C_{p+1}t\right)^{p-2} \tag{2.12}$$

and

$$g(t) = 6\varepsilon^{-2}C_4^3 t \left(\|\hat{u}\|_{L^4(\Omega)} + C_4 t\right), \tag{2.13}$$

respectively, where δ *and* K *are the constants in* (2.6) *and* (2.7) *for* $\hat{u} \in V$*. The proof can be found in* [14, Theorem 3.1].

Our objective includes a verified numerical computation for positive solutions to (1.1), i.e., solutions to (1.2). When $f(t) = |t|^{p-1}t$ $(p > 1)$, we derive numerical inclusions of the positive solutions to (2.5) on the basis of the following Theorem 2.3. Note that, we will describe an inclusion method for the case that $f(t) = \varepsilon^{-2}(t - t^3)$ $(\varepsilon > 0)$ in the next section, because this requires an additional consideration of inclusions in the sense of L^∞-error.

Theorem 2.3. *Let* $f(t) = |t|^{p-1}t$ $(p > 1)$ *(therefore,* $F(u) = |u|^{p-1}u$*). Moreover, let* $\hat{u} \in V$ *be some numerical approximation of a solution to* (2.5)*. Suppose that there exist* $\delta > 0$*,* $K > 0$*, and a non-decreasing function* g *satisfying* (2.6)–(2.9)*, and that* *some* $\alpha > 0$ *exists satisfying* (2.10)*. If we have*

$$\|\hat{u}_-\|_V + \alpha < C_{p+1}^{-\frac{p+1}{p-1}}, \tag{2.14}$$

then there exists a positive solution $u \in V$ *to* (2.5) *satisfying* (2.11)*. Furthermore, the solution is unique under the side condition* (2.11)*.*

Proof. The existence of solution u satisfying (2.11) is ensured on the basis of Theorem 2.1. Therefore, we prove the positivity of u.

Let us first prove

$$\|u_-\|_V \leq \|\hat{u}_-\|_V + \alpha. \tag{2.15}$$

We express $u \in V$ by the form $u = \hat{u} + \alpha\omega$ with $\omega \in V$ satisfying $\|\omega\|_V \leq 1$. Since $b \geq (a-b)_-(:= \max\{-(a-b), 0\})$ for nonnegative numbers $a, b \in \mathbb{R}$, we have

$$\begin{aligned} u_- &= (\hat{u} + \alpha\omega)_- \\ &= (\hat{u}_+ - \hat{u}_- + \alpha\omega_+ - \alpha\omega_-)_- \\ &= (\hat{u}_+ + \alpha\omega_+ - (\hat{u}_- + \alpha\omega_-))_- \\ &\leq \hat{u}_- + \alpha\omega_-, \end{aligned}$$

which implies (2.15).

We then prove the positivity of u. Since u satisfies that

$$(\nabla u, \nabla v) = \left(|u|^{p-1}u, v\right) \quad \text{for all } v \in V,$$

we have

$$\|u_-\|_V^2 = (\nabla u, \nabla u_-) = \left(|u|^{p-1}u, u_-\right) \tag{2.16}$$

by fixing $v = u_-$. Moreover, we have

$$\begin{aligned}\left(u^{p-1}|u|, u_-\right) &= \int_\Omega u_-(x)^{p+1}dx \\ &= \|u_-\|_{L^{p+1}(\Omega)}^{p+1} \\ &\leq C_{p+1}^{p+1} \|u_-\|_V^{p+1}; \end{aligned} \tag{2.17}$$

note that, since u satisfies the homogeneous Dirichlet boundary condition, $u_- = 0$ on the boundary of $\mathrm{supp}\,(u_-)$. Therefore, it follows from (2.16) and (2.17) that

$$\|u_-\|_V^2 \left(1 - C_{p+1}^{p+1} \|u_-\|_V^{p-1}\right) \leq 0.$$

Since (2.14) ensures that $1 - C_{p+1}^{p+1} \|u_-\|_V^{p-1} > 0$, we have $\|u_-\|_V = 0$, i.e., the solution u is nonnegative in Ω. Therefore, the maximum principle ensures that u is positive in Ω (because $|u|^{p-1}u \geq 0$ when u is nonnegative). □

2.2 Inclusion in $L^\infty(\Omega)$

In this subsection, we discuss a method that gives an L^∞-error bound for a solution to (1.1) from a known H_0^1-error bound, that is, we compute an explicit bound for $\|u - \hat{u}\|_{L^\infty(\Omega)}$ for a solution $u \in V$ to (1.1) satisfying

$$\|u - \hat{u}\|_V \leq \alpha \tag{2.18}$$

with $\alpha > 0$ and $\hat{u} \in V$. We assume that Ω is convex and polygonal to obtain such an error estimation; this condition gives the H^2-regularity of solutions to (1.1) (and therefore, ensures their boundedness) a priori. More precisely, when Ω is a convex polygonal domain, a weak solution $u \in V$ to the Poisson equation

$$(u, v)_V = (h, v)_{L^2(\Omega)} \quad \text{for all } v \in V \tag{2.19}$$

for $h \in L^2(\Omega)$ is H^2-regular (see, e.g., [7, Section 3.3]). A solution u satisfying (2.18) can be written in the form $u = \hat{u} + \alpha\omega$ with some $\omega \in V$, $\|\omega\|_V \le 1$. Moreover, ω satisfies

$$\begin{cases} -\Delta\alpha\omega = F(\hat{u} + \alpha\omega) + \Delta\hat{u} & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega, \end{cases}$$

and therefore is also H^2-regular if $\Delta\hat{u} \in L^2(\Omega)$. We then use the following theorem to obtain an L^∞ error estimation.

Theorem 2.4 ([18]). *For all $u \in H^2(\Omega)$,*

$$\|u\|_{L^\infty(\Omega)} \le c_0\|u\|_{L^2(\Omega)} + c_1\|\nabla u\|_{L^2(\Omega)} + c_2\|D^2u\|_{L^2(\Omega)}$$

with

$$c_j = \frac{\gamma_j}{|\overline{\Omega}|}\left[\max_{x_0\in\overline{\Omega}} \int_{\overline{\Omega}} |x - x_0|^{2j} dx\right]^{1/2}, \ (j = 0, 1, 2),$$

where D^2u denotes the Hesse matrix of u, $|\overline{\Omega}|$ is the measure of $\overline{\Omega}$, and

$$\gamma_0 = 1, \ \gamma_1 = 1.1548, \ \gamma_2 = 0.22361.$$

For $n = 3$, other values of γ_0, γ_1, and γ_2 have to be chosen (see [18]).

Remark 2.5. *The norm of the Hesse matrix of u is precisely defined by*

$$\|D^2u\|_{L^2(\Omega)} = \sqrt{\sum_{i,j=1}^{2} \left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{L^2(\Omega)}^2}.$$

Moreover, since Ω is polygonal, $\|D^2u\|_{L^2(\Omega)} = \|\Delta u\|_{L^2(\Omega)}$ for all $u \in H^2(\Omega) \cap V$ (see, e.g., [7]).

Remark 2.6. *Explicit values of each c_j are provided for some special domains Ω in* [18, 19]*. According to these papers, one can choose, for $\Omega=(0,1)^2$,*

$$c_0=\gamma_0,\ \ c_1=\sqrt{\frac{2}{3}}\gamma_1,\ \text{ and } c_2=\frac{\gamma_3}{3}\sqrt{\frac{28}{5}}.$$

For the application of Theorem 2.4 to the L^∞ estimation of solutions to (1.1), we consider the concrete nonlinearities $f(t)=|t|^{p-1}t$ and $f(t)=\varepsilon^{-2}(t-t^3)$, where $p\geq 2$ and $\varepsilon>0$. Recall that, for the H^2-regularity of solution u, we restrict $\Omega\subset\mathbb{R}^2$ to a convex polygonal domain in the following two theorems.

Theorem 2.7. *Let $f(t)=|t|^{p-1}t$ $(p>1)$, and let $\hat{u}\in V$ be some numerical approximation of a solution to* (2.5) *such that $\Delta\hat{u}\in L^2(\Omega)$. Moreover, let c_0, c_1, and c_2 be as in Theorem* 2.4*. Suppose that there exist $\delta>0$, $K>0$, and a non-decreasing function g satisfying* (2.6)*–*(2.9)*, and that some $\alpha>0$ exists satisfying* (2.10)*. Then, there exists a solution $u\in V\cap L^\infty(\Omega)$ to* (2.5) *satisfying*

$$\begin{aligned}&\|u-\hat{u}\|_{L^\infty(\Omega)}\leq c_0C_2\alpha+c_1\alpha+\\&c_2\left(2^{p-\frac{3}{2}}p\alpha C_3\sqrt{\|\hat{u}\|^{2(p-1)}_{L^{6(p-1)}(\Omega)}+\frac{\alpha^{2(p-1)}}{2p-1}C^{2(p-1)}_{6(p-1)}}+\left\|\Delta\hat{u}+|\hat{u}|^{p-1}\hat{u}\right\|_{L^2(\Omega)}\right).\end{aligned}\tag{2.20}$$

Proof. Owing to Theorem 2.4, we have

$$\begin{aligned}\|u-\hat{u}\|_{L^\infty(\Omega)}&=\alpha\|\omega\|_{L^\infty(\Omega)}\\&\leq\alpha\left(c_0\|\omega\|_{L^2(\Omega)}+c_1\|\omega\|_V+c_2\|\Delta\omega\|_{L^2(\Omega)}\right)\\&\leq\alpha\left(c_0C_2+c_1+c_2\|\Delta\omega\|_{L^2(\Omega)}\right).\end{aligned}$$

The last term $\|\Delta\omega\|_{L^2(\Omega)}$ is estimated by

$$\alpha\|\Delta\omega\|_{L^2(\Omega)}=\|F(\hat{u}+\alpha\omega)+\Delta\hat{u}\|_{L^2(\Omega)}$$

$$\begin{aligned}
&= \left\| F\left(\hat{u} + \alpha\omega\right) - F\left(\hat{u}\right) + F\left(\hat{u}\right) + \Delta\hat{u} \right\|_{L^2(\Omega)} \\
&\le \left\| F\left(\hat{u} + \alpha\omega\right) - F\left(\hat{u}\right) \right\|_{L^2(\Omega)} + \left\| \Delta\hat{u} + F\left(\hat{u}\right) \right\|_{L^2(\Omega)}.
\end{aligned}$$

Since the mean value theorem ensures that

$$\begin{aligned}
&\int_\Omega \left(F\left(\hat{u} + \alpha\omega\right) - F\left(\hat{u}\right)\right)^2 dx \\
=&\int_\Omega \left(\alpha\omega(x) \int_0^1 F'_{\hat{u}+t\alpha\omega}(x)\, dt\right)^2 dx \\
=&\int_\Omega \left(\alpha\omega(x) \int_0^1 p\left|\hat{u}(x) + \alpha t\omega(x)\right|^{p-1} dt\right)^2 dx \\
=&p^2\alpha^2 \int_\Omega \omega(x)^2 \left(\int_0^1 \left|\hat{u}(x) + \alpha t\omega(x)\right|^{p-1} dt\right)^2 dx \\
\le&p^2\alpha^2 \int_\Omega \omega(x)^2 \int_0^1 \left|\hat{u}(x) + \alpha t\omega(x)\right|^{2(p-1)} dtdx \\
\le&p^2\alpha^2 \left\|\omega\right\|^2_{L^3(\Omega)} \int_0^1 \left\| \left|\hat{u} + \alpha\omega t\right|^{2(p-1)} \right\|_{L^3(\Omega)} dt \\
=&p^2\alpha^2 \left\|\omega\right\|^2_{L^3(\Omega)} \int_0^1 \left\|\hat{u} + \alpha\omega t\right\|^{2(p-1)}_{L^{6(p-1)}(\Omega)} dt \\
\le&p^2\alpha^2 \left\|\omega\right\|^2_{L^3(\Omega)} \int_0^1 \left(\left\|\hat{u}\right\|_{L^{6(p-1)}(\Omega)} + t\alpha\left\|\omega\right\|_{L^{6(p-1)}(\Omega)}\right)^{2(p-1)} dt \\
\le&2^{2(p-1)-1}p^2\alpha^2 \left\|\omega\right\|^2_{L^3(\Omega)} \left\{ \left\|\hat{u}\right\|^{2(p-1)}_{L^{6(p-1)}(\Omega)} + \int_0^1 \left(t\alpha\left\|\omega\right\|_{L^{6(p-1)}(\Omega)}\right)^{2(p-1)} dt \right\} \\
=&2^{2p-3}p^2\alpha^2 \left\|\omega\right\|^2_{L^3(\Omega)} \left(\left\|\hat{u}\right\|^{2(p-1)}_{L^{6(p-1)}(\Omega)} + \frac{\alpha^{2(p-1)}}{2p-1} \left\|\omega\right\|^{2(p-1)}_{L^{6(p-1)}(\Omega)} \right) \\
\le&2^{2p-3}p^2\alpha^2 C_3^2 \left(\left\|\hat{u}\right\|^{2(p-1)}_{L^{6(p-1)}(\Omega)} + \frac{\alpha^{2(p-1)}}{2p-1} C^{2(p-1)}_{6(p-1)} \right).
\end{aligned}$$

it follows that

$$\alpha\left\|\Delta\omega\right\|_{L^2(\Omega)} \le 2^{p-\frac{3}{2}} p\alpha C_3 \sqrt{\left\|\hat{u}\right\|^{2(p-1)}_{L^{6(p-1)}(\Omega)} + \frac{\alpha^{2(p-1)}}{2p-1} C^{2(p-1)}_{6(p-1)}} + \left\|\Delta\hat{u} + F\left(\hat{u}\right)\right\|_{L^2(\Omega)}.$$

Consequently, the L^∞ error of u is estimated as asserted in (2.20). $\square$

Theorem 2.8. *Let $f(t) = \varepsilon^{-2}(t - t^3)$ ($\varepsilon > 0$), and let $\hat{u} \in V$ be some numerical approximation of a solution to* (2.5) *such that $\Delta\hat{u} \in L^2(\Omega)$. Moreover, let c_0, c_1, and c_2 be as in Theorem* 2.4. *Suppose that there exist $\delta > 0$, $K > 0$, and a non-decreasing function g satisfying* (2.6)*–*(2.9)*, and that some $\alpha > 0$ exists satisfying* (2.10)*. Then, there exists a solution $u \in V \cap L^\infty(\Omega)$ to* (2.5) *satisfying*

$$\begin{aligned}\|u - \hat{u}\|_{L^\infty(\Omega)} \le c_0 C_2 \alpha + c_1\alpha + c_2 \Big(\alpha\varepsilon^{-2} C_3 \Big(1 + 3\|\hat{u}\|^2_{L^{12}(\Omega)} \\ +3\alpha C_{12}\|\hat{u}\|_{L^{12}(\Omega)} + \alpha^2 C_{12}^2\Big) + \left\|\Delta\hat{u} + \varepsilon^{-2}(\hat{u} - \hat{u}^3)\right\|_{L^2(\Omega)}\Big). \qquad (2.21)\end{aligned}$$

Proof. Owing to Theorem 2.4, we have

$$\begin{aligned}\|u - \hat{u}\|_{L^\infty(\Omega)} &= \alpha\|\omega\|_{L^\infty(\Omega)} \\ &\le \alpha\left(c_0\|\omega\|_{L^2(\Omega)} + c_1\|\omega\|_V + c_2\|\Delta\omega\|_{L^2(\Omega)}\right) \\ &\le \alpha\left(c_0 C_2 + c_1 + c_2\|\Delta\omega\|_{L^2(\Omega)}\right).\end{aligned}$$

The last term $\|\Delta\omega\|_{L^2(\Omega)}$ is estimated by

$$\begin{aligned}\alpha\|\Delta\omega\|_{L^2(\Omega)} &= \|F(\hat{u} + \alpha\omega) + \Delta\hat{u}\|_{L^2(\Omega)} \\ &= \|F(\hat{u} + \alpha\omega) - F(\hat{u}) + F(\hat{u}) + \Delta\hat{u}\|_{L^2(\Omega)} \\ &\le \|F(\hat{u} + \alpha\omega) - F(\hat{u})\|_{L^2(\Omega)} + \|\Delta\hat{u} + F(\hat{u})\|_{L^2(\Omega)}.\end{aligned}$$

Since the mean value theorem ensures that

$$\begin{aligned}&\int_\Omega (F(\hat{u} + \alpha\omega) - F(\hat{u}))^2 dx \\ &= \int_\Omega \left(\alpha\omega(x)\int_0^1 F'(\hat{u}(x) + t\alpha\omega(x))\, dt\right)^2 dx \\ &= \int_\Omega \left(\alpha\omega(x)\int_0^1 \varepsilon^{-2}\left\{(1 - 3(\hat{u}(x) + t\alpha\omega(x))^2\right\} dt\right)^2 dx\end{aligned}$$

$$
\begin{aligned}
&=\alpha^2\varepsilon^{-4}\int_\Omega \omega(x)^2\left(\int_0^1\left\{1-3(\hat{u}(x)+t\alpha\omega(x))^2\right\}dt\right)^2 dx\\
&=\alpha^2\varepsilon^{-4}\int_\Omega \omega(x)^2\left(\int_0^1\left(1-3\hat{u}(x)^2-6t\alpha\omega(x)\hat{u}(x)-3t^2\alpha^2\omega(x)^2\right)dt\right)^2 dx\\
&\le\alpha^2\varepsilon^{-4}\left\|\omega\right\|_{L^3(\Omega)}^2\left\|\left(\int_0^1\left(1-3\hat{u}^2-6t\alpha\omega\hat{u}-3t^2\alpha^2\omega^2\right)dt\right)^2\right\|_{L^3(\Omega)}\\
&=\alpha^2\varepsilon^{-4}\left\|\omega\right\|_{L^3(\Omega)}^2\left\|\int_0^1\left(1-3\hat{u}^2-6t\alpha\omega\hat{u}-3t^2\alpha^2\omega^2\right)dt\right\|_{L^6(\Omega)}^2\\
&\le\alpha^2\varepsilon^{-4}\left\|\omega\right\|_{L^3(\Omega)}^2\left(1+3\left\|\hat{u}^2\right\|_{L^6(\Omega)}+3\alpha\left\|\omega\hat{u}\right\|_{L^6(\Omega)}+\alpha^2\left\|\omega^2\right\|_{L^6(\Omega)}\right)^2\\
&\le\alpha^2\varepsilon^{-4}\left\|\omega\right\|_{L^3(\Omega)}^2\left(1+3\left\|\hat{u}\right\|_{L^{12}(\Omega)}^2+3\alpha\left\|\hat{u}\right\|_{L^{12}(\Omega)}\left\|\omega\right\|_{L^{12}(\Omega)}+\alpha^2\left\|\omega\right\|_{L^{12}(\Omega)}^2\right)^2\\
&\le\alpha^2\varepsilon^{-4}C_3^2\left(1+3\left\|\hat{u}\right\|_{L^{12}(\Omega)}^2+3\alpha C_{12}\left\|\hat{u}\right\|_{L^{12}(\Omega)}+\alpha^2C_{12}^2\right)^2
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\alpha\left\|\Delta\omega\right\|_{L^2(\Omega)}\\
\le&\alpha\varepsilon^{-2}C_3\left(1+3\left\|\hat{u}\right\|_{L^{12}(\Omega)}^2+3\alpha C_{12}\left\|\hat{u}\right\|_{L^{12}(\Omega)}+\alpha^2C_{12}^2\right)+\left\|\Delta\hat{u}+F\left(\hat{u}\right)\right\|_{L^2(\Omega)}.
\end{aligned}
$$

Consequently, the L^∞ error of u is estimated as asserted in (2.21). □

We derive numerical inclusions of the positive solutions to (2.5) on the basis of the following theorem, when $f(t)=\varepsilon^{-2}(t-t^3)$ $(\varepsilon>0)$.

Theorem 2.9. *Let* $f(t)=\varepsilon^{-2}(t-t^3)$ $(\varepsilon>0)$. *Moreover, let* $\hat{u}\in V$ *be some numerical approximation of a solution to* (2.5) *such that* $\Delta\hat{u}\in L^2(\Omega)$. *Suppose that there exist* $\delta>0,\ K>0$, *and a non-decreasing function* g *satisfying* (2.6)–(2.9), *and that some* $\alpha>0$ *exists such that* (2.10). *There exists a positive solution* $u\in V\cap L^\infty(\Omega)$ *to* (2.5) *satisfying* (2.11) *and* (2.21), *if we have*

$$
\varepsilon^{-2}<\lambda_1(\operatorname{supp}(\hat{u}-\beta)_-), \tag{2.22}
$$

where $\beta > 0$ *is the right side of* (2.21) *and* $\lambda_1(\mathrm{supp}\,(\hat{u}-\beta)_-)$ *is the minimum eigenvalue of* $-\Delta$ *on* $H_0^1(\mathrm{supp}\,(\hat{u}-\beta)_-)$.

Proof. The existence of solution u satisfying (2.11) and (2.21) is ensured on the basis of Theorem 2.1 and 2.8, respectively. Therefore, we prove the positivity of u. Since

$$(\nabla u, \nabla u_-) = \varepsilon^{-2}\left(u - u^3, u_-\right),$$

we have

$$\begin{aligned}
\|\nabla u_-\|^2_{L^2(\Omega)} =& \varepsilon^{-2}\left(u - u^3, u_-\right)\\
=& \varepsilon^{-2}\int_\Omega \left(u_-(x) - u_-(x)^3\right) u_-(x)dx\\
\leq& \varepsilon^{-2}\|u_-\|^2_{L^2(\Omega)}\\
\leq& \frac{\varepsilon^{-2}}{\lambda_1(\mathrm{supp}\ u_-)}\|\nabla u_-\|^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore, it follows that

$$1 - \frac{\varepsilon^{-2}}{\lambda_1(\mathrm{supp}\ u_-)} \geq 1 - \frac{\varepsilon^{-2}}{\lambda_1(\mathrm{supp}\,(\hat{u}-\beta)_-)} > 0,$$

since $\mathrm{supp}\ u_- \subset \mathrm{supp}\,(\hat{u}-\beta)_-$. Hence, if (2.22), we have $\|\nabla u_-\|_{L^2(\Omega)} = 0$, i.e., the solution u is nonnegative in Ω. The maximum principle moreover ensures that u is positive in Ω. □

Remark 2.10. *For a domain* Ω_- *such that* $\mathrm{supp}\,(\hat{u}-\beta)_- \subset \Omega_-$, *we have*

$$\lambda_1(\Omega_-) \leq \lambda_1(\mathrm{supp}\,(\hat{u}-\beta)_-). \tag{2.23}$$

Therefore, in an actual computation, we choose such Ω_- *with simple shape and compute* $\lambda_1(\Omega_-)$ *as a lower bound of* $\lambda_1(\mathrm{supp}\,(\hat{u}-\beta)_-)$ (*see Section* 5.1 *for an explicit* Ω_-).

2.3 Simple upper bounds for embedding constant

In this section, we show two formulas for deriving rough upper bounds for the embedding constant C_p satisfying (1.3), with a simple computation.

We prepare the following theorem, which provides the best constant in the classical Sobolev inequality with critical exponents on $\mathbb{R}^2$.

Theorem 2.11 (T. Aubin [1] and G. Talenti [22]). *Let u be any function in $W^{1,q}(\mathbb{R}^n)$ $(n \geq 2)$, where q is any real number such that $1 < q < n$. Moreover, set $p = nq/(n-q)$. Then, $u \in L^p(\mathbb{R}^n)$ and*

$$\left(\int_{\mathbb{R}^n} |u(x)|^p \, dx\right)^{\frac{1}{p}} \leq T_p \left(\int_{\mathbb{R}^n} |\nabla u(x)|_2^q \, dx\right)^{\frac{1}{q}}$$

holds for

$$T_p = \pi^{-\frac{1}{2}} n^{-\frac{1}{q}} \left(\frac{q-1}{n-q}\right)^{1-\frac{1}{q}} \left\{\frac{\Gamma\left(1+\frac{n}{2}\right)\Gamma(n)}{\Gamma\left(\frac{n}{q}\right)\Gamma\left(1+n-\frac{n}{q}\right)}\right\}^{\frac{1}{n}}, \tag{2.24}$$

where $|\nabla u|_2 = \left((\partial u/\partial x_1)^2 + (\partial u/\partial x_2)^2 + \cdots + (\partial u/\partial x_n)^2\right)^{1/2}$, and Γ denotes the gamma function.

The following corollary, obtained from Theorem 2.11, provides a simple bound for C_p for a bounded domain Ω.

Corollary 2.12. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain. Let p be a real number such that $p \in (n/(n-1), 2n/(n-2)]$ if $n \geq 3$ and $p \in (n/(n-1), \infty)$ if $n = 2$. Moreover, set $q = np/(n+p)$. Then, (1.3) holds for*

$$C_p = |\Omega|^{\frac{2-q}{2q}} T_p,$$

where T_p is the constant in (2.24).

Proof. By zero extension outside Ω, we may regard $u \in H_0^1(\Omega)$ as an element $u \in W^{1,q}(\mathbb{R}^n)$; note that $q < 2$. Therefore, from Theorem 2.11,

$$\|u\|_{L^p(\Omega)} \le T_p \left(\int_\Omega |\nabla u(x)|_2^q \, dx \right)^{\frac{1}{q}}. \tag{2.25}$$

Hölder's inequality gives

$$\begin{aligned}\int_\Omega |\nabla u(x)|_2^q \, dx &\le \left(\int_\Omega |\nabla u(x)|_2^{q \cdot \frac{2}{q}} \, dx \right)^{\frac{q}{2}} \left(\int_\Omega 1^{\frac{2}{2-q}} \, dx \right)^{\frac{2-q}{2}} \\ &= |\Omega|^{\frac{2-q}{2}} \left(\int_\Omega |\nabla u(x)|_2^2 \, dx \right)^{\frac{q}{2}},\end{aligned}$$

that is,

$$\left(\int_{\mathbb{R}^n} |\nabla u(x)|_2^q \, dx \right)^{\frac{1}{q}} \le |\Omega|^{\frac{2-q}{2q}} \|\nabla u\|_{L^2(\Omega)}, \tag{2.26}$$

where $|\Omega|$ is the measure of Ω. From (2.25) and (2.26), it follows that

$$\|u\|_{L^p(\Omega)} \le |\Omega|^{\frac{2-q}{2q}} T_p \|\nabla u\|_{L^2(\Omega)}.$$

□

Remark 2.13. *The case that* $p = 2$ *is ruled out in Corollary* 2.12*, but it is well known that*

$$\|u\|_{L^2(\Omega)} \le \frac{1}{\sqrt{\lambda_1}} \|u\|_V,$$

where λ_1 *is the first eigenvalue of the following problem:*

$$(\nabla u, \nabla v) = \lambda (u, v) \quad \text{for all } v \in V. \tag{2.27}$$

Note that, when $\Omega = (0,1)^2$, $\lambda_1 = 2\pi^2$.

Using the following theorem, an upper bound of the embedding constant can be obtained when the minimal point of the spectrum of $-\Delta$ on V is explicitly estimated.

Theorem 2.14 ([20])**.** *Let* $\lambda_1 \in [0,\infty)$ *denote the minimal point of the spectrum of* $-\Delta$ *on* V *for a bounded domain* $\Omega \subset \mathbb{R}^n$ $(n = 2, 3, \cdots)$.

a) Let $n = 2$ *and* $p \in [2,\infty)$. *With the largest integer* ν *satisfying* $\nu \leq p/2$, (1.3) *holds for*

$$C_p = \left(\frac{1}{2}\right)^{\frac{1}{2}+\frac{2\nu-3}{p}} \left[\frac{p}{2}\left(\frac{p}{2}-1\right)\cdots\left(\frac{p}{2}-\nu+2\right)\right]^{\frac{2}{p}} \lambda_1^{-\frac{1}{p}},$$

where $\frac{p}{2}\left(\frac{p}{2}-1\right)\cdots\left(\frac{p}{2}-\nu+2\right) = 1$ *if* $\nu = 1$.

b) Let $n \geq 3$ *and* $p \in [2, 2n/(n-2)]$. *With* $s := n(p^{-1} - 2^{-1} + n^{-1}) \in [0,1]$, (1.3) *holds for*

$$C_p = \left(\frac{n-1}{\sqrt{n}\,(n-2)}\right)^{1-s} \lambda_1^{-\frac{s}{2}}.$$

Chapter 3

Norm of inverse of linearized operator

In this chapter, we discuss the invertibility of the linearized operator $\mathcal{F}'_{\hat{u}}$, and an explicit estimation of the operator norm $\left\|\mathcal{F}'^{-1}_{\hat{u}}\right\|_{B(V^*,V)}$. We check the invertibility of $\mathcal{F}'_{\hat{u}}$ by confirming that the point spectrum of an operator does not contain zero (see Theorem 3.2). The eigenvalues (in the point spectrum) are evaluated by the theory which originates from Liu-Oishi's theorm [12].

3.1 Verification for invertibility

We compute a bound K in (2.7) for the operator norm of $\mathcal{F}'^{-1}_{\hat{u}}$ by the following theorem, proving simultaneously that this inverse operator exists and is defined on the whole of V^*. In this chapter, for estimating the inverse norm $\left\|\mathcal{F}'^{-1}_{\hat{u}}\right\|_{B(V^*,V)}$, we endow V with inner product

$$(\cdot,\cdot)_V = (\nabla\cdot,\nabla\cdot)_{L^2(\Omega)} + \tau\,(\cdot,\cdot)_{L^2(\Omega)} \tag{3.1}$$

and norm $\|\cdot\|_V := \sqrt{(\cdot,\cdot)_V}$, where τ is a nonnegative number chosen as

$$\tau > -f'(\hat{u}(x)) \quad (x\in\Omega). \tag{3.2}$$

Remark 3.1. *We endow the weighted inner product* (3.1) *only when we compute the inverse norm* $\left\|\mathcal{F}'^{-1}_{\hat{u}}\right\|_{B(V^*,V)}$. *We denote the* V *with the usual inner product* $(\nabla\cdot,\nabla\cdot)_{L^2(\Omega)}$ *and the* τ*-weighted inner product* (3.1) *by* V_0 *and* V_τ, *respectively. Since*

$$\left\|\mathcal{F}'^{-1}_{\hat{u}}\right\|_{B(V_0^*,V_0)} \le \left\|\mathcal{F}'^{-1}_{\hat{u}}\right\|_{B(V_\tau^*,V_\tau)},$$

for any nonnegative τ, *we employ the value of* $\left\|\mathcal{F}'^{-1}_{\hat{u}}\right\|_{B(V_\tau^*,V_\tau)}$ *as an upper bound of* $\left\|\mathcal{F}'^{-1}_{\hat{u}}\right\|_{B(V_0^*,V_0)}$. *We use the all verification theorems provided in Chapter* 1 *with endowing* V *with the usual inner product* $(\nabla\cdot,\nabla\cdot)_{L^2(\Omega)}$.

Theorem 3.2. *Let* $\Phi : V \to V^*$ *be the canonical isometric isomorphism, i.e.,* Φ *is* *given by*

$$\langle \Phi u, v \rangle := (u, v)_V \quad \text{for } u, v \in V.$$

If the point spectrum of $\Phi^{-1}\mathcal{F}'_{\hat{u}}$ *(denoted by* $\sigma_p(\Phi^{-1}\mathcal{F}'_{\hat{u}})$*) does not contain zero, then the inverse of* $\mathcal{F}'_{\hat{u}}$ *exists and*

$$\left\| \mathcal{F}'^{-1}_{\hat{u}} \right\|_{B(V^*,V)} \leq \mu_0^{-1}, \tag{3.3}$$

where

$$\mu_0 = \min \left\{ |\mu| \ : \ \mu \in \sigma_p \left(\Phi^{-1}\mathcal{F}'_{\hat{u}} \right) \cup \{1\} \right\}. \tag{3.4}$$

Proof. We prove this theorem by adapting a theory of Fredholm operators, i.e., we have recourse to the fact that the injectivity and the surjectivity of a Fredholm operator are equivalent.

The operator $N := \Phi - \mathcal{F}'_{\hat{u}}$ from V to V^* is given by $\langle Nu, v \rangle = ((\tau + F'_{\hat{u}})u, v)$ for all $u, v \in V$. Thus, actually N maps V into $L^2(\Omega)$. Hence, $N : V \to V^*$ is compact, owing to the compactness of the embedding $L^2(\Omega) \hookrightarrow V^*$; note that $\Omega \subset \mathbb{R}^2$. Therefore, $\mathcal{F}'_{\hat{u}}$ is a Fredholm operator, and the spectrum $\sigma\left(\Phi^{-1}\mathcal{F}'_{\hat{u}}\right)$ of $\Phi^{-1}\mathcal{F}'_{\hat{u}}$ is given by

$$\sigma\left(\Phi^{-1}\mathcal{F}'_{\hat{u}}\right) = 1 - \sigma\left(\Phi^{-1}N\right) = 1 - \left\{ \sigma_p\left(\Phi^{-1}N\right) \cup \{0\} \right\} = \sigma_p\left(\Phi^{-1}\mathcal{F}'_{\hat{u}}\right) \cup \{1\}.$$

Since $\Phi^{-1}\mathcal{F}'_{\hat{u}}$ is self-adjoint, we have, for all $u \in V$,

$$\begin{aligned} \|\mathcal{F}'_{\hat{u}} u\|^2_{V^*} = \left\|\Phi^{-1}\mathcal{F}'_{\hat{u}}\right\|^2_V = & \int_{-\infty}^{\infty} \mu^2 d\left(E_\mu u, u\right)_V \\ \geq & \mu_0^2 \int_{-\infty}^{\infty} d\left(E_\mu u, u\right)_V = \mu_0^2 \left\|u\right\|^2_V, \end{aligned}$$

where E_μ is the resolution of the identity of $\Phi^{-1}\mathcal{F}'_{\hat{u}}$. Hence, $\mathcal{F}'_{\hat{u}}$ is one to one, and therefore is also onto. This implies (3.3). $\square$

3.2 Related eigenvalue problem

The eigenvalue problem $\Phi^{-1}\mathcal{F}'_{\hat{u}}u = \mu u$ in V is equivalent to

$$(\nabla u, \nabla v) - (F'_{\hat{u}}u, v) = \mu\,(u, v)_V \quad \text{for all } v \in V.$$

Since $\mu = 1$ is already known to be in $\sigma\,(\Phi^{-1}\mathcal{F}'_{\hat{u}})$, it suffices to look for eigenvalues $\mu \neq 1$. By setting $\lambda = (1-\mu)^{-1}$, we further transform this eigenvalue problem into

$$\text{Find } u \in V \text{ and } \lambda \in \mathbb{R} \text{ s.t. } (u, v)_V = \lambda\,((\tau + F'_{\hat{u}})u, v) \quad \text{for all } v \in V. \tag{3.5}$$

Owing to (3.2), (3.5) is an eigenvalue problem, the spectrum of which consists of a sequence $\{\lambda_k\}_{k=1}^{\infty}$ of eigenvalues converging to $+\infty$. In order to compute K on the basis of Theorem 3.2, we concretely enclose the eigenvalue λ of (3.5) that minimizes the corresponding absolute value of $|\mu|\,(=|1-\lambda^{-1}|)$, by considering the following approximate eigenvalue problem

$$\begin{aligned}&\text{Find } u \in V_N \text{ and } \lambda^N \in \mathbb{R}\\ &\qquad\qquad \text{s.t. } (u_N, v_N)_V = \lambda^N\,((\tau + F'_{\hat{u}})u_N, v_N) \quad \text{for all } v_N \in V_N,\end{aligned} \tag{3.6}$$

where V_N is a finite-dimensional subspace of V.

We estimate the error between the kth eigenvalue λ_k of (3.5) and the kth eigenvalue λ_k^N of (3.6), by considering the weak formulation of the Poisson equation (2.19) for given $h \in L^2\,(\Omega)$; it is well known that this equation has a unique solution $u \in V$ for each $h \in L^2\,(\Omega)$. Moreover, we introduce the orthogonal projection $P_N^\tau : V \to V_N$

defined by

$$(P_N^\tau u - u, v_N)_V = 0 \quad \text{for all } u \in V \text{ and } v_N \in V_N.$$

The following theorem enables us to estimate the error between λ_k and λ_k^N.

Theorem 3.3 ([23, 11])**.** *Suppose that there exists a positive number C_N^τ such that*

$$\|u_h - P_N^\tau u_h\|_V \leq C_N^\tau \|h\|_{L^2(\Omega)} \tag{3.7}$$

for any $h \in L^2(\Omega)$ and the corresponding solution $u_h \in V$ to (2.19)*. Then,*

$$\frac{\lambda_k^N}{\lambda_k^N (C_N^\tau)^2 \|\tau + f'(\hat{u}(\cdot))\|_{L^\infty(\Omega)} + 1} \leq \lambda_k \leq \lambda_k^N,$$

The inequality on the right is well known as a Rayleigh-Ritz bound, which is derived from the min-max principle:

$$\lambda_k = \min_{H_k \subset V} \left(\max_{v \in H_k \setminus \{0\}} \frac{\|v\|_V^2}{\|av\|_{L^2(\Omega)}^2} \right) \leq \lambda_k^N,$$

where we set $a = \sqrt{\tau + f'(\hat{u}(\cdot))}$ and the minimum is taken over all k-dimensional subspaces H_k of V. Moreover, proofs of the inequality on the left can be found in [23, 11]. Assuming the H^2-regularity of solutions to (2.19) (e.g., when $\Omega \subset \mathbb{R}^2$ is convex [7, Section 3.3]), [23, Theorem 4] ensures the left inequality. A more general statement, that does not require the H^2-regularity, can be found in [11, Theorem 2.1]. Bath theorems were proved on the basis of Liu-Oishi's theorem [12].

Remark 3.4. *When the H^2-regularity of solutions to* (2.19) *is confirmed a priori, e.g., when Ω is a convex polygonal domain* [7, Section 3.3], (3.7) *can be replaced by*

$$\|u - P_N^\tau u\|_V \leq C_N^\tau \|-\Delta u + \tau u\|_{L^2(\Omega)} \quad \text{for all } u \in H^2(\Omega) \cap V. \tag{3.8}$$

An explicit values of C_N^τ *for a given subspace* V_N *will be discussed in Section 3.3.*

3.3 Interpolation constant

In this section, we discuss the interpolation constant C_N^τ satisfying (3.8), where we select $\Omega = (0,a)^n$ $(a > 0)$ and the finite dimensional subspace V_N of V is spanned by a Legendre polynomial basis $\{\phi_{i_1}\phi_{i_2}\cdots\phi_{i_n}\}_{i_1,i_2,\cdots,i_n=1}^N$. Each ϕ_i is defined by

$$\phi_m(x) = \frac{1}{m(m+1)} x(a-x)\frac{dQ_m}{dx}(x), \quad m = 1,2,3,\cdots \tag{3.9}$$

with the Legendre polynomials Q_m defined by

$$Q_m = \frac{(-1)^m}{a^m m!}\left(\frac{d}{dx}\right)^m x^m(a-x)^m, \quad m = 0,1,2,\cdots . \tag{3.10}$$

The following theorem, which was previously proposed in [9], gives an explicit value of C_N^0 with $\tau = 0$ for the one-dimensional case.

Lemma 3.5 ([9]). *Let* $\Omega = (0,a)$ $(a > 0)$ *and let* V *be endowed with the inner product* (3.1) *with* $\tau = 0$. *Moreover, let the finite dimensional subspace* V_N *of* V *be spanned by a Legendre polynomial basis* $\{\phi_i\}_{i=1}$ *defined by* (3.9). *Then, we may select*

$$C_N^0 = a\sqrt{\max\{\Lambda_{N,1}, \Lambda_{N,2}\}}, \tag{3.11}$$

for satisfying (3.8), *where*

$$\Lambda_{N,1} = \frac{1}{2(2N+1)(2N+5)} + \frac{1}{4(2N+5)\sqrt{2N+3}\sqrt{2N+7}},$$

and

$$\Lambda_{N,2} = \frac{1}{4(2N+5)\sqrt{2N+3}\sqrt{2N+7}} + \frac{1}{2(2N+5)(2N+9)} + \\ + \frac{1}{4(2N+9)\sqrt{2N+7}\sqrt{2N+11}}.$$

Remark 3.6. *We may use the same constant C_N^0 in the one-dimensional case for higher-dimensional cases in which $\Omega = (0,a)^n$ (see,* [16]*).*

Theorem 3.7. *Let $\Omega = (0,a)^n$ $(a > 0)$ and let V be endowed with the inner product* (3.1) *with any $\tau \geq 0$. Moreover, let the finite dimensional subspace V_N of V be spanned by a Legendre polynomial basis $\{\phi_{i_1}\phi_{i_2}\cdots\phi_{i_n}\}_{i_1,i_2,\cdots,i_n=1}^{N}$, where each ϕ_m is defined by* (3.9)*. Then, we may select*

$$C_N^\tau = C_N^0\sqrt{1+\tau(C_N^0)^2},$$

for satisfying (3.8)*, where C_N^0 is computed by* (3.11)*.*

Proof. Let $\{(\lambda_i,\phi_i)\}_{i=1}^\infty$ be the set of eigenpairs of the problem:

$$-\Delta\phi = \lambda\phi \quad \text{in} \quad V,$$

where the derivatives on the left side are understood in the sense of distributions. Since the set of the eigenvalues $\{\phi_i\}_{i=1}^\infty$ forms an orthonormal basis of $L^2(\Omega)$, any $u \in H^2(\Omega)\cap V$ is expressed in the form $u = \sum_{i=1}^{\infty} a_i\phi_i$, where $a_i = (u,\phi_i)$. Therefore, it follows that

$$\begin{aligned}
\|-\Delta u + \tau u\|^2 &= \sum_{i=1}^{\infty} a_i^2\left\{(-\Delta\phi_i,-\Delta\phi_i) - 2\tau\left(-\Delta\phi_i,\phi_i\right) + \tau^2\left(\phi_i,\phi_i\right)\right\}\\
&= \sum_{i=1}^{\infty} a_i^2\left(\lambda_i^2 + 2\tau\lambda_i + \tau^2\right)(\phi_i,\phi_i)\\
&= \sum_{i=1}^{\infty} a_i^2\left(\lambda_i+\tau\right)^2.
\end{aligned}$$

This ensures that, for any $\tau \geq 0$,

$$\|-\Delta u\| \leq \|-\Delta u + \tau u\| \quad \text{for all } u \in H^2(\Omega)\cap V \tag{3.12}$$

Moreover, because of the definition of the projection P_N^τ, we have

$$\|u - P_N^\tau u\|_\tau \leq \left\|u - P_N^0 u\right\|_\tau . \tag{3.13}$$

Using Aubin-Nitsche's trick, we have

$$\|u - P_N^0 u\| \leq C_N^0 \|\nabla \left(u - P_N^0 u\right)\|. \tag{3.14}$$

From (3.12), (3.13), and (3.14), it follow that

$$\begin{aligned}
\|u - P_N^\tau u\|_\tau^2 \leq& \|\nabla(u - P_N^0 u)\|^2 + \tau \left\|u - P_N^0 u\right\|^2 \\
&\leq \left(1 + \tau (C_N^0)^2\right) \|\nabla \left(u - P_N^0 u\right)\|^2 \\
&\leq \left(1 + \tau (C_N^0)^2\right) (C_N^0)^2 \|-\Delta u\|^2 \\
&\leq \left(1 + \tau (C_N^0)^2\right) (C_N^0)^2 \|-\Delta u + \tau u\|^2 .
\end{aligned}$$

□

Chapter 4

The best constant for embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$

In this chapter, we consider the best constant for the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, i.e., the smallest constant C_p that satisfies (1.3) with $2 < p < \infty$.

Such constants are important in studies on partial differential equations (PDEs). In particular, verified numerical computation methods of our interest require explicit bounds for the embedding constant corresponding to a target equation at various points within them. Moreover, the precision in evaluating the embedding constants directly affects the precision of the verification results for the target equation. Occasionally, rough estimates of the embedding constants lead to failure in the verification. Therefore, accurately estimating such embedding constants is essential.

It is well known that the best constant in the classical Sobolev inequality has been proposed [1, 22] (see Theorem 2.11). A rough upper bound of C_p for a bounded domain $\Omega \subset \mathbb{R}^n$ can be obtained from the best constant by considering zero extension outside Ω (see Corollary 2.12). Moreover, Plum [20] proposed another estimation formula that requires not the boundedness of Ω but an explicit lower bound for the minimum eigenvalue of $-\Delta$ (see Theorem 2.14). Although these formulas enable us to easily compute the upper bound of C_p, little is known about the best constant.

In this chapter, we consider a numerical method for obtaining a verified sharp inclusion of the best constant C_p that satisfies (1.3) for $\Omega = (0,1)^2$. For the sake of convenience, we replace the notation C_p with C_{p+1} ($1 < p < \infty$). The smallest value of C_{p+1} can be written as

$$C_{p+1} = \sup_{u \in H_0^1(\Omega) \setminus \{0\}} \Phi(u), \tag{4.1}$$

where $\Phi(u) = \|u\|_{L^{p+1}(\Omega)} / \|u\|_{H_0^1(\Omega)}$.

The boundedness of C_{p+1} in (4.1) is ensured by considering zero extension outside Ω (see Corollary 2.12). In addition, it is true that the supremum C_{p+1} in (4.1) can be realized by an extremal function in $H_0^1(\Omega)$. A proof of this fact is sketched as follows. Let $\{u_i\} \in H_0^1(\Omega)$ be a sequence such that $\|u_i\|_{H_0^1(\Omega)} = 1$ and $\|u_i\|_{L^{p+1}(\Omega)} \to C_{p+1}$ as

$i \to \infty$. The Rellich–Kondrachov compactness theorem (see, e.g., [6, Theorem 7.22]) ensures that there exists a subsequence $\{u_{i_j}\}$ that converges to some u^* in $L^{p+1}(\Omega)$. Moreover, there exists a subsequence $\{u_{i_k}\} \subset \{u_{i_j}\}$ that converges to some $u' \in H_0^1(\Omega)$ in the weak topology of $H_0^1(\Omega)$ because $H_0^1(\Omega)$ is a Hilbert space. Since $\{u_{i_k}\}$ converges to u^* in $L^{p+1}(\Omega)$, it follows that $u^* = u'$. Hence, $u^* \in H_0^1(\Omega)(\subset L_{p+1}(\Omega))$ and $\|u^*\|_{L^{p+1}(\Omega)} = C_{p+1}$.

Since $|u| \in H^1(\Omega)$ for all $u \in H^1(\Omega)$ (see, e.g., [6, Lemma 7.6]) and $\Phi(u^*) = \Phi(|u^*|)$, we are looking for the extremal function u^* such that $u^* \geq 0$ (in fact, the later discussion additionally proves that $u^* > 0$ in Ω). The Euler-Lagrange equation for the variational problem is

$$\begin{cases} -\Delta u = lu^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

with some positive constant l (see, e.g., [3] for a detailed proof). Since Φ is scale-invariant (i.e., $\Phi(ku^*) = \Phi(u^*)$ for any $k > 0$), it suffices to consider the case that $l = 1$ for finding an extremal function u^* of Φ (recall that we consider the case that $p > 1$). Moreover, the strong maximum principle ensures that nontrivial solutions u to (4.2) such that $u \geq 0$ in Ω are positive in Ω. Therefore, in order to find the extremal function u^*, we consider the problem of finding weak solutions to the following problem:

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

This problem has a unique solution when $\Omega = (0,1)^2$. Therefore, we can obtain an inclusion of C_{p+1} as $\|u^*\|_{L^{p+1}(\Omega)} / \|u^*\|_{H_0^1(\Omega)}$ by enclosing the solution u^* to (4.3) using the method provided in Chapter 2.

As a verified result, we prove the following theorem by using our method through a computer-assisted technique:

Theorem 4.1. *For the square $\Omega = (0,1)^2$, the smallest values of C_p ($p = 3, 4, 5, 6, 7$) that satisfy* (1.3) *are enclosed as follows:*

$$
\begin{aligned}
C_3 &\in [0.25712475017618,\ 0.25712475017620];\\
C_4 &\in [0.28524446071925,\ 0.28524446071929];\\
C_5 &\in [0.31058015094505,\ 0.31058015094512];\\
C_6 &\in [0.33384042151102,\ 0.33384042151112];\\
C_7 &\in [0.35547994288611,\ 0.35547994288634].
\end{aligned}
$$

Remark 4.2. *Since it follows from a simple variable transformation that*

$$C_p((a,b)^2) = (b-a)^{\frac{2}{p}} C_p((0,1)^2), \tag{4.4}$$

the values in Theorem 4.1 *can be directly used for all squares $(a,b)^2$ $(-\infty < a < b < \infty)$ by multiplying them with $(b-a)^{2/p}$. Moreover, these values can be applied to deriving an explicit upper bound of $C_p(\Omega)$ for a general domain $\Omega \subset (a,b)^2$ by considering zero extension outside Ω, while the precision of the upper bound depends on the shape of Ω.*

4.1 Method for estimating the best embedding constant

In this section, we propose a method for estimating the embedding constant C_{p+1} defined in (4.1) for the square $\Omega = (0,1)^2$. The following theorem provides an explicit estimation of the embedding constant from a verified solution to (4.3).

Theorem 4.3. *Let* $\Omega = (0,1)^2$. *If there exists a solution to* (4.3) *in a closed ball* $B(\hat{u}, \alpha\ ;\ \|\cdot\|_{H_0^1(\Omega)})$ *with* $\hat{u} \in H_0^1(\Omega)$ *satisfying* $\|\hat{u}\|_{H_0^1(\Omega)} > 2\alpha$, *then the embedding constant* $C_{p+1}(\Omega)$ *defined in* (4.1) *is estimated as*

$$\frac{\|\hat{u}\|_{L^{p+1}(\Omega)}}{\|\hat{u}\|_{H_0^1(\Omega)}} \leq C_{p+1} \leq \frac{\|\hat{u}\|_{L^{p+1}(\Omega)}}{\|\hat{u}\|_{H_0^1(\Omega)} - 2\alpha}.$$

Proof. It is clear that $\|\hat{u}\|_{L^{p+1}(\Omega)} / \|\hat{u}\|_{H_0^1(\Omega)}$ is a lower bound of C_{p+1}. A solution to (4.3) is unique when $\Omega = (0,1)^2$. This was proved in [4], whereas the symmetric result [5] which was used in the proof, has to be replaced with [2]. Therefore, the ratio $\|u\|_{L^{p+1}(\Omega)} / \|u\|_{H_0^1(\Omega)}$ is maximized by the solution u to (4.3). By writing the solution to (4.3) as $\hat{u} + \alpha v$ with $v \in H_0^1(\Omega)$, $\|v\|_{H_0^1(\Omega)} \leq 1$, we have that

$$C_{p+1} = \frac{\|\hat{u} + \alpha v\|_{L^{p+1}(\Omega)}}{\|\hat{u} + \alpha v\|_{H_0^1(\Omega)}} \leq \frac{\|\hat{u}\|_{L^{p+1}(\Omega)} + \alpha C_{p+1}}{\|\hat{u}\|_{H_0^1(\Omega)} - \alpha}.$$

In other words, it follows that

$$\left(\|\hat{u}\|_{H_0^1(\Omega)} - 2\alpha\right) C_{p+1} \leq \|\hat{u}\|_{L^{p+1}(\Omega)}.$$

Hence, when $\|\hat{u}\|_{H_0^1(\Omega)} > 2\alpha$, $\|\hat{u}\|_{L^{p+1}(\Omega)} / (\|\hat{u}\|_{H_0^1(\Omega)} - 2\alpha)$ becomes an upper bound of C_{p+1}. □

4.2 Numerical result of the best constant

In this section, we present some numerical examples where the best values of the embedding constants on the square domain $\Omega = (0,1)^2$ are estimated to yield Theorem 4.1. The upper bounds for the embedding constants on the L-shaped domain $(0,2)^2 \backslash [1,2]^2$ through the application of Theorem 4.1 are also presented. All computations were carried out on a computer with Intel Xeon E7-4830 2.20 GHz×40 processors, 2 TB RAM, CentOS 6.6, and MATLAB 2012b. All rounding errors were strictly estimated by using toolboxes for the verified numerical computations: the INTLAB version 9 [21] and KV library version 0.4.16 [8]. Therefore, the accuracy of all results was guaranteed mathematically.

We consider the cases where $p = 2$, 3, 4, 5, and 6, which correspond to the critical point problems for embedding constants C_{p+1}. We computed approximate solutions $\hat{u}$ to (4.3), which are displayed in Fig. 4.1, with Legendre polynomials, i.e., we constructed $\hat{u}$ as

$$\hat{u} = \sum_{i,j=1}^{N} u_{i,j}\phi_i\phi_j, \quad u_{i,j} \in \mathbb{R}, \tag{4.5}$$

where each ϕ_i is defined by

$$\phi_n(x) = \frac{1}{n(n+1)} x(1-x)\frac{dP_n}{dx}(x), \quad n = 1, 2, 3, \cdots \tag{4.6}$$

with the Legendre polynomials P_n defined by

$$P_n = \frac{(-1)^n}{n!}\left(\frac{d}{dx}\right)^n x^n(1-x)^n, \quad n = 0, 1, 2, \cdots . \tag{4.7}$$

We then proved the existence of solutions u to (4.3) in an H_0^1-ball $B(\hat{u}, \alpha_1;\ \|\cdot\|_{H_0^1(\Omega)})$ and an L^∞-ball $B(\hat{u}, \alpha_2;\ \|\cdot\|_{L^\infty(\Omega)})$, both centered around the approximations $\hat{u}$. This was done on the basis of Theorems 2.3 and 2.7. The bound K for the inverse norm $\left\|\mathcal{F}_{\hat{u}}'^{-1}\right\|_{B(V^*,V)}$ was estimated by the method described in Chapter 3.

Table 4.2 presents the verification results, where we can find the condition (2.14) was satisfied. The last column in the table presents intervals containing $C_{p+1}\left((0,1)^2\right)$, e.g., 1.23_{456}^{789} represents the interval [1.23456,1.23789]. These intervals yield the results in Theorem 4.1. Table 4.2 compares the lower and upper bounds derived by our method, the upper bounds derived by Corollary 2.12, and the upper bounds derived by Plum's formula [20] (Theorem 2.14).

In addition, we applied the results of Theorem 4.1 to estimate the upper bounds of the embedding constants on $(0,2)^2\backslash[1,2]^2$. Since $(0,2)^2\backslash[1,2]^2 \subset (0,2)^2$, which is the smallest square that encloses $(0,2)^2\backslash[1,2]^2$, $C_p((0,2)^2\backslash[1,2]^2)$ is bounded by $2^{2/p}C_p((0,1)^2)$ owing to the discussion in Remark 4.2. Table 4.3 compares $2^{2/p}C_p((0,1)^2)$ derived by our method, the upper bounds for $C_p((0,2)^2\backslash[1,2]^2)$ derived by Corollary 2.12, and the upper bounds for $C_p((0,2)^2\backslash[1,2]^2)$ derived by Theorem 2.14. Theorem 2.14 requires a concrete value for the minimum eigenvalue of $-\Delta$. Therefore, we employed the result of $\lambda_1 \geq 9.5585$ presented in [13, Table 5.1].

Table 4.1: Verification results for the cases p = 2, 3, 4, 5, and 6 on $\Omega = (0,1)^2$. Except for the last column, these values represent the upper bounds for the corresponding constants. The upper bound for $C_{p+1}^{-\frac{p+1}{p-1}}$ was computed using the rough upper bound for C_{p+1} derived by Corollary 2.12.

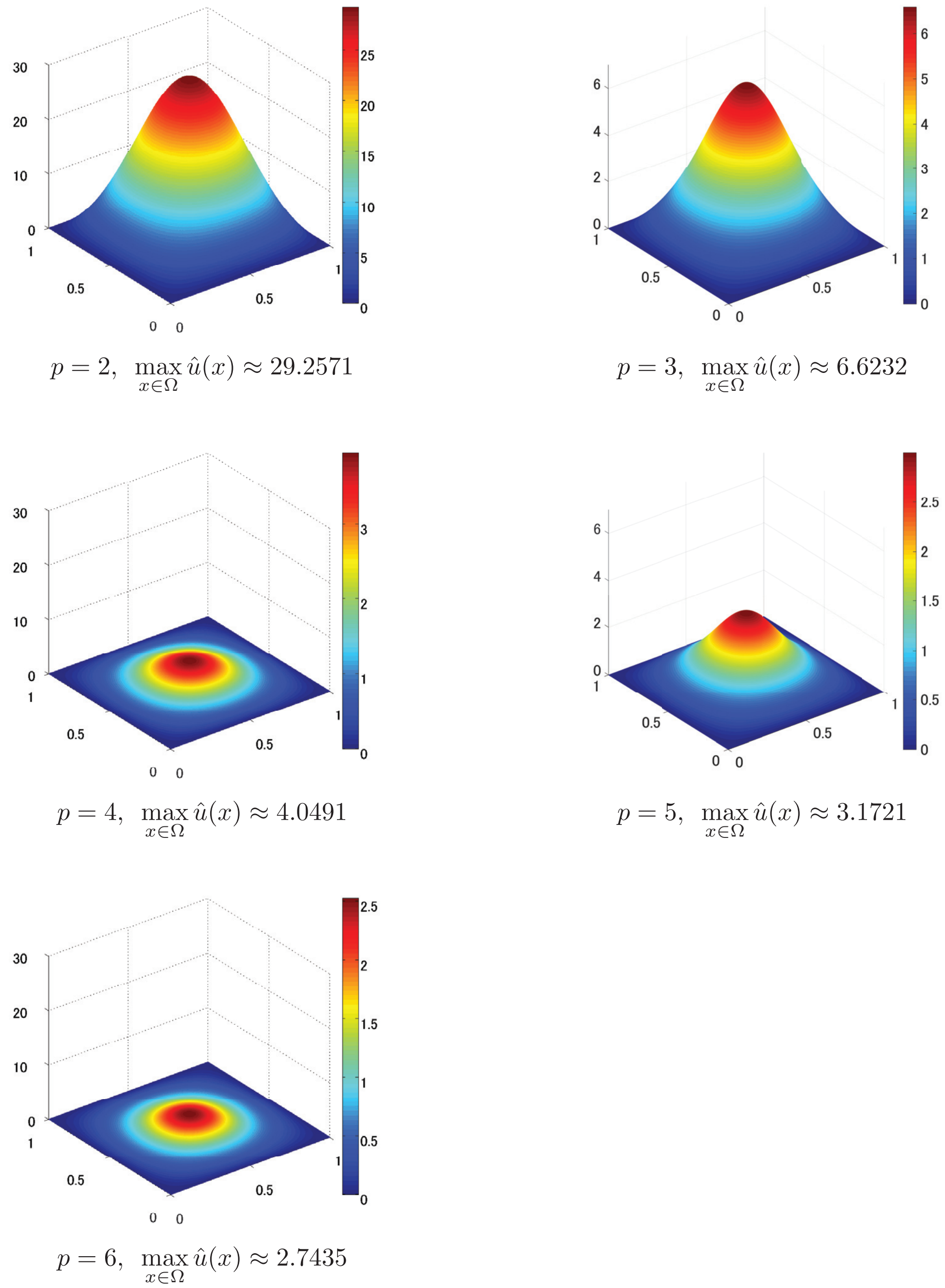
p	N	δ	K	α	β	$C_{p+1}^{-\frac{p+1}{p-1}}$ (rough)	C_{p+1} (best)
2	100	8.8360e-13	1.4589	1.2891e-12	3.6431e-12	-	0.257124750176^{20}_{18}
3	150	3.9872e-13	1.6644	6.6365e-13	4.3638e-12	9.8697	0.28524446071929^{9}_{5}
4	150	3.0202e-13	1.9342	5.8413e-13	2.0029e-11	-	0.310580150945^{12}_{05}
5	150	3.1562e-13	2.2451	7.0884e-13	1.7246e-10	4.0152	0.333840421511^{12}_{02}
6	200	4.8054e-13	2.7255	1.3106e-12	4.7697e-08	-	0.355479942886^{34}_{11}

Table 4.2: Estimates of C_p derived by our method, Corollary 2.12, and Theorem 2.14 for square $\Omega = (0,1)^2$.

C_p	Our method	Corollary 2.12	Theorem 2.14
C_3	0.257124750176^{20}_{18}	0.27991104681668	0.32964899322075
C_4	0.28524446071929^{9}_{5}	0.31830988618380	0.39894228040144
C_5	0.310580150945^{12}_{05}	0.35780388458051	0.48909030972535
C_6	0.333840421511^{12}_{02}	0.39585399866620	0.55266945714001
C_7	0.355479942886^{34}_{11}	0.43211185419351	0.63763213907292

Table 4.3: Same as Table 4.2 but for $\Omega = (0,2)^2 \backslash [1,2]^2$.

C_p	Our method ($2^{2/p} C_p((0,1)^2)$)	Corollary 2.12	Theorem 2.14
C_3	0.40816009891676	0.40370158699565	0.41978967493887
C_4	0.40339658494102	0.41891936927236	0.47823908300428
C_5	0.40981296610112	0.44572736933656	0.56542767015609
C_6	0.42061257436764	0.47539569585243	0.62367087563741
C_7	0.43333490417428	0.50554097277928	0.70723155088841



$p = 2,\ \max\limits_{x\in\Omega} \hat{u}(x) \approx 29.2571$

$p = 3,\ \max\limits_{x\in\Omega} \hat{u}(x) \approx 6.6232$

$p = 4,\ \max\limits_{x\in\Omega} \hat{u}(x) \approx 4.0491$

$p = 5,\ \max\limits_{x\in\Omega} \hat{u}(x) \approx 3.1721$

$p = 6,\ \max\limits_{x\in\Omega} \hat{u}(x) \approx 2.7435$

Figure 4.1: Approximate solutions to (4.3) on $\Omega = (0,1)^2$ for $p = 2, 3, 4, 5$, and 6.

Chapter 5

Numerical result for stationary problem of Allen-Cahn equation

In this Chapter, we consider the application of the verification method described in Chapters 2 and 3 to the stationary problems of the Allen-Cahn equation:

$$\begin{cases} -\Delta u = \varepsilon^{-2}(u - u^3) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

and also

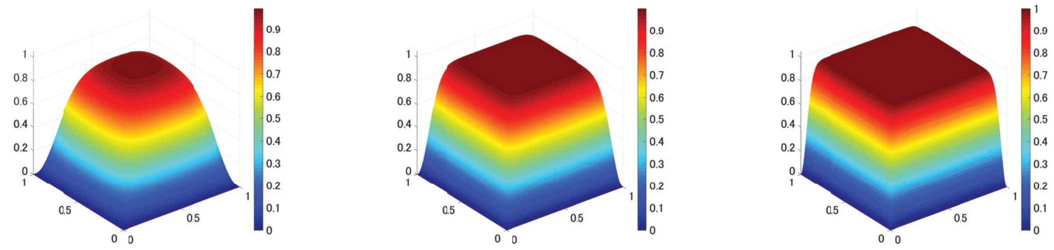
$$\begin{cases} -\Delta u = \varepsilon^{-2}(u - u^3) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where $\varepsilon > 0$. The small parameter $\varepsilon > 0$ causes the singular perturbation of solutions to the above problems, which makes their verification difficult. Indeed, we observed that small ε makes the constants required in the verification process (δ, K, and so on) large.

All computations were carried out on a computer with Intel Xeon E7-4830 2.20 GHz$\times$40 processors, 2 TB RAM, CentOS 6.6, and MATLAB 2012b. All rounding errors were strictly estimated by using toolboxes for the verified numerical computations: the INTLAB version 9 [21] and KV library version 0.4.16 [8]. Therefore, the accuracy of all results was guaranteed mathematically.

5.1 Positive solutions

In this section, we present verification results for positive solutions to (5.1), i.e., solutions to (5.2). We constructed approximate solutions $\hat{u}$ to problem (5.2) using a Legendre polynomial basis. These solutions are displayed in Fig. 5.1. On the basis of Theorem 2.9, we verified the existence of solutions to (5.2) in the balls $B(\hat{u}, \alpha; \|\nabla \cdot\|_{L^2(\Omega)})$ and $B(\hat{u}, \beta; \|\cdot\|_{L^\infty(\Omega)})$. We present the verification results for $\varepsilon = 0.1$, 0.05, and 0.025 in Table 5.1. To check the condition required in Theorem 2.9, we set $\Omega_- = (0,1)^2 \backslash [0.009765625, 0.990234375]^2$ and proved $\lambda_1(\mathrm{supp}\,(\hat{u}-\beta)_-) \subset \Omega_-$ in all the cases, for computing the lower bounds of $\lambda_1(\mathrm{supp}\,(\hat{u}-\beta)_-)$ (see again Remark 2.10). The upper and lower bounds for the first eigenvalue $\lambda_1(\Omega_-)$ were rigorously computed using the method in [12, 11] with a piecewise linear finite element basis.



$\varepsilon = 0.1$ $\varepsilon = 0.05$ $\varepsilon = 0.025$

Figure 5.1: Approximate solutions to (5.2) on $\Omega = (0,1)^2$.

Table 5.1: Verification results for (5.2) on $\Omega = (0,1)^2$.

ε	N	δ	K	α	β	ε^{-2}	$\lambda_1(\Omega_-) \in$
0.1	80	3.4571e-16	2.7081	5.8208e-16	4.5702e-15	1.0e+02	$[0.9585, 1.0032]$e+05
0.05	80	2.6679e-14	3.5469	9.4879e-14	3.9127e-12	4.0e+02	″
0.025	80	3.5439e-09	3.9098	1.3856e-08	2.6113e-06	1.6e+03	″

5.2 Nonpositive solutions

In this section, we present verification results for nonpositive solutions to (5.1). We again constructed approximate solutions $\hat{u}$ to problem (5.1) using a Legendre polynomial basis, which are displayed in Fig. 5.2. On the basis of the method described in Chapters 2 and 3, we verified the existence of solutions to (5.1) in the balls

$B(\hat{u}, \alpha; \|\nabla \cdot\|_{L^2(\Omega)})$ and $B(\hat{u}, \beta; \|\cdot\|_{L^\infty(\Omega)})$. We present the verification results for $\varepsilon = 0.1$, 0.08, 0.06, and 0.04 in Table 5.2.

Table 5.2: Verification results for (5.1) on $\Omega = (0,1)^2$.

ID	ε	N	δ	K	α	β
(A)	0.1	100	3.5313e-17	1.1129e+03	3.9582e-14	1.4449e-13
	0.08	100	4.8566e-14	1.0333e+01	5.0234e-13	1.0207e-11
	0.06	100	1.2760e-09	5.3305e+01	6.8014e-08	3.1047e-06
	0.04	150	1.2163e-09	3.1060e+03	3.8993e-06	4.9775e-04
(B)	0.1	100	3.2618e-17	2.6031e+02	8.7312e-15	3.1113e-14
	0.08	100	2.3806e-16	1.4712e+01	4.0746e-15	5.4577e-14
	0.06	100	3.0612e-14	1.2236e+01	3.7486e-13	1.3881e-11
	0.04	120	7.0273e-11	1.6153e+03	1.1357e-07	1.4317e-05
(C)	0.1	80	3.2642e-17	2.6279e+02	8.7312e-15	3.1120e-14
	0.08	80	9.0597e-14	1.2806e+01	1.1607e-12	1.8513e-11
	0.06	80	6.4217e-10	1.3308e+01	8.5457e-09	3.2865e-07
	0.04	120	6.9481e-10	8.1587e+02	5.6752e-07	7.1235e-05

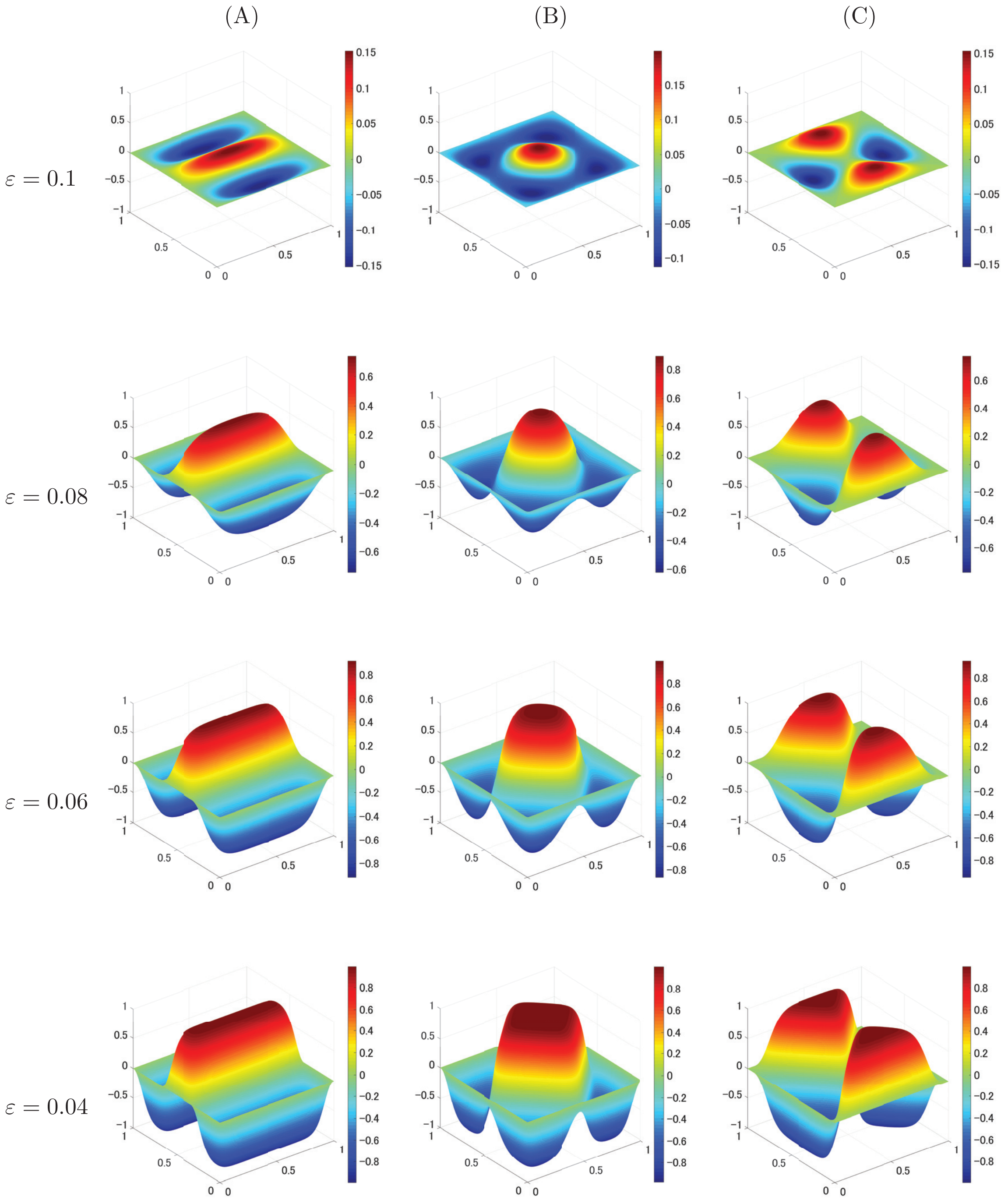


Figure 5.2: Approximate (nonpositive) solutions to (5.1) on $\Omega = (0,1)^2$.

Chapter 6

Conclusion

In this thesis, we proposed verified numerical computation methods for solutions to the problem (1.1) and (1.2) on bounded polygonal domain Ω. We treated the cases in which $f(t) = |t|^{p-1}t$ $(p > 1)$ and $f(t) = \varepsilon^{-2}(t - t^3)$ with a small parameter $\varepsilon > 0$ related with the so called singular perturbation phenomenon. In Chapter 2, we introduced a verification theory for deriving H_0^1- and L^∞-estimations of a given numerical approximation of a solution to (1.1). With imposing some additional condition on the numerical approximation, this theory can be extended to the verification of a positive solution to (1.1), i.e., a solution to (1.2). When we consider the L^∞-estimation of an approximate solution, the convexity of Ω is additionally required. In Chapter 3, we proposed a method for estimating the norm bound $\left\|\mathcal{F}_{\hat{u}}'^{-1}\right\|_{B(V^*,V)}$. This method is based on Theorem 3.2, and the problem of estimating this norm bound is reduced to the eigenvalue problem (3.5). We estimated the eigenvalues of the problem (3.5) on the basis of Theorem 3.3. In Chapter 4, we proposed a method of evaluating the best constant $C_p(\Omega)$ for the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ with $\Omega = (0,1)^2$. The best constant of $C_p(\Omega)$ is achieved by a solution (4.3), and actually (4.3) has a unique solution when $\Omega = (0,1)^2$. We derived sharp inclusions of the best constant by verifying the solution to (4.3) using the method provided in Chapters 2 and 3. In Chapter 5, we applied the methods proposed in this thesis to the verified numerical computation for stationary problem of Allen-Cahn equation.

In future work, we would like to extend our verification method to more general problems, e.g., problem (1.1) with more complicated domain Ω (including unbounded one), parabolic and hyperbolic partial differential equations, and partial differential equations of higher order.

Bibliography

[1] Thierry Aubin. Problèmes isopérimétriques et espaces de Sobolev. *Journal of Differential Geometry*, 11(4):573–598, 1976.

[2] Henri Berestycki and Louis Nirenberg. On the method of moving planes and the sliding method. *Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society*, 22(1):1–37, 1991.

[3] Tom Carroll and Jesse Ratzkin. Interpolating between torsional rigidity and principal frequency. *Journal of Mathematical Analysis and Applications*, 379(2):818–826, 2011.

[4] Lucio Damascelli, Massimo Grossi, and Filomena Pacella. Qualitative properties of positive solutions of semilinear elliptic equations in symmetric domains via the maximum principle. In *Annales de l'IHP Analyse non linéaire*, volume 16, pages 631–652, 1999.

[5] Basilis Gidas, Wei-Ming Ni, and Louis Nirenberg. Symmetry and related properties via the maximum principle. *Communications in Mathematical Physics*, 68(3):209–243, 1979.

[6] David Gilbarg and Neil S Trudinger. *Elliptic partial differential equations of second order*, volume 224. Springer Science & Business Media, 2001.

[7] Pierre Grisvard. *Elliptic problems in nonsmooth domains*, volume 69. SIAM, 2011.

[8] Masahide Kashiwagi. KV library, 2016. `http://verifiedby.me/kv/`.

[9] Seiji Kimura and Nobito Yamamoto. On explicit bounds in the error for the H_0^1-projection into piecewise polynomial spaces. *Bulletin of informatics and cybernetics*, 31(2):109–115, 1999.

[10] Pierre Louis Lions. On the existence of positive solutions of semilinear elliptic equations. *SIAM review*, 24(4):441–467, 1982.

[11] Xuefeng Liu. A framework of verified eigenvalue bounds for self-adjoint differential operators. *Applied Mathematics and Computation*, 267:341–355, 2015.

[12] Xuefeng Liu and Shin'ichi Oishi. Verified eigenvalue evaluation for the laplacian over polygonal domains of arbitrary shape. *SIAM Journal on Numerical Analysis*, 51(3):1634–1654, 2013.

[13] Xuefeng Liu, Tomoaki Okayama, and Shin'ichi Oishi. High-precision eigenvalue bound for the laplacian with singularities. In *Computer Mathematics*, pages 311–323. Springer, 2014.

[14] P.J. McKenna, F. Pacella, M. Plum, and D. Roth. A uniqueness result for a semilinear elliptic problem: A computer-assisted proof. *Journal of Differential Equations*, 247(7):2140–2162, 2009.

[15] Mitsuhiro T Nakao. A numerical approach to the proof of existence of solutions for elliptic problems. *Japan Journal of Applied Mathematics*, 5(2):313–332, 1988.

[16] Mitsuhiro T Nakao, Nobito Yamamoto, and Seiji Kimura. On the best constant in the error bound for the H_0^1-projection into piecewise polynomial spaces. *Journal of Approximation Theory*, 93(3):491–500, 1998.

[17] Michael Plum. Computer-assisted existence proofs for two-point boundary value problems. *Computing*, 46(1):19–34, 1991.

[18] Michael Plum. Explicit H^2-estimates and pointwise bounds for solutions of second-order elliptic boundary value problems. *Journal of Mathematical Analysis and Applications*, 165(1):36–61, 1992.

[19] Michael Plum. Computer-assisted enclosure methods for elliptic differential equations. *Linear Algebra and its Applications*, 324(1):147–187, 2001.

[20] Michael Plum. Existence and multiplicity proofs for semilinear elliptic boundary value problems by computer assistance. *Jahresbericht der Deutschen Mathematiker Vereinigung*, 110(1):19–54, 2008.

[21] Siegfried M. Rump. INTLAB - INTerval LABoratory. In Tibor Csendes, editor, *Developments in Reliable Computing*, pages 77–104. Kluwer Academic Publishers, Dordrecht, 1999. `http://www.ti3.tuhh.de/rump/`.

[22] Giorgio Talenti. Best constant in Sobolev inequality. *Annali di Matematica pura ed Applicata*, 110(1):353–372, 1976.

[23] Kazuaki Tanaka, Akitoshi Takayasu, Xuefeng Liu, and Shin'ichi Oishi. Verified norm estimation for the inverse of linear elliptic operators using eigenvalue evaluation. *Japan Journal of Industrial and Applied Mathematics*, 31(3):665–679, 2014.

早稲田大学　博士（工学）　学位申請　研究業績書

氏 名　　田中　一成　印

（2016 年　12 月　現在）

種 類 別	題名、　発表・発行掲載誌名、　発表・発行年月、　連名者（申請者含む）
論文	[1] Akitoshi Takayasu, Kaname Matsue, Takiko Sasaki, Kazuaki Tanaka, Makoto Mizuguchi, Shin'ichi Oishi: Numerical validation of blow-up solutions for ODEs, to appear in Journal of Computational and Applied Mathematics.
	[2] Kazuaki Tanaka, Kouta Sekine, Makoto Mizuguchi, and Shin'ichi Oishi: Sharp numerical inclusion of the best constant for embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ on bounded convex domain, Journal of Computational and Applied Mathematics, 311, 306–313 (2017), to appear. Electronically published in doi.org/10.1016/j.cam.2016.07.021.
	[3] Kazuaki Tanaka, Kouta Sekine, Makoto Mizuguchi, and Shin'ichi Oishi: Estimation of the Sobolev embedding constant on domains with minimally smooth boundary using extension operator, Journal of Inequalities and Applications, 1, 1-23 (2015).
	[4] Kazuaki Tanaka, Kouta Sekine, Makoto Mizuguchi, and Shin'ichi Oishi: Numerical verification of positiveness for solutions to semilinear elliptic problems, JSIAM Letters 7, 73-76 (2015).
	[5] Kazuaki Tanaka, Akitoshi Takayasu, Xuefeng Liu, and Shin'ichi Oishi: Verified norm estimation for the inverse of linear elliptic operators using eigenvalue evaluation, Japan Journal of Industrial and Applied Mathematics, 31, 665-679 (2014).
講演	[6] 田中一成, 関根晃太, 大石進一：楕円型微分方程式の正値解に対する精度保証付き数値計算法（Verified numerical computation method for positive solutions to elliptic differential equations）, RIMS 研究集会「現象解明に向けた数値解析学の新展開 II」, 2016 年 10 月 19 日～10 月 21 日.
	[7] 関根晃太, 田中一成, 大石進一：ある無限次元固有値を用いた楕円型偏微分方程式の解の存在性に対する計算機援用証明法（Computer-assisted proof for existence of solutions to PDEs using an infinite eigenvalue）, RIMS 研究集会「現象解明に向けた数値解析学の新展開 II」, 2016 年 10 月 19 日～10 月 21 日.
	[8] 関根晃太, 田中一成, 大石進一：有界な凸領域における連立楕円型偏微分方程式の解の計算機援用存在証明法（Computer assisted existence proof of solutions to system of partial differential equations with bounded convex polygonal domains）, The Twenty-Eighth RAMP Symposium, 2016 年 10 月 13 日～10 月 14 日.
	[9] Kazuaki Tanaka, Kouta Sekine, Shin'ichi Oishi: On verified numerical computation for positive solutions to elliptic boundary value problems, Computer Arithmetic and Validated Numerics, SCAN2016, Sep. 26-29, 2016.
	[10] Kouta Sekine, Kazuaki Tanaka, Shin'ichi Oishi: A norm estimation for an inverse of linear operator using a minimal eigenvalue, Computer Arithmetic and Validated Numerics, SCAN2016, Sep. 26-29, 2016.
	[11] Akitoshi Takayasu, Kaname Matsue, Takiko Sasaki, Kazuaki Tanaka, Makoto Mizuguchi, and Shin'ichi Oishi: Verified numerical computations for blow-up solutions of ODEs, Computer Arithmetic and Validated Numerics, SCAN2016, Sep. 26-29, 2016.

早稲田大学　博士（工学）　学位申請　研究業績書

種 類 別	題名、　発表・発行掲載誌名、　発表・発行年月、　連名者（申請者含む）
講演	[12] Kazuaki Tanaka, Shin'ichi Oishi: On verified numerical computation for elliptic Dirichlet boundary value problems using sub- and super-solution method, The fifth Asian conference on Nonlinear Analysis and Optimization, Toki Messe, Niigata, Japan, August 1-6, 2016. [13] Kouta Sekine, Kazuaki Tanaka, Shin'ichi Oishi: Estimation for optimal constant satisfying an inequality for linear operator using minimal eigenvalue, The fifth Asian conference on Nonlinear Analysis and Optimization, Toki Messe, Niigata, Japan, August 1-6, 2016. [14] Kaname Matsue, Akitoshi Takayasu, Takiko Sasaki, Kazuaki Tanaka, Makoto Mizuguchi, and Shin’ichi Oishi: Rigorous numerics of blowup solutions for ODEs, The 11th AIMS Conference on Dynamical Systems, Differential Equations and Applications, July 1-5, 2016. [15] Kazuaki Tanaka, Kouta Sekine, Shin'ichi Oishi: Numerically verifiable condition for positivity of solution to elliptic equation, The 11th East Asia SIAM. June, 20-22, 2016. [16] 高安亮紀, 松江要, 佐々木多希子, 田中一成, 水口信, 大石進一：放物面コンパクト化を用いる常微分方程式の爆発解の数値的検証法, 日本応用数理学会 2015 年度連合発表会，神戸学院大学ポートアイランドキャンパス，2016 年 3 月 4 日～3 月 5 日. [17] Kazuaki Tanaka, Kouta Sekine, Makoto Mizuguchi, Shin'ichi Oishi: Numerical verification for positiveness of solutions to self-adjoint elliptic problems, JSST 2015 International Conference on Simulation Technology, Oct. 12-14, 2015. [18] Kazuaki Tanaka and Shin'ichi Oishi: Computer-assisted analysis for solutions to nonlinear elliptic Neumann problems, JSST 2014 International Conference on Simulation Technology, Oct. 29-31, 2014. [19] Kazuaki Tanaka and Shin'ichi Oishi: Numerical verification for periodic stationary solutions to the Allen-Cahn equation, The16th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics, SCAN2014, Sep. 21-26, 2014. [20] 高安亮紀，松江要，佐々木多希子，田中一成，水口信，大石進一：Verified numerical enclosure of blow-up time for ODEs，日本数学会 2015 年度年会，京都産業大学，2015 年 9 月 13 日～9 月 16 日. [21] 田中一成，関根晃太，水口信，大石進一：楕円型偏微分方程式の解の正値性に対する数値的検証法，日本応用数理学会 2015 年度年会，金沢大学，2015 年 9 月 9 日～9 月 11 日. [22] 若山馨太，田中一成，関根晃太，尾崎克久，大石進一：逐次添加法による三角形分割の Delaunay 性に対する数値的検証法，日本応用数理学会 2015 年度年会，金沢大学，2015 年 9 月 9 日～9 月 11 日. [23] 高安亮紀，松江要，佐々木多希子，田中一成，水口信，大石進一：常微分方程式の爆発解に対する精度保証付き数値計算，日本応用数理学会 2015 年度年会，金沢大学，2015 年 9 月 9 日～9 月 11 日.

早稲田大学　博士（工学）　学位申請　研究業績書

種 類 別	題名、　発表・発行掲載誌名、　発表・発行年月、　連名者（申請者含む）
講演	[24] 関根晃太，田中一成，高安亮紀，山崎憲：シグマノルムを利用した精度保証付き数値計算法の連立楕円型偏微分方程式への応用，第４７回日本大学生産工学部学術講演会，日本大学，2014 年 12 月 6 日.
	[25] 田中一成, 水口信, 関根晃太, 大石進一 : An a priori estimation of the Sobolev embedding constant and its application to numerical verification for solutions to PDEs, 第 10 回日本応用理学会研究部会連合発表会，京都大学吉田キャンパス総合研究８号館， 2014 年 3 月 19 日～3 月 20 日.
	[26] Kazuaki Tanaka and Shin'ichi Oishi: Numerical verification for stationary solutions to the Allen-Cahn equation, The International Workshop on Numerical Verification and its Applications, Waseda Univ. Nishiwaseda campus, Japan, March, 2014.
	[27] Kazuaki Tanaka, Makoto Mizuguchi, Kouta Sekine, Akitoshi Takayasu, Shin'ichi Oishi: Estimation of an embedding constant on Lipschitz domains using extension operators, JSST 2013 International Conference on Simulation Technology. Sep. 11-13, 2013.
	[28] 田中一成, 高安亮紀, 劉雪峰, 大石進一 : 線形楕円型作用素の Neumann 条件下における精度保証付き逆作用素ノルム評価, 日本応用数理学会 2013 年度年会, アクロス福岡, 福岡県福岡市, 2013 年 9 月 9 日～9 月 11 日.
	[29] Kazuaki Tanaka, Akitoshi Takayasu, Xuefeng Liu, Shin'ichi Oishi: Verified norm estimation for the inverse of linear elliptic operators and its application, The 9th East Asia SIAM. June 18-20, 2013.
	[30] 田中一成，高安亮紀，劉雪峰，大石進一：逆作用素ノルム評価を用いた楕円型 Neumann 境界値問題の解に対する精度保証付き数値計算，日本応用数理学会 2013 年度連合発表会，東洋大学白山キャンパス, 2013 年 3 月 14 日～3 月 15 日.
	[31] 田中一成, 高安亮紀, 劉雪峰, 大石進一 : 線形楕円型作用素の Neumann 条件下における精度保証付き逆作用素ノルム評価, 日本応用数理学会 2012 年度年会, 稚内全日空ホテル, 北海道稚内市, 2012 年 8 月 28 日～9 月 2 日.　（その他講演５件）
その他（ポスター発表）	[32] 若山馨太，田中一成，関根晃太，尾崎克久，大石進一：Delaunay 三角形分割の精度保証付き数値計算手法に対する考察，日本応用数理学会 2016 年度年会，北九州国際会議場，2016 年 9 月 12 日～9 月 14 日.　（その他ポスター発表２件）
その他（受賞）	[33] 若山馨太，田中一成，関根晃太，尾崎克久，大石進一，日本応用数理学会 2016 年度年会，優秀ポスター賞 受賞.
	[34] 田中一成，2016 年度大川功記念特別優秀賞 受賞.
	[35] Kazuaki Tanaka, JSST 2014 International Conference, Student Presentation Award 受賞.
	[36] Kazuaki Tanaka, JSST 2013 International Conference, Student Presentation Award 受賞.