Doctoral Dissertation

Essays on variants of the Shapley value: Axiomatization and relationship to the core

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Chapter 1

Introduction

1.1 Outline of cooperative game theory

This Ph.D. thesis aims to contribute to *cooperative game theory*, as originally proposed by von Neumann and Morgenstern (1944). "Game theory aims to help us understand situations in which decision-makers interact" (Osborne (2009), p.1). Non-cooperative game theory targets the interactions between players that result from their chosen strategies. In contrast, cooperative game theory targets interactions caused by *conflicts among coalitions*. Several situations can result in conflict, such as cost-allocation problems, voting, matching problems, and markets, all of which fall within the scope of the theory.

Among the many problems studied using cooperative game theory, this thesis focuses on the following problem: "What is a fair assessment of individual responsibilities in the formation of total cost (or surplus)?" (Moulin (2004), p.139). As a first step to tackle this problem, we construct an economic model that is "intended to be a simplified description of the part of the economy that is relevant for the analysis" (Hindriks and Myles (2006), p.4). We abstract the following two components from the target situation: the set of agents involved, and the attainable outcomes for each coalition. Considered together, these two components are called a *game*.

A distinctive feature of the theory is that it focuses on *coalitions*, which often play an important role in allocation problems. To illustrate this, we refer to the cost-allocation problem studied by Moulin (2004):

> *This four-story building has one apartment on each of the second, third, and fourth floors $\cdots$. The manager of the building wishes to split fairly the cost of running an elevator to the three apartments. The cost of an elevator serving only the second floor is \$5,000. That of an elevator serving the second and third floors is \$10,000. An elevator serving all floors would cost \$40,000 $\cdots$.*
>
> (Moulin (2004), p.11)

How should the manager split the total cost of \$40,000? To solve this problem, we need to consider the coalition comprising apartments 2 and 3. If the total imputed cost is higher than \$10,000, then this is unfair because "each one of apartments 2 and 3 pays more than the full cost of an elevator stopping at its own floor" (Moulin (2004), p.12).

There are many examples of fair-division problems[1] in which individuals and coalitions

[1] For other examples, see Chapter 5 of Moulin (2004).

play key roles. Cooperative game theory offers a simple and general model for analyzing such problems.

In cooperative game theory, we describe the attainable outcomes for each coalition using attainable utility profiles. Following the terminology of cooperative game theory, we refer to utility as a *payoff* in the remainder of the paper. Here, we do not focus on the process of deriving a game from a target situation. Instead, assuming games are already given, we investigate a universal rule that describes how to distribute the total payoff obtained as a result of cooperation among all players. Such a rule is called a *solution*, and the resulting payoff distribution is called a *payoff vector*. There are two types of solutions: single-valued solutions, and set-valued solutions. A single-valued solution describes a single payoff vector for each game, while a set-valued solution describes a set of payoff vectors for each game. An example of a single-valued solution is the *Shapley value*, developed by Shapley (1953). The value determines each player's final payoff based on his/her contributions to the attainable payoffs for each coalition. An example of a set-valued solution is the *core*. This describes a set of payoff vectors that no coalition can improve upon on its own, representing "stable" outcomes against coalitional deviations.

A major contribution of cooperative game theory to economics is that it reveals the theoretical properties of the two solutions, enabling them to be applied to a variety of problems. Scarf (1967) identified a sufficient condition for the nonemptiness of the core, which Kaneko (1982) then applied to markets with indivisible goods. A line of literature on the Shapley value has uncovered its desirability as a distribution rule, and the value has subsequently provided a guide for fair division in many problems, such as sharing the cost of constructing an airport runway (Littlechild and Thompson (1977)), allocating the cost of cleaning a polluted river (Ni and Wang (2007)), and sharing the cost of damage caused jointly by several tortfeasors (Dehez and Ferey (2013)).

The remainder of this section explains several technical terms used in cooperative game theory.

TU game and NTU game

Depending on a target situation, games can take one of two forms, described here based on the work of Kaneko and Wooders (2004). The first is *transferable utility games*, abbreviated as TU games. To derive a TU game from a target situation, we implicitly assume that the players have quasi-linear utility functions and that monetary transfers are possible among the players. Under these assumptions, the set of attainable payoffs is described by a real number,[2] which makes the analysis simple and tractable. The second form is *non-transferable utility games*, abbreviated as NTU games. In an NTU game, we describe the attainable payoffs by a subset of a vector space. An NTU game is constructed without the two assumptions of TU games, and targets problems that are more general than those of TU games.

The core is defined for both TU and NTU games, but the Shapley value is defined only for TU games. A typical way of defining a single-valued solution for an NTU game is to extend the Shapley value. In this case, we introduce a solution that assigns a payoff vector to each NTU game and, in the class of TU games, assigns the same payoff vector as the Shapley value. These extensions are defined formally in Chapter 4.

[2]For a formal proof of this statement, see, for example, Proposition 2.1 in Kaneko and Wooders (2004).

Axiomatization

Axiomatization (or axiomatic characterization) is a method used to characterize solutions. It consists of the following two steps:

(1) We introduce some desirable properties (called *axioms*) that should be satisfied by a solution.

(2) We identify a unique solution satisfying the axioms.

In (1), we judge desirability based on our sense of equity or fairness. For example, an axiom called *symmetry* states that if two players make the same contributions in a game, then they should receive the same payoff.

In game theory, axiomatization was first studied by Nash (1950) in the context of bargaining problems. In terms of TU games, Shapley (1953) provided an axiomatization of the Shapley value using the four classical axioms of efficiency, symmetry, the null player property, and additivity.[3] Axiomatization helps us understand the difference between solutions from the viewpoint of the axioms that characterize them.

1.2 Remarks on "fairness" or "desirability"

The purpose of this section is to clarify what kind of information underlies the justification for fairness or desirability.

As detailed in Section 1.1, we describe a game in terms of attainable utility profiles and then discuss fairness/desirability *at the utility level*. Personal characteristics (e.g., physical conditions, historical backgrounds), which often have significant meaning when considering fairness, cannot be discussed unless they are reflected faithfully in utility functions.

To illustrate this point, we consider the income distribution problem studied by Sen (1997).

> *Consider two income distributions x and y, with identical total, over a collection of n people who are symmetric in all respects except that person 1 works in a nasty coal mine and has tougher working conditions than persons 3 to n, while person 2 works under pleasant working conditions than these other persons.*
>
> (Sen (1997), p.80)

For simplicity, we assume that the n people are identical in terms of their attainable income. If we formulate this situation as a TU game (i.e., quasi-linearity is imposed on the utility functions), then the utilities of agents 1 and 2 are measured by money and, therefore, we cannot reflect the difference in their working conditions. The symmetry axiom in cooperative game theory concludes that an equal split of the total income is "desirable." However, the "assumption of symmetry in the *evaluation* of income distributions may, therefore, have to be rejected $\cdots$ because of differences in non-income characteristics (e.g., particular working conditions)" (Sen (1997), p. 80).

[3] These axioms are defined formally in Section 1.4.1.

A game is an extreme simplification of a target situation. On the one hand, this simplification has the disadvantage of making it difficult to incorporate information not directly related to attainable payoffs. On the other hand, this simplification makes it possible to expand the applicability of cooperative game theory, as explained in Section 1.1.

In Section 4.2, we introduce the notion of a *weight*, which enables us to take a step closer to a target situation.

1.3 Overview of this thesis

The main contribution of this Ph.D. thesis is to provide new results on the relationship between the Shapley value and other solutions, both in TU games and in NTU games.

The first part of this thesis is devoted to analyses in TU games. To analyze the relationship between the Shapley value and other solutions, it is essential to understand the Shapley value itself. Thus, in Chapter 2, we develop new mathematical tools for analyzing the value. Mathematically, the Shapley value is written as a linear function from a linear space (the set of TU games) to another linear space (the set of payoff vectors). We introduce a new basis for the domain of the value, called the *commander games*, and examine its properties. The new basis identifies the set of TU games to which the Shapley value assigns the 0 vector, and clarifies how the Shapley value is determined in a TU game. We further extend the commander games to introduce new bases that have desirable properties related to the Shapley value.

In Chapter 3, we provide new axiomatizations of solutions by using *monotonicity*.[4] A monotonicity axiom in cooperative games states an increase in some parameters of a game as a hypothesis, and states an increase in a player's payoff as a conclusion. Young (1985) first introduced this type of axiom, called *strong monotonicity*. This axiom states that if a player's contributions weakly increase, then the player's final payoff should also weakly increase. Later, van den Brink et al. (2013) introduced a weakened axiom called *weak monotonicity*. This axiom states that if a player's contributions and the attainable payoffs for all players weakly increase, then the player's payoff also weakly increases. Under efficiency and symmetry, Young (1985) proved that strong monotonicity characterizes the Shapley value, and Casajus and Huettner (2014) proved that weak monotonicity characterizes the class of egalitarian Shapley values introduced by Joosten (1996). An egalitarian Shapley value takes a convex combination of the Shapley value payoff and an equal share of the total attainable payoff. Here, we develop the above line of literature further. We introduce new monotonicity axioms, and show that a monotonicity axiom and standard axioms characterize various solutions. More specifically, we characterize (i) four linear solutions in the literature, namely, the Shapley value, the equal division value, the CIS value, and the ENSC value, and (ii) a class of solutions obtained by taking a convex combination of the above solutions. With our characterizations, the differences between solutions can be explained comprehensively using the differences between the monotonicity axioms. In the proof of theorems, we utilize a basis developed in Chapter 2.

In Chapter 4, we focus on the relationship between the Shapley value and the core. A seminal paper by Monderer et al. (1992) describes a critical relationship between the two: in the class of TU games, any element of the core is attainable as the outcome of a

[4]See Sprumont (2008) for a survey on monotonicity in economics.

weighted Shapley value. A weighted Shapley value is an extension of the Shapley value that incorporates players' asymmetric characteristics. The core is a widely accepted solution in many research fields (e.g., market theory and matching theory). Thus, its relationship to weighted Shapley values seems to represent a bridge between cooperative game theory and other research fields. However, this is not necessarily the case, because of the underlying assumptions behind TU games. As discussed in Section 1.1, we implicitly assume that agents have quasi-linear utility functions and that monetary transfers are allowed. Quasi-linearity implies there is no income effect, which is a severe restriction in economic models. To circumvent this difficulty, we extend Monderer et al.'s (1992) result to NTU games. As an extension of the weighted Shapley value to NTU games, we focus on the *weighted egalitarian solutions* introduced by Kalai and Samet (1985). We prove that, in the class of NTU games, any element of the core is attainable as the outcome of a weighted egalitarian solution. Because weighted egalitarian solutions are supported by normative axioms, our result provides a normative foundation for the core. We further examine the relationship between the core and other extensions of weighted Shapley values to NTU games.

Structure of this thesis

Section 1.4 deals with preliminaries. In Chapter 2, we introduce a new basis for the set of TU games, and discuss its extensions. In Chapter 3, we provide new axiomatizations of solutions in TU games using monotonicity. In Chapter 4, we show the relationship between the core and the Shapley value in NTU games. Then, Chapter 5 discusses possible areas of future research and concludes this thesis.

Original published papers

Each chapter in this thesis is based on a paper published in a peer-reviewed journal. Chapter 2 is based on Yokote et al. (2016), published in *Mathematical Social Sciences*. Chapter 3 is based on Yokote and Funaki (2017), published in *Social Choice and Welfare*. Chapter 4 is based on Yokote (2017), published in *International Journal of Game Theory*.

1.4 Preliminaries

Let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{Q}$ denote the set of rational numbers, and $\mathbb{R}$ be the set of real numbers. For two sets A and B, $A \subseteq B$ means that A is a subset of B, and $A \subset B$ means that $A \subseteq B$ and $A \neq B$. Let $|A|$ denote the cardinality of A.

Let $N = \{1, \cdots, n\}$ denote a finite set of *players*. In the standard terminology, a game is defined as the pair of a player set and a *characteristic function* that describes the attainable payoffs for each coalition. However , in this thesis, we fix a player set N, and focus on the characteristic functions. Hence, we identify a characteristic function with a game. As discussed in Section 1.1, the description of a game takes two different forms: TU games and NTU games.

1.4.1 TU game

A *TU game* is a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. For each $S \subseteq N$, $v(S)$ represents the attainable payoff for S and is called the *worth* of coalition S.

Let Γ denote the set of all TU games. To conduct a mathematical analysis, it is useful to define addition and scalar multiplication in Γ. For any $v, w \in \Gamma$ and $\alpha \in \mathbb{R}$, we define $v + w$ and αv by $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$, and $(\alpha v)(S) = \alpha v(S)$ for all $S \subseteq N$. Because a game assigns a real number to each non-empty subset, we can identify a game as a $2^n - 1$-dimensional vector. Together with the operation defined above, we can identify Γ as a linear space $\mathbb{R}^{2^n-1}$. We say that a finite set of games $\{v_k\}_{k=1}^{\ell} \subseteq \Gamma$ *spans* $X \subseteq \Gamma$ if

$$X = \Big\{ \sum_{k=1}^{\ell} \alpha_k v_k : \alpha_k \in \mathbb{R} \text{ for all } k = 1, \cdots, \ell \Big\}.$$

For a finite set of games $\{v_k\}_{k=1}^{\ell} \subseteq \Gamma$, let $\mathrm{Sp}(\{v_k\}_{k=1}^{\ell})$ denote the set of games spanned by $\{v_k\}_{k=1}^{\ell}$.

Let $v \in \Gamma$ and $i, j \in N$. For each $T \subseteq N \backslash i$,[5] we define the *contribution* of player i to coalition T as $\Delta_i v(T) = v(T \cup i) - v(T)$.

A (single-valued) *solution* is a function from Γ to $\mathbb{R}^n$. In other words, a solution assigns to each game an n-dimensional payoff vector, representing the payoff for each player. We define the *Shapley value*, introduced by Shapley (1953), as follows:

$$Sh_i(v) = \sum_{T \subseteq N \backslash i} \frac{|T|!(n - |T| - 1)!}{n!} \cdot \Delta_i v(T) \text{ for all } v \in \Gamma, i \in N.$$

The Shapley value determines player i's final payoff based on the expected value of his/her contributions. An equivalent formula is given as follows: for any $v \in \Gamma$,

$$Sh_i(v) = \sum_{T \subseteq N : i \in T} \frac{1}{|T|} D(T, v) \text{ for all } i \in N, \tag{1.1}$$

where

$$D(T, v) = \sum_{S \subseteq T} (-1)^{|T \backslash S|} v(S) \text{ for all } T \subseteq N, T \neq \emptyset. \tag{1.2}$$

For each $T \subseteq N$, $T \neq \emptyset$, $D(T, v)$ is called the *dividend* of T.

For each $T \subseteq N$, $T \neq \emptyset$, we define the T-*unanimity game* u_T, introduced by Shapley (1953), as follows:

$$u_T(S) = \begin{cases} 1 \text{ if } T \subseteq S, \\ 0 \text{ otherwise.} \end{cases} \tag{1.3}$$

The set of unanimity games forms a basis for the linear space Γ. Moreover, when we express a game $v \in \Gamma$ as a linear combination of $\{u_T\}_{\emptyset \neq T \subseteq N}$, the coefficient of u_T is equal to the dividend $D(T, v)$. In other words, for any $v \in \Gamma$,

$$v = \sum_{T \subseteq N : T \neq \emptyset} D(T, v) \cdot u_T.$$

Mathematically, the Shapley value Sh is a surjective linear mapping from $\mathbb{R}^{2^n-1}$ to $\mathbb{R}^n$.

[5] For simplicity, we denote a singleton set $\{i\}$ simply by i.

The mapping Sh has the *null space* defined by

$$\{v \in \Gamma : Sh(v) = \mathbf{0}\}. \tag{1.4}$$

The null space is the set of all games to which the Shapley value assigns the 0 vector. As is well known in linear algebra, the dimension of the space is equal to $2^n - 1 - n$.[6]

Let $v \in \Gamma$ and $i, j \in N$, $i \neq j$. We say that i and j are *substitutes* in v if $\Delta_i v(S) = \Delta_j v(S)$ for all $S \subseteq N \backslash \{i, j\}$. We say that i is a *null player in* v if $\Delta_i v(S) = 0$ for all $S \subseteq N \backslash i$. Shapley (1953) characterized the Shapley value using the following axioms imposed on a solution ψ:

Efficiency $\sum_{i \in N} \psi_i(v) = v(N)$ for all $v \in \Gamma$.

Symmetry If i and j are substitutes in $v \in \Gamma$, then $\psi_i(v) = \psi_j(v)$.

Null Player Property If i is a null player in $v \in \Gamma$, then $\psi_i(v) = 0$.

Additivity For any $v, w \in \Gamma$, we have $\psi(v + w) = \psi(v) + \psi(w)$.

Efficiency states that the total payoff should be fully distributed among the players. Symmetry states that if i and j make the same contributions, then they should receive the same payoff. The null player property states that if player i does not make any contribution, then i should receive nothing. Additivity states that the payoff vector in $v + w$ is obtained by calculating the payoff vectors in games v and w independently.

Note that the Shapley value satisfies the following axiom, which is stronger than additivity:

Linearity For any $v, w \in \Gamma$ and $\alpha, \beta \in \mathbb{R}$, we have $\psi(\alpha v + \beta w) = \alpha\psi(v) + \beta\psi(w)$.

A solution satisfying linearity is called a *linear solution*.

1.4.2 NTU game

To define an NTU game, we first introduce some mathematical preliminaries. For each $S \subseteq N$, $S \neq \emptyset$, let $\mathbb{R}^S$ denote the $|S|$-dimensional Euclidean space; an element $(x_i)_{i \in S} \in \mathbb{R}^S$ is indexed by the players in S. For each $S \subseteq N$, $S \neq \emptyset$, we identify $\mathbb{R}^S$ as the subspace of $\mathbb{R}^N$. Let $\mathbb{R}^S_+$ denote the space with non-negative coordinates and $\mathbb{R}^S_{++}$ denote the space with positive coordinates. We define vector inequalities as follows: for each $S \subseteq N$, $S \neq \emptyset$, and $x, y \in \mathbb{R}^S$, $x \gg y$ means $x_i > y_i$ for all $i \in S$; $x \geq y$ means $x_i \geq y_i$ for all $i \in S$; $x > y$ means $x \geq y$ and $x \neq y$. For each $x \in \mathbb{R}^N$ and $S \subsetneq N$, $S \neq \emptyset$, let $x_S \in \mathbb{R}^S$ denote the *projection of* x *on* $\mathbb{R}^S$ (i.e., $(x_S)_i = x_i$, for all $i \in S$). For a subset X of $\mathbb{R}^S$, let ∂X denote the *boundary of* X, and let clX denote the *closure of* X. For each $x, y \in \mathbb{R}^S$, let $x \cdot y$ denote the *inner product of* x *and* y, i.e.,

$$x \cdot y = \sum_{i \in N} x_i \cdot y_i.$$

An *NTU game* is a function V that associates a subset of $\mathbb{R}^S$ with each coalition $S \subseteq N$, $S \neq \emptyset$. If $x \in V(S)$, this means that x is attainable through cooperation of players in S. We make the following assumptions on V: for each $S \subseteq N$, $S \neq \emptyset$, $V(S)$ is

[6] See, for example, Theorem 2 of Lax (2007).

N1: a non-empty proper subset of $\mathbb{R}^S$;

N2: closed, convex, and comprehensive (i.e., $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$); and

N3: uniformly non-leveled. There exists a real number $\delta > 0$ such that for every normalized vector $\lambda \in \mathbb{R}^S$ (i.e., $\sum_{i \in S} \lambda_i = 1$), the following condition holds:

$$\sup_{x \in V(S)} \lambda \cdot x < +\infty \text{ implies } \lambda_i \geq \delta \text{ for all } i \in S.$$

For each $S \subseteq N$, $S \neq \emptyset$, $\partial V(S)$ is called the *Pareto frontier for S* (or simply the Pareto frontier, when it is clear which coalition is alluded to).

The meaning of N1 is clear. In N2, comprehensiveness means *free disposal*; that is, if x is an attainable payoff, then any "worse" payoff vector y with $y \leq x$ is also attainable. N3 was first introduced by Maschler and Owen (1992), and then later employed by other researchers, such as Hinojosa et al. (2012). To see the meaning of this condition, we provide an example in which N3 is *not* satisfied. Let $N = \{1, 2\}$ and suppose that $V(N)$ is represented as in Fig. 1.

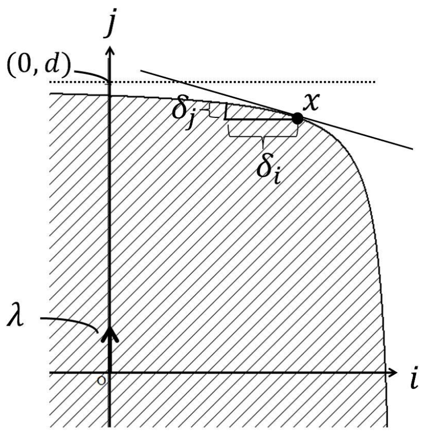

Fig. 1 Description of $V(N)$.

In this example, $V(N)$ is contained in the closed half-space $\{x \in \mathbb{R}^N : x_j \leq d\}$. Let $\lambda = (0, 1)$, as shown in Fig. 1. Then, $\sup_{x \in V(N)} \lambda \cdot x \leq d < +\infty$, but $\lambda_i = 0$. Hence, we cannot take a positive real number $\delta > 0$ with $\lambda_i \geq \delta$, which is a violation of N3. As this example illustrates, N3 excludes the case in which $\partial V(N)$ has a leveled part. This is why N3 is called uniformly "non-leveled."

To understand N3 in the context of allocation problems, we introduce the notion of the *transfer rates* of utilities between i and j.[7] Consider the Pareto optimal point x, which is depicted in Fig. 1. From this point, if player i gives up δ_i, then player j gets δ_j. Taking the limit of the quotient δ_j / δ_i yields the slope at x, which is interpreted as the local transfer rates of utilities. In Fig. 1, this slope asymptotically approaches to 0 as x moves to the upper-left corner of $\partial V(N)$. If the slope is 0, decreasing i's payoff does not improve j's payoff. N3 excludes these extreme transfer rates of utilities.

Let $\hat{\Gamma}$ denote the set of all NTU games. We can embed a TU game $v : 2^N \to \mathbb{R}$ in $\hat{\Gamma}$ by defining an NTU game V as follows: for each $S \subseteq N$, $S \neq \emptyset$,

$$V(S) = \Big\{ x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S) \Big\}. \tag{1.5}$$

[7] The explanation of transfer rates is borrowed from Peleg and Sudhölter (2007) (see p. 236).

To see the difference between a TU game and an NTU game, consider a two-player coalition $S = \{i, j\}$. In the case of a TU game, (1.5) means that the Pareto frontier for coalition S (i.e., the boundary of the set of attainable payoffs for S) is described by the hyperplane $\{y \in \mathbb{R}^S : y_i + y_j \leq v(S)\}$. In the case of an NTU game, the Pareto frontier is not necessarily a hyperplane. The two cases are compared in Fig. 2.

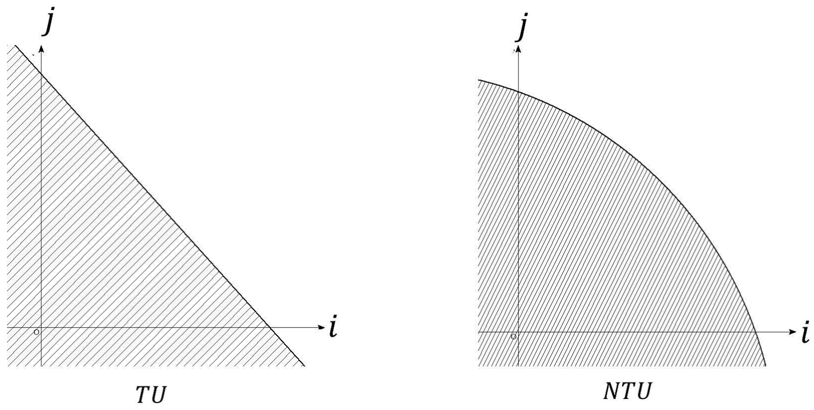

Fig. 2 Descriptions of the attainable payoffs for TU and NTU games.

A solution is defined in the same way as in the class of TU games. In other words, a (single-valued) *solution* is a function from the set of all NTU games $\hat{\Gamma}$ to the set of payoff vectors $\mathbb{R}^N$. A *set-valued solution* is a function that assigns a set of payoff vectors to each NTU game.

Chapter 2

Basis and the Shapley value

2.1 Introduction

The basis consisting of the unanimity games (Shapley (1953)) has long been recognized as a useful tool for analyses in TU games. The basis is often used in the proof of axiomatization of the Shapley value; see Young (1985), Chun (1989), Kalai and Samet (1987) or van den Brink (2002). The coefficients in the linear combination of the unanimity games are called the dividends (see (1.2)). The class of games with nonnegative dividends was investigated by Llerena and Rafels (2006) or van den Brink et al. (2014). As the set of TU games has a linear structure, to consider a basis is an essential task. The purpose of this chapter is to introduce new bases and explore their properties.

In the T-unanimity game (see (1.3)), the cooperation of **all** players in T yields payoff. We introduce a new game, termed the T-*commander game*, in which **only one** player in T yields payoff. The set of the commander games forms a basis and has two properties. First, when we express a game by a linear combination of the new basis, the coefficients related to singletons coincide with the Shapley value. Second, the basis induces the null space of the Shapley value (see (1.4)). The payoff vector of each commander game is uniquely determined by using three axioms: efficiency, symmetry and the null player property. Moreover, by using the two properties of the new basis, we can fully answer the *inverse problem*, i.e., the problem of how to characterize the class of games to which the Shapley value assigns a fixed vector.

The unanimity games and the commander games describe two extreme cases: the former requires the cooperation of all players, while the latter requires that of only one player. We consider intermediate games between them. More specifically, for a coalition T and $k \in \{1, \cdots, |T|\}$, we introduce the T^k-*intermediate game* in which the cooperation of k players in T yields payoff. We show that, under some conditions, the set of intermediate games forms a basis and preserves desirable properties of the commander games.

The new bases developed in this chapter are applicable to several research topics of the Shapley value. In Chapter 3, we apply a new basis to axiomatization of linear solutions.[1] Moreover, the basis consisting of the commander games can be used to investigate the coincidence between the Shapley value and other solutions; see Yokote et al. (2017).

This chapter is organized as follows. In Section 2,2, we define the commander games and show that the set of these games is a basis. In Section 2.3, we discuss the new basis from Shapley's (1953) axioms. In Section 2.4, we discuss intermediate games.

[1] As a relevant work, Yokote (2015) applied the commander games to axiomatization of the weighted Shapley values.

2.2 New basis

Let $T \subseteq N$, $T \neq \emptyset$. We define $\bar{u}_T$ as follows:

$$\bar{u}_T(S) = \begin{cases} 1 & \text{if } |S \cap T| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

We call $\bar{u}_T$ the T*-commander game*. The interpretation of this game is as follows. Each member in T is a commander and has authority to control other players. If a coalition including only one member in T forms, then the member behaves as a commander. The coalition obtains power, which results in the payoff of 1. In contrast, if a coalition including two or more members in T forms, then they compete with each other and the coalition obtains nothing.

To show that the set of the commander games is a basis, we prove a lemma.

Lemma 1. *Let $T \subseteq N$, $T \neq \emptyset$. Then, we have*

$$|T|u_T = \sum_{S \subseteq T: S \neq \emptyset} (-1)^{|S|-1} \bar{u}_S.$$

Proof. Let $R \subseteq N$. We calculate the worth of coalition R in the right-hand side. By definition of the commander games, we only need to consider a coalition $S \subseteq T$ such that $|S \cap R| = 1$. Such a coalition S can be determined by choosing one player from $T \cap R$ and k players from $T \backslash R$, where $0 \leq k \leq |T \backslash R|$. Hence,

$$\sum_{S \subseteq T: S \neq \emptyset} (-1)^{|S|-1} \bar{u}_S(R) = |T \cap R| \cdot \sum_{k=0}^{|T \backslash R|} \binom{|T \backslash R|}{k} (-1)^k = \begin{cases} |T| & \text{if } T \subseteq R, \\ 0 & \text{otherwise,} \end{cases}$$

where the second equality follows from the binomial theorem. It follows that the worth of coalition R in the right-hand side is equal to $|T|u_T(R)$. □

Theorem 1. *The set of games $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$ is a basis for Γ.*

Proof. Let $v \in \Gamma$. Since the dividends are the coefficients in the linear combination of the unanimity games, we have

$$\begin{aligned} v &= \sum_{R \subseteq N: R \neq \emptyset} \frac{D(R,v)}{|R|} \cdot |R|u_R \\ &= \sum_{R \subseteq N: R \neq \emptyset} \frac{D(R,v)}{|R|} \cdot \sum_{S \subseteq R: S \neq \emptyset} (-1)^{|S|-1} \bar{u}_S \\ &= \sum_{S \subseteq N: S \neq \emptyset} (-1)^{|S|-1} \sum_{R \subseteq N: S \subseteq R} \frac{D(R,v)}{|R|} \bar{u}_S, \end{aligned} \tag{2.2}$$

where the second equality follows from Lemma 1. Hence, any game $v \in \Gamma$ can be expressed by a linear combination of the games $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$. Mathematically, the set $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$ spans the linear space Γ. If the set $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$ is linearly dependent, then there exist a coalition $T \subseteq N$, $T \neq \emptyset$, and a vector $(\alpha_S)_{\emptyset \neq S \subseteq N, S \neq T}$ such that

$$\bar{u}_T = \sum_{S \subseteq N: S \neq \emptyset, S \neq T} \alpha_S \bar{u}_S.$$

Together with (2.2), the set Γ can be spanned by vectors with less than $2^n - 1$ vectors, which is a contradiction to the fact that the dimension of Γ is $2^n - 1$.[2] □

By Theorem 1, any game $v \in \Gamma$ is uniquely represented by a linear combination of $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$. Let $d(T, v)$ denote the coefficient of $\bar{u}_T$ in the linear combination, namely, $v = \sum_{T \subseteq N: T \neq \emptyset} d(T, v)\bar{u}_T$. By (1.1) and (2.2), we obtain the following theorem:

Theorem 2. *For any $v \in \Gamma$,*

$$d(\{i\}, v) = \sum_{R \subseteq N: i \in R} \frac{D(R, v)}{|R|} = Sh_i(v) \text{ for all } i \in N.$$

Theorem 2 states that the coefficients related to singletons coincide with the Shapley value.

2.3 Basis and Shapley's axioms

The unanimity games are useful in the sense that the payoff vector of each game can be uniquely determined by using the standard axioms. In this section, we show that the commander games have the same property.

Consider the four axioms in Section 1.4.1. Using efficiency, symmetry and the null player property, we can calculate the Shapley value in $\bar{u}_T$ for $T \subseteq N$, $|T| \geq 2$. First, consider a player $j \in N \backslash T$. Then, for any $S \subseteq N \backslash j$, we have $|S \cap T| = |(S \cup j) \cap T|$. Thus, j is a null player, which implies $Sh_j(\bar{u}_T) = 0$ by the null player property. By efficiency, $\sum_{i \in T} Sh_i(\bar{u}_T) = 0$. Since any two players in T are substitutes, by symmetry, we have $Sh_i(\bar{u}_T) = 0$ for all $i \in T$. It follows that $Sh(\bar{u}_T) = \mathbf{0}$. Note that we can determine the payoff vector in the game $\bar{u}_{\{i\}}$, $i \in N$, in the same way.

Since the set $\{\bar{u}_T : T \subseteq N, |T| \geq 2\}$ consists of $2^n - 1 - n$ linearly independent vectors, we obtain the following theorem:

Theorem 3. *The set $\{\bar{u}_T : T \subseteq N, |T| \geq 2\}$ spans the null space of Sh, i.e.,*

$$\{v \in \Gamma : Sh(v) = \mathbf{0}\} = Sp(\{\bar{u}_T : T \subseteq N, |T| \geq 2\}).$$

Theorem 3 states that the Shapley value does not depend on the coefficients $d(T, v)$, $T \subseteq N$, $|T| \geq 2$. Together with Theorem 2, we obtain the following corollary:

Corollary 1. *Let $x \in \mathbf{R}^n$. Then, $Sh(v) = x$ if and only if there exists a vector $(\alpha_T)_{T \subseteq N: |T| \geq 2} \in \mathbb{R}^{2^n - 1 - n}$ such that*

$$v = \sum_{i \in N} x_i \bar{u}_{\{i\}} + \sum_{T \subseteq N: |T| \geq 2} \alpha_T \bar{u}_T.$$

[2] Here, we implicitly use the following result in linear algebra: if vectors $x_1, \cdots, x_n$ span a linear space X and the vectors $y_1, \cdots, y_j$ in X are linearly independent, then $j \leq n$. See Lax (2007), Lemma 1 on page 5.

Corollary 1 solves the *inverse problem*, i.e., we characterize the set of all games to which the Shapley value assigns a fixed vector. It is worth stressing that the Shapley value is determined only by the coefficients of $\bar{u}_i$ for $i \in N$ and is silent about the change in the coefficients of $\bar{u}_T$ for $T \subseteq N$, $|T| \geq 2$. This result clarifies how the Shapley value determines the players' payoffs.

Example 1. We apply Corollary 1 to 3-person games. Let $N = \{1, 2, 3\}$ and $v \in \Gamma$. Then, $Sh(v) = x$ if and only if there exists $(y_{12}, y_{13}, y_{23}, y_N) \in \mathbb{R}^4$ such that $v(N) = x_1 + x_2 + x_3$ and

$$v(\{i, j\}) = x_i + x_j + y_{ik} + y_{jk},\ v(\{k\}) = x_k + y_{ik} + y_{jk} + y_N,$$

where i, j, k are distinct players in N. The above equations imply

$$v(\{i, j\}) = x_i + x_j + v(\{k\}) - x_k - y_N.$$

As a result, we obtain the following: let $N = \{1, 2, 3\}$ and $v \in \Gamma$ be a game such that $v(\{k\}) = 0$ for all $k \in N$. Then, $Sh(v) = x$ if and only if there exists $y \in \mathbb{R}$ such that $v(N) = x_1 + x_2 + x_3$ and

$$v(\{i, j\}) = x_i + x_j - x_k + y,$$

where i, j, k are distinct players in N. The only-if part says that, given an arbitrary vector x, we can always find an identical amount y for all coalitions with 2 players.

2.4 Intermediate games and new bases

In this section, we extend the basis consisting of the commander games.[3] For each $T \subseteq N$, $T \neq \emptyset$, the T-commander game assigns 1 to coalitions including only one member in T and 0 otherwise. In contrast, the T-unanimity game assigns 1 to a coalition including all players in T and 0 otherwise. These two games describe two extreme cases for obtaining a payoff: only one player or all players in T. In this section, we consider intermediate cases between them.

Let $T \subseteq N$ and $k \in \mathbb{N}$, $1 \leq k \leq |T|$. We define the T^k-*intermediate game* $\bar{u}_T^k$ by

$$\bar{u}_T^k(S) = \begin{cases} 1 & \text{if } |S \cap T| = k, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Note that $\bar{u}_T^1 = \bar{u}_T$ and $\bar{u}_T^{|T|} = u_T$. We show that, if there is a certain relationship between the size of coalition T and k, then we can construct a basis.

Consider a function $\ell : \{1, \cdots, n\} \to \{1, \cdots, n\}$ satisfying the following conditions:

C1: $\ell(1) = 1$.

C2: $\ell(t) = \ell(t-1)$ or $\ell(t-1) + 1$ or $\ell(t-1) - 1$ for all $t = 2, \cdots, n$.

Theorem 4. *Let ℓ be a function satisfying C1 and C2. Then, the set of games $\{\bar{u}_T^{\ell(|T|)}\}_{\emptyset \neq T \subseteq N}$ is a basis for Γ.*

[3]The results in this section are based on Yokote and Funaki (2015).

Special cases of interest are when $\ell(k) = 1$ for all $k = 1, \cdots, n$ and $\ell(k) = k$ for all $k = 1, \cdots, n$. The former coincides with the basis consisting of the commander games and the latter coincides with the basis consisting of the unanimity games. Thus, Theorem 4 generalizes the results by Shapley (1953) and Theorem 1.

To prove Theorem 4, we first prove a lemma.

Lemma 2. *Let $T \subseteq N$, $|T| \geq 2$, and $k \in \mathbb{N}$, $2 \leq k \leq |T|$. Then, we have*

$$\bar{u}_T^k = \frac{1}{k}\left(\textstyle\sum_{i \in T} \bar{u}_{T \backslash i}^{(k-1)} - (|T| - k + 1)\bar{u}_T^{(k-1)}\right).$$

Proof. Let $S \subseteq N$, $S \neq \emptyset$. We calculate the worth of S in both sides.

Case 1 $0 \leq |T \cap S| \leq k - 2$.

By definition of $\bar{u}_T^k$, we have $\bar{u}_T^k(S) = \bar{u}_T^{(k-1)}(S) = 0$. Consider the game $\bar{u}_{T\backslash i}^{(k-1)}$, $i \in T$.

If $i \in S$, $|(T\backslash i) \cap S| \leq k - 3$.
If $i \notin S$, $|(T\backslash i) \cap S| \leq k - 2$.

It follows that $\bar{u}_{T\backslash i}^{(k-1)}(S) = 0$ for all $i \in T$.

Case 2 $k + 1 \leq |T \cap S| \leq |T|$.

By definition of $\bar{u}_T^k$, we have $\bar{u}_T^k(S) = \bar{u}_T^{(k-1)}(S) = 0$. Consider the game $\bar{u}_{T\backslash i}^{(k-1)}$, $i \in T$.

If $i \in S$, $|(T\backslash i) \cap S| \geq k$.
If $i \notin S$, $|(T\backslash i) \cap S| \geq k + 1$.

It follows that $\bar{u}_{T\backslash i}^{(k-1)}(S) = 0$ for all $i \in T$.

Case 3 $|T \cap S| = k - 1$.

By definition of $\bar{u}_T^k$, we have $\bar{u}_T^k(S) = 0$. Let $i \in T$.

If $i \in S$, $|(T\backslash\{i\}) \cap S| = k - 2$.
If $i \notin S$, $|(T\backslash\{i\}) \cap S| = k - 1$.

That is, if $i \in S \cap T$, then $\bar{u}_{T\backslash i}^{(k-1)}(S) = 0$. As a result,

$$\begin{aligned}\sum_{i \in T} \bar{u}_{T\backslash i}^{(k-1)}(S) &= \sum_{i \in T\backslash S} \bar{u}_{T\backslash i}^{(k-1)}(S) \\ &= \sum_{i \in T\backslash S} \bar{u}_T^{(k-1)}(S) \\ &= |T| - (k - 1),\end{aligned}$$

where the second equality follows from $(T\backslash i) \cap S = T \cap S$ for $i \in T\backslash S$. Together with $-(|T| - k + 1)\bar{u}_T^{(k-1)}(S) = -(|T| - k + 1)$, the right-hand side is equal to 0, which is equal to the left-hand side.

Case 4 $|T \cap S| = k$.

By definition of $\bar{u}_T^k$, we have $\bar{u}_T^{(k-1)}(S) = 0$. Let $i \in T$.

If $i \in S$, $|(T \backslash i) \cap S| = k - 1$,
If $i \notin S$, $|(T \backslash i) \cap S| = k$.

Hence, $\sum_{i \in T} \bar{u}_{T \backslash i}^{(k-1)}(S) = k$, which implies that the right-hand side is equal to 1. Since the left-hand side is also equal to 1, the proof completes

□

Proof of Theorem 4 **.** Throughout the proof, we refer to functions ℓ satisfying C1 and C2. For a function ℓ, we define

$$K(\ell) = \sum_{k=1}^{n} \ell(k),$$
$$M(\ell) = \max\{\ell(k) : k = 1, \cdots, n\},$$
$$Q(\ell) = \{k : \ell(k) = M(\ell)\}.$$

Induction base: Suppose $K(\ell) = n$, namely, $\ell(k) = 1$ for all $k = 1, \cdots, n$. In this case, Theorem 1 completes the proof.
Induction step: Suppose the result holds for all l with $n \leq K(\ell) \leq p$, and we prove the result for ℓ with $K(\ell) = p + 1$, where $n \leq p \leq \frac{n(n+1)}{2} - 1$.

Suppose to the contrary that $\{\bar{u}_T^{\ell(|T|)}\}_{\emptyset \neq T \subseteq N}$ is not a basis. Then, there exists a vector $(\lambda_T)_{\emptyset \neq T \subseteq N} \neq \mathbf{0}$ such that

$$\sum_{T \subseteq N : T \neq \emptyset} \lambda_T \bar{u}_T^{\ell(|T|)} = \mathbf{0}. \tag{2.4}$$

Let $q \geq 2$ be such that $\ell(q) = M(\ell)$ and $q \leq k$ for all $k \in Q(\ell)$. Then,

$$\ell(q-1) = \ell(q) - 1 \geq 1.$$

By (2.4),

$$\sum_{T \subseteq N : T \neq \emptyset, |T| \neq q} \lambda_T \bar{u}_T^{\ell(|T|)} + \sum_{T \subseteq N : |T| = q} \lambda_T \bar{u}_T^{\ell(q)} = \mathbf{0}.$$

By Lemma 2,

$$\sum_{T \subseteq N : T \neq \emptyset, |T| \neq q} \lambda_T \bar{u}_T^{\ell(|T|)} + \sum_{T \subseteq N : |T| = q} \frac{\lambda_T}{\ell(q)} \left(\sum_{i \in T} \bar{u}_{T \backslash i}^{(\ell(q)-1)} - (q - \ell(q) + 1) \bar{u}_T^{(\ell(q)-1)} \right) = \mathbf{0}. \tag{2.5}$$

We define ℓ' by

$$\ell'(|T|) = \begin{cases} \ell(|T|) & \text{if } |T| \neq q, \\ \ell(q) - 1 & \text{if } |T| = q. \end{cases}$$

We show that ℓ' satisfies C1 and C2. Since $q \geq 2$, $\ell'(1) = \ell(1) = 1$, which proves C1. Since $\ell(q) = M(\ell)$, we have

$$\ell(q+1) = \ell(q) \text{ or } \ell(q) - 1.$$

If $\ell(q+1) = \ell(q)$, then $\ell'(q+1) = \ell(q+1) = \ell'(q) + 1$. If $\ell(q+1) = \ell(q) - 1$, then $\ell'(q+1) = \ell(q+1) = \ell'(q)$. Namely, $\ell'(q+1) = \ell'(q) + 1$ or $\ell'(q)$. In addition, $\ell'(q) = M(\ell) - 1 = \ell(q-1) = \ell'(q-1)$, which proves C2.

Using the function ℓ', (2.5) can be written as follows:

$$\begin{aligned}
&\sum_{T \subseteq N: T \neq \emptyset, |T| \neq q} \lambda_T \bar{u}_T^{\ell'(|T|)} + \sum_{T \subseteq N: |T| = q} \frac{\lambda_T}{\ell(q)} \left(\textstyle\sum_{i \in T} \bar{u}_{T \setminus i}^{\ell'(|T \setminus i|)} - (q - \ell'(q)) \bar{u}_T^{\ell'(q)} \right) \\
= &\sum_{T \subseteq N: T \neq \emptyset, |T| \leq q-2, |T| \geq q+1} \lambda_T \bar{u}_T^{\ell'(|T|)} \\
&+ \sum_{T \subseteq N: |T| = q-1} \left(\lambda_T + \sum_{j \in N \setminus T} \frac{\lambda_{T \cup j}}{\ell(q)} \right) \bar{u}_T^{\ell'(|T|)} \\
&- \sum_{T \subseteq N: |T| = q} \frac{\lambda_T (q - \ell'(q))}{\ell(q)} \bar{u}_T^{\ell'(q)} \\
= &\mathbf{0}.
\end{aligned} \tag{2.6}$$

Since $K(\ell') \leq p$, by the induction hypothesis, all the coefficients in the above equation are 0. We obtain

$$\lambda_T = 0 \text{ for all } T \subseteq N, T \neq \emptyset, |T| \leq q-2, |T| \geq q+1,$$

and, together with $\ell'(q) = \ell(q) - 1 \leq q - 1$,

$$\lambda_T = 0 \text{ for all } T \subseteq N, |T| = q.$$

Substituting this equation into the coefficients in (2.6),

$$\lambda_T = 0 \text{ for all } T \subseteq N, |T| = q - 1.$$

We obtain a contradiction to $(\lambda_T)_{\emptyset \neq T \subseteq N} \neq \mathbf{0}$. □

As proven in Section 2.2, the set $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$ has two desirable properties. First, when we express a game v by a linear combination of the basis, the coefficients related to singletons coincide with the Shapley value. Second, the set of games $\{\bar{u}_T\}_{T \subseteq N: |T| \geq 2}$ spans the null space of the Shapley value. We prove that, by making an additional assumption on ℓ, the basis consisting of the intermediate games preserves the desirable properties.

Theorem 5. *Let ℓ be a function satisfying C1, C2 and $\ell(2) = 1$. Then,*

(1): The set of games $\{\bar{u}_T^{\ell(|T|)}\}_{\emptyset \neq T \subseteq N}$ is a basis for Γ.

(2): When we express a game $v \in \Gamma$ by a linear combination of this basis, the coefficient of $\bar{u}_{\{i\}}^1$ is equal to $Sh_i(v)$ for all $i \in N$.

(3): The set $\{\bar{u}_T^{\ell(|T|)} : T \subseteq N, |T| \geq 2\}$ spans the null space of Sh.

Proof **.** The first statement (1) follows from Theorem 4. Let $T \subseteq N$, $|T| \geq 2$, and $j \in N \backslash T$. Then, for any $S \subseteq N \backslash j$, we have $|S \cap T| = |(S \cup j) \cap T|$. It follows that j is a null player. By the null player property, we obtain $Sh_j(\bar{u}_T^{\ell(|T|)}) = 0$. By symmetry and $\bar{u}_T^{\ell(|T|)}(N) = 0$, we have $Sh_i(\bar{u}_T^{\ell(|T|)}) = 0$ for all $i \in N$. As a result, (3) holds. It remains to prove (2). Let $v \in \Gamma$ and $(\alpha_T)_{\emptyset \neq T \subseteq N}$ be the coefficients in the linear combination of v by $\{\bar{u}_T^{\ell(|T|)}\}_{\emptyset \neq T \subseteq N}$. Then, for any $i \in N$,

$$
\begin{aligned}
Sh_i(v) &= Sh_i\Big(\sum_{T \subseteq N : T \neq \emptyset} \alpha_T \bar{u}_T^{\ell(|T|)}\Big) \\
&= \sum_{T \subseteq N : T \neq \emptyset} \alpha_T Sh_i(\bar{u}_T^{\ell(|T|)}) \\
&= \sum_{j \in N} \alpha_{\{j\}} Sh_i(\bar{u}_{\{j\}}^1) \\
&= \alpha_{\{i\}},
\end{aligned}
$$

where the first equality follows from linearity of the Shapley value, and the fourth equality follows from $Sh_i(\bar{u}_{\{i\}}^1) = 1$ and $Sh_i(\bar{u}_{\{j\}}^1) = 0$ for all $j \in N \backslash i$.[4] □

Using a new basis developed in this chapter, we provide new axiomatizations of solutions in the next chapter.

[4]These equations follow from the null player property and efficiency.

Chapter 3

Monotonicity and axiomatization of linear solutions

3.1 Introduction

Monotonicity has been intensively discussed not only in specific fair allocation problems but also in the general class of TU games. A major finding in the literature is that, for entire class of TU games, monotonicity and standard axioms imply linearity. Under efficiency and symmetry, Young (1985) proved that strong monotonicity characterizes the Shapley value, and Casajus and Huettner (2014) proved that weak monotonicity characterizes the egalitarian Shapley values, the convex combinations of the Shapley value and the equal division value. Linearity as an axiom is often criticized as merely being a technical condition, while monotonicity can be regarded as a fairness criterion. Therefore, the above results support the desirability of linear solutions.

The above results enable us to conjecture that other linear solutions can also be characterized by monotonicity. If such a characterization is possible, then the difference among them can be comprehensively explained using the difference in monotonicity axioms. This is the underlying motivation of this study, and we show that a variety of solutions can be characterized by efficiency, symmetry and monotonicity.

A monotonicity axiom states an increase in certain parameters of a game as a hypothesis and states an increase in a player's payoff as a conclusion. We focus on various parameters of a game and introduce new axioms. Combined with previous results, we prove that efficiency, symmetry and a monotonicity axiom characterize (i) four linear solutions in the literature, namely, the Shapley value, the equal division value, the CIS value and the ENSC value, and (ii) a class of solutions obtained by taking a convex combination of the above solutions.

To see the relationship between the characterization of a single solution and that of a class of solutions, consider two parameters x and y of a game (for example, x represents a player's contributions, and y is the worth of the grand coalition). An axiom called x-monotonicity states that if parameter x increases, then a player's payoff weakly increases, and y-monotonicity is defined analogously. Suppose that x-monotonicity and y-monotonicity, when combined with efficiency and symmetry, characterize a solution X and a solution Y, respectively. Here, we consider a new axiom, $x+y$-monotonicity, which states that if *both* x and y increase, then a player's payoff weakly increases. We show that $x+y$-monotonicity, together with efficiency and symmetry, characterizes a class of solutions obtained by taking a convex combination of solutions X and Y.

Our methodological contribution is to provide a new linear algebraic approach for characterizing solutions by monotonicity. Using a new basis developed in Chapter 2, we decompose the whole space into three subspaces. This decomposition enables us to identify a set of games in which a solution that satisfies monotonicity is linear. Our approach provides some intuition for why monotonicity implies linearity.

This study is part of the literature on the characterization of convex combinations of solutions. For other approaches to this problem, see van den Brink and Funaki (2009), van den Brink et al. (2013) or Casajus and Huettner (2013). Casajus (2015) applied an approach based on monotonicity to the problem of redistributing income in a society.

The remainder of this chapter is organized as follows. Section 3.2 deals with preliminaries. In Section 3.3, we introduce new monotonicity axioms and provide a comprehensive characterization of linear solutions by monotonicity. In Section 3.4, we revisit a basis in Chapter 2 and explain why the basis is useful. In Section 3.5, we illustrate a sketch of the proof of our main result. Section 3.6 shows the independence of the axioms employed in our characterizations. All proofs are provided in Section 3.7.

3.2 Preliminaries

We define additional solutions. The *equal division value* is defined by

$$ED_i(v) = \frac{v(N)}{n} \text{ for all } i \in N, v \in \Gamma.$$

The *CIS value* and the *ENSC value* (Driessen and Funaki (1991)) are defined by

$$CIS_i(v) = v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n} \text{ for all } i \in N, v \in \Gamma,$$

$$ENSC_i(v) = v(N) - v(N \backslash i) + \frac{v(N) - \sum_{j \in N} \{v(N) - v(N \backslash j)\}}{n} \text{ for all } i \in N, v \in \Gamma.$$

For each $\alpha \in [0,1]$, we define the *α-egalitarian Shapley value* (Joosten (1996)) by

$$Sh^\alpha(v) = \alpha Sh(v) + (1-\alpha)ED(v) \text{ for all } v \in \Gamma.$$

For each $\alpha \in [0,1]$, we define the *α-consensus value* (Ju et al. (2007)) Ψ^α by

$$\Psi^\alpha(v) = \alpha Sh(v) + (1-\alpha)CIS(v) \text{ for all } v \in \Gamma.$$

3.3 Monotonicity

We first revisit previous monotonicity axioms. Young (1985) introduced the following axiom:

Strong monotonicity (Young (1985)). Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i v(S) \geq \Delta_i w(S)$ for all $S \subseteq N \backslash i$, then $\psi_i(v) \geq \psi_i(w)$.

As argued by van den Brink et al. (2013), strong monotonicity is a very strong axiom in the sense that i's payoff could increase irrespective of what is to be allocated. Hence, van den Brink et al. (2013) introduced the following weakened form:

Weak monotonicity (van den Brink et al. (2013)). Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i v(S) \geq \Delta_i w(S)$ for all $S \subseteq N \backslash i$ and $v(N) \geq w(N)$, then $\psi_i(v) \geq \psi_i(w)$.

Under efficiency and symmetry, strong monotonicity characterizes the Shapley value and weak monotonicity characterizes the egalitarian Shapley values, as proven by Young (1985) and Casajus and Huettner (2014), respectively.[1] The difference between the two axioms lies in the parameters stated in the hypothesis; strong monotonicity states an increase in a player's contributions, while weak monotonicity states an increase in a player's contributions *and* the grand coalition worth. Note that increasing the number of parameters stated in the hypothesis of an axiom weakens the axiom.

Following the above line of research, we establish a comprehensive characterization of linear solutions by monotonicity. To define new monotonicity axioms, we consider the following four parameters of a game v, with their abbreviations in parentheses:

(i) $v(i)$, a player's individual worth (id);

(ii) $(\Delta_i v(S))_{S \subseteq N \backslash i}$, a player's contributions (cont);

(iii) $v(N)$, the grand coalition worth (gr); and

(iv) $v(N) - \sum_{i \in N} v(i)$, the cooperative surplus (sur).

We use as prefix to "monotonicity" the list of parameters that increase. For example, strong monotonicity and weak monotonicity are reformulated as follows:

cont-monotonicity. Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i v(S) \geq \Delta_i w(S)$ for all $S \subseteq N \backslash i$, then $\psi_i(v) \geq \psi_i(w)$.

cont+gr-monotonicity. Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i v(S) \geq \Delta_i w(S)$ for all $S \subseteq N \backslash i$ and $v(N) \geq w(N)$, then $\psi_i(v) \geq \psi_i(w)$.

Different configurations of parameters yield new axioms. We start from the weakest axiom and then consider stronger axioms.

cont+gr+sur-monotonicity. Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i v(S) \geq \Delta_i w(S)$ for all $S \subseteq N \backslash i$, $v(N) \geq w(N)$ and $v(N) - \sum_{i \in N} v(i) \geq w(N) - \sum_{i \in N} w(i)$, then $\psi_i(v) \geq \psi_i(w)$.

One can check that this axiom is satisfied by the Shapley value, the equal division value and the CIS value. Our first result shows that efficiency, symmetry and this axiom characterize the convex combinations of the three solutions.

Theorem 6. *Let $n \geq 6$. Then, ψ satisfies efficiency, symmetry and cont+gr+sur-monotonicity if and only if there exist $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$ such that*

$$\psi = \alpha Sh + \beta CIS + (1 - \alpha - \beta) ED. \tag{3.1}$$

[1] We remark that Young's (1985) result is valid for an arbitrary number of players, while Casajus and Huettner's (2014) result is valid except for 2-person games.

We defer the formal proof to Section 3.7.1. Since the proof is involved, in Section 3.5, we illustrate a sketch of the proof and explain why we need at least 6 players.

In (3.1), by letting $\beta = 0$, we obtain an egalitarian Shapley value, which is interpreted as a compromise between a player's productivity and egalitarian principles (see van den Brink et al. (2013)). We interpret (3.1) as an extension of this idea. To be more precise about what we mean by "extension," we provide two equivalent formulas for (3.1).

The first equivalent formula is given as follows: there exist $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$ such that

$$\psi_i(v) = \alpha Sh_i(v) + \beta v(i) + \frac{v(N) - \big(\alpha v(N) + \sum_{j \in N} \beta v(j)\big)}{n} \text{ for all } v \in \Gamma, i \in N.$$

In this solution, we assign to players uniform fractions α and β in their singleton worth and their Shapley value payoff, respectively, and the remainder is equally divided. Compared to the α-egalitarian Shapley values, the above formula allows a solution to be more dependent on a player's individual worth. As individual worth often plays an important role in allocation problems,[2] it is more appealing to create further room for it.

The second equivalent formula is given as follows: there exist $\alpha', \beta' \in [0,1]$ such that

$$\psi(v) = \alpha' Sh(v) + (1 - \alpha')\big(\beta' CIS(v) + (1 - \beta')ED(v)\big) \text{ for all } v \in \Gamma.$$

In this solution, we assign to players an uniform fraction α in their Shapley payoff and the reminder is distributed according to a convex combination of the CIS value and the ENSC value. The equal division value and the CIS value evenly distribute the grand coalition worth and the cooperative surplus, respectively, both of which seem to capture the egalitarian principle. We allow for any convex combination of the equal division value and the CIS value. In this sense, the above formula creates further room for considering the egalitarian principle.

By replacing *cont + gr + sur*-monotonicity with a stronger axiom, we obtain new characterizations. Consider the following axioms:

cont+sur-monotonicity. Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i v(S) \geq \Delta_i w(S)$ for all $S \subseteq N \backslash i$ and $v(N) - \sum_{i \in N} v(i) \geq w(N) - \sum_{i \in N} w(i)$, then $\psi_i(v) \geq \psi_i(w)$.

id+gr+sur-monotonicity. Let $v, w \in \Gamma$ and $i \in N$. If $v(i) \geq w(i)$, $v(N) \geq w(N)$ and $v(N) - \sum_{i \in N} v(i) \geq w(N) - \sum_{i \in N} w(i)$, then $\psi_i(v) \geq \psi_i(w)$.

id+sur-monotonicity. Let $v, w \in \Gamma$ and $i \in N$. If $v(i) \geq w(i)$ and $v(N) - \sum_{i \in N} v(i) \geq w(N) - \sum_{i \in N} w(i)$, then $\psi_i(v) \geq \psi_i(w)$.

Theorem 7. *Let $n \geq 6$. Then, ψ satisfies efficiency, symmetry and cont+sur-monotonicity if and only if there exists $\alpha \in [0,1]$ such that $\psi = \Psi^\alpha$.*

Theorem 8. *Let $n \geq 6$. Then, ψ satisfies efficiency, symmetry and id+gr+sur-monotonicity if and only if there exists $\alpha \in [0,1]$ such that $\psi = \alpha CIS + (1 - \alpha)ED$.*

Theorem 9. *Let $n \geq 6$. Then, ψ satisfies efficiency, symmetry and id+sur-monotonicity if and only if $\psi = CIS$.*

[2]For example, in public good provision problems (Moulin (1995)), the voluntary participation axiom states that a player should receive no less than his individual worth.

We prove Theorems 7 to 9 in Section 3.7.2.

We emphasize the correspondence between axioms and solutions. Note that the four parameters (i) to (iv) are used in the definitions of solutions; a player's contributions are in the Shapley value, the grand coalition worth is in the equal division value, and the individual worth and the cooperative surplus are in the CIS value. In Theorems 6 to 9, the parameters stated in an axiom correspond to the parameters used in the definitions of the solutions characterized by the axiom.

Remark 1. In Theorems 7 and 8, we employ stronger axioms than *cont* + *gr* + *sur*-monotonicity. As a result, a coefficient in (3.1) degenerates to 0 and we obtain a smaller class of solutions. In Theorem 9, we employ an even stronger axiom, which results in a new characterization of the CIS value. Note that the CIS value is determined only by the worth of 1-person and n-person coalitions, and *id* + *sur*-monotonicity requires this property. Based on this, one may argue that *id* + *sur*-monotonicity is too strong. The main message of Theorem 9, however, is the following: among many solutions depending only on the worth of 1-person and n-person coalitions, the CIS value is the *only* solution satisfying efficiency, symmetry and *id* + *sur*-monotonicity. Thus, *id* + *sur*-monotonicity clarifies the difference between the CIS value and other solutions. ∎

Another direction for extending Theorem 6 is to take the dual of the new axiom. For $v \in \Gamma$, we define the *dual game* $v^* \in \Gamma$ by

$$v^*(S) = v(N) - v(N \backslash S) \text{ for all } S \subseteq N, S \neq \emptyset.$$

Regarding a player's individual worth and the cooperative surplus, taking the dual yields new parameters of a game. A player's individual worth in the dual game, $v^*(i) = v(N) - v(N \backslash i)$, represents the player's contribution to the grand coalition. The cooperative surplus in the dual game,

$$v^*(N) - \sum_{i \in N} v^*(i) = v(N) - \sum_{i \in N} \big\{ v(N) - v(N \backslash i) \big\},$$

represents the difference between the grand coalition worth and the sum of the contributions to the grand coalition.

To capture the above parameters, we introduce the dual of an axiom, and characterize the duals of solutions. For a detailed discussion on the dual approach, we refer the reader to Oishi et al. (2016).

Given a monotonicity axiom, its dual is defined by replacing an increase in $v(i)$ by an increase in $v^*(i)$ and replacing an increase in $v(N) - \sum_{i \in N} v(i)$ by an increase in $v(N) - \sum_{i \in N} v^*(i)$; the other two parameters, a player's contributions and the grand coalition worth, remain the same. We attach the superscript $*$ to a parameter to represent the fact that the parameter is considered in the dual game. For example, the dual of *cont* + *gr* + *sur*-monotonicity is given as follows:

cont+gr+sur**-monotonicity.* Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i v(S) \geq \Delta_i w(S)$ for all $S \subseteq N \backslash i$, $v(N) \geq w(N)$ and $v(N) - \sum_{i \in N} v^*(i) \geq w(N) - \sum_{i \in N} w^*(i)$, then $\psi_i(v) \geq \psi_i(w)$.

The consequence of taking the dual of an axiom can be seen by the duality of solutions. Let ψ, ψ^* be two solutions. We say that ψ^* *is the dual of* ψ if

$$\psi^*(v) = \psi(v^*) \text{ for all } v \in \Gamma.$$

We say that ψ is *self-dual* if ψ is the dual of ψ.

One can easily verify that the CIS value is the dual of the ENSC value (and vice versa) and that the Shapley value and the equal division are self-dual. Moreover, the duality relationship is preserved under convex operations. To be more precise, let ψ, ϕ be solutions, and ψ^*, ϕ^* be their duals, respectively. Then, for any $\alpha \in [0, 1]$, $\alpha\psi^* + (1-\alpha)\phi^*$ is the dual of $\alpha\psi + (1 - \alpha)\phi$. Together with the duality of parameters, it can be proved that the dual of a monotonicity axiom characterizes the convex combinations of the duals of solutions. We exemplify one theorem obtained by the duality approach.

Theorem 6* . *Let $n \geq 6$. Then, ψ satisfies efficiency, symmetry and cont+gr+sur*-monotonicity if and only if there exist $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$ such that $\psi = \alpha Sh + \beta ENSC + (1 - \alpha - \beta)ED$.*

Thus far, we have introduced several characterizations of the convex combinations of solutions by monotonicity. We refer to one more axiom, *grand coalition monotonicity* (Casajus and Huettner (2014)), which is formulated as follows:

gr-monotonicity. Let $v, w \in \Gamma$ and $i \in N$. If $v(N) \geq w(N)$, then $\psi_i(v) \geq \psi_i(w)$.

As proven by Casajus and Huettner (2014), the above axiom, together with efficiency and symmetry, characterizes the equal division value.

Remark 2. The following *coalitional monotonicity* (van den Brink (2007)), which is weaker than gr-monotonicity, also characterizes the equal division value: for any $v, w \in N$ and $i \in N$, if $v(S) \geq w(S)$ for all $S \ni i$, then $\psi_i(v) \geq \psi_i(w)$. ∎

Characterizations of solutions by monotonicity are summarized in Table 1 below. Below each axiom, we indicate by ↑ the parameters that are stated in the axiom.

Axioms \ Solutions	Sh	ED	CIS	ENSC
cont + *gr* + *sur*-monotonicity (Th. 6) $\Delta_i v(S) \uparrow$, $v(N) \uparrow$, $v(N) - \sum_{i \in N} v(i) \uparrow$	$\surd$	$\surd$	$\surd$	
cont + *gr* + *sur**-monotonicity (Th. 6*) $\Delta_i v(S) \uparrow$, $v(N) \uparrow$, $v(N) - \sum_{i \in N} v^*(i) \uparrow$	$\surd$	$\surd$		$\surd$
cont + *gr*-monotonicity $\Delta_i v(S) \uparrow$, $v(N) \uparrow$	$\surd$	$\surd$		
cont + *sur*-monotonicity (Th. 7) $\Delta_i v(S) \uparrow$, $v(N) - \sum_{i \in N} v(i) \uparrow$	$\surd$		$\surd$	
cont + *sur**-monotonicity $\Delta_i v(S) \uparrow$, $v(N) - \sum_{i \in N} v^*(i) \uparrow$	$\surd$			$\surd$
id + *gr* + *sur*-monotonicity (Th. 8) $v(i) \uparrow$, $v(N) \uparrow$, $v(N) - \sum_{i \in N} v(i) \uparrow$		$\surd$	$\surd$	
id* + *gr* + *sur**-monotonicity $v^*(i) \uparrow$, $v(N) \uparrow$, $v(N) - \sum_{i \in N} v^*(i) \uparrow$		$\surd$		$\surd$
cont-monotonicity $\Delta_i v(S) \uparrow$	$\surd$			
id + *sur*-monotonicity (Th. 9) $v(i) \uparrow$, $v(N) - \sum_{i \in N} v(i) \uparrow$			$\surd$	
id + *sur**-monotonicity $v^*(i) \uparrow$, $v(N) - \sum_{i \in N} v^*(i) \uparrow$				$\surd$
gr-monotonicity $v(N) \uparrow$		$\surd$		

Table 1. Summary of characterizations of solutions.

Table 1 is read as follows: each axiom in a row characterizes the convex combinations of solutions marked by $\surd$.[3] As a concrete example, let us focus on the first row. This row means that *cont* + *gr* + *sur*-monotonicity characterizes the convex combinations of the solutions marked by $\surd$, namely the Shapley value, the equal division value and the CIS value.

3.4 Basis and monotonicity

The proof of Theorem 6 relies heavily on a basis introduced in Chapter 2. The basis enables us to identify a class of games in which a solution satisfying monotonicity is linear. Our proof method based on the basis is different from Casajus and Huettner's (2014) proof and offers a new approach for characterizing solutions.

We first argue that the basis consisting of the unanimity games (see (1.3)) is not suitable in view of applying *cont* + *gr* + *sur*-monotonicity. To apply this axiom, it is important to focus on the addition of games after which the parameters do not change; in such cases, monotonicity has a strong implication that a player's payoff does not change. From this perspective, the unanimity games are not suitable because after adding u_T for $|T| \geq 2$, both the grand coalition worth and the cooperative surplus increase. This observation leads us to search for an alternative basis.

[3]By our characterization results, a solution in a column does not satisfy the axiom with an empty cell.

An ideal basis would be the set of games $\{w_T\}_{\emptyset \neq T \subseteq N}$ with the following properties: for any $T \subseteq N$, $T \neq \emptyset$,

(a) any two players in T are substitutes in w_T;

(b) all players outside T are null players in w_T;

(c) $w_T(N) = 0$; and

(d) $w_T(N) - \sum_{i \in N} w_T(i) = 0$.

However, for $|T| = 1, 2$, (a) to (d) are incompatible. For $|T| = 1$, (b) implies that $w_T = \lambda u_T$ for some λ and (c) implies $w_T = \mathbf{0}$, which never constitutes a basis. For $|T| = 2$, (b) and (c) imply $w_T(S) = 0$ for all $S \supseteq T$, which is equivalent to saying that $w_T(S) = 0$ for all $S \subseteq N$ with $|S \cap T| = 2$. Moreover, (b) implies $w_T(S) = 0$ for all $S \subseteq N$ with $|S \cap T| = 0$. Together with (a), there exists $\lambda \in \mathbb{R}$ such that $w_T(S) = \lambda$ if $|S \cap T| = 1$, 0 otherwise. However by (c) and (d), $\sum_i w_T(i) = 2\lambda = 0$. This means that $w_T = \mathbf{0}$.

The above discussion leads us to require (a) to (d) only for $|T| \geq 3$. This requirement turns out to be achievable. For each $T \subseteq N$ with $|T| \geq 3$, we consider the game $\bar{u}_T^2$ defined by (2.3), namely,

$$\bar{u}_T^2(S) = \begin{cases} 1 & \text{if } |S \cap T| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

One can easily verify that for $|T| \geq 3$, $\bar{u}_T^2$ satisfies (a) to (d). By Theorem 4,[4] the following set

$$\{u_T : T \subseteq N, 1 \leq |T| \leq 2\} \cup \{\bar{u}_T^2 : T \subseteq N, |T| \geq 3\} \tag{3.2}$$

is a basis of Γ. Regarding $|T| = 1, 2$, we modify the games $\{u_T : T \subseteq N, 1 \leq |T| \leq 2\}$ in a way that the grand coalition worth and the cooperative surplus are zero for as many games as possible. To preserve linear independence, we use the following result in linear algebra:[5] Let $A = \{w_T\}_{\emptyset \neq T \subseteq N}$ be a basis of Γ. If B is a set of games obtained from A

- by multiplying each game by a nonzero constant, or
- by replacing a game w_T with itself plus a scalar multiple of some other game $w_{T'}$,

then B is also a basis.

Since (3.2) is a basis, the following set

$$\begin{aligned} &\{u_1\} \cup \{u_1 - u_i : i \in N, i \neq 1\} \cup \{u_{12}\} \\ &\cup \{u_{12} - u_T : T \subseteq N, |T| = 2, T \neq \{1,2\}\} \cup \{\bar{u}_T^2 : |T| \geq 3\} \end{aligned}$$

is a basis. Define $u^1 = \sum_{i \in N} u_i$, $u^2 = \sum_{T \subseteq N : |T| = 2} u_T$. Since

$$u^1 = nu_1 - \sum_{i \in N : i \neq 1} (u_1 - u_i), u^2 = \frac{n(n-1)}{2} u_{12} - \sum_{T \subseteq N : |T| = 2, T \neq \{1,2\}} (u_{12} - u_T),$$

[4] Theorem 4 refers to a function $\ell : \{1, \cdots, n\} \to \{1, \cdots, n\}$. Here, we define ℓ as follows: $\ell(1) = 1$, $\ell(t) = 2$ for all $t \geq 2$.

[5] See, for example, Theorem 1.42 of Sundaram (1996).

the following set

$$\begin{aligned}&\left\{u^1\right\} \cup \{u_1 - u_i : i \in N, i \neq 1\} \cup \{u^2\}\\ &\cup \{u_{12} - u_T : T \subseteq N, |T| = 2, T \neq \{1,2\}\} \cup \{\bar{u}^2_T : |T| \geq 3\}\end{aligned}$$

is also a basis. We provide an example for $n = 3$:

$$\begin{array}{c}\\ 1\\2\\3\\12\\13\\23\\123\end{array}\quad \begin{array}{c}u^1\\ \begin{pmatrix}1\\1\\1\\2\\2\\2\\3\end{pmatrix}\end{array}\quad \begin{array}{c}u_1 - u_2\\ \begin{pmatrix}1\\-1\\0\\0\\1\\-1\\0\end{pmatrix}\end{array}\quad \begin{array}{c}u_1 - u_3\\ \begin{pmatrix}1\\0\\-1\\1\\0\\-1\\0\end{pmatrix}\end{array}\quad \begin{array}{c}u^2\\ \begin{pmatrix}0\\0\\0\\1\\1\\1\\3\end{pmatrix}\end{array}\quad \begin{array}{c}u_{12} - u_{13}\\ \begin{pmatrix}0\\0\\0\\1\\-1\\0\\0\end{pmatrix}\end{array}\quad \begin{array}{c}u_{12} - u_{23}\\ \begin{pmatrix}0\\0\\0\\1\\0\\-1\\0\end{pmatrix}\end{array}\quad \begin{array}{c}\bar{u}^2_N\\ \begin{pmatrix}0\\0\\0\\1\\1\\1\\0\end{pmatrix}\end{array}$$

Define

$$\begin{aligned}
V^1 &= \{u^1\} \cup \{u^2\} \cup \{\bar{u}_T : |T| \geq 3\}, & \Gamma^1 &= \mathrm{Sp}(V^1),\\
V^2 &= \{u_{12} - u_T : T \subseteq N, |T| = 2, T \neq \{1,2\}\}, & \Gamma^2 &= \mathrm{Sp}(V^2),\\
V^3 &= \{u_1 - u_i : i \in N, i \neq 1\}, & \Gamma^3 &= \mathrm{Sp}(V^3).
\end{aligned}$$

Note that $\mathrm{Sp}(V^1 \cup V^2 \cup V^3) = \Gamma$.

As it turns out, the payoff vector of a solution satisfying efficiency, symmetry and *cont* + *gr* + *sur*-monotonicity is tractable in each subspace. Section 3.5 provides some intuition for this result, and Section 3.7 provides a formal proof.

3.5 Proof sketch of Theorem 6

We illustrate a sketch of the proof of Theorem 6. We provide some intuition for why the three axioms imply linearity and explain why our proof does not cover the cases $n < 6$.

Let ψ be a solution satisfying efficiency, symmetry and *cont* + *gr* + *sur*-monotonicity. The key step of the proof involves showing that ψ is linear with respect to the addition of games in Γ^3. Suppose for simplicity that $N = \{1, 2, 3\}$. To see the key idea of proving linearity, we prove the simplest equation, $\psi_1(2u_1 - 2u_2) = 2\psi_1(u_1 - u_2)$.

$$\begin{aligned}
\psi_1(2u_1 - 2u_2) &= \psi_1\big(2u_1 - u_2 - u_3\big)\\
&= -\psi_2\big(2u_1 - u_2 - u_3\big) - \psi_3\big(2u_1 - u_2 - u_3\big)\\
&= -\psi_2(u_1 - u_2) - \psi_3(u_1 - u_3)\\
&= \psi_1(u_1 - u_2) + \psi_1(u_1 - u_3)\\
&= \psi_1(u_1 - u_2) + \psi_1(u_1 - u_2),
\end{aligned}$$

where the first equality follows from *cont* + *gr* + *sur*-monotonicity, the second equality follows from efficiency, the third equality follows from *cont* + *gr* + *sur*-monotonicity, the fourth equality follows from efficiency and $\psi_3(u_1 - u_2) = \psi_2(u_1 - u_3) = 0$,[6] and the last

[6]We can prove $\psi_3(u_1 - u_2) = 0$ in the following way: by efficiency and symmetry, $\psi_3(\mathbf{0}) = 0$, and by

equality follows from $cont + gr + sur$-monotonicity.

The key idea of the above calculation is to replace $-2u_2$ with $-u_2 - u_3$. Note that after this change, player 1's contributions, the grand coalition worth and the cooperative surplus are invariant. Hence by $cont + gr + sur$-monotonicity, 1's payoff is invariant. In general, this replacement yields a relationship between an integer q and $q-1$. Using an induction argument, we prove the following lemma:

Lemma 3. *Let $v \in \Gamma$, $i, j \in N$, $i \neq j$, and $\lambda \in \mathbb{R}$. Then,*

$$\psi\big(v + \lambda(u_i - u_j)\big) = \psi(v) + \lambda\psi(u_i - u_j).$$

Next, we prove that ψ is linear with respect to the addition of games in Γ^2. Again, we focus on the simplest equation, $\psi_1(2u_{12} - 2u_{34}) = 2\psi_1(u_{12} - u_{34})$. To apply the same idea of the proof of $\psi_1(2u_1 - 2u_2) = 2\psi_1(u_1 - u_2)$, we construct a game $2u_{12} - u_{34} - u_{56}$. Mimicking the proof of Lemma 3, we obtain the following lemma:

Lemma 4. *Let $v \in \Gamma$, $\lambda \in \mathbb{R}$, and $S, T \subseteq N$ with $|S| = |T| = 2$, $S \neq T$. If $v(i) = v(j)$ for all $i, j \in N$, then*

$$\psi\big(v + \lambda(u_S - u_T)\big) - \psi(v) = \lambda\psi(u_S - u_T).$$

The remaining task is to determine the payoff vector of a game in Γ^1. Using the properties (a) to (d) discussed in Section 3.4 and mimicking Young's (1985) proof, we obtain the following lemma:

Lemma 5. *Let $v \in \Gamma^1$. Then,*

$$\psi_i(v) = \frac{v(N)}{n} \text{ for all } i \in N.$$

Combining Lemmas 3 to 5, we prove that ψ is a convex combination of the three solutions on the whole class of games.

Remark 3. As we discussed in Section 3.4, our new basis is derived by focusing on the parameters (a) to (d). When we employ stronger axioms than $cont + gr + sur$-monotonicity, we can decrease the number of parameters we consider. Thus, there is a possibility to find a "better" basis in the sense that having fewer players suffices for the proof. This is indeed true in the case of $cont + gr$-monotonicity (namely, weak monotonicity). For the commander games $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$ (see (2.1)), $\bar{u}_T$ satisfies (a) to (c) for all $T \subseteq N$ with $|T| \geq 2$. Using this basis, we can provide another proof for Casajus and Huettner's (2014) theorem for the cases $n \geq 3$; see Appendix A. Regarding $cont+sur$-monotonicity, we could not obtain an alternative basis. However, it is worth mentioning that there exists a solution for $n = 3$ satisfying efficiency, symmetry and $cont + sur$-monotonicity, but is not a convex combination of linear solutions; see appendix B. Thus, we can at least say that the characterization of the consensus values requires more players than that of the egalitarian Shapley values. ∎

Remark 4. Theorem 6 is concerned only with the cases $n \geq 6$. For $n = 1$, efficiency uniquely determines the solution. For $n = 2$, Casajus and Huettner (2014) provided a

$cont + gr + sur$-monotonicity, $\psi_3(\mathbf{0}) = \psi_3(u_1 - u_2)$.

counterexample that satisfies efficiency, symmetry and weak monotonicity, but is not a convex combination of linear solutions. This solution is also a counterexample to Theorem 6. For $n = 3$, we provide a counterexample in Appendix B. For $n = 4, 5$, the validity of Theorem 6 is an open question. ∎

3.6 Independence of axioms

We show the logical independence of the axioms employed in our characterizations.
Efficiency: The solution $\bar{\psi}$ defined by $\bar{\psi}(v) = \mathbf{0}$ for all $v \in \Gamma$ satisfies symmetry and all monotonicity axioms, but violates efficiency.
Symmetry: Choose one monotonicity axiom and let ψ be the solution that satisfies efficiency, symmetry and the monotonicity axiom. Define $\tilde{\psi}$ as follows:

$$\tilde{\psi}_i(v) = \begin{cases} \psi_i(v) + 1 & \text{if } i = 1, \\ \psi_i(v) - 1 & \text{if } i = 2, \\ \psi_i(v) & \text{otherwise,} \end{cases} \qquad \text{for all } v \in \Gamma.$$

Then, $\tilde{\psi}$ satisfies efficiency and the monotonicity axiom, but violates symmetry.
Monotonicity: We define $\hat{\psi}$ by

$$\hat{\psi}_i(v) = \begin{cases} \dfrac{v(N)}{n} & \text{if } v(j) = v(k) \text{ for some } j, k \in N, \\ v(N) & \text{if } v(j) \neq v(k) \text{ for all } j, k \in N, i = 1, \\ 0 & \text{if } v(j) \neq v(k) \text{ for all } j, k \in N, i \neq 1, \end{cases} \qquad \text{for all } v \in \Gamma.$$

Then, $\hat{\psi}$ satisfies efficiency and symmetry. Consider the following two games $w, w' \in \Gamma$:

$$w = \sum_{j=2}^{n} j \cdot u_j, \;\; w' = \sum_{j=2}^{n} j \cdot u_2.$$

One can check that $\Delta_1 w(S) = \Delta_1 w'(S)$ for all $S \subseteq N \backslash \{1\}$, $w(N) = w'(N)$, $w(1) = w'(1)$, $w^*(1) = w'^*(1)$, $\sum_{i \in N} w(i) = \sum_{i \in N} w'(i)$ and $\sum_{i \in N} w^*(i) = \sum_{i \in N} w'^*(i)$. Thus, all monotonicity axioms state the invariance of 1's payoff between w and w'. However,

$$\hat{\psi}_1(w) = w(N) \neq \frac{w'(N)}{n} = \hat{\psi}_1(w').$$

3.7 Proofs

3.7.1 Proof of Theorem 6

We use the following abbreviations:

$$\text{Claim} \to \text{C}, \; \text{Induction hypothesis} \to \text{IH}, \; \text{Lemma} \to \text{L},$$
$$\textit{cont} + \textit{gr} + \textit{sur}\text{-monotonicity} \to \text{M}, \; \text{Efficiency} \to \text{E}, \; \text{Symmetry} \to \text{S}.$$

The proof consists of 5 steps.
Step 1: We show that ψ is linear with respect to the addition of $v \in \Gamma^3$.

Claim 1. *Let $i \in N$, $v \in \Gamma$. Then, for any $p, q \in \mathbb{N}$ and $j \in N$, $j \neq i$,*

$$\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v) = q\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big].$$

Proof. Let p be fixed.
Induction base: If $q = 1$, the result holds.
Induction step: Suppose that the result holds for $q = r - 1$, and we prove the result for $q = r$, where $r \geq 2$. Let $k \in N \backslash \{i, j\}$. Define $w = v + \frac{r}{p}u_i - \frac{r-1}{p}u_j - \frac{1}{p}u_k$. Then,

$$\begin{aligned}
\psi_j(w) - \psi_j(v) &\overset{\text{M}}{=} \psi_j\Big(v + \frac{r-1}{p}(u_i - u_j)\Big) - \psi_j(v) \\
&\overset{\text{E,M}}{=} -\Big[\psi_i\Big(v + \frac{r-1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big] \\
&\overset{\text{IH}}{=} -(r-1)\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big], \\
\psi_k(w) - \psi_k(v) &\overset{\text{M}}{=} \psi_k\Big(v + \frac{1}{p}(u_i - u_k)\Big) - \psi_k(v) \\
&\overset{\text{E,M}}{=} -\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_k)\Big) - \psi_i(v)\Big] \\
&\overset{\text{M}}{=} -\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
\psi_i(w) - \psi_i(v) &\overset{\text{E,M}}{=} -\{\psi_j(w) - \psi_j(v)\} - \{\psi_k(w) - \psi_k(v)\} \\
&= (r-1)\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big] \\
&\quad + \psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v) \\
&= r\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big]. \qquad (3.3)
\end{aligned}$$

On the other hand,

$$\psi_i(w) - \psi_i(v) \overset{\text{M}}{=} \psi_i\Big(v + \frac{r}{p}(u_i - u_j)\Big) - \psi_i(v). \qquad (3.4)$$

Equations (3.3) and (3.4) complete the proof. □

Claim 2. *Let $i \in N$, $v \in \Gamma$. Then, for any $p, q \in \mathbb{N}$ and $j \in N$, $j \neq i$,*

$$\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v) = \frac{q}{p}\Big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\Big].$$

Proof. Let p be fixed. Then, by letting $q = p$ in C1,

$$\begin{aligned}
\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v) &\overset{\text{C1}}{=} p\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big], \\
\frac{1}{p}\Big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\Big] &= \psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v).
\end{aligned}$$

It follows that, for any $q \in \mathbb{N}$,

$$\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v) \overset{\text{C1}}{=} q\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big]$$
$$= \frac{q}{p}\Big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\Big].$$

□

Claim 3. *Let $v \in \Gamma$, $i, j \in N$, $i \neq j$, and $p, q \in \mathbb{N}$. Then,*

$$\psi_i\Big(v - \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v) = -\Big[\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v)\Big].$$

Proof. Let $k \in N \backslash \{i, j\}$. Define $w = v + \frac{q}{p}(u_i + u_j) - \frac{2q}{p}u_k$. Then,

$$\begin{aligned}
\psi_j(w) - \psi_j(v) &\overset{\text{M}}{=} \psi_j\Big(v - \frac{q}{p}(u_i - u_j)\Big) - \psi_j(v) \\
&\overset{\text{E,M}}{=} -\Big[\psi_i\Big(v - \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v)\Big], \\
\psi_k(w) - \psi_k(v) &\overset{\text{M}}{=} \psi_k\Big(v + \frac{2q}{p}(u_i - u_k)\Big) - \psi_k(v) \\
&\overset{\text{C2}}{=} 2 \cdot \frac{q}{p}\Big[\psi_k\big(v + (u_i - u_k)\big) - \psi_k(v)\Big] \\
&\overset{\text{C2}}{=} 2\Big[\psi_k\Big(v + \frac{q}{p}(u_i - u_k)\Big) - \psi_k(v)\Big] \\
&\overset{\text{E,M}}{=} -2\Big[\psi_i\Big(v + \frac{q}{p}(u_i - u_k)\Big) - \psi_i(v)\Big] \\
&\overset{\text{M}}{=} -2\Big[\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v)\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
\psi_i(w) - \psi_i(v) &\overset{\text{E,M}}{=} -\{\psi_j(w) - \psi_j(v)\} - \{\psi_k(w) - \psi_k(v)\} \\
&= \Big[\psi_i\Big(v - \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v)\Big] \\
&\quad + 2\Big[\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v)\Big].
\end{aligned} \tag{3.5}$$

On the other hand,

$$\psi_i(w) - \psi_i(v) \overset{\text{M}}{=} \psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v). \tag{3.6}$$

Equations (3.5) and (3.6) complete the proof. □

Claim 4. *Let $v \in \Gamma$, $i, j \in N$, $i \neq j$. Then, for any $s \in \mathbb{Q}$,*

$$\psi_i\big(v + s(u_i - u_j)\big) - \psi_i(v) = s\big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\big].$$

Proof. If $s \geq 0$, C2 completes the proof. Suppose that $s < 0$. Then, we can write $s = -\frac{q}{p}$

for some $p, q \in \mathbb{N}$.

$$\begin{aligned}\psi_i\big(v+s(u_i-u_j)\big)-\psi_i(v) &= \psi_i\Big(v-\frac{q}{p}(u_i-u_j)\Big)-\psi_i(v)\\ &\overset{\text{C3}}{=} -\Big[\psi_i\Big(v+\frac{q}{p}(u_i-u_j)\Big)-\psi_i(v)\Big]\\ &\overset{\text{C2}}{=} -\frac{q}{p}\Big[\psi_i\Big(v+(u_i-u_j)\Big)-\psi_i(v)\Big]\\ &= s\big[\psi_i\big(v+(u_i-u_j)\big)-\psi_i(v)\big].\end{aligned}$$

□

Claim 5. *Let $i \in N$, $v \in \Gamma$. Then, for any $\lambda \in \mathbb{R}$ and $j \in N$, $j \neq i$,*

$$\psi_i\big(v+\lambda(u_i-u_j)\big)-\psi_i(v) = \lambda\Big[\psi_i\big(v+(u_i-u_j)\big)-\psi_i(v)\Big].$$

Proof. Let $\lambda \in \mathbb{R}$. Choose sequences of rational numbers $\{r_t\}$ and $\{s_t\}$ that converge to λ from below and above, respectively. Then,

$$\begin{aligned}r_t\Big[\psi_i\big(v+(u_i-u_j)\big)-\psi_i(v)\Big] &\overset{\text{C4}}{=} \psi_i\Big(v+r_t(u_i-u_j)\Big)-\psi_i(v)\\ &\overset{\text{M}}{\leq} \psi_i\Big(v+\lambda(u_i-u_j)\Big)-\psi_i(v)\\ &\overset{\text{M}}{\leq} \psi_i\Big(v+s_t(u_i-u_j)\Big)-\psi_i(v)\\ &\overset{\text{C4}}{=} s_t\Big[\psi_i\big(v+(u_i-u_j)\big)-\psi_i(v)\Big].\end{aligned}$$

Letting $t \to \infty$, we obtain the result. □

Claim 6. *Let $v, w \in \Gamma$. Then,*

$$\psi_i\big(v+(u_i-u_j)\big)-\psi_i(v) = \psi_i\big(w+(u_i-u_j)\big)-\psi_i(w).$$

Proof. Let $z \in \Gamma$ be such that

$$\begin{aligned}&z(N) \geq v(N), z(N)-\sum_{m\in N} z(m) \geq v(N)-\sum_{m\in N} v(m), \Delta_i z(S) \geq \Delta_i v(S)\\ &\qquad\qquad\text{for all } S \subseteq N\backslash i,\\ &z(N) \geq w(N), z(N)-\sum_{m\in N} z(m) \geq w(N)-\sum_{m\in N} w(m), \Delta_i z(S) \geq \Delta_i w(S)\\ &\qquad\qquad\text{for all } S \subseteq N\backslash i.\end{aligned}$$

Suppose that

$$\psi_i\big(v+(u_i-u_j)\big)-\psi_i(v) \neq \psi_i\big(z+(u_i-u_j)\big)-\psi_i(z).$$

For any $\lambda \in \mathbb{R}$,

$$\begin{aligned}
&\psi_i\big(v+\lambda(u_i-u_j)\big)-\psi_i(v)-\Big\{\psi_i\big(z+\lambda(u_i-u_j)\big)-\psi_i(z)\Big\}\\
\overset{\text{C5}}{=}&\lambda\Big[\psi_i\big(v+(u_i-u_j)\big)-\psi_i(v)-\Big\{\psi_i\big(z+(u_i-u_j)\big)-\psi_i(z)\Big\}\Big],\\
&\psi_i\big(v+\lambda(u_i-u_j)\big)-\psi_i\big(z+\lambda(u_i-u_j)\big)\\
=&\lambda\Big[\psi_i\big(v+(u_i-u_j)\big)-\psi_i(v)-\Big\{\psi_i\big(z+(u_i-u_j)\big)-\psi_i(z)\Big\}\Big]\\
&+\psi_i(v)-\psi_i(z).
\end{aligned}$$

So, by appropriately choosing λ, we obtain

$$\psi_i\big(v+\lambda(u_i-u_j)\big)-\psi_i\big(z+\lambda(u_i-u_j)\big)>0.$$

This contradicts M. So, we must have

$$\psi_i\big(v+(u_i-u_j)\big)-\psi_i(v)=\psi_i\big(z+(u_i-u_j)\big)-\psi_i(z).$$

Applying the same argument to the games w and z, we obtain the desired equation. □

Lemma 3 . *Let $v \in \Gamma$, $i, j \in N$, $i \neq j$, and $\lambda \in \mathbb{R}$. Then,*

$$\psi\big(v+\lambda(u_i-u_j)\big)=\psi(v)+\lambda\psi(u_i-u_j).$$

Proof. By E and S, $\psi_i(\mathbf{0}) = 0$ for all $i \in N$. With this equation in hand, by letting $w = \mathbf{0}$ in C6,

$$\begin{aligned}
\lambda\psi_i(u_i-u_j)&\overset{\text{C6}}{=}\lambda\Big[\psi_i\big(v+(u_i-u_j)\big)-\psi_i(v)\Big]\\
&\overset{\text{C5}}{=}\psi_i\big(v+\lambda(u_i-u_j)\big)-\psi_i(v).
\end{aligned}$$

For $k \in N\backslash\{i, j\}$,

$$\psi_k\big(v+\lambda(u_i-u_j)\big)-\psi_k(v)\overset{\text{M}}{=}0=\lambda\psi_k(\mathbf{0})\overset{\text{M}}{=}\lambda\psi_k(u_i-u_j).$$

E completes the proof. □

Step 2: We show ψ is linear with respect to the addition of $v \in \Gamma^2$. For any $v \in \Gamma$ and $i, j \in N$, $i \neq j$, define

$$\psi_{ij}(v)=\psi_i(v)+\psi_j(v).$$

Claim 7. *Let $v \in \Gamma$ and $i, j \in N$, $i \neq j$, $k, l \in N\backslash\{i, j\}$, $k \neq l$. Then, for any $p, q \in \mathbb{N}$,*

$$\psi_{ij}\Big(v+\frac{q}{p}(u_{ij}-u_{kl})\Big)-\psi_{ij}(v)=q\Big[\psi_{ij}\Big(v+\frac{1}{p}(u_{ij}-u_{kl})\Big)-\psi_{ij}(v)\Big].$$

Proof. Let p be fixed.
Induction base: If $q = 1$, the result holds.
Induction step: Suppose that the result holds for $q = r - 1$, and we prove the result

for $q = r$, where $r \geq 2$. Without loss of generality, suppose that $1 = i$, $2 = j$, $3 = k$ and $4 = l$. Define $w = v + \frac{r}{p}u_{12} - \frac{r-1}{p}u_{34} - \frac{1}{p}u_{56}$. Then,

$$
\begin{aligned}
\psi_{34}(w) - \psi_{34}(v) &\overset{\text{M}}{=} \psi_{34}\Big(v + \frac{r-1}{p}(u_{12} - u_{34})\Big) - \psi_{34}(v) \\
&\overset{\text{E,M}}{=} -\Big[\psi_{12}\Big(v + \frac{r-1}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big] \\
&\overset{\text{IH}}{=} -(r-1)\Big[\psi_{12}\Big(v + \frac{1}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big], \\
\psi_{56}(w) - \psi_{56}(v) &\overset{\text{M}}{=} \psi_{56}\Big(v + \frac{1}{p}(u_{12} - u_{56})\Big) - \psi_{56}(v) \\
&\overset{\text{E,M}}{=} -\Big[\psi_{12}\Big(v + \frac{1}{p}(u_{12} - u_{56})\Big) - \psi_{12}(v)\Big] \\
&\overset{\text{M}}{=} -\Big[\psi_{12}\Big(v + \frac{1}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\psi_{12}(w) - \psi_{12}(v) &\overset{\text{E,M}}{=} -\{\psi_{34}(w) - \psi_{34}(v)\} - \{\psi_{56}(w) - \psi_{56}(v)\} \\
&= (r-1)\Big[\psi_{12}\Big(v + \frac{1}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big] \\
&+ \Big[\psi_{12}\Big(v + \frac{1}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big] \\
&= r\Big[\psi_{12}\Big(v + \frac{1}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big]. \qquad (3.7)
\end{aligned}
$$

On the other hand,

$$
\psi_{12}(w) - \psi_{12}(v) \overset{\text{M}}{=} \psi_{12}\Big(v + \frac{r}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v). \qquad (3.8)
$$

Equations (3.7) and (3.8) complete the proof. □

Claim 8. *Let $v \in \Gamma$ and $i, j \in N$, $i \neq j$, $k, l \in N \backslash \{i, j\}$, $k \neq l$. Then, for any $p, q \in \mathbb{N}$,*

$$
\psi_{ij}\Big(v + \frac{q}{p}(u_{ij} - u_{kl})\Big) - \psi_{ij}(v) = \frac{q}{p}\Big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\Big].
$$

Proof. Let p be fixed. Then, by letting $q = p$ in C7,

$$
\begin{aligned}
&\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v) \overset{\text{C7}}{=} p\Big[\psi_{ij}\Big(v + \frac{1}{p}(u_{ij} - u_{kl})\Big) - \psi_{ij}(v)\Big], \\
&\frac{1}{p}\Big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\Big] = \psi_{ij}\Big(v + \frac{1}{p}(u_{ij} - u_{kl})\Big) - \psi_{ij}(v).
\end{aligned}
$$

It follows that, for any $q \in \mathbb{N}$,

$$
\begin{aligned}
\psi_{ij}\Big(v + \frac{q}{p}(u_{ij} - u_{kl})\Big) - \psi_{ij}(v) &\overset{\text{C7}}{=} q\Big[\psi_{ij}\Big(v + \frac{1}{p}(u_{ij} - u_{kl})\Big) - \psi_{ij}(v)\Big] \\
&= \frac{q}{p}\Big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\Big].
\end{aligned}
$$

□

Claim 9. *Let $v \in \Gamma$, $i, j \in N$, $i \neq j$, $k, l \in N \backslash \{i, j\}$, $k \neq l$, and $p, q \in \mathbb{N}$. Then,*

$$\psi_{ij}\Big(v - \frac{q}{p}(u_{ij} - u_{kl})\Big) - \psi_{ij}(v) = -\Big[\psi_{ij}\Big(v + \frac{q}{p}(u_{ij} - u_{kl})\Big) - \psi_{ij}(v)\Big].$$

Proof. Without loss of generality, suppose that $1 = i$, $2 = j$, $3 = k$ and $4 = l$. Define $w = v + \frac{q}{p}(u_{12} + u_{34}) - \frac{2q}{p}u_{56}$. Then,

$$\begin{aligned}
\psi_{34}(w) - \psi_{34}(v) &\overset{\text{M}}{=} \psi_{34}\Big(v - \frac{q}{p}(u_{12} - u_{34})\Big) - \psi_{34}(v) \\
&\overset{\text{E,M}}{=} -\Big[\psi_{12}\Big(v - \frac{q}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big], \\
\psi_{56}(w) - \psi_{56}(v) &\overset{\text{M}}{=} \psi_{56}\Big(v + \frac{2q}{p}(u_{12} - u_{56})\Big) - \psi_{56}(v) \\
&\overset{\text{C8}}{=} 2 \cdot \frac{q}{p}\Big[\psi_{56}\Big(v + (u_{12} - u_{56})\Big) - \psi_{56}(v)\Big] \\
&\overset{\text{C8}}{=} 2\Big[\psi_{56}\Big(v + \frac{q}{p}(u_{12} - u_{56})\Big) - \psi_{56}(v)\Big] \\
&\overset{\text{E,M}}{=} -2\Big[\psi_{12}\Big(v + \frac{q}{p}(u_{12} - u_{56})\Big) - \psi_{12}(v)\Big] \\
&\overset{\text{M}}{=} -2\Big[\psi_{12}\Big(v + \frac{q}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
\psi_{12}(w) - \psi_{12}(v) &\overset{\text{E,M}}{=} -\{\psi_{34}(w) - \psi_{34}(v)\} - \{\psi_{56}(w) - \psi_{56}(v)\} \\
&= \Big[\psi_{12}\Big(v - \frac{q}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big] \\
&\quad + 2\Big[\psi_{12}\Big(v + \frac{q}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big]. \qquad (3.9)
\end{aligned}$$

On the other hand,

$$\psi_{12}(w) - \psi_{12}(v) \overset{\text{M}}{=} \psi_{12}\Big(v + \frac{q}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v). \qquad (3.10)$$

Equations (3.9) and (3.10) complete the proof. □

Claim 10. *Let $v \in \Gamma$, $i, j \in N$, $i \neq j$, $k, l \in N \backslash \{i, j\}$, $k \neq l$. Then, for any $s \in \mathbb{Q}$,*

$$\psi_{ij}\big(v + s(u_{ij} - u_{kl})\big) - \psi_{ij}(v) = s\big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\big].$$

Proof. If $s \geq 0$, C8 completes the proof. Suppose that $s < 0$. Then, we can write $s = -\frac{q}{p}$

for some $p, q \in \mathbb{N}$.

$$\begin{aligned}\psi_{ij}\big(v + s(u_{ij} - u_{kl})\big) - \psi_{ij}(v) &= \psi_{ij}\Big(v - \frac{q}{p}(u_{ij} - u_{kl})\Big) - \psi_{ij}(v) \\ &\overset{\text{C9}}{=} -\Big[\psi_{ij}\Big(v + \frac{q}{p}(u_{ij} - u_{kl})\Big) - \psi_{ij}(v)\Big] \\ &\overset{\text{C8}}{=} -\frac{q}{p}\Big[\psi_{ij}\Big(v + (u_{ij} - u_{kl})\Big) - \psi_{ij}(v)\Big] \\ &= s\big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\big].\end{aligned}$$

□

Claim 11. *Let $v \in \Gamma$, $\lambda \in \mathbb{R}$ and $i, j \in N$, $i \neq j$, $k, l \in N \backslash \{i, j\}$, $k \neq l$. Then,*

$$\psi_{ij}\big(v + \lambda(u_{ij} - u_{kl})\big) - \psi_{ij}(v) = \lambda\big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\big].$$

Proof. Let $\lambda \in \mathbb{R}$. Choose sequences of rational numbers $\{r_t\}$ and $\{s_t\}$ that converge to λ from below and above, respectively. Then,

$$\begin{aligned}r_t\Big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\Big] &\overset{\text{C10}}{=} \psi_{ij}\Big(v + r_t(u_{ij} - u_{kl})\Big) - \psi_{ij}(v) \\ &\overset{\text{M}}{\leq} \psi_{ij}\Big(v + \lambda(u_{ij} - u_{kl})\Big) - \psi_{ij}(v) \\ &\overset{\text{M}}{\leq} \psi_{ij}\Big(v + s_t(u_{ij} - u_{kl})\Big) - \psi_{ij}(v) \\ &\overset{\text{C10}}{=} s_t\Big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\Big].\end{aligned}$$

Letting $t \to \infty$, we obtain the result. □

Claim 12. *Let $v, w \in \Gamma$ and $i, j \in N$, $i \neq j$, $k, l \in N \backslash \{i, j\}$, $k \neq l$. Then,*

$$\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v) = \psi_{ij}\big(w + (u_{ij} - u_{kl})\big) - \psi_{ij}(w).$$

Proof. Let $z \in \Gamma$ be such that

$$\begin{aligned}&z(N) \geq v(N), z(N) - \sum_{m \in N} z(m) \geq v(N) - \sum_{m \in N} v(m), \\ &\Delta_i z(S) \geq \Delta_i v(S) \text{ for all } S \subseteq N \backslash i, \Delta_j z(S) \geq \Delta_j v(S) \text{ for all } S \subseteq N \backslash j, \\ &z(N) \geq w(N), z(N) - \sum_{m \in N} z(m) \geq w(N) - \sum_{m \in N} w(m), \\ &\Delta_i z(S) \geq \Delta_i w(S) \text{ for all } S \subseteq N \backslash i, \Delta_j z(S) \geq \Delta_j w(S) \text{ for all } S \subseteq N \backslash j.\end{aligned}$$

Suppose that

$$\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v) \neq \psi_{ij}\big(z + (u_{ij} - u_{kl})\big) - \psi_{ij}(z).$$

For any $\lambda \in \mathbb{R}$,

$$\begin{aligned}
&\psi_{ij}\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_{ij}(v)-\Big\{\psi_{ij}\big(z+\lambda(u_{ij}-u_{kl})\big)-\psi_{ij}(z)\Big\}\\
&\overset{\text{C11}}{=}\lambda\Big[\psi_{ij}(v+(u_{ij}-u_{kl}))-\psi_{ij}(v)-\Big\{\psi_{ij}\big(z+(u_{ij}-u_{kl})\big)-\psi_{ij}(z)\Big\}\Big],\\
&\psi_{ij}\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_{ij}\big(z+\lambda(u_{ij}-u_{kl})\big)\\
&=\lambda\Big[\psi_{ij}(v+(u_{ij}-u_{kl}))-\psi_{ij}(v)-\Big\{\psi_{ij}\big(z+(u_{ij}-u_{kl})\big)-\psi_{ij}(z)\Big\}\Big]\\
&+\psi_{ij}(v)-\psi_{ij}(z).
\end{aligned}$$

So, by appropriately choosing λ, we obtain

$$\psi_{ij}\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_{ij}\big(z+\lambda(u_{ij}-u_{kl})\big)>0.$$

This contradicts M. So, we must have

$$\psi_{ij}\big(v+(u_{ij}-u_{kl})\big)-\psi_{ij}(v)=\psi_{ij}\big(z+(u_{ij}-u_{kl})\big)-\psi_{ij}(z).$$

Applying the same argument to the games w and z, we obtain the desired equation. □

Claim 13. *Let $v \in \Gamma$, $\lambda \in \mathbb{R}$, and $i,j \in N$, $i \neq j$, $k,l \in N\backslash\{i,j\}$, $k \neq l$. If $v(i)=v(j)$, then,*

$$\psi_i\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_i(v)=\lambda\psi_i(u_{ij}-u_{kl}).$$

Proof. **Case 1:** Suppose that i and j are substitutes in v. Then,

$$\begin{aligned}
2\Big[\psi_i\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_i(v)\Big] &\overset{\text{S}}{=}\psi_{ij}\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_{ij}(v)\\
&\overset{\text{C11}}{=}\lambda\Big[\psi_{ij}\big(v+(u_{ij}-u_{kl})\big)-\psi_{ij}(v)\Big]\\
&\overset{\text{C12}}{=}\lambda\psi_{ij}(u_{ij}-u_{kl})\\
&\overset{\text{S}}{=}2\lambda\psi_i(u_{ij}-u_{kl}).
\end{aligned}$$

Case 2: Suppose that i and j are not substitutes in v. By Pintér (2015), there exists $w \in \Gamma$ such that

$$\begin{aligned}
&\Delta_i v(S)=\Delta_i w(S) \text{ for all } S\subseteq N\backslash i,\\
&i \text{ and } j \text{ are substitutes in } w.
\end{aligned}$$

Note that $w(j)=w(i)=v(i)=v(j)$. Let $k,l \in N\backslash\{i,j\}$, $k \neq l$, and define $z \in \Gamma$ by

$$\begin{aligned}
z=w&+\sum_{m\in N\backslash\{i,j\}}\big(v(m)-w(m)\big)u_m\\
&+\Big(v(N)-w(N)-\sum_{m\in N\backslash\{i,j\}}\big(v(m)-w(m)\big)\Big)u_{kl}.
\end{aligned}$$

Then, $\Delta_i v(S)=\Delta_i z(S)$ for all $S\subseteq N\backslash i$, $v(m)=z(m)$ for all $m \in N$, and $v(N)=z(N)$.

Moreover, i and j are substitutes in z. Hence,

$$\begin{aligned}\psi_i\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_i(v) &\overset{\text{M}}{=} \psi_i\big(z+\lambda(u_{ij}-u_{kl})\big)-\psi_i(z)\\ &\overset{\text{Case1}}{=} \lambda\psi_i(u_{ij}-u_{kl}).\end{aligned}$$

□

Lemma 4 . *Let $v \in \Gamma$, $\lambda \in \mathbb{R}$, and $S, T \subseteq N$, $|S| = |T| = 2$, $S \neq T$. If $v(i) = v(j)$ for all $i, j \in N$, then*

$$\psi\big(v+\lambda(u_S-u_T)\big)-\psi(v)=\lambda\psi(u_S-u_T).$$

Proof. **Case 1:** Suppose that $S \cap T = \emptyset$. Let $S = \{i, j\}$, $T = \{k, l\}$. Then, for $m = i, j$,

$$\psi_m\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_m(v) \overset{\text{C13}}{=} \lambda\psi_m(u_{ij}-u_{kl}).$$

For $m = k, l$,

$$\begin{aligned}\psi_m\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_m(v) &= \psi_m\big(v-\lambda(u_{kl}-u_{ij})\big)-\psi_m(v)\\ &\overset{\text{C13}}{=} -\lambda\psi_m(u_{kl}-u_{ij})\\ &= \lambda\psi_m(u_{ij}-u_{kl}),\end{aligned}$$

where the last equality follows from C13 and $\psi_m(\mathbf{0}) \overset{\text{E,S}}{=} 0$. For player $m \in N\backslash\{i, j, k, l\}$,

$$\psi_m\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_m(v) \overset{\text{M}}{=} 0 \overset{\text{E,S}}{=} \psi_m(\mathbf{0}) \overset{\text{M}}{=} \lambda\psi_m(u_{ij}-u_{kl}).$$

Case 2: Suppose that $S \cap T \neq \emptyset$. Let $S = \{i, j\}$, $T = \{j, k\}$. Let $l \in N\backslash\{i, j, k\}$. For player i,

$$\begin{aligned}\psi_i\big(v+\lambda(u_{ij}-u_{jk})\big)-\psi_i(v) &\overset{\text{M}}{=} \psi_i\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_i(v)\\ &\overset{\text{C13}}{=} \lambda\psi_i(u_{ij}-u_{kl})\\ &\overset{\text{M}}{=} \lambda\psi_i(u_{ij}-u_{jk}).\end{aligned}$$

For player k,

$$\begin{aligned}\psi_k(v+\lambda(u_{ij}-u_{jk}))-\psi_k(v) &= \psi_k\big(v-\lambda(u_{jk}-u_{ij})\big)-\psi_k(v)\\ &\overset{\text{M}}{=} \psi_k\big(v-\lambda(u_{jk}-u_{il})\big)-\psi_k(v)\\ &\overset{\text{C13}}{=} -\lambda\psi_k(u_{jk}-u_{il})\\ &\overset{\text{C13}}{=} \lambda\psi_k(u_{il}-u_{jk})\\ &\overset{\text{M}}{=} \lambda\psi_k(u_{ij}-u_{jk}).\end{aligned}$$

For player $m \in N\backslash\{i, j, k\}$,

$$\psi_m\big(v+\lambda(u_{ij}-u_{jk})\big)-\psi_m(v) \overset{\text{M}}{=} 0 \overset{\text{E,S}}{=} \psi_m(\mathbf{0}) \overset{\text{M}}{=} \lambda\psi_m(u_{ij}-u_{jk}).$$

E completes the proof. □

Step 3 We endogenously derive coefficients of Sh and ED.

Claim 14. *For distinct players* $i, j, k \in N$*,*

$$\psi_j(u_{ij} - u_{jk}) = 0.$$

Proof. Without loss of generality, suppose that $1 = i$, $2 = j$, $3 = k$. Then,

$$\begin{aligned}
\psi_1(u_{12} - u_{23}) &\overset{\text{M}}{=} \psi_1(u_{12} + u_{23} - 2u_{45}),\\
\psi_3(u_{23} - u_{12}) &\overset{\text{M}}{=} \psi_3(u_{12} + u_{23} - 2u_{45}),\\
\psi_1(u_{12} + u_{23} - 2u_{45}) &\overset{\text{S}}{=} \psi_3(u_{12} + u_{23} - 2u_{45}).
\end{aligned}$$

The above equations imply

$$\psi_1(u_{12} - u_{23}) = \psi_3(u_{23} - u_{12}). \tag{3.11}$$

For player $m \in N\backslash\{1, 2, 3\}$,

$$\psi_m(u_{12} - u_{23}) \overset{\text{M}}{=} \psi_m(\mathbf{0}) \overset{\text{E,S}}{=} 0.$$

Thus,

$$\begin{aligned}
\psi_2(u_{12} - u_{23}) &\overset{\text{E}}{=} -\psi_1(u_{12} - u_{23}) - \psi_3(u_{12} - u_{23})\\
&= -\psi_1(u_{12} - u_{23}) + \psi_3(u_{23} - u_{12})\\
&= 0,
\end{aligned}$$

where the second equality follows from L2 and $\psi_3(\mathbf{0}) \overset{\text{E,S}}{=} 0$, and the last equality follows from (3.11). □

Define

$$y = \psi_3(nu_1), \tag{3.12}$$
$$z = \psi_3(nu_{12}). \tag{3.13}$$

Then,

$$\begin{aligned}
0 \overset{\text{E,S}}{=} \psi_3(\mathbf{0}) \overset{\text{M}}{\leq} y \overset{\text{M}}{\leq} \psi_3(nu_N) \overset{\text{E,S}}{=} 1,\\
0 \overset{\text{E,S}}{=} \psi_3(\mathbf{0}) \overset{\text{M}}{\leq} z \overset{\text{M}}{\leq} \psi_3(nu_N) \overset{\text{E,S}}{=} 1.
\end{aligned}$$

In addition,

$$y = \psi_3(nu_1) \overset{\text{M}}{\leq} \psi_3(nu_{12}) = z.$$

Define $x = 1 - z$. Note that $0 \leq x \leq 1$, $0 \leq x + y \leq 1$. Define a new solution $\Phi^{x,y}$ by

$$\Phi^{x,y}(v) = xSh(v) + yED(v) + (1 - x - y)CIS(v) \text{ for all } v \in \Gamma. \tag{3.14}$$

Claim 15. *Let $i \in N$, $i \neq 1$. Then,*

$$\psi(u_1 - u_i) = \Phi^{x,y}(u_1 - u_i).$$

Proof. Recall that $u^1 = \sum_{j \in N} u_j$.

$$\begin{aligned}
1 &\overset{\text{E,S}}{=} \psi_1(u^1) \\
&= \psi_1\Big(nu_1 - \sum_{j \neq 1}(u_1 - u_j)\Big) \\
&\overset{\text{L1}}{=} \psi_1(nu_1) - \sum_{j \neq 1} \psi_1(u_1 - u_j) \\
&\overset{\text{E,S}}{=} \big(n - (n-1)y\big) - \sum_{j \neq 1} \psi_1(u_1 - u_j) \\
&\overset{\text{M}}{=} \big(n - (n-1)y\big) - (n-1)\psi_1(u_1 - u_i).
\end{aligned}$$

By rearranging,

$$\psi_1(u_1 - u_i) = 1 - y.$$

Moreover, since $Sh_1(u_1 - u_i) = CIS_1(u_1 - u_i) = 1$ and $ED_1(u_1 - u_i) = 0$, we obtain $\Phi_1^{x,y}(u_1 - u_i) = 1 - y$. Together with the above equation,

$$\psi_1(u_1 - u_i) = \Phi_1^{x,y}(u_1 - u_i).$$

For player $j \in N \backslash \{1, i\}$,

$$\psi_j(u_1 - u_i) \overset{\text{M}}{=} \psi_j(\mathbf{0}) \overset{\text{E,S}}{=} 0 = \Phi_j^{x,y}(u_1 - u_i).$$

E completes the proof. □

Claim 16. *Let $S \subseteq N$, $|S| = 2$, $S \neq \{1, 2\}$. Then,*

$$\psi(u_{12} - u_S) = \Phi^{x,y}(u_{12} - u_S).$$

Proof. **Case 1:** Suppose that $2 \notin S$. Since $u^2 = \sum_{T \subseteq N : |T|=2} u_T$,

$$u^2(N) = \frac{n(n-1)}{2} \text{ and } u^2 = \frac{n(n-1)}{2} u_{12} - \sum_{T \subseteq N : |T|=2, T \neq \{1,2\}} (u_{12} - u_T).$$

With this in mind,

$$\begin{aligned}
1 &\overset{\text{E,S}}{=} \psi_2\Big(\frac{2}{n-1}u^2\Big) \\
&= \psi_2\Big(nu_{12} - \frac{2}{n-1}\sum_{T\subseteq N:|T|=2,T\neq\{1,2\}}(u_{12}-u_T)\Big) \\
&\overset{\text{L2}}{=} \psi_2(nu_{12}) - \frac{2}{n-1}\sum_{T\subseteq N:|T|=2,T\neq\{1,2\}}\psi_2(u_{12}-u_T) \\
&\overset{\text{E,S}}{=} \frac{1}{2}\big\{n-(n-2)z\big\} - \frac{2}{n-1}\sum_{T\subseteq N:|T|=2,T\neq\{1,2\}}\psi_2(u_{12}-u_T) \\
&\overset{\text{C14}}{=} \frac{1}{2}\big\{n-(n-2)z\big\} - \frac{2}{n-1}\sum_{T\subseteq N:|T|=2,2\notin T}\psi_2(u_{12}-u_T) \\
&\overset{\text{M}}{=} \frac{1}{2}\big\{n-(n-2)z\big\} - \frac{2}{n-1}\cdot\frac{(n-1)(n-2)}{2}\psi_2(u_{12}-u_S) \\
&= \frac{1}{2}\big\{n-(n-2)z\big\} - (n-2)\psi_2(u_{12}-u_S).
\end{aligned}$$

By rearranging,

$$\psi_2(u_{12}-u_S) = \frac{1-z}{2} = \frac{x}{2} = \Phi_2^{x,y}(u_{12}-u_S).$$

If $1\notin S$,

$$\psi_1(u_{12}-u_S) \overset{\text{S}}{=} \psi_2(u_{12}-u_S) = \Phi_2^{x,y}(u_{12}-u_S) = \Phi_1^{x,y}(u_{12}-u_S).$$

If $1\in S$,

$$\psi_1(u_{12}-u_S) \overset{\text{C14}}{=} 0 = \Phi_1^{x,y}(u_{12}-u_S).$$

For $m\in N\backslash(S\cup\{1,2\})$,

$$\psi_m(u_{12}-u_S) \overset{\text{M}}{=} \psi_m(\mathbf{0}) \overset{\text{E,S}}{=} 0 = \Phi_m^{x,y}(u_{12}-u_S).$$

E and S establish the desired equation.
Case 2: Suppose that $2\in S$. We can write $S=\{2,i\}$ for some $i\in N\backslash\{1,2\}$.

$$\begin{aligned}
&\psi_1(u_{12}-u_{2i}) \overset{\text{M}}{=} \psi_1(u_{12}-u_{34}) \overset{\text{Case1}}{=} \Phi_1^{x,y}(u_{12}-u_{34}) = \Phi_1^{x,y}(u_{12}-u_{2i}), \\
&\psi_2(u_{12}-u_{2i}) \overset{\text{C14}}{=} 0 = \Phi_2^{x,y}(u_{12}-u_{2i}).
\end{aligned}$$

For $m\in N\backslash\{1,2,i\}$,

$$\psi_m(u_{12}-u_{2i}) \overset{\text{M}}{=} \psi_m(\mathbf{0}) \overset{\text{E,S}}{=} 0 = \Phi_m^{x,y}(u_{12}-u_{2i}).$$

E completes the proof. □

Step 4 We show that ψ coincides with ED on Γ^1.

Lemma 5 . *Let $v \in \Gamma^1$. Then,*

$$\psi_i(v) = \Phi_i^{x,y}(v) = \frac{v(N)}{n} \text{ for all } i \in N.$$

Proof. Since $v \in \Gamma^1$, there exist unique real numbers α, β, γ_T, $T \subseteq N$, $|T| \geq 3$, such that

$$v = \alpha u^1 + \beta u^2 + \sum_{T \subseteq N : |T| \geq 3} \gamma_T \bar{u}_T.$$

Let $\mathcal{C} = \{T \subseteq N : |T| \geq 3, \gamma_T \neq 0\}$. We proceed by induction.
Induction base: If $|\mathcal{C}| = 0$, then $v = \alpha u^1 + \beta u^2$. In this game, any two players are substitutes in v. Hence, the result follows from E and S.
Induction step: Suppose that the result holds for $|\mathcal{C}| = t - 1$, and we prove the result for $|\mathcal{C}| = t$, where $t \geq 1$.
Let $j \in N \backslash (\cap_{R \in \mathcal{C}} R)$. Let R be such that $R \in \mathcal{C}$ and $j \notin R$. Then, we have

$$\psi_j(v) \stackrel{\text{M}}{=} \psi_j(v - \gamma_R \bar{u}_R) \stackrel{\text{IH}}{=} \frac{v(N)}{n}.$$

Hence, the payoff of player $j \in N \backslash (\cap_{R \in \mathcal{C}} R)$ is determined. Since any two players in $\cap_{R \in \mathcal{C}} R$ are substitutes in v, E and S uniquely determine the payoffs of all players. □

Step 5 We show that $\psi(v) = \Phi^{x,y}(v)$ for all $v \in \Gamma$.

Claim 17. *Let $v^1 \in \Gamma^1$, $v^2 \in \Gamma^2$. Then, $\psi(v^1 + v^2) = \Phi^{x,y}(v^1 + v^2)$.*

Proof. Since $v^2 \in \Gamma^2$, there exist unique real numbers γ_T for $T \subseteq N$, $|T| = 2$, $T \neq \{1, 2\}$, such that

$$v^2 = \sum_{T \subseteq N : |T| = 2, T \neq \{1,2\}} \gamma_T (u_{12} - u_T).$$

Since $v^1(i) = v^1(j)$ for all $i, j \in N$,[7]

$$\begin{aligned} \psi(v^1 + v^2) &\stackrel{\text{L4}}{=} \psi(v^1) + \sum_{T \subseteq N : |T| = 2, T \neq \{1,2\}} \gamma_T \psi(u_{12} - u_T) \\ &\stackrel{\text{L5,C16}}{=} \Phi^{x,y}(v^1) + \sum_{T \subseteq N : |T| = 2, T \neq \{1,2\}} \gamma_T \Phi^{x,y}(u_{12} - u_T) \\ &= \Phi^{x,y}(v^1 + v^2). \end{aligned}$$

□

We resume the proof of Theorem 6. Let $v \in \Gamma$. Then, there exist $v^1 \in \Gamma^1$, $v^2 \in \Gamma^2$ and $v^3 \in \Gamma^3$ such that

$$v = v^1 + v^2 + v^3.$$

Since $v^3 \in \Gamma^3$, there exist unique real numbers γ_i, $i \in N$, such that

$$v^3 = \sum_{i \in N} \gamma_i (u_1 - u_i).$$

[7]Note that $(u^1 + u^2)(i) = (u^1 + u^2)(j)$ for all $i, j \in N$ and $\bar{u}_T(i) = 0$ for all $T \subseteq N$, $|T| \geq 3$, $i \in N$.

Then,

$$\begin{aligned}\psi(v^1+v^2+v^3) &= \psi\Big(v^1+v^2+\sum_{i\in N}\gamma_i(u_1-u_i)\Big)\\ &\overset{\text{L3}}{=} \psi(v^1+v^2)+\sum_{i\in N}\gamma_i\psi(u_1-u_i)\\ &\overset{\text{C17, C15}}{=} \Phi^{x,y}(v^1+v^2)+\sum_{i\in N}\gamma_i\Phi^{x,y}(u_1-u_i)\\ &= \Phi^{x,y}(v^1+v^2+v^3).\end{aligned}$$

Thus, $\psi=\Phi^{x,y}$.

Remark 5. In some part of the proof, we borrowed ideas of previous works. The proofs of Claims 6 and 12 are based on the proof of Claim 5 of Casajus and Huettner (2014). The proof of Lemma 3 is based on the proof of Theorem 2 of Young (1985). ∎

3.7.2 Proofs of Theorems 7, 8 and 9

Note that the axioms employed in Theorems 7, 8 and 9 are stronger than *cont* + *gr* + *sur*-monotonicity. Thus, we can follow the same line of the proof of Theorem 6. We only change the proof of Step 3 in which we endogenously derive the coefficients of a convex combination.

Proof of Theorem 7 **.** Define y and z as we did in equations (3.12), (3.13) and let $x=1-z$. Then,

$$0 \overset{\text{E,S}}{=} \psi_3(\mathbf{0})=\psi_3(nu_1)=y,$$

where the second equality follows from *cont* + *sur*-monotonicity. Thus, equation (3.14) reduces to

$$\Phi^{x,y}(v)=xSh(v)+(1-x)CIS(v) \text{ for all } v\in\Gamma.$$

□

Proof of Theorem 8 **.** Define y and z as we did in equations (3.12), (3.13) and let $x=1-z$. Then,

$$z=\psi_3(nu_{12})=\psi_3(nu_N)=1,$$

where the second equality follows from *id* + *gr* + *sur*-monotonicity. It follows that $x=0$. Thus, equation (3.14) reduces to

$$\Phi^{x,y}(v)=yCIS(v)+(1-y)ED(v) \text{ for all } v\in\Gamma.$$

□

Proof of Theorem 9 **.** Define y and z as we did in equations (3.12), (3.13) and let $x=1-z$. Then,

$$0 \overset{\text{E,S}}{=} \psi_3(\mathbf{0})=\psi_3(nu_1)=y,$$

where the second equality follows from $id + sur$-monotonicity. In addition,

$$z = \psi_3(nu_{12}) = \psi_3(nu_N) = 1,$$

where the second equality follows from $id + sur$-monotonicity. It follows that $x = y = 0$. Thus, equation (3.14) reduces to

$$\Phi^{x,y}(v) = CIS(v) \text{ for all } v \in \Gamma.$$

□

Chapter 4

The Shapley value and the core in NTU games

4.1 Introduction

The core and the Shapley value are two central solutions in the class of TU games. The two solutions are typically discussed separately, but there is a general relationship between them. Monderer et al. (1992) proved that any core element is attainable as the outcome of a weighted Shapley value, an asymmetric extension of the Shapley value.[1] To be more precise, let V be an NTU game and x be a core element of V. Then, there exists a weight w such that the weighted Shapley value with weight w in game V coincides with x. This is a notable result "in light of the difference in concept behind these solutions (Monderer et al. (1992))."

Monderer et al.'s (1992) result, however, relies on the underlying assumptions behind TU games. As the core has been applied to allocation problems without the quasi-linearity assumption, e.g., exchange economies or matching markets, its relationship needs to be studied in NTU games. The purpose of this chapter is to extend Monderer et al.'s (1992) result to NTU games.

We can extend the core to NTU games in a straightforward way, while an extension of the weighted Shapley value is not unique. We focus on several extensions of the weighted Shapley values. We first show that the core is included in the closure of the outcomes of the weighted egalitarian solutions introduced by Kalai and Samet (1985). This result provides a normative foundation to the core. In contrast, we show that the core is not always included in the closure of the outcomes of the weighted Shapley NTU values (Shapley (1969, 1988)). Similarly we show that the core is not always included in the closure of the weighted MC values (Otten et al. (1998)).

The above results offer new insight into the problem of extending the weighted Shapley value to NTU games. In view of the relationship to the core, the weighted egalitarian solutions are a more desirable extension.

As a byproduct of our approach, we study the relationship between the core and contributions of players in NTU games. We show that, if the attainable payoffs for the grand coalition is represented as a closed half-space, then any element of the core is attainable as the expected value of contributions.

[1]Here, the word "asymmetric" is intended to mean that all the weighted Shapley values (except for the original Shapley value) do not satisfy the symmetry axiom (see Section 1.4.1).

The remainder of this chapter is organized as follows. Section 4.2 deals with preliminaries. In Section 4.3, we study the relationship between the core and extensions of the weighted Shapley value to NTU games. All proofs are provided in Section 4.4.

4.2 Preliminaries

Let $V \in \hat{\Gamma}$, $x, y \in V(N)$ and $S \subseteq N$, $S \neq \emptyset$. We say that y *dominates* x *via* S if $y_i > x_i$ for all $i \in S$ and $y_S \in V(S)$. We say that y *dominates* x if there exists a non-empty coalition S such that y dominates x via S. We define the core of V by

$$C(V) = \{x \in \mathbb{R}^N : \text{there exists no } y \in \mathbb{R}^N \text{ that dominates } x\}.$$

The core describes the set of outcomes that no coalition can improve upon on its own.

For each $S \subseteq N$, $S \neq \emptyset$, we define

$$\Delta_{++}^S = \Big\{x \in \mathbb{R}_{++}^S : \sum_{i \in S} x_i = 1\Big\}.$$

In this chapter, we consider solutions that do not satisfy the symmetry axiom. Such solutions are formulated by using the notion of a weight. An element $w = (w_i)_{i \in N} \in \Delta_{++}^N$ is called a *weight* and is interpreted as follows:[2]

> These weights can be interpreted as “a-priori measures of importance;” they are taken to reflect considerations *not* captured by the game.
>
> (Hart and Mas-Colell (1989), p.603)

For example, “when players are of unequal ‘size’ (e.g. a player may represent a ‘group’, a ‘department’, and so on)” (Hart (1989)), a weight captures the asymmetry inherited in the difference of size.

A *weighted solution* is a function from $\Delta_{++}^N \times \hat{\Gamma}$ to $\mathbb{R}^N$. Namely, a weighted solution assigns a payoff vector to each pair of a game and a weight. All the weighted solutions discussed in this chapter coincide with a weighted Shapley value on the class of TU games $\Gamma \subseteq \hat{\Gamma}$.[3] For each $w \in \Delta_{++}^N$ and $v \in \Gamma$, we define the *w-weighted Shapley value* by

$$Sh_i^w(v) = \sum_{T \subseteq N : i \in T} D(T, v) \cdot \frac{w_i}{\sum_{j \in T} w_j} \text{ for all } i \in N.$$

4.3 The core and weighted solutions

We first provide a lemma that identifies a sufficient condition under which the closure of a family of values contains the core. Let $\psi : \Delta_{++}^N \times \hat{\Gamma} \to \mathbb{R}^N$ be an arbitrary weighted solution. Let $V \in \hat{\Gamma}$ be fixed and we restrict the domain of ψ to Δ_{++}^N, i.e., we consider the function $\psi^w(V) : \Delta_{++}^N \to \mathbb{R}^N$. Consider the following three conditions on $\psi^w(V)$:

C1: $\psi^w(V) \in \partial V(N)$ for all $w \in \Delta_{++}^N$.

[2] We replace “characteristic function” in the original text with “game”.

[3] As discussed in Section 1.4.2, any TU game is represented as an NTU game.

C2: $\psi^w(V)$ is continuous with respect to w.

C3: Let $\{w^k\}_{k=1}^{\infty}$ be a convergent sequence such that there exists $T \subsetneq N$, $T \neq \emptyset$, satisfying $\lim_{k\to\infty} w_i^k = 0$ for all $i \in T$ and $\lim_{k\to\infty} w_j^k > 0$ for all $j \in N\backslash T$. Then, $\{\psi^{w^k}(V)\}_{k=1}^{\infty}$ has a convergent subsequence satisfying

$$\lim_{k\to\infty} \psi_T^{w^k}(V) \in V(T).$$

C1 states that the outcome $\psi^w(V)$ is always Pareto optimal. C2 states that if a weight vector w slightly changes, then the final outcome also slightly changes. C3 states that, if the weights of players in T go to 0, then the players receive a payoff vector in $V(T)$. In other words, if importance of players in T becomes increasingly small, then the players in T only receive the attainable payoff on their own.

Lemma 6. *Let $V \in \hat{\Gamma}$ and ψ be a weighted solution. If the function $\psi^w(V) : \Delta_{++}^N \to \mathbb{R}^N$ satisfies C1 to C3, then*

$$C(V) \subseteq cl\{\psi^w(V) : w \in \Delta_{++}^N\}.$$

Proof. See Section 4.4.1. □

Some intuition for this lemma can be seen in the following example:

Example 2. Consider the following 2-person game V^1:

$$V^1(\{i\}) = \{x \in \mathbb{R} : x \leq 0\} \text{ for } i = 1, 2,$$

and $V^1(\{1,2\})$ is depicted by Fig. 3.

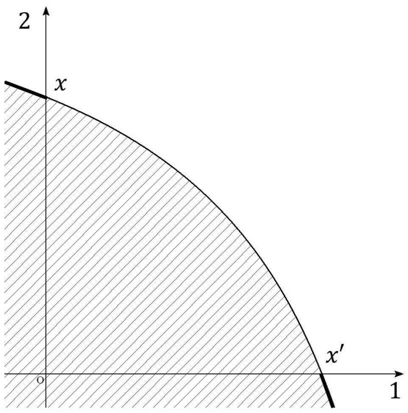

Fig. 3 Description of $V^1(\{1,2\})$

The core is the set of payoff vectors represented by the arc from x to x'. Consider now a function $\psi^w(V^1)$ that satisfies C1 to C3. We focus on the locus of $\psi^w(V^1)$ as w changes. By C1 and C2, $\psi^w(V^1)$ continuously moves along the boundary of $V^1(\{1,2\})$. Moreover, by C3, $\psi^w(V^1)$ reaches the left-hand side of point x (emphasized by the bold line) when $w_1 \to 0$. Similarly, $\psi^w(V^1)$ reaches the right-hand side of point x' (emphasized by the bold line) when $w_2 \to 0$. These observations indicate that the range of $\psi^w(V^1)$ includes the core as w changes.

Lemma 6 is used to prove theorems in the remainder of this section.

4.3.1 Weighted egalitarian solutions

We focus on the weighted egalitarian solutions introduced by Kalai and Samet (1985). Let $V \in \hat{\Gamma}$ and $w \in \Delta^N_{++}$. We define $D^w(V,S) \in \mathbb{R}^S$ and $Z^w(V,S) \in \mathbb{R}^S$ for $S \subseteq N$, $S \neq \emptyset$, inductively on the size of S as follows: for each $i \in N$,

$$Z^w(V,\{i\}) = D^w(V,\{i\}) = \max\{t \in \mathbb{R} : t \in V(\{i\})\}.$$

For each $S \subseteq N$, $|S| \geq 2$,

$$\begin{aligned} Z_i^w(V,S) &= \sum_{T \subsetneq S : i \in T} D_i^w(V,T) \text{ for all } i \in S. \\ D_i^w(V,S) &= w_i \max\{t : (Z^w(V,S) + t w_S) \in V(S)\} \text{ for all } i \in S. \end{aligned} \tag{4.1}$$

For each $w \in \Delta^N_{++}$, we define the w-weighted egalitarian solution $\xi^w : \hat{\Gamma} \to \mathbb{R}^N$ as follows:

$$\xi_i^w(V) = \sum_{S \subseteq N : i \in S} D_i^w(V,S) \text{ for all } i \in N, V \in \hat{\Gamma}.$$

We explain what kind of payoff vector is assigned by ξ^w. For 1-person coalitions, $Z^w(V,\{i\})$ (equivalently $D^w(V,\{i\})$) represents the maximum attainable payoff for i. Next, consider the 2-person coalition $\{1,2\}$. The vector $Z^w(V,\{1,2\}) \in \mathbb{R}^2$ represents the attainable payoff for each player, i.e.,

$$Z_1^w(V,\{1,2\}) = Z^w(V,\{1\}),\ Z_2^w(V,\{1,2\}) = Z^w(V,\{2\}).$$

From the reference point $Z^w(V,\{1,2\})$, we draw a line in the direction of w until we reach the Pareto frontier and obtain $D^w(\{1,2\},V)$. See Fig. 4 below:

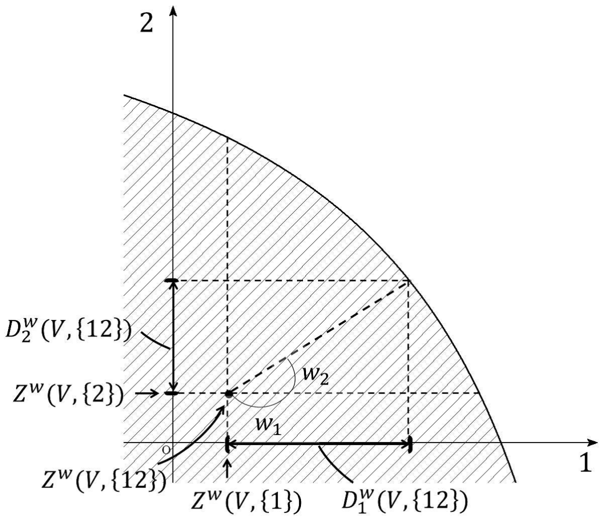

Fig. 4 Graphical representations of Z^w and D^w for $\{1,2\}$

In this way, we obtain $D^w(V,S)$ for all 2-person coalitions, which enables us to derive $Z^w(V,S)$ for 3-person coalitions. Again, from the reference point $Z^w(V,S)$, we draw a line in the direction of w and obtain $D^w(V,S)$ for 3-person coalitions. Repeating this procedure, we obtain $D^w(V,S)$ for all coalitions. The w-weighted egalitarian solution assigns the sum of $D_i^w(V,S)$, $S \subseteq N$, $i \in S$, to player i.

In the class of TU games, the w-weighted egalitarian solution ξ^w coincides with the w-weighted Shapley value; see Theorem 4 of Kalai and Samet (1985). For each $V \in \hat{\Gamma}$, we define the set of weighted egalitarian solutions $\Xi(V)$ as follows:

$$\Xi(V) = \{\xi^w(V) : w \in \Delta^N_{++}\}.$$

Applying Lemma 6, we obtain the following theorem:

Theorem 10. *For any $V \in \hat{\Gamma}$, $C(V) \subseteq cl\Xi(V)$.*

Proof. See Section 5.2. □

The weighted egalitarian solutions are supported by desirable axioms; see Kalai and Samet (1985) or Hart and Mas-Colell (1989). Theorem 10 states that any element of the core is "almost" attainable as the outcome of the solution. This result strengthens the elements of the core as reasonable outcomes in NTU games.

4.3.2 Shapley NTU value

The Shapley NTU value was first introduced by Shapley (1969, 1988). To calculate this value, we first derive a TU game from the original NTU game and then apply the Shapley value to the derived game. We can define the weighted version by using the weighted Shapley value instead of the (symmetric) Shapley value.

In this section, we provide a counterexample in which the weighted version of the Shapley NTU value does not contain the core. Intuitively, this negative result is obtained because an element of the core in the original NTU game is not necessarily an element of the core in the derived TU game.

We revisit definition of the Shapley NTU value by following notations of Peleg and Sudhölter (2007). Let $V \in \hat{\Gamma}$. For each $\lambda \in \Delta^N_{++}$, we define $v_\lambda : 2^N \to \mathbb{R} \cup \{+\infty\}$ by

$$v_\lambda(S) = \sup\{\lambda_S \cdot x : x \in V(S)\} \text{ for all } S \subseteq N, S \neq \emptyset. \tag{4.2}$$

We say that $\lambda \in \Delta^N_{++}$ is viable in V if $v_\lambda(S) \in \mathbb{R}$ for all $S \subseteq N$, $S \neq \emptyset$. For $y, z \in \mathbb{R}^N$, we define $y * z = (y_i z_i)_{i \in N} \in \mathbb{R}^N$. We say that $x \in V(N)$ is a weighted Shapley NTU value with positive weight $w \in \Delta^N_{++}$ if there exists λ such that λ is viable in V and $\lambda * x = Sh^w(v_\lambda)$.

Example 3. Consider the following 3-person game V^2:

$$V^2(N) = \{x \in \mathbb{R}^N : x_1 + x_2 + x_3 \leq 1\},$$
$$V^2(S) = \Big\{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq 0\Big\} \text{ if } S \neq N, S \neq \{1,2\}.$$

$V^2(\{1,2\})$ is the shaded area of Fig. 5.

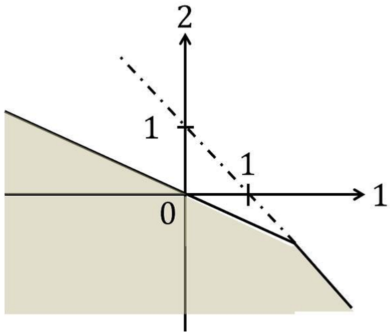

Fig. 5 Description of $V^2(\{1,2\})$.

Consider $(0,0,1) \in C(V^2)$. We show that any convergent sequence of the weighted Shapley NTU values cannot attain this vector as the limit point.

For λ to be viable in V^2, we must have $\lambda = (1/3, 1/3, 1/3)$; otherwise, $v_\lambda(N) = +\infty$. Set $\mu = (1/3, 1/3, 1/3)$. By definitions of V^2 and v_μ,

$$v_\mu(S) = \begin{cases} \frac{1}{3} & \text{if } S \in \{N, \{1,2\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

If $x \in \mathbb{R}^N$ is a weighted Shapley NTU value, we must have $\mu * x = Sh^w(v_\mu)$, which is equivalent to saying that $x = Sh^w(3 \cdot v_\mu)$. As player 3's contribution is always equal to 0 in $3 \cdot v_\mu$, we conclude that $(0,0,1)$ is not attainable as the outcome of a weighted Shapley NTU value.

Comparing the results in Subsections 4.3.1 and 4.3.2 provides new insight into the problem of extending the weighted Shapley value to NTU games. When we extend a value in TU games to NTU games, an important criterion is whether the extension preserves desirable properties of the original version.[4] In view of the relationship to the core, the weighted egalitarian solutions are a more desirable extension.

4.3.3 The core and contributions of players

In the class of TU games, Theorem 10 states that the core is included in the closure of the weighted Shapley values, as proven by Monderer et al. (1992). This inclusion theorem provides another interpretation of the core: if a payoff vector is not dominated by any coalition, then the vector can be represented as the expected value of contributions. The question here is whether this interpretation of the core remains valid even in NTU games.[5]

In this section, we provide a positive answer to the above question in a restricted class of $\hat{\Gamma}$. We show that, if $V(N)$ is a closed half-space, then any element of the core is represented as the expected value of contributions.

We introduce some notation. Let $\mathbf{R}(N)$ denote the set of orders of players in N. For any $R \in \mathbf{R}(N)$ and $i \in N$, let $\mathbf{B}(R,i)$ denote the set of players preceding i in R. For each $R \in \mathbf{R}(N)$ and $V \in \hat{\Gamma}$, we define the contribution vector $m^R(V) \in \mathbb{R}^N$ as follows: for player $i \in N$ with $\mathbf{B}(R,i) = \emptyset$,

$$m_i^R(V) = \max\{t \in \mathbb{R} : t \in V(\{i\})\} \text{ if } \mathbf{B}(R,i) = \emptyset.$$

[4] Section 1 of Chang and Chen (2013) discusses this criterion in detail.

[5] Otten et al. (1998) also raised this open question in their concluding remarks.

Assume that $m_j^R(V)$ are defined for all $j \in \mathbf{B}(R,i)$ and we define

$$m_i^R(V) = \max\big\{t \in \mathbb{R} : \big(t, (m_j^R(V))_{j\in\mathbf{B}(R,i)}\big) \in V\big(\mathbf{B}(R,i) \cup \{i\}\big)\big\}.$$

In words, $m_i^R(V)$ represents the maximum attainable payoff for $i \in N$ given the payoff of preceding players in the order R.

Remark 6. Under the assumptions N1 to N3 (see Section 1.4.2), the contribution vector is always well-defined. We prove this result in Appendix C.

For $X \subseteq \mathbb{R}^N$, let coX denote the convex hull of X.

Theorem 11. *Let $V \in \hat{\Gamma}$ be such that $V(N)$ is a closed half-space. Then, $C(V) \subseteq co\{m^R(V) : R \in \mathbf{R}(N)\}$.*

Theorem 11 says that, if $V(N)$ is a closed half-space, then the relationship between the core and contributions in TU games is preserved in NTU games.

4.3.4 MC value

We focus on the MC value developed by Otten et al. (1998). The basic idea of this value is to rescale the expected value of contributions in such a way that the resulting vector is Pareto optimal. We consider the weighted version of this value.

We introduce additional notations. For each $R = (i_1, \cdots, i_n) \in \mathbf{R}(N)$ and $w = (w_{i_1}, \cdots, w_{i_n}) \in \Delta^N_{++}$, we define $P_w(R)$ by[6]

$$P_w(R) = \prod_{m=1}^{n} \Big(w_{i_m} / \sum_{t=1}^{m} w_{i_t}\Big). \tag{4.3}$$

For each $V \in \hat{\Gamma}$ and $w \in \Delta^N_{++}$, we define $\phi^w(V)$ as follows:

$$\phi^w(V) = \sum_{R\in\mathbf{R}(N)} P_w(R) m^R(V). \tag{4.4}$$

Let $V \in \hat{\Gamma}$. The weighted MC value with positive weight w, denoted as $MC^w(V)$, is the unique payoff vector that satisfies the following:

1: $MC^w(V) = \alpha\phi^w(V)$ for some $\alpha \in \mathbb{R}$.

2: $MC^w(V) \in \partial V(N)$.

We provide a counterexample in which the weighted MC values do not contain the core.

Example 4. Consider the following 3-person game V^3:

$$V^3(S) = \Big\{x \in \mathbb{R}^S : \sum_{i\in S} x_i \leq 0\Big\} \text{ for all } S \subseteq N, S \neq \emptyset, S \neq N, S \neq \{1,2\},$$

$$V^3(\{1,2\}) = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 1\},$$

[6] If we calculate the expected value of contributions by using this probability in TU games, then we obtain the weighted Shapley value; see Kalai and Samet (1987).

and $V^3(N)$ is the set of payoff vectors in $\mathbb{R}^3$ such that the cross-sectional view along each plane is represented by Fig. 6.

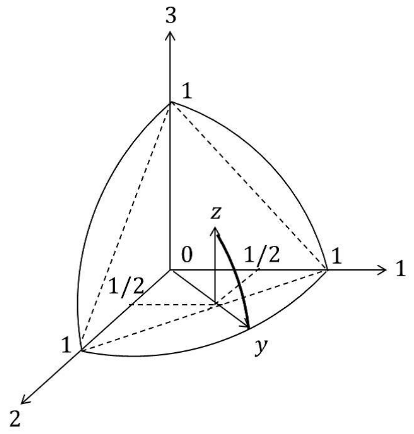

Fig. 6 Description of $V^3(N)$

Consider the payoff vector $z = (1/2, 1/2, a)$ such that $a > 0$ and $z \in \partial V^3(N)$. Then, we have $z \in C(V^3)$. In this game, we can verify that $m_3^R(V^3) = 0$ for all $R \in \mathbf{R}(N)$, which implies that $MC_3^w(V^3) = 0$ for all $w \in \Delta_{++}^N$. Thus, the vector z cannot be represented as the outcome of $MC^w(V^3)$.

In view of Proposition 1, $MC^w(V^3)$ violates one of the three conditions C1, C2 and C3. We show that it violates C3. To see this, consider a sequence of weights $w^k \in \Delta_{++}^N$ such that $w^k = (1/k, 1/k, 1 - 2/k)$, $k = 3, 4, \cdots$. Then, we can verify that

$$\lim_{k \to \infty} MC^{w^k}(V^3) = y,$$

where y is depicted in Fig. 6. Because $y \notin V^3(\{1, 2\})$, $MC^w(V^3)$ violates C3.

Remark 7. Monderer et al. (1992) proved the following two theorems in the class of TU games:

(I) For any core element, there exists a weight such that the corresponding weighted Shapley value coincides with the element.

(II) A TU game is convex if and only if the core and the set of weighted Shapley values coincide.

In this paper, we proved that (I) can be extended to NTU games. However, it remains as an open question whether we can extend (II) to NTU games. Convexity in NTU games has been defined in previous works: cardinal convexity by Sharkey (1981) and strongly ordinal convexity by Masuzawa (2012). It may be the case that these notions are related to the coincidence between the solution concepts discussed in this paper. ∎

4.4 Proofs

In this section, we prove Lemma 6 and Theorems 10 and 11.

4.4.1 Proof of Lemma 6

We provide a remark on N3 (uniformly non-levelled). In view of proving Lemma 6, the following weaker condition than N3 is sufficient: for each $S \subseteq N$, $S \neq \emptyset$,

N3': $x, y \in \partial V(S)$ and $x \leq y$ imply $x = y$.

We say that V is *non-levelled* if it satisfies N3'.

We prove Lemma 6 by using the fixed point theorem. Since Δ^N_{++} is an open set, we cannot directly apply the theorem. To overcome the difficulty, we approximate the open set by a sequence of closed sets. We borrow this idea from the proof of Theorem 5.3 by Jehle and Reny (2011).

We introduce some notation. For each $x, y \in \mathbb{R}^N$, we define $\min\{x, y\}$ by

$$\min\{x, y\} = (z_1, \cdots, z_n) \in \mathbb{R}^N, \text{ where } z_k = \min\{x_k, y_k\} \text{ for all } k = 1, \cdots, n.$$

We define $\max\{x, y\}$ in a parallel manner.

***Proof of Lemma 6* .** Suppose $C(V) \neq \emptyset$ and let $x \in C(V)$. Let $\Psi : \Delta^N_{++} \to \mathbb{R}^N$ denote the following function:

$$\Psi(w) = x - \psi^w(V) \text{ for all } w \in \Delta^N_{++}. \tag{4.5}$$

Since $\psi^w(V)$ is continuous by C2, Ψ is also continuous. For any $w \in \Delta^N_{++}$, let $\tilde{\Psi}(w) = \min\{\Psi(w), \mathbf{1}\}$. For any $\epsilon \in (0, 1)$, we define

$$S_\epsilon = \Big\{ w \in \Delta^N_{++} : w_i \geq \frac{\epsilon}{1 + 2n} \text{ for all } i \in N \Big\}.$$

The set is compact and convex. We can also check that the set is non-empty; for any $\epsilon \in (0, 1)$, we define $d \in \Delta^N_{++}$ by

$$d_i = \frac{2 + \frac{1}{n}}{1 + 2n} \text{ for all } i = 1, \cdots, n.$$

Then, $d \in S_\epsilon$.

We define $g : S_\epsilon \to \mathbb{R}^N$ as follows:[7]

$$g_i(w) = \frac{\epsilon + w_i + \max\{0, \tilde{\Psi}_i(w)\}}{n\epsilon + 1 + \sum_{j \in N} \max\{0, \tilde{\Psi}_j(w)\}} \text{ for all } i \in N. \tag{4.6}$$

Note that

$$g_i(w) \geq \frac{\epsilon}{n\epsilon + 1 + n} \geq \frac{\epsilon}{1 + 2n} \text{ for all } i \in N.$$

Hence, $g : S_\epsilon \to S_\epsilon$ is a continuous function from the compact, convex and non-empty set to itself. By Brouwer's fixed point theorem, there exists a fixed point. For any $k \in \mathbb{N}$, $k \geq 2$, let $w^{\frac{1}{k}} \in S_{\frac{1}{k}}$ denote the vector that satisfies $g(w^{\frac{1}{k}}) = w^{\frac{1}{k}}$.

[7] We explain the motivation for this function. Our final goal is to find a sequence of weights with which the weighted values converge to x. Suppose that $\tilde{\Psi}_i(w) > 0$, i.e., $x_i > \psi^w(V)$. This means that the value assigned by a given weight vector to player i falls short of i's payoff in the core outcome x. Then, the function g requires that the weight assigned to i should be increased.

Now, consider the sequence $\{w^{\frac{1}{k}}\}_{k=2}^{\infty}$. Since the sequence is bounded, there exists a convergent subsequence. Take a convergent subsequence $\{w^{\frac{1}{k}}\} \subseteq \{w^{\frac{1}{k}}\}_{k=2}^{\infty}$. By (4.6), we have, for any k,

$$w_i^{\frac{1}{k}}\left[\frac{n}{k}+\sum_{j\in N}\max\big\{0,\tilde{\Psi}_j(w^{\frac{1}{k}})\big\}\right]=\frac{1}{k}+\max\big\{0,\tilde{\Psi}_i(w^{\frac{1}{k}})\big\} \text{ for all } i\in N. \tag{4.7}$$

Let w^* be the limit point of $\{w^{\frac{1}{k}}\}$, i.e., $w^{\frac{1}{k}} \to w^*$. Since $\sum_{i\in N} w_i^* = 1$ and $w_i^* \geq 0$ for all $i = 1, \cdots, n$, there is at least one player i such that $w_i^* > 0$. Without loss of generality, assume that

$$\begin{aligned} w_i^* &> 0 \text{ for } i = 1,\cdots,t,\\ w_j^* &= 0 \text{ for } j = t+1,\cdots,n. \end{aligned}$$

We define $S = \{1, \cdots, t\}$ and $T = \{t + 1, \cdots, n\}$. Note that T might be empty.

Case 1: If $S = N$, we have $w_i^* > 0$ for all $i \in N$. By C2, $\lim_{k\to\infty} \tilde{\Psi}_i(w^{\frac{1}{k}}) = \tilde{\Psi}_i(w^*)$ for all $i \in N$. Taking the limit $k \to \infty$ of both sides of (4.7),

$$w_i^*\Big[\sum_{j\in N}\max\big\{0,\tilde{\Psi}_j(w^*)\big\}\Big]=\max\big\{0,\tilde{\Psi}_i(w^*)\big\} \text{ for all } i\in N. \tag{4.8}$$

Suppose to the contrary that $\Big[\sum_{j\in N}\max\big\{0,\tilde{\Psi}_j(w^*)\big\}\Big] > 0$. Then, by (4.8), $\tilde{\Psi}_i(w^*) > 0$ for all $i \in N$. In this case, there exists a sufficiently large k' such that $\tilde{\Psi}_i(w^{\frac{1}{k'}}) > 0$ for all $i \in N$. It follows that

$$\begin{aligned} \tilde{\Psi}_i(w^{\frac{1}{k'}}) = \min\{\Psi_i(w^{\frac{1}{k'}}), 1\} &> 0 \text{ for all } i\in N,\\ \Psi_i(w^{\frac{1}{k'}}) &> 0 \text{ for all } i\in N,\\ x_i &> \psi_i^{w^{\frac{1}{k'}}}(V) \text{ for all } i\in N. \end{aligned}$$

Since $x \in C(V)$, we have $x \in \partial V(N)$. By C1, $\psi^{w^{\frac{1}{k'}}}(V) \in \partial V(N)$. By N3', $x = \psi(w^{\frac{1}{k'}})$, contradicting $x_i > \psi_i(w^{\frac{1}{k'}})$ for all $i \in N$.

As a result, we must have $\Big[\sum_{j\in N}\max\big\{0,\tilde{\Psi}_j^*\big\}\Big] = 0$. By (4.8),

$$\begin{aligned} \tilde{\Psi}_i(w^*) &\leq 0 \text{ for all } i\in N,\\ \lim_{k\to\infty}\min\{\Psi_i(w^{\frac{1}{k}}),1\} &\leq 0 \text{ for all } i\in N,\\ \lim_{k\to\infty}\Psi_i(w^{\frac{1}{k}}) &\leq 0 \text{ for all } i\in N. \end{aligned}$$

By (4.5),

$$\begin{aligned} \lim_{k\to\infty}\Big\{x_i-\psi_i^{w^{\frac{1}{k}}}(V)\Big\} &\leq 0 \text{ for all } i\in N,\\ x_i &\leq \psi_i^{w^*}(V) \text{ for all } i\in N. \end{aligned}$$

Since $x, \psi^{w^*}(V) \in \partial V(N)$, by N3', $x = \psi^{w^*}(V)$. It follows that $x = \lim_{k\to\infty} \psi^{w^{\frac{1}{k}}}(V)$.

Case 2: The remaining possibility is that $1 \leq |S| < n$. Consider the sequence

$\psi^{w^{\frac{1}{k}}}(V)$. By C3, $\psi^{w^{\frac{1}{k}}}(V)$ has a convergent subsequence that satisfies the condition stated in C3. To simplify the notation, suppose that $\psi^{w^{\frac{1}{k}}}(V)$ itself converges and let ψ^* denote the limit point. Since $\psi^{w^{\frac{1}{k}}}(V)$ converges, $\Psi(w^{\frac{1}{k}})$ and $\tilde{\Psi}(w^{\frac{1}{k}}) = \min\{\Psi(w^{\frac{1}{k}}), \mathbf{1}\}$ also converge. Let $\tilde{\Psi}^*$ denote the limit point of $\tilde{\Psi}(w^{\frac{1}{k}})$. Taking the limit $k \to \infty$ of both sides of (4.7),

$$w_i^* \Big[\sum_{j \in N} \max\big\{0, \tilde{\Psi}_j^*\big\}\Big] = \max\big\{0, \tilde{\Psi}_i^*\big\} \text{ for all } i \in N. \tag{4.9}$$

Suppose to the contrary that $\Big[\sum_{j \in N} \max\big\{0, \tilde{\Psi}_j^*\big\}\Big] > 0$. Then, by (4.9),

$$\begin{cases} \tilde{\Psi}_i^* > 0 & \text{for all } i \in S, \\ \tilde{\Psi}_j^* \leq 0 & \text{for all } j \in T. \end{cases}$$

By definition of $\tilde{\Psi}_i^*$, we have

$$\begin{cases} \tilde{\Psi}_i^* = \lim_{k \to \infty} \tilde{\Psi}_i(w^{\frac{1}{k}}) = \lim_{k \to \infty} \min\{\Psi_i(w^{\frac{1}{k}}), 1\} > 0 & \text{for all } i \in S, \\ \tilde{\Psi}_j^* = \lim_{k \to \infty} \tilde{\Psi}_j(w^{\frac{1}{k}}) = \lim_{k \to \infty} \min\{\Psi_j(w^{\frac{1}{k}}), 1\} \leq 0 & \text{for all } j \in T. \end{cases}$$

The above two conditions imply

$$\begin{cases} \lim_{k \to \infty} \Psi_i(w^{\frac{1}{k}}) > 0 & \text{for all } i \in S, \\ \lim_{k \to \infty} \Psi_j(w^{\frac{1}{k}}) \leq 0 & \text{for all } j \in T. \end{cases}$$

Let us focus on the sequence $\Psi_j(w^{\frac{1}{k}})$ for $j \in T$. Since $\lim_{k \to \infty} \Psi_j(w^{\frac{1}{k}}) \leq 0$,

$$\begin{aligned} x_j - \lim_{k \to \infty} \psi_j^{w^{\frac{1}{k}}}(V) \leq 0 \text{ for all } j \in T, \\ x_j \leq \psi_j^* \text{ for all } j \in T. \end{aligned}$$

By C3, $\psi_T^* \in V(T)$. Since $x_T \leq \psi_T^*$, together with N2 (comprehensive), we have $x_T \in V(T)$. Since $x \in C(V)$, we have $x_T \in \partial V(T)$. By N3', $x_T = \psi_T^*$. On the other hand, for each $i \in S$, we have

$$\begin{aligned} x_i - \lim_{k \to \infty} \psi_i^{w^{\frac{1}{k}}}(V) > 0 \text{ for all } i \in S, \\ x_i > \psi_i^* \text{ for all } i \in S. \end{aligned}$$

It follows that $x \geq \psi^*$. Since $x, \psi^* \in \partial V(N)$, by N3', $x = \psi^*$, contradicting $x_i > \psi_i^*$ for all $i \in S$.

As a result, we must have $\Big[\sum_{i \in N} \max\big\{0, \tilde{\Psi}_i^*\big\}\Big] = 0$. By (4.9),

$$\begin{aligned} \tilde{\Psi}_i^* \leq 0 \text{ for all } i \in N, \\ \lim_{k \to \infty} \min\{\Psi_i(w^{\frac{1}{k}}), 1\} \leq 0 \text{ for all } i \in N, \\ \lim_{k \to \infty} \Psi_i(w^{\frac{1}{k}}) \leq 0 \text{ for all } i \in N. \end{aligned}$$

By (4.5),

$$\lim_{k\to\infty}\left\{x_i - \psi_i^{w^{\frac{1}{k}}}(V)\right\} \leq 0 \text{ for all } i \in N,$$
$$x_i \leq \psi_i^* \text{ for all } i \in N.$$

Since $x, \psi^* \in \partial V(N)$, by N3', $x = \psi^*$. Thus, we have $x = \lim_{k\to\infty} \psi^{w^{\frac{1}{k}}}(V)$. □

4.4.2 Proof of Theorem 10

Let $V \in \hat{\Gamma}$ and we go back to definition of $\xi^w(V)$. In (4.1), there always exists a unique real number t that attains the maximum by N1 (proper subset) and N2 (closed, comprehensive). We introduce a function that assigns the unique real number t to each $w \in \Delta_{++}^N$ and $S \subseteq N$, $S \neq \emptyset$. Let $Q : \Delta_{++}^N \times 2^N \backslash \emptyset \to \mathbb{R}$ denote the function such that

$$D_i^w(V, S) = w_i Q(w, S) \text{ for all } w \in \Delta_{++}^N, S \in 2^N \backslash \emptyset, i \in S.$$

We briefly explain the proof method. To apply Lemma 6, we prove that $\xi^w(V)$ satisfies C2 and C3. In both conditions, we consider a sequence of weights $\{w^k\}_{k=1}^\infty$. For each $S \subseteq N$, $S \neq \emptyset$, a sequence of weights yields the sequence $D^{w^k}(V, S) = w^k Q(w^k, S)$, $k = 1, 2, \cdots$. If $Q(w^k, S)$ diverges, then it becomes difficult to capture the behavior of $w^k Q(w^k, S)$. To avoid this difficulty, we provide a claim (Claim 20) which shows that the function $D^w(V, S)$ is bounded from both sides.

Consider the following two conditions on $V(S)$, $S \subseteq N$, $S \neq \emptyset$:

N4: $\{x^k\}_{k=1}^\infty \subseteq \partial V(S)$ and $x_i^k \to +\infty$ for some $i \in S$ implies $x_j \to -\infty$ for some $j \in S$.

N5: $\{x^k\}_{k=1}^\infty \subseteq \partial V(S)$ and $x_i^k \to -\infty$ for some $i \in S$ implies $x_j \to +\infty$ for some $j \in S$.

Claim 18. *Let $V \in \hat{\Gamma}$. Then, V satisfies N4.*

Proof **.** Let $x \in \partial V(S)$. Then, by N2 (convex), there exists a normal vector $\lambda(x)$ such that

$$V(S) \subseteq \{y \in \mathbb{R}^S : y \cdot \lambda(x) \leq x \cdot \lambda(x)\}.$$

Since $\sup_{y \in V(S)} y \cdot \lambda(x) = x \cdot \lambda(x) < +\infty$, by N3, there exists $\delta > 0$ such that $\lambda_i(x) \geq \delta$ for all $i \in N$. Now, consider a sequence $\{x^k\}_{k=1}^\infty \subseteq \partial V(S)$ such that $x_i^k \to +\infty$ for some $i \in S$. Then, $x^k \cdot \lambda(x) \leq x \cdot \lambda(x)$ for all $k = 1, 2, \cdots$. Since $\lambda(x) \in \Delta_{++}^N$ and $x_i^k \to +\infty$, $x_i^k \cdot \lambda_i(x) \to +\infty$. Since the sequence $x^k \cdot \lambda(x)$ is bounded from above, there must be a player $j \in N$ such that $x_j^k \to -\infty$. □

Claim 19. *Let $V \in \hat{\Gamma}$. Then, V satisfies N5.*

Proof **.** Consider a sequence $\{x^k\}_{k=1}^\infty \subseteq \partial V(S)$ such that $x_i^k \to -\infty$ for some $i \in S$. By N2 (convex), for each x^k, there exists a normal vector $\lambda(x^k)$ such that

$$V(S) \subseteq \{y \in \mathbb{R}^S : y \cdot \lambda(x^k) \leq x^k \cdot \lambda(x^k)\}.$$

As a result, for each $k = 1, 2, \cdots$, $\sup_{x \in V(S)} x \cdot \lambda(x^k) = x^k \cdot \lambda(x^k) < +\infty$. By N3, there exists $\delta > 0$ such that $\lambda_i(x^k) \geq \delta$ for all $i \in S$, $k = 1, 2, \cdots$. It follows that

$x_i^k \cdot \lambda_i(x^k) \to -\infty$. Since the sequence $x^k \cdot \lambda(x^k)$ is bounded from below,[8] there must be a player $j \in N$ such that $x_j^k \to +\infty$. □

Using the above two lemmas, we prove a lemma which shows that the range of the function $\xi^w(V) : \Delta_{++}^N \to \mathbb{R}^N$ is bounded.

Claim 20. *Let $V \in \hat{\Gamma}$, $i \in N$ and $S \subseteq N$, $i \in S$. Then, there exist $M > 0$ and $m < 0$ such that for all $w \in \Delta_{++}^N$, $m \leq D_i^w(V, S) \leq M$.*

Proof . We proceed by induction. If $S = \{i\}$, then for all $w \in \Delta_{++}^N$, $D_i^w(V, \{i\}) = \max\{x_i : x \in V(\{i\})\}$ and the statement holds. Suppose that the result holds for $T \subseteq N$, $i \in T$, $|T| = r$, and we prove the result for $S \subseteq N$, $i \in S$, $|S| = r + 1$, where $r \geq 1$.

We first prove that there exists $M > 0$ such that $D_i^w(V, S) \leq M$ for all $w \in \Delta_{++}^N$. Assume to the contrary that for all $M > 0$, there exists $w \in \Delta_{++}^S$ such that

$$D_i^w(V, S) = w_i Q(w, S) > M.$$

Then, we have the following statement: for all $k = 1, 2, \cdots$, there exists $w^k \in \Delta_{++}^N$ such that

$$w_i^k Q(w^k, S) > k,$$
$$\left(Z^{w^k}(V, S) + w^k Q(w^k, S)\right) \in \partial V(S).$$

By the induction hypothesis, $Z^{w^k}(V, S)$ is bounded from below. Then, $z^k := Z^{w^k}(V, S) + w^k Q(w^k, S)$ is a sequence such that $z^k \in \partial V(S)$ for all $k = 1, 2, \cdots$, and $\lim_{k\to\infty} z_i^k = +\infty$. On the other hand, z^k is bounded from below for all $k = 1, 2, \cdots$, contradicting N4. We can prove that there exists $m < 0$ such that $m \leq D_i^w(V, S)$ for all $w \in \Delta_{++}^N$ in a parallel manner by using N5. □

Using Claim 20, we prove that $\xi^w(V)$ satisfies C2 and C3.

Claim 21. *Let $V \in \hat{\Gamma}$. Then, the function $\xi^w(V) : \Delta_{++}^N \to \mathbb{R}^N$ satisfies C3.*

Proof . Let w^k be a convergent sequence such that there exists a non-empty coalition $T \subsetneq N$ satisfying $\lim_{k\to\infty} w_j^k = 0$ for all $j \in T$ and $\lim_{k\to\infty} w_i^k > 0$ for all $i \in N \backslash T$.

Let $j \in T$ be fixed. Consider the sequence $D_j^{w^k}(V, S)$ for $S \subseteq N, j \in S$. By Claim 20, there exists a convergent subsequence of $D_j^{w^k}(V, S)$ for each $S \subseteq N, j \in S$. For notational convenience, we assume that $D_j^{w^k}(V, S)$ itself converges for all $S \subseteq N$, $j \in S$. Let $\bar{S} \subseteq N$ be such that $j \in \bar{S}$ and $\bar{S} \nsubseteq T$. We prove that $\lim_{k\to\infty} D_j^{w^k}(V, \bar{S}) = 0$. By definition of $Q(w, S)$, for any k,

$$D^{w^k}(V, \bar{S}) = w^k Q(w^k, \bar{S}).$$

Let $i \in \bar{S} \backslash T$. Suppose that $\lim_{k\to\infty} Q(w^k, \bar{S}) = +\infty$. Then, the sequence $z^k := Z^{w^k}(V, \bar{S}) + w^k Q(w^k, \bar{S})$ satisfies $z^k \in \partial V(\bar{S})$ for all k and $\lim_{k\to\infty} z_i^k = +\infty$. Since z^k is bounded from below by Claim 20, this result contradicts N4. Similarly, if we assume

[8] To check this fact, let $x \in V(S)$ be fixed and let $\bar{\lambda}$ be the vector such that $\bar{\lambda}_i = 1$ if $x_i < 0$ and $\bar{\lambda}_i = 0$ otherwise. Then, $x \cdot \bar{\lambda} \leq x \cdot \lambda(x^k) \leq x^k \cdot \lambda(x^k)$ for all $k = 1, 2, \cdots$.

$\lim_{k\to\infty} Q(w^k, \bar{S}) = -\infty$, we can obtain the result that contradicts N5. It follows that $Q(w^k, \bar{S})$ is a bounded sequence. Since $\lim_{k\to\infty} w_j^k = 0$, we have

$$\lim_{k\to\infty} D_j^{w^k}(V, \bar{S}) = 0.$$

As a result, we obtain

$$\lim_{k\to\infty} \xi_j^{w^k}(V) = \lim_{k\to\infty} \sum_{R\subseteq T: j\in R} D_j^{w^k}(V, R) \text{ for all } j \in T.$$

It follows that $\lim_{k\to\infty} \xi_T^{w^k}(V) \in \partial V(T)$. □

Claim 22. *Let $V \in \hat{\Gamma}$. Then, the function $\xi^w(V) : \Delta_{++}^N \to \mathbb{R}^N$ satisfies C2.*

Proof **.** We prove that $D^w(V, S)$ is continuous for each non-empty $S \subseteq N$. The result holds for $S = \{i\}$, $i \in N$. We proceed by induction.

Take an arbitrary sequence $\{w^k\}_{k=1}^{\infty} \subseteq \Delta_{++}^N$ such that $w^k \to w^* \in \Delta_{++}^N$. Let $i \in S$. By Claim 20, there exist $M > 0$ and $m < 0$ such that for all k,

$$m \le w_i^k Q(w^k, S) \le M.$$

Since $w_i^k Q(w^k, S)$ is a bounded sequence and i is an arbitrary player, $w_S^k Q(w^k, S)$ is also a bounded sequence. Thus, there exists a convergent subsequence. Take an arbitrary convergent subsequence $w_S^{l(k)} Q(w^{l(k)}, S) \to w_S^* Q^*$, where $l : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. By the induction hypothesis, $Z^w(V, S)$ is continuous, which implies

$$\left(Z^{w^{l(k)}}(V, S) + w_S^{l(k)} Q(w^{l(k)}, S)\right) \to \left(Z^{w^*}(V, S) + w_S^* Q^*\right) \in \partial V(S).$$

Since $Q(w^*, S)$ is unique, we obtain $Q^* = Q(w^*, S)$. Thus, $w_S^{l(k)} Q(w^{l(k)}, S) \to w_S^* Q(w^*, S)$. Since any convergent subsequence of $w_S^k Q(w^k, S)$ converges to $w_S^* Q(w^*, S)$, we have $w_S^k Q(w^k, S) \to w_S^* Q(w^*, S)$. That is, $D^{w^k}(V, S) \to D^{w^*}(V, S)$, which proves continuity of $D^w(V, S)$. □

Proof of Theorem 10 **.** By Claims 21 and 22, for any $V \in \hat{\Gamma}$, any weighted egalitarian solution satisfies C2 and C3. C1 follows from definition of the solution. Lemma 6 completes the proof. □

4.4.3 Proof of Theorem 11

Let $V \in \hat{\Gamma}$. We focus on the expected value of contributions $\phi^w(V)$ for $w \in \Delta_{++}^N$ defined by (4.4). We define $\Phi(V)$ by

$$\Phi(V) = \{\phi^w(V) : w \in \Delta_{++}^N\}.$$

By definition of $\phi^w(V)$, $\Phi(V) \subseteq co\{m^R(V) : R \in \mathbf{R}(N)\}$. Moreover, since $co\{m^R(V) : R \in \mathbf{R}(N)\}$ is a closed set, we have

$$cl\Phi(V) \subseteq co\{m^R(V) : R \in \mathbf{R}(N)\}. \tag{4.10}$$

We prove that, for any $V \in \hat{\Gamma}$, $\phi^w(V)$ satisfies C3. For any order $R \in \mathbf{R}(N)$, let $i \succ_R j$ mean that i is a successor of j in the order R.

Claim 23. *Let $V \in \hat{\Gamma}$. Then, the function $\phi^w(V) : \Delta^N_{++} \to \mathbb{R}^N$ satisfies C3.*

Proof **.** Let $\{w^k\}_{k=1}^{\infty}$ be a convergent sequence such that there exists a coalition $T \subsetneq N$, $T \neq \emptyset$, such that

$$\lim_{k\to\infty} w_i^k = 0 \text{ for all } i \in T,$$
$$\lim_{k\to\infty} w_j^k > 0 \text{ for all } j \in N\backslash T.$$

For any $R = (i_1, \cdots, i_n) \in \mathbf{R}(N)$ and $m \in \{1, \cdots, n\}$, the sequence

$$\frac{w_{i_m}^k}{\sum_{t=1}^{m} w_{i_t}^k}, k = 1, 2, \cdots \tag{4.11}$$

is bounded. Thus, there exists a convergent subsequence. Assume for simplicity that the sequence (4.11) converges for any $R = (i_1, \cdots, i_n) \in \mathbf{R}(N)$ and $m \in \{1, \cdots, n\}$. Then, the sequence $\{P_{w^k}(R)\}_{k=1}^{\infty}$ also converges. Let $P^*(R)$ denote its limit point.

Let $R = (i_1, \cdots, i_n) \in \mathbf{R}(N)$ be an order such that there exist $i \in N\backslash T$ and $j \in T$, $j \neq i$, such that $j \succ_R i$. Let $m' \in \{2, \cdots, n\}$ by such that $j = i_{m'}$. By (4.3), we have

$$P^*(R) = \lim_{k\to\infty} \prod_{m=1}^{n} \Big(w_{i_m}^k / \sum_{t=1}^{m} w_{i_t}^k \Big). \tag{4.12}$$

By assumption, $\lim_{k\to\infty} w_{i_{m'}}^k = 0$ and $\lim_{k\to\infty} \sum_{t=1}^{m'} w_{i_t}^k > 0$. It follows that

$$\lim_{k\to\infty} \Big(w_{i_{m'}}^k / \sum_{t=1}^{m'} w_{i_t}^k \Big) = 0.$$

Thus, (4.12) is equal to 0. As a result, we restrict our attention to the following set of orders:

$$\mathbf{R}'(N) = \{R \in \mathbf{R}(N) : i \succ_R j \text{ for all } i \in N\backslash T \text{ and } j \in T\}.$$

We calculate the limit of $\phi_T^{w^k}(V)$:

$$\lim_{k\to\infty} \phi_T^{w^k}(V) = \lim_{k\to\infty} \sum_{R\in\mathbf{R}(N)} P_{w^k}(R) m_T^R(V) = \sum_{R\in\mathbf{R}'(N)} P^*(R) m_T^R(V).$$

For each $R \in \mathbf{R}'(N)$, we have $m_T^R(V) \in \partial V(T)$. By N2 (convex), we have $\phi_T^{w^k}(V) \in V(T)$ for each k. By N2 (closed), $\lim_{k\to\infty} \phi_T^{w^k}(V) \in V(T)$. □

Proof of Theorem 11 **.** By Claim 23, $\phi^w(V)$ satisfies C3. It satisfies C2 by definition of the solution. Since $V(N)$ is a closed half-space, C1 holds. By Proposition 1, $C(V) \subseteq cl\Phi(V)$. Together with (4.10), the desired condition follows. □

Chapter 5

Conclusion

We first summarize the results obtained in each chapter and then discuss new insights obtained in this thesis.

- **Chapter 2:** This chapter studies bases of the linear space of TU games. The commander games are basically the opposite of the unanimity games in the sense that the former regards a single player as a productive unit, while the latter regards a whole coalition as a productive unit. The basis consisting of the commander games decomposes the whole game space into two subspaces. The first space, spanned by the games defined for singleton coalitions, essentially determines the Shapley value of a given TU game. The second space, spanned by the games defined for coalitions with no less than two players, forms the null space of the Shapley value. We also consider intermediate cases between the commander games and the unanimity games, and construct new bases. Our new bases contribute to the understanding of how the Shapley value is determined in a given TU game, as well as provide new tools for axiomatizing solutions.

- **Chapter 3:** This chapter provides new axiomatizations of linear solutions. Monotonicity basically states that if a player's productivity and/or the size of the pie for society as a whole increases, then the player should be better off. In combination with previous results, we prove that efficiency, symmetry, and a monotonicity axiom characterize (i) four linear solutions in the literature, namely, the Shapley value, the equal division value, the CIS value, and the ENSC value, and (ii) a class of solutions obtained by taking a convex combination of the above solutions. Remarkably, monotonicity and two of the standard axioms (efficiency, symmetry) are satisfied only by linear solutions. Our result extends the existing literature and supports the desirability of linear solutions, as well as clarifying the difference between them.

- **Chapter 4:** This chapter uncovers a striking relationship between the Shapley value and the core in NTU games. For an arbitrary NTU game, any element of the core is attainable as the outcome of an extension of the Shapley value, called a weighted egalitarian solution, as introduced by Kalai and Samet (1985). This result shows a close tie between stability against coalitional deviation and fairness inherited by the weighted egalitarian solutions. We further prove that the inclusion of the core is not a common property among extensions of the weighted Shapley values to NTU games; for example, the weighted Shapley NTU values (Shapley (1969, 1988)) do not always contain the core. We also discuss the relationship between the core and players' contributions.

The Shapley value has long been recognized as one of the most convincing formulae for fair division. "It is the most important contribution of game theory to distributive justice" (Moulin (2004), p.12). However, if we focus only on the Shapley value, then our understanding of fair division is limited. The Shapley value determines the final payoffs using only the contributions of players, and does not allow for other fairness principles (e.g., egalitarian principles). To correct this flaw, several variants of the Shapley value have been proposed. The solutions examined in this thesis, such as egalitarian Shapley values and weighted Shapley values, are examples of such variants.

A desirable variant of the Shapley value would be one that possesses a different normative property than that of the Shapley value. Such a variant enables a more fruitful discussion of fair division by comparing the payoff vector of the variant with that of the Shapley value. The results described in this thesis show that existing variants indeed possess different normative properties, in two separate dimensions: (i) monotonicity, and (ii) core stability. With regard to (i), we show that the Shapley value satisfies monotonicity with respect to players' contributions, while a convex combination of the Shapley value and other solutions satisfies monotonicity with respect to different parameters. With regard to (ii), we show that a weighted variant of the Shapley value satisfies core stability in NTU games.

To overcome conflicts in allocation problems, it is important to compare the desirability of various solutions. Our results achieve a clearer comparison of the Shapley value and its variants, thereby broadening the applicability of solutions in cooperative game theory.

Future works

We conclude this thesis by discussing future works. In this thesis, we ignore externalities. In other words, we assumed that the worth of coalition S is not affected by outside players. To allow for the existence of externalities, Thrall and Lucas (1963) introduced the notion of games in partition function form. A recent paper by Sanchez-Perez (2017) took a linear algebraic approach to games in partition function form, and analyzed linear and symmetric solutions. Extending our new bases to the class of games in partition function form might yield a new tool for analyzing existing solutions.

In resource allocation problems, Moulin (2004) studied a monotonicity axiom called *resource monotonicity*. This axiom states that if the total amount of resources in a society increases, then no one should end up with a lower payoff. Moulin (2004) studied which solution is compatible with this axiom. In future work, we will extend our monotonicity axioms in TU games to other problems.

Pérez-Castrillo and Wettstein (2006) extend the Shapley value to a general class of pure exchange economies, which is a subclass of NTU games. It remains as a topic for future work to study the relationship between the competitive equilibria and the weighted solutions in exchange economies. It is often observed in exchange economies that, in addition to attainable payoffs, players' bargaining positions play an important role. To illustrate this, consider the three-person *glove market*,[1] in which one owner of a left-hand glove (player 1) and two owners of right-hand gloves (players 2, 3) exchange gloves in an attempt to make an assembled pair. The unique core allocation coincides with the competitive equilibria and assigns all generated payoffs to player 1. Intuitively,

[1]For a detailed study on this market, see Shapley and Shubik (1969).

the left-hand glove is in short supply, and the owner (player 1) is in a much better bargaining position than those of players 2 and 3. Here, it seems possible to reflect the players' bargaining positions by introducing a weight $w = (1 - 2\epsilon, \epsilon, \epsilon)$, where ϵ is a small real number. Letting $\epsilon \to 0$, the corresponding weighted Shapley values converge to the core. In other words, the Shapley value, which is intended to be a fair distribution of the surplus, approaches the competitive equilibria. This observation appears to deepen our understanding of the discrepancy between fairness and asymmetry driven by the competition for initial endowments.

References

Casajus, A., 2015. Monotonic redistribution of performance-based allocations: A case for proportional taxation. Theoretical Economics 10 (3), 887–892.

Casajus, A., Huettner, F., 2013. Null players, solidarity, and the egalitarian Shapley values. Journal of Mathematical Economics 49 (1), 58–61.

Casajus, A., Huettner, F., 2014. Weakly monotonic solutions for cooperative games. Journal of Economic Theory 154, 162–172.

Chang, C., Chen, P., 2013. The lambda-nucleolus for non-transferable utility games and Shapley procedure. avaibale at http://www.saet.uiowa.edu/event-14/.

Chun, Y., 1989. A new axiomatization of the Shapley value. Games and Economic Behavior 1 (2), 119–130.

Dehez, P., Ferey, S., 2013. How to share joint liability: a cooperative game approach. Mathematical Social Sciences 66 (1), 44–50.

Driessen, T., Funaki, Y., 1991. Coincidence of and collinearity between game theoretic solutions. Operations Research Spektrum 13 (1), 15–30.

Hart, S., 1989. Shapley value. In: Game Theory, part of the series of The New Palgrave. Springer, pp. 210–216.

Hart, S., Mas-Colell, A., 1989. Potential, value, and consistency. Econometrica 57 (3), 589–614.

Hindriks, J., Myles, G. D., 2006. Intermediate public economics. MIT press.

Joosten, R., 1996. Dynamics, equilibria and values. dissertation, Maastricht University.

Ju, Y., Borm, P., Ruys, P., 2007. The consensus value: a new solution concept for cooperative games. Social Choice and Welfare 28 (4), 685–703.

Kalai, E., Samet, D., 1985. Monotonic solutions to general cooperative games. Econometrica, 307–327.

Kalai, E., Samet, D., 1987. On weighted Shapley values. International Journal of Game Theory 16 (3), 205–222.

Kaneko, M., 1982. The central assignment game and the assignment markets. Journal of Mathematical Economics 10 (2), 205–232.

Kaneko, M., Wooders, M. H., 2004. Utility theories in cooperative games. In: Handbook of Utility Theory. Springer, pp. 1065–1098.

Lax, P., 2007. Linear algebra and its applications, 2nd ed. Wiley, New Jersey.

Littlechild, S. C., Thompson, G., 1977. Aircraft landing fees: a game theory approach. The Bell Journal of Economics, 186–204.

Llerena, F., Rafels, C., 2006. The vector lattice structure of the n-person TU games. Games and Economic Behavior 54 (2), 373–379.

Maschler, M., Owen, G., 1992. The consistent Shapley value for games without side payments. In: Rational interaction. Springer, Berlin, pp 5–12.

Masuzawa, T., 2012. Strong convexity of NTU games. International Journal of Game Theory 41 (3), 699–705.

Monderer, D., Samet, D., Shapley, L. S., 1992. Weighted values and the core. International Journal of Game Theory 21 (1), 27–39.

Moulin, H., 1995. Cooperative microeconomics: a game-theoretic introduction. Prentice Hall.

Moulin, H., 2004. Fair division and collective welfare. MIT press.

Nash, J. F., 1950. The bargaining problem. Econometrica 18 (2), 155–162.

Ni, D., Wang, Y., 2007. Sharing a polluted river. Games and Economic Behavior 60 (1), 176–186.

Oishi, T., Nakayama, M., Hokari, T., Funaki, Y., 2016. Duality and anti-duality in TU games applied to solutions, axioms, and axiomatizations. Journal of Mathematical Economics 63, 44–53.

Osborne, M. J., 2009. An introduction to game theory, international ed. Oxford university press.

Otten, G.-J., Borm, P., Peleg, B., Tijs, S., 1998. The MC-value for monotonic NTU-games. International Journal of Game Theory 27 (1), 37–47.

Peleg, B., Sudhölter, P., 2007. Introduction to the theory of cooperative games, 2nd ed. Springer.

Pérez-Castrillo, D., Wettstein, D., 2006. An ordinal Shapley value for economic environments. Journal of Economic Theory 127 (1), 296–308.

Pintér, M., 2015. Young's axiomatization of the Shapley value: a new proof. Annals of Operations Research 235 (1), 665–673.

Sanchez-Perez, J., 2017. A decomposition for the space of games with externalities. International Journal of Game Theory 46 (1), 205–233.

Scarf, H. E., 1967. The core of an n-person game. Econometrica 35 (1), 50–69.

Sen, A. K., 1997. On Economic Inequality, expanded edition with a substantial annexe by James E. Foster and Amartya Sen. Oxford university press.

Shapley, L. S., 1953. A value for n-person games. in Contributions to the Theory of Games II, H kuhn and AW Tucker (eds), Annals of Mathematics Studies 28, Princeton University Press, Princeton, New Jersey, 307–317.

Shapley, L. S., 1969. Utility comparison and the theory of games. In: La Decision, aggregation et dynamique des ordres de preference, Editions du Centre National de la Recherche Scientifique, Paris, pp 251–263.

Shapley, L. S., 1988. Utility comparison and the theory of games. In: Roth AE (ed) The Shapley value. Cambridge University Press, Cambridge.

Shapley, L. S., Shubik, M., 1969. Pure competition, coalitional power, and fair division. International Economic Review 10 (3), 337–362.

Sharkey, W., 1981. Convex games without side payments. International Journal of Game Theory 10 (2), 101–106.

Sprumont, Y., 2008. Monotonicity and solidarity axioms in economics and game theory. In: Rational Choice and Social Welfare. Springer, pp. 71–94.

Sundaram, R. K., 1996. A first course in optimization theory. Cambridge university press.

Thrall, R. M., Lucas, W. F., 1963. N-person games in partition function form. Naval Research Logistics 10 (1), 281–298.

van den Brink, R., 2002. An axiomatization of the Shapley value using a fairness property. International Journal of Game Theory 30 (3), 309–319.

van den Brink, R., 2007. Null or nullifying players: the difference between the Shapley value and equal division solutions. Journal of Economic Theory 136 (1), 767–775.

van den Brink, R., Funaki, Y., 2009. Axiomatizations of a class of equal surplus sharing solutions for TU-games. Theory and Decision 67 (3), 303–340.

van den Brink, R., Funaki, Y., Ju, Y., 2013. Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values. Social Choice and Welfare 40 (3), 693–714.

van den Brink, R., van der Laan, G., Vasil'ev, V. A., 2014. Constrained core solutions for totally positive games with ordered players. International Journal of Game Theory 43 (2), 351–368.

von Neumann, J., Morgenstern, O., 1944. Theory of games and economic behavior. Princeton university press, Princeton.

Yokote, K., 2015. Weak addition invariance and axiomatization of the weighted Shapley value. International Journal of Game Theory 44 (2), 275–293.

Yokote, K., 2017. Weighted values and the core in NTU games. International Journal of Game Theory 46(3), 631–654.

Yokote, K., Funaki, Y., 2015. Several bases of a game space and an application to the Shapley value. WINPEC Working Paper Series No. E1419.

Yokote, K., Funaki, Y., 2017. Monotonicity implies linearity: characterizations of convex combinations of solutions to cooperative games. Social Choice and Welfare 49 (1), 171–203.

Yokote, K., Funaki, Y., Kamijo, Y., 2016. A new basis and the Shapley value. Mathematical Social Sciences 80, 21–24.

Yokote, K., Funaki, Y., Kamijo, Y., 2017. Coincidence of the Shapley value with other solutions satisfying covariance. Mathematical Social Sciences 89, 1–9.

Young, H. P., 1985. Monotonic solutions of cooperative games. International Journal of Game Theory 14 (2), 65–72.

Appendix A

Another proof of Casajus and Huettner's (2014) theorem

The purpose of this appendix is to provide another proof of Casajus and Huettner's (2014) theorem by using a basis. For the definition of the α-egalitarian Shapley value Sh^α, see Section 3.2. Weak monotonicity (see Section 3.3) is abbreviated as WM throughout this appendix.

Theorem (Casajus and Huettner (2014)) . *Let $n \geq 3$. Then, a solution ψ satisfies E, S and WM if and only if there exists $\alpha \in [0, 1]$ such that $\psi = Sh^\alpha$.*

Proof **.** By Theorem 1, the commander games $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$ form a basis for Γ, from which we obtain the following basis:

$$\left\{u^1\right\} \cup \{u_1 - u_i : i \in N, i \neq 1\} \cup \{\bar{u}_T : |T| \geq 2\}, \tag{A.1}$$

where $u^1 = \sum_{i \in N} u_i$.

Let $i \in N \backslash \{1\}$ and $x = \psi_1(u_1 - u_i)$. Here, x does not depend on the choice of $i \in N \backslash \{1\}$. To see this, let $i, j \in N \backslash \{1\}$. Then,

$$\psi_1(u_1 - u_i) \overset{\text{WM}}{=} \psi_1\Big(u_1 - \frac{1}{2}u_i - \frac{1}{2}u_j\Big) \overset{\text{WM}}{=} \psi_1(u_1 - u_j).$$

Note that x satisfies

$$0 \overset{\text{E,S}}{=} \psi_1(\mathbf{0}) \overset{\text{WM}}{\leq} x \overset{\text{WM}}{\leq} \psi_1(u^1) \overset{\text{E,S}}{=} 1.$$

The following equality holds: for any $i \in N \backslash \{1\}$,

$$\psi(u_1 - u_i) = Sh^x(u_1 - u_i). \tag{A.2}$$

Indeed, for player $j \in N \backslash \{1, i\}$,

$$\psi_j(u_1 - u_i) \overset{\text{WM}}{=} \psi_j(\mathbf{0}) \overset{\text{E,S}}{=} 0.$$

Since $\psi_1(u_1 - u_i) = Sh^x_1(u_1 - u_i)$, E implies $\psi(u_1 - u_i) = Sh^x(u_1 - u_i)$.

We go back to Step 1 in the proof of Theorem 6. To prove Lemma 3, it suffices to assume that $n \geq 3$. Since WM is stronger than *cont* + *gr* + *sur*-monotonicity, we can use the lemma:

$$\psi\big(v + \lambda(u_1 - u_i)\big) = \psi(v) + \lambda\psi(u_1 - u_i) \text{ for all } v \in \Gamma, i \neq 1, \text{ and } \lambda \in \mathbb{R}. \tag{A.3}$$

Define

$$\Gamma^4 = \text{Sp}(u^1 \cup \{\bar{u}_T : T \subseteq N, |T| \geq 2\}).$$

Following the same line of the proof of Lemma 5, we obtain

$$\psi_i(v) = Sh_i^x(v) = \frac{v(N)}{n} \text{ for all } i \in N, v \in \Gamma^4. \tag{A.4}$$

Let $v \in \Gamma$. As the set of games in (A.1) is a basis, there exist $\gamma_i \in \mathbb{R}$, $i \neq 1$, and $v^4 \in \Gamma^4$ such that

$$v = \sum_{i \in N : i \neq 1} \gamma_i(u_1 - u_i) + v^4.$$

We conclude that

$$\begin{aligned}\psi(v) &\overset{(A.3)}{=} \sum_{i \in N : i \neq 1} \gamma_i \psi(u_1 - u_i) + \psi(v^4) \\ &\overset{(A.2),\ (A.4)}{=} \sum_{i \in N : i \neq 1} \gamma_i \Phi^x(u_1 - u_i) + Sh^x(v^4) \\ &= Sh^x(v).\end{aligned}$$

□

Appendix B

Counterexample to Theorem 6 for $n = 3$

The purpose of this appendix is to introduce a solution φ for $n = 3$ that satisfies efficiency, symmetry, *cont* + *sur*-monotonicity, but is not a convex combination of linear solutions. Since *cont* + *sur*-monotonicity is stronger than *cont* + *gr* + *sur*-monotonicity, this solution is a counterexample to Theorems 6 and 7 for $n = 3$.

Let $N = \{1, 2, 3\}$. For $\lambda \in \mathbb{R}$ and $T \subseteq N$, $T \neq \emptyset$, we define $\varphi(\lambda u_T)$ as follows:

$$\varphi(\lambda u_T) = \begin{cases} ED(\lambda u_T) & \text{if } \lambda < 0 \text{ and } |T| = 2, \\ Sh(\lambda u_T) & \text{otherwise.} \end{cases}$$

For $v \in \Gamma$ and $T \subseteq N$, $T \neq \emptyset$, let $d_T^v = D(T, v)$ (see (1.1)). We define φ for a general game $v \in \Gamma$ as follows:

$$\varphi(v) = \sum_{T \subseteq N : T \neq \emptyset} \varphi(d_T^v u_T).$$

Let $v, w \in \Gamma$ be games such that $\Delta_1 v(S) \geq \Delta_1 w(S)$ for all $S \subseteq N \setminus \{1\}$ and $v(N) - \sum_{i \in N} v(i) \geq w(N) - \sum_{i \in N} w(i)$. Our goal is to prove that

$$\begin{aligned} \varphi_1(v) - \varphi_1(w) &= \sum_{T \in 2^N \setminus \emptyset} \varphi_1(d_T^v u_T) - \sum_{T \in 2^N \setminus \emptyset} \varphi_1(d_T^w u_T) \\ &= \sum_{T \in 2^N \setminus \{\emptyset, \{2\}, \{3\}\}} \left\{ \varphi_1(d_T^v u_T) - \varphi_1(d_T^w u_T) \right\} \geq 0. \end{aligned}$$

For each $T \subseteq N$, $T \neq \emptyset$, let $\delta_T = d_T^v - d_T^w$. We provide a claim that immediately follows from the definition of φ.

Claim 24. *The following two statements hold:*

(i) Let $i \in N \setminus \{1\}$. If $\delta_{1i} \geq 0$, then $\varphi_1(d_{1i}^v u_{1i}) - \varphi_1(d_{1i}^w u_{1i}) \geq \frac{\delta_{1i}}{3}$. If $\delta_{1i} \leq 0$, then $\varphi_1(d_{1i}^v u_{1i}) - \varphi_1(d_{1i}^w u_{1i}) \geq \frac{\delta_{1i}}{2}$.

(ii) If $\delta_{23} \geq 0$, then $\varphi_1(d_{23}^v u_{23}) - \varphi_1(d_{23}^w u_{23}) \geq 0$. If $\delta_{23} \leq 0$, then $\varphi_1(d_{23}^v u_{23}) - \varphi_1(d_{23}^w u_{23}) \geq \frac{\delta_{23}}{3}$.

In what follows, Claim 24 is abbreviated as C24.

If $\delta_1 < 0$, then $v(1) < w(1)$, which is a contradiction with $\Delta_1 v(S) \geq \Delta_1 w(S)$ for all $S \subseteq N \backslash 1$. Thus, $\delta_1 \geq 0$. Depending on the signs of δ_{12}, δ_{13}, δ_{23} and δ_{123}, we consider 16 cases.

If $\delta_T \geq 0$ for all $T \subseteq N$, $|T| \geq 2$, then $\varphi_1(v) \geq \varphi_1(w)$ immediately holds. If $\delta_T < 0$ for all $T \subseteq N$, $|T| \geq 2$, we obtain a contradiction with $v(N) - \sum_{i \in N} v(i) \geq w(N) - \sum_{i \in N} w(i)$. In what follows, we consider other 14 cases.

- **Case 1:** $\boldsymbol{\delta_{12} < 0}$, $\boldsymbol{\delta_{13} \geq 0}$, $\boldsymbol{\delta_{23} \geq 0}$, $\boldsymbol{\delta_{123} \geq 0}$.

  In order that $v(12) - v(2) \geq w(12) - w(2)$, we must have $\delta_1 \geq -\delta_{12}$. Then,

$$\sum_{T \in 2^N \backslash \{\emptyset, \{2\}, \{3\}\}} \left\{ \varphi_1(d_T^v u_T) - \varphi_1(d_T^w u_T) \right\} \overset{\text{C24}}{\geq} \delta_1 + \frac{\delta_{12}}{2} \geq \delta_1 + \delta_{12} \geq 0.$$

- **Case 2:** $\boldsymbol{\delta_{12} \geq 0}$, $\boldsymbol{\delta_{13} < 0}$, $\boldsymbol{\delta_{23} \geq 0}$, $\boldsymbol{\delta_{123} \geq 0}$.

  This case can be proved in the same way as Case 1.

- **Case 3:** $\boldsymbol{\delta_{12} \geq 0}$, $\boldsymbol{\delta_{13} \geq 0}$, $\boldsymbol{\delta_{23} < 0}$, $\boldsymbol{\delta_{123} \geq 0}$.

  In order that $v(N) - \sum_{i \in N} v(i) \geq w(N) - \sum_{i \in N} w(i)$, we must have $\delta_{12} + \delta_{13} + \delta_{123} \geq -\delta_{23}$. Then,

$$\sum_{T \in 2^N \backslash \{\emptyset, \{2\}, \{3\}\}} \left\{ \varphi_1(d_T^v u_T) - \varphi_1(d_T^w u_T) \right\} \overset{\text{C24}}{\geq} \frac{\delta_{12} + \delta_{13} + \delta_{123}}{3} + \frac{\delta_{23}}{3} \geq 0.$$

- **Case 4:** $\boldsymbol{\delta_{12} \geq 0}$, $\boldsymbol{\delta_{13} \geq 0}$, $\boldsymbol{\delta_{23} \geq 0}$, $\boldsymbol{\delta_{123} < 0}$.

  In order that $v(N) - v(23) \geq w(N) - w(23)$, we must have $\delta_1 + \delta_{12} + \delta_{13} \geq -\delta_{123}$. Then,

$$\begin{aligned} \sum_{T \in 2^N \backslash \{\emptyset, \{2\}, \{3\}\}} \left\{ \varphi_1(d_T^v u_T) - \varphi_1(d_T^w u_T) \right\} &\overset{\text{C24}}{\geq} \delta_1 + \frac{\delta_{12} + \delta_{13}}{3} + \frac{\delta_{123}}{3} \\ &\geq \frac{\delta_1 + \delta_{12} + \delta_{13}}{3} + \frac{\delta_{123}}{3} \geq 0. \end{aligned}$$

- **Case 5:** $\boldsymbol{\delta_{12} < 0}$, $\boldsymbol{\delta_{13} < 0}$, $\boldsymbol{\delta_{23} \geq 0}$, $\boldsymbol{\delta_{123} \geq 0}$.

  In order that $v(12) - v(2) \geq w(12) - w(2)$ and $v(13) - v(3) \geq w(13) - w(3)$, we must have $\delta_1 \geq -\min\{\delta_{12}, \delta_{13}\}$. Then,

$$\begin{aligned} \sum_{T \in 2^N \backslash \{\emptyset, \{2\}, \{3\}\}} \left\{ \varphi_1(d_T^v u_T) - \varphi_1(d_T^w u_T) \right\} &\overset{\text{C24}}{\geq} \delta_1 + \frac{\delta_{12} + \delta_{13}}{2} \\ &\geq \delta_1 + \min\{\delta_{12}, \delta_{13}\} \geq 0. \end{aligned}$$

- **Case 6:** $\boldsymbol{\delta_{12} < 0}$, $\boldsymbol{\delta_{13} \geq 0}$, $\boldsymbol{\delta_{23} < 0}$, $\boldsymbol{\delta_{123} \geq 0}$.

In order that $v(12) - v(2) \geq w(12) - w(2)$, we must have $\delta_1 \geq -\delta_{12}$. In order that $v(N) - \sum_{i \in N} v(i) \geq w(N) - \sum_{i \in N} w(i)$, we must have $\delta_{13} + \delta_{123} \geq -\delta_{12} - \delta_{23} \geq -\delta_{23}$. Then,

$$\sum_{T \in 2^N \setminus \{\emptyset, \{2\}, \{3\}\}} \left\{\varphi_1(d_T^v u_T) - \varphi_1(d_T^w u_T)\right\} \overset{\text{C24}}{\geq} \delta_1 + \frac{\delta_{12}}{2} + \frac{\delta_{13}}{3} + \frac{\delta_{23}}{3} + \frac{\delta_{123}}{3}$$
$$\geq \delta_1 + \delta_{12} + \frac{\delta_{13} + \delta_{23} + \delta_{123}}{3} \geq 0.$$

- **Case 7: $\boldsymbol{\delta_{12} \geq 0}$, $\boldsymbol{\delta_{13} < 0}$, $\boldsymbol{\delta_{23} < 0}$, $\boldsymbol{\delta_{123} \geq 0}$.**

  This case can be proved in the same way as Case 6.

- **Case 8: $\boldsymbol{\delta_{12} < 0}$, $\boldsymbol{\delta_{13} \geq 0}$, $\boldsymbol{\delta_{23} \geq 0}$, $\boldsymbol{\delta_{123} < 0}$.**

  In order that $v(12) - v(2) \geq w(12) - w(2)$ and $v(N) - v(23) \geq w(N) - w(23)$, we must have $\delta_1 \geq -\delta_{12}$ and $\delta_1 + \delta_{13} \geq -\delta_{12} - \delta_{123} \geq -\delta_{123}$. Then,

$$\sum_{T \in 2^N \setminus \{\emptyset, \{2\}, \{3\}\}} \left\{\varphi_1(d_T^v u_T) - \varphi_1(d_T^w u_T)\right\} \overset{\text{C24}}{\geq} \delta_1 + \frac{\delta_{12}}{2} + \frac{\delta_{13}}{3} + \frac{\delta_{123}}{3}$$
$$\geq \frac{\delta_1 + \delta_{12}}{2} + \frac{\delta_1 + \delta_{13} + \delta_{123}}{3} \geq 0.$$

- **Case 9: $\boldsymbol{\delta_{12} \geq 0}$, $\boldsymbol{\delta_{13} < 0}$, $\boldsymbol{\delta_{23} \geq 0}$, $\boldsymbol{\delta_{123} < 0}$.**

  This case can be proved in the same way as Case 8.

- **Case 10: $\boldsymbol{\delta_{12} \geq 0}$, $\boldsymbol{\delta_{13} \geq 0}$, $\boldsymbol{\delta_{23} < 0}$, $\boldsymbol{\delta_{123} < 0}$.**

  In order that $v(N) - \sum_{i \in N} v(i) \geq w(N) - \sum_{i \in N} w(i)$, we must have $\delta_{12} + \delta_{13} \geq -\delta_{23} - \delta_{123}$. Then,

$$\sum_{T \in 2^N \setminus \{\emptyset, \{2\}, \{3\}\}} \left\{\varphi_1(d_T^v u_T) - \varphi_1(d_T^w u_T)\right\} \overset{\text{C24}}{\geq} \frac{\delta_{12}}{2} + \frac{\delta_{13}}{2} + \frac{\delta_{23}}{3} + \frac{\delta_{123}}{3}$$
$$\geq \frac{\delta_{12} + \delta_{13} + \delta_{23} + \delta_{123}}{3} \geq 0.$$

- **Case 11: $\boldsymbol{\delta_{12} < 0}$, $\boldsymbol{\delta_{13} \geq 0}$, $\boldsymbol{\delta_{23} < 0}$, $\boldsymbol{\delta_{123} < 0}$.**

  In order that $v(12) - v(2) \geq w(12) - w(2)$, we must have $\delta_1 \geq -\delta_{12}$. In order that $v(N) - \sum_{i \in N} v(i) \geq w(N) - \sum_{i \in N} w(i)$, we must have $\delta_{13} \geq -\delta_{12} - \delta_{23} - \delta_{123} \geq -\delta_{23} - \delta_{123}$. Then,

$$\sum_{T \in 2^N \setminus \{\emptyset, \{2\}, \{3\}\}} \left\{\varphi_1(d_T^v u_T) - \varphi_1(d_T^w u_T)\right\} \overset{\text{C24}}{\geq} \delta_1 + \frac{\delta_{12}}{2} + \frac{\delta_{13}}{3} + \frac{\delta_{23}}{3} + \frac{\delta_{123}}{3}$$
$$\geq \delta_1 + \delta_{12} + \frac{\delta_{13} + \delta_{23} + \delta_{123}}{3} \geq 0.$$

- **Case 12: $\boldsymbol{\delta_{12} \geq 0}$, $\boldsymbol{\delta_{13} < 0}$, $\boldsymbol{\delta_{23} < 0}$, $\boldsymbol{\delta_{123} < 0}$.**

This case can be proved in the same way as Case 11.

- **Case 13:** $\boldsymbol{\delta_{12} < 0}$, $\boldsymbol{\delta_{13} < 0}$, $\boldsymbol{\delta_{23} \geq 0}$, $\boldsymbol{\delta_{123} < 0}$.

  In order that $v(12) - v(2) \geq w(12) - w(2)$ and $v(13) - v(3) \geq w(13) - w(3)$, we must have $\delta_1 \geq -\min\{\delta_{12}, \delta_{13}\}$. In order that $v(N) - v(23) \geq w(N) - w(23)$, we must have $\delta_1 \geq -\delta_{12} - \delta_{13} - \delta_{123}$. Then,

$$\sum_{T \in 2^N \setminus \{\emptyset, \{2\}, \{3\}\}} \left\{\varphi_1(d_T^v u_T) - \varphi_1(d_T^w u_T)\right\} \overset{\text{C24}}{\geq} \delta_1 + \frac{\delta_{12}}{2} + \frac{\delta_{13}}{2} + \frac{\delta_{123}}{3}$$
$$\geq \frac{\delta_1}{3} + \frac{\delta_{12}}{6} + \frac{\delta_{13}}{6} + \frac{\delta_1 + \delta_{12} + \delta_{13} + \delta_{123}}{3}$$
$$\geq \frac{\delta_1}{3} + \frac{\min\{\delta_{12}, \delta_{13}\}}{3} \geq 0.$$

- **Case 14:** $\boldsymbol{\delta_{12} < 0}$, $\boldsymbol{\delta_{13} < 0}$, $\boldsymbol{\delta_{23} < 0}$, $\boldsymbol{\delta_{123} \geq 0}$.

  In order that $v(12) - v(2) \geq w(12) - w(2)$ and $v(13) - v(3) \geq w(13) - w(3)$, we must have $\delta_1 \geq -\min\{\delta_{12}, \delta_{13}\}$. In order that $v(N) - \sum_{i \in N} v(i) \geq w(N) - \sum_{i \in N} w(i)$, we must have $\delta_{123} \geq -\delta_{12} - \delta_{13} - \delta_{23} \geq -\delta_{23}$. Then,

$$\sum_{T \in 2^N \setminus \{\emptyset, \{2\}, \{3\}\}} \left\{\varphi_1(d_T^v u_T) - \varphi_1(d_T^w u_T)\right\} \overset{\text{C24}}{\geq} \delta_1 + \frac{\delta_{12}}{2} + \frac{\delta_{13}}{2} + \frac{\delta_{23}}{3} + \frac{\delta_{123}}{3}$$
$$\geq \delta_1 + \min\{\delta_{12}, \delta_{13}\} + \frac{\delta_{23} + \delta_{123}}{3}$$
$$\geq 0.$$

Appendix C

Proof of well-definedness of the contribution vector in NTU games

The purpose of this appendix is to prove that the contribution vector $m^R(V)$ (see Section 4.3.3) is well-defined.

Proof . Let $V \in \hat{\Gamma}$, $S \subseteq N$, $|S| \geq 2$, $j \in S$ and $z \in \mathbb{R}^{S\backslash j}$. We define $Y \subseteq \mathbb{R}$ by

$$Y = \left\{ r \in \mathbb{R} : \left((z_k)_{k \in S\backslash j}, r \right) \in V(S) \right\}$$

By N2 (closed), Y is closed. Thus, to prove Claim 1, it suffices to prove that Y is bounded from above and non-empty.

Let $x \in \partial V(S)$. By N2 (convex), there exists a normal vector λ such that

$$V(S) \subseteq \{ y \in \mathbb{R}^S : y \cdot \lambda \leq x \cdot \lambda \}.$$

Moreover, by N3, there exists $\delta > 0$ such that $\lambda_j \geq \delta$. It follows that Y is bounded from above.

We prove $Y \neq \emptyset$, i.e., there exists $r \in \mathbb{R}$ such that $\left((z_k)_{k \in S\backslash j}, r \right) \in V(S)$. Assume, by way of contradiction, that for any $r \in \mathbb{R}$, $\left((z_k)_{k \in S\backslash j}, r \right) \notin V(S)$. We define $Z \subseteq \mathbb{R}^S$ by

$$Z = \left\{ \left((z_k)_{k \in S\backslash j}, r \right) : r \in \mathbb{R} \right\}.$$

The two sets $V(S)$ and Z are convex, and the intersection between the two sets is empty. By the separation theorem, there exists $p \in \mathbb{R}^S$, $p \neq \mathbf{0}$, such that

$$p \cdot x \leq p \cdot y \text{ for all } x \in V(S), y \in Z. \tag{C.1}$$

Assume $p_j > 0$. Then, by taking an arbitrary sequence $\{y^k\}_{k=1}^{\infty} \subseteq Z$ such that $y_j^k \to -\infty$, we have $p \cdot y^k \to -\infty$, which contradicts (C.1). Similarly, if we assume $p_j < 0$, we have a contradiction with (C.1). As a result, $p_j = 0$.

Assume $p_i < 0$ for some $i \in S\backslash j$. Consider the sequence $\{x^l\}_{l=1}^{\infty} \subseteq V(S)$ such that $x_i^l \to -\infty$ and $x_h^l = x_h^{l+1}$ for all $h \in S$, $h \neq i$, $l = 1, 2, \cdots$. By N2 (comprehensive), such a sequence always exists. Then, $p \cdot x^l \to +\infty$, which contradicts (C.1). It follows that $p \geq \mathbf{0}$. Since $p \neq \mathbf{0}$, there exists a player $i' \in S\backslash j$ such that $p_{i'} > 0$.

Let $x \in \partial V(S)$ be arbitrarily given. For any $m \in \mathbb{N}$, let $\tilde{x}^m$ denote the following vector:

$$\tilde{x}_j^m = x_j - m, \tilde{x}_i^m = x_i \text{ for all } i \in S \backslash j.$$

By N2 (comprehensive), $\tilde{x}^m \in V(S)$. Then, for any $m \in \mathbb{N}$, there exists x^m such that

$$\begin{aligned}
&x_j^m = \tilde{x}_j^m, \\
&x_h^m = x_h \text{ for all } h \in S, h \neq i', h \neq j, \\
&x^m \in \partial V(S).
\end{aligned}$$

Consider the sequence $\{x^m\}_{m=1}^{\infty} \subseteq \partial V(S)$. Since $x_j^m \to -\infty$, by N4, we have $x_{i'}^m \to +\infty$. Since $p_j = 0$ and $p_{i'} > 0$, we have $p \cdot x^m \to +\infty$, which contradicts (C.1). □