Essays on individual and social choice theories with desirability

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To my parents

who always encourage me.

I proud of them from the bottom of my heart.

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Chapter 1

Prologue

1.1 Desirability and null alternatives

The aim of this thesis is to propose methods to apply the concept of 'likes and dislikes' (hereafter called *desirability*) to choice theories for making better decisions. To introduce desirability, we use 'choosing not to choose the alternatives' (hereafter called *null alternatives*).

In choice theories or Economics, we use the preference rankings of alternatives to analyse choice behaviours. However, preference relations have only the relative evaluation criterion for the alternatives, such as 'an apple is better than an orange'. In fact, we also consider whether each alternative is desirable, neutral,[1] or undesirable when we make choices. If we use the preference rankings of alternatives and null alternatives, we obtain the relative and absolute evaluation criterion for the alternatives.

[1]An alternative is neutral if and only if the alternative and its null alternative are indifferent.

The concept of null alternatives is similar to that of opportunity loss. Sartre (2007) expressed the essence of null alternatives as follows:

> *"In one sense, choice is possible; but what is impossible is not to choose. I can always choose, but I must also realize that, if I decide not to choose, that still constitutes a choice".* (Sartre, 2007, p. 44, translated by Macomber.)

We can find lots of examples showing that Sartre's (2007) words are correct. Consider the following two examples showing that the concept of null alternatives is typical since we often express 'choosing not to choose the alternatives'.

First, consider the trolley problem and assume the following situation: you are standing in front of a lever and there is a runaway trolley on the main t . If you do nothing, the trolley will kill five people, whereas if you pull the lever, the trolley will switch to another track and kill one person.[2] Some people might interpret that there is only one alternative, that is, 'pull the lever' and that they are free from responsibilities if they do nothing. However, in fact, there are two alternatives: 'do nothing' and 'pull the lever'. In other words, lose five lives or one life. Thus, we can interpret 'do nothing' as equivalent to 'choosing not to pull the lever' (i.e. the null alternative of 'pull the lever').

Second, consider cardinal utilities, which are often used in Economics. When we calculate our cardinal utilities, we express choosing not to choose the goods (or bads). If we do not consume a certain good (or bad), we put zero

[2] See, for example, Sharp (1908) and Foot (1967).

into the variable as its amount. However, we do not express null alternatives when we rank sets as final outcomes without cardinal utilities. This explains the large gap between preference rankings and cardinal utilities.

Before suggesting the methods to consider the desirability in social and individual choice theories, we discuss existing choice rules having similar concepts to desirability. We thus analyse the anti-plurality[3] and basic best-worst[4] rules, and improve their axiomatisation in Chapter 2.

We then apply the concept of desirability to the Borda rule for single–winner voting in Chapter 3[5] and the lexicographic preference extension rules for ranking all subsets of the finite alternative set in Chapter 4.[6]

The aim of each chapter is described in the following manner.

Anti-plurality and basic best-worst rules

There are several social choice rules having similar concepts to desirability, such as the basic best–worst[7] and dis&approval voting[8] rules. Strictly speaking, they do not include the concept of desirability. However, in almost every situation, we normally have likes and dislikes in the alternative set. Thus, we can consider desirability partially by using the above rules. They are developed based on the anti–plurality rule considering only the worst alternatives

[3]This study was published on *Economics Letters* 168, 110–111.

[4]This study was published on *Economics Letters* 172, 19-22.

[5]This study was presented at the 2nd Spain–Japan Meeting of Economic Theory (Tokyo). Furthermore, this study is based on a working paper with Professor Edith Elkind (University of Oxford).

[6]This study was presented at SSCW 2016 (Lund), AMES 2016 (Kyoto), COMSOC-2018 (Troy), and so on.

[7]We assign 1, -1, and 0 points to the best, worst, and other alternatives, respectively.

[8]We assign 1, -1, and 0 points to approval, disapproval, and other alternatives, respectively.

(which are mostly dislikes) for all individuals.[9]

Thus, it is crucial to discuss the axiomatisation of the above rules. In particular, we improve the existing axiomatisations of anti–plurality and basic best–worst rules since the previous studies simply applied the Young's (1975) axiomatisation of the class of scoring rules. We then show the direct characterisation of anti–plurality and basic best–worst rules.

Net Borda rule

The Borda rule only includes the relative evaluation criterion for the alternatives since we assign scores 0, 1, 2, ..., for example to all alternatives from the bottom to the top. We thus propose a new Borda rule considering the desirability of alternatives. We can find an advantage in considering desirability from the following example. Suppose that there are two alternatives a and b, and 100 individuals (or voters). If the number of individuals who prefer a (b) to b (a) is 51 (49), a is chosen as the social choice by using the original Borda rule. However, if 51 (49) individuals like a (b) and 49 (0) individuals dislike a (b), it is possible that b is more intuitive and appropriate as the social choice. In the case of single–winner voting rules, we simply try to find the social preference ranking of alternatives. Thus, null alternatives can be expressed as a representative one, namely, an *outside option* introduced by Roth and Sotomayor (1990). The outside option is equivalent to an empty set, and indicates that we do not choose any alternative.

In the case of the Borda rule with desirability, we employ the following

[9] Advantages of negative votes have been long time discussed. For instance, see Brams (1977).

method. First, we define a linear order over the set of all alternatives and the outside option. We then define desirability as follows: each alternative is (*un*)*desirable* if and only if it is better (worse) than the outside option. Additionally, Each individual assigns the Borda scores to the alternatives and outside option. Then, there is a gap of one point between desirable and undesirable alternatives. We provide the characterisation of this new Borda rule (hereafter called *net Borda rule*) by *neutrality*, *faithfulness*, *reinforcement*, and *total cancellation*. Note that the social choice is assumed to include at least one alternative in this study.

Furthermore, we show the advantage in using the net Borda rule by comparing with the original Borda rule.

Lexicographic preference extension rules

From decision making in daily lives to policymaking, we often choose a set of several alternatives based on a preference ranking of alternatives. We thus suppose that an individual chooses a subset of the finite alternative set by considering the preference ranking and desirability of alternatives when the individual does not know his/her accurate cardinal utility, or does not have an ability to construct his/her accurate utility function. We assume that the individual receives all elements in a subset which he/she chooses.[10]

To rank all subsets, we use the *null alternatives*, which are assumed to indicate 'choosing not to choose existing alternatives'. Addtionally, we employ a complete preorder over the (null) alternative set. Then, each alternative is

[10] A theory of ranking sets with this assumption is called 'sets as final outcomes' in Barberà et al. (2004).

defined as *(un)desirable* if and only if it is strictly better (worse) than its null alternative, and *neutral* if and only if it and its null alternative are indifferent.

As an example of this framework, Fishburn (1992) mentioned committee voting and assumed that each individual constructs his/her preference ranking of subsets based on his/her preference ranking of alternatives. For instance, 'a null alternative of a is better than b' indicates that 'it is more important to you that a not be on the committee than that b be on it'.

In particular, we characterise the leximax and leximin extension rules. By adding null alternatives into each subset in which existing alternatives are not included and rearranging the elements of each transformed subset in descending order, we can rank all subsets lexicographically.

However, if we simply use the complete preorder over the (null) alternative set, the leximax and leximin extension rules violate *extensibility* requiring that the preference ranking of a and b is equivalent to that of $\{a\}$ and $\{b\}$.[11]

We thus need some properties for the complete preorder over the (null) alternative set to obtain an intuitive preference ranking of subsets. In this study, we use three properties. Hereafter, we denote the null alternative of a by n_a. The first property is *asymmetry of desirability*, which requires that if a is better than b, n_b is better than n_a. The second property is *consistency of desirability*, which requires that if n_a is better than b, n_b is better than a, and if a is better than n_b, b is better than n_a. The third property is *self–reflecting* introduced by Fishburn (1992). This requires that (i) a is better than b if and only if n_b is better than n_a, (ii) n_a is better than b if and only if n_b is better

[11] *Extensibility* has been called the *congruence condition* in Fishburn (1992) or an *extension rule* in related fields, such as *complete uncertainty* and *opportunity sets* (see Barberà et al. (2004)).

than a, and (iii) a is better than n_b if and only if b is better than n_a.

We show the following relationship among the above properties:

(i) *asymmetry of desirability* $\Rightarrow$ *extensibility*;

(ii) *self–reflecting* $\Rightarrow$ *asymmetry of desirability* & *consistency of desirability*.

From (i), we characterise the leximax and leximin extension rules by *dominance* axioms for preference relations over the power set when the complete preorder over the (null) alternative set satisfies *asymmetry of desirability*. We also find that the leximax and leximin extension rules satisfy *monotonicity of desirability* and *extended independence*.

Furthermore, we show that if the complete preorder over the (null) alternative set satisfies *self–reflecting*, the leximax and leximin extension rules are equivalent. However, even if the complete preorder over the (null) alternative set satisfies *asymmetry of desirability* and *consistency of desirability*, the leximax and leximin extension rules are not equivalent.

1.2 The structure of this thesis

The remainder of this thesis is structured as follows.

Chapter 2 proposes a new characterisation of the anti–plurality rule and basic best–worst rules. Section 2.2 reviews a generalised scoring rule, which was analysed by Young (1975). Section 2.3 characterises the anti–plurality rule by *anonymity*, *neutrality*, *reinforcement*, *averseness*, and *bottoms–only*. Section 2.4 then characterises the basic best–worst rule by *neutrality*, *reinforcement*, *top–bottom cancellation*, and *top–bottom non–negativity*.

Chapter 3 considers the desirability of alternatives when we find the social choice by using the Borda rule. Section 3.2 introduces the original Borda rule. Section 3.3 reports our notations and axioms, which are related to the net Borda rule. Section 3.4 characterises the net Borda rule by *neutrality*, *faithfulness*, *reinforcement*, and *total cancellation*. Finally, Section 3.5 discusses the advantages of the net Borda rule by comparing with the original Borda rule.

Chapter 4 characterises the leximax and leximin extension rules for ranking sets as final outcomes. Section 4.2 reports our notations and definitions. Section 4.3 discusses the need for *asymmetry of desirability*. Section 4.4 introduces axioms, and Section 4.5 axiomatises the leximax and leximin extension rules. Additionally, the section clarifies more necessary conditions for deriving these extension rules. Section 4.6 introduces additional properties to obtain more intuitive preference rankings of subsets, that is, *consistency of desirability* and *self-reflecting*.

Finally, Chapter 5 provides concluding remarks and future research directions.

Chapter 2

Anti–plurality and basic best–worst rules

2.1 Introduction

After Young (1975) axiomatised a scoring social choice rule by *anonymity*, *neutrality*, *reinforcement*, and *continuity*,[1] it became easier for us to characterise specific scoring rules, such as the plurality, anti–plurality, best–worst, and the Borda rules. For instance, Baharad and Nitzan (2005) and Garciá-Lapresta et al. (2010) characterised the anti–plurality and best–worst rules by using Young's (1975) theorem.

However, we need to check whether the axioms for the characterisation can be relaxed or there could be an alternative approach. In fact, Sekiguchi (2012) and Young (1974) provided direct characterisations of the plurality and Borda rules.

[1]See also Hansson and Sahlquist (1976).

We then propose direct characterisations of the anti–plurality and 'basic' best–worst rules[2] with a variable electorate and fixed alternatives.[3]

In our characterisation of the anti–plurality rule, we used *anonymity*, *neutrality*, *reinforcement*, *averseness*, and *bottoms–only*, as with Sekiguchi's (2012) characterisation of the plurality rule by *anonymity*, *neutrality*, *reinforcement*, *faithfulness*, and *tops–only*. Note that *averseness* is a weaker axiom than *faithfulness* and *bottoms–only* is the opposite of *tops–only*.

Additionally, we propose two characterisations of the basic best–worst rule. First, we show that *reinforcement* and *top–bottom cancellation* imply *anonymity*. This result indicates that we can characterise the basic best–worst rule by *neutrality*, *continuity*, *reinforcement*, and *top–bottom cancellation*. Second, we propose a direct characterisation of the basic best–worst rule by *neutrality*, *reinforcement*, *top–bottom cancellation*, and *top–bottom non–negativity*. Thus, we do not apply Young's (1975) theorem to the second characterisation. *Top–bottom non–negativity* requires that if the difference in number between the individuals preferring a certain alternative as their best and worst alternatives respectively is strictly negative, then that alternative is not included in the social choice.[4]

[2] There are two types of best–worst rules. The first is the basic best–worst rule. In this rule, we assign 1, -1, and 0 points for the best, worst, and other alternatives, respectively. There exists a similar voting rule called the dis&approval voting rule, which was discussed by Felsenthal (1989) and Alcantud and Laruelle (2014). Alcantud and Laruelle (2014) assumes that we assign 1, -1, and 0 points for approval, disapproval, and other alternatives, respectively by applying the dis&approval voting rule. The second is the weighted best–worst rule. In this rule, we give α, -β, and 0 points ($\alpha, \beta > 0$ and $\alpha \neq \beta$) for the best and worst alternatives, and the others, respectively (see Garciá-Lapresta et al. (2010)).

[3] Bossert and Suzumura (2016) characterised the anti–plurality rule directly by *anonymity*, *neutrality*, *reinforcement*, *individual–equality independence*, *single–agent monotonicity*, and *single–agent expansion* with a variable electorate and variable alternatives.

[4] Note that we describe definitions of the above axioms in the following sections.

The remainder of this chapter is structured as follows. Section 2.2 reviews a generalised scoring rule, which was analysed by Young (1975). Section 2.2.1 reports our notations and definitions. Section 2.2.2 introduces axioms and shows Young's (1975) theorem. Next, Section 2.3 characterises the anti–plurality rule. Sections 2.3.1 and 2.3.2 introduces our notations and axioms, respectively. Section 2.3.3 suggests direct axiomatisation of the anti–plurality rule. Finally, Section 2.4 argues the basic best–worst rule. As with Section 2.3, Sections 2.4.1 and 2.4.2 introduces our notations and axioms, respectively. Then, Section 2.4.3 proposes two axiomatisations of the basic best–worst rule.

2.2 Single–winner social choice scoring rules

2.2.1 Preliminaries

Let V be a finite set of individuals such that $V \subset \mathbb{Z}_+$ and $|V| \geq 1$, where $\mathbb{Z}_+$ is the set of positive integers and $|V|$ indicates the cardinality of V. Now, suppose that X is the finite alternative set and $|X| \geq 2$. The alternatives in X will be denoted by $a, b, c,$ etc. Next, suppose that $P_i \in \mathscr{D}_i \subseteq \mathscr{P}$ is a linear order over X for each $i \in V$, where $\mathscr{P}$ is the set of all preference relations over X and $\mathscr{D}_i$ is the set of feasible preference relations over X for $i \in V$. Also, let $\mathcal{P} = (P_i)_{i \in V} \in \mathscr{D}$ be the profile of all P_i's such that $\mathscr{D} = \Pi_{i \in V} \mathscr{D}_i \subseteq \mathscr{P}^{|V|}$. We assume the following property in this study:

Unrestricted domain: $\mathscr{D} = \mathscr{P}^{|V|}$.

Let $C : \mathscr{P}^{|V|} \rightrightarrows X$ be a *social choice correspondence*.[5] Additionally, let $n_{ar}(\mathcal{P})$ be the number of individuals whose rth most preferred alternatives are a. Then, suppose that $\boldsymbol{n}_a(\mathcal{P}) = (n_{a1}(\mathcal{P}), ..., n_{a|X|}(\mathcal{P}))$.

We then introduce the *generalised scoring vector*: $\boldsymbol{s} = (s_1, ..., s_{|X|})$ such that $s_1 > s_{|X|}$, $s_r \geq s_{r+1}$ for all $r \in \{1, ..., |X| - 1\}$. Using $\boldsymbol{s}$ and $\boldsymbol{n}_a(\mathcal{P})$, we can define the *generalised score function* of each alternative $f_a : \mathscr{P}^{|V|} \to \mathbb{R}$: $f_a(\mathcal{P}) = \boldsymbol{s} \cdot \boldsymbol{n}_a(\mathcal{P})$,[6] where $\mathbb{R}$ is the set of real numbers. We then define the *generalised scoring rule* as given below.

Definition 2.1. *Generalised scoring rule*: $C_s(\mathcal{P}) = \{a \in X \mid a \in argmax_{b \in X} f_b(\mathcal{P})\}$.

From Definition 2.1, the winner(s) is the alternative(s) which receives the highest score according to the above generalised scoring system.

2.2.2 Axioms and Young's (1975) theorem

We introduce the necessary and sufficient conditions for deriving C_s, which was clarified by Young (1975).

First, C is *anonymous* if

$$C(\mathcal{P}) = C((P_{\pi(i)})_{i \in V})$$

for any $\pi \in \Pi$ and for any $\mathcal{P} \in \mathscr{P}^{|V|}$, where π is a permutation such that $\pi : V \to V$, and Π is the set of all permutations on the individuals in V.

[5] The correspondence C outputs a non–empty subset of X.

[6] The inner product is used in the formula.

Thus, *anonymity* requires that no social choice should depend on the names of individuals.

Second, C is *neutral* if

$$C(\psi(\mathcal{P})) = \{\lambda(a) \in X \mid a \in C(\mathcal{P})\}$$

for any $\lambda \in \Lambda$ and for any $\mathcal{P} \in \mathscr{P}^{|V|}$, where λ is a permutation such that $\lambda : X \to X$, Λ is the set of all permutations on X, ψ is a permutation such that $\psi : \mathscr{P}^{|V|} \to \mathscr{P}^{|V|}$, and Ψ is the set of all permutations on $\mathscr{P}^{|V|}$. Note that for an arbitrary $V \subset \mathbb{Z}_+$, every $\lambda \in \Lambda$ induces $\psi \in \Psi$, in other words, there exists a bijection whose domain and co–domain are Λ and Ψ, respectively.[7] Intuitively, *neutrality* requires that no social choice should depend on the names of alternatives.

Third, C is *continuous* if $C((P_i)_{i \in nV_1}) = \{a\}$, $V_1 \subset \mathbb{Z}_+$, $a \in X$, and there exists a sufficiently large positive integer $m \in \mathbb{Z}_+$ such that, for all $n > m$, $n \in \mathbb{Z}_+$,

$$C((P_i)_{i \in nV_1 \cup V_2}) = \{a\}$$

for all $V_2 \subset \mathbb{Z}_+$ such that V_2 is disjoint from V_1. Note that nV_1 indicates the union of n 'clone' sets of V_1. Assume that every clone set has the same preference profile as V_1. Additionally, let each clone set be disjoint from V_1 and other clone sets. From the above statement, *continuity* requires that if the social choice of a certain society $V \subset \mathbb{Z}_+$ includes only one alternative, it is equal to that of a unified society consisting of the following societies: (i)

[7] For example, assume that $X = \{a, b, c\}$, $V = \{1, 2\}$, and society has the following preference profile: $(P_1, P_2) \in \mathscr{P}^2$ such that aP_1bP_1c and bP_2cP_2a. If $\lambda(a) = b$, $\lambda(b) = c$, and $\lambda(c) = a$, we obtain $\psi((P_1, P_2)) = (P_1', P_2') \in \mathscr{P}^2$ such that $bP_1'cP_1'a$ and $cP_2'aP_2'b$.

sufficiently many clone societies of V, and (ii) an arbitrary society.

Fourth, C satisfies *reinforcement* if

$$C(\mathcal{P}_1) \cap C(\mathcal{P}_2) \neq \emptyset \Rightarrow C(\mathcal{P}_1) \cap C(\mathcal{P}_2) = C(\mathcal{P}_1 + \mathcal{P}_2)$$

for any $\mathcal{P}_1 = (P_i)_{i \in V_1} \in \mathscr{P}^{|V_1|}$ and for any $\mathcal{P}_2 = (P_i)_{i \in V_2} \in \mathscr{P}^{|V_2|}$ such that $V_1, V_2 \subset \mathbb{Z}_+$ and V_1 and V_2 are disjoint, where $\mathcal{P}_1 + \mathcal{P}_2 = (P_i)_{i \in V_1 \cup V_2} \in \mathscr{P}^{|V_1|+|V_2|}$ for all $V_1, V_2 \subset \mathbb{Z}_+$. Thus, this axiom requires that if the intersection of the social choices for any two disjoint subsets of individuals is not empty, then the intersection is equal to the social choice for the union of those two subsets.

Then, Young (1975) shows the following characterisation of C^s.

Theorem 2.1. (Theorem 1, Young (1975)) $C = C_s$ if and only if C satisfies *anonymity*, *neutrality*, *continuity*, and *reinforcement*.

Proof. See the proof of Theorem 1 in Young (1975). □

2.3 The anti–plurality rule

2.3.1 Preliminaries

We use the same setting as in Chapter 2. Furthermore, let $\boldsymbol{s}^{ap} = (0, ..., 0, -1)$ be the *anti–plurality scoring vector*, and let $f_a^{ap}(\mathcal{P}) = \boldsymbol{s}^{ap} \cdot \boldsymbol{n}_a(\mathcal{P})$ be the *anti–plurality score function* of each alternative. We then define the *anti–plurality rule* as follows:

Definition 2.2. *Anti–plurality rule*: $C_{ap}(\mathcal{P}) = \{a \in X \mid a \in argmax_{b \in X} f_b^{ap}(\mathcal{P})\}$.

From Definition 2.2, the winner(s) is the alternative(s) which receives the highest score according to the anti–plurality scoring system.

2.3.2 Axioms

We will use *anonymity*, *neutrality*, and *reinforcement* to characterise C_{ap}. We then introduce the following two additional axioms.

First, C satisfies *averseness* if

$$V = \{1\} \Rightarrow \check{a}(P_1) \notin C(P_1)$$

for all $P_1 \in \mathscr{P}$, where $\check{a}(P_i)$ is the worst alternative of $i \in V$. Thus, *averseness* requires that if there exists only one individual in a society, then the social choice does not include the worst alternative of the individual. *Averseness* is a weaker axiom than *faithfulness*.[8]

Second, C satisfies *bottoms–only* if

$$[\check{a}(P_i) = \check{a}(P_i') \; \forall i \in V] \Rightarrow C(\mathcal{P}) = C(\mathcal{P}')$$

for any $\mathcal{P}, \mathcal{P}' \in \mathscr{P}^{|V|}$. From the definition, this axiom requires that every social choice depends only on the worst alternatives of all individuals. It is the opposite of *tops–only*.[9]

[8] $V = \{1\} \Rightarrow C(P_1) = \{\hat{a}(P_1)\} \; \forall P_1 \in \mathscr{P}$, where $\{\hat{a}(P_i)\}$ is i's best alternative in X.
[9] $\forall \mathcal{P}, \mathcal{P}' \in \mathscr{P}^{|V|}$, $[\hat{a}(P_i) = \hat{a}(P_i') \; \forall i \in V] \Rightarrow C(\mathcal{P}) = C(\mathcal{P}')$.

2.3.3 Characterisation

Baharad and Nitzan (2005) applied Young's (1975) theorem and showed that $C_s = C_{ap}$ if and only if it satisfies *minimal veto*.[10] Furthermore, Bossert and Suzumura (2016) characterised the anti–plurality rule directly by *anonymity*, *neutrality*, *reinforcement*, *individual–equality independence*,[11] *single–agent monotonicity*,[12] and *single–agent expansion*[13] with variable alternatives.

Now, Theorem 2.2 shows a direct characterisation of C_{ap} by *anonymity*, *neutrality*, *reinforcement*, *averseness*, and *bottoms–only* with the fixed set of alternatives X. We prove Theorem 2.2 by a method similar to that for the proof of Sekiguchi's (2012) theorem.[14] A difference between our and Sekiguchi's (2012) proofs is the assumptions in Case (ii) of Theorem 2.2 and Lemma 1 in Sekiguchi (2012). Both of them are prepared by way of contradiction. We do not consider a choice of an individual in $\mathbb{Z}_+ \setminus V$ in Case (ii) of Theorem 2.2. The proof is appeared in Appendix A.1.

Theorem 2.2. C=C_{ap} if and only if C satisfies *anonymity*, *neutrality*, *reinforcement*, *averseness*, and *bottoms–only*.

Finally, we show the independence of *anonymity*, *neutrality*, *reinforcement*, *averseness*, and *bottoms–only*.

[10]This requires that C assigns a veto power to every minority group of $\lceil |V|/|X| \rceil$ members. Thus, even $\hat{a}(P_i) = a$ for all $i \in V' \subset V$, $a \notin C(\mathcal{P})$ if $V'' = V \setminus V'$, $|V''| \geq \lceil |V|/|X| \rceil$, $\hat{a}(P_j) \neq a$ for all $j \in V''$, and $a \notin C((P_j)_{j \in V''})$.

[11]Let $S \subseteq X$ be a non–empty feasible set of alternatives. For any $\mathcal{P}, \mathcal{P}' \in \mathscr{P}^{|V|}$, if i's choices from S based on P_i and P'_i are the same for all $i \in V$, the social choice based on $\mathcal{P}$ will not changed by replacing only one preference P_i by P'_i for each $i \in V$ without changing alternatives.

[12]When $S = \{a, b\}$ and $|V| = 1$, aP_ib implies that i's choice will be a.

[13]For each $i \in \mathbb{Z}_+$, for any $a \notin S$ such that $|S| > 1$, aP_ib for all $b \in S$ implies that i's choice from $S \cup \{a\}$ is equal to the union of $\{a\}$ and i's choice from S.

[14]See also Ching (1996) and Yeh (2008).

A *priority rule* C_{pr} violates *anonymity* and satisfies other four axioms. Suppose that $\succ$ is a linear order over V, and '$i \succ j$' indicates that the society sets i above j. Then, $C_{pr}(\mathcal{P}) = X \setminus \{\breve{a}(P_i)\}$, $i \succ j$ for all $j \in V \setminus \{i\}$. The non–Bottom rule C_{nb} violates *reinforcement* and satisfies other four axioms. Suppose that $C_{nb}(\mathcal{P}) = X \setminus \cap_{i \in N}\{\breve{a}(P_i)\}$. The fixed–order non–bottom rule C_{fnb} violates *neutrality* and satisfies other four axioms. Suppose that i' is called an 'arbitrator' in the outside of V. Then, $C_{fnb}(\mathcal{P}) = \{a'\}$, where a' is the arbitrator's best alternative in $C_{nb}(\mathcal{P})$. The feasibility rule C_{fe}, which always outputs X, violates *averseness* and satisfies other four axioms. The Borda rule violates *bottoms-only* and satisfies other four axioms.

2.4 The basic best–worst rule

2.4.1 Preliminaries

As with Section 2.2, we use the same notations as in Section 2.1. Additionally, suppose that $\boldsymbol{s}^{bw} = (1, 0, ..., 0, -1)$ is the *basic best–worst scoring vector*, and $f_a^{bw} = \boldsymbol{s}^{bw} \cdot \boldsymbol{n}_a(\mathcal{P})$ is the *basic best–worst score function* of each alternative. We then define the *basic best–worst rule* in the following manner:

Definition 2.3. *Basic best–worst rule*: $C_{bw}(\mathcal{P}) = \{a \in X \mid a \in argmax_{b \in X} f_b^{bw}(\mathcal{P})\}$.

From Definition 2.3, the winner(s) is the alternative(s) which receives the highest score according to the basic best–worst scoring system.

2.4.2 Axioms

We will use *anonymity*, *neutrality*, *continuity*, and *reinforcement* for the characterisation of C_{bw}. We then introduce the following two additional axioms.

First, C satisfies *top–bottom non–negativity* if

$$n_{a1}(\mathcal{P}) < n_{a|X|}(\mathcal{P}) \Rightarrow a \notin C(\mathcal{P})$$

for all $a \in X$ and for any $P \in \mathscr{P}$. This axiom requires that if the difference in number between the individuals preferring a certain alternative as their best and worst alternatives respectively is strictly negative, then that alternative is not included in the social choice. *Top–bottom non–negativity* implies *averseness*, as proposed in Kurihara (2018b).[15]

Second, C satisfies *top–bottom cancellation* if

$$[n_{a1}(\mathcal{P}) = n_{a|X|}(\mathcal{P})\ \forall a \in X] \Rightarrow C(\mathcal{P}) = X$$

for any $\mathcal{P} \in \mathscr{P}^{|V|}$. Hence, *top–bottom cancellation* requires that the social choice is X if the number of individuals preferring a as their best alternatives is cancelled out by the number of individuals preferring a as their worst alternatives for every $a \in X$. This axiom was introduced by Garciá-Lapresta et al. (2010).

[15]*Averseness* requires that if we assume that only one individual exists, then the individual's social choice will not include the worst alternative.

2.4.3 Characterisation

Theorem 2.3 is the first major result, which shows that C^{bw} can be characterised by *neutrality*, *continuity*, *reinforcement*, and *top–bottom cancellation*. A proof of Theorem 2.3 is appeared in Appendix A.2.

Theorem 2.3. $C = C_{bw}$ if and only if C satisfies *neutrality*, *continuity*, *reinforcement*, and *top–bottom cancellation*.

Next, Theorem 2.4 shows another characterisation of C_{bw} without using Young's (1975) theorem. In the characterisation, we use *top–bottom non–negativity* instead of *continuity*. A method of proof is similar to that of the theorem proofs in Sekiguchi (2012) and Kurihara (2018b). A proof of Theorem 2.4 is appeared in Appendix A.3.

Theorem 2.4. $C = C_{bw}$ if and only if C satisfies *neutrality*, *reinforcement*, *top–bottom non–negativity*, and *top–bottom cancellation*.

Finally, we show the independence of axioms used in Theorem 2.4.

The top–bottom rule C_{tb} violates *reinforcement* and satisfies other three axioms. Let $C_{tb}(\mathcal{P}) = \cup_{i\in N}\{\hat{a}(P_i)\} \setminus \cup_{i\in N}\{\check{a}(P_i)\}$ if $\cup_{i\in N}\{\hat{a}(P_i)\} \neq \cup_{i\in N}\{\check{a}(P_i)\}$, and let $C_{tb}(\mathcal{P}) = X$ if $\cup_{i\in N}\{\hat{a}(P_i)\} = \cup_{i\in N}\{\check{a}(P_i)\}$. The fixed–order top–bottom rule C_{ftb} violates *neutrality* and satisfies other three axioms. Suppose that i' is called an 'arbitrator' in the outside of V. Then, $C_{ftb}(\mathcal{P}) = \{a'\}$, where a' is the arbitrator's best alternative in $C_{tb}(\mathcal{P})$ when $C_{tb}(\mathcal{P}) \neq X$. Additionally, if $C_{tb}(\mathcal{P}) = X$, assume that $C_{ftb}(\mathcal{P}) = X$. The worst–best rule C_{wb} violates *top–bottom non–negativity* and satisfies other three axioms. In this rule, each individual assigns -1, 1, and 0 points to the best, worst, and other alternatives, respectively. The best–worst–Borda rule violates *top–bottom cancellation* and satisfies other three axioms. This rule is equivalent to the Borda rule over $C_{wb}(\mathcal{P})$, not X.

Chapter 3

Net Borda rule

3.1 Introduction

In Chapter 2, we analyse two scoring rules having similar concepts to desirability. Except for theses rules, there are social choice rules considering desirability, that is, the preference approval voting and fallback voting rules, analysed by Brams and Sanver (2009). Brams and Sanver (2009) expressed preference–approval as follows: $ab \mid c$ indicating that a is better than b, b is better than c, a and b are approval, and c is disapproval. However, both voting rules violate *Pareto efficiency*.[1]

We thus propose the *net Borda* rule considering the desirability of alternatives since the Borda rule includes only the relative evaluation criterion (preference rankings) for the alternatives. We can find an advantage in considering desirability from the example in Chapter 1. By using the information

[1]If there are two alternatives a and b, disapproved by all individuals, we simply apply the approval voting rule in the cases of the preference approval and fallback voting rules. Thus, even if every individual prefers a to b, both are included in the social choice.

about desirability, we can obtain more intuitive social choice by using the Borda rule. In t –winner voting rules, we simply try to find the social preference ranking of alternatives. Thus, null alternatives can be expressed as an *outside option* introduced by Roth and Sotomayor (1990). The outside option is equivalent to an empty set, and indicates 'choosing nothing'.

In the framework of the net Borda rule, we employ a linear order over the set of all alternatives and the outside option. We then define desirability as follows: each alternative is (*un*)*desirable* if and only if it is better (worse) than the outside option Additionally, we assign the Borda scores to the alternatives and outside option. Then, there is a gap of one point between desirable and undesirable alternatives.

The remainder of this chapter is structured as follows. Section 3.2 introduces the original Borda rule. Section 3.3 reports our notations and axioms, which are related to the net Borda rule. Section 3.4 characterises the net Borda rule by *neutrality*, *faithfulness*, *reinforcement*, and *total cancellation*. Finally, Section 3.5 discusses the advantage of the net Borda rule by comparing with the original Borda rule.

3.2 Borda rule

3.2.1 Preliminaries

We use the same notations in Chapter 2. We then report additional notations.

Let $n_{ab}(\mathcal{P})$ be the number of individuals who prefer a to b for all $a \neq b$. For a given $\mathcal{P} \in \mathscr{P}^{|V|}$ and for all $a \in X$, the *Borda score* over X is denoted by

$$B_a(\mathcal{P}) = \Sigma_{b\in X\setminus\{a\}}(n_{ab}(\mathcal{P}) - n_{ba}(\mathcal{P})).$$

We then define the *Borda rule* as follows:

Definition 3.1. *Borda rule*: $C_{br}(\mathcal{P}) = \{a \in X \mid a \in argmax_{b\in X} B_b(\mathcal{P})\}$.

From Definition 3.1, the winner(s) is the alternative(s) which receives the highest Borda score.

3.2.2 Axioms and Young's (1974) theorem

We introduce the necessary and sufficient conditions for deriving C_{br}, which was clarified by Young (1974) and Hansson and Sahlquist (1976). Note that *neutral*, *reinforcement*, and *faithfulness* are introduced in Chapter 2. We thus introduce an additional axiom.

C satisfies *cancellation* if

$$[n_{ab}(\mathcal{P}) = n_{ba}(\mathcal{P}) \ \ \forall a, b \in X] \Rightarrow C(\mathcal{P}) = X,$$

for any $\mathcal{P} \in \mathscr{P}^{|V|}$, where $n_{ab}(\mathcal{P})$ is the number of individuals who prefers a to b for all $a, b \in X$. This requires that if $n_{ab}(\mathcal{P})$ is cancelled out by $n_{ba}(\mathcal{P})$ for all $a, b \in X$, X should be the social choice.

Then, Young (1974) and Hansson and Sahlquist (1976) show the following characterisation of C_{br}.

Theorem 3.1. (Theorem 1, Young, 1974 or Hansson and Sahlquist, 1976) $C = C_{br}$ if and only if C satisfies *neutrality*, *reinforcement*, *faithfulness*, and *cancellation*.

Proof. See the proof of Theorem 1 in Young (1974) or Hansson and Sahlquist (1976). □

3.3 Net Borda rule

3.3.1 Preliminaries

A part of notations are the same as that of Chapter 2. We then report additional notations.

We employ a linear order over $X \cup \{\emptyset\}$, $P_i^* \in \mathscr{D}_i^* \subseteq \mathscr{P}^*$ for each $i \in V$, where $\mathscr{P}^*$ is the set of all preference relations over $X \cup \{\emptyset\}$ and $\mathscr{D}_i^*$ is the set of feasible preference relations over $X \cup \{\emptyset\}$ for $i \in V$. Additionally, we assume that each alternative $a \in X$ is (*un*)*desirable* for individual i if and only if $a P_i^* \emptyset$ ($\emptyset P_i^* a$). We call the empty set an 'outside option'. It was introduced by Roth and Sotomayor (1990). Let $\mathcal{P}^* = (P_i^*)_{i \in V} \in \mathscr{D}^*$ be a preference profile of all P_i^* such that $\mathscr{D}^* = \Pi_{i \in V} \mathscr{D}_i^* \subseteq (\mathscr{P}^*)^{|V|}$. We then assume the following condition.

*Unrestricted domain**: $\mathscr{D}^* = (\mathscr{P}^*)^{|V|}$.

Let $C^* : (\mathscr{P}^*)^{|V|} \rightrightarrows X$ be a *social choice correspondence*. Furthermore, let $n_{ab}^*(\mathcal{P}^*)$ be the number of individuals who prefer a to b for all $a \neq b$. For a given $\mathcal{P}^* \in (\mathscr{P}^*)^{|V|}$ and for all $a \in X \cup \{\emptyset\}$, the *net Borda score* over $X \cup \{\emptyset\}$ is denoted by

$$B_a^*(\mathcal{P}^*) = \Sigma_{b \in (X \cup \{\emptyset\}) \setminus \{a\}} (n_{ab}^*(\mathcal{P}^*) - n_{ba}^*(\mathcal{P}^*)).$$

We then define the *net Borda rule* as follows:

Definition 3.2. *Net Borda rule*: $C^*_{br}(\mathcal{P}^*) = \{a \in X \mid a \in argmax_{b \in X} B^*_b(\mathcal{P}^*)\}$.

From Definition 3.2, the winner(s) is the alternative(s) which receives the highest net Borda score.

3.3.2 Axioms

We introduce axioms to characterise C^*_{br} in the following manner. Note that the interpretations of *neutrality** and *reinforcement** are almost the same as those of *neutrality* and *reinforcement* in Chapter 2.

First, C^* satisfies *neutrality** if

$$C^*(\psi^*(\mathcal{P}^*)) = \{\lambda(a) \in X \mid a \in C^*(\mathcal{P}^*)\}$$

for any $\lambda \in \Lambda$ and for any $\mathcal{P}^* \in (\mathscr{P}^*)^{|V|}$, where ψ^* is a permutation of preference profiles such that $\psi^* : (\mathscr{P}^*)^{|V|} \to (\mathscr{P}^*)^{|V|}$, and Ψ^* is the set of all permutations on $(\mathscr{P}^*)^{|V|}$. For an arbitrary $V \subset \mathbb{Z}_+$, every $\lambda \in \Lambda$ induces $\psi^* \in \Psi^*$, in other words, there exists a bijection whose domain and co–domain are Λ and Ψ^*, respectively. Suppose that the position of $\emptyset$ is fixed for each individual even if each profile is transformed into another profile by ψ^*.

Second, C^* satisfies *reinforcement** if

$$C^*(\mathcal{P}^*_1) \cap C^*(\mathcal{P}^*_2) \neq \emptyset \Rightarrow C^*(\mathcal{P}^*_1) \cap C^*(\mathcal{P}^*_2) = C^*(\mathcal{P}^*_1 + \mathcal{P}^*_2)$$

for any $\mathcal{P}^*_1 = (P^*_i)_{i \in V_1} \in (\mathscr{P}^*)^{|V_1|}$ and for any $\mathcal{P}^*_2 = (P^*_i)_{i \in V_2} \in (\mathscr{P}^*)^{|V_2|}$

such that $V_1, V_2 \subset \mathbb{Z}_+$ and V_1 and V_2 are disjoint, where $\mathcal{P}_1^* + \mathcal{P}_2^* = (P_i^*)_{i \in V_1 \cup V_2} \in (\mathscr{P}^*)^{|V_1|+|V_2|}$ for all $V_1, V_2 \subset \mathbb{Z}_+$.

Third, C^* satisfies *faithfulness** if

$$V = \{1\} \Rightarrow C^*(P_1^*) = \{\hat{a}(P_1^*)\},$$

for all $P_1^* \in \mathscr{P}^*$, where $\hat{a}(P_i^*)$ is the best alternative for each $i \in V$. This axiom requires that if there exists only one individual in the society, the social choice will be the best alternative of the individual. Thus, if the outside option is the best for the individual, the social choice should be the second best, that is, the best alternative, for the individual.

We then introduce four *cancellation* conditions.

First, C^* satisfies *cancellation** if

$$[n_{ab}^*(\mathcal{P}^*) = n_{ba}^*(\mathcal{P}^*) \ \ \forall a, b \in X \cup \{\emptyset\}] \Rightarrow C^*(\mathcal{P}^*) = X.$$

This axiom requires that if $n_{ab}^*(\mathcal{P}^*)$ is cancelled out by $n_{ba}^*(\mathcal{P}^*)$ for all $a, b \in X \cup \{\emptyset\}$, X should be the social choice.

Second, C^* satisfies *cancellation*** if

$$[n_{ab}^*(\mathcal{P}^*) = n_{ba}^*(\mathcal{P}^*) \wedge n_{a\emptyset}^*(\mathcal{P}^*) = n_{b\emptyset}^*(\mathcal{P}^*) \ \ \forall a, b \in X] \Rightarrow C^*(\mathcal{P}^*) = X.$$

This axiom requires that if $n_{ab}^*(\mathcal{P}^*)$ is cancelled out by $n_{ba}^*(\mathcal{P}^*)$ for all $a, b \in X$ and $n_{a\emptyset}^*(\mathcal{P}^*)$ is the same for every $a \in X$, X should be the social choice. Trivially, *cancellation*** implies *cancellation**.

Third, C^* satisfies *cyclic cancellation** if

$$[n^*_{ar}(\mathcal{P}^*) = n^*_{br}(\mathcal{P}^*)\ \ \forall a, b \in X,\ \forall r \in \{1, ..., |X| + 1\}] \Rightarrow C^*(\mathcal{P}^*) = X,$$

where $n^*_{ar}(\mathcal{P}^*)$ is the number of individuals whose rth best element is a. This axiom requires that if every $a \in X$ is appeared at each rank the same times, X should be the social choice.

Finally, C^* satisfies *total cancellation** if

$$[B^*_a(\mathcal{P}^*) = B^*_b(\mathcal{P}^*)\ \ \forall a, b \in X] \Rightarrow C^*(\mathcal{P}^*) = X.$$

This axiom requires that if the net Borda score of each $a \in X$ is the same, X should be the social choice. *Total cancellation** implies *cancellation*** and *cyclic cancellation**.

3.4 Characterisation

First, we consider to characterise C^*_{br} by *neutrality**, *reinforcement**, *faithfulness**, and *cancellation**. However, another social choice rule satisfies them. We define another Borda score as follows: $B'_a(\mathcal{P}^*) = \Sigma_{b\in X\setminus\{a\}}(n^*_{ab}(\mathcal{P}^*) - n^*_{ba}(\mathcal{P}^*))$.[2] Thus, each individual is assumed to have a preference ranking of the alternatives and outside option, but ignore (or do not assign a Borda score to) the outside option. Then, the social choice, denoted by $C'_{br}(\mathcal{P}^*) : (\mathscr{P}^*)^{|V|} \to X$, is the set of winners whose $B'_a(\mathcal{P}^*)$ is the highest. Furthermore, $C'_{br}(\mathcal{P}^*)$ satisfies *cancellation***, and violates *cyclic cancella-*

[2]Trivially, $B'_a(\mathcal{P}^*) = B_a(\mathcal{P})$ for all $a \in X$.

*tion**.

Second, we consider to characterise C^*_{br} by *neutrality**, *reinforcement**, *faithfulness**, *cancellation**, and *cyclic cancellation**. Even if we use *cancellation** and *cyclic cancellation**, there still exists another social choice rule satisfying the above five axioms. Suppose that the social choice rule is denoted by $C^*_{bi} : (\mathscr{P}^*)^{|V|} \rightarrow X$ and called the *bisection* rule. Let s^{bi}_r be a score of the rth best element in $X \cup \{\emptyset\}$ for every individual. Suppose that $\boldsymbol{s}^{bi} = (s^{bi}_1, ..., s^{bi}_{|X|/2}, s^{bi}_{|X|/2+1}, s^{bi}_{|X|/2+2}, ..., s^{bi}_{|X|+1}) = (|X|/2, ..., 1, 0, -1, ..., -|X|/2)$ if $|X| + 1$ is an odd number, and $\boldsymbol{s}^{bi} = (s^{bi}_1, ..., s^{bi}_{(|X|+1)/2}, s^{bi}_{(|X|+3)/2}, ..., s^{bi}_{|X|+1}) = ((|X|+1)/2, ..., -1, 1, ..., -(|X|+1)/2)$ if $|X| + 1$ is an even number. Then, C^*_{bi} includes elements whose aggregated scores are the highest according to the above scoring system $\boldsymbol{s}^{bi}$. Additionally, C^*_{bi} violates *cancellation***.[3]

Finally, Theorem 3.2 shows the necessary and sufficient conditions to derive C^*_{br}. A proof of Theorem 3.2 is appeared in Appendix A.4.

Theorem 3.2. $C^* = C^*_{br}$ if and only if C^* satisfies *neutrality**, *reinforcement**, *faithfulness**, and *total cancellation**. [4]

Additionally, Table 3.1 shows a summary of this section.

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[3]Suppose that $V = \{1, 2\}$, $X = \{a, b, c\}$, $aP^*_1 bP^*_1 cP^*_1 \emptyset$, and $cP^*_1 bP^*_1 aP^*_1 \emptyset$. Then, $C^*_{bi}(\mathcal{P}^*) = \{b\}$.

[4]We are still considering to characterise C^*_{br} by *neutrality**, *reinforcement**, *faithfulness**, *cancellation***, and *cyclic cancellation**, or checking whether there exists another social choice rule satisfying the five axioms.

Table 3.1: Summary of characterisation

Cancellation conditions	C'_{br}	C^*_{bi}	C^*_{br}
*Cancellation**	✓	✓	✓
*Cancellation***	✓		✓
*Cyclic cancellation**		✓	✓
*Total cancellation**			✓

*Cancellation**⇒cancellation*.*
Total cancellation⇒cancellation*** and *cyclic cancellation*.*
Each rule satisfies *neutrality**, *reinforcement**, and *faithfulness**

3.5 Borda and net Borda rules

First, we compare the axiomatisation of Borda and net Borda rules. If we consider the original Borda rule over $X \cup \{\emptyset\}$, denoted by $C^\dagger_{br}$, and $\emptyset$ can be also included in the social choice, $C^\dagger_{br}(\mathcal{P}^*) = \{a \in X \cup \{\emptyset\} \mid a \in argmax_{b \in X \cup \{\emptyset\}} B^*_b(\mathcal{P}^*)\}$. Thus, $C^\dagger_{br}$ violates *neutrality**, *faithfulness** and *total cancellation**.

Next, we compare the social choices by using the original Borda and net Borda rules. Theoretically, it might be impossible to discuss any advantage in considering desirability when we know that individuals do not know their likes and dislikes for the alternatives. However, if we use $\mathcal{P}$ over X without any consideration for the desirability, we might obtain a non–intuitive social choice, and as a matter of fact, we often do that in the real world.

We then argue this matter in the following setting: $X = \{a, b\}$, $|V| = 5$. The outputs of C_{br} and C^*_{br} are reported in Tables 3.2 and 3.3. Table 3.2 shows the possible scores of a and b when $C_{br}(\mathcal{P}) = \{a\}$. If $C_{br}(\mathcal{P}) = \{b\}$, we can obtain the other results by switching a and b in Table 3.2. Thus, it is enough to consider only Cases I–III. Note that ab and $ab\emptyset$ indicate aPb and $aP^*bP^*\emptyset$, respectively. The same applies to other preference rankings.

In Case I, $C^*_{br}(\mathcal{P}^*) = C_{br}(\mathcal{P})$ because aP^*b for all individuals. Similarly, $C^*_{br}(\mathcal{P}^*) = C_{br}(\mathcal{P})$ in Case II. Assume that $\emptyset P^*aP^*b$ for four individuals and $bP^*\emptyset P^*a$ for one individual. Then, the scores of a and b are -1 and -1, respectively. This example shows that $B^*_a(\mathcal{P}^*) > B^*_b(\mathcal{P}^*)$ even if a is undesirable for the all five individuals, because four individuals strictly prefer a to b.

However, in Case III, $C^*_{br}(\mathcal{P}^*)$ becomes $\{a, b\}$ or $\{b\}$ with some preference profiles. Table 3.3 shows the outputs of C^*_{br}, corresponding to Case III. Assume that $\emptyset P^*aP^*b$ for three individuals and $bP^*\emptyset P^*a$ for two individuals. Then, the scores of a and b are -4 and -2, respectively, and $C^*_{br}(\mathcal{P}^*) = \{b\}$. This result makes sense because a is undesirable for the all five individuals, two individuals strictly prefer b to a, and b is desirable for the same two individuals. Thus, the outputs of C^*_{br} is more intuitive than that of C_{br}.

Form these results, it is crucial to confirm whether individuals have likes and dislikes before deciding a social choice rule. Furthermore, if there exist individuals who have likes and dislikes, we should employ C^*_{br}, rather than C_{br}.

Table 3.2: The possible scores of a and b when $C^{br} = \{a\}$

	preferences		scores	
Case	ab	ba	B_a	B_b
I	5	0	5	0

II	4	1	3	-3
III	3	2	1	-1

Table 3.3: The outputs of C^*_{br}, corresponding to Case III

preferences						scores			
$ab\emptyset$	$a\emptyset b$	$\emptyset ab$	$ba\emptyset$	$b\emptyset a$	$\emptyset ba$	B^*_a	B^*_b	C^*_{br}	$C^*_{br} \neq C^{br}$
3	0	0	2	0	0	6	4	$\{a\}$	
3	0	0	1	1	0	4	4	$\{a, b\}$	✓
3	0	0	1	0	1	4	2	$\{a\}$	
3	0	0	0	2	0	2	4	$\{b\}$	✓
3	0	0	0	1	1	2	2	$\{a, b\}$	✓
3	0	0	0	0	2	2	0	$\{a\}$	
2	1	0	2	0	0	6	2	$\{a\}$	
2	1	0	1	1	0	4	2	$\{a\}$	
2	1	0	1	0	1	4	0	$\{a\}$	
2	1	0	0	2	0	2	2	$\{a, b\}$	✓
2	1	0	0	1	1	2	0	$\{a\}$	
2	1	0	0	0	2	2	-2	$\{a\}$	
2	0	1	2	0	0	4	2	$\{a\}$	
2	0	1	1	1	0	2	2	$\{a, b\}$	✓
2	0	1	1	0	1	2	0	$\{a\}$	
2	0	1	0	2	0	0	2	$\{b\}$	✓
2	0	1	0	1	1	0	0	$\{a, b\}$	✓

$ab\emptyset$	$a\emptyset b$	$\emptyset ab$	$ba\emptyset$	$b\emptyset a$	$\emptyset ba$	B_a^*	B_b^*	C_{br}^*	$C_{br}^* \neq C^{br}$
2	0	1	0	0	2	0	-2	$\{a\}$	
1	2	0	2	0	0	6	0	$\{a\}$	
1	2	0	1	1	0	4	0	$\{a\}$	
1	2	0	1	0	1	4	-2	$\{a\}$	
1	2	0	0	2	0	2	0	$\{a\}$	
1	2	0	0	1	1	2	-2	$\{a\}$	
1	2	0	0	0	2	2	-4	$\{a\}$	
1	1	1	2	0	0	4	0	$\{a\}$	
1	1	1	1	1	0	2	0	$\{a\}$	
1	1	1	1	0	1	2	-2	$\{a\}$	
1	1	1	0	2	0	0	0	$\{a, b\}$	✓
1	1	1	0	1	1	0	-2	$\{a\}$	
1	1	1	0	0	2	0	-4	$\{a\}$	
1	0	2	2	0	0	2	0	$\{a\}$	
1	0	2	1	1	0	0	0	$\{a, b\}$	✓
1	0	2	1	0	1	0	-2	$\{a\}$	
1	0	2	0	2	0	-2	0	$\{b\}$	✓
1	0	2	0	1	1	-2	-2	$\{a, b\}$	✓
1	0	2	0	0	2	-2	-4	$\{a\}$	
0	3	0	2	0	0	6	-2	$\{a\}$	
0	3	0	1	1	0	4	-2	$\{a\}$	
0	3	0	1	0	1	4	-4	$\{a\}$	
0	3	0	0	2	0	2	-2	$\{a\}$	

$ab\emptyset$	$a\emptyset b$	$\emptyset ab$	$ba\emptyset$	$b\emptyset a$	$\emptyset ba$	B^*_a	B^*_b	C^*_{br}	$C^*_{br} \neq C^{br}$
0	3	0	0	1	1	2	-4	$\{a\}$	
0	3	0	0	0	2	2	-6	$\{a\}$	
0	2	1	2	0	0	4	-2	$\{a\}$	
0	2	1	1	1	0	2	-2	$\{a\}$	
0	2	1	1	0	1	2	-4	$\{a\}$	
0	2	1	0	2	0	0	-2	$\{a\}$	
0	2	1	0	1	1	0	-4	$\{a\}$	
0	2	1	0	0	2	0	-6	$\{a\}$	
0	1	2	2	0	0	2	-2	$\{a\}$	
0	1	2	1	1	0	0	-2	$\{a\}$	
0	1	2	1	0	1	0	-4	$\{a\}$	
0	1	2	0	2	0	-2	-2	$\{a, b\}$	✓
0	1	2	0	1	1	-2	-4	$\{a\}$	
0	1	2	0	0	2	-2	-6	$\{a\}$	
0	0	3	2	0	0	0	-2	$\{a\}$	
0	0	3	1	1	0	-2	-2	$\{a, b\}$	✓
0	0	3	1	0	1	-2	-4	$\{a\}$	
0	0	3	0	2	0	-4	-2	$\{b\}$	✓
0	0	3	0	1	1	-4	-4	$\{a, b\}$	✓
0	0	3	0	0	2	-4	-6	$\{a\}$	

Chapter 4

Leximax and leximin extension rules

4.1 Introduction

Three classes have been proposed in the theories of *ranking sets of alternatives* (or *objects*).[1] The first and second classes are *complete uncertainty*[2] and *opportunity sets*,[3] respectively. In these classes, an individual is assumed to receive only one alternative from a subset chosen by the individual. The third class is *sets as final outcomes*.[4] In this class, an individual is assumed to receive all alternatives in a subset chosen by the individual.

[1] See Barberà et al. (2004).

[2] See Arrow and Hurwicz (1972); Barrett and Pattanaik (1994); Bossert et al. (2000); Cohen and Jaffray (1980); Kannai and Peleg (1984); Pattanaik and Peleg (1984); Schmeidler (1989).

[3] See Bossert (1997); Bossert et al. (1994); Carter (1999); Dutta and Sen (1996); Foster (2011); Gravel (1998); Pattanaik and Xu (2000); Puppe (1996); Sen (2001).

[4] See Bossert, 1995; Fishburn, 1992; Yunfeng et al., 1996; Nitzan and Pattanaik, 1984; Roth, 1985.

In this study, which belongs to the third class, we analyse the individual, not collective, preference extension rules for ranking sets. In particular, we characterise the leximax and leximin extension rules[5] for the situations that we do not know our accurate cardinal utilities, and do not have an ability to construct our accurate utility functions.

Bossert (1995) characterised the class of lexicographic extension rules including the leximax, leximin, median–based, and other lexicographic rules. However, subsets can be ranked if and only if they have the same cardinality in the framework of Bossert (1995).

To solve this restriction, we can use a *disjoint copy* of the alternative set, which was introduced by Fishburn (1992). Each element in the disjoint copy corresponds to each alternative, and it can be interpreted as the complement of the alternative. We then call the elements in the disjoint copy *null alternatives*, which are assumed to indicate 'choosing not to choose existing alternatives'. Each alternative is defined as (*un*)*desirable* if and only if it is strictly better (worse) than its null alternative, and *neutral* if and only if it and its null alternative are indifferent.

Note that we set the simplest framework with no compatibility within alternatives and no category (or only one category) in this study. As an example of this framework, Fishburn (1992) mentioned committee voting since familiar scoring rules, such as the k–Borda rule, do not consider compatibility. Fishburn (1992) assumed that each individual constructs his/her preference

[5] We do not research a median–based rule such as the Nitzan and Pattanaik (1984) rule, which belongs to a subgroup, including rules such as the maximax and maximin ones. Even if a leximedian rule could be defined, it would differ from the leximax and leximin rules because we need an additional rule to rank the pairs of both the outsides of the medians. Thus, such a rule is excluded from this study.

ranking of subsets based on that of (null) alternatives. Thus, for instance, 'n_a (the null alternative of a) is better than b' indicates that 'it is more important to you that a not be on the committee than that b be on it'.

According to the framework of Fishburn (1992), Yunfeng et al. (1996) also proposed and characterised the lexicographic extension rule based on a *signed order* over the (null) alternative set. The *signed order* is a complete preorder satisfying *self-reflecting*,[6] which was introduced by Fishburn (1992). However, *self-reflecting* restricts the scope of considerable situations because it does not allow that null alternatives are indifferent if the existing alternatives are not indifferent. We thus find the sufficient condition making the leximax and leximin extension rules satisfy a desirable property called *extensibility*[7] requiring that the preference ranking of a and b is equivalent to that of $\{a\}$ and $\{b\}$.

First, we employ a complete preorder over the (null) alternative set, and assume *asymmetry of desirability*. This implies *extensibility* and requires that if a is better than b, n_b is better than n_a. We then characterise the leximax and leximin extension rules with *asymmetry of desirability* by the *dominance* axioms. We also find that the leximax and leximin extension rules satisfy *monotonicity of desirability* and *extended independence* in this framework.

Second, we consider an additional condition called *consistency of desirability*, which requires that if n_a is better than b, n_b is better than a, and

[6]This requires that (i) a is better than b if and only if n_b is better than n_a, (ii) n_a is better than b if and only if n_b is better than a, and (iii) a is better than n_b if and only if b is better than n_a.

[7]*Extensibility* has been called the *congruence condition* in Fishburn (1992) or an *extension rule* in related fields, such as *complete uncertainty* and *opportunity sets* (see Barberà et al. (2004)).

if a is better than n_b, b is better than n_a. We then show that *asymmetry of desirability* and *consistency of desirability* are reasonable and weaker than *self-reflecting*. However, we find that the leximax and leximin extension rules are not equivalent even if the complete preorder over the (null) alternative set satisfies *asymmetry of desirability* and *consistency of desirability*.

Finally, we show that if the complete preorder over the (null) alternative set satisfies *self-reflecting*, the leximax and leximin extension rules are equivalent.

Section 4.2 reports our notations and definitions. Section 4.3 discusses the need for *asymmetry of desirability*. Section 4.4 introduces axioms, and Section 4.5 characterises the leximax and leximin extension rules with *asymmetry of desirability*. Section 4.6 introduces *consistency of desirability* and discuss the relationship between *self-reflecting*, *asymmetry of desirability*, and *consistency of desirability*. Furthermore, we show that the leximax and leximin extension rules are equivalent if the complete preorder over the (null) alternative set satisfies *self-reflecting*.

4.2 Preliminaries

Let X be the finite set of all alternatives with cardinality $|X| \geq 2$. The power set of X is denoted by $\mathscr{X}$. As with the framework of Fishburn (1992), let $N = \cup_{a \in X}\{n_a\}$ be the finite set of null alternatives such that 'choosing not to choose a' is equivalent to 'choosing n_a' for all $a \in X$. Furthermore, each subset of N is denoted by $N_A = \cup_{a \in A}\{n_a\}$, corresponding to each subset $A \in \mathscr{X}$. A preference relation over $X \cup N$ is assumed to be a complete preorder denoted by $R \in \mathscr{R}$, where $\mathscr{R}$ is the set of all preference relations over $X \cup N$. The asymmetric and symmetric components are denoted by P and I, respectively. In this step, Fishburn (1992) assumed *extensibility* and *self-reflecting*. *Extensibility* requires that the preference ranking of any two alternatives and that of their singleton sets are the same. *Self-reflecting* does not allow n_a and n_b to be indifferent when a and b are not indifferent.

Extensibility: $\forall a, b \in X$, $aRb \Leftrightarrow \{a\}\bar{R}\{b\}$.

Self-reflecting: $\forall a, b \in X$, $aRb \Leftrightarrow n_aRn_b \wedge aRn_b \Leftrightarrow bRn_a \wedge n_aRb \Leftrightarrow n_bRa$.

However, *self-reflecting* implies *extensibility*,[8] and is a strong assumption. Thus, we do not assume them here, and argue whether *self-reflecting* can be relaxed to obtain an intuitive preference ranking of subsets by using the lexicographic extension rules.

We define the desirability of alternatives as follows: a is (un)desirable if and only if aPn_a (n_aPa), and neutral if and only if aIn_a for each $a \in X$. Additionally, let $\bar{R} \in \bar{\mathscr{R}}$ be a preference relation over $\mathscr{X}$, where $\bar{\mathscr{R}}$ is the set of all preference relations over $\mathscr{X}$. The asymmetric and symmetric components are denoted by $\bar{P}$ and $\bar{I}$, respectively.

Next, we discuss the method for ranking all subsets lexicographically. In Bossert (1995), any two subsets can be ranked if and only if they have the same cardinality. Several methods are used to solve the restriction. Roth and Sotomayor (1990) assumed a situation similar to a *college admissions problem*, and introduced the concept of empty slots.[9] Empty slots are added

[8]Thus, *self-reflecting* and *extensibility* are not independent.

[9]The outside option and threshold in Chapter 3 have similar concepts. However, neither

to each subset whose cardinality is smaller than a given quota. However, we also consider problems in more general choice theories. Furthermore, empty slots are the same as null alternatives assumed to satisfy *null indifference*.

Null indifference: $\forall a, b \in X$, $n_a I n_b$.

This condition is stronger than *self-reflecting*: $n_a I n_b$ might be non–intuitive if an individual hates a and likes b. Thus, we do not assume *null indifference* in this study.

We then introduce *transformed* subsets by adding null alternatives. For each $A \in \mathscr{X}$, let f_A: $X \to A \cup (N \setminus N_A)$ be a bijection such that $f_A(a) = a$ if $a \in A$ and $f_A(a) = n_a$ if $a \notin A$ for all $a \in X$. Let $A^* = \cup_{a \in X}\{f_A(a)\}$ be the transformed subset of A and $\mathscr{X}^* = \cup_{A \in \mathscr{X}}\{A^*\}$ be the transformed power set of X. Furthermore, all (null) alternatives are assumed to be rearranged in descending order: for all $A^* = \{a_1^*, ..., a_{|X|}^*\} \in \mathscr{X}^*$, $a_i^* R a_{i+1}^*$ for all $i \in \{1, ..., |X| - 1\}$. Thus, all subsets can be ranked even if they have different cardinalities by using transformed subsets.

Then, the *leximax* and *leximin extension rules* are defined in the following manner:

Definition 4.1. *Leximax extension rule* $\bar{R}_{lmax}$: $\forall A, B \in \mathscr{X}$,

$$A \bar{P}_{lmax} B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a_i^* P b_i^* \wedge a_j^* I b_j^* \ \forall j < i;$$

$$A \bar{I}_{lmax} B \Leftrightarrow a_i^* I b_i^* \ \forall i \in \{1, 2, ..., |X|\}.$$

is suitable for ranking all subsets because we cannot frame the cardinalities of all subsets by using them.

Definition 4.2. *Leximin extension rule* $\bar{R}_{lmin}$: $\forall A, B \in \mathscr{X}$,

$$A\bar{P}_{lmin}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a_i^* P b_i^* \wedge a_j^* I b_j^* \; \forall j > i;$$

$$A\bar{I}_{lmin}B \Leftrightarrow a_i^* I b_i^* \; \forall i \in \{1, 2, ..., |X|\}.$$

From Definition 4.2, we compare the pair of alternatives in A^* and B^* from the top to the bottom. If we find a strict preference relation and the alternative of A^* (B^*) is strictly better that of B^* (A^*), $A\bar{P}_{lmax}B$ ($B\bar{P}_{lmax}A$). From Definition 4.3, we compare the pair of alternatives in A^* and B^* from the bottom to the top in the case of $\bar{R}_{lmin}$.

We can avoid some non–intuitive preference rankings by using null alternatives. Suppose that $X = \{a, b, c, d\}$, $aIbPn_aIn_bIn_cIn_dPcPd$, $A = \{a, c, d\}$, and $B = \{b, d\}$. Without transformed subsets, they are forcibly ranked according to the leximax criteria as follows: $A\bar{P}B$. However, the third and second alternatives of A and B are d and n_cPc. Thus, by transforming A and B into $A^* = \{a, n_b, c, d\}$ and $B^* = \{b, n_a, n_c, d\}$, respectively, we find that $B\bar{P}_{lmax}A$.

However, $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ still have a serious problem. In the following example, $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ violate *extensibility*: $X = \{a, b\}$ and $n_aPaPbPn_b$. Even if aPb, $\{b\}\bar{P}_{lmax}\{a\}$ and $\{b\}\bar{P}_{lmin}\{a\}$ because $\{a\}^* = \{a, n_b\}$ and $\{b\}^* = \{n_a, b\}$. However, $n_aPaPbPn_b$ is non–intuitive because b is desirable and a is undesirable.

4.3 Asymmetry of desirability

To solve the above problem, we introduce the properties for R and check whether they make $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ satisfy *extensibility*.

First, *transitive desirability* requires that every desirable alternative is strictly better than every neutral or undesirable alternative, every neutral alternative is strictly better than every undesirable alternative, and any two neutral alternatives are indifferent.[10]

Transitive desirability: $\forall a, b \in X$, $[[aPn_a \wedge n_bRb] \vee [aIn_a \wedge n_bPb]] \Rightarrow aPb$; $[aIn_a \wedge bIn_b] \Rightarrow aIb$.

Transitive desirability is suitable to the meaning of null alternatives, but insufficient to imply *extensibility*. Suppose that $X = \{a, b\}$ and n_aPn_bPaIb. This preference ranking does not violate *transitive desirability*. However, $\{b\}\bar{P}_{lmax}\{a\}$ and $\{b\}\bar{P}_{lmin}\{a\}$ even if aIb. We thus need to introduce a stronger property than *transitive desirability*.

Now, we introduce *asymmetry of desirability*. This requires that the preference ranking of any two alternatives and that of their null alternatives are opposite.

Asymmetry of desirability: $\forall a, b \in X$, $aRb \Rightarrow n_bRn_a$.

First, Propositions 4.1 and 4.2 show that *asymmetry of desirability* is a weaker property than *null indifference* and *self-reflecting*, respectively. These propositions are trivial from the statements of the above three properties.

[10]This condition is similar to *consistency* introduced in Fishburn (1992).

Proposition 4.1. R satisfies *asymmetry of desirability* if R satisfies *null indifference*.

Proposition 4.2. R satisfies *asymmetry of desirability* if R satisfies *self-reflecting*.

From Proposition 4.3, *asymmetry of desirability* implies *transitive desirability*. A proof of Proposition 4.3 is appeared in Appendix A.5.

Proposition 4.3. R satisfies *transitive desirability* if R satisfies *asymmetry of desirability*.

From Proposition 4.4, *asymmetry of desirability* is the sufficient condition for $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ to satisfy *extensibility*. A proof of Proposition 4.4 is also appeared in Appendix A.6.

Proposition 4.4. $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ satisfy *extensibility* if R satisfies *asymmetry of desirability*.

However, *extensibility* does not imply *asymmetry of desirability*. For instance, suppose that $X = \{a, b, c\}$ and $aPn_cPn_aPn_bPbPc$ for an individual. In this case, aPb, $\{a\}\bar{P}_{lmax}\{b\}$, and $\{a\}\bar{P}_{lmin}\{b\}$, but aPb does not imply n_bRn_a.

Thus, we should discuss the strength of *asymmetry of desirability*. Take any two alternatives $a, b \in X$. In total, there are 75 preference rankings of $a, b, n_a, n_b \in X \cup N$ since R is a complete preorder over $X \cup N$. In 45 of these 75 orders, both $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ satisfy *extensibility*. Furthermore, in 39 of the 45 orders, R satisfies *asymmetry of desirability*. If the individual has one of the following six of the 45 orders, $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ satisfy *extensibility*, but violate *asymmetry of desirability*: aPn_aPn_bPb, aPn_aPn_bIb, aIn_aPn_bPb, bPn_bPn_aPa, bPn_bPn_aIa, and bIn_bPn_aPa. Thus, *asymmetry of desirability* is not the necessary condition to make only $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ satisfy *extensibility*. However, there are more lexicographic extension rules such as the median–based and leximedian ones. From this discussion, *asymmetry of desirability* might be insufficiently strong to consider *extensibility*. Additionally, we can relax the strong restrictions of *null indifference* and *self-reflecting*, and obtain the consistent preference rankings of singleton sets based on R by *asymmetry of desirability*.

We then suppose that $R^\dagger$ denotes R satisfying *asymmetry of desirability*. Thus, $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ are redefined by using $R^\dagger$ instead of R as follows:

Definition 4.3. *Leximax extension rule* $\bar{R}^\dagger_{lmax}$: $\forall A, B \in \mathscr{X}$,

$$A\bar{P}^\dagger_{lmax}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a^*_i P^\dagger b^*_i \wedge a^*_j I^\dagger b^*_j \; \forall j < i;$$

$$A\bar{I}^\dagger_{lmax}B \Leftrightarrow a^*_i I^\dagger b^*_i \; \forall i \in \{1, 2, ..., |X|\}.$$

Definition 4.4. *Leximin extension rule* $\bar{R}^\dagger_{lmin}$: $\forall A, B \in \mathscr{X}$,

$$A\bar{P}^\dagger_{lmin}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a^*_i P^\dagger b^*_i \wedge a^*_j I^\dagger b^*_j \; \forall j > i;$$

$$A\bar{I}^\dagger_{lmin}B \Leftrightarrow a^*_i I^\dagger b^*_i \; \forall i \in \{1, 2, ..., |X|\}.$$

Finally, we discuss one advantage in employing $R^\dagger$ and relaxing *null indifference*. Suppose that $\{a, b, c\}$ and $aPbPc$ for an individual. Furthermore, assume that the individual likes a and b intermediately, but hates c

enormously. We then consider $\{a, c\}$ and $\{b\}$. If R satisfies *null indifference*, $\{a, c\}^* = \{a, n_b, c\}$, $\{b\}^* = \{b, n_a, n_c\}$, and $\{a, c\} \bar{P}_{lmax} \{b\}$. However, from the setting, aPn_c is non–intuitive. If we allow that $n_c P^\dagger a$, then $\{a, c\}^* = \{a, n_b, c\}$, $\{b\}^* = \{n_c, b, n_a\}$, and we find that $\{b\} \bar{P}^\dagger_{lmax} \{a, c\}$. At first sight, $n_c P^\dagger a$ seems a strange statement; however, it simply indicates that the degree of the undesirability of choosing c is strictly larger than that of the desirability of choosing a. Thus, we can express the degree of desirability and adapt a part of the leximin (leximax) criteria in the leximax (leximin) extension rule by using $R^\dagger$. In this sense, an ordinal preference ranking can come close to an order of cardinal utilities by using $R^\dagger$.

4.4 Axioms

Bossert (1995) introduced two axioms with a fixed cardinality of subsets to characterise the class of lexicographic extension rules: *responsiveness*[11] and *neutrality*.[12] However, we characterise $\bar{R}^\dagger_{lmax}$ and $\bar{R}^\dagger_{lmin}$ separately. Thus, we employ another approach to characterise the leximax and leximin extension rules by the following three axioms.

First, *indifference dominance* requires that, for all $A, B \in \mathscr{X}$, $A \bar{I} B$ if all elements in A^* and B^* are indifferent for every rank.

Indifference dominance: $\forall A, B \in \mathscr{X}$, $[a^*_i I b^*_i \ \forall i \in \{1, 2, ..., |X|\}] \Rightarrow A \bar{I} B$.

The second (third) axiom is *prior* (*posterior*) *strict dominance*. This re-

[11] $\forall A \in \mathscr{X}_k = \{A \subseteq X \mid |A| = k\}, \forall b \in X, \forall c \in X \setminus A, bRc \Leftrightarrow A\bar{R}(A \setminus \{b\}) \cup \{c\}$.

[12] $\forall A, B \in \mathscr{X}_k, \forall \sigma : X \to X, [aRb \Leftrightarrow \sigma(a) R \sigma(b) \ \forall a \in A, \forall b \in B] \Rightarrow [A\bar{R}B \Leftrightarrow \{\sigma(a)\}_{a \in A} \bar{R} \{\sigma(b)\}_{b \in B}]$, where σ is a one–to–one mapping.

quires that $A\bar{P}B$ for all $A, B \in \mathscr{X}$ if each element of A^* dominates each element of B^* for every rank, and there is at least one strict preference ranking in certain prior (posterior) parts of them.

Prior strict dominance: $\forall A, B \in \mathscr{X}$, $[\exists j \in \{1, 2, ..., |X|\}$ s.t. $[a_i^* R b_i^*\ \forall i \in \{1, 2, ..., j\}] \wedge [\exists k \in \{1, 2, ..., j\}$ s.t. $a_k^* P b_k^*]] \Rightarrow A\bar{P}B$.

Posterior strict dominance: $\forall A, B \in \mathscr{X}$, $[\exists j \in \{1, 2, ..., |X|\}$ s.t. $[a_i^* R b_i^*\ \forall i \in \{j, j+1, ..., |X|\}] \wedge [\exists k \in \{j, j+1, ..., |X|\}$ s.t. $a_k^* P b_k^*]] \Rightarrow A\bar{P}B$

4.5 Characterisation

From Lemma 4.1, $\bar{R}^\dagger_{lmax}$ and $\bar{R}^\dagger_{lmin}$ are complete preorders over $\mathscr{X}$. A proof of Lemma 4.1 is shown in Appendix A.7.

Lemma 4.1. $\bar{R}^\dagger_{lmax}$ and $\bar{R}^\dagger_{lmin}$ satisfy *reflexivity*, *completeness*, and *transitivity*.

The major result is Theorem 4.1 that describes the necessary and sufficient conditions to derive $\bar{R}^\dagger_{lmax}$ and $\bar{R}^\dagger_{lmin}$. A proof of Theorem 4.1 is shown in Appendix A.8.

Theorem 4.1. $\bar{R} = \bar{R}^\dagger_{lmax}$ ($\bar{R} = \bar{R}^\dagger_{lmin}$) if and only if $R = R^\dagger$ and $\bar{R}$ satisfies *indifference dominance* and *prior* (*posterior*) *strict dominance*.

Theorem 4.1 shows the axiomatisation of $\bar{R}^\dagger_{lmax}$ and $\bar{R}^\dagger_{lmin}$. However, they seem predictable following *indifference dominance*, *prior strict dominance*, and *posterior strict dominance*. We need these critical axioms to characterise $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$ because the lexicographic extension rules have the restrictions such that we must begin to compare from the best or worst (null) alternatives and stop comparing alternatives if we find a strict preference ranking. Thus, it is crucial to discuss whether $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$ satisfy other axioms. Indeed, some axioms should be satisfied by them because we assume no compatibility within alternatives in this study. We thus introduce additional axioms.

First, *monotonicity of desirability* requires that the rank of each subset increases (decreases) by adding every (un)desirable alternative, but does not change by adding any neutral alternative. This is formulated for the relationship between alternatives and their null alternatives. This condition was introduced by Barberà et al. (1991).

Monotonicity of desirability: $\forall A \in \mathscr{X}, \forall a \in X \setminus A$, $aRn_a \Leftrightarrow A \cup \{a\}\bar{R}A$.

Next, *extended independence* requires that the preference ranking of any two subsets is not affected by adding another disjoint subset to both subsets.

Extended independence: $\forall A, B \in \mathscr{X}, \forall C \subseteq X \setminus (A \cup B)$, $A\bar{R}B \Leftrightarrow (A \cup C)\bar{R}(B \cup C)$.

Extended independence implies *independence*[13], *extended responsiveness*[14], and other weaker related axioms. *Extended monotonicity*[15] is also a weaker axiom than *extended independence*. However, This is formulated for

[13] $\forall a, b \in X, \forall C \subseteq X \setminus \{a, b\}$, $aRb \Leftrightarrow \{a\} \cup C\bar{R}\{b\} \cup C$.
[14] $\forall A \in \mathscr{X}$, $B \subseteq A$, $\forall C \subseteq X \setminus A$, $B\bar{R}C \Leftrightarrow A\bar{R}(A \setminus B) \cup C$.
[15] $\forall A \in \mathscr{X}, \forall B \subseteq X \setminus A$, $B\bar{R}\emptyset \Leftrightarrow A \cup B\bar{R}A$.

the relationship between subsets and an empty set, not their null subsets. Additionally, we cannot consider another *monotonicity*[16] that is weaker than *extended monotonicity* because R is defined over $X \cup N$, which does not include an empty set. Thus, we consider *monotonicity of desirability* and *extended independence* separately.

Finally, Theorem 4.2 shows that $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$ satisfy the above axioms. A proof of Theorem 4.2 is stated in Appendix A.9.

Theorem 4.2. $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$ satisfy *monotonicity of desirability* and *extended independence*.

4.6 Lexicographic extension rule

In the above sections, we define and characterise the leximax and leximin extension rules which can rank all subsets. However, the leximax (leximin) rule includes the leximin (leximax) criteria.

We then have the following question: when the leximax and leximin extension rules output the same preference ranking of subsets? We answer the question by introducing another property of R and discussing the signed order over $X \cup N$, which was introduced by Fishburn (1992).

4.6.1 Consistency of desirability

Even R satisfies *asymmetry of desirability*, $R^{\dagger}$ still have non–intuitive preference rankings of (null) alternatives. For example, suppose that

[16] $\forall a \in X, \forall B \subseteq X \setminus \{a\}$, $aR'\emptyset \Leftrightarrow \{a\} \cup B\bar{R}B$, where R' is a preference relation over $X \cup N \cup \{\emptyset\}$.

$aP^{\dagger}n_bP^{\dagger}n_aP^{\dagger}b$. This preference ranking satisfies *asymmetry of desirability*. However, aPn_b means that the degree of the desirability of a is greater than that of the undesirability of b. It should be imply that the degree of the desirability of b is greater than that of the undesirability of a, that is, bPn_a. Thus, aPn_bPbPn_a is more intuitive. In other words, if a is enormously (intermediately) desirable, n_a should be enormously (intermediately) undesirable, and if a is enormously (intermediately) undesirable, n_a should be enormously (intermediately) desirable for each $a \in X$.

We then introduce an additional property of R, called *consistency of desirability* as given below.

Consistency of desirability: $\forall a, b \in X$, $[aRn_b \Rightarrow bRn_a] \wedge [n_aRb \Rightarrow n_bRa]$.

Consistency of desirability and *asymmetry of desirability* are independent. For arbitrary two alternatives $a, b \in X$, suppose that an individual has the following preference ranking: $aPbPn_aPn_b$. This satisfies *consistency of desirability*, but violates *asymmetry of desirability*. Furthermore, if the individual has the following preference ranking: aPn_bPn_aPb, this ranking satisfies *asymmetry of desirability*, but violates *consistency of desirability*.

Proposition 4.5 shows that *self-reflecting* is stronger than *asymmetry of desirability* and *consistency of desirability*. This proposition is trivially true.

Proposition 4.5. If R satisfies *self-reflecting*, it also satisfies *asymmetry of desirability* and *consistency of desirability*.

We then introduce another property of R called *null symmetry* to clarify the preference ranking of (null) alternatives when R satisfies *asymmetry of* *desirability* and *consistency of desirability*. Let $X = \{a_1, ..., a_{|X|}\}$ such that $a_i R a_{i+1}$ for all $i \in \{1, ..., |X| - 1\}$, and $X \cup N = \{a'_1, ..., a'_{2|X|}\}$ such that $a'_i R a'_{i+1}$ for all $i \in \{1, ..., 2|X| - 1\}$. *Null symmetry* then requires that each alternative and its null alternative are located at the symmetric positions.

Null symmetry: $\forall a \in X$, $a = a'_i \Rightarrow n_a = a'_{2|X|+1-i}$.

Proposition 4.6 shows the relationship between *asymmetry of desirability*, *consistency of desirability*, and *null symmetry*. A proof of Proposition 4.6 is shown in Appendix A.10.

Proposition 4.6. R satisfies *null symmetry* if and only if R satisfies *asymmetry of desirability* and *consistency of desirability*.

We denote R satisfying *asymmetry of desirability* and *consistency of desirability* by $R^\ddagger$, and redefine $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ by using $R^\ddagger$ as follows:

Definition 4.5. *Leximax extension rule* $\bar{R}^\ddagger_{lmax}$: $\forall A, B \in \mathscr{X}$,

$$A\bar{P}^\ddagger_{lmax}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a^*_i P^\ddagger b^*_i \wedge a^*_j I^\ddagger b^*_j \; \forall j < i;$$

$$A\bar{I}^\ddagger_{lmax}B \Leftrightarrow a^*_i I^\ddagger b^*_i \; \forall i \in \{1, 2, ..., |X|\}.$$

Definition 4.6. *Leximin extension rule* $\bar{R}^\ddagger_{lmin}$: $\forall A, B \in \mathscr{X}$,

$$A\bar{P}^\ddagger_{lmin}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a^*_i P^\ddagger b^*_i \wedge a^*_j I^\ddagger b^*_j \; \forall j > i;$$

$$A\bar{I}^\ddagger_{lmin}B \Leftrightarrow a^*_i I^\ddagger b^*_i \; \forall i \in \{1, 2, ..., |X|\}.$$

However, Theorem 4.3 shows that $\bar{R}^{\ddagger}_{lmax}$ and $\bar{R}^{\ddagger}_{lmin}$ are not equivalent when $R^{\ddagger}$.

Theorem 4.3. $\bar{R}_{lmax} \neq \bar{R}_{lmin}$ even if $R = R^{\ddagger}$.

Consider the following example to prove Theorem 4.3: $X = \{a, b, c\}$ and $aP^{\ddagger}cI^{\ddagger}n_bP^{\ddagger}bI^{\ddagger}n_cI^{\ddagger}n_a$. This preference ranking of alternatives satisfies *null symmetry*. However, $\{a, b\}\bar{P}^{\ddagger}_{lmax}\{c\}$ and $\{c\}\bar{P}^{\ddagger}_{lmin}\{a, b\}$. Thus, we need a stronger property than *asymmetry of desirability* and *consistency of desirability* to make the leximax and leximin extension rules output the same preference ranking of all subsets.

4.6.2 Signed order

As stated in Section 4.2, Fishburn (1992) introduced *self-reflecting*, and it is stronger than *asymmetry of desirability* and *consistency of desirability* from Proposition 4.5. Then, R is the signed order if and only if R satisfies *self-reflecting*.

Self-reflecting: $\forall a, b \in X$, $[aRb \Leftrightarrow n_aRn_b] \wedge [aRn_b \Leftrightarrow bRn_a] \wedge [n_aRb \Leftrightarrow n_bRa]$.

As with $R^{\ddagger}$, we distinguish the preference ranking of alternatives when R satisfies *self-reflecting*. We introduce *perfect null symmetry*. Let $f_R : \{1, ..., 2|X|\}^2 \rightarrow \{P, I\}$ be a function such that $f_R(i, j)$ indicates the preference relation between a'_i and a'_j for all $i, j \in \{1, ..., 2|X|\}$ such that $i < j$. *Perfect null symmetry* requires that each alternative and its null alternative are located at symmetric positions, and preference relations are also symmetric.

Perfect null symmetry: $[\forall a \in X,\ a = a'_i \Rightarrow n_a = a'_{2|X|+1-i}] \wedge [\forall i \in \{1, ..., 2|X|-1\},\ f_R(i, i+1) = f_R(2|X|-i, 2|X|+1-i)]$.

Proposition 4.7 shows that *self-reflecting* and *perfect null symmetry* are equivalent. This proposition holds true from the proof of Proposition 4.6.

Proposition 4.7. R satisfies *perfect null symmetry* if and only if R satisfies *self-reflecting*.

We then denote R satisfying *self-reflecting* (the signed order) by R^*, and redefine $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ by using R^* as follows:

Definition 4.7. *Leximax extension rule* $\bar{R}^*_{lmax}$: $\forall A, B \in \mathscr{X}$,

$$A\bar{P}^*_{lmax}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a^*_i P^* b^*_i \wedge a^*_j I^* b^*_j\ \forall j < i;$$

$$A\bar{I}^*_{lmax}B \Leftrightarrow a^*_i I^* b^*_i\ \forall i \in \{1, 2, ..., |X|\}.$$

Definition 4.8. *Leximin extension rule* $\bar{R}^*_{lmin}$: $\forall A, B \in \mathscr{X}$,

$$A\bar{P}^*_{lmin}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a^*_i P^* b^*_i \wedge a^*_j I^* g b^*_j\ \forall j > i;$$

$$A\bar{I}^*_{lmin}B \Leftrightarrow a^*_i I^* b^*_i\ \forall i \in \{1, 2, ..., |X|\}.$$

Theorem 4.4 shows that $\bar{R}^*_{lmax}$ and $\bar{R}^*_{lmin}$ output the same preference ranking of all subsets. A proof of Theorem 4.4 is shown in Appendix A.11.

Theorem 4.4. $\bar{R}^*_{lmax} = \bar{R}^*_{lmin}$.

Let $\bar{R}^*_{lexi}$ denote $\bar{R}^*_{lmax}$ or $\bar{R}^*_{lmin}$ hereafter. From the definition of $\bar{R}^*_{lexi}$, $\bar{R}^*_{lexi}$ and the lexicographic extension rule introduced by Yunfeng et al. (1996)

are equivalent. Yunfeng et al. (1996) provided the characterisation of $\bar{R}^*_{lexi}$ by the following axioms.

(*M*++): $\forall A, B, C.D \in \mathscr{X}$ s.t. $(A \cup B) \cap (A \cup B) \cap (X^+ \cup X^-) = \emptyset \wedge C \cap (X^+ \cup X^-) = D \cap (X^+ \cup X^-)$, $A\bar{R}B \Leftrightarrow A \cup C\bar{R}B \cup D$.

(*K*+): $\forall A, B \in \mathscr{X} \setminus \{\emptyset\}$, $[aPb\ \forall a \in A^+,\ \forall b \in N_{A^-} \cup B] \Rightarrow A\bar{P}B$; $[[aPb\ \forall a \in A,\ \forall b \in B^-] \wedge [aPb\ \forall a \in N_{B^-},\ \forall b \in B^+]] \Rightarrow A\bar{P}B$.

(*M*++) is a similar condition of (*M*+) introduced by Fishburn (1992), and (*M*+) is completely the same as *extended independence*.[17] (*K*+), which was introduced by Heiner and Packard (1983), is an extended condition of (*K*)[18] introduced by Kelly (1977). Then, Yunfeng et al. (1996) characterised $\bar{R}^*_{lexi}$ by (*M*++) and (*K*+).

Theorem 4.5. $\bar{R} = \bar{R}^*_{lexi}$ if and only if $R = R^*$ and $\bar{R}$ satisfies (*M*++) and (*K*+).

Proof. See the proof of Theorem 2 in Yunfeng et al. (1996). □

Finally, we summarise the relationship among the properties of R and the leximax and leximin extension rules over the power set of X in Table 4.1.

From the above discussion, $R^\ddagger$ and R^* are reasonable preference relations for the alternatives. As stated earlier, *self-reflecting* implies *asymmetry of desirability* and *consistency of desirability*. Thus, if an individual finds that

[17] Yunfeng et al. (1996) said that (*M*++) is a generalised condition of *estended independence* because if $X^0 = \emptyset$, (*M*++) becomes (*M*+). However, this result simply indicates that (*M*++) and (*M*+) are equivalent when $X^0 = \emptyset$.

[18] $\forall A, B \in \mathscr{X} \setminus \{\emptyset\}$, $[aPb\ \forall a \in A,\ \forall b \in B] \Rightarrow A\bar{P}B$.

self-reflecting is too strong, $R^{\ddagger}$ is the most reasonable one for the individual. In this case, the individual has to have his/her priority order of the leximax and leximin criteria. However, if an individual finds that *self-reflecting* is intuitive, and *asymmetry of desirability* and *consistency of desirability* do not guarantee rational choices, R^{*} is the most reasonable one for the individual.

Table 4.1: The properties of R and leximax and leximin extension rules

	Properties	Extensibility	Leximax and Leximin
R	reflexivity, completeness, transitivity	×	$\bar{R}_{lmax} \neq \bar{R}_{lmin}$
$R^{\dagger}$	reflexivity, completeness, transitivity, asymmetry of desirability	✓	$\bar{R}^{\dagger}_{lmax} \neq \bar{R}^{\dagger}_{lmin}$
$R^{\ddagger}$	reflexivity, completeness, transitivity, asymmetry of desirability, consistency of desirability	✓	$\bar{R}^{\ddagger}_{lmax} \neq \bar{R}^{\ddagger}_{lmin}$
R^{*}	reflexivity, completeness, transitivity, self-reflecting	✓	$\bar{R}^{*}_{lmax} = \bar{R}^{*}_{lmin}$

* Asymmetry of desirability & consistency of desirability⇔null symmetry.
* Self-reflecting⇒asymmetry of desirability & consistency of desirability.
* Self-reflecting⇔perfect null symmetry.

Chapter 5

Epilogue

5.1 Summary

We summarise the results of each chapter in the following manner.

Chapter 2 improves the existing axiomatisations of the anti–plurality and basic best–worst rules. We introduce three new axioms, that is, *averseness*, *bottoms–only*, and *top–bottom cancellation*. In particular, *averseness* might be convenient to characterise other rules, because it is a weaker condition than *faithfulness*, which has been used in characterisations of several scoring social choice rules as a common-sensible assumption.

Chapter 3 introduces and characterises the net Borda rule by using *neutrality**, *faithfulness**, *reinforcement**, and *total cancellation**. Furthermore, we find that the net Borda rule outputs more intuitive social choice than the Borda rule by considering the example (two alternatives and five individuals).

Chapter 4 introduces the null alternatives allowed their heterogeneous positions, to rank all subsets. Additionally, we require *asymmetry of desirability* to R over $X \cup N$ to make the leximax and leximin extension rules satisfy *extensibility*. According to this framework, we define the leximax and leximin extension rules over $\mathscr{X}$, and axiomatise them by *indifference dominance*, *prior strict dominance*, and *posterior strict dominance*. We also find that the leximax and leximin extension rules satisfy *monotonicity of desirability* and *extended independence*. Furthermore, we find that the leximax and leximin extension rules are not equivalent if R satisfies *asymmetry of desirability* and *consistency of desirability*, but they are equivalent if R satisfies *self-reflecting*.

5.2 Remaining problems

In this section, we show the major remaining problem for each chapter.

In Chapter 2, we used *averseness* and *faithfulness* to characterise the anti-plurality and basic best-worst rules, respectively. However, if there exists only one individual, it is not a society intuitively. Thus, if we assume that $|V| > 1$, we cannot use the above two axioms. In this case, Baharad and Nitzan (2005)[1] and Theorem 2.3 can be the axiomatisation of anti-plurality and basic best-worst rules, respectively. Furthermore, there exists another approach. Yeh (2008) used *efficiency* to characterise the plurality rule.[2] If C satisfies *efficiency*, for each $a \in C(\mathcal{P})$,

$$\nexists b \in X \setminus \{a\} \text{ s.t. } bP_i a \; \forall i \in V.$$

[1] We can say that they assumed $|V| > 3$ in Baharad and Nitzan (2002), and reffered it in the proof of their theorems and lemmas in Baharad and Nitzan (2005).

[2] Sekiguchi (2012) replaced *efficiency* by *faithfulness* since *faithfulness* is weaker than *efficiency*. Additionally, Yeh (2008) assumed that $|V| \geq 1$ even if *efficiency* was used in his characterisation of the plurality rule.

Thus, *efficiency* is equivalent to *Pareto efficiency*. However, *faithfulness* and *reinforcement* imply *efficiency*, and *efficiency* implies *faithfulness*. Thus, even if we use *efficiency* to characterise the basic-best worst rules with the assumption, $|V| > 1$, it is not a novel approach. Furthermore, in the case of the anti-plurality rule, we should find an alternative axiom[3] since *efficiency* does not imply *averseness*.

Next, we still have to check whether the net Borda can be characterised by *neutrality**, *reinforcement**, *faithfulness**, *cancellation***, and *cyclic cancellation**, or there exists another social choice rule satisfying these five axioms. Additionally, it is possible that C^* satisfies *cancellation*** and *cyclic cancellation** if and only if C^* satisfies *total cancellation**. This is the remaining part of Chapter 3.

Finally, we need to discuss the necessary and sufficient conditions which make the leximax and leximin extension rules be equivalent. In Chapter 4, we show that if R satisfies *self-reflecting* ($R = R^*$), the leximax and leximin extension rules are equivalent, but do not prove that the opposite direction holds true.

5.3 Future research directions

We provide several applications of the studies in Chapters 3 and 4 to show the advantages in considering desirability and null alternatives (or the outside option).

[3]For instance, we can consider an axiom which requires that there is no $a \in C(\mathcal{P})$ such that bP_ia for all $i \in V$ and for all $b \in X \setminus \{a\}$.

First, we can extend the study in Chapter 3 to multi–winner voting. There are two extension rules of the Borda rule in the multi–winner voting system. The first is the k–Borda rule, which outputs k alternatives having higher Borda scores than remaining alternatives. Debord (1992) characterised the k–Borda rule based on Young's (1947) theorem. Thus, we can employ a linear order over the set of all alternatives and the outside option, and define and characterise the *net* k*–Borda rule*. The second extension is the Borda rule over the power set of the alternative set. If we must choose k alternatives, then, we consider the Borda rule over k–subsets. In the case of this extended Borda rule, we assign the Borda scores to all (k–)subsets, and choose the best subset having the highest Borda score.

Next, we can more generally apply the leximax and leximin extension rules over the power set with null alternatives to various fields of choice theories, such as theories of consumer behaviour with a budget constraint. Assume that the budget constraint and prices of alternatives are given. We then create the feasible power set, which is a subset of the power set. For every subset in the feasible power set, the sum of prices does not exceed the budget. Thereafter, we simply rank all subsets in the feasible power set by using leximax and leximin extension rules. This is one way of carrying out the discrete optimisation for consumer behaviours with the finite alternative set. This discrete optimisation is more intuitive than utility maximisation when we purchase books in an antiquarian bookshop (each book is one of kinds), for example, because we do not need to assign our cardinal utilities to all books or construct our utility functions accurately. If we are in a situation that we should consider the compatibility of alternatives, the discrete optimization will be more complicated. This is also an extension of the work in Chapter 4. By analysing these problems, we might be able to establish the theories of consumer behaviour based on Discrete Mathematics.

Appendix A

Proofs

A.1 Proof of Theorem 2.2

It is trivial that C^{ap} satisfies the five axioms. We then prove that the axioms are the sufficient conditions for deriving C^{ap}.

Assume that C satisfies the five axioms. By *bottoms–only*, C is determined from $\check{a}(P_i) \in X$. Thus, by *anonymity* and *neutrality*, $a \in C(\mathcal{P})$ if and only if $b \in C(\mathcal{P})$ when $n_{a|X|}(\mathcal{P}) = n_{b|X|}(\mathcal{P})$ for all $a.b \in X$, for all $\mathcal{P} \in \mathscr{P}^{|V|}$.

We consider the following three cases: (i) $| \cup_{i \in V} \{\check{a}(P_i)\}| = |V| = |X|$, (ii) $| \cup_{i \in V} \{\check{a}(P_i)\}| = |V| < |X|$, and (iii) otherwise ($| \cup_{i \in V} \{\check{a}(P_i)\}| < |V|$).

Case (i): $C(\mathcal{P}) = X$ because $n_{a|X|}(\mathcal{P}) = 1$ for all $a \in X$.

Case (ii): We prove that $C(\mathcal{P}) = X \setminus \cup_{i \in V}\{\check{a}(P_i)\}$. By way of contradiction, assume the existence of $a \in C(\mathcal{P})$ such that $a = \check{a}(P_i)$, $i \in V$. In this case, $n_{a|X|}(P_i) = 1$ and $n_{b|X|}(P_i) = 0$ for all $b \in X \setminus \{a\}$. Thus, $C(P_i)$ is equal to X, $X \setminus \{a\}$, or $\{a\}$. By *averseness*, we obtain that $C(P_i) = X \setminus \{a\}$.

Since $n_{b|X|}(\mathcal{P}_{-i})$ is equal to 1 or 0 for all $b \in X \setminus \{a\}$ and $a \neq \check{a}(P_j)$ for all $j \in V_{-i}$, $C(\mathcal{P}_{-i})$ is equal to X, $\cup_{j\in V_{-i}}\{\check{a}(P_j)\}$, or $X \setminus \cup_{j\in V_{-i}}\{\check{a}(P_j)\}$, where $V_{-i} = N \setminus \{i\}$ and $\mathcal{P}_{-i} = (P_j)_{j\in V_{-i}}$. By *reinforcement*, $C(\mathcal{P})$ is equal to $X \setminus \{a\}$, $\cup_{j\in V_{-i}}\{\check{a}(P_j)\}$, or $X \setminus \cup_{i\in V}\{\check{a}(P)\}$. However, this result contradicts the assumption that $a \in C(\mathcal{P})$. We then obtain that $C(\mathcal{P}) = X \setminus \cup_{i\in V}\{\check{a}(P_i)\}$.

Case (iii): Let $n = max_{a\in X} n_{a|X|}(\mathcal{P})$, $n > 1$. Suppose that $V = \{V_1, ..., V_n\}$ such that $n_{a|X|}((P_i)_{i\in V_k})$ is equal to 1 or 0 for all k, $1 \leq k \leq n$. From the previous cases, $C((P_i)_{i\in V_k}) = X$ if $|V_k| = |X|$, and $C((P_i)_{i\in V_k}) = X \setminus \cup_{i\in V_k}\{\check{a}(P_i)\}$ if $|V_k| < |X|$ for each k, $1 \leq k \leq n$. By *reinforcement*, $C(\mathcal{P}) = X \setminus \cup_{i\in V}\{\check{a}(P_i)\}$ if $|V_k| < |X|$ for all k, $1 \leq k \leq n$, and $C(\mathcal{P}) = X \setminus \cup_{i\in V\setminus V_l}\{\check{a}(P_i)\}$, where $|V_l| = |X|$ and $1 \leq l \leq n$.

From the above results, $C = C^{ap}$ if C satisfies *anonymity*, *neutrality*, *reinforcement*, *averseness*, and *bottoms–only*. □

A.2 Proof of Theorem 2.3

First, Lemma A.1 shows that *reinforcement* and *top–bottom cancellation* imply *anonymity*. A method of proof is similar to that of Young's (1974) proofs: *reinforcement* and *cancellation*[1] imply *anonymity*.

Lemma A.1. If C satisfies *reinforcement* and *top–bottom cancellation*, then C is determined based only on values of $n_{a1}(\mathcal{P}) - n_{a|X|}(\mathcal{P})$, $a \in X$.

Proof. Suppose that C satisfies *reinforcement* and *top–bottom cancellation*.

[1] Young (1974) characterised the Borda rule by *neutrality*, *reinforcement*, *faithfulness*, and *cancellation*. *Cancellation* requires that, for all $\mathcal{P} \in \mathscr{P}^{|V|}$, $n_{ab}(\mathcal{P}) = n_{ba}(\mathcal{P})$ for all $a, b \in X$ implies that $C(\mathcal{P}) = X$, where $n_{ab}(\mathcal{P}) = |\{i \in V \mid aP_ib\}|$ for all $a, b \in X$.

Let $\mathcal{P}_1 = (P_i)_{i \in V_1}$ and $\mathcal{P}_2 = (P_i)_{i \in V_2}$, where $V_1, V_2 \subseteq \mathbb{N}_+$, such that

$$n_{a1}(\mathcal{P}_1) - n_{a|X|}(\mathcal{P}_1) = n_{a1}(\mathcal{P}_2) - n_{a|X|}(\mathcal{P}_2)$$

for all $a \in X$, and for all $a \in X$,

$$q_a = n_{a1}(\mathcal{P}_1) - n_{a1}(\mathcal{P}_2).$$

Thus, $q_a = 0$ implies that $n_{a1}(\mathcal{P}_1) = n_{a1}(\mathcal{P}_2)$ and $n_{a|X|}(\mathcal{P}_1) = n_{a|X|}(\mathcal{P}_2)$. Additionally, suppose that

$$A_+ = \{a \in X \mid q_a > 0\},$$
$$A_0 = \{a \in X \mid q_a = 0\},$$
$$A_- = \{a \in X \mid q_a < 0\}.$$

Let $\mathcal{P}_3 = (P_i)_{i \in V_3}$ such that V_3 is disjoint from $V_1 \cup V_2$ for all $a \in X$. Furthermore, assume that for all $a \in A_+ \cup A_0$,

$$n_{a1}(\mathcal{P}_3) = n_{a|X|}(\mathcal{P}_1) \wedge n_{a|X|}(\mathcal{P}_3) = n_{a1}(\mathcal{P}_1),$$

and for all $a \in A_- \cup A_0$,

$$n_{a1}(\mathcal{P}_3) = n_{a|X|}(\mathcal{P}_2) \wedge n_{a|X|}(\mathcal{P}_3) = n_{a1}(\mathcal{P}_2).$$

From the above settings, for all $a \in A_+ \cup A_0$,

$$n_{a1}(\mathcal{P}_1 + \mathcal{P}_3) = n_{a|X|}(\mathcal{P}_3) + n_{a1}(\mathcal{P}_3),$$
$$n_{a|X|}(\mathcal{P}_1 + \mathcal{P}_3) = n_{a1}(\mathcal{P}_3) + n_{a|X|}(\mathcal{P}_3),$$

and for all $a \in A_- \cup A_0$,

$$n_{a1}(\mathcal{P}_1 + \mathcal{P}_3) = n_{a1}(\mathcal{P}_1) + n_{a|X|}(\mathcal{P}_2),$$
$$n_{a|X|}(\mathcal{P}_1 + \mathcal{P}_3) = n_{a|X|}(\mathcal{P}_1) + n_{a1}(\mathcal{P}_2).$$

Thus, for all $a \in X$, $n_{a1}(\mathcal{P}_1 + \mathcal{P}_3) = n_{a|X|}(\mathcal{P}_1 + \mathcal{P}_3)$. By *top–bottom cancellation*, we obtain that $C(\mathcal{P}_1 + \mathcal{P}_3) = X$. Similarly,

$$n_{a1}(\mathcal{P}_2 + \mathcal{P}_3) = n_{a1}(\mathcal{P}_2) + n_{a|X|}(\mathcal{P}_1),$$
$$n_{a|X|}(\mathcal{P}_2 + \mathcal{P}_3) = n_{a|X|}(\mathcal{P}_2) + n_{a1}(\mathcal{P}_1)$$

for all $a \in A_+ \cup A_0$, and

$$n_{a1}(\mathcal{P}_2 + \mathcal{P}_3) = n_{a|X|}(\mathcal{P}_3) + n_{a1}(\mathcal{P}_3),$$
$$n_{a|X|}(\mathcal{P}_1 + \mathcal{P}_3) = n_{a1}(\mathcal{P}_3) + n_{a|X|}(\mathcal{P}_3)$$

for all $a \in A_- \cup A_0$. Thus, for all $a \in X$, $n_{a1}(\mathcal{P}_2 + \mathcal{P}_3) = n_{a|X|}(\mathcal{P}_2 + \mathcal{P}_3)$. By *top–bottom cancellation*, we obtain that $C(\mathcal{P}_2 + \mathcal{P}_3) = X$.

Finally, from the above results and *reinforcement*,

$$C(\mathcal{P}_1) = C(\mathcal{P}_1) \cup X = C(\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3) = X \cup C(\mathcal{P}_2) = C(\mathcal{P}_2).$$

Thus, if C satisfies *reinforcement* and *top–bottom cancellation*, C depends only on $n_{a1}(\mathcal{P}) - n_{a|X|}(\mathcal{P})$, with $a \in X$ for all $\mathcal{P} \in \mathscr{P}^{|V|}$. □

As stated earlier, Garciá-Lapresta et al. (2010) proposed Lemma A.2 by using Young's (1975) theorem.

Lemma A.2. $C = C^{bw}$ if and only if C satisfies *anonymity*, *neutrality*, *continuity*, *reinforcement*, and *top–bottom cancellation*.

Proof. See the proof of Theorem 2 in Garciá-Lapresta et al. (2010). □

Finally, from Lemmas A.1 and A.2, we can remove *anonymity* from the necessary and sufficient conditions for deriving C^{bw} in Lemma A.2, and obtain Theorem 2.3. □

A.3 Proof of Theorem 2.4

It is trivial to show that C^{bw} satisfies the four axioms. Thus, we can prove that they are the sufficient conditions for deriving C^{bw}.

Assume that C satisfies the four axioms. From Lemma 1 and *neutrality*, for all $a, b \in X$ and all $\mathcal{P} \in \mathscr{P}^{|V|}$, $a \in C(\mathcal{P})$ if and only if $b \in C(\mathcal{P})$ when $n_{a1}(\mathcal{P}) - n_{a|X|}(\mathcal{P}) = n_{b1}(\mathcal{P}) - n_{b|X|}(\mathcal{P})$. Note that $\Sigma_{a\in X}(n_{a1}(\mathcal{P}) - n_{a|X|}(\mathcal{P})) = 0$.

Assume that $A_n(\mathcal{P}) = \{a \in X \mid n_{a1}(\mathcal{P}) - n_{a|X|}(\mathcal{P}) = n\}$. We then consider the following four cases:

(1) $X = A_0(\mathcal{P})$,

(2) $X = A_1(\mathcal{P}) \cup A_{-1}(\mathcal{P}) \cup A_{-2}(\mathcal{P}) \cup \cdots$ such that $A_1(\mathcal{P}) \neq \emptyset$,

(3) $X = A_1(\mathcal{P}) \cup A_0(\mathcal{P}) \cup A_{-1}(\mathcal{P}) \cup A_{-2}(\mathcal{P}) \cup \cdots$ such that $A_1(\mathcal{P}) \neq \emptyset$ and $A_0(\mathcal{P}) \neq \emptyset$, and

(4) otherwise.

Case (1): $C(\mathcal{P}) = X$ by *top–bottom cancellation*.

Case (2): $C(\mathcal{P}) = A_1(\mathcal{P})$ by *top–bottom non–negativity*.

Case (3): We show that $C(\mathcal{P}) = A_1(\mathcal{P})$. By *top–bottom non–negativity*, $A_{-m}(\mathcal{P}) \cap C(\mathcal{P}) = \emptyset$ for all $m \in \mathbb{N}_+$. By way of contradiction, assume the existence of $a \in C(\mathcal{P})$ such that $a \in A_0(\mathcal{P})$. Suppose that $a = \hat{a}(P_i)$ and $b = \check{a}(P_i)$, $i \notin V$. Thus, $C(\mathcal{P} + P) = \{a\}$ by *reinforcement*. We then consider the following three cases: (3-1) $b \in A_1(\mathcal{P})$, (3-2) $b \in A_0(\mathcal{P})$, and (3-3) $b \in A_{-1}(\mathcal{P})$.

Case (3-1): If $A_1(\mathcal{P}) \setminus \{b\} \neq \emptyset$, $c \in A_1(\mathcal{P})$, should also be included in $C(\mathcal{P} + P)$. Additionally, even if $A_1(\mathcal{P}) \setminus \{b\} = \emptyset$, there is no alternative in $C(\mathcal{P} + P) \cap A_0(\mathcal{P} + P)$. This result contradicts $a \in C(\mathcal{P})$.

Cases (3-2) and (3-3): Similarly, $c \in A_1(\mathcal{P})$ should also be included in $C(\mathcal{P} + P)$. This result contradicts $a \in C(\mathcal{P})$.

We thus have $C(\mathcal{P}) = A_1(\mathcal{P})$.

Case (4): Assume that $m = max_{a \in X} n_{a1}(\mathcal{P}) - n_{a|X|}(\mathcal{P})$ and $a^* \in A_m(\mathcal{P})$. Then, we can divide V into m subsets $V_1, ..., V_m$ such that $n_{a^*1}(\mathcal{P}) - n_{a^*|X|}(\mathcal{P}) = 1$ for all $a^* \in A_m(\mathcal{P})$. From the results of Cases (2) and

(3), $C(\mathcal{P}) = \cap_{n=1}^{m} A_1(\mathcal{P}_n)$, where $\mathcal{P}_n = (P_i)_{i \in V_n} \in \mathscr{P}^{|V_n|}$. This implies that $C(\mathcal{P}) = A_m(\mathcal{P})$.

From the results of Cases (1)–(4), $C = C^{bw}$ if and only if C satisfies *neutrality*, *reinforcement*, *top–bottom non–negativity*, and *top–bottom cancellation*. □

A.4 Proof of Theorem 3.2

It is trivial that C^*_{br} satisfies *neutrality**, *reinforcement**, *faithfulness**, and *total cancellation**. We then assume that C^* satisfies the four axioms, and prove two lemmas.

Lemma A.3. $B^*_a(\mathcal{P}^*_1) = B^*_a(\mathcal{P}^*_2)$ for all $a \in X$ implies that $C^*(\mathcal{P}^*_1) = C^*(\mathcal{P}^*_2)$, where $\mathcal{P}^*_1 \in (\mathscr{P}^*)^{|V_1|}$ and $\mathcal{P}^*_2 \in (\mathscr{P}^*)^{|V_2|}$ such that $V_1, V_2 \subset \mathbb{Z}_+$.

Proof. Let $\mathcal{P}^*_1 \in (\mathscr{P}^*)^{|V_1|}$ and $\mathcal{P}^*_2 \in (\mathscr{P}^*)^{|V_2|}$ such that $V_1, V_2 \subset \mathbb{Z}_+$. Additionally, assume that V_1 and V_2 are disjoint, and $B^*_a(\mathcal{P}^*_1) = B^*_a(\mathcal{P}^*_2)$ for all $a \in X \cup \{\emptyset\}$. Let $\mathcal{P}^*_3 \in (\mathscr{P}^*)^{|V_3|}$ such that V_3 and $V_1 \cup V_2$ are disjoint, and $B^*_a(\mathcal{P}^*_1 + \mathcal{P}^*_3) = B^*_b(\mathcal{P}^*_1 + \mathcal{P}^*_3)$ for all $a, b \in X$. From *reinforcement** and *total cancellation**, $C^*(\mathcal{P}^*_1 + \mathcal{P}^*_3) = C^*(\mathcal{P}^*_2 + \mathcal{P}^*_3) = X$ and

$$C^*(\mathcal{P}^*_1) = C^*(\mathcal{P}^*_1) \cap X = C^*(\mathcal{P}^*_1 + \mathcal{P}^*_2 + \mathcal{P}^*_3) = X \cap C^*(\mathcal{P}^*_2) = C^*(\mathcal{P}^*_2).$$

Furthermore, let $\mathcal{P}^*_4 \in (\mathscr{P}^*)^{|V_4|}$ such that V_4 is a clone of V_1, including different individuals who have the same preference profile as individuals in V_1. Thus, V_4 is assumed to be disjoint from $V_1 \cup V_2$. Then, if V_1 and V_2 are not disjoint, $C^*(\mathcal{P}_4^*) = C^*(\mathcal{P}_2^*)$ from the above result. Since $C^*(\mathcal{P}_1^*) = C^*(\mathcal{P}_4^*)$, $C^*(\mathcal{P}_1^*) = C^*(\mathcal{P}_2^*)$. □

Before proving the next lemma, we introduce additional notations.

First, $t\mathcal{P}^*$ denotes a preference profile of $t(\in \mathbb{Z}_+)$ clone sets of V from the notations in Section 3.2. We extend the population of t to the set of non–negative rational numbers $\mathbb{Q}_0$. Then, let $q\mathcal{P}^*$ be a preference profile of $q(\in \mathbb{Q}_0)$ clone sets of V, and $C^*(q\mathcal{P}^*) = C^*(\mathcal{P}^*)$ for all $q \in \mathbb{Q}_0 \setminus \{0\}$. Furthermore, assume that $C^*(q\mathcal{P}^*) = X$ if $q = 0$ for any $\mathcal{P}^* \in (\mathscr{P}^*)^{|V|}$.

Second, let $a^m_{\mathcal{P}^*}$ be an alternative which has the mth highest net Borda score $B^*_{a_m}(\mathcal{P}^*)$ with a given preference profile $\mathcal{P}^*$. Thus, $B^*_{a_1}(\mathcal{P}^*) \geq \cdots \geq B^*_{a_{|X|+1}}(\mathcal{P}^*)$. Then, suppose that $\boldsymbol{B}^*(\mathcal{P}^*) = (B^*_{a_1}(\mathcal{P}^*), \ldots, B^*_{a_{|X|+1}}(\mathcal{P}^*))$ be the *net Borda scoring vector*.

Third, suppose that $\mathcal{P}^*_{2,a^m_{\mathcal{P}^*}} = (P_1, P_2)$ such that $a^m_{\mathcal{P}^*}(\in X \cup \{\emptyset\})$ is the best alternative for both individuals, and their preference rankings of remaining alternatives are opposite. Thus, $C^*(\mathcal{P}^*_{2,a^m_{\mathcal{P}^*}}) = \{a^m_{\mathcal{P}^*}\}$ if $a^m_{\mathcal{P}^*} \in X$ and $C^*(\mathcal{P}^*_{2,a^m_{\mathcal{P}^*}}) = X$ if $a^m_{\mathcal{P}^*} = \emptyset$ from Lemma A.3, *faithfulness**, and *reinforcement**. We also have that

$$
\begin{aligned}
\boldsymbol{B}^*(\mathcal{P}^*_{2,a^1_{\mathcal{P}^*}}) &= (2|X|, -2, \ldots, -2), \\
\boldsymbol{B}^*(\mathcal{P}^*_{2,a^2_{\mathcal{P}^*}}) &= (-2, 2|X|, \ldots, -2), \\
&\vdots \\
\boldsymbol{B}^*(\mathcal{P}^*_{2,a^{|X|+1}_{\mathcal{P}^*}}) &= (-2, \ldots, -2, 2|X|).
\end{aligned}
$$

From the above setting, we can define $\boldsymbol{B}^*(\mathcal{P}^*)$ as a linear combination of

$\boldsymbol{B}^*(\mathcal{P}^*_{2,a^m_{\mathcal{P}*}})$'s as follows:

$$\boldsymbol{B}^*(\mathcal{P}^*) = \Sigma_{m=1}^{|X|+1} q_m \boldsymbol{B}^*(\mathcal{P}^*_{2,a^m_{\mathcal{P}*}}),$$

where $q_m \in \mathbb{Q}$ for all $m \in \{1, ..., |X|+1\}$ and $\mathbb{Q}$ is the set of rational numbers. Additionally, $B^*_{a_m}(\mathcal{P}^*) = 2q_m|X| - 2\Sigma_{l \neq m} q_l$. Thus,

$$B^*_{a_m}(\mathcal{P}^*) - B^*_{a_n}(\mathcal{P}^*) = 2|X|(q_m - q_n),$$

and since $m < n$ implies that $B^*_{a_m}(\mathcal{P}^*) - B^*_{a_n}(\mathcal{P}^*) \geq 0$,

$$m < n \Rightarrow q_m - q_n \geq 0.$$

Furthermore, we find that

$$B^*_{a_m}(\mathcal{P}^*) = B^*_{a_n}(\mathcal{P}^*) \Leftrightarrow q_m = q_n.$$

Since $\Sigma_{m=1}^{|X|+1} \boldsymbol{B}^*(\mathcal{P}^*_{2,a^m_{\mathcal{P}*}}) = \mathbf{0}$, we can redefine $\boldsymbol{B}^*(\mathcal{P}^*)$ in another way.

$$\begin{aligned}\boldsymbol{B}^*(\mathcal{P}^*) &= \Sigma_{m=1}^{|X|}[(q_m - q_{m+1})\Sigma_{l \leq m} \boldsymbol{B}(\mathcal{P}^*_{2,a^l_{\mathcal{P}*}})] \\ &= \boldsymbol{B}[\Sigma_{m=1}^{|X|}((q_m - q_{m+1})\Sigma_{l \leq m} \mathcal{P}^*_{2,a^l_{\mathcal{P}*}})].\end{aligned}$$

This holds true because $q_m - q_{m+1} \geq 0$ for all $m \in \{1, ..., |X|\}$.

By using these results, we prove Lemma A.4, which shows that $C^*(\mathcal{P}^*) = \{a^m_{\mathcal{P}^*} \in X \mid B^*_{a_m}(\mathcal{P}^*) = B^*_{a_1}(\mathcal{P}^*)\}$ if C^* depends on the net Borda scores.

Lemma A.4. C^* depends on the net Borda scores if and only if $C^* = C^*_{br}$.

Proof. The 'if' part is trivial. Thus, we prove the 'only if' part in the following manner.

First, we should prove that

$$C^*(\Sigma_{l\le m}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}}) = \{a^1_{\mathcal{P}*}, ..., a^m_{\mathcal{P}*}\} \setminus \{\emptyset\}.$$

If $m = |X|+1$, $C^*(\Sigma_{l\le m}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}}) = X$ because all Borda scores are the same. Then, we consider cases of $m < |X|+1$.

By way of contradiction, assume that there exists $n > m$ such that $a^n_{\mathcal{P}^*} \in C^*(\Sigma_{l\le m}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}})$. Since C^* depends only on the Borda scores and $B_{a^1_{\mathcal{P}^*}}(\Sigma_{l\le m}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}}) = \cdots = B_{a^m_{\mathcal{P}^*}}(\Sigma_{l\le m}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}}) > B_{a^{m+1}_{\mathcal{P}^*}}(\Sigma_{l\le m}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}}) = \cdots = B_{a^{|X|+1}_{\mathcal{P}^*}}(\Sigma_{l\le m}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}})$,

$$a^{m+1}_{\mathcal{P}^*} \in C^*(\Sigma_{l\le m}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}})$$

if $a^{m+1}_{\mathcal{P}^*} \neq \emptyset$. From *reinforcement*,

$$C^*(\Sigma_{l\le m+1}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}}) = C^*(\Sigma_{l\le m}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}}) \cap C^*(\mathcal{P}^*_{2,a^{m+1}_{\mathcal{P}^*}}) = \{a^{m+1}_{\mathcal{P}^*}\}.$$

However, $a^{m+1}_{\mathcal{P}^*} \in C^*(\Sigma_{l\le m+1}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}})$ must imply that $\{a^1_{\mathcal{P}*}, ..., a^m_{\mathcal{P}*}\} \subset C^*(\Sigma_{l\le m+1}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}})$: there is a contradiction. Similarly, if $a^{m+1}_{\mathcal{P}^*} = \emptyset$, we can obtain the same contradiction by considering $a^{m+2}_{\mathcal{P}^*}$ and $C^*(\Sigma_{l\le m+2}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}})$. Thus, $C^*(\Sigma_{l\le m}\mathcal{P}^*_{2,a^l_{\mathcal{P}*}}) = \{a^1_{\mathcal{P}*}, ..., a^m_{\mathcal{P}*}\} \setminus \{\emptyset\}$.

From the above result and *reinforcement*, we obtain that

$$
\begin{aligned}
C^*(\mathcal{P}^*) &= C[\Sigma_{m=1}^{|X|}((q_m - q_{m+1})\Sigma_{l\leq m}\mathcal{P}^*_{2,a^l_{\mathcal{P}^*}})] \\
&= X \cap_{m\leq |X|} \{a^1_{\mathcal{P}^*}, ..., a^m_{\mathcal{P}^*}\} \\
&= \{a^m_{\mathcal{P}^*} \in X \mid q_m = q_1\}.
\end{aligned}
$$

Thus, $C^*(\mathcal{P}^*) = \{a^m_{\mathcal{P}^*} \in X \mid B^*_{a_m}(\mathcal{P}^*) = B^*_{a_1}(\mathcal{P}^*)\}$ if all societies, which have the following preference profiles: $(q_m - q_{m+1})\Sigma_{l\leq m}\mathcal{P}^*_{2,a^l_{\mathcal{P}^*}}$, $m = 1, ..., |X|$, are disjoint. Furthermore, even if some of them are not disjoint, consider their disjoint clone societies, and by the same method of Lemma A.4, we obtain that $C^*(\mathcal{P}^*) = \{a^m_{\mathcal{P}^*} \in X \mid B^*_{a_m}(\mathcal{P}^*) = B^*_{a_1}(\mathcal{P}^*)\}$. □

From Lemmas A.3 and A.4, $C^* = C^*_{br}$ if C^* satisfies *neutrality**, *reinforcement**, *faithfulness**, and *total cancellation**. □

A.5 Proof of Proposition 4.3

Let R satisfy *asymmetry of desirability*. By way of contradiction, take any two alternatives $a, b \in X$ and assume that bRa when (i) aPn_a and n_bRb, or (ii) aIn_a and n_bPb. From *asymmetry of desirability*, bRa implies that n_aRn_b. We thus obtain aPb by *transitivity* in cases (i) and (ii), but that is a contradiction.

Next, assume that (iii) aPb or (iv) bPa when aIn_a and bIn_b. From *asymmetry of desirability*, aPb and bPa implies n_bRn_a and n_aRn_b, respectively. Thus, we obtain bRa in case (iii) and aRb in case (iv) by *transitivity*. These results are contradictions.

Thus, R satisfies *transitive desirability* if R satisfies *asymmetry of desirability*. □

A.6 Proof of Proposition 4.4

Assume that R satisfies *asymmetry of desirability*. First, we prove that $\{a\}\bar{R}_{lmax}\{b\}$ implies aRb for all $a, b \in X$. Take any two alternatives $a, b \in X$ such that $\{a\}\bar{R}_{lmax}\{b\}$. The difference between $\{a\}^*$ and $\{b\}^*$ is that $a, n_b \notin \{b\}^*$ and $n_a, b \notin \{a\}^*$. From Definition 4.1 and *asymmetry of desirability*, $\{a\}\bar{R}_{lmax}\{b\}$ if and only if

(i) $[aRn_b \wedge n_aRb] \Rightarrow [aPn_a \vee [aIn_a \wedge n_bRb]]$;

(ii) $[aRn_b \wedge bRn_a] \Rightarrow aRb$;

(iii) $[n_bRa \wedge n_aRb] \Rightarrow [n_bPn_a \vee [n_bIn_a \wedge aRb]]$; and

(iv) $[n_bRa \wedge bRn_a] \Rightarrow [n_bPb \vee [n_bIb \wedge aRn_a]]$.

In cases (i) and (ii), aRb holds true. In cases (iii) and (iv), by way of contradiction, suppose that bPa, implying n_aRn_b from *asymmetry of desirability*. However, the assumption contradicts n_bPn_a in both cases, and aRb in case (iii). Thus, aRb holds true in all four cases.

Next, we prove that aRb implies $\{a\}\bar{R}_{lmax}\{b\}$ for all $a, b \in X$. Take any two alternatives $a, b \in X$ such that aRb. From *asymmetry of desirability*, aRb implies n_bRn_a. We then obtain all four results, (i)–(iv); in other words, $\{a\}\bar{R}_{lmax}\{b\}$.

Thus, $\bar{R}_{lmax}$ satisfies *extensibility* if R satisfies *asymmetry of desirability*. Similarly, $\bar{R}_{lmin}$ satisfies *extensibility* if R satisfies *asymmetry of desirability*.

□

A.7 Proof of Lemma 4.1

From Definitions 4.3 and 4.4, $\bar{R}^{\dagger}_{lmax}$ satisfies *reflexivity* and *completeness*. Thus, we prove that $\bar{R}^{\dagger}_{lmax}$ satisfies *transitivity*.

For all $A, B, C \in \mathscr{X}$, $A\bar{P}^{\dagger}_{lmax}B$ and $B\bar{P}^{\dagger}_{lmax}C$ if and only if there exist $i, k \in \{1, 2, ..., |X|\}$ such that $a^*_i P^{\dagger} b^*_i$ and $a^*_j I^{\dagger} b^*_j$ for all $j < i$, and $b^*_k P^{\dagger} c^*_k$ and $b^*_l I^{\dagger} c^*_l$ for all $l < k$. Then, $k \leq i$ implies that $a^*_k P^{\dagger} c^*_k$ and $a^*_l I^{\dagger} c^*_l$ for all $l < k$, and $k > i$ implies that $a^*_i P^{\dagger} c^*_i$ and $a^*_j I^{\dagger} c^*_j$ for all $j < i$ from the *transitivity* of $R^{\dagger}$. Thus, if $A\bar{P}^{\dagger}_{lmax}B$ and $B\bar{P}^{\dagger}_{lmax}C$, then $A\bar{P}^{\dagger}_{lmax}C$ for all $A, B, C \in \mathscr{X}$.

Next, $A\bar{P}^{\dagger}_{lmax}B$ and $B\bar{I}^{\dagger}_{lmax}C$ if and only if there exists $i \in \{1, 2, ..., |X|\}$ such that $a^*_i P^{\dagger} b^*_i$ and $a^*_j I^{\dagger} b^*_j$ for all $j < i$, and $b^*_k I^{\dagger} c^*_k$ for all $k \in \{1, 2, ..., |X|\}$. Then, $a^*_i P^{\dagger} c^*_i$ and $a^*_j I^{\dagger} c^*_j$ for all $j < i$ from the *transitivity* of $R^{\dagger}$. Thus, if $A\bar{P}^{\dagger}_{lmax}B$ and $B\bar{I}^{\dagger}_{lmax}C$, $A\bar{P}^{\dagger}_{lmax}C$ for all $A, B, C \in \mathscr{X}$. Similarly, if $A\bar{I}^{\dagger}_{lmax}B$ and $B\bar{P}^{\dagger}_{lmax}C$, $A\bar{P}^{\dagger}_{lmax}C$ for all $A, B, C \in \mathscr{X}$.

Finally, $A\bar{I}^{\dagger}_{lmax}B$ and $B\bar{I}^{\dagger}_{lmax}C$ if and only if $a^*_i I^{\dagger} b^*_i I^{\dagger} c^*_i$ for all $i \in \{1, 2, ..., |X|\}$. Thus, if $A\bar{I}^{\dagger}_{lmax}B$ and $B\bar{I}^{\dagger}_{lmax}C$, then $A\bar{I}^{\dagger}_{lmax}C$ for all $A, B, C \in \mathscr{X}$.

From these results, $\bar{R}^{\dagger}_{lmax}$ satisfies *transitivity*. Similarly, $\bar{R}^{\dagger}_{lmin}$ satisfies *reflexivity*, *completeness*, and *transitivity*. □

A.8 Proof of Theorem 4.1

From Lemma 4.1 and Definition 4.3, $R = R^{\dagger}$, $\bar{R}^{\dagger}_{lmax}$ is a complete preorder over $\mathscr{X}$, and trivially satisfies *indifference dominance* and *prior strict dominance*.

We then prove that the following two propositions hold true if $R = R^\dagger$ and $\bar{R}$ satisfies *reflexivity*, *completeness*, *transitivity*, *indifference dominance*, and *prior strict dominance*:

(i) $A\bar{I}B \Leftrightarrow a_i^* I^\dagger b_i^* \; \forall i \in \{1, 2, ..., |X|\}$;

(ii) $A\bar{P}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\}$ s.t. $a_i^* P^\dagger b_i^* \wedge a_j^* I^\dagger b_j^* \; \forall j < i$.

The 'if' parts of (i) and (ii): They are trivial from $R = R^\dagger$, *indifference dominance*, and *prior strict dominance*.

The 'only if ' part of (i): By way of contradiction, let $A\bar{I}B$ imply the existence of some $i \in \{1, 2, ..., |X|\}$ such that $a_i^* P^\dagger b_i^*$ or $b_i^* P^\dagger a_i^*$. Suppose that i' is the argument of the minimum of i such that $a_i^* P^\dagger b_i^*$ or $b_i^* P^\dagger a_i^*$. From *prior strict dominance*, $A\bar{P}B$ if $a_{i'}^* P^\dagger b_{i'}^*$ and $B\bar{P}A$ if $b_{i'}^* P^\dagger a_{i'}^*$. These contradict $A\bar{I}B$ because $\bar{R}$ is a complete preorder over $\mathscr{X}$.

The 'only if' part of (ii): By way of contradiction, let $A\bar{P}B$ imply that $a_i^* I^\dagger b_i^*$ for all $i \in \{1, 2, ..., |X|\}$ or there exists $i \in \{1, 2, ..., |X|\}$ such that $b_i^* P^\dagger a_i^*$ and $a_j^* I^\dagger b_j^*$ for all $j < i$. Each case respectively implies that $A\bar{I}B$ or $B\bar{P}A$ from the 'if' parts of (i) and (ii). These contradict $A\bar{P}B$ since $\bar{R}$ is a complete preorder over $\mathscr{X}$.

Thus, $\bar{R} = \bar{R}^\dagger_{lmax}$ if and only if $R = R^\dagger$ and $\bar{R}$ is a complete preorder satisfying *indifference dominance* and *prior strict dominance*.

Similarly, $\bar{R} = \bar{R}^\dagger_{lmin}$ if and only if $R = R^\dagger$ and $\bar{R}$ is a complete preorder satisfying *indifference dominance* and *posterior strict dominance*. □

A.9 Proof of Theorem 4.2

First, we prove that $\bar{R}^{\dagger}_{lmax}$ satisfies *monotonicity of desirability*. Suppose that $a_i^* = n_a \in A^*$ and $B = A \cup \{a\}$. Then, $aR^{\dagger}n_a$ if and only if $a = b_j^*$, where $1 \leq j \leq i$. In this case, $b_k^* = a_k^*$ for all $k \in \{1, 2, ..., j-1, i+1, ..., |X|\}$ and $b_{l+1}^* = a_l^*$ for all $l \in \{j, 2, ..., i\}$. Thus, we find that $aR^{\dagger}n_a$ if and only if $aR^{\dagger}a_j^*$, which implies $B\bar{R}^{\dagger}_{lmax}A$.

Second, we prove that $\bar{R}^{\dagger}_{lmax}$ satisfies *extended independence*. Take any three subsets: $A, B \in \mathscr{X}$, and $C \subseteq X \setminus (A \cup B)$. From Definition 4.3, $A\bar{P}^{\dagger}_{lmax}B$ if and only if $a_i^* P^{\dagger} b_i^*$, $i \in \{1, 2, ..., |X|\}$, and $a_j^* I^{\dagger} b_j^*$ for all $j < i$. Suppose that there are k alternatives in $\tilde{C} = \{\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_k\} \subseteq C$ such that $a_i^* P^{\dagger} \tilde{c}_1 R^{\dagger} \tilde{c}_2 R^{\dagger} \cdots R^{\dagger} \tilde{c}_k P^{\dagger} b_i^*$. Then, $k = 0$ implies that $A \cup C\bar{P}^{\dagger}_{lmax}B \cup C$ because $a_i^* P^{\dagger} b_i^*$ and the *transitivity* of $R^{\dagger}$. Furthermore, $k \geq 1$ also implies that $A \cup C\bar{P}^{\dagger}_{lmax}B \cup C$ because $a_i^* P^{\dagger} \tilde{c}_1$ and the *transitivity* of $R^{\dagger}$. From these results, we obtain that $A\bar{P}^{\dagger}_{lmax}B$ if and only if $A \cup C\bar{P}^{\dagger}_{lmax}B \cup C$. Next, from Definition 4.3, $A\bar{I}^{\dagger}_{lmax}B$ if and only if $a_i^* I^{\dagger} b_i^*$ for all $i \in \{1, 2, ..., |X|\}$. In this case, the positions of $c_k \in C$ in $(A \cup C)^*$ and $(B \cup C)^*$ have to be the same for all $k \in \{1, 2, ..., |C|\}$. Finally, we obtain that $A\bar{I}^{\dagger}_{lmax}B$ if and only if $A \cup C\bar{I}^{\dagger}_{lmax}B \cup C$.

Similarly, $\bar{R}^{\dagger}_{lmin}$ satisfies *monotonicity of desirability* and *extended independence*. □

A.10 Proof of Proposition 4.6

First, it is trivial that *null symmetry* implies *asymmetry of desirability* and *consistency of desirability*. Thus, we prove that *asymmetry of desirability* and *consistency of desirability* imply *null symmetry*.

Assume that R satisfies *asymmetry of desirability* and *consistency of desirability*. Additionally, suppose that $X^+ = \{a \in X \mid aPn_a\}$, $X^0 = \{a \in X \mid aIn_a\}$ and $X^- = \{a \in X \mid n_aPa\}$. Furthermore, assume that $|X^+ \cup X^0| = m \leq |X|$ without loss of generality. Thus, $X^+ \cup X^0 = \{a_1, ..., a_m\}$ and $X^- = \{a_{m+1}, ..., a_{|X|}\}$. By *asymmetry of desirability*, $(X^+ \cup X^0)^* = X^+ \cup X^0 \cup N_{X^-}$ and $(X^-)^* = X^- \cup N_{X^+ \cup X^0}$. Additionally, $n_{a_{i+1}} R n_{a_i}$ for all $i \in \{1, ..., |X| - 1\}$.

Fix the ranking of all elements in $(X^+ \cup X^0)^*$ as follows: $(X^+ \cup X^0)^* = \{a'_1, ..., a'_{|X|}\}$. By *consistency of desirability* and *transitivity*, the position of each element in $(X^-)^*$ is determined. For example, if $a_m R n_{a_{m+1}}$ and $n_{a_m} R a_{m+1}$, $a'_{|X|} = n_{a_{m+1}}$, and $a'_{|X|+1} = a_m$. If $a'_{|X|-1} I a'_{|X|}$, $a'_{|X|-1} = n_{a_{m+1}}$ is also possible. However, it is important that $a'_{|X|} = n_{a_{m+1}}$ can hold true. The same applies to other (null) alternatives. Thus, the ranking of all elements in $(X^+ \cup X^0)^*$ and $(X^-)^*$ must be decided to make it satisfy *null symmetry*.

Similarly, if we fix the ranking of all elements in $(X^-)^*$, the position of each element in $(X^+ \cup X^0)^*$ is determined as with the above case. □

A.11 Proof of Theorem 4.4

Assume that $R = R^*$ and consider the ranking of all subsets according to $\bar{R}^*_{lmax}$. The best and worst subsets are $X^+ \cup A^0$ and $X^- \cup A^0$, respectively, where A is an arbitrary subset of X and $A^0 = A \cap X^0$.

Then, the transformed subsets are reported from the best subset A^*_1 to the worst subset $A^*_{|\mathscr{X}|}$ in the following manner. Note that $A^*_i \bar{R}^*_{lmax} A^*_{i+1}$ for all $i \in \{1, ..., |\mathscr{X}| - 1\}$, and without loss of generality, suppose that $A^*_1 = (X^+ \cup X^0)^*$. Additionally, a'_i and $a'_{2|X|+1-i}$ cannot be included in the same transformed subset for all $i \in \{1, ..., |X|\}$. Thus, A^*_1 is indifferent to the following subsets until all elements in X^0 are replaced by the elements in N_{X^0}.

We then obtain the following preference ranking of all subsets:

$$
\begin{aligned}
A^*_1 &= \{a'_1, ..., a'_{|X|-3}, a'_{|X|-2}, a'_{|X|-1}, a'_{|X|}\},\\
A^*_2 &= \{a'_1, ..., a'_{|X|-3}, a'_{|X|-2}, a'_{|X|-1}, a'_{|X|+1}\},\\
A^*_3 &= \{a'_1, ..., a'_{|X|-3}, a'_{|X|-2}, a'_{|X|}, a'_{|X|+2}\},\\
A^*_4 &= \{a'_1, ..., a'_{|X|-3}, a'_{|X|-2}, a'_{|X|+1}, a'_{|X|+2}\},\\
A^*_5 &= \{a'_1, ..., a'_{|X|-3}, a'_{|X|-1}, a'_{|X|}, a'_{|X|+3}\},\\
&\vdots\\
A^*_{|\mathscr{X}|} &= \{a'_{|X|+1}, ..., a'_{|2X|-3}, a'_{2|X|-2}, a'_{2|X|-1}, a'_{2|X|}\},
\end{aligned}
$$

From the above ranking of subsets, $A^*_i \bar{R}^*_{lmin} A^*_{i+1}$ also holds true for all $i \in \{1, ..., |\mathscr{X}| - 1\}$. We thus find that $\bar{R}_{lmax} = \bar{R}_{lmin}$ if $R = R^*$, in other words, $\bar{R}^*_{lmax} = \bar{R}^*_{lmin}$. $\square$

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