Two geometries arising from Poisson geometry and their applications

Poisson幾何学に起因する2つの幾何学とその応用

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Contents

Chapter 1

Introduction

The origin of Poisson geometry is the Poisson bracket which was introduced as an important operator in Hamiltonian mechanics on smooth manifolds. Recently, however studies of properties and dynamical systems of Poisson structures (Poisson brackets) themselves on smooth manifolds are done actively and Poisson geometry is one of important fields in Geometry. In addition, geometrical objects having properties analogous to Poisson structures, for example, twisted Poisson [37], Jacobi [17], Nambu-Poisson structures [40] and so on are also studied actively. In this thesis, we deal with quasi-Poisson and pseudo-Poisson-Nijenhuis structures, which are such geometrical objects. The former structures generalize Poisson structures with Hamiltonian-Poisson actions, i.e., Poisson actions with moment maps. We show that a symplectic structure on a smooth manifold M is naturally deformable to another symplectic structure on M via the quasi-Poisson theory [31]. The latter structures were introduced in [32] and extend the notion of Poisson-Nijenhuis structures, defined by Magri and Morosi [25], in terms of the relationship with Lie algebroids.

1.1 Deformations of symplectic structures by moment maps

In the context of symplectic geometry, deformation-equivalence assumptions and conditions are often appeared, for example, in the statement of Moser's theorem [28], Donaldson's four-six conjecture [35] and so on. However, it seems that a method of constructing deformation-equivalent symplectic structures specifically is not well known. In this thesis, we construct a method of producing new symplectic structures deformation-equivalent to a given symplectic structure with a Hamiltonian action. Our approach to deformations of symplectic structures is to use quasi-Poisson theory which was introduced by Alekseev and Kosmann-Schwarzbach [1], and this approach is carried out by using the fact that a moment map for a symplectic-Hamiltonian action σ is also a moment map for a quasi-Poisson action σ. The former moment map satisfies conditions for only one symplectic structure, whereas the latter does conditions for a family of quasi-Poisson structures parametrized by elements in $\Lambda^2\mathfrak{g}$. From here we call these elements *twists*. We regard a symplectic structure as a quasi-Poisson strucutre with twist 0, which is denoted by π_0. Then we can find different quasi-Poisson structures π_t which induce symplectic structures ω^t by the choice of "good" twists t. The quasi-Poisson structure inducing a symplectic structure must be a non-degenerate Poisson structure. We describe the conditions for the quasi-Poisson structure with a twist t to be a non-degenerate Poisson structure. Our method of using the family of quasi-Poisson structures is one of interesting geometry frameworks [1].

From here, we explain briefly the difference among moment maps for symplectic, Poisson and quasi-Poisson actions on a smooth manifold. We will explain these theories in detail in Section 2.1, 2.2 and 2.3.

(I) Symplectic-Hamiltonian actions

In symplectic geometry, a moment map $\mu : M \to \mathfrak{g}^*$ for a symplectic action σ of a Lie group G on a symplectic manifold (M, ω) is defined with two conditions: one is for the symplectic structure ω,

$$d\mu^X = \iota_{X_\sigma}\omega \ (X \in \mathfrak{g}). \tag{1.1}$$

Here $\mu^X(p) := \langle \mu(p), X \rangle$ and X_σ is a vector field on M defined by

$$X_{\sigma,p} := \frac{d}{dt}\sigma_{\exp tX}(p)\bigg|_{t=0} \tag{1.2}$$

for p in M. The other is the G-equivariance condition with respect to the action σ on M and the coadjoint action Ad^* on $\mathfrak{g}^*$,

$$\mu \circ \sigma_g = \mathrm{Ad}_g^* \circ \mu \tag{1.3}$$

for all g in G. In this thesis, we call symplectic actions with moment maps *symplectic-Hamiltonian actions* to distinguish it from other actions with moment maps.

(II) Poisson-Hamiltonian actions

A Poisson Lie group, which was introduced by Drinfel'd [8], is a Lie group with a Poisson structure π compatible with the group structure. Namely, the structure π satisfies

$$\pi_{gh} = \mathrm{L}_{g*}\pi_h + \mathrm{R}_{h*}\pi_g \tag{1.4}$$

for any g and h in G, where L_g and R_h are the left and right translations in G by g and h, respectively. Such a structure is called *multiplicative*. Then the simply connected Lie group G^* called the dual Poisson Lie group is obtained uniquely from a Poisson-Lie group (G, π) and a local action λ of G on G^* is defined naturally. We call a multiplicative Poisson structure π on G *complete* if the action λ is global. Then (G, π) is called a *complete Poisson-Lie group*. A moment map $\mu : M \to G^*$ for a Poisson action σ of a Poisson Lie group (G, π) on a Poisson manifold (M, π_M) is defined with a condition

$$X_\sigma = -\pi_M^\sharp(\mu^*(X^{\mathrm{R}})) \tag{1.5}$$

for any X in $\mathfrak{g}$, where X^{R} is the right-invariant 1-form on G^* with value X at e. In this thesis, we call Poisson actions with moment maps *Poisson-Hamiltonian actions*. If (G, π) is complete, we can also consider the G-equivariance of a moment map with respect to σ and λ. An equivariant moment map for a Poisson action of a complete Poisson Lie group on a Poisson manifold is a generalization of a moment map for a symplectic-Hamiltonian action on a symplectic manifold, which was given by Lu in [18].

(III) Quasi-Poisson-Hamiltonian actions

Quasi-Poisson theory, which was originated with [1] by Alekseev and Kosmann-Schwarzbach, is a generalization of Poisson theory with Poison actions. More specifically, the theory gives an unified view for various moment map theories [3], [18], [21], [28]. In quasi-Poisson geometry, quasi-triples $(D, G, \mathfrak{h})$ and its infinitesimal version, Manin quasi-triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, play important roles. A quasi-triple $(D, G, \mathfrak{h})$ defines a quasi-Poisson Lie group $G_D^{\mathfrak{h}}$ and we can obtain the notion of a quasi-Poisson action of such a quasi-Poisson-Lie group $G_D^{\mathfrak{h}}$. A moment map μ for the action is a map from M into the quotient D/G and satisfies a condition not for one quasi-Poisson structure but for a family of quasi-Poisson structures parametrized by elements in $\Lambda^2\mathfrak{g}$. In this thesis, we call quasi-Poisson actions with moment maps *quasi-Poisson-Hamiltonian actions*. An equivariant moment map for a Poisson action in (II) is an example of a moment map for a quasi-Poisson action if the Lie group is connected and simply connected. We use the moment map theory for quasi-Poisson actions to deform symplectic structures on a smooth manifold.

Now we state the first our main theorem in this thesis.

Theorem 1.1.1. Let (M, ω) be a symplectic manifold on which a connected Lie group G with the Lie algebra $\mathfrak{g}$ acts by a symplectic-Hamiltonian action σ, $\mu : M \to \mathfrak{g}^*$ a moment map for σ and π the Poisson structure induced by ω. Then the following holds:

1. If a twist t in $\Lambda^2\mathfrak{g}$ satisfies that $[t, t]_M = 0$, then t deforms the Poisson structure π to a Poisson structure $\pi^t_M := \pi - t_M$. Moreover, if t is an r-matrix, then σ is a Poisson action of (G, π^t_G) on (M, π^t_M), where $\pi^t_G = t^{\mathrm{L}} - t^{\mathrm{R}}$.

2. For a twist t in $\Lambda^2\mathfrak{g}$, if the isotropic complement $\mathfrak{g}^*_t$ is admissible on $\mu(M)$, then t deforms the non-degenerate 2-vector field π to a non-degenerate 2-vector field π^t_M. This condition is equivalent to that the matrix $A_t(\xi)$ defined by (3.7) is regular for any ξ in $\mu(M)$.

Therefore, if a twist t satisfies the assumptions of both 1 and 2, then t deforms ω to a symplectic structure ω^t induced by the non-degenerate Poisson structure π^t_M. In other words, ω and ω^t are deformation-equivalent.

Theorem 1.1.2. Let (M, ω) be a symplectic manifold on which an n-dimensional connected Lie group G acts by a symplectic-Hamiltonian action σ. Assume that X, Y in $\mathfrak{g}$ satisfy $[X, Y] = 0$. Then the twist $t = \frac{1}{2}X \wedge Y$ in $\Lambda^2\mathfrak{g}$ deforms the symplectic structure ω to a symplectic structure ω_t. For example, a twist t in $\Lambda^2\mathfrak{h}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, satisfies the assumption of the theorem.

In Section 3.2, we give examples of deformations of symplectic structures on $\mathbb{R}^{2n}, \mathbb{C}\mathrm{P}^1$ and the complex Grassmannian $\mathrm{Gr}_{\mathbb{C}}(n, r)$. In Section 3.3, we study deformations on symplectic toric manifolds. Under certain assumption, we show that our deformations give canonical transformations on a symplectic toric manifold.

1.2 Pseudo-Poisson-Nijenhuis manifolds

Poisson-Nijenhuis structures were defined by Magri and Morosi [25] to study bi-Hamiltonian systems. A pair of a Poisson structure π and a Nijenhuis structure N on a C^∞-manifold M is said to be a *Poisson-Nijenhuis structure* on M if π and N have a compatibility condition, i.e., they satisfy

$$N \circ \pi^\sharp = \pi^\sharp \circ N^*, \tag{1.6}$$

and the $(2, 1)$-tensor C^N_π given by

$$C^N_\pi(\alpha, \beta) := [\alpha, \beta]_{N\pi^\sharp} - [\alpha, \beta]^{N^*}_\pi \tag{1.7}$$

for any α and β in $\Omega^1(M)$ vanishes. It is known that Poisson-Nijenhuis manifolds (i.e., manifolds with Poisson-Nijenhuis structures) are related with various mathematical objects [15], [16], [25].

Kosmann-Schwarzbach [15] showed that there is a one-to-one correspondence between the Poisson-Nijenhuis manifolds (M, π, N) and the Lie bialgebroids $((TM)_N, (T^*M)_\pi)$, where $(TM)_N$ is a Lie algebroid deformed by the Nijenhuis structure N and $(T^*M)_\pi$ is the cotangent bundle equipped with the standard Lie algebroid structure induced by the Poisson structure π. On the other hand, Stiénon and Xu [38] introduced the concept of a Poisson-quasi-Nijenhuis manifold (M, π, N, ϕ), and showed that a Poisson-quasi-Nijenhuis manifold corresponds to a quasi-Lie bialgebroid $((T^*M)_\pi, d_N, \phi)$. Here a Lie bialgebroid [29], [30] consists of a pair (A, A^*), where A is a Lie algebroid, and A^* is the dual bundle equipped with a Lie algebroid structure, together with the following condition: for any D_1 and D_2 in $\Gamma(\Lambda^* A)$,

$$d_{A^*}[D_1, D_2]_A = [d_{A^*}D_1, D_2]_A + (-1)^{\deg D_1 + 1}[D_1, d_{A^*}D_2]_A, \tag{1.8}$$

where a bracket $[\cdot, \cdot]_A$ is the Schouten bracket of the Lie bracket of A, and d_{A^*} is the Lie algebroid differential determined from the Lie algebroid structure of A^* [22]. Since the Lie algebroid structure on A^* can be recovered from the derivation d_{A^*}, a Lie bialgebroid (A, A^*) is also denoted by (A, d_{A^*}). A quasi-Lie bialgebroid [12] is a Lie algebroid $(A, [\cdot, \cdot]_A, a)$ equipped with a degree-one derivation δ of the Gerstenhaber algebra $(\Gamma(\Lambda^* A), \wedge, [\cdot, \cdot]_A)$, i.e., δ satisfies (1.8), and a 3-section of A, ϕ_A in $\Gamma(\Lambda^3 A)$ such that $\delta^2 = [\phi_A, \cdot]_A$ and $\delta\phi_A = 0$.

Some of our main purposes in Chapter 4 in this thesis are to define a pseudo-Poisson-Nijenhuis manifold (M, π, N, Φ) and to show that there is a one-to-one correspondence between the pseudo-Poisson-Nijenhuis manifolds (M, π, N, Φ) and the quasi-Lie bialgebroids $((TM)_N, d_\pi, \Phi)$. A quasi-Lie bialgebroid $((TM)_N, d_\pi, \Phi)$ is, so to speak, "the opposite side" of a quasi-Lie bialgebroid $((T^*M)_\pi, d_N, \phi)$. Here d_N and d_π are operators on $\Omega^*(M) := \Gamma(\Lambda^* T^*M)$ and $\mathfrak{X}^*(M) := \Gamma(\Lambda^* TM)$ determined from a 2-vector field π and a $(1,1)$-tensor N, respectively.

Definition 1. Let M be a C^∞-manifold, π a 2-vector field on M, a $(1,1)$-tensor N a Nijenhuis structure on M compatible with π, and Φ a 3-vector field on M. Then a triple (π, N, Φ) is a *pseudo-Poisson Nijenhuis structure* on M if the following holds:

$$\begin{aligned}
&\text{(i)} && [\pi,\Phi]=0,\\
&\text{(ii)} && \frac{1}{2}\iota_{\alpha\wedge\beta}[\pi,\pi]=N\iota_{\alpha\wedge\beta}\Phi,\\
&\text{(iii)} && N\iota_{\alpha\wedge\beta}\mathcal{L}_X\Phi-\iota_{\alpha\wedge\beta}\mathcal{L}_{NX}\Phi-\iota_{(\mathcal{L}_XN^*)(\alpha\wedge\beta)}\Phi=0
\end{aligned}$$

for any X in $\mathfrak{X}(M)$, α and β in $\Omega^1(M)$, where $\iota_{\alpha\wedge\beta}:=\iota_\beta\iota_\alpha$ and $(\mathcal{L}_XN^*)(\alpha\wedge\beta):=(\mathcal{L}_XN^*)\alpha\wedge\beta+\alpha\wedge(\mathcal{L}_XN^*)\beta$.

Furthermore, since quasi-Lie bialgebroids (of course, Lie bialgebroids also) construct Courant algebroids [22], [34], we can obtain a new Courant algebroid structure on $TM\oplus T^*M$ from a pseudo-Poisson-Nijenhuis structure on M. Therefore a pseudo-Poisson-Nijenhuis structure on M complements the bottom left of the correspondence table below:

a Courant algebroid structure [22] on $TM\oplus T^*M$		
a quasi-Lie bialgebroid [34] $((TM)_N, d_\pi, \Phi)$	a Lie bialgebroid [29] $((TM)_N, (T^*M)_\pi)$	a quasi-Lie bialgebroid [34] $((T^*M)_\pi, d_N, \phi)$
a pseudo-Poisson Nijenhuis (π, N, Φ) π : a 2-vector field N : Nijenhuis Φ : a 3-vector field	a Poisson Nijenhuis [25] (π, N) π : Poisson N : Nijenhuis	a Poisson quasi-Nijenhuis [38] (π, N, ϕ) π : Poisson N : a $(1,1)$-tensor ϕ : a 3-form

All of the pairs (π, N) of the bottom of the correspondence table above are compatible. The condition that a 2-vector field π and a $(1,1)$-tensor N on M are compatible is very important in studying Poisson-Nijenhuis, pseudo-Poisson-Nijenhuis and Poisson-quasi-Nijenhuis manifolds. In Section 4.1, we prove several properties related to the compatibility under minimum assumptions, for example, Poisson-Nijenhuis hierarchy [16], [26] and a relation with a brackets on the tangent and the cotangent bundle [15], [38] and so on.

In Section 4.3, under the assumption that a 2-vector field π is nondegenerate, we show that we can reduce one of the conditions for a triple (π, N, Φ) to be a pseudo-Poisson-Nijenhuis structure. In this case, since there is a unique nondegenerate 2-form ω corresponding to π, we can rewrite the definition of pseudo-Poisson-Nijenhuis structures by words of the differential forms.

Definition 2. Let M be a C^∞-manifold, ω a nondegenerate 2-form on M, a $(1,1)$-tensor N a Nijenhuis structure on M compatible with π corresponding to ω, and ϕ a closed 3-form on M. Then a triple (ω, N, ϕ) is a *pseudo-symplectic-Nijenhuis structure* on M if

$$\iota_{X\wedge Y} d\omega = N^* \iota_{X\wedge Y} \phi \qquad (X, Y \in \mathfrak{X}(M)).$$

Moreover we show that the above triple (ω, N, ϕ) induces a twisted Poisson structure (π_N, ϕ) [37]. Twisted Poisson structures arose from the study of topological sigma models by Park [33]. It is known that twisted Poisson structures on M engender certain quasi-Lie bialgebroid structures on T^*M [34].

Chapter 2

Preliminaries

In this chapter, as preliminalies of the main theorems in Section 3 and Section 4, we recall symplectic, Poisson, quasi-Poisson geometries and Lie algebroid theory.

2.1 Symplectic geometry

In this section, we recall symplectic geometry and moment map theory for a symplectic action [28], [36]. Let M be a C^∞-manifold.

Definition 3. Let ω be a 2-form on M. Then the 2-form ω is a *symplectic structure* or *symplectic form* on M if ω is closed and ω_p is nondegenerate on T_pM for all $p \in M$. A pair (M, ω) is a *symplectic manifold* if ω is a symplectic structure on M.

The nondegeneracy of a symplectic structure ω on M means that $\dim M (= \dim T_pM)$ must be even and that $\omega^n := \omega \wedge \overset{(n)}{\cdots} \wedge \omega$ is a volume form on M. The form $\frac{\omega^n}{n!}$ is called the *symplectic volume form* or *Liouville volume form* of (M, ω) and the integral

$$\int_M \frac{\omega^n}{n!} \tag{2.1}$$

is called the *symplectic volume* of (M, ω), denoted by $\mathrm{Vol}(M, \omega)$.

Example 1. We consider $M = \mathbb{R}^{2n}$ with linear coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$. The 2-form

$$\omega_0 := \sum_{k=1}^{n} dx_k \wedge dy_k \tag{2.2}$$

is a symplectic structure on $\mathbb{R}^{2n}$.

Example 2. We consider $M = \mathbb{C}^n$ with linear coordinates $(z_1, \ldots, z_n)$. The 2-form

$$\omega_0 := \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k \tag{2.3}$$

is a symplectic structure on $\mathbb{C}^n$. Under the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}, z_k = x_k + iy_k$, this structure coincides with the structure (2.2) in Example 1.

Example 3. On the $2n$-torus $\mathbb{T}^{2n}$ with angle coordinates $(\theta_1, \ldots, \theta_{2n})$, we consider the 2-form

$$\omega := \sum_{i=1}^{n} d\theta_{2i-1} \wedge d\theta_{2i}.$$

Then ω is a symplectic structure on $\mathbb{T}^{2n}$.

Example 4. The complex projective space $\mathbb{C}\mathrm{P}^n$ has the standard coordinate neighborhood system $\{(U_i, \varphi_i)\}_i$ consisting of $n+1$ open sets U_i given by

$$\begin{aligned} U_i &:= \{[z_1 : \cdots : z_{n+1}] \in \mathbb{C}\mathrm{P}^n | z_i \neq 0\}, \\ \varphi_i &: U_i \to \mathbb{C}^n \cong \mathbb{R}^{2n}, \\ [z_1 : \cdots : z_{n+1}] &\mapsto \left(\frac{z_1}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_{n+1}}{z_i} \right) \\ &\mapsto \left(\mathrm{Re}\frac{z_1}{z_i}, \mathrm{Im}\frac{z_1}{z_i}, \ldots, \mathrm{Re}\frac{z_{n+1}}{z_i}, \mathrm{Im}\frac{z_{n+1}}{z_i} \right), \end{aligned}$$

for $i = 1, \ldots, n+1$. By using this coordinate system, the 2-form ω_{FS} on $\mathbb{C}\mathrm{P}^n$ defined by setting

$$\varphi_j^* \left(\frac{i}{2} \partial\bar{\partial} \log \left(\sum_k |z_k|^2 + 1 \right) \right)$$

on each U_j is a symplectic structure on $\mathbb{C}\mathrm{P}^n$. The 2-form ω_{FS} is called the *Fubini-Study form.*

In the case of $n = 1$, The complex projective line $\mathbb{C}\mathrm{P}^1$ has the standard coordinate neighborhood system $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$. The Fubini-Study form ω_{FS} on $\mathbb{C}\mathrm{P}^1$ is

$$\omega_{\mathrm{FS}} = \frac{dx_1 \wedge dy_1}{(x_1^2 + y_1^2 + 1)^2}$$

on U_1, where $(x_1, y_1) := \left(\mathrm{Re}\frac{z_2}{z_1}, \mathrm{Im}\frac{z_2}{z_1} \right)$.

Definition 4. Let $(M_i, \omega_i), i = 1, 2$, be $2n$-dimensional symplectic manifolds and $\varphi : M_1 \to M_2$ a diffeomorphism. Then φ is a *symplectomorphism* if $\varphi^*\omega_2 = \omega_1$. We denote the symplectomorphisms of a symplectic manifold (M, ω) by $\mathrm{Sympl}(M, \omega)$, i.e.,

$$\mathrm{Sympl}(M, \omega) := \{f : M \xrightarrow{\cong} M | f^*\omega = \omega\}.$$

Let (M, ω) be symplectic manifold and $f : M \to \mathbb{R}$ a smooth function. For the 1-form df, there exists a unique vector field X_f in $\mathfrak{X}(M)$ such that $\iota_{X_f}\omega = df$ by nondegeneracy of ω.

Definition 5. A vector field X_f on (M, ω) as above is called the *Hamiltonian vector field* with a function f.

If X in $\mathfrak{X}(M)$ is Hamiltonian with a function f, we obtain

$$\mathcal{L}_X\omega = d\iota_X\omega + \iota_X d\omega = d^2 f + 0 = 0. \tag{2.4}$$

Therefore Hamiltonian vector fields on (M, ω) preserve ω.

Definition 6. A vector field X on (M, ω) preserving ω, i.e., satisfying $\mathcal{L}_X\omega = 0$, is called a *symplectic vector field*.

By Definition 5 and the calculation (2.4), we can summarize the following:

$$\begin{cases} X \text{ in } \mathfrak{X}(M) \text{ is symplectic} \Leftrightarrow \iota_X\omega \text{ is closed} \\ X \text{ in } \mathfrak{X}(M) \text{ is Hamiltonian} \Leftrightarrow \iota_X\omega \text{ is exact.} \end{cases} \tag{2.5}$$

Definition 7. Let G be a Lie group. An *action* of G or a G-*action* on M is a group homomorphism

$$\psi : G \to \mathrm{Diff}(M),\ g \mapsto \psi_g.$$

The *evaluation map* associated with an action $\psi : G \to \mathrm{Diff}(M)$ is

$$G \times M \to M,\ (g, p) \mapsto \psi_g(p).$$

The action ψ is *smooth* if the evaluation map is smooth.

Example 5. Let X in $\mathfrak{X}(M)$ be a complete vector field. Then

$$\psi : \mathbb{R} \to \mathrm{Diff}(M),\ t \mapsto \psi_t := \mathrm{Exp} tX$$

is a smooth action of $\mathbb{R}$ on M.

In this thesis, we simply call a smooth action as an action. In addition, we identify an action with the evaluation map associated with it and denote the evaluation map associated with an action by the same symbol as the action.

We introduce some types of actions of Lie groups.

Definition 8. Let $\sigma : G \times M \to M$ be an action of a Lie group G on a manifold M. Then,

(i) σ is *transitive* if for any x and y in M, there exists g in G such that $\sigma_g(x) = y$.

(ii) σ is *effective* if for any two distinct g and h in G, there exists x in M such that $\sigma_g(x) \neq \sigma_h(x)$.

Next, we define symplectic and symplectic-Hamiltonian actions, which are actions on symplectic manifolds.

Definition 9. Let (M, ω) be a symplectic manifold and G a Lie group. An action σ of G on M is *symplectic* if

$$\sigma : G \to \mathrm{Sympl}(M, \omega) \subset \mathrm{Diff}(M).$$

Definition 10. Let (M, ω) be a symplectic manifold, G a Lie group, $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$ and σ a symplectic action of G on M. Then the action σ is a *symplectic-Hamiltonian action* if there exists a map

$$\mu : M \to \mathfrak{g}^*$$

satisfying the followings:

1. For each X in $\mathfrak{g}$,
$$d\mu^X = \iota_{X_\sigma}\omega. \tag{2.6}$$
Here $\mu^X(p) := \langle \mu(p), X \rangle$ and X_σ is the *fundamental vector field* of X for σ on M defined by
$$X_{\sigma,p} := \frac{d}{dt}\sigma_{\exp tX}(p)\bigg|_{t=0} \tag{2.7}$$
for p in M.

2. the G-equivariancy with respect to the action σ on M and the coadjoint action Ad^* on $\mathfrak{g}^*$,
$$\mu \circ \sigma_g = \mathrm{Ad}_g^* \circ \mu \tag{2.8}$$
for all g in G.

Then the quadruple (M, ω, G, μ) is called a *Hamiltonian G-space* and μ is called a *moment map*.

In the special case of $G \cong \mathbb{R}$ (resp. S^1), since $\mathfrak{g} \cong \mathfrak{g}^* = \mathbb{R}$, a moment map $\mu : M \to \mathbb{R}$ for an action σ on (M, ω) satisfies the following:

1. For the generator $X = 1$ of $\mathfrak{g} \cong \mathbb{R}$, we obtain $\mu^X(p) = \mu(p) \cdot 1$ for any p in M, i.e., $\mu^X = \mu$. The fundamental vector field X_σ is just the vector field generated by the 1-parameter group of transformation $\{\sigma_t\}_{t \in \mathbb{R}}$. Hence the condition (2.6) is
$$d\mu = \iota_{X_\sigma} \omega;$$
2. Since the coadjoint action Ad_g^* for any g in $G = \mathbb{R}$ (or S^1) is equal to the identity map, the condition (2.8) is
$$\sigma_g^* \mu = \mu \tag{2.9}$$
for all g in $G = \mathbb{R}$ (or S^1), i.e., $\mathcal{L}_{X_\sigma} \mu = 0$.

Therefore σ is a symplectic-Hamiltonian action of $\mathbb{R}$ or S^1 on (M, ω) with a moment map μ if and only if the vector field generated by the 1-parameter group of transformation $\{\sigma_t\}_{t \in \mathbb{R}}$ is Hamiltonian with a function μ.

Example 6. We consider $\mathbb{R}^{2n}$ with $\omega_0 := \sum_{k=1}^n dx_k \wedge dy_k$ (Example 1) and set $X := -\frac{\partial}{\partial y_1}$. Let $\{\sigma_t\}_{t \in \mathbb{R}}$ be the 1-parameter group of transformation generated by X. Since $X = X_{x_1}$ is Hamiltonian with the linear coordinate function x_1, $\mathbb{R}$-action σ is symplectic-Hamiltonian action on $\mathbb{R}^{2n}$.

Example 7. We consider $\mathbb{R}^{2n}$ with $\omega_0 := \sum_{k=1}^n dx_k \wedge dy_k$ (Example 1). An additive group $\mathbb{R}^n$ acts on $\mathbb{R}^{2n}$ by the parallel transformation:
$$a \cdot (x, y) := (x + a, y) \ (x, y, a \in \mathbb{R}^n).$$
This action is symplectic. Then the infinitesimal action $a_{\mathbb{R}^{2n}}$ of $a = {}^{\mathrm{t}}(a_1, a_2, a_3)$ in $\mathbb{R}^n = \mathrm{Lie}(\mathbb{R}^n)$ is
$$a_{\mathbb{R}^{2n}} = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathbb{R}^{2n}).$$

Moreover, a map

$$\mu : \mathbb{R}^{2n} \to \mathbb{R}^n, \; (x, y) \mapsto y$$

is a moment map for this action.

Example 8. We consider the case that $\mathrm{SU}(n+1)$ acts on $(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}})$ given by

$$g \cdot [z_1 : \cdots : z_{n+1}] := \left[\sum_{j=1}^{n+1} a_{1j} z_j : \cdots : \sum_{j=1}^{n+1} a_{n+1,j} z_j \right],$$

for any $[z_1 : \cdots : z_{n+1}]$ in $\mathbb{C}\mathrm{P}^n$ and $g = (a_{ij})$ in $\mathrm{SU}(n+1)$.

The isotropic subgroup of $[1 : 0 : \cdots : 0]$ is

$$\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)) := \left\{ \begin{pmatrix} e^{i\theta} & O \\ O & B \end{pmatrix} \in \mathrm{SU}(n+1) \,\middle|\, \theta \in \mathbb{R}, B \in \mathrm{U}(n) \right\}.$$

Therefore it follows

$$\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)) \cong \mathbb{C}\mathrm{P}^n.$$

The action of $\mathrm{SU}(n+1)$ on $(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}})$ is a symplectic-Hamiltonian action and its moment map μ is defined by

$$\langle \mu([z_1 : \cdots : z_{n+1}]), X \rangle := \frac{1}{2} \mathrm{Im} \frac{\langle {}^t(z_1, \ldots, z_{n+1}), X\, {}^t(z_1, \ldots, z_{n+1}) \rangle}{\langle {}^t(z_1, \ldots, z_{n+1}), {}^t(z_1, \ldots, z_{n+1}) \rangle}$$

for any $[z_1 : \cdots : z_{n+1}]$ in $\mathbb{C}\mathrm{P}^n$ and X in $\mathfrak{su}(n+1)$.

We consider the case of $n = 1$. The matrices

$$e_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; e_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; e_3 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

form a basis of $\mathfrak{su}(2)$. Let $\{\varepsilon^i\}$ be the dual basis of $\mathfrak{su}(2)^*$. We obtain

$$\mu(x_1, y_1) \;=\; \frac{y_1}{1 + x_1^2 + y_1^2} \varepsilon^1 + \frac{x_1}{1 + x_1^2 + y_1^2} \varepsilon^2 + \frac{1 - x_1^2 - y_1^2}{2(1 + x_1^2 + y_1^2)} \varepsilon^3$$

on the standard neighborhood U_1. Hence $\mu(\mathbb{C}\mathrm{P}^1) \subset \mathfrak{su}(2)^* \cong \mathbb{R}^3$ is the 2-sphere with center at the origin and with radius $\frac{1}{2}$.

Example 9 (Coadjoint orbits)**.** Let G be a compact Lie group, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. We consider the adjoint representation Ad and the coadjoint representation Ad^* of G on $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively. For any ξ in $\mathfrak{g}^*$, the set

$$\mathcal{O}_\xi := \{\mathrm{Ad}_g^*\xi \in \mathfrak{g}^* |\ g \in G\}$$

is a submanifold of $\mathfrak{g}^*$ called a *coadjoint orbit* through ξ. Since the restriction to $\mathcal{O}_\xi$ of Ad^* is transitive, we obtain

$$\mathcal{O}_\xi \cong G/G_\xi,$$

where

$$G_\xi := \{g \in G |\ \mathrm{Ad}_g^*\xi = \xi\}$$

is *stabilizer* of ξ. Let $X_\mathfrak{g}$ and $X_{\mathfrak{g}^*}$ be the fundamental vector fields of X in $\mathfrak{g}$ for the adjoint and coadjoint representations of G, respectively. We show that

$$\begin{aligned} X_{\mathfrak{g},Y} &= [X,Y] = \mathrm{ad}_X Y, \\ \langle X_{\mathfrak{g}^*,\xi}, Y\rangle &= \langle \xi, -\mathrm{ad}_X Y\rangle = \langle \mathrm{ad}_X^*\xi, Y\rangle. \end{aligned}$$

for any Y in $\mathfrak{g}$ and ξ in $\mathfrak{g}^*$. Hence we obtain $X_{\mathfrak{g}^*,\xi} = \mathrm{ad}_X^*\xi$ in $T_\xi\mathfrak{g}^* \cong \mathfrak{g}^*$. We define for any ξ in $\mathfrak{g}^*$, a skew-symmetric bilinear form $\tilde{\omega}_\xi$ on $\mathfrak{g}$ by

$$\tilde{\omega}_\xi(X,Y) := \langle \xi, [X,Y]\rangle.$$

Then it follows that the kernel of $\tilde{\omega}_\xi$ is the Lie algebra $\mathfrak{g}_\xi$ of the stabilizer of ξ for the coadjoint representation by the fact that

$$\begin{aligned} \mathfrak{g}_\xi &= \{X \in \mathfrak{g} |\ \mathrm{ad}_X^*\xi = 0\} \\ &= \{X \in \mathfrak{g} |\ \langle \xi, [X,Y]\rangle = 0\ ({}^\forall Y \in \mathfrak{g})\}. \end{aligned}$$

We show that ω_ξ defines a nondegenerate 2-form on $T_\xi\mathcal{O}_\xi$. In fact, since the map

$$\mathfrak{g} \to T_\xi\mathcal{O}_\xi,\ X \mapsto X_{\mathfrak{g}^*,\xi}$$

is surjective and the kernel of this map is just $\mathfrak{g}_\xi$, we obtain $T_\xi\mathcal{O}_\xi \cong \mathfrak{g}/\mathfrak{g}_\xi$. Since for any vector v in $T_\xi\mathcal{O}_\xi$, there exists an element X in $\mathfrak{g}$ such that $v = X_{\mathfrak{g}^*,\xi}$, we define a skew-symmetric bilinear form ω_ξ on $T_\xi\mathcal{O}_\xi$ as

$$\omega_\xi(X_{\mathfrak{g}^*,\xi}, Y_{\mathfrak{g}^*,\xi}) := \langle \xi, [X,Y]\rangle (= \tilde{\omega}_\xi(X,Y)).$$

This definition is well-defined since $\ker \tilde{\omega}_\xi = \mathfrak{g}_\xi$ and $X_{\mathfrak{g}^*,\xi} = \mathrm{ad}_X^*\xi = 0$ for any X in $\mathfrak{g}_\xi$, i.e., this definition does not depend on the choice of a representative. The nondegeneracy of ω_ξ follows from the fact that $\ker \tilde{\omega}_\xi = \mathfrak{g}_\xi$.

For any g in G, we set

$$\omega_{\mathrm{Ad}_g^*\xi}(X_{\mathfrak{g}^*,\mathrm{Ad}_g^*\xi}, Y_{\mathfrak{g}^*,\mathrm{Ad}_g^*\xi}) := \langle \mathrm{Ad}_g^*\xi, [X, Y]\rangle.$$

Then it follows that the 2-form ω on $\mathcal{O}_\xi$ with value ω_η at each point η in $\mathcal{O}_\xi$ is a symplectic structure on $\mathcal{O}_\xi$. It is obvious that ω is G-invariant. The 2-form ω on $\mathcal{O}_\xi$ is called *canonical*, the *Lie-Poisson* or the *Kirillov-Kostant form.* Then the restriction to $\mathcal{O}_\xi$ of the coadjoint representation is a symplectic-Hamiltonian G-action on $(\mathcal{O}_\xi, \omega)$ with a moment map

$$\iota : \mathcal{O}_\xi \hookrightarrow \mathfrak{g}^*, \ \xi \mapsto \xi.$$

In the case of $G = \mathrm{SU}(n+1)$, it is well known that the Grassmannian manifold $\mathrm{Gr}_{\mathbb{C}}(n, r)$ is included in $\mathfrak{su}(n+1)^*$ as a coadjoint orbit and that the Kirillov-Kostant form on the projective space $\mathbb{C}\mathrm{P}^n = \mathrm{Gr}_{\mathbb{C}}(n, 1)$ coincide with the Fubini-Study form ω_{FS} on $\mathbb{C}\mathrm{P}^n = \mathrm{Gr}_{\mathbb{C}}(n, 1)$.

We define a symplectic toric manifold, which is an example of symplectic manifolds with symplectic Hamiltonian torus-actions.

Definition 11. A *symplectic toric manifold* is a $2n$-dimensional compact connected symplectic manifold (M, ω) on which the n-dimensional torus $\mathbb{T}^n$ acts effectively as a symplectic-Hamiltonian action with a moment map $\mu : M \to \mathbb{R}^n (= \mathrm{Lie}(\mathbb{T}^n))$.

In general, there exists no classification of symplectic manifolds. However there exists a classification of symplectic toric manifolds, which is well known as the Delzant theorem. The Delzant theorem is one of the applications of the moment theory. To describe the Delzant theorem, we define Delzant polytopes.

Definition 12. A polytope Δ in $\mathbb{R}^n$ is a *Delzant polytope* if Δ is convex and satisfies the followings:

(i) it is *simple*, i.e., there exist n edges meeting at each vertex;

(ii) it is *rational*, i.e., the edges meeting at the vertex p are rational in the sense that each edge is of the form $p + tu_i$ $(t \geqq 0)$, where u_i in $\mathbb{Z}^n$;

(iii) it is *smooth*, i.e., for each vertex, the corresponding $u_1, \ldots, u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Theorem 2.1.1 (the Delzant Theorem [7]). Symplectic toric manifolds are classified by Delzant polytopes. More precisely, there exists the following one-to-one correspondence:

$$\begin{array}{rcl} \{\text{symplectic toric manifolds}\} & \xrightarrow{\text{1-1}} & \{\text{Delzant polytopes}\} \\ (M, \omega, \mathbb{T}^n, \mu) & \mapsto & \mu(M). \end{array}$$

2.2 Poisson geometry

In this section, we review Poisson geometry [27], [43]. Moreover we review Poisson-Lie group theory, Poisson action and its moment map theory [18], [19], [21].

2.2.1 Poisson manifolds

We begin with the definition of a Poisson bracket.

Definition 13. A *Poisson bracket* on the C^∞-functions $C^\infty(M)$ on M is the bilinear operator $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying the following:

(i) $(C^\infty(M), \{\cdot,\cdot\})$ is a Lie algebra;

(ii) $\{\cdot,\cdot\}$ is a derivation in each factor, that is, for any f, g and h in $C^\infty(M)$,

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

A pair $(M, \{\cdot,\cdot\})$ is called a *Poisson manifold*.

Example 10. Any manifold M has the *trivial Poisson structure*, which is defined by

$$\{f, g\} := 0$$

for any f and g in $C^\infty(M)$.

Example 11. Let (M, ω) be a symplectic manifold. Then we can define a Poisson bracket $\{\cdot,\cdot\}_\omega$ on M by

$$\{f, g\}_\omega := \omega(X_f, X_g)$$

for any f and g in $C^\infty(M)$, where X_f and X_g in $\mathfrak{X}(M)$ are the Hamiltonian vector fields for ω with functions f and g, respectively. Therefore any symplectic manifold has the Poisson bracket induced by the symplectic structure.

Example 12. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Then $\mathfrak{g}^*$ is a Poisson manifold with the *Lie-Poisson bracket* defined by, for any f, g in $C^\infty(M)$ and ξ in $\mathfrak{g}^*$,

$$\{f, g\}(\xi) := \langle \xi, [df_\xi, dg_\xi] \rangle,$$

where we regard the differentials $df_\xi : T_\xi\mathfrak{g}^* \cong \mathfrak{g}^* \to T_{f(\xi)}\mathbb{R} \cong \mathbb{R}$ and $dg_\xi : T_\xi\mathfrak{g}^* \cong \mathfrak{g}^* \to T_{g(\xi)}\mathbb{R} \cong \mathbb{R}$ as elements in $(\mathfrak{g}^*)^* \cong \mathfrak{g}$. A pair $(\mathfrak{g}, \{\cdot, \cdot\})$ is called a *Lie-Poisson space.*

Example 13. Let $(M_i, \{\cdot, \cdot\}_i)$, $i = 1, 2$, be Poisson manifolds. Then a Poisson bracket $\{\cdot, \cdot\}$ on the product manifold $M_1 \times M_2$ is given by

$$\{f, g\}(x, y) := \{f(x, \cdot), g(x, \cdot)\}_2(y) + \{f(\cdot, y), g(\cdot, y)\}_1(x)$$

for any f, g in $C^\infty(M_1 \times M_2)$, x in M_1 and y in M_2. Here $f(x, \cdot),\ g(x, \cdot) : M_2 \to \mathbb{R}$ are in $C^\infty(M_2)$ and $f(\cdot, y),\ g(\cdot, y) : M_1 \to \mathbb{R}$ are in $C^\infty(M_2)$. Hence $(M_1 \times M_2, \{\cdot, \cdot\})$ is a Poisson manifold.

Proposition 2.2.1. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold. Then for f in $C^\infty(M)$, there exists a unique vector field X_f on M such that

$$X_f g = \{g, f\}$$

for any g in $C^\infty(M)$. The vector field X_f is called the *Hamiltonian vector field* of f.

Obviously, a Hamiltonian vector field for a Poisson bracket $\{\cdot, \cdot\}_\omega$ induced by a symplectic structure ω on M coincides with a Hamiltonian vector field for the symplectic structure ω on M.

Definition 14. Let $(M_i, \{\cdot, \cdot\}_i)$, $i = 1, 2$, be Poisson manifolds. Then a map $\varphi : M_1 \to M_2$ is *Poisson map* between $(M_1, \{\cdot, \cdot\}_1)$ and $(M_2, \{\cdot, \cdot\}_2)$ if

$$\varphi^*\{f, g\}_2 = \{\varphi^* f, \varphi^* g\}_1$$

for any f and g in $C^\infty(M_2)$.

If (M_i, ω_i), $i = 1, 2$ are symplectic manifolds, then a map $\varphi : M_1 \to M_2$ is symplectomorphism if and only if φ is Poisson with respect to Poisson brackets $\{\cdot, \cdot\}_{\omega_1}$ on M_1 and $\{\cdot, \cdot\}_{\omega_2}$ on M_2 induced by ω_1 and ω_2 respectively.

Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold. Then there exists a 2-vector field π on M such that

$$\pi_p(df_p, dg_p) = \{f, g\}(p)$$

for any f, g in $C^\infty(M)$ and p in M.

Definition 15. The above 2-vector field π on a Poisson manifold $(M, \{\cdot,\cdot\})$ is called the *Poisson structure* induced by $\{\cdot,\cdot\}$ on M.

Theorem 2.2.2 (Pauli-Jost Theorem)**.** We assume that the Poisson structure π induced by a Poisson bracket $\{\cdot,\cdot\}$ on M is nondegenerate, i.e., it defines an isomorphism $\pi_p^\sharp : T_p^*M \to T_pM$ for any p in M given by

$$\langle \pi_p^\sharp \alpha_p, \beta_p \rangle := \pi_p(\alpha_p, \beta_p)$$

for any α and β in $\Omega^1(M)$. Then π induces a symplectic structure on M. The symplectic form ω is defined by the formula

$$\omega(X_f, X_g) := \{f, g\}$$

for any locally defined Hamiltonian vector fields X_f and X_g.

Let $(M_i, \{\cdot,\cdot\}_i)$ be Poisson manifolds, π_i the Poisson structures induced by $\{\cdot,\cdot\}_i,\ i = 1, 2$. Then a map $\varphi : M_1 \to M_2$ is Poisson if and only if, for any p in M_1,

$$\varphi_* \pi_{1,p} = \pi_{2,\varphi(p)}.$$

To describe a necessary and sufficient condition for a given 2-vector field on M to a Poisson structure on M, we define the Schouten bracket on the multi-vector fields $\mathfrak{X}^*(M)$ generalized the Lie bracket on the vector field $\mathfrak{X}(M)$.

Theorem 2.2.3 ([27])**.** Let M be a manifold. Then there exists a unique anti-symmetric bilinear operator $[\cdot,\cdot] : \mathfrak{X}^*(M) \times \mathfrak{X}^*(M) \longrightarrow \mathfrak{X}^*(M)$, called *the Schouten bracket* on $\mathfrak{X}^*(M)$, that satisfies the following properties:

(i) It is a biderivation of degree -1, that is, it is bilinear,

$$\deg[D_1, D_2] = \deg D_1 + \deg D_2 - 1, \tag{2.10}$$

and

$$[D_1, D_2 \wedge D_3] = [D_1, D_2] \wedge D_3 + (-1)^{(\deg D_1 + 1)\deg D_2} D_2 \wedge [D_1, D_3] \tag{2.11}$$

for D_i in $\mathfrak{X}^*(M)$,

(ii) It is determined on $C^\infty(M)$ and $\mathfrak{X}^*(M)$ by

(a) $[f,g]=0$ $(f,g\in C^\infty(M))$;
(b) $[X,f]=Xf$ $(f\in C^\infty(M), X\in\mathfrak{X}(M))$;
(c) $[X,Y]$ $(X,Y\in\mathfrak{X}(M))$ is the standard Lie bracket on $\mathfrak{X}(M)$.

(iii) $[D_1,D_2]=-(-1)^{(\deg D_1-1)(\deg D_2-1)}[D_2,D_1]$.

In addition, the Schouten bracket satifsies the *graded Jacobi identity*

$$(-1)^{(\deg A-1)(\deg C-1)}[[A,B],C]+(-1)^{(\deg B-1)(\deg A-1)}[[B,C],A] \\ +(-1)^{(\deg C-1)(\deg A-1)}[[C,A],B]=0 \tag{2.12}$$

for A,B and C in $\mathfrak{X}^*(M)$.

The following formulas are very useful for computing with the Schouten bracket:

(i) $\mathcal{L}_X A=[X,A]$ $(X\in\mathfrak{X}(M), A\in\mathfrak{X}^*(M))$;

(ii) (the derivation property of the Lie derivative relative to the Schouten bracket)
$\mathcal{L}_X[A,B]=[\mathcal{L}_X A,B]+[A,\mathcal{L}_X B]$ $(X\in\mathfrak{X}(M), A,B\in\mathfrak{X}^*(M))$;

(iii) For X_i in $\mathfrak{X}(M)$ and A in $\mathfrak{X}^*(M)$,

$$[X_1\wedge\cdots\wedge X_r,A]=\sum_{i=1}^r(-1)^{i+1}X_i\wedge\cdots\wedge[X_i,A]\wedge\cdots\wedge X_r;$$

(iv) For X_i and Y_j in $\mathfrak{X}(M)$,

$$[X_1\wedge\cdots\wedge X_r, Y_1\wedge\cdots\wedge Y_s] \\ =(-1)^{r+1}\sum_{i,j}(-1)^{i+j}[X_i,Y_j]\wedge X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge X_r \\ \wedge Y_1\wedge\cdots\wedge\hat{Y}_j\wedge\cdots\wedge Y_s.$$

A necessary and sufficient condition for a 2-vector field π on M to be Poisson is the following.

Proposition 2.2.4. (i) For a 2-vector field π on M satisfying

$$[\pi,\pi]=0,$$

the bracket $\{f,g\}_\pi$ given by

$$\{f,g\}_\pi := \pi(df,dg) \tag{2.13}$$

for any f and g in $C^\infty(M)$ is Poisson.

(ii) Let π be the Poisson structure induced by the Poisson bracket $\{\cdot,\cdot\}$ on M. Then

$$\{f,g\} = \{f,g\}_\pi, \tag{2.14}$$

where the bracket of the right hand side is given by (2.13).

From here on, using π in $\mathfrak{X}^2(M)$ satisfying $[\pi,\pi]=0$, we denote a Poisson manifold by (M,π) and the induced bracket by $\{\cdot,\cdot\}_\pi$. This is justified by Proposition 2.2.4.

Proposition 2.2.5. We assume that the Poisson structure π on M is nondegenerate. Then the symplectic structure ω on M induced by π is determined by

$$\omega^\flat := -(\pi^\sharp)^{-1}.$$

In general, for any 2-form ω in M, the map $\omega^\flat_p$ is a homomorphism $T_pM \to T^*_pM$ for any p in M defined by

$$\langle \omega^\flat_p X_p, Y_p\rangle := \omega_p(X_p,Y_p)$$

for any X and Y in $\mathfrak{X}(M)$.

For any 2-vector field π on M, the followings hold:

Proposition 2.2.6 ([43])**.** For a local coordinate (x_i) around a point p in M, we set

$$\pi = \sum_{i,j} \pi^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}.$$

Then we obtain

$$[\pi,\pi] = \sum_{i,j,k}\sum_{l}\left(\pi^{li}\frac{\partial\pi^{jk}}{\partial x^l} + \pi^{lj}\frac{\partial\pi^{ki}}{\partial x^l} + \pi^{lk}\frac{\partial\pi^{ij}}{\partial x^l}\right)\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}\wedge\frac{\partial}{\partial x^k}. \tag{2.15}$$

We define the characteristic distributions of Poisson structures on M.

Definition 16. Let π be a Poisson structure on M. We call a subset $\pi^\sharp(T^*M)$ of TM the *characteristic distribution* of the Poisson structure π. In general, $\pi^\sharp(T^*M)$ is not a subbundle of TM. The *rank* of the Poisson structure π at a point p in M is defined as the rank of $\pi^\sharp_p : T^*_pM \to T_pM$, i.e., the dimension of the vector space $\pi^\sharp_p(T^*_pM)$. A Poisson structure π is *regular* if the rank of π is everywhere equal.

Example 14. On the n-torus $\mathbb{T}^n$ ($n \geqq 3$) with angle coordinates $(\theta_1, \ldots, \theta_n)$, a 2-vector field

$$\pi_\lambda := \frac{\partial}{\partial\theta_a} \wedge \left(\frac{\partial}{\partial\theta_b} + \lambda\frac{\partial}{\partial\theta_c}\right),$$

where λ is in $\mathbb{R}$ and a, b and c are three distinct numbers, is a regular Poisson structure with rank 2 (see [20]).

2.2.2 Poisson-Lie groups and Poisson actions

We review the definitions of Poisson-Lie groups and Poisson actions [19].

Definition 17 ([19])**.** Let G be a Lie group and π a Poisson structure on G. Then a pair (G, π) is a *Poisson-Lie group* if the multiplication map $m : G \times G \to G$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure (see Example 13). In this case, we call the Poisson structure π on G *multiplicative*.

Let π be a Poisson structure on a Lie group G. Then π is multiplicative if and only if

$$\pi_{gh} = \mathrm{L}_{g*}\pi_h + \mathrm{R}_{h*}\pi_g \tag{2.16}$$

for any g and h in G, where maps $\mathrm{L}_g : G \to G$ and $\mathrm{R}_h : G \to G$ are the left and the right translations in G by g and h respectively, as well as their differential maps extended to multi-vector fields. By the formula 2.16, we notice that a non-zero multiplicative Poisson structure is in general neither left nor right invariant 2-vector field.

Example 15 ([19])**.** The trivial 2-vector field $\pi = 0$ is obviously multiplicative, so that any Lie group G with $\pi = 0$ is a Poisson-Lie group.

Example 16 ([19])**.** The direct product of two Poisson-Lie groups with the product Poisson structure is a Poisson-Lie group again.

Example 17 ([19])**.** Let G be an Abelian Lie group, $\mathfrak{g}$ the Lie algebra of G and π a 2-vector field on G. Then π is multiplicative if and only if

$$\pi_{\mathrm{R}} : G \to \mathfrak{g} \wedge \mathfrak{g}, \ g \mapsto \mathrm{R}_{g^{-1}*}\pi_g \tag{2.17}$$

is a Lie group homomorphism from G to the Abelian group $\mathfrak{g}\wedge\mathfrak{g}$. Lie-Poisson space $\mathfrak{g}^*$ is a Poisson-Lie group when considered as an Abelian group. In fact, let π be the Poisson structure on $\mathfrak{g}^*$ induced by the Lie-Poisson bracket, $\{e_i\}_i$ a basis of $\mathfrak{g}$, $\{e^i\}_i$ the dual basis of $\mathfrak{g}^*$ and (ξ_i) the linear coordinates for $\{e^i\}$ on $\mathfrak{g}^*$. Then by the formula

$$e^i \mapsto \left(\frac{\partial}{\partial \xi_i}\right)_{\xi},$$

we identify $\mathfrak{g}^*$ with $T_\xi \mathfrak{g}^*$. It follows that

$$\mathrm{R}_{\xi*}\left(\frac{\partial}{\partial \xi_i}\right)_{\eta} = \left(\frac{\partial}{\partial \xi_i}\right)_{\xi+\eta}, \ \mathrm{R}^*_{\xi}(d\xi_i)_\eta = (d\xi_i)_{\xi+\eta}$$

for any ξ and η in $\mathfrak{g}^*$. Regarding e_i as $(d\xi_i)_\xi$, we obtain

$$\begin{aligned}
(\mathrm{R}_{-(\xi+\eta)*}\pi_{\xi+\eta})((d\xi_i)_0, (d\xi_j)_0) &= \pi_{\xi+\eta}(\mathrm{R}^*_{\xi+\eta}(d\xi_i)_0, \mathrm{R}^*_{\xi+\eta}(d\xi_j)_0) \\
&= \pi_{\xi+\eta}((d\xi_i)_{\xi+\eta}, (d\xi_j)_{\xi+\eta}) \\
&= \langle \xi+\eta, [(d\xi_i)_{\xi+\eta}, (d\xi_j)_{\xi+\eta}]\rangle \\
&= \langle \xi+\eta, [e_i, e_j]\rangle \\
&= \langle \xi, [e_i, e_j]\rangle + \langle \eta, [e_i, e_j]\rangle \\
&= \langle \xi, [(d\xi_i)_\xi, (d\xi_j)_\xi]\rangle + \langle \eta, [(d\xi_i)_\eta, (d\xi_j)_\eta]\rangle \\
&= \pi_\xi((d\xi_i)_\xi, (d\xi_j)_\xi) + \pi_\eta((d\xi_i)_\eta, (d\xi_j)_\eta) \\
&= \pi_\xi(\mathrm{R}^*_\xi(d\xi_i)_0, \mathrm{R}^*_\xi(d\xi_j)_0) \\
&\qquad + \pi_\eta(\mathrm{R}^*_\eta(d\xi_i)_0, \mathrm{R}^*_\eta(d\xi_j)_0) \\
&= (\mathrm{R}_{-\xi*}\pi_\xi)((d\xi_i)_0, (d\xi_j)_0) \\
&\qquad + (\mathrm{R}_{-\eta*}\pi_\eta)((d\xi_i)_0, (d\xi_j)_0)
\end{aligned}$$

for any ξ and η in $\mathfrak{g}^*$. Therefore the map (2.17) is a Lie group homomorphism.

We define Poisson actions for Poisson manifolds.

Definition 18 ([19])**.** Let (G, π_G) be a Poisson-Lie group, (M, π_M) a Poisson manifold and a map $\sigma : G \times M \to M$ a G-action on M. Then σ is a *Poisson G-action* if σ is a Poisson map, where $G \times M$ is equipped with the product Poisson structure.

By Definition 17, the action of a Poisson-Lie group G on itself by the left translation is a Poisson action. For a given action $\sigma : G \times M \to M$ of a Poisson-Lie group G on a Poisson manifold M, we set

$$\sigma_g : M \to M, p \mapsto \sigma_g(p),$$
$$\sigma_p : G \to M, g \mapsto \sigma_g(p)$$

for any g in G and p in M. Then the action σ is Poisson if and only if

$$\pi_{M,\sigma_g(p)} = \sigma_{g*}\pi_{M,p} + \sigma_{p*}\pi_{G,g} \tag{2.18}$$

for any g in G and p in M.

Example 18 ([19])**.** Let (M, ω) be a symplectic manifold, G a Lie group and σ a symplectic action of G on (M, ω). Let π_ω be a Poisson structure on M induced by ω. Then by regarding G as a trivial Poisson-Lie group $(G, 0)$, the action σ of $(G, 0)$ on (M, π_ω) is Poisson.

Example 19 ([19])**.** Let G be a Lie group, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Then an action of the Lie-Poisson space $\mathfrak{g}^*$ on the cotangent bundle $T^*G \cong G \times \mathfrak{g}^*$ defined by

$$(\xi, (g, \eta)) \mapsto (g, \xi + \eta)$$

for any ξ, η in $\mathfrak{g}^*$ and g in G is a Poisson action, where $T^*G \cong G \times \mathfrak{g}^*$ is equipped with the Lie group structure of a semi-direct product with respect to coadjoint action Ad of G on $\mathfrak{g}^*$ and with the product Poisson structure of the trivial Poisson structure on G and the Lie-Poisson structure on $\mathfrak{g}^*$.

Let Λ be an arbitrary 2-bector in $\mathfrak{g} \wedge \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of a Lie group G. We define a 2-vector field π_Λ on G by

$$\pi_{\Lambda,g} := \mathrm{L}_{g*}\Lambda - \mathrm{R}_{g*}\Lambda$$

for any g in G. The 2-vector field π_Λ satisfies (2.16). Setting a left- and a right-invariant 2-vector fields on G by

$$\Lambda^{\mathrm{L}}_g := \mathrm{L}_{g*}\Lambda,$$
$$\Lambda^{\mathrm{R}}_g := \mathrm{R}_{g*}\Lambda$$

for any g in G, we obtain $\pi_{\Lambda,g} = \Lambda^{\mathrm{L}}_g - \Lambda^{\mathrm{R}}_g$. The Schouten bracket of π_Λ with itself is computed as

$$\begin{aligned}
[\pi_\Lambda, \pi_\Lambda] &= [\Lambda^L - \Lambda^R, \Lambda^L - \Lambda^R] \\
&= [\Lambda^{\mathrm{L}}, \Lambda^{\mathrm{L}}] - 2[\Lambda^{\mathrm{L}}, \Lambda^{\mathrm{R}}] + [\Lambda^{\mathrm{R}}, \Lambda^{\mathrm{R}}] \\
&= [\Lambda^{\mathrm{L}}, \Lambda^{\mathrm{L}}] + [\Lambda^{\mathrm{R}}, \Lambda^{\mathrm{R}}],
\end{aligned}$$

where we use the fact that the Lie bracket of a left- and a right-invariant vector field vanishes. Furthermore, $[\Lambda^{\mathrm{L}}, \Lambda^{\mathrm{L}}]$ is a left-invariant and $[\Lambda^{\mathrm{R}}, \Lambda^{\mathrm{R}}]$ is a right-invariant by a property of the Lie bracket. By setting $g = h = e$, where e is the identity in G, in (2.16), we obtain $\pi_{\Lambda,e} = 0$. If we write

$$\pi_\Lambda = \sum_{i,j} \pi_\Lambda^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

for a local coordinate (x_i) around e in G, then $\pi_{\Lambda,e}^{ij} = 0$ for any i and j. Since it follows from (2.15) that

$$[\pi_\Lambda, \pi_\Lambda] = \sum_{i,j,k} \sum_l \left(\pi_\Lambda^{li} \frac{\partial \pi_\Lambda^{jk}}{\partial x^l} + \pi_\Lambda^{lj} \frac{\partial \pi_\Lambda^{ki}}{\partial x^l} + \pi_\Lambda^{lk} \frac{\partial \pi_\Lambda^{ij}}{\partial x^l} \right) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k},$$

we obtain $[\pi_\Lambda, \pi_\Lambda]_e = 0$. Therefore it follows that

$$[\Lambda^{\mathrm{L}}, \Lambda^{\mathrm{L}}]_e = -[\Lambda^{\mathrm{R}}, \Lambda^{\mathrm{R}}]_e.$$

We denote $[\Lambda^{\mathrm{L}}, \Lambda^{\mathrm{L}}]_e$ in $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ by $[\Lambda, \Lambda]$ and obtain

$$[\pi_\Lambda, \pi_\Lambda] = [\Lambda, \Lambda]^{\mathrm{L}} - [\Lambda, \Lambda]^{\mathrm{R}}.$$

Then for any g in G, we compute

$$\begin{aligned} [\pi_\Lambda, \pi_\Lambda]_g = 0 &\Longleftrightarrow [\Lambda, \Lambda]_g^{\mathrm{L}} - [\Lambda, \Lambda]_g^{\mathrm{R}} = 0 \\ &\Longleftrightarrow \mathrm{L}_{g*}[\Lambda, \Lambda] = \mathrm{R}_{g*}[\Lambda, \Lambda] \\ &\Longleftrightarrow \mathrm{Ad}_g[\Lambda, \Lambda] = [\Lambda, \Lambda], \end{aligned}$$

so that π_Λ is Poisson if and only if $[\Lambda, \Lambda]$ is Ad-invariant. We call such an element Λ an *r-matrix*. Using the definition of the Schouten bracket for $[\Lambda^L, \Lambda^L]$, we obtain the explicit formula for $[\Lambda, \Lambda]$ as

$$[\Lambda, \Lambda](\xi, \eta, \zeta) = -2 \sum_{\mathrm{Cycl}(\xi,\eta,\zeta)} \langle \xi, [\Lambda^\sharp \eta, \Lambda^\sharp \zeta] \rangle$$

for any ξ, η and ζ in $\mathfrak{g}^*$, where the linear map $\Lambda^\sharp : \mathfrak{g}^* \to \mathfrak{g}$ is defined by $\langle \Lambda^\sharp \xi, \eta \rangle := \Lambda(\xi, \eta)$ for any ξ and η in $\mathfrak{g}^*$. Here the symbol $\sum_{\mathrm{Cycl}(\xi,\eta,\zeta)}$ means the sum of the remaining cyclic permutations of ξ, η and ζ.

Definition 19 ([19])**.** We say that a 2-vector Λ in $\mathfrak{g} \wedge \mathfrak{g}$ satisfies the *classical Yang-Baxter equation* if

$$[\Lambda, \Lambda] = 0. \tag{2.19}$$

Drinfeld proved the following theorem [8].

Theorem 2.2.7 ([8], [19])**.** Let G be a Lie group, $\mathfrak{g}$ the Lie algebra of G and Λ in $\mathfrak{g} \wedge \mathfrak{g}$ arbitrary. We define a 2-vector field π_Λ on G by

$$\pi_{\Lambda,g} := \mathrm{L}_{g*}\Lambda - \mathrm{R}_{g*}\Lambda \tag{2.20}$$

for any g in G. Then (G, π_Λ) is a Poisson-Lie group if and only if Λ is an r-matrix. In particular, if Λ satisfies the classical Yang-Baxter equation, then a 2-vector field π_Λ defined by (2.20) is a multiplicative Poisson structure on G.

Example 20 ([19])**.** Let $G = \mathrm{SL}(2, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. The matrices

$$e_1 := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ e_2 := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ e_3 := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

form a basis of $\mathfrak{sl}(2, \mathbb{R})$. Then it follows that

$$[e_1, e_2] = e_3, \ [e_2, e_3] = e_1, \ [e_3, e_1] = -e_2.$$

Any element Λ in $\mathfrak{g} \wedge \mathfrak{g}$ is of the form $\Lambda = \lambda_1 e_1 \wedge e_2 + \lambda_2 e_2 \wedge e_3 + \lambda_3 e_3 \wedge e_1$. Since a vector space $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is 1-dimensional and $e_1 \wedge e_2 \wedge e_3$ is Ad-invariant, $[\Lambda, \Lambda]$ is also Ad-invariant.

Example 21 ([19])**.** Let $G = \mathrm{SU}(2)$ and $\mathfrak{g} = \mathfrak{su}(2)$. The matrices

$$e_1 := \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ e_2 := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ e_3 := \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

form a basis of $\mathfrak{su}(2)$. Then it follows that

$$[e_1, e_2] = e_3, \ [e_2, e_3] = e_1, \ [e_3, e_1] = e_2.$$

Similar to in the case of $\mathrm{SL}(2, \mathbb{R})$, any element Λ in $\mathfrak{g} \wedge \mathfrak{g}$ is an r-matrix.

We shall define the multiplicativity of general multi-vector fields.

Definition 20 ([19])**.** Let G be a Lie group and K a multi-vector field on G. Then K is *multiplicative* if

$$K_{gh} = \mathrm{L}_{g*}K_h + \mathrm{R}_{h*}K_g \tag{2.21}$$

for any g and h in G.

A (1-)vector field X is multiplicative if and only if it generates the 1-parameter group of transformation of group automorphism of G. In fact, let φ_t be the 1-parameter group of transformation of X. For any g and h in G, we obtain

$$X_{gh} = \frac{d}{dt}\varphi_t(gh)\Big|_{t=0},$$
$$\mathrm{L}_{g*}X_h + \mathrm{R}_{h*}X_g = \frac{d}{dt}\varphi_t(g)\varphi_t(h)\Big|_{t=0}.$$

Therefore the equivalence follows from existence and uniqueness of solutions of ordinary differential equations.

In general, any K_0 in $\Lambda^k\mathfrak{g}$ defines a multiplicative k-vector field K on G by $K_g := \mathrm{L}_{g*}K_0 + \mathrm{R}_{g*}K_0$ for any g in G. We will show later, when G is compact or semi-simple, these are all the possible multiplicative K-vector fields.

For any k-vector field K on G, we define a map $K_\mathrm{R} : G \to \Lambda^k\mathfrak{g}$ by $g \mapsto \mathrm{R}_{g^{-1}*}K_g$ for any g in G. Then K is multiplicative if and only if K_R satisfies the cocycle condition

$$K_{\mathrm{R},gh} = K_{\mathrm{R},g} + \mathrm{Ad}_{g*}K_{\mathrm{R},h} \tag{2.22}$$

for any g and h in G.

Lemma 2.2.8 ([19])**.** Let G be a connected Lie group and $\mathfrak{g}$ the Lie algebra of G. Let $\rho : G \times V \to V$ be a representation of G on a vector space V and $d\rho : \mathfrak{g} \times V \to V$ the differential representation of ρ. Then:

(i) If a map $\phi : G \to V$ is a 1-cocycle on G relative to ρ, i.e., for any g and h in G,

$$\phi(gh) = \phi(g) + \rho(g)(\phi(h)),$$

then the differential $\varepsilon := (d\phi)_e : \mathfrak{g} \to V$ at e is a 1-cocycle on $\mathfrak{g}$ relative to $d\rho$, i.e., for any X and Y in $\mathfrak{g}$,

$$\varepsilon([X,Y]) = d\rho(X)(\varepsilon(Y)) - d\rho(Y)(\varepsilon(X)),$$

and $d\phi = 0$ means that $\phi = 0$.

(ii) When G is simply connected, any 1-cocycle ε on $\mathfrak{g}$ relative to $d\rho$ can be integrated to a 1-cocycle ϕ on G relative to ρ, i.e., $d\phi = \varepsilon$.

(iii) When G is semi-simple, any 1-cocycle $\varepsilon : \mathfrak{g} \to V$ on $\mathfrak{g}$ relative to $d\rho$ is a coboundary, i.e., $\varepsilon(X) = d\rho(X)(v_0)$ for some v_0 in V.

(iv) When G is compact, any 1-cocycle $\phi : G \to V$ on G relative to ρ is a coboundary, i.e., $\phi(g) = v_0 - \rho(g)(v_0)$ for some v_0 in V.

For a k-vector fields K on G with $K_e = 0$, we call the differential $(dK_{\mathrm{R}})_e$ of K_{R} at e the *derivative* of K at e and donote by d_eK. In general, for k-vector field K on any manifold M with $K_{x_0} = 0$ for some x_0 in M, we can define the *derivative* $d_{x_0}K$ of K at x_0 as a linear map given by

$$T_{x_0}M \to \Lambda^k T_{x_0}M, \ X \mapsto (\mathcal{L}_{\bar{X}}K)_{x_0},$$

where $\bar{X}$ is any vector field on M with $\bar{X}_{x_0} = X$. From the fact that $K_{x_0} = 0$, it follows that the value of $(\mathcal{L}_{\bar{X}}K)_{x_0}$ does not depend on the choice of $\bar{X}$. We call the dual map of $d_{x_0}K$ the *linearization* of K at x_0. It is a linear map given by

$$\Lambda^k T^*_{x_0}M \to T^*_{x_0}M, \ \alpha_1 \wedge \cdots \wedge \alpha_k \mapsto d(K(\bar{\alpha}_1, \ldots \bar{\alpha}_k))_{x_0},$$

where $\bar{\alpha}_i$ is any 1-form on M with value α at x_0.

Proposition 2.2.9 ([19])**.** Let G be a Lie group, e the identity in G and $\mathfrak{g}$ the Lie algebra of G.

(i) the derivative of a multiplicative k-vector field on G at e is a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\Lambda^k\mathfrak{g}$.

(ii) When G is connected, a multiplicative k-vector field K on G is uniquely determined by its derivative at e.

(iii) When G is connected and simply connected, there exists a one-to-one correspondence between multiplicative k-vector field on G and 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation ad of $\mathfrak{g}$ on $\Lambda^k\mathfrak{g}$.

(iv) When G is connected and semi-simple, or compact, for any multiplicative k-vector field K on G, there exists an element K_0 in $\Lambda^k\mathfrak{g}$ such that

$$K_g = \mathrm{L}_{g*}K_0 + \mathrm{R}_{g*}K_0 \tag{2.23}$$

for any g in G.

Corollary 2.2.10 ([19])**.** Let G be a connected and semi-simple, or compact Lie group and $\mathfrak{g}$ the Lie algebra of G. Then any multiplicative Poisson structure π on G is of the form

$$\pi_g = \mathrm{L}_{g*}\Lambda - \mathrm{R}_{g*}\Lambda \tag{2.24}$$

for any g in G, where Λ in $\mathfrak{g} \wedge \mathfrak{g}$ is an r-matrix.

Proposition 2.2.11 ([19])**.** Let G be a connected Lie group, e the identity in G and K a multi-vector field on G. The following conditions are equivalent:

(i) K is multiplicative;

(ii) $K_e = 0$ and $\mathcal{L}_X K$ is left-invariant for any left-invariant vector field X in G;

(iii) $K_e = 0$ and $\mathcal{L}_X K$ is right-invariant for any right-invariant vector field X in G.

By Proposition 2.2.11, we obtain the following proposition.

Proposition 2.2.12 ([19])**.** Let G be a connected Lie group and K and L two multiplicative multi-vector fields on G. Then their Schouten bracket $[K, L]$ is multiplicative.

From now on, we recall the infinitesimal version of Poisson-Lie group, namely Lie bialgebra. First, by Proposition 2.2.9 and Proposition 2.2.12, the following holds.

Proposition 2.2.13 ([19])**.** Let G be a connected Lie group and π a multiplicative 2-vector field on G. We denote the linearization of π at e by $[\cdot,\cdot]^\pi : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$. It is a skew-symmetric bilinear form given by, for any ξ and η in $\mathfrak{g}^*$,

$$[\xi, \eta]^\pi = d(\pi(\bar{\xi}, \bar{\eta}))_e,$$

where $\bar{\xi}$ and $\bar{\eta}$ are any 1-forms on G with values ξ and η at e respectively. Then π is Poisson if and only if the skew-symmetric bilinear form $[\cdot,\cdot]^\pi$ satisfies the Jacobi identity, that is, a pair $(\mathfrak{g}^*, [\cdot,\cdot]^\pi)$ is a Lie algebra. we call such a Lie algebra the *linearization* of the Poisson structure at e.

Example 22 ([19])**.** Let G be a connected Lie group, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Let Λ in $\mathfrak{g} \wedge \mathfrak{g}$ be an r-matrix. Then a 2-vector field π_Λ defined by (2.20) is a multiplicative Poisson structure on G. The Lie bracket $[\cdot,\cdot]^{\pi_\Lambda}$ on $\mathfrak{g}^*$ defined by the linearization of π_Λ at e is given by

$$[\xi,\eta]^{\pi_\Lambda} = \mathrm{ad}^*_{\Lambda^\sharp\xi}\eta - \mathrm{ad}^*_{\Lambda^\sharp\eta}\xi \tag{2.25}$$

for any ξ and η in $\mathfrak{g}^*$.

Definition 21 ([8], [19]). Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. We assume that $\mathfrak{g}^*$ has a Lie algebra structure $[\cdot,\cdot]_*$. A pair $(\mathfrak{g},\mathfrak{g}^*)$ is a *Lie bialgebra* if the dual map $\delta : \mathfrak{g} \to \mathfrak{g}\wedge\mathfrak{g}$ of the Lie bracket map $[\cdot,\cdot]_* : \mathfrak{g}^*\wedge\mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$ is a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}\wedge\mathfrak{g}$, i.e.,

$$\delta([X,Y]) = \mathrm{ad}_X\delta(Y) - \mathrm{ad}_Y\delta(X) \tag{2.26}$$

for any X and Y in $\mathfrak{g}$. Sometimes we denote the Lie bialgebra by $(\mathfrak{g},\delta)$.

The following theorem follows immediately from Proposition 2.2.9.

Theorem 2.2.14 ([8], [19]). Let (G,π) be a Poisson-Lie group, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Then the linearization of π at e defines a Lie algebra structure on $\mathfrak{g}^*$ such that $(\mathfrak{g},\mathfrak{g}^*)$ is a Lie bialgebra, called the *tangent Lie bialgebra* to (G,π). Conversely, if G is connected and simply connected, any Lie bialgebra $(\mathfrak{g},\mathfrak{g}^*)$ defines a unique multiplicative Poisson structure π on G such that $(\mathfrak{g},\mathfrak{g}^*)$ is the tangent Lie bialgebra to the Poisson-Lie group (G,π).

The following theorem is a characterization of Poisson actions of connected Poisson-Lie groups on Poisson manifolds [21].

Theorem 2.2.15 ([19], [21]). Let (M,π_M) be a Poisson manifold, (G,π_G) be a connected Poisson-Lie group, $\mathfrak{g}$ the Lie algebra of G and σ an action of G on M. Then the action σ is a Poisson action of (G,π_G) on (M,π_M) if for each X in $\mathfrak{g}$,

$$\mathcal{L}_{X_\sigma}\pi_M = \delta(X)_\sigma, \tag{2.27}$$

where x_σ is a fundamental multi-vector field for any x in $\wedge^*\mathfrak{g}$. Here δ is the 1-cocycle belonging to the tangential Lie bialgebra $(\mathfrak{g},\mathfrak{g}^*)$ to (G,π_G).

Theorem 2.2.16 (Manin Theorem [8], [19]). Let $\mathfrak{g}$ be a Lie algebra with a bracket $[\cdot,\cdot]$ and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$ with a Lie algebra structure $[\cdot,\cdot]_*$.

We define the nondegenerate symmetric bilinear scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ on the vector space $\mathfrak{g}\oplus\mathfrak{g}^*$ given by

$$\langle\langle X+\xi, Y+\eta\rangle\rangle := \frac{1}{2}(\langle\xi, Y\rangle + \langle\eta, X\rangle) \tag{2.28}$$

for any X, Y in $\mathfrak{g}$, ξ and η in $\mathfrak{g}^*$. We denote the coadjoint representations of $\mathfrak{g}$ on $\mathfrak{g}^*$ and of $\mathfrak{g}^*$ on $\mathfrak{g}=(\mathfrak{g}^*)^*$ by $\mathrm{ad}^*_X\xi$ and $\mathrm{ad}^*_\xi X$ for any X in $\mathfrak{g}$ and ξ in $\mathfrak{g}^*$ respectively. Then there exists a unique skew-symmetric bracket operation $[\![\cdot,\cdot]\!]$ on $\mathfrak{g}\oplus\mathfrak{g}^*$ such that

(i) it restricts to the given brackets on $\mathfrak{g}$ and $\mathfrak{g}^*$;

(ii) the scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathfrak{g}\oplus\mathfrak{g}^*$ is invariant, i.e.,

$$\langle\langle [\![X+\xi, Y+\eta]\!], Z+\zeta\rangle\rangle + \langle\langle Y+\eta, [\![X+\xi, Z+\zeta]\!]\rangle\rangle = 0 \tag{2.29}$$

for any X, Y, Z in $\mathfrak{g}$, ξ, η and ζ in $\mathfrak{g}^*$.

This operation is given by

$$[\![X+\xi, Y+\eta]\!] := [X,Y] - \mathrm{ad}^*_\eta X + \mathrm{ad}^*_\xi Y + [\xi,\eta]_* + \mathrm{ad}^*_X\eta - \mathrm{ad}^*_Y\xi \tag{2.30}$$

for any X, Y in $\mathfrak{g}$, ξ and η in $\mathfrak{g}^*$. Moreover, it is a Lie bracket on $\mathfrak{g}\oplus\mathfrak{g}^*$ if and only if $(\mathfrak{g},\mathfrak{g}^*)$ is a Lie bialgebra.

Definition 22 ([19])**.** For a Lie bialgebra $(\mathfrak{g},\mathfrak{g}^*)$, we call the space $\mathfrak{g}\oplus\mathfrak{g}^*$ equipped with the Lie bracket given by Theorem 2.2.16 the *double Lie algebra* of the Lie bialgebra $(\mathfrak{g},\mathfrak{g}^*)$ and denote by $\mathfrak{g}\bowtie\mathfrak{g}^*$.

By Theorem 2.2.16, the following holds.

Corollary 2.2.17 ([19])**.** If $(\mathfrak{g},\mathfrak{g}^*)$ is a Lie bialgebra, then so is $(\mathfrak{g}^*,\mathfrak{g})$.

Definition 23 ([19])**.** Let (G,π_G) be a Poisson-Lie group, $(\mathfrak{g},\mathfrak{g}^*)$ the tangent Lie bialgebra to (G,π_G) and G^* the connected and simply connected Lie group with the Lie algebra $\mathfrak{g}^*$. Then by Corollary 2.2.17, G^* has a unique multiplicative Poisson structure π_{G^*} such that $(\mathfrak{g}^*,\mathfrak{g})$ is the tangent Lie algebra to (G^*,π_{G^*}). A pair (G^*,π_{G^*}) is called the *dual Poisson-Lie group* of (G,π_G).

Example 23 ([19])**.** Let G be a Lie group equipped with the trivial Poisson structure, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Then the dual Poisson-Lie group G^* of $(G,0)$ is the Abelian Lie group $\mathfrak{g}^*$ with the Lie-Poisson structure. The double Lie algebra is the Lie algebra of the cotangent bundle $T^*G\cong G\times\mathfrak{g}^*$ equipped with the Lie group structure of a semi-direct product with respect to coadjoint action Ad of G on $\mathfrak{g}^*$ (cf. Example 19).

Definition 24 ([19])**.** Let $\mathfrak{g}$ be a Lie algebra equipped with a nondegenerate invariant symmetric scalar product $\langle\cdot,\cdot\rangle$, $\mathfrak{g}_\pm$ Lie subalgebras of $\mathfrak{g}$. Then a triple $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$ is *Manin triple* if the following conditions hold:

(i) $\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_-$ as vector spaces;

(ii) both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic with respect to the scalar product $\langle\cdot,\cdot\rangle$, i.e., $\langle\mathfrak{g}_+,\mathfrak{g}_+\rangle=\langle\mathfrak{g}_-,\mathfrak{g}_-\rangle=0$.

The correspondence between Lie bialgebra and Manin triple is constructed as follows: If $(\mathfrak{g},\mathfrak{g}^*)$ is a Lie bialgebra, then a triple $(\mathfrak{g}\oplus\mathfrak{g}^*,\mathfrak{g},\mathfrak{g}^*)$ with $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ in Theorem 2.2.16 is a Manin triple. Conversely, if $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$ is a Manin triple, then $\mathfrak{g}_-$ is naturally isomorphic to $\mathfrak{g}_+^*$ under $\langle\cdot,\cdot\rangle$, that is, the map $\mathfrak{g}_-\to\mathfrak{g}_+^*$, $\xi\mapsto 2\left.\langle\xi,\cdot\rangle\right|_{\mathfrak{g}_+}$ is an isomorphism. Therefore $\mathfrak{g}\cong\mathfrak{g}_+\oplus\mathfrak{g}_+^*$ as vector spaces, and $\langle\cdot,\cdot\rangle$ coincides with the natural scalar product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ on $\mathfrak{g}_+\oplus\mathfrak{g}_+^*$ in Theorem 2.2.16. Again by Theorem 2.2.16, a pair $(\mathfrak{g}_+,\mathfrak{g}_+^*)$ is a Lie bialgebra.

Let (M,π) be a Poisson manifold. Then we define a Lie bracket $[\cdot,\cdot]_\pi$ on the 1-forms $\Omega^1(M)$ induced by π by

$$[\xi,\eta]_\pi:=\mathcal{L}_{\pi^\sharp\xi}\eta-\mathcal{L}_{\pi^\sharp\eta}\xi-d\langle\pi^\sharp\xi,\eta\rangle \tag{2.31}$$

for any ξ and η in $\Omega^1(M)$. The Lie bracket has the following properties:

$$[\xi,f\eta]_\pi=f[\xi,\eta]_\pi+((\pi^\sharp\xi)f)\eta, \tag{2.32}$$

$$\pi^\sharp[\xi,\eta]_\pi=[\pi^\sharp\xi,\pi^\sharp\eta] \tag{2.33}$$

for any ξ,η in $\Omega^1(M)$ and f in $C^\infty(M)$. Then the triple $(T^*M,\pi^\sharp,[\cdot,\cdot]_\pi)$ forms a *Lie algebroid* over M, which we will recall in Section 2.4. Let (G,π) be a Poisson-Lie group with the Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$, equipped with the Lie bracket $[\cdot,\cdot]^\pi$ induced by the linearization of π at e. Then the left- and right-invariant 1-forms $\Omega^1(M)^{\mathrm{L}}$ and $\Omega^1(M)^{\mathrm{R}}$ are closed under the above Lie bracket $[\cdot,\cdot]_\pi$. We identifies $\mathfrak{g}^*$ with $\Omega^1(M)^{\mathrm{L}}$ and $\Omega^1(M)^{\mathrm{R}}$ by $X\mapsto X^{\mathrm{L}}$ and $X\mapsto X^{\mathrm{R}}$ respectively. Both the above brackets $[\cdot,\cdot]_\pi|_{\Omega^1(M)^{\mathrm{L}}}$ and $[\cdot,\cdot]_\pi|_{\Omega^1(M)^{\mathrm{R}}}$ coincide with the opposite of the linearization of π at e by the fact that $\pi_e=0$. We define a map $\lambda^\pi:\mathfrak{g}^*\to\mathfrak{X}(G)$ by

$$\lambda^\pi(\xi):=-\pi^\sharp(\xi^{\mathrm{R}}) \tag{2.34}$$

for any ξ in $\mathfrak{g}^*$. Then λ^π is a Lie algebra anti-homomorphism by the property (2.33) and the fact $[\xi,\eta]^\pi=[\xi^{\mathrm{R}},\eta^{\mathrm{R}}]_{\pi,e}$.

Definition 25 ([19])**.** We call a vector field $\lambda^{\pi}(\xi)$ a *dressing vector field* on (G, π) and the map λ^{π} the *infinitesimal dressing action* of $\mathfrak{g}^*$ on G. By integrating λ^{π}, we obtain a local (and global if all dressing vector fields are complete) action of G^* on G. This action is called the *dressing action* of G^* on G, denoted by the same symbol λ_{π}.

Remark 1. In [18], [19], [21], the dressing action and its infinitesimal one are defined by $\lambda^{\pi}(\xi) := -\pi^{\sharp}(\xi^{\mathrm{L}})$ instead of (2.34). However all the results on the dressing action and its infinitesimal one hold by some correction, so that there is no problem even if (2.34) is used. Rather, adopting the definition given by (2.34) is more natural in terms of quasi-Poisson theory, which we will review in next section.

Definition 26 ([19])**.** Let G be a Lie group and π a multiplicative Poisson structure on G. Then π is *complete* if all dressing vector fields on (G, π) are complete.

Example 24. The trivial Poisson structure 0 on any Lie group G is complete. In fact, by a computation $\lambda^0(\xi) = -0^{\sharp}(\xi^{\mathrm{R}}) = 0$ for any ξ in $\mathfrak{g}^*$, the dressing action is a global action $\lambda^0_u = \mathrm{id}_G$ for any u in G^*.

The following propositions are properties for the completeness of a multiplicative Poisson structure on a Lie group.

Proposition 2.2.18 ([19], [24])**.** A Poisson-Lie group (G, π) is complete if and only if the dual Poisson-Lie group G^* of (G, π) is complete.

Proposition 2.2.19 ([19])**.** Let (G, π) be a Poisson-Lie group with the tangential Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. Let $(\tilde{G}, \tilde{\pi})$ be the universal covering group of G with the multiplicative Poisson structure induced by $(\mathfrak{g}, \mathfrak{g}^*)$. Then (G, π) is complete if and only if $(\tilde{G}, \tilde{\pi})$ is complete.

The completeness of a multiplicative Poisson structure π on a Lie group G with the Lie algebra $\mathfrak{g}$ also can be described by using the Lie group with the Lie algebra $\mathfrak{g} \bowtie \mathfrak{g}^*$. Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra and $\mathfrak{g} \bowtie \mathfrak{g}^*$ the double Lie algebra of $(\mathfrak{g}, \mathfrak{g}^*)$. Let G, G^* and $G \bowtie G^*$ be the connected and simply connected Lie groups with the Lie algebras $\mathfrak{g}, \mathfrak{g}^*$ and $\mathfrak{g} \bowtie \mathfrak{g}^*$ respectively. We call $G \bowtie G^*$ the *double Lie group* induced by $(\mathfrak{g}, \mathfrak{g}^*)$. We set Lie group homomorphisms obtained by integrating including maps $\mathfrak{g} \hookrightarrow \mathfrak{g} \bowtie \mathfrak{g}^*$ and $\mathfrak{g}^* \hookrightarrow \mathfrak{g} \bowtie \mathfrak{g}^*$ by

$$\phi_G : G \to G \bowtie G^*, \quad \phi_{G^*} : G^* \to G \bowtie G^*,$$

respectively, and define $\phi_{G,G^*} : G \times G^* \to G \bowtie G^*$ by

$$\phi_{G,G^*}(g, u) := \phi_G(g)\phi_{G^*}(u).$$

Proposition 2.2.20 ([19])**.** Let the notations be as above. Then a connected and simply connected Poisson-Lie group (G, π) is complete if and only if the map ϕ_{G,G^*} is a diffeomorphism.

2.2.3 Moment maps for Poisson actions

The definition of a moment map for Poisson action is the following:

Definition 27 ([19])**.** Let (M, π_M) be a Poisson manifold, (G, π_G) a Poisson-Lie group, G^* the dual Poisson-Lie group of (G, π_G) and $\sigma : G \times M \to M$ a Poisson action of (G, π_G) on (M, π_M). Let X^{R} be a right-invariant 1-form on G^* with value X at e for any X in $\mathfrak{g} = (\mathfrak{g}^*)^*$ and Y_σ a vector field on M generating the action $\sigma_{\exp tY}$ for any Y in $\mathfrak{g}$. Then a map $\mu : M \to G^*$ is a *moment map* for the Poisson action σ if for any X in $\mathfrak{g}$,

$$X_\sigma = -\pi_M^\sharp(\mu^* X^{\mathrm{R}}). \tag{2.35}$$

Example 25 ([18], [19])**.** Let (G, π) be a connected complete Poisson-Lie group and G^* the dual Poisson-Lie group of (G, π). Then the dressing action λ^π of G^* on (G, π) is Poisson (for the detail, see [18]). Moreover, the identity map $\mathrm{id} : G \to G$ is a moment map for the dressing action λ^π. In fact, we compute that for any ξ in $\mathfrak{g}^*$ and g in G,

$$\begin{aligned} \xi_{\lambda^\pi, g} &= \frac{d}{dt}\lambda^\pi_{\exp t\xi}(g)\Big|_{t=0} \\ &= \lambda(\xi)_g \\ &= (-\pi^\sharp(\xi^{\mathrm{R}}))_g \\ &= (-\pi^\sharp(\mathrm{id}^*\xi^{\mathrm{R}}))_g \end{aligned}$$

Definition 28 ([19])**.** Let (M, π_M) be a Poisson manifold, (G, π_G) a complete Poisson-Lie group, (G^*, π_{G^*}) the dual Poisson-Lie group of (G, π_G), $\lambda^{\pi_{G^*}} : G \times G^* \to G^*$ the dressing action of (G^*, π_{G^*}) on (G, π_G) and $\sigma : G \times M \to M$ a Poisson action of (G, π_G) on (M, π_M) with a moment map $\mu : M \to G^*$. Then μ is G-*equivariant* if for any g in G,

$$\mu \circ \sigma_g = \lambda_g^{\pi_{G^*}} \circ \mu. \tag{2.36}$$

In other words, μ is G-equivariant if for any g in G, the diagram

$$\begin{array}{ccc} M & \xrightarrow{\sigma_g} & M \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ G^* & \xrightarrow[\lambda_g^{\pi_{G^*}}]{} & G^* \end{array} \tag{2.37}$$

is commutative.

Remark 2. Since the dual Poisson-Lie group G^* is complete by Proposition 2.2.18, the dressing action of (G, π_G) on G^* is global. Therefore the above definition is well-defined.

Example 26 (moment maps for symplectic-Hamiltonian actions)**.** Let (M, ω, G, μ) be a Hamiltonian G-space with a symplectic-Hamiltonian action σ and π_ω the Poisson structure induced by ω. By Example 18, the action σ is a Poisson action of a trivial Poisson-Lie group $(G, 0)$ on a Poisson manifold (M, π_ω). Obviously, the trivial Poisson structure is complete. By Example 23, the dual Poisson-Lie group G^* of $(G, 0)$ is the Abelian Lie group $\mathfrak{g}^*$ with the Lie-Poisson structure $\pi_{\mathfrak{g}^*}$, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$. By the definition of a moment map for a symplectic-Hamiltonian action, the condition (2.6) follows for $\mu : M \to \mathfrak{g}^*$. Let $\{e_i\}_i$ be a basis of $\mathfrak{g}$, $\{e^i\}_i$ the dual basis of $\mathfrak{g}^*$ and (ξ_i) the linear coordinates for $\{e^i\}$ on $\mathfrak{g}^*$. Regarding e_i as $(d\xi_i)_\xi$, for any $X = \sum_i X^i e_i$, we obtain $X^{\mathrm{R}} = \sum_i X^i d\xi_i$ by Example 17. For the moment map μ and any point p in M, setting $\mu(p) = \sum_i \mu_i(p) e^i$ for μ_i in $C^\infty(M)$, we compute

$$
\begin{aligned}
\mu^* X^{\mathrm{R}} = \mu^* \left(\sum_i X^i d\xi_i \right) &= \sum_i X^i \mu^* d\xi_i = \sum_i X^i d(\mu^* \xi_i) = \sum_i X^i d(\xi_i \circ \mu) \\
&= \sum_i X^i d\mu_i = \sum_i X^i d\langle \mu, e_i \rangle = d \left\langle \mu, \sum_i X^i e_i \right\rangle = d\langle \mu, X \rangle \\
&= d\mu^X.
\end{aligned}
$$

Hence it follows that the condition (2.6) is equivalent with $\mu^* X^{\mathrm{R}} = \iota_{X_\sigma} \omega$. Moreover we have

$$
\begin{aligned}
\mu^* X^{\mathrm{R}} = \iota_{X_\sigma} \omega &\Longleftrightarrow \mu^* X^{\mathrm{R}} = \omega^\flat X_\sigma \\
&\Longleftrightarrow (\omega^\flat)^{-1} (\mu^* X^{\mathrm{R}}) = X_\sigma \\
&\Longleftrightarrow -\pi^\sharp (\mu^* X^{\mathrm{R}}) = X_\sigma.
\end{aligned}
$$

Therefore $\mu : M \to \mathfrak{g}^*$ is a moment map for the Poisson $(G, 0)$-action on (M, π_ω). Moreover, the dressing action $\lambda^{\pi_{\mathfrak{g}^*}}$ of G on $\mathfrak{g}^*$ coincides with the coadjoint representation Ad^* of G on $\mathfrak{g}^*$. In fact, for any $X = \sum_i X^i e_i$ in $\mathfrak{g}$ and $\xi = \sum_j \xi_j e^j$ in $\mathfrak{g}^*$, regarding e_i and e^j as $(d\xi_i)_\xi$ and $\left(\frac{\partial}{\partial \xi_j}\right)_\xi$ respectively, we compute

$$\begin{aligned}
\left\langle \frac{d}{dt}\mathrm{Ad}^*_{\exp tX}\xi\Big|_{t=0}, (d\xi_k)_\xi \right\rangle &= \langle \mathrm{ad}^*_X\xi, e_k\rangle \\
&= \langle \xi, -\mathrm{ad}_X e_k\rangle \\
&= -\langle \xi, [X, e_k]\rangle \\
&= -\sum_i X^i \langle \xi, [e_i, e_k]\rangle \\
&= -\sum_i X^i \langle \xi, [(d\xi_i)_\xi, (d\xi_k)_\xi]\rangle \\
&= -\sum_i X^i \pi_{\mathfrak{g}^*}(d\xi_i, d\xi_k)_\xi \\
&= -\sum_i X^i \left\langle \pi^\sharp_{\mathfrak{g}^*}(d\xi_i)_\xi, (d\xi_k)_\xi \right\rangle \\
&= -\left\langle \pi^\sharp_{\mathfrak{g}^*}\left(\sum_i X^i d\xi_i\right)_\xi, (d\xi_k)_\xi \right\rangle \\
&= \left\langle -\pi^\sharp_{\mathfrak{g}^*}\left(X^{\mathrm{R}}\right)_\xi, (d\xi_k)_\xi \right\rangle \\
&= \langle \lambda^{\pi_{\mathfrak{g}^*}}(X)_\xi, (d\xi_k)_\xi\rangle ,
\end{aligned}$$

so that $\mathrm{Ad}^* = \lambda^{\pi_{\mathfrak{g}^*}}$ holds. Since μ satisfies the condition (2.8), therefore we obtain

$$\mu \circ \sigma_g = \lambda_g^{\pi_{\mathfrak{g}^*}} \circ \mu \tag{2.38}$$

for any g in G, and $\mu : M \to \mathfrak{g}^*$ is a G-equivariant moment map for the Poisson $(G, 0)$-action on (M, π_ω). From the above, a G-equivariant moment map for a Poisson action of a complete Poisson-Lie group is a generalization of a moment map for a symplectic-Hamiltonian action of a Lie group.

The G-equivariance of a moment map for a Poisson action is described as follows.

Theorem 2.2.21 ([19])**.** Let (M, π_M) be a Poisson manifold, (G, π_G) a connected complete Poisson-Lie group and (G^*, π_{G^*}) the dual Poisson-Lie group of (G, π_G). Then a moment map $\mu : M \to G^*$ for a Poisson (G, π_G)-action on (M, π_M) is G-equivariant if and only if μ is a Poisson map.

2.3 Quasi-Poisson geometry

In this section, we shall recall the quasi-Poisson theory [1]. We start with the definition of quasi-Poisson-Lie groups, which is a generalization of Poisson-Lie groups.

Definition 29 ([1], [13], [14])**.** Let G be a Lie group with the Lie algebra $\mathfrak{g}$. Then a pair (π, φ) is a *quasi-Poisson structure* on G if a multiplicative 2-vector field π on G and an element φ of $\Lambda^3\mathfrak{g}$ satisfy

$$\frac{1}{2}[\pi,\pi] = \varphi^{\mathrm{R}} - \varphi^{\mathrm{L}}, \tag{2.39}$$

$$\left[\pi, \varphi^{\mathrm{L}}\right] = \left[\pi, \varphi^{\mathrm{R}}\right] = 0, \tag{2.40}$$

where the bracket $[\cdot,\cdot]$ is the Schouten bracket on the multi-vector fields on G, and φ^{L} and φ^{R} denote the left- and right-invariant 2-vector fields on G with value φ at e respectively. A triple (G, π, φ) is called a *quasi-Poisson-Lie group*.

Example 27. Let (G, π_G) be a Poisson-Lie group. Then by setting $\varphi := 0$, a triple (G, π_G, φ) is a quasi-Poisson-Lie group.

We deal with quasi-Poisson-Lie groups induced by "quasi-triples" to define quasi-Poisson actions and its moment maps. To define quasi-triples, we need to describe its infinitesimal version, a Manin quasi-triples.

Definition 30 ([1])**.** Let $\mathfrak{d}$ be a $2n$-dimensional Lie algebra with an invariant non-degenerate symmetric bilinear form of signature (n, n), which is denoted by $(\cdot|\cdot)$. Let $\mathfrak{g}$ be an n-dimensional Lie subalgebra of $\mathfrak{d}$ and $\mathfrak{h}$ be an n-dimensional vector subspace of $\mathfrak{d}$. Then a pair $(\mathfrak{d}, \mathfrak{g})$ is a *Manin pair* if $\mathfrak{g}$ is a maximal isotropic subspace with respect to $(\cdot|\cdot)$. A triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a *Manin quasi-triple* if $(\mathfrak{d}, \mathfrak{g})$ is a Manin pair and $\mathfrak{h}$ is an isotropic complement subspace of $\mathfrak{g}$ in $\mathfrak{d}$.

Remark 3. For a given Manin pair $(\mathfrak{d}, \mathfrak{g})$, a choice of an isotropic complement subspace $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{d}$ is not unique.

A Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ defines the decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$. Then the linear isomorphism

$$j : \mathfrak{g}^* \to \mathfrak{h},\ \ (j(\xi)|x) := \langle \xi, x\rangle\ \ (\xi \in \mathfrak{g}^*, x \in \mathfrak{g}) \tag{2.41}$$

is determined by the decomposition. We denote the projections from $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ to $\mathfrak{g}$ and $\mathfrak{h}$ by $p_{\mathfrak{g}}$ and $p_{\mathfrak{h}}$ respectively. We introduce an element $\varphi_{\mathfrak{h}}$ in $\Lambda^3\mathfrak{g}$ which is defined by the map from $\Lambda^2\mathfrak{g}^*$ to $\mathfrak{g}$, denoted by the same letter,

$$\varphi_{\mathfrak{h}}(\xi, \eta) := p_{\mathfrak{g}}([j(\xi), j(\eta)]), \tag{2.42}$$

for any ξ, η in $\mathfrak{g}^*$. We introduce the *cobracket*. It is the linear map $F_{\mathfrak{h}} : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ by setting

$$F_{\mathfrak{h}}^*(\xi, \eta) := j^{-1}(p_{\mathfrak{h}}([j(\xi), j(\eta)])) \tag{2.43}$$

for any ξ, η in $\mathfrak{g}^*$, where $F_{\mathfrak{h}}^* : \Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$ is the dual map of $F_{\mathfrak{h}}$. These elements will give important examples of quasi-Poisson structures and be used later to define quasi-Poisson actions. The Lie algebra $\mathfrak{g}$ with the cobracket F and the element φ in $\Lambda^3\mathfrak{g}^*$ is called a *quasi-Lie bialgebra*.

Conversely, It is well known that any Manin quasi-triple is obtained from a quasi-Lie bialgebra [9].

Example 28 ([1])**.** Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. By setting $\varphi = 0$, we obtain a Manin pair $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g})$, where the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{g} \oplus \mathfrak{g}^*$ coincides with the bracket (2.30). Therefore by choosing the canonical isotropic complement $\mathfrak{h} = \mathfrak{g}^*$, a Manin quasi-triple $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ is just a Manin triple corresponding with a Lie bialgebra $(\mathfrak{g}, F)$. Moreover, in the case of $F = 0$, we call a Manin pair $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g})$ the *standard Manin pair* associated $\mathfrak{g}$.

Example 29 ([1], [39])**.** Let $\mathfrak{g}$ be a Lie algebra with an invariant nondegenerate symmetric bilinear form K. Then we can construct a Manin pair $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g})$ as follows: The Lie bracket $[\cdot, \cdot]$ on $\mathfrak{g} \oplus \mathfrak{g}$ and the bilinear form $(\cdot|\cdot)$ on $\mathfrak{g} \oplus \mathfrak{g}$ are given by

$$[(X_1, X_2), (Y_1, Y_2)] := ([X_1, Y_1], [X_2, Y_2]), \tag{2.44}$$

$$((X_1, X_2)|(Y_1, Y_2)) := K(X_1, Y_1) - K(X_2, Y_2) \tag{2.45}$$

for any X_1, X_2, Y_1 and Y_2 in $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is embedded into $\mathfrak{g} \oplus \mathfrak{g}$ by the diagonal map $\mathrm{diag} : X \mapsto (X, X)$. We can choose an isotropic complement $\mathfrak{g}_- := \frac{1}{2}\mathrm{diag}_-(\mathfrak{g})$, where $\mathrm{diag}_- : X \mapsto (X, -X)$ is the anti-diagonal map. In general, the isotropic complement $\mathfrak{g}_-$ is not a Lie subalgebra, so that $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}, \mathfrak{g}_-)$ is a Manin quasi-triple but not a Manin triple.

Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ be a Manin quasi-triple. Using the inverse $j^{-1} : \mathfrak{h} \to \mathfrak{g}^*$ of the linear isomorphism (2.41), we identify $\mathfrak{d}$ with $\mathfrak{g} \oplus \mathfrak{g}^*$. Consider the map

$$r_{\mathfrak{h}} : \mathfrak{d}^* \to \mathfrak{d},\ \xi + X \mapsto \xi, \tag{2.46}$$

for any ξ in $\mathfrak{g}^*$ and X in $\mathfrak{g}$. This map defines an element $r_{\mathfrak{h}} \in \mathfrak{d} \otimes \mathfrak{d}$ which we denote by the same letter. Let $\{e_i\}$ be a basis on $\mathfrak{g}$ and $\{\varepsilon^i\}$ the dual basis of $\{e_i\}$ on $\mathfrak{g}^*$. Then it follows that

$$r_{\mathfrak{h}} = \sum_i e_i \otimes j(\varepsilon^i). \tag{2.47}$$

Definition 31 ([1])**.** The above element $r_{\mathfrak{h}}$ in $\mathfrak{d} \otimes \mathfrak{d}$ is called the *canonical r-matrix* for the Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$.

Next we define a twist between isotropic complement subspaces $\mathfrak{h}$ and $\mathfrak{h}'$ of $\mathfrak{g}$ in $\mathfrak{d}$. Twists play an important role in the moment map theory for quasi-Poisson actions defined later. Let j and j' be the linear isomorphism (2.41) defined by Manin quasi-triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h}')$ respectively. Consider the map

$$t := j' - j : \mathfrak{g}^* \to \mathfrak{d}.$$

It is easy to show that t takes values in $\mathfrak{g}$ and that it is anti-symmetric, so that the map t defines an element t in $\Lambda^2\mathfrak{g}$ which we denote by the same letter. The element t is called the *twist* from $\mathfrak{h}$ to $\mathfrak{h}'$. Fix a Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. For any t in $\Lambda^2\mathfrak{g}$, there exists the isotropic complement $\mathfrak{h}'$ of $\mathfrak{g}$ to which the twist from $\mathfrak{h}$ is t. In fact, let $\{e_i\}$ be a basis on $\mathfrak{g}$ and $\{\varepsilon^i\}$ be the basis on $\mathfrak{h}$ identified with the dual basis of $\{e_i\}$ on $\mathfrak{g}^*$ by j^{-1}. We set $t := \frac{1}{2}\sum_{i,j} t^{ij} e_i \wedge e_j$ and

$$\varepsilon'^i := \varepsilon^i + \sum_j t^{ij} e_j. \tag{2.48}$$

Then the set $\{\varepsilon'^i\}$ spans an isotropic complement subspace of $\mathfrak{g}$. We can set $\mathfrak{h}' := \mathrm{span}\{\varepsilon'^i\}$. From now on, we denote $\mathfrak{h}'$ by $\mathfrak{h}_t$. Then we can represent the canonical r-matrix $r_{\mathfrak{h}_t}$, the elements $\varphi_{\mathfrak{h}_t}$ and $F_{\mathfrak{h}_t}$ defined by a Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h}_t)$ as follows:

$$r_{\mathfrak{h}_t} = r_{\mathfrak{h}} + t \tag{2.49}$$

$$\varphi_{\mathfrak{h}_t} = \varphi_{\mathfrak{h}} + \frac{1}{2}[t,t] + \varphi_t, \tag{2.50}$$

$$F_{\mathfrak{h}_t} = F_{\mathfrak{h}} + F_t, \tag{2.51}$$

where $[t,t] := [t^{\mathrm{L}}, t^{\mathrm{L}}]_e$, $\varphi_t(\xi) := \overline{\mathrm{ad}^*_\xi t}$ and $F_t(X) := \mathrm{ad}_X t$. Here ad denotes the adjoint action of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$ and $\overline{\mathrm{ad}^*_\xi t}$ denotes the projection of $\mathrm{ad}^*_\xi t$ onto

$\Lambda^2\mathfrak{g} \subset \Lambda^2\mathfrak{d}$, where $\mathfrak{d}^*$ including $\mathfrak{g}^*$ acts on $\Lambda^2\mathfrak{d}$ by the coadjoint action (see [1]). Moreover components of φ_t with respect to the basis $\{\varepsilon^i\}$ on $\mathfrak{h}$ are written as

$$\varphi_t^{ijk} = \sum_l \left((F_{\mathfrak{h}})_l^{jk} t^{il} - (F_{\mathfrak{h}})_l^{ik} t^{jl} \right). \tag{2.52}$$

This expression is useful later.

Next we define a group pair (D, G) and a quasi-triple $(D, G, \mathfrak{h})$.

Definition 32 ([1])**.** Let D be a connected Lie group with a bi-invariant scalar product with the Lie algebra $\mathfrak{d}$ and G be a connected closed Lie subgroup of D with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a vector subspace of $\mathfrak{d}$. Then a pair (D, G) is a *group pair* if $(\mathfrak{d}, \mathfrak{g})$ is a Manin pair. A triple $(D, G, \mathfrak{h})$ is a *quasi-triple* if $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin quasi-triple.

Example 30 ([1])**.** Let $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g})$ be the standard Manin pair associated a Lie algebra $\mathfrak{g}$. Then a group pair corresponding to $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g})$ is (T^*G, G), where the cotangent bundle $T^*G \cong G \times \mathfrak{g}^*$ is equipped with the Lie group structure of a semi-direct product with respect to coadjoint action Ad of G on $\mathfrak{g}^*$ (cf. Example 23). The Lie group G is embedded into T^*G as the zero section.

A method of constructing a quasi-Poisson structure by a quasi-triple is as follows. Let $(D, G, \mathfrak{h})$ be a quasi-triple with a Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ and $r_{\mathfrak{h}}$ in $\mathfrak{d} \otimes \mathfrak{d}$ the canonical r-matrix for $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. We set

$$\pi_D^{\mathfrak{h}} := r_{\mathfrak{h}}^{\mathrm{L}} - r_{\mathfrak{h}}^{\mathrm{R}}, \tag{2.53}$$

where $r_{\mathfrak{h}}^{\mathrm{L}}$ and $r_{\mathfrak{h}}^{\mathrm{R}}$ is denoted as the left- and right-invariant 2-tensors on D with value $r_{\mathfrak{h}}$ at the identity element e in D respectively, and we can see that it is a multiplicative 2-vector field on D. In fact, a pair $(\pi_D^{\mathfrak{h}}, \varphi_{\mathfrak{h}})$, where the element $\varphi_{\mathfrak{h}}$ defined by (2.42), is a quasi-Poisson structure on G.

Proposition 2.3.1 ([1])**.** The above $\pi_D^{\mathfrak{h}}$ and $\varphi_{\mathfrak{h}}$ satisfy the following properties:

$$\frac{1}{2}\left[\pi_D^{\mathfrak{h}}, \pi_D^{\mathfrak{h}}\right] = \varphi_{\mathfrak{h}}^{\mathrm{R}} - \varphi_{\mathfrak{h}}^{\mathrm{L}}, \tag{2.54}$$

$$\left[\pi_D^{\mathfrak{h}}, \varphi_{\mathfrak{h}}^{\mathrm{L}}\right] = \left[\pi_D^{\mathfrak{h}}, \varphi_{\mathfrak{h}}^{\mathrm{R}}\right] = 0, \tag{2.55}$$

$$\mathcal{L}_{X^{\mathrm{R}}} \pi_D^{\mathfrak{h}} = F_{\mathfrak{h}}(X)^{\mathrm{R}} \tag{2.56}$$

for any X in $\mathfrak{g}$. Due to (2.54) and (2.55), $(D, \pi_D^{\mathfrak{h}}, \varphi_{\mathfrak{h}})$ is a quasi-Poisson-Lie group.

The bivector field $\pi_D^{\mathfrak{h}}$ on D can be restricted to on a subgroup G naturally. For the decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$, we obtain another decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathrm{Ad}_g\mathfrak{h}$ for any g in G by the facts that $\mathrm{Ad}_g\mathfrak{d} = \mathfrak{d}$ and $\mathrm{Ad}_g\mathfrak{g} = \mathfrak{g}$. Then the map j_g identifying $\mathfrak{h}_g := \mathrm{Ad}_g\mathfrak{h}$ with $\mathfrak{g}^*$ can be written as

$$j_g = \mathrm{Ad}_g \circ j \circ \mathrm{Ad}^*_{g^{-1}}. \tag{2.57}$$

In fact, for any x in $\mathfrak{g}$ and ξ in $\mathfrak{g}^*$,

$$\begin{aligned}(j(\mathrm{Ad}^*_{g^{-1}}\xi)|x) &= \langle \mathrm{Ad}^*_{g^{-1}}\xi, x\rangle \\ &= \langle \xi, \mathrm{Ad}_g x\rangle \\ &= (j_g(\xi)|\mathrm{Ad}_g x) \\ &= (\mathrm{Ad}_{g^{-1}}(j_g(\xi))|x)\end{aligned}$$

by (2.41) and the invariance of $(\cdot|\cdot)$. By $\mathrm{Ad}_{g^{-1}}(j_g(\xi))$ being in $\mathrm{Ad}_{g^{-1}}\mathfrak{h}_g = \mathrm{Ad}_{g^{-1}}\mathrm{Ad}_g\mathfrak{h} = \mathfrak{h}$ and the isotropy of $\mathfrak{h}$ with respect to $(\cdot|\cdot)$, we have

$$(j(\mathrm{Ad}^*_{g^{-1}}\xi)|x + j(\eta)) = (\mathrm{Ad}_{g^{-1}}(j_g(\xi))|x + j(\eta)) \tag{2.58}$$

for any η in $\mathfrak{g}^*$. Hence, by the nondegeneracy of $(\cdot|\cdot)$, we obtain $j(\mathrm{Ad}^*_{g^{-1}}\xi) = \mathrm{Ad}_{g^{-1}}(j_g(\xi))$. Therefore it follows that

$$\mathrm{Ad}_g(j(\mathrm{Ad}^*_{g^{-1}}\xi)) = j_g(\xi). \tag{2.59}$$

The canonical r-matrix $r_{\mathfrak{h}_g}$ for the Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h}_g)$ satisfies

$$r_{\mathfrak{h}_g} = \mathrm{Ad}_g r_{\mathfrak{h}}, \tag{2.60}$$

where Ad_g is the adjoint action of D on $\mathfrak{d} \otimes \mathfrak{d}$. In order to show the relation (2.60), first we prove that for any $x + j(\xi)$ in $\mathfrak{d}$, there exists the element x' in $\mathfrak{g}$ such that $x + j(\xi) = x' + j_g(\xi)$ (this claim holds for the map $j' : \mathfrak{g}^* \to \mathfrak{h}'$ determined by any Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h}')$). In fact, setting $x + j(\xi) = x' + j_g(\xi')$ for x, x' in $\mathfrak{g}$, ξ and ξ' in $\mathfrak{g}^*$, we obtain

$$x - x' = j_g(\xi') - j(\xi) \in \mathfrak{g}. \tag{2.61}$$

By the isotropy of $\mathfrak{g}$,

$$\begin{aligned}0 &= (x - x'|y) \\ &= (j_g(\xi') - j(\xi)|y) \\ &= (j_g(\xi')|y) - (j(\xi)|y) \\ &= \langle \xi', y\rangle - \langle \xi, y\rangle \\ &= \langle \xi' - \xi, y\rangle\end{aligned}$$

for any y in $\mathfrak{g}$. Since y in $\mathfrak{g}$ is arbitrary, we obtain $\xi' - \xi = 0$. Hence $\xi' = \xi$. Then, by the definition, we have

$$r_{\mathfrak{h}_g}(x' + j_g(\xi)) = j_g(\xi). \tag{2.62}$$

On the other hand, we compute

$$\begin{aligned}
(\mathrm{Ad}_g r_{\mathfrak{h}})(x' + j_g(\xi)) &= \left(\mathrm{Ad}_g \left(\sum_i e_i \otimes j(\varepsilon^i)\right)\right)(x + j(\xi)) \\
&= \sum_i (\mathrm{Ad}_g e_i \otimes \mathrm{Ad}_g(j(\varepsilon^i)))(x + j(\xi)) \\
&= \sum_i (\mathrm{Ad}_g e_i | x + j(\xi)) \mathrm{Ad}_g(j(\varepsilon^i)) \\
&= \sum_i (\mathrm{Ad}_g e_i | j(\xi)) \mathrm{Ad}_g(j(\varepsilon^i)) \\
&= \sum_i \langle \mathrm{Ad}_g e_i, \xi \rangle \mathrm{Ad}_g(j(\varepsilon^i)) \\
&= \sum_i \langle e_i, \mathrm{Ad}^*_{g^{-1}} \xi \rangle \mathrm{Ad}_g(j(\varepsilon^i)) \\
&= \sum_i (\mathrm{Ad}^*_{g^{-1}} \xi)_i \mathrm{Ad}_g(j(\varepsilon^i)) \\
&= \mathrm{Ad}_g \left(j \left(\sum_i (\mathrm{Ad}^*_{g^{-1}} \xi)_i \varepsilon^i\right)\right) \\
&= \mathrm{Ad}_g (j(\mathrm{Ad}^*_{g^{-1}} \xi)) \\
&= j_g(\xi).
\end{aligned}$$

Therefore we obtain the relation (2.60). Let t_g be a twist corresponding to $\mathfrak{h}_g$. By (2.49), we obtain $r_{\mathfrak{h}_g} = r_{\mathfrak{h}} + t_g$, so that it follows that $\mathrm{Ad}_g r_{\mathfrak{h}} - r_{\mathfrak{h}} = r_{\mathfrak{h}_g} - r_{\mathfrak{h}} = t_g$ in $\Lambda^2 \mathfrak{g}$. Setting

$$\pi^{\mathfrak{h}}_{G,g} := \left(\pi^{\mathfrak{h}}_D\Big|_G\right)_g \tag{2.63}$$

for any g in G, we can see that

$$\begin{aligned}
\pi^{\mathfrak{h}}_{G,g} = \left(\pi^{\mathfrak{h}}_D\Big|_G\right)_g &= r^{\mathrm{L}}_{\mathfrak{h},g} - r^{\mathrm{R}}_{\mathfrak{h},g} = \mathrm{L}_{g*} r_{\mathfrak{h}} - \mathrm{R}_{g,*} r_{\mathfrak{h}} \\
&= \mathrm{R}_{g*} \mathrm{R}_{g^{-1}*} \mathrm{L}_{g*} r_{\mathfrak{h}} - \mathrm{R}_{g,*} r_{\mathfrak{h}} = \mathrm{R}_{g*} \mathrm{Ad}_g r_{\mathfrak{h}} - \mathrm{R}_{g*} r_{\mathfrak{h}} \\
&= \mathrm{R}_{g*} (\mathrm{Ad}_g r_{\mathfrak{h}} - r_{\mathfrak{h}}) \\
&= \mathrm{R}_{g*} t_g
\end{aligned}$$

is in $\mathrm{R}_{g*}\Lambda^2\mathfrak{g} = \Lambda^2 T_g G$. Therefore $\pi_G^{\mathfrak{h}}$ is a well-defined 2-vector field on G. Obviously $\pi_G^{\mathfrak{h}}$ inherits the multiplicativity of $\pi_D^{\mathfrak{h}}$. Moreover, it is clear that $\pi_G^{\mathfrak{h}}$ and $\varphi_{\mathfrak{h}}$ defined by (2.42) also satisfy (2.39) and (2.40). Therefore $(G, \pi_G^{\mathfrak{h}}, \varphi_{\mathfrak{h}})$ is a quasi-Poisson-Lie group. We denote a Lie group with such a structure by $G_D^{\mathfrak{h}}$. It follows from (2.53) and (2.49) that a twist t in $\Lambda^2\mathfrak{g}$ deforms $\pi_D^{\mathfrak{h}}$ and $\pi_G^{\mathfrak{h}}$ to $\pi_D^{\mathfrak{h}_t}$ and $\pi_G^{\mathfrak{h}_t}$ respectively, by the following ways:

$$\pi_D^{\mathfrak{h}_t} = \pi_D^{\mathfrak{h}} + t^{\mathrm{L}} - t^{\mathrm{R}}, \tag{2.64}$$

$$\pi_G^{\mathfrak{h}_t} = \pi_G^{\mathfrak{h}} + t^{\mathrm{L}} - t^{\mathrm{R}}, \tag{2.65}$$

For any group pair (D, G), since G is a closed subgroup of D, the quotient space D/G is a smooth manifold, which is the range of moment maps for quasi-Poisson actions defined later. We use this moment maps to carry out the deformation of symplectic structures in Section 3.1. The action of D on itself by left multiplication induces an action of D on D/G. We call it *dressing action* of D on D/G and denote the corresponding infinitesimal action by $X \mapsto X_{D/G}$ for X in $\mathfrak{d}$. Let $p_{D/G} : D \to D/G$ be the natural projection. By the definition, it follows that $X_{D/G} = p_{D/G*}X^{\mathrm{R}}$. The following definition is one of the important notions to define moment maps.

Definition 33 ([1])**.** Let (D, G) be a group pair with a Manin pair $(\mathfrak{d}, \mathfrak{g})$. An isotropic complement $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{d}$ is called *admissible* at a point s in D/G if the infinitesimal dressing action restricted to $\mathfrak{h}$ defines an isomorphism from $\mathfrak{h}$ onto $T_s(D/G)$, that is, the map $\mathfrak{h} \to T_s(D/G)$, $\xi \mapsto \xi_{D/G,s}$ is an isomorphism. A quasi-triple $(D, G, \mathfrak{h})$ is *complete* if $\mathfrak{h}$ is admissible everywhere on D/G.

Any isotropic complement $\mathfrak{h}$ of $\mathfrak{g}$ is admissible at eG in D/G. In fact, for any ξ in $\mathfrak{h}$,

$$\xi_{D/G,eG} = (p_{D/G*}\xi^{\mathrm{R}})_{eG} = p_{D/G*}\xi_e^{\mathrm{R}} = p_{D/G*}\xi \tag{2.66}$$

and the projection $p_{D/G} : \mathfrak{h} \to \mathfrak{d}/\mathfrak{g} (= T_{eG}(D/G))$ is an isomorphism. If the complement $\mathfrak{h}$ is admissible at a point s in D/G, then it is also admissible on some open neighborhood U of s. In fact, since there exists elements $X_1, \dots, X_n$ in $\mathfrak{h}$ such that $\left\{X_{1,D/G,s}, \dots, X_{n,D/G,s}\right\}$ is a basis of $T_s(D/G)$ and $X_{1,D/G}, \dots, X_{n,D/G}$ in $\mathfrak{X}(D/G)$ are C^∞-vector fields, $\left\{X_{1,D/G}, \dots, X_{n,D/G}\right\}$ forms a local frame on some open neighborhood U of s.

Proposition 2.3.2 ([1])**.** Let (D, G) be a group pair. Then at any point s in D/G, there exists an admissible complement $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{d}$.

Let $(D, G, \mathfrak{h})$ be a quasi-triple such that $\mathfrak{h}$ is admissible on an open subset U of D/G. Then for any X in $\mathfrak{g}$, we define the 1-form $\hat{X}_{\mathfrak{h}}$ on U by the fomula

$$\langle \hat{X}_{\mathfrak{h}}, \xi_{D/G} \rangle = (X|\ \xi) \tag{2.67}$$

for any ξ in $\mathfrak{h}$. If a quasi-triple $(D, G, \mathfrak{h})$ is complete, then $\hat{X}_{\mathfrak{h}}$ is a global 1-form on D/G.

Example 31 ([1])**.** Let (T^*G, G) be a group pair with the standard Manin pair $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g})$ associated a Lie algebra $\mathfrak{g}$ with the dual space $\mathfrak{g}^*$. Then $T^*G/G \cong \mathfrak{g}^*$ as a manifold. Let $\{e_i\}_i$ be a basis of $\mathfrak{g}$, $\{e^i\}_i$ the dual basis of $\mathfrak{g}^*$ and (ξ_i) the linear coordinates for $\{e^i\}$ on $\mathfrak{g}^*$. Then the vector fields generating the dressing action are

$$e^i_{D/G} = \frac{\partial}{\partial \xi_i}, \tag{2.68}$$

$$e_{i,D/G,\xi} = \mathrm{ad}^*_{e_i}\xi = -c^k_{ij}\xi_k \frac{\partial}{\partial \xi_j} \tag{2.69}$$

for any ξ in $\mathfrak{g}^* \cong T_\xi \mathfrak{g}^*$, where c^k_{ij}'s are the structure constants of $\mathfrak{g}$ for $\{e_i\}_i$. From the map

$$\mathfrak{g}^* \to T_\xi \mathfrak{g}^*,\ e^i \mapsto e^i_{D/G} = \left(\frac{\partial}{\partial \xi_i}\right)_\xi \tag{2.70}$$

for any ξ in $\mathfrak{g}^*$, it follows that a Manin quasi-triple $(T^*G, G, \mathfrak{g}^*)$ is complete. Hence we can define global 1-forms $\hat{e}_i$ corresponding to the elements e_i in $\mathfrak{g}$. Then $\hat{e}_i = d\xi_i$ holds.

Example 32 ([1])**.** Let (G, π_G) be a connected and simply connected Poisson-Lie group, G^* the dual Poisson-Lie group of (G, π_G) and $G \bowtie G^*$ the double Lie group of (G, π_G). Then $(G \bowtie G^*, G, \mathfrak{g}^*)$ is a quasi-triple. Moreover, if (G, π_G) is complete, $(G \bowtie G^*, G, \mathfrak{g}^*)$ is complete. In fact, since π_G is complete, the double Lie group $G \bowtie G^*$ is diffeomorphic to $G \times G^*$ as a manifold by Proposition 2.2.20, and $G \bowtie G^*/G \cong G^*$ as a manifold holds. Then $\xi_{D/G}$ on $G \bowtie G^*/G$ for ξ in $\mathfrak{g}^*$ is identified with the right-invariant vector field ξ^{R} on G^* with value ξ at the identity. Therefore a map

$$\mathfrak{g}^* \to T_s(G \bowtie G^*/G) \cong T_u G^*,\ \xi \mapsto \xi_{D/G,s} = \xi^{\mathrm{R}}_u \tag{2.71}$$

for any $s = (g, u)G$ in $G \bowtie G^*/G$ is an isomorphism, and $(G \bowtie G^*, G, \mathfrak{g}^*)$ is complete. Hence a global 1-form $\hat{X}_{\mathfrak{g}^*}$ on $G \bowtie G^*/G$ is identifies with a right-invariant 1-form X^{R} on G^* with value X at the identity due to (2.67).

For any quasi-triple $(D, G, \mathfrak{h})$, we can define a 2-vector field on D/G as follows: The 2-vector field $\pi_D^{\mathfrak{h}}$ defined by (2.53) on D is projectable by the natural projection $p_{D/G} : D \to D/G$, i.e., for any g and h in D, if $gG = hG$, then $p_{D/G*}\pi_{D,g}^{\mathfrak{h}} = p_{D/G*}\pi_{D,h}^{\mathfrak{h}}$. Hence we define

$$\pi_{D/G}^{\mathfrak{h}} := p_{D/G*}\pi_D^{\mathfrak{h}}. \tag{2.72}$$

Since all left-invariant vector fields generated by $\mathfrak{g}$ are projected to zero, it follows that $p_{D/G*}r_{\mathfrak{h}}^{\mathrm{L}}$ vanishes. Therefore we obtain

$$\pi_{D/G}^{\mathfrak{h}} = -p_{D/G*}r_{\mathfrak{h}}^{\mathrm{R}} \tag{2.73}$$

$$= -r_{\mathfrak{h},D/G}. \tag{2.74}$$

From here, we consider only connected quasi-Poisson-Lie group $G_D^{\mathfrak{h}}$ defined as above by a quasi-triple $(D, G, \mathfrak{h})$. For a smooth manifold M with a 2-vector field π_M, a quasi-Poisson action is defined as follows. It is a generalization of Poisson actions of connected Poisson-Lie groups by using Theorem 2.2.15.

Definition 34 ([1])**.** Let (M, π_M) be a manifold with a 2-vector field π_M, $G_D^{\mathfrak{h}} = (G, \pi_G^{\mathfrak{h}}, \varphi_{\mathfrak{h}})$ be a connected quasi-Poisson Lie group induced by a quasi-triple $(D, G, \mathfrak{h})$, $\mathfrak{g}$ the Lie algebra of G and σ an action of G on M. Then the action σ is a *quasi-Poisson action* of $G_D^{\mathfrak{h}}$ on (M, π_M) if for each X in $\mathfrak{g}$,

$$\frac{1}{2}[\pi_M, \pi_M] = (\varphi_{\mathfrak{h}})_\sigma, \tag{2.75}$$

$$\mathcal{L}_{X_\sigma}\pi_M = F_{\mathfrak{h}}(X)_\sigma, \tag{2.76}$$

where x_σ is a fundamental multi-vector field for any x in $\wedge^*\mathfrak{g}$. Here $F_{\mathfrak{h}}$ is the dual of the map (2.43). Then a 2-vector field π_M is called a *quasi-Poisson $G_D^{\mathfrak{h}}$-structure* on M and (M, π_M) is called a *quasi-Poisson $G_D^{\mathfrak{h}}$-manifold* or simply a *quasi-Poisson manifold*.

Remark 4. A connected quasi-Poisson-Lie group $G_D^{\mathfrak{h}}$ with the natural left action is not a quasi-Poisson $G_D^{\mathfrak{h}}$-manifold. In fact, $(\varphi_{\mathfrak{h}})_G = \varphi_{\mathfrak{h}}^R$.

Example 33 ([1])**.** We consider the dressing action on D/G restricted to

G. Then it follows that $\pi^{\mathfrak{h}}_{D/G}$ satisfies (2.75) and (2.76). In fact,

$$\begin{aligned}
\left[\pi^{\mathfrak{h}}_{D/G}, \pi^{\mathfrak{h}}_{D/G}\right] &= \left[p_{D/G*}\pi^{\mathfrak{h}}_{D}, p_{D/G*}\pi^{\mathfrak{h}}_{D}\right] \\
&= p_{D/G*}\left[\pi^{\mathfrak{h}}_{D}, \pi^{\mathfrak{h}}_{D}\right] \\
&= p_{D/G*}\left(2\left(\varphi^{\mathrm{R}}_{\mathfrak{h}} - \varphi^{\mathrm{L}}_{\mathfrak{h}}\right)\right) \\
&= 2\left(p_{D/G*}\varphi^{\mathrm{R}}_{\mathfrak{h}} - p_{D/G*}\varphi^{\mathrm{L}}_{\mathfrak{h}}\right) \\
&= 2p_{D/G*}\varphi^{\mathrm{R}}_{\mathfrak{h}} \\
&= 2\left(\varphi_{\mathfrak{h}}\right)_{D/G}, \\
\mathcal{L}_{X_{D/G}}\pi^{\mathfrak{h}}_{D/G} &= \left[X_{D/G}, \pi^{\mathfrak{h}}_{D/G}\right] \\
&= \left[p_{D/G*}X^{\mathrm{R}}, p_{D/G*}\pi^{\mathfrak{h}}_{D}\right] \\
&= p_{D/G*}\left[X^{\mathrm{R}}, \pi^{\mathfrak{h}}_{D}\right] \\
&= p_{D/G*}\mathcal{L}_{X^{\mathrm{R}}}\pi^{\mathfrak{h}}_{D} \\
&= p_{D/G*}F_{\mathfrak{h}}(X)^{\mathrm{R}} \\
&= F_{\mathfrak{h}}(X)_{D/G}
\end{aligned}$$

for any X in $\mathfrak{g}$, where we use the formulas (2.54) and (2.56). Therefore $(D/G, \pi^{\mathfrak{h}}_{D/G})$ is a quasi-Poisson $G^{\mathfrak{h}}_{D}$-manifold.

Let $(M, \pi^{\mathfrak{h}}_{M})$ be a quasi-Poisson $G^{\mathfrak{h}}_{D}$-manifold. We set

$$\pi^{\mathfrak{h}_t}_{M} := \pi^{\mathfrak{h}}_{M} - t_M. \tag{2.77}$$

Then $(M, \pi^{\mathfrak{h}_t}_{M})$ is a quasi-Poisson $G^{\mathfrak{h}_t}_{D}$-manifold. This shows that we can consider a family of quasi-Poisson $G^{\mathfrak{h}_t}_{D}$-manifolds $(M, \pi^{\mathfrak{h}_t}_{M})$. A moment map for a quasi-Poisson action is defined as a map with a condition not for one quasi-Poisson $G^{\mathfrak{h}}_{D}$-structure but for a family of quasi-Poisson $G^{\mathfrak{h}_t}_{D}$-structure parametrized by twists t in $\Lambda^2\mathfrak{g}$.

Definition 35 ([1])**.** Let $G^{\mathfrak{h}}_{D}$ be a connected quasi-Poisson Lie group defined by a quasi-triple $(D, G, \mathfrak{h})$ and $(M, \pi^{\mathfrak{h}}_{M})$ a quasi-Poisson $G^{\mathfrak{h}}_{D}$-manifold. Then a map $\mu : M \to D/G$ which is G-equivariant with respect to the G-action σ on M and the dressing action of G on D/G is a *moment map* for the quasi-Poisson action σ of $G^{\mathfrak{h}}_{D}$ on $(M, \pi^{\mathfrak{h}}_{M})$ if on any open subset $U \subset M$,

$$X_\sigma = -(\pi^{\mathfrak{h}'}_{M})^{\sharp}(\mu^*(\hat{X}_{\mathfrak{h}'})) \tag{2.78}$$

for any isotropic complement $\mathfrak{h}'$ admissible on $\mu(U)$ and X in $\mathfrak{g}$. Here $\langle(\pi_M^{\mathfrak{h}'})^\sharp\alpha, \beta\rangle := \pi_M^{\mathfrak{h}'}(\alpha, \beta)$. We call a quasi-Poisson action with a moment map a *quasi-Poisson-Hamiltonian action*.

Actually we need not impose the equation (2.78) on all admissible complements because we have the following proposition.

Proposition 2.3.3 ([1])**.** Let $\mathfrak{h}$ and $\mathfrak{h}'$ be two complements admissible at a point s in D/G, and p in M be such that $\mu(p) = s$. Then, at the point p, conditions (2.78) for $\mathfrak{h}$ and $\mathfrak{h}'$ are equivalent, namely

$$(\pi_M^{\mathfrak{h}})^\sharp(\mu^*(\hat{X}_{\mathfrak{h}}))_p = (\pi_M^{\mathfrak{h}'})^\sharp(\mu^*(\hat{X}_{\mathfrak{h}'}))_p. \tag{2.79}$$

In particular, if there exists a isotropic complement $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{d}$ such that a quasi-triple $(D, G, \mathfrak{h})$ is complete, it is sufficient that the equation holds (2.78) for $\mathfrak{h}$.

Now we show important examples for moment maps for quasi-Poisson-Hamiltonian actions.

Example 34 (Poisson manifolds [1],[4],[21])**.** Let (M, π) be a Poisson manifold on which a connected and simply connected Poisson-Lie group (G, π_G) acts by a Poisson action σ. Then (M, π) is a quasi-Poisson $(G, \pi_G, 0)$-manifold and σ is a quasi-Poisson action on (M, π). In fact, the Manin triple $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ corresponding to (G, π_G) is a Manin quasi-triple and the multiplicative 2-vector field π_G on G coincides with the 2-vector field $\pi_G^{\mathfrak{g}^*}$ defined by the corresponding quasi-triple $(G \bowtie G^*, G, \mathfrak{g}^*)$ given by (2.63). Since σ is a Poisson (G, π_G)-action of (M, π), it follows that $[\pi, \pi] = 0$ and that for any X in $\mathfrak{g}$,

$$\mathcal{L}_{X_\sigma}\pi = \delta(X)_\sigma, \tag{2.80}$$

where δ is the 1-cocycle belonging to the tangential Lie bialgebra of $(\mathfrak{g}, \delta)$. Since the cobracket $F_{\mathfrak{g}^*}$ coincides with δ, the action σ is a quasi-Poisson action by Definition 34.

We assume that π_G is complete and that there exists a G-equivariant moment map $\mu : M \to G^*$ for the Poisson action σ, where G^* is the dual Poisson-Lie group of (G, π_G). Then σ is a quasi-Poisson-Hamiltonian action with a moment map μ. Actually, by the definition, the map μ satisfies

$$X_\sigma = -\pi^\sharp(\mu^*(X^{\mathrm{R}})) \tag{2.81}$$

for any X in $\mathfrak{g}$, where X^L is a left-invariant 1-form on G^* with value X at e in G^*. The quasi-triple $(G \bowtie G^*, G, \mathfrak{g}^*)$ is complete since π_G is complete and the global 1-form $\hat{X}_{\mathfrak{g}^*}$ on $G \bowtie G^*/G \cong G^*$ coincides with X^{R} (Example 32). The complement $\mathfrak{g}^*$ is admissible at any point in $G \bowtie G^*/G$, so that the map $\mu : M \to G^* \cong G \bowtie G^*/G$ is a moment map for the quasi-Poisson action σ because of (2.81) and Proposition 2.3.3.

Example 35 (symplectic manifolds [1],[36])**.** Let (M, ω) be a symplectic manifold on which a connected Lie group G acts by a symplectic-Hamiltonian action σ. Since the symplectic structure ω induces a Poisson structure π, the pair (M, π) is a Poisson manifold. Then the action σ is a Poisson action of a trivial Poisson Lie group $(G, 0)$ on (M, π). The trivial Poisson structure 0 on G is complete (Example 24) and a quasi-triple $(T^*G, G, \mathfrak{g}^*)$ corresponding to $(G, 0)$ is also complete (Example 31). The dual group G^* of $(G, 0)$ is the Abelian group $\mathfrak{g}^*$ and the moment map μ for symplectic action σ is G-equivariant with respect to σ on M and the dressing action Ad^* on $G^* = \mathfrak{g}^*$ by Example 26. Thus the map $\mu : M \to G^* = \mathfrak{g}^*$ is a moment map for the Poisson action σ. Therefore, similarly to Example 34, the map $\mu : M \to \mathfrak{g}^* = G^* \cong T^*G/G$ is a moment map for the quasi-Poisson action σ on the quasi-Poisson $(G, 0, 0)$-manifold (M, π).

For a quasi-Poisson manifold with a quasi-Poisson-Hamiltonian action, the following theorem holds.

Theorem 2.3.4 ([1])**.** Let $(M, \pi_M^{\mathfrak{h}})$ be a quasi-Poisson manifold on which a quasi-Poisson Lie group $G_D^{\mathfrak{h}}$ defined by a quasi-triple $(D, G, \mathfrak{h})$ acts by a quasi-Poisson-Hamiltonian action σ . For any p in M, if both $\mathfrak{h}'$ and $\mathfrak{h}''$ are admissible at $\mu(p)$ in D/G, then

$$\mathrm{Im}(\pi_M^{\mathfrak{h}'})_p^\sharp = \mathrm{Im}(\pi_M^{\mathfrak{h}''})_p^\sharp,$$

where μ is a moment map for σ.

2.4 Lie algebroids

In this section, we recall Lie algebroid, Lie bialgebroid and quasi-Lie bialgebroid theory. In addition, we also recall relations between Lie bialgebroids and Poisson-Nijenhuis structures.

Definition 36. Let M be a manifold, A a vector bundle over M, $[\cdot, \cdot]_A$ a Lie bracket on the space $\Gamma(A)$ of the global sections of A and $\rho_A : A \to TM$ a bundle map over M. Then a triple $(A, [\cdot, \cdot]_A, \rho_A)$ is a *Lie algebroid* over M if the followings hold:

(i) $\rho_A([X,Y]_A) = [\rho_A(X), \rho_A(Y)]$;

(ii) $[X, fY]_A = f[X,Y]_A + (\rho_A(X)f)Y$

for any X, Y in $\Gamma(A)$ and f in $C^\infty(M)$. We call the bundle map ρ_A the *anchor map*.

Remark 5. The condition (i) in Definition 36 is induced by a Lie bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ and the conditions (ii) in Definition 36. In fact, for any X, Y, Z in $\Gamma(A)$ and f in $C^\infty(M)$, we compute that

$$\begin{aligned} 0 &= [[X,Y]_A, fZ]_A + [[Y, fZ]_A, X]_A + [[fZ, X]_A, Y]_A \\ &= f([[X,Y]_A, Z]_A + [[Y,Z]_A, X]_A + [[Z,X]_A, Y]_A) \\ &\quad\quad + (\rho_A([X,Y]_A)f - \rho_A(X)(\rho_A(Y)f) + \rho_A(Y)(\rho_A(X)f))Z \\ &= 0 + (\rho_A([X,Y]_A)f - [\rho_A(X), \rho_A(Y)]f)Z \\ &= (\rho_A([X,Y]_A)f - [\rho_A(X), \rho_A(Y)]f)Z. \end{aligned}$$

Since Z in $\Gamma(A)$ and f in $C^\infty(M)$ are arbitrary, the condition (i) in Definition 36 holds.

Example 36. Any finite dimensional Lie algebra $\mathfrak{g}$ is a Lie algebroid over a point.

Example 37. Let M be a manifold. Then the tangent bundle TM of M is a Lie algebroid over M, where a Lie bracket is the ordinal Lie bracket $[\cdot,\cdot]$ on $\mathfrak{X}(M)$ and an anchor map is the identity map $\mathrm{id} : TM \to TM$. We call this Lie algebroid the *standard Lie algebroid* and denote by the same symbol TM.

Remark 6. It is well known that some Lie algebroids are constructed from *Lie groupoids* similarly to the fact that the tangent spaces at the identities of Lie groups has Lie algebra structures (see [23] for a detailed definition and properties of a Lie groupoid). By regarding Lie groups as Lie groupoids over a point, Lie algebras are Lie algebroids constructed from the groupoids. However, any Lie algebroid is not always constructed from a Lie groupoid. A Lie algebroid constructed from a Lie groupoid is called a *integrable Lie algebroid*.

The following examples are important.

Example 38. Let E be any vector bundle over a manifold M. By setting a bracket and an anchor map by $[\cdot,\cdot]_E := 0$ and $\rho_E := 0$ respectively, the triple $(E, [\cdot,\cdot]_E, \rho_E)$ is a Lie algebroid. The pair $([\cdot,\cdot]_E, \rho_E)$ is called the *trivial Lie algebroid structure* over E.

Example 39 (Nijenhuis structures)**.** Let M be a manifold and N a $(1,1)$-tensor on M. Then N is *Nijenhuis* if N satisfies $\mathcal{T}_N = 0$, where the $(2,1)$-tensor $\mathcal{T}_N$ is called the *Nijenhuis torsion* of N and defined by

$$\mathcal{T}_N(X,Y) := [NX,NY] - N[NX,Y] - N[X,NY] + N^2[X,Y] \qquad (2.82)$$

for any X and Y in $\mathfrak{X}(M)$. A bracket $[\cdot,\cdot]_N$ defined by, for any X and Y in $\mathfrak{X}(M)$,

$$[X,Y]_N := [NX,Y] + [X,NY] - N[X,Y] \qquad (2.83)$$

is a Lie bracket on $\Gamma(TM) = \mathfrak{X}(M)$. We set an anchor map as $N : TM \to TM$ and the Leibniz rule

$$[X,fY]_N = f[X,Y]_N + ((NX)f)Y \qquad (2.84)$$

for any X, Y in $\mathfrak{X}(M)$ and f in $C^\infty(M)$ holds. Therefore $[\cdot,\cdot]_N$ and N make the tangent bundle TM of M a Lie algebroid over M. We denote the Lie algebroid $(TM, [\cdot,\cdot]_N, N)$ by $(TM)_N$.

Example 40 (Poisson structures)**.** Let M be a manifold and π a Poisson structure on M. Then a bracket $[\cdot,\cdot]_\pi$ defined by (2.31) and an anchor map $\pi^\sharp$ make the cotangent bundle T^*M of M a Lie algebroid over M due to (2.32). We denote the Lie algebroid $(T^*M, [\cdot,\cdot]_\pi, \pi^\sharp)$ by $(T^*M)_\pi$.

Example 41 (twisted Poisson structures)**.** Let M be a manifold, π a 2-vector field on M and ϕ a closed 3-form on M. Then a pair (π,ϕ) is a *twisted Poisson structure* [37] if the pair satisfies

$$\frac{1}{2}[\pi,\pi] = \pi^\sharp\phi. \qquad (2.85)$$

Then setting a bracket $[\cdot,\cdot]_\pi^\phi$ on $\Omega^1(M)$ by

$$\begin{aligned}[\alpha,\beta]_\pi^\phi &:= \mathcal{L}_{\pi^\sharp\xi}\eta - \mathcal{L}_{\pi^\sharp\eta}\xi - d\langle\pi^\sharp\xi,\eta\rangle + \phi(\pi^\sharp\alpha,\pi^\sharp\beta,\cdot)\\ &= [\xi,\eta]_\pi + \phi(\pi^\sharp\alpha,\pi^\sharp\beta,\cdot)\end{aligned}$$

for any α and β in $\Omega(M)$, we obtain a Lie algebroid $(T^*M)_{\pi,\phi} := (T^*M, [\cdot,\cdot]_\pi^\phi, \pi^\sharp)$.

We define the differential and the Lie derivative of the Lie algebroid A.

Definition 37. Let M be a manifold, $(A, [\cdot,\cdot]_A, \rho_A)$ a Lie algebroid over M. Then an operator $d_A : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^{k+1} A^*)$ is the *differential of the Lie algebroid* A if for any ω in $\Gamma(\Lambda^k A^*)$ and $X_0, \ldots, X_k$ in $\Gamma(A)$,

$$\begin{aligned}(d_A\omega)(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i \rho_A(X_i)(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j]_A, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).\end{aligned} \tag{2.86}$$

For any X in $\Gamma(A)$, the *Lie derivative* $\mathcal{L}_X^A : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^k A^*)$ is defined by the *Cartan formula*

$$\mathcal{L}_X^A = d_A \iota_X + \iota_X d_A \tag{2.87}$$

and are extended on $\Gamma(\Lambda^* A)$ in the same way as the usual Lie derivative $\mathcal{L}_X$ respectively. For example, it follows that

$$\mathcal{L}_X^A Y = [X, Y]_A \tag{2.88}$$

for any X and Y in $\Gamma(A)$.

Example 42. We consider the standard Lie algebroid $TM = (TM, [\cdot,\cdot], \mathrm{id})$. Then the differential of the Lie algebroid TM is just the usual exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$. The Lie derivative is also the usual Lie derivetive $\mathcal{L}$.

Similarly to the usual exterior derivative d, we notice that $d_A^2 = 0$ for any Lie algebroid A. Conversely, an operator $\delta : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$ satisfying $\delta^2 = 0$ and the Leibniz rule

$$\delta(\omega \wedge \eta) = \delta\omega \wedge \eta + (-1)^k \omega \wedge \delta\eta \tag{2.89}$$

for any ω in $\Gamma(\Lambda^k A^*)$ and η in $\Gamma(\Lambda^* A^*)$ constructs a Lie algebroid structure $([\cdot,\cdot]_A, \rho_A)$ on A. In fact, we may define

$$\rho_A(X)f := \langle \delta f, X \rangle \tag{2.90}$$

$$\langle \alpha, [X, Y]_A \rangle := \rho_A(X)\langle \alpha, Y \rangle - \rho_A(Y)\langle \alpha, X \rangle - (\delta\alpha)(X, Y) \tag{2.91}$$

for any α in $\Gamma(A^*)$, X and Y in $\Gamma(A)$.

The Schouten bracket on $\Gamma(\Lambda^* A)$ is defined similarly to the Schouten bracket $[\cdot,\cdot]$ on $\mathfrak{X}^*(M)$. That is, the Schouten bracket $[\cdot,\cdot]_A : \Gamma(\Lambda^k A) \times$

$\Gamma(\Lambda^l A) \to \Gamma(\Lambda^{k+l-1} A)$ is defined as the unique extension of the Lie bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ such that

$[f, g]_A = 0$;
$[X, f]_A = \rho_A(X)f$;
$[X, Y]_A$ is the Lie bracket on $\Gamma(A)$;
$[D_1, D_2 \wedge D_3] = [D_1, D_2] \wedge D_3 + (-1)^{(\deg D_1+1)\deg D_2} D_2 \wedge [D_1, D_3]$;
$[D_1, D_2]_A = -(-1)^{(\deg D_1-1)(\deg D_2-1)}[D_2, D_1]_A$

for any f, g in $C^\infty(M)$, X, Y in $\Gamma(A)$, D_i in $\Gamma(\Lambda^* A)$.

In addition, the Schouten bracket satisfies the *graded Jacobi identity* (2.12).

The same relation between the standard Schouten bracket $[\cdot,\cdot]$ and the usual Lie derivative $\mathcal{L}$ holds for the Schouten bracket $[\cdot,\cdot]_A$ and the usual Lie derivative $\mathcal{L}^A$ on a Lie algebroid A.

Example 43. Let $(T^*M)_\pi$ be a Lie algebroid over M defined by a Poisson structure π on M. We denote the differential of the Lie algebroid $(T^*M)_\pi$ by d_π. Then it follows that

$$d_\pi = [\pi, \cdot].$$

We denote by $\mathcal{L}^\pi$ the Lie derivative induced by d_π.

Example 44. Let $(TM)_N$ be a Lie algebroid over M defined by a Nijenhuis structure N on M. We denote the differential of the Lie algebroid $(TM)_N$ by d_N. Then it follows that

$$d_N = \iota_N \circ d - d \circ \iota_N,$$

where ι_N is the degree 0 derivation of $(\Omega^*(M), \wedge)$ defined by

$$(\iota_N \alpha)(X_1, \ldots, X_k) := \sum_i \alpha(X_1, \ldots, NX_i, \ldots, X_k) \tag{2.92}$$

for any α in $\Omega^k(M)$. We denote by $\mathcal{L}^N$ the Lie derivative induced by d_N.

We define Lie bialgebroids.

Definition 38 ([34])**.** A *Lie bialgebroid* over M is a dual pair (A, A^*) of vector bundles over M equipped with Lie algebroid structures such that the differential d_{A^*} on $\Gamma(\Lambda^* A)$ coming from the structure on A^* is a derivation of the Schouten bracket $[\cdot,\cdot]_A$ on $\Gamma(\Lambda^* A)$ obtained by extention of the structure on A. That is,

$$d_{A^*}[D_1, D_2]_A = [d_{A^*}D_1, D_2]_A + (-1)^{\deg D_1 + 1}[D_1, d_{A^*}D_2]_A \qquad (2.93)$$

for any D_i in $\Gamma(\Lambda^* A)$. Sometimes we denote the Lie bialgebroid by (A, d_{A^*}) since the Lie algebroid structure of A^* is decided by the differential d_{A^*}.

Example 45. Any Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebroid over a point. Then the corresponding differential is a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation of g on $\mathfrak{g} \wedge \mathfrak{g}$, i.e.,

$$\delta([X, Y]) = \mathrm{ad}_X \delta(Y) - \mathrm{ad}_Y \delta(X) \qquad (2.94)$$

for any X and Y in $\mathfrak{g}$.

Example 46. Let TM be the tangent bundle TM of M with the ordinal Lie bracket $[\cdot,\cdot]$ on $\mathfrak{X}(M)$ and an anchor map id and $(T^*M)_0$ the cotangent bundle of M with the trivial Lie algebroid structure. Then a pair $(TM, (T^*M)_0)$ is a Lie bialgebroid over M. In fact, the differential d_{A^*} on $\Gamma(\Lambda^* TM) = \mathfrak{X}^*(M)$ is zero, so that for any D_i in $\mathfrak{X}^*(M)$,

$$d_{A^*}[D_1, D_2] = [d_{A^*}D_1, D_2] = [D_1, d_{A^*}D_2] = 0. \qquad (2.95)$$

Example 47. Let π be a Poisson structure on M. Then a pair of the standard Lie algebroid TM and the Lie algebroid $(T^*M)_\pi = (T^*M, [\cdot,\cdot]_\pi, \pi^\sharp)$ is a Lie bialgebroid.

Now we define Poisson-Nijenhuis structures, which was introduced by Magri and Morosi [26] in their strudy of completely integrable systems.

Definition 39 ([16], [26])**.** Let π be a Poisson structure on M and N a Nijenhuis structure on M. Then π and N are *compatible* if they satisfy

$$N \circ \pi^\sharp = \pi^\sharp \circ N^*, \qquad (2.96)$$

and the $(2,1)$-tensor C_π^N given by

$$C_\pi^N(\alpha, \beta) := [\alpha, \beta]_{N\pi^\sharp} - [\alpha, \beta]_\pi^{N^*} \qquad (2.97)$$

for any α and β in $\Omega^1(M)$ vanishes, where for any α and β in $\Omega^1(M)$,

$$[\alpha, \beta]_{N\pi^\sharp} := \mathcal{L}_{N\pi^\sharp\alpha}\beta - \mathcal{L}_{N\pi^\sharp\beta}\alpha - d\langle N\pi^\sharp\alpha, \beta\rangle, \qquad (2.98)$$

$$[\alpha, \beta]_\pi^{N^*} := [N^*\alpha, \beta]_\pi + [\alpha, N^*\beta]_\pi - N^*[\alpha, \beta]_\pi. \qquad (2.99)$$

A pair (π, N) is a *Poisson-Nijenhuis structure* on M if π and N is compatible. The triple (M, π, N) is called a *Poisson-Nijenhuis manifold*. A pair (ω, N), where ω is a symplectic structure on M and N is Nijenhuis, is a *symplectic-Nijenhuis structure* on M if for the corresponding Poisson structure π_ω, a pair (π_ω, N) is a Poisson-Nijenhuis structure on M. The triple (M, ω, N) is called a *symplectic-Nijenhuis manifold*.

Let (π, N) be a Poisson-Nijenhuis structure and set $\pi_N(\alpha, \beta) := \langle N\pi^\sharp\alpha, \beta\rangle$. Then it follows from (2.96) that π_N is a 2-vector field on M. Hence under the assumption (2.96), the bracket $[\cdot,\cdot]_{N\pi^\sharp}$ can be rewritten as $[\cdot,\cdot]_{\pi_N}$. Moreover, then the three brackets $[\cdot,\cdot]_{\pi_N}, [\cdot,\cdot]_\pi^{N^*}$ and $[\cdot,\cdot]_{N,\pi}$ coincide, where for any α and β in $\Omega^1(M)$,

$$[\alpha, \beta]_{N,\pi} := \mathcal{L}^N_{\pi^\sharp\alpha}\beta - \mathcal{L}^N_{\pi^\sharp\beta}\alpha - d_N\langle\pi^\sharp\alpha, \beta\rangle. \tag{2.100}$$

Let (ω, N) be a symplectic-Nijenhuis structure on M and set $\omega_N(X, Y) := \langle\omega^\flat NX, Y\rangle$ for any X and Y in $\mathfrak{X}(M)$. Then it follows from (2.96) that ω_N is a 2-form on M.

The main result of the theory of Poisson-Nijenhuis structures is that they admit the following iteration process:

Theorem 2.4.1 ([16], [26])**.** Let (π, N) be a Poisson-Nijenhuis structure on M. We set $\pi_0 := \pi$ and define a 2-vector field π_{k+1} by the condition $\pi^\sharp_{k+1} = N \circ \pi^\sharp_k$ inductively. Then all pairs (π_k, N^p) $(k, p \geq 0)$ are Poisson-Nijenhuis structures on M. Furthermore for any $k, l \geq 0$, it follows that $[\pi_k, \pi_l] = 0$. The set of Poisson-Nijenhuis structures $\{(\pi_k, N^p)\}$ is called the *hierarchy of Poisson-Nijenhuis structures* of (M, π, N).

The following theorem describes a relation between Poisson-Nijenhuis structures on M and Lie bialgebroids on M.

Theorem 2.4.2 ([15])**.** Let π be a Poisson structure on M and N a Nijenhuis structure on M. Then a pair (π, N) is a Poisson-Nijenhuis structure on M if and only if a pair $((TM)_N, (T^*M)_\pi)$ is a Lie bialgebroid over M.

We recall the definition of Courant algebroids.

Definition 40 ([22])**.** A *Courant algebroid* is a vector bundle $E \longrightarrow M$ equipped with a nondegenerate symmetric bilinear form $\langle\langle\cdot,\cdot\rangle\rangle$ (called the *pairing*) on the bundle, a skew-symmetric bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(E)$ and a bundle map $\rho : E \longrightarrow TM$ such that the following properties are satisfied: for any e, e_1, e_2, e_3 in $\Gamma(E)$, any f and g in $C^\infty(M)$,

(i) $\sum_{\text{Cycl}(e_1,e_2,e_3)}[\![[\![e_1,e_2]\!],e_3]\!] = \frac{1}{3}\sum_{\text{Cycl}(e_1,e_2,e_3)}\mathcal{D}\langle\!\langle[\![e_1,e_2]\!],e_3\rangle\!\rangle$;

(ii) $\rho([\![e_1,e_2]\!]) = [\rho(e_1),\rho(e_2)]$;

(iii) $[\![e_1, fe_2]\!] = f[\![e_1,e_2]\!] + (\rho(e_1)f)e_2 - \langle\!\langle e_1,e_2\rangle\!\rangle\mathcal{D}f$;

(iv) $\rho\circ\mathcal{D} = 0$, i.e., $\langle\!\langle\mathcal{D}f,\mathcal{D}g\rangle\!\rangle = 0$;

(v) $\rho(e)\langle\!\langle e_1,e_2\rangle\!\rangle = \langle\!\langle[\![e,e_1]\!]+\mathcal{D}\langle\!\langle e,e_1\rangle\!\rangle,e_2\rangle\!\rangle + \langle\!\langle e_1,[\![e,e_2]\!]+\mathcal{D}\langle\!\langle e,e_2\rangle\!\rangle\rangle\!\rangle$,

where $\mathcal{D}: C^\infty(M)\longrightarrow\Gamma(E)$ is the smooth map defined by

$$\langle\!\langle\mathcal{D}f,e\rangle\!\rangle = \frac{1}{2}\rho(e)f.$$

The map ρ and the operator $[\![\cdot,\cdot]\!]$ are called an *anchor map* and a *Courant bracket*, respectively.

A Courant algebroid is not a Lie algebroid since the Jacobi identity is not satisfied due to (i).

Definition 41 ([22])**.** Let $(E,[\![\cdot,\cdot]\!],\rho,\langle\!\langle\cdot,\cdot\rangle\!\rangle)$ be a Courant algebroid over M. A subbundle L of E is *isotropic* if it is isotropic under the pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle$. A subbundle L is *integrable* if $\Gamma(L)$ is closed under the bracket $[\![\cdot,\cdot]\!]$. A subbundle L is *Dirac structure* or *Dirac subbundle* if it is maximally isotropic and integrable.

Proposition 2.4.3 ([22])**.** Let $(E,[\![\cdot,\cdot]\!],\rho,\langle\!\langle\cdot,\cdot\rangle\!\rangle)$ be a Courant algebroid over M and a subbundle L a Dirac subbundle. Then $(L,[\![\cdot,\cdot]\!]|_L,\rho|_L)$ is a Lie algebroid over M.

The following two theorems show that Lie bialgebroids and Courant algebroids are a generalization of Lie bialgebras and double Lie algebra.

Theorem 2.4.4 ([22])**.** If (A,A^*) is a Lie bialgebroid, then $E := A\oplus A^*$ equipped with $([\![\cdot,\cdot]\!],\rho,\langle\!\langle\cdot,\cdot\rangle\!\rangle)$ is a Courant algebroid, where

$$\begin{aligned}[\![X+\xi,Y+\eta]\!] := &\left([X,Y]_A + \mathcal{L}^{A^*}_\xi Y - \mathcal{L}^{A^*}_\eta X - \frac{1}{2}d_{A^*}(\langle\xi,Y\rangle - \langle\eta,X\rangle)\right)\\ &+\left([\xi,\eta]_{A^*} + \mathcal{L}^A_X\eta - \mathcal{L}^A_Y\xi + \frac{1}{2}d_A(\langle\xi,Y\rangle - \langle\eta,X\rangle)\right)\end{aligned} \tag{2.101}$$

$$\rho(X+\xi) := \rho_A(X) + \rho_{A^*}(\xi) \tag{2.102}$$

$$\langle\!\langle X+\xi,Y+\eta\rangle\!\rangle := \frac{1}{2}(\langle\xi,Y\rangle + \langle\eta,X\rangle) \tag{2.103}$$

for any X,Y in $\Gamma(A)$, ξ and η in $\Gamma(A^*)$.

Theorem 2.4.5 ([22])**.** Let $(E, [\![\cdot,\cdot]\!], \rho, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$ be a Courant algebroid over M, L_1 and L_2 Dirac subbundles transversal to each other, i.e., $E = L_1 \oplus L_2$. Then a pair (L_1, L_2) is a Lie bialgebroid, where L_2 is identified with the dual bundle of L_1 under the pairing $2\langle\!\langle\cdot,\cdot\rangle\!\rangle$.

An immediate consequence of the above theorems is the duality of Lie bialgebroids, which is a generalization of that of Lie bialgebras.

Corollary 2.4.6 ([22])**.** If (A, A^*) is a Lie bialgebroid, so is (A^*, A).

The following example is fundamental.

Example 48 ([22])**.** Let $(TM, (T^*M)_0)$ be a Lie bialgebroid in Example 46. Then the direct sum $TM \oplus T^*M$ on a C^∞-manifold M is a Courant algebroid by Theorem 2.4.4. Here the anchor map ρ, the pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ and the Courant bracket $[\![\cdot,\cdot]\!]$ are given by

$$\rho(X+\xi) = X, \tag{2.104}$$

$$\langle\!\langle X+\xi, Y+\eta\rangle\!\rangle = \frac{1}{2}(\langle\xi, Y\rangle + \langle\eta, X\rangle), \tag{2.105}$$

$$[\![X+\xi, Y+\eta]\!] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + \frac{1}{2}d(\langle\xi, Y\rangle - \langle\eta, X\rangle) \tag{2.106}$$

respectively, where X and Y are in $\mathfrak{X}(M)$, and ξ and η are in $\Omega^1(M)$. This is called *the standard Courant algebroid* over M.

Next we shall recall the definition of quasi-Lie bialgebroids.

Definition 42 ([34])**.** A *quasi-Lie bialgebroid* is a Lie algebroid $(A, [\cdot,\cdot]_A, a)$ equipped with a degree-one derivation δ of the Gerstenhaber algebra $(\Gamma(\Lambda^*A), \wedge, [\cdot,\cdot]_A)$ and a 3-section of A, ϕ_A in $\Gamma(\Lambda^3 A)$ such that

$$\delta^2 = [\phi_A, \cdot]_A, \tag{2.107}$$

$$\delta\phi_A = 0. \tag{2.108}$$

If the 3-section ϕ_A is equal to 0, the quasi-Lie bialgebroid (A, δ, ϕ_A) is just a Lie bialgebroid (A, δ).

Theorem 2.4.7 ([34])**.** Let (A, δ, ϕ_A) be a quasi-Lie bialgebroid, where $A = (A, [\cdot,\cdot]_A, a)$, and $d_A : \Gamma(\Lambda^*A^*) \to \Gamma(\Lambda^{*+1}A^*)$ be the Lie algebroid derivative of A. Then its double $E = A \oplus A^*$ has naturally a Courant algebroid structure. Namely, it is equipped with an anchor map ρ, a pairing

$\langle\!\langle\cdot,\cdot\rangle\!\rangle$ and a Courant bracket $[\![\cdot,\cdot]\!]$ given by the following: for any X, Y in $\Gamma(A)$, any ξ and η in $\Gamma(A^*)$,

$$\begin{aligned}
\rho(X+\xi) &= a(X) + a_*(\xi),\\
\langle\!\langle X+\xi, Y+\eta\rangle\!\rangle &= \frac{1}{2}(\langle \xi, Y\rangle + \langle \eta, X\rangle),\\
[\![X,Y]\!] &= [X,Y]_A\\
[\![\xi,\eta]\!] &= [\xi,\eta]_{A^*} + \phi_A(X,Y,\cdot)\\
[\![X,\xi]\!] &= \left(\iota_X d_A\xi + \frac{1}{2}d_A\langle\xi,X\rangle\right) - \left(\iota_\xi d_* X + \frac{1}{2}d_*\langle\xi,X\rangle\right),
\end{aligned}$$

where the map $a_* : A^* \longrightarrow TM$ and the bracket $[\cdot,\cdot]_{A^*}$ are defined by

$$\begin{aligned}
a_*(\xi)f &:= \langle \xi, d_* f\rangle,\\
\langle [\xi,\eta]_{A^*}, X\rangle &:= a_*(\xi)\langle\eta, Y\rangle - a_*(\eta)\langle\xi,X\rangle - (d_*X)(\xi,\eta),
\end{aligned}$$

respectively.

Taking $\phi_A = 0$, we obtain the Courant algebroid structure of a double of a Lie bialgebroid (Theorem 2.4.4).

Example 49 ([6])**.** Let M be a manifold, (π,ϕ) a twisted Poisson structure on M and $(T^*M)_{\pi,\phi}$ the corresponding Lie algebroid with (π,ϕ) (Example 41). We set

$$\begin{aligned}
d'f &:= df\\
d'\alpha &:= d\alpha - \iota_{\pi^\sharp\alpha}\phi
\end{aligned}$$

for any f in $C^\infty(M)$ and α in $\Omega^1(M)$. Then the triple $((T^*M)_{\pi,\phi}, d', \phi)$ is a quasi-Lie bialgebroid.

We obtain the definition of Poisson-quasi-Nijenhuis structures as corresponding structures with quasi-Lie bialgebroids by generalizing Poisson-Nijenhuis structures corresponding with Lie bialgebroids.

Definition 43 ([38])**.** Let π be a Poisson structure on M, N a $(1,1)$-tensor on M and ϕ a closed 3-form on M. Then a triple (π, N, ϕ) is a *Poisson-quasi-Nijenhuis structure* on M if the following conditions hold:

(i) $N \circ \pi^\sharp = \pi^\sharp \circ N^*$;

(ii) C_π^N defined by (2.97) vanishes;

(iii) $\mathcal{T}_N(X,Y) = \pi^\sharp(\iota_{X\wedge Y}\phi)$ for any X and Y in $\mathfrak{X}(M)$;

(iv) $\iota_N \phi$ is closed,

where $\iota_{X \wedge Y}\omega := \omega(X, Y, \dots)$ for any ω in $\Omega^*(M)$, X and Y in $\mathfrak{X}(M)$, and ι_N is defined by (2.92). A quadruple (M, π, N, ϕ) is called *Poisson-quasi-Nijenhuis manifold.*

Theorem 2.4.8 ([38])**.** Let π be a Poisson structure on M, N a $(1,1)$-tensor on M and ϕ a closed 3-form on M. Then a triple (π, N, ϕ) is a Poisson-quasi-Nijenhuis structure on M if and only if a triple $((T^*M)_\pi, d_N, \phi)$ is a quasi-Lie bialgebroid over M, where d_N is a degree 1 derivation defined by the formula (2.86) using $N : TM \to TM$ and $[\cdot,\cdot]_N$ defined by (2.83) instead of ρ_A and $[\cdot,\cdot]_A$ respectively.

We can generalize the definition of Poisson-quasi-Nijenhuis structures on manifolds to on Lie algebroids.

Definition 44 ([5])**.** Let $(A, [\cdot,\cdot]_A, \rho_A)$ be a Lie algebroid on M. Let π be a Poisson structure on A, i.e., it satisfies $[\pi, \pi]_A = 0$, $N : A \to A$ a bundle map over M and ϕ a d_A-closed 3-section on A, i.e., ϕ in $\Gamma(\Lambda^3 A)$ and $d_A\phi = 0$. Then a quadruple (A, π, N, ϕ) is a *Poisson-quasi-Nijenhuis Lie algebroid* if the following conditions hold:

(i) $N \circ \pi^\sharp = \pi^\sharp \circ N^*$;

(ii) C_π^N defined by (2.97) for $[\cdot,\cdot]_A$ vanishes;

(iii) $\mathcal{T}_N(X, Y) = \pi^\sharp(\iota_{X \wedge Y}\phi)$ for any X and Y in $\Gamma(A)$;

(iv) $\iota_N \phi$ is d_A-closed,

where $\iota_{X \wedge Y}\omega := \omega(X, Y, \dots)$ for any ω in $\Gamma(\Lambda^* A^*)$, X and Y in $\Gamma(A)$, and ι_N is the degree 0 derivation of $(\Gamma(\Lambda^* A^*), \wedge)$ defined by (2.92).

Chapter 3

Deformations of symplectic structures by moment maps

In this chapter, we carry out deformations of symplectic structures on a smooth manifold. We use the moment map theory for quasi-Poisson actions to do.

3.1 Main result

A moment map for the quasi-Poisson action on a quasi-Poisson $G_D^{\mathfrak{h}}$-manifold $(M, \pi_M^{\mathfrak{h}})$ are defined with the conditions for the family of quasi-Poisson $G_D^{\mathfrak{h}'}$-structures $\left\{\pi_M^{\mathfrak{h}'}\right\}_{\mathfrak{h}'}$ on M. For each complement $\mathfrak{h}'$, there exists a twist t in $\Lambda^2\mathfrak{g}$ such that $\mathfrak{h}' = \mathfrak{h}_t$, so that the family $\left\{\pi_M^{\mathfrak{h}'}\right\}_{\mathfrak{h}'}$ is regarded as the family $\left\{\pi_M^{\mathfrak{h}_t}\right\}_{t\in\Lambda^2\mathfrak{g}}$ parametrized by the twists. When the quasi-Poisson $G_D^{\mathfrak{h}}$-structure $\pi_M^{\mathfrak{h}}$ is induced by a given symplectic structure, we will give the method of finding a quasi-Poisson $G_D^{\mathfrak{h}_t}$-structure which induce a symplectic structure in $\left\{\pi_M^{\mathfrak{h}_t}\right\}_t$. That is, we can deform a given symplectic structure to a new one by a twist t. This deformation can be carried out due to using the family $\left\{\pi_M^{\mathfrak{h}_t}\right\}_t$ with moment map conditions for quasi-Poisson actions. In this regard, it is described as follows in [1]: It would be interesting to find a geometric framework for considering the family $\left\{\pi_M^{\mathfrak{h}_t}\right\}_t$. Our deformation is one of the answers for this proposal.

Let (M, ω) be a symplectic manifold on which an n-dimensional connected Lie group G acts by symplectic-Hamiltonian action σ with a moment map $\mu : M \to \mathfrak{g}^*$. Let π be the non-degenerate Poisson structure on M induced by ω. Then μ is a moment map for the quasi-Poisson-Hamiltonian action σ of $(G, 0, 0)$ on (M, π) by Example 35 in Section 2.3.

Let $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ be the Manin triple corresponding to the trivial Poisson Lie group $(G, 0)$, where $\mathfrak{g} \oplus \mathfrak{g}^*$ has the Lie bracket

$$[X, Y] = [X, Y]_{\mathfrak{g}}, \quad [X, \xi] = \mathrm{ad}^*_X \xi, \quad [\xi, \eta] = [\xi, \eta]_{\mathfrak{g}^*} = 0 \tag{3.1}$$

for any X, Y in $\mathfrak{g}$ and ξ, η in $\mathfrak{g}^*$. Here the bracket $[\cdot, \cdot]_{\mathfrak{g}}$ and $[\cdot, \cdot]_{\mathfrak{g}^*}$ are the brackets on $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. Then the Manin (quasi-)triple $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ defines $F := F_{\mathfrak{g}^*} = 0$ and $\varphi := \varphi_{\mathfrak{g}^*} = 0$ (see (2.42) and (2.43)). Since the corresponding quasi-triple $(T^*G, G, \mathfrak{g}^*)$ is complete by Example 34 and 35, an isotropic complement $\mathfrak{g}^*$ is admissible at any ξ in $\mathfrak{g}^*$ by Definition 33, and hence it is admissible at any ξ in $\mu(M)$.

Let $\mathfrak{g}^*_t$ be an isotropic complement of $\mathfrak{g}$ in $\mathfrak{g} \oplus \mathfrak{g}^*$ to which the twist in $\Lambda^2\mathfrak{g}$ from $\mathfrak{g}^*$ is t. When we deform π to $\pi^t_M := \pi - t_M$ by a twist t, the quasi-Poisson Lie group $(G, 0, 0)$ is deformed to $(G, \pi^t_G, \varphi_{\mathfrak{g}^*_t})$, where $\pi^t_G = t^L - t^R$ and $\varphi_{\mathfrak{g}^*_t} = \frac{1}{2}[t, t] + \varphi_t$ by (2.50) and (2.65). Moreover it follows from $F = 0$ and (2.52) that $\varphi_t = 0$. So $\varphi_{\mathfrak{g}^*_t} = \frac{1}{2}[t, t]$.

On the other hand, it follows from Definition 34 that the quasi-Poisson $(G, \pi^t_G, \varphi_{\mathfrak{g}^*_t})$-manifold (M, π^t_M) satisfies

$$\frac{1}{2}\left[\pi^t_M, \pi^t_M\right] = (\varphi_{\mathfrak{g}^*_t})_M, \tag{3.2}$$

$$\mathcal{L}_{X_M}\pi^t_M = F_{\mathfrak{g}^*_t}(X)_M. \tag{3.3}$$

If $(\varphi_{\mathfrak{g}^*_t})_M = 0$, i.e., $[t, t]_M = 0$, then the 2-vector field π^t_M is a Poisson structure on M by (3.2).

Assume that a twist t in $\Lambda^2\mathfrak{g}$ satisfies $[t, t]$ is ad-invariant. Then $\pi^t_G = t^L - t^R$ is a multiplicative Poisson structure (see [21]). Therefore (G, π^t_G) is a Poisson Lie group. Then it follows that $F_{\mathfrak{g}^*_t}$ coincides with the dual of the bracket map $[\cdot, \cdot]^{\pi^t_G} : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$ defined by the Poisson Lie group (G, π^t_G). In fact, let $j_t : \mathfrak{g}^* \to \mathfrak{g}^*_t$ be the linear isomorphism (2.41) determined by $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*_t)$ and we obtain

$$j_t(\varepsilon^i) = \varepsilon^i + \sum_k t^{ik} e_k \tag{3.4}$$

by (2.48), where $\{e_i\}$ is a basis on $\mathfrak{g}$, a set $\{\varepsilon^i\}$ is the dual basis of $\{e_i\}$ on $\mathfrak{g}^*$ and $t = \frac{1}{2}\sum_{i,j} t^{ij} e_i \wedge e_j$. By using the result of Example 22, formulas

(3.1) and the fact that $j_t^{-1} \circ p_{\mathfrak{g}_t^*} = p_{\mathfrak{g}^*}$, we compute

$$\begin{aligned}
F_{\mathfrak{g}_t^*}^*(\varepsilon^i, \varepsilon^j) &= j_t^{-1}\left(p_{\mathfrak{g}_t^*}\left([j_t(\varepsilon^i), j_t(\varepsilon^j)]\right)\right) \\
&= p_{\mathfrak{g}^*}\left(\left[\varepsilon^i + \sum_k t^{ik} e_k, \varepsilon^j + \sum_l t^{jl} e_l\right]\right) \\
&= p_{\mathfrak{g}^*}\left(\sum_{k,l} t^{ik} t^{jl} [e_k, e_l]_{\mathfrak{g}} + \mathrm{ad}^*_{\sum_k t^{ik} e_k} \varepsilon^j - \mathrm{ad}^*_{\sum_l t^{jl} e_l} \varepsilon^i\right) \\
&= \mathrm{ad}^*_{\sum_k t^{ik} e_k} \varepsilon^j - \mathrm{ad}^*_{\sum_l t^{jl} e_l} \varepsilon^i \\
&= \mathrm{ad}^*_{t^\sharp \varepsilon^i} \varepsilon^j - \mathrm{ad}^*_{t^\sharp \varepsilon^j} \varepsilon^i \\
&= [\varepsilon^i, \varepsilon^j]^{\pi_G^t}.
\end{aligned}$$

Therefore, since G is connected, the condition (3.3) means that the action σ is a Poisson action of (G, π_G^t) on (M, π_M^t) under the assumption that t is an r-matrix and that $[t, t]_M = 0$.

Next, we can write by (2.48),

$$\mathfrak{g}_t^* = \mathrm{span}\left\{ \varepsilon^i + \sum_j t^{ij} e_j \;\middle|\; i = 1, \ldots, n \right\}. \tag{3.5}$$

If $\mathfrak{g}_t^*$ is admissible at any point in $\mu(M)$, then it satisfies $\mathrm{Im}\pi_p^\sharp = \mathrm{Im}(\pi_M^t)_p^\sharp$ for any p in M by Theorem 2.3.4. The non-degeneracy of π means that $\mathrm{Im}\pi_p^\sharp = T_pM$ for any p in M. Therefore, by the fact that $\mathrm{Im}(\pi_M^t)_p^\sharp = T_pM$ for any p in M, a quasi-Poisson structure π_M^t is also non-degenerate.

Here we shall examine the condition for a isotropic complement to be admissible at a point in $\mathfrak{g}^*$ in more detail. Let (ξ_i) be the linear coordinates for $\{\varepsilon^i\}$. Then it follows that for $i = 1, \ldots, n$,

$$\begin{aligned}
\left(\varepsilon^i + \sum_j t^{ij} e_j\right)_{\mathfrak{g}^*} &= \frac{\partial}{\partial \xi_i} - \sum_{j,k,l} t^{ij} c_{jl}^k \xi_k \frac{\partial}{\partial \xi_l} \\
&= \sum_{j,k} \sum_{l \neq i} t^{ij} c_{lj}^k \xi_k \frac{\partial}{\partial \xi_l} + \left(1 + \sum_{j,k} t^{ij} c_{ij}^k \xi_k\right) \frac{\partial}{\partial \xi_i},
\end{aligned} \tag{3.6}$$

where $X \mapsto X_{\mathfrak{g}^*}$, for X in $\mathfrak{g} \oplus \mathfrak{g}^*$, is the infinitesimal action of the dressing action on $\mathfrak{g}^* \cong T^*G/G$ (Example 31). The isotropic complement $\mathfrak{g}_t^*$ is admissible at $\xi = (\xi_1, \dots, \xi_n)$ in $\mathfrak{g}^*$ if and only if the elements (3.6) form a basis on $T_\xi(\mathfrak{g}^*) \cong \mathfrak{g}^*$. Hence this means that the matrix

$$A_t(\xi) := \begin{pmatrix} 1 + \sum_{j,k} t^{1j} c^k_{1j} \xi_k & \sum_{j,k} t^{1j} c^k_{2j} \xi_k & \cdots & \sum_{j,k} t^{1j} c^k_{nj} \xi_k \\ \sum_{j,k} t^{2j} c^k_{1j} \xi_k & 1 + \sum_{j,k} t^{2j} c^k_{2j} \xi_k & \cdots & \sum_{j,k} t^{2j} c^k_{nj} \xi_k \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j,k} t^{nj} c^k_{1j} \xi_k & \sum_{j,k} t^{nj} c^k_{2j} \xi_k & \cdots & 1 + \sum_{j,k} t^{nj} c^k_{nj} \xi_k \end{pmatrix} \tag{3.7}$$

is regular. Therefore $\mathfrak{g}_t^*$ is admissible at ξ in $\mu(M)$ if and only if the matrix $A_t(\xi)$ for ξ in $\mu(M)$ is regular. Since any non-degenerate Poisson structure on M defines a symplectic structure on M, the following theorem holds.

Theorem 3.1.1 (Nakamura [31])**.** Let (M, ω) be a symplectic manifold on which a connected Lie group G with the Lie algebra $\mathfrak{g}$ acts by a symplectic-Hamiltonian action σ, $\mu : M \to \mathfrak{g}^*$ a moment map for σ and π the Poisson structure induced by ω. Then the following holds:

1. If a twist t in $\Lambda^2\mathfrak{g}$ satisfies that $[t, t]_M = 0$, then t deforms the Poisson structure π to a Poisson structure $\pi^t_M := \pi - t_M$. Moreover, if t is an r-matrix, then σ is a Poisson action of (G, π^t_G) on (M, π^t_M), where $\pi^t_G = t^{\mathrm{L}} - t^{\mathrm{R}}$.

2. For a twist t in $\Lambda^2\mathfrak{g}$, if the isotropic complement $\mathfrak{g}_t^*$ is admissible on $\mu(M)$, then t deforms the non-degenerate 2-vector field π to a non-degenerate 2-vector field π^t_M. This condition is equivalent to that the matrix $A_t(\xi)$ defined by (3.7) is regular for any ξ in $\mu(M)$.

Therefore, if a twist t satisfies the assumptions of both 1 and 2, then t deforms ω to a symplectic structure ω^t induced by the non-degenerate Poisson structure π^t_M. In other words, ω and ω^t are deformation-equivalent.

Remark 7. (i) In Section 3.2, we will show that the condition in Theorem 3.1.1 is not a necessary condition for π^t_M to be a non-degenerate Poisson structure.
(ii) If a twist t satisfies the assumptions of both 4.28 and 4.29 and is an r-matrix, then the Poisson action σ of (G, π^t_G) on a symplectic manifold (M, ω^t) has a moment map (although not necessarily G-equivariant) due to Theorem 3.16 in [19].

The following theorem gives a sufficient condition for a twist to deform a symplectic structure in the sense of Theorem 3.1.1.

Theorem 3.1.2 (Nakamura [31])**.** Let (M, ω) be a symplectic manifold on which an n-dimensional connected Lie group G acts by a symplectic-Hamiltonian action σ. Assume that X, Y in $\mathfrak{g}$ satisfy $[X, Y] = 0$. Then the twist $t = \frac{1}{2} X \wedge Y$ in $\Lambda^2 \mathfrak{g}$ deforms the symplectic structure ω to a symplectic structure ω_t. For example, a twist t in $\Lambda^2 \mathfrak{h}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, satisfies the assumption of the theorem.

Proof. For X and Y in $\mathfrak{g}$, we set

$$X = \sum_i X^i e_i, \; Y = \sum_j Y^j e_j,$$

where $\{e_i\}_{i=1}^n$ is a basis on the Lie algebra $\mathfrak{g}$. Then since $[X, Y] = \sum_{i,j,k} X^i Y^j c_{ij}^k e_k = 0$, we obtain the following conditions:

$$\sum_{i,j} X^i Y^j c_{ij}^k = 0$$

for any k, where c_{ij}^k's are the structure constants of $\mathfrak{g}$ with respect to the basis $\{e_i\}$. Moreover, since we have

$$[t, t] = \left[\frac{1}{2} X \wedge Y, \frac{1}{2} X \wedge Y \right] = \frac{1}{2} X \wedge [X, Y] \wedge Y = 0,$$

the twist t is an r-matrix such that $[t, t]_M = 0$ obviously. Hence π_M^t is a Poisson structure, and if π_M^t is non-degenerate, then the twist t induces the symplectic structure ω_t.

We shall show the non-degeneracy of π_M^t. Let μ be the moment map for a given symplectic-Hamiltonian action ψ. We must show that $\mathfrak{g}_t^*$ is admissible at any point in $\mu(M)$. We prove a stronger condition that the quasi-triple $(T^* G, G, \mathfrak{g}_t^*)$ is complete.

Let $\{\varepsilon^i\}$ be the dual basis of $\{e_i\}$ on $\mathfrak{g}^*$ and (ξ_i) be the linear coordinates for $\{\varepsilon^i\}$. Since $t = \frac{1}{2} \sum_{i,j} X^i Y^j e_i \wedge e_j$,

$$\mathfrak{g}_t^* = \operatorname{span} \left\{ \varepsilon^i + \sum_{i,j} X^i Y^j e_j \;\middle|\; i = 1, \cdots n \right\}.$$

Then it follows that for $i = 1, \ldots, n$,

$$\left(\varepsilon^i + \sum_{i,j} X^i Y^j e_j \right)_{\mathfrak{g}^*} = \sum_{i,j,k} \sum_{l \neq i} X^i Y^j c_{lj}^k \xi_k \frac{\partial}{\partial \xi_l} + \left(1 + \sum_{i,j,k} X^i Y^j c_{ij}^k \xi_k \right) \frac{\partial}{\partial \xi_i}. \tag{3.8}$$

The quasi-triple $(T^*G, G, \mathfrak{g}_t^*)$ is complete if and only if the elements (3.8) form a basis on $T_\xi(\mathfrak{g}^*) \cong \mathfrak{g}^*$ for any $\xi = (\xi_1, \ldots, \xi_n)$. Therefore we shall prove that the matrix

$$\begin{pmatrix} 1+\sum_{j,k} X^1 Y^j c_{1j}^k \xi_k & \sum_{j,k} X^1 Y^j c_{2j}^k \xi_k & \cdots & \sum_{j,k} X^1 Y^j c_{nj}^k \xi_k \\ \sum_{j,k} X^2 Y^j c_{1j}^k \xi_k & 1+\sum_{j,k} X^2 Y^j c_{2j}^k \xi_k & \cdots & \sum_{j,k} X^2 Y^j c_{nj}^k \xi_k \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j,k} X^n Y^j c_{1j}^k \xi_k & \sum_{j,k} X^n Y^j c_{2j}^k \xi_k & \cdots & 1+\sum_{j,k} X^n Y^j c_{nj}^k \xi_k \end{pmatrix} \tag{3.9}$$

is regular. In the case of $X = 0$, this matrix is equal to the identity matrix, so that it is regular. In the case of $X \neq 0$, using $\sum_{i,j} X^i Y^j c_{ij}^k = 0$, we can transform the matrix to the identity matrix. Thus the matrix (3.9) is regular. Therefore $\mathfrak{g}_t^*$ is admissible at any point in $\mathfrak{g}^*$. That is, $(T^*G, G, \mathfrak{g}_t^*)$ is complete. □

Remark 8. We try to generalize the assumption of Theorem 3.1.2 and consider X, Y in $\mathfrak{g}$ such that $[X, Y] = aX + bY$ $(a, b \in \mathbb{R})$, that is, the subspace spanned by X, Y is also a Lie subalgebra. We set $t = \frac{1}{2} X \wedge Y$ in $\Lambda^2\mathfrak{g}$. Since $[t, t] = 0$, the twist t is an r-matrix such that $[t, t]_M = 0$. Therefore the symplectic action ψ is a Poisson action of (G, π_G^t) on (M, π_M^t). Then we research whether $\mathfrak{g}_t^*$ is admissible at each point in $\mathfrak{g}^*$. Similarly to the proof of Theorem 3.1.2, a matrix to check the regularity can be deformed to

$$\begin{pmatrix} 1+\sum_k (aX^k + bY^k)\xi_k & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Therefore this matrix is regular if and only if

$$1 + \sum_k (aX^k + bY^k)\xi_k \neq 0.$$

In the case of $[X, Y] = 0$, by Theorem 3.1.2, the space $\mathfrak{g}_t^*$ is admissible at all points in $\mathfrak{g}^*$. In the case of $[X, Y] \neq 0$, the above condition means

$$\langle [X, Y], \xi \rangle \neq -1.$$

Let ξ' be an element satisfying that $\langle [X, Y], \xi' \rangle \neq 0$. By setting

$$\xi := -\frac{\xi'}{\langle [X, Y], \xi' \rangle},$$

we obtain $\langle [X,Y],\xi\rangle = -1$, so that $\mathfrak{g}_t^*$ is not admissible at ξ. Eventually, to make sure of the admissibility of $\mathfrak{g}_t^*$, we need check whether such a point ξ is included in $\mu(M)$.

3.2 Examples

In this section, we compute specifically which element t in $\Lambda^2\mathfrak{g}$ defines a new symplectic structure ω_t from given one ω on a smooth manifold.

Example 50. We consider $(\mathbb{R}^{2n},\omega_0)$ with a symplectic-Hamiltonian action by the parallel transformation (Example 7). The Lie algebra $\mathbb{R}^n$ of an additive group $\mathbb{R}^n$ has the commutative bracket $[\cdot,\cdot]$. Hence since $[a,b]=0$ for any a and b in $\mathbb{R}^n$, an element $t := a\wedge b$ in $\Lambda^2\mathbb{R}^n$ deforms ω_0 by Theorem 3.1.2.

Next we consider the complex projective space $(\mathbb{C}\mathrm{P}^n,\omega_{\mathrm{FS}})$ on which the special unitary group $\mathrm{SU}(n+1)$ acts naturally as a symplectic-Hamiltonian action with a moment map μ (Example 4 and Example 8).

We use

$$\begin{aligned} X_{jk} &: \text{the } (j,k)\text{-element is } 1, \text{ the } (k,j)\text{-element is } -1, \text{ and the rest are } 0,\\ Y_{jk} &: \text{the } (j,k)\text{- and } (k,j)\text{-elements are } i, \text{ and the rest are } 0,\\ Z_l &: \text{the } (l,l)\text{–element is } i, \text{ the } (n+1,n+1)\text{-element is } -i,\\ &\quad \text{and the rest are } 0 \end{aligned}$$

for $1\le j<k\le n+1$ and $l=1,\dots,n$, as a basis of $\mathfrak{su}(n+1)$ which is defined by a Chevalley basis of the complexified Lie algebra $\mathfrak{sl}(n+1,\mathbb{C})$ of $\mathfrak{su}(n+1)$. The subspace spanned by Z_l's is a Cartan subalgebra of $\mathfrak{su}(n+1)$.

In the case of $n=1$, denoting the dual basis of $\{X_{12},Y_{12},Z_1\}$ by $\{\varepsilon^i\}$, We obtain

$$\mu(x_1,y_1) = \frac{y_1}{1+x_1^2+y_1^2}\varepsilon^1 + \frac{x_1}{1+x_1^2+y_1^2}\varepsilon^2 + \frac{1-x_1^2-y_1^2}{2(1+x_1^2+y_1^2)}\varepsilon^3,$$

i.e., $\mu(\mathbb{C}\mathrm{P}^1)\subset\mathfrak{su}(2)^*$ is the 2-sphere with center at the origin and with radius $\frac{1}{2}$ (Example 8).

Let (ξ_i) be the linear coordinates for $\{\varepsilon^i\}$. We set $\mathfrak{g} := \mathfrak{su}(2)$. Any twist t is an r-matrix (Example 21). Since $\mathbb{C}\mathrm{P}^1$ is 2-dimensional, it follows that $[t,t]_{\mathbb{C}\mathrm{P}^1}=0$. Therefore we can deform the Poisson structure π_{FS} induced by ω_{FS} to a Poisson structure π_{FS}^t on $\mathbb{C}\mathrm{P}^1$ by t and the natural action is a Poisson action of $(\mathrm{SU}(2),t^L-t^R)$.

Let $\mathfrak{g}_t^*$ be the space twisted $\mathfrak{g}^*$ by t in $\Lambda^2\mathfrak{g}$. We consider what is the condition for t under which $\mathfrak{g}_t^*$ is admissible on $\mu(\mathbb{C}\mathrm{P}^1)$. For any twist

$$t = \sum_{i<j} \frac{1}{2}\lambda_{ij} e_i \wedge e_j \in \Lambda^2\mathfrak{g}\ (\lambda_{ij} \in \mathbb{R}),$$

we obtain

$$\mathfrak{g}_t^* = \mathrm{span}\{\varepsilon^1 + \lambda_{12}e_2 + \lambda_{13}e_3,\ \varepsilon^2 - \lambda_{12}e_1 + \lambda_{13}e_3,\ \varepsilon^3 - \lambda_{13}e_1 - \lambda_{23}e_2\}.$$

Then $\mathfrak{g}_t^*$ is admissible at $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathfrak{g}^*$ if and only if the matrix

$$A_t(\xi) = \begin{pmatrix} 1 + 2\lambda_{12}\xi_3 - 2\lambda_{13}\xi_2 & 2\lambda_{13}\xi_1 & -2\lambda_{12}\xi_1 \\ -2\lambda_{23}\xi_2 & 1 + 2\lambda_{12}\xi_3 + 2\lambda_{23}\xi_1 & -2\lambda_{12}\xi_2 \\ -2\lambda_{23}\xi_3 & 2\lambda_{13}\xi_3 & 1 - 2\lambda_{13}\xi_2 + 2\lambda_{23}\xi_1 \end{pmatrix}$$

is regular. By computing the determinant of the matrix, we have

$$\det A_t(\xi) = (1 + 2\lambda_{23}\xi_1 - 2\lambda_{13}\xi_2 + 2\lambda_{12}\xi_3)^2.$$

So the complement $\mathfrak{g}_t^*$ is admissible at $\xi = (\xi_1, \xi_2, \xi_3)$ if and only if $1 + 2\lambda_{23}\xi_1 - 2\lambda_{13}\xi_2 + 2\lambda_{12}\xi_3 \neq 0$.

Therefore $\mathfrak{g}_t^*$ is admissible on $\mu(\mathbb{C}\mathrm{P}^1)$ if and only if the "non-admissible surface" $\{\xi = (\xi_1, \xi_2, \xi_3) \in \mathfrak{g}^* \mid 1 + 2\lambda_{23}\xi_1 - 2\lambda_{13}\xi_2 + 2\lambda_{12}\xi_3 \neq 0\}$ for $\mathfrak{g}_t^*$ and the image $\mu(\mathbb{C}\mathrm{P}^1)$ have no common point. Since $\mu(\mathbb{C}\mathrm{P}^1)$ is the 2-sphere with center at the origin and with radius $\frac{1}{2}$, we can see that this condition is equivalent to the condition

$$\lambda_{12}^2 + \lambda_{13}^2 + \lambda_{23}^2 < 1.$$

From the above discussion, we obtain the following theorem.

Theorem 3.2.1 (Nakamura [31])**.** If a twist $t := \sum_{i<j} \frac{1}{2}\lambda_{ij} e_i \wedge e_j$ satisfies $\lambda_{12}^2 + \lambda_{13}^2 + \lambda_{23}^2 < 1$, then the Fubini-Study form ω_{FS} on $\mathbb{C}\mathrm{P}^1$ can be deformed by t in the sense of Section 3.1.

We shall see an example of a concrete twists on $\mathbb{C}\mathrm{P}^1$.

Example 51 ([31])**.** We use a twist $t = \frac{1}{2}X_{12} \wedge Y_{12}$ in $\Lambda^2\mathfrak{su}(2)$ and a real number λ, where $-1 < \lambda < 1$. The symplectic structure $\omega_{\mathrm{FS}}^{\lambda t}$ deformed ω_{FS} by λt is written by

$$\omega_{\mathrm{FS}}^{\lambda t} = \left\{ \left(1 + \frac{1}{2}\lambda\right)(x_1^2 + y_1^2)^2 + 2(x_1^2 + y_1^2) + \left(1 - \frac{1}{2}\lambda\right) \right\}^{-1} dx_1 \wedge dy_1 \tag{3.10}$$

on U_1. Then it follows from an elementary calculation that the symplectic volume $\mathrm{Vol}(\mathbb{C}\mathrm{P}^1, \omega_{\mathrm{FS}}^{\lambda t})$ of $(\mathbb{C}\mathrm{P}^1, \omega_{\mathrm{FS}}^{\lambda t})$ is

$$\mathrm{Vol}(\mathbb{C}\mathrm{P}^1, \omega_{\mathrm{FS}}^{\lambda t}) = \begin{cases} \pi & (\lambda = 0) \\ \frac{\pi}{\lambda} \log \left| \frac{2+\lambda}{2-\lambda} \right| & (\lambda \neq 0). \end{cases} \tag{3.11}$$

Next, we consider a cohomology class of each $\omega_{\mathrm{FS}}^{\lambda t}$. Since $H^2_{\mathrm{DR}}(\mathbb{C}\mathrm{P}^1) = \mathbb{R}$, there exists a real number k_λ in $\mathbb{R}$ such that $\left[\omega_{\mathrm{FS}}^{\lambda t}\right] = k_\lambda \left[\omega_{\mathrm{FS}}\right]$. By integrating, we obtain

$$k_\lambda = \frac{1}{\lambda} \log \left| \frac{2+\lambda}{2-\lambda} \right|,$$

where $\lambda \neq 0$. Since the function k_λ of λ is smooth, even and strictly monotone increasing when λ is positive, $\omega_{\mathrm{FS}}^{\lambda t}$ and $\omega_{\mathrm{FS}}^{-\lambda t}$ are cohomologous. This means that we obtain a lot of non-trivial symplectic structures different from original ω_{FS} and non-trivial symplectomorphisms $(M, \omega_{\mathrm{FS}}^{\lambda t}) \longrightarrow (M, \omega_{\mathrm{FS}}^{-\lambda t})$.

In the above example, the condition $-1 < \lambda < 1$ is *not* a necessary condition for $\omega_{\mathrm{FS}}^{\lambda t}$ to be a symplectic structure. In fact, it follows that $\omega_{\mathrm{FS}}^{\lambda t}$ is a symplectic structure for $-2 < \lambda < 2$. Therefore in general, the non-degeneracy for π^t is not equivalent to that the isotropic complement $\mathfrak{g}_t^*$ is admissible on $\mu(M)$.

The next example is the complex Grassmannian $\mathrm{Gr}_{\mathbb{C}}(n,r) := \mathrm{SU}(n)/(\mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(n-r)))$ with the Kirillov-Kostant form ω_{KK}. With respect to ω_{KK}, the natural $\mathrm{SU}(n)$-action is symplectic-Hamiltonian (Example 9).

Then we consider the following r-matrix of $\mathfrak{su}(n)$:

$$t = \frac{1}{4n} \sum_{1 \leq i < j \leq n} X_{ij} \wedge Y_{ij},$$

where the r-matrix t is the canonical one defined on any compact semi-simple Lie algebra over $\mathbb{R}$ (for example, see [8]). This is an r-matrix such that $[t,t] \neq 0$. We show that it satisfies $[t,t]_M = 0$, where $M := \mathrm{Gr}_{\mathbb{C}}(n,r)$. Since t is an r-matrix, the element $[t,t]$ is Ad-invariant. By the definition of the $\mathrm{SU}(n)$-action on $\mathrm{Gr}_{\mathbb{C}}(n,r)$, it follows that

$$[t,t]_M = p_*[t,t]^R,$$

where $p : \mathrm{SU}(n) \to \mathrm{Gr}_{\mathbb{C}}(n,r) = \mathrm{SU}(n)/(\mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(n-r)))$ is the natural projection. Since any point m in $\mathrm{Gr}_{\mathbb{C}}(n,r)$ is represented by gH, where g is in $\mathrm{SU}(n)$ and $H := \mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(n-r))$, we compute

$$[t,t]_{M,m} = p_*[t,t]^R_g = p_* R_{g*}[t,t].$$

Because of the Ad-invariance of $[t,t]$, we obtain

$$p_* R_{g*}[t,t] = p_* L_{g*} L_{g^{-1}*} R_{g*}[t,t] = p_* L_{g*} \mathrm{Ad}_{g^{-1}}[t,t] = p_* L_{g*}[t,t].$$

Let $\mathfrak{h}$ be the Lie algebra of H. For any X in $\mathfrak{h}$ and g in $\mathrm{SU}(n)$, we compute

$$p_* L_{g*} X = p_* L_{g*} \frac{d}{ds} \exp sX \bigg|_{s=0} = \frac{d}{ds}(g \exp sX)H \bigg|_{s=0} = \frac{d}{ds} gH \bigg|_{s=0} = 0,$$

where we have used that $\exp sX$ is in H in the third equality. Therefore it holds that $[t,t]_M = 0$ if each term of $[t,t]$ includes elements in $\mathfrak{h}$. We notice that

$$\mathfrak{h} = \mathrm{span}_{\mathbb{R}}\{X_{ij}, Y_{ij}, Z_k | 1 \le i < j \le r \text{ or } r+1 \le i < j \le n, \\ \text{and } k = 1, \ldots, n-1\}.$$

If $X_{ij}, Y_{ij} \in \mathfrak{h}$, then

$$[\,\cdot\,, X_{ij} \wedge Y_{ij}] = [\,\cdot\,, X_{ij}] \wedge Y_{ij} - X_{ij} \wedge [\,\cdot\,, Y_{ij}]\,.$$

So these terms include an element in $\mathfrak{h}$. Hence we investigate terms of the form

$$\begin{aligned}[X_{ij} \wedge Y_{ij}, X_{kl} \wedge Y_{kl}] = &- [X_{ij}, X_{kl}] \wedge Y_{ij} \wedge Y_{kl} - X_{ij} \wedge [Y_{ij}, X_{kl}] \wedge Y_{kl} \\ &- Y_{ij} \wedge [X_{ij}, Y_{kl}] \wedge X_{kl} - X_{ij} \wedge X_{kl} \wedge [Y_{ij}, Y_{kl}]\,,\end{aligned}$$

where X_{ij}, Y_{ij}, X_{kl} and Y_{kl} are not in $\mathfrak{h}$. In the case of $i = k$ and $j = l$, we get

$$\begin{aligned}[X_{ij}, X_{ij}] &= [Y_{ij}, Y_{ij}] = 0, \\ [X_{ij}, Y_{ij}] &= 2(Z_i - Z_j) \in \mathfrak{h},\end{aligned}$$

where we set $Z_n := 0$. In the case of $i = k$ and $j < l$ (resp. $l < j$), since it follows that $r+1 \le j,\ l \le n$, we obtain

$$\begin{aligned}[X_{ij}, X_{kl}] = [Y_{ij}, Y_{kl}] &= -X_{jl} (\text{resp. } X_{lj}) \in \mathfrak{h}, \\ [Y_{ij}, X_{kl}] = [Y_{kl}, X_{ij}] &= -Y_{jl} (\text{resp. } Y_{lj}) \in \mathfrak{h}.\end{aligned}$$

We can also show the case of $i < k$ (resp. $k < i$) and $j = l$ in the similar way. Therefore all terms of $[t,t]$ include elements in $\mathfrak{h}$, so that $[t,t]_M = 0$. Therefore π^t_{KK} is Poisson by Theorem 3.1.1, where π_{KK} is the Poisson structure induced by ω_{KK}. Since $\mathrm{Gr}_{\mathbb{C}}(n,r)$ is compact, for sufficiently small $|\lambda|$, the Poisson structure $\pi^{\lambda t}_{\mathrm{KK}}$ is non-degenerate. Example 51 is the special case of this example.

3.3 Symplectic toric manifolds

In this section, we consider deformations of symplectic toric manifolds. Then our deformations give canonical transformations for symplecyic toric manifolds.

Theorem 3.3.1 (Nakamura [31])**.** For any $2n$-dimensional symplectic toric manifold (M, ω) and any twist t in $\Lambda^2\mathbb{R}^n$, the manifold (M, ω^t) deformed by t in the sense of Section 3.1 is a symplectic toric manifold with the same action as on (M, ω). Moreover (M, ω^t) is isomorphic to (M, ω) as a symplectic toric manifold.

Proof. We denote the symplectic-Hamiltonian action and the moment map for it by σ and μ, respectively. Since $\mathbb{T}^n$ is commutative, the brackets $[X_i, X_j]$ vanish for all i and j. Hence for any λ_{12} in $\mathbb{R}$, the twist $t_{12} := \lambda_{12}X_1 \wedge X_2$ deforms ω to a symplectic structure $\omega^{t_{12}}$ induced by a Poisson structure $\pi^{t_{12}} := \pi - (t_{12})_M$ by Theorem 3.1.2. On the other hand it follows $\pi^t_{\mathbb{T}^n} := t^L - t^R = 0$ for any twist t by the commutativity of $\mathbb{T}^n$. Therefore, after deformation, the multiplicative Poisson structure 0 on $\mathbb{T}^n$ is invariant and the action σ is a symplectic action. Then this action is symplectic-Hamiltonian with a moment map μ. In fact, the map μ is a moment map for σ on $(M, \omega^{t_{12}})$ if and only if $d\mu^X = \iota_{X_\sigma}\omega^{t_{12}}$. Moreover,

$$\begin{aligned} d\mu^X = \iota_{X_\sigma}\omega^{t_{12}} &\iff d\mu^X = (\omega^{t_{12}})^\flat X_\sigma \\ &\iff (\pi^{t_{12}})^\sharp d\mu^X = (\pi^{t_{12}})^\sharp(\omega^{t_{12}})^\flat X_\sigma \\ &\iff (\pi^{t_{12}})^\sharp d\mu^X = -X_\sigma \\ &\iff X_\sigma = -(\pi^\sharp - (t_{12})^\sharp_\sigma)d\mu^X \\ &\iff X_\sigma = -\pi^\sharp d\mu^X + (t_{12})^\sharp_\sigma d\mu^X \\ &\iff X_\sigma = X_\sigma + (t_{12})^\sharp_\sigma d\mu^X \\ &\iff (t_{12})^\sharp_\sigma d\mu^X = 0 \end{aligned} \tag{3.12}$$

for any X in $\mathbb{R}^n$ since μ satisfies (1.1) with respect to ω. Then we calculate

$$\begin{aligned} (t_{12})^\sharp_\sigma d\mu^X &= (\lambda_{12}X_1 \wedge X_2)^\sharp_\sigma \iota_{X_\sigma}\omega \\ &= \lambda_{12}(X_{1,\sigma} \wedge X_{2,\sigma})^\sharp \omega^\flat X_\sigma \\ &= \lambda_{12}(\langle X_{1,\sigma}, \omega^\flat X_\sigma\rangle X_{2,\sigma} - \langle X_{2,\sigma}, \omega^\flat X_\sigma\rangle X_{1,\sigma}) \\ &= \lambda_{12}(\omega(X_{1,\sigma}, X_\sigma)X_{2,\sigma} - \omega(X_{2,\sigma}, X_\sigma)X_{1,\sigma}) \end{aligned}$$

Using the facts that for any Hamiltonian G-space (M, ω, G, μ),

$$\omega(Y_\sigma, Z_\sigma) = \mu^{[Y,Z]}$$

for any Y and Z in $\mathfrak{g}$, and that the Lie algebra $\mathbb{R}^n$ is commutative, we obtain the condition (3.12). Therefore $(M, \omega^{t_{12}})$ is a symplectic toric manifold on which a moment map for the symplectic-Hamiltonian action σ is μ. By Delzant theorem, (M, ω) and $(M, \omega^{t_{12}})$ are isomorphic as a symplectic toric manifold. Similarly, for $t_{13} = \lambda_{13} X_1 \wedge X_2$ $(\lambda_{13} \in \mathbb{R}, X_i \in \mathbb{R}^n)$, $(M, \omega^{t_{12}})$ and $(M, (\omega^{t_{12}})^{t_{13}}) = (M, \omega^{t_{12}+t_{13}})$ are isomorphic as a symplectic toric manifold. By repeating this operation, it follows that (M, ω) and (M, ω^t) are isomorphic as a symplectic toric manifold for any twist $t = \sum_{i<j} \lambda_{ij} X_i \wedge X_j$.
□

Remark 9. In Theorem 21 in [31], the results of Theorem 3.3.1 are proved under the assumptions that a symplectic toric manifold (M, ω) is compact, connected and satisfying a condition with respect to a symplectic structure. However Theorem 3.3.1 states that these assumptions are not necessary.

Example 52 ([31])**.** A symplectic toric manifold $(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}})$ has the torus action σ:

$$(e^{i\theta_2}, e^{i\theta_3}, \ldots, e^{i\theta_{n+1}}) \cdot [z_1 : \cdots : z_{n+1}] := [z_1 : e^{i\theta_2} z_2 : \cdots : e^{i\theta_{n+1}} z_{n+1}]$$

for any θ_i in $\mathbb{R}$. The moment map $\mu : \mathbb{C}\mathrm{P}^n \to \mathbb{R}^n$ for this action on $(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}})$ is

$$\mu([z_1 : \cdots : z_{n+1}]) := -\frac{1}{2}\left(\frac{|z_2|^2}{|z|^2}, \ldots, \frac{|z_{n+1}|^2}{|z|^2}\right),$$

where $z = (z_1, \ldots, z_{n+1})$ in $\mathbb{C}^n$. We set $X_1 := (1, 0, \ldots, 0), \ldots, X_n := (0, \ldots, 0, 1)$. On U_1, since for any $i = 1, \ldots, n$,

$$X_{i, \mathbb{C}\mathrm{P}^n} = -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}$$

we obtain

$$\begin{aligned}(X_i \wedge X_j)_{\mathbb{C}\mathrm{P}^n} = {} & y_i y_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} - y_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} \\ & - x_i y_j \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial x_j} + x_i x_j \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \ (1 \leq i < j \leq n),\end{aligned}$$

where $x_i := \mathrm{Re}\frac{z_{i+1}}{z_1}$ and $y_i := \mathrm{Im}\frac{z_{i+1}}{z_1}$, for example on $\mathbb{C}\mathrm{P}^n$, it follows that

$$\begin{aligned}\omega_{\mathrm{FS}}^{t_{12}} = \omega_{\mathrm{FS}} + {} & \frac{\lambda_{12}\left\{(x_1^2 + y_1^2)(x_2^2 + y_2^2) - 1\right\}}{(x_1^2 + y_1^2 + x_2^2 + y_2^2 + 1)^4}(x_1 x_2 dx_1 \wedge dx_2 \\ & + x_1 y_2 dx_1 \wedge dy_2 + y_1 x_2 dy_1 \wedge dx_2 + y_1 y_2 dy_1 \wedge dy_2).\end{aligned}$$

By Theorem 3.3.1, $(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}}^{t_{12}})$ is a symplectic toric manifold isomorphic to $(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}})$ as a symplectic toric manifold.

Chapter 4

Pseudo-Poisson-Nijenhuis manifolds

In this chapter, we study pseudo-Poisson-Nijenhuis manifolds.

4.1 Compatible pairs

In this section, we consider the compatibility of a 2-vector field and a $(1,1)$-tensor on a C^∞-manifold, which plays an important role to define not only a Poisson-Nijenhuis and Poisson-quasi-Nijenhuis manifold but also a pseudo-Poisson-Nijenhuis manifold, which is defined in Section 4.2. For that reason, first we begin with the definitions and properties of brackets defined by a 2-vector field and a $(1,1)$-tensor. We generalize several properties of a Poisson-Nijenhuis structure to that of a compatible pair of a 2-vector field and a $(1,1)$-tensor. Moreover we show that the brackets gives a characterization of the compatibility of a 2-vector field and a $(1,1)$-tensor, which is the main theorem of this subsection.

Let M be a C^∞-manifold, π a 2-vector field and N a $(1,1)$-tensor. Similarly as in the case that π is Poisson and that N is Nijenhuis, we define brackets $[\cdot,\cdot]_\pi$ and $[\cdot,\cdot]_N$ by (2.31) and (2.83) respectively. It is easy to see that these brackets are bilinear and anti-symmetry, and satisfy the Leibniz rule (ii) of Definition 36. From this, we obtain the derivation $d_\pi : \mathfrak{X}^*(M) \longrightarrow \mathfrak{X}^{*+1}(M)$ and $d_N : \Omega^*(M) \longrightarrow \Omega^{*+1}(M)$ defined by the formula (2.86) respectively and Lie derivatives $\mathcal{L}^\pi$ and $\mathcal{L}^N$ defined by (2.87) respectively. Then it follows that $d_\pi D = [\pi, D]$ for any D in $\mathfrak{X}^k(M)$, and that

$$\mathcal{L}^\pi_\alpha \beta = [\alpha, \beta]_\pi, \qquad \mathcal{L}^N_X Y = [X, Y]_N$$

for any α, β in $\Omega^1(M)$, X and Y in $\mathfrak{X}(M)$.

In general, any 2-vector field on M and $(1,1)$-tensor on M satisfy the followings.

Proposition 4.1.1 ([16],[42])**.** Let π be a 2-vector field on M. For any α, β, γ in $\Omega^1(M)$ and X in $\mathfrak{X}(M)$,

$$\frac{1}{2}\iota_{\alpha\wedge\beta}[\pi,\pi] = [\pi^\sharp\alpha, \pi^\sharp\beta] - \pi^\sharp[\alpha,\beta]_\pi, \tag{4.1}$$

$$\sum_{\mathrm{Cycl}(\alpha,\beta,\gamma)} \langle[[\alpha,\beta]_\pi,\gamma]_\pi, X\rangle = \frac{1}{2}\left(\mathcal{L}_X[\pi,\pi]\right)(\alpha,\beta,\gamma) + \frac{1}{2}\sum_{\mathrm{Cycl}(\alpha,\beta,\gamma)} [\pi,\pi](\alpha,\beta,d\langle\gamma,X\rangle). \tag{4.2}$$

Proposition 4.1.2 ([41])**.** Let N be a $(1,1)$-tensor on M. For any X, Y and Z in $\mathfrak{X}(M)$,

$$\sum_{\mathrm{Cycl}(X,Y,Z)} [[X,Y]_N, Z]_N = -\sum_{\mathrm{Cycl}(X,Y,Z)} ([\mathcal{T}_N(X,Y),Z] + \mathcal{T}_N([X,Y],Z)). \tag{4.3}$$

Remark 10. The above brackets are *not* Lie brackets in general. By Proposition 4.1.1, if the 2-vector field π on M is Poisson, i.e., $[\pi,\pi]=0$, then the bracket $[\cdot,\cdot]_\pi$ is a Lie bracket on $\Omega^1(M)$. By Proposition 4.1.2, if N is Nijenhuis, i.e., the Nijenhuis torsion $\mathcal{T}_N$ vanishes, then the bracket $[\cdot,\cdot]_N$ is a Lie bracket on $\mathfrak{X}(M)$.

The existence and uniqueness theorem of the Schouten bracket of a Lie bracket on the sections $\Gamma(A)$ of a Lie algebroid A is extended to the following situation:

Theorem 4.1.3 ([32])**.** Let (A,a) be an *anchored vector bundle* over M, i.e., $a: A \longrightarrow TM$ is a bundle map over M, and $[\cdot,\cdot]_A$ an anti-symmetric bilinear bracket on $\Gamma(A)$ satisfying the Leibniz rule

$$[X, fY]_A = (a(X)f)Y + f[X,Y]_A \tag{4.4}$$

for any X, Y in $\Gamma(A)$ and f in $C^\infty(M)$. Then there is a unique bilinear operator $[\cdot,\cdot]_A : \Gamma(\Lambda^* A)\times\Gamma(\Lambda^* A) \longrightarrow \Gamma(\Lambda^* A)$, called *the generalized Schouten bracket* or simply *the Schouten bracket*, that satisfies the following properties:

(i) It is a biderivation of degree -1, that is, it is bilinear,

$$\deg[D_1, D_2]_A = \deg D_1 + \deg D_2 - 1, \tag{4.5}$$

and

$$[D_1, D_2 \wedge D_3]_A = [D_1, D_2]_A \wedge D_3 + (-1)^{(\deg D_1+1)\deg D_2} D_2 \wedge [D_1, D_3]_A. \tag{4.6}$$

for D_i in $\Gamma(\Lambda^* A)$,

(ii) It is determined on $C^\infty(M)$ and $\Gamma(A)$ by

(a) $[f, g]_A = 0$ $(f, g \in C^\infty(M))$;

(b) $[X, f]_A = a(X)f$ $(f \in C^\infty(M), X \in \Gamma(A))$;

(c) $[X, Y]_A$ $(X, Y \in \Gamma(A))$ is the original bracket on $\Gamma(A)$.

(iii) $[D_1, D_2]_A = -(-1)^{(\deg D_1 - 1)(\deg D_2 - 1)}[D_2, D_1]_A$.

Remark 11. In general, the Schouten bracket of a bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ does not satisfy the graded Jacobi identity because $[\cdot,\cdot]_A$ does not satisfy the Jacobi identity.

Since (TM, N) and $(T^*M, \pi^\sharp)$ are anchored vector bundles over M and brackets $[\cdot,\cdot]_\pi$ and $[\cdot,\cdot]_N$ satisfy the Leibniz rule (ii) of Definition 36 respectively, by Theorem 4.1.3, $[\cdot,\cdot]_\pi$ and $[\cdot,\cdot]_N$ are extended to the Schouten bracket on $\Omega^*(M)$ and on $\mathfrak{X}^*(M)$ respectively.

We define the concept related to a 2-vector field and a $(1,1)$-tensor, called *the compatibility* of these.

Definition 45 ([16], [26], [32])**.** The 2-vector field π on M and the $(1,1)$-tensor N on M are *compatible* if they satisfy (2.96) and the $(2,1)$-tensor C_π^N defined by (2.97) vanishes.

Let (π, N) be a compatible pair and set $\pi_N(\alpha, \beta) := \langle N\pi^\sharp\alpha, \beta\rangle$. Then it follows from (2.96) that π_N is a 2-vector field on M. Hence under the assumption (2.96), the bracket $[\cdot,\cdot]_{N\pi^\sharp}$ can be rewritten as $[\cdot,\cdot]_{\pi_N}$. Moreover, then the three brackets $[\cdot,\cdot]_{\pi_N}$, $[\cdot,\cdot]_\pi^{N^*}$ and $[\cdot,\cdot]_{N,\pi}$ defined by (2.100) coincide.

For any 2-vector field π and $(1,1)$-tensor N satisfying (2.96), the three brackets $[\cdot,\cdot]_{\pi_N}$, $[\cdot,\cdot]_\pi^{N^*}$ and $[\cdot,\cdot]_{N,\pi}$ on an anchored vector bundle $(T^*M, \pi_N^\sharp)$ satisfy the Leibniz rule (ii) of Definition 36 respectively. Therefore we obtain the derivations defined by the formula (2.86) respectively. In particular, we denote the derivations defined by the bracket $[\cdot,\cdot]_{N,\pi}$ by d_π^N, and obtain the formula

$$d_\pi^N = [\pi, \cdot]_N. \tag{4.7}$$

For any compatible pair (π, N), we set $\pi_0 := \pi$ and define a 2-vector field π_{k+1} by the condition $\pi_{k+1}^\sharp = N \circ \pi_k^\sharp$ inductively. In the case of a compatible pair (π, N) of which N is Nijenhuis, the following proposition corresponding to the existence theorem of the hierarchy of Poisson-Nijenhuis structures (Theorem 2.4.1) can be shown in the same way.

Proposition 4.1.4 (Nakamura [32], the hierarchy of compatible pairs)**.** Let (π, N) be a compatible pair on M such that N is Nijenhuis. Then all pairs (π_k, N^p) $(k, p \geq 0)$ are compatible pairs on M such that $N^p = N \circ \cdots \circ N$ (p times) are Nijenhuis. Furthermore for any $k, l \geq 0$ and Q in $\mathfrak{X}^*(M)$, it follows that $[\pi_k, Q]_{N^{l+1}} = [\pi_{k+1}, Q]_{N^l}$.

Proof. For any $(1,1)$-tensor N, the Nijenhuis torsion $\mathcal{T}_N$ of N can be defined equivalently by

$$\iota_X \mathcal{T}_N = \mathcal{L}_{NX} N - N \circ \mathcal{L}_X N \tag{4.8}$$

for any X in $\mathfrak{X}(M)$. Then by induction, we obtain

$$\mathcal{L}_{N^p X} N = N^p \circ \mathcal{L}_X N + \sum_{h=0}^{p-1} N^h \circ \iota_X \mathcal{T}_N \tag{4.9}$$

If N is Nijenhuis, i.e., $\mathcal{T}_N = 0$, it follows that

$$\mathcal{L}_{N^p X} N^p = N^p \circ \mathcal{L}_X N^p. \tag{4.10}$$

Therefore

$$\iota_X \mathcal{T}_{N^p} = \mathcal{L}_{N^p X} N^p - N^p \circ \mathcal{L}_X N^p = 0 \tag{4.11}$$

and we see that N^p is Nijenhuis.

Moreover for any $(1,1)$-tensors A and B satisfying $AB = BA$ and $A\pi^\sharp = \pi^\sharp A$, $B\pi^\sharp = \pi^\sharp B$, we have

$$C_\pi^{AB}(\alpha, \beta) = B^*(C_\pi^A(\alpha, \beta)) + C_\pi^B(A^*\alpha, \beta) + (\mathcal{L}_{\pi^\sharp \alpha} B)^* A^* \beta - (\mathcal{L}_{A\pi^\sharp \alpha} B)^* \beta. \tag{4.12}$$

If (4.12) is applied for $A = N^p$ and $B = N$, where N is Nijenhuis, then we obtain

$$C_\pi^{N^{p+1}}(\alpha,\beta) = N^*(C_\pi^{N^p}(\alpha,\beta)) + C_\pi^N((N^p)^*\alpha,\beta) + \sum_{h=0}^{p-1}(N^h \circ \iota_{\pi^\sharp\alpha}\mathcal{T}_N)^*\beta. \tag{4.13}$$

Therefore we see inductively that all the pair (π, N^p) is compatible.

In order to prove the first part of Proposition 4.1.4, it is sufficient to prove that (π_k, N) are compatible. We obtain

$$C_{\pi_{k+1}}^N(\alpha,\beta) = C_{\pi_k}^N(\alpha, N^*\beta) + (\iota_{\pi_k^\sharp\alpha}\mathcal{T}_N)^*\beta \tag{4.14}$$

for any α and β in $\Omega^1(M)$. By (4.14), $C_\pi^N = 0$ and $\mathcal{T}_N = 0$, it holds that $C_{\pi_{k+1}}^N = 0$. Hence (π_{k+1}, N) is compatible.

Finally, if a pair (π, N) is compatible, then $[\cdot,\cdot]_{N,\pi} = [\cdot,\cdot]_{\pi_N}$ holds. Hence the corresponding derivations d_π^N and d_{π_N} coincide. Then $d_\pi^N = d_{\pi_N}$ means

$$[\pi, Q]_N = [\pi_N, Q] \tag{4.15}$$

for any Q in $\mathfrak{X}^*(M)$. Since (π_k, N^{l+1}) and (π_{k+1}, N^l) are compatible, then we obtain

$$[\pi_k, Q]_{N^{l+1}} = [\pi_{k+l+1}, Q] \tag{4.16}$$

$$= [\pi_{k+1}, Q]_{N^l} \tag{4.17}$$

for any Q in $\mathfrak{X}^*(M)$. □

The compatibility of a 2-vector field π and a $(1,1)$-tensor N is equivalent to the following equations using the Schouten brackets of $[\cdot,\cdot]_\pi$ and $[\cdot,\cdot]_N$.

Theorem 4.1.5 (Nakamura [32])**.** Let M be a C^∞-manifold, π a 2-vector field on M and N a $(1,1)$-tensor on M. Then the following conditions are equivalent:

(i) π and N are compatible;

(ii) the operator d_N is a derivation of the Schouten bracket $[\cdot,\cdot]_\pi$:

$$d_N[\xi_1,\xi_2]_\pi = [d_N\xi_1,\xi_2]_\pi + (-1)^{\deg\xi_1+1}[\xi_1, d_N\xi_2]_\pi; \tag{4.18}$$

(iii) the operator d_π is a derivation of the Schouten bracket $[\cdot,\cdot]_N$:

$$d_\pi[D_1,D_2]_N = [d_\pi D_1, D_2]_N + (-1)^{\deg D_1+1}[D_1, d_\pi D_2]_N, \tag{4.19}$$

where ξ_i's are in $\Omega^*(M)$ and D_i's are in $\mathfrak{X}^*(M)$.

In the case of that π is Poisson, Theorem 4.1.5 coincides with Lemma 3.6 in [38]. Moreover, if N is Nijenhuis, then Theorem 4.1.5 coincides with Proposition 3.2 in [15]. However to prove Proposition 3.2 in [15], properties for a Lie bialgebroid [22] were used since $((TM)_N, (T^*M)_\pi)$ is a Lie bialgebroid, and Lemma 3.6 in [38] does not mention the equivalence of (i) and (iii) in Theorem 4.1.5. Therefore Theorem 4.1.5 is worthy in the sense that these equivalence holds without an assumption that π is Poisson or N is Nijenhuis. To prove Theorem 4.1.5, we need the following lemma.

Lemma 4.1.6 (Nakamura [32])**.** Let π be a 2-vector field on M and N a $(1,1)$-tensor on M. Assume that π and N satisfy the condition (2.96). Then the pair (π, N) is compatible if and only if for any f in $C^\infty(M)$ and X in $\mathfrak{X}(M)$,

$$\mathcal{L}^\pi_{d_N f} X = -[d_\pi f, X]_N. \tag{4.20}$$

Proof. For any ξ in $\Omega^1(M)$, we calculate

$$\begin{aligned}
\langle \mathcal{L}^\pi_{d_N f} X, \xi\rangle &= \mathcal{L}^\pi_{d_N f}\langle X, \xi\rangle - \langle X, \mathcal{L}^\pi_{d_N f}\xi\rangle \\
&= (\pi^\sharp N^* df)\langle X, \xi\rangle - \langle X, [N^* df, \xi]_\pi\rangle \\
&= (\pi^\sharp_N df)\langle X, \xi\rangle - \langle X, [df, \xi]^{N^*}_\pi\rangle \\
&\quad + \langle X, [df, N^*\xi]_\pi\rangle - \langle X, N^*[df, \xi]_\pi\rangle \\
&= (\pi^\sharp_N df)\langle X, \xi\rangle - \langle X, [df, \xi]^{N^*}_\pi\rangle \\
&\quad + \langle X, \mathcal{L}_{\pi^\sharp df}(N^*\xi)\rangle - \langle NX, \mathcal{L}_{\pi^\sharp df}\xi\rangle \\
&= (\pi^\sharp_N df)\langle X, \xi\rangle - \langle X, [df, \xi]^{N^*}_\pi\rangle \\
&\quad + \langle N[d_\pi f, X], \xi\rangle - \langle [d_\pi f, NX], \xi\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle [d_\pi f, X]_N, \xi\rangle &= \langle [N d_\pi f, X] + [d_\pi f, NX] - N[d_\pi f, X], \xi\rangle \\
&= \langle [d_{\pi_N} f, X], \xi\rangle + \langle [d_\pi f, NX] - N[d_\pi f, X], \xi\rangle \\
&= \langle [[\pi_N, f], X], \xi\rangle + \langle [d_\pi f, NX] - N[d_\pi f, X], \xi\rangle \\
&= \langle -[[X, \pi_N], f] - [\pi_N, [f, X]], \xi\rangle \\
&\quad + \langle [d_\pi f, NX] - N[d_\pi f, X], \xi\rangle \\
&= -\langle [d_{\pi_N} X, f], \xi\rangle + \langle [\pi_N, Xf], \xi\rangle \\
&\quad + \langle [d_\pi f, NX] - N[d_\pi f, X], \xi\rangle
\end{aligned}$$

$$
\begin{aligned}
&= -(d_{\pi_N} X)(df, \xi) + \pi_N(d(Xf), \xi) \\
&\quad + \langle [d_\pi f, NX] - N[d_\pi f, X], \xi \rangle \\
&= -(\pi_N^\sharp df)\langle X, \xi \rangle + (\pi_N^\sharp \xi)\langle X, df \rangle \\
&\quad + \langle X, [df, \xi]_{\pi_N} \rangle - (\pi_N^\sharp \xi)(Xf) \\
&\quad + \langle [d_\pi f, NX] - N[d_\pi f, X], \xi \rangle \\
&= -(\pi_N^\sharp df)\langle X, \xi \rangle + \langle X, [df, \xi]_{\pi_N} \rangle \\
&\quad + \langle [d_\pi f, NX], \xi \rangle - \langle N[d_\pi f, X], \xi \rangle.
\end{aligned}
$$

Therefore we find

$$
\begin{aligned}
\langle \mathcal{L}^\pi_{d_N f} X + [d_\pi f, X]_N, \xi \rangle &= \langle X, [df, \xi]_{\pi_N} - [df, \xi]_\pi^{N^*} \rangle \\
&= \langle X, C_\pi^N(df, \xi) \rangle.
\end{aligned}
$$

Because the exact 1-forms generate locally the 1-forms as a $C^\infty(M)$-module and C_π^N is tensorial, we obtain the equivalence to prove. □

Proof of Theorem 4.1.5. The equivalence of (i) and (ii) can be proved similarly as Proposition 3.2 in [15]. In fact, we set for any ξ_1 and ξ_2 in $\Omega^*(M)$,

$$A_{N,\pi}(\xi_1, \xi_2) := d_N[\xi_1, \xi_2]_\pi - [d_N \xi_1, \xi_2]_\pi - (-1)^{\deg \xi_1 + 1}[\xi_1, d_N \xi_2]_\pi. \quad (4.21)$$

Then for any f, g in $C^\infty(M)$, α, β and γ in $\Omega^*(M)$, we obtain

$$
\begin{aligned}
A_{N,\pi}(f, g) &= \langle (N\pi^\sharp - \pi^\sharp N^*) df, dg \rangle, && (4.22) \\
A_{N,\pi}(df, g) &= C_N^\pi(df, dg), && (4.23) \\
A_{N,\pi}(df, dg) &= -d(C_N^\pi(df, dg)), && (4.24) \\
A_{N,\pi}(\alpha, \beta \wedge \gamma) &= A_{N,\pi}(\alpha, \beta) \wedge \gamma + (-1)^{\deg \alpha \deg \beta} \beta \wedge A_{N,\pi}(\alpha, \gamma), && (4.25) \\
A_{N,\pi}(\alpha, \beta) &= -(-1)^{(\deg \alpha - 1)(\deg \beta - 1)} A_{N,\pi}(\beta, \alpha), && (4.26)
\end{aligned}
$$

so that the conclusion follows from these equations.

We shall prove the equivalence of (i) and (iii). We set for any D_1 and D_2 in $\mathfrak{X}^*(M)$,

$$A_{\pi,N}(D_1, D_2) := d_\pi[D_1, D_2]_N - [d_\pi D_1, D_2]_N - (-1)^{\deg D_1 + 1}[D_1, d_\pi D_2]_N. \quad (4.27)$$

Then for any f and g in $C^\infty(M)$, we calculate

$$\begin{aligned}
d_\pi[f,g]_N &= 0,\\
[d_\pi f,g]_N &= (Nd_\pi f)g = \langle Nd_\pi f, dg\rangle\\
&= \langle -N\pi^\sharp df, dg\rangle,\\
[f,d_\pi g]_N &= -[d_\pi g,f]_N = -\langle -N\pi^\sharp dg, df\rangle = \langle \pi^\sharp dg, N^*df\rangle = \langle dg, -\pi^\sharp N^* df\rangle\\
&= \langle -\pi^\sharp N^* df, dg\rangle,
\end{aligned}$$

so that we obtain

$$\begin{aligned}
A_{\pi,N}(f,g) &= d_\pi[f,g]_N - [d_\pi f,g]_N + [f,d_\pi g]_N\\
&= 0 - \langle -N\pi^\sharp df, dg\rangle + \langle -\pi^\sharp N^* df, dg\rangle\\
&= \langle (N\pi^\sharp - \pi^\sharp N^*)df, dg\rangle.
\end{aligned}$$

Therefore $A_{\pi,N}(f,g) = 0$ is equivalent with the condition (2.96). For any f, g in $C^\infty(M)$ and X in $\mathfrak{X}(M)$, we calculate

$$\begin{aligned}
\langle d_\pi[X,g]_N, df\rangle &= \langle d_\pi((NX)g), df\rangle = \langle -\pi^\sharp d\langle dg, NX\rangle, df\rangle\\
&= \langle d\langle dg, NX\rangle, \pi^\sharp df\rangle\\
&= (\pi^\sharp df)\langle NX, dg\rangle,\\
\langle [d_\pi X, g]_N, df\rangle &= \langle -\iota_{d_N g} d_\pi X, df\rangle = -(d_\pi X)(d_N g, df)\\
&= -(\pi^\sharp d_N g)\langle X, df\rangle + (\pi^\sharp df)\langle X, d_N g\rangle + \langle X, [d_N g, df]_\pi\rangle\\
&= -(\pi^\sharp N^* dg)\langle X, df\rangle + (\pi^\sharp df)\langle X, N^* dg\rangle + \langle X, [N^* dg, df]_\pi\rangle\\
&= -(\pi^\sharp N^* dg)\langle X, df\rangle + (\pi^\sharp df)\langle NX, dg\rangle + \langle X, [N^* dg, df]_\pi\rangle,\\
\langle [X, d_\pi g]_N, df\rangle &= \langle [NX, d_\pi g] + [X, Nd_\pi g] - N[X, d_\pi g], df\rangle\\
&= \langle [NX, [\pi, g]], df\rangle + \langle [X, -N\pi^\sharp dg], df\rangle - \langle [X, [\pi, g]], N^* df\rangle\\
&= \langle -[\pi, [g, NX]] - [g, [\pi, NX]], df\rangle + \langle \mathcal{L}_{N\pi^\sharp dg} X, df\rangle\\
&\quad + \langle [\pi, [g, X]] + [g, [X, \pi]], N^* df\rangle\\
&= \langle d_\pi((NX)g), df\rangle + \langle \iota_{dg} d_\pi(NX), df\rangle + \mathcal{L}_{N\pi^\sharp dg}\langle X, df\rangle\\
&\quad - \langle X, \mathcal{L}_{N\pi^\sharp dg} df\rangle - \langle d_\pi(Xg), N^* df\rangle - \langle \iota_{dg} d_\pi X, N^* df\rangle\\
&= (\pi^\sharp df)\langle dg, NX\rangle + (d_\pi(NX))(dg, df) + (N\pi^\sharp dg)\langle X, df\rangle\\
&\quad - \langle X, \mathcal{L}_{N\pi^\sharp dg} df\rangle - (\pi^\sharp N^* df)\langle X, dg\rangle - (d_\pi X)(dg, N^* df)\\
&= (\pi^\sharp df)\langle dg, NX\rangle + (\pi^\sharp dg)\langle NX, df\rangle - (\pi^\sharp df)\langle NX, dg\rangle\\
&\quad - \langle NX, [dg, df]_\pi\rangle + (N\pi^\sharp dg)\langle X, df\rangle - \langle X, \mathcal{L}_{N\pi^\sharp dg} df\rangle
\end{aligned}$$

$$
\begin{aligned}
&\quad - (\pi^\sharp N^* df)\langle X, dg\rangle - (\pi^\sharp dg)\langle X, N^* df\rangle + (\pi^\sharp N^* df)\langle X, dg\rangle \\
&\quad + \langle X, [dg, N^* df]_\pi\rangle \\
&= (\pi^\sharp dg)\langle NX, df\rangle - \langle X, N^*[dg, df]_\pi\rangle + (N\pi^\sharp dg)\langle X, df\rangle \\
&\quad - \langle X, \mathcal{L}_{N\pi^\sharp dg} df\rangle - (\pi^\sharp dg)\langle NX, df\rangle + \langle X, [dg, N^* df]_\pi\rangle \\
&= \langle X, [dg, N^* df]_\pi - N^*[dg, df]_\pi\rangle \\
&\quad + (N\pi^\sharp dg)\langle X, df\rangle - \langle X, \mathcal{L}_{N\pi^\sharp dg} df\rangle,
\end{aligned}
$$

so that we obtain

$$
\begin{aligned}
\langle A_{\pi,N}(X,g), df\rangle &= \langle d_\pi[X,g]_N - [d_\pi X, g]_N - [X, d_\pi g]_N, df\rangle \\
&= (\pi^\sharp df)\langle NX, dg\rangle - (-(\pi^\sharp N^* dg)\langle X, df\rangle + (\pi^\sharp df)\langle NX, dg\rangle \\
&\quad + \langle X, [N^* dg, df]_\pi\rangle) - (\langle X, [dg, N^* df]_\pi - N^*[dg, df]_\pi\rangle \\
&\quad + (N\pi^\sharp dg)\langle X, df\rangle - \langle X, \mathcal{L}_{N\pi^\sharp dg} df\rangle) \\
&= ((\pi^\sharp N^* - N\pi^\sharp)dg)\langle X, df\rangle \\
&\quad - \langle X, [N^* dg, df]_\pi + [dg, N^* df]_\pi - N^*[dg, df]_\pi\rangle \\
&\quad + \langle X, \mathcal{L}_{N\pi^\sharp dg} df - \mathcal{L}_{N\pi^\sharp df} dg - d\langle N\pi^\sharp dg, df\rangle\rangle \\
&\quad + \langle X, \mathcal{L}_{N\pi^\sharp df} dg + d\langle N\pi^\sharp dg, df\rangle\rangle \\
&= ((\pi^\sharp N^* - N\pi^\sharp)dg)\langle X, df\rangle \\
&\quad + \langle X, -[dg, df]_\pi^{N^*}\rangle + \langle X, [dg, df]_{N\pi^\sharp}\rangle \\
&\quad + \langle X, d\iota_{N\pi^\sharp df} dg + \iota_{N\pi^\sharp df} d^2 g + d\langle \pi^\sharp dg, N^* df\rangle\rangle \\
&= ((\pi^\sharp N^* - N\pi^\sharp)dg)\langle X, df\rangle + \langle X, [dg, df]_{N\pi^\sharp} - [dg, df]_\pi^{N^*}\rangle \\
&\quad + \langle X, d\langle N\pi^\sharp df, dg\rangle + 0 - d\langle dg, \pi^\sharp N^* df\rangle\rangle \\
&= ((\pi^\sharp N^* - N\pi^\sharp)dg)\langle X, df\rangle + \langle X, C_N^\pi(dg, df)\rangle \\
&\quad + \langle X, d\langle (N\pi^\sharp - \pi^\sharp N^*)df, dg\rangle\rangle.
\end{aligned}
$$

For any X, Y in $\mathfrak{X}(M)$, f and g in $C^\infty(M)$, we calculate

$$
\begin{aligned}
(d_\pi[X,Y]_N)(df, dg) &= (\pi^\sharp df)\langle [X,Y]_N, dg\rangle \\
&\quad - (\pi^\sharp dg)\langle [X,Y]_N, df\rangle - \langle [X,Y]_N, [df, dg]_\pi\rangle, \\
[d_\pi X, Y]_N(df, dg) &= -[Y, d_\pi X]_N(df, dg) = -(\mathcal{L}_Y^N d_\pi X)(df, dg) \\
&= -\mathcal{L}_Y^N((d_\pi X)(df, dg)) \\
&\quad + (d_\pi X)(\mathcal{L}_Y^N df, dg) + (d_\pi X)(df, \mathcal{L}_Y^N dg) \\
&= -\mathcal{L}_Y^N((\pi^\sharp df)\langle X, dg\rangle - (\pi^\sharp dg)\langle X, df\rangle - \langle X, [df, dg]_\pi\rangle)
\end{aligned}
$$

$$
\begin{aligned}
&+ (\pi^\sharp \mathcal{L}^N_Y df)\langle X, dg\rangle \\
&- (\pi^\sharp dg)\langle X, \mathcal{L}^N_Y df\rangle - \langle X, [\mathcal{L}^N_Y df, dg]_\pi\rangle \\
&+ (\pi^\sharp df)\langle X, \mathcal{L}^N_Y dg\rangle \\
&- (\pi^\sharp \mathcal{L}^N_Y dg)\langle X, df\rangle - \langle X, [df, \mathcal{L}^N_Y dg]_\pi\rangle \\
= &-\mathcal{L}^N_Y(\mathcal{L}^\pi_{df}\langle X, dg\rangle - \mathcal{L}^\pi_{dg}\langle X, df\rangle - \langle X, [df, dg]_\pi\rangle) \\
&+ \langle \mathcal{L}^N_Y df, d_\pi\langle X, dg\rangle\rangle - \mathcal{L}^\pi_{dg}\langle X, \mathcal{L}^N_Y df\rangle \\
&- \langle X, [\mathcal{L}^N_Y df, dg]_\pi\rangle + \mathcal{L}^\pi_{df}\langle X, \mathcal{L}^N_Y dg\rangle \\
&- \langle \mathcal{L}^N_Y dg, d_\pi\langle X, df\rangle\rangle - \langle X, [df, \mathcal{L}^N_Y dg]_\pi\rangle \\
= &-\mathcal{L}^N_Y\mathcal{L}^\pi_{df}\langle X, dg\rangle + \mathcal{L}^N_Y\mathcal{L}^\pi_{dg}\langle X, df\rangle + \mathcal{L}^N_Y\langle X, [df, dg]_\pi\rangle \\
&+ \mathcal{L}^N_Y\langle df, d_\pi\langle X, dg\rangle\rangle - \langle df, \mathcal{L}^N_Y d_\pi\langle X, dg\rangle\rangle \\
&- \langle \mathcal{L}^\pi_{dg}X, \mathcal{L}^N_Y df\rangle - \langle X, \mathcal{L}^\pi_{dg}\mathcal{L}^N_Y df\rangle + \langle X, \mathcal{L}^\pi_{dg}\mathcal{L}^N_Y df\rangle \\
&+ \langle \mathcal{L}^\pi_{df}X, \mathcal{L}^N_Y dg\rangle + \langle X, \mathcal{L}^\pi_{df}\mathcal{L}^N_Y dg\rangle \\
&- \mathcal{L}^N_Y\langle dg, d_\pi\langle X, df\rangle\rangle + \langle dg, \mathcal{L}^N_Y d_\pi\langle X, df\rangle\rangle \\
&- \langle X, \mathcal{L}^\pi_{df}\mathcal{L}^N_Y dg\rangle \\
= &-\mathcal{L}^N_Y\mathcal{L}^\pi_{df}\langle X, dg\rangle + \mathcal{L}^N_Y\mathcal{L}^\pi_{dg}\langle X, df\rangle + (NY)\langle X, [df, dg]_\pi\rangle \\
&+ \mathcal{L}^N_Y\mathcal{L}^\pi_{df}\langle X, dg\rangle - \langle df, [Y, d_\pi\langle X, dg\rangle]_N\rangle \\
&- \langle \mathcal{L}^\pi_{dg}X, \mathcal{L}^N_Y df\rangle + \langle \mathcal{L}^\pi_{df}X, \mathcal{L}^N_Y dg\rangle \\
&- \mathcal{L}^N_Y\mathcal{L}^\pi_{dg}\langle X, df\rangle + \langle dg, [Y, d_\pi\langle X, df\rangle]_N\rangle \\
= &(NY)\langle X, [df, dg]_\pi\rangle - \langle df, [Y, d_\pi\langle X, dg\rangle]_N\rangle \\
&- \langle \mathcal{L}^\pi_{dg}X, \mathcal{L}^N_Y df\rangle + \langle \mathcal{L}^\pi_{df}X, \mathcal{L}^N_Y dg\rangle \\
&+ \langle dg, [Y, d_\pi\langle X, df\rangle]_N\rangle,
\end{aligned}
$$

and similarly

$$
\begin{aligned}
[X, d_\pi Y]_N(df, dg) &= -[d_\pi Y, X]_N(df, dg) \\
&= -(NX)\langle Y, [df, dg]_\pi\rangle + \langle df, [X, d_\pi\langle Y, dg\rangle]_N\rangle \\
&\quad + \langle \mathcal{L}^\pi_{dg}Y, \mathcal{L}^N_X df\rangle - \langle \mathcal{L}^\pi_{df}Y, \mathcal{L}^N_X dg\rangle \\
&\quad - \langle dg, [X, d_\pi\langle Y, df\rangle]_N\rangle.
\end{aligned}
$$

On the other hand, by the same calculations, we obtain

$$
\begin{aligned}
(d_N[df, dg]_\pi)(X, Y) = &(NX)\langle [df, dg]_\pi, Y\rangle \\
&- (NY)\langle [df, dg]_\pi, X\rangle - \langle [df, dg]_\pi, [X, Y]_N\rangle,
\end{aligned}
$$

$$\begin{aligned}[d_N df, dg]_\pi(X,Y) &= (\pi^\sharp dg)\langle df, [X,Y]_N\rangle - \langle X, [dg, d_N\langle df, Y\rangle]_\pi\rangle \\ &\quad - \langle \mathcal{L}^N_Y df, \mathcal{L}^\pi_{dg} X\rangle + \langle \mathcal{L}^N_X df, \mathcal{L}^N_{dg} Y\rangle \\ &\quad + \langle Y, [dg, d_N\langle df, X\rangle]_\pi\rangle, \\ [df, d_N dg]_\pi(X,Y) &= -(\pi^\sharp df)\langle dg, [X,Y]_N\rangle + \langle X, [df, d_N\langle dg, Y\rangle]_\pi\rangle \\ &\quad + \langle \mathcal{L}^N_Y dg, \mathcal{L}^\pi_{df} X\rangle - \langle \mathcal{L}^N_X dg, \mathcal{L}^N_{df} Y\rangle \\ &\quad - \langle Y, [df, d_N\langle dg, X\rangle]_\pi\rangle.\end{aligned}$$

Therefore by the equation (4.25), we obtain for any X, Y in $\mathfrak{X}(M)$, f and g in $C^\infty(M)$,

$$\begin{aligned}(A_{\pi,N}(X,Y))(df,dg) &= (A_{\pi,N}(X,Y))(df,dg) \\ &\quad + (d(C^\pi_N(df,dg)))(X,Y) - (d(C^\pi_N(df,dg)))(X,Y) \\ &= (A_{\pi,N}(X,Y))(df,dg) - (A_{N,\pi}(df,dg))(X,Y) \\ &\quad - (d(C^\pi_N(df,dg)))(X,Y) \\ &= (d_\pi[X,Y]_N - [d_\pi X, Y]_N(df,dg) - [X, d_\pi Y]_N)(df,dg) \\ &\quad - (d_N[df,dg]_\pi - [d_N df, dg]_\pi - [df, d_N dg]_\pi)(X,Y) \\ &\quad - (d(C^\pi_N(df,dg)))(X,Y) \\ &= (\pi^\sharp df)\langle [X,Y]_N, dg\rangle \\ &\quad - (\pi^\sharp dg)\langle [X,Y]_N, df\rangle - \langle [X,Y]_N, [df,dg]_\pi\rangle \\ &\quad - (NY)\langle X, [df,dg]_\pi\rangle + \langle df, [Y, d_\pi\langle X, dg\rangle]_N\rangle \\ &\quad + \langle \mathcal{L}^\pi_{dg} X, \mathcal{L}^N_Y df\rangle - \langle \mathcal{L}^\pi_{df} X, \mathcal{L}^N_Y dg\rangle \\ &\quad - \langle dg, [Y, d_\pi\langle X, df\rangle]_N\rangle \\ &\quad + (NX)\langle Y, [df,dg]_\pi\rangle - \langle df, [X, d_\pi\langle Y, dg\rangle]_N\rangle \\ &\quad - \langle \mathcal{L}^\pi_{dg} Y, \mathcal{L}^N_X df\rangle + \langle \mathcal{L}^\pi_{df} Y, \mathcal{L}^N_X dg\rangle \\ &\quad + \langle dg, [X, d_\pi\langle Y, df\rangle]_N\rangle \\ &\quad - (NX)\langle [df,dg]_\pi, Y\rangle \\ &\quad + (NY)\langle [df,dg]_\pi, X\rangle + \langle [df,dg]_\pi, [X,Y]_N\rangle \\ &\quad + (\pi^\sharp dg)\langle df, [X,Y]_N\rangle - \langle X, [dg, d_N\langle df, Y\rangle]_\pi\rangle \\ &\quad - \langle \mathcal{L}^N_Y df, \mathcal{L}^\pi_{dg} X\rangle + \langle \mathcal{L}^N_X df, \mathcal{L}^N_{dg} Y\rangle \\ &\quad + \langle Y, [dg, d_N\langle df, X\rangle]_\pi\rangle \\ &\quad - (\pi^\sharp df)\langle dg, [X,Y]_N\rangle + \langle X, [df, d_N\langle dg, Y\rangle]_\pi\rangle \\ &\quad + \langle \mathcal{L}^N_Y dg, \mathcal{L}^\pi_{df} X\rangle - \langle \mathcal{L}^N_X dg, \mathcal{L}^N_{df} Y\rangle \\ &\quad - \langle Y, [df, d_N\langle dg, X\rangle]_\pi\rangle\end{aligned}$$

$$
\begin{aligned}
&\quad - (d(C_N^\pi(df,dg)))(X,Y)\\
&= \langle df, [Y, d_\pi\langle X, dg\rangle]_N\rangle - \langle dg, [Y, d_\pi\langle X, df\rangle]_N\rangle\\
&\quad - \langle df, [X, d_\pi\langle Y, dg\rangle]_N\rangle + \langle dg, [X, d_\pi\langle Y, df\rangle]_N\rangle\\
&\quad - \langle X, [dg, d_N\langle df, Y\rangle]_\pi\rangle + \langle Y, [dg, d_N\langle df, X\rangle]_\pi\rangle\\
&\quad + \langle X, [df, d_N\langle dg, Y\rangle]_\pi\rangle - \langle Y, [df, d_N\langle dg, X\rangle]_\pi\rangle\\
&\quad - (d(C_N^\pi(df,dg)))(X,Y)\\
&= -\langle df, \mathcal{L}^\pi_{d_N\langle X,dg\rangle}Y + [d_\pi\langle X,dg\rangle, Y]_N\rangle\\
&\quad + \langle dg, \mathcal{L}^\pi_{d_N\langle X,df\rangle}Y + [d_\pi\langle X,df\rangle, Y]_N\rangle\\
&\quad + \langle df, \mathcal{L}^\pi_{d_N\langle Y,dg\rangle}X + [X, d_\pi\langle Y,dg\rangle]_N\rangle\\
&\quad - \langle dg, \mathcal{L}^\pi_{d_N\langle Y,df\rangle}X + [X, d_\pi\langle Y,df\rangle]_N\rangle\\
&\quad + \langle X, \mathcal{L}^\pi_{d_N\langle df,Y\rangle}dg\rangle - \langle Y, \mathcal{L}^\pi_{d_N\langle df,X\rangle}dg\rangle\\
&\quad - \langle X, \mathcal{L}^\pi_{d_N\langle dg,Y\rangle}df\rangle + \langle Y, \mathcal{L}^\pi_{d_N\langle dg,X\rangle}df\rangle\\
&\quad - (d(C_N^\pi(df,dg)))(X,Y)\\
&\quad + \langle df, \mathcal{L}^\pi_{d_N\langle X,dg\rangle}Y\rangle - \langle dg, \mathcal{L}^\pi_{d_N\langle X,df\rangle}Y\rangle\\
&\quad - \langle df, \mathcal{L}^\pi_{d_N\langle Y,dg\rangle}X\rangle + \langle dg, \mathcal{L}^\pi_{d_N\langle Y,df\rangle}X\rangle\\
&= -\langle df, \mathcal{L}^\pi_{d_N(Xg)}Y + [d_\pi(Xg), Y]_N\rangle\\
&\quad + \langle dg, \mathcal{L}^\pi_{d_N(Xf)}Y + [d_\pi(Xf), Y]_N\rangle\\
&\quad + \langle df, \mathcal{L}^\pi_{d_N(Yg)}X + [X, d_\pi(Yg)]_N\rangle\\
&\quad - \langle dg, \mathcal{L}^\pi_{d_N(Yf)}X + [X, d_\pi(Yf)]_N\rangle\\
&\quad + \mathcal{L}^\pi_{d_N(Yf)}\langle X, dg\rangle - \mathcal{L}^\pi_{d_N(Xf)}\langle Y, dg\rangle\\
&\quad - \mathcal{L}^\pi_{d_N(Yg)}\langle X, df\rangle + \mathcal{L}^\pi_{d_N(Xg)}\langle Y, df\rangle\\
&\quad - (d(C_N^\pi(df,dg)))(X,Y)\\
&= -\langle df, \mathcal{L}^\pi_{d_N(Xg)}Y + [d_\pi(Xg), Y]_N\rangle\\
&\quad + \langle dg, \mathcal{L}^\pi_{d_N(Xf)}Y + [d_\pi(Xf), Y]_N\rangle\\
&\quad + \langle df, \mathcal{L}^\pi_{d_N(Yg)}X + [X, d_\pi(Yg)]_N\rangle\\
&\quad - \langle dg, \mathcal{L}^\pi_{d_N(Yf)}X + [X, d_\pi(Yf)]_N\rangle\\
&\quad + (\pi^\sharp N^* d(Yf))(Xg) - (\pi^\sharp N^* d(Xf))(Yg)\\
&\quad - (\pi^\sharp N^* d(Yg))(Xf) + (\pi^\sharp N^* d(Xg))(Yf)\\
&\quad - (d(C_N^\pi(df,dg)))(X,Y)\\
&= -\langle df, \mathcal{L}^\pi_{d_N(Xg)}Y + [d_\pi(Xg), Y]_N\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+ \langle dg, \mathcal{L}^{\pi}_{d_N(Xf)} Y + [d_\pi(Xf), Y]_N \rangle \\
&+ \langle df, \mathcal{L}^{\pi}_{d_N(Yg)} X + [X, d_\pi(Yg)]_N \rangle \\
&- \langle dg, \mathcal{L}^{\pi}_{d_N(Yf)} X + [X, d_\pi(Yf)]_N \rangle \\
&+ \langle (\pi^\sharp N^* - N\pi^\sharp) d(Xg), d(Yf) \rangle \\
&- \langle (\pi^\sharp N^* - N\pi^\sharp) d(Xf), d(Yg) \rangle \\
&- (d(C^\pi_N(df, dg)))(X, Y).
\end{aligned}
$$

For any D_i in $\mathfrak{X}^*(M)$, $i = 1, 2, 3$, we calculate

$$
\begin{aligned}
A_{\pi,N}(D_1, D_2 \wedge D_3) &= d_\pi [D_1, D_2 \wedge D_3]_N - [d_\pi D_1, D_2 \wedge D_3]_N \\
&\quad - (-1)^{\deg D_1 + 1} [D_1, d_\pi (D_2 \wedge D_3)]_N \\
&= d_\pi ([D_1, D_2]_N \wedge D_3 \\
&\quad + (-1)^{(\deg D_1 + 1)\deg D_2} D_2 \wedge [D_1, D_3]_N) \\
&\quad - ([d_\pi D_1, D_2]_N \wedge D_3 \\
&\quad + (-1)^{(\deg D_1 + 2)\deg D_2} D_2 \wedge [d_\pi D_1, D_3]_N) \\
&\quad - (-1)^{\deg D_1 + 1} [D_1, d_\pi D_2 \wedge D_3 \\
&\quad + (-1)^{\deg D_2} D_2 \wedge d_\pi D_3]_N \\
&= d_\pi [D_1, D_2]_N \wedge D_3 \\
&\quad + (-1)^{\deg D_1 + \deg D_2 - 1} [D_1, D_2]_N \wedge d_\pi D_3 \\
&\quad + (-1)^{(\deg D_1 + 1)\deg D_2} d_\pi D_2 \wedge [D_1, D_3]_N \\
&\quad + (-1)^{(\deg D_1 + 1)\deg D_2 + \deg D_2} D_2 \wedge d_\pi [D_1, D_3]_N \\
&\quad - [d_\pi D_1, D_2]_N \wedge D_3 \\
&\quad - (-1)^{\deg D_1 \deg D_2} D_2 \wedge [d_\pi D_1, D_3]_N \\
&\quad + (-1)^{\deg D_1} [D_1, d_\pi D_2]_N \wedge D_3 \\
&\quad + (-1)^{\deg D_1 + (\deg D_1 + 1)(\deg D_2 + 1)} d_\pi D_2 \wedge [D_1, D_3]_N \\
&\quad + (-1)^{\deg D_1 + \deg D_2} [D_1, D_2]_N \wedge d_\pi D_3 \\
&\quad + (-1)^{\deg D_1 + \deg D_2 + (\deg D_1 + 1)\deg D_2} D_2 \wedge [D_1, d_\pi D_3]_N \\
&= (d_\pi [D_1, D_2]_N - [d_\pi D_1, D_2]_N \\
&\quad - (-1)^{\deg D_1 + 1} [D_1, d_\pi D_2]_N) \wedge D_3 \\
&\quad + (-1)^{\deg D_1 \deg D_2} D_2 \wedge (d_\pi [D_1, D_3]_N - [d_\pi D_1, D_3]_N
\end{aligned}
$$

$$
\begin{aligned}
&\quad - (-1)^{\deg_{D_1}+1}[D_1, d_\pi D_3]_N) \\
&= A_{\pi,N}(D_1, D_2) \wedge D_3 \\
&\quad + (-1)^{\deg_{D_1}\deg_{D_2}} D_2 \wedge A_{\pi,N}(D_1, D_3)
\end{aligned}
$$

and

$$
\begin{aligned}
A_{\pi,N}(D_1, D_2) &= d_\pi[D_1, D_2]_N - [d_\pi D_1, D_2]_N - (-1)^{\deg_{D_1}+1}[D_1, d_\pi D_2]_N \\
&= d_\pi(-(-1)^{(\deg_{D_1}-1)(\deg_{D_2}-1)}[D_2, D_1]_N) \\
&\quad - (-1)^{\deg_{D_1}(\deg_{D_2}-1)}[D_2, d_\pi D_1]_N \\
&\quad + (-1)^{\deg_{D_1}+1+(\deg_{D_1}-1)\deg_{D_2}}[d_\pi D_2, D_1]_N \\
&= -(-1)^{(\deg_{D_1}-1)(\deg_{D_2}-1)}(d_\pi[D_2, D_1]_N \\
&\quad - (-1)^{\deg_{D_2}+1}[D_2, d_\pi D_1]_N - [d_\pi D_2, D_1]_N) \\
&= -(-1)^{(\deg_{D_1}-1)(\deg_{D_2}-1)} A_{\pi,N}(D_2, D_1).
\end{aligned}
$$

From the above, the conclusion follows from these equations and Lemma 4.1.6. □

4.2 The definition and properties of pseudo-Poisson-Nijenhuis manifolds

In this section, we define Pseudo-Poisson-Nijenhuis manifolds and investigate properties of them.

Definition 46 ([32]). Let M be a C^∞-manifold, π a 2-vector field on M, a $(1,1)$-tensor N on M a Nijenhuis structure compatible with π, and Φ a 3-vector field on M. Then a triple (π, N, Φ) is a *pseudo-Poisson-Nijenhuis structure* on M if the following conditions hold:

$$
\text{(i)} \quad [\pi, \Phi] = 0, \tag{4.28}
$$

$$
\text{(ii)} \quad \frac{1}{2}\iota_{\alpha\wedge\beta}[\pi, \pi] = N\iota_{\alpha\wedge\beta}\Phi, \tag{4.29}
$$

$$
\text{(iii)} \quad N\iota_{\alpha\wedge\beta}\mathcal{L}_X\Phi - \iota_{\alpha\wedge\beta}\mathcal{L}_{NX}\Phi - \iota_{(\mathcal{L}_X N^*)(\alpha\wedge\beta)}\Phi = 0, \tag{4.30}
$$

for any X in $\mathfrak{X}(M)$, α and β in $\Omega^1(M)$, where $\iota_{\alpha\wedge\beta} := \iota_\beta\iota_\alpha$ and $(\mathcal{L}_X N^*)(\alpha\wedge\beta) := (\mathcal{L}_X N^*)\alpha\wedge\beta + \alpha\wedge(\mathcal{L}_X N^*)\beta$. The quadruple (M, π, N, Φ) is called a *pseudo-Poisson-Nijenhuis manifold.*

Remark 12. The reason why we use not "quasi-" but "pseudo-" is to avoid confusion with a quasi-Poisson manifold in Subsection 2.3.

Now we describe the main theorem in this section. This is one of the fundamental properties of pseudo-Poisson-Nijenhuis manifolds. A similar result for Poisson-quasi-Nijenhuis manifolds is also known [38].

Theorem 4.2.1 (Nakamura [32])**.** Let M be a C^∞-manifold, π a 2-vector field on M, N a Nijenhuis structure on M compatible with π and Φ a 3-vector field on M. Then a quadruple (M, π, N, Φ) is a pseudo-Poisson-Nijenhuis manifold if and only if $((TM)_N, d_\pi, \Phi)$ is a quasi-Lie bialgebroid.

Proof. Since a $(1,1)$-tensor N is Nijenhuis, the Lie algebroid $(TM)_N$ is well-defined. A triple $((TM)_N, d_\pi, \Phi)$ is a quasi-Lie bialgebroid if and only if the following three conditions hold: i) d_π is a degree-one derivation of the Gerstenhaber algebra $(\mathfrak{X}^*(M), \wedge, [\cdot,\cdot]_N)$, ii) $d_\pi^2 = [\Phi, \cdot]_N$ and iii) $d_\pi \Phi = 0$ by the definition.

i) means that (4.19) holds. This condition is equivalent to the compatibility of π and N by Theorem 4.1.5.

Next, For any f in $C^\infty(M)$, any α and β in $\Omega^1(M)$, we compute

$$\begin{aligned}(d_\pi^2 f)(\alpha,\beta) &= [\pi,[\pi,f]](\alpha,\beta) = \frac{1}{2}[[\pi,\pi],f](\alpha,\beta)\\ &= \frac{1}{2}\iota_{df}[\pi,\pi](\alpha,\beta) = \frac{1}{2}[\pi,\pi](df,\alpha,\beta)\\ &= \frac{1}{2}[\pi,\pi](\alpha,\beta,df) = \frac{1}{2}\iota_{\alpha\wedge\beta}[\pi,\pi](df),\end{aligned}$$

where we use the graded Jacobi identity of the Schouten bracket $[\cdot,\cdot]$ and the facts that $d_\pi = [\pi,\cdot]$ and that $[D,f] = (-1)^{k+1}\iota_{df}D$ for any D in $\mathfrak{X}^k(M)$. On the other hand, we have

$$\begin{aligned}[\Phi,f]_N(\alpha,\beta) &= \iota_{N^*df}\Phi(\alpha,\beta) = \Phi(N^*df,\alpha,\beta)\\ &= \Phi(\alpha,\beta,N^*df) = \iota_{\alpha\wedge\beta}\Phi(N^*df)\\ &= (N\iota_{\alpha\wedge\beta}\Phi)(df).\end{aligned}$$

Therefore it follows that $d_\pi^2 = [\Phi,\cdot]_N$ on $C^\infty(M)$ if and only if the equality (4.29) holds as a linear map on the exact 1-forms. By $C^\infty(M)$-linearity of (4.29) and the fact that the exact 1-forms generate locally the 1-forms as a $C^\infty(M)$-module, the equality (4.29) holds on $\Omega^1(M)$ if and only if $d_\pi^2 = [\Phi,\cdot]_N$ holds on $C^\infty(M)$.

Next, under the assumption that the equality (4.29) holds on $\Omega^1(M)$, for any X in $\mathfrak{X}(M)$, any α, β and γ in $\Omega^1(M)$, we obtain

$$\begin{aligned}
(d_\pi^2 X)(\alpha,\beta,\gamma) &= [\pi,[\pi,X]](\alpha,\beta,\gamma) = \frac{1}{2}[[\pi,\pi],X](\alpha,\beta,\gamma) \\
&= -\frac{1}{2}[X,[\pi,\pi]](\alpha,\beta,\gamma) = -\frac{1}{2}(\mathcal{L}_X[\pi,\pi])(\alpha,\beta,\gamma) \\
&= -\frac{1}{2}\{\mathcal{L}_X([\pi,\pi](\alpha,\beta,\gamma)) - [\pi,\pi](\mathcal{L}_X\alpha,\beta,\gamma) \\
&\qquad - [\pi,\pi](\alpha,\mathcal{L}_X\beta,\gamma) - [\pi,\pi](\alpha,\beta,\mathcal{L}_X\gamma)\} \\
&= -\mathcal{L}_X\left(\frac{1}{2}\iota_{\alpha\wedge\beta}[\pi,\pi](\gamma)\right) + \frac{1}{2}\iota_{\mathcal{L}_X\alpha\wedge\beta}[\pi,\pi](\gamma) \\
&\quad + \frac{1}{2}\iota_{\alpha\wedge\mathcal{L}_X\beta}[\pi,\pi](\gamma) + \frac{1}{2}\iota_{\alpha\wedge\beta}[\pi,\pi](\mathcal{L}_X\gamma) \\
&= -\mathcal{L}_X\left((N\iota_{\alpha\wedge\beta}\Phi)(\gamma)\right) + (N\iota_{\mathcal{L}_X\alpha\wedge\beta}\Phi)(\gamma) \\
&\quad + (N\iota_{\alpha\wedge\mathcal{L}_X\beta}\Phi)(\gamma) + (N\iota_{\alpha\wedge\beta}\Phi)(\mathcal{L}_X\gamma) \\
&= -\mathcal{L}_X\left(\iota_{\alpha\wedge\beta}\Phi(N^*\gamma)\right) + \iota_{\mathcal{L}_X\alpha\wedge\beta}\Phi(N^*\gamma) \\
&\quad + \iota_{\alpha\wedge\mathcal{L}_X\beta}\Phi(N^*\gamma) + \iota_{\alpha\wedge\beta}\Phi(N^*\mathcal{L}_X\gamma) \\
&= -\mathcal{L}_X\left(\Phi(\alpha,\beta,N^*\gamma)\right) + \Phi(\mathcal{L}_X\alpha,\beta,N^*\gamma) \\
&\quad + \Phi(\alpha,\mathcal{L}_X\beta,N^*\gamma) + \Phi(\alpha,\beta,N^*\mathcal{L}_X\gamma) \\
&= -\mathcal{L}_X\left(\Phi(\alpha,\beta,N^*\gamma)\right) + \Phi(\mathcal{L}_X\alpha,\beta,N^*\gamma) \\
&\quad + \Phi(\alpha,\mathcal{L}_X\beta,N^*\gamma) + \Phi(\alpha,\beta,\mathcal{L}_X(N^*\gamma) - (\mathcal{L}_XN^*)\gamma) \\
&= -\mathcal{L}_X\left(\Phi(\alpha,\beta,N^*\gamma)\right) + \Phi(\mathcal{L}_X\alpha,\beta,N^*\gamma) \\
&\quad + \Phi(\alpha,\mathcal{L}_X\beta,N^*\gamma) + \Phi(\alpha,\beta,\mathcal{L}_X(N^*\gamma)) \\
&\quad - \Phi(\alpha,\beta,(\mathcal{L}_XN^*)\gamma) \\
&= -\left(\mathcal{L}_X\Phi\right)(\alpha,\beta,N^*\gamma) - \Phi(\alpha,\beta,(\mathcal{L}_XN^*)\gamma),
\end{aligned}$$

where we use the graded Jacobi identity of $[\cdot,\cdot]$. On the other hand, we obtain

$$\begin{aligned}
[\Phi,X]_N(\alpha,\beta,\gamma) &= -[X,\Phi]_N(\alpha,\beta,\gamma) = -(\mathcal{L}_X^N\Phi)(\alpha,\beta,\gamma) \\
&= -\mathcal{L}_X^N(\Phi(\alpha,\beta,\gamma)) + \Phi(\mathcal{L}_X^N\alpha,\beta,\gamma) \\
&\quad + \Phi(\alpha,\mathcal{L}_X^N\beta,\gamma) + \Phi(\alpha,\beta,\mathcal{L}_X^N\gamma) \\
&= -\mathcal{L}_{NX}(\Phi(\alpha,\beta,\gamma)) + \Phi(\mathcal{L}_{NX}\alpha - (\mathcal{L}_XN^*)\alpha,\beta,\gamma) \\
&\quad + \Phi(\alpha,\mathcal{L}_{NX}\beta - (\mathcal{L}_XN^*)\beta,\gamma) \\
&\quad + \Phi(\alpha,\beta,\mathcal{L}_{NX}\gamma - (\mathcal{L}_XN^*)\gamma) \\
&= -\mathcal{L}_{NX}(\Phi(\alpha,\beta,\gamma)) + \Phi(\mathcal{L}_{NX}\alpha,\beta,\gamma)
\end{aligned}$$

$$
\begin{aligned}
&+\Phi(\alpha,\mathcal{L}_{NX}\beta,\gamma)+\Phi(\alpha,\beta,\mathcal{L}_{NX}\gamma)\\
&-\Phi((\mathcal{L}_X N^*)\alpha,\beta,\gamma)-\Phi(\alpha,(\mathcal{L}_X N^*)\beta,\gamma)\\
&-\Phi(\alpha,\beta,(\mathcal{L}_X N^*)\gamma)\\
=&-(\mathcal{L}_{NX}\Phi)(\alpha,\beta,\gamma)-\Phi((\mathcal{L}_X N^*)\alpha,\beta,\gamma)\\
&-\Phi(\alpha,(\mathcal{L}_X N^*)\beta,\gamma)-\Phi(\alpha,\beta,(\mathcal{L}_X N^*)\gamma),
\end{aligned}
$$

where we use the property that $\mathcal{L}^N_X\alpha = \mathcal{L}_{NX}\alpha - (\mathcal{L}_X N^*)\alpha$ for any X in $\mathfrak{X}(M)$ and any α in $\Omega^1(M)$. Therefore, we obtain

$$
\begin{aligned}
(d_\pi^2-[\Phi,X]_N)(\alpha,\beta,\gamma)=&-(\mathcal{L}_X\Phi)(\alpha,\beta,N^*\gamma)-\Phi(\alpha,\beta,(\mathcal{L}_X N^*)\gamma)\\
&+(\mathcal{L}_{NX}\Phi)(\alpha,\beta,\gamma)+\Phi((\mathcal{L}_X N^*)\alpha,\beta,\gamma)\\
&+\Phi(\alpha,(\mathcal{L}_X N^*)\beta,\gamma)+\Phi(\alpha,\beta,(\mathcal{L}_X N^*)\gamma)\\
=&-(\mathcal{L}_X\Phi)(\alpha,\beta,N^*\gamma)+(\mathcal{L}_{NX}\Phi)(\alpha,\beta,\gamma)\\
&+\Phi((\mathcal{L}_X N^*)\alpha,\beta,\gamma)+\Phi(\alpha,(\mathcal{L}_X N^*)\beta,\gamma)\\
=&-(N\iota_{\alpha\wedge\beta}\mathcal{L}_X\Phi-\iota_{\alpha\wedge\beta}\mathcal{L}_{NX}\Phi-\iota_{(\mathcal{L}_X N^*)(\alpha\wedge\beta)}\Phi)(\gamma).
\end{aligned}
$$

Hence, under the assumption of (4.29), it follows that $d_\pi^2 = [\Phi,\cdot]_N$ on $\mathfrak{X}(M)$ if and only if the equality (4.30) holds.

Since d_π^2 and $[\Phi,\cdot]_N$ are derivatives on $(\Gamma(\Lambda^* TM),\wedge)$, it follows that $d_\pi^2 = [\Phi,\cdot]_N$ on $C^\infty(M)\oplus\mathfrak{X}(M)$ if and only if $d_\pi^2 = [\Phi,\cdot]_N$ on $\mathfrak{X}^*(M)$. Therefore ii) is equivalent to that (4.29) and (4.30) hold.

Finally, iii) is equivalent to (4.28) due to that $d_\pi\Phi = [\pi,\Phi]$. Therefore the proof has been completed. □

By the theorem, we have the following result (Theorem 2.4.2) of Kosmann-Schwarzbach [15].

Corollary 4.2.2 (Theorem 2.4.2, [32])**.** Under the same assumption as Theorem 4.2.1, the triple (M,π,N) is a Poisson-Nijenhuis manifold if and only if $((TM)_N, d_\pi)$ is a Lie bialgebroid.

As in the case of Poisson-quasi-Nijenhuis Lie algebroids (Definition 44), we can consider a straightforward generalization of pseudo-Poisson-Nijenhuis manifolds.

Definition 47 ([32])**.** A *pseudo-Poisson-Nijenhuis Lie algebroid* (A,π,N,Φ) is a Lie algebroid A equipped with a 2-section π in $\Gamma(\Lambda^2 A)$, a Nijenhuis structure $N : A \longrightarrow A$ compatible with π in the sense of Definition 45 and a 3-section Φ in $\Gamma(\Lambda^3 A)$ satisfying the conditions (4.28), (4.29) and (4.30) replaced $[\cdot,\cdot]$ and $\mathcal{L}$ with $[\cdot,\cdot]_A$ and $\mathcal{L}^A$, respectively.

Theorem 4.2.3 (Nakamura [32])**.** *If a quadruple (A, π, N, Φ) is a pseudo-Poisson-Nijenhuis Lie algebroid, then (A_N, d_π, Φ) is a quasi-Lie bialgebroid, where A_N is a Lie algebroid deformed by the Nijenhuis structure N.*

Now we show three simple and important examples of pseudo-Poisson-Nijenhuis manifolds.

Example 53 ([32])**.** A triple (π, N, Φ), where $\Phi = 0$, is a pseudo-Poisson-Nijenhuis structure if (π, N) is a Poisson-Nijenhuis structure.

Example 54 ([32])**.** Let (M, π) be a Poisson manifold and set $N = 0$. For any d_π-closed 3-vector field Φ, the triple (π, N, Φ) is a pseudo-Poisson-Nijenhuis structure. In fact, in this case, a pair $(\pi, 0)$ is compatible obviously and the conditions (i)-(iii) in Definition 46 are satisfied by

$$\begin{aligned}
&\text{(i)} \quad [\pi, \Phi] = d_\pi \Phi = 0;\\
&\text{(ii)} \quad \frac{1}{2}\iota_{\alpha\wedge\beta}[\pi, \pi] = 0 = N\iota_{\alpha\wedge\beta}\Phi;\\
&\text{(iii)} \quad N\iota_{\alpha\wedge\beta}\mathcal{L}_X\Phi - \iota_{\alpha\wedge\beta}\mathcal{L}_{NX}\Phi - \iota_{(\mathcal{L}_X N^*)(\alpha\wedge\beta)}\Phi\\
&\qquad = 0 - \iota_{\alpha\wedge\beta}\mathcal{L}_0\Phi - \iota_{(\mathcal{L}_X 0)(\alpha\wedge\beta)}\Phi\\
&\qquad = 0
\end{aligned}$$

for any α and β in $\Omega^1(M)$. Therefore, by Theorem 4.2.1 and Example 2.4.7, $((TM)_N, d_\pi, \Phi)$ is a quasi-Lie bialgebroid and $((TM)_N \oplus (T^*M)_\pi, \langle\!\langle \cdot, \cdot \rangle\!\rangle, [\![\cdot, \cdot]\!]_\pi^\Phi, \rho)$ is a Courant algebroid, where the Courant bracket $[\![\cdot, \cdot]\!]_\pi^\Phi$ is defined by

$$\begin{aligned}
[\![X, Y]\!]_\pi^\Phi &= [X, Y]_0 = 0,\\
[\![\xi, \eta]\!]_\pi^\Phi &= [\xi, \eta]_\pi + \Phi(\xi, \eta, \cdot),\\
[\![X, \xi]\!]_\pi^\Phi &= \left(\iota_X d_0 \xi + \frac{1}{2} d_0 \langle \xi, X\rangle\right) - \left(\iota_\xi d_\pi X + \frac{1}{2} d_\pi \langle \xi, X\rangle\right)\\
&= -\iota_\xi d_\pi X - \frac{1}{2} d_\pi \langle \xi, X\rangle,
\end{aligned}$$

the anchor map ρ satisfies $\rho(X + \xi) = NX + \pi^\sharp \xi = \pi^\sharp \xi$ and the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is given by (2.105) for any X, Y in $\mathfrak{X}(M)$, any ξ and η in $\Omega^1(M)$.

Example 55 ([32])**.** Let M be a C^∞-manifold and set $N = a \cdot \mathrm{id}_{TM}$, where a is a non-zero real number. For any 2-vector field π in $\mathfrak{X}^2(M)$, the triple (π, N, Φ), where $\Phi = \frac{1}{2a}[\pi, \pi]$, is a pseudo-Poisson-Nijenhuis structure. In fact, in this case, a pair $(\pi, a\cdot \mathrm{id}_{TM})$ is compatible obviously and the conditions in Definition 46 are satisfied by

$$
\begin{aligned}
&\text{(i)} \quad [\pi,\Phi] = \frac{1}{2a}\,[\pi,[\pi,\pi]] = 0;\\
&\text{(ii)} \quad N\iota_{\alpha\wedge\beta}\Phi = a\iota_{\alpha\wedge\beta}\left(\frac{1}{2a}[\pi,\pi]\right) = \frac{1}{2}\iota_{\alpha\wedge\beta}[\pi,\pi];\\
&\text{(iii)} \quad N\iota_{\alpha\wedge\beta}\mathcal{L}_X\Phi - \iota_{\alpha\wedge\beta}\mathcal{L}_{NX}\Phi - \iota_{(\mathcal{L}_X N^*)(\alpha\wedge\beta)}\Phi\\
&\qquad = a\iota_{\alpha\wedge\beta}\mathcal{L}_X\left(\frac{1}{2a}[\pi,\pi]\right) - \iota_{\alpha\wedge\beta}\mathcal{L}_{aX}\left(\frac{1}{2a}[\pi,\pi]\right)\\
&\qquad\qquad - \iota_{(\mathcal{L}_X(a\cdot\mathrm{id}_{TM})^*)(\alpha\wedge\beta)}\left(\frac{1}{2a}[\pi,\pi]\right)\\
&\qquad = \frac{1}{2}\left(\iota_{\alpha\wedge\beta}\mathcal{L}_X[\pi,\pi] - \iota_{\alpha\wedge\beta}\mathcal{L}_X[\pi,\pi] - \iota_0[\pi,\pi]\right)\\
&\qquad = 0
\end{aligned}
$$

for any α and β in $\Omega^1(M)$. Therefore $((TM)_N, d_\pi, \Phi)$ is a quasi-Lie bialgebroid and $(TM\oplus T^*M, \langle\!\langle\cdot,\cdot\rangle\!\rangle, [\![\cdot,\cdot]\!]_\pi^\Phi, \rho)$ is a Courant algebroid, where the Courant bracket $[\![\cdot,\cdot]\!]_\pi^\Phi$ is defined by

$$
\begin{aligned}
[\![X,Y]\!]_\pi^\Phi &= [X,Y]_{a\cdot\mathrm{id}_{TM}} = a[X,Y],\\
[\![\xi,\eta]\!]_\pi^\Phi &= [\xi,\eta]_\pi + \frac{1}{2a}\,[\pi,\pi]\,(\xi,\eta,\cdot),\\
[\![X,\xi]\!]_\pi^\Phi &= \left(\iota_X d_{a\cdot\mathrm{id}_{TM}}\xi + \frac{1}{2}d_{a\cdot\mathrm{id}_{TM}}\langle\xi,X\rangle\right) - \left(\iota_\xi d_\pi X + \frac{1}{2}d_\pi\langle\xi,X\rangle\right)\\
&= a\left(\iota_X d\xi + \frac{1}{2}d\langle\xi,X\rangle\right) - \left(\iota_\xi d_\pi X + \frac{1}{2}d_\pi\langle\xi,X\rangle\right),
\end{aligned}
$$

the anchor map ρ satisfies $\rho(X+\xi) = aX + \pi^\sharp\xi$ and the pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is given by (2.105) for any X, Y in $\mathfrak{X}(M)$, ξ and η in $\Omega^1(M)$.

Example 55 is an example of not a Poisson-Nijenhuis manifold but a pseudo-Poisson-Nijenhuis manifold.

The following proposition means that two given pseudo-Poisson-Nijenhuis manifolds generate a new one.

Proposition 4.2.4 (Nakamura [32])**.** Let $(M_i, \pi_i, N_i, \Phi_i)$, $i = 1, 2$, be pseudo-Poisson-Nijenhuis manifolds. Then the product $(M_1\times M_2, \pi_1+\pi_2, N_1\oplus N_2, \Phi_1+\Phi_2)$ is a pseudo-Poisson-Nijenhuis manifold.

Proof. Using the fact that $[X_1, X_2] = 0$ for any X_i in $\mathfrak{X}(M_i)$, $i = 1, 2$, we obtain

$$d_{\pi_1+\pi_2}(D_1 + D_2) = d_{\pi_1} D_1 + d_{\pi_2} D_2,$$
$$\mathcal{T}_{N_1 \oplus N_2}(f^1 X_1 + f^2 X_2, g^1 Y_1 + g^2 Y_2) = f^1 g^1 \mathcal{T}_{N_1}(X_1, Y_1) + f^2 g^2 \mathcal{T}_{N_2}(X_2, Y_2),$$
$$C^{N_1 \oplus N_2}_{\pi_1+\pi_2}(f^1 \alpha_1 + f^2 \alpha_2, g^1 \beta_1 + g^2 \beta_2) = f^1 g^1 C^{N_1}_{\pi_1}(\alpha_1, \beta_1) + f^2 g^2 C^{N_2}_{\pi_2}(\alpha_2, \beta_2)$$

for any 2-vector fields π_i in $\mathfrak{X}(M_i)$, $(1,1)$-tensors N_i on M_i, k-vector fields D_i in $\mathfrak{X}^k(M)$, functions f^i, g^i in $C^\infty(M_1 \times M_2)$, vector fields X_i, Y_i in $\mathfrak{X}(M)$, 1-forms α_i and β_i in $\Omega^1(M_i)$, $i = 1, 2$.

Therefore by the assumptions that $(M_i, \pi_i, N_i, \Phi_i)$, $i = 1, 2$, are pseudo-Poisson-Nijenhuis manifolds and straightforward calculations, we can see that the quadruple $(M_1 \times M_2, \pi_1 + \pi_2, N_1 \oplus N_2, \Phi_1 + \Phi_2)$ is a pseudo-Poisson-Nijenhuis manifold. □

4.3 Pseudo-symplectic-Nijenhuis manifolds

In this section, we always assume that a 2-vector field π is nondegenerate. Then we can reduce one of the conditions for a triple (π, N, Φ) to be a pseudo-Poisson-Nijenhuis structure. This fact is important in the sense that we can find pseudo-Poisson-Nijenhuis structures easily. Moreover we rewrite a pseudo-Poisson-Nijenhuis structure (π, N, Φ) with the nondegenerate 2-vector field π using differential forms, and investigate properties of the structure.

Theorem 4.3.1 (Nakamura [32])**.** Let π be a nondegenerate 2-vector field, N a Nijenhuis structure compatible with π, and Φ a 3-vector field. If a triple (π, N, Φ) satisfies the conditions (4.28) and (4.29) in Definition 46, then (π, N, Φ) is a pseudo-Poisson-Nijenhuis structure, i.e., (π, N, Φ) satisfies the condition (4.30).

Proof. We shall prove (4.30). By the nondegeneracy of π, the map $\pi^\sharp : T^*M \longrightarrow TM$ is a bundle isomorphism. Therefore a set $\{\pi^\sharp df \mid f \in C^\infty(M)\}$ generates locally the vector fields $\mathfrak{X}(M)$ as a $C^\infty(M)$-module. We have proved in Theorem 4.2.1 that the equality (4.29) holds if and only if $d_\pi^2 = [\Phi, \cdot]$ holds on $C^\infty(M)$. Thus we compute, for any f in $C^\infty(M)$,

$$\begin{aligned} d_\pi^2(\pi^\sharp df) &= d_\pi^2(-d_\pi f) = -d_\pi(d_\pi^2 f) = -d_\pi[\Phi, f]_N \\ &= -\left([d_\pi \Phi, f]_N + [\Phi, d_\pi f]_N\right) \\ &= -[\Phi, d_\pi f]_N = [\Phi, \pi^\sharp df]_N, \end{aligned}$$

where we use $\pi^\sharp df = -d_\pi f$ in the first and the last step, the fourth equality follows from (4.19) and the fifth equality does from (4.28). Therefore $d_\pi^2 = [\Phi, \cdot]$ holds on the set $\{\pi^\sharp df |\ f \in C^\infty(M)\}$. Since $d_\pi^2 = [\Phi, \cdot]$ holds on $C^\infty(M) \oplus \{\pi^\sharp df |\ f \in C^\infty(M)\}$ and since both d_π^2 and $[\Phi, \cdot]_N$ are derivatives on $(\Gamma(\Lambda^* TM), \wedge)$, we obtain that $d_\pi^2 = [\Phi, \cdot]$ holds on $\mathfrak{X}(M)$. This is equivalent to the condition (4.30) under the assumption of (4.29), so that the proof has been completed. □

In general, it is easier to deal with differential forms than multi-vector fields. Since a 2-vector field π is nondegenerate, there is a unique 2-form ω corresponding with π. Hence it is convenient to translate conditions (4.28) and (4.29) for π into those for ω. We compute

$$\begin{aligned}
\left\langle \frac{1}{2}\iota_{\alpha\wedge\beta}[\pi,\pi], \gamma \right\rangle &= \langle [\pi^\sharp\alpha, \pi^\sharp\beta] - \pi^\sharp[\alpha,\beta]_\pi, \gamma\rangle \\
&= \langle [\pi^\sharp\alpha, \pi^\sharp\beta], \gamma\rangle \\
&\quad - \langle \mathcal{L}_{\pi^\sharp\alpha}\beta - \mathcal{L}_{\pi^\sharp\beta}\alpha - d\langle \pi^\sharp\alpha, \beta\rangle, \pi^\sharp\gamma\rangle \\
&= \langle [\pi^\sharp\alpha, \pi^\sharp\beta], \gamma\rangle + \mathcal{L}_{\pi^\sharp\alpha}\langle \beta, \pi^\sharp\gamma\rangle - \langle \beta, \mathcal{L}_{\pi^\sharp\alpha}(\pi^\sharp\gamma)\rangle \\
&\quad - \mathcal{L}_{\pi^\sharp\beta}\langle \alpha, \pi^\sharp\gamma\rangle + \langle \alpha, \mathcal{L}_{\pi^\sharp\beta}(\pi^\sharp\gamma)\rangle - (\pi^\sharp\gamma)\langle \pi^\sharp\alpha, \beta\rangle \\
&= \langle [\pi^\sharp\alpha, \pi^\sharp\beta], -\omega^\flat\pi^\sharp\gamma\rangle + (\pi^\sharp\alpha)\langle -\omega^\flat\pi^\sharp\beta, \pi^\sharp\gamma\rangle \\
&\quad - \langle -\omega^\flat\pi^\sharp\beta, [\pi^\sharp\alpha, \pi^\sharp\gamma]\rangle - (\pi^\sharp\beta)\langle -\omega^\flat\pi^\sharp\alpha, \pi^\sharp\gamma\rangle \\
&\quad + \langle -\omega^\flat\pi^\sharp\alpha, [\pi^\sharp\beta, \pi^\sharp\gamma]\rangle - (\pi^\sharp\gamma)\langle \pi^\sharp\alpha, -\omega^\flat\pi^\sharp\beta\rangle \\
&= \omega([\pi^\sharp\alpha, \pi^\sharp\beta], \pi^\sharp\gamma) - (\pi^\sharp\alpha)(\omega(\pi^\sharp\beta, \pi^\sharp\gamma)) \\
&\quad - \omega([\pi^\sharp\alpha, \pi^\sharp\gamma], \pi^\sharp\beta) + (\pi^\sharp\beta)(\omega(\pi^\sharp\alpha, \pi^\sharp\gamma)) \\
&\quad + \omega([\pi^\sharp\beta, \pi^\sharp\gamma], \pi^\sharp\alpha) - (\pi^\sharp\gamma)(\omega(\pi^\sharp\alpha, \pi^\sharp\beta)) \\
&= -d\omega(\pi^\sharp\alpha, \pi^\sharp\beta, \pi^\sharp\gamma) \\
&= \langle -\iota_{\pi^\sharp\alpha\wedge\pi^\sharp\beta} d\omega, \pi^\sharp\gamma\rangle
\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}
\langle N\iota_{\alpha\wedge\beta}\Phi, \gamma\rangle &= \langle \iota_{\alpha\wedge\beta}\Phi, N^*\gamma\rangle = \Phi(\alpha, \beta, N^*\gamma) \\
&= \Phi(-\omega^\flat\pi^\sharp\alpha, -\omega^\flat\pi^\sharp\beta, -\omega^\flat\pi^\sharp N^*\gamma) \\
&= (\omega^\flat\Phi)(\pi^\sharp\alpha, \pi^\sharp\beta, N\pi^\sharp\gamma) \\
&= \langle \iota_{\pi^\sharp\alpha\wedge\pi^\sharp\beta}(\omega^\flat\Phi), N\pi^\sharp\gamma\rangle \\
&= \langle N^*\iota_{\pi^\sharp\alpha\wedge\pi^\sharp\beta}(\omega^\flat\Phi), \pi^\sharp\gamma\rangle
\end{aligned} \tag{4.32}$$

for any α, β and γ in $\Omega^1(M)$, where a bundle map $\omega^\flat : TM \longrightarrow T^*M$ is defined by $\langle \omega^\flat X, Y\rangle := \omega(X,Y)$. Therefore setting $\phi := -\omega^\flat \Phi$, we obtain the equivalence of the condition (4.29) and

$$\iota_{X\wedge Y} d\omega = N^* \iota_{X\wedge Y}\phi \; (X, Y \in \mathfrak{X}(M)) \tag{4.33}$$

due to the nondegeneracy of π. Under the assumption of (4.33), we calculate

$$\begin{aligned}
[\pi,&\Phi](\alpha_1,\alpha_2,\alpha_3,\alpha_4) \\
&= (d_\pi \Phi)(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \\
&= (\pi^\sharp\alpha_1)(\Phi(\alpha_2,\alpha_3,\alpha_4)) - (\pi^\sharp\alpha_2)(\Phi(\alpha_1,\alpha_3,\alpha_4)) \\
&\quad + (\pi^\sharp\alpha_3)(\Phi(\alpha_1,\alpha_2,\alpha_4)) - (\pi^\sharp\alpha_4)(\Phi(\alpha_1,\alpha_2,\alpha_3)) \\
&\quad - \Phi([\alpha_1,\alpha_2]_\pi,\alpha_3,\alpha_4) + \Phi([\alpha_1,\alpha_3]_\pi,\alpha_2,\alpha_4) \\
&\quad - \Phi([\alpha_1,\alpha_4]_\pi,\alpha_2,\alpha_3) - \Phi([\alpha_2,\alpha_3]_\pi,\alpha_1,\alpha_4) \\
&\quad + \Phi([\alpha_2,\alpha_4]_\pi,\alpha_1,\alpha_3) - \Phi([\alpha_3,\alpha_4]_\pi,\alpha_1,\alpha_2) \\
&= (\pi^\sharp\alpha_1)((\pi^\sharp\phi)(\alpha_2,\alpha_3,\alpha_4)) - (\pi^\sharp\alpha_2)((\pi^\sharp\phi)(\alpha_1,\alpha_3,\alpha_4)) \\
&\quad + (\pi^\sharp\alpha_3)((\pi^\sharp\phi)(\alpha_1,\alpha_2,\alpha_4)) - (\pi^\sharp\alpha_4)((\pi^\sharp\phi)(\alpha_1,\alpha_2,\alpha_3)) \\
&\quad - (\pi^\sharp\phi)([\alpha_1,\alpha_2]_\pi,\alpha_3,\alpha_4) + (\pi^\sharp\phi)([\alpha_1,\alpha_3]_\pi,\alpha_2,\alpha_4) \\
&\quad - (\pi^\sharp\phi)([\alpha_1,\alpha_4]_\pi,\alpha_2,\alpha_3) - (\pi^\sharp\phi)([\alpha_2,\alpha_3]_\pi,\alpha_1,\alpha_4) \\
&\quad + (\pi^\sharp\phi)([\alpha_2,\alpha_4]_\pi,\alpha_1,\alpha_3) - (\pi^\sharp\phi)([\alpha_3,\alpha_4]_\pi,\alpha_1,\alpha_2) \\
&= (\pi^\sharp\alpha_1)(-\phi(\pi^\sharp\alpha_2,\pi^\sharp\alpha_3,\pi^\sharp\alpha_4)) - (\pi^\sharp\alpha_2)(-\phi(\pi^\sharp\alpha_1,\pi^\sharp\alpha_3,\pi^\sharp\alpha_4)) \\
&\quad + (\pi^\sharp\alpha_3)(-\phi(\pi^\sharp\alpha_1,\pi^\sharp\alpha_2,\pi^\sharp\alpha_4)) - (\pi^\sharp\alpha_4)(-\phi(\pi^\sharp\alpha_1,\pi^\sharp\alpha_2,\pi^\sharp\alpha_3)) \\
&\quad + \phi(\pi^\sharp[\alpha_1,\alpha_2]_\pi,\pi^\sharp\alpha_3,\pi^\sharp\alpha_4) - \phi(\pi^\sharp[\alpha_1,\alpha_3]_\pi,\pi^\sharp\alpha_2,\pi^\sharp\alpha_4) \\
&\quad + \phi(\pi^\sharp[\alpha_1,\alpha_4]_\pi,\pi^\sharp\alpha_2,\pi^\sharp\alpha_3) + \phi(\pi^\sharp[\alpha_2,\alpha_3]_\pi,\pi^\sharp\alpha_1,\pi^\sharp\alpha_4) \\
&\quad - \phi(\pi^\sharp[\alpha_2,\alpha_4]_\pi,\pi^\sharp\alpha_1,\pi^\sharp\alpha_3) + \phi(\pi^\sharp[\alpha_3,\alpha_4]_\pi,\pi^\sharp\alpha_1,\pi^\sharp\alpha_2) \\
&= -(\pi^\sharp\alpha_1)(\phi(\pi^\sharp\alpha_2,\pi^\sharp\alpha_3,\pi^\sharp\alpha_4)) + (\pi^\sharp\alpha_2)(\phi(\pi^\sharp\alpha_1,\pi^\sharp\alpha_3,\pi^\sharp\alpha_4)) \\
&\quad - (\pi^\sharp\alpha_3)(\phi(\pi^\sharp\alpha_1,\pi^\sharp\alpha_2,\pi^\sharp\alpha_4)) + (\pi^\sharp\alpha_4)(\phi(\pi^\sharp\alpha_1,\pi^\sharp\alpha_2,\pi^\sharp\alpha_3)) \\
&\quad + \phi\left(\frac{1}{2}\iota_{\alpha_1\wedge\alpha_2}[\pi,\pi] + [\pi^\sharp\alpha_1,\pi^\sharp\alpha_2],\pi^\sharp\alpha_3,\pi^\sharp\alpha_4\right) \\
&\quad - \phi\left(-\frac{1}{2}\iota_{\alpha_1\wedge\alpha_3}[\pi,\pi] + [\pi^\sharp\alpha_1,\pi^\sharp\alpha_3],\pi^\sharp\alpha_2,\pi^\sharp\alpha_4\right) \\
&\quad + \phi\left(-\frac{1}{2}\iota_{\alpha_1\wedge\alpha_4}[\pi,\pi] + [\pi^\sharp\alpha_1,\pi^\sharp\alpha_4],\pi^\sharp\alpha_2,\pi^\sharp\alpha_3\right) \\
&\quad + \phi\left(-\frac{1}{2}\iota_{\alpha_2\wedge\alpha_3}[\pi,\pi] + [\pi^\sharp\alpha_2,\pi^\sharp\alpha_3],\pi^\sharp\alpha_1,\pi^\sharp\alpha_4\right)
\end{aligned}$$

$$
\begin{aligned}
&- \phi\left(-\frac{1}{2}\iota_{\alpha_2\wedge\alpha_4}[\pi,\pi] + [\pi^\sharp\alpha_2,\pi^\sharp\alpha_4],\pi^\sharp\alpha_1,\pi^\sharp\alpha_3\right)\\
&+ \phi\left(-\frac{1}{2}\iota_{\alpha_3\wedge\alpha_4}[\pi,\pi] + [\pi^\sharp\alpha_3,\pi^\sharp\alpha_4],\pi^\sharp\alpha_1,\pi^\sharp\alpha_2\right)\\
= &-(d\phi)(\pi^\sharp\alpha_1,\pi^\sharp\alpha_2,\pi^\sharp\alpha_3,\pi^\sharp\alpha_4)\\
&+ \phi\left(\pi^\sharp\omega^\flat\left(\frac{1}{2}\iota_{\alpha_1\wedge\alpha_2}[\pi,\pi]\right),\pi^\sharp\alpha_3,\pi^\sharp\alpha_4\right)\\
&- \phi\left(\pi^\sharp\omega^\flat\left(\frac{1}{2}\iota_{\alpha_1\wedge\alpha_3}[\pi,\pi]\right),\pi^\sharp\alpha_2,\pi^\sharp\alpha_4\right)\\
&+ \phi\left(\pi^\sharp\omega^\flat\left(\frac{1}{2}\iota_{\alpha_1\wedge\alpha_4}[\pi,\pi]\right),\pi^\sharp\alpha_2,\pi^\sharp\alpha_3\right)\\
&+ \phi\left(\pi^\sharp\omega^\flat\left(\frac{1}{2}\iota_{\alpha_2\wedge\alpha_3}[\pi,\pi]\right),\pi^\sharp\alpha_1,\pi^\sharp\alpha_4\right)\\
&- \phi\left(\pi^\sharp\omega^\flat\left(\frac{1}{2}\iota_{\alpha_2\wedge\alpha_4}[\pi,\pi]\right),\pi^\sharp\alpha_1,\pi^\sharp\alpha_3\right)\\
&+ \phi\left(\pi^\sharp\omega^\flat\left(\frac{1}{2}\iota_{\alpha_3\wedge\alpha_4}[\pi,\pi]\right),\pi^\sharp\alpha_1,\pi^\sharp\alpha_2\right)\\
= &-(d\phi)(\pi^\sharp\alpha_1,\pi^\sharp\alpha_2,\pi^\sharp\alpha_3,\pi^\sharp\alpha_4)\\
&- (\pi^\sharp\phi)\left(\omega^\flat\left(\frac{1}{2}\iota_{\alpha_1\wedge\alpha_2}[\pi,\pi]\right),\alpha_3,\alpha_4\right)\\
&+ (\pi^\sharp\phi)\left(\omega^\flat\left(\frac{1}{2}\iota_{\alpha_1\wedge\alpha_3}[\pi,\pi]\right),\alpha_2,\alpha_4\right)\\
&- (\pi^\sharp\phi)\left(\omega^\flat\left(\frac{1}{2}\iota_{\alpha_1\wedge\alpha_4}[\pi,\pi]\right),\alpha_2,\alpha_3\right)\\
&+ (\pi^\sharp\phi)\left(\omega^\flat\left(\frac{1}{2}\iota_{\alpha_2\wedge\alpha_3}[\pi,\pi]\right),\alpha_1,\alpha_4\right)\\
&+ (\pi^\sharp\phi)\left(\omega^\flat\left(\frac{1}{2}\iota_{\alpha_2\wedge\alpha_4}[\pi,\pi]\right),\alpha_1,\alpha_3\right)\\
&- (\pi^\sharp\phi)\left(\omega^\flat\left(\frac{1}{2}\iota_{\alpha_3\wedge\alpha_4}[\pi,\pi]\right),\alpha_1,\alpha_2\right)\\
= &-(d\phi)(\pi^\sharp\alpha_1,\pi^\sharp\alpha_2,\pi^\sharp\alpha_3,\pi^\sharp\alpha_4)\\
&- \Phi\left(\omega^\flat N\iota_{\alpha_1\wedge\alpha_2}\Phi,\alpha_3,\alpha_4\right) + \Phi\left(\omega^\flat N\iota_{\alpha_1\wedge\alpha_3}\Phi,\alpha_2,\alpha_4\right)\\
&- \Phi\left(\omega^\flat N\iota_{\alpha_1\wedge\alpha_4}\Phi,\alpha_2,\alpha_3\right) - \Phi\left(\omega^\flat N\iota_{\alpha_2\wedge\alpha_3}\Phi,\alpha_1,\alpha_4\right)\\
&+ \Phi\left(\omega^\flat N\iota_{\alpha_2\wedge\alpha_4}\Phi,\alpha_1,\alpha_3\right) - \Phi\left(\omega^\flat N\iota_{\alpha_3\wedge\alpha_4}\Phi,\alpha_1,\alpha_2\right)
\end{aligned}
$$

$$\begin{aligned}
&= -(d\phi)(\pi^\sharp\alpha_1, \pi^\sharp\alpha_2, \pi^\sharp\alpha_3, \pi^\sharp\alpha_4) \\
&\quad - \langle \iota_{\alpha_3\wedge\alpha_4}\Phi, \omega_N^\flat \iota_{\alpha_1\wedge\alpha_2}\Phi\rangle + \langle \iota_{\alpha_2\wedge\alpha_4}\Phi, \omega_N^\flat \iota_{\alpha_1\wedge\alpha_3}\Phi\rangle \\
&\quad - \langle \iota_{\alpha_2\wedge\alpha_3}\Phi, \omega_N^\flat \iota_{\alpha_1\wedge\alpha_4}\Phi\rangle - \langle \iota_{\alpha_1\wedge\alpha_4}\Phi, \omega_N^\flat \iota_{\alpha_2\wedge\alpha_3}\Phi\rangle \\
&\quad + \langle \iota_{\alpha_1\wedge\alpha_3}\Phi, \omega_N^\flat \iota_{\alpha_2\wedge\alpha_4}\Phi\rangle - \langle \iota_{\alpha_1\wedge\alpha_2}\Phi, \omega_N^\flat \iota_{\alpha_3\wedge\alpha_4}\Phi\rangle \\
&= -(d\phi)(\pi^\sharp\alpha_1, \pi^\sharp\alpha_2, \pi^\sharp\alpha_3, \pi^\sharp\alpha_4) \\
&\quad + \omega_N(\iota_{\alpha_3\wedge\alpha_4}\Phi, \iota_{\alpha_1\wedge\alpha_2}\Phi) - \omega_N(\iota_{\alpha_2\wedge\alpha_4}\Phi, \iota_{\alpha_1\wedge\alpha_3}\Phi) \\
&\quad + \omega_N(\iota_{\alpha_2\wedge\alpha_3}\Phi, \iota_{\alpha_1\wedge\alpha_4}\Phi) + \omega_N(\iota_{\alpha_1\wedge\alpha_4}\Phi, \iota_{\alpha_2\wedge\alpha_3}\Phi) \\
&\quad - \omega_N(\iota_{\alpha_1\wedge\alpha_3}\Phi, \iota_{\alpha_2\wedge\alpha_4}\Phi) + \omega_N(\iota_{\alpha_1\wedge\alpha_2}\Phi, \iota_{\alpha_3\wedge\alpha_4}\Phi) \\
&= -(d\phi)(\pi^\sharp\alpha_1, \pi^\sharp\alpha_2, \pi^\sharp\alpha_3, \pi^\sharp\alpha_4) \qquad (4.34)
\end{aligned}$$

for any α_i in $\Omega^1(M)$, where ω_N is given by $\omega_N(X,Y) := \langle \omega^\flat NX, Y\rangle$ for any X and Y in $\mathfrak{X}(M)$ and we see that ω_N is a 2-form on M by (2.96). From the above, we see that the conditions (4.28) and (4.29) are equivalent to the condition (4.33) and the closedness of ϕ if π is nondegenerate. Therefore we define as follows:

Definition 48 ([32])**.** Let M be a C^∞-manifold, ω a nondegenerate 2-form on M, a $(1,1)$-tensor N a Nijenhuis structure compatible with the nondegenerate 2-vector field π corresponding to ω, and ϕ a closed 3-form on M. Then a triple (ω, N, ϕ) is a *pseudo-symplectic-Nijenhuis structure* on M if the condition (4.33) holds. The quadruple (M, ω, N, ϕ) is called a *pseudo-symplectic-Nijenhuis manifold.*

The following corollary states that we can construct new pseudo-symplectic-Nijenhuis structures from a symplectic-Nijenhuis structure.

Corollary 4.3.2 (Nakamura [32])**.** Let (M, ω, N) be a symplectic-Nijenhuis manifold and ϕ a closed 3-form satisfying $\iota_{NX}\phi = 0$ for any X in $\mathfrak{X}(M)$. Then (M, ω, N, ϕ) is a pseudo-symplectic Nijenhuis manifold.

Proof. In this case, the condition (4.33) to prove is

$$N^*\iota_{X\wedge Y}\phi = 0 \ (X, Y \in \mathfrak{X}(M)) \qquad (4.35)$$

because of $d\omega = 0$. By computing that, for any Z in $\mathfrak{X}(M)$,

$$\begin{aligned}
\langle N^*\iota_{X\wedge Y}\phi, Z\rangle &= \phi(X, Y, NZ) = \phi(NZ, X, Y) \\
&= (\iota_{NZ}\phi)(X, Y) = 0,
\end{aligned}$$

where we use $\iota_{NX}\phi = 0$, we conclude that (4.35) holds. Hence (ω, N, ϕ) is a pseudo-symplectic Nijenhuis structure. □

Example 56 ([32]). On the 6-torus $\mathbb{T}^6$ with angle coordinates $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$, we consider the standard symplectic structure $\omega := d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4 + d\theta_5 \wedge d\theta_6$ and a regular Poisson structure with rank 2,

$$\pi_\lambda := \frac{\partial}{\partial\theta_a} \wedge \left(\frac{\partial}{\partial\theta_b} + \lambda \frac{\partial}{\partial\theta_c} \right),$$

where λ is in $\mathbb{R}$ and a, b and c are three distinct numbers (Example 3 and Example 14). Setting $N_\lambda := \pi_\lambda^\sharp \circ \omega^\flat$, we obtain a symplectic-Nijenhuis structure (ω, N_λ) on $\mathbb{T}^6$ (see [42] for a general theory of constructing symplectic-Nijenhuis structures from symplectic and Poisson structures). Since the rank of N_λ is 2 at each points, the kernel of N_λ^* is a subbundle with rank 4 of the cotangent bundle of $\mathbb{T}^6$. Hence for any closed 3-form ϕ in $\Gamma(\Lambda^3 \mathrm{Ker} N_\lambda^*)$, a triple $(\omega, N_\lambda, \phi)$ is a pseudo-symplectic-Nijenhuis structure on $\mathbb{T}^6$ by Corollary 4.3.2.

The following simple example is of (ω, N, ϕ) being a pseudo-symplectic Nijenhuis structure but not of (ω, N) being a symplectic Nijenhuis structure.

Example 57 ([32]). Let (x^1, x^2, x^3, x^4) be the canonical coordinates in $\mathbb{R}^4$ and $f(x), g(x)$ in $C^\infty(\mathbb{R})$ not constants but non-vanishing functions. We set

$$N := \begin{pmatrix} N_1^1 & \frac{(N_1^1 - N_3^3)^2}{N_2^1} & 0 & 0 \\ N_2^1 & N_1^1 & 0 & 0 \\ 0 & 0 & N_3^3 & \frac{(N_1^1 - N_3^3)^2}{N_4^3} \\ 0 & 0 & N_4^3 & N_3^3 \end{pmatrix},$$

where N_j^i's are in $\mathbb{R}^\times$ and satisfy that $N_1^1 \neq N_3^3$,

$$\omega := f(a_3 x^3 + a_4 x^4) dx^1 \wedge dx^2 + g(a_1 x^1 + a_2 x^2) dx^3 \wedge dx^4,$$

where a_i's satisfy $a_3 : a_4 = N_4^3 : (N_1^1 - N_3^3)$ and $a_1 : a_2 = N_2^1 : (N_1^1 - N_3^3)$, and

$$\begin{aligned} \phi := &(N_1^1)^{-1} f'(a_3 x^3 + a_4 x^4) dx_1 \wedge dx_2 \wedge (a_3 dx^3 + a_4 dx^4) \\ &+ (N_3^3)^{-1} g'(a_1 x^1 + a_2 x^2)(a_1 dx^1 + a_2 dx^2) \wedge dx_3 \wedge dx_4. \end{aligned}$$

Then (ω, N, ϕ) is a pseudo-symplectic Nijenhuis structure on $\mathbb{R}^4$. A pair (ω, N) is not symplectic-Nijenhuis by the fact that $d\omega \neq 0$.

Finally we describe a property of pseudo-symplectic Nijenhuis structures. This is the main theorem in this section.

Theorem 4.3.3 (Nakamura [32])**.** Let (ω, N, ϕ) be a pseudo-symplectic Nijenhuis structure on M and π the nondegenerate 2-vector field corresponding to ω. Then (π_N, ϕ) is a twisted Poisson structure, i.e., the pair satisfies

$$\frac{1}{2}[\pi_N, \pi_N] = \pi_N^\sharp \phi,$$
$$d\phi = 0.$$

We need the following lemma.

Lemma 4.3.4 ([12])**.** Let $(A, [\cdot,\cdot]_A, \rho_A)$ be a Lie algebroid over M equipped with a degree-one derivation δ of the Gerstenhaber algebra $(\Gamma(\Lambda^* A), \wedge, [\cdot,\cdot]_A)$. Then there exists a 2-vector field π_M on M given by

$$\pi_M(df_1, df_2) := -\langle \rho_A(\delta f_1), df_2 \rangle$$

for any f_1 and f_2 in $C^\infty(M)$. Moreover if (A, δ, ϕ) is a quasi-Lie bialgebroid over M, Then the bivector field π_M on M satisfies

$$\frac{1}{2}[\pi_M, \pi_M] = \phi_M,$$
$$[\pi_M, \phi_M] = 0,$$

where ϕ_M is the 3-vector field $\phi_M = \rho_A(\phi)$, and $\rho_A : \Gamma(\wedge A) \to \mathfrak{X}^3(M)$ is the extension of the anchor map ρ_A on A given by the formula

$$\rho_A(X_1 \wedge X_2 \wedge X_3) := \rho_A(X_1) \wedge \rho_A(X_2) \wedge \rho_A(X_3)$$

for any X_i in $\Gamma(A)$.

Proof of Theorem 4.3.3. By Definition 48, we obtain $d\phi = 0$. By Theorem 4.2.1, $((TM)_N, d_\pi, \Phi)$, where $\Phi := \pi^\sharp \phi$, is a quasi-Lie bialgebroid. Because of Lemma 4.3.4, the 2-vector field induced by d_π coincides with π_N. In fact, we calculate

$$\begin{aligned}
\pi_M(df_1, df_2) &= -\langle N(d_\pi f_1), df_2 \rangle = -\langle d_\pi f_1, N^* df_2 \rangle \\
&= -\langle (\pi^\sharp N^* df_2) f_1 = -(\pi_N^\sharp df_2) f_1 \\
&= -\langle df_1, \pi_N^\sharp df_2 \rangle = \langle \pi_N^\sharp df_1, df_2 \rangle \\
&= \pi_N(df_1, df_2)
\end{aligned}$$

for any f_1 and f_2 in $C^\infty(M)$. Moreover, since we obtain

$$\begin{aligned}(N\Phi)(\alpha_1, \alpha_2, \alpha_3) &= \Phi(N^*\alpha_1, N^*\alpha_2, N^*\alpha_3) \\ &= (\pi^\sharp\phi)(N^*\alpha_1, N^*\alpha_2, N^*\alpha_3) \\ &= -\phi(\pi^\sharp N^*\alpha_1, \pi^\sharp N^*\alpha_2, \pi^\sharp N^*\alpha_3) \\ &= -\phi(\pi_N^\sharp\alpha_1, \pi_N^\sharp\alpha_2, \pi_N^\sharp\alpha_3) \\ &= (\pi_N^\sharp\phi)(\alpha_1, \alpha_2, \alpha_3)\end{aligned}$$

for any α_1, α_2 and α_3 in $\Omega^1(M)$, by Lemma 4.3.4 again, we have

$$\begin{aligned}\frac{1}{2}[\pi_N, \pi_N] &= N\Phi \\ &= \pi_N^\sharp\phi.\end{aligned}$$

Therefore (π_N, ϕ) is a twisted Poisson structure on M. □

This theorem means that we can construct twisted Poisson structures from pseudo-symplectic Nijenhuis structures. Moreover such a twisted Poisson structures (π_N, ϕ) is compatible with the Nijenhuis structure N due to Proposition 4.1.4.

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Index

List of original papers

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