博士論文

The Geometry of Non-commutative Crepant Resolutions
非可換クレパント解消の幾何学

February 2019

Wahei HARA
原 和平

The Geometry of Non-commutative Crepant Resolutions
非可換クレパント解消の幾何学

February 2019

Waseda University
Graduate School of Fundamental Science and Engineering
Department of Pure and Applied Mathematics,
Research on Algebraic Geometry

Wahei HARA
原 和平

Contents

Articles of Wahei Hara

[H17a] W. Hara, *Non-commutative crepant resolution of minimal nilpotent orbit closures of type A and Mukai flops*, Adv. Math., **318** (2017), 355–410.

[H17b] W. Hara, *On derived equivalence for Abuaf flop: mutation of non-commutative crepant resolutions and spherical twists*, preprint 2017, `https://arxiv.org/abs/1706.04417`.

[H17c] W. Hara, *Strong full exceptional collections on certain toric varieties with Picard number three via mutations*, Le Matematiche, Vol. LXXII(2) (2017), 3–24.

[H18a] W. Hara, *Deformation of tilting-type derived equivalences for crepant resolutions*, to appear in IMRN.

[H18b] W. Hara, *On the Abuaf-Ueda flop via non-commutative crepant resolution*, preprint (2018).

Chapter 1

Introduction

In this thesis we develop the theory of non-commutative crepant resolutions (=NCCRs) of higher dimension. The notion of NCCRs is firstly introduced by M. Van den Bergh in relation to the study of the derived category of algebraic varieties, and now the theory of NCCRs in dimension three is quite well established by Van den Bergh, O. Iyama, W. Donovan, M. Wemyss, and many other researchers (both of algebraists and algebraic geometers). However, in contrast to the three dimensional cases, it seems that there are few works on NCCRs of higher dimension.

In this thesis, we study NCCRs that appear in the context of higher dimensional simple flops.

In Chapter 2, we recall definitions of notions we deal with in this thesis. We also recall basic properties of those notions.

In Chapter 3, we study Mukai flops from the point of view of NCCRs. Namely, we show some results from moduli problems and derived equivalences. In particular, we study mutations of NCCRs that was introduced by Iyama and Wemyss in the case of Mukai flop, and show that we can interpret P-twists in terms of mutations of NCCRs.

In Chapter 4, we study Abuaf's five dimensional flop (which we call the *Abuaf flop*) from the viewpoint of NCCRs. We show that the framework of Toda and Uehara for constructing tilting bundles works for this setting, and prove that we can produce a derived equivalence using those tilting bundles. Our derived equivalence is different from the one constructed by Segal, and we also show that the difference between our derived equivalence and Segal's one can be described as a spherical twist. We also study mutations of NCCR as in the case of Mukai flops.

In Chapter 5, we study a seven dimensional flop that was found independently by Abuaf and Ueda. Although the derived equivalence for this flop was already proved by Ueda, we provide an alternative proof in which we use tilting bundles. First we prove both sides of the flop admit tilting bundles and then show that those tilting bundles give derived equivalences for the flop. In this case, it is much more difficult to find tilting bundles explicitly than the cases of

Mukai flops or the Abuaf flop.

In Chapter 6, we study mutations of NCCRs of Gorenstein cyclic quotient singularities. We show a similar result as in the cases of Mukai flops or the Abuaf flop also holds for this setting.

In Chapter 7, we study deformations of derived equivalences for crepant resolutions constructed by using tilting bundles. If a derived equivalence between two crepant resolutions is constructed by using tilting bundles, we say that the equivalence is of tilting-type. We show that *tilting-type* derived equivalence lift to an equivalence for deformations of resolutions under certain good conditions. As an application, we study a relation between tilting-type derived equivalences for stratified Mukai flops and tilting-type derived equivalences for stratified Atiyah flops.

1.1 Notations and Conventions

Throughout this thesis, we adopt the following notations.

- $\mathrm{Sym}^k_R M$ (resp. $\mathrm{Sym}^k_X \mathcal{E}$) : the k-th symmetric product of an R-module M (resp. a vector bundle $\mathcal{E}$ on X).
- $\mathbb{P}(V) := V \setminus \{0\}/\mathbb{C}^\times$: the projectivization of a vector space V.
- $\mathrm{Tot}(\mathcal{E}) := \mathrm{Spec}_X \mathrm{Sym}^\bullet_X \mathcal{E}^*$: the total space of a vector bundle $\mathcal{E}$ on X.
- $\mathrm{Qcoh}(X)$: the category of quasi-coherent sheaves on a scheme X.
- $\mathrm{coh}(X)$: the category of coherent sheaves on a Noetherian scheme X.
- $\mathrm{mod}(A)$: the category of finitely generated right A-modules.
- $\mathrm{add}(M)$: the additive closure of M.
- $\mathrm{D}^*(\mathcal{A})$, $(* = \emptyset$ or $-$ or b) : the (unbounded or bounded above or bounded) derived category of an abelian category $\mathcal{A}$.
- $\mathrm{D}^*(X) := \mathrm{D}^*(\mathrm{coh}(X))$, $(* = -$ or b) : the (bounded above or bounded) derived category of $\mathrm{coh}(X)$.
- $\mathrm{D}^*(A) := \mathrm{D}^*(\mathrm{mod}(A))$, $(* = -$ or b) : the (bounded above or bounded) derived category of $\mathrm{mod}(A)$.
- $\mathrm{FM}_{\mathcal{P}}$, $\mathrm{FM}^{X \to Y}_{\mathcal{P}}$: A Fourier-Mukai functor from $\mathrm{D}^\mathrm{b}(X)$ to $\mathrm{D}^\mathrm{b}(Y)$ whose kernel is $\mathcal{P} \in \mathrm{D}^\mathrm{b}(X \times Y)$.
- $\mathrm{T}_{\mathcal{E}}$: the spherical twist around a spherical object $\mathcal{E}$.
- $\mu_N(M)$: the left (Iyama-Wemyss) mutation of M at N.
- $\Phi_N : \mathrm{D}^\mathrm{b}(\mathrm{End}_R(M)) \to \mathrm{D}^\mathrm{b}(\mathrm{End}_R(\mu_N(M)))$: the (Iyama-Wemyss) mutation functor.

In addition, we refer to the bounded derived category $\mathrm{D}^\mathrm{b}(X)$ of coherent sheaves on X as *the derived category of X*.

By CCR we mean commutative crepant resolution and by NCCR non-commutative crepant resolution.

1.2 Acknowledgements

First of all the author would like to express his sincere gratitude to his supervisor Professor Yasunari Nagai and Professor Michel Van den Bergh for many suggestions, discussions, their warm encouragement, and support. Moreover, I would like to thank a lot of professors and friends, including: Professors Roland Abuaf, Will Donovan, Akishi Ikeda, Osamu Iyama, Hajime Kaji, Manfred Lehn, Travis Schedler, Ed Segal, Taku Suzuki, Hokuto Uehara, Michael Wemyss, Doctors Takeru Fukuoka, Francesco Genovese, Yuki Hirano, Hiromi Ishii, Daizo Ishikawa, Naoki Koseki, Hayato Morimura, Genki Ouchi, Wenqing Tao, and Ryo Yamagishi for their encouragement, their support, and discussions.

A part of this work was done during the author's stay in Hasselt University. The author would like to thank Hasselt University for the hospitality and excellent working condition. This work is supported by Grant-in-Aid for JSPS Research Fellow 17J00857.

Finally, I would also like to thank my family for continued support and encouragement.

Wahei Hara
Department of Pure and Applied Mathematics
Graduate School of Fundamental Science and Engineering
Waseda University
3-4-1 Ohkubo Shinjuku
Tokyo 169-8555
Japan
E-mail: waheyhey@ruri.waseda.jp

Chapter 2

Preliminaries

2.1 Non-commutative crepant resolution

2.1.1 Non-commutative crepant resolution

Definition 2.1.1. Let R be a Cohen-Macaulay (commutative) algebra, and M a non-zero reflexive R-module. We set $\Lambda := \mathrm{End}_R(M)$. We say that the R-algebra Λ is a *non-commutative crepant resolution (=NCCR)* of R, or M gives an NCCR of R if

$$\mathrm{gldim}\,\Lambda_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$$

for all $\mathfrak{p} \in \mathrm{Spec}\, R$ and Λ is a maximal Cohen-Macaulay R-module.

If we assume that R is Gorenstein, then we can relax the definition of NCCRs.

Lemma 2.1.2 ([IW14a]). *Let us assume that R is Gorenstein and M is a non-zero reflexive R-module. In this case, an R-algebra $\Lambda := \mathrm{End}_R(M)$ is an NCCR of R if and only if* $\mathrm{gldim}\,\Lambda < \infty$ *and Λ is a maximal Cohen-Macaulay R-module.*

There is a criterion given by Iyama and Wemyss [IW14a] that does not refer to finiteness of global dimension.

Lemma 2.1.3. *Let R be a local normal Gorenstein ring. Let M be a reflexive R-module such that $\Lambda = \mathrm{End}_R(M)$ is a maximal Cohen-Macaulay R-module. Then Λ is an NCCR of R if and only if*

$$\mathrm{add}(M) = \{N \in \mathrm{ref}(R) \mid \mathrm{Hom}_R(M, N) \in \mathrm{CM}(R)\}.$$

Here, $\mathrm{add}(M)$ is the additive closure of M, that is, the category of finitely generated R-modules that are direct summands of $M^{\oplus N}$ for some integer N.

We regard NCCRs as a non-commutative analog of (commutative) crepant resolutions (= CCR). The following conjecture is due to Bondal, Orlov, and Van den Bergh.

Conjecture 2.1.4 ([VdB04b, Conjecture 4.6]). *Let R be a reduced Gorenstein $\mathbb{C}$-algebra of finite type. Then all crepant resolutions of R and all NCCRs of R are derived equivalent. This means that, if we have two crepant resolutions Y and Y' of $X = \operatorname{Spec} R$ and two NCCRs Λ and Λ' of R, then we have exact equivalences*

$$\mathrm{D^b}(Y) \simeq \mathrm{D^b}(Y') \simeq \mathrm{D^b}(\Lambda) \simeq \mathrm{D^b}(\Lambda').$$

Van den Bergh showed that Conjecture 2.1.4 holds if R is of dimension 3 and has only terminal singularities [VdB04a, VdB04b]. The existence of an NCCR and a derived equivalence between crepant resolutions and NCCRs are studied in many literatures [Boc12, BLV10, Dao10, HN17, Kal08, ŠV15a, ŠV15b, ŠV17, TU10].

The following theorem due to Iyama and Wemyss indicates that NCCR is a non-commutative analog of CCR.

Theorem 2.1.5 (Iyama-Wemyss). *Let X be an affine variety and $f : Y \to X$ be a resolution of X. Assume that there exists an algebra Λ which is derived equivalent to Y. Then Y is a CCR of X if and only if Λ is isomorphic to an NCCR of X.*

Example 2.1.6. If R is regular, then R gives an NCCR $R = \operatorname{End}_R(R)$ of R.

Example 2.1.7. Next we give a non-trivial example. Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup and $S = \mathbb{C}[x_1, ..., x_n]$ a polynomial ring. Then the group G acts on S. We define a skew group ring $S * G$ as follows. $S * G$ is $S \otimes_{\mathbb{C}} \mathbb{C}\, G$ as a $\mathbb{C}$-vector space. The multiplication is defined as

$$(f \otimes g) \cdot (f' \otimes g') := fg(f') \otimes gg'.$$

Let S^G be an invariant ring. Then a morphism

$$S * G \to \operatorname{End}_{S^G}(S),\ \ f \otimes g \mapsto (f' \to fg(f'))$$

gives an algebra isomorphism. It is known that $S * G$ has finite global dimension and S and $\operatorname{End}_{S^G}(S)$ are maximal Cohen-Macaulay S^G-modules. Thus, in this case, an S^G-module S gives an NCCR $S * G$ of S^G.

2.1.2 Tilting bundles

The notion of tilting bundles provides a geometric way to construct NCCRs. Tilting bundles are defined as a generator with certain properties. First we recall the definition of generators.

Definition 2.1.8. *For a triangulated category $\mathcal{D}$, we say that an object $E \in \mathcal{D}$ is a generator of $\mathcal{D}$ (or E generates $\mathcal{D}$) if for $F \in \mathcal{D}$, $\operatorname{Hom}_X(E, F[i]) = 0$ (for all $i \in \mathbb{Z}$) implies $F = 0$. We also say that E is a classical generator of $\mathcal{D}$ (or E classically generates $\mathcal{D}$) if the smallest thick subcategory of $\mathcal{D}$ containing E is the whole category $\mathcal{D}$.*

It is easy to see that a classical generator is a generator.

In the following, a *vector bundle* on a scheme X means a locally free sheaf of finite rank on X.

Definition 2.1.9. A vector bundle T on a scheme X is said to be *partial tilting* if

$$\mathrm{Ext}^p_X(T,T) = 0$$

for all $p > 0$. A partial tilting bundle T is called a *tilting* bundle if T is in addition a generator of the category $\mathrm{D}(\mathrm{Qcoh}(X))$. We say that a tilting bundle T is *good* if it contains a trivial line bundle as a direct summand.

In some literatures (eg. [Kal08, TU10]), tilting bundles are defined as a partial tilting bundle that is a generator of $\mathrm{D}^-(X) := \mathrm{D}^-(\mathrm{coh}(X))$. The following lemma resolves this ambiguity of the definition of tilting bundles.

Lemma 2.1.10. *Let X be a scheme that is projective over an affine scheme $S = \operatorname{Spec} R$, and E a partial tilting bundle on X. Then the following are equivalent.*

(i) E is a classical generator of the category of perfect complexes $\mathrm{Perf}(X)$ of X.

(ii) E is a generator of $\mathrm{D}^-(X)$

(iii) E is a generator of $\mathrm{D}^-(\mathrm{Qcoh}(X))$

(iv) E is a generator of the unbounded derived category $\mathrm{D}(\mathrm{Qcoh}(X))$ of quasi-coherent sheaves.

Here perfect complex means a complex that is locally isomorphic to a bounded complex of vector bundles on X.

Proof. Since $\mathrm{D}^-(X) \subset \mathrm{D}^-(\mathrm{Qcoh}(X)) \subset \mathrm{D}(\mathrm{Qcoh}(X))$, we only have to prove that (ii) implies (i) and that (i) implies (iv).

According to Theorems 3.1.1 and 2.1.2 of [BVdB03] (and also by Theorem 3.1.3 of *loc. cit.*), E is a classical generator of $\mathrm{Perf}(X)$ if and only if E is a generator of $\mathrm{D}(\mathrm{Qcoh}(X))$.

Let us assume that E is a generator of $\mathrm{D}^-(X)$ and put $\Lambda := \mathrm{End}_X(E)$. Then the following Proposition 2.1.12 implies that there is an equivalence of categories

$$\Psi : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(\Lambda)$$

such that $\Psi(E) = \Lambda$ (In [TU10], Toda and Uehara proved this equivalence under the assumption that the partial tilting bundle E is a generator of $\mathrm{D}^-(X)$). Let $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}(\Lambda))$ be a full subcategory of $\mathrm{D}^{\mathrm{b}}(\Lambda)$ consisting of complexes that are quasi-isomorphic to bounded complexes of projective modules. Since complexes in $\mathrm{Perf}(X)$ or $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}(\Lambda))$ are characterized as homologically finite objects (see [Orl06, Proposition 1.11]), the equivalence above restricts to an equivalence $\mathrm{Perf}(X) \simeq \mathrm{K}^{\mathrm{b}}(\mathrm{proj}(\Lambda))$. Since the category $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}(\Lambda))$ is classically generated by Λ, the smallest thick subcategory containing E should be $\mathrm{Perf}(X)$. ∎

We adopt the definition for tilting bundles in Definition 2.1.9 because in some parts of discussions in this thesis we need to deal with complexes of quasi-coherent sheaves.

Corollary 2.1.11. *Let X be a scheme that is projective over an affine scheme $\operatorname{Spec} R$. Assume that X admits a tilting bundle E. Then the dual $E^\vee$ of E is also a tilting bundle.*

Proof. It is clear that $E^\vee$ is a partial tilting bundle. Since E is a tilting bundle, E is a classical generator of $\operatorname{Perf}(X)$. Thus $E^\vee$ is also a classical generator of $\operatorname{Perf}(X)$. □

If we find a tilting bundle on a projective scheme over an affine variety, we have an equivalence between the derived category of the scheme and the derived category of a non-commutative ring.

Proposition 2.1.12 ([TU10, Lemma 3.3])**.** *Let Y be a scheme that is projective over an affine scheme $X = \operatorname{Spec} R$. Assume that there is a tilting bundle T on Y. Then, the functor,*

$$\Psi := \operatorname{RHom}(T, -) : \mathrm{D}^-(Y) \to \mathrm{D}^-(\operatorname{End}_Y(T)),$$

gives an equivalence of triangulated categories. Furthermore, this equivalence restricts an equivalence between $\mathrm{D}^{\mathrm{b}}(Y)$ and $\mathrm{D}^{\mathrm{b}}(\operatorname{End}_Y(T))$.

Remark 2.1.13. *Let us assume that we work over a field $\Bbbk$ of characteristic zero. Let X be a $\Bbbk$-scheme and T a partial tilting bundle on X. Then the trace map $T^\vee \otimes T \to \mathcal{O}_X$ is a split surjective morphism, and hence $\mathcal{O}_X$ is a direct summand of $T^\vee \otimes T$. Therefore we have $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.*

In particular, we have the following consequence. Let X be an affine variety, and $f : Y \to X$ a resolution of X. If the variety Y admits a partial tilting bundle, then X has only rational singularities.

Remark 2.1.14 (The reason why tilting bundles are important rather than tilting complexes)**.** If there is an exact equivalence $\Psi : \mathrm{D}^{\mathrm{b}}(Y) \to \mathrm{D}^{\mathrm{b}}(\Lambda)$ between the derived category of a scheme and the derived category of a non-commutative algebra then for the object $T := \Psi^{-1}(\Lambda)$ we can write $\Psi := \operatorname{RHom}_Y(T, -)$ (and this T is a *tilting complex*). Indeed for an object $F \in \mathrm{D}^{\mathrm{b}}(Y)$ we have

$$\operatorname{RHom}_Y(T, F) = \operatorname{RHom}_Y(\Psi^{-1}(\Lambda), F) = \operatorname{RHom}_\Lambda(\Lambda, \Psi(F)) = \Psi(F).$$

In addition, the following two conditions are equivalent.

(1) For any point $y \in Y$ we have $\Psi(\mathcal{O}_y) \in \operatorname{mod}(\Lambda)$.

(2) T is a vector bundle on Y.

To see this, we note that the first condition is equivalent to $\operatorname{Ext}^i_Y(T, \mathcal{O}_y) = 0$ for any $y \in Y$ and $i \neq 0$. Then, using [TU10, Lemma 4.3], we can show that the two conditions above are equivalent.

If we have an equivalence $\mathrm{D^b}(Y) \simeq \mathrm{D^b}(\Lambda)$, then as an analog of derived McKay correspondence, it is natural to expect that we can recover the scheme Y as a moduli space of modules over the algebra Λ. In particular, since we can regard Y as a moduli space of sheaves of the form $\mathcal{O}_y$ for $y \in Y$, it is natural to expect that we can choose the equivalence $\Psi : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(\Lambda)$ such that the equivalence Ψ satisfies the first condition above. These facts and expectations explain the importance of the notion of tilting bundles.

Next we recall some basic properties of tilting bundles. The following lemma is well-known.

Lemma 2.1.15. *Let $X = \operatorname{Spec} R$ be a normal Gorenstein affine variety and $\phi : Y \to X$ be a crepant resolution. Let F be a coherent sheaf on Y such that*

$$H^i(Y, F) = 0 = \operatorname{Ext}^i_Y(F, \mathcal{O}_Y)$$

for all $i > 0$. Then the R-module $\phi_* F$ is maximal Cohen-Macaulay.

Proof. Put $M := \phi_* F$. Since the resolution $\phi : Y \to X = \operatorname{Spec} R$ is crepant, we have $\phi^! \mathcal{O}_X \simeq \mathcal{O}_Y$. Thus, we have

$$\begin{aligned} \operatorname{Ext}^i_Y(F, \mathcal{O}_Y) &\simeq \operatorname{Ext}^i_Y(F, \phi^! \mathcal{O}_X) \\ &\simeq \operatorname{Ext}^i_R(R\phi_* F, R) \\ &\simeq \operatorname{Ext}^i_R(M, R) \end{aligned}$$

and hence we have

$$\operatorname{Ext}^i_R(M, R) = 0$$

for $i > 0$. Let $\mathfrak{p} \subset R$ be a prime ideal, $(\widehat{R}, \mathfrak{p})$ the $\mathfrak{p}$-adic completion of $(R_\mathfrak{p}, \mathfrak{p})$, and M the $\mathfrak{p}$-adic completion of $M_\mathfrak{p}$. Put $d := \dim R_\mathfrak{p}$. Since the local algebra R is Gorenstein, the canonical module $_R$ is isomorphic to R as an R-module. Thus, by Grothendieck's local duality theorem (see [BH93, Theorem 3.5.8]), we have

$$H^i_\mathfrak{p}(M) = \operatorname{Hom}_R \ \operatorname{Ext}^{d-i}_R(M, R), E(R/\mathfrak{p})$$

where $E(R/\mathfrak{p})$ is an injective hull of the residue field $R/\mathfrak{p}$. Therefore, we have

$$H^i_\mathfrak{p}(M) = 0$$

for $i < d = \dim R_\mathfrak{p}$ and hence M is a maximal Cohen-Macaulay R-module (see [BH93, Theorem 3.5.7]). □

The following is a direct corollary of the lemma above.

Corollary 2.1.16. *Let $X = \operatorname{Spec} R$ be a normal Gorenstein affine variety and $\phi : Y \to X$ a crepant resolution. Assume that Y admits a tilting bundle T. Then $\operatorname{End}_Y(T)$ is Cohen-Macaulay as an R-module.*

Moreover if T is a good tilting bundle, then the R-module $\phi_* T$ is Cohen-Macaulay.

Proof. Since $\mathrm{Ext}^i(T,T) = \mathrm{Ext}^i(T^\vee \otimes T, \mathcal{O}_Y) = H^i(Y, T^\vee \otimes T)$, we have the first result. If T contains a trivial line bundle $\mathcal{O}_Y$ as a direct summand then $H^i(Y,T) = \mathrm{Ext}^i(\mathcal{O}_Y, T) = 0$ and $\mathrm{Ext}^i(T, \mathcal{O}_Y) = 0$ for $i \neq 0$. □

The following lemma is also important in relation to NCCRs.

Lemma 2.1.17. *Let $X = \operatorname{Spec} R$ be a normal Gorenstein affine variety, and $\phi : Y \to X$ a crepant resolution. Assume that Y admits a tilting bundle T. Then there is an algebra isomorphism $\mathrm{End}_Y(T) \simeq \mathrm{End}_X(\phi_* T)$.*

Proof. Put $M := \phi_* T$, and let U be the smooth locus of X. Note that $\mathrm{End}_Y(T)$ and $\mathrm{End}_R(M)$ are isomorphic on U, and that $\mathrm{End}_Y(T)$ is a reflexive R-module by 2.1.16. Therefore we have that the R-module $\mathrm{End}_R(M)$ contains $\mathrm{End}_Y(T)$ as a direct summand, and that the quotient $\mathrm{End}_R(M)/\mathrm{End}_Y(T)$ is a submodule of $\mathrm{End}_R(M)$ whose support is contained in $\mathrm{Sing}(X)$.

On the other hand, since M is torsion free, its endomorphism ring $\mathrm{End}_R(M)$ is also torsion free as an R-module. Thus the module $\mathrm{End}_R(M)/\mathrm{End}_Y(T)$ should be zero, and hence we have

$$\mathrm{End}_Y(T) \simeq \mathrm{End}_R(M)$$

as desired. □

The relation between tilting bundles and NCCRs is given as follows.

Lemma 2.1.18. *Let $X = \operatorname{Spec} R$ be a normal Gorenstein affine scheme, and $\phi : Y \to X$ a crepant resolution. If T is a good tilting bundle on Y, then the module $M := \phi_* T$ gives an NCCR $\mathrm{End}_Y(T)$ of R.*

If the resolution ϕ is small (i.e., the exceptional locus $\mathrm{exc}(\phi)$ does not contain a divisor), then the same thing holds without the assumption that T is good.

Proof. This result follows from Lemma 2.1.2, Proposition 2.1.12, Corollary 2.1.16 and Lemma 2.1.17. □

In the rest of the present subsection, we recall some basic algebraic facts that we use implicitly in this thesis.

Lemma 2.1.19 ([BH93], Proposition 1.4.1)**.** *Let R be a noetherian ring and M a finitely generated R-module. Then the following are equivalent.*

(1) The module M is reflexive.

(2) For each $\mathfrak{p} \in \operatorname{Spec} R$, one of the following happens

- *(a)* $\mathrm{depth}(R_\mathfrak{p}) \leq 1$ *and $M_\mathfrak{p}$ is a reflexive $R_\mathfrak{p}$-module, or*
- *(b)* $\mathrm{depth}(R_\mathfrak{p}) \geq 2$ *and* $\mathrm{depth}(M_\mathfrak{p}) \geq 2$.

Proposition 2.1.20. *Let R be a normal Cohen-Macaulay domain and M a (maximal) Cohen-Macaulay R-module. Then, M is reflexive.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R. If $\dim R_{\mathfrak{p}} \leq 1$, then the ring $R_{\mathfrak{p}}$ is regular and hence $M_{\mathfrak{p}}$ has finite projective dimension. Therefore, by the Auslander-Buchsbaum formula ([BH93, Theorem 1.3.3]),

$$\text{proj.dim}(M_{\mathfrak{p}}) + \operatorname{depth} M_{\mathfrak{p}} = \dim R_{\mathfrak{p}},$$

$M_{\mathfrak{p}}$ is projective and hence free. If $\dim R_{\mathfrak{p}} \geq 2$, we have $\operatorname{depth}(M_{\mathfrak{p}}) \geq 2$ by the assumption. □

Proposition 2.1.21. *Let R be a normal Cohen-Macaulay domain and M, N (maximal) Cohen-Macaulay R-modules. Then, the R-module* $\operatorname{Hom}_R(N, M)$ *is reflexive.*

Proof. If $\dim R_{\mathfrak{p}} \leq 1$, then $M_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$ are free and hence $\operatorname{Hom}_R(N, M)_{\mathfrak{p}}$ is also free. Next we assume that R is local and $\dim R \geq 2$. Then, it is enough to show that the depth of $\operatorname{Hom}_R(N, M)$ is greater than or equal to 2. Let us consider the resolution of N

$$R^{\oplus N_1} \xrightarrow{\varphi} R^{\oplus N_0} \to N \to 0.$$

By applying the functor $\operatorname{Hom}_R(-, M)$, we have an exact sequence

$$0 \to \operatorname{Hom}_R(N, M) \to M^{\oplus N_0} \xrightarrow{\varphi^*} M^{\oplus N_1} \to \operatorname{coker}(\varphi^*) \to 0.$$

Then, by using Lemma 2.1.19, we have the result. □

2.1.3 Mutation of non-commutative crepant resolutions

In the present subsection, we recall some basic definitions and properties about Iyama-Wemyss's mutation. Flops provide a method to compare two crepant resolutions, and Iyama-Wemyss's mutations provide a basic tool to compare two different NCCRs.

Definition 2.1.22. Let R be a d-singular Calabi-Yau ring[1] (d-sCY, for short). A reflexive R-module M is said to be a *modifying module* if $\operatorname{End}_R(M)$ is a maximal Cohen-Macaulay R-module.

Definition 2.1.23. Let A be a ring, M, N A-modules, and $N_0 \in \operatorname{add} N$. A morphism $f : N_0 \to M$ is called a *right* $(\operatorname{add} N)$*-approximation* if the map

$$\operatorname{Hom}_A(N, N_0) \xrightarrow{f\circ} \operatorname{Hom}_A(N, M)$$

is surjective.

Let R be a normal d-sCY ring and M a modifying R-module. For $0 \neq N \in \operatorname{add} M$, we consider

(1) a right $(\operatorname{add} N)$-approximation of M, $a : N_0 \to M$.

[1] We do not give the definition here but note that this is equivalent to say that R is Gorenstein and $\dim R_{\mathfrak{m}} = d$ for all maximal ideal $\mathfrak{m} \subset R$ [IR08].

(2) a right $(\operatorname{add} N^*)$-approximation of M^*, $b : N_1^* \to M^*$.

Let $K_0 := \operatorname{Ker}(a)$ and $K_1 := \operatorname{Ker}(b)$.

Definition 2.1.24. With notations as above, we define the *right mutation* of M at N to be $\mu_N^R(M) := N \oplus K_0$ and the *left mutation* of M at N to be $\mu_N^L(M) := N \oplus K_1^*$.

Note that, the right mutation (or left mutation) is well-defined up to additive closure [IW14a, Lemma 6.3].

In this thesis, we also deal with mutations of exceptional collections (see Section 2.2.2). Thus we call this mutation a (left or right) *IW mutation*.

In [IW14a], Iyama and Wemyss proved the following theorem.

Theorem 2.1.25 ([IW14a])**.** *Let R be a normal d-sCY ring and M a modifying module. Assume that $0 \neq N \in \operatorname{add} M$. Then*

(1) R-algebras $\operatorname{End}_R(M)$, $\operatorname{End}_R(\mu_N^R(M))$, and $\operatorname{End}_R(\mu_N^L(M))$ are derived equivalent.

(2) If M gives an NCCR of R, so do its mutations $\mu_N^R(M)$ and $\mu_N^L(M)$.

The equivalence between $\operatorname{End}_R(M)$ and $\operatorname{End}_R(\mu_N^L(M))$ is given as follows. Let $Q := \operatorname{Hom}_R(M, N)$ and

$$C := \operatorname{Image}\left(\operatorname{Hom}_R(M, N_1) \to \operatorname{Hom}_R(M, K_1^*)\right).$$

Then, one can show that $V \oplus Q$ is a tilting $\Lambda := \operatorname{End}_R(M)$-module and there is an isomorphism of R-algebras

$$\operatorname{End}_R(\mu_N^L(M)) \simeq \operatorname{End}_\Lambda(C \oplus Q).$$

Thus, we have an equivalence

$$\Phi_N := \operatorname{RHom}(C \oplus Q, -) : \mathrm{D^b}(\operatorname{End}_R(M)) \to \mathrm{D^b}(\operatorname{End}_R(\mu_N^L(M))).$$

In this thesis, we only use left IW mutations and hence we call them simply *IW mutations* and write $\mu_N(M)$ instead of $\mu_N^L(M)$. We also call the functor Φ_N an *IW mutation functor*.

The following lemma is useful to find an approximation.

Lemma 2.1.26 ([IW14a, Lemma 6.4, (3)])**.** *Let us consider an exact sequence*

$$0 \to K \xrightarrow{b} N_0 \xrightarrow{a} M,$$

where a is a right $(\operatorname{add} N)$-approximation of M. Then, the dual of the above sequence

$$0 \to M^* \xrightarrow{a^*} N_0^* \xrightarrow{b^*} K^*$$

is also exact and b^* is a right $(\operatorname{add} N^*)$-approximation of K^*.

2.2 Derived categories

In this section we recall some notions in relation to triangulated categories.

2.2.1 Spherical twist and P-twist

Spherical twist

In this subsection we recall the definition of spherical twists.

Definition 2.2.1. Let X be an n-dimensional smooth variety.

(1) We say that an object $\mathcal{E} \in \mathrm{D^b}(X)$ is a *spherical object* if $\mathcal{E} \otimes \omega_X \simeq \mathcal{E}$ and

$$\mathrm{RHom}_X(\mathcal{E}, \mathcal{E}) \simeq \mathbb{C} \oplus \mathbb{C}[-n].$$

(2) Let $\mathcal{E}$ be a spherical object. Then a *spherical twist* $\mathrm{T}_{\mathcal{E}}$ around $\mathcal{E}$ is defined as

$$\mathrm{T}_{\mathcal{E}}(\mathcal{F}) := \mathrm{Cone}(\mathrm{RHom}_X(\mathcal{E}, \mathcal{F}) \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{F}).$$

Example 2.2.2. The following are important examples of spherical objects.

(1) Let X be a (smooth proper) Calabi-Yau variety. Then the structure sheaf (= trivial line bundle) $\mathcal{O}_X$ of X is a spherical object.

(2) Let X be a smooth projective surface that contains a (-2)-curve C. Then the structure sheaf $\mathcal{O}_C$ of C is a spherical object in $\mathrm{D^b}(X)$. Note that this is also an example of P-objects.

Theorem 2.2.3 ([ST01])**.** *A spherical twist is an autoequivalence of* $\mathrm{D^b}(X)$.

Spherical twists on the derived category of an algebraic variety is expected to be a mirror of symplectic monodromies.

P-twists

In this subsection we recall the definition of P-twists and their basic properties.

Definition 2.2.4. An object E in the derived category $\mathrm{D^b}(X)$ of a variety X of dimension $2n$ is called a $\mathbb{P}$*-object* if we have $E \otimes \omega_X \simeq E$ and an algebra isomorphism

$$\mathrm{Hom}^*(E, E) \simeq H^*(\mathbb{P}^n; \mathbb{C}).$$

For a $\mathbb{P}$-object E, the *P-twist* $P_E : \mathrm{D^b}(X) \to \mathrm{D^b}(X)$ by E is defined as follows

$$P_E(F) := \mathrm{Cone}\left(\mathrm{Cone}(E \otimes \mathrm{RHom}(E, F)[-2] \to E \otimes \mathrm{RHom}(E, F)) \xrightarrow{\mathrm{ev}} F\right).$$

See Lemma 3.5.15 for a basic example of $\mathbb{P}$-objects.

Proposition 2.2.5 ([HT06])**.** *A P-twist gives an autoequivalence of* $\mathrm{D^b}(X)$.

The notion of P-twist was first introduced by Huybrechts and Thomas in their paper [HT06] as an analogue of the notion of spherical twist. Spherical twists give an important class of autoequivalences on the derived category of a Calabi-Yau variety. In contrast, P-twists give a significant class of autoequivalences on the derived category of a (holomorphic) symplectic variety. In Section 3.5.2, we study P-twists on a symplectic variety $Y = \mathrm{Tot}(\Omega_{\mathbb{P}(V)})$ from the point of view of NCCRs.

2.2.2 Exceptional objects

Let $\mathcal{D}$ be a triangulated category with finite dimensional Hom spaces. The derived category $\mathrm{D}^{\mathrm{b}}(X)$ of a proper variety X satisfies this property.

Definition 2.2.6. (i) An object $\mathcal{E} \in \mathcal{D}$ is called an *exceptional object* if

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{E}, \mathcal{E}[i]) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

(ii) A sequence of exceptional objects $\mathcal{E}_1, \dots, \mathcal{E}_r$ is called an *exceptional collection* if $\mathrm{RHom}_{\mathcal{D}}(\mathcal{E}_l, \mathcal{E}_k) = 0$ for all $1 \leq k < l \leq r$.

(iii) An exceptional collection $\mathcal{E}_1, \dots, \mathcal{E}_r$ is *full* if it classically generates the whole category $\mathcal{D}$. In such case, we write

$$\mathcal{D} = \langle \mathcal{E}_1, \dots, \mathcal{E}_r \rangle.$$

(iv) We say that an exceptional collection $\mathcal{E}_1, \dots, \mathcal{E}_r$ is *strong* if

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_l, \mathcal{E}_k[i]) = 0$$

for all $1 \leq l < k \leq r$ and $i \neq 0$.

Remark 2.2.7. Let X be a smooth projective variety and assume that the derived category $\mathrm{D}^{\mathrm{b}}(X)$ of X admits a full strong exceptional collection

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{E}_1, \dots, \mathcal{E}_r \rangle$$

consisting of vector bundles $\mathcal{E}_k$. Then their direct sum

$$T = \bigoplus_{k=1}^{r} \mathcal{E}_k$$

is a tilting bundle on X. Indeed, since X is smooth projective, we have $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{Perf}(X)$, and hence T is a generator of $\mathrm{D}(\mathrm{Qcoh}(X))$ by Lemma 2.1.10. In particular, we have an equivalence

$$\mathrm{RHom}_X(T, -) : \mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{End}_X(T))$$

between the derived category of X and the derived category of a finite dimensional algebra $\mathrm{End}_X(T)$.

Example 2.2.8 ([Bei79], [Kuz08]). (1) An n-dimensional projective space $\mathbb{P}^n$ has a full strong exceptional collection consisting of line bundles called Beilinson collection

$$\mathrm{D^b}(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \ldots, \mathcal{O}(n) \rangle.$$

(2) Let us consider n-dimensional quadric hypersurface Q_n. If n is odd, there is a full strong exceptional collection

$$\mathrm{D^b}(Q_n) = \langle S, \mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n-1) \rangle,$$

where S is a spinor bundle. If n is even, spinor bundle has two indecomposable components S^+ and S^-, and we have a full strong exceptional collection

$$\mathrm{D^b}(Q_n) = \langle S^+, S^- \mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n-1) \rangle.$$

(3) Let V be a four dimensional symplectic vector space and $\mathrm{LGr}(V)$ the Lagrangian Grassmannian of V. there is a full strong exceptional collection

$$\mathrm{D^b}(\mathrm{LGr}(V)) = \langle \mathcal{S}, \mathcal{O}_{\mathrm{LGr}}, \mathcal{O}_{\mathrm{LGr}}(1), \mathcal{O}_{\mathrm{LGr}}(2) \rangle,$$

where $\mathcal{S}$ is the universal subbundle. It is not difficult to see that $\mathrm{LGr}(V)$ is isomorphic to Q_3, and that universal subbundle coincides with a spinor bundle via this isomorphism.

(4) Let us consider G_2-Grassmannian Gr_{G_2}. Gr_{G_2} is a closed subscheme of $\mathrm{Gr}(2, 7)$ and hence we can consider a restriction of the universal subbundle R of rank two. Gr_{G_2} admits a full strong exceptional collection

$$\mathrm{D^b}(\mathrm{Gr}_{G_2}) = \langle R(-2), \mathcal{O}(-2), R(-1), \mathcal{O}(-1), R, \mathcal{O} \rangle.$$

For an object $\mathcal{E} \in \mathcal{D}$, we define subcategories $\mathcal{E}^\perp, {}^\perp\mathcal{E} \subset \mathcal{D}$ by

$$\mathcal{E}^\perp := \{\mathcal{F} \in \mathcal{D} \mid \mathrm{RHom}_{\mathcal{D}}(\mathcal{E}, \mathcal{F}) = 0\},$$
$${}^\perp\mathcal{E} := \{\mathcal{F} \in \mathcal{D} \mid \mathrm{RHom}_{\mathcal{D}}(\mathcal{F}, \mathcal{E}) = 0\}.$$

Lemma 2.2.9. *Let*

$$\mathcal{D} = \langle \mathcal{E}_1, \ldots, \mathcal{E}_r \rangle = \langle \mathcal{E}'_1, \ldots, \mathcal{E}'_r \rangle$$

be two full exceptional collections of the same length. Let $1 \leq i \leq r$ and assume that $\mathcal{E}_j = \mathcal{E}'_j$ holds for all $j \neq i$. Then, we have

$$\mathcal{E}_i = \mathcal{E}'_i$$

up to shift.

Proof. This lemma follows from the fact

$${}^\perp\mathcal{E}_1 \cap \cdots \cap {}^\perp\mathcal{E}_{i-1} \cap \mathcal{E}_{i+1}^\perp \cap \cdots \cap \mathcal{E}_r^\perp = \mathrm{D^b}(\mathrm{Spec}\,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{E}_i,$$

see [Bon90]. □

Definition 2.2.10. Let $\mathcal{E} \in \mathcal{D}$ be an exceptional object. For an object $\mathcal{F}$, we define the *left mutation of $\mathcal{F}$ over $\mathcal{E}$* as an object $\mathbb{L}_{\mathcal{E}}(\mathcal{F})$ in $\mathcal{E}^{\perp}$ that lies in an exact triangle

$$\mathrm{RHom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathbb{L}_{\mathcal{E}}(\mathcal{F}).$$

Similarly, for an object $\mathcal{G}$, we define the *right mutation of $\mathcal{G}$ over $\mathcal{E}$* as an object $\mathbb{R}_{\mathcal{E}}(\mathcal{G})$ in ${}^{\perp}\mathcal{E}$ that lies in an exact triangle

$$\mathbb{R}_{\mathcal{E}}(\mathcal{G}) \longrightarrow \mathcal{G} \longrightarrow \mathrm{RHom}(\mathcal{G}, \mathcal{E})^{\vee} \otimes \mathcal{E}.$$

Lemma 2.2.11 ([Bon90])**.** *Let $\mathcal{E}_1, \mathcal{E}_2$ be an exceptional pair (i.e. an exceptional collection consisting of two objects).*

(i) The left (resp. right) mutated object $\mathbb{L}_{\mathcal{E}_1}(\mathcal{E}_2)$ (resp. $\mathbb{R}_{\mathcal{E}_2}(\mathcal{E}_1)$) is again an exceptional object.

(ii) The pairs of exceptional objects $\mathcal{E}_1, \mathbb{R}_{\mathcal{E}_1}(\mathcal{E}_2)$ and $\mathbb{L}_{\mathcal{E}_2}(\mathcal{E}_1), \mathcal{E}_2$ are again exceptional pairs.

Let $\mathcal{E}_1, \dots, \mathcal{E}_r$ be a full exceptional collection in $\mathcal{D}$. Then

(iii) The collection

$$\mathcal{E}_1, \dots, \mathcal{E}_{i-1}, \mathbb{L}_{\mathcal{E}_i}(\mathcal{E}_{i+1}), \mathcal{E}_i, \mathcal{E}_{i+2}, \dots, \mathcal{E}_r$$

is again full exceptional for each $1 \le i \le r-1$. Similarly, the collection

$$\mathcal{E}_1, \dots, \mathcal{E}_{i-2}, \mathcal{E}_i, \mathbb{R}_{\mathcal{E}_i}(\mathcal{E}_{i-1}), \mathcal{E}_{i+1}, \dots, \mathcal{E}_r$$

is again full exceptional for each $2 \le i \le r$.

(iv) Assume in addition that the category $\mathcal{D}$ admits the Serre functor $S_{\mathcal{D}}$. Then the collections

$$\mathcal{E}_2, \dots, \mathcal{E}_{r-1}, \mathcal{E}_r, S_{\mathcal{D}}^{-1}(\mathcal{E}_1) \quad \text{and} \quad S_{\mathcal{D}}(\mathcal{E}_r), \mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_{r-1}$$

are full exceptional collections on $\mathcal{D}$.

Example 2.2.12. Let us consider the Beilinson collection

$$\mathrm{D^b}(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n) \rangle.$$

Then $\mathrm{RHom}(\mathcal{O}, \mathcal{O}(1)) \simeq \mathbb{C}^{n+1}$ and hence we have an triangle

$$\mathbb{R}_{\mathcal{O}(1)}(\mathcal{O}) \to \mathcal{O} \to \mathcal{O}(1)^{\oplus n+1}.$$

Thus we have $\mathbb{R}_{\mathcal{O}(1)}(\mathcal{O}) \simeq T_{\mathbb{P}^n}[-1]$, and we obtain a new full exceptional collection

$$\mathrm{D^b}(\mathbb{P}^n) = \langle \mathcal{O}(1), T_{\mathbb{P}^n}, \cdots, \mathcal{O}(n) \rangle.$$

Example 2.2.13. By taking mutations, we have the following different full exceptional collections for $\mathrm{D^b}(\mathrm{LGr}(V))$:

$$\begin{aligned}
\mathrm{D^b}(\mathrm{LGr}(V)) &= \langle \mathcal{O}_{\mathrm{LGr}}, \mathcal{S}(1), \mathcal{O}_{\mathrm{LGr}}(1), \mathcal{O}_{\mathrm{LGr}}(2) \rangle \\
&= \langle \mathcal{O}_{\mathrm{LGr}}, \mathcal{O}_{\mathrm{LGr}}(1), \mathcal{S}(2), \mathcal{O}_{\mathrm{LGr}}(2) \rangle \\
&= \langle \mathcal{O}_{\mathrm{LGr}}, \mathcal{O}_{\mathrm{LGr}}(1), \mathcal{O}_{\mathrm{LGr}}(2), \mathcal{S}(3) \rangle \\
&= \langle \mathcal{S}, \mathcal{O}_{\mathrm{LGr}}, \mathcal{O}_{\mathrm{LGr}}(1), \mathcal{O}_{\mathrm{LGr}}(2) \rangle.
\end{aligned}$$

2.2.3 Relative Serre functors and Calabi-Yau categories

Let R be a commutative algebra of finite type over a field $\Bbbk$, and D an R-linear triangulated category.

Definition 2.2.14. An *R-triangulated category structure* on D is a functor

$$\mathrm{RHom}_{D/R} : D^{\mathrm{op}} \times D \to \mathrm{D^b}(\mathrm{mod}(R))$$

together with a functorial isomorphism

$$\mathrm{Hom}_D(F, G) \simeq \mathcal{H}^0(\mathrm{RHom}_{D/R}(G, F)).$$

Example 2.2.15. Let X be a smooth variety equipped with a projective morphism $\pi : X \to \operatorname{Spec} R$. Then the functor $\mathrm{RHom}_{D/R} := R\pi_* \,\mathrm{R}\mathcal{H}om_X$ gives a R-triangulated category structure on $D = \mathrm{D^b}(X) = \mathrm{D^b}(\mathrm{coh}(X))$.

Let $\mathbb{D}_Y$ be the dualizing functor on the derived category $\mathrm{D^b}(\mathrm{mod}(Y))$ of a quasi-projective scheme Y. If Y is an affine scheme $\operatorname{Spec} R$, we also write $\mathbb{D}_R$ instead of $\mathbb{D}_Y$.

Definition 2.2.16. Let D be an R-triangulated category. We say that an autoequivalence $S \in \mathrm{Auteq}(D)$ is the *R-Serre functor* if there is a functorial isomorphism

$$\mathrm{RHom}_{D/R}(F, G) \simeq \mathbb{D}_R(\mathrm{RHom}_{D/R}(G, S(F))).$$

An easy argument using Yoneda lemma shows that the R-Serre functor is unique up to isomorphism if it exists.

Definition 2.2.17. An R-triangulated category D is *(relative) Calabi-Yau category* if a shift functor $[n]$ for certain $n \in \mathbb{Z}$ gives the R-Serre functor on D.

Example 2.2.18. Let X be a smooth scheme projective over R and we regard the derived category $D = \mathrm{D^b}(X)$ of X as an R-triangulated category as in Example 2.2.15. Then the functor $S := - \otimes \omega_X[\dim X]$ gives a R-Serre functor on $\mathrm{D^b}(X)$. Indeed,

$$\begin{aligned}
\mathbb{D}_R(\mathrm{RHom}_{D/R}(G, S(F))) &= \mathbb{D}_R(R\pi_* \,\mathrm{R}\mathcal{H}om_{D/R}(G, S(F))) \\
&\simeq R\pi_*(\mathbb{D}_X \,\mathrm{R}\mathcal{H}om_{D/R}(G, S(F))) \\
&\simeq R\pi_* \,\mathrm{R}\mathcal{H}om_X(S(F), S(G)) \\
&\simeq \mathrm{RHom}_{D/R}(F, G).
\end{aligned}$$

Thus if X is local Calabi-Yau (i.e. $\omega_X \simeq \mathcal{O}_X$) then $D = \mathrm{D^b}(X)$ is a Calabi-Yau category.

If an indecomposable category is relative Calabi-Yau, then it does not have a non-trivial admissible subcategory:

Lemma 2.2.19 ([BMR08, Lemma 3.5.2]). *Let D be a Calabi-Yau R-triangulated category and $\Phi : C \to D$ be an exact functor. Then Φ gives an equivalence of categories if the following two conditions hold.*

(i) Φ has a right adjoint functor $\Phi^!$ and the adjunction morphism $\mathrm{id} \to \Phi^! \circ \Phi$ *is an isomorphism.*

(ii) D is indecomposable.

Using this result, we have the following.

Corollary 2.2.20. *Let $Y \to X = \operatorname{Spec} R$ be a crepant resolution and T a partial tilting bundle on Y. Then T is a tilting bundle if and only if the global dimension of* $\operatorname{End}_Y(T)$ *is finite.*

This corollary means that we can characterize tilting bundles on a crepant resolution without referring to its derived category.

2.3 Nilpotent orbit closures and stratified flops

In this section, we recall some basic properties of nilpotent orbit closures. The singularities of nilpotent orbit closures give an important class of symplectic singularities (see [Bea00]).

2.3.1 Symplectic singularity

First we recall the notion of symplectic singularity.

Definition 2.3.1 ([Bea00]). Let X be an algebraic variety. We say that X is a *symplectic variety* if

(i) X is normal.

(ii) The smooth part X_{sm} of X admits a symplectic 2-form ω.

(iii) For every resolution $f : Y \to X$, the pull back of ω to $f^{-1}(X_{\mathrm{sm}})$ extends to a global holomorphic 2-form on Y.

Let X be an algebraic variety. We say that a point $x \in X$ is a *symplectic singularity* if there is an open neighborhood U of x such that U is a symplectic variety.

Symplectic singularities belong to a good class of singularities that appears in minimal model theory.

Proposition 2.3.2 ([Bea00]). *A symplectic singularity is Gorenstein canonical.*

Example 2.3.3 (Du Val singularities). Since symplectic singularities are Gorenstein canonical, a two dimensional symplectic singularity is Du Val singularity. On the other hand, since a Du Val singularity is a quotient singularity by a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, and since in dimension two we have $\mathrm{SL}(2, \mathbb{C}) = \mathrm{Sp}(2, \mathbb{C})$, a Du Val singularity is symplectic. Thus two dimensional symplectic singularities are precisely Du Val singularities.

For symplectic singularities, we can consider the following reasonable class of resolutions.

Definition 2.3.4. Let X be a symplectic variety. A resolution $\phi : Y \to X$ of X is called *symplectic* if the extended 2-form ω on Y is non-degenerate. In other words, the 2-form ω defines a symplectic structure on Y.

Proposition 2.3.5. *Let X be a symplectic variety and $\phi : Y \to X$ a resolution. Then, the following statements are equivalent*

(1) ϕ is a crepant resolution,

(2) ϕ is a symplectic resolution,

(3) the canonical divisor K_Y of Y is trivial.

2.3.2 Nilpotent orbit closure

Next, we recall the definition of nilpotent orbit closures and some basic properties of them. Let $\mathfrak{g}$ be a complex Lie algebra. For $u \in \mathfrak{g}$, we define a linear map $\mathrm{ad}_u : \mathfrak{g} \to \mathfrak{g}$ by $x \mapsto [u, x]$. In the following, we assume that the Lie algebra $\mathfrak{g}$ is semi-simple, i.e. the bilinear form $\kappa(u, v) := \mathrm{trace}(\mathrm{ad}_u \circ \mathrm{ad}_v)$ is non-degenerate. An element $v \in \mathfrak{g}$ is *nilpotent* if the corresponding linear map ad_v is nilpotent. Let G be the adjoint algebraic group of $\mathfrak{g}$. Then, G acts on $\mathfrak{g}$ via the adjoint representation. An orbit $\mathcal{O} = G \cdot v \subset \mathfrak{g}$ of v under this action is called a *nilpotent orbit* if the element v is nilpotent.

Proposition 2.3.6 ([Pan91])**.** *The normalization $\widetilde{\mathcal{O}}$ of a nilpotent orbit closure $\overline{\mathcal{O}}$ in a complex semi-simple Lie algebra $\mathfrak{g}$ has only symplectic singularities. In particular the singularity of $\widetilde{\mathcal{O}}$ is Gorenstein canonical.*

For example, the following theorem by Beauville suggests that the singularity of nilpotent orbit closures constitutes an important class among symplectic singularities.

Theorem 2.3.7 ([Bea00])**.** *Let (X, o) be a germ of isolated symplectic singularity, whose projective tangent cone is smooth. Then (X, o) is analytically isomorphic to the germ $(\overline{\mathcal{O}}_{\min}, 0)$, where $\overline{\mathcal{O}}_{\min}$ is a (non-zero) smallest nilpotent orbit closure in some simple Lie algebra.*

Let $V = \mathbb{C}^N$ be a N-dimensional vector space and

$$B(r) := \{X \in \mathrm{End}_{\mathbb{C}}(V) \mid X^2 = 0, \mathrm{rank}(X) = r.\} \subset \mathfrak{sl}(V) \simeq \mathfrak{sl}_N.$$

This is an example of a nilpotent orbit of type A. We note that we have

$$\overline{B(r)} = \{X \in \mathrm{End}_{\mathbb{C}}(V) \mid X^2 = 0, \mathrm{rank}(X) \leq r\} = \bigcup_{k=0}^{r} B(r).$$

If we consider nilpotent orbit closures of type A, we need not to take the normalization to obtain symplectic singularities.

Proposition 2.3.8 ([KF79]). *Let $r \geq 1$. Then, the variety $\overline{B(r)}$ is normal, and hence is an affine symplectic variety. In particular, the variety $\overline{B(r)}$ is Gorenstein, and has only canonical (equivalently, rational) singularities.*

Moreover, we can show that the variety $\overline{B(r)}$ has symplectic (equivalently, crepant) resolutions. See the next section.

2.3.3 Stratified flops

Stratified Mukai flops and Stratified Atiyah flops on $\mathrm{Gr}(r, N)$

Let $V = \mathbb{C}^N$ be a N-dimensional vector space, r an integer such that $1 \leq r \leq N - 1$, and $\mathrm{Gr}(r, N)$ the Grassmannian of r-dimensional linear subspaces of V. For each r such that $2r \leq N$, we consider the following three varieties

$$\begin{aligned} Y_0 &:= \{(L, A) \in \mathrm{Gr}(r, N) \times \mathrm{End}(V) \mid A(V) \subset L, A(L) = 0\}, \\ Y_0' &:= \{(L', A') \in \mathrm{Gr}(N - r, N) \times \mathrm{End}(V) \mid A'(V) \subset L', A'(L') = 0\}, \\ X_0 &:= \overline{B(r)} := \{A \in \mathrm{End}(V) \mid A^2 = 0, \dim \mathrm{Ker}\, A \geq N - r\}. \end{aligned}$$

The variety Y_0 has two projections $\phi_0 : Y_0 \to X$ and $\pi_0 : Y_0 \to \mathrm{Gr}(r, N)$. Via the second projection π_0, we can identify Y_0 with the total space of the cotangent bundle on the Grassmannian $\mathrm{Gr}(r, N)$.

Similarly, Y_0' has two projections $\phi_0' : Y_0 \to X$ and $\pi_0' : Y_0 \to \mathrm{Gr}(N - r, N)$, and the second projection allows us to identify Y_0' with the total space of the cotangent bundle on the Grassmannian $\mathrm{Gr}(N - r, N)$.

The affine variety X_0 is an example of a nilpotent orbit closure of type A that we explained in the above subsection.

It is easy to see that two morphisms $\phi_0 : Y_0 \to X_0$ and $\phi_0' : Y_0' \to X_0$ give resolutions of X_0. Since Y_0 and Y_0' are algebraic symplectic varieties, these two resolutions are crepant resolutions.

If $N \geq 3$ and $2r < N$, the diagram

$$\begin{array}{ccc} Y_0 & & Y_0' \\ & \searrow^{\phi_0} \quad {}^{\phi_0'}\swarrow & \\ & X_0 & \end{array}$$

is a flop and this flop is called a *stratified Mukai flop* on $\mathrm{Gr}(r, N)$.

Note that Y_0, Y_0', and X_0 have natural $\mathbb{G}_{\mathrm{m}}$-actions and ϕ_0 and ϕ_0' are $\mathbb{G}_{\mathrm{m}}$-equivariant.

These three varieties Y_0, Y_0' and X_0 have natural one-parameter $\mathbb{G}_{\mathrm{m}}$-equivariant deformations as follows.

$$\begin{aligned} Y &:= \{(L, A, t) \in \mathrm{Gr}(r, N) \times \mathrm{End}(V) \times \mathbb{C} \mid (A - t \cdot \mathrm{id})(V) \subset L, (A + t \cdot \mathrm{id})(L) = 0\}, \\ Y' &:= \{(L', A', t') \in \mathrm{Gr}(N - r, N) \times \mathrm{End}(V) \times \mathbb{C} \mid (A' - t' \cdot \mathrm{id})(V) \subset L', (A' + t' \cdot \mathrm{id})(L') = 0\}, \\ X &:= \{(A, t) \in \mathrm{End}(V) \times \mathbb{C} \mid A^2 = t^2 \cdot \mathrm{id}, \dim \mathrm{Ker}(A - t \cdot \mathrm{id}) \geq N - r\}. \end{aligned}$$

Note that the variety Y is isomorphic to the total space of a bundle $\widetilde{\Omega}_{\mathrm{Gr}(r,N)}$ on $\mathrm{Gr}(r,N)$ that lies on an exact sequence

$$0 \to \Omega_{\mathrm{Gr}(r,N)} \to \widetilde{\Omega}_{\mathrm{Gr}(r,N)} \to \mathcal{O}_{\mathrm{Gr}(r,N)} \to 0$$

that gives a generator of

$$H^1(\mathrm{Gr}(r,N), \Omega_{\mathrm{Gr}(r,N)}) = \mathbb{C}\,.$$

The corresponding statement holds for Y'. Put

$$\phi : Y \ni (L, A, t) \mapsto (A, t) \in X \text{ and } \phi' : Y' \ni (L', A', t') \mapsto (A', -t') \in X.$$

Then the morphisms $\phi : Y \to X$ and $\phi' : Y' \to X$ give two crepant resolutions of X, and the diagram

$$\begin{array}{ccc} Y & & Y' \\ & \searrow^{\phi} \quad {}^{\phi'}\swarrow & \\ & X & \end{array}$$

is a flop called a *stratified Atiyah flop* on $\mathrm{Gr}(r,N)$.

In [CKL10, CKL13], Cautis, Kamnitzer, and Licata proved that there are derived equivalences for stratified Mukai flops $\mathrm{D^b}(Y_0) \simeq \mathrm{D^b}(Y'_0)$ (later we refer to this equivalence as CKL's equivalence). Their equivalence was given as a Fourier-Mukai transform and was obtained as a corollary of their framework of *categorical* $\mathfrak{sl}_2$ *action*. In [Cau12a], Cautis studied an explicit description of the Fourier-Mukai kernel, and as an application of it, he showed that CKL's equivalence for a stratified Mukai flop extends to an equivalence for a stratified Atiyah flop.

Chapter 3

Non-commutative crepant resolution of minimal nilpotent orbit closures of type A and Mukai flops

This chapter is based on the author's work

[H17a] W. Hara, *Non-commutative crepant resolution of minimal nilpotent orbit closures of type A and Mukai flops*, Adv. Math., **318** (2017), 355–410.

3.1 Introduction

The aim of this chapter is to study *non-commutative crepant resolutions (=NCCR)* of a minimal nilpotent orbit closure $\overline{B(1)}$ of type A. The notion of NCCR was first introduced by Van den Bergh [VdB04b] in relation to the study of the derived categories of algebraic varieties. We can regard the concept of NCCR as a generalization of the notion of *crepant resolution*. Van den Bergh introduced it with an expectation that all crepant resolutions, whether commutative or not, have equivalent derived categories. This expectation is a special (and non-commutative) version of a more general conjecture that K-equivalence implies derived equivalence. We note that the study of NCCR is also motivated by theoretical physics (see Introduction of [Leu12]).

An NCCR of a Gorenstein algebra R is defined as an endomorphism ring $\mathrm{End}_R(M)$ of a (maximal) Cohen-Macaulay R-module M such that $\mathrm{End}_R(M)$ is a (maximal) Cohen-Macaulay R-module and has finite global dimension. In relation to NCCR, it is natural to ask the following questions.

(1) Construct an NCCR of R and characterize a module M that gives the NCCR.

(2) Construct a derived equivalence between the NCCR and a (commutative) crepant resolution.

(3) Construct a (commutative) crepant resolution as a moduli space of modules over the NCCR.

For example, in [BLV10], Buchweitz, Leuschke, and Van den Bergh studied about these problems for a determinantal variety. In this chapter, we deal with the above problems for a minimal nilpotent orbit closure $\overline{B(1)}$ of type A. We also study about the derived equivalences for Mukai flops from the point of view of NCCR.

3.1.1 NCCR of minimal nilpotent orbit closures of type A

Let $V = \mathbb{C}^N$ be a complex vector space of dimension $N \geq 2$. Let us consider a subset $B(1)$ of $\mathrm{End}_{\mathbb{C}}(V)$ that is given by

$$B(1) := \{X \in \mathrm{End}_{\mathbb{C}}(V) \mid X^2 = 0, \dim \operatorname{Ker} X = N - 1\}.$$

This is a minimal nilpotent orbit of type A. It is well known that the closure $\overline{B(1)}$ of the orbit $B(1)$ is normal and has only symplectic singularities, and thus the affine coordinate ring R of $\overline{B(1)}$ is normal and Gorenstein. Since $\operatorname{codim}_{\overline{B(1)}}(\partial B(1)) \geq 2$, we have a $\mathbb{C}$-algebra isomorphism $R \simeq H^0(B(1), \mathcal{O}_{B(1)})$. Let H be a subgroup of SL_N such that $\mathrm{SL}_N / H \simeq B(1)$. It is easy to see that H is isomorphic to a subgroup of SL_N

$$H \simeq \left\{ A = \left(\begin{array}{c|ccc|c} c & 0 & \cdots & 0 & 0 \\ \hline & & & & 0 \\ * & & A' & & \vdots \\ & & & & 0 \\ \hline * & & * & & c \end{array} \right) \;\middle|\; \begin{array}{c} A' \in \mathrm{GL}_{N-2}, \\ c \in \mathbb{C}^\times, \\ c^2 \cdot \det(A') = 1 \end{array} \right\}.$$

Let $\mathcal{M}_a$ be a homogeneous line bundle on $B(1)$ that corresponds to the character $H \ni A \mapsto c^{-a} \in \mathbb{C}^\times$ and we set $M_a := H^0(B(1), \mathcal{M}_a)$. We prove that a direct sum of R-modules $(M_a)_a$ gives an NCCR of R.

Theorem 3.1.1 (see 3.3.3 and 3.2.3). *(a) M_a is a Cohen-Macaulay R-module for $-N+1 \leq a \leq N-1$.*

(b) For $0 \leq k \leq N-1$, the R module $\bigoplus_{a=-N+k+1}^{k} M_a$ gives an NCCR $\mathrm{End}_R \left(\bigoplus_{a=-N+k+1}^{k} M_a \right)$ of R.

The proof of Theorem 3.1.1 is based on the theory of *tilting bundles* on the crepant resolutions Y and Y^+. We note that the two crepant resolutions Y and Y^+ of $\overline{B(1)}$ are the total spaces of the cotangent bundles on $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, respectively. Let $\pi : Y \to \mathbb{P}(V)$ and $\pi' : Y^+ \to \mathbb{P}(V^*)$ be the projections. We show that, for all $k \in \mathbb{Z}$, the bundles

$$\mathcal{T}_k := \bigoplus_{a=-N+k+1}^{k} \pi^* \mathcal{O}_{\mathbb{P}(V)}(a) \quad \text{and} \quad \mathcal{T}_k^+ := \bigoplus_{a=-N+k+1}^{k} \pi'^* \mathcal{O}_{\mathbb{P}(V^*)}(a)$$

are tilting bundles on Y and Y^+, respectively (Theorem 3.3.3). We also show that there is a canonical isomorphism of R-algebras

$$\Lambda_k := \mathrm{End}_Y(\mathcal{T}_k) \simeq \mathrm{End}_{Y^+}(\mathcal{T}_{N-k-1}^+)$$

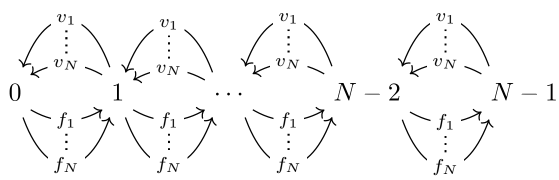
and that this algebra is isomorphic to the one that appears in Theorem 3.1.1 (b). Moreover, by the theory of tilting bundles, we have an equivalence of categories $\mathrm{D^b}(Y) \simeq \mathrm{D^b}(\mathrm{mod}(\Lambda_k))$ between the derived category of a crepant resolution and of an NCCR. In Section 3.3.3, we provide another NCCR Λ' of R that is not isomorphic to Λ_k but is derived equivalent to Λ_k.

3.1.2 NCCR as the path algebra of a quiver

Next, we describe an NCCR Λ_k of R as the path algebra of the *double Beilinson quiver* with some relations. We note that similar results for non-commutative resolutions of determinantal varieties are obtained by Buchweitz, Leuschke, and Van den Bergh [BLV10], and Weyman and Zhao [WZ12].

Let $S = \mathrm{Sym}^\bullet(V \otimes V^*)$ be the symmetric algebra of a vector space $V \otimes_{\mathbb{C}} V^*$. Let $v_1, \ldots, v_N$ be the standard basis of $V = \mathbb{C}^N$ and $f_1, \ldots, f_N$ the dual basis of V^*. We regard $x_{ij} = v_j \otimes f_i \in S$ as the variables of the affine coordinate ring of an affine variety $\mathrm{End}_{\mathbb{C}}(V) \simeq V^* \otimes_{\mathbb{C}} V$. Since $\overline{B(1)}$ is a closed subvariety of $\mathrm{End}_{\mathbb{C}}(V)$, R is a quotient of S.

Theorem 3.1.2 (= Thm. 3.3.7)**.** *As an S-algebra, the non-commutative algebra Λ_k is isomorphic to the path algebra $S\widetilde{\Gamma}$ of the double Beilinson quiver $\widetilde{\Gamma}$ with N vertices*



with relations

$$v_i v_j = v_j v_i, \quad f_i f_j = f_j f_i, \quad v_j f_i = f_i v_j = x_{ij} \quad \textit{for all } 1 \le i, j \le N,$$
$$\textit{and} \quad \sum_{i=1}^{N} f_i v_i = 0 = \sum_{i=1}^{N} v_i f_i.$$

Building on this theorem, we can also show that the two crepant resolutions Y and Y^+ are recovered from the quiver $\widetilde{\Gamma}$ as moduli spaces of representations (Theorem 3.4.1). The idea of the proof is based on the fact that crepant resolutions Y and Y^+ are moduli spaces that parametrizes representations of Nakajima's quiver of type A_1. We show that there is a natural correspondence between stable representations of Nakajima's quiver of type A_1 and representations of $\widetilde{\Gamma}$. At the end of Section 3.4, we also characterize simple representations of the quiver, namely we show that a simple representation corresponds to a point of a crepant resolution that lies over a non-singular point of $\overline{B(1)}$ (see Theorem 3.4.13).

We note that these relations between a crepant resolution Y (or Y^+) and an NCCR Λ_k can be considered as a generalization of *McKay correspondence*. Classical McKay correspondence states that, for a finite subgroup $G \subset \mathrm{SL}_2$, there are many relations between the geometry of a quotient variety $\mathbb{C}^2/G$ and representations of the group G. In the modern context, McKay correspondence is understood as relationships (e.g. a derived equivalence) between the crepant resolution $\widetilde{\mathbb{C}^2/G}$ of $\mathbb{C}^2/G$ and a quotient stack $[\mathbb{C}^2/G]$. We often say that the crepant resolution $\widetilde{\mathbb{C}^2/G}$ is a "geometric resolution" of $\mathbb{C}^2/G$. On the other hand, since a coherent sheaf on a quotient stack $[\mathbb{C}^2/G]$ is canonically identified with a module over the skew group algebra $\mathbb{C}[x,y]\sharp G$, we say that a smooth stack $[\mathbb{C}^2/G]$ is an "algebraic resolution" of $\mathbb{C}^2/G$. Thus, we can interpret McKay correspondence as a correspondence between geometric and algebraic resolutions. In our case, a geometric resolution of $\overline{B(1)}$ is Y (or Y^+) and an algebraic resolution is the NCCR Λ_k.

3.1.3 Mukai flops, P-twists and mutations

It is well-known that the diagram of two crepant resolutions

$$\begin{array}{ccc} Y & & Y^+ \\ & \searrow^{\phi} \quad {}^{\phi^+}\swarrow & \\ & \overline{B(1)} & \end{array}$$

is a local model of a class of flops that are called *Mukai flop*. Let $\widetilde{Y}$ be a blowing-up of Y along the zero-section $j(\mathbb{P}(V)) \subset Y$. Then, the exceptional divisor $E \subset \widetilde{Y}$ is naturally identified with the universal hyperplane in $\mathbb{P}(V) \times \mathbb{P}(V^*)$. Let $\widehat{Y} := \widetilde{Y} \cup_E \mathbb{P}(V) \times \mathbb{P}(V^*)$ and $\mathcal{L}_k$ a line bundle on $\widehat{Y}$ such that $\mathcal{L}_k|_{\widetilde{Y}} = \mathcal{O}_{\widetilde{Y}}(kE)$ and $\mathcal{L}_k|_{\mathbb{P}(V)\times\mathbb{P}(V^*)} = \mathcal{O}(-k,-k)$. By using a correspondence $Y \xleftarrow{\hat{q}} \widehat{Y} \xrightarrow{p} Y^+$, we define functors

$$\begin{aligned} \mathrm{KN}_k &:= R\hat{p}_*(L\hat{q}^*(-) \otimes \mathcal{L}_k) : \mathrm{D^b}(Y) \to \mathrm{D^b}(Y^+) \\ \text{and} \quad \mathrm{KN}'_k &:= R\hat{q}_*(L\hat{p}^*(-) \otimes \mathcal{L}_k) : \mathrm{D^b}(Y^+) \to \mathrm{D^b}(Y). \end{aligned}$$

According to the result of Kawamata and Namikawa [Kaw02, Nam03], the functors KN_k and KN'_k give equivalences between $\mathrm{D^b}(Y)$ and $\mathrm{D^b}(Y^+)$. On the other hand, by using tilting bundles $\mathcal{T}_k$ and $\mathcal{T}^+_{N-k-1}$ above, we get equivalences

$$\Psi_k : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_k) \;\text{ and }\; \Psi^+_{N-k-1} : \mathrm{D^b}(Y^+) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_k).$$

By composing Ψ_k and the inverse of Ψ^+_{N-k-1}, we have an equivalence $\mathrm{D^b}(Y) \to \mathrm{D^b}(Y^+)$. Although this functor seems to be different from the functor KN_k of Kawamata and Namikawa at a glance, we prove the following.

Theorem 3.1.3 (= Thm. 3.5.3)**.** *Our functor*

$$(\Psi^+_{N-k-1})^{-1} \circ \Psi_k \;\textit{(resp. } (\Psi_{N-k-1})^{-1} \circ \Psi^+_k\textit{)}$$

coincides with the Kawamata-Namikawa's functor KN_k *(resp.* KN'_k*).*

We note that our proof of Theorem 3.1.3 gives an alternative proof for the result of Kawamata and Namikawa that states the functors KN_k and KN'_k give equivalences of categories.

The R-algebras Λ_k and Λ_{k-1} are related by the operation that we call *multi-mutation*. We introduce a multi-mutation functor

$$\nu^-_k : \mathrm{D^b}(\Lambda_k) \to \mathrm{D^b}(\Lambda_{k-1})$$

(see Definition 3.5.6) as an analog of Iyama-Wemyss's mutation functor [IW14a] (we call it IW mutation, for short) that Wemyss applied to his framework of "Homological MMP" for 3-folds (see [Wem17]). We show that a multi-mutation functor ν^-_k gives an equivalence of categories. Moreover, we prove that our multi-mutation functor is obtained by composing IW mutation functors $N-1$ times (Theorem 3.5.9[1]). Dually, we introduce a multi-mutation functor $\nu^+_k : \mathrm{D^b}(\Lambda_k) \to \mathrm{D^b}(\Lambda_{k+1})$ and show that a multi-mutation ν^+_k is also a composition of $N-1$ IW mutation functors. Whereas, it is well-known that the derived category $\mathrm{D^b}(Y)$ of a crepant resolution Y has a non-trivial auto-equivalence called *P-twist* (see Definition 2.2.4). We show that a composition of multi-mutations corresponds to a P-twist on $\mathrm{D^b}(Y)$ in the following sense:

Theorem 3.1.4 (= Thm. 3.5.18)**.** *Let*

$$\nu^-_{N+k} : \mathrm{D^b}(\Lambda_{N+k}) \to \mathrm{D^b}(\Lambda_{N+k-1}) \quad \textit{and}$$
$$\nu^+_{N+k-1} : \mathrm{D^b}(\Lambda_{N+k-1}) \to \mathrm{D^b}(\Lambda_{N+k})$$

be multi-mutation functors. Then we have the following diagram of equivalence functors commutes

$$\begin{array}{ccc}
\mathrm{D^b}(Y) & \xrightarrow{\Psi_{N+k}} & \mathrm{D^b}(\Lambda_{N+k}) \\
\downarrow P_k & & \downarrow \nu^-_{N+k} \\
\mathrm{D^b}(Y) & \xrightarrow{\Psi_{N+k-1}} & \mathrm{D^b}(\Lambda_{N+k-1}) \\
\| & & \downarrow \nu^+_{N+k-1} \\
\mathrm{D^b}(Y) & \xrightarrow{\Psi_{N+k}} & \mathrm{D^b}(\Lambda_{N+k}),
\end{array}$$

[1]This statement is suggested by Michael Wemyss in our private communication.

where $P_k : \mathrm{D^b}(Y) \to \mathrm{D^b}(Y)$ *is the P-twist defined by a* $\mathbb{P}^{N-1}$*-object* $j_*\mathcal{O}_{\mathbb{P}(V)}(k)$*.*

This theorem means, under the identification $\Psi_{N+k} : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_{N+k})$, a composition of two multi-mutation functors

$$\nu^+_{N+k-1} \circ \nu^-_{N+k} \in \mathrm{Auteq}(\mathrm{D^b}(\Lambda_{N+k}))$$

corresponds to a P-twist $P_k \in \mathrm{Auteq}(\mathrm{D^b}(Y))$. Donovan and Wemyss proved that, in the case of three dimensional flops, a composition of two IW mutation functors corresponds to a spherical-like twist [DW16]. Our theorem says, in the case of Mukai flops, a composition of **many** IW mutations corresponds to a P-twist.

As a corollary of the theorem above, we can prove the following functor isomorphism that was first proved by Cautis [Cau12b] and later by Addington-Donovan-Meachan [ADM15]. This result gives an example of "flop-flop=twist" results that are widely observed [Tod07, DW16, DW15].

Corollary 3.1.5 (= 3.5.20, cf. [ADM15, Cau12b])**.** *We have a functor isomorphism*

$$\mathrm{KN}_{N+k} \circ \mathrm{KN}'_{-k} \simeq P_k$$

for all $k \in \mathbb{Z}$*.*

3.1.4 Plan of this chapter

In Section 3.2, we provide some basic definitions and recall some fundamental results that we need in later sections. In Section 3.3, we construct an NCCR of a minimal nilpotent orbit closure of type A, and interpret it as the path algebra of a quiver. In Section 3.4, we reconstruct the crepant resolutions from the quiver that gives the NCCR as moduli spaces of representations of the quiver. Furthermore, we study simple representations of the quiver. In Section 3.5, we study derived equivalences of the Mukai flop and P-twists on a crepant resolution via an NCCR.

3.1.5 Notations.

In this chapter, we always work over the complex number field $\mathbb{C}$. Moreover, we adopt the following notations.

- $V = \mathbb{C}^N$: N-dimensional vector space over $\mathbb{C}$ ($N \geq 2$).

3.2 Preliminaries

In the present chapter we study the minimal nilpotent orbit closure $\overline{B(1)}$ of type A.

3.2.1 The variety $\overline{B(1)}$ and its crepant resolutions Y and Y^+

Let $V = \mathbb{C}^N$ be an N-dimensional vector space and $\mathrm{End}_{\mathbb{C}}(V)$ an endomorphism ring of V. Then, the $\mathrm{SL}_N := \mathrm{SL}(N, \mathbb{C})$ acts on $\mathrm{End}_{\mathbb{C}}(V)$ via the adjoint representation

$$\mathrm{Adj} : \mathrm{SL}_N \to \mathrm{GL}(\mathrm{End}_{\mathbb{C}}(V)), \quad A \mapsto (X \mapsto AXA^{-1}).$$

Let X_0 be an matrix in $B(1)$ such that

$$X_0 := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{End}_{\mathbb{C}}(V).$$

Then, we have

$$\mathrm{SL}_N \cdot X_0 = B(1).$$

In the following, we consider homogeneous vector bundles on the orbit $B(1)$. They correspond to linear representations of the stabilizer subgroup $\mathrm{Stab}_{\mathrm{SL}_N}(X_0)$ of SL_N.

Lemma 3.2.1. *The stabilizer subgroup* $\mathrm{Stab}_{\mathrm{SL}_N}(X_0)$ *is given by*

$$\mathrm{Stab}_{\mathrm{SL}_N}(X_0) = \left\{ \left(\begin{array}{c|ccc|c} c & 0 & \cdots & 0 & 0 \\ \hline & & & & 0 \\ * & & A & & \vdots \\ & & & & 0 \\ \hline * & & * & & c \end{array} \right) \;\middle|\; \begin{array}{c} A \in \mathrm{GL}_{N-2}, \\ c \in \mathbb{C} \setminus \{0\}, \\ c^2 \cdot \det(A) = 1 \end{array} \right\}.$$

Proof. Let $A = (a_{ij}) \in \mathrm{SL}_N$. Then, we have

$$AX_0 = \begin{pmatrix} a_{1N} & 0 & \cdots & 0 \\ a_{2N} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{NN} & 0 & \cdots & 0 \end{pmatrix}, \quad X_0A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{11} & a_{12} & \cdots & a_{1N} \end{pmatrix}.$$

Thus, if $AX_0 = X_0A$, we have $a_{11} = a_{NN}$, $a_{12} = \cdots = a_{1N} = 0$, and $a_{2N} = \cdots = a_{N-1,N} = 0$. □

Definition 3.2.2. (1) For $a \in \mathbb{Z}$, we define a character $m_a : \mathrm{Stab}_{\mathrm{SL}_N}(X_0) \to \mathbb{C}^\times$ as

$$\mathrm{Stab}_{\mathrm{SL}_N}(X_0) \ni \left(\begin{array}{c|ccc|c} c & 0 & \cdots & 0 & 0 \\ \hline & & & & 0 \\ * & & A & & \vdots \\ & & & & 0 \\ \hline * & & * & & c \end{array} \right) \mapsto c^{-a} \in \mathbb{C}^\times.$$

(2) Let $\mathcal{M}_a$ be a line bundle on $B(1)$ that corresponds to the character m_a.

(3) We set $M_a := H^0(B(1), \mathcal{M}_a)$. Then M_a is a reflexive R-module.

Next, let us consider a resolution Y of $\overline{B(1)}$. The resolution Y is given by

$$Y := \{(X, L) \in \mathrm{End}_{\mathbb{C}}(V) \times \mathbb{P}(V) \mid X(V) \subset L, X^2 = 0\}$$

and a left SL_N-action on Y is given by

$$A \cdot (X, L) := (AXA^{-1}, AL)$$

for $A \in \mathrm{SL}_N$ and $(X, L) \in Y$. Via the second projection $\pi : Y \to \mathbb{P}(V)$, one can see that Y is isomorphic to the total space of the cotangent bundle $\Omega_{\mathbb{P}(V)}$ on $\mathbb{P}(V)$. Note that the embedding $Y \subset \mathrm{End}_{\mathbb{C}}(V) \times \mathbb{P}(V)$ is determined by a composition of injective bundle maps

$$\Omega_{\mathbb{P}(V)} \subset V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(-1) \subset V^* \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}.$$

Let $j : \mathbb{P}(V) \to Y$ be the zero-section, and then $j(\mathbb{P}(V))$ is given by

$$j(\mathbb{P}(V)) = \{(0, L) \in \mathrm{End}_{\mathbb{C}}(V) \times \mathbb{P}(V)\}.$$

On the other hand, the image of the first projection $\phi : Y \to \mathrm{End}_{\mathbb{C}}(V)$ is just $\overline{B(1)}$, and if we set $U := Y \setminus j(\mathbb{P}(V))$, then, ϕ contracts $j(\mathbb{P}(V))$ to a point $0 \in \overline{B(1)}$, and U is isomorphic to $B(1)$ via the morphism $\phi : Y \to \overline{B(1)}$. Thus, the first projection ϕ gives a resolution of $\overline{B(1)}$. Since the affine variety $\overline{B(1)}$ is a symplectic variety, the canonical divisor of $\overline{B(1)}$ is trivial. On the other hand, since Y is isomorphic to the total space of the cotangent bundle on a projective space, the canonical divisor of Y is also trivial. Thus, the resolution of singularities $\phi : Y \to \overline{B(1)}$ is a crepant resolution, and in this case, is symplectic resolution of $\overline{B(1)}$.

Let us set $\mathcal{O}_Y(a) := \pi^*\mathcal{O}_{\mathbb{P}(V)}(a)$.

Lemma 3.2.3. *Under the identification $U \simeq B(1)$, the homogeneous vector bundle $\mathcal{M}_a$ is isomorphic to $\mathcal{O}_Y(a)|_U$.*

Proof. We first note that $\mathcal{O}_Y(a)|_U$ is a homogeneous line bundle on U. Let $L_0 := X_0(V)$, then L_0 is a line in V. Let $y_0 := (X_0, L_0) \in U$ be a point. The fiber of the line bundle $\mathcal{O}_Y(a)|_U$ at $y_0 \in U$ is canonically isomorphic to $L_0^{\otimes -a}$. Note that the action of $\mathrm{Stab}_{\mathrm{SL}_N}(X_0)$ on L_0 is given by

$$\left(\begin{array}{c|ccc|c} c & 0 & \cdots & 0 & 0 \\ \hline & & & & 0 \\ * & & A & & \vdots \\ & & & & 0 \\ \hline * & & * & & c \end{array}\right) \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ ca_N \end{pmatrix}$$

Therefore, the character $\mathrm{Stab}_{\mathrm{SL}_N}(X_0) \to \mathrm{GL}(L_0^{\otimes -a})$ that determines $\mathcal{O}_Y(a)|_U$ coincides with the one that defines $\mathcal{M}_a$. □

Next, we study the other crepant resolution Y^+ of $\overline{B(1)}$. Let $\mathbb{P}(V^*)$ be a dual projective space, that is

$$\mathbb{P}(V^*) = \{H \subset V \mid H \text{ is a hyperplane in } V\}.$$

The variety Y^+ is defined by

$$Y^+ := \{(X, H) \in \mathrm{End}_{\mathbb{C}}(V) \times \mathbb{P}(V^*) \mid X(V) \subset H, X(H) = 0\}.$$

An SL_N-action on Y^+ is given by $A \cdot (X, H) = (AXA^{-1}, AH)$. Let $\phi^+ : Y \to \overline{B(1)}$ be the first projection and $\pi' : Y^+ \to \mathbb{P}(V^*)$ the second projection. As in the case of Y, Y^+ is isomorphic to the total space of the cotangent bundle $\Omega_{\mathbb{P}(V^*)}$ on $\mathbb{P}(V^*)$ via the second projection $\pi' : Y^+ \to \mathbb{P}(V^*)$, and the first projection $\phi^+ : Y^+ \to \overline{B(1)}$ gives a crepant resolution of $\overline{B(1)}$. The morphism $\phi^+ : Y \to \overline{B(1)}$ contracts the zero section $j' : \mathbb{P}(V^*) \hookrightarrow Y^+$. Let $U^+ := Y^+ \setminus j'(\mathbb{P}(V^*))$ and $\mathcal{O}_{Y^+}(a) := (\pi')^* \mathcal{O}_{\mathbb{P}(V^*)}(a)$.

As in the above, we can show the following.

Lemma 3.2.4. *Under the identification $U^+ \simeq B(1)$, the homogeneous vector bundle $\mathcal{M}_a$ is isomorphic to $\mathcal{O}_{Y^+}(-a)|_{U^+}$.*

3.3 Non-commutative crepant resolutions of $\overline{B(1)}$

3.3.1 The existence of NCCRs of $\overline{B(1)}$ and relations between CRs

In this section, we study non-commutative crepant resolutions of a minimal nilpotent closure $\overline{B(1)} \subset \mathrm{End}(V)$ where $V = \mathbb{C}^N$. We always assume $N \geq 2$. Let R be the affine coordinate ring of $\overline{B(1)}$. By Proposition 2.3.8, the $\mathbb{C}$-algebra R is Gorenstein and normal. Note that $\overline{B(1)} = B(1) \cup \{0\}$ as set and hence we have

$$\mathrm{codim}_{\overline{B(1)}}(\overline{B(1)} \setminus B(1)) = N \geq 2.$$

Thus, we have a $\mathbb{C}$-algebra isomorphism

$$R \simeq H^0(B(1), \mathcal{O}_{B(1)}).$$

Lemma 3.3.1. *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}(V)$ such that*

$$H^i(\mathbb{P}(V), \mathcal{E}(k)) = 0$$

for all $i > 0$ and $k \geq 0$. Then, we have

$$H^i(Y, \pi^* \mathcal{E}) = 0$$

for all $i > 0$.

Proof. Let Z be a total space of a vector bundle $V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(-1)$. Then, Y is embedded in Z via the Euler sequence

$$0 \to \Omega_{\mathbb{P}(V)} \to V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)} \to 0.$$

Since $(V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(-1))/\Omega_{\mathbb{P}(V)} \simeq \mathcal{O}_{\mathbb{P}(V)}$, the ideal sheaf $I_{Y/Z}$ is isomorphic to $\mathcal{O}_Z$. Thus, we have an exact sequence on Z

$$0 \to \mathcal{O}_Z \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0.$$

Let $\pi_Z : Z \to \mathbb{P}(V)$ be the projection. Then, we have

$$\begin{aligned} H^i(Z, \pi_Z^*\mathcal{E}) &\simeq H^i(\mathbb{P}(V), \mathcal{E} \otimes R\pi_{Z*}\mathcal{O}_Z) \\ &\simeq H^i(\mathbb{P}(V), \mathcal{E} \otimes \pi_{Z*}\mathcal{O}_Z) \quad \text{(since } \pi_Z \text{ is affine)} \\ &\simeq \bigoplus_{k \geq 0} \operatorname{Sym}^k V \otimes_{\mathbb{C}} H^i(\mathbb{P}(V), \mathcal{E}(k)) \end{aligned}$$

and this is zero for $i > 0$ by the assumption. Thus we have

$$H^i(Z, \pi_Z^*\mathcal{E} \otimes \mathcal{O}_Y) = H^i(Y, \pi^*\mathcal{E}) = 0$$

for $i > 0$. ☐

Definition 3.3.2. For an integer $k \in \mathbb{Z}$, let $\mathcal{T}_k := \bigoplus_{a=-N+k+1}^{k} \mathcal{O}_Y(a)$ be a vector bundle on Y and $\Lambda_k := \operatorname{End}_Y(\mathcal{T}_k)$ the endomorphism ring of $\mathcal{T}_k$.

Note that the R-algebra structure of Λ_k does not depend on the choice of the integer k. Nevertheless, we adopt this notation to emphasize that the algebra Λ_k is given as the endomorphism ring of a bundle $\mathcal{T}_k$.

Theorem 3.3.3. *The following hold.*

(1) For all $k \in \mathbb{Z}$, the vector bundle $\mathcal{T}_k$ is a tilting bundle on Y.

(2) For all $-N+1 \leq a \leq N-1$, we have

$$\phi_*\mathcal{O}_Y(a) = M_a,$$

and M_a is a (maximal) Cohen-Macaulay R-module.

(3) If $0 \leq k \leq N-1$, then we have an isomorphism

$$\operatorname{End}_Y(\mathcal{T}_k) \simeq \operatorname{End}_R\left(\bigoplus_{a=-N+k+1}^{k} M_a\right)$$

(4) The R-module

$$\bigoplus_{a=-N+k+1}^{k} M_a$$

gives an NCCR Λ_k of R for $0 \leq k \leq N-1$.

(5) There is an equivalence of categories

$$\mathrm{RHom}_Y(\mathcal{T}_k, -) : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_k).$$

We note that (1) and (5) of Theorem 3.3.3 are also obtained by Toda and Uehara in [TU10]. They also study the perverse heart of $\mathrm{D^b}(Y)$ that corresponds to $\mathrm{mod}(\Lambda_0)$ via the derived equivalence.

Proof. Let $T = \bigoplus_{a=0}^{N-1} \mathcal{O}_{\mathbb{P}(V)}(a)$ is a tilting bundle on $\mathbb{P}(V)$. Then, we have

$$H^i(\mathbb{P}(V), T^* \otimes T \otimes \mathcal{O}_{\mathbb{P}(V)}(k)) = 0$$

for all $i > 0$ and $k \geq 0$. Thus, by Lemma 3.3.1, we have

$$H^i(Y, \mathcal{T}_0^* \otimes \mathcal{T}_0) = 0$$

and hence $\mathcal{T}_0$ is a tilting bundle on Y. Since other bundles $\mathcal{T}_k$ $(k \in \mathbb{Z})$ are obtained from $\mathcal{T}_0$ by twisting $\mathcal{O}_Y(k)$, $\mathcal{T}_k$ $(k \in \mathbb{Z})$ are also tilting bundles on Y. This shows (1).

On the other hand, by Lemma 3.3.1, we have

$$H^i(Y, \mathcal{O}_Y(a)) = 0 \quad \text{for } i > 0$$

if $a \geq -N + 1$. Therefore, if $-N + 1 \leq a \leq N - 1$, we have

$$H^i(Y, \mathcal{O}_Y(a)) = 0,$$
$$\mathrm{Ext}_Y^i(\mathcal{O}_Y(a), \mathcal{O}_Y) = 0$$

for all $i > 0$. Thus, by Lemma 2.1.15, we have the R-module $\phi_*\mathcal{O}_Y(a) = H^0(Y, \mathcal{O}_Y(a))$ is Cohen-Macaulay if $-N + 1 \leq a \leq N - 1$. In particular, if $-N + 1 \leq a \leq N - 1$, $\phi_*\mathcal{O}_Y(a)$ is a reflexive R-module by Proposition 2.1.20. By Lemma 3.2.3, $\phi_*\mathcal{O}_Y(a)$ and M_a are isomorphic outside the unique singular point $0 \in \overline{B(1)}$. Thus, we have $\phi_*\mathcal{O}_Y(a) \simeq M_a$ for $-N + 1 \leq a \leq N - 1$ and hence M_a is (maximal) Cohen-Macaulay as an R-module if $-N+1 \leq a \leq N-1$. This shows (2).

The statement of (3) follows from Lemma 2.1.17 and (1).

Finally, (4) follows from (1), (2), and (3). (5) follows from (1). □

It is easy to see that the dual statements hold for Y^+.

Theorem 3.3.4. *Let $\mathcal{T}_k^+ := \bigoplus_{a=-N+k+1}^{k} \mathcal{O}_{Y^+}(a)$. Then, the following hold.*

(1) For all $k \in \mathbb{Z}$, the vector bundle $\mathcal{T}_k^+$ is a tilting bundle on Y^+.

(2) For all $-N + 1 \leq a \leq N - 1$, we have

$$\phi_*^+\mathcal{O}_{Y^+}(a) = M_{-a}.$$

(3) If $0 \leq k \leq N-1$, then we have an isomorphism

$$\mathrm{End}_{Y^+}(\mathcal{T}_k^+) \simeq \mathrm{End}_R\left(\bigoplus_{a=-N+k+1}^{k} M_{-a}\right).$$

(4) For all $k \in \mathbb{Z}$, there is a canonical isomorphism

$$\mathrm{End}_{Y^+}(\mathcal{T}_k^+) \simeq \Lambda_{N-k-1}.$$

(5) There is an equivalence of categories

$$\mathrm{RHom}_{Y^+}(\mathcal{T}_k^+, -) : \mathrm{D^b}(Y^+) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_{N-k-1}).$$

Proof. We only show (4). By Lemma 3.2.3 and Lemma 3.2.4, we have $\Lambda_k = \mathrm{End}_Y(\mathcal{T}_k)$ and $\mathrm{End}_{Y^+}(\mathcal{T}_{N-k-1}^+)$ are isomorphic to each other on the smooth locus $B(1)$. Since both algebras are Cohen-Macaulay as R-modules and hence are reflexive, we have an isomorphism

$$\Lambda_k = \mathrm{End}_Y(\mathcal{T}_k) \simeq \mathrm{End}_{Y^+}(\mathcal{T}_{N-k-1}^+).$$

This is what we want. □

3.3.2 NCCRs as the path algebra of a quiver

The aim of this subsection is to describe the NCCR Λ_k of $\overline{B(1)}$ as the path algebra of a quiver with relations.

As in the above subsection, let Z be the total space of a vector bundle $V^* \otimes \mathcal{O}_{\mathbb{P}(V)}(-1)$. Let $\pi_Z : Z \to \mathbb{P}(V)$ the projection, and we set $\mathcal{O}_Z(a) := \pi_Z^* \mathcal{O}_{\mathbb{P}(V)}(a)$, $\mathcal{T}_Z := \bigoplus_{a=-N+1}^{0} \mathcal{O}_Z(a)$, and $\Lambda_Z := \mathrm{End}_Z(\mathcal{T}_Z)$. Then, the algebra Λ_k is a quotient algebra of Λ_Z. First, we describe the non-commutative algebra Λ_Z as the path algebra of a quiver with certain relations.

Note that Z is a crepant resolution of an affine variety $\mathrm{Spec}\, H^0(Z, \mathcal{O}_Z)$. We set $\widetilde{R} := H^0(Z, \mathcal{O}_Z)$. Then, the algebra $\widetilde{R}$ is described as follows.

$$\begin{aligned}\widetilde{R} &:= H^0(Z, \mathcal{O}_Z)\\ &\simeq H^0(\mathbb{P}(V), \bigoplus_{k\geq 0} \mathrm{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(k))\\ &\simeq \bigoplus_{k\geq 0} \mathrm{Sym}^k V \otimes_{\mathbb{C}} \mathrm{Sym}^k V^*\end{aligned}$$

Let S be the affine coordinate ring of $\mathrm{End}_{\mathbb{C}}(V)$, i.e.

$$S := \bigoplus_{k\geq 0} \mathrm{Sym}^k (V \otimes_{\mathbb{C}} V^*).$$

Let $v_1, \cdots, v_N$ be the standard basis of $V = \mathbb{C}^N$ and $f_1, \ldots, f_N \in V^*$ the dual basis. If we set $x_{ij} := v_j \otimes f_i$, the algebra S is isomorphic to the polynomial ring with N^2 variables

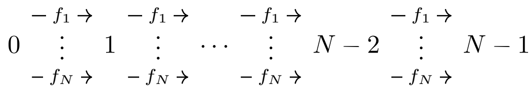
$$S \simeq \mathbb{C}[(x_{ij})_{i,j=1,\ldots,N}].$$

The affine variety $\operatorname{Spec} \widetilde{R}$ is embedded in $\operatorname{End}_{\mathbb{C}}(V) = \operatorname{Spec} S$ via the canonical surjective homomorphism of algebras

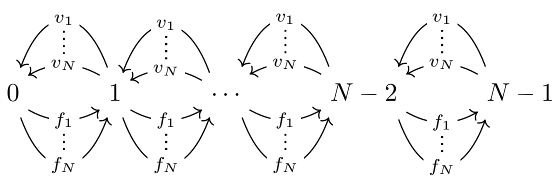
$$S := \bigoplus_{k \geq 0} \operatorname{Sym}^k (V \otimes_{\mathbb{C}} V^*) \twoheadrightarrow \bigoplus_{k \geq 0} \operatorname{Sym}^k V \otimes_{\mathbb{C}} \operatorname{Sym}^k V^*.$$

Next, we define quivers that we use later.

Definition 3.3.5. Let Γ be the *Beilinson quiver*



and $\widetilde{\Gamma}$ the *double Beilinson quiver*



Here, v_i, f_j serve as the label for N different arrows.

Next, we show that the non-commutative algebra Λ_Z has a description as the path algebra of the double Beilinson quiver with certain relations.

Theorem 3.3.6. *The non-commutative algebra Λ_Z is isomorphic to the path algebra $S\widetilde{\Gamma}$ of the double Beilinson quiver $\widetilde{\Gamma}$ over S with relations*

$$\begin{aligned} v_i v_j = v_j v_i \quad &\textit{for all } 1 \leq i, j \leq N \\ f_i f_j = f_j f_i \quad &\textit{for all } 1 \leq i, j \leq N \\ v_j f_i = f_i v_j = x_{ij} \quad &\textit{for all } 1 \leq i, j \leq N. \end{aligned}$$

Proof. First, for $a, b \in \mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned} \operatorname{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(b)) &\simeq \operatorname{Hom}_{\mathbb{P}(V)}(\mathcal{O}_{\mathbb{P}(V)}(a), (\pi_Z)_* \mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}(V)}(b)) \\ &\simeq \operatorname{Hom}_{\mathbb{P}(V)} \left(\mathcal{O}_{\mathbb{P}(V)}(a), \left(\bigoplus_{k \geq 0} \operatorname{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(1) \right) \otimes \mathcal{O}_{\mathbb{P}(V)}(b) \right). \\ &\simeq \operatorname{Hom}_{\mathbb{P}(V)} \left(\mathcal{O}_{\mathbb{P}(V)}(a), \bigoplus_{k \geq 0} \operatorname{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(b+k) \right). \end{aligned}$$

Moreover, if $b \geq a$, we have

$$\mathrm{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(b)) \simeq \bigoplus_{k \geq 0} \mathrm{Sym}^k V \otimes_{\mathbb{C}} \mathrm{Sym}^{k+b-a} V^*,$$

and if $b \leq a$, we have

$$\mathrm{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(b)) \simeq \bigoplus_{k \geq 0} \mathrm{Sym}^{k+a-b} V \otimes_{\mathbb{C}} \mathrm{Sym}^k V^*.$$

We define the action $v : \mathcal{O}_Z(a) \to \mathcal{O}_Z(a-1)$ of $v \in V$ on $\mathcal{T}_Z$ as a morphism that correspond to a morphism

$$\mathcal{O}_{\mathbb{P}(V)}(a) \to v \otimes \mathcal{O}_{\mathbb{P}(V)}(a) \subset V \otimes \mathcal{O}_{\mathbb{P}(V)}(a) \subset \bigoplus_{k \geq 0} \mathrm{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k-1)$$

via the adjunction. This morphism $v : \mathcal{O}_Z(a) \to \mathcal{O}_Z(a-1)$ corresponds to an element

$$v \otimes 1 \in V \otimes_{\mathbb{C}} \mathbb{C} \subset \bigoplus_{k \geq 0} \mathrm{Sym}^{k+1} V \otimes_{\mathbb{C}} \mathrm{Sym}^k V^*$$

via the isomorphism

$$\mathrm{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(a-1)) \simeq \bigoplus_{k \geq 0} \mathrm{Sym}^{k+1} V \otimes_{\mathbb{C}} \mathrm{Sym}^k V^*.$$

We also define the action $f : \mathcal{O}_Z(a) \to \mathcal{O}_Z(a+1)$ of $f \in V^*$ on $\mathcal{T}_Z$ as the morphism that is the pull-back of the morphism

$$f : \mathcal{O}_{\mathbb{P}(V)}(a) \to \mathcal{O}_{\mathbb{P}(V)}(a+1)$$

by $\pi_Z : Z \to \mathbb{P}(V)$. Note that this morphism corresponds to a morphism

$$\mathcal{O}_{\mathbb{P}(V)}(a) \xrightarrow{f} \mathcal{O}_{\mathbb{P}(V)}(a+1) \subset \bigoplus_{k \geq 0} \mathrm{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k+1)$$

via the adjunction, and also corresponds to an element

$$1 \otimes f \in \mathbb{C} \otimes_{\mathbb{C}} V^* \subset \bigoplus_{k \geq 0} \mathrm{Sym}^k V \otimes_{\mathbb{C}} \mathrm{Sym}^{k+1} V^*$$

via the isomorphism

$$\mathrm{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(a+1)) \simeq \bigoplus_{k \geq 0} \mathrm{Sym}^k V \otimes_{\mathbb{C}} \mathrm{Sym}^{k+1} V^*.$$

Now, it is clear that $v_1, \dots, v_N$ and $f_1, \dots, f_N$ generate Λ_Z as a S-algebra and satisfy the commutative relation

$$v_i v_j = v_j v_i$$
$$f_i f_j = f_j f_i$$

for any $i, j = 1, \dots, N$.

Next, we check that the relation

$$f_i v_j = v_j f_i = x_{ij}$$

is satisfied. By adjunction, the map

$$f_i v_j : \mathcal{O}_Z(a) \to \mathcal{O}_Z(a)$$

corresponds to the composition

$$\mathcal{O}_{\mathbb{P}(V)}(a) \to v_j \otimes \mathcal{O}_{\mathbb{P}(V)}(a) \subset \bigoplus_{k \geq 0} \operatorname{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k-1)$$
$$\xrightarrow{(\pi_Z)_* f_i} \bigoplus_{k \geq 0} \operatorname{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k),$$

where the map $(\pi_Z)_* f_i$ is the direct sum of the maps

$$\operatorname{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k-1) \xrightarrow{\mathrm{id} \otimes f_i} \operatorname{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k).$$

Thus, this map factors through as

$$\mathcal{O}_{\mathbb{P}(V)}(a) \to v_j \otimes \mathcal{O}_{\mathbb{P}(V)}(a) \xrightarrow{\mathrm{id} \otimes f_i} v_j \otimes \mathcal{O}_{\mathbb{P}(V)}(a+1) \subset \bigoplus_{k \geq 0} \operatorname{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k).$$

Similarly, the map

$$v_j f_i : \mathcal{O}_Z(a) \to \mathcal{O}_Z(a)$$

corresponds to the composition

$$\mathcal{O}_{\mathbb{P}(V)}(a) \xrightarrow{f_i} \mathcal{O}_{\mathbb{P}(V)}(a+1) \subset \bigoplus_{k \geq 0} \operatorname{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k+1)$$
$$\xrightarrow{(\pi_Z)_* v_j} \bigoplus_{k \geq 0} \operatorname{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k)$$

by adjunction, where the map $(\pi_Z)_* v_j$ is the direct sum of maps

$$\operatorname{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k) \xrightarrow{v_j \otimes \mathrm{id}} \operatorname{Sym}^{k+1} V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k).$$

Thus, this map factors through as

$$\mathcal{O}_{\mathbb{P}(V)}(a) \xrightarrow{f_i} \mathcal{O}_{\mathbb{P}(V)}(a+1) \xrightarrow{v_j \otimes \mathrm{id}} v_j \otimes \mathcal{O}_{\mathbb{P}(V)}(a+1) \subset \bigoplus_{k \geq 0} \operatorname{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k).$$

Thus, $f_i v_j$ and $v_j f_i$ defines the same element in $\operatorname{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(a))$, and they correspond to an element

$$x_{ij} = v_j \otimes f_i \in V \otimes V^* \subset \bigoplus_{k \geq 0} \operatorname{Sym}^k V \otimes \operatorname{Sym}^k V^* (= \widetilde{R})$$

via the isormorpshim

$$\mathrm{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(a)) \simeq \bigoplus_{k\geq 0} \mathrm{Sym}^k V \otimes \mathrm{Sym}^k V^*.$$

Thus, we have the relation

$$f_i v_j = v_j f_i = x_{ij}.$$

It is clear that $v_1, \ldots, v_N$ and $f_1, \ldots, f_N$ do not have other relations. Therefore, we have the result. □

The following is one of main theorems in this chapter.

Theorem 3.3.7. *The non-commutative algebra Λ_k is isomorphic to the path algebra $S\widetilde{\Gamma}$ of the double Beilinson quiver $\widetilde{\Gamma}$ with relations*

$$\begin{aligned} v_i v_j &= v_j v_i \quad \textit{for all } 1 \leq i, j \leq N, \\ f_i f_j &= f_j f_i \quad \textit{for all } 1 \leq i, j \leq N, \\ v_j f_i &= f_i v_j = x_{ij} \quad \textit{for all } 1 \leq i, j \leq N, \\ \textit{and} \quad \sum_{i=1}^N f_i v_i &= 0 = \sum_{i=1}^N v_i f_i \end{aligned}$$

Proof. By the exact sequence

$$0 \to \mathcal{O}_Z \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0,$$

we have an exact sequence

$$0 \to \Lambda_Z \xrightarrow{\iota} \Lambda_Z \to \Lambda_k \to 0.$$

Note that the map $\iota : \Lambda_Z \to \Lambda_Z$ is given by the multiplication of $\sum_{i=1}^N x_{ii} = \sum_{i=1}^N v_i \otimes f_i \in S$. Thus, the result follows from Theorem 3.3.6. □

Remark 3.3.8. If we work over the base field $\mathbb{C}$ instead of S, we have

$$\Lambda_k \simeq \mathbb{C}\widetilde{\Gamma}/J'$$

and J' is an ideal that is generated by

$$\begin{gathered} v_i v_j = v_j v_i, \ f_i f_j = f_j f_i, \ v_j f_i = f_i v_j, \\ f_k v_j f_i = f_i v_j f_k, \ v_j f_i v_l = v_l f_i v_j \\ \sum_{i=1}^N f_i v_i = 0 = \sum_{i=1}^N v_i f_i. \end{gathered}$$

The isomorphism $S\widetilde{\Gamma}/J \to \mathbb{C}\widetilde{\Gamma}/J'$ is given by $v_i \mapsto v_i, \ f_i \mapsto f_i, \ x_{ij} \mapsto v_j f_i$.

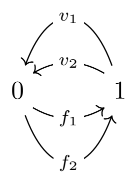
Example 3.3.9. Let us consider the case $N = 2$. In this case, the affine surface $\overline{B(1)}$ is given by

$$\overline{B(1)} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a^2 + bc = 0 \right\},$$

and hence has a Du Val singularity of type A_1 at the origin. The resolution $Y = |\Omega_{\mathbb{P}^1}| \to \overline{B(1)}$ is the minimal resolution, and the NCCR Λ_k is isomorphic to the smash product $\mathbb{C}[x, y]\sharp G$, where G is a subgroup of SL_2

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \subset \mathrm{SL}_2 .$$

The quiver that gives the NCCR Λ_k is given by



and the relations (over $\mathbb{C}$) are given by $f_1 v_1 + f_2 v_2 = 0$, $v_1 f_1 + v_2 f_2 = 0$.

This quiver (with relations) coincides with the one that is described in Weyman and Zhao's paper [WZ12, Example 6.15]. In [WZ12, Section 6], Weyman and Zhao studied a description of an NCCR of a (maximal) determinantal variety of symmetric matrices as the path algebra of a quiver. Since the surface $\overline{B(1)}$ is isomorphic to a (maximal) determinantal variety of symmetric matrices

$$\left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid ac - b^2 = 0 \right\},$$

they obtained the above description of Λ_k as a special case.

3.3.3 Remark: Alternative NCCRs of $\overline{B(1)}$

The NCCR Λ_k of $\overline{B(1)}$ that is constructed in the above subsection came from the Beilinson collection of $\mathbb{P}(V)$

$$\mathrm{D^b}(\mathbb{P}(V)) = \langle \mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(N-1) \rangle.$$

In this subsection, we construct an NCCR of R of another type from the different Beilinson collection

$$\mathrm{D^b}(\mathbb{P}(V)) = \langle \Omega^{N-1}(N), \Omega^{N-2}(N-1), \cdots, \Omega^1(2), \mathcal{O}(1) \rangle.$$

Definition 3.3.10. (1) We define a representation $n_1 : \mathrm{Stab}_{\mathrm{SL}_N}(X_0) \to \mathrm{SL}_{N-1}$ as

$$\mathrm{Stab}_{\mathrm{SL}_N}(X_0) \ni \left(\begin{array}{c|ccc|c} c & 0 & \cdots & 0 & 0 \\ \hline & & & & 0 \\ * & & A & & \vdots \\ & & & & 0 \\ \hline * & & * & & c \end{array}\right) \mapsto \left(\begin{array}{c|ccc} c & 0 & \cdots & 0 \\ \hline & & & \\ * & & A & \\ & & & \end{array}\right) \in \mathrm{SL}_{N-1}.$$

For $0 \le a \le N-1$, we define a representation n_a by

$$n_a := \bigwedge^a n_1.$$

(2) Let $\mathcal{N}_a$ be a vector bundle on $B(1)$ that corresponds to the representation n_a.

(3) We set $N_a := H^0(B(1), \mathcal{N}_a)$. Then N_a is a reflexive R-module.

(4) We define an R-algebra Λ' by

$$\Lambda' := \mathrm{End}_R \left(\bigoplus_{a=0}^{N-1} N_a \right).$$

As in Lemma 3.2.3, we can relate the homogeneous vector bundle $\mathcal{N}_a$ with a (co)tangent bundle on a projective space. We note that we have an isomorphism between vector bundles on $\mathbb{P}(V)$

$$\begin{aligned} \bigwedge^a (T_{\mathbb{P}(V)}(-1)) &\simeq (\Omega^a_{\mathbb{P}(V)})^*(-a) \\ &\simeq \Omega^{N-a-1}_{\mathbb{P}(V)}(N) \otimes \mathcal{O}(-a) \\ &\simeq \Omega^{N-a-1}_{\mathbb{P}(V)}(N-a). \end{aligned}$$

Here, $T_{\mathbb{P}(V)}$ is the tangent bundle on $\mathbb{P}(V)$ and $\Omega_{\mathbb{P}(V)}$ is the cotangent bundle on $\mathbb{P}(V)$.

Lemma 3.3.11. *We have* $\pi^* \Omega^{a-1}_{\mathbb{P}(V)}(a)|_U \simeq \mathcal{N}_{N-a}$.

The proof is completely same as in Lemma 3.2.3.

We want to show that the algebra Λ' is an NCCR of R. In order to show this, we need the following lemma.

Lemma 3.3.12 ([BLV10], Corollary 3.24)**.** *Put*

$$\mathcal{M}^b_a(-c) := \mathcal{H}om_{\mathbb{P}(V)}(\Omega^{b-1}_{\mathbb{P}(V)}(b), \Omega^{a-1}_{\mathbb{P}(V)}(a))(-c).$$

Then, the cohomology $H^d(\mathbb{P}(V), \mathcal{M}^b_a(-c))$ *is not zero only in the following cases:*

(1) *If* $d - c > 0$*, then* $d = 0$ *and, necessarily,* $c < 0$.

(2) *If* $d - c = 0$*, then* $c + b \in [\max\{a, b\}, \min\{N, a + b - 1\}]$.

(3) *If* $d - c = -1$*, then* $c - a \in [\max\{0, N - a - b - 1\}, \min\{N - b, N - a\}]$.

(4) *If* $d - c < -1$*, then* $d = N - 1$*, and necessarily,* $c > N$.

In particular, if $c \leq 0$*, we have* $H^d(\mathbb{P}(V), \mathcal{M}_a^b(-c)) = 0$ *for all* $d > 0$.

From this lemma, we can obtain the following corollaries.

Corollary 3.3.13. *For* $0 \leq a \leq N - 1$*, we have* $N_a \simeq \phi_* \pi^* \Omega_{\mathbb{P}(V)}^{N-a-1}(N - a)$ *and* N_a *is Cohen-Macaulay.*

Proof. Let $k \geq 0$ be a non-negative integer. Note that, $\Omega_{\mathbb{P}(V)}^{N-a-1}(N-a) \otimes \mathcal{O}(k) \simeq \mathcal{M}_{N-a}^N(k)$ and

$$\begin{aligned}(\Omega_{\mathbb{P}(V)}^{N-a-1}(N-a))^* \otimes \mathcal{O}(k) &\simeq \Omega_{\mathbb{P}(V)}^a(N) \otimes \mathcal{O}(-N+a) \otimes \mathcal{O}(k) \\ &\simeq \Omega_{\mathbb{P}(V)}^a(a+1) \otimes \mathcal{O}(k-1) \\ &\simeq \mathcal{M}_{a+1}^1(k).\end{aligned}$$

Thus, by Lemma 3.3.12 and Lemma 3.3.1, we have

$$H^i(Y, \pi^* \Omega_{\mathbb{P}(V)}^{N-a-1}(N-a)) = 0 = H^i(Y, \pi^*(\Omega_{\mathbb{P}(V)}^{N-a-1}(N-a))^*)$$

for $i > 0$, and hence by Lemma 2.1.15, we have the R-module $\phi_* \pi^* \Omega_{\mathbb{P}(V)}^{N-a-1}(N-a)$ is maximal Cohen-Macaulay. In particular, $\phi_* \pi^* \Omega_{\mathbb{P}(V)}^{N-a-1}(N-a)$ is reflexive and hence we have the desired isomorphism. □

Corollary 3.3.14. *The bundle*

$$\mathcal{T}' := \bigoplus_{a=1}^{N} \pi^* \Omega_{\mathbb{P}(V)}^{a-1}(a)$$

is a tilting bundle on Y *and there is an isomorphism as* R*-algebras*

$$\Lambda' \simeq \mathrm{End}_Y(\mathcal{T}').$$

In particular, the R*-module* $\bigoplus_{a=0}^{N-1} N_a$ *gives an NCCR* Λ' *of* R.

Proof. The bundle $(\mathcal{T}')^* \otimes \mathcal{T}'$ is the direct sum of $\pi^* \mathcal{M}_a^b(0)$. By Lemma 3.3.12 and Lemma 3.3.1, we have

$$H^i(Y, \pi^* \mathcal{M}_a^b(0)) = 0$$

for $i > 0$ and hence we have

$$\mathrm{Ext}_Y^i(\mathcal{T}', \mathcal{T}') = H^i(Y, (\mathcal{T}')^* \otimes \mathcal{T}') = 0$$

for $i > 0$. It is clear that the bundle generates the category $\mathrm{D}(\mathrm{Qcoh}(Y))$. Therefore, the bundle $\mathcal{T}'$ is tilting. □

Corollary 3.3.15. *Let us assume $N \geq 3$. In this case, although the two NCCRs Λ_k, Λ' of R are not isomorphic to each other, there is an equivalence of categories*

$$\mathrm{D^b}(Y) \simeq \mathrm{D^b}(\Lambda_k) \simeq \mathrm{D^b}(\Lambda').$$

Proof. The R-rank of the first NCCR Λ_k is just $2N$ and the R-rank of the second NCCR Λ' is

$$2 \sum_{a=1}^{N} \operatorname{rank} \Omega_{\mathbb{P}(V)}^{a-1} = 2^N.$$

Thus, if $N \geq 3$, Λ_k and Λ' are not isomorphic to each other but have the equivalent derived categories, where the equivalence is given by the composition

$$\mathrm{D^b}(\Lambda') \xrightarrow{- \otimes_{\Lambda'} \mathcal{T}'} \mathrm{D^b}(Y) \xrightarrow{\mathrm{RHom}_Y(\mathcal{T}_k, -)} \mathrm{D^b}(\Lambda_k).$$

This shows the result. □

At the end of this subsection, we give another type of tilting bundles that we use in the later section (Section 3.5.2).

Proposition 3.3.16. *The vector bundle*

$$\mathcal{S}_k = \bigoplus_{a=-N+2}^{0} \mathcal{O}_Y(a) \oplus \left(\pi^* \Omega_{\mathbb{P}(V)}^{k} \otimes \mathcal{O}_Y(1) \right)$$

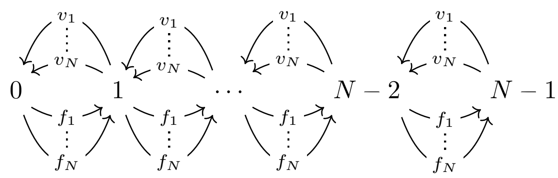
and its dual vector bundle $\mathcal{S}_k^*$ are tilting bundle on Y for all $0 \leq k \leq N-1$.

Proof. As in Lemma 3.3.14, the claim follows from direct computations using Lemma 3.3.12. □

3.4 From an NCCR to crepant resolutions

3.4.1 Main theorem

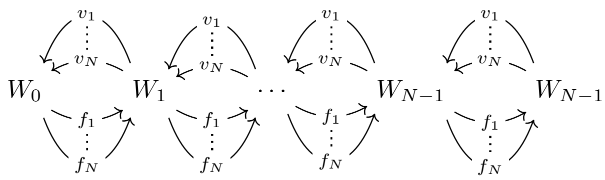
In this section, we recover the crepant resolutions Y and Y^+ of $\overline{B(1)}$ from the NCCR Λ_k. Again, let $\widetilde{\Gamma}$ be the double Beilinson quiver



with relations

$$\begin{aligned} v_iv_j &= v_jv_i \quad \text{for all } 1 \le i,j \le N, \\ f_if_j &= f_jf_i \quad \text{for all } 1 \le i,j \le N, \\ v_jf_i &= f_iv_j = x_{ij} \quad \text{for all } 1 \le i,j \le N, \\ \text{and} \quad \sum_{i=1}^{N} f_iv_i &= 0 = \sum_{i=1}^{N} v_if_i. \end{aligned}$$

For a commutative $\mathbb{C}$-algebra A, let $\widetilde{\mathcal{R}}(A)$ the set of representations W of the quiver $\widetilde{\Gamma}$ (with the above relations)



such that, for each i, W_i is a (constant) rank 1 projective A-module and W is generated by $W_0 = A$.

The goal of this section is to show the following theorem.

Theorem 3.4.1 (cf. [VdB04b], Section 6)**.** *Y is the fine moduli space of the functor $\widetilde{\mathcal{R}}$. The universal bundle is $\mathcal{T}_{N-1}$.*

Recall that the NCCR Λ_k is isomorphic to the path algebra $S\widetilde{\Gamma}/J$ where J is the ideal generated by the above relations. Therefore, Theorem 3.4.1 means that we can recover a crepant resolution Y of $\overline{B(1)}$ (and a tilting bundle on Y) from the NCCR Λ_k as a moduli space of Λ_k-modules (and its universal bundle). The other crepant resolution Y^+ is also recovered as the fine moduli space of another functor $\widetilde{\mathcal{R}}^+$ (see Remark 3.4.10).

3.4.2 Projective module of rank 1

Let A be a (commutative, noetherian) $\mathbb{C}$-algebra. In this subsection, we recall some basic properties of projective A-modules of (constant) rank 1. First, we recall the following fundamental result for projective modules. One can find the following proposition in Chapter II, §5, 2, Theorem 1 of [Bourbaki].

Proposition 3.4.2. *Let M be a finitely generated A-module. Then, the following are equivalent.*

(i) M is projective.

(ii) For all $\mathfrak{p} \in \operatorname{Spec} A$, there exists a non-negative integer $r(\mathfrak{p}) \in \mathbb{Z}_{\ge 0}$ such that $M_\mathfrak{p} \simeq A_\mathfrak{p}^{r(\mathfrak{p})}$.

(iii) There exist $f_1, \ldots, f_r \in A$ such that they generate the unite ideal of A and M_{f_i} is a free A_{f_i}-module for each i.

From this proposition, we have the following.

Corollary 3.4.3. *Let M be a finitely generated A-module. Then, the following are equivalent.*

(1) The sheaf on $\operatorname{Spec} A$ that associates to M is an invertible sheaf.

(2) M is a projective A-module of constant rank 1.

Thus, if we consider projective modules of constant rank 1, the symmetric product of them coincides with the tensor product.

Lemma 3.4.4. *Let P be a (finitely generated) projective A-module of (constant) rank 1. Let $\mathfrak{S}_k$ be a group of permutations of the set $\{1, 2, \ldots, k\}$. Then, for any $m_1, m_2, \ldots, m_k \in P$ and any $\sigma \in \mathfrak{S}_k$, we have*

$$m_1 \otimes m_2 \otimes \cdots \otimes m_k = m_{\sigma(1)} \otimes m_{\sigma(2)} \otimes \cdots \otimes m_{\sigma(k)}$$

in $P^{\otimes k}$. In particular, we have

$$P^{\otimes k} \simeq \operatorname{Sym}_A^k P$$

as an A-module.

Proof. This is the direct consequence of Proposition 3.4.2 (iii) and the gluing property of sheaves. □

Corollary 3.4.5. *Let P be a (finitely generated) projective A-module of (constant) rank 1. For any $u \in P^\vee = \operatorname{Hom}_A(P, A)$ and $m_1 \ldots m_k \in \operatorname{Sym}_A^k P$, we have*

$$u(m_i) \cdot m_1 \cdots \widehat{m_i} \cdots m_j \cdots m_k = u(m_j) \cdot m_1 \cdots m_i \cdots \widehat{m_j} \cdots m_k$$

in $\operatorname{Sym}_A^{k-1} P$ for all $1 \leq i < j \leq k$. In particular, the map

$$\operatorname{Sym}_A^k P \to \operatorname{Sym}_A^{k-1} P, \qquad m_1 \ldots m_k \mapsto u(m_i) \cdot m_1 \cdots \widehat{m_i} \cdots m_k$$

is well-defined and does not depend on the choice of i.

Corollary 3.4.5 will be used in Section 3.4.4 to construct a representation of $\widetilde{\Gamma}$ from a projective module P of constant rank 1.

3.4.3 An easy case

In order to prove Theorem 3.4.1, we first study an easier functor $\mathcal{R}$. For commutative $\mathbb{C}$-algebra A, let $\mathcal{R}(A)$ be the set of representation W of Beilinson quiver Γ

$$W_0 \;\begin{matrix} \xrightarrow{f_1} \\ \vdots \\ \xrightarrow{f_N} \end{matrix}\; W_1 \;\begin{matrix} \xrightarrow{f_1} \\ \vdots \\ \xrightarrow{f_N} \end{matrix}\; \cdots \;\begin{matrix} \xrightarrow{f_1} \\ \vdots \\ \xrightarrow{f_N} \end{matrix}\; W_{N-2} \;\begin{matrix} \xrightarrow{f_1} \\ \vdots \\ \xrightarrow{f_N} \end{matrix}\; W_{N-1}$$

with usual relations

$$f_i f_j = f_j f_i \quad (i, j = 1, \ldots, N)$$

such that each W_i is rank 1 projective A-module and W is generated by $W_0 = A$.

Let us consider a rank 1 projective A-module P and split injective morphism $\alpha : P \to V \otimes_{\mathbb{C}} A$. For the pair (P, α), we define a representation W_α of Γ as follows. Let $(W_\alpha)_k := \mathrm{Sym}_A^k P^\vee$ where $P^\vee := \mathrm{Hom}_A(P, A)$ is the dual of P. The action of $f \in V$ is defined by

$$f : \mathrm{Sym}_A^k P^\vee \to \mathrm{Sym}_A^{k+1} P^\vee, \quad u^1 \cdots u^k \mapsto \alpha^\vee(f) u^1 \cdots u^k.$$

By construction, we have $W_\alpha \in \mathcal{R}(A)$.

Proposition 3.4.6. *For any $W \in \mathcal{R}(A)$, there exists a unique pair (P, α) as above such that $W \simeq W_\alpha$.*

Proof. Let $W \in \mathcal{R}(A)$. Since W is generated by the first component $W_0 = A$, we have a surjective morphism

$$\pi : V^* \otimes_{\mathbb{C}} A \to W_1.$$

Since W_1 is a projective A-module, the morphism π is split surjection. If $W = W_\alpha$ for some (P, α), then we have $P = W_1^\vee = \mathrm{Hom}_A(W_1, A)$ and $\alpha = \pi^\vee = \mathrm{Hom}_A(\alpha, -)$. This shows the uniqueness of (P, α).

For arbitrary W, since W is generated by W_0, W is a quotient of a A-module

$$\bigoplus_{i=0}^{N-1} \mathrm{Sym}_A^i (V^* \otimes_{\mathbb{C}} A / \operatorname{Ker} \pi) \simeq \bigoplus_{i=0}^{N-1} \mathrm{Sym}_A^i P^\vee.$$

However, W and $\bigoplus_{i=0}^{N-1} \mathrm{Sym}_A^i P^\vee$ have the same A-rank $N-1$, we have

$$W \simeq \bigoplus_{i=0}^{N-1} \mathrm{Sym}_A^i P^\vee.$$

This shows the lemma. □

Thus, we have the next result.

Corollary 3.4.7. *The functor $\mathcal{R}$ is represented by the projective space $\mathbb{P}(V)$ and the universal sheaf is $\bigoplus_{a=0}^{N-1} \mathcal{O}(a)$.*

In the next subsection, we prove Theorem 3.4.1 by using Proposition 3.4.6.

3.4.4 Proof of Theorem 3.4.1

Let us consider a projective A-module P of rank 1 and a pair of morphisms (α, β), where

$$\alpha : P \hookrightarrow V \otimes_{\mathbb{C}} A$$
$$\beta : P^\vee \to V^* \otimes_{\mathbb{C}} A$$

that satisfies $\beta^\vee \circ \alpha = 0$ (equivalently, $\alpha^\vee \circ \beta = 0$) and α is injective and split. We note that the triple $(P, \alpha, \beta^\vee)$ is a (stable) representation of Nakajima's quiver of type A over the commutative algebra A. Via the basis $v_1, \dots, v_N$ of V, we set the matrix

$$(a_{ij}) := \alpha \circ \beta^\vee : V \otimes_{\mathbb{C}} A \to V \otimes_{\mathbb{C}} A.$$

For a triple (P, α, β) as above, we define a representation $W_{\alpha\beta}$ as follows. We set $(W_{\alpha\beta})_a := \operatorname{Sym}_A^a P^\vee$. The action of $f \in V^*$ is given by

$$f : \operatorname{Sym}_A^a P^\vee \to \operatorname{Sym}_A^{a+1} P^\vee, \quad u^1 \cdots u^a \mapsto \alpha^\vee(f) u^1 \cdots u^a.$$

The action of $v \in V$ is given by

$$v : \operatorname{Sym}_A^a P^\vee \to \operatorname{Sym}_A^{a-1} P^\vee, \quad u^1 \cdots u^a \mapsto u^j(\beta^\vee(v)) \cdot u^1 \cdots \widehat{u^j} \cdots u^a.$$

This map is well-defined and does not depends the choice of j by Corollary 3.4.5.

First, we need to check the following

Lemma 3.4.8. *For a triple (P, α, β) as above, we have $W_{\alpha\beta} \in \widetilde{\mathcal{R}}(A)$.*

Proof. We need to check the following.

(1) $v_i v_j = v_j v_i$ and $f_i f_j = f_j f_i$.

(2) $v_j f_i = f_i v_j = a_{ij}$.

(3) $\sum_{i=1}^N f_i v_i = 0 = \sum_{i=1}^N v_i f_i$.

(4) $W_{\alpha\beta}$ is generated by $(W_{\alpha\beta})_0$.

(1) and (4) trivially follows from the construction of $W_{\alpha\beta}$. We need to check (2) and (3). First, we check (2). The action on $f_i v_j$ on $(W_{\alpha\beta})_k = \operatorname{Sym}_A^k P^\vee$ is given by

$$f_i v_j (u^1 \cdots u^k) = u^l(\beta^\vee(v_j)) \cdot \alpha^\vee(f_i) u^1 \cdots \widehat{u^l} \cdots u^k,$$

for some l. On the other hand, $v_j f_i$ acts on $(W_{\alpha\beta})_k$ by

$$\begin{aligned} & v_j f_i(u^1 \cdots u^k) \\ =& v_j(\alpha^\vee(f_i) u^1 \dots u^k) \\ =& u^l(\beta^\vee(v_j)) \cdot \alpha^\vee(f_i) u^1 \cdots \widehat{u^l} \cdots u^k \\ =& f_i v_i (u^1 \cdots u^k). \end{aligned}$$

We note that we also have

$$v_j f_i(u^1 \cdots u^k) = \alpha^\vee(f_i)(\beta^\vee(v_j)) \cdot u^1 \cdots u^k$$

and $\alpha^\vee(f_i)(\beta^\vee(v_j)) = f_i((\alpha \circ \beta^\vee)(v_j)) = a_{ij} \in A$. Hence we have

$$(v_j f_i)(u^1 \cdots u^k) = (f_i v_j)(u^1 \cdots u^k) = a_{ij} \cdot u^1 \cdots u^k.$$

This shows (2). Next, we check (3). From the above computation, we have

$$(\sum_{i=1}^{N} f_i v_i)(u^1 \cdots u^k) = \left(\sum_{i=1}^{N} u^l(\beta^\vee(v_i)) \cdot \alpha^\vee(f_i) \right) \cdot u^1 \cdots \widehat{u^l} \cdots u^k.$$

Thus, we have to show that

$$\sum_{i=1}^{N} u^l(\beta^\vee(v_i)) \cdot \alpha^\vee(f_i) = 0.$$

Let us consider the composition

$$P^\vee \xrightarrow{\beta} V^* \otimes_{\mathbb{C}} A \xrightarrow{\alpha^\vee} P^\vee.$$

Note that $\beta(u^l) = \sum_{i=1}^{N} (\beta(u^l))(v_i) \cdot f_i = \sum_{i=1}^{N} u^l(\beta^\vee(v_i)) \cdot f_i$. Hence we have $\sum_{i=1}^{N} u^l(\beta^\vee(v_i)) \cdot \alpha^\vee(f_i) = (\alpha^\vee \circ \beta)(u^l) = 0$. The same argument shows that we have

$$\sum_{i=1}^{N} v_i f_i = 0.$$

This shows (3). □

Next, we show the next proposition.

Proposition 3.4.9. *For any $W \in \widetilde{\mathcal{R}}(A)$, there exists a unique (P, α, β) as above such that $W \simeq W_{\alpha\beta}$.*

Proof. By forgetting the action of V, we can regard W as an object in $\mathcal{R}$. Thus, by Proposition 3.4.6, there exist a projective A-module P and a split injective morphism $\alpha : P \to V \otimes_{\mathbb{C}} A$ such that $W \simeq W_\alpha$. We want to construct the morphism $\beta : P^\vee \to V^* \otimes_{\mathbb{C}} A$.

The action of $v_i \in V$ on $W_1 = P^\vee$

$$v_i : P^\vee \to W_0 = A$$

is an element in $\mathrm{Hom}_A(P^\vee, A) \simeq P$. Let $p_i \in P$ an element in P that corresponds to $v_i \in V$ via the above isomorphism. By using this, we set a morphism

$$\gamma : V \otimes_{\mathbb{C}} A \to P$$

by

$$v_i \otimes 1 \mapsto p_i,$$

and we set $\beta := \gamma^\vee$. In order to complete the proof, we need to check the next two properties.

(1) $\alpha^\vee \circ \beta = 0$,

(2) The given action of V on W coincides with the one that is determined by β.

First, we check (1). For $u \in P^\vee$, we have

$$\beta(u) = \sum_{i=1}^{N} (\beta(u))(v_i) \cdot f_i.$$

Therefore, we have

$$(\alpha^\vee \circ \beta)(u) = \sum_{i=1}^{N} (\beta(u))(v_i) \cdot \alpha^\vee(f_i) = \sum_{i=1}^{N} f_i((\beta(u))(v_i)) = \sum_{i=1}^{N} (f_i v_i)(u) = 0.$$

The last equality follows from the relation $\sum_{i=1}^{N} f_i v_i = 0$. This shows (1). Next, we check (2). We show that the action

$$v_i : W_k \to W_{k-1}$$

coincides with the desired one by induction on k. For $k = 1$, this is true by the construction of β. Let us assume $k > 1$. By definition, we have

$$(v_i f_j)(u^1 \cdots u^k) = v_j(\alpha^\vee(f_j) u^1 \cdots u^k).$$

On the other hand, by the relation and the induction hypothesis, we have

$$\begin{aligned}(v_i f_j)(u^1 \cdots u^k) =& a_{ji} \cdot u^1 \cdots u^k \\ =& \alpha^\vee(f_j)(\beta^\vee(v_i)) \cdot u^1 \cdots u^k\end{aligned}$$

Since $\alpha^\vee : V^* \otimes_{\mathbb{C}} A \to P^\vee$ is surjective, we can replace $\alpha^\vee(f_j)$ in the above equation by arbitrary $u \in P^\vee$, and hence we have

$$v_j(u u^1 \cdots u^k) = u(\beta^\vee(v_i)) \cdot u^1 \cdots u^k.$$

This shows (2) and the proof is completed. □

The triple $(P, \alpha, \beta^\vee)$ gives a representation of the Nakajima's quiver $\overline{Q^\heartsuit}$ over A

$$P \underset{\alpha}{\overset{\beta^\vee}{\rightleftarrows}} V \otimes_{\mathbb{C}} A$$

of dimension vector $(1, N)$, where Q is the A_1 quiver (i.e. a point). As it was explained above, the variety Y is given by

$$Y = \{(L, X) \in \mathbb{P}(V) \times \mathrm{End}_{\mathbb{C}}(V) \mid X(V) \subset L, X(L) = 0\}.$$

This is a description of Y as the Nakajima's quiver variety of type A_1 with dimension vector $(1, N)$. From this presentation of Y, we find that Y represents the functor $\widetilde{\mathcal{R}}$. Moreover, since Nakajima's quiver varieties admit a natural symplectic structure, we can say that a symplectic structure of Y can be recovered from the NCCR as well.

For the details of Nakajima's quiver variety and the notation that we used above, see [Gin09].

Remark 3.4.10. Let $\widetilde{\mathcal{R}}^+(A)$ be a set consists of the representations of $\widetilde{\Gamma}$ with the relations in Theorem 3.3.7 of dimension vector $(1, 1, \ldots, 1)$ and generated by the last component W_{N-1}. Then, the dual argument shows that the functor $\widetilde{\mathcal{R}}^+$ represented by the variety Y^+.

3.4.5 Simple representations

In the rest of this section, we determine simple representations that are contained in $\widetilde{\mathcal{R}}(\mathbb{C})$.

Lemma 3.4.11. *A representation $W = (W_k)_k \in \widetilde{\mathcal{R}}(\mathbb{C})$ is simple if and only if it is generated by the last component W_{N-1}.*

Proof. If W is not generated by W_{N-1}, the subrepresentation W' that is generated by W_{N-1} defines a non-trivial subrepresentation of W, and hence W is not simple.

On the other hand, let $W' = (W'_k)_k$ be a non-zero subrepresentation of W. Then, the last part of subrepresentation W'_{N-1} coincides with the one W_{N-1} of W. Indeed, since W' is non-zero, there exists k such that $W'_k = \operatorname{Sym}^k_{\mathbb{C}} P$, where P is a one-dimensional vector space over $\mathbb{C}$. As the map $\alpha^\vee : V^* \to P$ is surjective, there exists $f \in V$ such that the image of the map

$$\alpha^\vee(f)^{N-k-1} : \operatorname{Sym}^k_{\mathbb{C}} P \to \operatorname{Sym}^{N-1}_{\mathbb{C}} P$$

is non-zero. Therefore, we have $W'_{N-1} \neq 0$ and hence we have $W'_{N-1} = W_{N-1}$. Thus, if W is generated by the last component W_{N-1}, the subrepresentation W' should be W itself. $\square$

Corollary 3.4.12. *A representation $W = (W_k)_k \in \widetilde{\mathcal{R}}(\mathbb{C})$ is simple if and only if the map $\beta : P^\vee \to V$ is injective.*

Proof. Let W be a simple representation. Then, by Lemma 3.4.11, W is generated by the last part W_{N-1}. Thus, for at least one i, the map $v_i : W_1 = P^\vee \to W_0 = \mathbb{C}$ is non-zero. Therefore, if we set an element $p_i \in P$ that corresponds v_i via the identification $P \simeq \operatorname{Hom}_{\mathbb{C}}(P^\vee, \mathbb{C})$, the map $\gamma : V \to P, \quad v_i \mapsto p_i$ is non-zero and hence surjective. Recall that the morphism $\beta : P \to V$ is defined as the dual map of γ. Thus, we have that the map β is injective.

On the other hand, if β is injective, we have that the representation W is generated by W_{N-1} from the construction. $\square$

Let $W \in \widetilde{\mathcal{R}}(\mathbb{C})$ and (P, α, β) a triple that defines W. Then, $\alpha(P) \subset V$ defines a line in V and the composition $\alpha \circ \beta^\vee$ defines an element in $\operatorname{End}_{\mathbb{C}}(V)$. Moreover, a pair $(\alpha(P), \alpha \circ \beta^\vee) \in \mathbb{P}(V) \times \operatorname{End}_{\mathbb{C}}(V)$ defines a point of Y that corresponds to W via the identification $\widetilde{\mathcal{R}}(\mathbb{C}) \simeq Y(\mathbb{C})$.

If β is not injective, β must be zero and hence the corresponding point of Y belongs to the zero section

$$j(\mathbb{P}(V)) = \{(L, 0) \in \mathbb{P}(V) \times \operatorname{End}_{\mathbb{C}}(V)\}.$$

Conversely, if the point $(\alpha(P), \alpha \circ \beta^\vee) \in Y$ lies on the zero section, the map β must be zero and hence not injective.

By summarizing the above discussion, we have the following theorem.

Theorem 3.4.13. *Let W be a representation that belongs to the set $\widetilde{\mathcal{R}}(\mathbb{C})$. Then, the following are equivalent.*

(1) W is simple.

(2) W is generated by the last component W_{N-1}.

(3) W is corresponds to a point of Y that lies over the non-singular part of $\overline{B(1)}$ via the identification $\widetilde{\mathcal{R}}(\mathbb{C}) \simeq Y(\mathbb{C})$.

Of course, the corresponding argument holds for Y^+ and $\widetilde{\mathcal{R}}^+$.

3.5 Kawamata-Namikawa's equivalence for Mukai flops and P-twists

In this section, we always assume $N \geq 3$.

3.5.1 Kawamata-Namikawa's equivalence

Recall that the map $\phi : Y \to \overline{B(1)}$ contracts the zero section $j : \mathbb{P}(V) \hookrightarrow Y$ to $0 \in \overline{B(1)}$. This is a flopping contraction and the flop is $Y^+ = \mathrm{Tot}(\Omega_{\mathbb{P}(V^*)}) \xrightarrow{\phi^+} \overline{B(1)}$, where $\mathbb{P}(V^*)$ is the dual projective space of $\mathbb{P}(V)$. In the following, we write $\mathbb{P} := \mathbb{P}(V)$ and $\mathbb{P}^\vee := \mathbb{P}(V^*)$ for short.

$$\mathbb{P} \xhookrightarrow{\ j\ } Y \xrightarrow{\ \phi\ } \overline{B(1)} \xleftarrow{\ \phi^+\ } Y^+ \xhookleftarrow{\ j'\ } \mathbb{P}^\vee$$

As in the above sections, let $\pi : Y \to \mathbb{P}$ and $\pi' : Y^+ \to \mathbb{P}^\vee$ be the projections, and we set $\mathcal{O}_Y(1) := \pi^*\mathcal{O}_{\mathbb{P}}(1)$ and $\mathcal{O}_{Y^+}(1) := (\pi')^*\mathcal{O}_{\mathbb{P}^\vee}(1)$. Then, the vector bundles

$$\mathcal{T}_k := \bigoplus_{a=-N+k+1}^{k} \mathcal{O}_Y(a),$$
$$\mathcal{T}_k^+ := \bigoplus_{a=-N+k+1}^{k} \mathcal{O}_{Y^+}(a)$$

on Y, Y^+, respectively, are tilting bundles. Moreover, we have an R-algebra isomorphism

$$\Lambda_k := \mathrm{End}_Y(\mathcal{T}_k) \simeq \mathrm{End}_{Y^+}(\mathcal{T}_{N-k-1}^+),$$

by Theorem 3.3.4 (4).

By using the above tilting bundles, we have equivalences of categories

$$\Psi_k := \mathrm{RHom}_Y(\mathcal{T}_k, -) : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_k),$$
$$(\Psi^+_{N-k-1})^{-1} := - \otimes^L_{\Lambda_k} \mathcal{T}^+_{N-k-1} : \mathrm{D^b}(\Lambda_k) \xrightarrow{\sim} \mathrm{D^b}(Y^+),$$

and by compositing these equivalences, we have an equivalence

$$\mathrm{nKN}_k := \mathrm{RHom}_Y(\mathcal{T}_k, -) \otimes^L_{\Lambda_k} \mathcal{T}^+_{N-k-1} : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(Y^+).$$

By construction, the inverse of the equivalence nKN_k is given by

$$(\mathrm{nKN}_k)^{-1} \simeq \mathrm{nKN}'_{N-k-1} := \mathrm{RHom}_{Y^+}(\mathcal{T}^+_{N-k-1}, -) \otimes^L_{\Lambda_k} \mathcal{T}_k\,.$$

On the other hand, the equivalence between $\mathrm{D^b}(Y)$ and $\mathrm{D^b}(Y^+)$ is first given by Kawamata and Namikawa in terms of the Fourier-Makai transform. We recall their construction of Fourier-Makai type equivalences. Let $\widetilde{Y}$ be a blowing-up of Y at the zero section $\mathbb{P}$. Then, $\widetilde{Y}$ is also a blowing-up of Y^+ at $\mathbb{P}^\vee$. Since the normal bundle of $j : \mathbb{P} \hookrightarrow Y$ is isomorphic to $\Omega^1_{\mathbb{P}}$, the exceptional divisor $E = \mathbb{P}_{\mathbb{P}}(\Omega^1_{\mathbb{P}}) \subset \widetilde{Y}$ can be embedded in the fiber product $\mathbb{P} \times \mathbb{P}^\vee$ by the Euler sequence. We set $\hat{Y} := \widetilde{Y} \cup_E (\mathbb{P} \times \mathbb{P}^\vee)$, and let $\hat{q} : \hat{Y} \to Y$ and $\hat{p} : \hat{Y} \to Y^+$ be projections.

$$\begin{array}{ccc} & \hat{Y} & \\ {}^{\hat{q}}\swarrow & & \searrow^{\hat{p}} \\ Y & & Y^+ \end{array}$$

Let $\mathcal{L}_k$ be a line bundle on $\hat{Y}$ such that $\mathcal{L}_k|_{\widetilde{Y}} = \mathcal{O}_{\widetilde{Y}}(kE)$ and $\mathcal{L}_k|_{\mathbb{P}\times\mathbb{P}^\vee} = \mathcal{O}(-k,-k)$. The Kawamata-Namikawa's functors are given by

$$\mathrm{KN}_k := R\hat{p}_*(L\hat{q}^*(-) \otimes \mathcal{L}_k) : \mathrm{D^b}(Y) \to \mathrm{D^b}(Y^+),$$
$$\mathrm{KN}'_k := R\hat{q}_*(L\hat{p}^*(-) \otimes \mathcal{L}_k) : \mathrm{D^b}(Y^+) \to \mathrm{D^b}(Y).$$

The following result is due to Kawamata and Namikawa.

Theorem 3.5.1 ([Kaw02, Nam03])**.** *The functors* KN_k *and* KN'_k *are equivalences.*

Remark 3.5.2. By the definition of the functor KN_k, the following diagram commutes

$$\begin{array}{ccc} \mathrm{D^b}(Y) & \xrightarrow{\mathrm{KN}_k} & \mathrm{D^b}(Y^+) \\ {\scriptstyle -\otimes\mathcal{O}_Y(1)}\downarrow & & \downarrow{\scriptstyle -\otimes\mathcal{O}_{Y^+}(-1)} \\ \mathrm{D^b}(Y) & \xrightarrow[\mathrm{KN}_{k+1}]{} & \mathrm{D^b}(Y^+). \end{array}$$

The same holds for our equivalence nKN_k :

$$\begin{array}{ccc} \mathrm{D^b}(Y) & \xrightarrow{\mathrm{nKN}_k} & \mathrm{D^b}(Y^+) \\ {\scriptstyle -\otimes\mathcal{O}_Y(1)}\downarrow & & \downarrow{\scriptstyle -\otimes\mathcal{O}_{Y^+}(-1)} \\ \mathrm{D^b}(Y) & \xrightarrow[\mathrm{nKN}_{k+1}]{} & \mathrm{D^b}(Y^+). \end{array}$$

Theorem 3.5.3. *Our functor* nKN_k *(resp.* nKN'_k*) coincides with the Kawamata-Namikawa's functor* KN_k *(resp.* KN'_k*).*

Note that in the proof of Theorem 3.5.3, we does not use the fact that the functors KN_k and KN'_k are equivalences. Thus, our proof of Theorem 3.5.3 gives an alternative proof for Theorem 3.5.1 in this local model of the Mukai flop.

Proof. It is easy to see that KN'_{N-k-1} is the left and right adjoint of KN_k. Thus, it is enough to show that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{D^b}(Y) & \xleftarrow{\mathrm{KN}'_{N-k-1}} & \mathrm{D^b}(Y^+) \\ & {}_{\Psi_k}\searrow \quad \swarrow_{\Psi^+_{N-k-1}} & \\ & \mathrm{D^b}(\Lambda_k) & \end{array}$$

We note that the composition $\Psi_k \circ \mathrm{KN}'_{N-k-1}$ is given by

$$\mathrm{RHom}_{Y^+}(\mathrm{KN}_k(\mathcal{T}_k), -) : \mathrm{D^b}(Y^+) \to \mathrm{D^b}(\Lambda_k).$$

Now, Theorem 3.5.3 follows from Lemma 3.5.4. □

Lemma 3.5.4. *Let* $k \in \mathbb{Z}$ *a fixed integer. Then we have*

$$\mathrm{KN}_k(\mathcal{O}_Y(a)) \simeq \mathcal{O}_{Y^+}(-a)$$

for all $-N+k+1 \leq a \leq k$ *and hence we have an isomorphism*

$$\mathrm{KN}_k(\mathcal{T}_k) \simeq \mathcal{T}^+_{N-k-1}.$$

Proof. By Remark 3.5.2, it is enough to show the isomorphism of functors for $k = 0$. Recall that the correspondence $\widehat{Y}$ is given by $\widehat{Y} = \widetilde{Y} \cup_E \mathbb{P} \times \mathbb{P}^\vee$. Hence, we have an exact sequence on $Y \times Y^+$

$$0 \to \mathcal{O}_{\widehat{Y}} \to \mathcal{O}_{\widetilde{Y}} \oplus \mathcal{O}_{\mathbb{P}\times\mathbb{P}^\vee} \to \mathcal{O}_E \to 0.$$

We use this sequence to compute the Fourier-Mukai functor $\mathrm{KN}_0 := \mathrm{FM}_{\mathcal{O}_{\widehat{Y}}}$. First, we have

$$\begin{aligned} \mathrm{FM}_{\mathcal{O}_{\mathbb{P}\times\mathbb{P}^\vee}}(\mathcal{O}_Y(a)) &= R\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(a)) \otimes j'_*\mathcal{O}_{\mathbb{P}^\vee} \\ &= \begin{cases} j'_*\mathcal{O}_{\mathbb{P}^\vee} & (\text{if } a = 0) \\ 0 & (\text{if } -N+1 \leq a < 0). \end{cases} \end{aligned}$$

The exceptional divisor $E \subset \widetilde{Y}$ is a universal hyperplane section over P and hence a divisor on $P \times P^\vee$ of bi-degree $(1,1)$. Thus, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}^\vee}(-1,-1) \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}^\vee} \to \mathcal{O}_E \to 0.$$

From the same computation as above, we have

$$\begin{aligned}
\mathrm{FM}_{\mathcal{O}_{\mathbb{P}\times\mathbb{P}^\vee}(-1,-1)}(\mathcal{O}_Y(a)) &= R\Gamma(\mathbb{P},\mathcal{O}_{\mathbb{P}}(a-1))\otimes j'_*\mathcal{O}_{\mathbb{P}^\vee}(-1) \\
&= \begin{cases} 0 & (\text{if } -N+1<a\le 0) \\ j'_*\mathcal{O}_{\mathbb{P}^\vee}(-1)[-N+1] & (\text{if } a=-N+1), \end{cases}
\end{aligned}$$

and hence we have

$$\mathrm{FM}_{\mathcal{O}_E}(\mathcal{O}_Y(a)) = \begin{cases} j'_*\mathcal{O}_{\mathbb{P}^\vee} & (\text{if } a=0) \\ 0 & (\text{if } -N+1<a<0) \\ j'_*\mathcal{O}_{\mathbb{P}^\vee}(-1)[-N+2] & (\text{if } a=-N+1). \end{cases}$$

Furthermore, since we have

$$\mathcal{O}_{\tilde{Y}}(E)\simeq \tilde{q}^*\mathcal{O}_Y(-1)\otimes\tilde{p}^*\mathcal{O}_{Y^+}(-1),$$

and

$$R\tilde{p}_*\mathcal{O}_E(kE) = \begin{cases} 0 & \text{for all } k=1,\dots,N-2, \\ j'_*\mathcal{O}_{\mathbb{P}^\vee}(-N)[-N+2] & \text{for } k=N-1, \end{cases}$$

we have

$$\begin{aligned}
\mathrm{FM}_{\mathcal{O}_{\tilde{Y}}}(\mathcal{O}_Y(a)) &= R\tilde{p}_*(\mathcal{O}_{\tilde{Y}}(-aE))\otimes\mathcal{O}_{Y^+}(-a) \\
&= \mathcal{O}_{Y^+}(-a)
\end{aligned}$$

for $-N+1<a\le 0$, and $\mathrm{FM}_{\mathcal{O}_{\tilde{Y}}}(\mathcal{O}_Y(-N+1))$ lies on the exact triangle

$$\mathcal{O}_{Y^+}(N-1)\to\mathrm{FM}_{\mathcal{O}_{\tilde{Y}}}(\mathcal{O}_Y(-N+1))\to\mathcal{O}_{P^\vee}(-1)[-N+2].$$

From the above, we can compute $\mathrm{KN}_0(\mathcal{O}_Y(a))$ for $-N+1\le a\le 0$. If $a=0$, $\mathrm{KN}_0(\mathcal{O}_Y)$ lies on the exact triangle

$$\mathrm{KN}_0(\mathcal{O}_Y)\to\mathcal{O}_{Y^+}\oplus j'_*\mathcal{O}_{\mathbb{P}^\vee}\to j'_*\mathcal{O}_{\mathbb{P}^\vee},$$

and hence we have

$$\mathrm{KN}_0(\mathcal{O}_Y)\simeq\mathcal{O}_{Y^+}.$$

If $-N+1<a<0$, we have

$$\mathrm{KN}_0(\mathcal{O}_Y(a))\simeq\mathcal{O}_{Y^+}(-a).$$

Finally, if $a=-N+1$, $\mathrm{KN}_0(\mathcal{O}_Y(-N+1))$ lies on the exact triangle

$$\mathrm{KN}_0(\mathcal{O}_Y(-N+1))\to\mathrm{FM}_{\mathcal{O}_Y}(\mathcal{O}_Y(-N+1))\to j'_*\mathcal{O}_{\mathbb{P}^\vee}(-1)[-N+2].$$

This triangle coincides with the above one that gives the object $\mathrm{FM}_{\mathcal{O}_{\tilde{Y}}}(\mathcal{O}_Y(-N+1))$ and hence we have

$$\mathrm{KN}_0(\mathcal{O}_Y(-N+1))\simeq\mathcal{O}_{Y^+}(N-1).$$

Thus, we have the isomorphism $\mathrm{KN}_0(\mathcal{T}_0)\simeq\mathcal{T}_N^+$ that we want. □

3.5.2 P-twists and Mutations

In this section, we introduce equivalences ν^-_{N+k} and ν^+_{N+k-1} between the derived categories of non-commutative algebras Λ_{N+k} and Λ_{N+k-1}. We show that a composition of multi-mutation functors $\nu^+_{N+k-1} \circ \nu^-_{N+k}$ corresponds to an autoequivalence P_k of $\mathrm{D^b}(Y)$ that is a *P-twist* defined by a $\mathbb{P}^{N-1}$-object $j_*\mathcal{O}_{\mathbb{P}}(k)$.

Definition of multi-mutation

First, we define a multi-mutation functor $\nu^-_{N-1} : \mathrm{D^b}(\Lambda_{N-1}) \to \mathrm{D^b}(\Lambda_{N-2})$. Recall that the algebras Λ_{N-1} is given by

$$\Lambda_{N-1} = \mathrm{End}_R\left(\bigoplus_{a=0}^{N-1} M_a\right).$$

Let us consider the canonical surjective morphism $R^{\oplus N} \twoheadrightarrow M_{-1}$. Note that this morphism is given by the push-forward of the canonical surjection $V \otimes_{\mathbb{C}} \mathcal{O}_{Y^+} \twoheadrightarrow \mathcal{O}_{Y^+}(1)$ by ϕ^+. Then, we define a Λ_{N-1}-module C as

$$C := \mathrm{Image}\left(\mathrm{Hom}_R(\bigoplus_{a=0}^{N-1} M_a, R^{\oplus N}) \to \mathrm{Hom}_R(\bigoplus_{a=0}^{N-1} M_a, M_{-1})\right),$$

and set a Λ_{N-1}-module S as

$$S := \mathrm{Hom}_{\Lambda_{N-1}}(\bigoplus_{a=0}^{N-1} M_a, \bigoplus_{a=0}^{N-2} M_a) \oplus C.$$

Lemma 3.5.5. *The following hold.*

(i) There exists an isomorphism of Λ_{N-1}-modules

$$S \simeq \mathrm{RHom}_{Y^+}(\mathcal{T}^+_0, \mathcal{T}^+_1).$$

(ii) The Λ_{N-1}-module S defined above is a tilting generator of the category $\mathrm{D^b}(\Lambda_{N-1})$.

(iii) We have an isomorphism between R-algebras

$$\mathrm{End}_{\Lambda_{N-1}}(S) \simeq \Lambda_{N-2}.$$

Proof. The (ii) and (iii) follow from (i). First, we have

$$\mathrm{RHom}_{Y^+}(\mathcal{T}^+_0, \mathcal{T}^+_1) = \mathrm{RHom}_{Y^+}(\mathcal{T}^+_0, \bigoplus_{a=-N+2}^{0} \mathcal{O}_{Y^+}(a)) \oplus \mathrm{RHom}_{Y^+}(\mathcal{T}^+_0, \mathcal{O}_{Y^+}(1)).$$

As explained above, we have

$$M_{-a} = \phi^+_* \mathcal{O}_{Y^+}(a)$$

for all $-N+1 \leq a \leq N-1$, and we have

$$\begin{aligned}\mathrm{RHom}_{Y^+}(\mathcal{T}^+_0, \bigoplus_{a=-N+2}^{0} \mathcal{O}_{Y^+}(a)) &= \mathrm{Hom}_{Y^+}(\mathcal{T}^+_0, \bigoplus_{a=-N+2}^{0} \mathcal{O}_{Y^+}(a)) \\ &= \mathrm{Hom}_R(\bigoplus_{a=0}^{N-1} M_a, \bigoplus_{a=0}^{N-2} M_a).\end{aligned}$$

Next, since the sheaf $\mathcal{H}om_{Y^+}(\mathcal{T}^+_0, \mathcal{O}_{Y^+}(1))$ on Y^+ is a vector bundle and hence is torsion free, the R-module $\mathrm{Hom}_{Y^+}(\mathcal{T}^+_0, \mathcal{O}_{Y^+}(1)) = \phi^+_* \mathcal{H}om_{Y^+}(\mathcal{T}^+_0, \mathcal{O}_{Y^+}(1))$ is also torsion free. Since two R-modules $\mathrm{Hom}_R(\bigoplus_{a=0}^{N-1} M_a, M_{-1})$ and $\mathrm{Hom}_{Y^+}(\mathcal{T}^+_0, \mathcal{O}_{Y^+}(1))$ are isomorphic in codimension one, the natural map

$$\mathrm{Hom}_{Y^+}(\mathcal{T}^+_0, \mathcal{O}_{Y^+}(1)) \to \mathrm{Hom}_R(\bigoplus_{a=0}^{N-1} M_a, M_{-1})$$

is injective. Let us consider the surjective morphism $V \otimes_{\mathbb{C}} \mathcal{O}_{Y^+} \to \mathcal{O}_{Y^+}(1)$. We note that the map

$$\mathrm{Hom}_{Y^+}(\mathcal{T}^+_0, V \otimes_{\mathbb{C}} \mathcal{O}_{Y^+}) \to \mathrm{Hom}_{Y^+}(\mathcal{T}^+_0, \mathcal{O}_{Y^+}(1))$$

is surjective because we have a vanishing of an extension

$$\mathrm{Ext}^1_{Y^+}(\mathcal{T}^+_0, \pi'^* \Omega_{\mathbb{P}^\vee}(1)) = H^1(Y^+, \bigoplus_{a=0}^{N-1} \pi'^* \Omega_{\mathbb{P}^\vee}(a+1)) = 0$$

from the same argument as in the proof of Corollary 3.3.13. Thus, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{Y^+}(\mathcal{T}^+_0, V \otimes_{\mathbb{C}} \mathcal{O}_{Y^+}) & \xrightarrow{\simeq} & \mathrm{Hom}_R(\bigoplus_{a=0}^{N-1} M_a, V \otimes_{\mathbb{C}} M_0) \\ \downarrow\!\!\downarrow & & \downarrow \\ \mathrm{Hom}_{Y^+}(\mathcal{T}^+_0, \mathcal{O}_{Y^+}(1)) & \hookrightarrow & \mathrm{Hom}_R(\bigoplus_{a=0}^{N-1} M_a, M_{-1}) \end{array}$$

and hence we have $\mathrm{RHom}_{Y^+}(\mathcal{T}^+_0, \mathcal{O}_{Y^+}(1)) = \mathrm{Hom}_{Y^+}(\mathcal{T}^+_0, \mathcal{O}_{Y^+}(1)) = C$. □

From the above lemma, we can define the equivalence:

Definition 3.5.6. We set

$$\nu^-_{N-1} := \mathrm{RHom}_{\Lambda_{N-1}}(S, -) : \mathrm{D^b}(\Lambda_{N-1}) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_{N-2}).$$

We call this functor ν^-_{N-1} the *multi-mutation functor*. By Lemma 3.5.5, multi-mutation ν^-_{N-1} coincides with the functor

$$\mathrm{RHom}_{\Lambda_{N-1}}(\mathrm{RHom}_{Y^+}(\mathcal{T}^+_0, \mathcal{T}^+_1), -) : \mathrm{D^b}(\Lambda_{N-1}) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_{N-2}),$$

and hence the following diagram commutes

$$\begin{array}{ccc} \mathrm{D^b}(Y^+) & \xrightarrow{\Psi^+_{N-1}} & \mathrm{D^b}(\Lambda_k) \\ & \searrow^{\Psi^+_{N-2}} & \downarrow{\scriptstyle \nu^-_{N-1}} \\ & & \mathrm{D^b}(\Lambda_{N-2}). \end{array}$$

We also define a multi-mutation functor $\nu^-_k : \mathrm{D^b}(\Lambda_k) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_{k-1})$ by using the following commutative diagram.

$$\begin{array}{ccccccc} \mathrm{D^b}(Y^+) & \xrightarrow[\otimes\mathcal{O}(-N+k+1)]{} & \mathrm{D^b}(Y^+) & = & \mathrm{D^b}(Y^+) & \xrightarrow[\otimes\mathcal{O}(N-k-1)]{} & \mathrm{D^b}(Y^+) \\ \downarrow{\scriptstyle \Psi^+_{N-k-1}} & & \downarrow{\scriptstyle \Psi^+_0} & & \downarrow{\scriptstyle \Psi^+_1} & & \downarrow{\scriptstyle \Psi^+_{N-k}} \\ \mathrm{D^b}(\Lambda_k) & \xrightarrow{F^k_{N-1}} & \mathrm{D^b}(\Lambda_{N-1}) & \xrightarrow{\nu^-_{N-1}} & \mathrm{D^b}(\Lambda_{N-2}) & \xrightarrow{F^{N-2}_{k-1}} & \mathrm{D^b}(\Lambda_{k-1}), \end{array}$$

(with the top row composing to id and the bottom row composing to ν^-_k), where the functor $F^i_j : \mathrm{D^b}(\Lambda_i) \to \mathrm{D^b}(\Lambda_j)$ is given by the composition

$$F^i_j : \mathrm{D^b}(\Lambda_i) \xrightarrow{-\otimes_{\Lambda_i}\mathcal{T}^+_{N-i-1}} \mathrm{D^b}(Y^+) \xrightarrow{-\otimes\mathcal{O}_Y(i-j)} \mathrm{D^b}(Y^+) \xrightarrow{\mathrm{RHom}_Y(\mathcal{T}^+_{N-j-1},-)} \mathrm{D^b}(\Lambda_j).$$

Applying the same argument for the side of Y, we can define a multi-mutation functor $\nu^+_k : \mathrm{D^b}(\Lambda_k) \to \mathrm{D^b}(\Lambda_{k+1})$. Again, by construction, we can show that there is a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{D^b}(Y) & \xrightarrow{\Psi_k} & \mathrm{D^b}(\Lambda_k) \\ & \searrow^{\Psi_{k+1}} & \downarrow{\scriptstyle \nu^+_k} \\ & & \mathrm{D^b}(\Lambda_{k+1}). \end{array}$$

Connection between multi-mutations and IW mutations

In the following, we explain the multi-mutation functor ν^-_{N-1} is given by a composition of IW mutations. For definitions and basic properties of IW mutations. Let us consider the long Euler sequence on $\mathbb{P}^\vee = \mathbb{P}(V^*)$

$$0 \to \mathcal{O}_{\mathbb{P}^\vee}(-N+1) \to V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^\vee}(-N+2) \to \bigwedge^{N-2} V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^\vee}(-N+3) \to \cdots$$
$$\to \bigwedge^{2} V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^\vee}(-1) \to V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^\vee} \to \mathcal{O}_{\mathbb{P}^\vee}(1) \to 0.$$

By applying the functor $(\phi^+)_* \circ (\pi')^*$ to the above sequence, we have a resolution of the module M_{-1} by other modules $M_0, \dots, M_{N-1}$:

$$0 \to M_{N-1} \to V^* \otimes_{\mathbb{C}} M_{N-2} \to \bigwedge^{N-2} V \otimes_{\mathbb{C}} M_{N-3} \to \cdots \to \bigwedge^{2} V \otimes_{\mathbb{C}} M_1 \to V \otimes_{\mathbb{C}} M_0 \to M_{-1} \to 0.$$

We splice this sequence into short exact sequences

$$0 \to M_{N-1} \to V^* \otimes_{\mathbb{C}} M_{N-2} \to L_{N-2} \to 0$$
$$0 \to L_{N-2} \to \bigwedge^{N-2} V \otimes_{\mathbb{C}} M_{N-3} \to L_{N-3} \to 0$$
$$\vdots$$
$$0 \to L_k \to \bigwedge^{k} V \otimes_{\mathbb{C}} M_{k-1} \to L_{k-1} \to 0$$
$$\vdots$$
$$0 \to L_1 \to V \otimes_{\mathbb{C}} M_0 \to M_{-1} \to 0$$

and set $L_{N-1} := M_{N-1}$, $L_0 := M_{-1}$, $W := \bigoplus_{a=0}^{N-2} M_a$, and $E_k := W \oplus L_k$. By dualizing above morphisms, we have a map $\bigwedge^k V^* \otimes_{\mathbb{C}} M_{k-1}^* \to L_k^*$. Since the module M_a is reflexive, the above map is surjective. Then, applying the functor $- \oplus W^*$, we have a surjective map

$$\left(\bigwedge^{k} V^* \otimes_{\mathbb{C}} M_{k-1}^* \right) \oplus W^* \twoheadrightarrow E_k^*.$$

First, we prove the following

Lemma 3.5.7. *The map* $\left(\bigwedge^k V^* \otimes_{\mathbb{C}} M_{k-1}^* \right) \oplus W^* \to E_k^*$ *is a right* $(\operatorname{add} W^*)$*-approximation.*

Proof. Let us consider the exact sequence

$$0 \to L_{k-1}^* \to \bigwedge^{k} V^* \otimes_{\mathbb{C}} M_{k-1}^* \to L_k^* \to 0.$$

We have to show that the map

$$\operatorname{Hom}_R(W^*, \bigwedge^{k} V^* \otimes_{\mathbb{C}} M_{k-1}^*) \to \operatorname{Hom}_R(W^*, L_k^*)$$

is surjective. First, by definition, we have $M_{k-1}^* \simeq M_{-k+1} \simeq (\phi^+)_* \mathcal{O}_{Y^+}(k-1)$. On Y^+, there is a canonical short exact sequence

$$0 \to (\pi')^* \bigwedge^{k-1} T_{\mathbb{P}^\vee} \otimes \mathcal{O}_{Y^+}(-1) \to \bigwedge^{k} V^* \otimes_{\mathbb{C}} \mathcal{O}_{Y^+}(k-1) \to (\pi')^* \bigwedge^{k} T_{\mathbb{P}^\vee} \otimes \mathcal{O}_{Y^+}(-1) \to 0,$$

where $T_{\mathbb{P}^\vee}$ is the tangent bundle on Y^+. Put $\mathcal{H}_k^* := (\pi')^* \bigwedge^k T_{\mathbb{P}^\vee} \otimes \mathcal{O}_{Y^+}(-1)$ and $\mathcal{H}_k := \mathcal{H}_k^{**}$. Since the first non-trivial term of the above exact sequence does not have higher cohomology, we have an isomorphism

$$L_k^* \simeq (\phi^+)_* (\mathcal{H}_k^*)$$

by induction on k. Furthermore, since the third non-trivial term of the above exact sequence and its dual have no higher cohomology, it follows from Lemma 2.1.15 that the module L_k^* is (maximal) Cohen-Macaulay, and hence, the module $\mathrm{Hom}_R(W^*, L_k^*)$ is reflexive by Proposition 2.1.21. In addition, by Proposition 2.1.20, Lemma 2.1.15, and Proposition 3.3.16, the module

$$\mathrm{Hom}_{Y^+}\left(\bigoplus_{a=0}^{N-2} \mathcal{O}_{Y^+}(a), \mathcal{H}_k^*\right)$$

is also reflexive. Therefore, we have an isomorphism

$$\mathrm{Hom}_{Y^+}\left(\bigoplus_{a=0}^{N-2} \mathcal{O}_{Y^+}(a), \mathcal{H}_k^*\right) \simeq \mathrm{Hom}_R(W^*, L_k^*).$$

On the other hand, again by Proposition 3.3.16, we have the vanishing of an extension group

$$\mathrm{Ext}^1_{Y^+}\left(\bigoplus_{a=0}^{N-2} \mathcal{O}_{Y^+}(a), \mathcal{H}_{k-1}^*\right) = 0.$$

This vanishing says that the map

$$\mathrm{Hom}_{Y^+}\left(\bigoplus_{a=0}^{N-2} \mathcal{O}_{Y^+}(a), \bigwedge^k V^* \otimes_{\mathbb{C}} \mathcal{O}_{Y^+}(k-1)\right) \to \mathrm{Hom}_{Y^+}\left(\bigoplus_{a=0}^{N-2} \mathcal{O}_{Y^+}(a), \mathcal{H}_k^*\right)$$

is surjective. Thus, we have the morphism

$$\mathrm{Hom}_R(W^*, \bigwedge^k V^* \otimes_{\mathbb{C}} M_{k-1}^*) \to \mathrm{Hom}_R(W^*, L_k^*)$$

is also surjective. □

Since the kernel of the approximation $\left(\bigwedge^k V^* \otimes_{\mathbb{C}} M_{k-1}^*\right) \oplus W^* \to E_k^*$ is isomorphic to L_{k-1}^*, the R-module E_{k-1} is isomorphic to a (left) IW mutation $\mu_W^L(E_k)$ of E_k at W. Thus, by Theorem 2.1.25, we have a derived equivalence

$$\Phi_W : \mathrm{D^b}(\mathrm{End}_R(E_k)) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{End}_R(E_{k-1})).$$

However, in this case, we can show directly that the functor Φ_W actually gives an equivalence of categories. As in the proof of Lemma 3.5.7, put $\mathcal{H}_k := (\pi')^* \Omega^k_{\mathbb{P}^\vee} \otimes \mathcal{O}_{Y^+}(1)$.

Lemma 3.5.8. *(1) We have an isomorphism of R-algebras $\mathrm{End}_R(E_k) \simeq \mathrm{End}_{Y^+}(\mathcal{S}_k^+)$, where $\mathcal{S}_k^+ := \bigoplus_{-N+2}^{0} \mathcal{O}_{Y^+}(a) \oplus \mathcal{H}_k$ is a tilting bundle on Y^+ that is given in Proposition 3.3.16.*

(2) We have an isomorphism of functors

$$\Phi_W \simeq \mathrm{RHom}_{\mathrm{End}_R(E_k)}(\mathrm{RHom}_{Y^+}(\mathcal{S}_k^+, \mathcal{S}_{k-1}^+), -).$$

(3) In particular, IW mutation functor Φ_W gives an equivalence of categories, and the following diagram of functors commutes

$$\begin{array}{ccc} \mathrm{D^b}(Y^+) & \xrightarrow{S_k} & \mathrm{D^b}(\mathrm{End}_R(E_k)) \\ & \searrow^{S_{k-1}} & \downarrow \Phi_W \\ & & \mathrm{D^b}(\mathrm{End}_R(E_{k-1})), \end{array}$$

where $S_k := \mathrm{RHom}_{Y^+}(\mathcal{S}_k^+, -) : \mathrm{D^b}(Y^+) \to \mathrm{D^b}(\mathrm{End}_R(E_k))$.

Proof. We can prove this lemma by using almost same arguments as in Lemma 3.5.5. The different point from Lemma 3.5.5 is that the vanishing of $\mathrm{Ext}^i_{Y^+}(\mathcal{S}_k^+, \mathcal{S}_{k-1}^+)$ for $i > 0$ is non-trivial. However, this vanishing follows from direct computations using Proposition 3.3.12. □

Now we ready to prove the following result that gives a correspondence between multi-mutations and IW mutations.

Theorem 3.5.9. *An equivalence obtained by composing $N-1$ IW mutation functors*

$$\Phi_W \circ \Phi_W \circ \cdots \circ \Phi_W : \mathrm{D^b}(\Lambda_{N-1}) \to \mathrm{D^b}(\Lambda_{N-2})$$

is isomorphic to a multi-mutation functor ν_{N-1}^-.

Here, we note that $\mathrm{End}_R(E_{N-1}) = \Lambda_{N-1}$ and $\mathrm{End}_R(E_0) = \Lambda_{N-2}$.

Proof. By Lemma 3.5.5 (3), we have a commutative diagram

$$\begin{array}{ccccccc} \mathrm{D^b}(Y^+) & & \xrightarrow{\qquad\qquad S_0 \qquad\qquad} & & & & \\ \downarrow S_{N-1} & \searrow^{S_{N-2}} & & & & \searrow & \\ \mathrm{D^b}(\mathrm{End}_R(E_{N-1})) & \xrightarrow[\Phi_W]{} & \mathrm{D^b}(\mathrm{End}_R(E_{N-2})) & \xrightarrow[\Phi_W]{} & \cdots & \xrightarrow[\Phi_W]{} & \mathrm{D^b}(\mathrm{End}_R(E_0)), \end{array}$$

Hence, we have $\Phi_W \circ \Phi_W \circ \cdots \circ \Phi_W \simeq S_0 \circ S_{N-1}^{-1} = \Psi_1^+ \circ (\Psi_0^+)^{-1} \simeq \nu_{N-1}^-$. □

Remark 3.5.10. Applying the same argument, we can prove that a multi-mutation functor

$$\nu_k^- : \mathrm{D^b}(\Lambda_k) \to \mathrm{D^b}(\Lambda_{k-1})$$

is written as a composition of IW mutation functors if $1 \le k \le N-1$. In other cases, the above argument cannot be applied because we only know that the module $\bigoplus_{a=-N+k+1}^{k} M_a$ gives an NCCR if $0 \le k \le N-1$ (see Theorem 3.3.3).

Next, we discuss the case of multi-mutations ν_k^+.

Theorem 3.5.11. *A multi-mutation functor*

$$\nu_{N-2}^+ : \mathrm{D^b}(\Lambda_{N-2}) \to \mathrm{D^b}(\Lambda_{N-1})$$

can be written as a composition of $N-1$ IW mutation functors.

Proof. Let us consider the long Euler sequence on $\mathbb{P}$

$$0 \to \mathcal{O}_{\mathbb{P}}(-1) \to V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}} \to \bigwedge^{N-2} V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}}(1) \to \cdots$$
$$\to \bigwedge^{2} V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}}(N-3) \to V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}}(N-2) \to \mathcal{O}_{\mathbb{P}}(N-1) \to 0.$$

Applying a functor $\phi_* \circ \pi^*$, we have a long exact sequence

$$0 \to M_{-1} \to V \otimes_{\mathbb{C}} R \to \bigwedge^{N-2} V^* \otimes_{\mathbb{C}} M_1 \to \cdots$$
$$\to \bigwedge^{2} V^* \otimes_{\mathbb{C}} M_{N-3} \to V^* \otimes_{\mathbb{C}} M_{N-2} \to M_{N-1} \to 0.$$

Using completely same argument as in the proof of Theorem 3.5.9, we have an equivalence of categories

$$\Phi_W \circ \Phi_W \circ \cdots \circ \Phi_W : \mathrm{D^b}(\Lambda_{N-2}) \to \mathrm{D^b}(\Lambda_{N-1})$$

and this functor is isomorphic to the functor $\Psi_{N-1} \circ \Psi_{N-2}^{-1} \simeq \nu_{N-2}^+$ under the above identification of algebras. □

Remark 3.5.12. As in Remark 3.5.10, we can show that a multi-mutation functor $\nu_k^+ : \mathrm{D^b}(\Lambda_k) \to \mathrm{D^b}(\Lambda_{k+1})$ can be described as a composition of IW mutation functors if $0 \leq k \leq N-2$.

Remark 3.5.13. From the proof of theorems, we notice that the object

$$\mu_W^L(\mu_W^L(\cdots(\mu_W^L(\bigoplus_{a=0}^{N-1} M_a))\cdots)),$$

which obtained from $\bigoplus_{a=0}^{N-1} M_a$ after taking IW mutations at W $(2N-2)$-times, coincides with the original module $\bigoplus_{a=0}^{N-1} M_a$:

$$\mu_W^L(\mu_W^L(\cdots(\mu_W^L(\bigoplus_{a=0}^{N-1} M_a))\cdots)) = \bigoplus_{a=0}^{N-1} M_a.$$

If the ring R is complete normal 3-sCY and M is a maximal modifying module[2], Iyama and Wemyss proved that two times mutation $\mu_N^L\mu_N^L(M)$ of M at an indecomposable summand N coincides with M [IW14a, Summary 6.25]:

$$\mu_N^L\mu_N^L(M) = M.$$

[2]For the definition of maximal modifying R-modules, see [IW14a, Definition 4.1]. We note that a module that gives an NCCR is a maximal modifying module if R is a normal d-sCY ring [IW14a, Proposition 4.5].

Although the module W that we used for mutations is not indecomposable, I think we can regard our equality of modules as a generalization of Iyama-Wemyss's one. The number of mutations we need seems to be related to the dimension of a fiber of a crepant resolution (or $\mathbb{Q}$-factorial terminalization).

Corollary 3.5.14. *The equivalence from* $\mathrm{D^b}(Y)$ *to* $\mathrm{D^b}(Y^+)$ *obtained by the composition*

$$\mathrm{D^b}(Y) \xrightarrow{\Psi_0} \mathrm{D^b}(\Lambda_0) \xrightarrow{\nu_{N-2}^+\circ\cdots\circ\nu_0^+} \mathrm{D^b}(\Lambda_{N-1}) \xrightarrow{(\Psi_0^+)^{-1}} \mathrm{D^b}(Y^+)$$

is the inverse of the (original) Kawamata-Namikawa's functor KN_0'.

By the above remark, the functor $\nu_{N-2}^+ \circ \cdots \circ \nu_0^+$ can be written as the composition of $(N-1)^{N-1}$ IW mutation functors. On the other hand, two tilting bundles $\mathcal{T}_0$ and $\mathcal{T}_0^+$ provide projective generators of the perverse hearts ${}^0\mathrm{Per}(Y/A_{N-2})$ and ${}^0\mathrm{Per}(Y^+/A_{N-2}^o)$ respectively (see [TU10, Example 5.3]). Please compare this corollary with [Wem17, Theorem 4.2].

Multi-mutations and P-twists

Next, we explain that a composition of two multi-mutation functors corresponds to a P-twist on $\mathrm{D^b}(Y)$. First, we recall that the object $j_*\mathcal{O}_{\mathbb{P}}(k)$ is a $\mathbb{P}^{N-1}$-object in $\mathrm{D^b}(Y)$. This fact is well-known but I give the proof here for reader's convenience.

Lemma 3.5.15. $j_*\mathcal{O}_{\mathbb{P}}(k)$ *is a* $\mathbb{P}^{N-1}$*-object in* $\mathrm{D^b}(Y)$.

Proof. It is enough to show the case if $k = 0$. Let us consider the spectral sequence

$$E_2^{p,q} = H^p(Y, \mathcal{E}xt_Y^q(j_*\mathcal{O}_{\mathbb{P}}, j_*\mathcal{O}_{\mathbb{P}})) \Rightarrow \mathrm{Ext}_Y^{p+q}(j_*\mathcal{O}_{\mathbb{P}}, j_*\mathcal{O}_{\mathbb{P}}).$$

Since there is an isomorphism

$$\mathcal{E}xt_Y^q(j_*\mathcal{O}_{\mathbb{P}}, j_*\mathcal{O}_{\mathbb{P}}) \simeq j_* \bigwedge^q \mathcal{N}_{\mathbb{P}/Y} \simeq j_*\Omega_{\mathbb{P}}^q,$$

we have

$$\begin{aligned} E_2^{p,q} &= H^p(\mathbb{P}, \Omega_{\mathbb{P}}^q) \\ &= \begin{cases} \mathbb{C} & \text{if } 0 \le p = q \le N-1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, we have

$$\mathrm{Ext}_Y^i(j_*\mathcal{O}_{\mathbb{P}}, j_*\mathcal{O}_{\mathbb{P}}) = \begin{cases} \mathbb{C} & \text{if } i = 2k \text{ and } k = 0, \dots, N-1, \\ 0 & \text{otherwise.} \end{cases}$$

This shows the lemma. □

Definition 3.5.16. Let P_k be a P-twist that is defied by the $\mathbb{P}^{N-1}$-object $j_*\mathcal{O}_P(k)$. More explicitly, the functor P_k is given by

$$P_k(E) = \mathrm{Cone}\Big(\mathrm{Cone}\,(j_*\mathcal{O}_{\mathbb{P}}(k)[-2] \to j_*\mathcal{O}_{\mathbb{P}}(k)) \otimes_{\mathbb{C}} \mathrm{RHom}_Y(j_*\mathcal{O}_{\mathbb{P}}(k), E) \xrightarrow{\mathrm{ev}} E\Big).$$

Remark 3.5.17. By the definition of the functor P_k, the following diagram commutes.

$$\begin{array}{ccc} \mathrm{D^b}(Y) & \xrightarrow{P_k} & \mathrm{D^b}(Y) \\ {\scriptstyle -\otimes\mathcal{O}_Y(-1)}\downarrow & & \downarrow{\scriptstyle -\otimes\mathcal{O}_Y(-1)} \\ \mathrm{D^b}(Y) & \xrightarrow{P_{k-1}} & \mathrm{D^b}(Y) \end{array}$$

The following is one of main results in this chapter.

Theorem 3.5.18. *The following diagram of equivalence functors commutes*

$$\begin{array}{ccc} \mathrm{D^b}(Y) & \xrightarrow{\Psi_{N+k}} & \mathrm{D^b}(\Lambda_{N+k}) \\ \downarrow{\scriptstyle P_k} & & \downarrow{\scriptstyle \nu^-_{N+k}} \\ \mathrm{D^b}(Y) & \xrightarrow{\Psi_{N+k-1}} & \mathrm{D^b}(\Lambda_{N+k-1}) \\ \| & & \downarrow{\scriptstyle \nu^+_{N+k-1}} \\ \mathrm{D^b}(Y) & \xrightarrow{\Psi_{N+k}} & \mathrm{D^b}(\Lambda_{N+k}). \end{array}$$

In particular, if we fix the identification $\Psi_{N+k} : \mathrm{D^b}(Y) \to \mathrm{D^b}(\Lambda_{N+k})$*, a composition of two multi-mutation functors*

$$\nu^+_{N+k-1} \circ \nu^-_{N+k} \in \mathrm{Auteq}(\mathrm{D^b}(\Lambda_{N+k}))$$

corresponds to a P-twist $P_k \in \mathrm{Auteq}(\mathrm{D^b}(Y))$.

Remark 3.5.19. If $1 \le N + k \le N - 1$ (i.e. if $-N + 1 \le k \le -1$), multi-mutation functors ν^-_{N+k} and ν^+_{N+k-1} are can be written as compositions of IW mutation functors. Thus, in the case of Mukai flops, we can interpret a P-twist on Y as a composition of many IW mutation functors. This is a higher dimensional generalization of the result of Donovan and Wemyss [DW16].

Proof of Theorem 3.5.18. It is enough to show the theorem for one k. Here, we prove the case if $k = -1$. Recall that the composition

$$\mathrm{D^b}(Y) \xrightarrow{\Psi_{N-1}} \mathrm{D^b}(\Lambda_{N-1}) \xrightarrow{\nu^-_{N-1}} \mathrm{D^b}(\Lambda_{N-2})$$

coincides with the functor

$$\mathrm{RHom}_Y(S \otimes_{\Lambda_{N-1}} \mathcal{T}_{N-1}, -) : \mathrm{D^b}(Y) \to \mathrm{D^b}(\Lambda_{N-2}).$$

By Theorem 3.5.3 and Lemma 3.5.5, we have $S \otimes_{\Lambda_{N-1}} \mathcal{T}_{N-1} \simeq \mathrm{KN}'_0(\mathcal{T}_1^+)$. On the other hand, the equivalence that is given by the composition of functors

$$\mathrm{D^b}(Y) \xrightarrow{P_{-1}} \mathrm{D^b}(Y) \xrightarrow{\Psi_{N-2}} \mathrm{D^b}(\Lambda_{N-2})$$

coincides with the functor that is given by

$$\mathrm{RHom}_Y\left((P_{-1})^{-1}(\mathcal{T}_{N-2}), -\right) : \mathrm{D^b}(Y) \to \mathrm{D^b}(\Lambda_{N-2}).$$

Thus, we have to show that

$$P_{-1}(\mathrm{KN}'_0(\mathcal{T}_1^+)) \simeq \mathcal{T}_{N-2}\,.$$

Recall that the tilting bundles are given by

$$\mathcal{T}_{N-2} = \bigoplus_{a=-1}^{N-2} \mathcal{O}_Y(a), \quad \mathcal{T}_1^+ = \bigoplus_{a=-N+2}^{1} \mathcal{O}_{Y^+}(a).$$

By Lemma 3.5.4, we have

$$\mathrm{KN}'_0(\mathcal{O}_{Y^+}(a)) \simeq \mathcal{O}_Y(-a)$$

for $-N+2 \le a \le 0$. Therefore, we have to compute the object $\mathrm{KN}'_0(\mathcal{O}_{Y^+}(1))$. As in Lemma 3.5.4, we use the exact sequence

$$0 \to \mathcal{O}_{\hat{Y}} \to \mathcal{O}_{\widetilde{Y}} \oplus \mathcal{O}_{\mathbb{P}\times\mathbb{P}^\vee} \to \mathcal{O}_E \to 0.$$

An easy computation shows that we have

$$\begin{aligned}
\mathrm{FM}^{Y^+\to Y}_{\mathcal{O}_{\mathbb{P}\times\mathbb{P}^\vee}}(\mathcal{O}_{Y^+}(1)) &\simeq V \otimes_{\mathbb{C}} j_*\mathcal{O}_{\mathbb{P}},\\
\mathrm{FM}^{Y^+\to Y}_{\mathcal{O}_E}(\mathcal{O}_{Y^+}(1)) &\simeq j_*T_{\mathbb{P}}(-1),\\
\mathrm{FM}^{Y^+\to Y}_{\mathcal{O}_{\widetilde{Y}}}(\mathcal{O}_{Y^+}(1)) &\simeq I_{\mathbb{P}/Y}(-1),
\end{aligned}$$

where $T_{\mathbb{P}}$ is the tangent bundle on $\mathbb{P} = \mathbb{P}(V)$ and $I_{\mathbb{P}/Y}$ the ideal sheaf of $j : \mathbb{P} \subset Y$. Thus, we have the following exact triangle

$$\mathrm{KN}'_0(\mathcal{O}_{Y^+}(1)) \to I_{\mathbb{P}/Y}(-1) \oplus (V \otimes_{\mathbb{C}} j_*\mathcal{O}_{\mathbb{P}}) \to j_*T_{\mathbb{P}}(-1).$$

By combining this triangle with the split triangle

$$V \otimes_{\mathbb{C}} j_*\mathcal{O}_{\mathbb{P}} \to I_{\mathbb{P}/Y}(-1) \oplus (V \otimes_{\mathbb{C}} j_*\mathcal{O}_{\mathbb{P}}) \to I_{\mathbb{P}/Y}(-1),$$

we have the following diagram

$$\begin{array}{ccccc}
j_*\mathcal{O}_{\mathbb{P}}(-1) & \longrightarrow & V \otimes_{\mathbb{C}} j_*\mathcal{O}_{\mathbb{P}} & \longrightarrow & j_*T_{\mathbb{P}}(-1)\\
\downarrow & & \downarrow & & \|\\
\mathrm{KN}'_0(\mathcal{O}_{Y^+}(1)) & \longrightarrow & I_{\mathbb{P}/Y}(-1) \oplus (V \otimes_{\mathbb{C}} j_*\mathcal{O}_{\mathbb{P}}) & \longrightarrow & j_*T_{\mathbb{P}}(-1)\\
\downarrow & & \downarrow & & \\
I_{\mathbb{P}/Y}(-1) & = & I_{\mathbb{P}/Y}(-1) & &
\end{array}$$

Hence, the object $\mathrm{KN}'_0(\mathcal{O}_{Y^+}(1)) \in \mathrm{D^b}(Y)$ is a sheaf, and if we set $\mathcal{F} = \mathrm{KN}'_0(\mathcal{O}_{Y^+}(1))$, the sheaf $\mathcal{F}$ lies on the exact sequence

$$0 \to j_*\mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{F} \to \mathcal{O}_Y(-1) \to j_*\mathcal{O}_{\mathbb{P}}(-1) \to 0.$$

Recall that $j_*\mathcal{O}_{\mathbb{P}}(-1)$ is a $\mathbb{P}^{N-1}$ object that defines the P-twist P_{-1}. In particular, $\mathrm{Ext}^2_Y(j_*\mathcal{O}_{\mathbb{P}}(-1), j_*\mathcal{O}_{\mathbb{P}}(-1)) = \mathbb{C}\cdot h$. Let $C(h)$ be an object in $\mathrm{D^b}(Y)$ that lies on the exact triangle

$$j_*\mathcal{O}_{\mathbb{P}}(-1)[-2] \xrightarrow{h} j_*\mathcal{O}_{\mathbb{P}}(-1) \to C(h).$$

Then, we have an exact triangle

$$\mathcal{O}_Y(-1)[-1] \to C(h) \to \mathcal{F}.$$

Let $e : C(h) \to \mathcal{F}$ be the morphism that appears in the above triangle.

Next, we compute the objects $P_{-1}(\mathrm{KN}'_0(\mathcal{O}_{Y^+}(a)))$ for $-N+2 \le a \le 1$. Recall that the P-twist P_{-1} is given by

$$P_{-1}(E) := \mathrm{Cone}(C(h) \otimes_{\mathbb{C}} \mathrm{RHom}_Y(j_*\mathcal{O}_{\mathbb{P}}(-1), E) \to E).$$

Since we have

$$\begin{aligned}\mathrm{RHom}_Y(j_*\mathcal{O}_{\mathbb{P}}(-1), \mathcal{O}_Y(b)) &\simeq \mathrm{RHom}_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(-1), j^!\mathcal{O}_Y(b))\\ &\simeq R\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-N+b+1))[-N+1]\end{aligned}$$

by adjunction, we have

$$\mathrm{RHom}_Y(j_*\mathcal{O}_{\mathbb{P}}(-1), \mathcal{O}_Y(b)) = 0$$

for $0 \le b \le N-2$, and hence we have

$$P_{-1}(\mathrm{KN}'_0(\mathcal{O}_{Y^+}(a))) \simeq P_{-1}(\mathcal{O}_Y(-a)) = \mathcal{O}_Y(-a)$$

for $-N+2 \le a \le 0$. It is remaining to compute the object $P_{-1}(\mathrm{KN}'_0(\mathcal{O}_{Y^+}(1))) = P_{-1}(\mathcal{F})$. From the above computation, we have

$$\mathrm{RHom}_Y(j_*\mathcal{O}_{\mathbb{P}}(-1), \mathcal{O}_Y(-1)) \simeq R\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-N))[-N+1] \simeq \mathbb{C}[-2N+2].$$

On the other hand, by the exact triangle

$$j_*\mathcal{O}_{\mathbb{P}}(-1)[-2] \xrightarrow{h} j_*\mathcal{O}_{\mathbb{P}}(-1) \to C(h)$$

that defines $C(h)$ and the computation

$$\mathrm{RHom}_Y(j_*\mathcal{O}_{\mathbb{P}}(-1), j_*\mathcal{O}_{\mathbb{P}}(-1)) = \bigoplus_{i=0}^{N-1} \mathbb{C}[-2i],$$

we have

$$\mathrm{RHom}_Y(j_*\mathcal{O}_{\mathbb{P}}(-1), C(h)) = \mathbb{C} \oplus \mathbb{C}[-2N+1].$$

Hence, by the exact triangle

$$\mathcal{O}_Y(-1)[-1] \to C(h) \xrightarrow{e} \mathcal{F}$$

that we obtained above, we have

$$\mathrm{RHom}_Y(j_*\mathcal{O}_{\mathbb{P}}(-1), \mathcal{F}) = \mathbb{C},$$

and thus, the object $P_{-1}(\mathcal{F})$ lies on the exact triangle

$$C(h) \xrightarrow{\mathrm{ev}} \mathcal{F} \to P_{-1}(\mathcal{F}).$$

Since we have

$$\mathrm{Hom}_Y(C(h), \mathcal{F}) \simeq \mathbb{C}$$

from the above, we have the map $\mathrm{ev} : C(h) \to \mathcal{F}$ coincides with the map $e : C(h) \to \mathcal{F}$ up to non-zero scaler. Therefore, we have

$$P_{-1}(\mathcal{F}) \simeq \mathcal{O}_Y(-1)$$

and hence $P_{-1}(\mathrm{KN}'_0(\mathcal{T}_1^+)) \simeq \mathcal{T}_{N-2}$. This is what we want. □

Theorem 3.5.18 recovers the following result that was first proved by Cautis, and later Addington-Donovan-Meachan in different ways. Our approach that uses non-commutative crepant resolutions and their mutations gives a new alternative proof for their result.

Corollary 3.5.20 ([Cau12b, ADM15])**.** *We have a functor isomorphism*

$$\mathrm{KN}'_{-k} \circ \mathrm{KN}_{N+k} \simeq P_k$$

for all $k \in \mathbb{Z}$.

Proof. Let us consider the next diagram

$$\begin{array}{ccccc}
\mathrm{D^b}(Y) & \xrightarrow{\Psi_{N+k}} & \mathrm{D^b}(\Lambda_{N+k}) & \xleftarrow{\Psi^+_{-k-1}} & \mathrm{D^b}(Y^+) \\
\downarrow{\scriptstyle P_k} & & \downarrow{\scriptstyle \nu^-_{N+k}} & \swarrow{\scriptstyle \Psi^+_{-k}} & \\
\mathrm{D^b}(Y) & \xrightarrow[\Psi_{N+k-1}]{} & \mathrm{D^b}(\Lambda_{N+k-1}). & &
\end{array}$$

Since $(\Psi^+_{-k-1})^{-1} \circ \Psi_{N+k} \simeq \mathrm{KN}_{N+k}$ and $(\Psi_{N+k-1})^{-1} \circ \Psi^+_{-k} \simeq \mathrm{KN}'_{-k}$ by Theorem 3.5.3, we have $\mathrm{KN}'_{-k} \circ \mathrm{KN}_{N+k} \simeq P_k$. □

We note that, in order to prove this corollary, Cautis used an elaborate framework "categorical $\mathfrak{sl}_2$-action" that is established by Cautis, Kamnitzer, and Licata [CKL10, CKL13]. Addington, Donovan, and Meachan provided two different proofs. The first one uses a technique of semi-orthogonal decomposition, and the second one uses the variation of GIT quotients and "window shifts".

Chapter 4

On derived equivalence for Abuaf flop via non-commutative crepant resolutions

This chapter is based on the author's work

[H17b] W. Hara, *On derived equivalence for Abuaf flop: mutation of non-commutative crepant resolutions and spherical twists*, preprint, `https://arxiv.org/abs/1706.04417`.

4.1 Introduction

4.1.1 Motivation

In [Seg16], Segal studied an interesting flop provided by Abuaf, which we call the *Abuaf flop.* Let V be a four dimensional symplectic vector space and $\mathrm{LGr}(V)$ the Lagrangian Grassmannian. Let Y be a total space of a rank 2 bundle $\mathcal{S}(-1)$ on $\mathrm{LGr}(V)$, where $\mathcal{S}$ is the rank 2 subbundle and $\mathcal{O}_{\mathrm{LGr}(V)}(-1) := \bigwedge^2 \mathcal{S}$. Then, Y is a local Calabi-Yau 5-fold. On the other hand, let us consider a projective space $\mathbb{P}(V)$ and put $\mathcal{L} := \mathcal{O}_{\mathbb{P}(V)}(-1)$. By using the symplectic form on V, we have an injective bundle map $\mathcal{L} \hookrightarrow \mathcal{L}^{\perp}$. Let Y' be the total space of a bundle $(\mathcal{L}^{\perp} / \mathcal{L}) \otimes \mathcal{L}^2$. Then, Y' is also a local Calabi-Yau 5-fold and we have an isomorphism $H^0(Y, \mathcal{O}_Y) \simeq H^0(Y', \mathcal{O}_{Y'}) =: R$ of $\mathbb{C}$-algebras. Put $X := \operatorname{Spec} R$. Abuaf observed that the correspondence $Y \to X \leftarrow Y'$ gives an example of 5-dimensional flops. This flop has the nice feature that the contracting loci on either side are not isomorphic. In [Li17], Li proved that a simple flop of dimension at most five is one of the following

(1) a (locally trivial deformation of) standard flop,

(2) a (locally trivial deformation of) Mukai flop,

(3) the Abuaf flop.

Standard flops and Mukai flops are well-studied. Thus it is important to study Abuaf flop from the point of view of Li's classification.

Based on the famous conjecture by Bondal, Orlov, and Kawamata, we expect that Y and Y' are derived equivalent. Segal proved that this expectation is correct. The method of his proof is as follows. He constructed tilting bundles $\mathcal{T}_{\mathrm{S}}$ and $\mathcal{T}'_{\mathrm{S}}$ on Y and Y' respectively, and proved that there is an isomorphism

$$\mathrm{End}_Y(\mathcal{T}_{\mathrm{S}}) \simeq \mathrm{End}_{Y'}(\mathcal{T}'_{\mathrm{S}}).$$

Then, by using a basic theorem for tilting objects, we have a derived equivalence

$$\mathrm{Seg}' : \mathrm{D^b}(Y') \xrightarrow{\sim} \mathrm{D^b}(Y).$$

On the other hand, in [TU10], Toda and Uehara provided a method to construct a tilting bundle under some assumptions (Assumption 4.2.2 and Assumption 4.2.3). The difficulties to use Toda-Uehara's method are the following:

(a) There are few examples known to satisfy their assumptions.

(b) Since Toda-Uehara's construction consists of complicated inductive step, it is difficult to find an explicit description of the resulting tilting bundle in general.

However, we can show the following.

Theorem 4.1.1 (see Section 4.3.2)**.** *Y and Y' satisfy Toda-Uehara's assumptions.*

Hence we obtain new tilting bundles $\mathcal{T}_{\mathrm{T}}$ and $\mathcal{T}'_{\mathrm{T}}$ on Y and Y' respectively. Moreover, fortunately, we can compute the resulting tilting bundles explicitly in this case. By using this explicit description of the tilting bundle, we can show that there is another tilting bundle $\mathcal{T}_{\mathrm{U}}$ on Y that satisfies

$$\mathrm{End}_Y(\mathcal{T}_{\mathrm{U}}) \simeq \mathrm{End}_{Y'}(\mathcal{T}'_{\mathrm{T}}).$$

Therefore, by applying the basic theorem for tilting objects again, we have a new derived equivalence

$$\mathrm{TU}' : \mathrm{D^b}(Y') \to \mathrm{D^b}(Y).$$

Note that a tilting bundle constructed by using Toda-Uehara's method is a canonical one because it provides a projective generator of a perverse heart of the derived category. Thus, it is quite natural to ask the following questions.

Question 4.1.2. (1) What is the relation among three tilting bundles on Y, $\mathcal{T}_{\mathrm{S}}$, $\mathcal{T}_{\mathrm{T}}$, and $\mathcal{T}_{\mathrm{U}}$?

(2) What is the relation between two tilting bundles on Y', $\mathcal{T}'_{\mathrm{S}}$ and $\mathcal{T}'_{\mathrm{T}}$?

(3) What is the relation between two equivalences Seg' and TU'?

The aim of this chapter is to answer these questions.

4.1.2 NCCRs and Iyama-Wemyss's mutations

Set

$$\begin{aligned}\Lambda_{\mathrm{S}} &:= \mathrm{End}_Y(\mathcal{T}_{\mathrm{S}}) = \mathrm{End}_{Y'}(\mathcal{T}'_{\mathrm{S}}),\\ \Lambda_{\mathrm{T}} &:= \mathrm{End}_Y(\mathcal{T}_{\mathrm{T}}),\\ \Lambda_{\mathrm{U}} &:= \mathrm{End}_Y(\mathcal{T}_{\mathrm{U}}) = \mathrm{End}_{Y'}(\mathcal{T}'_{\mathrm{T}}).\end{aligned}$$

Then, these algebras are *non-commutative crepant resolutions* (=NCCRs) of $X = \operatorname{Spec} R$. The notion of NCCR was first introduced by Van den Bergh as a non-commutative analog of crepant resolutions. An NCCR of a Gorenstein domain R is defined as the endomorphism ring $\Lambda := \mathrm{End}_R(M)$ of a reflexive R-module M such that Λ is Cohen-Macaulay as an R-module and its global dimension is finite. As in the commutative case, a Gorenstein domain R may have many different NCCRs. One of the basic ways to compare some NCCRs is to use *Iyama-Wemyss's mutations* (= IW mutations).

Let A be a d-singular Calabi-Yau algebra and M an A-module whose endomorphism ring $\mathrm{End}_A(M)$ is an NCCR of A. Let $N \in \operatorname{add} M$ and consider a right $(\operatorname{add} N^*)$-approximation of M^*

$$a : N_0^* \to M^*$$

(see Definition 2.1.23). Then IW mutation of M at N is defined as $\mu_N(M) := N \oplus \mathrm{Ker}(a)^*$. In [IW14a], Iyama and Wemyss proved that the endomorphism ring of $\mu_N(M)$ is also an NCCR of A and there is a derived equivalence

$$\Phi_N : \mathrm{D^b}(\operatorname{mod} \mathrm{End}_A(M)) \xrightarrow{\sim} \mathrm{D^b}(\operatorname{mod} \mathrm{End}_A(\mu_N(M)))$$

(see Theorem 2.1.25 for more detail).

In many cases, it is observed that important NCCRs are connected by multiple IW mutations. For example, Nakajima proved that, in the case of three dimensional Gorenstein toric singularities associated with reflexive polygons, all *splitting* NCCRs are connected by repeating IW mutations [Nak16]. In addition, the author studied IW mutations of certain NCCRs of the minimal nilpotent orbit closure of type A (Section 3). Also in the case of the Abuaf flop, we can show the following.

Theorem 4.1.3 (= Theorem 4.3.13, Theorem 4.3.15)**.** *The above three NCCRs Λ_{S}, Λ_{T}, and Λ_{U} are connected by multiple IW mutations.*

This result provides an answer to Question 4.1.2 (1) and (2). We prove this theorem by relating IW mutations with *mutations of full exceptional collections* on $\mathrm{D^b}(\mathrm{LGr}(V))$ (see Appendix 4.6).

4.1.3 Flop-Flop=Twist result

Recall that Y and Y' are 5-dimensional local Calabi-Yau varieties. It is known that the derived category of a Calabi-Yau variety normally admits an interesting autoequivalence called a *spherical twist*. Spherical twists arise naturally in mathematical string theory and homological mirror symmetry.

On the other hand, it is widely observed that spherical twists also appear in the context of birational geometry. For example, let us consider a threefold Z that contains a $\mathbb{P}^1$ whose normal bundle is $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. Then, we can contract the curve $\mathbb{P}^1 \subset Z$ and get a diagram of the standard flop

$$Z \xleftarrow{q} \widetilde{Z} \xrightarrow{p} Z'.$$

Bondal and Orlov showed that the functors $Rp_*Lq^* : \mathrm{D^b}(Z) \to \mathrm{D^b}(Z')$ and $Rq_*Lp^* : \mathrm{D^b}(Z') \to \mathrm{D^b}(Z)$ give equivalences of categories. Furthermore, it is known that an autoequivalence obtained by composing two equivalences $(Rq_*Lp^*) \circ (Rp_*Lq^*) \in \mathrm{Auteq}(\mathrm{D^b}(Z))$ is isomorphic to the inverse of the spherical twist associated to $\mathcal{O}_{\mathbb{P}^1}(-1)$. This means that we can obtain a spherical twist by composing two derived equivalences for a flop.

In many other cases, we can also observe "flop-flop=twist" results like the above [ADM15, BB15, Cau12a, H17a, DW16, DW15, Tod07]. We can also show the following "flop-flop=twist" results for the Abuaf flop:

Theorem 4.1.4 (= Theorem 4.4.3, Theorem 4.4.6)**.** *(1) Let us consider a spherical twist $\mathrm{T}_{\mathcal{S}[2]}$ around a 1-term complex $\mathcal{S}[2] = \mathcal{S}|_{\mathrm{LGr}}[2]$ on the zero section $\mathrm{LGr} \subset Y$. Then, we have a functor isomorphism*

$$\mathrm{Seg}' \circ \mathrm{TU}'^{-1} \simeq \mathrm{T}_{\mathcal{S}[2]} \in \mathrm{Auteq}(\mathrm{D^b}(Y)).$$

(2) An autoequivalence $\mathrm{TU}'^{-1} \circ \mathrm{Seg}'$ of $\mathrm{D^b}(Y')$ is isomorphic to a spherical twist $\mathrm{T}_{\mathcal{O}_{PP}(-3)}$ associated to a sheaf $\mathcal{O}_{\mathbb{P}}(-3)$ on the zero-section $\mathbb{P} \subset Y'$:

$$\mathrm{TU}'^{-1} \circ \mathrm{Seg}' \simeq \mathrm{T}_{\mathcal{O}_{\mathbb{P}}(-3)} \in \mathrm{Auteq}(\mathrm{D^b}(Y')).$$

To prove the first statement of the theorem, we provide an explicit description of a Fourier-Mukai kernel of an equivalence TU'. Let $\widetilde{Y}$ be a blowing-up of Y along the zero section $\mathrm{LGr} = \mathrm{LGr}(V)$. Then, the exceptional divisor E of $\widetilde{Y}$ is isomorphic to $\mathbb{P}_{\mathrm{LGr}}(\mathcal{S}(-1))$. Thus we can embed E into the product $\mathrm{LGr}(V) \times \mathbb{P}(V)$ via an injective bundle map $\mathcal{S}(-1) \subset V \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{LGr}}(-1)$. Set $\widehat{Y} := \widetilde{Y} \cup_E (\mathrm{LGr}(V) \times \mathbb{P}(V))$. We prove the following.

Theorem 4.1.5 (= Theorem 4.4.5)**.** *The Fourier-Mukai kernel of the equivalence TU' is given by the structure sheaf of $\widehat{Y}$.*

Note that $\widehat{Y} = Y \times_X Y'$. This is very close to the case of Mukai flops [Kaw02, Nam03].

4.1.4 Multi-mutation=twist result.

We also study a spherical twist from the point of view of NCCRs. Namely, we can understand a spherical twist as a composition of IW mutations in the following way. Let us consider a bundle on Y

$$\mathcal{T}_{\mathrm{U},1} := \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y \oplus \mathcal{O}_Y(1) \oplus \mathcal{S}(1).$$

We can show that this bundle is also a tilting bundle on Y. Put

$$\begin{aligned} M &:= H^0(Y, \mathcal{T}_{\mathrm{U},1}), \\ W' &:= H^0(Y, \mathcal{O}_Y \oplus \mathcal{O}_Y(1) \oplus \mathcal{S}(1)), \text{ and} \\ \Lambda_{\mathrm{U},1} &:= \mathrm{End}_Y(\mathcal{T}_{\mathrm{U},1}) \simeq \mathrm{End}_R(M). \end{aligned}$$

We show that there is an isomorphism of R-modules

$$\mu_{W'}(\mu_{W'}(\mu_{W'}(\mu_{W'}(M)))) \simeq M$$

(Proposition 4.4.8). Furthermore, by using Iyama-Wemyss's theorem, we get an autoequivalence of $\mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U},1})$

$$\nu_{W'} := \Phi_{W'} \circ \Phi_{W'} \circ \Phi_{W'} \circ \Phi_{W'} \in \mathrm{Auteq}(\mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U},1})).$$

This autoequivalence corresponds to a spherical twist in the following sense:

Theorem 4.1.6 (= Theorem 4.4.9)**.** *The autoequivalence $\nu_{W'}$ of $\mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U},1})$ corresponds to a spherical twist*

$$\mathrm{T}_{\mathcal{O}_{\mathrm{LGr}}(-1)} \in \mathrm{Auteq}(\mathrm{D^b}(Y))$$

under the identification $\mathrm{RHom}_Y(\mathcal{T}_{\mathrm{U},1}, -) : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U},1})$.

Donovan and Wemyss proved that, in the case of 3-fold flops, a composition of **two** IW mutation functors corresponds to a spherical-like twist [DW16]. In the case of Mukai flops, the author observed that a composition of **many** IW mutation functors corresponds to a P-twist [H17a].

The author expects that we can observe these "multi-mutation=twist" results for any higher dimensional crepant resolutions. The above theorem provides a "multi-mutation=twist" result for the Abuaf flop.

4.2 Preliminaries

4.2.1 Abuaf flop

First, we explain the geometry of the Abuaf flop briefly. For more details, see [Seg16]. Let V be a four dimensional symplectic vector space. Let $\mathrm{LGr}(V)$ be the Lagrangian Grassmannian of V and $\mathcal{S} \subset V \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{LGr}(V)}$ the rank two universal subbundle. Note that $\mathcal{O}_{\mathrm{LGr}(V)}(1) := \bigwedge^2 \mathcal{S}^*$ is the ample generator of $\mathrm{Pic}(\mathrm{LGr}(V))$, and by this polarization, we can identify the Lagrangian Grassmannian $\mathrm{LGr}(V)$ with the quadric threefold $Q_3 \subset \mathbb{P}^4$. We also note that the canonical embedding $\mathrm{LGr}(V) \subset \mathrm{Gr}(2, V)$ corresponds to a hyperplane cut $Q_3 = Q_4 \cap H \subset Q_4$. Let us consider the total space Y of a vector bundle $\mathcal{S}(-1)$:

$$Y := \mathrm{Tot}(\mathcal{S}(-1)) \xrightarrow{\pi} \mathrm{LGr}(V).$$

Since $\bigwedge^2(\mathcal{S}(-1)) \simeq \mathcal{O}_{\mathrm{LGr}(V)}(-3) \simeq \omega_{\mathrm{LGr}(V)}$, the variety Y is a five dimensional (local) Calabi-Yau variety.

Let $\mathrm{LGr} \subset Y$ be the zero section. Then, we can contract the locus LGr and have a flopping contraction $\phi : Y \to X$. Let $R := \phi_* \mathcal{O}_Y$ and then $X = \operatorname{Spec} R$.

Next, let us consider the 3-dimensional projective space $\mathbb{P}(V)$. By using the symplectic form on V, we can embed the universal line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}(V)}(-1)$ into $\Omega^1_{\mathbb{P}(V)}(1) \simeq \mathcal{L}^{\perp}$. Let us consider a vector bundle $(\mathcal{L}^{\perp} / \mathcal{L}) \otimes \mathcal{L}^2$ and its total space

$$Y' := \mathrm{Tot}((\mathcal{L}^{\perp} / \mathcal{L}) \otimes \mathcal{L}^2) \xrightarrow{\pi'} \mathbb{P}(V).$$

As in the case of Y, we can easily see that Y' is a five dimensional (local) Calabi-Yau variety. If we denote the zero section by $\mathbb{P} \subset Y'$, then we can contract $\mathbb{P}$ and have a flopping contraction $\phi' : Y' \to X$. By combining Y and Y', we have a diagram of a flop

$$\begin{array}{ccc} Y & & Y'. \\ & \searrow^{\phi} \quad {}^{\phi'}\swarrow & \\ & X & \end{array}$$

Let $o \in X$ be the unique singular point of X. Then, in contrast to the case of Atiyah flops or Mukai flops, two fibers $\phi^{-1}(o) = \mathrm{LGr}$ and $\phi'^{-1}(o) = \mathbb{P}$ are not isomorphic to each other. Since this interesting flop was first provided by Abuaf, we call this flop the *Abuaf flop*.

Remark 4.2.1. Note that X is Gorenstein. Indeed, since Y is Calabi-Yau, we have

$$\omega_X \simeq H^0(\omega_{X_{\mathrm{reg}}}) = H^0(\mathcal{O}_{X_{\mathrm{reg}}}) \simeq \mathcal{O}_X.$$

4.2.2 Toda-Uehara's construction for tilting bundles and perverse hearts

Van den Bergh showed in [VdB04a, VdB04b] that if $f : Y \to X$ is a morphism with at most one dimensional fibers and satisfies $Rf_* \mathcal{O}_Y \simeq \mathcal{O}_X$ (e.g. 3-fold flopping contraction), then there is a tilting bundle on Y that is a projective generator of a perverse heart ${}^0\,\mathrm{Per}(Y/X)$. By generalizing his result, Toda and Uehara provided a method to construct a tilting bundle in higher dimensional cases with certain assumptions [TU10]. They also provided a perverse heart ${}^0\,\mathrm{Per}(Y/A_{n-1})$ that contains the tilting bundle as a projective generator. In the present subsection, we recall the construction of Toda-Uehara's tilting bundle.

Let $f : Y \to X = \operatorname{Spec} R$ be a projective morphism from a Noetherian scheme Y to an affine scheme X of finite type. Assume that $Rf_*\mathcal{O}_Y \simeq \mathcal{O}_X$ and $\dim f^{-1}(x) \leq n$ for all $x \in X$. Further, let us assume the following condition holds for Y:

Assumption 4.2.2. *There exists an ample and globally generated line bundle $\mathcal{O}_Y(1)$ such that*

$$H^i(Y, \mathcal{O}_Y(-j)) = 0$$

for $i \geq 2$, $0 < j < n$.

Step 1. In this setting, we inductively define partial tilting bundles $\mathcal{E}_k$ for $0 \leq k \leq n-1$ as follows. First, set $\mathcal{E}_0 := \mathcal{O}_Y$. Assume that $0 < k \leq n-1$. Let r_{k-1} be a minimal number of generators of $\operatorname{Ext}^1_Y(\mathcal{E}_{k-1}, \mathcal{O}_Y(-k))$ over $\operatorname{End}_Y(\mathcal{E}_{k-1})$. Take a r_{k-1} generators of $\operatorname{Ext}^1_Y(\mathcal{E}_{k-1}, \mathcal{O}_Y(-k))$ and consider an exact sequence corresponding to the generators:

$$0 \to \mathcal{O}_Y(-k) \to \mathcal{N}_{k-1} \to \mathcal{E}_{k-1}^{\oplus r_{k-1}} \to 0.$$

If we set $\mathcal{E}_k := \mathcal{E}_{k-1} \oplus \mathcal{N}_{k-1}$, then we can show that $\mathcal{E}_k$ is a partial tilting bundle [TU10, Claim 4.4]. Finally, we obtain a partial tilting bundle $\mathcal{E}_{n-1}$ but this is not a generator in general.

Step 2. Put $A_{n-1} := \operatorname{End}_Y(\mathcal{E}_{n-1})$ and consider the following functors

$$F := \operatorname{RHom}_Y(\mathcal{E}_{n-1}, -) : \mathrm{D^b}(Y) \to \mathrm{D^b}(\operatorname{mod} A_{n-1}),$$
$$G := - \otimes^{\mathrm{L}}_{A_{n-1}} \mathcal{E}_{n-1} : \mathrm{D^b}(\operatorname{mod} A_{n-1}) \to \mathrm{D^b}(Y).$$

Note that G is the left adjoint functor of F. Let us consider an object $F(\mathcal{O}_Y(-n)) := \operatorname{RHom}_Y(\mathcal{E}_{n-1}, \mathcal{O}_Y(-n))$. Let P be a projective A_{n-1}-resolution of $F(\mathcal{O}_Y(-n))$ and $\sigma_{\geq 1}(P)$ the sigma stupid truncation of P. Then, there is a canonical morphism $\sigma_{\geq 1}(P) \to P$. Further, we have a morphism

$$G(\sigma_{\geq 1}(P)) \to G(P) \simeq G(F(\mathcal{O}_Y(-n))) \xrightarrow{\mathrm{adj}} \mathcal{O}_Y(-n).$$

Set

$$\mathcal{N}_{n-1} := \operatorname{Cone}(G(\sigma_{\geq 1}(P)) \to \mathcal{O}_Y(-n))$$

and $\mathcal{E}_n := \mathcal{E}_{n-1} \oplus \mathcal{N}_{n-1}$. This $\mathcal{E}_n$ is a generator of $\mathrm{D^b}(Y)$ but we cannot conclude that $\mathcal{E}_n$ is tilting [TU10, Lemma 4.6].

Step 3. Under the following assumption, we can conclude that $\mathcal{E}_n$ is tilting.

Assumption 4.2.3. For an object $\mathcal{K} \in \mathrm{D}(Y)$, if we have

$$\operatorname{RHom}_Y\left(\bigoplus_{i=0}^{n-1} \mathcal{O}_Y(-i), \mathcal{K}\right) = 0,$$

the equality

$$\operatorname{RHom}_Y\left(\bigoplus_{i=0}^{n-1} \mathcal{O}_Y(-i), \mathcal{H}^k(\mathcal{K})\right) = 0$$

holds for all k.

Theorem 4.2.4 ([TU10])**.** *Assume that the above assumption is satisfied. Then, $\mathcal{E}_n$ is a tilting vector bundle on Y.*

Remark 4.2.5 ([TU10], Remark 4.7)**.** We can also conclude that the object $\mathcal{E}_n$ is a tilting bundle if we assume the vanishing

$$H^{>1}(Y, \mathcal{O}_Y(-n)) = 0$$

instead of Assumption 4.2.3. In this case, the bundle $\mathcal{N}_{n-1}$ lies on an exact sequence

$$0 \to \mathcal{O}_Y(-n) \to \mathcal{N}_{n-1} \to \mathcal{E}_{n-1}^{\oplus r_{n-1}} \to 0,$$

where r_{n-1} is the minimal number of generators of $\mathrm{Ext}^1_Y(\mathcal{E}_{n-1}, \mathcal{O}_Y(-n))$ over A_{n-1}.

Perverse heart. Set $\mathcal{E} := \mathcal{E}_n$ and $A := \mathrm{End}_Y(\mathcal{E})$. By using the above tilting bundle, we have a derived equivalence

$$\Psi_{\mathcal{E}} := \mathrm{RHom}_Y(\mathcal{E}, -) : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\, A).$$

In [TU10], Toda and Uehara also studied the perverse heart

$$^0\,\mathrm{Per}(Y/A_{n-1}) \subset \mathrm{D^b}(Y)$$

that corresponds to $\mathrm{mod}\, A$ under the equivalence $\Psi_{\mathcal{E}}$. The construction of $^0\,\mathrm{Per}(Y/A_{n-1})$ is as follows. First, let us consider a subcategory of $\mathrm{D}(Y)$

$$\mathrm{D}^\dagger(Y) := \{\mathcal{K} \in \mathrm{D}(Y) \mid F(\mathcal{K}) \in \mathrm{D^b}(\mathrm{mod}\, A_{n-1})\}$$

and set

$$\begin{aligned} \mathcal{C} &:= \{\mathcal{K} \in \mathrm{D}(Y) \mid F(\mathcal{K}) = 0\}, \\ \mathcal{C}^{\leq 0} &:= \mathcal{C} \cap \mathrm{D}(Y)^{\leq 0}, \\ \mathcal{C}^{\geq 0} &:= \mathcal{C} \cap \mathrm{D}(Y)^{\geq 0}. \end{aligned}$$

By definition, there is an inclusion $i : \mathcal{C} \hookrightarrow \mathrm{D}^\dagger(Y)$. The advantage to consider the subcategory $\mathrm{D}^\dagger(Y)$ is that we can consider the left and right adjoint of i:

$$i^* : \mathrm{D}^\dagger(Y) \to \mathcal{C}, \quad i^! : \mathrm{D}^\dagger(Y) \to \mathcal{C}.$$

By using these functors, we define the perverse heart $^0\,\mathrm{Per}(Y/A_{n-1})$.

$$^0\,\mathrm{Per}(Y/A_{n-1}) := \{\mathcal{K} \in \mathrm{D}^\dagger(Y) \mid F(\mathcal{K}) \in \mathrm{mod}\, A_{n-1}, i^*\mathcal{K} \in \mathcal{C}^{\leq 0}, i^!\mathcal{K} \in \mathcal{C}^{\geq 0}\}.$$

Theorem 4.2.6 ([TU10] Theorem 5.1)**.** *Under the Assumption 4.2.3, the abelian category $^0\,\mathrm{Per}(Y/A_{n-1})$ is the heart of a bounded t-structure on $\mathrm{D^b}(Y)$, and $\Psi(^0\,\mathrm{Per}(Y/A_{n-1})) = \mathrm{mod}\, A$. In particular, $\mathcal{E}$ is a projective generator of $^0\,\mathrm{Per}(Y/A_{n-1})$.*

4.2.3 Iyama-Wemyss's mutation via tilting bundles

Lemma 4.2.7. *Let $\phi : Y \to X = \operatorname{Spec} R$ be a crepant resolution of an affine Gorenstein normal variety X. Let $\mathcal{W}$ be a vector bundle on Y and*

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{C} \to 0$$

an exact sequence of vector bundles on Y. Assume that

(a) $\mathcal{E} \in \operatorname{add}(\mathcal{W})$,

(b) $\mathcal{W} \oplus \mathcal{K}$ *and* $\mathcal{W} \oplus \mathcal{C}$ *are tilting bundles, and*

(c) $\mathcal{W}$ *contains* $\mathcal{O}_Y$ *as a direct summand.*

Then,

(1) The sequence

$$0 \to f_*\mathcal{K} \to f_*\mathcal{E} \to f_*\mathcal{C} \to 0$$

is exact and provides a right $(\operatorname{add} f_*\mathcal{W})$*-approximation of* $f_*\mathcal{C}$.

(2) The IW mutation functor

$$\Phi_{f_*\mathcal{W}} : \mathrm{D^b}(\operatorname{mod} \operatorname{End}_Y(\mathcal{W} \oplus \mathcal{K})) \xrightarrow{\sim} \mathrm{D^b}(\operatorname{mod} \operatorname{End}_Y(\mathcal{W} \oplus \mathcal{C}))$$

coincides with the functor $\operatorname{RHom}(\operatorname{RHom}_Y(\mathcal{W} \oplus \mathcal{K}, \mathcal{W} \oplus \mathcal{C}), -)$.

Proof. First, note that we have isomorphisms of R-algebras

$$\operatorname{End}_Y(\mathcal{W} \oplus \mathcal{K}) \simeq \operatorname{End}_R(f_*\mathcal{W} \oplus f_*\mathcal{K})$$
$$\operatorname{End}_Y(\mathcal{W} \oplus \mathcal{C}) \simeq \operatorname{End}_R(f_*\mathcal{W} \oplus f_*\mathcal{C})$$

by Lemma 2.1.17.

By the assumption (b) and (c), we have $H^1(Y, \mathcal{K}) = 0$ and thus the sequence

$$0 \to f_*\mathcal{K} \to f_*\mathcal{E} \to f_*\mathcal{C} \to 0$$

is exact. Moreover, as in the proof of Lemma 2.1.17, we have

$$\operatorname{Hom}_Y(\mathcal{W}, \mathcal{E}) \simeq \operatorname{Hom}_R(f_*\mathcal{W}, f_*\mathcal{E}),$$
$$\operatorname{Hom}_Y(\mathcal{W}, \mathcal{C}) \simeq \operatorname{Hom}_R(f_*\mathcal{W}, f_*\mathcal{C}).$$

Since $\operatorname{Ext}^1_Y(\mathcal{W}, \mathcal{K}) = 0$, we have the map

$$\operatorname{Hom}_R(f_*\mathcal{W}, f_*\mathcal{E}) \to \operatorname{Hom}_R(f_*\mathcal{W}, f_*\mathcal{C})$$

is surjective. This shows (1).

Let $V := \operatorname{Hom}_R(f_*\mathcal{W} \oplus f_*\mathcal{K}, f_*\mathcal{W})$ and

$$Q := \operatorname{Image}(\operatorname{Hom}_R(f_*\mathcal{W} \oplus f_*\mathcal{K}, f_*\mathcal{E}) \to \operatorname{Hom}_R(f_*\mathcal{W} \oplus f_*\mathcal{K}, f_*\mathcal{C})).$$

Then, the IW mutation functor is defined as

$$\Phi_{f_*\mathcal{W}} := \mathrm{RHom}(V \oplus Q, -).$$

First we have

$$V := \mathrm{Hom}_R(f_*\mathcal{W} \oplus f_*\mathcal{K}, f_*\mathcal{W}) \simeq \mathrm{Hom}_Y(\mathcal{W} \oplus \mathcal{K}, \mathcal{W})$$

and

$$\mathrm{Hom}_R(f_*\mathcal{W} \oplus f_*\mathcal{K}, f_*\,\mathcal{E}) \simeq \mathrm{Hom}_Y(\mathcal{W} \oplus \mathcal{K}, \mathcal{E}).$$

Since the R-module $\mathrm{Hom}_Y(\mathcal{W}\oplus\mathcal{K}, \mathcal{C})$ is torsion free and isomorphic to $\mathrm{Hom}_R(f_*\mathcal{W}\oplus f_*\mathcal{K}, f_*\mathcal{C})$ in codimension one, the natural map

$$\mathrm{Hom}_Y(\mathcal{W} \oplus \mathcal{K}, \mathcal{C}) \to \mathrm{Hom}_R(f_*\mathcal{W} \oplus f_*\mathcal{K}, f_*\mathcal{C})$$

is injective. Thus, we have the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_Y(\mathcal{W} \oplus \mathcal{K}, \mathcal{E}) & = & \mathrm{Hom}_R(f_*\mathcal{W} \oplus f_*\mathcal{K}, f_*\,\mathcal{E}) \\ \downarrow\!\!\downarrow & & \downarrow \\ \mathrm{Hom}_Y(\mathcal{W} \oplus \mathcal{K}, \mathcal{C}) & \hookrightarrow & \mathrm{Hom}_R(f_*\mathcal{W} \oplus f_*\mathcal{K}, f_*\mathcal{C}). \end{array}$$

Therefore, we have

$$Q = \mathrm{Hom}_Y(\mathcal{W} \oplus \mathcal{K}, \mathcal{C})$$

and hence

$$V \oplus Q \simeq \mathrm{RHom}_Y(\mathcal{W} \oplus \mathcal{K}, \mathcal{W} \oplus \mathcal{C}).$$

This shows (2). □

4.3 Toda-Uehara's tilting bundles and Segal's tilting bundles

4.3.1 Notations

From now on, we fix the following notations.

- $Y := \mathrm{Tot}(\mathcal{S}(-1)) \xrightarrow{\pi} \mathrm{LGr}(V)$.
- $Y' := \mathrm{Tot}((\mathcal{L}^{\perp} / \mathcal{L}) \otimes \mathcal{L}^2) \xrightarrow{\pi'} \mathbb{P}(V)$.
- $\iota : \mathrm{LGr} \hookrightarrow Y$, $\iota' : \mathbb{P} \hookrightarrow Y'$: the zero sections.
- $Y^o := Y \setminus \mathrm{LGr} = Y' \setminus \mathbb{P} = X_{\mathrm{sm}}$.
- $\phi : Y \to X$, $\phi' : Y' \to X$: two crepant resolutions.
- $\mathcal{O}_Y(1) := \pi^*\mathcal{O}_{\mathrm{LGr}(V)}(1)$, $\mathcal{O}_{Y'}(1) := \pi'^*\mathcal{O}_{\mathbb{P}(V)}(1)$.
- We write $\mathcal{S}$ instead of $\pi^*\,\mathcal{S}$.

4.3.2 Toda-Uehara's assumptions for Y and Y'

In the present subsection, we check that Toda-Uehara's assumptions (Assumption 4.2.2 and Assumption 4.2.3) hold for Y and Y'.

First, we check Assumption 4.2.2 holds. This follows from Segal's computation.

Lemma 4.3.1 ([Seg16]). *We have*

(1) $H^{\geq 1}(Y, \mathcal{O}_Y(j)) = 0$ *for* $j \geq -2$.

(2) $H^{>1}(Y', \mathcal{O}_{Y'}(j)) = 0$ *for* $j \geq -3$. *Further, we have* $H^1(Y', \mathcal{O}_{Y'}(j)) = 0$ *for* $j \geq -2$ *and* $H^1(Y', \mathcal{O}_{Y'}(-3)) \simeq \mathbb{C}$.

In particular, pairs $(Y, \mathcal{O}_Y(1))$ *and* $(Y', \mathcal{O}_{Y'}(1))$ *satisfy Assumption 4.2.2.*

Next, we prove the following. Note that the proof is almost same as in the one provided in [TU10, Section 6.2].

Lemma 4.3.2. Y *and* Y' *satisfy Assumption 4.2.3.*

Proof. First, we provide a proof for Y. By using the bundle $\mathcal{O}_Y(1)$, we can embed Y into $\mathbb{P}^4_R$:

$$h : Y \to \mathbb{P}^4_R.$$

Let $g : \mathbb{P}^4_R \to X = \operatorname{Spec} R$ be a projection. Note that the derived category $\mathrm{D^b}(\mathbb{P}^4_R)$ has a semi-orthogonal decomposition

$$\mathrm{D^b}(\mathbb{P}^4_R) = \langle g^* \mathrm{D^b}(X) \otimes \mathcal{O}_{\mathbb{P}^4}(-4), g^* \mathrm{D^b}(X) \otimes \mathcal{O}_{\mathbb{P}^4}(-3), \cdots, g^* \mathrm{D^b}(X) \otimes \mathcal{O}_{\mathbb{P}^4} \rangle.$$

Let $\mathcal{K} \in \mathrm{D}(Y)$ and assume that

$$\mathrm{RHom}_Y \left(\bigoplus_{i=0}^{2} \mathcal{O}_Y(-i), \mathcal{K} \right) = 0.$$

Then, we have

$$h_* \mathcal{K} \in \langle g^* \mathrm{D^b}(X) \otimes \mathcal{O}_{\mathbb{P}^4}(-4), g^* \mathrm{D^b}(X) \otimes \mathcal{O}_{\mathbb{P}^4}(-3) \rangle$$

and hence there is an exact triangle

$$g^* W_{-3} \otimes \mathcal{O}_Y(-3) \to h_* \mathcal{K} \to g^* W_{-4} \otimes \mathcal{O}_Y(-4),$$

where $W_l \in \mathrm{D^b}(X)$. Note that the support of $\mathcal{H}^k(h_*\mathcal{K})$ is contained in Y and the support of $\mathcal{H}^k(W_{-4}) \otimes_R \mathcal{O}_{\mathbb{P}^4_R}(-4)$ is the inverse image of a closed subset of X by g. Thus, the map

$$\mathcal{H}^k(h_*\mathcal{K}) \to \mathcal{H}^k(W_{-4}) \otimes_R \mathcal{O}_{\mathbb{P}^4_R}(-4)$$

should be zero and we have an exact sequence

$$0 \to \mathcal{H}^{k-1}(W_{-4}) \otimes_R \mathcal{O}_{\mathbb{P}^4_R}(-4) \to \mathcal{H}^k(W_{-3}) \otimes_R \mathcal{O}_{\mathbb{P}^4_R}(-3) \to \mathcal{H}^k(h_*\mathcal{K}) \to 0.$$

By using this sequence, we have

$$\mathrm{RHom}_Y\left(\bigoplus_{i=0}^{2}\mathcal{O}_Y(-i),\mathcal{H}^k(\mathcal{K})\right)=0.$$

Next, we prove the lemma for Y'. Let $\mathcal{K}'\in \mathrm{D}(Y')$ and assume that

$$\mathrm{RHom}_{Y'}\left(\bigoplus_{i=0}^{2}\mathcal{O}_{Y'}(-i),\mathcal{K}\right)=0.$$

In this case, by embedding Y' into $\mathbb{P}^3_R$

$$h' : Y' \hookrightarrow \mathbb{P}^3_R,$$

we have

$$h'_*\mathcal{K}' \in \langle \mathrm{D^b}(R)\otimes_R \mathcal{O}_{\mathbb{P}^3_R}(-3)\rangle.$$

Thus, we also have

$$\mathcal{H}^k(h'_*\mathcal{K}') \in \langle \mathrm{D^b}(R)\otimes_R \mathcal{O}_{\mathbb{P}^3_R}(-3)\rangle,$$

and hence we have the result. □

Corollary 4.3.3. *Y (resp. Y') admits a tilting bundle that is a projective generator of the perverse heart ${}^0\operatorname{Per}(Y/A_2)$ (resp. ${}^0\operatorname{Per}(Y'/A'_2)$).*

In the next subsection, we give explicit descriptions of the tilting bundles.

4.3.3 Tilting bundles on Y and Y'

In this subsection, we provide some tilting bundles on Y and Y' explicitly.

Tilting bundles on Y.

Theorem 4.3.4. *For $-2 \le k \le 1$, let $\mathcal{T}_k$ be a vector bundle*

$$\mathcal{T}_k := \mathcal{O}_Y \oplus \mathcal{O}_Y(-1)\oplus\mathcal{O}_Y(-2)\oplus\mathcal{S}(k).$$

Then, $\mathcal{T}_k$ is a tilting bundle on Y.

Proof. By Lemma 4.3.1 (1), the direct sum of line bundles $\mathcal{O}_Y \oplus \mathcal{O}_Y(-1)\oplus \mathcal{O}_Y(-2)$ is a partial tilting bundle on Y. Further, it is easy to see that $\mathcal{S}$ is also a partial tilting bundle. Since $\mathcal{S}^*\simeq\mathcal{S}(1)$, it is enough to show that

$$H^i(Y',\mathcal{S}(j))=0$$

for $j \geq -2$. By adjunction, we have

$$\begin{aligned} H^i(Y', \mathcal{S}(j)) &\simeq H^i(\mathrm{LGr}(V), \bigoplus_{l\geq 0} \mathrm{Sym}^l(\mathcal{S}^*(1)) \otimes \mathcal{S}(j)) \\ &\simeq \bigoplus_{l\geq 0} H^i(\mathrm{LGr}(V), \mathrm{Sym}^l(\mathcal{S}) \otimes \mathcal{S} \otimes \mathcal{O}(2l+j)) \\ &\simeq \bigoplus_{l\geq 0} H^i\left(\mathrm{LGr}(V), \mathrm{Sym}^{l+1}(\mathcal{S})(2l+j) \oplus \mathrm{Sym}^{l-1}(\mathcal{S})(2l+j-1)\right) \end{aligned}$$

By using Borel-Bott-Weil theorem, we can check the vanishing of this cohomology. □

Proposition 4.3.5. *Let us consider*

$$\mathcal{T}_{\mathrm{T}} := \mathcal{T}_{-2} = \mathcal{O}_Y \oplus \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(-2) \oplus \mathcal{S}(-2).$$

Then, $\mathcal{T}_{\mathrm{T}}$ coincides with the bundle on Y constructed by Toda-Uehara's method (up to additive closure), and hence is a projective generator of the perverse heart ${}^0\,\mathrm{Per}(Y/A_2)$.

Proof. Let $\mathcal{E}_k$ $(0 \leq k \leq 2)$ be a partial tilting constructed in Toda-Uehara's inductive steps. By Lemma 4.3.1, we have $\mathcal{E}_k = \bigoplus_{i=0}^k \mathcal{O}_Y(-i)$. Put $A_2 := \mathrm{End}_Y(\mathcal{E}_2)$ and

$$F := \mathrm{RHom}_Y(\mathcal{E}_2, -) : \mathrm{D^b}(Y) \to \mathrm{D^b}(\mathrm{mod}\, A_2).$$

Since there is a semi-orthogonal decomposition

$$\mathrm{D^b}(\mathrm{LGr}(V)) = \langle \mathcal{S}(-2), \mathcal{O}_{\mathrm{LGr}}(-2), \mathcal{O}_{\mathrm{LGr}}(-1), \mathcal{O}_{\mathrm{LGr}} \rangle,$$

we have an exact triangle in $\mathrm{D^b}(\mathrm{LGr}(V))$

$$\mathcal{G} \to \mathcal{O}_{\mathrm{LGr}}(-3) \to \mathcal{S}(-2)^{\oplus 4} \to \mathcal{G}[1],$$

where $\mathcal{G} \in \langle \mathcal{O}_{\mathrm{LGr}}(-2), \mathcal{O}_{\mathrm{LGr}}(-1), \mathcal{O}_{\mathrm{LGr}} \rangle$. Moreover, we have a quasi-isomorphism

$$\mathcal{G}[1] \simeq_{\mathrm{qis}} (\cdots \to 0 \to \mathcal{O}_{\mathrm{LGr}}(-2)^{\oplus 11} \to \mathcal{O}_{\mathrm{LGr}}(-1)^{\oplus 5} \to \mathcal{O}_{\mathrm{LGr}} \to 0 \to \cdots)$$

(note that the degree zero term is $\mathcal{O}_{\mathrm{LGr}}(-2)^{\oplus 11}$, see Lemma 4.6.1 for the proof). Pulling back the above triangle to Y by π, we have an exact triangle

$$\pi^*\mathcal{G} \to \mathcal{O}_Y(-3) \to \mathcal{S}(-2)^{\oplus 4} \to \pi^*\mathcal{G}[1].$$

Now if we denote the left adjoint functor of F by

$$G = - \otimes^{\mathbf{L}}_{A_2} \mathcal{E}_2 : \mathrm{D}^-(A_2) \to \mathrm{D}^-(Y),$$

we have that

$$\pi^*\mathcal{G}[1] \simeq G\left(\cdots \to 0 \to F(\mathcal{O}_Y(-2))^{\oplus 11} \to F(\mathcal{O}_Y(-1))^{\oplus 5} \to F(\mathcal{O}_Y) \to 0 \to \cdots\right).$$

Since $F \circ G \simeq \mathrm{id}$, the complex

$$\big(\cdots \to 0 \to F(\mathcal{O}_Y(-2))^{\oplus 11} \to F(\mathcal{O}_Y(-1))^{\oplus 5} \to F(\mathcal{O}_Y) \to 0 \to \cdots\big)$$

is a projective resolution of a complex $F(\pi^*\mathcal{G}[1])$. On the other hand, since $F(\mathcal{S}(-2))^{\oplus 4} \in \operatorname{mod} A_2$, there is a projective resolution $(P')^\bullet$ of $F(\mathcal{S}(-2))^{\oplus 4}$ such that $(P')^i = 0$ for $i \geq 1$.

From now on, we construct a projective resolution of $F(\mathcal{O}_Y(-3))$ explicitly. First there is not only a morphism in $\mathrm{D^b}(Y)$ but also a complex morphism

$$\mathcal{S}(-2)^{\oplus 4} \to \big(\cdots \to 0 \to \mathcal{O}_Y(-2)^{\oplus 11} \to \mathcal{O}_Y(-1)^{\oplus 5} \to \mathcal{O}_Y \to 0 \to \cdots\big)$$

whose cone is $\mathcal{O}_Y(-3)[1]$. Applying a functor F, we have a morphism

$$F(\mathcal{S}(-2))^{\oplus 4} \to \big(\cdots \to 0 \to F(\mathcal{O}_Y(-2))^{\oplus 11} \to F(\mathcal{O}_Y(-1))^{\oplus 5} \to F(\mathcal{O}_Y) \to 0 \to \cdots\big)$$

that is also a morphism of complexes. Therefore we have a morphism of complexes

$$(P')^\bullet \to \big(\cdots \to 0 \to F(\mathcal{O}_Y(-2))^{\oplus 11} \to F(\mathcal{O}_Y(-1))^{\oplus 5} \to F(\mathcal{O}_Y) \to 0 \to \cdots\big)$$

whose cone is quasi-isomorphic to $F(\mathcal{O}_Y(-3))[1]$. If we have a morphism of complexes, we can compute the cone (in the derived category $\mathrm{D}(\operatorname{mod} A_2)$ explicitly. Using that formula, we have that $F(\mathcal{O}_Y(-3))$ is quasi-isomorphic to a complex $P^\bullet$ such that

$$P^i = \begin{cases} (P')^i & \text{if } i \leq 0 \\ F(\mathcal{O}_Y(-2))^{\oplus 11} & \text{if } i = 1 \\ F(\mathcal{O}_Y(-1))^{\oplus 5} & \text{if } i = 2 \\ F(\mathcal{O}_Y) & \text{if } i = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Since all terms of $P^\bullet$ are projective A_2-modules, $P^\bullet$ is a projective resolution of $F(\mathcal{O}_Y(-3))$, and in particular we have $\sigma_{\geq 1} P^\bullet \simeq F(\mathcal{G})$ and hence we obtain

$$\begin{aligned} G(\sigma_{\geq 1} P^\bullet) &\simeq GF(\pi^*\mathcal{G}) \\ &\simeq GFG\big(\cdots \to 0 \to F(\mathcal{O}_Y(-2))^{\oplus 11} \to F(\mathcal{O}_Y(-1))^{\oplus 5} \to F(\mathcal{O}_Y) \to 0 \to \cdots\big)[-1] \\ &\simeq G\big(\cdots \to 0 \to F(\mathcal{O}_Y(-2))^{\oplus 11} \to F(\mathcal{O}_Y(-1))^{\oplus 5} \to F(\mathcal{O}_Y) \to 0 \to \cdots\big)[-1] \\ &\simeq (\pi^*\mathcal{G}[1])[-1] \simeq \pi^*\mathcal{G}. \end{aligned}$$

Thus the resulting bundle obtained by Toda-Uehara's construction is

$$\mathcal{O}_Y \oplus \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(-2) \oplus \mathcal{S}(-2)^{\oplus 4}.$$

□

Definition 4.3.6. We call the bundle

$$\mathcal{T}_{\mathrm{T}} := \mathcal{T}_{-2} = \mathcal{O}_Y \oplus \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(-2) \oplus \mathcal{S}(-2)$$

Toda-Uehara's tilting bundle on Y. On the other hand, let us consider a bundle

$$\mathcal{T}_{\mathrm{S}} := (\mathcal{T}_0)^* \simeq \mathcal{O}_Y \oplus \mathcal{O}_Y(1) \oplus \mathcal{O}_Y(2) \oplus \mathcal{S}(1).$$

This tilting bundle coincides with the one found by Segal [Seg16]. Thus we call this bundle *Segal's tilting bundle* on Y.

Tilting bundles on Y'.

By Lemma 4.3.1 (2), we have

$$H^1(Y', \mathcal{O}_{Y'}(-3)) \simeq \mathbb{C}\,.$$

Let Σ be a rank 2 vector bundle on Y' that lies on an exact sequence

$$0 \to \mathcal{O}_{Y'}(-1) \to \Sigma \to \mathcal{O}_{Y'}(2) \to 0$$

corresponding to a generator of

$$\mathrm{Ext}^1_{Y'}(\mathcal{O}_{Y'}(2), \mathcal{O}_{Y'}(-1)) \simeq H^1(Y', \mathcal{O}_{Y'}(-3)) \simeq \mathbb{C}\,.$$

Segal's tilting bundle on Y' is given as follows.

Proposition 4.3.7 ([Seg16])**.** *Put*

$$\mathcal{T}'_{\mathrm{S}} := \mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(-1) \oplus \mathcal{O}_{Y'}(-2) \oplus \Sigma(-1),$$

Then, $\mathcal{T}'_{\mathrm{S}}$ is a tilting bundle on $\mathrm{D^b}(Y')$.

On the other hand, by using Toda-Uehara's construction, we have a new tilting bundle.

Proposition 4.3.8. *Put*

$$\mathcal{T}'_{\mathrm{T}} := \mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(-1) \oplus \mathcal{O}_{Y'}(-2) \oplus \Sigma(-2).$$

Then, $\mathcal{T}'_{\mathrm{T}}$ is the Toda-Uehara's tilting bundle on Y', and hence is a projective generator of the perverse heart ${}^0\mathrm{Per}(Y'/A'_2)$, where A'_2 is the endomorphism ring of a vector bundle $\mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(-1) \oplus \mathcal{O}_{Y'}(-2)$.

Proof. If $k \leq 2$, then $H^i(Y', \mathcal{O}_{Y'}(-k)) = 0$ for all $i \geq 1$. Furthermore, we have $H^1(Y', \mathcal{O}_{Y'}(-3)) \simeq \mathbb{C}$ and $H^i(Y', \mathcal{O}_{Y'}(-3)) = 0$ for $i \geq 2$. Recall that the vector bundle $\Sigma(-2)$ lies in an exact sequence

$$0 \to \mathcal{O}_{Y'}(-3) \to \Sigma(-2) \to \mathcal{O}_{Y'} \to 0$$

that corresponds to the generator of $H^1(Y', \mathcal{O}_{Y'}(-3))$. Since

$$\mathrm{Ext}^1_{Y'}(\mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(-1) \oplus \mathcal{O}_{Y'}(-2), \mathcal{O}_{Y'}(-3)) \simeq H^1(Y', \mathcal{O}_{Y'}(-3)),$$

the bundle $\mathcal{T}'_{\mathrm{T}}$ is the Toda-Uehara's tilting bundle on Y' and a projective generator of the perverse heart ${}^0\mathrm{Per}(Y'/A'_2)$ by Remark 4.2.5. $\square$

4.3.4 Derived equivalences for Abuaf flop

In this section, we define derived equivalences induced by tilting bundles.

Put
$$\mathcal{T}_{\mathrm{U}} := (\mathcal{T}_{-1})^* \simeq \mathcal{O}_Y \oplus \mathcal{O}_Y(1) \oplus \mathcal{O}_Y(2) \oplus \mathcal{S}(2).$$

Lemma 4.3.9. *We have the following isomorphism of tilting bundles on Y^o.*

(1) $\mathcal{T}_{\mathrm{S}}|_{Y^o} \simeq \mathcal{T}'_{\mathrm{S}}|_{Y^o}$.

(2) $\mathcal{T}_{\mathrm{U}}|_{Y^o} \simeq \mathcal{T}'_{\mathrm{T}}|_{Y^o}$.

Thus, we have the following isomorphism of R-algebras.

(i) $\mathrm{End}_Y(\mathcal{T}_{\mathrm{S}}) \simeq \mathrm{End}_{Y'}(\mathcal{T}'_{\mathrm{S}})$.

(ii) $\mathrm{End}_Y(\mathcal{T}_{\mathrm{U}}) \simeq \mathrm{End}_{Y'}(\mathcal{T}'_{\mathrm{T}})$.

Proof. In [Seg16], Segal proved that
$$\mathcal{O}_Y(a)|_{Y^o} \simeq \mathcal{O}_{Y'}|_{Y^o} \quad \text{and} \quad \mathcal{S}|_{Y^o} \simeq \Sigma|_{Y^o}.$$
The result follows from these isomorphisms. □

Remark 4.3.10. The vector bundle $\mathcal{T}_{\mathrm{T}}|_{Y^o}$ on Y^o extends to an bundle
$$\mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(1) \oplus \mathcal{O}_{Y'}(2) \oplus \Sigma(2)$$
on Y'. Unfortunately, this bundle is not tilting.

Definition 4.3.11. We set
$$\begin{aligned} \Lambda_{\mathrm{T}} &:= \mathrm{End}_Y(\mathcal{T}_{\mathrm{T}}), \\ \Lambda_{\mathrm{S}} &:= \mathrm{End}_Y(\mathcal{T}_{\mathrm{S}}) = \mathrm{End}_{Y'}(\mathcal{T}'_{\mathrm{S}}), \\ \Lambda_{\mathrm{U}} &:= \mathrm{End}_Y(\mathcal{T}_{\mathrm{U}}) = \mathrm{End}_{Y'}(\mathcal{T}'_{\mathrm{T}}), \end{aligned}$$
and
$$\begin{aligned} \Psi_{\mathrm{T}} &:= \mathrm{RHom}_Y(\mathcal{T}_{\mathrm{T}}, -) : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{T}}), \\ \Psi_{\mathrm{S}} &:= \mathrm{RHom}_Y(\mathcal{T}_{\mathrm{S}}, -) : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{S}}), \\ \Psi_{\mathrm{U}} &:= \mathrm{RHom}_Y(\mathcal{T}_{\mathrm{U}}, -) : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U}}), \\ \Psi'_{\mathrm{T}} &:= \mathrm{RHom}_{Y'}(\mathcal{T}'_{\mathrm{T}}, -) : \mathrm{D^b}(Y') \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U}}), \\ \Psi'_{\mathrm{S}} &:= \mathrm{RHom}_{Y'}(\mathcal{T}'_{\mathrm{S}}, -) : \mathrm{D^b}(Y') \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{S}}). \end{aligned}$$

Definition 4.3.12. Let us consider equivalences of categories that are given as
$$\begin{aligned} \mathrm{Seg} &:= (\Psi'_{\mathrm{S}})^{-1} \circ \Psi_{\mathrm{S}} : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(Y'), \\ \mathrm{Seg}' &:= \mathrm{Seg}^{-1} = (\Psi_{\mathrm{S}})^{-1} \circ \Psi'_{\mathrm{S}} : \mathrm{D^b}(Y') \xrightarrow{\sim} \mathrm{D^b}(Y). \end{aligned}$$

These equivalences are introduced by Segal [Seg16]. Hence we call these functors *Segal's equivalences.*

On the other hand, let us consider the following equivalences

$$\begin{aligned}\mathrm{TU}' &:= \Psi_{\mathrm{U}}^{-1} \circ \Psi'_{\mathrm{T}} := \mathrm{D}^{\mathrm{b}}(Y') \to \mathrm{D}^{\mathrm{b}}(Y)\\ \mathrm{UT} &:= \mathrm{TU}'^{-1} = \Psi_{\mathrm{T}}'^{-1} \circ \Psi_{\mathrm{U}} : \mathrm{D}^{\mathrm{b}}(Y) \to \mathrm{D}^{\mathrm{b}}(Y').\end{aligned}$$

Since we construct these equivalence by using Toda-Uehara's tilting bundle on Y', we call these equivalences TU' and UT *Toda-Uehara's equivalences.*

4.3.5 Segal's tilting vs Toda-Uehara's tilting

In this subsection, we compare Toda-Uehara's tilting bundles with Segal's by using IW mutations. First, we fix the following notations:

$$\begin{aligned}M_a &:= \phi_* \mathcal{O}_Y(a),\\ S_a &:= \phi_* \mathcal{S}(a).\end{aligned}$$

Note that $M_0 = R$. First, we compare two NCCRs Λ_{T} and Λ_{S}.

Theorem 4.3.13. *A derived equivalence of NCCRs*

$$\Psi_{\mathrm{S}} \circ \Psi_{\mathrm{T}}^{-1} \simeq \mathrm{RHom}_{\Lambda_{\mathrm{T}}}(\mathrm{RHom}_Y(\mathcal{T}_{\mathrm{T}}, \mathcal{T}_{\mathrm{S}}), -) : \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_{\mathrm{T}}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_{\mathrm{S}})$$

can be written as a composition of nine IW mutation functors.

To prove the theorem above, we use the following lemma.

Lemma 4.3.14. *Let $\mathcal{W}$ be a vector bundle on a smooth variety Z and*

$$0 \to \mathcal{E}_0 \xrightarrow{a_0} \mathcal{E}_1 \xrightarrow{a_1} \mathcal{E}_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{m-2}} \mathcal{E}_{m-1} \xrightarrow{a_{m-1}} \mathcal{E}_m \to 0$$

a long exact sequence consisting of vector bundles $\mathcal{E}_k$ $(0 \leq k \leq m)$ on Z. Assume that

(a) $\mathcal{W} \oplus \mathcal{E}_0$ and $\mathcal{W} \oplus \mathcal{E}_m$ are tilting bundles.

(b) $\mathcal{E}_k \in \mathrm{add}(\mathcal{W})$ for $1 \leq k \leq m-1$.

Then, $\mathcal{W} \oplus \mathrm{Image}(a_k)$ is a tilting bundle for all $0 \leq k \leq m-1$.

Proof. Since $\mathcal{W} \oplus \mathcal{E}_0$ is a tilting bundle, we have $\mathrm{Ext}^i_Z(\mathcal{W}, \mathcal{E}_0) = \mathrm{Ext}^i_Z(\mathcal{W}, \mathrm{Image}(a_0)) = 0$ for $i \geq 1$. Let $k > 0$ and assume $\mathrm{Ext}^i(\mathcal{W}, \mathrm{Image}(a_{k-1})) = 0$ for $i \geq 1$. Then, by the exact sequence

$$0 \to \mathrm{Image}(a_{k-1}) \to \mathcal{E}_k \to \mathrm{Image}(a_k) \to 0,$$

and the assumption (b), we have $\mathrm{Ext}^i_Z(\mathcal{W}, \mathrm{Image}(a_k)) = 0$ for $i \geq 1$. Thus, we have $\mathrm{Ext}^i_Z(\mathcal{W}, \mathrm{Image}(a_k)) = 0$ for $i \geq 1$ and $0 \leq k \leq m-1$.

Similarly, by using the assumption that $\mathcal{W} \oplus \mathcal{E}_m$ is a tilting bundle, we have $\mathrm{Ext}^i_Z(\mathrm{Image}(a_k), \mathcal{W}) = 0$ for $i \geq 1$ and $0 \leq k \leq m-1$.

Next, let us assume $\mathrm{Image}(a_{k-1})$ is partial tilting. let us consider the exact sequence

$$0 \to \mathrm{Image}(a_{k-1}) \to \mathcal{E}_k \to \mathrm{Image}(a_k) \to 0$$

and apply the functor $\mathrm{RHom}(-, \mathrm{Image}(a_{k-1}))$:

$$\mathrm{RHom}_Z(\mathrm{Image}(a_k), \mathrm{Image}(a_{k-1})) \to \mathrm{RHom}_Z(\mathcal{E}_k, \mathrm{Image}(a_{k-1})) \to \mathrm{RHom}_Z(\mathrm{Image}(a_{k-1}), \mathrm{Image}(a_{k-1})).$$

By the assumption (b) and the above arguments, we have $\mathrm{Ext}^i_Z(\mathcal{E}_k, \mathrm{Image}(a_{k-1})) = 0$ for $i \geq 1$. Therefore, we have $\mathrm{Ext}^i_Z(\mathrm{Image}(a_k), \mathrm{Image}(a_{k-1})) = 0$ for $i \geq 2$.

Consider the sequence

$$0 \to \mathrm{Image}(a_{k-1}) \to \mathcal{E}_k \to \mathrm{Image}(a_k) \to 0$$

again and apply the functor $\mathrm{RHom}_Z(\mathrm{Image}(a_k), -)$:

$$\mathrm{RHom}_Z(\mathrm{Image}(a_k), \mathrm{Image}(a_{k-1})) \to \mathrm{RHom}_Z(\mathrm{Image}(a_k), \mathcal{E}_k) \to \mathrm{RHom}_Z(\mathrm{Image}(a_k), \mathrm{Image}(a_k)).$$

By the assumption (b) and the above arguments, we have $\mathrm{Ext}^i(\mathrm{Image}(a_k), \mathcal{E}_k) = 0$ for $i \geq 1$. Thus, from the above computation, we have $\mathrm{Ext}^i_Z(\mathrm{Image}(a_k), \mathrm{Image}(a_k)) = 0$ for $i \geq 1$.

It is clear that $\mathcal{W} \oplus \mathrm{Image}(a_k)$ is a generator. Thus, the bundle $\mathcal{W} \oplus \mathrm{Image}(a_k)$ is tilting. □

Proof of Theorem 4.3.13. Put

$$\nu\mathcal{T} := \mathcal{O}_Y \oplus \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(-2) \oplus \mathcal{S}(1)$$
$$\nu^2\mathcal{T} := \mathcal{O}_Y \oplus \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(1) \oplus \mathcal{S}(1).$$

By Theorem 4.3.4, these bundles are tilting. Set

$$W_1 := R \oplus M_{-1} \oplus M_{-2}$$
$$W_2 := R \oplus M_{-1} \oplus S_1$$
$$W_3 := R \oplus S_1 \oplus M_1.$$

We will show there are three isomorphisms

$$\mu_{W_1}\mu_{W_1}\mu_{W_1}(\mathrm{End}_Y(\mathcal{T}_{\mathrm{T}})) \simeq \mathrm{End}_Y(\nu\mathcal{T})$$
$$\mu_{W_2}\mu_{W_2}\mu_{W_2}(\mathrm{End}_Y(\nu\mathcal{T})) \simeq \mathrm{End}_Y(\nu^2\mathcal{T})$$
$$\mu_{W_3}\mu_{W_3}\mu_{W_3}(\mathrm{End}_Y(\nu^2\mathcal{T})) \simeq \mathrm{End}_Y(\mathcal{T}_{\mathrm{S}}),$$

and each IW mutation functors can be written as

$$\Phi^3_{W_1} \simeq \mathrm{RHom}_{\Lambda_{\mathrm{T}}}(\mathrm{RHom}(\mathcal{T}_{\mathrm{T}}, \nu\mathcal{T}), -) : \mathrm{D^b}(\Lambda_{\mathrm{T}}) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\mathrm{End}_Y(\nu\mathcal{T}))$$
$$\Phi^3_{W_2} \simeq \mathrm{RHom}_{\mathrm{End}_Y(\nu\mathcal{T})}(\mathrm{RHom}(\nu\mathcal{T}, \nu^2\mathcal{T}), -) : \mathrm{D^b}(\mathrm{mod}\,\mathrm{End}_Y(\nu\mathcal{T})) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\mathrm{End}_Y(\nu^2\mathcal{T}))$$
$$\Phi^3_{W_3} \simeq \mathrm{RHom}_{\mathrm{End}_Y(\nu^2\mathcal{T})}(\mathrm{RHom}(\nu^2\mathcal{T}, \mathcal{T}_{\mathrm{S}}), -) : \mathrm{D^b}(\mathrm{mod}\,\mathrm{End}_Y(\nu^2\mathcal{T})) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_{\mathrm{S}}).$$

First we provide the proof for mutations at W_1. Let us consider an exact sequence

$$0 \to \mathcal{S}(-2) \xrightarrow{a_{-2}} \mathcal{O}_Y(-2)^{\oplus 4} \xrightarrow{a_{-1}} \mathcal{O}_Y(-1)^{\oplus 4} \xrightarrow{a_0} \mathcal{O}_Y^{\oplus 4} \xrightarrow{a_1} \mathcal{S}(1) \to 0.$$

Note that the image of the map a_i is $\mathcal{S}(i)$, and this exact sequence comes from the right mutation of $\mathcal{S}(-2)$ over an (partial) exceptional collection

$$\mathcal{O}_{\mathrm{LGr}}(-2), \mathcal{O}_{\mathrm{LGr}}(-1), \mathcal{O}_{\mathrm{LGr}}$$

of $\mathrm{D^b}(\mathrm{LGr}(V))$. By pushing this exact sequence to X, we have an exact sequence

$$0 \to S_{-2} \xrightarrow{a_{-2}} M_{-2}^{\oplus 4} \xrightarrow{a_{-1}} M_{-1}^{\oplus 4} \xrightarrow{a_0} M_0^{\oplus 4} \xrightarrow{a_1} S_1 \to 0.$$

Splicing this sequence, we have short exact sequences

$$0 \to S_i \xrightarrow{a_i} M_i^{\oplus 4} \xrightarrow{a_{i+1}} S_{i+1} \to 0.$$

for $-2 \leq i \leq 0$. By Lemma 4.2.7, this morphism a_{i+1} is a right $(\operatorname{add} W_1)$-approximation of S_{i+1} for $-2 \leq i \leq 0$ and

$$\mu_{W_1}(W_1 \oplus S_i) = W_1 \oplus S_{i+1}.$$

Let $Q_i := \mathrm{Hom}_R(W_1 \oplus S_i, W_1)$ and

$$C_i := \mathrm{Image}(\mathrm{Hom}_R(W_1 \oplus S_i, M_i^{\oplus 4}) \to \mathrm{Hom}_R(W_1 \oplus S_i, S_{i+1})).$$

Then, IW mutation functor

$$\Phi_{W_1} : \mathrm{D^b}(\operatorname{mod} \mathrm{End}_R(W_1 \oplus S_i)) \to \mathrm{D^b}(\operatorname{mod} \mathrm{End}_R(W_1 \oplus S_{i+1}))$$

is given by

$$\Phi_{W_1}(-) := \mathrm{RHom}_{\mathrm{End}_R(W_1 \oplus S_i)}(Q_i \oplus C_i, -).$$

Again, by Lemma 4.2.7, there is an isomorphism

$$\mathrm{RHom}_Y(\mathcal{W}_1 \oplus \mathcal{S}(i), \mathcal{W}_1 \oplus \mathcal{S}(i+1)) \simeq Q_i \oplus C_i$$

for $-2 \leq i \leq 0$ and hence the following diagram commutes

$$\begin{array}{ccc} \mathrm{D^b}(Y) & \xrightarrow{\Psi_{i+1}} & \\ \downarrow{\scriptstyle \Psi_i} & \searrow & \\ \mathrm{D^b}(\operatorname{mod} \mathrm{End}_Y(\mathcal{W}_1 \oplus \mathcal{S}(i))) & \xrightarrow[\Phi_{W_1}]{} & \mathrm{D^b}(\operatorname{mod} \mathrm{End}_Y(\mathcal{W}_1 \oplus \mathcal{S}(i+1))), \end{array}$$

where $\Psi_i := \mathrm{RHom}_Y(\mathcal{W}_1 \oplus \mathcal{S}(i), -)$. Therefore we have

$$\Phi_{W_1}^3 \simeq \Psi_1 \circ \Psi_{-2}^{-1} \simeq \mathrm{RHom}_{\Lambda_{\mathrm{T}}}(\mathrm{RHom}(\mathcal{T}_{\mathrm{T}}, \nu \mathcal{T}), -).$$

To show the result for W_2, we use an exact sequence

$$0 \to \mathcal{O}_Y(-2) \xrightarrow{b_1} \mathcal{O}_Y(-1)^{\oplus 5} \xrightarrow{b_2} \mathcal{O}_Y^{\oplus 11} \xrightarrow{b_3} \mathcal{S}(1)^{\oplus 4} \xrightarrow{b_4} \mathcal{O}_Y(1) \to 0.$$

Note that this exact sequence coming from a right mutation of $\mathcal{O}_{\mathrm{LGr}}(-2)$ over an (partial) exceptional collection

$$\mathcal{O}_{\mathrm{LGr}}(-1), \mathcal{O}_{\mathrm{LGr}}, \mathcal{S}(1)$$

of $\mathrm{D^b}(\mathrm{LGr}(V))$ (cf. Lemma 4.6.1). Put $\mathcal{W}_2 := \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y \oplus \mathcal{S}(1)$. Then, $\mathcal{W}_2 \oplus \mathcal{O}_Y(-2)$ and $\mathcal{W}_2 \oplus \mathcal{O}_Y(1)$ are tilting bundles by Theorem 4.3.4. Therefore, by Lemma 4.3.14, the bundle $\mathcal{W}_2 \oplus \mathrm{Cok}(b_j)$ is also a tilting bundle for all $1 \leq j \leq 4$. Then the same argument as in the case of W_1 shows the result.

One can show for W_3 by using the same argument. We note that the exact sequence we use in this case is

$$0 \to \mathcal{O}_Y(-1) \xrightarrow{c_1} \mathcal{O}_Y^{\oplus 5} \xrightarrow{c_2} \mathcal{S}(1)^{\oplus 4} \xrightarrow{c_3} \mathcal{O}_Y(1)^{\oplus 5} \xrightarrow{c_4} \mathcal{O}_Y(2) \to 0$$

and

$$\mathrm{Cok}(c_1) \simeq \pi^*(T_{\mathbb{P}^4}(-1)|_{\mathrm{LGr}})$$
$$\mathrm{Cok}(c_2) \simeq \pi^*(\Omega^1_{\mathbb{P}^4}(2)|_{\mathrm{LGr}}).$$

This exact sequence comes from the right mutation of $\mathcal{O}_{\mathrm{LGr}}(-1)$ over an (partial) exceptional collection

$$\mathcal{O}_{\mathrm{LGr}}, \mathcal{S}(1), \mathcal{O}_{\mathrm{LGr}}(1).$$

□

Next, we compare Λ_{S} with Λ_{U}. The IW mutation that connects Λ_{S} and Λ_{U} is much simpler than the one that connects Λ_{T} and Λ_{S}.

Theorem 4.3.15. *Let $W_4 := M_0 \oplus M_1 \oplus M_2$. Λ_{U} is a left IW mutation of Λ_{S} at W_4. Furthermore, if we set the IW functor*

$$\Phi_{W_4} : \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{S}}) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U}}),$$

then the following diagram commutes

$$\begin{array}{ccc} \mathrm{D^b}(Y) & \xrightarrow{\Psi_{\mathrm{S}}} & \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{S}}) \\ & \searrow^{\Psi_{\mathrm{U}}} & \downarrow \Phi_{W_4} \\ & & \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U}}). \end{array}$$

Proof. Let us consider an exact sequence

$$0 \to S_1 \to V \otimes_{\mathbb{C}} M_1 \to S_2 \to 0$$

obtained by pushing an exact sequence

$$0 \to \mathcal{S}(1) \to V \otimes_{\mathbb{C}} \mathcal{O}_Y(1) \to \mathcal{S}(2) \to 0$$

on Y by ϕ. Then, by Lemma 4.2.7, this sequence is a right (add W_4)-approximation of S_2 and we have $\mu_{W_4}(W_4 \oplus S_1) = W_4 \oplus S_2$. The commutativity of the diagram also follows from Lemma 4.2.7. □

Summarizing the above results, we have the following corollary.

Corollary 4.3.16. *Let* Φ *be an equivalence between* $\mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{T}})$ *and* $\mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U}})$ *obtained by composing ten IW mutation functors:*

$$\Phi := \Phi_{W_4} \circ \Phi_{W_3} \circ \Phi_{W_3} \circ \Phi_{W_3} \circ \Phi_{W_2} \circ \Phi_{W_2} \circ \Phi_{W_2} \circ \Phi_{W_1} \circ \Phi_{W_1} \circ \Phi_{W_1}.$$

The equivalence between $\mathrm{D^b}(Y)$ *and* $\mathrm{D^b}(Y')$ *obtained by a composition*

$$\mathrm{D^b}(Y) \xrightarrow{\Psi_{\mathrm{T}}} \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{T}}) \xrightarrow{\Phi} \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U}}) \xrightarrow{\Psi_{\mathrm{T}}^{-1}} \mathrm{D^b}(Y')$$

is the inverse of the functor TU'.

Later, we show that the Fourier-Mukai kernel of the functor TU' is the structure sheaf of $\widetilde{Y} \cup_E (\mathrm{LGr} \times \mathbb{P})$, where $\widetilde{Y}$ is the blowing up of Y (or Y') along the zero section and E is the exceptional divisor. Please compare this corollary with [Wem17, Theorem 4.2] and Corollary 3.5.14.

4.4 Flop-Flop=Twist results and Multi-mutation=Twist result

In this section, we show "flop-flop=twist" results and "multi-mutation=twist" results for the Abuaf flop.

4.4.1 spherical objects

First, we study spherical objects on Y and Y'. For the definition of spherical objects and spherical twists.

Lemma 4.4.1. *(1) Let* $\iota : \mathrm{LGr} \hookrightarrow Y$ *be the zero section. Then, an object* $\iota_* \mathcal{O}_{\mathrm{LGr}} \in \mathrm{D^b}(Y)$ *is a spherical object.*

(2) The universal subbundle $\iota_* \mathcal{S}|_{\mathrm{LGr}}$ *on* LGr *is also a spherical object in* $\mathrm{D^b}(Y)$.

(3) Let $\iota' : \mathbb{P} \hookrightarrow Y'$ *be the zero section. Then, an object* $\iota'_* \mathcal{O}_{\mathbb{P}} \in \mathrm{D^b}(Y')$ *is a spherical object.*

Proof. Here we provide the proof of (2) and (3) only, but one can show (1) by using the same argument. First, we prove (3). The normal bundle $\mathcal{N}_{\mathbb{P}/Y'}$ of the zero section is isomorphic to $(\mathcal{L}^{\perp}/\mathcal{L}) \otimes \mathcal{L}^2$. Note that this bundle lies on the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-3) \to \Omega^1_{\mathbb{P}}(-1) \to \mathcal{N}_{\mathbb{P}/Y'} \to 0.$$

Thus, we have

$$\begin{aligned}
&\mathrm{R}\Gamma(\mathbb{P}, \bigwedge^0 \mathcal{N}_{\mathbb{P}/Y'}) \simeq \mathbb{C}, \\
&\mathrm{R}\Gamma(\mathbb{P}, \bigwedge^1 \mathcal{N}_{\mathbb{P}/Y'}) \simeq 0, \quad \text{and} \\
&\mathrm{R}\Gamma(\mathbb{P}, \bigwedge^2 \mathcal{N}_{\mathbb{P}/Y'}) \simeq \mathrm{R}\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-4)) \simeq \mathbb{C}[-3].
\end{aligned}$$

Let us consider a spectral sequence

$$E_2^{p,q} := H^p(Y', \mathcal{E}xt^q_{Y'}(\iota'_*\mathcal{O}_{\mathbb{P}}, j'_*\mathcal{O}_{\mathbb{P}})) \Rightarrow E^{p+q} = \mathrm{Ext}^{p+q}_{Y'}(\iota'_*\mathcal{O}_{\mathbb{P}}, \iota'_*\mathcal{O}_{\mathbb{P}}).$$

Since we have an isomorphism

$$\mathcal{E}xt^q_{Y'}(\iota'_*\mathcal{O}_{\mathbb{P}}, \iota'_*\mathcal{O}_{\mathbb{P}}) \simeq \iota'_* \bigwedge^q \mathcal{N}_{\mathbb{P}/Y'},$$

we have

$$E_2^{p,q} = \begin{cases} \mathbb{C} & \text{if } p = q = 0 \text{ or } p = 3, q = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\mathrm{Ext}^i_{Y'}(\iota'_*\mathcal{O}_{\mathbb{P}}, \iota'_*\mathcal{O}_{\mathbb{P}}) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ or } i = 5, \\ 0 & \text{otherwise.} \end{cases}$$

Since Y' is Calabi-Yau, the condition $\iota'_*\mathcal{O}_{\mathbb{P}} \otimes \omega_{Y'} \simeq \iota'_*\mathcal{O}_{\mathbb{P}}$ is trivially satisfied. Hence the object $\iota'_*\mathcal{O}_{\mathbb{P}}$ is a spherical object.

Next, we prove (2). Note that we have

$$\begin{aligned}
\mathcal{E}xt^i_Y(\iota_*\mathcal{S}|_{\mathrm{LGr}}, \iota_*\mathcal{S}|_{\mathrm{LGr}}) &\simeq \mathcal{E}xt^i_Y(\iota_*\mathcal{O}_{\mathrm{LGr}}, \iota_*\mathcal{O}_{\mathrm{LGr}}) \otimes \mathcal{S}^* \otimes \mathcal{S} \\
&\simeq \begin{cases} (\mathrm{Sym}^2 \mathcal{S})(1)|_{\mathrm{LGr}} \oplus \mathcal{O}_{\mathrm{LGr}} & \text{if } i = 0 \\ \left(\mathrm{Sym}^3(\mathcal{S}) \oplus \mathcal{S}(-1)^{\oplus 2}\right)|_{\mathrm{LGr}} & \text{if } i = 1 \\ (\mathrm{Sym}^2 \mathcal{S})(-2)|_{\mathrm{LGr}} \oplus \mathcal{O}_{\mathrm{LGr}}(-3) & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

By using Borel-Bott-Weil theorem and a spectral sequence

$$E_2^{p,q} = H^p(Y, \mathcal{E}xt^q_Y(\iota_*\mathcal{S}|_{\mathrm{LGr}}, \iota_*\mathcal{S}|_{\mathrm{LGr}})) \Rightarrow E^{p+q} = \mathrm{Ext}^{p+q}_Y(\iota_*\mathcal{S}|_{\mathrm{LGr}}, \iota_*\mathcal{S}|_{\mathrm{LGr}}),$$

we have the result. ∎

4.4.2 On the side of Y'

In the present subsection, we prove a "flop-flop=twist" result on the side of Y'. The next lemma is a key of the proof of Theorem 4.4.3, which provides a "flop-flop=twist" result.

Lemma 4.4.2. *There is an exact sequence*

$$0 \to \Sigma(-1) \to V \otimes_{\mathbb{C}} \mathcal{O}_{Y'}(-1) \to \Sigma(-2) \to \iota'_* \mathcal{O}_{\mathbb{P}}(-3) \to 0$$

on Y'.

Proof. On Y, there is a canonical exact sequence

$$0 \to \mathcal{S}(1) \to V \otimes_{\mathbb{C}} \mathcal{O}_Y(1) \to \mathcal{S}(2) \to 0.$$

By restricting this sequence on Y^o and then extending it to Y', we have a left exact sequence

$$0 \to \Sigma(-1) \to V \otimes_{\mathbb{C}} \mathcal{O}_{Y'}(1) \xrightarrow{a} \Sigma(2).$$

Thus, it is enough to show that $\mathrm{Cok}(a) \simeq \mathcal{O}_{\mathbb{P}}(-3)$.

Let us consider two open immersions $j' : Y^o \hookrightarrow Y'$ and $\tilde{j} : Y^o \hookrightarrow \widetilde{Y}$. Since $\tilde{j}$ is an affine morphism, we have an exact sequence

$$0 \to \tilde{j}_* \Sigma(-1)|_{Y^o} \to V \otimes_{\mathbb{C}} \tilde{j}_* \mathcal{O}_{Y'}(1)|_{Y^o} \to \tilde{j}_* \Sigma(2)|_{Y^o} \to 0$$

and an isomorphism

$$Rj'_*(\Sigma(-1)|_{Y^o}) \simeq R\bar{p}_* \tilde{j}_* \Sigma(-1)|_{Y^o}.$$

On the other hand, we have an exact sequence

$$0 \to \mathcal{O}_{\widetilde{Y}} \to \tilde{j}_* \mathcal{O}_{Y^o} \to \bigoplus_{d=1}^{\infty} \mathcal{O}_E(dE) \to 0.$$

From this exact sequence and the projection formula, we have

$$\begin{aligned} R^1 j'_*(\Sigma(-1)|_{Y^o}) &\simeq R^1 \bar{p}_* \tilde{j}_* \Sigma(-1)|_{Y^o} \\ &\simeq \Sigma(-1)|_{\mathbb{P}} \otimes \bigoplus_{d \geq 1} R^1 \bar{p}_* \mathcal{O}_E(dE) \\ &\simeq \Sigma(-1)|_{\mathbb{P}} \otimes \bigoplus_{d \geq 1} \mathrm{Sym}^{d-2}(\mathcal{L}^{\perp} / \mathcal{L}) \otimes \mathcal{L}^{2d} \\ &\simeq \bigoplus_{d \geq 1} \left(\mathrm{Sym}^{d-2}(\mathcal{L}^{\perp} / \mathcal{L}) \otimes \mathcal{L}^{2d+2} \right) \oplus \left(\mathrm{Sym}^{d-2}(\mathcal{L}^{\perp} / \mathcal{L}) \otimes \mathcal{L}^{2d-1} \right). \end{aligned}$$

Since the sheaf $\mathrm{Cok}(a)$ is a subsheaf of $R^1 j'_*(\Sigma(-1)|_{Y^o})$, the map $\Sigma(-2) \to R^1 j'_*(\Sigma(-1)|_{Y^o})$ factors through as

$$\Sigma(-2) \to \Sigma(-2)|_{\mathbb{P}} \twoheadrightarrow \mathrm{Cok}(a) \hookrightarrow R^1 j'_*(\Sigma(-1)|_{Y^o}).$$

Note that $\Sigma(-2)|_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(-3)$. It is easy to observe that two sheaves (on $\mathbb{P}$)

$$\mathrm{Sym}^{d-2}(\mathcal{L}^{\perp} / \mathcal{L}) \otimes \mathcal{L}^{2d+2} \quad \text{and} \quad \mathrm{Sym}^{d-2}(\mathcal{L}^{\perp} / \mathcal{L}) \otimes \mathcal{L}^{2d-1}$$

do not have global sections for all $d \geq 1$. Thus, $\mathrm{Cok}(a)$ is a torsion free sheaf on $\mathbb{P}$ that can be written as a quotient of $\mathcal{O}_{\mathbb{P}}(-3)$. This means we have $\mathrm{Cok}(a) \simeq \mathcal{O}_{\mathbb{P}}(-3)$. □

Theorem 4.4.3. *We have a functor isomorphism*

$$\mathrm{UT} \circ \mathrm{Seg}' \simeq \mathrm{T}_{\iota'_* \mathcal{O}_{\mathbb{P}}(-3)} \in \mathrm{Auteq}(\mathrm{D^b}(Y')).$$

Proof. We have to show the following diagram commutes

$$\begin{array}{ccc} \mathrm{D^b}(Y') & \xrightarrow{\mathrm{T}^{-1}_{\iota'_*\mathcal{O}_{\mathbb{P}}(-3)}} & \mathrm{D^b}(Y') \\ {\scriptstyle \mathrm{TU}'}\downarrow & & \downarrow{\scriptstyle \Psi'_{\mathrm{S}}} \\ \mathrm{D^b}(Y) & \xrightarrow{\Psi_{\mathrm{S}}} & \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{S}}) \end{array}$$

Note that we have

$$\Psi'_{\mathrm{S}} \circ \mathrm{T}^{-1}_{\iota'_*\mathcal{O}_{\mathbb{P}}(-3)} \simeq \mathrm{RHom}_{Y'}(\mathrm{T}_{\iota'_*\mathcal{O}_{\mathbb{P}}(-3)}(\mathcal{T}'_{\mathrm{S}}), -)$$
$$\Psi_{\mathrm{S}} \circ \mathrm{TU}' \simeq \mathrm{RHom}_{Y'}(\mathrm{TU}'^{-1}(\mathcal{T}_{\mathrm{S}}), -)$$

and

$$\mathrm{T}_{\iota'_*\mathcal{O}_{\mathbb{P}}(-3)}(\mathcal{T}'_{\mathrm{S}}) \simeq \mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(-1) \oplus \mathcal{O}_{Y'}(-2) \oplus \mathrm{T}_{\iota'_*\mathcal{O}_{\mathbb{P}}(-3)}(\Sigma(-1))$$
$$\mathrm{UT}(\mathcal{T}_{\mathrm{S}}) \simeq \mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(-1) \oplus \mathcal{O}_{Y'}(-2) \oplus \mathrm{UT}(\Sigma(-1)).$$

Thus, it is enough to show that

$$\mathrm{T}_{j'_*\mathcal{O}_{\mathbb{P}}(-3)}(\Sigma(-1)) \simeq \mathrm{UT}(\mathcal{S}(1)).$$

Applying the functor UT to the exact sequence

$$0 \to \mathcal{S}(1) \to V \otimes_{\mathbb{C}} \mathcal{O}_Y(1) \to \mathcal{S}(2) \to 0,$$

we have an exact triangle on $\mathrm{D^b}(Y')$

$$\mathrm{UT}(\mathcal{S}(1)) \to V \otimes_{\mathbb{C}} \mathcal{O}_{Y'}(-1) \to \Sigma(-2) \to \mathrm{UT}(\mathcal{S}(1))[1].$$

On the other hand, by using an exact sequence

$$0 \to \mathcal{O}_{Y'}(-2) \to \Sigma(-1) \to \mathcal{O}_{Y'}(1) \to 0,$$

we have

$$\begin{aligned} \mathrm{RHom}_{Y'}(\iota'_*\mathcal{O}_{\mathbb{P}}(-3), \Sigma(-1)) &\simeq \mathrm{RHom}_{Y'}(\iota'_*\mathcal{O}_{\mathbb{P}}(-3), \mathcal{O}_{Y'}(1)) \\ &\simeq \mathrm{RHom}_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(-3), \mathcal{O}_{Y'}(1) \otimes \omega_{\mathbb{P}})[-2] \\ &\simeq \mathbb{C}[-2]. \end{aligned}$$

The non-trivial extension that corresponds to a generator of $\mathrm{Ext}^2_{Y'}(\iota'_*\mathcal{O}_{\mathbb{P}}(-3), \Sigma(-1))$ is the one that was given in Lemma 4.4.2. Thus the object $\mathrm{T}_{\iota'_*\mathcal{O}_{\mathbb{P}}(-3)}(\Sigma(-1))$ defined by a triangle

$$\iota'_*\mathcal{O}_{\mathbb{P}}(-3)[-2] \to \Sigma(-1) \to \mathrm{T}_{\iota'_*\mathcal{O}_{\mathbb{P}}(-3)}(\Sigma(-1))$$

is quasi-isomorphic to a complex

$$(\cdots \to 0 \to 0 \to V \otimes_{\mathbb{C}} \mathcal{O}_{Y'}(-1) \to \Sigma(-2) \to 0 \to 0 \to \cdots)$$

whose degree zero part is $V \otimes_{\mathbb{C}} \mathcal{O}_{Y'}(-1)$. Hence there is an exact triangle

$$\mathrm{T}_{\iota'_*\mathcal{O}_{\mathbb{P}}(-3)}(\Sigma(-1)) \to V \otimes_{\mathbb{C}} \mathcal{O}_{Y'}(-1) \to \Sigma(-2) \to \mathrm{T}_{\iota'_*\mathcal{O}_{\mathbb{P}}(-3)}(\Sigma(-1))[1].$$

Therefore, we have the desired isomorphism

$$\mathrm{UT}(\mathcal{S}(1)) \simeq \mathrm{T}_{\iota'_*\mathcal{O}_{\mathbb{P}}(-3)}(\Sigma(-1)).$$

□

4.4.3 The kernel of the equivalence TU'

In the same way as in Theorem 4.4.3, we can prove a "flop-flop=twist" result on Y. However, to prove this, we need the geometric description of the equivalence TU'. In the present subsection, we provide a Fourier-Mukai kernel of the equivalence TU'.

Lemma 4.4.4. *There is an exact sequence*

$$0 \to \mathcal{O}_Y(3) \to \mathcal{S}(2) \xrightarrow{b} \mathcal{O}_Y \to \mathcal{O}_{\mathrm{LGr}} \to 0$$

on Y.

Proof. On Y', there is an exact sequence

$$0 \to \mathcal{O}_{Y'}(-3) \to \Sigma(-2) \to \mathcal{O}_{Y'} \to 0.$$

Restricting on Y^o and then extending to Y, we have a left exact sequence

$$0 \to \mathcal{O}_Y(3) \to \mathcal{S}(2) \xrightarrow{b} \mathcal{O}_Y.$$

Thus, it is enough to show that $\mathrm{Cok}(b) \simeq \mathcal{O}_{\mathrm{LGr}}$.

Note that this sequence cannot be right exact. Indeed, if this is a right exact sequence, then the sequence is locally split. This contradicts to the fact that there is no non-trivial morphism from $\mathcal{S}(2)$ to $\mathcal{O}_{\mathrm{LGr}}$ on LGr.

Let $j : Y^o \hookrightarrow Y$ be an open immersion. As in the proof of Lemma 4.4.2, we have

$$R^1 j_*\mathcal{O}_Y(3)|_{Y^o} \simeq \mathcal{O}_{\mathrm{LGr}}(3) \otimes \bigoplus_{d \geq 1} \mathrm{Sym}^{d-2}(\mathcal{S}(-1)) \otimes \omega_{\mathrm{LGr}}.$$

In particular, $R^1 j_* \mathcal{O}_Y(3)|_{Y^o}$ is a vector bundle on the zero section LGr, and hence its subsheaf $\mathrm{Cok}(b)$ is a torsion free sheaf on LGr. In particular, the surjective morphism $\mathcal{O}_Y \to \mathrm{Cok}(b)$ factors through a morphism $\mathcal{O}_{\mathrm{LGr}} \to \mathrm{Cok}(b)$ which is also surjective. Since the sheaf $\mathrm{Cok}(b)$ is torsion free sheaf on LGr, the surjective morphism $\mathcal{O}_{\mathrm{LGr}} \to \mathrm{Cok}(b)$ should be an isomorphism. $\square$

Let $\widetilde{Y}$ be a blowing up of Y along the zero section LGr (or equivalently, of Y' along the zero section $\mathbb{P}$). Then, the exceptional divisor E is isomorphic to $\mathbb{P}_{\mathrm{LGr}}(\mathcal{S}(-1))$ and can be embedded into $\mathrm{LGr} \times \mathbb{P}$ via an injective bundle map $\mathcal{S}(-1) \hookrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{LGr}}(-1)$. Put

$$\widehat{Y} := \widetilde{Y} \cup_E (\mathrm{LGr} \times \mathbb{P}).$$

The aim of the present section is to prove that a Fourier-Mukai functor from $\mathrm{D^b}(Y')$ to $\mathrm{D^b}(Y)$ whose kernel is $\mathcal{O}_{\widehat{Y}}$ gives the equivalence TU'. Note that $\widehat{Y} \simeq Y \times_X Y'$. This is very close to the case of Mukai flops ([Kaw02, Nam03]. See also [TU10, Example 5.3] and [H17a]).

Theorem 4.4.5. *The Fourier-Mukai kernel of the equivalence*

$$\mathrm{TU}' : \mathrm{D^b}(Y') \to \mathrm{D^b}(Y)$$

is given by the structure sheaf of $\widehat{Y}$*:*

$$\mathrm{TU}' \simeq \mathrm{FM}^{Y' \to Y}_{\mathcal{O}_{\widehat{Y}}}.$$

Proof. Let $\mathrm{FM}_{\mathcal{O}_{\widehat{Y}}} : \mathrm{D^b}(Y') \to \mathrm{D^b}(Y)$ be a Fourier-Mukai functor whose kernel is $\mathcal{O}_{\widehat{Y}}$. What we want to show is the commutativity of the following diagram

$$\begin{array}{ccc} \mathrm{D^b}(Y) & \xrightarrow{\mathrm{FM}^!_{\mathcal{O}_{\widehat{Y}}}} & \mathrm{D^b}(Y') \\ & {\scriptstyle \Psi_{\mathrm{U}}} \searrow \quad \swarrow {\scriptstyle \Psi'_{\mathrm{T}}} & \\ & \mathrm{D^b}(\Lambda_{\mathrm{U}}) & \end{array}$$

where $\mathrm{FM}^!_{\mathcal{O}_{\widehat{Y}}}$ is the right adjoint functor of $\mathrm{FM}_{\mathcal{O}_{\widehat{Y}}}$. Since there is a functor isomorphism

$$\begin{aligned} \Psi'_{\mathrm{T}} \circ \mathrm{FM}^!_{\mathcal{O}_{\widehat{Y}}} &= \mathrm{RHom}_{Y'}(\mathcal{T}'_{\mathrm{T}}, \mathrm{FM}^!_{\mathcal{O}_{\widehat{Y}}}(-)) \\ &\simeq \mathrm{RHom}_Y(\mathrm{FM}_{\mathcal{O}_{\widehat{Y}}}(\mathcal{T}'_{\mathrm{T}}), -), \end{aligned}$$

it is enough to show that

$$\mathrm{FM}_{\mathcal{O}_{\widehat{Y}}}(\mathcal{T}'_{\mathrm{T}}) \simeq \mathcal{T}_{\mathrm{U}}.$$

By computations using an exact sequence

$$0 \to \mathcal{O}_{\widehat{Y}} \to \mathcal{O}_{\widetilde{Y}} \oplus \mathcal{O}_{\mathrm{LGr} \times \mathbb{P}} \to \mathcal{O}_E \to 0,$$

we have

$$\mathrm{FM}_{\mathcal{O}_{\widehat{Y}}}(\mathcal{O}_{Y'}(-a)) \simeq \mathcal{O}_Y(a)$$

for $0 \le a \le 2$ and

$$\mathcal{H}^i(\mathrm{FM}_{\mathcal{O}_{\widehat{Y}}}(\mathcal{O}_{Y'}(-3))) \simeq \begin{cases} \mathcal{O}_Y(3) & \text{if } i = 0 \\ \mathcal{O}_{\mathrm{LGr}} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the following exact sequence

$$0 \to \mathcal{O}_{Y'}(-3) \to \Sigma(-2) \to \mathcal{O}_{Y'} \to 0.$$

Applying the functor $\mathrm{FM}_{\mathcal{O}_{\widehat{Y}}}$ to this sequence and taking the cohomology long exact sequence, we have that $\mathrm{FM}_{\mathcal{O}_{\widehat{Y}}}(\Sigma(-2))$ is a coherent sheaf on Y appearing in a sequence

$$0 \to \mathcal{O}_Y(3) \to \mathrm{FM}_{\mathcal{O}_{\widehat{Y}}}(\Sigma(-2)) \to \mathcal{O}_Y \to \mathcal{O}_{\mathrm{LGr}} \to 0.$$

Since $\mathrm{Ext}^1_Y(I_{\mathrm{LGr}/Y}, \mathcal{O}_Y(3)) \simeq \mathrm{Ext}^2_Y(\mathcal{O}_{\mathrm{LGr}}, \mathcal{O}_Y(3)) \simeq \mathbb{C}$, this exact sequence coincides with the one given in Lemma 4.4.4. Therefore, we have $\mathrm{FM}_{\mathcal{O}_{\widehat{Y}}}(\Sigma(-2)) \simeq \mathcal{S}(2)$. □

4.4.4 On the side of Y

Finally, we prove the "flop-flop=twist" result for Y.

Theorem 4.4.6. *Let us consider a spherical twist* $\mathrm{T}_{\iota_*(\mathcal{S})[2]} \in \mathrm{Auteq}(\mathrm{D^b}(Y))$ *around a sheaf* $\iota_*(\mathcal{S})[2] = \iota_*(\mathcal{S}|_{\mathrm{LGr}})[2]$ *on* LGr*. Then, we have a functor isomorphism*

$$\mathrm{Seg}' \circ \mathrm{UT} \simeq \mathrm{T}_{\iota_*(\mathcal{S})[2]} \in \mathrm{Auteq}(\mathrm{D^b}(Y)).$$

Proof. By Theorem 4.4.3, it is enough to show that

$$\mathrm{TU}'(\mathcal{O}_{\mathbb{P}}(-3)) \simeq \mathcal{S}|_{\mathrm{LGr}}[2].$$

By the proof of Theorem 4.4.3, we have a distinguished triangle

$$\Sigma(-1) \to \mathrm{UT}(\mathcal{S}(1)) \to \mathcal{O}_{\mathbb{P}}(-3)[-1] \to \Sigma(-1)[1].$$

Applying a functor TU' to this sequence, we have

$$\mathrm{TU}'(\Sigma(-1)) \to \mathcal{S}(1) \to \mathrm{TU}'(\mathcal{O}_{\mathbb{P}}(-3))[-1] \to \mathrm{TU}'(\Sigma(-1))[1].$$

Thus, we have to compute the object $\mathrm{TU}'(\Sigma(-1))$. Let us consider the exact sequence

$$0 \to \mathcal{O}_{Y'}(-2) \to \Sigma(-1) \to \mathcal{O}_{Y'}(1) \to 0$$

on Y'. By applying the functor TU', we have an exact triangle

$$\mathcal{O}_Y(2) \to \mathrm{TU}'(\Sigma(-1)) \to \mathrm{TU}'(\mathcal{O}_{Y'}(1)) \to \mathcal{O}_Y(2)[1].$$

Then, by a computation using Theorem 4.4.5, we obtain that $\mathrm{TU}'(\mathcal{O}_{Y'}(1))$ lies in the following triangle

$$\mathrm{TU}'(\mathcal{O}_{Y'}(1)) \to I_{\mathrm{LGr}\,/Y}(-1) \oplus (V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{LGr}}) \to \mathcal{S}^*|_{\mathrm{LGr}} \to \mathrm{TU}'(\mathcal{O}_{Y'}(1))[1].$$

Moreover, by considering the following diagram

$$\begin{array}{ccccc} & & V \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{LGr}} & = & V \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{LGr}} \\ & & \downarrow & & \downarrow \\ \mathrm{TU}'(\mathcal{O}_{Y'}(1)) & \longrightarrow & I_{\mathrm{LGr}\,/Y}(-1) \oplus (V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{LGr}}) & \longrightarrow & \mathcal{S}^*|_{\mathrm{LGr}} \\ \| & & \downarrow & & \downarrow \\ \mathrm{TU}'(\mathcal{O}_{Y'}(1)) & \longrightarrow & I_{\mathrm{LGr}\,/Y}(-1) & \longrightarrow & \mathcal{S}|_{\mathrm{LGr}}[1], \end{array}$$

we have that $\mathrm{TU}'(\mathcal{O}_{Y'}(1))$ lies on the following sequence

$$\mathrm{TU}'(\mathcal{O}_{Y'}(1)) \to I_{\mathrm{LGr}\,/Y} \to \mathcal{S}|_{\mathrm{LGr}}[1] \to \mathrm{TU}'(\mathcal{O}_{Y'}(1))[1].$$

On the other hand, by Lemma 4.4.4 and the construction of morphisms, we have the following morphism between exact triangles

$$\begin{array}{ccccccc} \mathcal{O}_Y(2) & \longrightarrow & \mathrm{TU}'(\Sigma(-1)) & \longrightarrow & \mathrm{TU}'(\mathcal{O}_{Y'}(1)) & \longrightarrow & \mathcal{O}_Y(2)[1] \\ \| & & \downarrow & & \downarrow & & \| \\ \mathcal{O}_Y(2) & \longrightarrow & \mathcal{S}(1) & \longrightarrow & I_{\mathrm{LGr}\,/Y}(-1) & \longrightarrow & \mathcal{O}_Y(2)[1]. \end{array}$$

Summarizing the above computations, we have

$$\begin{aligned} \mathrm{TU}'(\mathcal{O}_{\mathbb{P}}(-3)) &\simeq \mathrm{Cone}(\mathrm{TU}'(\Sigma(-1)) \to \mathcal{S}(1))[1] \\ &\simeq \mathrm{Cone}(\mathrm{TU}'(\mathcal{O}_{Y'}(1)) \to I_{\mathrm{LGr}\,/Y})[1] \\ &\simeq \mathcal{S}|_{\mathrm{LGr}}[2]. \end{aligned}$$

□

4.4.5 Another Flop-Flop=twist result

Put

$$\begin{aligned} \mathcal{T}_{\mathrm{U},1} &:= \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y \oplus \mathcal{O}_Y(1) \oplus \mathcal{S}(1), \\ \mathcal{T}'_{\mathrm{T},1} &:= \mathcal{O}_{Y'}(1) \oplus \mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(-1) \oplus \Sigma(-1), \\ \Lambda_{\mathrm{U},1} &:= \mathrm{End}_Y(\mathcal{T}_{\mathrm{U},1}) = \mathrm{End}_{Y'}(\mathcal{T}'_{\mathrm{T},1}). \end{aligned}$$

Note that $\mathcal{T}_{\mathrm{U},1}$ was denoted by $\nu^2\mathcal{T}$ in Theorem 4.3.13. Let us consider derived equivalences

$$\begin{aligned} \Psi_{\mathrm{U},1} &:= \mathrm{RHom}_Y(\mathcal{T}_{\mathrm{U},1}, -) : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U},1}), \\ \Psi'_{\mathrm{T},1} &:= \mathrm{RHom}_{Y'}(\mathcal{T}'_{\mathrm{T},1}, -) : \mathrm{D^b}(Y') \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U},1}), \\ \mathrm{UT}_1 &:= (\Psi'_{\mathrm{T},1})^{-1} \circ \Psi_{\mathrm{U},1} : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(Y'), \\ \mathrm{TU}'_1 &:= \Psi_{\mathrm{U},1}^{-1} \circ \Psi'_{\mathrm{T},1} : \mathrm{D^b}(Y') \xrightarrow{\sim} \mathrm{D^b}(Y). \end{aligned}$$

Then $\mathrm{UT}_1^{-1} \simeq \mathrm{TU}_1'$ and the following diagram commutes

$$\begin{array}{ccc} \mathrm{D^b}(Y) & \xrightarrow{\mathrm{UT}_1} & \mathrm{D^b}(Y') \\ {\scriptstyle -\otimes\mathcal{O}_Y(1)}\downarrow & & \downarrow{\scriptstyle -\otimes\mathcal{O}_{Y'}(-1)} \\ \mathrm{D^b}(Y) & \xrightarrow{\mathrm{UT}} & D(Y') \end{array}$$

Theorem 4.4.7. *We have a functor isomorphism*

$$\mathrm{TU}_1' \circ \mathrm{Seg} \simeq \mathrm{T}_{\mathcal{O}_{\mathrm{LGr}}(-1)} \in \mathrm{Auteq}(\mathrm{D^b}(Y)).$$

Proof. We have to show the following diagram commutes:

$$\begin{array}{ccc} \mathrm{D^b}(Y) & \xrightarrow{\mathrm{Seg}} & \mathrm{D^b}(Y') \\ {\scriptstyle \mathrm{T}_{\mathcal{O}_{\mathrm{LGr}}(-1)}}\downarrow & & \downarrow{\scriptstyle \Psi'_{\mathrm{T},1}} \\ \mathrm{D^b}(Y) & \xrightarrow{\Psi_{\mathrm{U},1}} & \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathrm{U},1}). \end{array}$$

As in the proof of Theorem 4.4.3, it is enough to show that

$$\mathrm{Seg}'(\mathcal{O}_{Y'}(1)) \simeq \mathrm{T}^{-1}_{\mathcal{O}_{\mathrm{LGr}}(-1)}(\mathcal{O}_Y(-1)).$$

First, by using an exact sequence

$$0 \to I_{\mathrm{LGr}\,/Y}(-1) \to \mathcal{O}_Y(-1) \to \mathcal{O}_{\mathrm{LGr}}(-1) \to 0$$

and a computation

$$\mathrm{RHom}_Y(\iota_*\mathcal{O}_{\mathrm{LGr}}(-1), \mathcal{O}_Y(-1)) \simeq \mathrm{R\Gamma}(\mathrm{LGr}, \mathcal{O}_{\mathrm{LGr}}(-3))[-2] \simeq \mathbb{C}[-5],$$

we have

$$\mathrm{RHom}_Y(\iota_*\mathcal{O}_{\mathrm{LGr}}(-1), I_{\mathrm{LGr}\,/Y}(-1)) \simeq \mathbb{C}[1]$$

and hence we obtain

$$\mathrm{T}_{\mathcal{O}_{\mathrm{LGr}}(-1)}(I_{\mathrm{LGr}\,/Y}(-1)) = \mathcal{O}_Y(-1).$$

On the other hand, by applying the functor Seg' to the sequence

$$0 \to \mathcal{O}_{Y'}(-2) \to \Sigma(-1) \to \mathcal{O}_{Y'}(1) \to 0,$$

we have a triangle

$$\mathcal{O}_Y(2) \to \mathcal{S}(1) \to \mathrm{Seg}'(\mathcal{O}_{Y'}(1)) \to \mathcal{O}_Y(2)[1],$$

and by Lemma 4.4.4, we have

$$\mathrm{Seg}'(\mathcal{O}_{Y'}(1)) \simeq I_{\mathrm{LGr}\,/Y}(-1).$$

Thus we have the result. ☐

4.4.6 Multi-mutation=twist result

Note that $\Lambda_{\mathrm{U},1}$ is the endomorphism ring of an R-module

$$M_{-1} \oplus M_0 \oplus M_1 \oplus S_1.$$

Let $W' := M_0 \oplus M_1 \oplus S_1$. This W' was denoted by W_3 in Theorem 4.3.13. Recall that Λ_{S} is the endomorphism ring of $W' \oplus M_2$.

Proposition 4.4.8. *We have the following two isomorphism of R-modules:*

(1) $\mu_{W'}\mu_{W'}\mu_{W'}(W' \oplus M_{-1}) \simeq W' \oplus M_2$.

(2) $\mu_{W'}(W' \oplus M_2) \simeq W' \oplus M_{-1}$.

Moreover, the induced IW functor

$$\Phi_{W'} : \mathrm{D^b}(\operatorname{mod} \Lambda_{\mathrm{S}}) \to \mathrm{D^b}(\operatorname{mod} \Lambda_{\mathrm{U},1})$$

from (2) is isomorphic to $\Psi'_{\mathrm{U},1} \circ (\Psi'_{\mathrm{S}})^{-1}$.

Proof. (1) was proved in Theorem 4.3.13. One can show (2) by using Lemma 4.2.7. We only note that the exchange sequence for (2) is given by the dual of the exact sequence

$$0 \to M_2 \to S_1 \to M_{-1} \to 0.$$

This sequence is obtained by taking the global section of the sequence

$$0 \to \mathcal{O}_{Y'}(-2) \to \Sigma(-1) \to \mathcal{O}_{Y'}(1) \to 0.$$

□

The following is a "multi-mutation=twist" result for the Abuaf flop.

Theorem 4.4.9. *By Proposition 4.4.8, we have an autoequivalence of* $\mathrm{D^b}(\operatorname{mod} \Lambda_{\mathrm{U},1})$ *by composing four IW mutation functors at W':*

$$\Phi_{\mathrm{W'}} \circ \Phi_{\mathrm{W'}} \circ \Phi_{\mathrm{W'}} \circ \Phi_{\mathrm{W'}} \in \operatorname{Auteq}(\mathrm{D^b}(\operatorname{mod} \Lambda_{\mathrm{U},1})).$$

This autoequivalence corresponds to a spherical twist $\mathrm{T}_{\mathcal{O}_{\mathrm{LGr}}(-1)} \in \operatorname{Auteq}(\mathrm{D^b}(Y))$ *under the identification*

$$\Phi_{\mathrm{U},1} : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(\operatorname{mod} \Lambda_{\mathrm{U},1}).$$

Proof. By Theorem 4.3.13, Theorem 4.4.7, and Proposition 4.4.8, we have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{D^b}(Y) & = & \mathrm{D^b}(Y) & \xrightarrow{\mathrm{T}_{\mathcal{O}_{\mathrm{LGr}}(-1)}} & \mathrm{D^b}(Y) \\
\downarrow{\scriptstyle \Psi_{\mathrm{U},1}} & & \downarrow{\scriptstyle \Psi_{\mathrm{S}}} & & \downarrow{\scriptstyle \Psi_{\mathrm{U},1}} \\
\mathrm{D^b}(\operatorname{mod} \Lambda_{\mathrm{U},1}) & \xrightarrow{\Phi^3_{W'}} & \mathrm{D^b}(\operatorname{mod} \Lambda_{\mathrm{S}}) & \xrightarrow{\Phi_{W'}} & \mathrm{D^b}(\operatorname{mod} \Lambda_{\mathrm{U},1}) \\
 & & {\scriptstyle \Psi_{\mathrm{S}}}\uparrow & & {\scriptstyle \Psi_{\mathrm{T},1}}\uparrow \\
 & & \mathrm{D^b}(Y') & = & \mathrm{D^b}(Y'),
\end{array}$$

and the result follows from this diagram. □

Remark 4.4.10. Compare this result with [H17a, Theorem 5.18 and Remark 5.19]. There, the author proved that a P-twist on the cotangent bundle $T^* \mathbb{P}^n$ of $\mathbb{P}^n$ associated to the sheaf $\mathcal{O}_{\mathbb{P}}(-1)$ on the zero section $\mathbb{P} \subset T^* \mathbb{P}^n$ corresponds to a composition of $2n$ IW mutations of an NCCR.

Remark 4.4.11. By Theorem 4.4.9, we notice that an autoequivalence

$$\Phi_{\mathrm{W}'} \circ \Phi_{\mathrm{W}'} \circ \Phi_{\mathrm{W}'} \circ \Phi_{\mathrm{W}'} \in \mathrm{Auteq}(\mathrm{D^b}(\mathrm{mod}\, \Lambda_{\mathrm{U},1}))$$

corresponds to a spherical twist

$$\mathrm{T}_{\mathcal{F}} \in \mathrm{Auteq}(\mathrm{D^b}(Y'))$$

on Y' around an object $\mathcal{F} := \mathrm{UT}_1(\mathcal{O}_{\mathrm{LGr}}(-1))$, under the identification

$$\Psi'_{\mathrm{T},1} : \mathrm{D^b}(Y') \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\, \Lambda_{\mathrm{U},1}).$$

Note that $\mathcal{F}$ is also a spherical object on Y' because Y' has a trivial canonical bundle. However, in contrast to the case for Y, the object $\mathcal{F}$ is not contained in the subcategory $\iota'_* \mathrm{D^b}(\mathbb{P})$ of $\mathrm{D^b}(Y')$.

Indeed, we have

$$\begin{aligned}
&\mathrm{RHom}_{Y'}(\mathcal{F}, \mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(-1) \oplus \Sigma(-1)) \\
\simeq\, &\mathrm{RHom}_{Y}(\iota_*\mathcal{O}_{\mathrm{LGr}}(-1), \mathcal{O}_{Y} \oplus \mathcal{O}_{Y}(1) \oplus \mathcal{S}(1)) \\
\simeq\, &\mathrm{RHom}_{\mathrm{LGr}}(\mathcal{O}_{\mathrm{LGr}}(-1), (\mathcal{O}_{\mathrm{LGr}} \oplus \mathcal{O}_{\mathrm{LGr}}(1) \oplus \mathcal{S}(1)) \otimes \omega_{\mathrm{LGr}})[-2] \\
=\, &0.
\end{aligned}$$

Thus, if $\mathcal{F} \simeq \iota'_* F$ for some $F \in \mathrm{D^b}(\mathbb{P})$, we have

$$0 = \mathrm{RHom}_{Y'}(\iota'_* F, \mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(-1) \oplus \Sigma(-1)) \simeq \mathrm{RHom}_{\mathbb{P}}(F, \bigoplus_{k=-2}^{1} \mathcal{O}_{\mathbb{P}}(k) \otimes \omega_{\mathbb{P}})[-2].$$

Since the object $\bigoplus_{k=-2}^{1} \mathcal{O}_{\mathbb{P}}(k) \otimes \omega_{\mathbb{P}}$ spans the derived category $\mathrm{D^b}(\mathbb{P})$ of the three dimensional projective space, we have $F = 0$. This is contradiction.

4.5 Borel-Bott-Weil Theorem

In this section, we explain how to compute the cohomology of a homogeneous vector bundle on $\mathrm{LGr}(V)$. Since $\mathrm{LGr}(V)$ is a homogeneous variety of $\mathrm{Sp}(V)$, we can compute cohomologies on $\mathrm{LGr}(V)$ by using the representation theory of $\mathrm{Sp}(V)$. However, we provide a different method: we embed $\mathrm{LGr}(V)$ into $\mathrm{Gr}(2, V)$ and use Borel-Bott-Weil theorem for $\mathrm{GL}_4 = \mathrm{GL}(V)$.

Put

$$P := \left\{ \left(\begin{array}{c|c} A_1 & * \\ \hline O & A_2 \end{array} \right) \mid A_1, A_2 \in \mathrm{GL}_2 \right\}.$$

Then $\mathrm{Gr}(2,V) = \mathrm{Gr}(2,4) = \mathrm{GL}_4/P$. The category of homogeneous vector bundles on $\mathrm{Gr}(2,V)$ is equivalent to the category of P-modules. Since P contains $\mathrm{GL}_2 \times \mathrm{GL}_2$ (Levi subgroup), we can consider a P-module as a $\mathrm{GL}_2 \times \mathrm{GL}_2$-module. Thus, for a homogeneous vector bundle on $\mathrm{Gr}(2,4)$, we can consider its highest weight vector in the weight lattice of $\mathrm{GL}_2 \times \mathrm{GL}_2$. Let $\mathbb{T} \subset \mathrm{GL}(V)$ be a set diagonal matrices. Then $\mathbb{T}$ is a maximal torus of GL_4, and also provides a maximal torus of $\mathrm{GL}_2 \times \mathrm{GL}_2$. Hence $\mathrm{GL}(V)$ and $\mathrm{GL}_2 \times \mathrm{GL}_2$ have same weight lattice. Recall that the space of weights $X(\mathbb{T})$ of GL_4 is isomorphic to $\mathbb{Z}^4 = \bigoplus_{i=1}^4 \mathbb{Z} \cdot \varepsilon_i$, where

$$\varepsilon_i : \mathbb{T} \ni \mathrm{diag}(d_1, d_2, d_3, d_4) \mapsto d_i \in \mathbb{C}^\times.$$

Example 4.5.1. Let $\mathcal{S}$ be the universal subbundle on $\mathrm{Gr}(2,V)$. The highest weight of $(\mathrm{Sym}^k \mathcal{S})(l)$ is $(l, l-k, 0, 0)$.

The Weyl group of GL_4 is the symmetric group $\mathfrak{S}_4$ and it acts on the space of weights $\mathbb{Z}^4$ by permutation. To state Borel-Bott-Weil theorem, we need to define another action of $\mathfrak{S}_4$ on $\mathbb{Z}^4$, called tilde-action. Put $\rho = (3,2,1,0)$ and we call it fundamental weight. For $\omega \in \mathbb{Z}^4$ and $\sigma \in \mathfrak{S}_4$, we set

$$\widetilde{\sigma} \cdot \omega := \sigma \cdot (\omega + \rho) - \rho.$$

Example 4.5.2. Put $\sigma_{ij} = (i,j) \in \mathfrak{S}_4$. Then

$$\begin{aligned}
\widetilde{\sigma_{12}} \cdot (\omega_1, \omega_2, \omega_3, \omega_4) &= (\omega_2 - 1, \omega_1 + 1, \omega_3, \omega_4), \\
\widetilde{\sigma_{23}} \cdot (\omega_1, \omega_2, \omega_3, \omega_4) &= (\omega_1, \omega_3 - 1, \omega_2 + 1, \omega_4), \\
\widetilde{\sigma_{34}} \cdot (\omega_1, \omega_2, \omega_3, \omega_4) &= (\omega_1, \omega_2, \omega_4 - 1, \omega_3 + 1).
\end{aligned}$$

For $\omega \in \mathbb{Z}^4$, there is exist $\sigma \in \mathfrak{S}_4$ such that $\widetilde{\sigma} \cdot \omega = (\omega'_1, \omega'_2, \omega'_3, \omega'_4)$ is one of the following:

(1) The dominant weight of GL_4 i.e. $\omega'_1 \geq \omega'_2 \geq \omega'_3 \geq \omega'_4$.

(2) There exists i such that $\omega'_i = \omega'_{i+1} - 1$.

The Borel-Bott-Weil theorem is given as follows.

Theorem 4.5.3 (Borel-Bott-Weil theorem)**.** *Let $\mathcal{E}_\omega$ be a homogeneous vector bundle with highest weight ω, $\sigma \in \mathfrak{S}_4$ as above, and $l(\sigma)$ the length of σ. Then,*

(i) If $\widetilde{\sigma} \cdot \omega$ is as in (1), then we have

$$H^i(\mathrm{Gr}(2,V), \mathcal{E}_\omega) = \begin{cases} (V_{\widetilde{\sigma} \cdot \omega})^* & \text{if } i = l(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If $\widetilde{\sigma} \cdot \omega$ is as in (2), then

$$\mathrm{R}\Gamma(\mathrm{Gr}(2,V), \mathcal{E}_\omega) = 0.$$

By using this theorem, we can compute the cohomology of a homogeneous vector bundle on $\mathrm{Gr}(V)$.

Example 4.5.4. Set $\mathbb{G} := \mathrm{Gr}(2, 4)$.

(1) The highest weight of $\mathcal{O}_{\mathbb{G}}(-3)$ is $(-3, -3, 0, 0)$. Since

$$\widetilde{\sigma_{12}}\widetilde{\sigma_{34}}(-3, -3, 0, 0) = (-4, -2, -1, 1),$$

we have

$$\mathrm{R}\Gamma(\mathbb{G}, \mathcal{O}_{\mathbb{G}}(-3)) = 0.$$

(2) The highest weight of $\mathcal{O}_{\mathbb{G}}(-4)$ is $(-4, -4, 0, 0)$. Since

$$\widetilde{\sigma_{23}}\widetilde{\sigma_{12}}\widetilde{\sigma_{34}}\widetilde{\sigma_{23}}(-4, -4, 0, 0) = (-2, -2, -2, -2)$$

and the representation of GL_4 whose highest weight is $(-2, -2, -2, -2)$ is $(\det^{-2}, \mathbb{C})$. Thus we have

$$\mathrm{R}\Gamma(\mathbb{G}, \mathcal{O}_{\mathbb{G}}(-4)) \simeq \mathbb{C}[-4].$$

To go back to $\mathrm{LGr}(V)$, we use the exact sequence

$$0 \to \mathcal{O}_{\mathrm{Gr}(V)}(-1) \to \mathcal{O}_{\mathrm{Gr}(V)} \to \mathcal{O}_{\mathrm{LGr}(V)} \to 0.$$

Example 4.5.5. By the exact sequence

$$0 \to \mathcal{O}_{\mathbb{G}}(-4) \to \mathcal{O}_{\mathbb{G}}(-3) \to \mathcal{O}_{\mathrm{LGr}}(-3) \to 0,$$

we have

$$\mathrm{R}\Gamma(\mathrm{LGr}(V), \mathcal{O}_{\mathrm{LGr}}(-3)) \simeq \mathbb{C}[-3].$$

The following trivial proposition is also useful to compute the cohomologies.

Proposition 4.5.6 (cf. [FH91] Exercise 11.11)**.** *There is an isomorphism of vector bundles on* LGr

$$\mathrm{Sym}^a \mathcal{S} \otimes \mathrm{Sym}^b \mathcal{S} \simeq \bigoplus_{k=0}^{b} (\mathrm{Sym}^{a+b-2k} \mathcal{S})(-k).$$

for any $a \geq b$*.*

4.6 Mutation of exceptional objects

In Section 4.3, we construct a resolution of a sheaf by using the mutation of exceptional objects. In this section, we recall the definition of exceptional objects and mutation of them, and explain how to find resolutions that we used in Section 4.3.

4.6.1 Application for finding resolutions

Lemma 4.6.1. *There is an exact sequence on* $\mathrm{LGr}(V)$

$$0 \to \mathcal{O}_{\mathrm{LGr}}(-3) \to \mathcal{S}(-2)^{\oplus 4} \to \mathcal{O}_{\mathrm{LGr}}(-2)^{\oplus 11} \to \mathcal{O}_{\mathrm{LGr}}(-1)^{\oplus 5} \to \mathcal{O}_{\mathrm{LGr}} \to 0.$$

Proof. Let us consider a full exceptional collection

$$\mathrm{D^b}(\mathrm{LGr}(V)) = \langle \mathcal{O}_{\mathrm{LGr}}(-3), \mathcal{S}(-2), \mathcal{O}_{\mathrm{LGr}}(-2), \mathcal{O}_Y(-1)\rangle.$$

Then, by Lemma 2.2.11 and Lemma 2.2.9, we have an isomorphism

$$\mathbb{R}_{\mathcal{S}(-2)}(\mathcal{O}_{\mathrm{LGr}}(-3)) \simeq \mathbb{L}_{\mathcal{O}_{\mathrm{LGr}}(-2)}\mathbb{L}_{\mathcal{O}_{\mathrm{LGr}}(-1)}(\mathcal{O}_{\mathrm{LGr}}(-3) \otimes \omega_{\mathrm{LGr}}^{-1})$$

up to shift. Note that $\mathcal{O}_{\mathrm{LGr}}(-3) \otimes \omega_{\mathrm{LGr}}^{-1} \simeq \mathcal{O}_{\mathrm{LGr}}$.

First, we have

$$\mathrm{RHom}_{\mathrm{LGr}(V)}(\mathcal{O}_{\mathrm{LGr}}(-3), \mathcal{S}(-2)) \simeq \mathbb{C}^4$$

and hence the object $\mathbb{R}_{\mathcal{S}(-2)}(\mathcal{O}_{\mathrm{LGr}}(-3))[1]$ lies on an exact triangle

$$\mathcal{O}_{\mathrm{LGr}}(-3) \xrightarrow{\mathrm{ev}} \mathcal{S}(-2)^{\oplus 4} \to \mathbb{R}_{\mathcal{S}(-2)}(\mathcal{O}_{\mathrm{LGr}}(-3))[1] \to \mathcal{O}_{\mathrm{LGr}}(-3)[1].$$

Since $\mathcal{O}_{\mathrm{LGr}}(-3)$ and $\mathcal{S}(-2)^{\oplus 4}$ are vector bundles on $\mathrm{LGr}(V)$, the map ev should be injective and hence the object $\mathbb{R}_{\mathcal{S}(-2)}(\mathcal{O}_{\mathrm{LGr}}(-3))[1]$ is a sheaf on $\mathrm{LGr}(V)$. Thus, we put

$$\mathcal{F} := \mathbb{R}_{\mathcal{S}(-2)}(\mathcal{O}_{\mathrm{LGr}}(-3))[1].$$

Next, we have $\mathrm{RHom}_{\mathrm{LGr}(V)}(\mathcal{O}_{\mathrm{LGr}}(-1), \mathcal{O}_{\mathrm{LGr}}) \simeq \mathbb{C}^5$ and hence

$$\mathbb{L}_{\mathcal{O}_{\mathrm{LGr}}(-1)}(\mathcal{O}_{\mathrm{LGr}})[-1] \simeq \Omega^1_{\mathbb{P}^4}|_{\mathrm{LGr}}.$$

Moreover, an easy computation shows that $\mathrm{RHom}_{\mathrm{LGr}(V)}(\mathcal{O}_{\mathrm{LGr}}(-2), \Omega^1_{\mathbb{P}^4}|_{\mathrm{LGr}}) \simeq \mathbb{C}^{11}$ and hence the object $\mathbb{L}_{\mathcal{O}_{\mathrm{LGr}}(-2)}(\Omega^1_{\mathbb{P}^4}|_{\mathrm{LGr}})$ lies on the exact sequence

$$\mathcal{O}_{\mathrm{LGr}}(-2)^{\oplus 11} \xrightarrow{\mathrm{ev}} \Omega^1_{\mathbb{P}^4}|_{\mathrm{LGr}} \to \mathbb{L}_{\mathcal{O}_{\mathrm{LGr}}(-2)}(\Omega^1_{\mathbb{P}^4}|_{\mathrm{LGr}}) \to \mathcal{O}_{\mathrm{LGr}}(-2)^{\oplus 11}[1].$$

From the above computation, the object $\mathbb{L}_{\mathcal{O}_{\mathrm{LGr}}(-2)}(\Omega^1_{\mathbb{P}^4}|_{\mathrm{LGr}})$ should be a sheaf on $\mathrm{LGr}(V)$ (up to shift) whose generic rank is equal to 7. Thus, we have the map ev is surjective and $\mathbb{L}_{\mathcal{O}_{\mathrm{LGr}}(-2)}(\Omega^1_{\mathbb{P}^4}|_{\mathrm{LGr}})[-1] \simeq \mathcal{F}$.

Summarizing the above arguments, we have the following three exact sequences:

$$\begin{aligned}
&0 \to \mathcal{O}_{\mathrm{LGr}}(-3) \to \mathcal{S}(-2)^{\oplus 4} \to \mathcal{F} \to 0,\\
&0 \to \mathcal{F} \to \mathcal{O}_{\mathrm{LGr}}(-2)^{\oplus 11} \to \Omega^1_{\mathbb{P}^4}|_{\mathrm{LGr}} \to 0,\\
&0 \to \Omega^1_{\mathbb{P}^4}|_{\mathrm{LGr}} \to \mathcal{O}_{\mathrm{LGr}}(-1)^{\oplus 5} \to \mathcal{O}_{\mathrm{LGr}} \to 0.
\end{aligned}$$

By combining these three exact sequences, we have the desired long exact sequence. □

By using similar arguments, we can obtain the long exact sequences that we used in the proof of Theorem 4.3.13.

Chapter 5

On the Abuaf-Ueda flop via non-commutative crepant resolutions

This chapter is based on the author's work

[H18b] W. Hara, *On the Abuaf-Ueda flop via non-commutative crepant resolution*, preprint (2018).

5.1 Introduction

5.1.1 The Abuaf-Ueda flop

First of all we give the construction of the flop we study in this chapter. Let us consider the G_2 Dynkin diagram ◯⇛ . Then by the classification theory of homogeneous varieties, projective homogeneous varieties of the semi-simple algebraic group of type G_2 corresponds to a marked Dynkin diagram. The one ⟩≡◯ corresponds to the G_2-Grassmannian $\mathbf{G} = \mathrm{Gr}_{G_2}$. Another one ◯⇛× corresponds to the 5-dimensional quadric $\mathbf{Q} \subset \mathbb{P}^6$. The last one ×⇛× corresponds to the (full) flag variety $\mathbf{F}$ of type G_2. There are projections $\mathbf{F} \to \mathbf{G}$ and $\mathbf{F} \to \mathbf{Q}$, and both of them give $\mathbb{P}^1$-bundle structures of $\mathbf{F}$.

Let us consider the Cox ring of $\mathbf{F}$

$$C := \bigoplus_{a,b=0}^{\infty} H^0(\mathbf{F}, \mathcal{O}_{\mathbf{F}}(a,b)) \simeq \bigoplus_{a,b=0}^{\infty} V_{(a,b)}^{\vee},$$

where $\mathcal{O}_{\mathbf{F}}(a,b)$ (resp. $V_{(a,b)}^{\vee}$) is a line bundle on $\mathbf{F}$ (resp. the dual of an irreducible representation of G_2) that corresponds to the dominant weight (a,b).

Put $C_{a,b} := H^0(\mathbf{F}, \mathcal{O}_{\mathbf{F}}(a,b))$ and

$$C_n := \bigoplus_{a \in \mathbb{Z}} C_{n+a,a}.$$

Using them we can define a $\mathbb{Z}$-grading on C by

$$C = \bigoplus_{n \in \mathbb{Z}} C_n.$$

This grading corresponds to a $\mathbb{G}_{\mathrm{m}}$-action on $\operatorname{Spec} C$ that is obtained by a map $\mathbb{G}_{\mathrm{m}} \to (\mathbb{G}_{\mathrm{m}})^2, \alpha \mapsto (\alpha, \alpha^{-1})$ and the natural $(\mathbb{G}_{\mathrm{m}})^2$-action on $\operatorname{Spec} C$ coming from the original bi-grading.

Then we can take the geometric invariant theory quotients

$$Y_+ := \operatorname{Proj}(C_+),\ Y_- := \operatorname{Proj}(C_-), \text{ and } X := \operatorname{Proj} C_0,$$

where

$$C_+ := \bigoplus_{n \geq 0} C_n \text{ and } C_- := \bigoplus_{n \leq 0} C_n.$$

The projective quotients Y_+ and Y_- are the total spaces of rank two vector bundles on $\mathbf{G}$ and $\mathbf{Q}$ respectively. The affinization morphism $\phi_+ : Y_+ \to X$ and $\phi_- : Y_- \to X$ are small resolution of the singular affine variety X and they contract the zero-sections. Furthermore we can show that the birational map $Y_+ \dashrightarrow Y_-$ is a 7-dimensional simple flop with an interesting feature that the contraction loci are not isomorphic to each other.

The author first learned this interesting flop from Abuaf. Later the author noticed that the same flop was found by Ueda independently [Ued16]. Thus the author would like to attribute this new flop to both of them, and would like to call this flop *the Abuaf-Ueda flop*.

When there is a flop $Y_+ \dashrightarrow Y_-$ between two smooth varieties, it is important to compare their derived categories. According to a famous conjecture due to Bondal and Orlov [BO02], we expect that we have a derived equivalence $\mathrm{D^b}(Y_+) \simeq \mathrm{D^b}(Y_-)$. In the case of the Abuaf-Ueda flop, Ueda proved the derived equivalence using the theory of semi-orthogonal decomposition and its mutation. However, since there are many other methods to construct an equivalence between derived categories, it is still interesting problem to prove the derived equivalence using other methods.

5.1.2 Results in this chapter

The main purpose of this chapter to construct *tilting bundles* on both sides of the flop $Y_+ \dashrightarrow Y_-$, and construct equivalences between the derived categories of Y_+ and Y_- using those tilting bundles. A tilting bundle T_* on Y_* ($* \in \{+,-\}$) is a vector bundle on Y_* that gives an equivalence

$$\mathrm{RHom}_{Y_*}(T_*, -) : \mathrm{D^b}(Y_*) \to \mathrm{D^b}(\mathrm{End}_{Y_*}(T_*))$$

between two derived categories. In particular, if we find tilting bundles T_+ and T_- with the same endomorphism ring, then we have an equivalence $\mathrm{D^b}(Y_+) \simeq \mathrm{D^b}(Y_-)$ as desired.

The advantage of this method is that it enables us to study a flop from the point of view of the theory of *non-commutative crepant resolutions (= NCCRs)* that is first introduced by Van den Bergh [VdB04b]. In our case, an NCCR appears as the endomorphism algebra $\mathrm{End}_{Y_*}(T_*)$ of a tilting bundle T_*. Via the theory of NCCRs, we also study the Abuaf-Ueda flop from the moduli-theoretic point of view.

Recall that Y_+ and Y_- are the total spaces of rank two vector bundles on $\mathbf{G}$ and $\mathbf{Q}$ respectively. If there is a variety Z that gives a rational resolution of an affine singular variety and that is the total space of a vector bundle on a projective variety W admitting a tilting bundle T, it is natural to expect that the pull back of T via the projection $Z \to W$ gives a tilting bundle on Z. Indeed, in many known examples, we can produce tilting bundles in such a way [BLV10, H17a, WZ12].

However, in our case, we cannot obtain tilting bundles on Y_+ or Y_- as a pull back of known tilting bundles on $\mathbf{G}$ or $\mathbf{Q}$. Thus the situation is different from previous works. Nevertheless, by modifying bundles that are obtained from tilting bundles on the base $\mathbf{G}$ or $\mathbf{Q}$, we can find tilting bundles on Y_+ and Y_-. Namely, tilting bundles we construct are the direct sum of indecomposable bundles that are obtained by taking extensions of other bundles obtained from $\mathbf{G}$ or $\mathbf{Q}$. We can also check that they produce derived equivalences $\mathrm{D^b}(Y_+) \simeq \mathrm{D^b}(Y_-)$.

5.1.3 Related works

If we apply a similar construction to Dynkin diagrams A_2 or C_2, then we have the four-dimensional Mukai flop or the (five-dimensional) Abuaf flop [Seg16] respectively. Therefore this chapter is a sequel of chapters above.

Recently, Kanemitsu [Kan18] classified simple flops of dimension up to eight, which is a certain generalization of the theorem of Li [Li17]. It is interesting to prove the derived equivalence for all simple flops that appear in Kanemitsu's list using tilting bundles, and we can regard this chapter as a part of such a project.

This flop is also related to certain (compact) Calabi-Yau threefolds which are studied in [IMOU16a, IMOU16b, Kuz18]. Let us consider the (geometric) vector bundle $Y_+ \to \mathbf{G}$ over $\mathbf{G}$. Then as a zero-locus of a regular section of this bundle we have a smooth Calabi-Yau threefold V_+ in $\mathbf{G}$. Similarly, we can construct another Calabi-Yau threefold V_- in $\mathbf{Q}$. Papers [IMOU16b, Kuz18] show that Calabi-Yau threefolds V_+ and V_- are L-equivalent, derived equivalent but NOT birationally equivalent to each other. (L-equivalence and non-birationality is due to [IMOU16b], and derived equivalence is due to [Kuz18].) As explained in [Ued16], we can construct a derived equivalence $\mathrm{D^b}(V_+) \xrightarrow{\sim} \mathrm{D^b}(V_-)$ for Calabi-Yau threefolds from a derived equivalence $\mathrm{D^b}(Y_+) \xrightarrow{\sim} \mathrm{D^b}(Y_-)$ with a certain nice property.

5.1.4 Open questions

It would be interesting to compare the equivalences in this chapter and the one constructed by Ueda. It is also interesting to find Fourier-Mukai kernels that give equivalences. In the case of the Mukai flop or the Abuaf flop, the structure sheaf of the fiber product $Y_+ \times_X Y_-$ over the singularity X gives a Fourier-Mukai kernel of an equivalence (see [Kaw02, Nam03, H17b]). Thus it is interesting to ask whether this fact remains to hold or not for the Abuaf-Ueda flop.

Another interesting topic is to study the autoequivalence group of the derived category. Since we produce some derived equivalences that are different to each other in this chapter, we can find some non-trivial autoequivalences by combining them. It would be interesting to find an action of an interesting group on the derived category of Y_+ (and Y_-) that contains our autoequivalences.

5.2 Preliminaries

5.2.1 Tilting bundle and derived category

First we prepare some basic terminologies and facts about tilting bundles.

Definition 5.2.1. Let Y be a quasi-projective variety and T a vector bundle (of finite rank) on Y. Then we say that T is *partial tilting* if $\mathrm{Ext}_Y^{\geq 1}(T, T) = 0$. We say that a partial tilting bundle T on Y is *tilting* if T is a generator of the unbounded derived category $\mathrm{D}(\mathrm{Qcoh}(Y))$, i.e. if an object $E \in \mathrm{D}(\mathrm{Qcoh}(Y))$ satisfies $\mathrm{RHom}(T, E) \simeq 0$ then $E \simeq 0$.

If we find a tilting bundle on a projective scheme (over an affine variety), we can construct a derived equivalence between the derived category of the scheme and the derived category of a non-commutative algebra obtained as the endomorphism ring of the bundle.

Proposition 5.2.2. *Let Y be a projective scheme over an affine scheme* $\mathrm{Spec}(R)$. *Assume that Y admits a tilting bundle T. Then we have the following derived equivalence*

$$\mathrm{RHom}_Y(T, -) : \mathrm{D^b}(Y) \to \mathrm{D^b}(\mathrm{End}_Y(T)).$$

These equivalences coming from tilting bundles are very useful to construct equivalences between the derived categories of two crepant resolutions.

Lemma 5.2.3. *Let $X = \mathrm{Spec}\, R$ be a normal Gorenstein affine variety of dimension greater than or equal to two, and let $\phi : Y \to X$ and $\phi' : Y' \to X$ be two crepant resolutions of X. Put $U := X_{\mathrm{sm}} = Y \setminus \mathrm{exc}(\phi) = Y' \setminus \mathrm{exc}(\phi')$. Assume that there are tilting bundles T and T' on Y and Y', respectively, such that*

$$T|_U \simeq T'|_U.$$

Then there is a derived equivalence

$$\mathrm{D^b}(Y) \simeq \mathrm{D^b}(\mathrm{End}_Y(T)) \simeq \mathrm{D^b}(\mathrm{End}_{Y'}(T')) \simeq \mathrm{D^b}(Y').$$

Proof. See [H18a, Lemma 3.4]. □

The existence of a tilting bundle on a crepant resolution does not hold in general. For this fact, see [IW14b, Theorem 4.20]. In addition, even in the case that a tilting bundle exists, it is still non-trivial to construct a tilting bundle explicitly. The following lemma is very useful to find a tilting bundle.

Lemma 5.2.4. *Let $\{E_i\}_{i=1}^n$ be a collection of vector bundles on a quasi-projective scheme Y. Assume that*

(i) The direct sum $\bigoplus_{i=1}^n E_i$ is a generator of $\mathrm{D}(\mathrm{Qcoh}(Y))$.

(ii) There is no former $\mathrm{Ext}_Y^{\geq 1}$, i.e. $\mathrm{Ext}_Y^{\geq 1}(E_i, E_j) = 0$ for $i \leq j$. In particular, this assumption implies that E_i is a partial tilting bundle for any i.

(iii) There is no backward $\mathrm{Ext}_Y^{\geq 2}$, i.e. $\mathrm{Ext}_Y^{\geq 2}(E_i, E_j) = 0$ for $i > j$.

Then there exists a tilting bundle on Y.

Proof. We use an induction on n. If $n = 1$, the statement is trivial. Let $n > 1$. Choose generators of $\mathrm{Ext}_Y^1(E_1, E_2)$ as a right $\mathrm{End}_Y(E_1)$-module, and let r be the number of the generators. Then we can take the corresponding sequence

$$0 \to E_2 \to F \to E_1^{\oplus r} \to 0.$$

Then we have $\mathrm{Ext}_Y^{\geq 1}(E_1, F) = 0$. Indeed, if we apply the functor $\mathrm{Ext}_Y^i(E_1, -)$ to the sequence above, we have the long exact sequence

$$\cdots \to \mathrm{End}_Y(E_1)^{\oplus r} \xrightarrow{\delta} \mathrm{Ext}_Y^1(E_1, E_2) \to \mathrm{Ext}_Y^1(E_1, F) \to \mathrm{Ext}_Y^1(E_1, E_1^{\oplus r}) = 0 \to \cdots .$$

Now δ is surjective by construction, and hence we have

$$\mathrm{Ext}_Y^{\geq 1}(E_1, F) = 0.$$

Applying the derived functor $\mathrm{Ext}_Y^i(E_2, -)$ to the same short exact sequence, we have $\mathrm{Ext}^{\geq 1}(E_2, F) = 0$ from the assumption that there is no former $\mathrm{Ext}^{\geq 1}$. Thus we have $\mathrm{Ext}_Y^{\geq 1}(F, F) = 0$. One can also show that $\mathrm{Ext}_Y^{\geq 1}(F, E_1) = 0$, and therefore $E_1 \oplus F$ is a partial tilting bundle.

Put $E_1' = E_1 \oplus F$ and $E_i' = E_{i-1}$ for $1 < i < n$. Then it is easy to see that the new collection $\{E_i'\}_{i=1}^{n-1}$ satisfies the assumptions (i), (ii) and (iii). Note that the condition (i) holds since the new collection $\{E_i'\}_{i=1}^{n-1}$ split-generates the original collection $\{E_i\}_{i=1}^n$. Thus we have the result by the assumption of the induction. □

5.2.2 Geometry and representation theory

Next we recall the representation theory and the geometry of homogeneous varieties we need. We also explain the geometric aspect of the Abuaf-Ueda flop in the present subsection.

Representation of G_2

In the present subsection, we recall the representation theory of the semi-simple algebraic group of type G_2. We need the representation theory when we compute cohomologies of homogeneous vector bundles using Borel-Bott-Weil theorem in Section 5.3.

Let $V = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ be a hyperplane in $\mathbb{R}^3$. Then the G_2 *root system* in V is the following collection of twelve vectors in V.

$$\Delta = \{(0, \pm 1, \mp 1), (\pm 1, 0, \mp 1), (\pm 1, \mp 1, 0), (\pm 2, \mp 1, \mp 1), (\mp 1, \pm 2, \mp 1), (\mp 1, \mp 1, \pm 2)\}.$$

The vector in Δ is called *root*. Especially,

$$\alpha_1 = (1, -1, 0) \text{ and } \alpha_2 = (-2, 1, 1)$$

are called *simple roots*, and we say that a root $\alpha \in \Delta$ is a *positive root* if $\alpha = a\alpha_1 + b\alpha_2$ for some $a \geq 0$ and $b \geq 0$.

By definition, the fundamental weights $\{\pi_1, \pi_2\} \subset V$ are the set vectors in V such that

$$\langle \alpha_i, \pi_j \rangle = \delta_{ij}.$$

Here the parting $\langle -, - \rangle$ is a usual one

$$\langle (a, b, c), (x, y, z) \rangle := ax + by + cz.$$

An easy computation shows that

$$\pi_1 = (0, -1, 1) \text{ and } \pi_2 = (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}).$$

The lattice $L = \mathbb{Z}\pi_1 + \mathbb{Z}\pi_2$ in V generated by π_1 and π_2 is called the *weight lattice* of G_2, and a vector in this lattice is called a *weight*. We call an weight of the form $a\pi_1 + b\pi_2$ for $a, b \in \mathbb{Z}_{\geq 0}$ a *dominant weight*. The set of dominant weights plays a central role in the representation theory because they corresponds to irreducible representations.

Let $\alpha \in \Delta$ be a root. Then we can consider the reflection S_α defined by the root α. That is a linear map $\mathrm{S}_\alpha : V \to V$ defined as

$$\mathrm{S}_\alpha(v) := v - \frac{2\langle \alpha, v \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

The Weyl group W is defined by a subgroup of the orthogonal group $\mathrm{O}(V)$ generated by S_α for $\alpha \in \Delta$:

$$W := \langle \mathrm{S}_\alpha \mid \alpha \in \Delta \rangle \subset \mathrm{O}(V).$$

It is known that W is generated by two reflections S_{α_1} and S_{α_2} defined by simple roots. Using this generator, we define the length of an element in W as follow. The length $l(w)$ of an element $w \in W$ is the smallest number n so that w is a composition of n reflections by simple roots. In the case of G_2, the Weyl group

Table 5.1: Elements of Weyl group and their length

element	length
1	0
S_1	1
S_2	1
S_{12}	2
S_{21}	2
S_{121}	3
S_{212}	3
S_{1212}	4
S_{2121}	4
S_{12121}	5
S_{21212}	5
$S_{121212} = S_{212121}$	6

W has twelve elements. The Table 5.1 shows all elements in W and their length. In that table, we denote $S_{\alpha_{i_k}} \cdots S_{\alpha_{i_2}} S_{\alpha_{i_1}}$ by $S_{i_k \cdots i_2 i_1}$ for short.

Let ρ be the half of the sum of all positive weights. It is known that ρ also can be written as $\rho = \pi_1 + \pi_2$. Using this weight, we can define another action of the Weyl group W on the weight lattice L that is called *dot-action*. The dot-action is defined by

$$S_\alpha \cdot v := S_\alpha(v + \rho) - \rho.$$

In our G_2 case, the dot-action is the following affine transform.

$$S_{\alpha_1} \cdot (a\pi_1 + b\pi_2) = (-a - 2)\pi_1 + (3a + b + 3)\pi_2,$$
$$S_{\alpha_2} \cdot (a\pi_1 + b\pi_2) = (a + b + 1)\pi_1 + (-b - 2)\pi_2.$$

Geometry of G_2-homogeneous varieties

Next we recall the geometry of G_2-homogeneous varieties.

The G_2-Grassmannian $\mathbf{G} = \mathrm{Gr}_{G_2}$ is a 5-dimensional closed subvariety of $\mathrm{Gr}(2,7)$, and has Picard rank one. The Grassmannian $\mathrm{Gr}(2,7)$ admits the universal quotient bundle Q of rank 5 and G is the zero-locus of a regular section of the bundle $Q^\vee(1)$. Since $\det(Q^\vee(1)) \simeq \mathcal{O}_{\mathrm{Gr}(2,7)}(4)$ and $\omega_{\mathrm{Gr}(2,7)} \simeq \mathcal{O}_{\mathrm{Gr}(2,7)}(-7)$, we have $\omega_{\mathbf{G}} \simeq \mathcal{O}_{\mathbf{G}}(-3)$. Thus $\mathbf{G}$ is a five dimensional Fano variety of Picard rank one and of Fano index three. We denote the restriction of the universal subbundle on $\mathrm{Gr}(2,7)$ to $\mathbf{G}$ by R. The bundle R has rank two and $\det(R) \simeq \mathcal{O}_{\mathbf{G}}(-1)$. It is known that the derived category $\mathrm{D^b}(\mathbf{G})$ of $\mathbf{G}$ admits a full strong exceptional collection

$$\mathrm{D^b}(\mathbf{G}) = \langle R(-1), \mathcal{O}_{\mathbf{G}}(-1), R, \mathcal{O}_{\mathbf{G}}, R(1), \mathcal{O}_{\mathbf{G}}(1)\rangle$$

(see [Kuz06]). In particular, the variety $\mathbf{G}$ admits a tilting bundle

$$R(-1) \oplus \mathcal{O}_{\mathbf{G}}(-1) \oplus R \oplus \mathcal{O}_{\mathbf{G}} \oplus R(1) \oplus \mathcal{O}_{\mathbf{G}}(1).$$

The other G_2-homogeneous variety of Picard rank one is the five dimensional quadric variety $\mathbf{Q} = Q_5$. On $\mathbf{Q}$ there are two important vector bundles of higher rank. One is the *spinor bundle* S on $\mathbf{Q}$. The spinor bundle S has rank 4 and appear in a full strong exceptional collection

$$\mathrm{D^b}(\mathbf{Q}) = \langle \mathcal{O}_{\mathbf{Q}}(-2), \mathcal{O}_{\mathbf{Q}}(-1), S, \mathcal{O}_{\mathbf{Q}}, \mathcal{O}_{\mathbf{Q}}(1), \mathcal{O}_{\mathbf{Q}}(2) \rangle.$$

Lemma 5.2.5. *For the spinor bundle S on Q, we have*

(1) $S^\vee \simeq S(1)$ *and* $\det S \simeq \mathcal{O}_{\mathbf{Q}}(-2)$.

(2) There exists an exact sequence

$$0 \to S \to \mathcal{O}_{\mathbf{Q}}^{\oplus 8} \to S(1) \to 0.$$

This lemma should be well-known but we give the proof here for convenience.

Proof. To show this lemma, we use the theory of mutations of an exceptional collection.

First, by taking dual of the collection above, we have another exceptional collection

$$\mathrm{D^b}(\mathbf{Q}) = \langle \mathcal{O}_{\mathbf{Q}}(-2), \mathcal{O}_{\mathbf{Q}}(-1), \mathcal{O}_{\mathbf{Q}}, S^\vee, \mathcal{O}_{\mathbf{Q}}(1), \mathcal{O}_{\mathbf{Q}}(2) \rangle.$$

On the other hand, by applying a functor $(-) \otimes \mathcal{O}_{\mathbf{Q}}(1)$ to the original collection, we have

$$\mathrm{D^b}(\mathbf{Q}) = \langle \mathcal{O}_{\mathbf{Q}}(-1), \mathcal{O}_{\mathbf{Q}}, S(1), \mathcal{O}_{\mathbf{Q}}(1), \mathcal{O}_{\mathbf{Q}}(2), \mathcal{O}_{\mathbf{Q}}(3) \rangle.$$

Then by mutating $\mathcal{O}_{\mathbf{Q}}(3)$ to the left end, we have another collection

$$\mathrm{D^b}(\mathbf{Q}) = \langle \mathcal{O}_{\mathbf{Q}}(-2), \mathcal{O}_{\mathbf{Q}}(-1), \mathcal{O}_{\mathbf{Q}}, S(1), \mathcal{O}_{\mathbf{Q}}(1), \mathcal{O}_{\mathbf{Q}}(2) \rangle.$$

Therefore we have $S^\vee \simeq S(1)$ from a basic fact about exceptional collections. By taking det, we have $\det S \simeq \mathcal{O}_{\mathbf{Q}}(-2)$.

Let us show (2). From the exceptional collections above we have

$$\mathbb{L}_{\mathcal{O}_{\mathbf{Q}}}(S(1)) \simeq S[a]$$

for some $a \in \mathbb{Z}$, where $\mathbb{L}_{\mathcal{O}_{\mathbf{Q}}}$ is the left mutation over $\mathcal{O}_{\mathbf{Q}}$. By definition of a left mutation, we have an exact triangle

$$\mathrm{RHom}_{\mathbf{Q}}(\mathcal{O}_{\mathbf{Q}}, S(1)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{Q}} \xrightarrow{\mathrm{ev}} S(1) \to \mathbb{L}_{\mathcal{O}_{\mathbf{Q}}}(S(1)).$$

Since $S(1)$ and $\mathbb{L}_{\mathcal{O}_{\mathbf{Q}}}(S(1))$ are (some shifts of) sheaves, the integer a should be $a = -1$ and we have an exact triangle

$$0 \to S \to \mathrm{Hom}_{\mathbf{Q}}(\mathcal{O}_{\mathbf{Q}}, S(1)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{Q}} \to S(1) \to 0.$$

By computing the rank of bundles, we have $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbf{Q}}(\mathcal{O}_{\mathbf{Q}}, S(1)) = 8$. $\square$

Another important vector bundle on $\mathbf{Q}$ is the *Cayley bundle* C. The Cayley bundle C is a homogeneous vector bundle of rank two, and $\det C \simeq \mathcal{O}_{\mathbf{Q}}(-1)$. Historically, this bundle was first studied by Ottaviani [Ott90]. Later we will see that the variety Y_- that gives one side of the Abuaf-Ueda flop is the total space of $C(-2)$.

The G_2-flag variety $\mathbf{F}$ is a 6-dimensional variety of Picard rank two. There is a projection $p' : \mathbf{F} \to \mathbf{G}$, and via this projection, $\mathbf{F}$ is isomorphic to the projectivization of the universal subbundle $R(1)$ (with some line bundle twist):

$$\mathbf{F} \simeq \mathbb{P}_{\mathbf{G}}(R(-1)) := \operatorname{Proj}_{\mathbf{G}} \operatorname{Sym}^{\bullet}(R(-1))^{\vee}.$$

Similarly, via a projection $q' : \mathbf{F} \to \mathbf{Q}$, we have

$$\mathbf{F} \simeq \mathbb{P}_{\mathbf{Q}}(C(-2)) := \operatorname{Proj}_{\mathbf{Q}} \operatorname{Sym}^{\bullet}(C(-2))^{\vee}.$$

Fix general members $H \in |(p')^*\mathcal{O}_{\mathbf{G}}(1)|$ and $h \in |(q')^*\mathcal{O}_{\mathbf{Q}}(1)|$. Then we can write

$$\mathcal{O}_{\mathbf{F}}(aH + bh) \simeq \mathcal{O}_{\mathbf{G}}(a) \boxtimes \mathcal{O}_{\mathbf{Q}}(b).$$

Borel-Bott-Weil theorem

For homogeneous vector bundles on homogeneous varieties, we can compute their sheaf cohomologies using the Borel-Bott-Weil theorem.

Theorem 5.2.6 (Borel-Bott-Weil)**.** *Let E be a homogeneous vector bundle on a projective homogeneous variety Z that corresponds to a weight π. Then one of the following can happen.*

(i) There exists an element w of the Weyl group W such that $w \cdot \pi$ is a dominant weight.

(ii) There exists $w \in W$ such that $w \cdot \pi = \pi$.

Furthermore,

(I) In the case of (i), we have

$$H^i(Z, E) \simeq \begin{cases} (V_{S\cdot\pi})^{\vee} & \text{if } i = l(w) \\ 0 & \text{otherwise} \end{cases}$$

(II) In the case of (ii), we have

$$\mathrm{R}\Gamma(Z, E) \simeq 0.$$

Note that we use the dot-action in this theorem. We also note that the condition (ii) is equivalent to the condition (ii') in our case:

(ii') $\pi + \rho \in \mathbb{R} \cdot \alpha$ for some $\alpha \in \Delta$, where $\mathbb{R} \cdot \alpha$ is a line spanned by a root α.

On the G_2-Grassmannian $\mathbf{G}$, a homogeneous vector bundle corresponding to a weight $a\pi_1+b\pi_2$ exists if and only if $b \geq 0$, and that bundle is $\mathrm{Sym}^b(R^\vee)(a)$. On the five dimensional quadric $\mathbf{Q}$, a homogeneous vector bundle corresponding to a weight $a\pi_1+b\pi_2$ exists if and only if $a \geq 0$, and that bundle is $\mathrm{Sym}^a(C^\vee)(a+b)$. On the flag variety $\mathbf{F}$, a line bundle $\mathcal{O}_\mathbf{F}(aH + bh)$ corresponds to a weight $a\pi_1 + b\pi_2$. Thus we can compute the cohomology of these bundles using the Borel-Bott-Weil theorem.

Geometry of the Abuaf-Ueda flop

We explain the geometric description of the Abuaf-Ueda flop. First as explained in [Ued16], Y_+ is the total space of a vector bundle $R(-1)$ on $\mathbf{G}$. Since $\det(R(-1)) \simeq \mathcal{O}_\mathbf{G}(-3) \simeq \omega_\mathbf{G}$, the variety Y_+ is local Calabi-Yau of dimension seven.

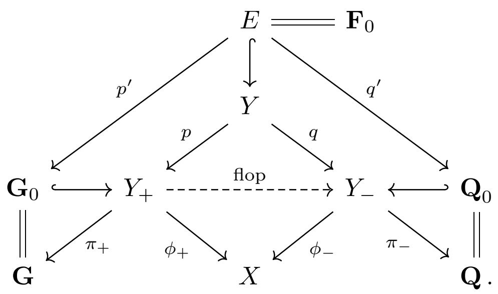
The other side of the flop Y_- is also a total space of a vector bundle of rank two on $\mathbf{Q}$. The bundle is $C(-2)$. Note that $\det(C(-2)) \simeq \mathcal{O}_\mathbf{Q}(-5) \simeq \omega_\mathbf{Q}$.

Let $\mathbf{G}_0 \subset Y_+$ and $\mathbf{Q}_0 \subset Y_-$ be the zero-sections. Then the blowing-ups of these zero-sections give the same variety

$$\mathrm{Bl}_{\mathbf{G}_0}(Y_+) \simeq \mathrm{Bl}_{\mathbf{Q}_0}(Y_-) =: Y,$$

and exceptional divisors of $p : Y \to Y_+$ and $q : Y \to Y_-$ are same, which we denote by E. There is a morphism $Y \to \mathbf{F}$, and via this morphism, Y is isomorphic to the total space of $\mathcal{O}_\mathbf{F}(-H - h)$. The zero-section $\mathbf{F}_0$ (via this description of Y) is the exceptional divisor E.

Thus we have the following diagram



Using projections $\pi_+ : Y_+ \to \mathbf{G}$ and $\pi_- : Y_- \to \mathbf{Q}$, we define vector bundles

$$\mathcal{O}_{Y_+}(a) := \pi_+^*\mathcal{O}_\mathbf{G}(a) \text{ and } \mathcal{R} := \pi_+^* R$$

on Y_+ and

$$\mathcal{O}_{Y_-}(a) := \pi_-^*\mathcal{O}_\mathbf{Q}(a) \text{ and } \mathcal{S} := \pi_-^* S$$

on Y_-. As for $\mathbf{F}$, we define

$$\mathcal{O}_Y(aH + bh) := \mathcal{O}_{Y_+}(a) \boxtimes \mathcal{O}_{Y_-}(b).$$

By construction a line bundle $\mathcal{O}_Y(aH+bh)$ coincides with the pull-back of $\mathcal{O}_{\mathbf{F}}(aH+bh)$ by the projection $Y \to \mathbf{F}$ and thus we have

$$\mathcal{O}_Y(E) \simeq \mathcal{O}_Y(-H-h).$$

5.3 Tilting bundles and derived equivalences

5.3.1 Tilting bundles on Y_+

First, we construct tilting bundles on Y_+. Recall that the derived category $\mathrm{D^b}(\mathbf{G})$ has an exceptional collection

$$R(-1), \mathcal{O}_{\mathbf{G}}(-1), R, \mathcal{O}_{\mathbf{G}}, R(1), \mathcal{O}_{\mathbf{G}}(1),$$

where R is the universal subbundle. Pulling back this collection, we have a collection of vector bundles on Y_+ that is

$$\mathcal{R}(-1), \mathcal{O}_{Y_+}(-1), \mathcal{R}, \mathcal{O}_{Y_+}, \mathcal{R}(1), \mathcal{O}_{Y_+}(1).$$

The direct sum of these vector bundles gives a generator of $\mathrm{D}(\mathrm{Qcoh}(Y_+))$ by the following Lemma 5.3.1. However, the following Proposition 5.3.2 shows that the direct sum of these vector bundles is NOT a tilting bundle on Y_+.

Lemma 5.3.1. *Let $\pi : Z \to W$ be an affine morphism and $E \in \mathrm{D}(\mathrm{Qcoh}(W))$ is a generator. Then the derived pull back $L\pi^*(E)$ is a generator of $\mathrm{D}(\mathrm{Qcoh}(Z))$.*

Proof. Let $F \in \mathrm{D}(\mathrm{Qcoh}(Z))$ be an object with $\mathrm{RHom}_Z(L\pi^*(E), F) = 0$. Then since $\mathrm{RHom}_Z(L\pi^*(E), F) = \mathrm{RHom}_W(E, R\pi_*(F))$ and E is a generator, we have $R\pi_*(F) = 0$. The affineness of the morphism π implies $F = 0$. □

Proposition 5.3.2. *We have*

(1) $H^{\geq 1}(Y_+, \mathcal{O}_{Y_+}(a)) = 0$ for all $a \geq -2$.

(2) $H^{\geq 1}(Y_+, \mathcal{R}(a)) = 0$ for $a \geq -2$.

(3) $\mathrm{Ext}^{\geq 1}_{Y_+}(\mathcal{R}, \mathcal{O}_{Y_+}(a)) = 0$ for $a \geq -3$.

(4) $\mathrm{Ext}^{\geq 1}_{Y_+}(\mathcal{R}, \mathcal{R}(a)) = 0$ for $a \geq -1$.

(5) $\mathrm{Ext}^{\geq 2}_{Y_+}(\mathcal{R}, \mathcal{R}(-2)) = 0$ and $\mathrm{Ext}^1_{Y_+}(\mathcal{R}, \mathcal{R}(-2)) \simeq \mathbb{C}$.

Proof. Here we prove (4) and (5) only. Other cases follow from similar (and easier) computations.

Let $a \geq -2$ and $i \geq 1$. Since there are irreducible decompositions

$$\mathcal{R}^\vee \otimes \mathcal{R}(a) \simeq (\mathrm{Sym}^2 \mathcal{R}^\vee)(a-1) \oplus \mathcal{O}_{Y_-}(a),$$

we have

$$\mathrm{Ext}^i_{Y_+}(\mathcal{R}, \mathcal{R}(a)) \simeq H^i(Y_+, \mathrm{Sym}^2 \mathcal{R}^\vee(a-1)) \oplus H^i(Y_+, \mathcal{O}_{Y_+}(a)).$$

The second term of this decomposition is zero by (1), and hence we have

$$
\begin{aligned}
\operatorname{Ext}^i_{Y_+}(\mathcal{R},\mathcal{R}(a)) &\simeq H^i(Y_+, \operatorname{Sym}^2\mathcal{R}^\vee(a-1)) \\
&\simeq H^i\left(\mathbf{G}, (\operatorname{Sym}^2 R^\vee)(a-1)\otimes \bigoplus_{k\geq 0}(\operatorname{Sym}^k R^\vee)(k)\right)
\end{aligned}
$$

by adjunction. To compute this cohomology, we use the following decomposition

$$
(\operatorname{Sym}^k R^\vee)(k)\otimes(\operatorname{Sym}^2 R^\vee)(a-1)\simeq
$$
$$
\begin{cases}
(\operatorname{Sym}^{k+2} R^\vee)(k+a-1)\oplus(\operatorname{Sym}^k R^\vee)(k+a)\oplus(\operatorname{Sym}^{k-2} R^\vee)(k+a+1) & \text{if } k\geq 2\\
(\operatorname{Sym}^3 R^\vee)(a)\oplus(\operatorname{Sym}^1 R^\vee)(a+1) & \text{if } k=1\\
(\operatorname{Sym}^2 R^\vee)(a-1) & \text{if } k=0.
\end{cases}
$$

According to this irreducible decomposition, it is enough to compute the cohomology of the following vector bundles.

(i) $(\operatorname{Sym}^{k+2} R^\vee)(k+a-1)$ for $k\geq 0$ and $a\geq -2$.

(ii) $(\operatorname{Sym}^k R^\vee)(k+a)$ for $k\geq 1$ and $a\geq -2$.

(iii) $(\operatorname{Sym}^{k-2} R^\vee)(k+a+1)$ for $k\geq 2$ and $a\geq -2$.

To compute the cohomology of these bundles, we use the Borel-Bott-Weil theorem. A bundle of type (i) corresponds to a weight $(k+a-1)\pi_1+(k+2)\pi_2$. This weight is dominant if and only if $k+a\geq 1$, i.e.

$$
(k,a)\notin\{(0,-2),(0,-1),(0,0),(1,-2),(1,-1),(2,-2)\}.
$$

In this case the bundle has no higher cohomology. If $(k,a)=(0,-2)$, then we have

$$
-3\pi_1+2\pi_2+\rho=-2\pi_1+3\pi_2=(0,2,-2)+(-1,-1,2)=(-1,1,0)
$$

and this vector is a root. Thus we have that the corresponding bundle is acyclic, i.e.

$$
\mathrm{R}\Gamma(\mathbf{G}, \operatorname{Sym}^2 R^\vee(-3))=0.
$$

One can show that the same things hold for $(k,a)=(0,-1),(0,0),(1,-1),(2,-2)$. Let us compute the case if $(k,a)=(1,-2)$. In this case we have

$$
\mathrm{S}_{\alpha_1}\cdot(-2\pi_1+3\pi_2)=0.
$$

Thus the Borel-Bott-Weil theorem implies

$$
\mathrm{R}\Gamma(\mathbf{G},(\operatorname{Sym}^3 R^\vee)(-2))\simeq\mathbb{C}[-1].
$$

Using the Borel-Bott-Weil theorem in the same way, we can show that bundles of type (ii) and (iii) have no higher cohomology. This shows (4) and (5). □

Definition 5.3.3. Let Σ be a rank 4 vector bundle on Y_+ that lies in the following unique non-trivial extension

$$0 \to \mathcal{R}(-1) \to \Sigma \to \mathcal{R}(1) \to 0.$$

Now we can show that the bundle Σ is partial tilting and that a bunde

$$\mathcal{O}_{Y_+}(-1) \oplus \mathcal{O}_{Y_+} \oplus \mathcal{O}_{Y_+}(1) \oplus \mathcal{R} \oplus \mathcal{R}(1) \oplus \Sigma$$

is a tilting bundle on Y_+ as in the proof of Lemma 5.2.4. We also note that the dual $\Sigma^\vee$ of Σ is isomorphic to $\Sigma(1)$. Indeed, the bundle $\Sigma^\vee$ lies in the sequence

$$0 \to \mathcal{R}^\vee(-1) \to \Sigma^\vee \to \mathcal{R}^\vee(1) \to 0.$$

The isomorphism $\mathcal{R}^\vee \simeq \mathcal{R}(1)$ and the uniqueness of such a non-trivial extension imply that $\Sigma^\vee \simeq \Sigma(1)$.

We can apply the same method to another collection

$$\mathcal{O}_{Y_+}(-1), \mathcal{R}, \mathcal{O}_{Y_+}, \mathcal{R}(1), \mathcal{O}_{Y_+}(1), \mathcal{R}(2),$$

and then we get another tilting bundle. As a consequence, we have the following.

Theorem 5.3.4. *The following vector bundles on Y_+ are tilting bundles.*

(1) $T_+^\spadesuit := \mathcal{O}_{Y_+}(-1) \oplus \mathcal{O}_{Y_+} \oplus \mathcal{O}_{Y_+}(1) \oplus \mathcal{R} \oplus \mathcal{R}(1) \oplus \Sigma$

(2) $T_+^\clubsuit := \mathcal{O}_{Y_+}(-1) \oplus \mathcal{O}_{Y_+} \oplus \mathcal{O}_{Y_+}(1) \oplus \mathcal{R} \oplus \mathcal{R}(1) \oplus \Sigma(1)$

(3) $T_+^\heartsuit := \mathcal{O}_{Y_+}(-1) \oplus \mathcal{O}_{Y_+} \oplus \mathcal{O}_{Y_+}(1) \oplus \mathcal{R}(-1) \oplus \mathcal{R} \oplus \Sigma$

(4) $T_+^\diamondsuit := \mathcal{O}_{Y_+}(-1) \oplus \mathcal{O}_{Y_+} \oplus \mathcal{O}_{Y_+}(1) \oplus \mathcal{R}(1) \oplus \mathcal{R}(2) \oplus \Sigma(1)$

Note that the pair $T_+^\spadesuit$ and $T_+^\clubsuit$ are dual to each other, and the pair $T_+^\heartsuit$ and $T_+^\diamondsuit$ are dual to each other.

5.3.2 Tilting bundles on Y_-

To find explicit tilting bundles on Y_-, we need to use not only the Borel-Bott-Weil theorem but also some geometry of the flop. Recall that the derived category $\mathrm{D^b}(\mathbf{Q})$ has an exceptional collection

$$\mathcal{O}_\mathbf{Q}(-2), \mathcal{O}_\mathbf{Q}(-1), S, \mathcal{O}_\mathbf{Q}, \mathcal{O}_\mathbf{Q}(1), \mathcal{O}_\mathbf{Q}(2),$$

where S is the rank 4 spinor bundle on the five dimensional quadric $\mathbf{Q}$. Pulling back this collection by the projection $\pi_- : Y_- \to \mathbf{Q}$, we have a collection of vector bundles on Y_-

$$\mathcal{O}_{Y_-}(-2), \mathcal{O}_{Y_-}(-1), \mathcal{S}, \mathcal{O}_{Y_-}, \mathcal{O}_{Y_-}(1), \mathcal{O}_{Y_-}(2).$$

The direct sum of these vector bundles is a generator of $\mathrm{D}(\mathrm{Qcoh}(Y))$ by Lemma 5.3.1, but does NOT give a tilting bundle on Y_-. First, we compute cohomologies of line bundles.

Proposition 5.3.5. *(1)* $H^{\geq 1}(Y_-, \mathcal{O}_{Y_-}(a)) = 0$ *for all* $a \geq -2$.

(2) $H^{\geq 2}(Y_-, \mathcal{O}_{Y_-}(a)) = 0$ *for all* $a \geq -4$.

(3) $H^1(Y_-, \mathcal{O}_{Y_-}(-3)) = \mathbb{C}$.

Proof. Let $a \geq -4$. We have the following isomorphism by adjunction

$$H^i(Y_-, \mathcal{O}_{Y_-}(a)) \simeq \bigoplus_{k \geq 0} H^i(\mathbf{Q}, (\mathrm{Sym}^k C^\vee)(2k + a)).$$

A bundle $(\mathrm{Sym}^k C^\vee)(2k + a)$ corresponds to a weight $k\pi_1 + (k + a)\pi_2$. This weight is dominant if and only if $k + a \geq 0$, i.e.

$$(k, a) \notin \{(0, -4), (0, -3), (0, -2), (0, -1), (1, -4), (1, -3), (1, -2), (2, -4), (2, -3), (3, -4)\}.$$

In this case, the corresponding vector bundle has no higher cohomologies. If $k = 0$ and $a \leq -1$, then the corresponding bundle is an acyclic line bundle $\mathcal{O}_{\mathbf{Q}}(a)$.

Let us consider remaining cases. If $(k, a) = (1, -4), (1, -2), (2, -3), (3, -4)$ then

$$k\pi_1 + (k+a)\pi_2 + \rho = (k+1)\pi_1 + (k+a+1)\pi_2 = \begin{cases} 2\pi_1 - 2\pi_2 = (\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}) & \text{if } (k, a) = (1, -4) \\ 2\pi_1 & \text{if } (k, a) = (1, -2) \\ 3\pi_1 & \text{if } (k, a) = (2, -3) \\ 4\pi_1 & \text{if } (k, a) = (3, -4) \end{cases}$$

and the weight lies in a line spanned by a root. Thus the corresponding bundle is acyclic in those cases. If $(k, a) = (1, -3)$ then we have

$$\mathrm{S}_{\alpha_2} \cdot (\pi_1 - 2\pi_2) = 0$$

and hence we obtain

$$\mathrm{R}\Gamma(\mathbf{Q}, C^\vee(-1)) \simeq \mathbb{C}[-1].$$

If $(k, a) = (2, -4)$ then we have

$$\mathrm{S}_{\alpha_2} \cdot (2\pi_1 - 2\pi_2) = \pi_1$$

and thus we get

$$\mathrm{R}\Gamma(\mathbf{Q}, \mathrm{Sym}^2 C^\vee) \simeq V_{\pi_1}^\vee[-1].$$

This shows the result. □

Definition 5.3.6. Let $\mathcal{P}$ be the rank 2 vector bundle on Y_- which lies in the following unique non-trivial extension

$$0 \to \mathcal{O}_{Y_-}(-2) \to \mathcal{P} \to \mathcal{O}_{Y_-}(1) \to 0.$$

One can show that the bundle $\mathcal{P}$ is partial tilting as in Lemma 5.2.4. Note that, by the uniqueness of such a non-trivial sequence, we have $\mathcal{P}^\vee \simeq \mathcal{P}(1)$.

Proposition 5.3.7. *We have* $H^{\geq 1}(Y_-, \mathcal{P}(a)) = 0$ *for* $a \geq -2$.

To prove this Proposition, we have to use the geometry of the flop. The following two lemmas are important.

Lemma 5.3.8. *On the full flag variety* $\mathbf{F}$, *there is an exact sequence of vector bundles*

$$0 \to \mathcal{O}_{\mathbf{F}}(-h) \to p'^{*}R \to \mathcal{O}_{\mathbf{F}}(-H + h) \to 0$$

Proof. See [Kuz18]. □

Lemma 5.3.9. *There is an isomorphism* $\mathcal{P} \simeq Rq_*(p^*\mathcal{R}(-h))$.

Proof. From the lemma above, we have an exact sequence on Y

$$0 \to \mathcal{O}_Y(-h) \to p^*\mathcal{R} \to \mathcal{O}_Y(-H + h) \to 0.$$

Since $\mathcal{O}_Y(E) \simeq \mathcal{O}_Y(-H - h)$, we have an exact sequence

$$0 \to \mathcal{O}_Y(-2h) \to p^*\mathcal{R}(-h) \to \mathcal{O}_Y(h + E) \to 0.$$

Using projection formula and $Rq_*\mathcal{O}_Y(E) \simeq \mathcal{O}_{Y_-}$, we have $Rq_*(p^*\mathcal{R}(-h)) \simeq q_*(p^*\mathcal{R}(-h))$, and this bundle lies in the exact sequence

$$0 \to \mathcal{O}_{Y_-}(-2) \to q_*(p^*\mathcal{R}(-h)) \to \mathcal{O}_{Y_-}(1) \to 0.$$

This sequence is not split. Indeed if it is split, the bundle $q_*(p^*\mathcal{R}(-h))|_{(Y_- \setminus \mathbf{Q}_0)}$ is also split. However, under the natural identification $Y_- \setminus \mathbf{Q}_0 \simeq Y_+ \setminus \mathbf{G}_0$, the bundle $q_*(p^*\mathcal{R}(-h))|_{(Y_- \setminus \mathbf{Q}_0)}$ is identified with $\mathcal{R}(1)|_{(Y_+ \setminus \mathbf{G}_0)}$. Since the zero-section $\mathbf{G}_0$ has codimension two in Y_+, if the bundle $\mathcal{R}(1)|_{(Y_+ \setminus \mathbf{G}_0)}$ is split, the bundle $\mathcal{R}(1)$ is also split. This is contradiction.

Thus, by Proposition 5.3.5, we have $\mathcal{P} \simeq Rq_*(p^*\mathcal{R}(-h))$. □

Proof of Proposition 5.3.7. First, we have

$$H^{\geq 1}(Y_-, \mathcal{P}(a)) = 0 \text{ for all } a \geq 0,$$

and

$$H^{\geq 2}(Y_-, \mathcal{P}(a)) = 0 \text{ for all } a \geq -2,$$

by the definition of $\mathcal{P}$ and Proposition 5.3.5. Thus the non-trivial parts are the vanishing of $H^1(Y_-, \mathcal{P}(-1))$ and $H^1(Y_-, \mathcal{P}(-2))$. The first part also follows from the definition of $\mathcal{P}$ using the same argument as in the proof of Lemma 5.2.4.

In the following, we show the vanishing of $H^1(Y_-, \mathcal{P}(-2))$. First by Lemma 5.3.9, we have $\mathcal{P} \simeq Rq_*(p^*\mathcal{R}(-h))$.

Therefore we can compute the cohomology as follows.

$$\begin{aligned}H^1(Y_-, \mathcal{P}(-2)) &\simeq H^1(Y_-, Rq_*(p^*\mathcal{R}(-h)) \otimes \mathcal{O}_{Y_-}(-2)) \\ &\simeq H^1(Y_-, Rq_*(p^*\mathcal{R}(-3h))) \\ &\simeq H^1(Y, p^*\mathcal{R}(-3h)) \\ &\simeq H^1(Y, p^*\mathcal{R}(3H+3E)) \\ &\simeq H^1(Y_+, \mathcal{R}(3) \otimes Rp_*\mathcal{O}_Y(3E)).\end{aligned}$$

To compute this cohomology, we use a spectral sequence

$$E_2^{k,l} = H^k(Y_+, \mathcal{R}(3) \otimes R^l p_*\mathcal{O}_Y(3E)) \Rightarrow H^{k+l}(Y_+, \mathcal{R}(3) \otimes Rp_*\mathcal{O}_Y(3E)).$$

Since $p_*\mathcal{O}_Y(3E) \simeq \mathcal{O}_{Y_+}$, we have

$$E_2^{k,0} = H^k(Y_+, \mathcal{R}(3)) = 0 \text{ for } k \geq 1.$$

This shows that there is an isomorphism of cohomologies

$$H^1(Y_+, \mathcal{R}(3) \otimes Rp_*\mathcal{O}_Y(3E)) \simeq H^0(Y_+, \mathcal{R}(3) \otimes R^1p_*\mathcal{O}_Y(3E)).$$

Let us consider the exact sequence

$$0 \to \mathcal{O}_Y(2E) \to \mathcal{O}_Y(3E) \to \mathcal{O}_E(3E) \to 0.$$

Now we have

$$\begin{aligned}&p_*(\mathcal{O}_E(3E)) = 0, \\ &R^1p_*\mathcal{O}_E(3E) \simeq R(-1) \otimes \det(R(-1)) \simeq R(-4), \text{ and} \\ &R^1p_*\mathcal{O}_Y(2E) \simeq R^1p_*\mathcal{O}_E(2E) \simeq \det(R(-1)) \simeq \mathcal{O}_{\mathbf{G}_0}(-3).\end{aligned}$$

Hence there is an exact sequence

$$0 \to \mathcal{O}_{\mathbf{G}_0}(-3) \to R^1p_*(\mathcal{O}_Y(3E)) \to R(-4) \to 0.$$

Since

$$H^0(Y_+, \mathcal{R}(3) \otimes \mathcal{O}_{\mathbf{G}_0}(-3)) \simeq H^0(\mathbf{G}, R) = 0$$

and

$$H^0(Y_+, \mathcal{R}(3) \otimes R(-4)) \simeq H^0(\mathbf{G}, R \otimes R(-1)) \simeq \mathrm{Hom}_{\mathbf{G}}(R^\vee, R(-1)) = 0,$$

we finally have the desired vanishing

$$H^1(Y_-, \mathcal{P}(-2)) \simeq H^0(Y_+, \mathcal{R}(3) \otimes R^1p_*\mathcal{O}_Y(3E)) = 0.$$

□

Corollary 5.3.10. *We have*

(1) $\mathrm{Ext}^{\geq 1}_{Y_-}(\mathcal{P}(1), \mathcal{P}) = 0$ *and*

(2) $\mathrm{Ext}^{\geq 1}_{Y_-}(\mathcal{P}, \mathcal{P}(1)) = 0$.

Proof. Let us consider the exact sequence

$$0 \to \mathcal{O}_{Y_-}(-2) \to \mathcal{P} \to \mathcal{O}_{Y_-}(1) \to 0$$

that defines the bundle $\mathcal{P}$. Applying functors $\mathrm{RHom}_{Y_-}(\mathcal{P}(1), -)$ and $\mathrm{RHom}_{Y_-}(-, \mathcal{P}(1))$, we get exact triangles

$$\mathrm{RHom}_{Y_-}(\mathcal{P}(1), \mathcal{O}_{Y_-}(-2)) \to \mathrm{RHom}_{Y_-}(\mathcal{P}(1), \mathcal{P}) \to \mathrm{RHom}_{Y_-}(\mathcal{P}(1), \mathcal{O}_{Y_-}(1)),$$
$$\mathrm{RHom}_{Y_-}(\mathcal{O}_{Y_-}(1), \mathcal{P}(1)) \to \mathrm{RHom}_{Y_-}(\mathcal{P}, \mathcal{P}(1)) \to \mathrm{RHom}_{Y_-}(\mathcal{O}_{Y_-}(-2), \mathcal{P}(1)).$$

Now the results follow from Proposition 5.3.7. □

Next we compute the cohomology of (the pull back of) the spinor bundle $\mathcal{S}$. For this computation, we use the geometry of the flop again.

The following lemma is due to Kuznetsov.

Lemma 5.3.11. *There is an exact sequence on the flag variety* **F**

$$0 \to p'^* R \to q'^* S \to p'^* R(H - h) \to 0.$$

Proof. See [Kuz18, Proposition 3 and Lemma 4]. □

Remark 5.3.12. Interestingly, to prove this geometric lemma, Kuznetsov used derived categories (namely, mutations of exceptional collections).

Using this lemma, we have the following.

Lemma 5.3.13. *An object* $Rq_*(p^* \mathcal{R}(H - h))$ *is a sheaf on* Y_- *and there exists an exact sequence on* Y_-

$$0 \to \mathcal{P}(1) \to \mathcal{S} \to Rq_*(p^* \mathcal{R}(H - h)) \to 0.$$

Proof. By Kuznetsov's lemma, there is an exact sequence on Y

$$0 \to p^* \mathcal{R} \to q^* \mathcal{S} \to p^* \mathcal{R}(H - h) \to 0.$$

Since $Rq_*(p^* \mathcal{R}) \simeq \mathcal{P}(1)$, the object $Rq_*(p^* \mathcal{R}(H - h))$ is a sheaf on Y_- and we have an exact sequence on Y_-

$$0 \to \mathcal{P}(1) \to \mathcal{S} \to Rq_*(p^* \mathcal{R}(H - h)) \to 0.$$

□

Using this exact sequence, we can do the following computations.

Lemma 5.3.14. *We have*

(1) $\mathrm{Ext}^{\geq 1}_{Y_-}(\mathcal{O}_{Y_-}(a), \mathcal{S}) = 0$ *for* $a \leq 1$*, and*

(2) $\mathrm{Ext}^{\geq 1}_{Y_-}(\mathcal{S}, \mathcal{O}_{Y_-}(b)) = 0$ *for* $b \geq -2$*.*

Proof. Since $\mathcal{S}^\vee \simeq \mathcal{S}(1)$, it is enough to show that $H^{\geq 1}(Y_-, \mathcal{S}(a)) = 0$ for $a \geq -1$. By Lemma 5.2.5, for any $a \in \mathbb{Z}$, there is an exact sequence

$$0 \to \mathcal{S}(a) \to \mathcal{O}_{Y_-}(a)^{\oplus 8} \to \mathcal{S}(a+1) \to 0.$$

Since $H^{\geq 1}(Y_-, \mathcal{O}_{Y_-}(a)) = 0$ if $a \geq -2$, it is enough to show the case if $a = -1$. Let us consider the exact sequence

$$0 \to \mathcal{P} \to \mathcal{S}(-1) \to Rq_*(p^* \mathcal{R}(H - 2h)) \to 0.$$

Now we have $H^{\geq 1}(Y_-, \mathcal{P}) = 0$, and therefore we can compute as

$$\begin{aligned} H^i(Y_-, \mathcal{S}(-1)) &\simeq H^i(Y_-, Rq_*(p^* \mathcal{R}(H - 2h))) \\ &\simeq H^i(Y, p^* \mathcal{R}(3H + 2E)) \\ &\simeq H^i(Y_+, \mathcal{R}(3) \otimes Rp_* \mathcal{O}_Y(2E)) \end{aligned}$$

for any $i \geq 1$.

Let us consider a spectral sequence

$$E_2^{k,l} = H^k(Y_+, \mathcal{R}(3) \otimes R^l p_* \mathcal{O}_Y(2E)) \Rightarrow H^{k+l}(Y_+, \mathcal{R}(3) \otimes Rp_* \mathcal{O}_Y(2E)).$$

Now since $p_* \mathcal{O}_Y(2E) \simeq \mathcal{O}_{Y_+}$ we have

$$E_2^{k,0} = H^k(Y_+, \mathcal{R}(3) \otimes p_* \mathcal{O}_Y(2E)) \simeq H^k(Y_+ \mathcal{R}(3)) = 0$$

for $k \geq 1$ and hence we have

$$H^i(Y_+, \mathcal{R}(3) \otimes Rp_* \mathcal{O}_Y(2E)) \simeq H^{i-1}(Y_+, \mathcal{R}(3) \otimes R^1 p_* \mathcal{O}_Y(2E))$$

for $i \geq 1$. Now we can compute as $R^1 p_* \mathcal{O}_Y(2E) \simeq Rp_* \mathcal{O}_E(2E) \simeq \mathcal{O}_{\mathbf{G}_0}(-3)$ and thus

$$H^{i-1}(Y_+, \mathcal{R}(3) \otimes R^1 p_* \mathcal{O}_Y(2E)) \simeq H^{i-1}(\mathbf{G}, \mathcal{R}) = 0$$

for all $i \geq 1$. This finishes the proof. □

From the lemma above, we have the following.

Theorem 5.3.15. *The following hold.*

(1) $\mathrm{Ext}^{\geq 1}_{Y_-}(\mathcal{S}, \mathcal{P}) = 0$ *and* $\mathrm{Ext}^{\geq 1}_{Y_-}(\mathcal{P}, \mathcal{S}) = 0$*.*

(2) $\mathrm{Ext}^{\geq 1}_{Y_-}(\mathcal{S}, \mathcal{P}(1)) = 0$*.*

(3) $\mathrm{Ext}^{\geq 1}_{Y_-}(\mathcal{P}(-1), \mathcal{S}) = 0$*.*

(4) $\mathcal{S}$ *is a partial tilting bundle.*

Proof. (1). First note that $\mathrm{Ext}^i_{Y_-}(\mathcal{S},\mathcal{P}) \simeq \mathrm{Ext}^i_{Y_-}(\mathcal{P},\mathcal{S})$ since $\mathcal{P}^\vee \simeq \mathcal{P}(1)$ and $\mathcal{S}^\vee \simeq \mathcal{S}(1)$. Let us consider the exact sequence

$$0 \to \mathcal{O}_{Y_-}(-2) \to \mathcal{P} \to \mathcal{O}_{Y_-}(1) \to 0.$$

Then, applying the functor $\mathrm{RHom}_{Y_-}(\mathcal{S}, -) \simeq \mathrm{R}\Gamma(Y_-, \mathcal{S}(1) \otimes -)$ we have an exact triangle

$$\mathrm{R}\Gamma(Y_-, \mathcal{S}(-1)) \to \mathrm{RHom}_{Y_-}(\mathcal{S},\mathcal{P}) \to \mathrm{R}\Gamma(Y_-, \mathcal{S}(2)).$$

Now the result follows from the lemma above. The proof of (2) is similar. (3) follows from (2). Indeed, since $\mathcal{S}^\vee \simeq \mathcal{S}(1)$ and $(\mathcal{P}(1))^\vee \simeq \mathcal{P}$, we have

$$\mathrm{Ext}^i_{Y_-}(\mathcal{P}(-1),\mathcal{S}) \simeq \mathrm{Ext}^i_{Y_-}(\mathcal{P},\mathcal{S}(1)) \simeq \mathrm{Ext}^i_{Y_-}(\mathcal{S},\mathcal{P}(1)).$$

Now let us prove (4). Recall that there is an exact sequence

$$0 \to \mathcal{P}(1) \to \mathcal{S} \to Rq_*(p^*\mathcal{R}(H-h)) \to 0.$$

By (2), we have

$$\mathrm{Ext}^i_{Y_-}(\mathcal{S},\mathcal{S}) \simeq \mathrm{Ext}^i_{Y_-}(\mathcal{S}, Rq_*(p^*\mathcal{R}(H-h))) \simeq \mathrm{Ext}^i_{Y}(q^*\mathcal{S}, p^*\mathcal{R}(H-h))$$

for $i \geq 1$. Let us consider an exact sequence

$$0 \to p^*\mathcal{R} \to q^*\mathcal{S} \to p^*\mathcal{R}(H-h) \to 0.$$

Then we have

$$\mathrm{Ext}^i_Y(p^*\mathcal{R}(H-h), p^*\mathcal{R}(H-h)) \simeq \mathrm{Ext}^i_{Y_+}(\mathcal{R},\mathcal{R}) = 0$$

for $i \geq 1$ and

$$\mathrm{Ext}^i_Y(p^*\mathcal{R}, p^*\mathcal{R}(H-h)) \simeq \mathrm{Ext}^i_Y(p^*\mathcal{R}, p^*\mathcal{R}(2H+E)) \simeq \mathrm{Ext}^i_{Y_+}(\mathcal{R},\mathcal{R}(2)) = 0$$

for $i \geq 1$. Thus we have $\mathrm{Ext}^i_Y(q^*\mathcal{S}, p^*\mathcal{R}(H-h)) = 0$ for $i \geq 1$. □

Next we show the following:

Lemma 5.3.16. *We have*

(1) $\mathrm{Ext}^{\geq 1}_{Y_-}(\mathcal{P}(1),\mathcal{S}) = 0$ *and*

(2) $\mathrm{Ext}^{\geq 1}_{Y_-}(\mathcal{S},\mathcal{P}(-1)) = 0$.

Proof. (2) follows from (1). Let us prove (1). Recall that $Rq_*(p^*\mathcal{R}) \simeq \mathcal{P}(1)$. Therefore by the Grothendieck duality we have

$$\mathrm{Ext}^i_{Y_-}(\mathcal{P}(1),\mathcal{S}) \simeq \mathrm{Ext}^i_Y(p^*\mathcal{R}, q^*\mathcal{S}(E)).$$

Let us consider the exact sequence

$$0 \to p^* \mathcal{R}(E) \to q^* \mathcal{S}(E) \to p^* \mathcal{R}(2H + 2E) \to 0.$$

First we have

$$\mathrm{Ext}^i_Y(p^* \mathcal{R}, p^* \mathcal{R}(E)) \simeq \mathrm{Ext}^i_{Y_+}(\mathcal{R}, \mathcal{R}) = 0$$

for $i \geq 1$. Hence it is enough to show the vanishing of

$$\mathrm{Ext}^i_Y(p^* \mathcal{R}, p^* \mathcal{R}(2H + 2E)) \simeq \mathrm{Ext}^i_{Y_+}(\mathcal{R}, \mathcal{R}(2) \otimes Rp_* \mathcal{O}_Y(2E)).$$

Let us consider a spectral sequence

$$E_2^{k,l} = \mathrm{Ext}^k_{Y_+}(\mathcal{R}, \mathcal{R}(2) \otimes R^l p_* \mathcal{O}_Y(2E)) \Rightarrow \mathrm{Ext}^{k+l}_{Y_+}(\mathcal{R}, \mathcal{R}(2) \otimes Rp_* \mathcal{O}_Y(2E)).$$

Note that

$$E_2^{k,0} = \mathrm{Ext}^k_{Y_+}(\mathcal{R}, \mathcal{R}(2)) = 0$$

for $k \geq 1$. Thus we have

$$\begin{aligned}\mathrm{Ext}^i_{Y_+}(\mathcal{R}, \mathcal{R}(2) \otimes Rp_* \mathcal{O}_Y(2E)) &\simeq \mathrm{Ext}^{i-1}_{Y_+}(\mathcal{R}, \mathcal{R}(2) \otimes R^1 p_* \mathcal{O}_Y(2E)) \\ &\simeq \mathrm{Ext}^{i-1}_{Y_+}(\mathcal{R}, \mathcal{R}(2) \otimes \mathcal{O}_{\mathbf{G}_0}(-3)) \\ &\simeq \mathrm{Ext}^{i-1}_{\mathbf{G}}(R, R(-1))\end{aligned}$$

for $i \geq 1$. This is zero. □

Combining all Ext-vanishings in the present subsection, we obtain the following consequence.

Theorem 5.3.17. *The following vector bundles on Y_- are tilting bundles.*

(1) $T_-^{\spadesuit} := \mathcal{O}_{Y_-}(-1) \oplus \mathcal{O}_{Y_-} \oplus \mathcal{O}_{Y_-}(1) \oplus \mathcal{P} \oplus \mathcal{P}(1) \oplus \mathcal{S}(1)$

(2) $T_-^{\clubsuit} := \mathcal{O}_{Y_-}(-1) \oplus \mathcal{O}_{Y_-} \oplus \mathcal{O}_{Y_-}(1) \oplus \mathcal{P} \oplus \mathcal{P}(1) \oplus \mathcal{S}$

(3) $T_-^{\heartsuit} := \mathcal{O}_{Y_-}(-1) \oplus \mathcal{O}_{Y_-} \oplus \mathcal{O}_{Y_-}(1) \oplus \mathcal{P}(1) \oplus \mathcal{P}(2) \oplus \mathcal{S}(1)$

(4) $T_-^{\diamondsuit} := \mathcal{O}_{Y_-}(-1) \oplus \mathcal{O}_{Y_-} \oplus \mathcal{O}_{Y_-}(1) \oplus \mathcal{P}(-1) \oplus \mathcal{P} \oplus \mathcal{S}$

We note that these bundles are generators of $\mathrm{D}(\mathrm{Qcoh}(Y_-))$ because they split-generate another generators

$$\begin{aligned}&\mathcal{O}_{Y_-}(-2) \oplus \mathcal{O}_{Y_-}(-1) \oplus \mathcal{O}_{Y_-} \oplus \mathcal{S}(1) \oplus \mathcal{O}_{Y_-}(1) \oplus \mathcal{O}_{Y_-}(2), \\ &\mathcal{O}_{Y_-}(-2) \oplus \mathcal{O}_{Y_-}(-1) \oplus \mathcal{S} \oplus \mathcal{O}_{Y_-} \oplus \mathcal{O}_{Y_-}(1) \oplus \mathcal{O}_{Y_-}(2), \\ &\mathcal{O}_{Y_-}(-1) \oplus \mathcal{O}_{Y_-} \oplus \mathcal{S}(1) \oplus \mathcal{O}_{Y_-}(1) \oplus \mathcal{O}_{Y_-}(2) \oplus \mathcal{O}_{Y_-}(3), \quad \text{and} \\ &\mathcal{O}_{Y_-}(-3) \oplus \mathcal{O}_{Y_-}(-2) \oplus \mathcal{O}_{Y_-}(-1) \oplus \mathcal{S} \oplus \mathcal{O}_{Y_-} \oplus \mathcal{O}_{Y_-}(1)\end{aligned}$$

respectively, that are obtained from tilting bundles on $\mathbf{Q}$. We also note that the pair $T_-^{\spadesuit}$ and $T_-^{\clubsuit}$ are dual to each other, and the pair $T_-^{\heartsuit}$ and $T_-^{\diamondsuit}$ are dual to each other.

5.3.3 Derived equivalences

According to Lemma 5.2.3, in order to show the derived equivalence between Y_+ and Y_-, it is enough to show that there are tilting bundles T_+ and T_- on Y_+ and Y_- respectively, such that they give the same vector bundle on the common open subset U of Y_+ and Y_-, which is isomorphic to the smooth locus of X. Using tilting bundles that we constructed in this chapter, we can give four derived equivalences for the Abuaf-Ueda flop.

Lemma 5.3.18. *On the common open subset U, we have the following.*

(1) $\mathcal{O}_{Y_+}(a)|_U \simeq \mathcal{O}_{Y_-}(-a)|_U$ for all $a \in \mathbb{Z}$.

(2) $\mathcal{R}|_U \simeq \mathcal{P}(1)|_U$.

(3) $\Sigma(1)|_U \simeq \mathcal{S}|_U$.

Proof. (1) follows from the fact that $\mathcal{O}_Y(E) \simeq \mathcal{O}_Y(-H-h)$ since $\mathcal{O}_Y(E)|_U \simeq \mathcal{O}_U$. (2) follows from the isomorphism $\mathcal{P}(1) \simeq Rq_*(p^*\mathcal{R})$.

Let us proof (3). To see this, we show that $Rp_*(q^*\mathcal{S}) \simeq \Sigma(1)$. By Lemma 5.3.11, there is an exact sequence

$$0 \to p^*\mathcal{R} \to q^*\mathcal{S} \to p^*\mathcal{R}(2H+E) \to 0$$

on Y. Since we have $Rp_*\mathcal{O}_Y(E) \simeq \mathcal{O}_{Y_+}$, by projection formula, we have an exact sequence

$$0 \to \mathcal{R} \to Rp_*(q^*\mathcal{S}) \to \mathcal{R}(2) \to 0$$

on Y_+. Note that this short exact sequence is not split. Thus the uniqueness of such a non-trivial sequence shows that the desired isomorphism $Rp_*(q^*\mathcal{S}) \simeq \Sigma(1)$. □

Corollary 5.3.19. *The pair of bundles T_+^* and T_-^* give the same bundle on the common open subset U for any $* \in \{\spadesuit, \clubsuit, \heartsuit, \diamondsuit\}$.*

As a consequence, we have the following theorem.

Theorem 5.3.20. *Let $* \in \{\spadesuit, \clubsuit, \heartsuit, \diamondsuit\}$ and put*

$$\Lambda^* := \mathrm{End}_{Y_+}(T_+^*) \simeq \mathrm{End}_{Y_-}(T_-^*).$$

Then we have derived equivalences

$$\Phi^* := \mathrm{RHom}_{Y_+}(T_+^*, -) \otimes_{\Lambda^*}^{\mathbb{L}} T_-^* : \mathrm{D^b}(Y_+) \xrightarrow{\sim} \mathrm{D^b}(Y_-)$$
$$\Psi^* := \mathrm{RHom}_{Y_-}(T_-^*, -) \otimes_{\Lambda^*}^{\mathbb{L}} T_+^* : \mathrm{D^b}(Y_-) \xrightarrow{\sim} \mathrm{D^b}(Y_+)$$

that are quasi-inverse to each other.

Remark 5.3.21. Composing Φ^* and $\Psi^\star$ for two different $*, \star \in \{\spadesuit, \clubsuit, \heartsuit, \diamondsuit\}$, we get some non-trivial autoequivalences on $\mathrm{D^b}(Y_+)$ (resp. $\mathrm{D^b}(Y_-)$) that fix line bundles $\mathcal{O}_{Y_+}(-1)$, $\mathcal{O}_{Y_+}$ and $\mathcal{O}_{Y_+}(1)$ (resp. $\mathcal{O}_{Y_-}(-1)$, $\mathcal{O}_{Y_-}$ and $\mathcal{O}_{Y_-}(1)$). It would be an interesting problem to find a (sufficiently large) subgroup of $\mathrm{Auteq}(\mathrm{D^b}(Y_+))$ ($\simeq \mathrm{Auteq}(\mathrm{D^b}(Y_-))$) that contains our autoequivalences.

5.4 Moduli problem

In this section we study the Abuaf-Ueda from the point of view of non-commutative crepant resolutions and moduli.

5.4.1 Non-commutative crepant resolution and moduli

Definition 5.4.1. Let R be a normal Gorenstein domain and M a reflexive R-module. Then we say that M gives a *non-commutative crepant resolution (= NCCR)* of R if the endomorphism ring $\operatorname{End}_R(M)$ of M is maximal Cohen-Macaulay as an R-module and $\operatorname{End}_R(M)$ has finite global dimension. When M gives an NCCR of R then the endomorphism ring $\operatorname{End}_R(M)$ is called an NCCR of R.

In many cases, an NCCR is constructed from a tilting bundle on a (commutative) crepant resolution using the following lemma.

Lemma 5.4.2. *Let $X = \operatorname{Spec} R$ be a normal Gorenstein affine variety that admits a (commutative) crepant resolution $\phi : Y \to X$. Then for a tilting bundle T on Y, the double-dual $(\phi_* T)^{\vee\vee}$ of the module $\phi_* T$ gives an NCCR $\operatorname{End}_Y(T) \simeq \operatorname{End}_R(\phi_* T)$ of R. If one of the following two conditions are satisfied, then $(\phi_* T)^{\vee\vee}$ is isomorphic to $\phi_* T$, i.e. we do not have to take the double-dual.*

(a) The tilting bundle T contains $\mathcal{O}_Y$ as a direct summand.

(b) The resolution ϕ is small, i.e. the exceptional locus of ϕ does not contain a divisor.

When we find an NCCR $\Lambda = \operatorname{End}_R(M)$ of an algebra R, we can consider the moduli spaces of modules over Λ.

In the following we recall the result of Karmazyn [Kar17]. Let $Y \to X = \operatorname{Spec} R$ be a projective morphism and T a tilting bundle on Y. Assume that T has a decomposition $T = \bigoplus_{i=0}^{n} E_i$ such that (i) E_i is indecomposable for any i, (ii) $E_i \neq E_j$ for $i \neq j$, and (iii) $E_0 = \mathcal{O}_Y$. Then we can regard the endomorphism ring $\Lambda := \operatorname{End}_Y(T)$ as a path algebra of a quiver with relations such that the summand E_i corresponds to a vertex i.

Now we define a dimension vector $d_T = (d_T(i))_{i=0}^{n}$ by

$$d_T(i) := \operatorname{rank} E_i.$$

Note that, since we assumed that $E_0 = \mathcal{O}_Y$, we have $d_T(0) = 1$. We also define a stability condition θ_T associated to the tilting bundle T by

$$\theta_T(i) := \begin{cases} -\Sigma_{i\neq 0} \operatorname{rank} E_i & \text{if } i = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Then we can consider King's moduli space $\mathcal{M}^{\mathrm{ss}}_{\Lambda, d_T, \theta_T}$ of θ-semistable (right) $\operatorname{End}_Y(T)$-modules with dimension vector d_T. It is easy to see that there is no strictly θ_T-semistable object with dimension vector d_T (see [Kar17]), and thus the moduli space $\mathcal{M}^{\mathrm{ss}}_{\Lambda,d_T,\theta_T}$ is isomorphic to a moduli space $\mathcal{M}^{\mathrm{s}}_{\Lambda,d_T,\theta_T}$ of θ_T-stable objects.

In this setting, Karmazyn [Kar17] proved the following.

Theorem 5.4.3. *Assume that a morphism $\mathcal{O}_Y \to \mathcal{O}_y$ is surjective for all closed points $y \in Y$ in the abelian category $\mathcal{A}$ that corresponds to* $\mathrm{mod}(\Lambda)$ *under the derived equivalence*

$$\mathrm{RHom}_Y(T, -) : \mathrm{D^b}(Y) \to \mathrm{D^b}(\Lambda).$$

Then there is a monomorphism $f : Y \to \mathcal{M}^{\mathrm{s}}_{\Lambda,d_T,\theta_T}$.

The condition in the theorem above can be interpreted as the following geometric condition for the bundle.

Lemma 5.4.4. *The assumption in Theorem 5.4.3 is satisfied if the dual $T^\vee$ of the tilting bundle T is globally generated.*

Proof. It is enough to show that $\mathrm{Ext}^{\geq 1}_Y(T, I_y) = 0$ for the ideal sheaf of any closed point $y \in Y$. Since we assumed that T contains $\mathcal{O}_Y$ as a direct summand, $\mathrm{Ext}^{\geq 1}_Y(T, \mathcal{O}_Y) = 0$. Thus the vanishing $\mathrm{Ext}^{\geq 2}_Y(T, I_y) = 0$ follows. To prove $\mathrm{Ext}^1_Y(T, I_y) = 0$, we show the surjectivity of the morphism $\mathrm{Hom}_Y(T, \mathcal{O}_Y) \to \mathrm{Hom}_Y(T, \mathcal{O}_y)$. This morphism coincides with $H^0(Y, T^\vee) \to T^\vee \otimes k(y)$.

By assumption, there is a surjective morphism $\mathcal{O}_Y^{\oplus r} \to T^\vee$ for some r. Now we have the following commutative diagram

$$\begin{array}{ccc} H^0(Y, \mathcal{O}_Y)^{\oplus r} & \longrightarrow & k(y)^{\oplus r} \\ \downarrow & & \downarrow \\ H^0(Y, T^\vee) & \longrightarrow & T^\vee \otimes k(y). \end{array}$$

Morphisms $H^0(Y, \mathcal{O}_Y)^{\oplus r} \to k(y)^{\oplus r}$ and $k(y)^{\oplus r} \to T^\vee \otimes k(y)$ are surjective. Thus we have the desired surjectivity. □

Let us discuss the moduli when $Y \to \operatorname{Spec} R$ is a crepant resolution. Then there is a unique irreducible component M of $\mathcal{M}^{\mathrm{s}}_{\Lambda,d_T,\theta_T}$ that dominates $\operatorname{Spec} R$ [VdB04b]. We call this component (with reduced scheme structure) the *main component*. As a corollary of results above, we have the following.

Corollary 5.4.5. *Let us assume that a crepant resolution Y of* $\operatorname{Spec} R$ *admits a tilting bundle T such that*

(a) T is a direct sum of non-isomorphic indecomposable bundles $T = \bigoplus_{i=0}^n E_i$.

(b) $E_0 = \mathcal{O}_Y$.

(c) The dual $T^\vee$ is globally generated.

Then the main component M of $\mathcal{M}^{\mathrm{s}}_{\Lambda,d_T,\theta_T}$ is isomorphic to Y.

Proof. Since Y and $\mathcal{M}^{\mathrm{s}}_{\Lambda,d_T,\theta_T}$ are projective over $\operatorname{Spec} R$ (see [VdB04b]), the monomorphism $Y \to \mathcal{M}^{\mathrm{s}}_{\Lambda,d_T,\theta_T}$ is proper. A proper monomorphism is a closed immersion. Since Y dominates $\operatorname{Spec} R$, the image of this monomorphism is contained in the main component M. Since Y and M are birational to $\operatorname{Spec} R$, they coincide with each other. □

5.4.2 Application to our situation

First, from the existence of tilting bundles, we have the following.

Theorem 5.4.6. *The affine variety $X = \operatorname{Spec} C_0$ that appears in the Abuaf-Ueda flop admits NCCRs.*

Let us consider bundles

$$T_+ := T_+^{\heartsuit} \otimes \mathcal{O}_{Y_+}(-1) = \mathcal{O}_{Y_+} \oplus \mathcal{O}_{Y_+}(-1) \oplus \mathcal{O}_{Y_+}(-2) \oplus \mathcal{R}(-1) \oplus \mathcal{R}(-2) \oplus \Sigma(-1)$$
$$T_- := T_-^{\diamondsuit} \otimes \mathcal{O}_{Y_-}(-1) = \mathcal{O}_{Y_-} \oplus \mathcal{O}_{Y_-}(-1) \oplus \mathcal{O}_{Y_-}(-2) \oplus \mathcal{P}(-1) \oplus \mathcal{P}(-2) \oplus \mathcal{S}(-1)$$

These bundles satisfy the assumptions in Corollary 5.4.5. Indeed the globally-generatedness of dual bundles follows from the following lemma.

Lemma 5.4.7. *(1) A bundle $\Sigma(a)$ is globally generated if and only if $a \geq 2$.*

(2) A bundle $\mathcal{P}(a)$ is globally generated if and only if $a \geq 2$.

Proof. First we note that $\mathcal{R}(a)$ is globally generated if and only if $a \geq 1$. Recall that $\Sigma(a)$ is defined by an exact sequence

$$0 \to \mathcal{R}(a-1) \to \Sigma(a) \to \mathcal{R}(a+1) \to 0.$$

If $a \geq 2$ then $\mathcal{R}(a-1)$ and $\mathcal{R}(a+1)$ are globally generated and $H^1(Y_+, \mathcal{R}(a-1)) = 0$. Hence we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & H^0(Y^+, \mathcal{R}(a-1)) \otimes_{\mathbb{C}} \mathcal{O}_{Y^+} & \to & H^0(Y^+, \Sigma(a)) \otimes_{\mathbb{C}} \mathcal{O}_{Y^+} & \to & H^0(Y^+, \mathcal{R}(a+1)) \otimes_{\mathbb{C}} \mathcal{O}_{Y^+} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{R}(a-1) & \to & \Sigma(a) & \to & \mathcal{R}(a+1) & \to & 0
\end{array}$$

Thus the five-lemma implies that the bundle $\Sigma(a)$ is also globally generated.

Next let us assume that $\Sigma(a)$ is globally generated for some a. Then the restriction $\Sigma(a)|_{\mathbf{G}_0}$ of $\Sigma(a)$ to the zero-section $\mathbf{G}_0$ is also globally generated. Since there is a splitting $\Sigma(a)|_{\mathbf{G}_0} = R(a-1) \oplus R(a+1)$ on $\mathbf{G}_0$, we have that $R(a-1)$ is also globally generated. Thus we have $a \geq 2$.

The proof for $\mathcal{P}(a)$ is similar. □

Corollary 5.4.8. *The bundles $T_+^{\vee}$ and $T_-^{\vee}$ are globally generated.*

Let us regard the endomorphism ring $\Lambda_+ := \operatorname{End}_{Y_+}(T_+)$ as a path algebra of a quiver with relations (Q_+, I_+). For $0 \leq i \leq 5$, let $E_{+,i}$ be the $(i+1)$-th indecomposable summand of T_+ with respect to the order

$$T_+ = \mathcal{O}_{Y_+} \oplus \mathcal{O}_{Y_+}(-1) \oplus \mathcal{O}_{Y_+}(-2) \oplus \mathcal{R}(-1) \oplus \mathcal{R}(-2) \oplus \Sigma(-1).$$

The vertex of the quiver (Q_+, I_+) corresponding to the summand $E_{+,i}$ is denoted by $i \in (Q_+)_0$. We define a dimension vector $d_+ \in \mathbb{Z}^6$ by

$$d_+ = (d_0, d_1, d_2, d_3, d_4, d_5) := (1, 1, 1, 2, 2, 4).$$

We also define a stability condition $\theta_+ \in \mathbb{R}^6$ by

$$\theta_+ = (\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5) := (-10, 1, 1, 1, 1, 1).$$

Then we have the following as a corollary.

Corollary 5.4.9. *The crepant resolution Y_+ of $X = \operatorname{Spec} C_0$ gives the main component of the moduli space $\mathcal{M}^{\mathrm{s}}_{\Lambda_+, d_+, \theta_+}$ of representations of an NCCR Λ_+ of X of dimension vector d_+ with respect to the stability condition θ_+.*

Similarly we define a quiver with relations (Q_-, I_-) with $(Q_-)_0 = \{0, 1, 2, 3, 4, 5\}$ whose path algebra is $\Lambda_- = \operatorname{End}_{Y_-}(T_-)$, using the order

$$T_- = \mathcal{O}_{Y_-} \oplus \mathcal{O}_{Y_-}(-1) \oplus \mathcal{O}_{Y_-}(-2) \oplus \mathcal{P}(-1) \oplus \mathcal{P}(-2) \oplus \mathcal{S}(-1)$$

and put

$$d_- := (1, 1, 1, 2, 2, 4) \in \mathbb{Z}^6$$
$$\theta_- := (-10, 1, 1, 1, 1, 1) \in \mathbb{R}^6.$$

Then we have

Corollary 5.4.10. *The crepant resolution Y_- of $X = \operatorname{Spec} C_0$ gives the main component of the moduli space $\mathcal{M}^{\mathrm{s}}_{\Lambda_-, d_-, \theta_-}$ of representations of an NCCR Λ_- of X of dimension vector d_- with respect to the stability condition θ_-.*

Finally we remark that there is an isomorphism of algebras

$$\Lambda_+ = \operatorname{End}_{Y_+}(T_+) \simeq \operatorname{End}_{Y_+}(T_+ \otimes \mathcal{O}_{Y_+}(2)) \simeq \operatorname{End}_{Y_-}(T_-^\vee) \simeq \operatorname{End}_{Y_-}(T_-)^{\mathrm{op}} = \Lambda_-^{\mathrm{op}}.$$

Note that this isomorphism does not preserve the order of vertices of the quivers that we used above.

Chapter 6

On non-commutative crepant resolutions for cyclic quotient singularities

This chapter is based on Appendix C of the author's article

[H17b] W. Hara, *On derived equivalence for Abuaf flop: mutation of non-commutative crepant resolutions and spherical twists*, preprint, `https://arxiv.org/abs/1706.04417`.

6.1 Resolution of cyclic quotient singularities

The aim of this chapter is to provide one instance for "multi-mutation=twist" result for Gorenstein isolated cyclic quotient singularities.

6.1.1 Summary of results in [KPS17]

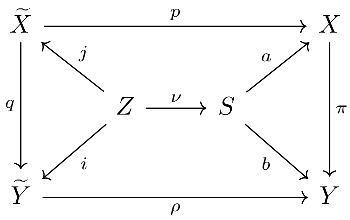
Let X be a smooth quasi-projective variety with an action of a cyclic group $G = \mu_n$. Let $S := \mathrm{Fix}(G)$. S is automatically smooth. Assume:

(i) For all $x \in X$, $\mathrm{Stab}_G(x)$ is 1 or G.

(ii) The generator g of G acts on the normal bundle $\mathcal{N}_{S/X}$ by multiplication with some fixed primitive n-th root of unity ζ.

(iii) $\mathrm{codim}_X(S) = n$.

Then, the G-Hilbert scheme $\widetilde{Y} := \mathrm{Hilb}^G(X)$ gives a crepant resolution of $Y = X/G$ and the exceptional divisor Z is isomorphic to $\mathbb{P}_S(\mathcal{N}_{S/X})$. Under the assumptions above, the affine variety X/G becomes Gorenstein (see [KPS17]). Let $\widetilde{X} := \mathrm{Bl}_S(X)$ be the blowing up of X along S.



Put

$$\Phi := Rp_* \circ Lq^* \circ \mathrm{triv} : \mathrm{D^b}(\widetilde{Y}) \to \mathrm{D^b}([X/G])$$
$$\Psi := (-)^G \circ q_* \circ Lp^* : \mathrm{D^b}([X/G]) \to \mathrm{D^b}(\widetilde{Y})$$

Lemma 6.1.1 ([KPS17], Lemma 4.10)**.** *$\mathcal{L}^{-1} := (q_*(\mathcal{O}_{\widetilde{X}} \otimes \chi^{-1}))^G$ is a line bundle on $\widetilde{Y}$. Furthermore, we have $\mathcal{L}^n \simeq \mathcal{O}_{\widetilde{Y}}(Z)$.*

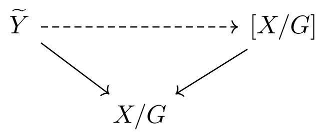
Proposition 6.1.2 ([KPS17], Corollary 4.11)**.** *$\Psi(\mathcal{O}_X \otimes \chi^\alpha) \simeq \mathcal{L}^\alpha$ for $-n+1 \leq \alpha \leq 0$ and $\Phi(\mathcal{L}^\alpha) = \mathcal{O}_X \otimes \chi^\alpha$ for $0 \leq \alpha \leq n-1$.*

Note that, if X is an affine variety, $\bigoplus_{\alpha=0}^{n-1} \mathcal{L}^\alpha$ (resp. $\mathcal{O}_X \otimes (\chi^0 \oplus \cdots \oplus \chi^{n-1})$) is a tilting bundle on $\widetilde{Y}$ (resp. $[X/G]$).

Theorem 6.1.3 ([KPS17])**.** *Φ and Ψ give equivalences of categories.*

Lemma 6.1.4 ([KPS17], Theorem 4.26)**.** *$\Theta := i_* \circ \nu^* : \mathrm{D^b}(S) \to \mathrm{D^b}(\widetilde{Y})$ is a spherical functor.*

Let us regard the setup of McKay correspondence as a flop of orbifolds.



Then we can regard the following functor isomorphism as the instance of an orbifold "flop-flop=twist" principle.

Theorem 6.1.5. *$\Psi \circ \Phi \simeq T_\Theta \circ (- \otimes \mathcal{L}^{-n})$.*

In the next subsection, we study this theorem from the point of view of mutations of NCCRs if $X = \mathbb{A}^n$.

6.1.2 The case if $X = \mathbb{A}^n$

Assume $X = \mathbb{A}^n$ and the action of G on X is diagonal. Then, $\widetilde{Y} \simeq \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^{n-1}}(-n))$ and $\mathcal{L}$ is the pull-back of $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. $S = \{o\}$ and $Z \simeq \mathbb{P}^{n-1}$ is the zero-section. Set

$$\mathcal{T}_k := \bigoplus_{\alpha=-n+k+1}^{k} \mathcal{L}^{\alpha}.$$

Since $\mathcal{L}^n \simeq \mathcal{O}_{\widetilde{Y}}(Z)$, we have $\mathcal{L}^n|_{\widetilde{Y}\setminus Z} \simeq \mathcal{O}_{\widetilde{Y}\setminus Z}$, and hence the module

$$M := \rho_* \mathcal{T}_k$$

and the algebra

$$\Lambda := \mathrm{End}_{X/G}(M) = \mathrm{End}_{\widetilde{Y}}(\mathcal{T}_k)$$

do not depend on k. Note that the algebra Λ is the toric NCCR of Y and there is a natural identification

$$\mathrm{D^b}(\mathrm{mod}\,\Lambda) \simeq \mathrm{D^b}([X/G]).$$

Put

$$M_k := \rho_* \mathcal{L}^k \simeq \rho_* \mathcal{L}^{k-n} =: M_{k-n}$$

and

$$W_k := \bigoplus_{\alpha=-n+k+1}^{k-1} M_\alpha$$

for $0 \le k \le n-1$. Note that

$$M = \bigoplus_{\alpha=0}^{n-1} M_\alpha = W_k \oplus M_k = W_k \oplus M_{k-n}.$$

The following is a "multi-mutation=twist" result for the resolution of a cyclic quotient singularity.

Proposition 6.1.6. *Let $1 \le k \le n-1$.*

(1) We have

$$\mu_{W_k}^{n-1}(M) \simeq M.$$

The IW mutation functor

$$\Phi_{W_k}^{n-1} \in \mathrm{Auteq}(\mathrm{D^b}(\Lambda))$$

coincides with the functor $\mathrm{RHom}_\Lambda(\mathrm{RHom}_{\widetilde{Y}}(\mathcal{T}_k, \mathcal{T}_{k-1}), -)$.

(2) Under the identification

$$\mathrm{RHom}_\Lambda(\mathcal{T}_{k-1}, -) : \mathrm{D^b}(\widetilde{Y}) \xrightarrow{\sim} \mathrm{D^b}(\Lambda),$$

the autoequivalence

$$\Phi_{W_k}^{n-1} \in \mathrm{Auteq}(\mathrm{D^b}(\Lambda))$$

corresponds to a spherical twist

$$\mathrm{T}_{\mathcal{O}_Z(-k)} \in \mathrm{Auteq}(\mathrm{D^b}(\Lambda)).$$

Proof of Proposition 6.1.6 (1). Let

$$\mathcal{W}_k := \bigoplus_{\alpha=-n+k+1}^{k-1} \mathcal{L}^{\alpha}.$$

and consider a long exact sequence

$$0 \to \mathcal{L}^k \xrightarrow{a_0} (\mathcal{L}^{k-1})^{\oplus n} \xrightarrow{a_1} \cdots \to (\mathcal{L}^{-n+k+1})^{\oplus n} \xrightarrow{a_{n-1}} \mathcal{L}^{-n+k} \to 0,$$

which we can get by taking pull back of the long Euler sequence on $\mathbb{P}^{n-1}$ (Recall that $\mathcal{L}$ is a pull back of $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$). By Lemma 4.3.14, a vector bundle $\mathcal{W}_k \oplus \operatorname{Image}(a_j)$ is a tilting bundle for $0 \le j \le n-1$.

Moreover, by Lemma 4.2.7, the sequence

$$0 \to \rho_* \operatorname{Image}(a_{j-1}) \to M_{k-j}^{\oplus c_{k-j}} \to \rho_* \operatorname{Image}(a_j) \to 0$$

gives a right approximation of $\rho_* \operatorname{Image}(a_j)$ and the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{D^b}(\widetilde{Y}) & = & \mathrm{D^b}(\widetilde{Y}) \\
\downarrow{\scriptstyle F_{j-1}} & & \downarrow{\scriptstyle F_j} \\
\mathrm{D^b}(\operatorname{mod}\operatorname{End}(\mathcal{W}_k \oplus \operatorname{Image}(a_{j-1}))) & \xrightarrow[\Phi_{W_k}]{} & \mathrm{D^b}(\operatorname{mod}\operatorname{End}(\mathcal{W}_k \oplus \operatorname{Image}(a_j))),
\end{array}$$

where

$$F_j := \operatorname{RHom}(\mathcal{W}_k \oplus \operatorname{Image}(a_j), -) : \mathrm{D^b}(\widetilde{Y}) \to \mathrm{D^b}(\operatorname{mod}\operatorname{End}(\mathcal{W}_k \oplus \operatorname{Image}(a_j))).$$

Thus, we have

$$\mu_{W_k}^{n-1}(M) \simeq M$$

and

$$\Phi_{W_k}^{n-1} \simeq F_{n-1} \circ F_0^{-1} \simeq \operatorname{RHom}_{\Lambda}(\operatorname{RHom}_{\widetilde{Y}}(\mathcal{T}_k, \mathcal{T}_{k-1}), -).$$

This shows the result. □

Lemma 6.1.7. *Let*

$$G_k := \operatorname{RHom}(\mathcal{T}_k, -) : \mathrm{D^b}(\widetilde{Y}) \to \mathrm{D^b}(\operatorname{mod}\Lambda).$$

Then, we have

$$G_k^{-1} \circ G_{k-1} \simeq \mathrm{T}_{\mathcal{O}_Z(-k)} \in \operatorname{Auteq}(\mathrm{D^b}(\widetilde{Y})).$$

Proof. First, we have

$$\begin{aligned}
\operatorname{RHom}_{\widetilde{Y}}(\mathcal{O}_Z(-k), E) &\simeq \operatorname{RHom}_Z(\mathcal{O}_Z(-k), Li^*E \otimes \mathcal{O}_Z(-n))[-1] \\
&\simeq \operatorname{RHom}_Z(\mathcal{O}_Z(-k+n), Li^*E)[-1].
\end{aligned}$$

Since $\mathcal{W}_k|_Z \simeq \bigoplus_{j=-k+1}^{n-k-1} \mathcal{O}_Z(j)$, we have

$$\mathrm{RHom}_{\widetilde{Y}}(\mathcal{O}_Z(-k), \mathcal{W}_k) = 0$$

and thus $\mathrm{T}_{\mathcal{O}_Z(-k)}(\mathcal{W}_k) = \mathcal{W}_k$.

On the other hand, since

$$\mathrm{RHom}_{\widetilde{Y}}(\mathcal{O}_Z(-k), \mathcal{L}^{-n+k}) \simeq \mathbb{C}[-1]$$

and $\mathcal{L}^{-n+k} \simeq \mathcal{L}^k \otimes \mathcal{O}_{\widetilde{Y}}(-Z)$, we have $\mathrm{T}_{\mathcal{O}_Z(-k)}(\mathcal{L}^{-n+k}) \simeq \mathcal{L}^k$:

$$\begin{array}{ccccccc}
\mathcal{O}_Z(-k)[-1] & \longrightarrow & \mathcal{L}^{-n+k} & \longrightarrow & \mathrm{T}_{\mathcal{O}_Z(-k)}(\mathcal{L}^{-n+k}) & \longrightarrow & \mathcal{O}_Z(-k) \\
\| & & \| & & \| & & \| \\
\mathcal{L}^k|_Z[-1] & \longrightarrow & \mathcal{L}^k \otimes \mathcal{O}_{\widetilde{Y}}(-Z) & \longrightarrow & \mathcal{L}^k & \longrightarrow & \mathcal{L}^k|_Z.
\end{array}$$

Thus, we have

$$\mathrm{T}_{\mathcal{O}_Z(-k)}(\mathcal{T}_{k-1}) = \mathrm{T}_{\mathcal{O}_Z(-k)}(\mathcal{W}_k \oplus \mathcal{L}^{-n+k}) \simeq \mathcal{W}_k \oplus \mathcal{L}^k = \mathcal{T}_k\,.$$

On the other hand, we have $(G_k^{-1} \circ G_{k-1})(\mathcal{T}_{k-1}) = G_k^{-1}(\Lambda) = \mathcal{T}_k$. Since $\mathcal{T}_{k-1}$ is a generator, we have the result. □

Proof of Proposition 6.1.6 (2). The result follows from the following diagram:

$$\begin{array}{ccccc}
\mathrm{D}^{\mathrm{b}}(\widetilde{Y}) & \xrightarrow{\mathrm{T}_{\mathcal{O}_Z(-k)}} & \mathrm{D}^{\mathrm{b}}(\widetilde{Y}) & = & \mathrm{D}^{\mathrm{b}}(\widetilde{Y}) \\
\downarrow{\scriptstyle G_{k-1}} & & \downarrow{\scriptstyle G_k} & & \downarrow{\scriptstyle G_{k-1}} \\
\mathrm{D}^{\mathrm{b}}(\Lambda) & = & \mathrm{D}^{\mathrm{b}}(\Lambda) & \xrightarrow[\Phi^{n-1}_{W_k}]{} & \mathrm{D}^{\mathrm{b}}(\Lambda)
\end{array}$$

The commutativity of this diagram follows from Proposition 6.1.6 (1) and Lemma 6.1.7. □

Corollary 6.1.8. *Let Φ and Ψ be the Krug-Ploog-Sosna's functors. Then, we have*

$$(\Psi \circ \Phi)^{-1} \simeq \mathrm{T}_{\mathcal{O}_Z(-n+1)} \circ \mathrm{T}_{\mathcal{O}_Z(-n+2)} \circ \cdots \circ \mathrm{T}_{\mathcal{O}_Z(-1)}\,.$$

Proof. Under the identification $\mathrm{D}^{\mathrm{b}}(\Lambda) = \mathrm{D}^{\mathrm{b}}([X/G])$, we have $\Psi^{-1} \simeq G_0$ and $\Phi \simeq G_{n-1}$. Thus, we have

$$\begin{aligned}
(\Psi \circ \Phi)^{-1} &\simeq G_{n-1}^{-1} \circ G_0 \\
&\simeq G_{n-1}^{-1} \circ (G_{n-2} \circ G_{n-2}^{-1}) \circ \cdots \circ (G_1 \circ G_1^{-1}) \circ G_0 \\
&\simeq (G_{n-1}^{-1} \circ G_{n-2}) \circ (G_{n-2}^{-1} \circ G_{n-3}) \circ \cdots \circ (G_2^{-1} \circ G_1) \circ (G_1^{-1} \circ G_0) \\
&\simeq \mathrm{T}_{\mathcal{O}_Z(-n+1)} \circ \mathrm{T}_{\mathcal{O}_Z(-n+2)} \circ \cdots \circ \mathrm{T}_{\mathcal{O}_Z(-1)}\,.
\end{aligned}$$

This is what we want. □

Lemma 6.1.9. *There is a functor isomorphism*

$$\mathrm{T}_{\mathcal{O}_Z(-n+1)} \circ \mathrm{T}_{\mathcal{O}_Z(-n+2)} \circ \cdots \circ \mathrm{T}_{\mathcal{O}_Z(-1)} \circ \mathrm{T}_{\mathcal{O}_Z} \simeq - \otimes \mathcal{L}^n \,.$$

Proof. Note that

$$\mathrm{T}_{\mathcal{O}_Z(-k)}(\mathcal{L}^j) = \mathcal{L}^j$$

if $-n+k+1 \leq j \leq k-1$ (or, equivalently, $j+1 \leq k \leq n+j-1$) and

$$\mathrm{T}_{\mathcal{O}_Z(-k)}(\mathcal{L}^{-n+k}) \simeq \mathcal{L}^k \,.$$

Thus, for $-n \leq k \leq 0$, we have

$$\begin{aligned}
&(\mathrm{T}_{\mathcal{O}_Z(-n+1)} \circ \cdots \circ \mathrm{T}_{\mathcal{O}_Z(-n+k-1)} \circ \mathrm{T}_{\mathcal{O}_Z(-n+k)} \circ \mathrm{T}_{\mathcal{O}_Z(-n+k+1)} \circ \cdots \circ \mathrm{T}_{\mathcal{O}_Z})(\mathcal{L}^k) \\
\simeq &(\mathrm{T}_{\mathcal{O}_Z(-n+1)} \circ \cdots \circ \mathrm{T}_{\mathcal{O}_Z(-n+k-1)} \circ \mathrm{T}_{\mathcal{O}_Z(-n+k)})(\mathcal{L}^k) \\
\simeq &(\mathrm{T}_{\mathcal{O}_Z(-n+1)} \circ \cdots \circ \mathrm{T}_{\mathcal{O}_Z(-n+k-1)})(\mathcal{L}^{n+k}) \\
\simeq &\,\mathcal{L}^{n+k} \,.
\end{aligned}$$

Therefore, we have

$$(\mathrm{T}_{\mathcal{O}_Z(-n+1)} \circ \cdots \circ \mathrm{T}_{\mathcal{O}_Z})(\mathcal{T}_0) \simeq \mathcal{T}_0 \otimes \mathcal{L}^n \,.$$

Since $\mathcal{T}_0$ is a generator, we have the result. □

From the above propositions, we can recover Krug-Ploog-Sosna's "flop-flop=twist" result (Theorem 6.1.5):

Proof of Theorem 6.1.5.

$$\begin{aligned}
\mathrm{T}_{\mathcal{O}_Z} \circ \mathcal{L}^{-n} &\simeq (\mathrm{T}_{\mathcal{O}_Z(-n+1)} \circ \mathrm{T}_{\mathcal{O}_Z(-n+2)} \circ \cdots \circ \mathrm{T}_{\mathcal{O}_Z(-1)})^{-1} \\
&\simeq \Psi \circ \Phi.
\end{aligned}$$

□

Chapter 7

Deformation of tilting-type derived equivalences for crepant resolutions

This chapter is based on the author's work

[H18a] W. Hara, *Deformation of tilting-type derived equivalences for crepant resolutions*, to appear in IMRN, `https://arxiv.org/abs/1709.09948`.

7.1 Introduction

7.1.1 Background

Let $X = \operatorname{Spec} R$ be a normal Gorenstein affine variety and assume that X admits a crepant resolution. Although X may have some different crepant resolutions, the following "uniqueness" is expected.

Conjecture 7.1.1 (Bondal-Orlov)**.** *Let $\phi : Y \to X$ and $\phi' : Y' \to X$ be two crepant resolutions of X. Then Y and Y' are derived equivalent to each other, i.e. there exists an exact equivalence*

$$\Phi : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(Y').$$

For given two crepant resolutions Y and Y' of X, there are various methods to construct a derived equivalence between $\mathrm{D^b}(Y)$ and $\mathrm{D^b}(Y')$ (eg. Fourier-Mukai transform, variation of GIT, or mutation of semi-orthogonal decomposition). In this chapter, we deal with the one using tilting bundles. A vector bundle E on a scheme Z is called *tilting bundle* if $\operatorname{Ext}^i_Z(E, E) = 0$ for $i \neq 0$ and if E is a generator of the category $\mathrm{D}^-(\mathrm{Qcoh}(Z))$. By using tilting bundles, we can construct equivalences of categories:

Lemma 7.1.2 (See also Proposition 2.1.12)**.** *If there are tilting bundles T and T' on Y and Y', respectively, with an R-algebra isomorphism*

$$\mathrm{End}_Y(T) \simeq \mathrm{End}_{Y'}(T'),$$

then we have equivalences of derived categories

$$\mathrm{D^b}(Y) \simeq \mathrm{D^b}(\mathrm{End}_Y(T)) \simeq \mathrm{D^b}(\mathrm{End}_{Y'}(T')) \simeq \mathrm{D^b}(Y').$$

We call a derived equivalence constructed in this way *tilting-type*. We say that a tilting-type equivalence $\mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(Y')$ is *strict* if the tilting bundles T and T' coincide with each other on the largest common open subset U of X, Y and Y'. We also say that a tilting-type equivalence is *good* if tilting bundles contain trivial line bundles as direct summands. Tilting-type equivalences that are good and strict constitute an important class of derived equivalences. For example, it is known that if X has only threefold terminal singularities, then there is a derived equivalence between Y and Y' that can be written as a composition of good and strict tilting-type equivalences. Indeed, since two crepant resolutions are connected by iterating flops, this fact follows from the result of Van den Bergh [VdB04a]. In addition, tilting-type equivalences for crepant resolutions have a strong relationship with the theory of non-commutative crepant resolutions, which was first introduced by Van den Bergh [VdB04b].

On the other hand, taking *a deformation of an algebraic variety* is one of standard methods to construct a new variety from the original one, and is studied in many branches of algebraic geometry. Taking deformations is also an important operation in Mirror Symmetry. According to Homological Mirror Symmetry, the derived categories of algebraic varieties are quite significant objects in the study of Mirror Symmetry. The aim of this chapter is to understand the behavior of (good or strict) tilting-type equivalences under deformations.

7.1.2 Results

Let X_0 be a normal Gorenstein affine variety, and let $\phi_0 : Y_0 \to X_0$ and $\phi'_0 : Y'_0 \to X_0$ be two crepant resolutions of X_0. In this chapter, we deal with three types of deformations: infinitesimal deformation, deformation over a complete local ring, and deformation with a $\mathbb{G}_m$-action.

• **Infinitesimal deformation.** First we study infinitesimal deformations of small resolutions. Assume that

$$\mathrm{codim}_{Y_0}(\mathrm{exc}(\phi_0)) \geq 3 \text{ and } \mathrm{codim}_{Y'_0}(\mathrm{exc}(\phi'_0)) \geq 3.$$

Then we have isomorphisms of deformation functors

$$\mathrm{Def}\, X_0 \simeq \mathrm{Def}\, Y_0 \simeq \mathrm{Def}\, Y'_0.$$

Let A be a local Artinian algebra with residue field $\mathbb{C}$ and choose an element

$$\xi \in (\mathrm{Def}\, X_0)(A) = (\mathrm{Def}\, Y_0)(A) = (\mathrm{Def}\, Y'_0)(A).$$

Let Y (resp. Y') be the infinitesimal deformation of Y_0 (resp. Y'_0) over A corresponding to ξ. Then we show the following.

Theorem 7.1.3 (= Theorem 7.4.9)**.** *Under the notations and assumptions above, any strict tilting-type equivalence between* $\mathrm{D^b}(Y_0)$ *and* $\mathrm{D^b}(Y'_0)$ *lifts to a strict tilting-type equivalence between* $\mathrm{D^b}(Y)$ *and* $\mathrm{D^b}(Y')$.

See Definition 7.3.7 for the precise meaning of the word *lift*.

• **Complete local or $\mathbb{G}_\mathrm{m}$-equivariant deformation.** We also study deformations over a complete local ring and deformations with $\mathbb{G}_\mathrm{m}$-actions. Let X_0, Y_0, Y'_0, ϕ_0 and ϕ'_0 as above. (Note that we do NOT assume the condition for the codimension of the exceptional locus here.) Consider a deformation of them

$$\begin{array}{ccccc} Y & \xrightarrow{\phi} & X & \xleftarrow{\phi'} & Y' \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle q} & \swarrow{\scriptstyle p'} & \\ & & (\operatorname{Spec} D, d) & & \end{array}$$

over a pointed affine scheme $(\operatorname{Spec} D, d)$, where ϕ and ϕ' are projective morphisms and $X = \operatorname{Spec} R$ is an affine scheme. Assume that an inequality

$$\operatorname{codim}_{X_0} \operatorname{Sing}(X_0) \geq 3$$

holds and that one of the following conditions holds.

(a) D is a complete local ring and $d \subset D$ is the maximal ideal.

(b) X, Y and Y' are $\mathbb{G}_\mathrm{m}$-varieties, ϕ and ϕ' are $\mathbb{G}_\mathrm{m}$-equivariant, and the action of $\mathbb{G}_\mathrm{m}$ on X is good. For a unique $\mathbb{G}_\mathrm{m}$-fixed point $x \in X$, we have $d = q(x)$.

See Definition 7.6.1 for the definition of good $\mathbb{G}_\mathrm{m}$-actions. Then we have a similar theorem as in the case of infinitesimal deformations.

Theorem 7.1.4 (= Theorem 7.5.1, 7.6.4)**.** *Under the conditions above, any good and strict tilting-type equivalence between* $\mathrm{D^b}(Y_0)$ *and* $\mathrm{D^b}(Y'_0)$ *lifts to a good tilting-type equivalence between* $\mathrm{D^b}(Y)$ *and* $\mathrm{D^b}(Y')$.

We note that we cannot generalize this theorem directly to the case when the codimension of the singular locus is two (see Section 7.7.2). As a direct corollary of the theorem above, we have the following.

Corollary 7.1.5 (= Corollary 7.6.9)**.** *Under the condition (b) above, assume that there exists a good and strict tilting-type equivalence between* $\mathrm{D^b}(Y_0)$ *and* $\mathrm{D^b}(Y'_0)$*. Then, for any closed point* $t \in \operatorname{Spec} D$*, there is a good tilting-type equivalence between* $\mathrm{D^b}(p^{-1}(t))$ *and* $\mathrm{D^b}(p'^{-1}(t))$.

• **Stratified Mukai flops and stratified Atiyah flops.** As an application of the theorems above, we study derived equivalences for stratified Mukai flops and stratified Atiyah flops. A *stratified Mukai flop* on $\mathrm{Gr}(r, N)$ is a birational map $Y_0 \dashrightarrow Y_0'$ between the cotangent bundles $Y_0 := T^* \mathrm{Gr}(r, N)$ and $Y_0' := T^* \mathrm{Gr}(N-r, N)$ of Grassmannian varieties, where r is an integer with $2r \leq N-1$. It is known that they have a natural one-parameter $\mathbb{G}_\mathrm{m}$-equivariant deformation $Y \dashrightarrow Y'$ called *stratified Atiyah flop* on $\mathrm{Gr}(r, N)$. Note that a stratified Atiyah flop on $\mathrm{Gr}(r, N)$ is also defined in the case if $2r = N$ (see Section 7.7.1 for more details). Stratified Mukai flops and stratified Atiyah flops form a fundamental class of higher dimensional flops.

The method to construct an equivalence for stratified Mukai flops from an equivalence for stratified Atiyah flops is well-established (eg. [Kaw02, Sze04]). On the other hand, our theorem provides a method to construct a tilting-type equivalence for stratified Atiyah flops from a tilting-type equivalence for stratified Mukai flops. More precisely:

Corollary 7.1.6 (= Theorem 7.7.2)**.** *Any good and strict tilting-type equivalence for the stratified Mukai flop on* $\mathrm{Gr}(r, N)$ *lifts to a good and strict tilting-type equivalence for the stratified Atiyah flop on* $\mathrm{Gr}(r, N)$.

We note that, if $2r \leq N-2$, we can remove the assumption that the tilting-type equivalence is good. Since we can construct a strict tilting-type equivalence for stratified Mukai flops using results of Kaledin [Kal08], we have the following corollary. Although there are some previous works on the derived equivalence for stratified Atiyah flops (eg. [Kaw05, Cau12a]), the following corollary is new to the best of the author's knowledge.

Corollary 7.1.7. *If* $2r \leq N-2$, *then there exists a strict tilting-type equivalence for the stratified Atiyah flop on* $\mathrm{Gr}(r, N)$.

7.1.3 Notations.

In this chapter, we always work over the complex number field $\mathbb{C}$. A *scheme* always means a Noetherian $\mathbb{C}$-scheme.

7.2 Local cohomologies and deformations

7.2.1 Local cohomologies

Let X be a topological space and $\mathcal{F}$ an abelian sheaf on X. For a closed subset Y of X, we define

$$\Gamma_Y(\mathcal{F}) = \Gamma_Y(X, \mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) \mid \mathrm{Supp}(s) \subset Y\}.$$

This Γ_Y defines a left exact functor

$$\Gamma_Y : \mathrm{Sh}(X) \to (\mathrm{Ab}),$$

from the category of abelian sheaves on X to the category of abelian groups, and we denote the right derived functor of Γ_Y by H^p_Y or $H^p_Y(X, -)$.

Similarly, we define an abelian sheaf $\underline{\Gamma}_Y(\mathcal{F})$ by

$$\underline{\Gamma}_Y(\mathcal{F})(U) := \Gamma_{U\cap Z}(\mathcal{F}|_U)$$

for an open subset $U \subset X$. Then, the functor $\underline{\Gamma}_Y : \mathrm{Sh}(X) \ni \mathcal{F} \mapsto \underline{\Gamma}_Y(\mathcal{F}) \in \mathrm{Sh}(X)$ is a left exact functor. We denote the right derived functor of $\underline{\Gamma}_Y$ by $\mathcal{H}^p_Y$.

In the rest of the present subsection, we provide some basic properties of local cohomologies.

Lemma 7.2.1 ([Har1, Corollary 1.9])**.** *Let X be a topological space, $Z \subset X$ a closed subset, $U := X \setminus Z$ the complement of Z, and $j : U \hookrightarrow X$ the open immersion. Then, for any abelian sheaf $\mathcal{F}$ on X, there are exact sequences*

$$0 \to \Gamma_Z(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}) \to \Gamma(U,\mathcal{F}) \to H^1_Z(X,\mathcal{F}) \to H^1(X,\mathcal{F}) \to H^1(U,\mathcal{F}) \to \cdots$$

and

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U) \to \mathcal{H}^1_Z(\mathcal{F}) \to 0.$$

In addition, there are functorial isomorphisms

$$R^p j_*(\mathcal{F}|_U) \simeq \mathcal{H}^{p+1}_Z(\mathcal{F})$$

for $p \geq 1$.

Definition 7.2.2. Let X be a locally Noetherian scheme and $Y \subset X$ a closed subset. If $\mathcal{F}$ is a coherent sheaf on X, then the Y*-depth* of $\mathcal{F}$ is defined by

$$\mathrm{depth}_Y(\mathcal{F}) := \inf_{x\in Y} \mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x).$$

Proposition 7.2.3 ([Har1, Theorem 3.8])**.** *Let X be a locally Noetherian scheme, $Y \subset X$ a closed subset, and $\mathcal{F}$ a coherent sheaf on X. Let n be a non-negative integer. Then, the following are equivalent*

(i) $\mathcal{H}^i_Y(\mathcal{F}) = 0$ *for all* $i < n$.

(ii) $\mathrm{depth}_Y(\mathcal{F}) \geq n$.

Corollary 7.2.4. *Let X be a locally Noetherian scheme, $Y \subset X$ a closed subset, and $\mathcal{F}$ a coherent sheaf on X. Assume that* $\mathrm{depth}_Y(\mathcal{F}) \geq n$*. Then we have* $H^i_Y(X,\mathcal{F}) = 0$ *for* $i < n$.

Proof. Let us consider a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}^q_Y(\mathcal{F})) \Rightarrow H^{p+q}_Y(X,\mathcal{F}).$$

By assumption we have $E_2^{p,q} = 0$ if $q < n$. Thus we have the result. □

Corollary 7.2.5. *Let X be a locally Noetherian scheme, $Y \subset X$ a closed subset, and $\mathcal{F}$ a coherent sheaf on X. Put $U := X \setminus Y$ and assume that* $\operatorname{depth}_Y(\mathcal{F}) \geq n$. *Then the canonical map*

$$H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F})$$

is an isomorphism for $i \leq n-2$ and

$$H^{n-1}(X, \mathcal{F}) \to H^{n-1}(U, \mathcal{F})$$

is injective.

Proof. This corollary follows from Lemma 7.2.1 and Corollary 7.2.4. □

7.2.2 Deformation of schemes and lift of coherent sheaves

Definition 7.2.6. Let X_0 be a scheme. A *deformation* of X_0 over a pointed scheme (S, s) is a flat morphism $\rho : X \to (S, s)$ such that $X \otimes_S \Bbbk(s) \simeq X_0$. Let $j_s : X_0 \to X$ the closed immersion. A deformation of X_0 will be denoted by $(\rho : X \to (S, s), j_s : X_0 \to X)$. If S is the spectrum of a local ring D and s is a closed point corresponding to the unique maximal ideal, we say that X is a deformation over D for short.

Let us consider the category (Art) of local Artinian $\mathbb{C}$-algebras with residue field $\mathbb{C}$. Note that, for any object $A \in$ (Art), A is finite dimensional as a $\mathbb{C}$-vector space.

We say that a deformation $(\rho : X \to (S, s), j_s : X_0 \to X)$ of X_0 is *infinitesimal* if S is the spectrum of $A \in$ (Art) and s is the closed point corresponding to a unique maximal ideal $\mathfrak{m}_A \subset A$.

Definition 7.2.7. For a scheme X, we define a functor

$$\operatorname{Def} X : (\mathrm{Art})^{\mathrm{op}} \to (\mathrm{Sets})$$

by

$$(\operatorname{Def} X)(A) := \{\text{isom class of deformation of } X_0 \text{ over } A\}$$

for $A \in$ (Art). It is easy to see that Def X actually defines a functor. We call this functor Def X the *deformation functor* of X (or the local moduli functor of X).

Definition 7.2.8. Let X_0 and Y_0 be schemes, $\phi_0 : X_0 \to Y_0$ a morphism, and (S, s) a pointed scheme. Let $(\rho : X \to (S, s), j_s : X_0 \to X)$ and $(\pi : Y \to (S, s), i_s : Y_0 \to Y)$ be deformations over (S, s). We say that a morphism of S-schemes $\phi : X \to Y$ is a *deformation* of ϕ_0 if the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{j_s} & X \\ {\scriptstyle \phi_0}\downarrow & & \downarrow{\scriptstyle \phi} \\ Y_0 & \xrightarrow{i_s} & Y \end{array}$$

is cartesian.

Definition 7.2.9. Let X_0 be a scheme and $(\rho : X \to (S, s), j_s : X_0 \to X)$ a deformation of X_0. Let $\mathcal{F}_0$ be a coherent sheaf on X_0. A *lift* of $\mathcal{F}_0$ to X is a coherent sheaf $\mathcal{F}$ on X that is flat over S and $j_s^* \mathcal{F} \simeq \mathcal{F}_0$.

For more information on deformations of schemes and lifts of coherent sheaves, see [Art, Har2, Ser].

7.3 Tilting-type equivalence

In this section, we provide a precise definition of (good or strict) tilting-type equivalences and study the properties of them.

Proposition 7.3.1. *Let Y and Y' be Noetherian schemes that are projective over an affine variety $X = \operatorname{Spec} R$. If there are tilting bundle T and T' on Y and Y', respectively, and an R-algebra isomorphism*

$$\operatorname{End}_Y(T) \simeq \operatorname{End}_{Y'}(T'),$$

then we have derived equivalences

$$\mathrm{D^b}(Y) \simeq \mathrm{D^b}(\operatorname{End}_Y(T)) \simeq \mathrm{D^b}(\operatorname{End}_{Y'}(T')) \simeq \mathrm{D^b}(Y').$$

The following is the definition of tilting-type equivalences.

Definition 7.3.2. Let Y and Y' are schemes that are projective over an affine variety $X = \operatorname{Spec} R$.

A *tilting-type equivalence* between $\mathrm{D^b}(Y)$ and $\mathrm{D^b}(Y')$ is an equivalence of derived categories constructed as in Proposition 7.3.1. To emphasize tilting bundles that are used to construct the equivalence, we also say that the tilting-type equivalence $\mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(Y')$ is determined by (T, T').

We say that a tilting-type equivalence between $\mathrm{D^b}(Y)$ and $\mathrm{D^b}(Y')$ is *good* if the tilting bundles T and T' are good.

Assume that the morphisms $Y \to X$ and $Y' \to X$ are birational and let U be the largest common open subset of X, Y, and Y'. Under these conditions, we say that a tilting-type equivalence between $\mathrm{D^b}(Y)$ and $\mathrm{D^b}(Y')$ is *strict* if we have

$$T|_U \simeq T'|_U.$$

Lemma 7.3.3. *Let Y and Y' be schemes that are projective over an affine variety $X = \operatorname{Spec} R$. Assume in addition that Y and Y' are Cohen-Macaulay, the morphisms $Y \to X$ and $Y' \to X$ are birational, and they have the largest common open subscheme U such that $U \neq \emptyset$ and*

$$\operatorname{codim}_Y(Y \setminus U) \geq 2, \operatorname{codim}_{Y'}(Y' \setminus U) \geq 2.$$

Suppose that there are tilting bundle T and T' on Y and Y', respectively, such that

$$T|_U \simeq T'|_U.$$

Then there is a strict tilting-type equivalence

$$\mathrm{D^b}(Y) \simeq \mathrm{D^b}(\operatorname{End}_Y(T)) \simeq \mathrm{D^b}(\operatorname{End}_{Y'}(T')) \simeq \mathrm{D^b}(Y').$$

Proof. By Proposition 2.1.12 we have equivalences

$$\mathrm{D^b}(Y) \simeq \mathrm{D^b}(\mathrm{End}_Y(T)) \text{ and } \mathrm{D^b}(Y') \simeq \mathrm{D^b}(\mathrm{End}_{Y'}(T')).$$

By Corollary 7.2.5, we have R-algebra isomorphisms

$$\mathrm{End}_Y(T) \simeq H^0(Y, T^\vee \otimes T) \simeq H^0(U, T^\vee \otimes T) \simeq H^0(Y', T'^\vee \otimes T') \simeq \mathrm{End}_{Y'}(T').$$

Combining the above, we have the result. □

Under the following nice condition, the same thing holds if $\mathrm{codim}_Y(Y \setminus U) = 1$.

Lemma 7.3.4. *Let $X = \operatorname{Spec} R$ be a normal Gorenstein affine variety of dimension greater than two, and let $\phi : Y \to X$ and $\phi' : Y' \to X$ be two crepant resolutions of X. Put $U := X_{\mathrm{sm}} = Y \setminus \mathrm{exc}(\phi) = Y' \setminus \mathrm{exc}(\phi')$. Assume that there are tilting bundles T and T' on Y and Y', respectively, such that*

$$T|_U \simeq T'|_U.$$

Then there is a strict tilting-type equivalence

$$\mathrm{D^b}(Y) \simeq \mathrm{D^b}(\mathrm{End}_Y(T)) \simeq \mathrm{D^b}(\mathrm{End}_{Y'}(T')) \simeq \mathrm{D^b}(Y').$$

Furthermore, if T and T' are good, then we have an R-module isomorphism

$$\phi_* T \simeq \phi'_* T'.$$

Proof. By Corollary 2.1.16, we have that $\mathrm{End}_Y(T) \simeq \phi_*(T^\vee \otimes T)$ is a Cohen-Macaulay R-module, and hence is a reflexive R-module. Therefore we have $\mathrm{End}_Y(T) \simeq j_*(T^\vee \otimes T)|_U$.

If T and T' are good, then $\phi_* T$ and $\phi'_* T'$ are Cohen-Macaulay and hence are reflexive. Since $\phi_* T$ and $\phi' T'$ are isomorphic in codimension one and thus we have $\phi_* T \simeq \phi_* T'$. □

Remark 7.3.5. In Lemma 7.3.4, assume in addition that

$$\mathrm{codim}_Y(Y \setminus U) \geq 2, \mathrm{codim}_{Y'}(Y' \setminus U) \geq 2.$$

Then the isomorphism $\phi_* T \simeq \phi_* T'$ holds without the assumption that T and T' are good.

On the other hand, if $\mathrm{codim}_Y(Y \setminus U) = 1$, then the isomorphism $\phi_* T \simeq \phi_* T'$ does not hold without the assumption that T and T' are good. Let us give an example. Put

$$R := \mathbb{C}[x, y, z]/(x^2 + y^2 + z^2).$$

Then $X := \operatorname{Spec} R$ admits an A_1-singularity at the origin $o \in X$. Let $\phi : Y \to X$ be the minimal resolution. The exceptional locus E of ϕ is an irreducible divisor of Y, which is a (-2)-curve. More explicitly, Y is the total space of a sheaf

$\mathcal{O}_{\mathbb{P}^1}(-2)$ on $\mathbb{P}^1$ and E is the zero section. If we put $\pi : Y \to \mathbb{P}^1$ be the projection and $\mathcal{O}_Y(1) := \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$, then it is easy to see that

$$\mathcal{O}_Y(E) \simeq \mathcal{O}_Y(-2).$$

Thus, if we put $U := Y \setminus E = X \setminus \{o\}$, then we have there is an isomorphism

$$\mathcal{O}_U \simeq \mathcal{O}_Y(-2)|_U.$$

Let $T := \mathcal{O}_Y \oplus \mathcal{O}_Y(1)$ and $T' := \mathcal{O}_Y(1) \oplus \mathcal{O}_Y(2)$. An easy computation shows that T and T' are tilting bundles on Y, and the discussion above shows that we have $T|_U \simeq T'|_U$. On the other hand, there is an exact sequence on Y

$$0 \to \mathcal{O}_Y(2) \oplus \mathcal{O}_Y(1) \xrightarrow{E \oplus \mathrm{id}} \mathcal{O}_Y \oplus \mathcal{O}_Y(1) \to \mathcal{O}_E \to 0.$$

Applying the functor ϕ_* to this sequence, we have a short exact sequence

$$0 \to \phi_* T' \to \phi_* T \to \Bbbk(o) \to 0,$$

where $\Bbbk(o)$ is the residue field at the origin $o \in X$. In particular, $\phi_* T'$ is not reflexive and hence we have $\phi_* T' \neq \phi_* T$.

In almost all known examples, a tilting bundle (on a crepant resolution) contains a line bundle as a direct summand. The following lemma suggests that the assumption that a tilting-type equivalence is good is not strong if the resolutions are small.

Lemma 7.3.6. *Let $X = \operatorname{Spec} R$ be a normal Gorenstein affine variety and assume that X admits two small resolutions $\phi : Y \to X$ and $\phi' : Y' \to X$. If there exists a strict tilting-type equivalence between $\mathrm{D^b}(Y)$ and $\mathrm{D^b}(Y')$ determined by (T, T') such that T contain a line bundle L on Y as a direct summand, then there is a good tilting-type equivalence between $\mathrm{D^b}(Y)$ and $\mathrm{D^b}(Y')$.*

Proof. Put $U := Y \setminus \operatorname{exc}(\phi) = Y' \setminus \operatorname{exc}(\phi')$ and let $j* : U \hookrightarrow Y'$ be the open immersion. Then, since ϕ' is small, $L' := j'_*(L|_U)$ is a divisorial sheaf. Moreover, since Y' is smooth, L' is a line bundle on Y'. Since L' is contained in T' as a direct summand, we have that

$$(T \otimes L^\vee)|_U \simeq (T' \otimes L'^\vee)|_U$$

and that $T \otimes L^\vee$ and $T' \otimes L'^\vee$ are good tilting bundles. Then, by Lemma 7.3.4, we have the result. ☐

Next we give the definition of the lift of a tilting-type equivalence.

Definition 7.3.7. Let $X_0 = \operatorname{Spec} R_0$ be an affine scheme, and let $\phi_0 : Y_0 \to X_0$ and $\phi'_0 : Y'_0 \to X_0$ be two projective morphisms. Let us consider deformations of varieties and morphisms above

$$\begin{array}{ccccc} Y & \xrightarrow{\phi} & X & \xleftarrow{\phi'} & Y' \\ & \searrow & \downarrow & \swarrow & \\ & & (S, s) & & \end{array}$$

over an pointed scheme (S, s), such that $X = \operatorname{Spec} R$ is affine and two morphisms ϕ and ϕ' are projective. Let $\Phi_0 : \mathrm{D}^{\mathrm{b}}(Y_0) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y_0')$ be a tilting-type equivalence determined by (T_0, T_0').

Under the set-up above, we say that a tilting-type equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(Y) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y')$ is a *lift* of Φ_0 if Φ is determined by (T, T') such that T (resp. T') is a lift of T_0 (resp. T_0') to Y (resp. Y'), and if the algebra isomorphism $\operatorname{End}_Y(T) \simeq \operatorname{End}_Y(T')$ coincides with the algebra isomorphism $\operatorname{End}_{Y_0}(T_0) \simeq \operatorname{End}_{Y_0'}(T_0')$ after we restrict it to X_0.

The following is an easy observation on a lift of a tilting-type equivalence.

Lemma 7.3.8. *Under the same condition as in Definition 7.3.7, let $j_s : Y_0 \hookrightarrow Y$ and $j_s' : Y_0' \hookrightarrow Y'$ be closed immersions associated to the deformations. Let $\Phi : \mathrm{D}^{\mathrm{b}}(Y) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y')$ be a tilting-type equivalence that is a lift of a tilting-type equivalence $\Phi_0 : \mathrm{D}^{\mathrm{b}}(Y_0) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y_0')$. Then the following diagram of functors commutes*

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}(Y_0) & \xrightarrow{(j_s)_*} & \mathrm{D}^{\mathrm{b}}(Y) \\ {\scriptstyle\Phi_0}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \mathrm{D}^{\mathrm{b}}(Y_0') & \xrightarrow{(j_s')_*} & \mathrm{D}^{\mathrm{b}}(Y'). \end{array}$$

Proof. By adjunction and the construction of the equivalences, we have

$$\begin{aligned} \operatorname{RHom}_{Y'}(T', (j_s')_*(\Phi_0(F))) &\simeq \operatorname{RHom}_{Y_0'}(T_0', \Phi_0(F)) \\ &\simeq \operatorname{RHom}_{Y_0}(T_0, F) \\ &\simeq \operatorname{RHom}_{Y}(T, (j_s)_*(F)) \\ &\simeq \operatorname{RHom}_{Y'}(T', \Phi((j_s)_*(F))) \end{aligned}$$

for all $F \in \mathrm{D}^{\mathrm{b}}(Y_0)$. Since the functor

$$\operatorname{RHom}_{Y'}(T', -) : \mathrm{D}^{\mathrm{b}}(Y') \to \mathrm{D}^{\mathrm{b}}(\operatorname{End}_{Y'}(T'))$$

gives an equivalence, we have a functorial isomorphism

$$(j_s')_*(\Phi_0(F)) \simeq \Phi((j_s)_*(F)).$$

This shows the result. □

7.4 Infinitesimal deformation of small resolutions

Let $X_0 = \operatorname{Spec} R_0$ be a normal Gorenstein affine variety and $\phi_0 : Y_0 \to X_0$ a crepant resolution. Throughout this section we always assume that

$$\operatorname{codim}_{Y_0} \operatorname{exc}(\phi_0) \geq 3.$$

Under this assumption, the deformation theory behaves very well. First, we observe the following proposition.

Proposition 7.4.1. *Under the conditions above, there is a functor isomorphism*

$$\operatorname{Def} Y_0 \simeq \operatorname{Def} X_0.$$

This proposition follows immediately from the following lemma.

Lemma 7.4.2. *Let Z_0 be a variety and $j : U_0 \hookrightarrow Z_0$ an open subset. Assume one of the following conditions.*

(1) Z_0 is affine, Cohen-Macaulay, $\dim Z_0 \geq 3$, and $U_0 = (Z_0)_{\mathrm{sm}}$.

(2) Z_0 is smooth and $\operatorname{codim}_{Z_0}(Z_0 \setminus U_0) \geq 3$.

Then, the restriction

$$\operatorname{Def} Z_0 \to \operatorname{Def} U_0$$

gives an isomorphism of functors.

Proof. The case (1) is proved in [Art, Proposition 9.2]. Let us prove the lemma under the condition (2). Let A be a local Artinian algebra with residue field $\mathbb{C}$ and U a deformation of U_0 over A. We will show that $\operatorname{Def}(Z_0)(A) = \operatorname{Def}(U_0)(A)$ by an induction on the dimension of A as a $\mathbb{C}$-vector space. Let

$$e : 0 \to (t) \to A \to A' \to 0$$

be a small extension and $U' := U \otimes_A A'$. By induction hypothesis, there is the unique deformation Z' of Z_0 over A' such that $Z'|_{U_0} = U'$.

By assumption, Lemma 7.2.1 and Proposition 7.2.3, we have an isomorphism

$$H^1(Z_0, \Theta_{Z_0}) \xrightarrow{\sim} H^1(U, \Theta_{U_0})$$

and an injective map

$$H^2(Z_0, \Theta_{Z_0}) \hookrightarrow H^2(U, \Theta_{U_0}).$$

Note that the second map is compatible with the obstruction map by construction. Since there is a lifting U of U' to A, the obstruction map sends e to 0, and hence the set

$$\{\text{isom class of lifts of } Z' \text{ to } A\}$$

is non-empty. By deformation theory, the first cohomology group $H^1(Z_0, \Theta_{Z_0}) = H^1(U, \Theta_{U_0})$ acts on the sets

$$\{\text{isom class of lifts of } Z' \text{ to } A\} \text{ and } \{\text{isom class of lifts of } U' \text{ to } A\}$$

transitively, and the restriction map

$$\{\text{isom class of lifts of } Z' \text{ to } A\} \to \{\text{isom class of lifts of } U' \text{ to } A\}$$

is compatible with these actions. Thus, the above restriction map is bijective and hence there is a unique lift of Z of Z' to A, which satisfies $Z|_{U_0} = U$.

Let Z^1 and Z^2 be two deformation of Z_0 over A such that $Z^i|_{U_0} = U$ $(i = 1, 2)$. If we set $Z'^i := Z^i \otimes_A A'$, then Z'^i is the extension of U' for $i = 1, 2$ and hence we have $Z'^1 = Z'^2$. Thus, the above argument shows that we have $Z^1 = Z^2$. This shows the result. ☐

Let A be a local Artinian algebra with residue field $\mathbb{C}$. Take an element

$$\xi \in (\operatorname{Def} Y_0)(A) = (\operatorname{Def} X_0)(A),$$

and let Y and X be deformations of Y_0 and X_0, respectively, over A that correspond to ξ.

It is easy to observe the following.

(1) X and Y are of finite type over A (or $\mathbb{C}$).

(2) The inclusions $X_0 \subset X$ and $Y_0 \subset Y$ are homeomorphisms.

For example, (1) is proved in [Art].

Lemma 7.4.3. *X and Y are Cohen-Macaulay schemes. Furthermore, if the local Artinian algebra A is Gorenstein, then so are X and Y.*

Proof. Note that Artinian algebras are zero-dimensional and hence are Cohen-Macaulay. In addition, for any point $x \in X$ (resp. $y \in Y$), the homomorphism $A \to \mathcal{O}_{X,x}$ (resp. $A \to \mathcal{O}_{Y,y}$) is automatically local. Then, the statement follows from [Mat, Corollary of Theorem 23.3 and Theorem 23.4]. □

Lemma 7.4.4. *Let us consider the map $\phi_0 : Y \to X$ of topological spaces. Then the direct image sheaf $(\phi_0)_*\mathcal{O}_Y$ is isomorphic to $\mathcal{O}_X$ as sheaves of A-algebras. In particular, there is a morphism of A-schemes $\phi : Y \to X$ whose pull back by $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec} A$ is ϕ_0.*

Proof. Let $j : U \to X$ and $i : U \to Y$ be open immersions. Since X and Y are Cohen-Macaulay, we have isomorphisms of sheaves of rings

$$(\phi_0)_*\mathcal{O}_Y \simeq (\phi_o)_* i_*\mathcal{O}_U \simeq j_*\mathcal{O}_U \simeq \mathcal{O}_X.$$

It is clear that all isomorphisms are A-linear. □

Lemma 7.4.5. *The morphism $\phi : Y \to X$ is projective.*

Proof. First, we note that the morphism ϕ is proper. Thus, it is enough to show that there is a ϕ-ample line bundle.

Let L_0 be a ϕ_0-ample line bundle. Since

$$\operatorname{Ext}^p_{Y_0}(L_0, L_0) \simeq H^p(Y_0, \mathcal{O}_{Y_0}) \simeq R^p\phi_{0*}(\mathcal{O}_{Y_0})$$

and ϕ_0 is a rational resolution of X_0, the line bundle L_0 is a partial tilting bundle on Y_0. Thus due to Proposition 7.4.6 below, L_0 has the unique lifting L on Y, which is invertible.

We show that this line bundle L on Y is ϕ-ample. Since $X_0 \hookrightarrow X$ is a homeomorphism, for $x \in X = X_0$, we have $\phi^{-1}(x) = \phi_0^{-1}(x)$. Thus, we have $L|_{\phi^{-1}(x)} \simeq L_0|_{\phi_0^{-1}(x)}$ and hence $L|_{\phi^{-1}(x)}$ is absolutely ample. Then, by [Laz, Theorem 1.2.17 and Remark 1.2.18], we have that L is ϕ-ample. □

Next we discuss tilting bundles on Y. First we recall the following result due to Karmazyn.

Proposition 7.4.6 ([Kar15, Theorem 3.4])**.** *Let $\pi_0 : Z_0 \to \operatorname{Spec} S_0$ be a projective morphisms of Noetherian schemes. Let $(A, \mathfrak{m})$ be a local Artinian algebra with residue field $\mathbb{C}$, Z and $\operatorname{Spec} S$ be deformations of Z_0 and $\operatorname{Spec} S_0$, respectively, over A. Assume that there is a A-morphism $\pi : Z \to \operatorname{Spec} S$ such that $\pi \otimes_A A/\mathfrak{m} = \pi_0$.*

Then, for any partial tilting bundle T_0 on Z_0, there is the unique lifting T of T_0 on Z, which is partial tilting. Moreover, if T_0 is tilting, then so is T.

The following lemma is a certain variation of the proposition above.

Proposition 7.4.7. *With the same condition as in Proposition 7.4.6, assume in addition that Z_0 is a Cohen-Macaulay variety of dimension greater than 4. Let T_0 be a partial tilting bundle on Z_0, $U_0 \subset Z_0$ an open subscheme of Z_0. Put $U := Z|_{U_0}$ and assume that*

$$\operatorname{codim}_{Z_0}(Z_0 \setminus U_0) \geq 3.$$

Then, the bundle $T_0|_{U_0}$ on U_0 lifts uniquely to a bundle on U.

Proof. Let $\mathfrak{m}$ be the maximal ideal of A and put

$$A_n := A/\mathfrak{m}^{n+1},\ Z_n := Z \otimes_A A_n,\ \text{and } U_n := U \otimes_A A_n$$

for $n \geq 0$. Since A is Artinian, we have $A = A_n$ for sufficiently large n. By Proposition 7.4.6, there is the unique lifting T_n of T_0 on Z_n, which is partial tilting. The existence of a lifting of $T_0|_{U_0}$ on U_n follows from this result.

We prove the uniqueness by an induction on n. Put $T'_n := T_n|_{U_n}$. Then the set

$$\{\text{isom class of liftings of } T'_n \text{ on } U_{n+1}\}$$

is non-empty and is a torsor under the action of $H^1(U_0, \mathcal{E}nd_{U_0}(T'_0)) \otimes_{\mathbb{C}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.

By adjunction, we have an isomorphism

$$H^1(U_0, \mathcal{E}nd_{U_0}(T'_0)) \simeq H^1(Z_0, \mathcal{E}nd_{Z_0}(T_0) \otimes_{Z_0} Rj_*\mathcal{O}_{U_0}),$$

where $j : U_0 \to Z_0$ is an open immersion. Thus, there is a spectral sequence

$$E_2^{p,q} := H^p(Z_0, \mathcal{E}nd_{Z_0}(T_0) \otimes_{Z_0} R^q j_*\mathcal{O}_{U_0}) \Rightarrow H^{p+q}(U_0, \mathcal{E}nd_{U_0}(T'_0)).$$

Since Z_0 is Cohen-Macaulay and T_0 is partial tilting on Z_0, we have

$$E_2^{p,0} = \operatorname{Ext}^p_{Z_0}(T_0, T_0) = 0$$

for $p > 0$. By assumption, we have $R^1 j_*\mathcal{O}_{U_0} = 0$ and hence we have $E_2^{p,1} = 0$ for all p. In particular, we have $E_2^{p,q} = 0$ for p, q with $p + q = 1$, and hence we have

$$H^1(U_0, \mathcal{E}nd_{U_0}(T'_0)) = 0.$$

Thus, the lifting of T'_n on U_{n+1} is unique. ☐

Remark 7.4.8. Put $E_0 := Z_0 \setminus U_0$ and assume that $\operatorname{codim}_{Z_0}(E_0) = 2$. In this case, analyzing the proof above, we notice that the vanishing of $H^0(Z_0, \mathcal{E}nd_{Z_0}(T_0)\otimes R^1 j_*\mathcal{O}_{U_0}) = 0$ is sufficient. If Z_0 and E_0 are smooth, one can easily show that there is an isomorphism

$$H^0(Z_0, \mathcal{E}nd_{Z_0}(T_0)\otimes R^1 j_*\mathcal{O}_{U_0}) \simeq \bigoplus_{k\geq 0} H^0(E_0, \mathcal{E}nd_{E_0}(T|_{E_0})\otimes \operatorname{Sym}^k N_{E_0/Z_0}\otimes \det(N_{E_0/Z_0})),$$

where N_{E_0/Z_0} is the normal bundle of $E_0 \subset Z_0$.

Theorem 7.4.9. *Let X_0 be a Gorenstein normal affine variety, $\phi_0 : Y_0 \to X_0$ and $\phi_0' : Y_0' \to X_0$ two crepant resolutions of X_0. Assume that*

$$\operatorname{codim}_{Y_0}(\operatorname{exc}(\phi_0)) \geq 3 \textit{ and } \operatorname{codim}_{Y_0'}(\operatorname{exc}(\phi_0')) \geq 3.$$

Then,

(1) *there is a functor isomorphism* $\operatorname{Def} Y_0 \simeq \operatorname{Def} Y_0'$.

(2) *Let A be a local Artinian algebra with residue field $\mathbb{C}$. Let Y and Y' be deformations of Y_0 and Y_0', respectively, over A that correspond to an element $\xi \in (\operatorname{Def} Y_0)(A) \simeq (\operatorname{Def} Y_0')(A)$. Then, any strict tilting-type equivalence $\mathrm{D^b}(Y_0) \xrightarrow{\sim} \mathrm{D^b}(Y_0')$ lifts to a strict tilting-type equivalence $\mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(Y')$.*

Proof. The first assertion follows from Proposition 7.4.1.

Put $U_0 := Y_0 \setminus \operatorname{exc}(\phi_0) = (X_0)_{\mathrm{sm}} = Y_0' \setminus \operatorname{exc}(\phi_0')$ and set $U := Y|_{U_0} = Y'|_{U_0}$. Let T_0 (resp. T_0') be a tilting bundle on Y_0 (resp. Y_0') such that $T_0|_{U_0} = T_0'|_{U_0}$, and T (resp. T') the unique lifting of T_0 (resp. T_0') on Y (resp. Y'). Then, by Proposition 7.4.7, we have $T|_U = T'|_U$. Since Y and Y' are projective over an affine variety, from Proposition 2.1.12, we have equivalences of categories

$$\mathrm{D^b}(Y) \simeq \mathrm{D^b}(\operatorname{End}_Y(T)) \simeq \mathrm{D^b}(\operatorname{End}_{Y'}(T')) \simeq \mathrm{D^b}(Y').$$

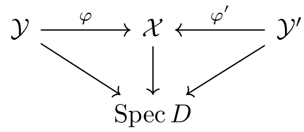
This is what we want. □

Remark 7.4.10. Compare our result with the result of Toda [Tod09, Theorem 7.1], where he proved a similar result for an equivalence for flops given as a Fourier-Mukai transform.

7.5 Deformation of crepant resolutions over a complete local ring

The goal of the present section is to prove the following theorem.

Theorem 7.5.1. *Let $X_0 := \operatorname{Spec} R_0$ be a normal Gorenstein affine variety, and $\phi_0 : Y_0 \to X_0$ and $\phi'_0 : Y'_0 \to X_0$ two crepant resolution of X_0. Let (D, d) be a Cohen-Macaulay complete local ring, and let a diagram*



be a deformation of varieties and morphisms above over (D, d), where $\mathcal{X} = \operatorname{Spec} \mathcal{R}$ is an affine scheme, and φ and φ' are projective morphisms. Assume that

$$\operatorname{codim}_X \operatorname{Sing}(X_0) \geq 3.$$

Then any good and strict tilting-type equivalence for Y_0 and Y'_0 lifts to a good tilting type equivalence for $\mathcal{Y}$ and $\mathcal{Y}'$.

In addition, if the lift is determined by tilting bundles $(\mathcal{T}, \mathcal{T}')$, then we have $\varphi_* \mathcal{T} \simeq \varphi'_* \mathcal{T}'$.

Remark 7.5.2. The assumption $\operatorname{codim}_{X_0} \operatorname{Sing}(X_0) \geq 3$ is essential and there is a counter-example of this theorem if we remove this assumption (see Section 7.7.2).

7.5.1 Preliminaries

Recall that a finitely generated S-module M is said to be *rigid* if $\operatorname{Ext}^1_S(M, M) = 0$, and to be *modifying* if M is reflexive and $\operatorname{End}_S(M)$ is a maximal Cohen-Macaulay S-module. By definition, if M gives an NCCR of S then M is modifying.

Lemma 7.5.3. *Let $Z := \operatorname{Spec} S$ be a Cohen-Macaulay affine variety and M is a modifying module over S that is locally free in codimension two. Assume that $\dim S \geq 3$ and $\operatorname{codim} \operatorname{Sing}(\operatorname{Spec} S) \geq 3$. Then the module M is rigid.*

Proof. Since M is reflexive, it satisfies (S_2) condition. In addition, since $\operatorname{End}_S(M)$ is maximal Cohen-Macaulay and $\dim S \geq 3$, $\operatorname{End}_S(M)$ satisfies (S_3) condition. Then the result follows from [Dao10, Lemma 2.3]. $\square$

Corollary 7.5.4. *Let $Z = \operatorname{Spec} S$ be a normal Gorenstein affine variety of dimension greater then three and $\psi : W \to Z$ be a crepant resolution. Assume that $\operatorname{codim}_Z \operatorname{Sing}(Z) \geq 3$, and that W admits a good tilting bundle T. Then the S-module $M := \psi_* T$ is rigid.*

If the resolution $\psi : W \to Z$ is small, we can remove the assumption that T is good.

Proof. Since we assumed that $\operatorname{codim}_Z \operatorname{Sing}(Z) \geq 3$, the module M is locally free in codimension two. In addition, since the tilting bundle T is good, the module M is Cohen-Macaulay and hence reflexive. Note that if the resolution

$\psi : W \to Z$ is small, we have that M is reflexive without the assumption that T is good. Since $\mathrm{End}_W(T) \simeq \mathrm{End}_Z(M)$ is Cohen-Macaulay, the module M is modifying. Thus we have the result. $\square$

The following proposition should be well-known, but we provide a sketch of the proof here because the author has no reference for this proposition.

Proposition 7.5.5. *Let S_0 be a $\mathbb{C}$-algebra and M_0 a finitely generated S_0-module. Let A a local Artinian algebra with residue field $\mathbb{C}$ and*

$$0 \to J \to A' \to A \to 0$$

an extension of A such that $J^2 = 0$. Let S be a deformation of S_0 over A, M a lift of M_0 over S, and S' a lift of S over A'. Assume that there exists a lift M' of M over S'. Then the lift of M over S' is a torsor of $\mathrm{Ext}^1_{S_0}(M_0, M_0) \otimes_{\mathbb{C}} J$.

Proof. Let M'_1 and M'_2 be two lifts of M over S'. Then there exist a free S'-module P' and surjective morphisms $P' \to M'_1$ and $P' \to M'_2$ whose restriction to S coincide with each other. Let us consider the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes_{\mathbb{C}} N_0 & \longrightarrow & N'_i & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes_{\mathbb{C}} P_0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes_{\mathbb{C}} M_0 & \longrightarrow & M'_i & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Take an element $x \in N$ and choose its lifts $x'_1 \in N'_1$ and $x'_2 \in N'_2$. Then we can consider the difference $x'_1 - x'_2$ in P' and we have $x'_1 - x'_2 \in J \otimes_{\mathbb{C}} P_0$. Let $y \in J \otimes_{\mathbb{C}} M_0$ be the image of $x'_1 - x'_2 \in J \otimes_{\mathbb{C}} P_0$. Then the correspondence $N \ni x \mapsto y \in J \otimes_{\mathbb{C}} M_0$ gives an S_0-module homomorphism $\gamma : N_0 \to J \otimes_{\mathbb{C}} M_0$. It is easy to see that $N'_1 = N'_2$ as submodules of P' if and only if $\gamma = 0$, and the similar argument shows that the ambiguity of the choice of a surjective morphism $P' \to M'$ is resolved by $\mathrm{Hom}_{S_0}(P_0, J \otimes_{\mathbb{C}} M_0)$. Thus we have the result. $\square$

Under the set-up of Theorem 7.5.1, put $(D_n, d_n) := (D/d^{n+1}, d/d^{n+1})$,

$$X_n := \mathcal{X} \otimes_D D_n,\ Y_n := \mathcal{Y} \otimes_D D_n,\ Y'_n := \mathcal{Y} \otimes_D D_n,$$

and let

$$\phi_n := \phi \otimes_D D_n : Y_n \to X_n,\ \phi'_n := \phi' \otimes_D D_n : Y'_n \to X_n$$

be the projections.

Lemma 7.5.6. *Let T_0 be a good tilting bundle on Y_0 and T_n the lift of T_0 to Y_n. Then a sheaf $M_n := (\phi_n)_* T_n$ on X_n is a lift of $M_0 := (\phi_0)_* T_0$ to X_n.*

Proof. Note that T_n is also good and hence we have $M_n \simeq R\phi_{n*}T_n$ and $M_0 \simeq R\phi_{0*}T_0$. Then the result follows from [Kar15, Corollary 2.11]. □

7.5.2 Proof of the theorem

Proof of Theorem 7.5.1. Let U_0 be the common open subscheme of X_0, Y_0 and Y'_0. Let T_0 and T'_0 be good tilting bundles on Y_0 and Y'_0, respectively, such that $T_0|_{U_0} \simeq T'_0|_{U_0}$. Put $U_n := X_n|_{U_0}$.

There exist good tilting bundles T_n and T'_n on Y_n and Y'_n, respectively, such that T_n (resp. T'_n) is a lift of T_0 (resp. T'_0). Then $(\phi_n)_* T_n$ and $(\phi'_n)_* T'_n$ are two lifts of $M_0 = H^0(T_0) = H^0(T'_0)$ by Lemma 7.5.6. By assumptions the module M_0 is rigid and hence we have

$$(\phi_n)_* T_n \simeq (\phi'_n)_* T'_n.$$

In particular, we have

$$T_n|_{U_n} \simeq T'_n|_{U_n}.$$

Next we show that we have an algebra isomorphism

$$\mathrm{End}_{Y_n}(T_n) \simeq \mathrm{End}_{Y'_n}(T'_n).$$

As in the proof of Lemma 7.5.6, we have that $H^0(Y_n, T_n^\vee \otimes T_n)$ and $H^0(Y'_n, T_n'^\vee \otimes T'_n)$ are coherent sheaves on X_n that are flat over D_n. On the other hand, we have

$$H^0(Y_n, T_n^\vee \otimes T_n) \otimes_{D_n} D_n/d_n \simeq H^0(Y_0, T_0^\vee \otimes T_0) \text{ and}$$
$$H^0(Y'_n, T_n'^\vee \otimes T'_n) \otimes_{D_n} D_n/d_n \simeq H^0(Y_0, T_0'^\vee \otimes T'_0)$$

are Cohen-Macaulay modules over X_0. Therefore the flat extension theorem [Mat, Theorem 23.3] implies that $H^0(Y_n, T_n^\vee \otimes T_n)$ and $H^0(Y'_n, T_n'^\vee \otimes T'_n)$ are Cohen-Macaulay modules over X_n. Thus we have algebra isomorphisms

$$H^0(Y_n, T_n^\vee \otimes T_n) \simeq H^0(U_n, T_n^\vee \otimes T_n) \simeq H^0(U_n, T_n'^\vee \otimes T'_n) \simeq H^0(Y'_n, T_n'^\vee \otimes T'_n).$$

Finally unwinding Grothendieck's existence theorem [Sta, Tag 088F] and the result of Karmazyn [Kar15, Lemma 3.3] imply that there exist good tilting bundles $\mathcal{T}$ and $\mathcal{T}'$ on $\mathcal{Y}$ and $\mathcal{Y}'$, respectively, such that

$$\mathrm{End}_{\mathcal{Y}}(\mathcal{T}) \simeq \lim \mathrm{End}_{Y_n}(T_n) \simeq \lim \mathrm{End}_{Y'_n}(T'_n) \simeq \mathrm{End}_{\mathcal{Y}'}(\mathcal{T}').$$

Note that we have $(\varphi_* \mathcal{T})|_{X_n} \simeq (\phi_n)_* T_n \simeq (\phi'_n)_* T'_n \simeq (\varphi'_* \mathcal{T}')|_{X_n}$ by construction. This implies that we have an isomorphism $\varphi_* \mathcal{T} \simeq \varphi'_* \mathcal{T}'$ (see [Sta, Tag 087W]). This shows the result.

In addition, if we put $\mathcal{M} := \varphi_* \mathcal{T} \simeq \varphi'_* \mathcal{T}$, we also have isomorphisms

$$\mathrm{End}_{\mathcal{Y}}(\mathcal{T}) \simeq \lim \mathrm{End}_{Y_n}(T_n) \simeq \lim \mathrm{End}_{X_n}((\phi_n)_* T_n) = \mathrm{End}_{\mathcal{R}}(\mathcal{M}),$$

which also follow from [Sta, Tag 087W]. □

The following corollary is a direct consequence of the proof of Theorem 7.5.1.

Corollary 7.5.7. *Under the same set-up of Theorem 7.5.1, assume in addition that*

$$\operatorname{codim}_Y(\operatorname{exc}(\phi_0)) \geq 3.$$

Then any strict tilting-type equivalence for Y_0 and Y_0' lifts to a tilting type equivalence for $\mathcal{Y}$ and $\mathcal{Y}'$.

7.6 Deformation with an action of $\mathbb{G}_{\mathrm{m}}$

7.6.1 $\mathbb{G}_{\mathrm{m}}$-action

First of all, we recall some basic definitions and properties of actions of a group $\mathbb{G}_{\mathrm{m}}$ on schemes.

Definition 7.6.1. We say that a $\mathbb{G}_{\mathrm{m}}$-action on an affine scheme $\operatorname{Spec} R$ is *good* if there is a unique $\mathbb{G}_{\mathrm{m}}$-fixed closed point corresponding to a maximal ideal $\mathfrak{m}$, such that $\mathbb{G}_{\mathrm{m}}$ acts on $\mathfrak{m}$ by strictly positive weights.

The advantage to consider good $\mathbb{G}_{\mathrm{m}}$-actions is that we can use the following useful theorems.

Theorem 7.6.2 ([Kal08], Theorem 1.8 (ii))**.** *Let Y be a scheme that is projective over an affine variety $X = \operatorname{Spec} R$, and assume that X admits a good $\mathbb{G}_{\mathrm{m}}$-action that lifts to a $\mathbb{G}_{\mathrm{m}}$-action on Y. Let $\widehat{X}$ be a completion of $X = \operatorname{Spec} R$ with respect to the maximal ideal $\mathfrak{m} \subset R$ that corresponds to a unique fixed point $x \in X$.*

Then any tilting bundle on $\widehat{Y} := Y \times_X \widehat{X}$ is obtained by a pull-back of a $\mathbb{G}_{\mathrm{m}}$-equivariant tilting bundle on Y.

In relation to the theorem above, see also [Nam08, Appendix A] and [Kar15, Proposition 5.1].

Lemma 7.6.3 ([Kal08], Lemma 5.3)**.** *Let R be a $\mathbb{C}$-algebra of finite type such that the corresponding affine scheme $\operatorname{Spec} R$ admits a good $\mathbb{G}_{\mathrm{m}}$-action. Let $\mathfrak{m} \subset R$ be the maximal ideal that corresponds to the unique fixed point of $\operatorname{Spec} R$, and $\widehat{R}$ the completion of R with respect to the maximal ideal $\mathfrak{m}$.*

Then, the $\mathfrak{m}$-adic completion functor gives an equivalence between the category of finitely generated $\mathbb{G}_{\mathrm{m}}$-equivariant R-modules and the category of complete Noetherian $\mathbb{G}_{\mathrm{m}}$-equivariant $\widehat{R}$-modules.

7.6.2 Result

Let $X_0 := \operatorname{Spec} R_0$ be a normal Gorenstein affine variety, and $\phi_0 : Y_0 \to X_0$ and $\phi_0' : Y_0' \to X_0$ two crepant resolutions of X_0. Let $(\operatorname{Spec} D, d)$ be a pointed affine variety, and let the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\phi} & X & \xleftarrow{\phi'} & Y' \\ & \searrow & \downarrow & \swarrow & \\ & & \operatorname{Spec} D & & \end{array}$$

be a deformation of varieties and morphisms above over $(\operatorname{Spec} D, d)$ such that the varieties Y, Y' and X are $\mathbb{G}_{\mathrm{m}}$-variety and the morphisms ϕ and ϕ' are projective and $\mathbb{G}_{\mathrm{m}}$-equivariant. Assume that $X = \operatorname{Spec} R$ is an affine *variety* and that the $\mathbb{G}_{\mathrm{m}}$-action on X is good whose fixed point $x \in X$ is a point over $d \in \operatorname{Spec} D$.

In this subsection, we prove the following theorem.

Theorem 7.6.4. *Under the set-up above, assume in addition that the inequality* $\operatorname{codim}_{X_0} \operatorname{Sing}(X_0) \geq 3$ *holds. Then any good and strict tilting-type equivalence between* $\mathrm{D}^{\mathrm{b}}(Y_0)$ *and* $\mathrm{D}^{\mathrm{b}}(Y_0')$ *lifts to a tilting type equivalence for* $\mathrm{D}^{\mathrm{b}}(Y)$ *and* $\mathrm{D}^{\mathrm{b}}(Y')$*.*

In addition, if the lift is determined by tilting bundles (T, T')*, then we have* $\phi_* T \simeq \phi'_* T'$*.*

Remark 7.6.5. Again we remark that the assumption $\operatorname{codim}_{X_0} \operatorname{Sing}(X_0) \geq 3$ is essential for this theorem (see Section 7.7.2).

Proof of Theorem 7.6.4. In the following, we provide the proof of the theorem that consists of three steps.

Step 1. Construction of tilting bundles.

Put $(D_n, d_n) := (D/d^{n+1}, d/d^{n+1})$, $\mathcal{D} := \lim D_n$, $\mathcal{X} := X \otimes_D \mathcal{D}$, $\mathcal{Y} := Y \otimes_D \mathcal{D}$, and $\mathcal{Y}' := Y' \otimes_D \mathcal{D}$. Let

$$\varphi : \mathcal{Y} \to \mathcal{X} \text{ and } \varphi' : \mathcal{Y}' \to \mathcal{X}$$

be the projection. Then we can apply Theorem 7.5.1 to the diagram

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{\varphi} & \mathcal{X} & \xleftarrow{\varphi'} & \mathcal{Y}' \\ & \searrow & \downarrow & \swarrow & \\ & & \operatorname{Spec} \mathcal{D} & & \end{array}$$

and have a tilting bundle $\mathcal{T}$ (resp. $\mathcal{T}'$) on $\mathcal{Y}$ (resp. $\mathcal{Y}'$) such that

$$\varphi_* \mathcal{T} \simeq \varphi'_* \mathcal{T}'$$

and

$$\operatorname{End}_{\mathcal{Y}}(\mathcal{T}) \simeq \operatorname{End}_{\mathcal{Y}'}(\mathcal{T}').$$

Let $x \in X$ be the unique fixed closed point and let $\widehat{X} := \operatorname{Spec} \widehat{R}$, where $\widehat{R} := \widehat{\mathcal{O}_{X,x}}$ is the completion of $\mathcal{O}_{X,x}$ with respect to the unique maximal ideal. Then the canonical morphism $\widehat{X} \to X$ factors through the morphism $\mathcal{X} \to X$ and hence we have a diagram

$$\begin{array}{ccccc} Y \times_X \widehat{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & Y \\ \downarrow{\scriptstyle \widehat{\phi}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \phi} \\ \widehat{X} & \xrightarrow{\iota} & \mathcal{X} & \longrightarrow & X \end{array}$$

with $\kappa : \widehat{X} \to X$ the composite of the bottom row.

such that all squares are cartesian. Thus the following Lemma 7.6.7 implies that the scheme $\widehat{Y} := Y \times_X \widehat{X}$ admits a tilting bundle $\widehat{T}$ such that $\widehat{\phi}_*\widehat{T} \simeq \iota^*\varphi_*\mathcal{T}$. Similarly, the scheme $\widehat{Y}' := Y' \times_X \widehat{X}$ admits a tilting bundle $\widehat{T}'$ such that $\widehat{\phi}'_*\widehat{T}' \simeq \iota^*\varphi'_*\mathcal{T}'$. In particular, we have

$$\widehat{\phi}_*\widehat{T} \simeq \widehat{\phi}'_*\widehat{T}'$$

and

$$\mathrm{End}_{\widehat{Y}}(\widehat{T}) \simeq \mathrm{End}_{\widehat{Y}'}(\widehat{T}').$$

Then Theorem 7.6.2 implies that there is a $\mathbb{G}_{\mathrm{m}}$-equivariant tilting bundle T (resp. T') on Y (resp. Y') such that the pull-back of T (resp. T') on $\widehat{Y}$ (resp. $\widehat{Y}'$) is isomorphic to $\widehat{T}$ (resp. $\widehat{T}'$). In addition, Kaledin constructed $\mathbb{G}_{\mathrm{m}}$-actions on $\widehat{\phi}_*\widehat{T}$ and $\widehat{\phi}'_*\widehat{T}'$ such that there are $\mathbb{G}_{\mathrm{m}}$-equivariant isomorphisms

$$\widehat{\phi}_*\widehat{T} \simeq \kappa^*\phi_*T \text{ and } \widehat{\phi}'_*\widehat{T}' \simeq \kappa^*\phi'_*T'.$$

To show that the $\mathbb{G}_{\mathrm{m}}$-equivariant structures of $\widehat{\phi}_*\widehat{T}$ and $\widehat{\phi}'_*\widehat{T}'$ are same under the isomorphism $\widehat{\phi}_*\widehat{T} \simeq \widehat{\phi}'_*\widehat{T}'$, we have to recall Kaledin's construction of $\mathbb{G}_{\mathrm{m}}$-equivariant structures. In the following, we provide the detail of his argument, because there is only a sketch of the proof in [Kal08]. Put $\widehat{M} := \widehat{\phi}_*\widehat{T}$.

Step 2. $\mathbb{G}_{\mathrm{m}}$-equivariant structure of $\widehat{M}$.

Let us consider the $\mathbb{G}_{\mathrm{m}}$-action $\mathbb{G}_{\mathrm{m}} \to \mathrm{Aut}(\widehat{R})$. It is well-known that we can regard $\mathrm{Aut}(\widehat{R})$ as an open subscheme of the Hilbert scheme $\mathrm{Hilb}(\mathrm{Spec}(\widehat{R} \otimes \widehat{R}))$ via the map taking the graph of an automorphism of $\mathrm{Spec}\,\widehat{R}$. Thus the tangent space of $\mathrm{Aut}(\widehat{R})$ at the identity $\mathrm{id} \in \mathrm{Aut}(\widehat{R})$ is isomorphic to

$$\mathrm{Ext}^1_{\widehat{X}\times\widehat{X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \simeq \mathrm{HH}^1(\widehat{R}) \simeq \mathrm{Der}_{\mathbb{C}}(\widehat{R}, \widehat{R}),$$

where $\Delta \subset \widehat{X} \times \widehat{X}$ is the diagonal and $\mathrm{HH}^1(\hat{R})$ is the 1st Hochschild cohomology of $\widehat{R}$. Thus, the $\mathbb{G}_{\mathrm{m}}$-action $\mathbb{G}_{\mathrm{m}} \to \mathrm{Aut}(\widehat{R})$ determines a (non-zero) derivation

$$\xi : \widehat{R} \to \widehat{R}$$

uniquely up to scalar multiplication. Let $\mathbb{C}[\varepsilon]$ be the ring of dual numbers and put

$$\widehat{R}^{(1)} := \widehat{R} \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon],\ \widehat{Y}^{(1)} := \widehat{Y} \times \mathrm{Spec}\,\mathbb{C}[\varepsilon],\ \text{and } \widehat{Y}'^{(1)} := \widehat{Y}' \times \mathrm{Spec}\,\mathbb{C}[\varepsilon].$$

By deformation theory, the derivation $\xi : \widehat{R} \to \widehat{R}$ determines an automorphism

$$f_\xi : \widehat{R}^{(1)} \xrightarrow{\sim} \widehat{R}^{(1)}$$

of $\widehat{R}^{(1)}$ such that $f_\xi \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C} \simeq \mathrm{id}_{\widehat{R}}$ (see [Ser, Lemma 1.2.6]), and then, by taking pull-back, we have isomorphisms of schemes

$$F_\xi : \widehat{Y}^{(1)} \xrightarrow{\sim} \widehat{Y}^{(1)} \text{ and } F'_\xi : \widehat{Y}'^{(1)} \xrightarrow{\sim} \widehat{Y}'^{(1)}.$$

Via an isomorphism

$$\mathrm{Der}_{\mathbb{C}[\varepsilon]}(\widehat{R}^{(1)}, \widehat{R}) \simeq \mathrm{Der}_{\mathbb{C}}(\widehat{R}, \widehat{R}),$$

we have a derivation $\tilde{\xi} : \widehat{R}^{(1)} \to \widehat{R}$ and then the automorphism is given by

$$f_\xi(\tilde{r}) = \tilde{r} + \tilde{\xi}(\tilde{r}) \otimes \varepsilon.$$

Let $\widehat{T}^{(1)}$ be the pull-back of $\widehat{T}$ by the projection $\widehat{Y}^{(1)} \to \widehat{Y}$, and we also define a bundle $\widehat{T}'^{(1)}$ on $\widehat{Y}'^{(1)}$ in the same way. Then $\widehat{T}^{(1)}$ and $F_{\xi_*}\widehat{T}^{(1)}$ are two lifts of $\widehat{T}$ to $\widehat{Y}^{(1)}$. Since $\widehat{T}$ is tilting, we have $\mathrm{Ext}^1_{\widehat{Y}}(\widehat{T}, \widehat{T}) = 0$ and hence there is an isomorphism

$$g : \widehat{T}^{(1)} \xrightarrow{\sim} F_{\xi_*}\widehat{T}^{(1)}$$

(see [Har2, Theorem 7.1 (c)]). If we put $\widehat{M}^{(1)} := H^0(\widehat{Y}^{(1)}, \widehat{T}^{(1)})$, then we have a $\mathbb{C}$-linear bijective morphism

$$g_\xi : \widehat{M}^{(1)} := H^0(\widehat{Y}^{(1)}, \widehat{T}^{(1)}) \xrightarrow[g]{\sim} H^0(\widehat{Y}^{(1)}, F_{\xi_*}\widehat{T}^{(1)}) \xrightarrow[\mathrm{adj}]{\sim} H^0(\widehat{Y}^{(1)}, \widehat{T}^{(1)}) = \widehat{M}^{(1)}$$

of $\widehat{M}^{(1)}$. Note that, by construction, we have $g_\xi \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C} = \mathrm{id}_{\widehat{M}}$ and

$$g_\xi(\tilde{r}\tilde{m}) = f_\xi(\tilde{r}) g_\xi(\tilde{m})$$

for $\tilde{r} \in \widehat{R}^{(1)}$ and $\tilde{m} \in \widehat{M}^{(1)}$. From the first equality, for any $\tilde{m} \in \widehat{M}^{(1)}$, there is an element $\tilde{\xi}_{\widehat{M}}(\tilde{m}) \in \widehat{M}$ such that

$$g_\xi(\tilde{m}) = \tilde{m} + \tilde{\xi}_{\widehat{M}}(\tilde{m}) \otimes \varepsilon.$$

This correspondence $\tilde{m} \mapsto \tilde{\xi}_{\widehat{M}}(\tilde{m})$ defines a $\mathbb{C}$-linear map

$$\tilde{\xi}_{\widehat{M}} : \widehat{M}^{(1)} \to \widehat{M}.$$

In addition, since we have

$$\begin{aligned} \tilde{r}\tilde{m} + \tilde{\xi}_{\widehat{M}}(\tilde{r}\tilde{m}) \otimes \varepsilon &= g_\xi(\tilde{r}\tilde{m}) \\ &= f_\xi(\tilde{r}) g_\xi(\tilde{m}) \\ &= (\tilde{r} + \tilde{\xi}(\tilde{r}) \otimes \varepsilon)(\tilde{m} + \tilde{\xi}_{\widehat{M}}(\tilde{m}) \otimes \varepsilon) \\ &= \tilde{r}\tilde{m} + (\tilde{r}\tilde{\xi}_{\widehat{M}}(\tilde{m}) + \tilde{\xi}(\tilde{r})\tilde{m}) \otimes \varepsilon, \end{aligned}$$

we have an equality

$$\tilde{\xi}_{\widehat{M}}(\tilde{r}\tilde{m}) = \tilde{r}\tilde{\xi}_{\widehat{M}}(\tilde{m}) + \tilde{\xi}(\tilde{r})\tilde{m}.$$

Thus, restricting $\tilde{\xi}_{\widehat{M}}$ to the subgroup $\widehat{M} \simeq \widehat{M} \otimes_{\mathbb{C}} \varepsilon \subset \widehat{M}^{(1)}$, we have a $\mathbb{C}$-linear endomorphism

$$\xi_{\widehat{M}} \in \mathrm{End}_{\mathbb{C}}(\widehat{M})$$

such that

$$\xi_{\widehat{M}}(rm) = \xi(r)m + r\xi_{\widehat{M}}(m)$$

for $r \in \widehat{R}$ and $m \in \widehat{M}$.

Let $\widehat{R} \oplus \widehat{M}$ be the square-zero extension, whose multiplication is given by

$$(r, m)(r', m') := (rr', rm' + r'm).$$

Then one can check that the map

$$\xi : \widehat{R} \oplus \widehat{M} \ni (r, m) \mapsto (\xi(r), \xi_{\widehat{M}}(m)) \in \widehat{R} \oplus \widehat{M}$$

defines a derivation of $\widehat{R} \oplus \widehat{M}$. Furthermore, by taking restriction, ξ also defines a derivation of the algebra

$$(\widehat{R}/\mathfrak{m}^n\widehat{R}) \oplus (\widehat{M}/\mathfrak{m}^n\widehat{M})$$

for $n > 0$. If $n = 1$, then the module $\widehat{M}/\mathfrak{m}\widehat{M}$ is finite dimensional, and thus there is a representation of $\mathbb{G}_{\mathrm{m}}$

$$\pi : \mathbb{G}_{\mathrm{m}} \to \mathrm{GL}(\widehat{M}/\mathfrak{m}\widehat{M})$$

whose differential is the restriction of $\xi_{\widehat{M}}$. Using this representation, we have an action

$$\mathbb{G}_{\mathrm{m}} \to \mathrm{Aut}_{\mathbb{C}}((\widehat{R}/\mathfrak{m}\widehat{R}) \oplus (\widehat{M}/\mathfrak{m}\widehat{M}))$$

of $\mathbb{G}_{\mathrm{m}}$ on the algebra $(\widehat{R}/\mathfrak{m}\widehat{R}) \oplus (\widehat{M}/\mathfrak{m}\widehat{M})$ such that

$$c \cdot (\bar{r}, \bar{m}) := (\bar{r}, \pi(c)\bar{m})$$

for $c \in \mathbb{G}_{\mathrm{m}}$ and $(\bar{r}, \bar{m}) \in (\widehat{R}/\mathfrak{m}\widehat{R}) \oplus (\widehat{M}/\mathfrak{m}\widehat{M})$.

Then by using [Kal08, Lemma 5.2] inductively on n, we have an action of $\mathbb{G}_{\mathrm{m}}$ on $\widehat{R} \oplus \widehat{M}$ and therefore we have a $\mathbb{G}_{\mathrm{m}}$-equivariant structure of $\widehat{M}$.

Step 3. Comparing $\mathbb{G}_{\mathrm{m}}$-equivariant structures.

From the construction above, we notice that the $\mathbb{G}_{\mathrm{m}}$-equivariant structure of $\widehat{M}$ depends on the choice of the isomorphism

$$g : \widehat{T}^{(1)} \xrightarrow{\sim} F_{\xi_*}\widehat{T}^{(1)}$$

such that $g \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C} = \mathrm{id}$, and the ambiguity of the choice of such isomorphisms is resolved by the group $\mathrm{End}_{\widehat{Y}}(\widehat{T})$ (see [Har2, Theorem 7.1 (a)]). Since we have

$$\mathrm{End}_{\widehat{Y}}(\widehat{T}) \simeq \mathrm{End}_{\widehat{R}}(\widehat{M}) \simeq \mathrm{End}_{\widehat{Y}'}(\widehat{T}'),$$

we can choose isomorphisms

$$g : \widehat{T}^{(1)} \xrightarrow{\sim} F_{\xi_*}\widehat{T}^{(1)} \text{ and } g' : \widehat{T}'^{(1)} \xrightarrow{\sim} F'_{\xi_*}\widehat{T}'^{(1)}$$

such that they give the same $\mathbb{C}$-linear map

$$g_\xi : \widehat{M}^{(1)} \to \widehat{M}^{(1)}$$

after taking the global sections. This means that we can take $\mathbb{G}_{\mathrm{m}}$-equivariant tilting bundles T and T' such that there are $\mathbb{G}_{\mathrm{m}}$-equivariant isomorphisms

$$\widehat{\phi}_*\widehat{T} \simeq \kappa^*\phi_*T \simeq \kappa^*\phi'_*T' \simeq \widehat{\phi}'_*\widehat{T}',$$

as desired.

By Lemma 7.6.3, the $\mathbb{G}_{\mathrm{m}}$-equivariant isomorphism $\widehat{\phi}_*\widehat{T} \simeq \widehat{\phi}'_*\widehat{T}'$ implies that we have

$$\phi_*T \simeq \phi'_*T$$

and the isomorphism $\mathrm{End}_{\widehat{Y}}(\widehat{T}) \simeq \mathrm{End}_{\widehat{Y}'}(\widehat{T}')$ implies that we have

$$\mathrm{End}_Y(T) \simeq \mathrm{End}_{Y'}(T').$$

Therefore we have the result. □

Remark 7.6.6. We also have an equivalence between $\mathbb{G}_{\mathrm{m}}$-equivariant derived categories of Y and Y' (see, for example, [Kar15, Section 4.2]).

Lemma 7.6.7. *Let $\psi : Z \to \mathrm{Spec}\, S$ be a projective morphism, $\mathfrak{p} \in \mathrm{Spec}\, S$ a point, $\mathcal{Z} := Z \times_{\mathrm{Spec}\, S} \mathrm{Spec}\, \widehat{S_{\mathfrak{p}}}$, and $\iota : \mathcal{Z} \to Z$ the canonical morphism. If T is a tilting bundle on Z, then ι^*T is a tilting bundle on $\mathcal{Z}$.*

Proof. Since $\mathrm{Spec}\, \widehat{S_{\mathfrak{p}}} \to \mathrm{Spec}\, S$ is flat and $\mathrm{Ext}^i_Z(T,T) \simeq R^i\psi_*(T^\vee \otimes T)$, it follows from the flat base change formula that ι^*T is a partial tilting bundle.

Next we show that ι^*T is a generator of $\mathrm{D}^-(\mathrm{Qcoh}(\mathcal{Z}))$. Let $F \in \mathrm{D}^-(\mathrm{Qcoh}(\mathcal{Z}))$ be a complex such that $\mathrm{RHom}_{\mathcal{Z}}(\iota^*T, F) = 0$. Since ι is quasi-compact and quasi-separated, we can define a functor $\iota_* : \mathrm{Qcoh}(\mathcal{Z}) \to \mathrm{Qcoh}(Z)$. In addition, since the morphism ι is affine, the functor ι_* is exact and hence we can consider a functor between derived categories

$$\iota_* : \mathrm{D}^-(\mathrm{Qcoh}(\mathcal{Z})) \to \mathrm{D}^-(\mathrm{Qcoh}(Z)),$$

which is the right adjoint of ι^*. Thus we have an isomorphism

$$\mathrm{RHom}_Z(T, \iota_*F) \simeq \mathrm{RHom}_{\mathcal{Z}}(\iota^*T, F) = 0,$$

and this shows that $\iota_*F = 0$. Since ι is affine, $\iota_*F = 0$ implies $F = 0$. □

As in Corollary 7.5.7, we can relax the assumption in Theorem 7.6.4 if the codimension of the exceptional locus is greater than or equal to three.

Corollary 7.6.8. *Under the same set-up of Theorem 7.6.4, assume in addition that*

$$\mathrm{codim}_Y(\mathrm{exc}(\phi_0)) \geq 3.$$

Then any strict tilting-type equivalence for Y_0 and Y'_0 lifts to a tilting type equivalence for Y and Y'.

Let $p : Y \to \mathrm{Spec}\, D$, $p' : Y' \to \mathrm{Spec}\, D$ and $q : X \to \mathrm{Spec}\, D$ be the morphisms associated to the deformation.

Corollary 7.6.9. *Under the same assumption as in Theorem 7.6.4, assume that there exists a good and strict tilting-type equivalence between* $\mathrm{D^b}(Y_0)$ *and* $\mathrm{D^b}(Y_0')$. *Then there is a good tilting-type equivalence between* $\mathrm{D^b}(p^{-1}(t))$ *and* $\mathrm{D^b}(p'^{-1}(t))$ *for any closed point* $t \in \operatorname{Spec} D$.

Proof. Let T_0 and T_0' are tilting bundle that determines the good and strict tilting-type equivalence between $\mathrm{D^b}(Y_0)$ and $\mathrm{D^b}(Y_0')$. Then by Theorem 7.6.4 we have a tilting bundle T (resp. T') on Y (resp. Y') such that

$$\operatorname{End}_Y(T) \simeq \operatorname{End}_{Y'}(T').$$

Let us consider the following diagram.

$$\begin{array}{ccccc} p^{-1}(t) & \xrightarrow{\phi} & q^{-1}(t) & \longrightarrow & \operatorname{Spec}\mathbb{C} \\ \downarrow & & \downarrow & & \downarrow t \\ Y & \xrightarrow{\phi} & X & \xrightarrow{q} & \operatorname{Spec} D. \end{array}$$

Since we have an isomorphism

$$(R\phi_* F)|_{q^{-1}(t)} \simeq \mathrm{R}\Gamma(p^{-1}(t), F|_{p^{-1}(t)})$$

for an vector bundle F on Y applying the (derived) flat base change formula, we have that $T_t := T|_{p^{-1}(t)}$ is a good partial tilting bundle on $p^{-1}(t)$. In addition it is clear that T_t is a generator and hence T_t is a good tilting bundle on $p^{-1}(t)$.

Similarly, a bundle $T_t' := T'|_{p'^{-1}(t)}$ is a tilting bundle on $p'^{-1}(t)$ such that

$$\operatorname{End}_{p^{-1}(t)}(T_t) \simeq \operatorname{End}_{p'^{-1}(t)}(T_t').$$

This shows the result. Note that we also have $\phi_*(T_t) \simeq \phi'_*(T_t')$. □

7.7 Examples

The aim of this section is to provide some applications and counter-examples of the theorem we established in the sections above.

7.7.1 Stratified Mukai flops and stratified Atiyah flops

In the present subsection, we add some results for derived equivalences for stratified Mukai flops and stratified Atiyah flops from the point of view of tilting bundles. We use notations in Section 2.3.3.

Theorem 7.7.1. *Assume that* $2r < N$. *Then there exists a strict tilting-type equivalence for a stratified Mukai flop on* $\operatorname{Gr}(r, N)$.

Proof. According to Theorem 7.8.2, it is enough to show that there is a good $\mathbb{G}_{\mathrm{m}}$-action on $X_0 = \overline{B(r)}$.

Let $e_1, \dots, e_N$ be a standard basis of $V = \mathbb{C}^N$ and $f_1, \dots, f_N$ the dual basis of $V^\vee$. We regard $x_{ij} := e_i \otimes f_j \in V \otimes_{\mathbb{C}} V^\vee$ as a variable of the affine coordinate ring of $\mathrm{End}(V) = V^\vee \otimes V$. Then the affine coordinate ring R_0 of X_0 is a quotient of a polynomial ring $\mathbb{C}[(x_{ij})_{i,j}]$ and the maximal ideal $\mathfrak{m}_o$ corresponding to the origin $o \in X_0$ is the ideal generated by the image of $\{x_{ij}\}_{i,j}$.

Let us consider an action of $\mathbb{G}_{\mathrm{m}}$ on X_0 given by $t \cdot A := t^{-1}A$ for $t \in \mathbb{G}_{\mathrm{m}}$ and $A \in X_0$. Clearly this action has a unique fixed point $o \in X$. As a $\mathbb{G}_{\mathrm{m}}$-representation, $\mathfrak{m}_o$ splits into the direct sum of lines spanned by a monomial. Let $r \in R_0$ be a monomial of degree $d \geq 1$. Then, for $t \in \mathbb{G}_{\mathrm{m}}$ and $A \in X_0$, we have

$$r^t(A) = r(t^{-1} \cdot A) = r(tA) = t^d r(A).$$

Thus the action of $\mathbb{G}_{\mathrm{m}}$ on $\mathfrak{m}_o$ is positive weight and hence the action of $\mathbb{G}_{\mathrm{m}}$ on X_0 is good. □

As an application of Theorem 7.6.4, we have the following result.

Theorem 7.7.2. *Assume that* $2r \leq N-1$. *Then, any good and strict tilting-type equivalence between* $\mathrm{D^b}(Y_0)$ *and* $\mathrm{D^b}(Y_0')$ *lifts to a good tilting-type equivalence between* $\mathrm{D^b}(Y)$ *and* $\mathrm{D^b}(Y')$.

In addition, if $2r \leq N-2$, *then any strict tilting-type equivalence between* $\mathrm{D^b}(Y_0)$ *and* $\mathrm{D^b}(Y_0')$ *lifts to a good tilting-type equivalence between* $\mathrm{D^b}(Y)$ *and* $\mathrm{D^b}(Y')$.

Proof. As in Theorem 7.7.1, we can check that there exists a good $\mathbb{G}_{\mathrm{m}}$-action on X which lifts to $\mathbb{G}_{\mathrm{m}}$-actions on Y and Y'. Note that $\dim X_0 = \dim \overline{B(r)} = 2r(N-r)$ and $\mathrm{Sing}(X_0) = \overline{B(r-1)}$. Thus we have

$$\operatorname{codim}_{X_0} \mathrm{Sing}(X_0) = 2r(N-r) - 2(r-1)(N-r+1) = 2(N-2r+1).$$

Therefore $\operatorname{codim}_{X_0} \mathrm{Sing}(X_0) \geq 3$ if and only if $2r < N$.

Furthermore, if we put

$$B(k) := \{A \in \mathrm{End}(V) \mid A^2 = 0, \dim \operatorname{Ker} A = N-k\},$$

the fiber $\phi_0^{-1}(A)$ of $A \in B(k)$ $(k < r)$ is isomorphic to $\mathrm{Gr}(r-k, \operatorname{Ker}(A)/\operatorname{Im}(A))$ and thus we have

$$\dim \phi_0^{-1}(A) = (r-k)(N-r-k).$$

Therefore we have

$$\dim \phi_0^{-1}(B(k)) = 2k(N-k) + (r-k)(N-r-k)$$

for $0 \leq k \leq r-1$ and hence we have

$$\begin{aligned} \operatorname{codim}_{Y_0} \overline{\phi_0^{-1}(B(k))} &= 2r(N-r) - 2k(N-k) - (r-k)(N-r-k) \\ &= \left(k - \frac{N}{2}\right)^2 - \frac{N^2}{4} + rN - r^2 \\ &\geq N - 2r + 1. \end{aligned}$$

and the equality holds if $k = r - 1$. This shows that $\operatorname{codim}_{Y_0}(\operatorname{exc}(\phi_0)) \geq 3$ if and only if $2r \leq N - 2$.

Let T and T' be tilting bundles that give the tilting-type equivalence between $\mathrm{D^b}(Y)$ and $\mathrm{D^b}(Y')$. Then by Theorem 7.6.4 we have $\phi_* T \simeq \phi'_* T'$. This means that the tilting-type equivalence is strict. □

In conclusion, we have the following result.

Theorem 7.7.3. *If $2r \leq N - 2$, then there exists a tilting-type equivalence for the stratified Atiyah flop on $\mathrm{Gr}(r, N)$.*

The author expects that the same statement holds for remaining cases:

Conjecture 7.7.4. *Assume that $2r = N - 1$ or $2r = N$. There exists a tilting-type equivalence for the stratified Atiyah flop on $\mathrm{Gr}(r, N)$.*

This conjecture is true in the following low dimensional cases:

Example 7.7.5. If $N = 2$ and $r = 1$, the stratified Atiyah flop is the 3-fold Atiyah flop. If $N = 3$ and $r = 1$, the stratified Atiyah flop on $\mathrm{Gr}(1, 3)$ is the usual standard flop of 4-folds. In these cases, the conjecture above is known to be true.

The author also expects that CKL's equivalence for a stratified Mukai flop is tilting-type. Indeed, if $r = 1$, CKL's equivalence for a Mukai flop is tilting-type (see [Cau12a] and Section 3). In addition, as noted above, Cautis proved that CKL's equivalence extends to an equivalence for a stratified Atiyah flop [Cau12a, Theorem 4.1] as in our Theorem 7.7.2.

Finally we note that the discussions above also show the following result.

Theorem 7.7.6. *The nilpotent orbit closure $X_0 := \overline{B(r)}$ (resp. its $\mathbb{G}_{\mathrm{m}}$-equivariant deformation X) admits an NCCR for all $2r \leq N$ that is derived equivalent to Y_0 and Y'_0 (resp. Y and Y').*

Proof. We can take a lift T of a tilting bundle T_0 on Y_0 to Y without assuming that T_0 is good or $\operatorname{codim}_{X_0} \operatorname{Sing}(X_0) \geq 3$. Then the result follows from Lemma 2.1.18. As noted above, the derived equivalence for Y and Y' was proved by Cautis. □

7.7.2 A counter-example

In the present subsection, we provide an example that suggests we cannot remove the assumption $\operatorname{codim}_{X_0} \operatorname{Sing}(X_0) \geq 3$ in Theorem 7.6.4.

Let us consider the case if $N = 2$ and $r = 1$ in the subsection above. Then $X_0 = \overline{B(1)}$ admits a du Val singularity of type A_1, and $Y_0 = Y'_0$ is the total space of a line bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$ on $\mathbb{P}^1$. Moreover X is a 3-fold ODP, and Y and Y' are isomorphic to the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ as abstract varieties. However there is no isomorphism $f : Y \xrightarrow{\sim} Y'$ such that $\phi = \phi' \circ f$. A bundle $T_0 = \mathcal{O}_{Y_0} \oplus \mathcal{O}_{Y_0}(-1)$, where $\mathcal{O}_{Y_0}(-1)$ is a pull-back of $\mathcal{O}_{\mathbb{P}^1}(-1)$ to Y_0, is a good tilting bundle on Y_0.

A pair of tilting bundles (T_0, T_0) provides a good and strict tilting type equivalence $\mathrm{D^b}(Y_0) \simeq \mathrm{D^b}(Y_0)$, which is identity. T_0 lifts to a good tilting bundle $T = \mathcal{O}_Y \oplus \mathcal{O}_Y(-1)$ on Y, where $\mathcal{O}_Y(-1)$ is a pull-back of $\mathcal{O}_{\mathbb{P}^1}(-1)$ to Y. Similarly, T_0 lifts to a tilting bundle $T' = \mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(-1)$ on Y', where $\mathcal{O}_{Y'}(-1)$ is a pull-back of $\mathcal{O}_{\mathbb{P}^1}(-1)$ on Y'.

However we have $T|_{Y^o} \neq T'|_{Y^o}$, where Y^o is the common open subset of Y and Y'. Indeed, we have $\phi_* \mathcal{O}_Y(-1) \neq \phi'_* \mathcal{O}_{Y'}(-1)$.

On the other hand, a pair of tilting bundles $(T_0, T_0^\vee)$ induces a good and strict tilting-type equivalence $\mathrm{D^b}(Y_0) \xrightarrow{\sim} \mathrm{D^b}(Y_0)$, which is a spherical twist around a sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$ on the zero-section $\mathbb{P}^1 \subset Y_0$.

Since $T_0^\vee$ lifts to a bundle $T'^\vee$ on Y' and one has $T|_{Y^o} \simeq T'^\vee|_{Y^o}$, the above equivalence lifts to a good and strict tilting-type equivalence

$$\mathrm{RHom}_Y(T, -) \otimes^{\mathrm{L}}_{\mathrm{End}_Y(T)} T'^\vee : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(Y').$$

7.8 Derived equivalence for symplectic resolutions

In this subsection, we discuss the derived equivalence for symplectic resolutions. First we recall the following theorem, which is the main theorem of Kaledin's paper [Kal08].

Theorem 7.8.1 ([Kal08], Theorem 1.6)**.** *Let $X = \operatorname{Spec} R$ be an affine symplectic variety, and Y and Y' two symplectic resolutions of X. Then every point $x \in X$ admits an étale neighborhood $U_x \to X$ such that there exists a strict tilting-type equivalence between $\mathrm{D^b}(Y \times_X U_x)$ and $\mathrm{D^b}(Y' \times_X U_x)$.*

The property that the equivalence is strict follows from his proof. Since the construction of tilting bundles is very complicated, it is not clear whether the tilting bundle he constructed is good or not (at least for the author).

Theorem 7.8.2. *Let $X = \operatorname{Spec} R$ be an affine symplectic variety, and $\phi : Y \to X$ and $\phi : Y' \to X$ two symplectic resolutions of X. Assume that X admits a good $\mathbb{G}_\mathrm{m}$-action. Then there exists a strict tilting-type equivalence between $\mathrm{D^b}(Y)$ and $\mathrm{D^b}(Y')$.*

Proof. First we note that the $\mathbb{G}_\mathrm{m}$-action on X lifts to a $\mathbb{G}_\mathrm{m}$-action on Y and Y' [Kal08, Theorem 1.8 (i)]. Let $\mathfrak{m} \subset R$ be the maximal ideal that corresponds to a unique fixed point of X. Let $\widehat{R}$ be the completion of R with respect to $\mathfrak{m} \subset R$. Put $\widehat{Y} := Y \times_X \operatorname{Spec} \widehat{R}$ and $\widehat{Y}' := Y' \times_X \operatorname{Spec} \widehat{R}$, and let $\widehat{\phi} : \widehat{Y} \to \operatorname{Spec} \widehat{R}$ and $\widehat{\phi}' : \widehat{Y}' \to \operatorname{Spec} \widehat{R}$ be the projections. Let $Y^o \subset Y$, Y' be the largest common open subscheme of Y and Y'. Put $\widehat{Y}^o := Y^o \times_Y \widehat{Y} = Y^o \times_{Y'} \widehat{Y}'$.

Then, by Theorem 7.8.1, there exist tilting bundles $\widehat{\mathcal{E}}$ and $\widehat{\mathcal{E}'}$ on $\widehat{Y}$ and $\widehat{Y}'$, respectively, such that $\widehat{\mathcal{E}}|_{\hat{Y}^o} \simeq \widehat{\mathcal{E}'}|_{\hat{Y}^o}$. Thus we have an isomorphism $\widehat{\phi}_* \widehat{\mathcal{E}} \simeq \widehat{\phi'}_* \widehat{\mathcal{E}'}$ of $\widehat{R}$-modules and an isomorphism $\mathrm{End}_{\widehat{Y}}(\widehat{\mathcal{E}}) \simeq \mathrm{End}_{\widehat{Y}'}(\widehat{\mathcal{E}'})$ of $\widehat{R}$-algebras.

By Theorem 7.6.2, there exist tilting bundles $\mathcal{E}$ and $\mathcal{E}'$ on Y and Y', respectively, such that $\mathcal{E} \otimes_R \widehat{R} \simeq \widehat{\mathcal{E}}$ and $\mathcal{E}' \otimes_R \widehat{R} \simeq \widehat{\mathcal{E}'}$. Since $\phi_* \mathcal{E} \otimes_R \widehat{R} \simeq \widehat{\phi}_* \widehat{\mathcal{E}}$ and $\phi'_* \mathcal{E}' \otimes_R \widehat{R} \simeq \widehat{\phi}'_* \widehat{\mathcal{E}'}$, Lemma 7.6.3 and the similar argument as in Step 3 of the proof of Theorem 7.6.4 imply that we have

$$\phi_* \mathcal{E} \simeq \phi'_* \mathcal{E}'$$

and hence we have $\mathcal{E}\,|_{Y^o} \simeq \mathcal{E}'\,|_{Y^o}$. Thus we have a strict tilting-type equivalence between $\mathrm{D^b}(Y)$ and $\mathrm{D^b}(Y')$. $\square$

Bibliography

[ADM15] N. Addington, W. Donovan, C. Meachan, *Mukai flops and P-twists*, preprint (2015), https://arxiv.org/abs/1507.02595.

[Art] M. Artin, *Lectures on Deformations of Singularities*, available from http://www.math.tifr.res.in/~publ/ln/tifr54.pdf.

[Bei79] A. Beilinson, *Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra*, Funct. Anal. Appl. **12**(3) (1978), 68–69.

[Bea00] A. Beauville, *Symplectic singularities*, Invent. Math. **139**(3) (2000), 541–549.

[BMR06] R. Bezrukavnikov, I. Mirković, D. Rumynin, *Singular localization and intertwining functors for reductive Lie algebras in prime characteristic*, Nagoya Math. J. **184** (2006), 1–55.

[BMR08] R. Bezrukavnikov, I. Mirković, D. Rumynin, *Localization of modules for a semisimple Lie algebra in prime characteristic*, Ann. of Math. (2) **167**(3) (2008), 945–991.

[Boc12] R. Bocklandt, *Generating toric noncommutative crepant resolutions*, J. Algebra, **364** (2012), 119–147.

[BB15] A. Bodzenta, A. Bondal, *Flops and spherical functors*, preprint (2015), https://arxiv.org/abs/1511.00665.

[Bon90] A. Bondal, *Representation of associative algebras and coherent sheaves*, Math. USSR Izv. **34**(1) (1990), 23–42.

[BO02] A. Bondal, D. Orlov, *Derived categories of coherent sheaves*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 2002, 47–56.

[BVdB03] A. Bondal, M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (1) (2003), 1–36, 258.

[Bourbaki] N. Bourbaki, *Commutative algebra. Chapters 1–7*, Elements of Mathematics (Berlin), Translated from the French, Reprint of the 1989 English translation, Springer-Verlag, Berlin, 1998.

[BH93] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, Cambridge, 1993.

[BLV10] R-o. Buchweitz, G. J. Leuschke, M. Van den Bergh, *Non-commutative desingularization of determinantal varieties I*, Invent. Math., **182**(1) (2010), 47–115.

[Cau12a] S. Cautis, *Equivalences and stratified flops*, Compos. Math., **148**(1) (2012), 185–208.

[Cau12b] S. Cautis, *Flops and about: a guide*, Derived categories in algebraic geometry, 61–101, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012.

[CKL10] S. Cautis, J. Kamnitzer, A. Licata, *Coherent sheaves and categorical $\mathfrak{sl}_2$ actions*, Duke Math. J., **154**(1) (2010), 135–179.

[CKL13] S. Cautis, J. Kamnitzer, A. Licata, *Derived equivalences for cotangent bundles of Grassmannians via categorical $\mathfrak{sl}_2$ actions*, J. Reine Angew. Math., **613** (2013), 53–99.

[Dao10] , H. Dao, *Remarks on non-commutative crepant resolutions of complete intersections*, Adv. Math., **224**(3) (2010), 1021–1030.

[DW15] W. Donovan, M. Wemyss, *Twists and braids for general 3-fold flops*, preprint (2015), `https://arxiv.org/abs/1504.05320`.

[DW16] W. Donovan, M. Wemyss, *Noncommutative deformations and flops*, Duke Math. J., **165**(8) (2016), 1397–1474.

[Fu07] B. Fu, *Extremal contractions, stratified Mukai flops and Springer maps*, Adv. Math. **213** (2007), no. 1, 165–182.

[FH91] W. Fulton, J. Harris, *Representation theory. A first course.*, GTM 129, 1991, Springer.

[Gin09] V. Ginzburg, *Lectures on Nakajima's quiver varieties*, `https://arxiv.org/abs/0905.0686`.

[H17a] W. Hara, *Non-commutative crepant resolution of minimal nilpotent orbit closures of type A and Mukai flops*, Adv. Math., **318** (2017), 355–410.

[H17b] W. Hara, *On derived equivalence for Abuaf flop: mutation of non-commutative crepant resolutions and spherical twists*, preprint, `https://arxiv.org/abs/1706.04417`.

[H17c] W. Hara, *Strong full exceptional collections on certain toric varieties with Picard number three via mutations*, Le Matematiche, Vol. LXXII(2) (2017), 3–24

[H18a] W. Hara, *Deformation of tilting-type derived equivalences for crepant resolutions*, to appear in International Mathematical Research Notice, https://arxiv.org/abs/1709.09948.

[H18b] W. Hara, *On the Abuaf-Ueda flop via non-commutative crepant resolution*, preprint (2018).

[Har1] R. Hartshorne, *Local Cohomology, A seminar given by A. Grothendieck*, Lecture Notes in Mathematics 41, Springer, 1967.

[Har2] R. Hartshorne, *Deformation theory*, Graduate Texts in Mathematics, 257. Springer, New York, 2010.

[HN17] A. Higashitani, Y. Nakajima, *Conic divisorial ideals of Hibi rings and their applications to non-commutative crepant resolutions*, preprint (2017), https://arxiv.org/abs/1702.07058.

[HT06] D. Huybrechts, R. Thomas, *$\mathbb{P}$-objects and autoequivalences of derived categories*, Math. Res. Lett., **13**(1) (2006), 87–98.

[HV07] L. Hille, M. Van den Bergh, *Fourier-Mukai transforms*, In: Handbook of Tilting Theory. London Math. Soc. Lecture Note Ser., vol.332, Cambridge University Press, Cambridge (2007), 147–177..

[IR08] O. Iyama, I. Reiten.: *Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras*, Am. J. Math. **130**(4) (2008), 1087–1149.

[IW14a] O. Iyama, M. Wemyss, *Maximal modifications and Auslander-Reiten duality for non-isolated singularities*, Invent. Math., **197**(3) (2014), 521–586.

[IW14b] O. Iyama, M. Wemyss, *Singular derived categories of $\mathbb{Q}$-factorial terminalizations and maximal modification algebras*, Adv. Math., **261** (2014), 85–121.

[IMOU16a] A. Ito, M. Miura, S. Okawa, K. Ueda, *Calabi-Yau complete intersections in G_2-Grassmannians*, preprint (2016), https://arxiv.org/abs/1606.04076.

[IMOU16b] A. Ito, M. Miura, S. Okawa, K. Ueda, *The class of the affine line is a zero divisor in the Grothendieck ring: via G_2-Grassmannians*, to appear in Journal of Algebraic Geometry, https://arxiv.org/abs/1606.04210.

[Kar15] J. Karmazyn, *Deformations of algebras defined by tilting bundles*, preprint, https://arxiv.org/abs/1505.04049v1.

[Kar17] J. Karmazyn, *Quiver GIT for varieties with tilting bundles*, Manuscripta Math., **157** (2017), 91–128.

[Kal08] D. Kaledin, *Derived equivalences by quantization*, Geom. Funct. Anal., **17**(6) (2008), 1968–2004.

[Kan18] A. Kanemitsu, *Mukai pairs and K-equivalences of simple type*, preprint (2018), https://arxiv.org/abs/1812.05392.

[Kaw02] Y. Kawamata, *D-equivalence and K-equivalence*, J. Differential Geom., **61**(1) (2002), 147–171.

[Kaw05] Y. Kawamata, *Derived equivalence for stratified Mukai flop on $G(2,4)$*, Mirror symmetry. V, 285–294, AMS/IP Stud. Adv. Math., 38, Amer. Math. Soc., Providence, RI, 2006. available from https://arxiv.org/abs/math/0503101v1.

[KF79] H. Kraft, C. Procesi, *Closures of conjugacy class of matrices are normel*, Invent. Math. **53** (1979), 227–247.

[KPS17] A. Krug, D. Ploog, P. Sosna, *Derived categories of resolutions of cyclic quotient singularities*, https://arxiv.org/abs/1701.01331.

[Kuz06] A. Kuznetsov, *Hyperplane sections and derived categories*, Izv. Math. **70**(3) (2006), 447–547. (English version)

[Kuz08] A. Kuznetsov, *Exceptional collections for Grassmannians of isotropic lines*, Proc. London Math. Soc. **97** (2008), 155–182.

[Kuz18] A. Kuznetsov, *Derived equivalence of Ito-Miura-Okawa-Ueda Calabi-Yau 3-folds*, J. Math. Soc. Japan 70 (2018), no. 3, 1007–1013.

[Laz] R. Lazarsfeld, *Positivity in algebraic geometry. I.* , Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 48. Springer-Verlag, Berlin, 2004.

[Leu12] G. J. Leuschke, *Non-commutative crepant resolutions: scenes from categorical geometry*, Progress in commutative algebra 1, 2012, 293–361.

[Li17] D. Li, *On certain K-equivalent birational maps*, preprint 2017, https://arxiv.org/abs/1701.04054.

[Mat] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, 1989.

[Nak16] Y. Nakajima, *Mutations of splitting maximal modifying modules: The case of reflexive polygons*, preprint (2016), https://arxiv.org/abs/1601.05203, to appear in IMRN.

[Nam03] Y. Namikawa, *Mukai flops and derived categories*, J. Reine Angew. Math., **560** (2003), 65–76.

[Nam08] Y. Namikawa, *Flops and Poisson deformations of symplectic varieties*, Publ. Res. Inst. Math. Sci., **44**(2) (2008), 259–314.

[Ott90] G. Ottaviani, *On Cayley bundles on the five-dimensional quadric*, Boll. Un. Mat. Ital. A (7), **4**(1) (1990), 87–100.

[Orl06] D. Orlov, *Triangulated categories of singularities, and equivalences between Landau-Ginzburg models*, (Russian. Russian summary) Mat. Sb. 197 (2006), no. 12, 117–132; translation in Sb. Math. 197 (2006), no. 11-12, 1827–1840.

[Pan91] D. Panyushev, *Rationality of singularities and the Gorenstein property for nilpotent orbits*, Funct. Appl. **25** (1991), 225–226.

[Seg16] E. Segal, *A new 5-fold flop and derived equivalence*, Bull. Lond. Math. Soc., **48**(3) (2016), 533–538.

[ST01] P. Seidel, R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke. Math. J., **108** (2001), 37–108.

[Ser] E. Sernesi, *Deformations of algebraic schemes*, Grundlehren der mathematischen Wissenschaften Vol. 334, Springer, 2000.

[ŠV15a] Š. Špenko, M. Van den Bergh, *Non-commutative resolutions of quotient singularities*, preprint (2015), `https://arxiv.org/abs/1502.05240`.

[ŠV15b] Š. Špenko, M. Van den Bergh, *Comparing the commutative and non-commutative resolutions for determinantal varieties of skew symmetric and symmetric matrices*, preprint (2015), `https://arxiv.org/abs/1511.07290`.

[ŠV17] Š. Špenko, M. Van den Bergh, *Non-commutative crepant resolutions for some toric singularities I*, preprint (2017), `https://arxiv.org/abs/1701.05255`.

[Sta] The Stacks Project authors, *Stacks project*, available from `https://stacks.math.columbia.edu`.

[Sze04] B. Szendröi, *Artin group actions on derived categories of threefolds*, J. Reine Angew. Math. **572** (2004), 139–166.

[Tod07] Y. Toda, *On certain generalization of spherical twists*, Bull. Soc. Math. France. **135**(1) (2007), 119–134.

[Tod09] Y. Toda, *Deformations and Fourier-Mukai transforms*, J. Differential Geom., **81**(1) (2009), 197–224.

[TU10] Y. Toda, H. Uehara, *Tilting generators via ample line bundles*, Adv. Math., **223**(1) (2010), 1–29.

[Ued16] K. Ueda, *G_2-Grassmannians and derived equivalences*, to appear in Manuscripta Mathematica, `https://arxiv.org/abs/1612.08443`.

[VdB04a] M. Van den Bergh, *Three-dimensional flops and noncommutative rings*, Duke Math. J., **122**(3) (2004), 423–455.

[VdB04b] M. Van den Bergh, *Non-commutative crepant resolutions*, The legacy of Niels Henrik Abel, 749–770, Springer, Berlin, 2004.

[Wem17] M. Wemyss, *Flops and Clusters in the Homological Minimal Model Program*, to appear Invent. Math, `https://arxiv.org/abs/1411.7189`.

[WZ12] J. Weyman, G. Zhao, *Noncommutative desingularization of orbit closures for some representations of* GL_n, preprint (2012), `https://arxiv.org/abs/1204.0488`.

We hereby admit that above thesis is worth enough to present as doctoral dissertation.

Referees:

Principal Referee

Yasunari Nagai, Doctor of
Professor, Waseda University

Referee

Hajime Kaji, Doctor of
Professor, Waseda University

Referee

Osamu Iyama, Doctor of
Professor, Nagoya University

早稲田大学　博士（理学）　学位申請　研究業績書

氏 名　原　和平　　　　　　印

（2018 年 11 月 8 日現在）

種 類 別	題名、　発表・発行掲載誌名、　発表・発行年月、　連名者（申請者含む）
出版論文	1.W. Hara, Non-commutative crepant resolution of minimal nilpotent orbit closures of type A and Mukai flops, Adv. Math., 318 (2017), 355–410.
	2.W. Hara, Strong full exceptional collections on certain toric varieties with Picard number three via mutations, Le Matematiche, Vol. LXXII(2) (2017), 3–24
	3.W. Hara, Deformation of tilting-type derived equivalences for crepant resolutions, to appear in IMRN（掲載決定）
報告集	4.W. Hara, Mukai flop from the point of view of noncommutative crepant resolutions, Intersections between Algebraic Geometry and Noncommutatieve algebra, Mathematisches Forschungsinstintut Oberwolfach, Report no. 24/2018, DOI: 10.4171/OWR/2018/24
	5.原和平, 高次元単純フロップと非可換クレパント解消, 城崎代数幾何学シンポジウム報告集2018.
口頭発表	6.「Dimension of derived categories of singular varieties and its singularity categories」埼玉代数幾何ミニ研究集会, 埼玉大学, 2016 年 3 月 10, 11 日.
	7.「Dimension of derived categories of varieties with rational singularity and its singularity categories」 特異点論セミナー, 日本大学, 2016 年 4 月 25 日.
	8.「Dimension of derived categories of varieties with rational singularity and its sin- gularity categories」 Triangulated Categories by the young researchers,首都大学東京, 2016 年 6 月 22, 23 日(英語発表)
	9.「Rouquier Dimension and Orlov Spectrum of Singular Varieties」日本数学会・2016 年度 秋総合分科会(一般講演), 関西大学, 2016 年 9 月 15–18 日.
	10.　「高々有理特異点のみを持つ代数多様体のルキエ次元について」グレブナー基底の展望 RIMS 隔週セミナー, 京都大学数理

	解析研究所， 2016 年 10 月 26 日 11. 「Rouquier Dimension and Orlov Spectrum of Singular Varieties」都の西北代数幾何学シンポジウム，早稲田大学, 2016 年 11 月 15–18 日 12. 「Categorical relation between lines and conics on cubic threefolds」特異点論セミナー，日本大学, 2016 年 12 月 26 日. 13. 「Rouquier Dimension and Orlov Spectrum of Singular Varieties」第 22 回大和郡山セミ ナー, 奈良高専, 2017 年 3 月 10–11 日 「Non-commutative crepant resolution of minimal nilpotent orbit closures of type A and Mukai flops」 Antwerp algebra seminar, University of Antwerp， 2017 年 6 月 2 日(英語発表) 14. 「Mukai flops and P-twists via non-commutative crepant resolutions」Matrix factorizations and related topics, International Center for Materials Science(Edinburgh Universiry), 2017 年 7 月 24–28 日 (英語発表) 15. 「Deformation of tilting-type derived equivalences for crepant resolutions」 KCL/UCL Geometry Seminar, King's College London, 2017 年 10 月 18 日(英語発表) 16. 「Mukai flops and P-twists via non-commutative crepant resolutions」Research Program "Symplectic geometry and Representation theory" Trimester Seminar, Hausdorff institute of Mathematics, 2017 年 12 月 12 日(英語発表) 17. 「Mukai flops and P-twists via non-commutative crepant resolutions」SFB/TRR45 Kolloquium, Johannes Gutenberg-Universitat Mainz, 2017 年 12 月 14 日(英語発表) 18. 「Mukai flops and P-twists from non-commutative point of view」Intersections between Algebraic geometry and non-commutative algebras, Mathematisches Forschungsinstitut Oberwolfach, 2018 年 5 月 30 日(英語発表) 19. 「Mukai flops and P-twists from non-commutative point of view」城崎代数幾何学シンポジウム， 城崎国際アートセンター, 2018 年 10 月 24 日 （英語発表）

	20.　「Stratified Mukai flops and tilting bundles」京都大学代数幾何学セミナー，京都大学, 2018年 11月 2日 21.　「On derived equivalences for Abuaf's 5-dimensional flop」McKay correspondence and noncommutative algebra，名古屋大学, 2018年 11月16－17日（発表予定）
ポスター発表	22.　「Strong full exceptional collections on certain toric varieties with Picard number three via mutations」城崎代数幾何学シンポジウム，城崎国際アートセンター,2015 年 10 月 20–23 日. 23.　「Rouquier Dimension and Orlov Spectrum of Singular Varieties」Non-commutative crepant resolutions, Ulrich modules and generalizations of the McKay correspondence，京都大学数 理解析研究所, 2016 年 6 月 13–17 日.
論文紹介	24.　「"Arend Bayer, Tom Bridgeland, Derived automorphism groups of K3 surfaces of Picard rank 1, arXiv:1310.8266" の紹介」サマースクール "Symplectic Geometry and Bridgeland stability"，北海道大学, 2015 年 8 月 24–28 日
研究プログラム参加	25.　Research Program "Symplectic geometry and Representation theory" Trimester Seminar, Haus- dorff institute of Mathematics, 2017 年 12 月 6–13 日.