Study on a class of finitely additive measures by the method of topological dynamics

ある有限加法的測度の族に関する位相力学系の手法による研究

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Chapter 1

Introduction

Let X be a set, $\mathcal{F}$ be an algebra of subsets of X. A set function $\mu : \mathcal{F} \to [-\infty, \infty]$ is called a *finitely additive measure* or *charge* if it satisfies the following conditions.

(1) $\mu(\emptyset) = 0$,

(2) $\mu(A \cup B) = \mu(A) + \mu(B)$ for every $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$.

Namely, it is a generalization of the ordinary notion of measure by replacing countable additivity with finitely additivity. The triple $(X, \mathcal{F}, \mu)$ is called a finitely additive measure space or charge space. In what follows, we will use the term ‘charge’ exclusively for the sake of simplicity. A charge μ is called *bounded* if $\sup_{A \in \mathcal{F}} |\mu(A)| < \infty$ and called *nonnegative* if $\mu(A) \geq 0$ for every $A \in \mathcal{F}$. If μ is positive and $\mu(X) = 1$ holds then μ is called a *probability charge*.

Charges arise quite naturally in many areas of mathematics and there exist a large number of studies over the past decades. In particular, based on researches which had been done before, K. P. S Bhaskara Rao and M. Bhaskara Rao developed a theory of charge spaces systematically in [3], in which various generalizations of notions and results in measure theory to charges are obtained. For example, the notion of measurable functions for charge spaces and their integrability, the construction of L^p spaces over charges and Hölder's inequality for them and a generalization of Lebesgue's dominated convergence theorem. In this thesis, we discuss a certain class of charges on the basis of this theory and it can be viewed as an application of the theory of charges. In particular, our main objectives are the notions of absolutely continuity and singularity (Chapter 4) and the additive property (Chapter 5).

We denote the set of natural numbers by $\mathbb{N}$, and the family of all subsets of $\mathbb{N}$ by $\mathcal{P}(\mathbb{N})$. For a set $A \in \mathcal{P}(\mathbb{N})$, $|A|$ stands for the cardinality of A. In particular, we use the symbol $|A \cap n| = |A \cap [1, n]|$ for each $n \in \mathbb{N}$. Recall that the *asymptotic density* $d(A)$ of a set $A \in \mathcal{P}(\mathbb{N})$ is defined as

$$d(A) = \lim_{n} \frac{|A \cap n|}{n}$$

if this limit exists. The asymptotic density d is obviously finitely additive on the class $\mathcal{D}$ of all subsets of $\mathbb{N}$ having the asymptotic density. This notion is one of the simplest examples of finitely additive set functions on a countable space and of particular importance for number theory.

Since $\mathcal{D}$ itself is not an algebra, we get a charge space by restricting d to some subclass of $\mathcal{D}$ which forms an algebra of subsets of $\mathbb{N}$. One of such examples is the class $\mathcal{A}$ generated by all arithmetic progressions $A = \{an + b : n \geq 0\}$, where a, b are nonnegative integers. The charge space $(\mathbb{N}, \mathcal{A}, d)$ is particularly important for probabilistic number theory, which is based on the analogy between $(\mathbb{N}, \mathcal{A}, d)$ and a probability space (as discussed in [9, 10]). Another way of constructing a charge space from the asymptotic density is extending d to some algebra of subsets of $\mathbb{N}$ containing the class $\mathcal{D}$. This leads to the notion of density measures.

A charge defined on $\mathcal{P}(\mathbb{N})$ extending the asymptotic density is called a *density measure*. Density measures have been studied by several authors from various points of view (see for instance [4, 5, 11, 12, 15, 17]). Our main interest is the density measures constructed from ultrafilters on $\mathbb{N}$. Recall that for a bounded function f on $\mathbb{N}$ and an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the *limit of* f *along* $\mathcal{U}$ is a number α such that $\{n \in \mathbb{N} : |f(n) - \alpha| < \varepsilon\} \in \mathcal{U}$ holds for every $\varepsilon > 0$ and denoted by $\mathcal{U}\text{-}\lim_n f(n)$ (see [5] for details). In particular, we say that an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is free if $\cap_{A \in \mathcal{U}} A \neq \emptyset$ holds. Then for a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we define the density measure $\nu^{\mathcal{U}}$ by

$$\nu^{\mathcal{U}}(A) = \mathcal{U}\text{-}\lim_{n} \frac{|A \cap n|}{n}, \quad A \in \mathcal{P}(\mathbb{N}).$$

We denote the set of all such density measures by $\tilde{\mathcal{C}}$. Remark that there are distinct free ultrafilters $\mathcal{U}$ and $\mathcal{U}'$ which give the same element of $\tilde{\mathcal{C}}$, thus $\tilde{\mathcal{C}}$ is isomorphic as a set to some quotient space of the set of free ultrafilters on $\mathbb{N}$. We show that each density measure in $\tilde{\mathcal{C}}$ is equal to some $\nu^{\mathcal{U}}$ for a certain kind of ultrafilter $\mathcal{U}$, which has a form convenient to investigate the associated density measure (Theorem 3.2.1).

Sometimes it is convenient to consider density measures as linear functionals on l^∞ of the Banach space of all real-valued bounded functions on $\mathbb{N}$. In general, the probability charges on $\mathcal{P}(\mathbb{N})$ and the normalized positive linear functionals on l^∞ can be identified in a natural way; namely, if a probability charge μ on $\mathcal{P}(\mathbb{N})$ is given, one can obtain a normalized positive linear functional φ on l^∞ by the integral with respect to μ. Conversely, if a normalized positive linear functional φ on l^∞ is given, we get a probability charge μ on $\mathcal{P}(\mathbb{N})$ by $\mu(A) = \varphi(I_A)$, where I_A is the characteristic function of $A \in \mathcal{P}(\mathbb{N})$. In this way, $\tilde{\mathcal{C}}$ can be identified with a subset of $(l^\infty)^*$ of the conjugate space of l^∞ and thus we can consider $\tilde{\mathcal{C}}$ to be a topological space endowed with the relative topology of the weak* topology of $(l^\infty)^*$. From now on we consider such a topology on $\tilde{\mathcal{C}}$.

Observe that $\tilde{\mathcal{C}}$ is a compact set and it is clear that for any Borel probability measure

μ on $\tilde{\mathcal{C}}$, the charge ν on $\mathcal{P}(\mathbb{N})$ defined as follows is also a density measure:

$$\nu(A) = \int_{\tilde{\mathcal{C}}} \nu^{\mathcal{U}}(A) d\mu, \quad A \in \mathcal{P}(\mathbb{N}).$$

We denote the set of all such density measures by $\mathcal{C}$. The relation between $\tilde{\mathcal{C}}$ and $\mathcal{C}$ can be understood simply in view of the theory of linear topological spaces, that is, it is shown that $\tilde{\mathcal{C}}$ is precisely the set of extreme points $ex(\mathcal{C})$ of $\mathcal{C}$. Also we show that each element of $\mathcal{C}$ can be expressed as an integral with respect to some unique probability measure supported by its extreme points. It seems to be an interesting example on Choquet's theory.

The thesis is organized as follows: Chapter 2 deals with preliminary results and notions which will be used throughout the paper. In Chapter 3, we investigate the space $\tilde{\mathcal{C}}$ and $\mathcal{C}$ in detail. We show in Section 3.1 that the space $\tilde{\mathcal{C}}$ is homeomorphic to a certain compact space Ω^* on which a continuous flow $\tau^s : \Omega^* \to \Omega^*,\ s \in \mathbb{R}$ is defined in a natural way. This flow $(\Omega^*, \{\tau^s\}_{s\in\mathbb{R}})$, which is defined in the following chapter, plays a very important role in studying density measures in $\tilde{\mathcal{C}}$ throughout the paper. In Section 3.2, we show the result that $\tilde{\mathcal{C}} = ex(\mathcal{C})$. Although the fact that $ex(\mathcal{C}) \subseteq \tilde{\mathcal{C}}$ followed from the Krein-Milman theorem with relative ease, it is rather difficult to prove that $\tilde{\mathcal{C}}$ is exactly $ex(\mathcal{C})$ and we have to prepare some amount of machinery. After that we show the representation theorem for general elements of $\mathcal{C}$.

In Chapter 4, we deal with absolute continuity and singularity. Such notions are well known for measures, we can define those notions for charges. In Section 4.1 we give definitions of absolutely continuity and singularity for charges in general setting. The relation between the existing notions of absolute continuity and singularity for measures and those of charges are also discussed.

In Section 4.2, we study absolute continuity and singularity for density measures in $\tilde{\mathcal{C}}$. For a given pair μ, ν in $\tilde{\mathcal{C}}$, we can regard them as elements ω, ω' in Ω^* through the above homeomorphism and give complete descriptions on absolute continuity and weak absolute continuity of the pair in terms of the continuous flow $(\Omega^*, \{\tau^s\}_{s\in\mathbb{R}})$ (Theorems 4.2.1 and 4.2.4). Also we give characterizations of singularity and strongly singularity of the pair by means of the continuous flow as well (Theorems 4.2.3 and 4.2.5).

In Chapter 5, we deal with the property of charges which is concerned with a weakening of countable additivity. One of the problems of developing the theory of charges is that some of the main theorems in measure theory, including the completeness of L^p-spaces and the Radon-Nikodym theorem, do not hold. This fact leads us to study the condition of charges under which these theorems hold. This condition is known as the additive property, whose definition is as follows: Let $(X, \mathcal{F}, \mu)$ be a charge space where μ is nonnegative and $\mathcal{F}$ is a σ-algebra. We say that μ has the *additive property* if for any increasing sequence $\{A_i\}_{i=1}^{\infty}$ of $\mathcal{F}$, there exists a set $B \in \mathcal{F}$ such that

(1) $\mu(B) = \lim_{i\to\infty} \mu(A_i)$,

(2) $\mu(A_i \setminus B) = 0$ for every $i = 1, 2, \cdots$.

This notion is studied systematically in [1]. In first three sections of Chapter 5, we discuss the general theory of the additive property. In Section 5.1 we introduce several conditions which are equivalent to the additive property. In Section 5.2 we show the condition of finite number of charges under which the sum of these charges has the additive property. This result is extended to the case of countable sums of charges in Section 5.3, which is used to prove the main result of the following section. In Section 5.4 we study the additive property of density measures in $\tilde{\mathcal{C}}$. It was shown in [4, Theorem 1] that there exists a density measure with the additive property; for a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ containing a set $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty,$$

the density measure $\nu^{\mathcal{U}}$ has the additive property. We shall generalize the result and prove a necessary and sufficient condition for density measures in $\tilde{\mathcal{C}}$ to have the additive property. We will also deal with the additive property of a more general form of density measures $\nu \in \mathcal{C}$ for a certain class of Borel probability measures μ on $\tilde{\mathcal{C}} = ex(\mathcal{C})$ in Section 5.5.

Chapter 2

Preliminaries

We consider $\tilde{\mathcal{C}}$ as a topological space endowed with the relative topology of the weak* topology of $(l^\infty)^*$. From this point of view, it is convenient to use the notion of the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. The Stone-Čech compactification of $\mathbb{N}$ is a compactification of $\mathbb{N}$ characterized by the following property: any continuous mapping of $\mathbb{N}$ into a compact space X can be extended continuously to $\beta\mathbb{N}$. Also it is noted that $\beta\mathbb{N}$ is unique in the following sense: If a compactification $\mathbb{N}_\infty$ of $\mathbb{N}$ satisfies the above condition, then there exists a homeomorphism of $\beta\mathbb{N}$ onto $\mathbb{N}_\infty$ that leaves $\mathbb{N}$ pointwise fixed.

Remark that $\beta\mathbb{N}$ can be identified with the set of all ultrafilters on $\mathbb{N}$ in which the topology is given by defining a basis of open sets by $\hat{A} = \{\mathcal{U} : A \in \mathcal{U}\}$, where $A \in \mathcal{P}(\mathbb{N})$. Recall that for any set $A \in \mathcal{P}(\mathbb{N})$, $\hat{A} = cl_{\beta\mathbb{N}}A$ holds and these subsets are exactly the clopen subsets of $\beta\mathbb{N}$. In particular, let us denote by $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ the set of all free ultrafilters on $\mathbb{N}$. Then the sets of the form $A^* = \hat{A} \cap \mathbb{N}^*$ gives the clopen subsets of $\mathbb{N}^*$ and these also form a topological basis of $\mathbb{N}^*$. In the sequel we identify a point of $\beta\mathbb{N}$ with a ultrafilter on $\mathbb{N}$. From this point of view, for any mapping $\iota : \mathbb{N} \to X$ of $\mathbb{N}$ into a compact space X, the continuous extension $\bar{\iota}$ of ι is given by the limit along an ultrafilter: for any $\mathcal{U} \in \beta\mathbb{N}$, one can define the limit of ι along $\mathcal{U}$ as a point $x \in X$ such that for every neighborhood U of x, it holds that $\{n \in \mathbb{N} : \iota(n) \in U\} \in \mathcal{U}$. In this case, we write $\mathcal{U}\text{-}\lim_n \iota(n) = x$ and then we have

$$\bar{\iota}(\mathcal{U}) = \mathcal{U}\text{-}\lim_n \iota(n).$$

Of particular importance is the case that ι is in l^∞. Notice that for any $f \in l^\infty$, by the above mentioned property of $\beta\mathbb{N}$, we can extend f to a continuous function $\overline{f}$ on $\beta\mathbb{N}$. This correspondence $f \mapsto \overline{f}$ gives an isomorphism between the Banach algebras l^∞ and $C(\beta\mathbb{N})$, the space of all real-valued continuous functions on $\beta\mathbb{N}$. In particular, this leads to the fact that $\beta\mathbb{N}$ is homeomorphic to the maximal ideal space of l^∞. As mentioned above, for each $f \in l^\infty$, the isomorphic image $\overline{f}$ in $C(\beta\mathbb{N})$ is given by the formula:

$$\overline{f}(\mathcal{U}) = \mathcal{U}\text{-}\lim_n f(n)$$

for every ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

Another notion pertaining to $\mathbb{N}^*$ which is important for our study is an extension of right translation on $\mathbb{N}$. We define a mapping $\tau_0 : \mathbb{N} \to \mathbb{N}$ by $\tau_0(n) = n + 1$. Regarding it as a mapping from $\mathbb{N}$ to $\beta\mathbb{N}$, we can extend it to a continuous mapping on $\beta\mathbb{N}$. We denote this extension by τ. The restriction of τ to $\mathbb{N}^*$ is a homeomorphism of $\mathbb{N}^*$ onto itself and we denote it by the same symbol τ as well. Then $(\mathbb{N}^*, \tau)$ is a topological dynamical system.

We use the following notations for the orbits of η in $\mathbb{N}^*$ under the action of $\{\tau^n\}_{n=-\infty}^{\infty}$:

$$o_+(\eta) = \{\tau^n \eta : n = 0, 1, 2, \ldots .\},$$
$$o_-(\eta) = \{\tau^{-n} \eta : n = 0, 1, 2, \ldots .\},$$
$$o(\eta) = \{\tau^n \eta : n \in \mathbb{Z}\}.$$

Furthermore we denote the closures in $\mathbb{N}^*$ of these orbits by $\bar{o}_+(\eta), \bar{o}_-(\eta)$ and $\bar{o}(\eta)$, respectively.

Recall that a point $\eta \in \mathbb{N}^*$ is called *wandering* if there is an open neighborhood U of η such that the sets $\tau^n U$, n is any integers, are mutually disjoint. Let $\mathcal{W}_d$ be the set of all wandering points.

We denote by $\mathcal{D}_{d,-}$ the subset of $\mathbb{N}^*$ consisting of all points that does not return arbitrarily close to the initial point under negative iteration by τ (i.e., $\eta \in \mathbb{N}^*$ is in $\mathcal{D}_d$ if and only if there exists an open neighborhood U of η such that $U \cap \{\tau^{-n}\eta : n \geq 1\} = \emptyset$). This is equivalent to the condition that the orbit $\{\tau^{-n}\eta : n \geq 0\}$ is a discrete space in its relative topology. $\mathcal{W}_d \subseteq \mathcal{D}_{d,-}$ is clear by the definitions.

We denote by $\mathcal{A}_d$ the set of all almost periodic points for the topological dynamical system $(\mathbb{N}^*, \tau)$. Namely, the set of those points whose orbit closures are minimal closed invariant sets.

Further, we consider the continuous flow $(\Omega^*, \{\tau^s\}_{s\in\mathbb{R}})$ of the suspension of the discrete flow $(\mathbb{N}^*, \tau)$, whose construction is well known in topological dynamics (for example see [18, Chapter 2]) and is given as follows. Let us consider a product space $\beta\mathbb{N} \times [0, 1]$ and construct the compact space Ω by identifying all the pairs of points $(\eta, 1)$ and $(\tau\eta, 0)$ for all $\eta \in \beta\mathbb{N}$. Also we denote by Ω^* the closed subspace of Ω consisting of all elements (η, t) in Ω with $\eta \in \mathbb{N}^*$. Then we define a continuous flow on Ω^* extending $(\mathbb{N}^*, \tau)$ as follows; for each $s \in \mathbb{R}$, we define the homeomorphism $\tau^s : \Omega^* \to \Omega^*$ by

$$\tau^s(\eta, t) = (\tau^{[t+s]}\eta, t + s - [t + s]),$$

where $[x]$ denotes the largest integer not exceeding x for a real number x.

Also we use similar notations for the orbits of ω in Ω^* under the action of $\{\tau^s\}_{s\in\mathbb{R}}$:

$$o_+(\omega) = \{\tau^s \omega : s \geq 0\},$$

$$o_-(\omega) = \{\tau^{-s}\omega : s \geq 0\},$$
$$o(\omega) = \{\tau^s\omega : s \in \mathbb{R}\},$$

and also $\overline{o}_+(\omega), \overline{o}_-(\omega)$ and $\overline{o}(\omega)$ represent their closures in Ω^*, respectively.

For a point $\omega \in \Omega^*$, ω is called *wandering* if there are open neighborhood U of ω and V of $0 \in \mathbb{R}$ such that $U \cap \tau^s U = \emptyset$ for every s in $\mathbb{R} \setminus V$. We denote the set of all wandering points by $\mathcal{W}$.

We denote by $\mathcal{D}$ all the points ω in Ω^* whose negative semi-orbit $o_-(\omega)$ is not recurrent. This means that there exist a neighborhood U of ω and a real number $L > 0$ such that $\tau^{-s}\omega$ does not enter U for every $s > L$. Note that this is equivalent to the condition that the orbit $\{\tau^{-s}\omega : s \geq 0\}$ is homeomorphic to $\mathbb{R}_+$ in its relative topology. In particular $\mathcal{W} \subseteq \mathcal{D}_-$ is obvious.

We denote by $\mathcal{A}$ the set of all almost periodic points in the flow $(\Omega^*, \{\tau^s\}_{s\in\mathbb{R}})$. Recall that we say that a point ω in Ω^* is almost periodic if the orbit closure $\overline{o}(\omega)$ is a minimal closed invariant set.

It is easy by the definitions to check that $\omega = (\eta, t) \in \mathcal{W}$ if and only if $\eta \in \mathcal{W}_d$, $\omega = (\eta, t) \in \mathcal{D}_-$ if and only if $\eta \in \mathcal{D}_{d,-}$ and $\omega = (\eta, t) \in \mathcal{A}$ if and only if $\eta \in \mathcal{A}_d$.

Chapter 3

Basic properties of the space $\mathcal{C}$

3.1 Density measures and functionals by Cesàro mean

Recall that the Cesàro mean of a function $f \in l^\infty$ is defined by

$$C(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(i)$$

if this limit exists. When f is the characteristic function I_A of a set $A \in \mathcal{P}(\mathbb{N})$, its Cesàro mean $C(I_A) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} I_A(i)$ coincides with the asymptotic density $d(A)$ of A. The relation between asymptotic density and Cesàro mean is analogous to that of measure and integral. As we have mentioned above, to each charge on $\mathcal{P}(\mathbb{N})$, there corresponds a normalized positive linear functional on l^∞. One can readily verify that the class of normalized positive linear functionals on l^∞ which corresponds to $\mathcal{C}$ is the linear functionals φ on l^∞ satisfying the following condition:

$$\varphi(f) \leq \overline{C}(f) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(i)$$

for each $f \in l^\infty$. We denote such functionals by the same symbol $\mathcal{C}$ as corresponding charges. It is remarked that such a functional φ is an extension of Cesàro mean, that is, $\varphi(f) = C(f)$ provided the limit exists. Since $\mathcal{C}$ is a compact convex set in the weak* topology, the Krein-Milman theorem shows that the set of extreme points $ex(\mathcal{C})$ of $\mathcal{C}$ is not an empty set. An example of such a functional is given by

$$\varphi^{\mathcal{U}}(f) = \mathcal{U}\text{-}\lim_{n} \frac{1}{n} \sum_{i=1}^{n} f(i),$$

where $f \in l^\infty$, which is obviously obtained by integration with respect to $\nu^{\mathcal{U}} \in \tilde{\mathcal{C}}$:

$$\varphi^{\mathcal{U}}(f) = \int_{\mathbb{N}} f(n) d\nu^{\mathcal{U}}(n).$$

Now let us consider the relation between $\mathcal{C}$ and the class of density measures. It is shown in [10] that functionals corresponding to density measures are precisely the positive functionals extending Cesàro mean. Let $\mathcal{P}$ be the set of all such functionals. Then $\mathcal{P}$ is a weak* compact convex subset of $(l^\infty)^*$ and the following result is known [11, Proposition 5.5]:

$$\overline{P}(f) = \sup_{\varphi\in\mathcal{P}} \varphi(f) = \lim_{\theta\to 1-} \limsup_{n\to\infty} \frac{\sum_{i\in[\theta n,n]} f(i)}{n-\theta n}$$

for each $f \in l^\infty$. This functional $\overline{P}$ is an extension of Pólya density for bounded sequences.

Since $\overline{C}(f) \leq \overline{P}(f)$ for every $f \in l^\infty$, we see that $\mathcal{C} \subseteq \mathcal{P}$. And it is known that there exists an element f of l^∞ such that $\overline{C}(f) < \overline{P}(f)$ (for example, see [6, P. 572]), so we have that $\mathcal{C} \subsetneqq \mathcal{P}$.

3.2 Topological structure on the space $\tilde{\mathcal{C}}$

In this section we will investigate details of the compact Hausdorff space $\tilde{\mathcal{C}}$. The main purpose of this section is to prove the following result, which was suggested by arguments in the proof of [5, Lemma 5]. In what follows, we denote a general element of $\beta\mathbb{N}$ by η and those of Ω by ω.

Theorem 3.2.1. *Each element of $\tilde{\mathcal{C}}$ can be expressed uniquely in the form*

$$\varphi_\omega(f) = \eta\text{-}\lim_n \frac{1}{\theta\cdot 2^n} \sum_{i=1}^{[\theta\cdot 2^n]} f(i)$$

for some $\omega = (\eta, t)$ in Ω^*, where $\theta = 2^t$. Also this correspondence of Ω^* to $\tilde{\mathcal{C}}$ is continuous, that is, $\tilde{\mathcal{C}}$ is homeomorphic to Ω^*.

This result plays an important role in proving our theorems in Chapters 4 and 5 and is interesting in its own right. It is helpful to introduce the notion of the image of an ultrafilter to understand the above limit. Let X and Y be arbitrary sets, and let $f : X \to Y$. For any ultrafilter $\mathcal{U}$ on X, one can define the ultrafilter on Y, denoted by $f(\mathcal{U})$ consisting of those $A \subseteq Y$ for which $f^{-1}(A) \in \mathcal{U}$. Then it is easy to see that

$$f(\mathcal{U})\text{-}\lim_y g(y) = \mathcal{U}\text{-}\lim_x g\circ f(x),$$

where g is any bounded function on Y.

Let us $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_+^\times = [1,\infty)$. We particularly consider the following three maps:

$$\mathbb{R}_+ \ni x \mapsto 2^x \in \mathbb{R}_+^\times,$$

$$\mathbb{R}_+^\times \ni x \mapsto [x] \in \mathbb{N},$$
$$\mathbb{R}_+^\times \ni x \mapsto \theta x \in \mathbb{R}_+^\times \ (\theta \geq 1).$$

The images of an ultrafilter $\mathcal{U}$ under the induced mappings defined above are denoted by $2^{\mathcal{U}}, [\mathcal{U}], \theta\mathcal{U}$, respectively. Notice that $2^{\mathcal{U}}$ is a ultrafilter on $\mathbb{R}_+^\times$ which does not contain any bounded set of $\mathbb{R}_+^\times$ if and only if $\mathcal{U}$ is a ultrafilter on $\mathbb{R}_+$ of the same kind, and those can be considered to be equal, then the map $\mathcal{U} \to 2^{\mathcal{U}}$ is a bijection of the set of all such ultrafilters on $\mathbb{R}_+^\times$ onto itself. Notice that with the notation above we can write $\varphi_\omega = \varphi^{[2^\omega]} = \varphi^{[\theta 2^\eta]}$.

We will need some more preparation to prove the theorem. Let $C_{ub}(\mathbb{R}_+^\times)$ be the space of all real-valued uniformly continuous bounded functions on $\mathbb{R}_+^\times$. Its maximal ideal space, denoted here by $\mathfrak{M}$, is a compact Hausdorff space and the space $C(\mathfrak{M})$ of all real-valued continuous functions on $\mathfrak{M}$ is isometric to $C_{ub}(\mathbb{R}_+^\times)$ as a Banach algebra. The following lemma is a consequence of [16, Lemma 2.1], but provides a proof, for the sake of completeness:

Lemma 3.2.1. *$\mathfrak{M}$ is homeomorphic to Ω.*

Proof. It is sufficient to show the algebraic isomorphism $C_{ub}(\mathbb{R}_+^\times) \cong C(\Omega)$. If we regard the points (n,t) in Ω with $n \in \mathbb{N}$ as the points $n+t$ in $\mathbb{R}_+^\times$ we can consider that Ω contains $\mathbb{R}_+^\times$ as a dense subspace, so that Ω is a compactification of $\mathbb{R}_+^\times$. Now given any $f \in C_{ub}(\mathbb{R}_+^\times)$, put $f_n(s) = f(n+s),\ s \in [0,1], n = 1,2,\cdots$. Then we have a sequence $\{f_n\}_{n=1}^\infty$ of $C([0,1])$. Since f is bounded and uniformly continuous on $\mathbb{R}_+^\times$, it follows that this sequence is uniformly bounded and equicontinuous. Hence by Arzelà-Ascoli's theorem, $\{f_n\}_{n=1}^\infty$ is relatively compact in $C([0,1])$ in its uniform topology. Therefore when we put

$$\Phi_f : \mathbb{N} \longrightarrow C([0,1]),\ \Phi_f(n) = f_n, n = 1,2,\cdots,$$

then we can extend it continuously to $\beta\mathbb{N}$. Then we define a continuous function $\overline{f}$ on Ω by

$$\overline{f}(\omega) = (\Phi_f(\eta))(t), \quad \omega = (\eta, t).$$

We denote this mapping $f \mapsto \overline{f}$ by $\Phi : C_{ub}(\mathbb{R}_+^\times) \to C(\Omega)$. Notice that $f = \overline{f}$ on $\mathbb{R}_+^\times$, so that $\overline{f}$ is a continuous extension of f to Ω. In particular, it is obvious that Φ is injective. We shall show that Φ is a algebraic isomorphism. It is trivial that Φ is a algebraic homomorphism. To show that Φ is surjective, it is sufficient to show that for every continuous function g on Ω its restriction to $\mathbb{R}_+^\times$ is uniformly continuous on $\mathbb{R}_+^\times$. Now we regard g as a mapping from $\beta\mathbb{N}$ to $C([0,1])$ with uniform topology:

$$\Phi_g : \beta\mathbb{N} \longrightarrow C([0,1]), \quad \Phi_g(\omega) = g(\omega, t),$$

then Φ_g is continuous. Since $\Phi_g(\beta\mathbb{N})$ is a compact subset of $C([0,1])$, $\Phi_g(\mathbb{N})$ is relatively compact in $C([0,1])$. Hence $\{\Phi_g(n)\}_{n=1}^\infty = \{g(n+t)\}_{n=1}^\infty$ is equicontinuous. Thus g is uniformly continuous on $\mathbb{R}_+^\times$.

Thus we can identify $\mathfrak{M}$ with Ω, so that in the sequel we will use only the symbol Ω. Notice that Ω is the compactification of $\mathbb{R}_+^\times$ to which any uniformly continuous bounded function $f(x)$ on $\mathbb{R}_+^\times$ can be extended continuously. In particular, we can see from the above proof that, for any $f \in C_{ub}(\mathbb{R}_+^\times)$ and $\omega = (\eta, t) \in \Omega$, its continuous extension $\overline{f}(\omega)$ is given by the formula

$$\overline{f}(\omega) = \omega\text{-}\lim_s f(s),$$

where ω is regarded as an ultrafilter on $\mathbb{R}_+^\times$ generated by the basis $\{A + t : A \in \eta\}$. From now on, we often identify a point $\omega = (\eta, t) \in \Omega$ with the above ultrafilter. An immediate consequence of these facts which will be used in the next section is that for any cluster point α of the set $\{f(x)\}_{x \in \mathbb{R}_+^\times}$, there exists a point $\omega \in \Omega$ such that $\overline{f}(\omega) = \alpha$. Since we are mainly interested in the extended values of $f(x) \in C_{ub}(\mathbb{R}_+^\times)$, that is, cluster points of $\{f(x)\}_{x \geq 1}$ as $x \to \infty$, we may often ignore the difference in values on bounded sets of $\mathbb{R}_+^\times$ among members in $C_{ub}(\mathbb{R}_+^\times)$; namely, we consider a member of $C_{ub}(\mathbb{R}_+^\times)$ modulo $C_0(\mathbb{R}_+^\times)$, where $C_0(\mathbb{R}_+^\times)$ is the ideal of $C_{ub}(\mathbb{R}_+^\times)$ consisting of all those members $f(x)$ which converges to zero as x tends to ∞. Then it holds that

$$C(\Omega^*) = C_{ub}(\mathbb{R}_+^\times)/C_0(\mathbb{R}_+^\times),$$

where $C(\Omega^*)$ is the space of all real-valued continuous functions on Ω^*.

$L^\infty(\mathbb{R}_+^\times)$ be the Banach space of all real-valued essentially bounded measurable functions on $\mathbb{R}_+^\times = [1, \infty)$. Now it is useful to introduce an integral analogy $\mathcal{M}$ of $\mathcal{C}$ which is a class of normalized positive linear functionals on $L^\infty(\mathbb{R}_+^\times)$ defined by using the sublinear functional $\overline{M}$ on $L^\infty(\mathbb{R}_+^\times)$ which adopts the integral with respect to the Haar measure of real line $\mathbb{R}$ in place of the summation: namely, $\mathcal{M}$ is the set of linear functionals ψ on $L^\infty(\mathbb{R}_+^\times)$ for which

$$\psi(f) \leq \overline{M}(f) = \limsup_{x \to \infty} \frac{1}{x} \int_1^x f(t)dt$$

holds for every $f \in L^\infty(\mathbb{R}_+^\times)$. Similarly we define a subclass $\tilde{\mathcal{M}}$ of $\mathcal{M}$ consisting of those $\psi^{\mathcal{U}}$ defined by

$$\psi^{\mathcal{U}}(f) = \mathcal{U}\text{-}\lim_x \frac{1}{x} \int_1^x f(t)dt,$$

where $f \in L^\infty(\mathbb{R}_+^\times)$ and again the limit means the limit along an ultrafilter $\mathcal{U}$ on $\mathbb{R}_+^\times$ which contains no bounded set of $\mathbb{R}_+^\times$. In fact, it turns out that $\mathcal{C}$ and $\mathcal{M}$ are isomorphic as a compact convex sets and that definitions and results obtained in the integral setting can be transferred to the summation setting with ease. Therefore in the rest of this chapter, where we study the structure of the convex set $\mathcal{C}$, we mainly work with the integral setting since arguments are simpler. In what follows, we shall show an affine homeomorphism between $\mathcal{C}$ and $\mathcal{M}$ and then introduce a version of

Theorem 3.2.1 which is formulated in the integral setting. For each $f \in l^\infty$, we define a function $\tilde{f} \in L^\infty(\mathbb{R}_+^\times)$ by $\tilde{f}(x) = f([x])$. Then we define an affine continuous mapping V as follows:

$$V : \mathcal{M} \longrightarrow \mathcal{C}, \quad (V\psi)(f) = \psi(\tilde{f}).$$

Theorem 3.2.2. *V is an affine homeomorphism between $\mathcal{C}$ and $\mathcal{M}$.*

Proof. First we show that V is surjective. It is noted that for each $f \in l^\infty$

$$\frac{1}{n}\sum_{i=1}^{n} f(i) = \frac{1}{n}\int_1^{n+1} \tilde{f}(t)dt.$$

Let $\tilde{l^\infty} = \{\tilde{f}(x) \in L^\infty(\mathbb{R}_+) : f \in l^\infty\}$. Given any $\varphi \in \mathcal{C}$, we define a functional ψ_0 on $\tilde{l^\infty}$ by $\psi_0(\tilde{f}) = \varphi(f)$ for every $f \in l^\infty$. Since

$$\psi_0(\tilde{f}) = \varphi(f) \leq \limsup_n \frac{1}{n}\sum_{i=1}^{n} f(i) = \limsup_x \frac{1}{x}\int_1^x \tilde{f}(t)dt$$

holds from above, we can extend ψ_0 to $\psi \in \mathcal{M}$ by the Hahn-Banach theorem. Then we have obviously that $V(\psi) = \varphi$, which shows that V is surjective. Next we show that V is injective. It is sufficient to show that for any $f \in L^\infty(\mathbb{R}_+^\times)$, there exists a function $g \in l^\infty$ such that $\psi(f) = \psi(\tilde{g})$ for every $\psi \in \mathcal{M}$. In fact, suppose that this holds and let ψ, ψ_1 be two distinct elements of $\tilde{\mathcal{M}}$ with $V\psi = V\psi_1$. Then there is some $f \in L^\infty(\mathbb{R}_+^\times)$ such that $\psi(f) \neq \psi_1(f)$. On the other hand, there exists some $g \in l^\infty$ such that $\psi(f) = \psi(\tilde{g}) = (V\psi)(g), \psi_1(f) = \psi_1(\tilde{g}) = (V\psi_1)(g)$, i.e., $\psi(f) = \psi_1(f)$, which is a contradiction. We can get such a function $g(n)$ simply by putting $g(n) = \int_n^{n+1} f(t)dt, n = 1, 2, \cdots$. Thus we obtain the desired result.

For any $\omega \in \Omega^*$ we define $\psi_\omega = \psi^{2^\omega}$, i.e.,

$$\psi_\omega(f) = 2^\omega\text{-}\lim_x \frac{1}{x}\int_1^x f(t)dt = \omega\text{-}\lim_x \frac{1}{2^x}\int_1^{2^x} f(t)dt.$$

We denote by Ψ this mapping of Ω^* to $\tilde{\mathcal{M}}$, $\omega \mapsto \psi_\omega$. The following lemma is obvious.

Lemma 3.2.2. *V maps $\tilde{\mathcal{M}}$ onto $\tilde{\mathcal{C}}$ and $V\psi_\omega = \varphi_\omega$ holds for every $\omega \in \Omega^*$.*

From this lemma, Theorem 3.2.1 is equivalent to the assertion that Ψ is a homeomorphism, which we will prove sequentially. For the sake of simplicity, we will use a linear operator $U : L^\infty(\mathbb{R}_+^\times) \longrightarrow L^\infty(\mathbb{R}_+^\times)$ defined as

$$Uf(x) = \frac{1}{x}\int_1^x f(t)dt, \quad x \geq 1.$$

Then we can write $\psi^{\mathcal{U}}(f) = \mathcal{U}\text{-}\lim_x (Uf)(x)$. Also let us define the linear operator W as follows:

$$W : L^\infty(\mathbb{R}_+^\times) \longrightarrow L^\infty(\mathbb{R}_+), \quad (Wf)(x) = f(2^x).$$

First, we will need the following elementary lemma.

Lemma 3.2.3. *If $f \in L^\infty(\mathbb{R}_+^\times)$, then $WUf \in C_{ub}(\mathbb{R}_+^\times)$.*

Proof. Let f be in $L^\infty(\mathbb{R}_+^\times)$, and let h be a positive real number, then we have

$$(Uf)(x+h) - (Uf)(x) = -\frac{h}{x+h}(Uf)(x) + \frac{1}{x+h}\int_x^{x+h} f(t)dt.$$

Hence we get that

$$|(Uf)(x+h) - (Uf)(x)| \leq \frac{2h\|f\|_\infty}{x+h}.$$

Let $s, \theta \in \mathbb{R}_+$ and put $x = 2^s,\ h = 2^{s+\theta} - 2^s$. Applying above results, we have

$$|(Uf)(2^{s+\theta}) - (Uf)(2^s)| \leq \frac{2 \cdot 2^s(2^\theta - 1)\|f\|_\infty}{2^{s+\theta}} = 2\|f\|_\infty \left(1 - \frac{1}{2^\theta}\right).$$

The right hand side of the equation tends to 0 monotonically as $\theta \to 0$, and that does not depend on s. Then $(WUf)(s)$ is uniformly continuous on $\mathbb{R}_+^\times$, so the proof is complete.

Notice that by the above result it can be written as follows:

$$\psi_\omega(f) = \omega\text{-}\lim_x (WUf)(x) = \overline{(WUf)}(\omega).$$

Lemma 3.2.4. Ψ *is continuous.*

Proof. Let $\{\omega_\alpha\}_{\alpha\in\Lambda}$ be a net in Ω^* which converges to ω. We will show that

$$\lim_\alpha \psi_{\omega_\alpha}(f) = \psi_\omega(f)$$

for every $f \in L^\infty(\mathbb{R}_+^\times)$. It follows from the assumption that for any $g \in C(\Omega^*)$

$$\lim_\alpha g(\omega_\alpha) = g(\omega).$$

Notice that $\overline{WUf}$ is in $C(\Omega^*)$ and then we have

$$\lim_\alpha \overline{WUf}(\omega_\alpha) = \overline{WUf}(\omega),$$

which implies that

$$\lim_\alpha \psi_{\omega_\alpha}(f) = \psi_\omega(f).$$

The proof is complete.

Lemma 3.2.5. Ψ *is surjective.*

Proof. We take any $\psi^{\mathcal{U}} \in \tilde{\mathcal{M}}$. Then we shall show that there exists a point $\omega = (\eta, t) \in \Omega^*$ such that $\psi_\omega = \psi^{\mathcal{U}}$. As we have mentioned before, since the mapping $\mathcal{U} \mapsto 2^{\mathcal{U}}$ is a bijection of the set of ultrafilters on $\mathbb{R}_+^\times$ not containing any bounded set of $\mathbb{R}_+^\times$ onto itself, we can get the inverse image $\mathcal{U}_0$ of $\mathcal{U}$, that is, $\mathcal{U} = 2^{\mathcal{U}_0}$. Then it follows that

$$\psi^{\mathcal{U}}(f) = \mathcal{U}\text{-}\lim_x (Uf)(x) = 2^{\mathcal{U}_0}\text{-}\lim_x (Uf)(x) = \mathcal{U}_0\text{-}\lim_x (WUf)(x)$$

for every $f \in L^\infty(\mathbb{R}_+^\times)$. Since $(WUf)(x) \in C_{ub}(\mathbb{R}_+^\times)$, $\mathcal{U}_0$ can be replaced by some $\omega = (\eta, t) \in \Omega^*$. Therefore we have that

$$\psi^{\mathcal{U}}(f) = \omega\text{-}\lim_x (WUf)(x) = 2^\omega\text{-}\lim_x (Uf)(x) = \psi_\omega(f).$$

The proof is complete.

Lemma 3.2.6. Ψ *is injective.*

Proof. It is sufficient to show that for any pair ω, ω' of distinct elements of Ω^*, there exists a set $X \in \mathcal{B}(\mathbb{R}_+^\times)$ such that $\psi_\omega(I_X) \neq \psi_{\omega'}(I_X)$, where $\mathcal{B}(\mathbb{R}_+^\times)$ denotes the set of Borel subsets of $\mathbb{R}_+^\times$ and I_X denotes the characteristic function of X. We divide the proof into two cases according to whether one is contained in the orbit of the other or not. Let us denote $o(\omega) = \{\tau^s\omega : s \in \mathbb{R}\}$, the orbit of ω under $\{\tau^s\}_{s\in\mathbb{R}}$.

<u>Case 1</u>. $\omega' \in o(\omega)$.

Without loss of generality, we can assume that $\omega' = \tau^s\omega (s > 0)$. Let $\omega = (\eta, t)$. We take a set $A \in \eta$ such that $|n - m| \geq [s] + 2$ whenever $n, m \in A, n \neq m$. Then we define a set $X \in \mathcal{B}(\mathbb{R}_+^\times)$ as $X = \cup_{n\in A}(2^{t+n-1}, 2^{t+n}]$. We will show that $\psi_\omega(I_X) \neq \psi_{\omega'}(I_X)$. Now assume oppositely that $\psi_\omega(I_X) = \psi_{\omega'}(I_X) = \alpha$. Let ε be a positive number with $\varepsilon < \frac{1-2^{-s}}{1+2^{-s}}\alpha$. Then there exists a set $B \in \eta$ such that $B \subseteq A$ and

$$\left|\frac{1}{2^{t+x}}\int_1^{2^{t+x}} I_X(y)dy - \alpha\right| < \varepsilon \quad \text{and} \quad \left|\frac{1}{2^{s+t+x}}\int_1^{2^{s+t+x}} I_X(y)dy - \alpha\right| < \varepsilon$$

whenever $x \in B$. Observing that by the assumption of A, $X \cap (2^{t+n}, 2^{s+t+n}] = \emptyset$ for any $n \in A$. We have then that if $x \in B$,

$$\int_1^{2^{t+x}} I_X(y)dy < 2^{t+x}(\alpha + \varepsilon) \Longrightarrow \int_1^{2^{s+t+x}} I_X(y)dy < 2^{t+x}(\alpha + \varepsilon)$$
$$\Longleftrightarrow \frac{1}{2^{s+t+x}}\int_1^{2^{s+t+x}} I_X(y)dy \leq 2^{-s}(\alpha + \varepsilon) < \alpha - \varepsilon,$$

which is a contradiction.

<u>Case2</u>. $\omega' \notin o(\omega)$.

Let us $\omega = (\eta, t)$ and $\omega' = (\eta', t')$. We take $A \in \eta$ such that $\tau^{-1}A \cup A \cup \tau A \cup \tau^2 A \notin \eta'$. We set $X = \cup_{n \in A}(2^{t+n-1}, 2^{t+n}]$, then it is obvious that

$$\begin{aligned}\psi_\omega(I_X) &= \omega\text{-}\lim_x \frac{1}{2^{t+x}} \int_1^{2^{t+x}} I_X(y)dy \\ &\geq \liminf_{x \in A} \frac{1}{2^{t+x}} \int_1^{2^{t+x}} I_X(y)dy \\ &\geq \frac{2^{t+x} - 2^{t+x-1}}{2^{t+x}} = \frac{1}{2}.\end{aligned}$$

Hence in order to show that $\psi_\omega(I_X) \neq \psi_{\omega'}(I_X)$, it is sufficient to show that $\psi_{\omega'}(I_X) < \frac{1}{2}$. Now we choose $B \in \eta'$ such that $(\tau^{-1}A \cup A \cup \tau A \cup \tau^2 A) \cap B = \emptyset$. Then for any $x' \in B$ we have $(2^{t'+x'-2}, 2^{t'+x'}] \cap X = \emptyset$. Then we obtain

$$\begin{aligned}\psi_{\omega'}(I_X) &\leq \limsup_{x' \in B} \frac{1}{2^{t'+x'}} \int_1^{2^{t'+x'}} I_X(y)dy \\ &\leq \frac{2^{t'+x'-2}}{2^{t'+x'}} = \frac{1}{4},\end{aligned}$$

which proves the theorem.

Therefore, since Ψ is a continuous bijective mapping from Ω^* to $\tilde{\mathcal{M}}$, it is a homeomorphism. We have completed the proof of Theorem 3.2.1.

Furthermore, in the integral setting, we can naturally consider the continuous flow on $\tilde{\mathcal{M}}$ induced by the action of the positive part of the multiplicative group $\mathbb{R}^\times = (0, \infty)$ of the real field $\mathbb{R}$ on $\mathbb{R}_+^\times$ defined as follows: let us consider a semiflow on $\mathbb{R}_+^\times$ as follows

$$\rho^s : \mathbb{R}_+^\times \longrightarrow \mathbb{R}_+^\times, \quad \rho^s x = 2^s x, \quad s \geq 0.$$

Then define linear operators P_s as

$$P_s : L^\infty(\mathbb{R}_+^\times) \longrightarrow L^\infty(\mathbb{R}_+^\times), \quad (P_s f)(x) = f(\rho^s x), \quad s \geq 0.$$

Let P_s^* be the adjoint operators of P_s, then

$$P_s^* : \tilde{\mathcal{M}} \longrightarrow \tilde{\mathcal{M}}, \quad s \geq 0$$

are homeomorphisms and $(\tilde{\mathcal{M}}, \{P_s^*\}_{s \in \mathbb{R}})$ is a continuous flow. Now we explain that the mapping $\Psi : \Omega^* \to \tilde{\mathcal{M}}$ carries over the structure of continuous flow on Ω^* defined in Chapter 2 into the continuous flow on $\tilde{\mathcal{M}}$ defined above; for any $r = 2^s$ with $s \geq 0$ we have that

$$
\begin{aligned}
(P_s^*\psi_\omega)(f) = \psi_\omega(P_s f) &= 2^\omega\text{-}\lim_x \frac{1}{x}\int_1^x f(rt)dt \\
&= 2^\omega\text{-}\lim_x \frac{1}{rx}\int_r^{rx} f(t)dt \\
&= r\cdot 2^\omega\text{-}\lim_x \frac{1}{x}\int_1^x f(t)dt \\
&= 2^{\tau^s\omega}\text{-}\lim_x \frac{1}{x}\int_1^x f(t)dt. \\
&= \psi_{\tau^s\omega}(f).
\end{aligned}
$$

Therefore, we have obtained the following result, which asserts that the two continuous flows $(\Omega^*, \{\tau^s\}_{s\in\mathbb{R}})$ and $(\tilde{\mathcal{M}}, \{P_s^*\}_{s\in\mathbb{R}})$ are isomorphic via Ψ.

Theorem 3.2.3. *$\Psi\circ\tau^s = P_s^*\circ\Psi$ holds for each $s\in\mathbb{R}$.*

3.3 Extreme points of $\mathcal{C}$

In this section we investigate the algebraic structure of $\mathcal{C}$ as a compact convex set. Since $\mathcal{C}$ is affinely homeomorphic to $\mathcal{M}$ by Theorem 3.2.2, we work with $\mathcal{M}$. We begin with showing an elementary result.

Theorem 3.3.1. *Let $ex(\mathcal{M})$ be the set of all extreme points of $\mathcal{M}$. Then $\tilde{\mathcal{M}}$ is weak* compact, and $ex(\mathcal{M})\subseteq\tilde{\mathcal{M}}$ holds.*

Proof. The compactness of $\tilde{\mathcal{M}}$ follows from Lemma 3.2.4. By the Krein-Milman theorem and Lemma 3.2.5, it is sufficient to show that

$$\sup_{\omega\in\Omega^*}\psi_\omega(f) = \overline{M}(f)$$

for every $f\in L^\infty(\mathbb{R}_+^\times)$. We have

$$
\begin{aligned}
\overline{M}(f) &= \limsup_{x\to\infty}(Uf)(x) = \limsup_{x\to\infty}(WUf)(x) = \overline{(WUf)}(\omega) \\
&= 2^\omega\text{-}\lim_x \frac{1}{x}\int_1^x f(t)dt = \psi_\omega(f).
\end{aligned}
$$

for some $\omega\in\Omega^*$. This completes the proof.

Let $P(\Omega^*)$ be the set of all probability Borel measures on Ω^*. Given any $\mu\in P(\Omega^*)$, the following integral yields a member ψ of $\mathcal{M}$;

$$\psi(f) = \int_{\Omega^*}\psi_\omega(f)d\mu(\omega).$$

Since for any $\omega \in \Omega^*$ the measure $\delta_\omega \in P(\Omega^*)$, the probability measure equals 1 on any Borel subset of Ω^* which contains ω and equals 0 otherwise, induces $\psi_\omega \in \mathcal{M}$, we can regard this mapping of $P(\Omega^*)$ to $\mathcal{M}$ as an extension of Ψ and we denote it by the symbol $\overline{\Psi}$. Together with the Krein-Milman theorem, Theorem 3.3.1 asserts that this mapping $\overline{\Psi}$ is surjective ([14, Section 1]):

Corollary 3.3.1. *Every member ψ of $\mathcal{M}$ can be expressed in the form*

$$\psi(f) = \int_{\Omega^*} \psi_\omega(f) d\mu(\omega).$$

for some probability measure μ on Ω^*.

In connection with the linear operator U, we introduce a subspace $\mathfrak{U}$ of $C_{ub}(\mathbb{R}_+^\times)$ as follows:

$$\mathfrak{U} = \{f(x) \in C_{ub}(\mathbb{R}_+^\times) : (xf(x))' \in L^\infty(\mathbb{R}_+^\times)\}.$$

In other words, $f(x) \in C_{ub}(\mathbb{R}_+^\times)$ is in $\mathfrak{U}$ if and only if derivative $(xf(x))'$ exists almost everywhere on $\mathbb{R}_+^\times$ and also it is an essentially bounded measurable function.

Lemma 3.3.1. *$\mathfrak{U}$ is a subalgebra of $C_{ub}(\mathbb{R}_+^\times)$.*

Proof. We show that it is closed under multiplication. For $f(x) \in \mathfrak{U}$, notice that $(xf(x))' = f(x) + xf'(x) \in L^\infty(\mathbb{R}_+^\times)$ and which implies that $xf'(x) \in L^\infty(\mathbb{R}_+^\times)$ since $f(x)$ is bounded on $\mathbb{R}_+^\times$. Now let us given arbitrary pair of elements f, g of $\mathfrak{U}$. Then we get by the product rule,

$$(x(fg)(x))' = f(x)g(x) + xf'(x)g(x) + xg'(x)f(x)$$

exists almost everywhere on $\mathbb{R}_+^\times$ and it is essentially bounded since as mentioned above, $xf'(x)$ and $xg'(x)$ are essentially bounded. Hence fg is in $\mathfrak{U}$.

Now we take up the relation between $\mathfrak{U}$ and the range UL^∞ of the operator U. A hat placed above the symbol for a subalgebra of $C_{ub}(\mathbb{R}_+^\times)$ will be used to indicate its quotient algebra modulo the ideal $C_0(\mathbb{R}_+^\times)$; for example, $\hat{\mathfrak{U}} = \mathfrak{U}/(\mathfrak{U} \cap C_0(\mathbb{R}_+^\times))$. Let us take any $f(x)$ in $\mathfrak{U}$ and put $(xf(x))' = \xi_f(x)$. Then

$$\begin{aligned} xf(x) - f(1) &= \int_1^x \xi_f(t)dt, \quad x \geq 1 \\ \iff f(x) &= \frac{1}{x}\int_1^x \xi_f(t)dt + \frac{f(1)}{x} = (U\xi_f)(x) + \frac{f(1)}{x}, \quad x \geq 1. \end{aligned}$$

Hence,

$$f(x) \equiv (U\xi_f)(x) \pmod{C_0(\mathbb{R}_+^\times)}.$$

Conversely, for any $f \in L^\infty(\mathbb{R}_+^\times)$ it is obvious that $Uf \in \mathfrak{U}$. This leads to the following lemma.

Lemma 3.3.2. $\hat{\mathfrak{U}} = (\widehat{UL^\infty})$.

The next result is essential to prove our main theorem.

Lemma 3.3.3. $(\widehat{WUL^\infty})$ *is uniformly dense in* $C(\Omega^*)$.

Proof. It is easy to see that $(\widehat{WUL^\infty})$ contains constants and separates the points in Ω^*. It follows from Lemma 3.3.1 and 3.3.2 that $(\widehat{UL^\infty})$ is an algebra and hence $(\widehat{WUL^\infty})$ is also an algebra. Applying the Stone-Weierstrass theorem, we obtain the desired result.

With the aid of these results, we now prove our main theorem.

Theorem 3.3.2. $\tilde{\mathcal{M}} = ex(\mathcal{M})$.

Proof. Since we have already shown that $ex(\mathcal{M}) \subseteq \tilde{\mathcal{M}}$ in Theorem 3.3.1, we have to prove only that $\tilde{\mathcal{M}} \subseteq ex(\mathcal{M})$. Assume that for an $\omega \in \Omega^*$ and some $\psi_1, \psi_2 \in \mathcal{M}$,

$$\psi_\omega = \alpha\psi_1 + (1-\alpha)\psi_2, \quad 0 < \alpha < 1.$$

By Corollary 3.3.1, there exist probability measures μ, ν on Ω^* such that

$$\psi_1(f) = \int_{\Omega^*} \psi_{\omega'}(f)d\mu(\omega'), \quad \psi_2(f) = \int_{\Omega^*} \psi_{\omega'}(f)d\nu(\omega')$$

for every $f \in L^\infty(\mathbb{R}_+^\times)$. Then if we put $\lambda = \alpha\mu + (1-\alpha)\nu \in P(\Omega^*)$, we have that

$$\begin{aligned}
&\psi_\omega(f) = \alpha\psi_1 + (1-\alpha)\psi_2 = \int_{\Omega^*} \psi_{\omega'}(f)d\lambda(\omega') \quad \textit{for every } f \in L^\infty(\mathbb{R}_+^\times) \\
&\Longleftrightarrow \overline{(WUf)}(\omega) = \int_{\Omega^*} \overline{(WUf)}(\omega')d\lambda(\omega') \quad \textit{for every } f \in L^\infty(\mathbb{R}_+^\times) \\
&\Longleftrightarrow \overline{g}(\omega) = \int_{\Omega^*} \overline{g}(\omega')d\lambda(\omega') \quad \textit{for every } g \in WUL^\infty \\
&\Longleftrightarrow h(\omega) = \int_{\Omega^*} h(\omega')d\lambda(\omega') \quad \textit{for every } h \in C(\Omega^*).
\end{aligned}$$

Therefore, $\lambda = \delta_\omega$ holds. Thus the support sets of μ and ν are $\{\omega\}$ and we conclude that $\mu = \nu = \delta_\omega$, that is, $\psi_1 = \psi_2 = \psi_\omega$. This completes the proof.

Next theorem is an immediate consequence of the proof of Theorem 3.3.2.

Theorem 3.3.3. *For any $\psi \in \mathcal{M}$, a probability measure μ on Ω^* which represents ψ is unique. Namely, ψ is uniquely expressed in the form*

$$\psi(f) = \int_{\Omega^*} \psi_\omega(f) d\mu(\omega).$$

for some $\mu \in P(\Omega^*)$.

A consequence of Theorem 3.3.3 is that $\overline{\Psi} : P(\Omega^*) \to \mathcal{M}$ is an affine homeomorphism. Also the isomorphism Ψ between the two flows $(\Omega^*, \{\tau^s\}_{s\in\mathbb{R}})$ and $(\tilde{\mathcal{M}}, \{P_s^*\}_{s\in\mathbb{R}})$ established in Section 3.2 can be extended to an isomorphism between their closed convex hulls; we define linear operators T_s in a similar way as P_s for each $s \geq 0$.

$$T_s : C_{ub}(\mathbb{R}_+^\times) \longrightarrow C_{ub}(\mathbb{R}_+^\times), \quad (T_s f)(x) = f(x+s), \quad s \geq 0.$$

Let T_s^* be their adjoint operators. Then we have that the two continuous flows $(P(\Omega^*), \{T_s^*\}_{s\in\mathbb{R}})$ and $(\mathcal{M}, \{P_s^*\}_{s\in\mathbb{R}})$ are isomorphic via $\overline{\Psi}$:

Theorem 3.3.4. $\overline{\Psi} \circ T_s^* = P_s^* \circ \overline{\Psi}$ *holds for each $s \in \mathbb{R}$.*

We give below our main results of this section formulated in the summation setting.

Corollary 3.3.2. $ex(\mathcal{C}) = \tilde{\mathcal{C}}$.

Corollary 3.3.3. *For any $\varphi \in \mathcal{C}$, there is a unique probability measure μ on Ω^* such that*

$$\varphi(f) = \int_{\Omega^*} \varphi_\omega(f) d\mu(\omega)$$

holds for every $f \in l^\infty$.

Chapter 4

Absolute continuity and singularity

4.1 Definitions

Following [3, Chapter 6] we introduce the notions of absolute continuity and singularity for charges. In the following, let μ and ν be any two nonnegative charges on $(X, \mathcal{F})$.

Definition 4.1.1. We say that ν is *absolutely continuous* with respect to μ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\nu(A) < \varepsilon$ whenever $\mu(A) < \delta$, where $A \in \mathcal{F}$. In this case, we write $\nu \ll \mu$.

We can consider a weak version of absolute continuity in a natural way as follows:

Definition 4.1.2. We say that ν is *weakly absolutely continuous* with respect to μ if $\nu(A) = 0$ whenever $\mu(A) = 0$, where $A \in \mathcal{F}$. In this case, we write $\nu \prec \mu$.

Next we define the notion of singularity.

Definition 4.1.3. We say that μ and ν are *singular* if for every $\varepsilon > 0$, there exists a set $D \in \mathcal{F}$ such that $\mu(D) < \varepsilon$ and $\nu(D^c) < \varepsilon$. In this case, we write $\mu \perp \nu$.

Also we can define a strong version of singularity in a way that seems to be natural.

Definition 4.1.4. We say that μ and ν are *strongly singular* if there exists a set $D \in \mathcal{F}$ such that $\mu(D) = 0$ and $\nu(D^c) = 0$. In this case, we write $\mu \perp\!\!\!\perp \nu$.

The distinctions between absolute continuity and weak absolute continuity, singularity and strong singularity are essential, as the following theorems show, when μ and ν are charges which are not countably additive. We refer to [3] for the proofs.

Theorem 4.1.1. *Let $\mathcal{F}$ be a σ-algebra of subsets of X, and μ and ν are nonnegative measures on $(X, \mathcal{F})$ such that ν is bounded. Then $\nu \prec \mu$ if and only if $\nu \ll \mu$.*

Theorem 4.1.2. *Let $\mathcal{F}$ be a σ-algebra of subsets of X, and μ and ν are nonnegative measures on $(X, \mathcal{F})$. Then $\nu \perp \mu$ if and only if $\mu \perp\!\!\!\perp \nu$.*

Now for a given charge space $(X, \mathcal{F}, \mu)$, we introduce an extension of μ to the Borel measure of the stone space of $\mathcal{F}$. This method plays an important role for formulating various notions concerning charges.

Let $(X, \mathcal{F}, \mu)$ be a charge space with μ is bounded and nonnegative. By the Stone representation theorem, For an algebra $\mathcal{F}$, there exists a compact space F and a natural Boolean isomorphism $\phi : \mathcal{F} \to \mathcal{C}$, where $\mathcal{C}$ is the algebra of the clopen subsets of F.

Now define a charge $\hat{\mu}$ on $\mathcal{C}$ by $\hat{\mu}(\phi(A)) = \mu(A)$ and we get a charge space $(F, \mathcal{C}, \hat{\mu})$. Since any union of infinite disjoint family of clopen subsets can not be a clopen subset, $\hat{\mu}$ is countably additive on $\mathcal{C}$ and thus by the E. Hopf extension theorem, we can extend it to a countable additive measure on the σ-algebra generated by $\mathcal{C}$, that is, the Baire σ-algebra of F. This can also be extended to the Borel σ-algebra $\mathcal{B}(F)$ of S as a countable additive measure in a unique way that $\hat{\mu}$ is regular. We still denote it by $\hat{\mu}$. We denote by supp μ the support of $\hat{\mu}$ in F.

In particular, for a charge space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is a bounded nonnegative charge, the Stone space of $\mathcal{P}(\mathbb{N})$ is $\beta\mathbb{N}$ and Boolean isomorphism of $\mathcal{P}(\mathbb{N})$ onto the clopen algebra of $\beta\mathbb{N}$ is given by $A \mapsto \hat{A}$ for every $A \in \mathcal{P}(\mathbb{N})$. Thus μ can be extended to a Borel measure $\hat{\mu}$ on $\beta\mathbb{N}$.

Now we can formulate these notions by extended measures. Below we use familiar notions of absolute continuity and singularity of countably additive measures.

Theorem 4.1.3. *Let μ and ν be bounded nonnegative charges on $(X, \mathcal{F})$. Then the following statements hold:*

1. *$\nu \ll \mu$ if and only if $\hat{\nu} \ll \hat{\mu}$.*
2. *$\nu \prec \mu$ if and only if supp $\nu \subseteq$ supp μ.*
3. *$\nu \perp \mu$ if and only if $\hat{\nu} \perp \hat{\mu}$.*
4. *$\nu \perp\!\!\!\perp \mu$ if and only if supp $\nu \cap$ supp $\mu = \emptyset$.*

Proof . (1) (1) Let us assume that $\nu \ll \mu$. Given $\varepsilon > 0$, take $\delta > 0$ as above. Let $E \in \mathcal{B}(F)$ with $\hat{\mu}(E) = 0$. Since $\hat{\mu}$ is regular, for any $\delta > \delta' > 0$, one can choose $A \in \mathcal{F}$ such that $\hat{\mu}(\phi(A) \triangle E) < \delta'$ and $\hat{\nu}(\phi(A) \triangle E) < \delta'$. Thus we have $\mu(A) = \hat{\mu}(\phi(A)) \leq \hat{\mu}(\phi(A) \triangle E) + \hat{\mu}(E) = \hat{\mu}(\phi(A) \triangle E) < \delta'$. On the other hand, $\hat{\nu}(E) \leq \hat{\nu}(\phi(A) \triangle E) + \hat{\nu}(\phi(A)) \leq \delta' + \varepsilon$. Since δ' and ε can be arbitrary small, we have $\hat{\nu}(E) = 0$. Hence we have $\hat{\nu} \ll \hat{\mu}$ in the sense of measure theory.

Conversely, suppose that $\hat{\nu} \ll \hat{\mu}$. This can be written as $\hat{\nu} \prec \hat{\mu}$ in the terms defined above. Since $\mathcal{B}(F)$ is a σ-algebra and $\hat{\mu}$ and $\hat{\nu}$ are measures, $\hat{\nu} \ll \hat{\mu}$ if and only if $\hat{\nu} \prec \hat{\mu}$ by Theorem 4.1.1. Thus through ϕ^{-1} we have $\nu \ll \mu$.

(2) is obvious.

(3) Let us assume that $\nu \perp \mu$. Let $\varepsilon_1 > 0$ and $D_{\varepsilon_1,i} \in \mathcal{F}$, $i \geq 1$ be such that $\mu(D_{\varepsilon_1,i}) < \frac{\varepsilon_1}{2^i}$ and $\nu(D^c_{\varepsilon_1,i}) < \frac{\varepsilon_1}{2^i}$. Notice that $\nu(D_{\varepsilon_1,i}) > 1 - \frac{\varepsilon_1}{2^i}$ for every $i \geq 1$. Hence we have $\hat{\nu}(\cup_{i\geq 1}\phi(D_{\varepsilon_1,i})) = 1$. On the other hand, $\hat{\mu}(\cup_{i\geq 1}\phi(D_{\varepsilon_1,i})) \leq \sum_{i=1}^{\infty} \frac{\varepsilon_1}{2^i} = \varepsilon_1$. Now we choose a decreasing sequence $\{\varepsilon_j\}_{j\geq 1}$ of positive numbers such that $\lim_{j\to\infty} \varepsilon_j = 0$. Then we put $E = \cap_{j\geq 1} \cup_{i\geq 1} \phi(D_{\varepsilon_j,i})$ and we get $\hat{\nu}(E^c) = 0$ and $\hat{\mu}(E) = 0$, which means that $\hat{\nu} \perp \hat{\mu}$ in the sense of measure theory.

Now suppose that $\hat{\nu} \perp \hat{\mu}$. Then there exists a set D in $\mathcal{B}(F)$ such that $\hat{\mu}(D) = 0$ and $\hat{\nu}(D^c) = 0$. By the regularity of $\hat{\mu}$ and $\hat{\nu}$, there exists a set C in $\mathcal{F}$ such that $\hat{\mu}(\phi(C)\triangle D) < \varepsilon$ and $\hat{\nu}(\phi(C)\triangle D) < \varepsilon$. Thus we have $\mu(C) = \hat{\mu}(\phi(C)) \leq \hat{\mu}(\phi(C)\triangle D) + \hat{\mu}(D) < \varepsilon$ and $\nu(C^c) = \hat{\nu}(\phi(C^c)) \leq \hat{\nu}(\phi(C^c)\triangle D^c) + \hat{\nu}(D^c) = \hat{\nu}(\phi(C)\triangle D) + \hat{\nu}(D^c) < \varepsilon$. This shows that $\nu \perp \mu$.

(4) is obvious.

Finally, concerning these notions, we denote a generalization of the Lebesgue decomposition theorem to charges.

Theorem 4.1.4. *For given bounded nonnegative charges μ and ν, there exist nonnegative charges ν_1 and ν_2 on $(X, \mathcal{F})$ such that*

1. $\nu = \nu_1 + \nu_2$.
2. $\nu_1 \ll \mu$.
3. $\nu_2 \perp \mu$.

Furthermore, a decomposition of ν satisfying (2) and (3) is unique.

Recall that by Theorem 3.2.1, each density measure $\nu^{\mathcal{U}}$ in $\tilde{\mathcal{C}}$ is equal to ν_ω for some $\omega = (\eta, t) \in \Omega^*$ defined as follows:

$$\nu_\omega(A) = \eta\text{-}\lim_n \frac{|A \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n}, \quad A \in \mathcal{P}(\mathbb{N}),$$

where $\theta = 2^t$. Arguments in the following sections are developed by using this result.

4.2 Absolute continuity and singularity of density measures in $\tilde{\mathcal{C}}$

Observe that by the above remark for a pair μ, ν in $\tilde{\mathcal{C}}$, there are points $\omega = (\eta, t)$ and $\omega' = (\eta', t')$ in Ω^* such that $\mu = \nu_\omega$ and $\nu = \nu_{\omega'}$. Without loss of generality, we assume $t, t' \in [0, 1)$. Now we begin with absolute continuity.

Theorem 4.2.1. $\nu_{\omega'} \ll \nu_\omega$ *if and only if* $\omega' \in o_-(\omega)$.

Proof. (Sufficiency) By the assumption, $\eta' = \tau^{-m}\eta$ holds for some nonnegative integer m. Fix any positive number $\delta > 0$ and let A be a set with $\nu_\omega(A) < \delta$. We take $X \in \eta$ such that

$$\frac{|A \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} < 2\delta$$

whenever $n \in X$. Remark that $\tau^{-m}X = \{n - m : n \in X\} \in \eta'$. We have that for any $n \in X$ (notice that in the case of $m = 0$, $\theta' \leq \theta$ holds),

$$\begin{aligned}\frac{|A \cap [\theta' \cdot 2^{n-m}]|}{\theta' \cdot 2^{n-m}} &\leq 2^m \cdot \frac{\theta}{\theta'} \cdot \frac{|A \cap [\theta' \cdot 2^{n-m}]|}{\theta \cdot 2^n} \\ &\leq 2^m \cdot \frac{\theta}{\theta'} \cdot \frac{|A \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} < 2^{m+1} \cdot \frac{\theta}{\theta'} \cdot \delta\end{aligned}$$

then

$$\begin{aligned}\nu_{\omega'}(A) &= \eta'\text{-}\lim_n \frac{|A \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \\ &\leq \limsup_{n \in \tau^{-m}X} \frac{|A \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \leq 2^{m+1} \cdot \frac{\theta}{\theta'} \cdot \delta.\end{aligned}$$

Hence for any given $\varepsilon > 0$, put $\delta < \frac{1}{2^{m+1}}\frac{\theta'}{\theta}\varepsilon$, then for any $A \in \mathcal{P}(\mathbb{N})$ we have

$$\nu_\omega(A) < \delta \Longrightarrow \nu_{\omega'}(A) < \varepsilon.$$

The proof is complete.

(Necessity) We shall show the contraposition. Now we assume that $\omega' \notin o_-(\omega)$ and then either (1) $\eta' \notin o_-(\eta)$ or (2) $\eta = \eta'$ and $t' > t$ holds.

(1) For any positive integer m, we can take $A \in \eta, B \in \eta'$ such that $(B \cup \tau B \cup \cdots \cup \tau^{m-1}B) \cap A = \emptyset$. Now we put $I_m = \cup_{n \in B}([\theta \cdot 2^{n-1}], 2^n]$, then we have

$$\begin{aligned}\nu_{\omega'}(I_m) &= \eta'\text{-}\lim_n \frac{|I_m \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \\ &\geq \liminf_{n \in B} \frac{|I_m \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \\ &\geq \frac{2^n - \theta \cdot 2^{n-1}}{\theta' \cdot 2^n} \geq \frac{1}{\theta'}\left(1 - \frac{\theta}{2}\right).\end{aligned}$$

On the other hand, notice that $n \in A$ implies that $B \cap [n - m + 1, n] = \emptyset$, then

$$\begin{aligned}\nu_\omega(I_m) &= \eta\text{-}\lim_n \frac{|I_m \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\ &\leq \limsup_{n \in A} \frac{|I_m \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\ &\leq \frac{2^{n-m}}{\theta \cdot 2^n} = \frac{1}{\theta \cdot 2^m}.\end{aligned}$$

Hence for any $\delta > 0$, Choose any positive integer m with $\frac{1}{\theta 2^m} < \delta$, we have

$$\nu_\omega(I_m) < \delta \quad and \quad \nu_{\omega'}(I_m) \geq \frac{1}{\theta'}\left(1 - \frac{\theta}{2}\right).$$

Thus we conclude that $\nu_{\omega'} \not\ll \nu_\omega$.

(2) In this case, for $m \geq 1$, take $B \in \eta$ such that $B \cap \tau^k B = \emptyset$, $k = 1, 2 \ldots, m$. Put $I_m = \cup_{n \in B}([2^n \cdot \theta], [2^n \cdot \theta'])$ and we have

$$\begin{aligned}\nu_{\omega'}(I_m) &= \eta'\text{-}\lim_n \frac{|I_m \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \\ &\geq \liminf_{n \in B} \frac{|I_m \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \\ &\geq \frac{\theta' \cdot 2^n - \theta \cdot 2^n}{\theta' \cdot 2^n} = 1 - \frac{\theta}{\theta'}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\nu_\omega(I_m) &= \eta\text{-}\lim_n \frac{|I_m \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\ &\leq \limsup_{n \in B} \frac{|I_m \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\ &\leq \frac{2^{n-m}}{\theta \cdot 2^n} = \frac{1}{\theta \cdot 2^m}.\end{aligned}$$

Hence for any $\delta > 0$, Choose any positive integer m with $\frac{1}{\theta 2^m} < \delta$, we have

$$\nu_\omega(I_m) < \delta \quad and \quad \nu_{\omega'}(I_m) \geq 1 - \frac{\theta}{\theta'}.$$

Thus we have $\nu_{\omega'} \not\ll \nu_\omega$.

Theorem 4.2.2. *If μ and ν are elements of $\tilde{\mathcal{C}}$ and mutually absolutely continuous, that is, $\nu \ll \mu$ and $\mu \ll \nu$, then $\mu = \nu$.*

Proof. Let $\mu = \nu_\omega$ and $\nu = \nu_{\omega'}$ for some $\omega, \omega' \in \Omega^*$. From the assumption of the theorem and Theorem 4.2.1, we can write $\omega' = \tau^{-s}\omega$ and $\omega = \tau^{-t}\omega'$ for some $s, t \geq 0$. Thus we have that $\omega = \tau^{-(s+t)}\omega$. As is well known, there are no periodic points in the discrete flow $(\mathbb{N}^*, \tau)$, which fact implies immediately that there are no periodic points in the continuous flow $(\Omega^*, \{\tau^s\}_{s \in \mathbb{R}})$. Thus we get $s + t = 0$, i.e., $s = t = 0$. Hence $\omega = \omega'$. We obtain the result.

Remark 4.2.1. Let ω, ω' be any two elements of Ω^*, then we define a partial order $\leq$ on Ω^* as follows:

$$\omega' \leq \omega \Longleftrightarrow \omega' \in o_-(\omega).$$

Building on Theorem 4.2.1 one can check that Φ is also an isomorphism between the partially ordered sets $(\Omega^*, \leq)$ and $(\tilde{\mathcal{C}}, \ll)$.

Next we prove the result of singularity.

Theorem 4.2.3. *ν_ω and $\nu_{\omega'}$ are singular if and only if $o(\omega) \cap o(\omega') = \emptyset$, i.e., $\omega' \notin o(\omega)$.*

Proof. Necessity is obvious by Theorem 4.2.1. Hence we shall prove sufficiency. By the assumption, $o(\eta) \cap o(\eta') = \emptyset$ also holds in $\mathbb{N}^*$. For any positive integer m, take a set B in η' such that $\{\tau^{-(m-1)}\eta, \cdots, \tau^{-1}\eta, \eta, \tau\eta, \cdots, \tau^m\eta\} \cap B^* = \emptyset$. Thus we have

$$\eta \notin \bigcup_{i=-m}^{m-1} \tau^i B^*.$$

Then take A in η such that $(\cup_{i=-m}^{m-1} \tau^i B) \cap A = \emptyset$. Then we define $J_m = \cup_{n \in B}([\theta' \cdot 2^{n-m}], [\theta' \cdot 2^n]],\ m \geq 1$. Hence

$$\begin{aligned}
\nu_{\omega'}(J_m) &= \eta'\text{-}\lim_n \frac{|J_m \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \\
&\geq \liminf_{n \in B} \frac{|J_m \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \\
&\geq \liminf_{n \in B} \frac{\theta' \cdot 2^n - \theta' \cdot 2^{n-m}}{\theta' \cdot 2^n} = 1 - \frac{1}{2^m}.
\end{aligned}$$

On the other hand, since if $n \in A$, then $[n - m + 1, n + m] \cap B = \emptyset$, we have

$$\begin{aligned}
\nu_\omega(J_m) &= \eta\text{-}\lim_n \frac{|J_m \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\
&\leq \limsup_{n \in A} \frac{|J_m \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\
&\leq \frac{\theta' \cdot 2^{n-m}}{\theta \cdot 2^n} = \frac{\theta'}{\theta} \cdot \frac{1}{2^m}.
\end{aligned}$$

Therefore for any $\varepsilon > 0$, take any positive integer m with $\frac{\theta'}{\theta} \frac{1}{2^{m-1}} < \varepsilon$, we get

$$\nu_\omega(J_m) \leq \varepsilon, \quad \textit{and} \quad \nu_{\omega'}({J_m}^c) \leq \varepsilon.$$

Thus we conclude that $\nu_\omega \perp \nu_{\omega'}$.

Next we concerned with weakly absolutely continuity and strong singularity.

Theorem 4.2.4. *$\nu_{\omega'} \prec \nu_\omega$ if and only if $\omega' \in \overline{o}_-(\omega)$.*

Proof. (Sufficiency) Remark that it suffices to show that

$$\nu_{\omega'}(A) > 0 \Longrightarrow \nu_\omega(A) > 0.$$

By the assumption, $\eta' \in \overline{o}_-(\eta)$ holds in $\mathbb{N}^*$. For any X' in η', there exists an integer $m \geq 0$ such that $\tau^{-m}\eta \in X'^*$, i.e., $\tau^m X' \in \eta$. Now we take any $A \in \mathcal{P}(\mathbb{N})$ such that

$$\nu_{\omega'}(A) = \eta'\text{-}\lim_n \frac{|A \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} = \delta > 0$$

and then take $X' \in \eta'$ such that

$$n \in X' \Longrightarrow \frac{|A \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} > \frac{\delta}{2}.$$

Then we have

$$\begin{aligned}\frac{|A \cap [\theta \cdot 2^{n+m}]|}{\theta \cdot 2^{n+m}} &= \frac{1}{2^m} \cdot \frac{\theta'}{\theta} \cdot \frac{|A \cap [\theta \cdot 2^{n+m}]|}{\theta' \cdot 2^n} \\ &\geq \frac{1}{2^m} \cdot \frac{\theta'}{\theta} \cdot \frac{|A \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \\ &> \frac{1}{2^{m+1}} \cdot \frac{\theta'}{\theta} \cdot \delta\end{aligned}$$

for any $n \in X$. Hence

$$\begin{aligned}\nu_\omega(A) &= \eta\text{-}\lim_n \frac{|A \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\ &\geq \liminf_{n \in \tau^m X'} \frac{|A \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\ &\geq \frac{1}{2^{m+1}} \cdot \frac{\theta'}{\theta} \cdot \delta > 0.\end{aligned}$$

This completes the proof.

(Necessity) We shall show the contraposition. Assume that $\omega' \notin \overline{o}_-(\omega)$ and it implies that either (1) $\eta' \notin \overline{o}_-(\eta)$ or (2) $\eta = \eta', t' > t$ holds.

(1) There exists a set $X' \in \eta'$ such that $X'^* \cap o_-(\eta) = \emptyset$, i.e., $\eta \notin \cup_{i \geq 0} \tau^i X'^*$. Hence for any fixed positive integer m, there is a set $A_m \in \eta$ with $A_m \cap (X' \cup \tau X' \cup \cdots \cup \tau^{m-1} X') = \emptyset$. Now we define $I = \cup_{n \in X}([\theta \cdot 2^{n-1}], 2^n]$. For any $n \in A_m$, since $X' \cap [n - m + 1, n] = \emptyset$ we have

$$\begin{aligned}\nu_\omega(I) &= \eta\text{-}\lim_n \frac{|I \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\ &\leq \limsup_{n \in A_m} \frac{|I \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\ &\leq \frac{2^{n-m}}{\theta \cdot 2^n} = \frac{1}{\theta \cdot 2^m}.\end{aligned}$$

Thus since $m \geq 1$ can be arbitrary, we have $\nu_\omega(I) = 0$. On the other hand,

$$\begin{aligned}
\nu_{\omega'}(I) &= \eta'\text{-}\lim_n \frac{|I \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \\
&\geq \liminf_{n \in X} \frac{|I \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \\
&\geq \frac{2^n - \theta \cdot 2^{n-1}}{\theta' \cdot 2^n} \geq \frac{1}{\theta'}\left(1 - \frac{\theta}{2}\right).
\end{aligned}$$

Hence we have shown that $I \subseteq \mathbb{N}$ satisfies $\nu_\omega(I) = 0$ and $\nu_{\omega'}(I) > 0$. Thus $\nu_{\omega'} \not\prec \nu_\omega$. We get the result.

(2) We take $X \in \eta$ as $\overline{o}_-(\eta) \setminus \{\eta\} \notin X^*$. Put $I = \cup_{n \in X}([\theta \cdot 2^n], [\theta' \cdot 2^n]]$. For any $m \geq 1$, take $A_m \in \eta$ such that $A_m \cap (\tau X \cup \tau^2 X \cup \ldots \cup \tau^{m-1} X) = \emptyset$ and notice that $n \in A_m$ implies $X \cap [n - m + 1, n - 1] = \emptyset$, we have

$$\begin{aligned}
\nu_\omega(I) &= \eta\text{-}\lim_n \frac{|I \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\
&\leq \limsup_{n \in A_m} \frac{|I \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\
&\leq \frac{\theta' \cdot 2^{n-m}}{\theta \cdot 2^n} = \frac{\theta'}{\theta} \cdot \frac{1}{2^m},
\end{aligned}$$

which shows that $\nu_{\omega'}(I) = 0$. On the other hand, we have

$$\begin{aligned}
\nu_{\omega'}(I) &= \eta'\text{-}\lim_n \frac{|I \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \\
&\geq \liminf_{n \in X} \frac{|I \cap [\theta' \cdot 2^n]|}{\theta' \cdot 2^n} \\
&\geq \frac{\theta' \cdot 2^n - \theta \cdot 2^n}{\theta' \cdot 2^n} = 1 - \frac{\theta}{\theta'}.
\end{aligned}$$

Hence we get $\nu_\omega(I) = 0$ and $\nu_{\omega'}(I) > 0$ and thus $\nu_{\omega'} \not\prec \nu_\omega$ holds.

Remark 4.2.2. Let ω, ω' be any two elements of Ω^*, then we define a preorder $\sqsubseteq$ on Ω^* as follows:

$$\omega' \sqsubseteq \omega \iff \omega' \in \overline{o}_-(\omega).$$

Building on Theorem 4.2.4 one can check that Φ is an isomorphism between the preordered sets $(\Omega^*, \sqsubseteq)$ and $(\tilde{\mathcal{C}}, \prec)$.

Finally, we shall show the following result about strong singularity.

Theorem 4.2.5. *ν_ω and $\nu_{\omega'}$ are strongly singular if and only if $\omega' \notin \overline{o}_-(\omega)$ and $\omega \notin \overline{o}_-(\omega')$.*

Proof. Necessity is obvious by Theorem 4.2.4. Hence we will show sufficiency. First, by the assumption, we can take disjoint sets $X \in \eta$ and $X' \in \eta'$ such that $X^* \cap \overline{o}_-(\eta') = \emptyset$, $X'^* \cap \overline{o}_-(\eta) = \emptyset$, i.e., $\cup_{i\geq 0}\tau^i X^* \not\ni \eta'$ and $\cup_{i\geq 0}\tau^i X'^* \not\ni \eta$. Now we take decreasing sequences $\{X_i\}_{i\geq 0}, \{X_i'\}_{i\geq 0}$ of η, η' with $X_0 \subseteq X$ and $X_0' \subseteq X'$ such that for every $m \geq 0$,

$$(X \cup \tau X \cup \dots \tau^{m+1} X) \cap X_m' = \emptyset$$

and

$$(X' \cup \tau X' \dots \tau^{m+1} X') \cap X_m = \emptyset$$

holds. For every $k \geq 0$ we define

$$I_k = \bigcup_{n \in \tau^{-k} X_k} ([\theta \ \cdot 2^{n-1}], [\theta \cdot 2^n]], \quad J_k = \bigcup_{n \in \tau^{-k} X_k'} ([\theta' \cdot 2^{n-1}], [\theta' \cdot 2^n]].$$

Let us $Y = \cup_{k=0}^\infty \tau^{-k} X_k$ and $Y' = \cup_{k=0}^\infty \tau^{-k} X_k'$ and put $I = \cup_{k=0}^\infty I_k = \cup_{n\in Y}([\theta \ \cdot 2^{n-1}], [\theta \cdot 2^n]]$ and $J = \cup_{k=0}^\infty J_k = \cup_{n\in Y'}([\theta' \cdot 2^{n-1}], [\theta' \cdot 2^n]]$. First we show that $I \cap J = \emptyset$. In fact, assume that $I \cap J \neq \emptyset$ and there exists a pair m, m' such that $I_m \cap I_{m'} \neq \emptyset$. This means that at least one of the sets $\tau^{-m} X_m \cap \tau^{-m'} X_{m'}$, $\tau^{-(m-1)} X_m \cap \tau^{-m'} X_{m'}$ or $\tau^{-m} X_m \cap \tau^{-(m'-1)} X_{m'}$ is not empty. But since we have assumed that $X_m \subseteq X$ and $X_{m'} \subseteq X'$, this contradicts the assumption above.

For every $m \geq 1$, $n \in X_{m-1}$ implies that $n - m + 1, \cdots, n \in Y$. Hence

$$n \in X_{m-1} \Longrightarrow ([\theta \cdot 2^{n-m}], [\theta \cdot 2^n]] \subseteq I.$$

Then we have

$$\begin{aligned} \nu_\omega(I) &= \eta\text{-}\lim_n \frac{|I \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\ &\geq \liminf_{n \in X_{m-1}} \frac{|I \cap [\theta \cdot 2^n]|}{\theta \cdot 2^n} \\ &\geq \frac{\theta \cdot 2^n - \theta \cdot 2^{n-m}}{\theta \cdot 2^n} = 1 - \frac{1}{2^m}. \end{aligned}$$

Since $m \geq 1$ is arbitrary, we get $\nu_\omega(I) = 1$. In a similar way, we also have $\nu_{\omega'}(J) = 1$. Thus we obtain $\nu_\omega \perp\!\!\!\perp \nu_{\omega'}$.

We remark that Theorem 4.2.5 is equivalent to the following.

Theorem 4.2.6. *ν_ω and $\nu_{\omega'}$ are strongly singular if and only if $\overline{o}_-(\omega) \cap \overline{o}_-(\omega') = \emptyset$.*

Proof. Sufficiency is obvious. However it seems that the necessity is seemingly stronger than the claim in Theorem 4.2.5. Assume that $\nu_\omega \perp\!\!\!\perp \nu_{\omega'}$ and $\overline{o}_-(\omega) \cap \overline{o}_-(\omega') \neq \emptyset$. Then there exists a $\omega'' \in \mathbb{N}^*$ such that $\omega'' \in \overline{o}_-(\omega)$ and $\omega'' \in \overline{o}_-(\omega')$. But from Theorem 4.2.4 we have that $\nu_{\omega''} \prec \nu_\omega$ and $\nu_{\omega''} \prec \nu_{\omega'}$. This implies that supp $\nu_{\omega''} \subseteq$ supp $\nu_\omega \cap$ supp $\nu_{\omega'} \neq \emptyset$, which contradicts the assumption.

4.3 Miscellany

In this section we shall give some consequences of the results of the previous section. First, by Theorem 4.2.1 and Theorem 4.2.3 we obtain immediately the following result:

Theorem 4.3.1. *For any two elements in $\tilde{\mathcal{C}}$, one is absolutely continuous with respect to the other or they are singular.*

Similarly, combining Theorem 4.2.4 and Theorem 3.2.5, we obtain the following result.

Theorem 4.3.2. *For any two elements in $\tilde{\mathcal{C}}$, one is weakly absolutely continuous with respect to the other or they are strongly singular.*

As we have seen in the previous section, properties of the orbit of ω under $\{\tau^s\}_{s\in\mathbb{R}}$ give rise to properties of the density measure ν_ω in $\tilde{\mathcal{C}}$. In what follows, we particularly pay attention to recurrence of the orbit of a point ω in Ω^*, and see how it affects properties of the density measure ν_ω.

Theorem 4.3.3. *(1) If $\omega \in \mathcal{D}$, then supp $\nu_{\omega'} \subsetneq$ supp ν_ω for any $\omega' \in \overline{o}_-(\omega) \setminus \{\omega\}$.*

(2) If $\omega \notin \mathcal{D}$ then supp $\nu_{\omega'} \subseteq$ supp ν_ω for any $\omega' \in \overline{o}(\omega)$. In particular, supp $\nu_{\omega'} =$ supp ν_ω for any $\omega' \in o(\omega)$.

(3) If $\omega \in \mathcal{A}$, then supp $\nu_{\omega'} =$ supp ν_ω for any $\omega' \in \overline{o}(\omega)$.

Proof. (1) By Theorem 4.2.4 for any $\omega' \in \overline{o}_-(\omega)$, supp $\nu_{\omega'} \subseteq$ supp ν_ω. Also $\omega \notin \overline{o}_-(\omega')$ since ω is not recurrent. Then again by Theorem 4.2.4 supp $\nu_\omega \subsetneq$ supp $\nu_{\omega'}$. Hence supp $\nu_{\omega'} \subsetneq$ supp ν_ω.

(2) Observe that if $\omega \in \mathcal{D}$ then $\overline{o}_-(\omega) = \overline{o}(\omega)$. Then first half of the claim follows immediately by Theorem 4.2.4. Without loss of generality, we can assume that $\omega' \in o_-(\omega)$. By Theorem 4.2.1 we get that supp $\nu_{\omega'} \subseteq$ supp ν_ω. On the other hand, $\omega \in \overline{o}_-(\omega')$ since $\omega \notin \mathcal{D}$. That is, by Theorem 4.2.4, supp $\nu_\omega \subseteq$ supp $\nu_{\omega'}$. Hence supp $\nu_\omega =$ supp $\nu_{\omega'}$.

(3) Let $\omega \in \mathcal{A}$, then notice that for any $\omega' \in \overline{o}(\omega) = \overline{o}_-(\omega)$, the negative semi-orbit $o_-(\omega')$ is dense in $\overline{o}(\omega)$, that is, $\omega \in \overline{o}_-(\omega')$ i.e., supp $\nu_\omega \subseteq$ supp $\nu_{\omega'}$ by Theorem 4.2.4. Then we get the result immediately.

Chapter 5

Additive property

5.1 Equivalent conditions to additive property

In this section, we introduce some equivalent assertions to the additive property. As we have mentioned in Chapter 1, one can generalize some of the main theorems in measure theory to charges having the additive property. In fact, conversely, the validity of these theorems are also sufficient conditions for charges to have the additive property. We begin with the completeness of L^p spaces over charges, which is the original motive of introducing the notion of the additive property.

Theorem 5.1.1. *For a nonnegative charge μ on $(X, \mathcal{F})$, μ has the additive property if and only if $L^p(\mu)$ is complete.*

The next result is a generalization of the Radon-Nikodym theorem to charges.

Theorem 5.1.2. *For a bounded nonnegative charge μ on $(X, \mathcal{F})$, μ has the additive property if and only if for every charge ν on $(X, \mathcal{F})$ with $\nu \ll \mu$ there exists some $f \in L^1(\mu)$ such that $\nu(A) = \int_A f d\mu$ holds for every $A \in \mathcal{F}$.*

The Hahn decomposition theorem can be generalized to charges as follows.

Theorem 5.1.3. *For a bounded nonnegative charge μ on $(X, \mathcal{F})$ where $\mathcal{F}$ is a σ-algebra, μ has the additive property if and only if for every charge ν on $(X, \mathcal{F})$ such that $\nu \ll \mu$, there exists some $A \in \mathcal{F}$ satisfying the following property; for each $B \in \mathcal{F}$ with $B \subseteq A$, $\nu(B) \geq 0$ holds, and each $B \in \mathcal{F}$ with $B \subseteq A^c$, $\nu(B) \leq 0$ holds.*

The additive property of μ can be characterized by the extended measure $\hat{\mu}$. As we have seen in Section 4.1, we extend a charge space $(X, \mathcal{F}, \mu)$ to $(F, \mathcal{B}(F), \hat{\mu})$. We will need the following formulation of the additive property, which is a version of [2, Theorem 2].

Theorem 5.1.4. *A bounded nonnegative charge μ has the additive property if and only if $\hat{\mu}(U) = \hat{\mu}(\overline{U})$ for every open sets U of supp μ, where $\overline{U}$ denotes the closure of U in supp μ.*

5.2 Additive property of finite sums of charges

In this section, we consider a necessary and sufficient condition that charges which are expressed by finite sums of charges have the additive property. Generally, if charges μ and ν on $(X, \mathcal{F})$ have the additive property, the sum $\mu + \nu$ need not have the additive property. First we have the following result.

Theorem 5.2.1. *Let μ, ν be bounded nonnegative charges on $(X, \mathcal{F})$ such that $\nu \ll \mu$. If μ has the additive property, then ν also has the additive property.*

From this result together with the Lebesgue decomposition theorem, It is sufficient to consider the condition for pairs of charges μ, ν which are mutually singular. It is given by the following.

Theorem 5.2.2. *For any nonnegative and mutually singular charges μ, ν on $(X, \mathcal{F})$ with $\mathcal{F}$ is a σ-algebra, $\mu + \nu$ has the additive property if and only if both μ and ν have the additive property and μ and ν are strongly singular.*

This result can be extended to the case of finite sums of charges immediately:

Theorem 5.2.3. *Let $\mu_1, \mu_2, \ldots, \mu_n$ be nonnegative charges on $(X, \mathcal{F})$ with $\mathcal{F}$ is a σ-algebra and they are mutually singular one another. Then $\mu_1 + \mu_2 + \ldots + \mu_n$ has the additive property if and only if every μ_i, $1 \leq i \leq n$, has the additive property and they are mutually strongly singular.*

5.3 Additive property of countable sums of charges

Now we consider an extension of Theorem 5.2.3. In the previous section, we considered finite sums and here we deal with the additive property of countable sums of charges.

Theorem 5.3.1. *Let $\{\mu_i\}_{i \geq 1}$ be a countable family of bounded nonnegative charges on $(X, \mathcal{F})$ such that they are mutually singular and $\mu = \sum_{i \geq 1} \mu_i$ exists. Let S_i be the support of μ_i and S be the support of μ. Then μ has the additive property if and only if each μ_i has the additive property and they are mutually strongly singular and*

$$\left(\limsup_i S_i\right) \bigcap \bigcup_{i \geq 1} S_i = \emptyset$$

holds, where $\limsup_i S_i = \cap_{i \geq 1} \overline{\cup_{j \geq i} S_j}$.

Proof . (Sufficiency) We prove the condition in Theorem 5.1.4. By the definition of μ, it holds that $\hat{\mu} = \sum_{i \geq 1} \hat{\mu}_i$ and thus $\hat{\mu}$ is on $\cup_{i \geq 1} S_i$. Also since $\hat{\mu}_i$ are mutually strongly singular, $\hat{\mu}(A) = \hat{\mu}(A \cap \cup_{i \geq 1} S_i) = \sum_{i \geq 1} \hat{\mu}(A \cap S_i)$ holds.

In what follows, For any Borel set X of F we denote the closure of $B \subseteq X$ in X by $\overline{B}^X$. In the case of $X = F$, we omit the superscript. Take any $A \in \mathcal{B}(F)$. By the assumption, notice that $S = \cup_{i\geq 1} S_i \cap \limsup_i S_i$ and $\limsup_i S_i \cap \cup_{i\geq 1} S_i = \emptyset$. Since each μ_i has the additive property, from Theorem 5.1.4 and the fact that charges μ_i are mutually strongly singular, we have $\hat{\mu}(\overline{A \cap S_i}) = \hat{\mu}_i(\overline{A \cap S_i}) = \hat{\mu}_i(\overline{A \cap S_i}^{S_i}) = \hat{\mu}_i(A \cap S_i) = \hat{\mu}(A \cap S_i)$.

On the other hand, it holds that $\overline{A \cap S} = \overline{A \cap (\cup_{i\geq 1} S_i \cup \limsup_i S_i)} = \overline{\cup_{i\geq 1}(A \cap S_i)} \cup \overline{A \cap \limsup_i S_i} = \cup_{i\geq 1} \overline{A \cap S_i} \cup \overline{A \cap \limsup_i S_i}$. Together with the fact that $\hat{\mu}(\limsup_i S_i) = 0$, we have

$$\begin{aligned}\hat{\mu}(\overline{A \cap S}^S) &= \hat{\mu}(\overline{A \cap S}) = \hat{\mu}(\cup_{i\geq 1} \overline{A \cap S_i}) + \hat{\mu}(\overline{A \cap \limsup_i S_i}) \\ &= \sum_{i\geq 1} \hat{\mu}(\overline{A \cap S_i}) = \sum_{i\geq 1} \hat{\mu}(A \cap S_i) = \hat{\mu}(A \cap S).\end{aligned}$$

Hence by Theorem 5.1.4 we see that μ has the additive property.

(Necessity) Suppose that $\{\mu_i\}_{i\geq 1}$ are singular and $\mu = \sum_{i\geq 1} \mu_i$ has the additive property. Let μ_n and μ_m be any pair of distinct charges. Put $\mu' = \sum_{i\geq 1, i\neq n} \mu_i$ and since $\mu' \perp \mu_n$ and $\mu = \mu' + \mu_n$, we have that by Theorem 5.2.2 μ' and μ_n have the additive property and they are strongly singular. Next assume to the contrary that $\limsup_i S_i \cap \cup_{i\geq 1} S_i \neq \emptyset$. Fix some $n \geq 1$ with $\limsup_i S_i \cap S_n \neq \emptyset$ and consider the charge $\mu' = \sum_{i\geq 1, i\neq n} \mu_i$. Since the support S' of μ' is $\cup_{i\geq 1, i\neq n} S_i \cup \limsup_i S_i$, $S' \cap S_n \neq \emptyset$ holds and thus μ' and μ_n are not strongly singular. But this contradicts Theorem 5.2.2 by the same arguments above, which completes the proof.

5.4 Additive property of density measures in $\tilde{\mathcal{C}}$

In this section we study the additive property of elements of $\tilde{\mathcal{C}}$. Recall that we say that $\mu \in \tilde{\mathcal{C}}$ has the additive property if for any increasing sequence $A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots$ of $\mathcal{P}(\mathbb{N})$, there exists a set $B \subseteq \mathbb{N}$ such that

(1) $\mu(B) = \lim_k \mu(A_k)$,

(2) $\mu(A_k \setminus B) = 0 \quad \textit{for every } k \in \mathbb{N}$.

Now we have the following theorem.

Theorem 5.4.1. *ν_ω has the additive property if and only if $\omega \in \mathcal{D}_-$.*

Proof. First we introduce the following auxiliary charges; let $\omega = (\eta, t) \in \Omega^*$ and $m = 0, 1, \ldots$.

$$\nu_{\omega,m}(A) = \eta\text{-}\lim_n \frac{|A \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}]]|}{\theta \cdot 2^{n-m-1}}, \quad A \in \mathcal{P}(\mathbb{N}).$$

Then it is obvious that

$$\nu_\omega = \sum_{m=0}^{\infty} \frac{1}{2^m} \nu_{\omega,m}.$$

First we show that every $\nu_{\omega,m}$ satisfy the additive property. Given an increasing sequence $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_i \subseteq \ldots$ of $\mathcal{P}(\mathbb{N})$. Put $\lim_{i\to\infty} \nu_{\omega,m}(A_i) = \alpha$. We take a decreasing sequence $\{X_i\}_{i\geq 1}$ of η such that

$$\left| \frac{|A_i \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}]]|}{\theta \cdot 2^{n-m-1}} - \nu_{\omega,m}(A_i) \right| < \frac{1}{i}$$

whenever $n \in X_i$. Then we define $B \subseteq \mathbb{N}$ as $B \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}]] = A_i \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}]]$ if $n \in X_i \setminus X_{i+1}$ and $B \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}]] = \emptyset$ otherwise. First we show that $\nu_{\omega,m}(B) = \alpha$. For any $\varepsilon > 0$, take $i \in \mathbb{N}$ with $\varepsilon > \frac{1}{i}$ and $\nu_{\omega,m}(A_i) - \alpha < \varepsilon$. Then for $n \in X_i$, there exists some $j \geq i$ such that $n \in X_j \setminus X_{j+1}$. Hence

$$\begin{aligned}
|\nu_{\omega,m}(B) - \alpha| &\leq \limsup_{n\in X_i} \left| \frac{|B \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}]]|}{\theta \cdot 2^{n-m-1}} - \nu_{\omega,m}(A_j) \right| + |\nu_{\omega,m}(A_j) - \alpha| \\
&\leq \limsup_{n\in X_i} \left| \frac{|A_j \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}]]|}{\theta \cdot 2^{n-m-1}} - \nu_{\omega,m}(A_j) \right| + |\nu_{\omega,m}(A_j) - \alpha| \\
&\leq \frac{1}{j} + \varepsilon \leq \frac{1}{i} + \varepsilon < 2\varepsilon.
\end{aligned}$$

Since $X_i \in \eta$, we have $\nu_{\omega,m}(B) = \alpha$. Next we show that $\nu_{\omega,m}(A_i \setminus B) = 0$ for every $i \geq 1$. For any $n \in X_i$, there exists some $j \geq i$ such that $n \in X_j \setminus X_{j+1}$. Hence we have

$$\nu_{\omega,m}(A_i \setminus B) \leq \limsup_{n\in X_i} \left| \frac{|(A_i \setminus B) \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}]]|}{\theta \cdot 2^{n-m-1}} \right| = 0$$

since $B \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}]] = A_j \cap ([\theta \cdot 2^{n-m-1}], [\theta \cdot 2^{n-m}]]$ and $A_i \subseteq A_j$. So we have the result.

Let $S_m = \text{supp } \nu_{\omega,m}$ and $S = \text{supp } \nu_\omega$. By Theorem 6.3.1, it is sufficient to show that $\limsup_m S_m \cap \cup_{i\geq 1} S_i = \emptyset$ if and only if $\omega \in \mathcal{D}_-$. Assume that $\omega = (\eta, t) \in \mathcal{D}_-$ and thus $\eta \in \mathcal{D}_{d,-}$. This means that there exists some $X \in \eta$ such that $X^* \cap \overline{o}_-(\eta) \setminus \{\eta\}$. Put $I_0 = \cup_{n\in X}([\theta \cdot 2^{n-1}], [\theta \cdot 2^n]]$ and it is obvious that $\nu_{\omega,0}(I_0) = 1$, which means that $S_0 \subseteq I_0^*$. Now we show that $I_0^* \cap S_m = \emptyset$ for every $m \geq 1$. In fact, take $X_m \in \eta$ such that $\tau^{-m} X_m \cap X = \emptyset$ and put $I_m = \cup_{n\in\tau^{-m}X_m}([\theta \cdot 2^{n-1}], [\theta \cdot 2^n]]$. Then we have $\nu_{\omega,m}(I_m) = 1$ and $I_0^* \cap I_m^* = \emptyset$. Hence we have that $I_0^* \cap S_m = \emptyset$ for all $m \geq 1$ and then $\limsup_m S_m \cap S_0 = \emptyset$. In a similar way, we can show that $\limsup_m S_m \cap S_i = \emptyset$ for every $i \geq 1$ and thus we have $\limsup_m S_m \cap \cup_{i\geq 1} S_i = \emptyset$.

On the other hand, assume that $\omega \notin \mathcal{D}_-$, i.e., $\eta \in \overline{o}_-(\eta)$. Notice that for any $X \in \eta$ and positive integer $N > 0$ there exists some $n \geq N$ such that $\tau^{-n}\eta \in X^*$.

We show that for any neighborhood I^* of S_0 and positive integer N, there is some $n \geq N$ such that $I^* \cap S_n \neq \emptyset$, which obviously implies $\limsup_m S_m \cap \cup_{i\geq 0} S_i \neq \emptyset$. Let $f(m) \in l^\infty$ be the function $|I \cap ([\theta \cdot 2^{m-1}], [\theta \cdot 2^m]]|/\theta \cdot 2^{m-1}$. Then $\nu_{\omega,0}(I) = \overline{f}(\eta)$ holds. Since $\overline{f}(\eta) = \nu_{\omega,0}(\mathbb{N}) > 0$, there exists a neighborhood X of η such that $\eta' \in X$ implies $\overline{f}(\eta') > 0$. Let $n \geq N$ be any integer such that $\tau^{-n}\eta \in X$. Then $\nu_{\omega,n}(I) = \overline{f}(\tau^{-n}\eta) > 0$, that is, $S_n \cap I^* \neq \emptyset$. We completes the proof.

In the remainder of the section, we consider the existence of a density measure in $\tilde{\mathcal{C}}$ with the additive property. As we have seen in Theorem 5.4.1, there is a close relation between the additive property of density measures in $\tilde{\mathcal{C}}$ and the topological dynamical system $(\mathbb{N}^*, \tau)$ or the flow $(\Omega^*, \{\tau^s\}_{s\in\mathbb{R}})$. Following Chou [7], we say a set $A \subseteq \mathbb{N}$ is *thin* if $A \cap \tau^n A$ is a finite set for each positive integer n. It is obvious that a point $\eta \in \mathbb{N}^*$ is in $\mathcal{W}_d$ if and only if ω is contained in the closure of a thin set A, that is, η contains a thin set A. Chou proved $\mathcal{W}_d$ is dense in $\mathbb{N}^*$ [7, Proposition 1.2]. In particular, together with our result of Theorem 5.4.1, the existence of a density measure ν_ω having the additive property follows immediately.

Lemma 5.4.1. *For a set $A = \{n_k\}_{k=1}^\infty$, A is a thin set if and only if*

$$\liminf_{k\to\infty}(n_k - n_{k-1}) = \infty.$$

Proof. Sufficiency is obvious. Suppose that A is thin and

$$\liminf_{k\to\infty}(n_k - n_{k-1}) = l_A < \infty$$

then $A \cap \tau^{l_A} A$ is an infinite set. It contradicts the assumption that A is thin.

We give the following characterization of a density measure ν_ω with ω in $\mathcal{W}$. Recall that $\nu_\omega = \nu^{[2^\omega]}$ for ω in Ω^*.

Theorem 5.4.2. *For $\omega = (\eta, t)$ in Ω^*, ν_ω has the additive property and the associated free ultrafilter $\mathcal{U} = [2^\omega]$ contains a set $X = \{n_k\}_{k=1}^\infty$ such that*

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty$$

if and only if $\omega \in \mathcal{W}$.

Proof. Note that a free ultrafilter $\mathcal{U} = [2^\omega] = [\theta 2^\eta]$ is generated by the basis $\{[\theta \cdot 2^A] : A \in \eta\}$, where $\theta = 2^t$ and $[\theta \cdot 2^A] = \{[\theta \cdot 2^n] : n \in A\}$ for each $A \in \eta$. First we prove sufficiency. Since $\omega \in \mathcal{W} \subseteq \mathcal{D}_-$, ν_ω has the additive property by Theorem 5.4.1. Take any thin set $A = \{n_k\}_{k=1}^\infty$ contained in η, put $X = [\theta \cdot 2^A] = \{m_k\}_{k=1}^\infty$, then $X \in \mathcal{U}$. By Lemma 5.4.1 we have that

$$\liminf_{k\to\infty} \frac{m_{k+1}}{m_k} = \liminf_{k\to\infty} \frac{[\theta \cdot 2^{n_{k+1}}]}{[\theta \cdot 2^{n_k}]} = 2^{\liminf_{k\to\infty}(n_{k+1}-n_k)} = \infty.$$

Conversely, assume that $\mathcal{U} = [2^\omega]$ contains a set $X = \{m_k\}_{k=1}^\infty$ with $\lim_{k\to\infty} \frac{m_{k+1}}{m_k} = \infty$. Since there is a set $A = \{n_k\}_{k=1}^\infty$ in η such that $[\theta \cdot 2^A] \subseteq X$, then

$$2^{\liminf_{k\to\infty}(n_{k+1}-n_k)} = \liminf_{k\to\infty} \frac{[\theta \cdot 2^{n_{k+1}}]}{[\theta \cdot 2^{n_k}]} \geq \liminf_{k\to\infty} \frac{m_{k+1}}{m_k} = \infty.$$

Hence $\liminf_{k\to\infty}(n_{k+1} - n_k) = \infty$, that is, by Lemma 5.4.1 A is a thin set. Then $\omega \in \mathcal{W}$.

In particular, this result is contained in [4, Theorem 1], which we remarked at Section 1. Then it is natural to ask that whether there exists a density measure $\nu_\omega \in \tilde{\mathcal{C}}$ with the additive property and the associated ultrafilter does not contain a set $\{n_k\}_{k=1}^\infty$ with $\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty$. The answer to this question is affirmative. Notice that from the above theorem, it is equivalent to $\mathcal{W} \subsetneq \mathcal{D}_-$ or, equivalently, $\mathcal{W}_d \subsetneq \mathcal{D}_{d,-}$.

Theorem 5.4.3. *We have* $\mathcal{W}_d \subsetneq \mathcal{D}_{d,-}$.

Proof. We put $\Gamma = \mathbb{N}^* \setminus \mathcal{W}_d$. Since $\mathcal{W}_d$ is an open invariant set, Γ is a closed invariant subset. For any $A \subseteq \mathbb{N}$, we denote $A^* \cap \Gamma$ by $\hat{A}$. Then it is sufficient to show that there exists a set $X \subseteq \mathbb{N}$ such that

$$\hat{X} \subsetneq \cup_{i=1}^l \tau^i \hat{X}$$

for every $l \geq 1$. Indeed, if it is true, it follows that by the compactness of $\hat{X}$, $\hat{X} \setminus (\cup_{i=1}^\infty \tau^i \hat{X}) \neq \emptyset$, and obviously any point in the set is contained in $\mathcal{D}_{d,-} \setminus \mathcal{W}_d$.

Take a set $X \subseteq \mathbb{N}$ and write $X = \{n_k\}_{k=1}^\infty$. We put

$$Y_X = \{m \in \mathbb{N} : |\{k \geq 2 : n_k - n_{k-1} = m\}| = \infty\}$$

and observe that

$$X \setminus (\cup_{i=1}^l \tau^i X) = \{n_i \in X : n_k - n_{k-1} > l\}$$

and

$$\begin{aligned} \hat{X} \setminus (\cup_{i=1}^l \tau^i \hat{X}) \neq \emptyset &\iff (X \setminus \cup_{i=1}^l \tau^i X)\hat{} \neq \emptyset \\ &\iff X \setminus \cup_{i=1}^l \tau^i X \not\subseteq \mathcal{W}^\tau \\ &\iff X \setminus \cup_{i=1}^l \tau^i X \textit{ is not a thin set.} \end{aligned}$$

Hence we obtain that $\hat{X} \subsetneq \cup_{i=1}^l \tau^i \hat{X}$ for any $l \geq 1$ if and only if $\{n_k \in X : n_k - n_{k-1} > l\}$ is not a thin set for any $l \geq 1$, i.e., Y_X is an infinite set. We can see easily that such a set X exists. For example, put $X = \{2^n + 2^k : n \geq 0, 0 \leq k < n\}$.

Therefore for any point $\omega \in \mathcal{D}_- \setminus \mathcal{W}$, the density measure ν_ω gives an example having the additive property but does not satisfy the sufficient condition of [4, Theorem 1].

5.5 Additive property of density measures of a more general form

From Theorems 4.2.1 and Theorem 5.4.1 we get the following result.

Theorem 5.5.1. *Let $\omega \in \mathcal{D}_-$ and μ be a countably additive probability measure on the space $\mathbb{R}_+ = [0, \infty)$ of nonnegative real numbers. Then the density measure ν in $\mathcal{C}$ defined by*

$$\nu(A) = \int_0^\infty \nu_{\tau^{-s}\omega}(A) d\mu(s), \quad A \in \mathcal{P}(\mathbb{N})$$

has the additive property.

Proof. Let $\{A_i\}_{i=1}^\infty$ be an increasing sequence of $\mathcal{P}(\mathbb{N})$. By Theorem 5.4.1 ν_ω has the additive property. Then there is a set $B \in \mathcal{P}(\mathbb{N})$ such that $\lim_i \nu_\omega(A_i) = \nu_\omega(B)$ and $\nu_\omega(A_i \setminus B) = 0$ for every $i \geq 1$. Notice that the latter condition yields that $\hat{A}_i \cap$ supp ν_ω $\subseteq \hat{B} \cap$ supp ν_ω for each $i \geq 1$, then we have that

$$\begin{aligned} \lim_i \nu_\omega(A_i) = \nu_\omega(B) &\Longleftrightarrow \hat{\nu}_\omega(\cup_{i=1}^\infty \hat{A}_i) = \hat{\nu}_\omega(\hat{B}) \\ &\Longleftrightarrow \hat{\nu}_\omega(\hat{B} \setminus (\cup_{i=1}^\infty \hat{A}_i)) = 0. \end{aligned}$$

For any $\omega' \in o_-(\omega)$ since $\nu_{\omega'} \ll \nu_\omega$, it follows that

$$\hat{\nu}_{\omega'}(\hat{B} \setminus (\cup_{i=1}^\infty \hat{A}_i)) = 0 \Longleftrightarrow \lim_i \nu_{\omega'}(A_i) = \nu_{\omega'}(B).$$

Also $\nu_{\omega'}(A_i \setminus B) = 0$ holds for each $i \geq 1$. Then it follows that

$$\nu(A_i \setminus B) = \int_0^\infty \nu_{\tau^{-s}\omega}(A_i \setminus B) d\mu(s) = 0$$

for each $i \geq 1$. Also we have that

$$\begin{aligned} \nu(B) &= \int_0^\infty \nu_{\tau^{-s}\omega}(B) d\mu(s) \\ &= \int_0^\infty \lim_i \nu_{\tau^{-s}\omega}(A_i) d\mu(s) \\ &= \lim_i \int_0^\infty \nu_{\tau^{-s}\omega}(A_i) d\mu(s) = \lim_i \nu(A_i). \end{aligned}$$

Hence ν has the additive property.

Next from Theorems 4.2.6, 5.2.2 and 5.4.1 we obtain the following result.

Theorem 5.5.2. *For any mutually singular finite number of density measures* $\nu_{\omega_1}, \nu_{\omega_2}, \cdots, \nu_{\omega_m}$*, their finite convex combinations*

$$\nu = \sum_{i=1}^{m} c_i \nu_{\omega_i}$$

has the additive property if and only if $\omega_i \in \mathcal{D}_-, i = 1, 2, \cdots, m$ *and orbit closures* $\overline{o}_-(\omega_i), i = 1, 2, \cdots, m$*, are pairwise disjoint.*

Combining Theorems 5.5.1 and 5.5.2, we have that

Theorem 5.5.3. *Take a finite number of points* $\omega_1, \omega_2, \cdots, \omega_m$ *in* $\mathcal{D}_-$ *such that orbit closures* $\overline{o}_-(\omega_1), \overline{o}_-(\omega_2), \cdots, \overline{o}_-(\omega_m)$ *are pairwise disjoint, and let* $\mu_i, i = 1, 2, \cdots, m$*, be countably additive probability measures on* $\mathbb{R}_+ = [0, \infty)$*. Then the density measure* ν *in* $\mathcal{C}$ *defined by*

$$\nu(A) = \sum_{i=1}^{m} c_i \int_0^{\infty} \nu_{\varphi^{-s}\omega_i}(A) d\mu_i(s), \quad A \in \mathcal{P}(\mathbb{N})$$

has the additive property, where $0 \leq c_i \leq 1, i = 1, 2, \cdots, m$ *with* $\sum_{i=1}^{m} c_i = 1$.

Bibliography

[1] A. Basile, K. P. S. Bhaskara Rao, *Completeness of L_p-Spaces in the Finitely Additive Setting and Related Stories,* J. Math. Anal. Appl. **248**, (2000) 588-624.

[2] W. C. Bell, J. W. Hagood, *Completeness of L_p spaces and Radon-Nikodym theorems for unbounded finitely additive measures,* J. Math. Anal. Appl. **218** (1998) 82-96.

[3] K. P. S. Bhaskara Rao, M. Bhaskara Rao, *Theory of charges,* Academic Press, London, 1983.

[4] A. Blass, R. Frankiewicz, G. Plebanek, C. Ryll-Nardzewski, *A note on extensions of asymptotic density,* Proc. Amer. Math. Soc. **129** (11) (2001) 3313-3320.

[5] M. Blümlinger, *Lèvy group action and invariant measures on $\beta\mathbb{N}$,* Trans. Amer. Math. Soc. **348** (12) (1996) 5087-5111.

[6] R. C. Buck, *The measure theoretic approach to density,* Amer. J. Math. **68** (1946) 560-580.

[7] C. Chou, *Minimal sets and ergodic measures for $\beta\mathbb{N}\backslash\mathbb{N}$,* Illinois J. Math. **13** (1969) 777-788.

[8] K. Hrbacek, T. Jech, *Introduction to Set Theory*, Marcel Dekker, New York, 1999.

[9] M. Kac, *Probability methods in some problems of analysis and number theory,* Bull. Amer. Math. Soc. **55** (1949), 641-665.

[10] M. Kac, *Statistical Independence in Probability, Analysis and Number Theory,* Wiley, New York, 1959.

[11] P. Letavaj, L. Mišík, M. Sleziak, *Extreme points of the set of density measures,* J. Math. Anal. Appl. **423** (2015) 1150-1165.

[12] D. Maharam, *Finitely additive measures on the integers,* Sankhya Ser. A **38** (1976) 44-59.

[13] A. H. Mekler, *Finitely additive measures on* $\mathbb{N}$ *and the additive property,* Porc. Amer. Math. Soc. **92** (1984) 439-444.

[14] R. R. Phelps, *Lectures on Choquet's Theorem, 2nd ed.,* Springer-Verlag, Berlin, 2001.

[15] M. Sleziak, *Lèvy group and density measures,* J. Number Theory **128** (12) (2008) 3005-3012.

[16] J. Tanaka, *Flows in fibers,* Trans. Amer. Math. Soc. **343**, (1994) 779-804.

[17] E. K. van Douwen, *Finitely additive measures on* $\mathbb{N}$*,* Topology Appl. **47** (1992) 223-268.

[18] J. de Vries, *Elements of Topological dynamics,* Kluwer Academic Publishers, Dordrecht, 1993.

List of papers by Ryoichi Kunisada

1. *Density measures and additive property*, J. Number Theory, **176** (2017), 184-203.
2. *On a relation between density measures and a certain flow*, Proc. Amer. Math. Soc, to appear.