Studies on nonlinear Schrödinger equations with derivative coupling

微分型相互作用を持つ非線形シュレディンガー方程式の研究

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Chapter 1

Introduction

1.1 Background

In this thesis we study the following equation

$$i\partial_t\psi + \partial_x^2\psi + i\partial_x(|\psi|^2\psi) = 0,\ (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1.1.1}$$

which is known as a derivative nonlinear Schrödinger equation. This equation appears in plasma physics as a model for the propagation of Alfvén waves in magnetized plasma (see [48, 49]) and it is known to be completely integrable (see [40]). The equation (1.1.1) is L^2-critical in the sense that the equation and L^2-norm are invariant under the scaling transformation

$$\psi_\gamma(t,x) := \gamma^{\frac{1}{2}}\psi(\gamma^2 t, \gamma x), \quad \gamma > 0.$$

There is a large literature on the Cauchy problem for the equation (1.1.1). Tsutsumi and Fukuda [68, 69] studied the well-posedness in $H^s(\mathbb{R})$ for $s > 3/2$ by classical energy method which depends on parabolic regularization. The well-posedness in the energy space $H^1(\mathbb{R})$ was first proved by Hayashi [31]. He introduced gauge transformation (see e.g. (1.1.2) or (1.1.13) below) to overcome the derivative loss, and combining with the Strichartz estimate, the well-posedness in $H^1(\mathbb{R})$ was proved. In a later work, Hayashi and Ozawa [32] proved the $H^1(\mathbb{R})$-solution is global if the initial data ψ_0 satisfies $\|\psi_0\|_{L^2}^2 < 2\pi$. Recently, Wu [73] improved this global result, more specifically, he proved that the solution is global if the initial data satisfies $\|\psi_0\|_{L^2}^2 < 4\pi$. We will discuss connection between these global results and solitons later.

For the Cauchy problem for (1.1.1) in $H^s(\mathbb{R})$ with $s < 1$, there are also many works. Takaoka [66] proved that (1.1.1) is locally well-posed in $H^s(\mathbb{R})$ when $s \geq 1/2$ by the Fourier restriction norm method. Biagioni and Linares [9] proved that the solution map from $H^s(\mathbb{R})$ to $C([-T,T] : H^s(\mathbb{R}))$ is not locally uniformly continuous when $s < 1/2$. Colliander, Keel, Staffilani, Takaoka, and Tao [19] proved by the so-called I-method that when $s > 1/2$ the $H^s(\mathbb{R})$-solution is global if the initial data satisfying $\|\psi_0\|_{L^2}^2 < 2\pi$ (see also [18]). Guo and Wu [28] improved their result, that is, they proved that the $H^{1/2}(\mathbb{R})$-solution is global if $\|\psi_0\|_{L^2}^2 < 4\pi$.

There are several forms of (1.1.1) that are equivalent under gauge transformation. By using the following gauge transformation to the solution of (1.1.1)

$$u(t,x) = \psi(t,x)\exp\left(\frac{i}{2}\int_{-\infty}^{x} |\psi(t,x)|^2 dx\right), \tag{1.1.2}$$

then u satisfies the following equation:

$$i\partial_t u + \partial_x^2 u + i|u|^2\partial_x u = 0, \ (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{DNLS}$$

This equation has the following conserved quantities:

$$E(u) := \frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2\, dx - \frac{1}{4}\mathrm{Re}\int_{\mathbb{R}} i|u|^2\partial_x u\bar{u}dx, \tag{Energy}$$

$$M(u) := \int_{\mathbb{R}} |u|^2 dx, \tag{Mass}$$

$$P(u) := \mathrm{Re}\int_{\mathbb{R}} i\partial_x u\bar{u}dx. \tag{Momentum}$$

We note that the equation (DNLS) can be rewritten as

$$i\partial_t u = E'(u). \tag{1.1.3}$$

The Hamiltonian form (1.1.3) is useful when one considers problems of orbital stability/instability of solitons. It is known that (DNLS) has a two-parameter family of solitons (see [40, 17])

$$u_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x-ct), \tag{1.1.4}$$

where (ω, c) satisfies $-2\sqrt{\omega} < c \le 2\sqrt{\omega}$, and

$$\phi_{\omega,c}(x) = \Phi_{\omega,c}(x)\exp\left(i\frac{c}{2}x - \frac{i}{4}\int_{-\infty}^{x}\Phi_{\omega,c}(y)^2 dy\right), \tag{1.1.5}$$

$$\Phi^2_{\omega,c}(x) = \begin{cases} \dfrac{4\omega - c^2}{\sqrt{\omega}\left(\cosh(\sqrt{4\omega - c^2}x) - \frac{c}{2\sqrt{\omega}}\right)} & \text{if } \omega > c^2/4, \\[2ex] \dfrac{4c}{(cx)^2+1} & \text{if } c = 2\sqrt{\omega}. \end{cases} \tag{1.1.6}$$

We note that $\Phi_{\omega,c}$ is the positive radial (even) solution of

$$-\Phi'' + \left(\omega - \frac{c^2}{4}\right)\Phi + \frac{c}{2}|\Phi|^2\Phi - \frac{3}{16}|\Phi|^4\Phi = 0, \tag{1.1.7}$$

and the complex-valued function $\phi_{\omega,c}$ is the solution of

$$-\phi'' + \omega\phi + ic\phi' - i|\phi|^2\phi' = 0. \tag{1.1.8}$$

The equation (1.1.8) can be rewritten as

$$S'_{\omega,c}(\phi) = 0,$$

where the functional $S_{\omega,c}(\phi)$ is defined by

$$S_{\omega,c}(\phi) := E(\phi) + \frac{\omega}{2}M(\phi) + \frac{c}{2}P(\phi).$$

The condition of two parameters (ω, c)

$$-2\sqrt{\omega} < c \leq 2\sqrt{\omega} \tag{1.1.9}$$

is a necessary and sufficient condition for the existence of non-trivial solutions of (1.1.7) vanishing at infinity (see [8]). As can be seen in the explicit formulae of the solitons, (DNLS) has two types of solitons; one has exponential decay and the other has algebraic decay. The latter corresponds to the soliton for the massless case.

Guo and Wu [27] proved that the soliton $u_{\omega,c}$ is orbitally stable when $\omega > c^2/4$ and $c < 0$ by applying the abstract theory of Grillakis, Shatah, and Strauss [24, 25]. Colin and Ohta [17] proved that the soliton $u_{\omega,c}$ is orbitally stable when $\omega > c^2/4$ by applying variational characterization of solitons as in Shatah [64]. The case of $c = 2\sqrt{\omega}$ (massless case) is treated[1] by Kwon and Wu [41], while the orbital stability or instability for the massless case is still an open problem.

From the explicit formulae (1.1.5) and (1.1.6) of solitons, we have

$$M(\phi_{\omega,c}) = M(\Phi_{\omega,c}) = 8\tan^{-1}\sqrt{\frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c}}, \tag{1.1.10}$$

where (ω, c) satisfies (1.1.9) (see [17, Lemma 5] or Section 4.2). If we consider the curve

$$c = 2s\sqrt{\omega} \tag{1.1.11}$$

for $\omega > 0$ and $s \in (-1, 1]$, we have

$$\Phi_{\omega,2s\sqrt{\omega}}(x) = \omega^{\frac{1}{4}}\Phi^2_{1,2s}(\sqrt{\omega}x).$$

This means that the curve (1.1.11) corresponds to the scaling which is invariant of the mass of the soliton. We note that the function

$$s \mapsto M(\Phi_{\omega,2s\sqrt{\omega}}) = 8\tan^{-1}\sqrt{\frac{1+s}{1-s}} \tag{1.1.12}$$

is a strictly increasing function from $(-1, 1]$ to $(0, 4\pi]$. Especially, the threshold value 4π corresponds to the mass of the algebraic soliton.

[1] The "orbital stability" discussed in [41] is different from usual definition. Their result does not contradict that finite time blow-up occurs to the initial data near the soliton for the massless case.

Here, let us review the global results in the energy space $H^1(\mathbb{R})$. We consider another gauge equivalent form of (1.1.1). By using the following gauge transformation to the solution of (DNLS)

$$v(t,x) = u(t,x)\exp\left(\frac{i}{4}\int_{-\infty}^{x}|u(t,x)|^2dx\right), \tag{1.1.13}$$

then v satisfies the following equation:

$$i\partial_t v + \partial_x^2 v + \frac{i}{2}|v|^2\partial_x v - \frac{i}{2}v^2\partial_x\overline{v} + \frac{3}{16}|v|^4 v = 0,\ (t,x)\in\mathbb{R}\times\mathbb{R}. \tag{1.1.14}$$

Conserved quantities of (DNLS) are transformed as follows;

$$\mathcal{E}(v) := \frac{1}{2}\int_{\mathbb{R}}|\partial_x v|^2dx - \frac{1}{32}\int_{\mathbb{R}}|v|^6dx, \tag{1.1.15}$$

$$\mathcal{M}(v) := \int_{\mathbb{R}}|v|^2dx, \tag{1.1.16}$$

$$\mathcal{P}(v) := \mathrm{Re}\int_{\mathbb{R}} i\partial_x v\overline{v}dx + \frac{1}{4}\int_{\mathbb{R}}|v|^4dx. \tag{1.1.17}$$

The gauge transformation (1.1.13) was first derived in [32] to cancel out the interaction term with derivative in the energy functional. Hayashi and Ozawa [32] used the following sharp Gagliardo–Nirenberg inequality

$$\|f\|_{L^6}^6 \le \frac{4}{\pi^2}\|f\|_{L^2}^4\|\partial_x f\|_{L^2}^2 \tag{1.1.18}$$

in order to obtain a priori estimate in $\dot{H}^1(\mathbb{R})$ by using conservation laws of the mass and the energy. They proved the $H^1(\mathbb{R})$-solution of (1.1.14) is global if the initial data u_0 satisfies

$$\mathcal{M}(u_0) < \mathcal{M}(Q) = 2\pi, \tag{1.1.19}$$

where Q is defined by $Q := \Phi_{1,0}$. We note that Q is an optimal function for the inequality (1.1.18). This result is closely related to the earlier work by Weinstein [71] for focusing L^2-critical nonlinear Schrödinger equations. Consider the following quintic nonlinear Schrödinger equation:

$$i\partial_t u + \partial_x^2 u + \frac{3}{16}|u|^4 u = 0,\ (t,x)\in\mathbb{R}\times\mathbb{R}. \tag{1.1.20}$$

The equation (1.1.20) has the same energy $\mathcal{E}(u)$ of (1.1.15) and the same standing wave $e^{it}Q$ as the equation (1.1.14). Furthermore, (1.1.14) and (1.1.20) are L^2-critical in the sense that the equation and L^2-norm are invariant under the scaling transformation

$$u_\gamma(t,x) := \gamma^{\frac{1}{2}}u(\gamma^2 t,\gamma x),\quad \gamma > 0. \tag{1.1.21}$$

Weinstein [71] proved that if the initial data of (1.1.20) satisfies the mass condition (1.1.19), then the $H^1(\mathbb{R})$-solution is global. In the case of (1.1.20), it is known that this mass condition is sharp, in the sense that for any $\rho \geq 2\pi$, there exists $u_0 \in H^1(\mathbb{R})$ such that $\mathcal{M}(u_0) = \rho$ and such that corresponding solution u to (1.1.20) blows up in finite time. From this analogy, Hayashi and Ozawa [32] conjectured that the mass condition (1.1.19) is also sharp for the equation (1.1.14) (equivalently (1.1.1) or (DNLS)).

A similar analogy can be seen for the quintic generalized Korteweg-de Vries equation:

$$\partial_t u + \partial_x^3 u + \frac{3}{16}\partial_x(u^5) = 0, \ (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1.1.22}$$

This equation is also the L^2-critical equation which has the same energy $\mathcal{E}(u)$ as (1.1.14) and the traveling wave solution $Q(x-t)$. Hence, if the initial data of (1.1.22) satisfies the mass condition (1.1.19), then the $H^1(\mathbb{R})$-solution is global. It is also known that the mass condition for (1.1.22) is sharp; more precisely, the $H^1(\mathbb{R})$-solution of (1.1.22) blows up in finite time to the initial data satisfying

$$\mathcal{E}(u_0) < 0, \ \mathcal{M}(Q) < \mathcal{M}(u_0) < \mathcal{M}(Q) + \varepsilon$$

for small $\varepsilon > 0$ and some decay condition; see [47, 46].

However, the mass condition (1.1.19) is not sharp to the equation (1.1.14) (equivalently (1.1.1) or (DNLS)). Wu [72, 73] took advantage of conservation law of the momentum as well as conservation laws of the mass and the energy. He used the following sharp Gagliardo–Nirenberg inequality

$$\|f\|_{L^6}^6 \leq 3(2\pi)^{-\frac{2}{3}} \|f\|_{L^4}^{\frac{16}{3}} \|\partial_x f\|_{L^2}^{\frac{2}{3}} \tag{1.1.23}$$

in his argument to connect the estimates obtained from the energy (1.1.15) and the momentum (1.1.17) (see also [28]). Then, he proved that the $H^1(\mathbb{R})$-solution of (1.1.14) is global if the initial data u_0 satisfies

$$\mathcal{M}(u_0) < \mathcal{M}(W) = 4\pi, \tag{1.1.24}$$

where W is defined by $W := \Phi_{1,2}$. We note that W is an optimal function for the inequality (1.1.23).

One of the main reason why the difference of global results as described above occurs is due to that the equation (1.1.14) has a two-parameter family of solitons. The algebraic soliton corresponds to the threshold for the existence of solitons, and the value 4π corresponds to the mass of the algebraic soliton. Hence, it is reasonable to conjecture that 4π is an optimal upper bound of the mass for the global existence of $H^1(\mathbb{R})$-solutions by the analogy with (1.1.20) and (1.1.22) as L^2-critical equations. However, existence of blow-up solutions for the derivative nonlinear Schrödinger equation is a large open problem. It is known that finite time blow-up occurs for the equation (1.1.1) on a bounded interval or on the half line, with Dirichlet boundary condition (see [67, 72]), but unfortunately one can not apply these proofs to the whole line case. We also refer to [44, 15] for numerical approaches to this problem.

Recently, in [37] it was proved by inverse scattering approach (see also [61, 62] for related works) that the equation (DNLS) is globally well-posed for any initial data belonging to weighted Sobolev space $H^{2,2}(\mathbb{R})$, where

$$H^{2,2}(\mathbb{R}) := \left\{ u \in H^2(\mathbb{R}) \ ; \ \langle \cdot \rangle^2 u \in L^2(\mathbb{R}) \right\}.$$

This is the strong result for the global well-posedness to (DNLS), however, the dynamics in the energy space $H^1(\mathbb{R})$ (especially above the mass threshold 4π) is still unclear. We note that the algebraic solitons do not contain in $H^{2,2}(\mathbb{R})$, but they contain in $H^1(\mathbb{R})$. Therefore, the difference of functional spaces is quite important for (DNLS) from the viewpoint of solitons. We also note that the results in [37] do not imply the nonexistence of blow-up solutions for (DNLS) in the energy space $H^1(\mathbb{R})$; see blow-up criteria in [41].

Our main aim of this thesis is to investigate the structure of the equation (DNLS) from the viewpoints of the solitons. One of the main theorem in this thesis is to establish a sufficient condition for global existence of the solutions to (DNLS) by variational approach. Our variational approach recovers Wu's global results and clarifies the connection between the 4π-mass condition and potential well generated by the ground states. Moreover we establish the new global result; if the initial data $u_0 \in H^1(\mathbb{R})$ of (DNLS) satisfies

$$M(u_0) = 4\pi \text{ and } P(u_0) < 0,$$

then the corresponding $H^1(\mathbb{R})$-solution exists in globally in time. This gives the first progress to investigate the dynamics around the algebraic soliton. Furthermore, we establish the global result for oscillating data which contains the initial data with arbitrarily large mass. We note that the proofs for these theorems are done by essentially using the properties of two-parameter of the solitons, and especially the algebraic soliton plays an important role in the proof.

One of the significant advantage of our variational approach is that we do not need any structure of integrability. This means that our arguments are applicable to more general equations. In this thesis we also study naturally generalized equations of (DNLS); see the next section for more details. The deep understanding of these generalized equations is expected to be useful for further progress to the study on (DNLS).

1.2 Organization of the thesis

We briefly state the organization of this thesis. In Chapter 2 we study the generalized derivative nonlinear Schrödinger equation:

$$\text{(gDNLS)} \qquad i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \ \sigma > 0,$$

which was introduced by Liu, Simpson, and Sulem [45] to understand the structural properties of (DNLS). The equation (gDNLS) is invariant under the scaling transformation

$$u_\gamma(t,x) := \gamma^{\frac{1}{2\sigma}} u(\gamma^2 t, \gamma x), \quad \gamma > 0,$$

which implies that the critical Sobolev exponent is $s_c = \frac{1}{2} - \frac{1}{2\sigma}$. We note that the case $0 < \sigma < 1$ corresponds to L^2-subcritical case and the case $\sigma > 1$ corresponds to L^2-supercritical case. In [45] they studied the orbital stability/instability of the solitary waves for (gDNLS), however the well-posedness in the energy space $H^1(\mathbb{R})$ was assumed. Before our work well-posedness results for (gDNLS) were partially known (see Chapter 2 for the details), but the well-posedness in the energy space was unsolved. In Chapter 2 we study the Cauchy problem for the equation (gDNLS) with a focus on the well-posedness in the energy space. In the L^2-supercritical case, we construct the solutions by proving that approximate solutions form a Cauchy sequence in appropriate Banach spaces, which gives a more constructive proof compared to the one by compactness arguments. We also study global existence for (gDNLS) in the energy space in the L^2-subcritical case.

In Chapter 3 we study global existence of solutions for (DNLS) and (gDNLS) in the L^2-supercritical setting. Based on the local well-posedness results in Chapter 2, we establish a sufficient condition for global existence of the solutions by variational approach. First we give a variational characterization of two types of the solitons. Then, combined with potential well theory, we give a sufficient condition for global existence in the energy space. The key step is to examine the invariant sets represented by potential well. Especially, in the case of (DNLS) we clarify the connection between the 4π-mass condition and potential well generated by the ground states, and reprove Wu's global results. Moreover, we prove that the $H^1(\mathbb{R})$-solution to (DNLS) is global if the initial data u_0 satisfies that $M(u_0) = 4\pi$ and the momentum $P(u_0)$ is negative. We also see that global results for arbitrarily large mass are obtained by variational approach.

In Chapter 4 we consider the nonlinear Schrödinger equation of derivative type:

$$\text{(DNLSb)} \qquad i\partial_t u + \partial_x^2 u + i|u|^2\partial_x u + b|u|^4 u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R},\ b \in \mathbb{R}.$$

If $b = 0$, or course, this equation is nothing but the equation (DNLS). The equation (DNLSb) can be considered as a generalized equation of (DNLS) while preserving both L^2-criticality and Hamiltonian structure. The main aim of this chapter is to investigate global well-posedness in the energy space $H^1(\mathbb{R})$ for the equation (DNLSb) from the viewpoints of the solitons. We extend the global results for (DNLS) to the equation (DNLSb) by variational approach developed in the Chapter 3. Interestingly, if $b < 0$, 4π-mass condition for (DNLS) is improved due to the defocusing effect from the quintic term. The orbital stability of the solitons is also studied. The stability of the solitons is closely related to the mass condition for global existence in the energy space. We see that the effect of the momentum plays an important role in the arguments on both global existence and stability of the solitons.

In Chapter 5 we study the periodic traveling wave solutions of (DNLS). To investigate further properties of the solitons, we construct exact periodic traveling wave solutions which yield the solitons on the whole line including the massless case in the long-period limit. Moreover, we study the regularity of the convergence of these exact solutions in the long-period limit. Throughout the chapter, the theory of elliptic functions and elliptic integrals is used in the calculation.

Chapter 2

The Cauchy problem for generalized derivative NLS equations

2.1 Introduction

We consider the Cauchy problem for the following generalized derivative nonlinear Schrödinger equation (gDNLS) with the Dirichlet boundary condition

$$\text{(2.1.1)} \qquad \begin{cases} i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0 & \text{in} \quad \mathbb{R}\times\Omega, \\ u = 0 & \text{on} \quad \mathbb{R}\times\partial\Omega, \\ u(0) = u_0 & \text{on} \quad \Omega, \end{cases}$$

where u is a complex valued function of $(t, x) \in \mathbb{R} \times \Omega$, $\sigma > 0$ and $\Omega \subset \mathbb{R}$ is an open interval. With $\sigma = 1$, (2.1.1) has appeared as a model for ultrashort optical pulses [50]. For simplicity we consider the case $\Omega = \mathbb{R}$ here, but we note that our approach is applicable to (2.1.1) with a general open interval Ω in the mostly same way. So we study the Cauchy problem for the following equation:

$$\text{(gDNLS)} \qquad i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0, \ (t, x) \in \mathbb{R}\times\mathbb{R}, \ \sigma > 0.$$

The solution of (gDNLS) obeys formally the following energy, mass and momentum conservation laws:

$$\text{(Energy)} \qquad E(u) := \frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 dx - \frac{1}{2\sigma+2}\mathrm{Re}\int_{\mathbb{R}} i|u|^{2\sigma}\partial_x u\bar{u}dx = E(u_0),$$

$$\text{(Mass)} \qquad M(u) := \int_{\mathbb{R}} |u|^2 dx,$$

$$\text{(Momentum)} \qquad P(u) := \mathrm{Re}\int_{\mathbb{R}} i\partial_x u\bar{u}dx.$$

There are only a few results for the equation (gDNLS) with general exponents $\sigma > 0$, as compared with $\sigma = 1$. Hao [30] proved local well-posedness in $H^{1/2}(\mathbb{R})$ intersected with an appropriate Strichartz space for $\sigma \geq 5/2$ by using the gauge transformation and the Littlewood-Paley decomposition. Liu, Simpson and Sulem [45] studied the orbital stability/instability of solitary waves for (gDNLS); see Chapter 3 for more details. We should note that in [45] they assumed the well-posedness in the energy space $H^1(\mathbb{R})$ for general $\sigma > 0$. Ambrose and Simpson [1] proved the existence and uniqueness of solutions $u \in C([0,T], H^2(\mathbb{T}))$ and the existence of solution $u \in L^\infty((0,T), H^1(\mathbb{T}))$ for $\sigma \geq 1$. The construction of solutions is done by a compactness argument and the uniqueness of $H^1(\mathbb{T})$-solutions is unsolved. Recently, Santos [63] proved the existence and uniqueness of solutions $u \in L^\infty((0,T), H^{3/2}(\mathbb{R}) \cap \langle x \rangle^{-1} H^{1/2}(\mathbb{R}))$ for sufficient small initial data in the case of $1/2 < \sigma < 1$. The proof of [63] is based on parabolic regularization and smoothing properties associated with the Schrödinger group, where the weighted Sobolev space is essential to control the mixed norm $L^p_x L^q_t$. He also proved the existence and uniqueness of solutions $u \in C([0,T], H^{1/2}(\mathbb{R}))$ for sufficient small initial data in the case of $\sigma > 1$.

The main aim of this chapter is to prove the well-posedness for (gDNLS) in $H^1(\mathbb{R})$ and $H^2(\mathbb{R})$ when $\sigma \geq 1/2$. In the case of $1/2 \leq \sigma < 1$, the nonlinear term $|u|^{2\sigma}$ is not even C^2, and therefore a delicate argument is necessary. Our first result is the local well-posedness in $H^2(\mathbb{R})$ when $\sigma \geq 1/2$.

Theorem 2.1.1. *Let $\sigma \geq 1/2$. For any $u_0 \in H^2(\mathbb{R})$, there exist $T > 0$ and a unique solution $u \in C([-T,T], H^2(\mathbb{R}))$ of* (gDNLS)*. Moreover, the solution u depends continuously on u_0 in the following sense: If $u_{0n} \to u_0$ in $H^2(\mathbb{R})$ as $n \to \infty$ and if u_n is the corresponding solution of* (gDNLS)*, then u_n is defined on the same interval $[-T,T]$ for n large enough and $u_n \to u$* in *$C([-T,T], H^s(\mathbb{R}))$ as $n \to \infty$ for all $0 \leq s < 2$.*

Remark 2.1.2. When $\sigma = 1/2$, the nonlinear term $i|u|\partial_x u$ is quadratic. Christ [16] considered the following Cauchy problem:

$$\begin{cases} i\partial_t u + \partial_x^2 u + iu\partial_x u = 0, \ t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}, \end{cases} \tag{2.1.2}$$

and it was proved that the flow map of (2.1.2) is not continuous in $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$. Theorem 2.1.1 tells us that the behavior of the solution of (gDNLS) is very different from that of the solution of (2.1.2) even though both equations have the quadratic nonlinear term with derivative.

The proof of Theorem 2.1.1 proceeds by four steps. We first employ a Yosida-type regularization and construct approximate solutions. Next, we follow an argument in [1] and obtain the uniform estimate on the approximate solutions in $H^1(\mathbb{R})$ by using the conservation laws. Under the uniform bound in $H^1(\mathbb{R})$, we obtain uniform estimates in $H^2(\mathbb{R})$ by estimating time derivative of approximate solutions. More precisely, we differentiate the equation once in time instead of differentiating twice the equation in space in order to obtain $H^2(\mathbb{R})$-estimates. This enables us to relax the smoothness condition of the nonlinear term. This idea comes from Kato [39]. Finally, we prove the sequence of approximate solutions forms a Cauchy sequence in $L^2(\mathbb{R})$ and construct the solution of (gDNLS) by the completeness of Banach space directly. We remark that our construction of solutions does not need any compactness theorem, for example, Ascoli-Arzelà's theorem, Rellich-Kondrachov's theorem, Banach-Alaoglu's theorem, etc.

Santos [63] proved the uniqueness in $L^\infty((0,T), H^{3/2}(\mathbb{R}) \cap \langle x\rangle^{-1} H^{1/2}(\mathbb{R}))$ for $1/2 < \sigma < 1$. We see that it is not necessary to use the weighted Sobolev space for the uniqueness as follows.

Theorem 2.1.3. *Let $\sigma \geq 1/2$. Let $u_0 \in H^{3/2}(\mathbb{R})$ and $T > 0$. If u and v are two solutions of* (gDNLS) *in $L^\infty((-T,T), H^{3/2}(\mathbb{R}))$ with the same initial data, then $u = v$.*

Our proof of Theorem 2.1.3 is based on Yudovich type argument [38]. Related proofs for nonlinear Schrödinger equations with pure power nonlinearities are given in [70], [55], [56].

The main result in this chapter is the local well-posedness in the energy space $H^1(\mathbb{R})$ for $\sigma \geq 1$.

Theorem 2.1.4. *Let $\sigma \geq 1$. Let $u_0 \in H^1(\mathbb{R})$. Then there exist $0 < T_{\min}, T_{\max} \leq \infty$ and a unique maximal solution $u \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R})) \cap L^4_{\mathrm{loc}}((-T_{\min}, T_{\max}), W^{1,\infty}(\mathbb{R}))$ of* (gDNLS). *Moreover, the following properties hold:*

(i) *If $T_{\max} < \infty$ (resp., if $T_{\min} < \infty$), then $\|\partial_x u(t)\|_{L^2} \to \infty$ as $t \uparrow T_{\max}$ (resp., as $t \downarrow -T_{\min}$).*

(ii) *$u \in L^q_{\mathrm{loc}}((-T_{\min}, T_{\max}), W^{1,r}(\mathbb{R}))$ for every admissible pair (q,r), i.e., (q,r) satisfying $0 \leq 2/q = 1/2 - 1/r \leq 1/2$.*

(iii) *$E(u(t)) = E(u_0)$, $M(u(t)) = M(u_0)$, and $P(u(t)) = P(u_0)$ for all $t \in (-T_{\min}, T_{\max})$.*

(iv) *Continuous dependence is satisfied in the following sense; if $u_{0n} \to u_0$ in $H^1(\mathbb{R})$ and if $I \subset (-T_{\min}(u_0), T_{\max}(u_0))$ is a closed interval, then the maximal solution u_n of* (gDNLS) *with $u_n(0) = u_{0n}$ is defined on I for n large enough and satisfies $u_n \to u$ in $C(I, H^1(\mathbb{R}))$.*

The proof of Theorem 2.1.4 depends on the gauge transformation and the Strichartz estimate. We employ $H^2(\mathbb{R})$-solutions constructed in Theorem 2.1.1 as approximate solutions. First, we derive the differential equation by using the gauge transformation that the spatial derivative of approximate solutions should satisfy. Next, we obtain the uniform estimate on approximate solutions in $L^q_t W^{1,r}_x$ for any admissible pair (q,r) by using the Strichartz estimate. Finally, we prove the sequence of approximate solutions forms a Cauchy sequence in $L^2(\mathbb{R})$ and construct the $H^1(\mathbb{R})$-solution of (gDNLS). The last step is similar to that of the proof of Theorem 2.1.1. This method is required that the nonlinear term belongs to C^2, so we need to assume $\sigma \geq 1$. We note that our approach gives alternative proof even for the case of $\sigma = 1$ since we do not covert the equation into some system of equations as can be seen in [31, 33, 34].

From the conservation of mass and energy, one can prove the global well-posedness for small initial data in $H^1(\mathbb{R})$.

Theorem 2.1.5. *Let $\sigma > 1$. Then there exists $\varepsilon_0 > 0$ such that if $u_0 \in H^1(\mathbb{R})$ satisfies $\|u_0\|_{H^1} \le \varepsilon_0$, then there exists a unique solution $u \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^4_{\mathrm{loc}}(\mathbb{R}, W^{1,\infty}(\mathbb{R}))$ of* (gDNLS)*. Moreover, the following properties hold:*

(i) *$u \in L^q_{\mathrm{loc}}(\mathbb{R}, W^{1,r}(\mathbb{R}))$ for every admissible pair (q, r).*

(ii) *$E(u(t)) = E(u_0)$, $M(u(t)) = M(u_0)$, and $P(u(t)) = P(u_0)$ for all $t \in \mathbb{R}$.*

(iii) *Continuous dependence is satisfied in the following sense; if $u_{0n} \to u_0$ in $H^1(\mathbb{R})$ as $n \to \infty$ and if u_n is the global $H^1(\mathbb{R})$-solution of* (gDNLS) *with $u_n(0) = u_{0n}$, then $u_n \to u$* in *$C([-T, T], H^1(\mathbb{R}))$ for all $T > 0$.*

For the case of $\sigma < 1$, we have the following result.

Theorem 2.1.6. *Let $0 < \sigma < 1$. Let $u_0 \in H^1(\mathbb{R})$. Then there exists a solution $u \in (C_w \cap L^\infty)(\mathbb{R}, H^1(\mathbb{R}))$ of* (gDNLS)*. Moreover, we have*

$$E(u(t)) \le E(u_0),\ M(u(t)) = M(u_0) \text{ and } P(u(t)) = P(u_0)$$

for all $t \in \mathbb{R}$.

When $0 < \sigma < 1$, we do not need to assume the smallness of the initial data for the global existence of the solution. This is not surprising since the case $0 < \sigma < 1$ corresponds to L^2-subcritical setting. The solution in Theorem 2.1.6 is constructed by a compactness argument, and we do not know whether the solution is unique or not. If uniqueness holds in $L^\infty(\mathbb{R}, H^1(\mathbb{R}))$, one can prove easily that $E(u(t)) = E(u_0)$ for all $t \in \mathbb{R}$ and that $u \in C(\mathbb{R}, H^1(\mathbb{R}))$.

The rest of this chapter is organized as follows. Section 2.2 is concerned with local well-posedness in $H^2(\mathbb{R})$ and Theorem 2.1.1 is proved there. In Section 2.3 we prove Theorem 2.1.3. In Section 2.4 we study the well-posedness in the energy space $H^1(\mathbb{R})$ and prove Theorem 2.1.4 and Theorem 2.1.5. Finally we prove Theorem 2.1.6 in Section 2.5.

2.2 Local well-posedness in $H^2(\mathbb{R})$

2.2.1 Approximate solutions

Let $g(u)$ and $G(u)$ be defined by

$$\begin{aligned} g(u) &= i|u|^{2\sigma}\partial_x u, \\ G(u) &= \frac{1}{2\sigma + 2}\mathrm{Re}\int_{\mathbb{R}} i|u|^{2\sigma}\partial_x u\bar{u}\,dx \end{aligned}$$

for $\sigma > 0$. We consider $L^2(\mathbb{R})$ as a real Hilbert space with the scalar product

$$(u, v) = \mathrm{Re}\int_{\mathbb{R}} u(x)\overline{v(x)}\,dx \quad \text{for } u, v \in L^2(\mathbb{R}).$$

Then we have

$$G \in C^1(H^1(\mathbb{R}), \mathbb{R}),\ G' = g,$$

with the following identification

$$H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \simeq L^2(\mathbb{R})^* \subset H^{-1}(\mathbb{R}).$$

For any $m \in \mathbb{N}$, we consider the following approximate problem:

$$\begin{cases} i\partial_t u_m + \partial_x^2 u_m + J_m g(J_m u_m) = 0, \\ u_m(0) = u_0, \end{cases} \tag{2.2.1}$$

where J_m is Yosida type approximation defined by

$$J_m = \left(I - \frac{1}{m}\partial_x^2\right)^{-1}. \tag{2.2.2}$$

We recall the following basic properties of J_m. For the proof we refer to [13].

Proposition 2.2.1. *Let X be any of the spaces $H^2(\mathbb{R})$, $H^1(\mathbb{R})$, $H^{-1}(\mathbb{R})$, and $L^p(\mathbb{R})$ with $1 < p < \infty$ and let X^* be its dual space. Then the following properties hold:*

(i) $\langle J_m f, g\rangle_{X,X^*} = \langle f, J_m g\rangle_{X,X^*}$ $\forall f \in X$ $\forall g \in X^*$.

(ii) $J_m \in \mathcal{L}(L^2(\mathbb{R}), H^2(\mathbb{R}))$.

(iii) $\|J_m\|_{\mathcal{L}(X,X)} \le 1$.

(iv) $J_m u \to u$ *in* X $(m \to \infty)$ $\forall u \in X$.

(v) $\sup_{m\in\mathbb{N}} \|u_m\|_X < \infty \ \Rightarrow J_m u_m - u_m \rightharpoonup 0$ *in* X *as* $m \to \infty$.

Let $\sigma \ge 1/2$. Given $u_0 \in H^2(\mathbb{R})$. By Proposition 2.2.1 and the Banach fixed-point theorem, for each $m \in \mathbb{N}$ there exists $T_m > 0$ and $u_m \in C([-T_m, T_m], H^2(\mathbb{R}))$ which is a solution of the initial value problem (2.2.1).

Next, we establish the uniform bounds on the solutions in $H^2(\mathbb{R})$ with respect to m. This will allow us to construct a solution of (gDNLS) in the limit as $m \to \infty$. We define the functions g_m and G_m by

$$g_m(u) = J_m(g(J_m u)) \quad \text{and} \quad G_m(u) = G(J_m u).$$

Then we see that

$$G_m \in C^1(H^1(\mathbb{R}), \mathbb{R}),\ G_m' = g_m.$$

The energy of the equation (2.2.1) is given by the following:

$$E_m(u) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 dx - G_m(u). \tag{2.2.3}$$

A standard calculation shows that conservation laws of energy, mass and momentum hold for the approximate problem.

Lemma 2.2.2. *For each $m \in \mathbb{N}$, the $H^2(\mathbb{R})$-solution u_m of* (2.2.1) *satisfies*

$$E_m(u_m(t)) = E_m(u_0), M(u_m(t)) = M(u_0) \text{ and } P(u_m(t)) = P(u_0)$$

for all $t \in [-T_m, T_m]$.

We need the following lemma to obtain the uniform $H^1(\mathbb{R})$-estimate of $\{u_m\}$.

Lemma 2.2.3. *For any $r \geq 1$ there exists $C > 0$ such that*

$$\frac{d}{dt}\int_{\mathbb{R}} |u_m|^{2r} dx \leq C\big(1 + \|u_m\|_{H^1}^2\big)^{r+\sigma},$$

where the positive constant C is independent of m.

Proof. By the equation (2.2.1) and Hölder's inequality, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} |u_m|^{2r} dx &= \int_{\mathbb{R}} 2r|u_m|^{2(r-1)}\mathrm{Re}(\partial_t u_m \overline{u_m}) \\
&= \int_{\mathbb{R}} 2r|u_m|^{2(r-1)}\mathrm{Im}\Big((-\partial_x^2 u - g_m(u_m))\overline{u_m}\Big) \\
&= \int_{\mathbb{R}} 2r\mathrm{Im}\Big(\partial_x u_m \partial_x(|u_m|^{2(r-1)}\overline{u_m}) - |u_m|^{2(r-1)} g_m(u_m)\overline{u_m}\Big) \\
&\leq C\big(\|u_m\|_{L^\infty}^{2(r-1)}\|\partial_x u_m\|_{L^2}^2 + \|u_m\|_{L^\infty}^{2(r+\sigma-1)}\|\partial_x u_m\|_{L^2}\|u_m\|_{L^2}\big) \\
&\leq C\big(1 + \|u_m\|_{H^1}^2\big)^{r+\sigma}.
\end{aligned}$$

This completes the proof. □

We derive the uniform bound in $H^1(\mathbb{R})$ for $\{u_m\}$ by Lemma 2.2.2 and Lemma 2.2.3. We note that

$$\begin{aligned}
\|u_m\|_{H^1}^2 &= \|u_m\|_{L^2}^2 + \|\partial_x u_m\|_{L^2}^2 \\
&= \|u_m\|_{L^2}^2 + 2\left(E_m(u_m) + G_m(u_m)\right).
\end{aligned}$$

By Cauchy-Schwarz's inequality and Young's inequality, we obtain that

$$2G_m(u_m) \leq \frac{1}{\sigma+1}\|\partial_x u_m\|_{L^2}\|u_m\|_{L^{4\sigma+2}}^{2\sigma+1} \leq \frac{1}{2}\|\partial_x u_m\|_{L^2}^2 + \frac{1}{2}\|u_m\|_{L^{4\sigma+2}}^{4\sigma+2}.$$

This yields that

$$\|u_m\|_{H^1}^2 \leq M(u_m) + 2E_m(u_m) + \frac{1}{2}\int_{\mathbb{R}} |u_m|^{4\sigma+2} dx + \frac{1}{2}\|\partial_x u_m\|_{L^2}^2.$$

Hence, we have

$$\|u_m\|_{H^1}^2 \leq 2M(u_m) + 4E_m(u_m) + \int_{\mathbb{R}} |u_m|^{4\sigma+2} dx. \tag{2.2.4}$$

We introduce the following energy:

$$\mathcal{E}_m(u) = 2M(u) + 4E_m(u) + \int_{\mathbb{R}} |u|^{4\sigma+2} dx.$$

By using Lemma 2.2.2, Lemma 2.2.3 and (2.2.4), we conclude that

$$\frac{d}{dt}\mathcal{E}_m(u_m) \leq C\big(1 + \mathcal{E}_m(u_m)\big)^{3\sigma+1}. \tag{2.2.5}$$

The estimates (2.2.4) and (2.2.5) imply that there exists $T_0 > 0$ such that for all $m \in \mathbb{N}$ such that u_m exists on the time interval $[-T_0, T_0]$ and

$$M_0 := \sup_{m\in\mathbb{N}} \|u_m\|_{C([-T_0,T_0],H^1)} < \infty. \tag{2.2.6}$$

We note that T_0 depends on $\|u_0\|_{H^1}$.

Next, we establish the uniform $H^2(\mathbb{R})$-estimate of $\{u_m\}$.

Lemma 2.2.4. *There exists $T = T(\|u_0\|_{H^2}) > 0$ which is independent of m such that $u_m \in C([-T,T], H^2(\mathbb{R}))$ for all $m \in \mathbb{N}$ and*

$$M := \sup_{m\in\mathbb{N}} \|u_m\|_{C([-T,T],H^2)} < \infty. \tag{2.2.7}$$

Proof. We estimate $L^2(\mathbb{R})$-norm of the time derivative of u_m as

$$\begin{aligned}
\frac{d}{dt}\|\partial_t u_m\|_{L^2}^2 &= 2\Big(\partial_t^2 u_m, \partial_t u_m\Big) \\
&= -2\Big(\partial_t(|u_m|^{2\sigma}\partial_x u_m), \partial_t u_m\Big) \\
&= -2\Big(\partial_t(|u_m|^{2\sigma})\partial_x u_m, \partial_t u_m\Big) - 2\Big(|u_m|^{2\sigma}\partial_x\partial_t u_m, \partial_t u_m\Big) \\
&\leq C\|u_m\|_{L^\infty}^{2\sigma-1}\|\partial_x u_m\|_{L^\infty}\|\partial_t u_m\|_{L^2}^2,
\end{aligned}$$

where in the last inequality we used integration by parts. By Sobolev's embedding and (2.2.6), we obtain that

$$\frac{d}{dt}\|\partial_t u_m\|_{L^2}^2 \leq CM_0^{2\sigma-1}\|\partial_x u_m\|_{L^\infty}\|\partial_t u_m\|_{L^2}^2. \tag{2.2.8}$$

From the equation (2.2.1), we obtain that

$$\begin{aligned}
\|\partial_x^2 u_m\|_{L^2} &\leq \|\partial_t u_m\|_{L^2} + \|J_m g_m(J_m u_m)\|_{L^2} \\
&\leq \|\partial_t u_m\|_{L^2} + CM_0^{2\sigma+1}.
\end{aligned} \tag{2.2.9}$$

By Sobolev's embedding and the conservation of mass,

$$\begin{aligned}\|\partial_x u_m\|_{L^\infty} &\le C\|u_m\|_{H^2}\\ &\le C(\|u_m\|_{L^2} + \|\partial_x^2 u_m\|_{L^2})\\ &\le C(\|u_0\|_{L^2} + \|\partial_t u_m\|_{L^2} + CM_0^{2\sigma+1}).\end{aligned}$$

Applying this estimate to (2.2.8), we deduce that

$$\begin{aligned}\frac{d}{dt}\|\partial_t u_m\|_{L^2}^2 &\le C(M_0)\big(1 + \|\partial_t u_m\|_{L^2}\big)\|\partial_t u_m\|_{L^2}^2\\ &\le C(M_0)\big(1 + \|\partial_t u_m\|_{L^2}^3\big).\end{aligned}$$

This inequality implies that there exists $T > 0$ which is independent of $m \in \mathbb{N}$ such that $T \le T_0$ and

$$\sup_{m\in\mathbb{N}} \|\partial_t u_m\|_{C([-T,T],L^2)} < \infty. \tag{2.2.10}$$

From (2.2.10) and (2.2.9), we obtain the uniform $H^2(\mathbb{R})$-estimate (2.2.7). □

2.2.2 Convergence of the approximating sequence

In this subsection we prove that $\{u_m\}$ is a Cauchy sequence in $C([-T,T], L^2(\mathbb{R}))$ under the uniform $H^2(\mathbb{R})$-estimate (2.2.7). We set $I = [-T,T]$. Before proceeding to the proof, we prepare the following lemma.

Lemma 2.2.5. *Let $m, n \in \mathbb{N}$. Let $\varphi,\ \psi \in C_c^\infty(\mathbb{R})$. Then the following properties hold:*

(i) $\|J_m\varphi - J_n\varphi\|_{L^2} \le \left(\frac{1}{m} + \frac{1}{n}\right)\|\partial_x^2\varphi\|_{L^2}$.

(ii) $|(J_m\varphi - J_n\varphi, \psi)| \le \left(\frac{1}{m} + \frac{1}{n}\right)\|\partial_x\varphi\|_{L^2}\|\partial_x\psi\|_{L^2}$.

Proof. Let $v_m = J_m\varphi$, $v_n = J_n\varphi$. From the definition of J_m, we have

$$\begin{aligned}v_m - \frac{1}{m}\partial_x^2 v_m &= \varphi,\\ v_n - \frac{1}{n}\partial_x^2 v_n &= \varphi.\end{aligned}$$

Therefore, we have

$$\begin{aligned}v_m - v_n &= \frac{1}{m}\partial_x^2 v_m - \frac{1}{n}\partial_x^2 v_n\\ &= \frac{1}{m}\partial_x^2(v_m - v_n) + \partial_x^2 v_n\left(\frac{1}{m} - \frac{1}{n}\right).\end{aligned}$$

Without loss of generality, we may assume that $m \geq n$. From Proposition 2.2.1, we have

$$\begin{aligned}\|v_m - v_n\|_{L^2} &\leq \frac{2}{m}\|\partial_x^2\varphi\|_{L^2} + \left(\frac{1}{n} - \frac{1}{m}\right)\|\partial_x^2\varphi\|_{L^2} \\ &= \left(\frac{1}{m} + \frac{1}{n}\right)\|\partial_x^2\varphi\|_{L^2}.\end{aligned}$$

This completes the proof of (i). The proof of (ii) is done similarly. □

Now we estimate $L^2(\mathbb{R})$-norm of the difference $u_m - u_n$. By a straightforward calculation we have

$$\begin{aligned}\frac{d}{dt}\|u_m - u_n\|_{L^2}^2 &= 2(\partial_t u_m - \partial_t u_n, u_m - u_n) \\ &= 2(iJ_m g(J_m u_m) - iJ_n g(J_n u_n), u_m - u_n) \\ &= 2\Big[(iJ_m g(J_m u_m) - iJ_n g(J_m u_m), u_m - u_n) \\ &\quad - \Big((|J_m u_m|^{2\sigma} - |J_n u_m|^{2\sigma})J_m\partial_x u_m, J_n(u_m - u_n)\Big) \\ &\quad - \Big((|J_n u_m|^{2\sigma} - |J_n u_n|^{2\sigma})J_m\partial_x u_m, J_n(u_m - u_n)\Big) \\ &\quad - \Big(|J_n u_n|^{2\sigma}(J_m\partial_x u_m - J_n\partial_x u_m), J_n(u_m - u_n)\Big) \\ &\quad - \Big(|J_n u_n|^{2\sigma}(J_n\partial_x u_m - J_n\partial_x u_n), J_n(u_m - u_n)\Big)\Big] \\ &= I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}$$

We are going to estimate each of terms I_1, I_2, I_3, I_4 and I_5. By Lemma 2.2.5, I_1 is estimated as

$$\begin{aligned}I_1 &\leq 2\left(\frac{1}{m} + \frac{1}{n}\right)\|\partial_x g(J_m u_m)\|_{L^2}\|\partial_x(u_m - u_n)\|_{L^2} \\ &\leq C(M)\left(\frac{1}{m} + \frac{1}{n}\right).\end{aligned}$$

By using an elementary inequality

$$||u|^{2\sigma} - |v|^{2\sigma}| \leq C(|u|^{2\sigma-1} + |v|^{2\sigma-1})\ |u - v|$$

and Lemma 2.2.5, I_2 is estimated as

$$\begin{aligned}I_2 &\leq C(M)(\|J_m u_m\|_{L^\infty}^{2\sigma-1} + \|J_n u_m\|_{L^\infty}^{2\sigma-1})\|J_m u_m - J_n u_m\|_{L^2} \\ &\leq C(M)\left(\frac{1}{m} + \frac{1}{n}\right)\|\partial_x^2 u_m\|_{L^2} \\ &\leq C(M)\left(\frac{1}{m} + \frac{1}{n}\right).\end{aligned}$$

A similar calculation yields that

$$\begin{aligned} I_3 &\le 2\|J_m\partial_x u_m\|_{L^\infty}\||J_n u_m|^{2\sigma} - |J_n u_m|^{2\sigma}\|_{L^2}\|J_n(u_m - u_n)\|_{L^2} \\ &\le C(M)\|u_m - u_n\|_{L^2}^2. \end{aligned}$$

By Lemma 2.2.5, I_4 is estimated as

$$\begin{aligned} I_4 &\le 2\left|(J_m\partial_x u_m - J_n\partial_x u_m, |J_n u_n|^{2\sigma} J_n(u_m - u_n))\right| \\ &\le 2\left(\frac{1}{m} + \frac{1}{n}\right)\|\partial_x^2 u_m\|_{L^2}\|\partial_x\big(|J_n u_n|^{2\sigma} J_n(u_m - u_n)\big)\|_{L^2} \\ &\le C(M)\left(\frac{1}{m} + \frac{1}{n}\right). \end{aligned}$$

Finally, by integration by parts, I_5 is estimated as

$$\begin{aligned} I_5 &= -2\left(|J_n u_n|^{2\sigma}(\partial_x J_n u_m - \partial_x J_n u_n), J_n u_m - J_n u_n\right) \\ &= \left(\partial_x(|J_n u_n|^{2\sigma}), |J_n u_m - J_n u_n|^2\right) \\ &\le C(M)\|u_m - u_n\|_{L^2}^2. \end{aligned}$$

Gathering these estimates, we obtain that

$$\frac{d}{dt}\|u_m - u_n\|_{L^2}^2 \le C(M)\left(\frac{1}{m} + \frac{1}{n}\right) + C(M)\|u_m - u_n\|_{L^2}^2. \tag{2.2.11}$$

Applying the Gronwall inequality, we deduce that

$$\sup_{t\in I}\|u_m(t) - u_n(t)\|_{L^2}^2 \le C(M)T\left(\frac{1}{m} + \frac{1}{n}\right). \tag{2.2.12}$$

Therefore, there exists $u \in C(I, L^2(\mathbb{R}))$ such that $u_m \to u$ in $C(I, L^2(\mathbb{R}))$. By using the elementary interpolation estimate

$$\|f\|_{H^s} \le c\|f\|_{L^2}^{1-s/2}\|f\|_{H^2}^{s/2} \text{ for } 0 < s < 2$$

and the uniform $H^2(\mathbb{R})$-estimate (2.2.7), we obtain $u \in C(I, H^s(\mathbb{R}))$ with $0 \le s < 2$ such that $u_m \to u$ in $C(I, H^s(\mathbb{R}))$. From this convergence and Lemma 2.2.2 we deduce that

$$E(u(t)) = E(u_0), M(u(t)) = M(u_0) \text{ and } P(u(t)) = P(u_0) \tag{2.2.13}$$

for $t \in I$.

2.2.3 Proof of Theorem 2.1.1

We shall prove that the function u actually satisfies (gDNLS) and lies in $C(I, H^2(\mathbb{R}))$. We note that u_m is a solution of the integral equation

$$u_m(t) = U(t)u_0 + i\int_0^t U(t - s)J_m g(J_m u_m(s))ds. \tag{2.2.14}$$

By Proposition 2.2.1 and $u_m(s) \to u(s)$ in $H^1(\mathbb{R})$, we have

$$\begin{aligned} J_m g(J_m u_m(s)) - g(u(s)) &= J_m \left[g(J_m u_m(s)) - g(J_m u(s))\right] \\ &\quad + J_m \left[g(J_m u(s)) - g(u(s))\right] + J_m g(u(s)) - g(u(s)) \\ &\longrightarrow 0 \quad \text{in } L^2(\mathbb{R}) \text{ as } m \to \infty \end{aligned}$$

for all $s \in I$. Taking the limit in the integral equation (2.2.14) as $m \to \infty$, we conclude that

$$u(t) = U(t)u_0 + i\int_0^t U(t-s)g(u(s))ds. \tag{2.2.15}$$

We set

$$v(t) = i\int_0^t U(t-s)g(u(s))ds.$$

Since $g(u) \in C(I, L^2(\mathbb{R}))$, it follows that $v \in C^1(I, L^2(\mathbb{R}))$. Since v satisfies the equation

$$i\partial_t v + \partial_x^2 v + g(u) = 0, \tag{2.2.16}$$

it follows that $\partial_x^2 v \in C(I, L^2(\mathbb{R}))$. Therefore, $u \in C(I, H^2(\mathbb{R}))$ follows from the integral equation (2.2.15). The uniqueness and continuous dependence are verified by the same argument as in [1]. We omit the detail.

2.3 Proof of Theorem 2.1.3

For the proof of Theorem 2.1.3, the following lemma is essential.

Lemma 2.3.1 ([56]). *Let $p \in [2, \infty)$. For any $u \in H^{1/2}(\mathbb{R})$, we have*

$$\|u\|_{L^p} \le C\sqrt{p}\|u\|_{H^{1/2}}, \tag{2.3.1}$$

where C is independent of p.

We set

$$M = \max\{\|u\|_{L^\infty((-T,T),H^{3/2})}, \|v\|_{L^\infty((-T,T),H^{3/2})}\}.$$

By using integration by parts and Hölder's inequality, we obtain that

$$\begin{aligned} \frac{d}{dt}\|u-v\|_{L^2}^2 &= 2(\partial_t u - \partial_t v, u-v) \\ &= -2\Big((|u|^{2\sigma} - |v|^{2\sigma})\partial_x u, u-v\Big) - 2\Big(|v|^{2\sigma}(\partial_x u - \partial_x v), u-v\Big) \\ &\le C(M)\int_{\mathbb{R}} (|\partial_x u| + |\partial_x v|)|u-v|^2 dx \\ &\le C(M)(\|\partial_x u\|_{L^p} + \|\partial_x v\|_{L^p})\|u-v\|_{L^{2p'}}^2 \end{aligned}$$

for any $p \in (2, \infty)$. By Hölder's inequality, we have

$$\|u - v\|_{L^{2p'}} \leq \|u - v\|_{L^2}^{1/p'} \|u - v\|_{L^\infty}^{1-1/p'},$$

By Sobolev's embedding and Lemma 2.3.1, we obtain that

$$\begin{aligned} \frac{d}{dt}\|u - v\|_{L^2}^2 &\leq C(M)\sqrt{p}(\|u\|_{H^{3/2}} + \|v\|_{H^{3/2}})\|u - v\|_{L^2}^{2(1-1/p)} \\ &\leq C(M)\sqrt{p}\|u - v\|_{L^2}^{2(1-1/p)}, \end{aligned} \tag{2.3.2}$$

where $C(M)$ is still independent of p. Applying the Gronwall type inequality to (2.3.2), we have

$$\frac{d}{dt}\|u - v\|_{L^2}^{2/p} \leq \frac{C(M)}{\sqrt{p}}.$$

By integration in time, we deduce that

$$\|u(t) - v(t)\|_{L^2}^2 \leq \left(\frac{C(M)T}{\sqrt{p}}\right)^p \tag{2.3.3}$$

for all $t \in (-T, T)$. Since the RHS of (2.3.3) goes to 0 as $p \to \infty$, we deduce that $u = v$.

2.4 Well-posedness in the energy space $H^1(\mathbb{R})$

In this section, we prove the local and global well-posedness of (gDNLS) in the energy space $H^1(\mathbb{R})$.

2.4.1 The gauge transformation

Assume that $\sigma \geq 1$. Let u is a solution of (gDNLS). We formally derive a differential equation of $\partial_x u$. To this end, we follow an idea in [59]. We define the differential operator by

$$L = i\partial_t + \partial_x^2.$$

A direct calculation shows that

$$e^{-i\Lambda}L(e^{i\Lambda}\partial_x u) = L\partial_x u + \Big(-(\partial_x\Lambda)^2 + iL\Lambda\Big)\partial_x u + 2i\partial_x\Lambda\partial_x^2 u, \tag{2.4.1}$$

where Λ is a real-valued function determined later. We note that

$$L\partial_x u = \partial_x L u = -i|u|^{2\sigma}\partial_x^2 u - i\partial_x(|u|^{2\sigma})\partial_x u. \tag{2.4.2}$$

To absorb the worst term $-i|u|^{2\sigma}\partial_x^2 u$ by means of $-2\partial_x\Lambda\partial_x^2 u$ on the RHS of (2.4.1), we define Λ by

$$\Lambda = \frac{1}{2}\int_{-\infty}^{x} |u(t,y)|^{2\sigma} dy. \tag{2.4.3}$$

By using the equation of (gDNLS), we compute $\partial_t\Lambda$ as

$$\begin{aligned}
\partial_t\Lambda &= \frac{1}{2}\int_{-\infty}^{x} 2\sigma|u|^{2(\sigma-1)}\mathrm{Re}(\overline{u}\partial_t u)dy \\
&= \sigma\int_{-\infty}^{x} |u|^{2(\sigma-1)}\mathrm{Im}(\overline{u}(-\partial_x^2 u - i|u|^{2\sigma}\partial_x u))dy \\
&= -\sigma\mathrm{Im}(|u|^{2(\sigma-1)}\overline{u}\partial_x u) + \sigma\mathrm{Im}\left[\int_{-\infty}^{x} \partial_x(|u|^{2(\sigma-1)}\overline{u})\partial_x u dy\right] \\
&\quad - \sigma\int_{-\infty}^{x} |u|^{2(2\sigma-1)}\mathrm{Re}(\overline{u}\partial_x u)dy \\
&= -\sigma\mathrm{Im}(|u|^{2(\sigma-1)}\overline{u}\partial_x u) + \sigma\mathrm{Im}\left[\int_{-\infty}^{x} \partial_x(|u|^{2(\sigma-1)}\overline{u})\partial_x u dy\right] - \frac{1}{4}|u|^{4\sigma}.
\end{aligned}$$

Therefore, we have

$$-(\partial_x\Lambda)^2 + iL\Lambda = \sigma\mathrm{Im}(|u|^{2(\sigma-1)}\overline{u}\partial_x u) - \sigma\mathrm{Im}\left[\int_{-\infty}^{x} \partial_x(|u|^{2(\sigma-1)}\overline{u})\partial_x u dy\right] + \frac{i}{2}\partial_x(|u|^{2\sigma}).$$

Collecting these calculations, we obtain that

$$e^{-i\Lambda}L(e^{i\Lambda}\partial_x u) = Q_1(u) + Q_2(u), \tag{2.4.4}$$

where

$$\begin{aligned}
Q_1(u) &= -\frac{i}{2}\partial_x(|u|^{2\sigma})\partial_x u + \sigma\mathrm{Im}(|u|^{2(\sigma-1)}\overline{u}\partial_x u)\partial_x u, \\
Q_2(u) &= -\sigma\int_{-\infty}^{x} \mathrm{Im}\Big(\partial_x(|u|^{2\sigma-2}\overline{u})\partial_x u\Big)dy\partial_x u.
\end{aligned}$$

We note that $Q_2(u)$ is well-defined if and only if $\sigma \geq 1$. To prove Theorem 2.1.4, we approximate the initial data $u_0 \in H^1(\mathbb{R})$ by a sequence $\{\varphi_n\}$ such that $\varphi_n \in H^2(\mathbb{R})$ and $\varphi_n \to u_0$ in $H^1(\mathbb{R})$. By Theorem 2.1.1, (gDNLS) has a unique solution

$$u_n \in C([-T_n, T_n], H^2(\mathbb{R}))$$

with $u_n(0) = \varphi_n$. We set $I_n = [-T_n, T_n]$. Since the formal calculation above is justified with u replaced by u_n, we obtain that

$$u_n(t) = U(t)\varphi_n + iG(g(u_n(t))), \tag{2.4.5}$$

$$e^{i\Lambda_n(t)}\partial_x u_n(t) = U(t)(e^{i\Lambda_n(0)}\partial_x\varphi_n) + iG\Big(e^{i\Lambda_n(t)}\big(Q_1(u_n(t)) + Q_2(u_n(t))\big)\Big) \tag{2.4.6}$$

for all $t \in I_n$, where

$$\Lambda_n = \frac{1}{2}\int_{-\infty}^{x} |u_n(t,y)|^{2\sigma} dy,$$

$$U(t) = e^{it\partial_x^2},\ G(v)(t) = \int_0^t U(t-s)v(s)ds.$$

2.4.2 The uniform estimate in $H^1(\mathbb{R})$

To derive the uniform estimate in $H^1(\mathbb{R})$ of approximate solutions, we use the following Strichartz estimate. The proofs can be found in [13].

Proposition 2.4.1. *Let $U(t) = e^{it\partial_x^2}$. Then, the following properties hold:*

(i) *For any (q,r) with $0 \le 2/q = 1/2 - 1/r \le 1/2$,*

$$\|U(\cdot)\varphi\|_{L^q(\mathbb{R},L^r(\mathbb{R}))} \le C\|\varphi\|_{L^2(\mathbb{R})}.$$

(ii) *For any (q_j, r_j) with $0 \le 2/q_j = 1/2 - 1/r_j \le 1/2,\ j = 1,2$ for any interval $I \subset \mathbb{R}$ with $0 \in \overline{I}$,*

$$\|G(v)\|_{L^{q_1}(I,L^{r_1})} \le C\|v\|_{L^{q_2'}(I,L^{r_2'})},$$

where the constant C is independent of I.

Before proceeding the proof, we introduce function spaces. For a time interval I, we define the function spaces $\mathcal{X}_0(I)$ and $\mathcal{X}(I)$ by

$$\mathcal{X}_0(I) = \bigcap_{0\le 2/q = 1/2-1/r \le 1/2} L^q(I, L^r(\mathbb{R})),$$

$$\mathcal{X}(I) = \bigcap_{0\le 2/q = 1/2-1/r \le 1/2} L^q(I, W^{1,r}(\mathbb{R})),$$

with norms

$$\|u\|_{\mathcal{X}_0(I)} = \sup_{0\le 2/q = 1/2-1/r \le 1/2} \|u\|_{L^q(I,L^r)},$$

$$\|u\|_{\mathcal{X}(I)} = \|u\|_{\mathcal{X}_0(I)} + \|\partial_x u\|_{\mathcal{X}_0(I)}.$$

Applying Proposition 2.4.1 to (2.4.5) and (2.4.6), and by Sobolev's embedding and Hölder's inequality, we obtain that

$$\begin{aligned}\|u_n\|_{\mathcal{X}_0(I_n)} &\le C\|\varphi_n\|_{L^2} + C\||u_n|^{2\sigma}\partial_x u_n\|_{L^1(I_n,L^2)}\\ &\le C\|\varphi_n\|_{L^2} + CT_n\|u_n\|_{\mathcal{X}(I_n)}^{2\sigma+1},\end{aligned}$$

$$\begin{aligned}\|\partial_x u_n\|_{\mathcal{X}_0(I_n)} &= \|e^{i\Lambda_n}\partial_x u_n\|_{\mathcal{X}_0(I_n)}\\ &\le C\|e^{i\Lambda_n(0)}\partial_x\varphi_n\|_{L^2} + C\Big(\|e^{i\Lambda_n}Q_1(u_n)\|_{L^{\frac{3}{4}}(I_n,L^1)} + \|e^{i\Lambda_n}Q_2(u_n)\|_{L^1(I_n,L^2)}\Big)\\ &\le C\|\partial_x\varphi_n\|_{L^2} + C(T_n^{\frac{3}{4}} + T_n)\|u_n\|_{\mathcal{X}(I_n)}^{2\sigma+1},\end{aligned}$$

where the constant C is independent of n. Hence we deduce that

$$\|u_n\|_{\mathcal{X}(I_n)} \leq CM + C(T_n + T_n^{\frac{3}{4}})\|u_n\|_{\mathcal{X}(I_n)}^{2\sigma+1}, \tag{2.4.7}$$

where M is given by

$$M := \sup_{n\in\mathbb{N}} \|\varphi_n\|_{H^1}.$$

From (2.4.7) we have the following uniform estimate of $\{u_n\}$.

Lemma 2.4.2. *There exists $T = T(M) > 0$ such that for all $m \in \mathbb{N}$ such that the $H^2(\mathbb{R})$-solution u_m exists on the time interval $I := [-T, T]$ and*

$$\sup_{m\in\mathbb{N}} \|u_m\|_{\mathcal{X}(I)} \leq 2CM, \tag{2.4.8}$$

where C is a constant in the inequality (2.4.7)*.*

Proof. We define $T(M) > 0$ by

$$C(T(M) + T(M)^{\frac{3}{4}})\,(2CM)^{2\sigma+1} = CM.$$

We also define T_n^* by

$$T_n^* = \left\{T > 0 : \|u_n\|_{\mathcal{X}([-T,T])} \leq 2CM,\ 0 < T \leq T_n\right\}.$$

If $T_n^* < T(M)$, from (2.4.7) we have

$$\begin{aligned}\|u_n\|_{\mathcal{X}([-T_n^*,T_n^*])} &\leq CM + C(T_n^* + T_n^{*\frac{3}{4}})(2CM)^{2\sigma+1}\\ &< CM + C(T(M) + T(M)^{\frac{3}{4}})\,(2CM)^{2\sigma+1}\\ &= 2CM.\end{aligned}$$

This yields that $T_n^* = T_n$. Especially we have

$$\max\left\{\|u_n\|_{L^\infty(I_n,H^1)}, \|u_n\|_{L^4(I_n,W^{1,\infty})}\right\} < 2CM. \tag{2.4.9}$$

Under the estimate (2.4.9), in the same way of the derivation of (2.2.8), we have

$$\frac{d}{dt}\|\partial_t u_n\|_{L^2}^2 \leq C(M)\|\partial_x u_m\|_{L^\infty}\|\partial_t u_m\|_{L^2}^2 \ \forall t \in I_n.$$

Applying the Gronwall inequality, for $0 < t \leq T_n$ (similarly for $-T_n \leq t < 0$) we have

$$\begin{aligned}\|\partial_t u_n(t)\|_{L^2}^2 &\leq \|\partial_t u_n(0)\|_{L^2}^2 \exp\left(C(M)\int_0^t \|\partial_x u_m(s)\|_{L^\infty} ds\right)\\ &\leq C(\|\varphi_n\|_{H^2}) \exp\left(C(M)T_n^{\frac{3}{4}}\|u_n\|_{L^4(I_n,W^{1,\infty})}\right)\\ &\leq C(\|\varphi_n\|_{H^2}) \exp\left(C(M)T_n^{\frac{3}{4}}\right).\end{aligned}$$

Hence we obtain that

$$\|u_n\|_{L^\infty(I_n,H^2)} \le C(\|\varphi_n\|_{H^2}, M). \tag{2.4.10}$$

By the estimate (2.4.10) we can extend the $H^2(\mathbb{R})$-solution u_n on the interval $[-T_n - \varepsilon, T_n + \varepsilon]$ for some $\varepsilon > 0$. By iterating the argument above, one can extend the existence interval of u_n at least to $[-T(M), T(M)]$, i.e., $T(M) \le T_n$. Hence we deduce that $T(M) \le T_n^*$. Indeed, if $T_n^* < T(M)$, from the argument above we have $T_n = T_n^*$, but this contradicts $T(M) \le T_n$. Therefore, from the definition of T_n^* we deduce that

$$\|u_m\|_{\mathcal{X}([-T(M),T(M)])} \le 2CM$$

for any $m \in \mathbb{N}$. This completes the proof. □

We note that the existence time $T = T(M)$ only depends on $\|u_0\|_{H^1}$.

2.4.3 Proof of Theorem 2.1.4

Firstly, we prove that $\{u_m\}$ forms a Cauchy sequence in $C(I, L^2(\mathbb{R}))$ under the uniform estimate (2.4.8). A straightforward calculation shows that

$$\begin{aligned}
\frac{d}{dt}\|u_n - u_m\|_{L^2}^2 &= 2(\partial_t u_n - \partial_t u_m, u_n - u_m) \\
&= -2(|u_n|^{2\sigma}\partial_x u_n - |u_m|^{2\sigma}\partial_x u_m, u_n - u_m) \\
&= -2\Big((|u_n|^{2\sigma} - |u_m|^{2\sigma})\partial_x u_n, u_n - u_m\Big) \\
&\quad - 2\Big(|u_m|^{2\sigma}(\partial_x u_n - \partial_x u_m), u_n - u_m\Big) \\
&\le C\Big(\|u_n\|_{L^\infty}^{2\sigma-1} + \|u_m\|_{L^\infty}^{2\sigma-1}\Big)\Big(\|\partial_x u_n\|_{L^\infty} + \|\partial_x u_m\|_{L^\infty}\Big)\|u_n - u_m\|_{L^2}^2 \\
&\le C(M)\Big(\|\partial_x u_n\|_{L^\infty} + \|\partial_x u_m\|_{L^\infty}\Big)\|u_n - u_m\|_{L^2}^2.
\end{aligned}$$

Applying the Gronwall inequality, we obtain that

$$\sup_{t\in I}\|u_n(t) - u_m(t)\|_{L^2}^2 \le \|\varphi_n - \varphi_m\|_{L^2}^2 \exp(C(M)T^{\frac{1}{4}}).$$

This implies that there exists $u \in C(I, L^2(\mathbb{R}))$ such that

$$u_m \to u \text{ in } C(I, L^2(\mathbb{R})). \tag{2.4.11}$$

By the interpolation inequality, we have

$$u_n \to u \quad \text{in } C(I, L^r(\mathbb{R})) \tag{2.4.12}$$

for any $r \in [2, \infty)$. Since $W^{1,r}(\mathbb{R})$ is reflexive if (q, r) satisfies $0 \le 2/q = 1/2 - 1/r < 1/2$, we obtain from (2.4.8) and (2.4.12) that

$$\|u\|_{L^q(I,W^{1,r})} \le \liminf_{n\to\infty} \|u_n\|_{L^q(I,W^{1,r})} \le 2CM \tag{2.4.13}$$

for any $r \in [2, \infty)$. Since the constant on the RHS of (2.4.13) is independent of (q, r), taking the limit as $r \to \infty$, we conclude that

$$\|u\|_{L^4(I,W^{1,\infty})} \le 2CM.$$

Therefore, $u \in \mathcal{X}(I)$. We see that u is a solution of (gDNLS) in the distribution sense. We note that the approximate solution u_m of (gDNLS) conserves energy, mass and momentum (see (2.2.13)). By (2.4.11), we obtain $M(u(t)) = M(u_0)$ and $P(u(t)) = P(u_0)$ for all $t \in I$. To prove the conservation of energy, we need the following lemma.

Lemma 2.4.3. *Let $\sigma > 0$. For every $M > 0$, there exists $C(M) > 0$, we have*

$$|G(u) - G(v)| \le C(M)\|u - v\|_{L^2} \tag{2.4.14}$$

for all $u, v \in H^1(\mathbb{R})$ such that $\|u\|_{H^1}, \|v\|_{H^1} \le M$.

Proof. Since $G'(u) = g(u)$, we have

$$\begin{aligned} G(u) - G(v) &= \int_0^1 \frac{d}{ds} G(su + (1-s)v)ds \\ &= \int_0^1 \big(g(su + (1-s)v), u - v\big)ds. \end{aligned}$$

From this identity and Sobolev's embedding, the inequality (2.4.14) follows. □

By (2.4.8) and (2.4.11), we note that $u_m(t) \rightharpoonup u(t)$ in $H^1(\mathbb{R})$ for any $t \in I$. By the weak lower semicontinuity of the norm, (2.4.11) and Lemma 2.4.3, we obtain that

$$\begin{aligned} E(u(t)) &\le \liminf_{m\to\infty} \left(\frac{1}{2}\|\partial_x u_m(t)\|_{L^2}^2 - G(u_m(t))\right) \\ &= \liminf_{m\to\infty} E(u_m(t)) = E(\varphi) \end{aligned} \tag{2.4.15}$$

for all $t \in I$.

Next, we prove that u is the unique solution of (gDNLS). Suppose that

$$v \in L^\infty(I, H^1(\mathbb{R})) \cap L^4(I, W^{1,\infty}(\mathbb{R}))$$

is also a solution of (gDNLS). We set

$$M = \max\{\|u\|_{L^\infty(I,H^1)} + \|u\|_{L^4(I,W^{1,\infty})}, \|v\|_{L^\infty(I,H^1)} + \|v\|_{L^4(I,W^{1,\infty})}\}.$$

By the same calculation as before, we obtain that

$$\frac{d}{dt}\|u-v\|_{L^2}^2 \le C(M)\Big(\|\partial_x u\|_{L^\infty} + \|\partial_x v\|_{L^\infty}\Big)\|u-v\|_{L^2}^2. \tag{2.4.16}$$

Applying the Gronwall inequality to (2.4.16), we conclude that $u = v$ on I. By uniqueness and (2.4.15), it is easily verified that

$$E(u(t)) = E(\varphi) \tag{2.4.17}$$

for all $t \in I$, and which yields that $u \in C(I, H^1(\mathbb{R}))$.

We recall that the existence time T only depends on the $H^1(\mathbb{R})$-norm of the initial data. Hence the property (i) (blowup alternative) is proved by a standard method; see e.g. the proof of Theorem 3.3.9 in [13].

Finally, we prove the continuous dependence. Let $I_{\max} := (-T_{\min}(u_0), T_{\max}(u_0))$ be a maximal interval of the solution u. Let $I \subset I_{\max}$ be a closed interval. Suppose that $u_{0n} \to u_0$ in $H^1(\mathbb{R})$ and let u_n be a solution of (gDNLS) with $u_n(0) = u_{0n}$. We note that u_n is defined on I for n large enough. In the same way as the first calculation in this subsection, we deduce that

$$u_n \to u \text{ in } C(I, L^2(\mathbb{R})). \tag{2.4.18}$$

By the conservation of mass and energy and Lemma 2.4.3, we obtain that

$$\|u_n(t)\|_{H^1} \to \|u(t)\|_{H^1} \tag{2.4.19}$$

uniformly on I. Therefore, we conclude that $u_n \to u$ in $C(I, H^1(\mathbb{R}))$.

2.4.4 Proof of Theorem 2.1.5

Let $\sigma > 1$. We assume that $u \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}))$ is a maximal solution of (gDNLS). We set $I_{\max} := (-T_{\min}, T_{\max})$. By the conservation of energy and Sobolev's embedding, we obtain that

$$\begin{aligned}\frac{1}{2}\|\partial_x u\|_{L^2}^2 &= E(u) + G(u)\\ &\le E(u_0) + \frac{1}{2\sigma+2}\|u\|_{L^{4\sigma+2}}^{2\sigma+1}\|\partial_x u\|_{L^2}\\ &\le E(u_0) + \frac{c}{2\sigma+2}\|u\|_{H^1}^{2\sigma+2}.\end{aligned}$$

By the conservation of mass, we obtain that

$$f_\sigma(\|u\|_{H^1}) := \|u\|_{H^1}^2 - \frac{c}{\sigma+1}\|u\|_{H^1}^{2\sigma+2} \le M(u_0) + 2E(u_0). \tag{2.4.20}$$

We note that f_σ has an unique local maximum at $\delta > 0$, where δ is given by $\delta^{2\sigma} = c^{-1}$. If $u_0 \in H^1(\mathbb{R})$ satisfies that

$$M(u_0) + 2E(u_0) < f_\sigma(\delta) \text{ and } \|u_0\|_{H^1} < \delta,$$

then, by (2.4.20) we have

$$f_\sigma(\|u(t)\|_{H^1}) \le M(u_0) + E(u_0) < f_\sigma(\delta) \tag{2.4.21}$$

for all $t \in I_{\max}$. From $\|u_0\|_{H^1} < \delta$ and the continuity $t \mapsto \|u(t)\|_{H^1}$, we deduce that

$$\sup_{t\in I_{\max}} \|u(t)\|_{H^1} < \delta. \tag{2.4.22}$$

From the a priori estimate (2.4.22) and Theorem 2.1.4, the claim of Theorem 2.1.5 follows.

2.5 Proof of Theorem 2.1.6

Let $u_0 \in H^1(\mathbb{R})$ be given. We recall the following approximate problem in Section 2.2:

$$\begin{cases} i\partial_t u_m + \partial_x^2 u_m + J_m g(J_m u_m) = 0, \\ u_m(0) = u_0. \end{cases} \tag{2.5.1}$$

For each $m \in \mathbb{N}$ it is easily verified that there exist $T_m > 0$, and a sequence $\{u_m\}$ of $C((-T_m, T_m), H^1(\mathbb{R}))$ such that satisfies (2.5.1) and

$$E_m(u_m(t)) = E_m(u_0), M(u_m(t)) = M(u_0) \text{ and } P(u_m(t)) = P(u_0) \tag{2.5.2}$$

for all $t \in (-T_m, T_m)$, where E_m is defined by (2.2.3). We use the conservation laws (2.5.2) in order to obtain uniform $H^1(\mathbb{R})$-estimates of $\{u_m\}$. We have

$$\begin{aligned} \|\partial_x u_m\|_{L^2}^2 &= 2\left(E_m(u_0) - G_m(u_m)\right) \\ &\le 2E_m(u_0) + \frac{1}{\sigma+1}\|J_m u_m\|_{L^{4\sigma+2}}^{2\sigma+1}\|\partial_x J_m u_m\|_{L^2}. \end{aligned}$$

By using Gagliardo-Nirenberg's inequality

$$\|f\|_{L^{4\sigma+2}}^{2\sigma+1} \le C\|f\|_{L^2}^{\sigma+1}\|\partial_x f\|_{L^2}^{\sigma}$$

and Proposition 2.2.1, we obtain that

$$\begin{aligned} \|\partial_x u_m\|_{L^2}^2 &\le 2E_m(u_0) + \frac{C}{\sigma+1}\|u_m\|_{L^2}^{\sigma+1}\|\partial_x u_m\|_{L^2}^{\sigma+1} \\ &= 2E_m(u_0) + \frac{C}{\sigma+1}\|u_0\|_{L^2}^{\sigma+1}\|\partial_x u_m\|_{L^2}^{\sigma+1}, \end{aligned} \tag{2.5.3}$$

where in the last equality we used the conservation of mass. Since $\sigma + 1 < 2$, applying Young's inequality to (2.5.3), we have the following estimate

$$\|\partial_x u_m(t)\|_{L^2}^2 \le C(\|u_0\|_{H^1})$$

for all $t \in (-T_m, T_m)$. This implies that $T_m = \infty$ for every $m \in \mathbb{N}$ and

$$M := \sup_{m\in\mathbb{N}} \|u_m\|_{C(\mathbb{R},H^1)} < \infty. \tag{2.5.4}$$

By the equation (2.5.1) and the estimate $\|g_m(u_m(t))\|_{L^2} \le C(M)$ for all $t \in \mathbb{R}$, we obtain

$$\sup_{m\in\mathbb{N}} \|\partial_t u_m\|_{C(\mathbb{R},H^{-1})} \le C(M). \tag{2.5.5}$$

By (2.5.4), (2.5.5) and the abstract version of Ascoli-Arzelà's theorem, we deduce that

$$u \in L^\infty(\mathbb{R}, H^1(\mathbb{R})) \cap W^{1,\infty}(\mathbb{R}, H^{-1}(\mathbb{R})),$$

and that there exists a subsequence, which we still denote by $\{u_m\}$, such that

$$u_m(t) \rightharpoonup u(t) \text{ in } H^1(\mathbb{R}) \tag{2.5.6}$$

for all $t \in \mathbb{R}$. To prove that u is a weak solution of (gDNLS), we need the following lemma.

Lemma 2.5.1. *For all $t \in \mathbb{R}$, $g_m(u_m(t)) \rightharpoonup g(u(t))$ in $L^2(\mathbb{R})$.*

Proof. Let $\psi \in C_c^\infty(\mathbb{R})$ and let $B = \operatorname{supp} \psi$. We write

$$\begin{aligned}
\Big(g_m(u_m) - g(u), \psi\Big) &= \Big(J_m g(J_m u_m) - g(J_m u_m), \psi\Big) \\
&\quad + \Big(i|Ju_m|^{2\sigma}\partial_x J_m u_m - i|u_m|^{2\sigma}\partial_x J_m u_m, \psi\Big) \\
&\quad + \Big(i|u_m|^{2\sigma}\partial_x J_m u_m - i|u|^{2\sigma}\partial_x J_m u_m, \psi\Big) \\
&\quad + \Big(i|u|^{2\sigma}\partial_x J_m u_m - i|u|^{2\sigma}\partial_x u_m, \psi\Big) \\
&\quad + \Big(i|u|^{2\sigma}\partial_x u_m - i|u|^{2\sigma}\partial_x u, \psi\Big) \\
&= K_1 + K_2 + K_3 + K_4 + K_5.
\end{aligned}$$

Since $g(J_m u_m)$ is bounded in $L^2(\mathbb{R})$ due to (2.5.4), $K_1 \to 0$ by Proposition 2.2.1 (v). In the case of $1/2 \le \sigma < 1$, we estimate K_2 as

$$\begin{aligned}
|K_2| &\le \|\psi\|_{L^\infty}\|\partial_x J_m u_m\|_{L^2}\| \,|J_m u_m|^{2\sigma} - |u_m|^{2\sigma}\|_{L^2(B)} \\
&\le C(M)\|J_m u_m - u_m\|_{L^2(B)}.
\end{aligned}$$

Since u_m is bounded in $H^1(\mathbb{R})$, it follows $J_m u_m - u_m \rightharpoonup 0$ in $H^1(\mathbb{R})$, hence $J_m u_m - u_m \to 0$ in $L^2(B)$ by Rellich-Kondrachov's theorem. Therefore, $K_2 \to 0$. In the case of $0 < \sigma < 1/2$, we estimate K_2 as

$$\begin{aligned}
|K_2| &\le C(M)\| \,|J_m u_m|^{2\sigma} - |u_m|^{2\sigma}\|_{L^2(B)} \\
&\le C(M)\|J_m u_m - u_m\|_{L^{4\sigma}(B)}^{2\sigma} \\
&\le C(M)|B|^{\frac{1-2\sigma}{2}}\|J_m u_m - u_m\|_{L^2(B)}^{2\sigma} \\
&\longrightarrow 0 \quad \text{as} \quad m \to \infty.
\end{aligned}$$

Here, we used an elementary inequality

$$||u|^{2\sigma} - |v|^{2\sigma}| \leq |u - v|^{2\sigma}$$

in the second inequality. Similarly, we can show that $K_3,\ K_4 \to 0$. Since $\partial_x u_m \rightharpoonup \partial_x u$ in $L^2(\mathbb{R})$, we deduce that $K_5 \to 0$. This completes the proof. □

It follows from (2.5.6) and Lemma 2.5.1 that u is a solution of (gDNLS) in the distribution sense. Taking the H^{-1}-H^1 duality product of the equation (gDNLS), we deduce that

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 0 \tag{2.5.7}$$

for all $t \in \mathbb{R}$, and so

$$M(u(t)) = M(u_0). \tag{2.5.8}$$

By (2.5.2), (2.5.8) and (2.5.6), we deduce that

$$u_m \to u \text{ in } C_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R})). \tag{2.5.9}$$

It follows from (2.5.2), (2.5.6), (2.5.9) and Lemma 2.4.3 that

$$E(u(t)) \leq E(u_0) \tag{2.5.10}$$

for all $t \in \mathbb{R}$. The conservation of the momentum easily follows from (2.5.2) and (2.5.9). This completes the proof.

Chapter 3

Global existence for the derivative NLS equation

3.1 Introduction

3.1.1 Background

In this chapter we study global existence for the derivative nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u + i|u|^2\partial_x u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{DNLS}$$

and the generalized derivative nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{gDNLS}$$

for $\sigma > 1$ (L^2-supercritical case). First we review solitary waves of (gDNLS). It is known that (gDNLS) has a two-parameter family of solitary waves

$$u_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x - ct),$$

where (ω, c) satisfies $-2\sqrt{\omega} < c \le 2\sqrt{\omega}$,

$$\phi_{\omega,c}(x) = \Phi_{\omega,c}(x)\exp\left(i\frac{c}{2}x - \frac{i}{2\sigma+2}\int_{-\infty}^{x}\Phi_{\omega,c}(y)^{2\sigma}dy\right), \tag{3.1.1}$$

$$\Phi_{\omega,c}^{2\sigma}(x) = \begin{cases} \dfrac{(\sigma+1)(4\omega - c^2)}{2\sqrt{\omega}\cosh(\sigma\sqrt{4\omega - c^2}x) - c}, & \text{if } \omega > c^2/4, \\[2ex] \dfrac{2(\sigma+1)c}{\sigma^2(cx)^2 + 1}, & \text{if } c = 2\sqrt{\omega}. \end{cases} \tag{3.1.2}$$

We note that $\Phi_{\omega,c}$ is the positive even solution of

$$-\Phi'' + \left(\omega - \frac{c^2}{4}\right)\Phi + \frac{c}{2}|\Phi|^{2\sigma}\Phi - \frac{2\sigma+1}{(2\sigma+2)^2}|\Phi|^{4\sigma}\Phi = 0, \quad x \in \mathbb{R}, \tag{3.1.3}$$

and the complex-valued function $\phi_{\omega,c}$ satisfies

$$-\phi'' + \omega\phi + ic\phi' - i|\phi|^{2\sigma}\phi' = 0, \quad x \in \mathbb{R}.$$

In [45], it was proved that the solitary waves $u_{\omega,c}$ are orbitally stable if $-2\sqrt{\omega} < c < 2z_0\sqrt{\omega}$, and orbitally unstable if $2z_0\sqrt{\omega} < c < 2\sqrt{\omega}$ when $1 < \sigma < 2$, where the constant $z_0 = z_0(\sigma) \in (-1,1)$ is the solution of

$$\begin{aligned} F_\sigma(z) := (\sigma-1)^2 \left\{ \int_0^\infty (\cosh y - z)^{-\frac{1}{\sigma}} dy \right\}^2 \\ - \left\{ \int_0^\infty (\cosh y - z)^{-\frac{1}{\sigma}-1}(z\cosh y - 1) dy \right\}^2 = 0. \end{aligned}$$

Moreover, it was proved that solitary waves for all $\omega > c^2/4$ are orbitally unstable when $\sigma \geq 2$ and orbitally stable when $0 < \sigma < 1$. Recently, Fukaya [20] proved that the solitary waves are orbitally unstable if $c = 2z_0\sqrt{\omega}$ when $1 < \sigma < 2$.

In Chapter 2 we proved local well-posedness in $H^1(\mathbb{R})$ when $\sigma \geq 1$, and that the following quantities are conserved

$$\begin{aligned} &\text{(Energy)} && E(u) := \frac{1}{2}\|\partial_x u\|_{L^2}^2 - \frac{1}{2\sigma+2}\mathrm{Re}\int_{\mathbb{R}} i|u|^{2\sigma}\partial_x u\bar{u}dx, \\ &\text{(Mass)} && M(u) := \|u\|_{L^2}^2, \\ &\text{(Momentum)} && P(u) := \mathrm{Re}\int_{\mathbb{R}} i\partial_x u\bar{u}dx. \end{aligned}$$

Moreover, we proved global well-posedness for small initial data in $H^1(\mathbb{R})$; see Theorem 2.1.5. In the case $0 < \sigma < 1$ (L^2-subcritical case) we constructed global solutions for any initial data in $H^1(\mathbb{R})$; see Theorem 2.1.6.

In this chapter we study the case $\sigma \geq 1$ (L^2-critical or supercritical case), and improve the global existence results in the energy space $H^1(\mathbb{R})$ in previous works. The main methodology in this chapter is variational method. First we give a variational characterization of two types of solitary waves including the massless case. Then, by applying the variational characterization, we establish a sufficient condition for global existence by using potential well theory inspired from the classical work by Payne and Sattinger [60]. Potential well generated by two-parameter family of solitary waves has a rich structure. Our main contribution here is to clarify the connection between the potential well and 4π-mass condition for (DNLS). Especially, our variational approach gives another simple proof of the global result by Wu [73]. Moreover, we prove that the solution of (DNLS) is global if the initial data u_0 satisfies $M(u_0) = 4\pi$ and $P(u_0) < 0$. This is the first global result in the mass threshold case.

Here we review the global result for (DNLS) in the energy space $H^1(\mathbb{R})$. By using the following gauge transformation to the solution of (DNLS)

$$w(t,x) = u(t,x)\exp\left(\frac{i}{4}\int_{-\infty}^x |u(t,x)|^2 dx\right), \tag{3.1.4}$$

then w satisfies the following equation:

$$i\partial_t w + \partial_x^2 w + \frac{i}{2}|w|^2\partial_x w - \frac{i}{2}w^2\partial_x\overline{w} + \frac{3}{16}|w|^4 w = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{3.1.5}$$

The conserved quantities are transformed as follows:

$$\begin{aligned}
\mathcal{E}(w) &= \frac{1}{2}\|\partial_x w\|_{L^2}^2 - \frac{1}{32}\|w\|_{L^6}^6, \\
\mathcal{M}(w) &= \|w\|_{L^2}^2, \\
\mathcal{P}(w) &= \mathrm{Re}\int_{\mathbb{R}} i\partial_x w\overline{w}dx + \frac{1}{4}\|w\|_{L^4}^4.
\end{aligned}$$

Hayashi and Ozawa [32] used the following sharp Gagliardo–Nirenberg inequality

$$\|f\|_{L^6}^6 \leq \frac{4}{\pi^2}\|f\|_{L^2}^4\|\partial_x f\|_{L^2}^2 \tag{3.1.6}$$

in order to obtain a priori estimate in $\dot{H}^1(\mathbb{R})$. We note that an optimizer for the inequality (3.1.6) is given by $Q := \Phi_{1,0}$ and Q satisfies the following elliptic equation:

$$-Q'' + Q - \frac{3}{16}Q^5 = 0.$$

In [32], it was proved that the $H^1(\mathbb{R})$-solution of (DNLS) is global if the initial data u_0 satisfies

$$M(u_0) = \mathcal{M}(w_0) < \mathcal{M}(Q) = 2\pi;$$

see also Weinstein [71] for related works. Wu [73] took advantage of conservation law of the momentum as well as conservation laws of the energy and the mass. He used the following sharp Gagliardo–Nirenberg inequality

$$\|f\|_{L^6}^6 \leq 3(2\pi)^{-\frac{2}{3}}\|f\|_{L^4}^{\frac{16}{3}}\|\partial_x f\|_{L^2}^{\frac{2}{3}} \tag{3.1.7}$$

instead of using (3.1.6). Then, it was proved that the $H^1(\mathbb{R})$-solution of (DNLS) is global if the initial data u_0 satisfies

$$M(u_0) = \mathcal{M}(w_0) < \mathcal{M}(W) = 4\pi,$$

where $W := \Phi_{1,2}$. We note that an optimizer for the inequality (3.1.7) is given by W and W satisfies the following elliptic equation:

$$-W'' + \frac{1}{2}W^3 - \frac{3}{16}W^5 = 0.$$

Wu's proof depends on contradiction argument as follows. Suppose that there exists a time sequence $\{t_n\}$ with $t_n \to T_{\max}$, or $-T_{\min}$ such that $\|\partial_x w(t_n)\|_{L^2} \to \infty$ as $n \to$

∞, where $(-T_{\min}, T_{\max})$ is the maximal time interval. He mainly proved that $X = \|w(t_n)\|_{L^4}^8 / \|w(t_n)\|_{L^6}^6$ satisfies

$$X^3 - \mathcal{M}(w_0)X^2 + 16\{3(2\pi)^{-\frac{2}{3}}\}^{-3}\mathcal{M}(w_0) < 0,$$

but this does not hold when $\mathcal{M}(w_0) < 4\pi$. This argument is more or less complicated and hard to see the naturality of 4π as a mass condition and the connection of the solitons, although there is a fact that the main part of the soliton gives an optimizer for the inequality (3.1.7). In our approach we give a close relation between global existence theory and solitons, and derive the 4π-mass condition more naturally and directly.

3.1.2 Setting

To state our main results, we introduce some notations. Let (ω, c) satisfy

$$-2\sqrt{\omega} < c \le 2\sqrt{\omega}. \tag{3.1.8}$$

For (ω, c) satisfying (3.1.8), we define

$$\begin{aligned} S_{\omega,c}(\varphi) &:= E(\varphi) + \frac{\omega}{2}M(\varphi) + \frac{c}{2}P(\varphi), \\ d(\omega, c) &:= S_{\omega,c}(\phi_{\omega,c}). \end{aligned}$$

We denote the nonlinear term in the energy functional by

$$N(\varphi) := \mathrm{Re} \int_{\mathbb{R}} i|\varphi|^{2\sigma}\partial_x\varphi\overline{\varphi}dx.$$

We define the functional $\widetilde{S}_{\omega,c}(\psi)$ by

$$\widetilde{S}_{\omega,c}(\psi) := S_{\omega,c}(e^{i\frac{cx}{2}}\psi). \tag{3.1.9}$$

By using the following identities

$$cP(\varphi) = -\|\partial_x\varphi\|_{L^2}^2 - \frac{c^2}{4}\|\varphi\|_{L^2}^2 + \left\|\partial_x(e^{-i\frac{cx}{2}}\varphi)\right\|_{L^2}^2, \tag{3.1.10}$$

$$N(\varphi) = -\frac{c}{2}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2} + N(e^{-i\frac{cx}{2}}\varphi), \tag{3.1.11}$$

$\widetilde{S}_{\omega,c}(\psi)$ has the following explicit formula

$$\widetilde{S}_{\omega,c}(\psi) := \frac{1}{2}\|\partial_x\psi\|_{L^2}^2 + \frac{1}{2}\left(\omega - \frac{c^2}{4}\right)\|\psi\|_{L^2}^2 + \frac{c}{2(2\sigma+2)}\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2} - \frac{1}{2\sigma+2}N(\psi).$$

We also introduce the following functionals

$$\begin{aligned} K_{\omega,c}(\varphi) &:= \partial_\lambda S_{\omega,c}(\lambda\psi)\Big|_{\lambda=1} \\ &= \|\partial_x\varphi\|_{L^2}^2 + \omega\|\varphi\|_{L^2}^2 + cP(\varphi) - N(\varphi), \\ \widetilde{K}_{\omega,c}(\psi) &:= \partial_\lambda\widetilde{S}_{\omega,c}(\lambda\psi)\Big|_{\lambda=1} \\ &= \|\partial_x\psi\|_{L^2}^2 + \left(\omega - \frac{c^2}{4}\right)\|\psi\|_{L^2}^2 + \frac{c}{2}\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2} - N(\psi). \end{aligned}$$

By using (3.1.10) and (3.1.11), we have the following relation

$$\widetilde{K}_{\omega,c}(\psi) = K_{\omega,c}(e^{i\frac{cx}{2}}\psi). \tag{3.1.12}$$

This is of course corresponding to the relation (3.1.9).

We define the following functional space

$$Z_{\omega,c} := \begin{cases} H^1(\mathbb{R}), & \text{if } \omega > c^2/4, \\ \dot{H}^1(\mathbb{R}) \cap L^{2\sigma+2}(\mathbb{R}), & \text{if } c = 2\sqrt{\omega}. \end{cases}$$

We consider the following minimization problem:

$$\begin{aligned} \mu(\omega, c) :=\ & \inf\{S_{\omega,c}(\varphi) : e^{-i\frac{cx}{2}}\varphi \in Z_{\omega,c} \setminus \{0\}, K_{\omega,c}(\varphi) = 0\} \\ =\ & \inf\{\widetilde{S}_{\omega,c}(\psi) : \psi \in Z_{\omega,c} \setminus \{0\}, \widetilde{K}_{\omega,c}(\psi) = 0\}. \end{aligned}$$

We note that if $\omega > c^2/4$, $\mu(\omega, c)$ is also rewritten as

$$\mu(\omega, c) = \inf\{S_{\omega,c}(\varphi) : \varphi \in H^1(\mathbb{R}) \setminus \{0\}, K_{\omega,c}(\varphi) = 0\},$$

since $\varphi \in H^1(\mathbb{R})$ if and only if $e^{-i\frac{cx}{2}}\varphi \in H^1(\mathbb{R})$. We introduce the sets $\mathscr{G}_{\omega,c}$ and $\mathscr{M}_{\omega,c}$ defined by

$$\begin{aligned} \mathscr{G}_{\omega,c} &:= \{\varphi : e^{-i\frac{cx}{2}}\varphi \in Z_{\omega,c} \setminus \{0\}, S'_{\omega,c}(\varphi) = 0\}, \\ \mathscr{M}_{\omega,c} &:= \{\varphi : e^{-i\frac{cx}{2}}\varphi \in Z_{\omega,c} \setminus \{0\}, S_{\omega,c}(\varphi) = \mu(\omega, c),\ K_{\omega,c}(\varphi) = 0\}. \end{aligned}$$

The element of $\mathscr{G}_{\omega,c}$ is called a ground state. We note that $\mathscr{M}_{\omega,c}$ is the set of minimizers of $\mathcal{S}_{\omega,c}$ on the Nehari manifold.

Remark 3.1.1. The function space $Z_{c^2/4,c}$ comes from the functional $\widetilde{S}_{c^2/4,c}$. We note that when $\sigma \geq 2$ the solitary waves $\phi_{c^2/4,c}$ do not belong to $L^2(\mathbb{R})$, but belong to $L^{2\sigma+2}(\mathbb{R})$. The two functionals $\widetilde{S}_{c^2/4,c}$ and $\widetilde{K}_{c^2/4,c}$ are useful to obtain the variational characterization of the solitary waves for the massless case.

Remark 3.1.2. The functional $S_{c^2/4,c}$ seems meaningless at first glance on the function space

$$Y_{c^2/4,c} := \{\varphi : e^{-i\frac{cx}{2}}\varphi \in Z_{c^2/4,c}\},$$

since $S_{c^2/4,c}$ contains L^2-norm. However, since $\widetilde{S}_{c^2/4,c}$ is defined on $\dot{H}^1(\mathbb{R}) \cap L^{2\sigma+2}(\mathbb{R})$, $S_{c^2/4,c}$ is well-defined on the function space $Y_{c^2/4,c}$ through the relation (3.1.9). Similarly, $K_{c^2/4,c}$ is well-defined on $Y_{c^2/4,c}$ by the relation (3.1.12).

3.1.3 Main results

First, we begin with the result about the variational characterization of the solitary waves.

Proposition 3.1.3. *Let* $\sigma \geq 1$ *and* (ω, c) *satisfy* (3.1.8). *Then, we have*

$$\mathscr{G}_{\omega,c} = \mathscr{M}_{\omega,c} = \{e^{i\theta_0}\phi_{\omega,c}(\cdot - x_0) : \theta_0 \in [0, 2\pi), x_0 \in \mathbb{R}\}, \tag{3.1.13}$$

and $d(\omega, c) = \mu(\omega, c)$.

Our main contribution here is to give the variational characterization of the solitary waves for the massless case.

We apply Proposition 3.1.3 to establish a sufficient condition for global existence in the energy space. We define the subsets of the energy space by

$$\mathscr{K}_{\omega,c} := \{\varphi \in H^1(\mathbb{R}) : S_{\omega,c}(\varphi) \leq \mu_{\omega,c}, K_{\omega,c}(\varphi) \geq 0\},$$
$$\mathscr{K} := \bigcup_{\substack{-2\sqrt{\omega}<c\leq 2\sqrt{\omega} \\ \omega>0}} \mathscr{K}_{\omega,c}.$$

By applying the variational characterization and potential well theory, we have the following global result.

Proposition 3.1.4. *Let* $\sigma \geq 1$ *and* (ω, c) *satisfy* (3.1.8). *If the initial data* u_0 *belongs to* $\mathscr{K}_{\omega,c}$, *then the* $H^1(\mathbb{R})$*-solution* u *of* (gDNLS) *with* $u(0) = u_0$ *exists globally in time, and we have*

$$\|\partial_x u\|^2_{L^\infty(\mathbb{R}:L^2(\mathbb{R}))} \leq 4\left(1 + \frac{1}{\sigma}\right) S_{\omega,c}(u_0) + \frac{c^2}{2\sigma} M(u_0). \tag{3.1.14}$$

Especially, if $u_0 \in \mathscr{K}$, *the* $H^1(\mathbb{R})$*-solution* u *of* (gDNLS) *with* $u(0) = u_0$ *exists globally in time.*

We show that Theorem 3.1.4 gives us some interesting corollaries for (DNLS). Our variational approach covers Wu's global result.

Theorem 3.1.5. *If the initial data* $u_0 \in H^1(\mathbb{R})$ *satisfies* $M(u_0) < 4\pi$, *or* $M(u_0) = 4\pi$ *and* $P(u_0) < 0$, *then the* $H^1(\mathbb{R})$*-solution of* (DNLS) *with* $u(0) = u_0$ *exists globally in time.*

Remark 3.1.6. The existence of blow-up solutions in finite time is still an open problem. It might be a very interesting problem whether finite time blow-up occurs when the initial data u_0 satisfies $M(u_0) = 4\pi$ and $P(u_0) > 0$.

Remark 3.1.7. When $\sigma = 1$, by applying variational characterization of solitons, we have

$$\begin{aligned} &\left\{u_0 \in H^1(\mathbb{R}) : M(u_0) = 4\pi, E(u_0) = P(u_0) = 0\right\} \\ &= \left\{e^{i\theta_0}\phi_{\omega,2\sqrt{\omega}}(\cdot - x_0) : \theta_0, x_0 \in \mathbb{R}, \omega > 0\right\}, \end{aligned}$$

see Remark 3.3.2 for the details.

We note that the proof of Theorem 3.1.5 gives the simple alternative proof of Wu's result. The global result for $M(u_0) = 4\pi$ and $P(u_0) < 0$ gives the global result for the threshold case. This is the first progress to investigate the dynamics around the algebraic solitons.

The global results in Proposition 3.1.4 contains the large data. Indeed, we have the following theorem.

Theorem 3.1.8. *Let $\sigma \geq 1$. Given $\psi \in H^1(\mathbb{R})$, and set the initial data as $u_{0,c} = e^{i\frac{cx}{2}}\psi$. Then, there exists $c_0 = c_0(\psi) > 0$ such that if $c \geq c_0$, then the corresponding solution u_c of* (gDNLS) *exists globally in time.*

Theorem 3.1.8 means that if we consider sufficiently oscillating data, there exist global solutions with any large mass. We note that the oscillating term $e^{i\frac{cx}{2}}$ gives the change of the momentum. The results for Theorem 3.1.8 gives the important difference to the dynamics to nonlinear Schrödinger equations with pure power nonlinearities; see also the comments below Theorem 4.1.11.

3.2 Variational Characterization

We introduce the following sets

$$\widetilde{\mathscr{G}}_{\omega,c} := \{\psi \in Z_{\omega,c} \setminus \{0\} : \widetilde{S}'_{\omega,c}(\psi) = 0\},$$
$$\widetilde{\mathscr{M}}_{\omega,c} := \{\psi \in Z_{\omega,c} \setminus \{0\} : \widetilde{S}_{\omega,c}(\psi) = \mu(\omega,c),\ \widetilde{K}_{\omega,c}(\psi) = 0\}.$$

In this section, we prove the following proposition.

Proposition 3.2.1. *Let (ω, c) satisfy* (3.1.8)*. Then, we have*

$$\widetilde{\mathscr{G}}_{\omega,c} = \widetilde{\mathscr{M}}_{\omega,c} = \{e^{i\theta}e^{-i\frac{cx}{2}}\phi_{\omega,c}(\cdot - y) : \theta \in [0, 2\pi), y \in \mathbb{R}\}.$$

Moreover, we have $d(\omega, c) = \mu(\omega, c)$.

By using the relation $\widetilde{S}'_{\omega,c}(e^{-i\frac{cx}{2}}\varphi) = e^{-i\frac{cx}{2}}S'_{\omega,c}(\varphi)$, we have

$$\begin{aligned} \varphi \in \mathscr{G}_{\omega,c} &\Leftrightarrow e^{-i\frac{cx}{2}}\varphi \in \widetilde{\mathscr{G}}_{\omega,c}, \\ \varphi \in \mathscr{M}_{\omega,c} &\Leftrightarrow e^{-i\frac{cx}{2}}\varphi \in \widetilde{\mathscr{M}}_{\omega,c}. \end{aligned} \tag{3.2.1}$$

From Proposition 3.2.1 and (3.2.1), we deduce that the claim of Proposition 3.1.3 follows..

To prove Proposition 3.2.1, we prepare some basic lemmas.

Lemma 3.2.2. *Let $p \geq 1$. Then we have*

$$\|f\|_{L^\infty}^{2p} \leq 2p \|f\|_{L^{4p-2}}^{2p-1} \|\partial_x f\|_{L^2}. \tag{3.2.2}$$

Proof. By Cauchy-Schwarz's inequality, we have

$$\begin{aligned}|f(x)|^{2p} &= \int_{-\infty}^{x} \frac{d}{dx}(|f(y)|^{2p})dy \\ &= \int_{-\infty}^{x} 2p|f(y)|^{2p-2}\mathrm{Re}(\overline{f(y)}(\partial_x f)(y))dy \\ &\leq 2p \left\| |f|^{2p-1} \right\|_{L^2} \|\partial_x f\|_{L^2} \\ &= 2p \|f\|_{L^{4p-2}}^{2p-1} \|\partial_x f\|_{L^2}.\end{aligned}$$

This completes the proof. □

By a direct calculation we have the following relation:

$$\widetilde{S}_{\omega,c}(\psi) = \frac{1}{2\sigma+2}\widetilde{K}_{\omega,c}(\psi) + \frac{\sigma}{2\sigma+2}\widetilde{L}_{\omega,c}(\psi), \tag{3.2.3}$$

where the functional $\widetilde{L}_{\omega,c}$ is defined by

$$\widetilde{L}_{\omega,c}(\psi) := \|\partial_x\psi\|_{L^2}^2 + \left(\omega - \frac{c^2}{4}\right)\|\psi\|_{L^2}^2.$$

Hereafter we only prove the claims in the massless case $c = 2\sqrt{\omega}$. The case $\omega > c^2/4$ is proved in the similar way. Actually the proof is easier since we can use the boundedness in $L^2(\mathbb{R})$; see the arguments in Colin and Ohta [17] for more details.

Lemma 3.2.3. *Let* (ω, c) *satisfy* (3.1.8). *Then, we have*

$$\widetilde{\mathscr{G}}_{\omega,c} = \{e^{i\theta_0}e^{-\frac{c}{2}ix}\phi_{\omega,c}(\cdot - x_0) : \theta_0 \in [0, 2\pi), x_0 \in \mathbb{R}\}.$$

Proof. Since $e^{-i\frac{cx}{2}}\phi_{\omega,c}$ satisfies $\widetilde{S}'_{\omega,c}(e^{-i\frac{cx}{2}}\phi_{\omega,c}) = e^{-i\frac{cx}{2}}S'_{\omega,c}(\phi_{\omega,c}) = 0$, we have

$$\widetilde{\mathscr{G}}_{\omega,c} \supset \{e^{i\theta_0}e^{-\frac{c}{2}ix}\phi_{\omega,c}(\cdot - x_0) : \theta_0 \in [0, 2\pi), x_0 \in \mathbb{R}\}.$$

Conversely, let $\psi \in \widetilde{\mathscr{G}}_{\omega,c}$. By using the following transformation

$$\Phi(x) = \psi(x)\exp\left(\frac{i}{2\sigma+2}\int_0^x |\psi(y)|^{2\sigma}dy\right), \tag{3.2.4}$$

or equivalently,

$$\psi(x) = \Phi(x)\exp\left(-\frac{i}{2\sigma+2}\int_0^x |\Phi(y)|^{2\sigma}dy\right), \tag{3.2.5}$$

then it is easily verified that Φ is a solution of

$$-\Phi'' + \frac{c}{2}|\Phi|^{2\sigma}\Phi - \frac{2\sigma+1}{(2\sigma+2)^2}|\Phi|^{4\sigma}\Phi + \frac{\sigma}{\sigma+1}|\Phi|^{2\sigma-2}\mathrm{Im}(\overline{\Phi}\Phi')\Phi = 0. \tag{3.2.6}$$

If we put $f := \mathrm{Re}\Phi$, and $g := \mathrm{Im}\Phi$, from (3.2.6) we obtain that

$$f'' = A(\Phi)f \text{ and } g'' = A(\Phi)g,$$

where the function $A(\Phi)$ is defined by

$$A(\Phi) := \frac{c}{2}|\Phi|^{2\sigma} - \frac{2\sigma+1}{(2\sigma+2)^2}|\Phi|^{4\sigma} + \frac{\sigma}{\sigma+1}|\Phi|^{2\sigma-2}\mathrm{Im}(\overline{\Phi}\Phi').$$

We note that

$$(fg' - gf')' = fg'' - gf'' = fA(\Phi)g - gA(\Phi)f = 0.$$

Since $f, g \in \dot{H}^1(\mathbb{R}) \cap L^{2\sigma+2}(\mathbb{R})$, we obtain that $fg' - gf' = 0$ for any $x \in \mathbb{R}$. On the other hand, we have

$$fg' - gf' = \mathrm{Re}\Phi\mathrm{Im}\Phi' - \mathrm{Im}\Phi\mathrm{Re}\Phi' = \mathrm{Im}(\overline{\Phi}\Phi').$$

Thus, $\mathrm{Im}(\overline{\Phi}\Phi') = 0$ for any $x \in \mathbb{R}$. Therefore, Φ satisfies

$$-\Phi'' + \frac{c}{2}|\Phi|^{2\sigma}\Phi - \frac{2\sigma+1}{(2\sigma+2)^2}|\Phi|^{4\sigma}\Phi = 0. \tag{3.2.7}$$

From the uniqueness of the equation (3.2.7), there exist $\theta_0 \in (0, 2\pi]$ and $x_0 \in \mathbb{R}$ such that $\Phi = e^{i\theta_0}\Phi_{\omega,c}(\cdot - x_0)$. Thus we see that $\Phi \in L^{2\sigma}(\mathbb{R})$. We modify the gauge of ψ as

$$\begin{aligned}\psi(x) &= \Phi(x)\exp\left(-\frac{i}{2\sigma+2}\int_{-\infty}^{x}|\Phi(y)|^{2\sigma}dy\right)\exp\left(\frac{i}{2\sigma+2}\int_{-\infty}^{0}|\Phi(y)|^{2\sigma}dy\right)\\ &= \Phi(x)\exp\left(-\frac{i}{2\sigma+2}\int_{-\infty}^{x}|\Phi(y)|^{2\sigma}dy\right)\cdot e^{i\theta_1},\end{aligned}$$

where θ_1 is defined by

$$\theta_1 := \frac{1}{2\sigma+2}\int_{-\infty}^{0}|\Phi_{\omega,c}(y - x_0)|^{2\sigma}dy = \frac{1}{2\sigma+2}\int_{-\infty}^{-x_0}|\Phi_{\omega,c}(y)|^{2\sigma}dy.$$

Hence, from the explicit formula (3.1.1), we have

$$\begin{aligned}\psi(x) &= e^{i\theta_1}\Phi_{\omega,c}(x - x_0)\exp\left(-\frac{i}{2\sigma+2}\int_{-\infty}^{x}|\Phi_{\omega,c}(y - x_0)|^{2\sigma}dy\right)\\ &= e^{i\theta_1}\Phi_{\omega,c}(x - x_0)\exp\left(-\frac{i}{2\sigma+2}\int_{-\infty}^{x-x_0}|\Phi_{\omega,c}(y)|^{2\sigma}dy\right)\\ &= e^{i\theta_1}e^{-i\frac{c}{2}(x-x_0)}\phi_{\omega,c}(x - x_0)\\ &= e^{i\theta}e^{-i\frac{c}{2}x}\phi_{\omega,c}(x - x_0),\end{aligned}$$

where θ_1 is defined by

$$\theta := \theta_1 + \frac{c}{2}x_0.$$

we deduce that $\psi(x) = e^{i\theta}e^{-i\frac{cx}{2}}\phi_{\omega,c}(x - x_0)$ for some $\theta \in \mathbb{R}$ (see also Remark 3.2.4). This completes the proof. □

Remark 3.2.4. At that moment taking $\psi \in \widetilde{\mathscr{G}}_{\omega,c}$, we do not know $\psi \in L^{2\sigma}(\mathbb{R})$. Hence we applied the gauge transformation (3.2.4) instead of the following transformation

$$\Phi(x) = \psi(x) \exp\left(\frac{i}{2\sigma+2}\int_{-\infty}^{x} |\psi(y)|^{2\sigma} dy\right).$$

Lemma 3.2.5. *Assume that $\widetilde{\mathscr{M}}_{\omega,c} \neq \emptyset$. Then, we have $\widetilde{\mathscr{M}}_{\omega,c} \subset \widetilde{\mathscr{G}}_{\omega,c}$.*

Proof. Let $\psi \in \widetilde{\mathscr{M}}_{\omega,c}$. Since ψ is a minimizer, there exists a Lagrange multiplier $\eta \in \mathbb{R}$ such that $\widetilde{S}'_{\omega,c}(\psi) = \eta \widetilde{K}'_{\omega,c}(\psi)$. Then, we have

$$0 = \widetilde{K}_{\omega,c}(\psi) = \left\langle \widetilde{S}'_{\omega,c}(\psi), \psi \right\rangle = \eta \left\langle \widetilde{K}'_{\omega,c}(\psi), \psi \right\rangle.$$

By $\widetilde{K}_{\omega,c}(\psi) = 0$, we have

$$\begin{aligned}\left\langle \widetilde{K}'_{\omega,c}(\psi), \psi \right\rangle &= 2\widetilde{L}_{\omega,c}(\psi) - (\sigma+1)c\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2} - (2\sigma+2)N(\psi) \\ &= 2\widetilde{L}_{\omega,c}(\psi) - (2\sigma+2)\widetilde{L}_{\omega,c}(\psi) \\ &= -2\sigma\widetilde{L}_{\omega,c}(\psi) < 0.\end{aligned}$$

Therefore, we deduce that $\eta = 0$. This implies that $\widetilde{S}'_{\omega,c}(\psi) = 0$ and hence $\psi \in \widetilde{\mathscr{G}}_{\omega,c}$. □

Lemma 3.2.6. *Assume that $\widetilde{\mathscr{M}}_{\omega,c} \neq \emptyset$. Then, we have $\widetilde{\mathscr{G}}_{\omega,c} = \widetilde{\mathscr{M}}_{\omega,c}$. Moreover, we have $d(\omega,c) = \mu(\omega,c)$.*

Proof. Let $\psi \in \widetilde{\mathscr{G}}_{\omega,c}$. By Lemma 3.2.3, there exist $\theta_0 \in [0,2\pi)$ and $x_0 \in \mathbb{R}$ such that

$$\psi = e^{i\theta_0} e^{-i\frac{cx}{2}} \phi_{\omega,c}(\cdot - x_0).$$

Since $\widetilde{\mathscr{M}}_{\omega,c} \neq \emptyset$, we can take $\varphi \in \widetilde{\mathscr{M}}_{\omega,c}$. By Lemmas 3.2.3 and 3.2.5, there exist $\theta_1 \in [0,2\pi)$ and $x_1 \in \mathbb{R}$ such that $\varphi = e^{i\theta_1} e^{-i\frac{cx}{2}} \phi_{\omega,c}(\cdot - x_1)$. Thus,

$$\widetilde{S}_{\omega,c}(\psi) = \widetilde{S}_{\omega,c}(e^{-i\frac{cx}{2}}\phi_{\omega,c}) = \widetilde{S}_{\omega,c}(e^{-i\frac{cx}{2}}\varphi) = \mu(\omega,c). \tag{3.2.8}$$

Since $\widetilde{K}_{\omega,c}(\psi) = \left\langle \widetilde{S}'_{\omega,c}(\psi), \psi \right\rangle = 0$, we deduce that $\psi \in \widetilde{\mathscr{M}}_{\omega,c}$. We note that

$$\widetilde{S}_{\omega,c}(e^{-i\frac{cx}{2}}\phi_{\omega,c}) = S_{\omega,c}(\phi_{\omega,c}) = d(\omega,c).$$

Combined with (3.2.8), we deduce that $d(\omega,c) = \mu(\omega,c)$. □

To complete the proof of Proposition 3.2.1, we need to prove that $\widetilde{\mathscr{M}}_{\omega,c} \neq \emptyset$. The assertion $\widetilde{\mathscr{M}}_{\omega,c} \neq \emptyset$ actually follows from the following proposition.

Proposition 3.2.7. *Let* $\{\psi_n\} \subset Z_{\omega,c}$ *satisfy*

$$\widetilde{S}_{\omega,c}(\psi_n) \to \mu(\omega,c) \text{ and } \widetilde{K}_{\omega,c}(\psi_n) \to 0.$$

Then, there exist $\{y_n\} \subset \mathbb{R}$ *and* $\psi \in \widetilde{\mathscr{M}_{\omega,c}}$ *such that* $\{\psi_n(\cdot - y_n)\}$ *has a subsequence which converges to* ψ *strongly in* $Z_{\omega,c}$.

At first, we note the following lemma.

Lemma 3.2.8. *Let* (ω, c) *satisfy* (3.1.8). *Then, we have* $\mu(\omega, c) > 0$.

Proof. We recall that $\mu(\omega,c) = \inf\{\widetilde{S}_{\omega,c}(\psi) : \psi \in Z_{\omega,c} \setminus \{0\}, \widetilde{K}_{\omega,c}(\psi) = 0\}$. By (3.2.3), it is trivial that $\mu(\omega, c) \geq 0$. We prove $\mu(\omega, c) > 0$ by contradiction. We assume that $\mu(\omega, c) = 0$. Taking the minimizing sequence $\{\psi_n\} \subset Z_{\omega,c}$ as

$$\widetilde{S}_{\omega,c}(\psi_n) \to \mu(\omega,c) = 0 \text{ and } \widetilde{K}_{\omega,c}(\psi_n) = 0,$$

then we have $\|\partial_x\psi_n\|_{L^2}^2 \to 0$ by (3.2.3). From $\widetilde{K}_{\omega,c}(\psi_n) = 0$ and $\|\partial_x\psi_n\|_{L^2}^2 \to 0$, we have

$$\frac{c}{2}\|\psi_n\|_{L^{2\sigma+2}} - N(\psi_n) \to 0. \tag{3.2.9}$$

Applying Gagliardo–Nirenberg's inequality and Young's inequality, we have

$$\begin{aligned} |N(\psi_n)| &\leq \|\partial_x\psi_n\|_{L^2}\|\psi_n\|_{L^{4\sigma+2}}^{2\sigma+1} \\ &\lesssim \|\partial_x\psi_n\|_{L^2}^{1+\theta}\|\psi_n\|_{L^{2\sigma+2}}^{2\sigma+1-\theta} \\ &\leq \frac{c}{4}\|\psi_n\|_{L^{2\sigma+2}}^{2\sigma+2} + C\|\partial_x\psi_n\|_{L^2}^{2\sigma+2}, \end{aligned}$$

where $\theta \in (0,1)$ in the second inequality. Combining this with (3.2.9), we deduce that $\|\psi_n\|_{L^{2\sigma+2}} \to 0$. By using (3.2.2) as $p = (\sigma+2)/2$, we obtain that $\|\psi_n\|_{L^\infty} \to 0$. From the following relation

$$-N(\psi) = -\|\partial_x\psi\|_{L^2}^2 - \frac{1}{4}\|\psi\|_{L^{4\sigma+2}}^{4\sigma+2} + \left\|\partial_x\psi + \frac{i}{2}|\psi|^{2\sigma}\psi\right\|_{L^2}^2, \tag{3.2.10}$$

we obtain that

$$\begin{aligned} \widetilde{K}(\psi_n) &= \|\partial_x\psi_n\|_{L^2}^2 + \frac{c}{2}\|\psi_n\|_{L^{2\sigma+2}}^{2\sigma+2} - N(\psi_n) \\ &= \frac{c}{2}\|\psi_n\|_{L^{2\sigma+2}}^{2\sigma+2} - \frac{1}{4}\|\psi_n\|_{L^{4\sigma+2}}^{4\sigma+2} + \left\|\partial_x\psi_n + \frac{1}{2}i|\psi_n|^{2\sigma}\psi_n\right\|_{L^2}^2 \\ &\geq \frac{c}{2}\|\psi_n\|_{L^{2\sigma+2}}^{2\sigma+2} - \frac{1}{4}\|\psi_n\|_{L^{4\sigma+2}}^{4\sigma+2} \\ &\geq \left(\frac{c}{2} - \frac{1}{4}\|\psi_n\|_{L^\infty}^{2\sigma}\right)\|\psi_n\|_{L^{2\sigma+2}}^{2\sigma+2} \\ &> 0, \end{aligned}$$

for large $n \in \mathbb{N}$ since $\|\psi_n\|_{L^\infty} \to 0$. However, this contradicts $\widetilde{K}_{\omega,c}(\psi_n) = 0$ for all $n \in \mathbb{N}$. □

For the proof of Proposition 3.2.7 we apply the concentration compactness argument. Here we recall Lieb's compactness lemma. See [43] for $p = 2$ and [7, Lemma 2.1] for more general setting.

Lemma 3.2.9. *Let $p \geq 2$. Let $\{f_n\}$ be a bounded sequence in $\dot{H}^1(\mathbb{R}) \cap L^p(\mathbb{R})$. Assume that there exists $q \in (p, \infty)$ such that $\limsup_{n\to\infty} \|f_n\|_{L^q} > 0$. Then, there exist $\{y_n\}$ and $f \in \dot{H}^1(\mathbb{R}) \cap L^p(\mathbb{R}) \setminus \{0\}$ such that $\{f_n(\cdot - y_n)\}$ has a subsequence that converges to f weakly in $\dot{H}^1(\mathbb{R}) \cap L^p(\mathbb{R})$.*

We also recall the Brezis–Lieb lemma (see [11]).

Lemma 3.2.10. *Let $1 \leq p < \infty$. Let $\{f_n\}$ be a bounded sequence in $L^p(\mathbb{R})$ and $f_n \to f$ a.e. in $\mathbb{R}$. Then we have*

$$\|f_n\|_{L^p}^p - \|f_n - f\|_{L^p}^p - \|f\|_{L^p}^p \to 0. \tag{3.2.11}$$

Remark 3.2.11. When $p = 2$, if $\{f_n\}$ is a bounded sequence in $L^2(\mathbb{R})$ and f_n converges to f weakly in $L^2(\mathbb{R})$, then (3.2.11) still holds.

Proof of Proposition 3.2.7. We consider $\{\psi_n\} \subset Z_{\omega,c}$ such that $\widetilde{S}_{\omega,c}(\psi_n) \to \mu(\omega, c)$ and $\widetilde{K}_{\omega,c}(\psi_n) \to 0$.

Step 1. By (3.2.3) we obtain that $\|\partial_x \psi_n\|_{L^2}^2$ is bounded. We recall that $\widetilde{K}_{\omega,c}$ has a following explicit formula

$$\widetilde{K}_{\omega,c}(\psi) = \|\partial_x \psi\|_{L^2}^2 + \left(\omega - \frac{c^2}{4}\right) \|\psi\|_{L^2}^2 + \frac{c}{2}\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2} - N(\psi). \tag{3.2.12}$$

As seen in the proof of Lemma 3.2.8, $N(\psi_n)$ is estimated as

$$|N(\psi_n)| \leq \frac{c}{4}\|\psi_n\|_{L^{2\sigma+2}}^{2\sigma+2} + C\|\partial_x \psi_n\|_{L^2}^{2\sigma+2}.$$

Combined with $\widetilde{K}_{\omega,c}(\psi_n) \to 0$ and boundedness of $\|\partial_x \psi_n\|_{L^2}^2$, we deduce that $\|\psi_n\|_{L^{2\sigma+2}}^{2\sigma+2}$ is bounded. Hence, $\{\psi_n\}$ is a bounded sequence in $Z_{\omega,c}$.

Step 2. $\limsup_{n\to\infty} \|\psi_n\|_{L^{4\sigma+2}} > 0$. Suppose that $\limsup_{n\to\infty} \|\psi_n\|_{L^{4\sigma+2}} = 0$. We note that

$$|N(\psi_n)| \leq \|\partial_x \psi_n\|_{L^2} \|\psi_n\|_{L^{4\sigma+2}}^{2\sigma+1} \to 0.$$

From (3.2.12) we obtain that $\|\partial_x \psi_n\|_{L^2}^2 \to 0$ and $\|\psi_n\|_{L^{2\sigma+2}}^{2\sigma+2} \to 0$. By (3.2.3), we deduce that $\widetilde{S}_{\omega,c}(\psi_n) \to 0$. This contradicts $\mu(\omega, c) > 0$.

Step 3. Since $\{\psi_n\}$ is bounded in $Z_{\omega,c} = \dot{H}^1(\mathbb{R}) \cap L^{2\sigma+2}(\mathbb{R})$ and $\limsup_{n\to\infty} \|\psi_n\|_{L^{4\sigma+2}} > 0$, by applying Lemma 3.2.9 as $f_n = \psi_n$ and $p = 2\sigma+2$, there exist $\{y_n\}$ and $v \in Z_{\omega,c} \setminus \{0\}$ such that $\{\psi_n(\cdot - y_n)\}$ (we denote this by v_n) has a subsequence that converges to v weakly in $Z_{\omega,c}$. Next we show that

$$\widetilde{K}_{\omega,c}(v_n) - \widetilde{K}_{\omega,c}(v - v_n) - \widetilde{K}_{\omega,c}(v) \to 0 \text{ as } n \to \infty, \tag{3.2.13}$$

$$\widetilde{L}_{\omega,c}(v_n) - \widetilde{L}_{\omega,c}(v - v_n) - \widetilde{L}_{\omega,c}(v) \to 0 \text{ as } n \to \infty. \tag{3.2.14}$$

The relation (3.2.14) follows from $v_n \rightharpoonup v$ in $Z_{\omega,c}$ easily. As for (3.2.13), first we note that (3.2.12) is rewritten as

$$(3.2.15) \qquad \widetilde{K}_{\omega,c}(\psi) = \frac{c}{2}\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2} - \frac{1}{4}\|\psi\|_{L^{4\sigma+2}}^{4\sigma+2} + \left\|\partial_x\psi + \frac{i}{2}|\psi|^{2\sigma}\psi\right\|_{L^2}^2$$

for any $\psi \in Z_{\omega,c}$. Since v_n converges to v weakly in $Z_{\omega,c}$, we have $v_n \to v$ a.e. in $\mathbb{R}$. Therefore, by Lemma 3.2.10, we have

$$\|v_n\|_{L^p}^p - \|v_n - v\|_{L^p}^p - \|v\|_{L^p}^p \to 0$$

for $2\sigma + 2 \le p < \infty$. Moreover, if we set

$$w_n := \partial_x v_n + \frac{i}{2}|v_n|^{2\sigma}v_n \text{ and } w = \partial_x v + \frac{i}{2}|v|^{2\sigma}v,$$

it is easily verified that w_n converges to w weakly in $L^2(\mathbb{R})$. Therefore, by (3.2.15), we deduce that (3.2.13).

Step 4. We prove that $\widetilde{K}_{\omega,c}(\psi) < 0 \Rightarrow (2\sigma+2)\mu(\omega,c) < \widetilde{L}_{\omega,c}(\psi)$. By (3.2.3) and the definition of $\mu(\omega,c)$, we have

$$(3.2.16) \qquad \mu(\omega,c) = \frac{1}{2\sigma+2}\inf\{\widetilde{L}_{\omega,c}(\psi) : \psi \in Z_{\omega,c}\setminus\{0\}, \widetilde{K}_{\omega,c}(\psi) = 0\}.$$

If $\psi \in Z_{\omega,c}$ satisfies $\widetilde{K}_{\omega,c}(\psi) < 0$, then there exists $\lambda_0 \in (0,1)$ such that $\widetilde{K}_{\omega,c}(\lambda_0\psi) = 0$ since $\widetilde{K}_{\omega,c}(\lambda\psi) > 0$ for small $\lambda \in (0,1)$. Therefore, we deduce that

$$(2\sigma+2)\mu(\omega,c) \le \widetilde{L}_{\omega,c}(\lambda_0\psi) < \widetilde{L}_{\omega,c}(\psi).$$

Step 5. $\widetilde{K}_{\omega,c}(v) \le 0$. Suppose that $\widetilde{K}_{\omega,c}(v) > 0$. Since $\widetilde{K}_{\omega,c}(v_n) \to 0$ and (3.2.13) holds, we have

$$\widetilde{K}_{\omega,c}(v - v_n) \to -\widetilde{K}_{\omega,c}(v) < 0.$$

This implies that $\widetilde{K}_{\omega,c}(v - v_n) < 0$ for large $n \in \mathbb{N}$. By Step 4, this implies that

$$(2\sigma+2)\mu(\omega,c) < \widetilde{L}_{\omega,c}(v - v_n)$$

for large $n \in \mathbb{N}$. Combined with (3.2.14), we have

$$\begin{aligned}\widetilde{L}_{\omega,c}(v) &= \lim_{n\to\infty}(\widetilde{L}_{\omega,c}(v_n) - \widetilde{L}_{\omega,c}(v - v_n)) \\ &\le \lim_{n\to\infty}\widetilde{L}_{\omega,c}(v_n) - (2\sigma+2)\mu(\omega,c) = 0\end{aligned}$$

where we have used that $\widetilde{L}_{\omega,c}(v_n) \to (2\sigma+2)\mu(\omega,c)$. Since $v \ne 0$, $\widetilde{L}_{\omega,c}(v) > 0$. This gives a contradiction.

Step 6. We prove that $v \in \widetilde{\mathscr{M}}_{\omega,c}$. By (3.2.16) and the weakly lower semicontinuity of $\widetilde{L}_{\omega,c}$, we obtain that

$$(2\sigma+2)\mu(\omega,c) \le \widetilde{L}_{\omega,c}(v) \le \liminf_{n\to\infty} \widetilde{L}_{\omega,c}(v_n) = (2\sigma+2)\mu(\omega,c).$$

Thus, $\widetilde{L}_{\omega,c}(v) = (2\sigma+2)\mu(\omega,c)$ and this implies that v_n converges to v strongly in $\dot{H}^1(\mathbb{R})$. By Step 4 and Step 5, we deduce that $\widetilde{K}_{\omega,c}(v) = 0$. Combined with (3.2.13), we have $\widetilde{K}_{\omega,c}(v_n - v) \to 0$. Since $N(v_n - v) \to 0$ from the convergence in $\dot{H}^1(\mathbb{R})$, by (3.2.12) we deduce that $\|v_n - v\|_{L^{2\sigma+2}} \to 0$. Hence we deduce that v_n converges to v strongly in $Z_{\omega,c}$. Combined with $\widetilde{S}_{\omega,c}(v_n) \to \mu(\omega,c)$ and $\widetilde{K}_{\omega,c}(v_n) \to 0$, we deduce that $\widetilde{S}_{\omega,c}(v) = \mu(\omega,c)$ and $\widetilde{K}_{\omega,c}(v) = 0$, i.e., $v \in \widetilde{M}_{\omega,c}$. This completes the proof. □

3.3 Global existence

In this section we prove the main theorems in Chapter 3. First we show Proposition 3.1.4.

Proof of Proposition 3.1.4. Let $u_0 \in \mathscr{K}_{\omega,c}$ and $u \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}))$ be a maximal solution of (gDNLS) with $u(0) = u_0$. First, we consider the case that $K_{\omega,c}(u_0) = 0$. Since $S_{\omega,c}(u_0) \le S_{\omega,c}(\phi_{\omega,c})$, by Proposition 3.1.3, we have $u_0 = 0$ or $u_0 = e^{i\theta_0}\phi_{\omega,c}(\cdot - x_0)$. By the uniqueness of solution to (gDNLS), we have $u(t) = 0$ or $u(t) = e^{i\theta_0}e^{i\omega t}\phi_{\omega,c}(x - ct - x_0)$, respectively. This implies that $K_{\omega,c}(u(t)) = 0$ for all $t \in \mathbb{R}$. Next, we consider the case that $K_{\omega,c}(u_0) > 0$. We suppose that there exists some time t_0 such that $K_{\omega,c}(u(t_0)) \le 0$. Then, there exists some t_* such that $K_{\omega,c}(u(t_*)) = 0$ by the continuity of the flow $t \mapsto u(t)$ in $H^1(\mathbb{R})$. By the above argument, $K_{\omega,c}(u(t)) = 0$ for all $t \in \mathbb{R}$. This gives a contradiction. Thus, $K_{\omega,c}(u(t)) > 0$ for all $t \in (-T_{\min}, T_{\max})$. Therefore, we deduce that $K_{\omega,c}$ is an invariant set under the flow.

Next, we prove that the solution is global if $u_0 \in \mathscr{K}_{\omega,c}$. From (3.2.3) we have

$$(2\sigma+2)S_{\omega,c}(\varphi) = K_{\omega,c}(\varphi) + \sigma \left\| \partial_x \varphi - \frac{c}{2} i\varphi \right\|_{L^2}^2 + \sigma\left(\omega - \frac{c^2}{4}\right) \|\varphi\|_{L^2}^2 . \tag{3.3.1}$$

Since $u(t) \in K_{\omega,c}$, we have

$$\begin{aligned}(2\sigma+2)S_{\omega,c}(u_0) &= (2\sigma+2)S_{\omega,c}(u(t)) \\ &= K_{\omega,c}(u(t)) + \sigma \left\| \partial_x u(t) - \frac{c}{2} iu(t) \right\|_{L^2}^2 + \sigma\left(\omega - \frac{c^2}{4}\right) \|u(t)\|_{L^2}^2 \\ &\ge \sigma \left\| \partial_x u(t) - \frac{c}{2} iu(t) \right\|_{L^2}^2\end{aligned}$$

for all $t \in (-T_{\min}, T_{\max})$. This implies that $T_{\min} = T_{\max} = \infty$. Moreover, we have the following estimate:

$$\sigma \|\partial_x u(t)\|_{L^2}^2 \le \left(\left\| \partial_x u(t) - \frac{c}{2} i u(t) \right\|_{L^2} + \frac{|c|}{2} \|u(t)\|_{L^2} \right)^2$$
$$\le 4(\sigma+1) S_{\omega,c}(u_0) + \frac{c^2}{2} M(u_0).$$

This completes the proof. □

When $\sigma = 1$, we can calculate the conserved quantities of the solitons explicitly. See [17] or Chapter 4 for the detail.

Lemma 3.3.1. *Let* $\sigma = 1$ *and* (ω, c) *satisfy* (3.1.8)*. Then, we have*

$$M(\phi_{\omega,c}) = 8 \tan^{-1} \sqrt{\frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c}},$$
$$P(\phi_{\omega,c}) = 2\sqrt{4\omega - c^2},$$
$$E(\phi_{\omega,c}) = -\frac{c}{2}\sqrt{4\omega - c^2}.$$

In particular, we have

$$d(\omega, c) = S_{\omega,c}(\phi_{\omega,c}) = 4\omega \tan^{-1} \sqrt{\frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c}} + \frac{c}{2}\sqrt{4\omega - c^2}.$$

Remark 3.3.2. When $\sigma = 1$, we have $M(\phi_{c^2/4,c}) = 4\pi$, $P(\phi_{c^2/4,c}) = 0$, and $E(\phi_{c^2/4,c}) = 0$ for all $c > 0$ by Lemma 3.3.1. On the other hand, if $M(\phi) = 4\pi$, $P(\phi) = 0$, and $E(\phi) \le 0$, then $\phi(x) = e^{i\theta_0}\phi_{c_0^2/4,c_0}(x - x_0)$ for some $\theta_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$, and $c_0 > 0$. Indeed, $M(\phi) = 4\pi$, $P(\phi) = 0$, and $E(\phi) \le 0$ imply that

$$K_{c^2/4,c}(\phi) \le -\|\partial_x \phi\|_{L^2}^2 + \frac{c^2}{4} \cdot 4\pi,$$

where we have used the relation $-N(\phi) = -2\|\partial_x\phi\|_{L^2}^2 + 4E(\phi)$. Since $K_{c^2/4,c}(\phi) < 0$ for small $c > 0$ and $K_{c^2/4,c}(\phi) \to +\infty$ as $c \to \infty$, there exists $c_0 > 0$ such that $K_{c_0^2/4,c_0}(\phi) = 0$. Therefore, Theorem 3.1.3 implies that $\phi(x) = e^{i\theta_0}\phi_{c_0^2/4,c_0}(x - x_0)$ for some $\theta_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$. Note that this means that there is no function satisfying $M(\phi) = 4\pi$, $P(\phi) = 0$, and $E(\phi) < 0$.

Now we give the proofs of Theorem 3.1.5 and Theorem 3.1.8.

Proof of Theorem 3.1.5. The statement is trivial if $u_0 = 0$. We assume that $u_0 \neq 0$. We consider the curve for massless case. We note that

$$K_{c^2/4,c}(u_0) = \|\partial_x u_0\|_{L^2}^2 + \frac{c^2}{4}\|u_0\|_{L^2}^2 + cP(u_0) - N(u_0) \to \infty$$

as $c \to \infty$. By Lemma 3.3.1, we have

$$\begin{aligned} &S_{c^2/4,c}(u_0) \le S_{c^2/4,c}(\phi_{c^2/4,c}) = d(c^2/4, c) \\ \iff &E(u_0) + \frac{c^2}{8}M(u_0) + \frac{c}{2}P(u_0) \le \frac{c^2}{8} \cdot 4\pi \\ \iff &E(u_0) + \frac{c}{2}P(u_0) \le \frac{c^2}{8}\big(4\pi - M(u_0)\big). \end{aligned}$$

The last inequality holds if $M(u_0) < 4\pi$, or $M(u_0) = 4\pi$ and $P(u_0) < 0$, if we take sufficiently large $c > 0$. Hence, we deduce that $u_0 \in \mathscr{K}_{c^2/4,c}$ for large $c > 0$ under the assumption of Theorem 3.1.5. By Proposition 3.1.4, the $H^1(\mathbb{R})$-solution u with $u(0) = u_0$ is global. This completes the proof. □

Proof of Theorem 3.1.8. Let $\sigma \ge 1$. We consider the curve for massless case again. We note that the curve $c \mapsto (c^2/4, c)$ corresponds to the scaling for the solitons. Since $\Phi_{c^2,4}(x) = c^{\frac{1}{2\sigma}}\Phi_{1/4,1}(cx)$, we have

$$\begin{aligned} &\|\partial_x \Phi_{c^2/4,c}\|_{L^2}^2 = c^{1+\frac{1}{\sigma}}\|\partial_x \Phi_{1/4,1}\|_{L^2}^2, \\ &\|\Phi_{c^2/4,c}\|_{L^{2\sigma+2}}^{2\sigma+2} = c^{\frac{1}{\sigma}}\|\Phi_{1/4,1}\|_{L^{2\sigma+2}}^{2\sigma+2}, \ \|\Phi_{c^2/4,c}\|_{L^{4\sigma+2}}^{4\sigma+2} = c^{1+\frac{1}{\sigma}}\|\Phi_{1/4,1}\|_{L^{4\sigma+2}}^{4\sigma+2}. \end{aligned}$$

From this relation, it is easily verified that

$$S_{c^2/4,c}(\phi_{c^2,4}) = c^{1+\frac{1}{\sigma}} S_{1/4,1}(\phi_{1/4,1}).$$

Since $u_{0,c} = e^{i\frac{cx}{2}}\psi$, we have

$$\begin{aligned} S_{c^2/4,c}(u_{0,c}) &= \widetilde{S}_{c^2/4,c}(\psi) \\ &= \frac{1}{2}\|\partial_x\psi\|_{L^2}^2 + \frac{c}{2(2\sigma+2)}\|\psi\|_{L^4}^4 - \frac{1}{2\sigma+2}N(\psi) \\ &\le S_{c^2/4,c}(\phi_{c^2/4,c}) = c^{1+\frac{1}{\sigma}} S_{1/4,1}(\phi_{1/4,1}), \\ K_{c^2/4,c}(u_{0,c}) &= \widetilde{K}_{c^2/4,c}(\psi) \\ &= \|\partial_x\psi\|_{L^2}^2 + \frac{c}{2}\|\psi\|_{L^4}^4 - N(\psi) \\ &\ge 0, \end{aligned}$$

for sufficiently large $c > 0$. Thus, $u_{0,c} \in \mathscr{K}_{c^2/4,c}$ for large $c > 0$. Hence, the claim follows from Proposition 3.1.4. □

As can be seen in our proof, we do not use a contradiction argument, the gauge transformation as (3.1.4), and any sharp Gagliardo–Nirenberg inequality.

Chapter 4

Variational approach to NLS equations of derivative type

4.1 Introduction

In this chapter, we consider the following nonlinear Schrödinger equation of derivative type:

(DNLSb) $$i\partial_t u + \partial_x^2 u + i|u|^2\partial_x u + b|u|^4 u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R},\ b \in \mathbb{R}.$$

(DNLSb) is L^2-critical in the sense that the equation and L^2-norm are invariant under the scaling transformation

(4.1.1) $$u_\lambda(t,x) := \lambda^{\frac{1}{2}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

This equation has the following conserved quantities:

(Energy) $$E(u) := \frac{1}{2}\|\partial_x u\|_{L^2}^2 - \frac{1}{4}\left(i|u|^2\partial_x u, u\right) - \frac{b}{6}\|u\|_{L^6}^6,$$

(Mass) $$M(u) := \|u\|_{L^2}^2,$$

(Momentum) $$P(u) := (i\partial_x u, u),$$

where $(\cdot,\cdot)$ is an inner product defined by

$$(v,w) := \mathrm{Re}\int_{\mathbb{R}} v(x)\overline{w(x)}dx \quad \text{for } v,w \in L^2(\mathbb{R}).$$

When $b = 0$, the equation

(DNLS) $$i\partial_t u + \partial_x^2 u + i|u|^2\partial_x u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}$$

is well-known as a standard derivative nonlinear Schrödinger equation. The equation (DNLSb) can be considered as a generalized equation of (DNLS) while preserving both

L^2-criticality and Hamiltonian structure. The aim of this chapter is to investigate the structure of (DNLSb) from the viewpoint of solitons.

First we note that (DNLSb) can be rewritten as

$$i\partial_t u = E'(u). \tag{4.1.2}$$

The Hamiltonian form (4.1.2) is useful when one discusses problems of orbital stability and instability of solitons. It is well-known that (DNLS) has a two-parameter family of solitons (see [40, 17]). Here we formulate the solitons of (DNLSb) following [57]. Consider solutions of (DNLSb) of the form

$$u_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x-ct), \tag{4.1.3}$$

where $(\omega, c) \in \mathbb{R}^2$, and assume that $\phi_{\omega,c} \in H^1(\mathbb{R})$. It is clear that $\phi_{\omega,c}$ must satisfy the following equation:

$$-\phi'' + \omega\phi + ic\phi' - i|\phi|^2\phi' - b|\phi|^4\phi = 0, \quad x \in \mathbb{R}. \tag{4.1.4}$$

We note that the equation (4.1.4) can be rewritten as $S'_{\omega,c}(\phi) = 0$, where

$$S_{\omega,c}(\phi) := E(\phi) + \frac{\omega}{2}M(\phi) + \frac{c}{2}P(\phi). \tag{4.1.5}$$

Applying the following gauge transformation to $\phi_{\omega,c}$

$$\phi_{\omega,c}(x) = \Phi_{\omega,c}(x)\exp\left(i\frac{c}{2}x - \frac{i}{4}\int_{-\infty}^{x} |\Phi_{\omega,c}(y)|^2\, dy\right), \tag{4.1.6}$$

it is easily verified (see [17, Lemma 2] for details) that $\Phi_{\omega,c}$ satisfies the following equation:

$$-\Phi'' + \left(\omega - \frac{c^2}{4}\right)\Phi + \frac{c}{2}|\Phi|^2\Phi - \frac{3}{16}\gamma|\Phi|^4\Phi = 0, \quad \gamma := 1 + \frac{16}{3}b. \tag{4.1.7}$$

The positive radial (even) solution of (4.1.7) is explicitly obtained as follows; if $\gamma > 0$ or equivalently $b > -3/16$,

$$\Phi^2_{\omega,c}(x) = \begin{cases} \dfrac{2(4\omega - c^2)}{\sqrt{c^2+\gamma(4\omega-c^2)}\cosh(\sqrt{4\omega-c^2}x) - c} & \text{if } -2\sqrt{\omega} < c < 2\sqrt{\omega}, \\[2ex] \dfrac{4c}{(cx)^2+\gamma} & \text{if } c = 2\sqrt{\omega}, \end{cases} \tag{4.1.8}$$

if $\gamma \le 0$ or equivalently $b \le -3/16$,

$$\Phi^2_{\omega,c}(x) = \frac{2(4\omega - c^2)}{\sqrt{c^2+\gamma(4\omega-c^2)}\cosh(\sqrt{4\omega-c^2}x) - c} \quad \text{if } -2\sqrt{\omega} < c < -2s_*\sqrt{\omega}, \tag{4.1.9}$$

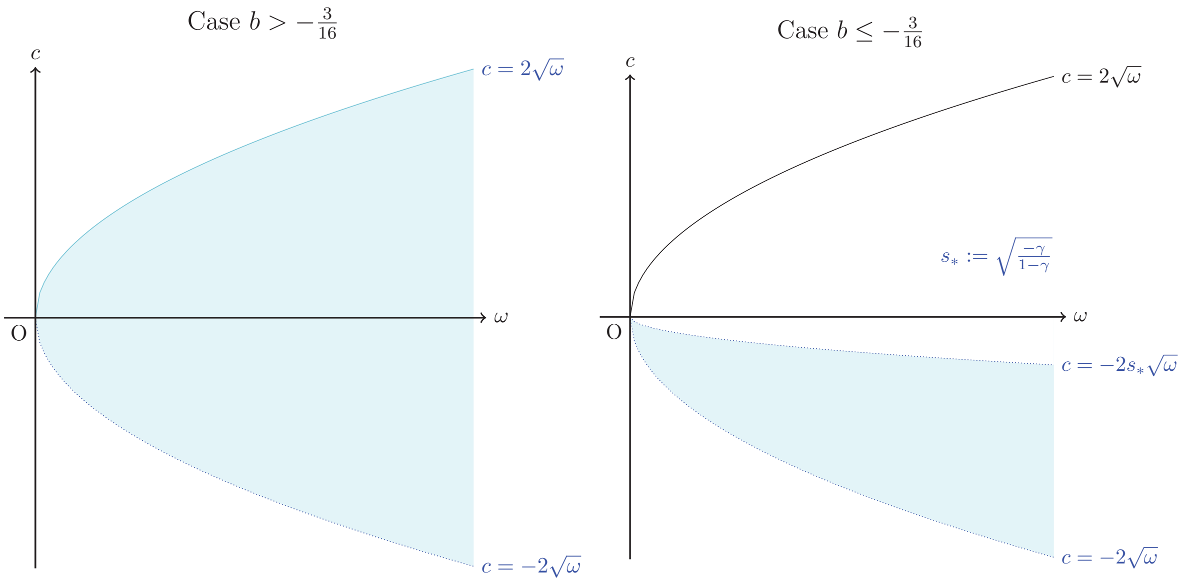

Figure 4.1: Existence region of solitons.

where $s_* = s_*(\gamma) = \sqrt{-\gamma/(1-\gamma)}$. From (4.1.3), (4.1.6), (4.1.8) and (4.1.9), we obtain the explicit formulae of solitons of (DNLSb).

We note that the condition of two parameters γ and (ω, c)

$$\text{(4.1.10)} \qquad \begin{aligned} &\text{if } \gamma > 0 \Leftrightarrow b > -3/16, \quad -2\sqrt{\omega} < c \le 2\sqrt{\omega}, \\ &\text{if } \gamma \le 0 \Leftrightarrow b \le -3/16, \quad -2\sqrt{\omega} < c < -2s_*\sqrt{\omega} \end{aligned}$$

is a necessary and sufficient condition for the existence of non-trivial solutions of (4.1.7) vanishing at infinity; see [8]. We note that the value $b = -3/16$ gives the turning point where the structure of the solitons of (DNLSb) changes. Especially algebraic solitons exist only for the case $b > -3/16$. In the case $b \le -3/16$ the solitons still exist, but their velocity must be negative. We note that $0 \le s_* < 1$ and $s_* \uparrow 1$ as $b \downarrow -\infty$. This means that as the defocusing effect is stronger, the existence region of solitons is narrower; see Figure 4.1.

When $b = 0$, Colin and Ohta [17] proved that the soliton $u_{\omega,c}$ is orbitally stable when $\omega > c^2/4$ by variational arguments, which are closely related to the work of Shatah [64]. See also [27] for partial results before [17]. The case $c = 2\sqrt{\omega}$ was treated[1] by Kwon and Wu [41], while the orbital stability or instability for this case is still an open problem.

When $b > 0$, the situation becomes different due to the focusing effect from the quintic term. Ohta [57] proved that for each $b > 0$ there exists a unique $s^* = s^*(b) \in (0,1)$ such that the soliton $u_{\omega,c}$ is orbitally stable if $-2\sqrt{\omega} < c < 2s^*\sqrt{\omega}$, and orbitally unstable if $2s^*\sqrt{\omega} < c < 2\sqrt{\omega}$. In [54] it was proved that the algebraic soliton $u_{\omega,2\sqrt{\omega}}$ is orbitally unstable when $b > 0$ is sufficiently small. If we observe the momentum of the solitons,

[1] The "orbital stability" discussed in [41] is different from usual definition as in Definition 4.1.12. Their result does not contradict that finite time blow-up occurs to the initial data near algebraic solitons.

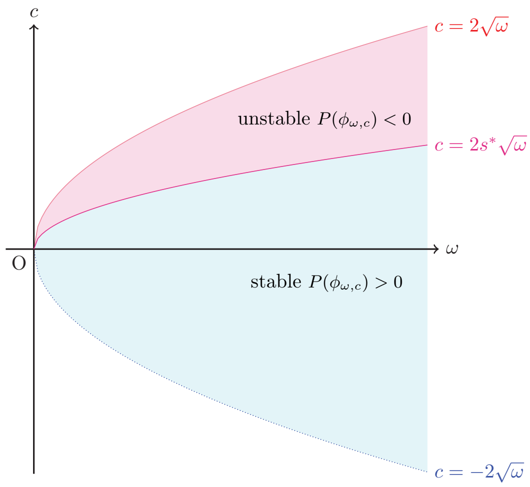

Figure 4.2: The stable/unstable region of solitons in the case $b > 0$.

the momentum is positive in the stable region, and negative in the unstable region; see Figure 4.2. This indicates that the momentum of the solitons has an important effect on the stability. In the borderline case $c = 2s^*\sqrt{\omega}$, the momentum of the solitons is zero, and the orbital stability or instability in this case remains an open problem.

The solitons in the defocusing case $b < 0$ are less well studied. In this chapter we study the properties of solitons of (DNLSb) including defocusing case. Our first theorem gives the connection between two types of solitons. To state the result, we introduce the set Ω defined by

$$\Omega := \left\{(\omega, c) \in \mathbb{R}^2 : -2\sqrt{\omega} < c < 2\sqrt{\omega}\right\}.$$

Then we have the following result.

Theorem 4.1.1. *Let $b > -3/16$. Suppose that (ω_0, c_0) satisfies $c_0 = 2\sqrt{\omega_0}$. Then, we have*

$$\lim_{\substack{(\omega,c)\to(\omega_0,c_0)\\(\omega,c)\in\Omega}} \|\phi_{\omega,c} - \phi_{\omega_0,c_0}\|_{H^m(\mathbb{R})} = 0$$

for any $m \in \mathbb{Z}_{\geq 0}$.

Remark 4.1.2. By Theorem 4.1.1 and Sobolev's embedding theorem, we obtain that

$$\lim_{\substack{(\omega,c)\to(\omega_0,c_0)\\(\omega,c)\in\Omega}} \|\phi_{\omega,c} - \phi_{\omega_0,c_0}\|_{W^{m,\infty}(\mathbb{R})} = 0$$

for any $m \in \mathbb{Z}_{\geq 0}$.

Theorem 4.1.1 shows that two types of solitons are connected in strong topology although each of the solitons has quite different decay. This relation is expected to be useful for further study on algebraic solitons. The key step for the proof is to prove the pointwise convergence. Since this limit corresponds to indeterminate forms, we need to calculate carefully so as to cancel out the singularity. Combining the pointwise convergence with the mass convergence

$$\lim_{\substack{(\omega,c)\to(\omega_0,c_0)\\(\omega,c)\in\Omega}} M(\phi_{\omega,c}) = M(\phi_{\omega_0,c_0}),$$

which is proved in Section 4.2.1, we obtain L^2-convergence. The regularity of the convergence is proved by using the equation (4.1.4) and a bootstrap argument.

Next we establish global existence for (DNLSb) in the energy space. The well-posedness in the energy space was studied in [33, 58]. In [58] it was proved that if the initial data $u_0 \in H^1(\mathbb{R})$ satisfies

$$\begin{aligned} &\text{if } b>0, \quad M(u_0) < \frac{2\pi}{\sqrt{\gamma}}, \\ &\text{if } b\le 0, \quad M(u_0) < 2\pi, \end{aligned} \tag{4.1.11}$$

then the corresponding $H^1(\mathbb{R})$-solution is global. This result is considered as an extension of that in [32]. The proof is done by gauge transformation, and by applying mass and energy conservation laws; see also Section 4.4 for details.

We note that the value $\frac{2\pi}{\sqrt{\gamma}}$ corresponds to the mass of the soliton $\phi_{\omega,0}$. If we take into account the effect of momentum, we can expect that the mass condition (4.1.11) is improved as in the case of (DNLS). Our main result in this chapter is the following.

Theorem 4.1.3. *Let $u_0 \in H^1(\mathbb{R})$ satisfy each of the following two cases:*

(i) *If $b>0$, $M(u_0) < M(\phi_{1,2s^*})$, or $M(u_0) = M(\phi_{1,2s^*})$ and $P(u_0)<0$.*

(ii) *If $-3/16 < b \le 0$, $M(u_0) < \frac{4\pi}{\gamma^{3/2}}$, or $M(u_0) = \frac{4\pi}{\gamma^{3/2}}$ and $P(u_0)<0$.*

Then the $H^1(\mathbb{R})$-solution u of (DNLSb) *with $u(0)=u_0$ exists globally in time. Moreover we have*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} \le C(\|u_0\|_{H^1}) < \infty.$$

Remark 4.1.4. One can establish explicit upper bound of $H^1(\mathbb{R})$-norm of the solution which is represented by the conserved quantities; see Lemma 4.6.1.

Remark 4.1.5. When $b>0$, by applying variational characterization of solitons, we have

$$\begin{aligned} &\left\{u_0 \in H^1(\mathbb{R}) : M(u_0) = M(\phi_{1,2s^*}), E(u_0) = P(u_0) = 0\right\} \\ &= \left\{e^{i\theta_0}\phi_{\omega,2s^*\sqrt{\omega}}(\cdot - x_0) : \theta_0, x_0 \in \mathbb{R},\, \omega>0\right\}. \end{aligned}$$

For the case of (DNLS), this relation corresponds to

$$\begin{aligned} &\left\{u_0 \in H^1(\mathbb{R}) : M(u_0) = 4\pi, E(u_0) = P(u_0) = 0\right\} \\ &= \left\{e^{i\theta_0}\phi_{\omega,2\sqrt{\omega}}(\cdot - x_0) : \theta_0, x_0 \in \mathbb{R}, \, \omega > 0\right\}, \end{aligned}$$

see Remark 3.1.7.

Remark 4.1.6. As seen in Section 4.2.1, when $\gamma > 0$ the function

$$(-1, 1] \ni s \mapsto M\left(\phi_{1,2s}\right) \in \left(0, \frac{4\pi}{\sqrt{\gamma}}\right]$$

is strictly increasing and surjective. Especially, when $b > 0$, we have

$$\frac{2\pi}{\sqrt{\gamma}} < M(\phi_{1,2s^*}) < \frac{4\pi}{\sqrt{\gamma}}.$$

Remark 4.1.7. When $b \leq -3/16$, by applying the suitable gauge transformation, one can easily prove that the $H^1(\mathbb{R})$-solution is global for any initial data $u_0 \in H^1(\mathbb{R})$; see Proposition 4.4.3. Especially the global result in the case $b = -3/16$ is compatible with Theorem 4.1.3, since $\frac{4\pi}{\gamma^{3/2}} \uparrow \infty$ as $b \downarrow -3/16$.

In the focusing case we recall that the soliton $\phi_{1,2s^*}$ corresponds to borderline case in the stable/unstable region of solitons as in Figure 4.2. In this sense the mass condition in Theorem 4.1.3 seems to be quite natural. We also note that

$$M(\phi_{1,2s^*}) \to 4\pi \text{ as } b \to 0,$$

which is proved in Section 4.2.3. This means that global results in Theorem 4.1.3 are compatible with the ones of (DNLS).

The global results in defocusing case are more interesting. When $-3/16 < b < 0$, since $0 < \gamma < 1$ in this case, the value $\frac{4\pi}{\gamma^{3/2}}$ is greater than 4π. This means that 4π-mass condition in (DNLS) is improved due to the defocusing effect from the quintic term. More surprisingly, the value $\frac{4\pi}{\gamma^{3/2}}$ is even greater than the mass of algebraic solitons. Indeed, we have the following relation:

$$M(\phi_{1,2}) = \frac{4\pi}{\sqrt{\gamma}} < \frac{4\pi}{\gamma^{3/2}} = M(\phi_{1,2}) + P(\phi_{1,2}),$$

which indicates that positive momentum of algebraic solitons improves the mass condition.

The proof of Theorem 4.1.3 is done by applying variational arguments developed in Chapter 3. First we give a variational characterization of the solitons in a unified way including the defocusing case. We note that in the case $b < 0$ the quintic term $b|u|^4u$ becomes an obstacle to characterize the solitons on the Nehari manifold with respect to the action functional $S_{\omega,c}$. To overcome that, we apply the suitable gauge transformation and consider the minimization problems on the Nehari manifold with respect to the transformed action functional. This approach enables us to give a variational characterization of the transformed solitons in the case $b \geq -3/16$.

Next, by applying the variational characterization and potential well theory, we give a sufficient condition for global existence in the energy space; see Lemma 4.6.1. This argument is closely related to the classical work of Payne and Sattinger [60]. The key step in the proof of Theorem 4.1.3 is to establish the connection between mass condition and a sufficient condition represented by potential well. To this end we prove the existence of the pair (ω, c) satisfying $-2\sqrt{\omega} < c \leq 2\sqrt{\omega}$ such that

$$S_{\omega,c}(u_0) \leq S_{\omega,c}(\phi_{\omega,c}), \tag{4.1.12}$$

under the assumption of Theorem 4.1.3. In this step we use the idea of taking the curve $c = 2s\sqrt{\omega}$ for $s \in (-1, 1]$ and large parameter $\omega > 0$, which was introduced for the case of (DNLS) in Chapter 3. Compared with (DNLS), we need to examine the effect of the momentum more carefully in our setting.

The condition (4.1.12) means that the initial data u_0 is below the ground state in the sense of action. We note that the mass condition in Theorem 4.1.3 is derived from the condition which expresses the initial data below the ground state.[2] The threshold value in the mass condition is optimal in the sense that if $b > 0$ (resp., if $-3/16 < b \leq 0$) for any $\rho \geq M(\phi_{1,2s^*})$ (resp., $\rho \geq \frac{4\pi}{\gamma^{3/2}}$) there exists $u_0 \in H^1(\mathbb{R})$ such that $M(u_0) = \rho$ and such that the condition (4.1.12) does not hold for any (ω, c) satisfying $-2\sqrt{\omega} < c \leq 2\sqrt{\omega}$, which means that u_0 is above the ground state.[3] Therefore, taking into account the L^2-critical structure of the equation, we conjecture the following:

Conjecture 4.1.8. *The mass condition in Theorem 4.1.3 is sharp for global existence of $H^1(\mathbb{R})$-solutions to* (DNLSb).

Remark 4.1.9. Related to the conjecture, it is an interesting problem whether the blow-up occurs in finite or infinite time for the initial data $u_0 \in H^1(\mathbb{R})$ satisfying that $M(u_0) = M(\phi_{1,2s^*})$ (resp., $M(u_0) = \frac{4\pi}{\gamma^{3/2}}$) and $P(u_0) \geq 0$ when $b > 0$ (resp., when $-3/16 < b \leq 0$).

Remark 4.1.10. Recently, in [37] it was proved by inverse scattering approach that (DNLS) is globally well-posed in weighted Sobolev space $H^{2,2}(\mathbb{R})$, where

$$H^{2,2}(\mathbb{R}) := \left\{ u \in H^2(\mathbb{R}) \ ; \ \langle \cdot \rangle^2 u \in L^2(\mathbb{R}) \right\}.$$

We note that algebraic solitons of (DNLS) do not contain in $H^{2,2}(\mathbb{R})$. We remark that our global results in this chapter treat the initial data in $H^1(\mathbb{R})$ which contain algebraic solitons. This difference of topology is quite important for (DNLS) from the viewpoint of solitons. We note that the results in [37] do not imply that Conjecture 4.1.8 is false in the case $b = 0$. We also note that inverse scattering approach works only for the case $b = 0$.

[2] See Proposition 4.6.4 for the case $b = -3/16$.

[3] As an example one can take a real-valued function $u_0 \in H^1(\mathbb{R})$ satisfying $M(u_0) = \rho$; see also the proof of Theorem 4.1.3.

If we consider sufficiently oscillating data, we obtain the global result for arbitrarily large mass:

Theorem 4.1.11. *Let $b > -3/16$. Given $\psi \in H^1(\mathbb{R})$, and set the initial data as $u_{0,c} = e^{i\frac{cx}{2}}\psi$. Then, there exists $c_0 = c_0(\psi) > 0$ such that if $c \geq c_0$, then the $H^1(\mathbb{R})$-solution u_c of* (DNLSb) *with $u_c(0) = u_{0,c}$ exists globally in time. Moreover we have*

$$\sup_{t\in\mathbb{R}} \|u_c(t)\|_{H^1} \leq C(\|u_{0,c}\|_{H^1}) < \infty.$$

This global result was proved in Chapter 3 for the case $b = 0$. For the proof of Theorem 4.1.11 we apply variational characterization for algebraic solitons.

Cazenave and Weissler [14] established global existence for the quadratic oscillating data on nonlinear Schrödinger equations with a pure power nonlinearity. One main difference with this result is that the oscillating term in Theorem 4.1.11 gives the change of the momentum. We note that (DNLSb) is not invariant under the Galilean transformation. Hence it is reasonable to consider that the momentum of initial data essentially influences global properties of the solutions to (DNLSb).

Finally, we study the orbital stability of the solitons as another application of variational arguments. First we give the precise definition of orbital stability.

Definition 4.1.12. *Let $u_{\omega,c}$ be a soliton of* (DNLSb) *defined by* (4.1.3)*. The soliton $u_{\omega,c}$ is said to be orbitally stable in $H^1(\mathbb{R})$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R})$ satisfies $\|u_0 - u_{\omega,c}(0)\|_{H^1} < \delta$, then the maximal solution $u(t)$ of* (DNLSb) *with $u(0) = u_0$ exists globally in time and satisfies*

$$\sup_{t\in\mathbb{R}} \inf_{(\theta,y)\in\mathbb{R}^2} \|u(t) - e^{i\theta}u_{\omega,c}(t, \cdot - y)\|_{H^1} < \varepsilon.$$

Otherwise, the soliton is said to be orbitally unstable.

We have the following theorem about the orbital stability of the solitons in the defocusing case.

Theorem 4.1.13. *Let $-3/16 \leq b < 0$. Assume that (ω_0, c_0) satisfies* (4.1.10)*. Then the soliton u_{ω_0,c_0} of* (DNLSb) *is orbitally stable.*

We note that Theorem 4.1.13 claims that algebraic solitons are orbitally stable in the case $-3/16 < b < 0$. This result gives the counterpart of the focusing case $b > 0$. For the proof of Theorem 4.1.13 we use variational argument inspired from the work developed in [64, 17, 57]. New perspective in our proof is to use the scaling curve $c = 2s\sqrt{\omega}$ effectively. To clarify our approach we first revisit the stability theory in the case $b \geq 0$. Our variational arguments along the scaling curve provide a simpler alternative proof in previous works. In our approach the positivity of the momentum of the soliton is used more directly to prove the stability. This is useful to tackle the stability in the defocusing case and enables us to prove the stability for two types of the solitons in a unified way. Unfortunately, our variational arguments do not cover the case $b < -3/16$ to prove the stability. However, if one takes spectral approach depending on the abstract theory of Grillakis, Shatah and Strauss [24, 25], one can recover the remaining cases; see the end of Section 4.7 for more details.

The rest of this chapter is organized as follows. In Section 4.2 we calculate the conserved quantities of the solitons. By using the explicit formulae of solitons, conserved quantities of solitons are also calculated explicitly. In Section 4.3 we study the connection of two types of solitons and give a proof of Theorem 4.1.1. In Section 4.4 we introduce the gauge transformation to (DNLSb). The local well-posedness theory in the energy space is also reviewed there. In Section 4.5 we study variational characterization of solitons. We give a unified proof for two types of solitons by applying concentration compactness arguments developed in [17]. In Section 4.6 we establish global existence in the energy space by applying variational characterization of the solitons. We show that a sufficient condition represented by potential well yields Theorem 4.1.3 and Theorem 4.1.11. In Section 4.7 we study orbital stability of the solitons and prove Theorem 4.1.13.

4.2 Conserved quantities of the solitons

4.2.1 Mass of the solitons

In this subsection we calculate the mass of the solitons. First we prepare the following elementary integration formulae.

Lemma 4.2.1. *Let $-1 < \alpha$. Then we have*

$$\int_{-\infty}^{\infty} \frac{dy}{\cosh y + \alpha} = \begin{cases} \dfrac{4}{\sqrt{1-\alpha^2}} \tan^{-1} \sqrt{\dfrac{1-\alpha}{1+\alpha}} & \text{if } \ |\alpha| < 1, \\ 2 & \text{if } \ \alpha = 1, \\ \dfrac{2}{\sqrt{\alpha^2-1}} \log\left(\alpha + \sqrt{\alpha^2 - 1}\right) & \text{if } \ \alpha > 1. \end{cases} \tag{4.2.1}$$

Proof. See the formula 3.513, 2 in [23]. □

By using Lemma 4.2.1, we have the following proposition.

Proposition 4.2.2. *Let γ and (ω, c) satisfy* (4.1.10)*. Then the following properties hold:*

(i) *When $\gamma > 0$, we have*

$$M\left(\phi_{\omega,c}\right) = \begin{cases} \dfrac{8}{\sqrt{\gamma}} \tan^{-1} \sqrt{\dfrac{1+\beta}{1-\beta}} & \text{if } \ -2\sqrt{\omega} < c < 2\sqrt{\omega}, \\ \dfrac{4\pi}{\sqrt{\gamma}} & \text{if } \ c = 2\sqrt{\omega}, \end{cases} \tag{4.2.2}$$

where β is defined by

$$\beta = \beta(\omega, c) := \frac{c}{\sqrt{c^2 + \gamma(4\omega - c^2)}}. \tag{4.2.3}$$

Moreover, the function

$$(-1,1] \ni s \mapsto M(\phi_{1,2s}) \in \left(0, \frac{4\pi}{\sqrt{\gamma}}\right]$$

is continuous, strictly increasing and surjective.

(ii) *When* $\gamma = 0$, *we have*

$$M(\phi_{\omega,c}) = \frac{4\sqrt{4\omega - c^2}}{-c} \quad \text{if } -2\sqrt{\omega} < c < 0. \tag{4.2.4}$$

Moreover, the function

$$(-1,0) \ni s \mapsto M(\phi_{1,2s}) \in (0,\infty)$$

is continuous, strictly increasing and surjective.

(iii) *When* $\gamma < 0$, *we have*

$$M(\phi_{\omega,c}) = \frac{4}{\sqrt{-\gamma}} \log\left(\alpha + \sqrt{\alpha^2 - 1}\right) \quad \text{if } -2\sqrt{\omega} < c < -2s_*\sqrt{\omega}, \tag{4.2.5}$$

where α *is defined by*

$$\alpha = \alpha(\omega, c) := \frac{-c}{\sqrt{c^2 + \gamma(4\omega - c^2)}}. \tag{4.2.6}$$

Moreover, the function

$$(-1,-s_*) \ni s \mapsto M(\phi_{1,2s}) \in (0,\infty)$$

is continuous, strictly increasing and surjective.

Proof. Let γ and (ω, c) satisfy (4.1.10). When $\omega > c^2/4$, from the explicit formulae of the solitons, we have

$$\begin{aligned} M(\phi_{\omega,c}) = M(\Phi_{\omega,c}) &= \int_{-\infty}^{\infty} \frac{2(4\omega - c^2)dx}{\sqrt{c^2 + \gamma(4\omega - c^2)}\cosh(\sqrt{4\omega - c^2}x) - c} \\ &= \frac{2\sqrt{4\omega - c^2}}{\sqrt{c^2 + \gamma(4\omega - c^2)}} \int_{-\infty}^{\infty} \frac{dy}{\cosh y + \alpha}, \end{aligned} \tag{4.2.7}$$

where α is defined by (4.2.6). We divide three cases to do calculations.

Case 1-1: $\gamma > 0$ and $-2\sqrt{\omega} < c < 2\sqrt{\omega}$. In this case we note that $|\alpha| < 1$ and

$$1 - \alpha^2 = 1 - \frac{c^2}{c^2 + \gamma(4\omega - c^2)} = \frac{\gamma(4\omega - c^2)}{c^2 + \gamma(4\omega - c^2)}. \tag{4.2.8}$$

Applying Lemma 4.2.1 to (4.2.7), we have

$$M\left(\phi_{\omega,c}\right)=\frac{2\sqrt{4\omega-c^2}}{\sqrt{c^2+\gamma(4\omega-c^2)}}\cdot\frac{4}{\sqrt{1-\alpha^2}}\tan^{-1}\sqrt{\frac{1-\alpha}{1+\alpha}}$$
$$=\frac{8}{\sqrt{\gamma}}\tan^{-1}\sqrt{\frac{1+\beta}{1-\beta}},\tag{4.2.9}$$

where β is defined by

$$\beta:=-\alpha=\frac{c}{\sqrt{c^2+\gamma(4\omega-c^2)}}.\tag{4.2.10}$$

We note that the function β is constant on each curve $c=2s\sqrt{\omega}$ for $s\in[-1,1]$. Then we have

$$\beta(s):=\beta(\omega,2s\sqrt{\omega})=\frac{s}{\sqrt{s^2+\gamma(1-s^2)}}$$
$$=\frac{\operatorname{sgn} s}{\sqrt{1+\gamma\left(\frac{1}{s^2}-1\right)}}.$$

This shows that the function

$$[-1,1]\ni s\mapsto\beta(s)\in[-1,1]\tag{4.2.11}$$

is continuous, strictly increasing and surjective. The function

$$(-1,1)\ni\beta\mapsto\frac{1+\beta}{1-\beta}\in(0,\infty)$$

also has the same property. Therefore, by (4.2.9) we obtain that the function

$$(-1,1)\ni s\mapsto M\left(\phi_{1,2s}\right)\in\left(0,\frac{4\pi}{\sqrt{\gamma}}\right)$$

is continuous, strictly increasing and surjective. We also note that

$$\lim_{s\to1-0}M\left(\phi_{1,2s}\right)=\frac{4\pi}{\sqrt{\gamma}}.\tag{4.2.12}$$

Case 1-2: $\gamma>0$ and $c=2\sqrt{\omega}$. From the explicit formulae of algebraic solitons, we have

$$M\left(\phi_{c^2/4,c}\right)=M\left(\Phi_{c^2/4,c}\right)=\int_{-\infty}^{\infty}\frac{4c}{c^2x^2+\gamma}dx$$
$$=\int_{-\infty}^{\infty}\frac{4dx}{x^2+\gamma}$$
$$=\frac{4\pi}{\sqrt{\gamma}}.\tag{4.2.13}$$

From (4.2.12) and (4.2.13), we obtain

$$\lim_{s\to 1-0} M\left(\phi_{1,2s}\right) = M\left(\phi_{1,2}\right), \tag{4.2.14}$$

which completes the proof of the case $\gamma > 0$.

Case 2: $\gamma = 0$ and $-2\sqrt{\omega} < c < 0$. In this case we note that $\alpha = 1$. From (4.2.7) and Lemma 4.2.1, we have

$$\begin{aligned} M\left(\phi_{\omega,c}\right) &= \frac{2\sqrt{4\omega - c^2}}{-c}\int_{-\infty}^{\infty}\frac{dy}{\cosh y + 1} \\ &= \frac{4\sqrt{4\omega - c^2}}{-c}. \end{aligned} \tag{4.2.15}$$

For $s \in (-1, 0)$, we have

$$\begin{aligned} M\left(\phi_{\omega,2s\sqrt{\omega}}\right) = M\left(\phi_{1,2s}\right) &= \frac{4\sqrt{1-s^2}}{-s} \\ &= 4\sqrt{\frac{1}{s^2} - 1}. \end{aligned}$$

This yields that the function

$$(-1, 0) \ni s \mapsto M\left(\phi_{1,2s}\right) \in (0, \infty) \tag{4.2.16}$$

is continuous, strictly increasing and surjective.

Case 3: $\gamma < 0$ and $-2\sqrt{\omega} < c < -2s_*\sqrt{\omega}$. In this case we note that $\alpha > 1$. From Lemma 4.2.1, (4.2.7) and (4.2.8), we have

$$\begin{aligned} M\left(\phi_{\omega,c}\right) &= \frac{2\sqrt{4\omega - c^2}}{\sqrt{c^2 + \gamma(4\omega - c^2)}}\int_{-\infty}^{\infty}\frac{dy}{\cosh y + \alpha} \\ &= \frac{2\sqrt{4\omega - c^2}}{\sqrt{c^2 + \gamma(4\omega - c^2)}}\cdot\frac{2}{\sqrt{\alpha^2 - 1}}\log\left(\alpha + \sqrt{\alpha^2 - 1}\right) \\ &= \frac{4}{\sqrt{-\gamma}}\log\left(\alpha + \sqrt{\alpha^2 - 1}\right) \end{aligned} \tag{4.2.17}$$

In the same way as β, the function α is constant on each curve $c = 2s\sqrt{\omega}$ for $s \in [-1, 1]$. We note that

$$\begin{aligned} \alpha(s) := \alpha(\omega, 2s\sqrt{\omega}) &= \frac{-s}{\sqrt{(1-\gamma)s^2 + \gamma}} \\ &= \frac{1}{\sqrt{1 - \gamma + \gamma s^{-2}}}. \end{aligned} \tag{4.2.18}$$

This yields that the function

$$(-1, -s_*) \ni s \mapsto \alpha(s) \in (1, \infty)$$

is continuous, strictly increasing and surjective. From the formula (4.2.17), we deduce that the function

$$(-1,-s_*) \ni s \mapsto M(\phi_{1,2s}) \in (0,\infty) \tag{4.2.19}$$

has the same property. This completes the proof. □

4.2.2 Momentum of the solitons

In this subsection we calculate the momentum of the solitons. From the formula (4.1.6) of the solitons, we have

$$\begin{aligned} P(\phi_{\omega,c}) &= \mathrm{Re}\int_{\mathbb{R}} i\phi'_{\omega,c}\overline{\phi_{\omega,c}}dx \\ &= \mathrm{Re}\int_{\mathbb{R}} i\left(\Phi'_{\omega,c}+\Phi_{\omega,c}\left(\frac{ic}{2}-\frac{i}{4}\Phi^2_{\omega,c}\right)\right)\overline{\Phi_{\omega,c}}dx \\ &= -\frac{c}{2}M(\Phi_{\omega,c})+\frac{1}{4}\|\Phi_{\omega,c}\|^4_{L^4} \end{aligned} \tag{4.2.20}$$

To calculate the L^4-norm, we prepare the following elementary integration formulae.

Lemma 4.2.3. *Let $-1<\alpha$. Then we have*

$$\int_{-\infty}^{\infty}\frac{dy}{(\cosh y+\alpha)^2} = \begin{cases} \frac{2}{1-\alpha^2}-\frac{4\alpha}{(1-\alpha^2)^{3/2}}\tan^{-1}\sqrt{\frac{1-\alpha}{1+\alpha}} & \text{if } |\alpha|<1, \\ \frac{2}{3} & \text{if } \alpha=1, \\ -\frac{2}{\alpha^2-1}+\frac{2\alpha}{(\alpha^2-1)^{3/2}}\log\left(\alpha+\sqrt{\alpha^2-1}\right) & \text{if } \alpha>1. \end{cases} \tag{4.2.21}$$

Proof. Change variables $t=e^y$ and apply the formula 3.252, 4 in [23]. □

By using Lemma 4.2.3, we have the following proposition.

Proposition 4.2.4. *The momentum of the solitons is represented as follows; if $\gamma>0$ and $-2\sqrt{\omega}<c\le 2\sqrt{\omega}$ or if $\gamma<0$ and $-2\sqrt{\omega}<c<-2s_*\sqrt{\omega}$, we have*

$$P(\phi_{\omega,c}) = \frac{c}{2}\left(-1+\frac{1}{\gamma}\right)M(\phi_{\omega,c})+\frac{2}{\gamma}\sqrt{4\omega-c^2}. \tag{4.2.22}$$

If $\gamma=0$ and $-2\sqrt{\omega}<c<0$, we have

$$P(\phi_{\omega,c}) = -\frac{2\omega+c^2}{3c}M(\phi_{\omega,c}). \tag{4.2.23}$$

Remark 4.2.5. The momentum is represented by the same formula in the cases $\gamma>0$ and $\gamma<0$ although each mass is represented by the different functions in these cases.

Proof. Let γ and (ω, c) satisfy (4.1.10). By Theorem 4.1.1, the momentum in the case $c = 2\sqrt{\omega}$ is obtained[4] by taking the limit

$$\lim_{s \to 1-0} P(\phi_{\omega, 2s\sqrt{\omega}}) = P(\phi_{\omega, 2\sqrt{\omega}}).$$

Hence we may consider the only case $\omega > c^2/4$. We note that $\Phi^2_{\omega,c}(x)$ is rewritten as

$$\Phi^2_{\omega,c}(x) = \frac{2(4\omega - c^2)}{\sqrt{c^2 + \gamma(4\omega - c^2)}} \cdot \frac{1}{\cosh(\sqrt{4\omega - c^2}x) + \alpha},$$

where α is defined by (4.2.6). Then we have

$$\|\Phi_{\omega,c}\|^4_{L^4} = \frac{4(4\omega - c^2)^{3/2}}{c^2 + \gamma(4\omega - c^2)} \int_{-\infty}^{\infty} \frac{dy}{(\cosh y + \alpha)^2}. \tag{4.2.24}$$

We divide three cases in the same way as the proof of Proposition 4.2.2.

Case 1: $\gamma > 0$ and $-2\sqrt{\omega} < c < 2\sqrt{\omega}$. In this case we note that $|\alpha| < 1$. By Lemma 4.2.3, (4.2.8) and (4.2.2), we obtain that

$$\begin{aligned} \|\Phi_{\omega,c}\|^4_{L^4} &= \frac{4(4\omega - c^2)^{3/2}}{c^2 + \gamma(4\omega - c^2)} \cdot \left[\frac{2}{1 - \alpha^2} - \frac{4\alpha}{(1 - \alpha^2)^{3/2}} \tan^{-1} \sqrt{\frac{1 - \alpha}{1 + \alpha}} \right] \\ &= \frac{8}{\gamma}\sqrt{4\omega - c^2} + \frac{16c}{\gamma^{3/2}} \tan^{-1} \sqrt{\frac{1 + \beta}{1 - \beta}} \\ &= \frac{8}{\gamma}\sqrt{4\omega - c^2} + \frac{2c}{\gamma} M(\Phi_{\omega,c}). \end{aligned} \tag{4.2.25}$$

From (4.2.20) and (4.2.25), we have

$$\begin{aligned} P(\phi_{\omega,c}) &= -\frac{c}{2} M(\Phi_{\omega,c}) + \frac{1}{4}\|\Phi_{\omega,c}\|^4_{L^4} \\ &= \frac{c}{2}\left(-1 + \frac{1}{\gamma}\right) M(\Phi_{\omega,c}) + \frac{2}{\gamma}\sqrt{4\omega - c^2}. \end{aligned}$$

Case 2: $\gamma < 0$ and $-2\sqrt{\omega} < c < 0$. In this case we note that $\alpha = 1$. By Lemma 4.2.3 and (4.2.4), we obtain that

$$\begin{aligned} \|\Phi_{\omega,c}\|^4_{L^4} &= \frac{4(4\omega - c^2)^{3/2}}{c^2} \int_{-\infty}^{\infty} \frac{dy}{(\cosh y + 1)^2} \\ &= \frac{8(4\omega - c^2)^{3/2}}{3c^2} \\ &= -\frac{2(4\omega - c^2)}{3c} M(\Phi_{\omega,c}). \end{aligned} \tag{4.2.26}$$

[4] The proof of Theorem 4.1.1 is proved in Section 4.3, which is independent of the proof of Proposition 4.2.4. One can also calculate the momentum in the case $c = 2\sqrt{\omega}$ directly.

From (4.2.20) and (4.2.26), we have

$$\begin{aligned} P(\phi_{\omega,c}) &= -\frac{c}{2}M(\Phi_{\omega,c}) + \frac{1}{4}\|\Phi_{\omega,c}\|_{L^4}^4 \\ &= -\frac{2\omega + c^2}{3c}M(\Phi_{\omega,c}). \end{aligned}$$

Case 3: $\gamma > 0$ and $-2\sqrt{\omega} < c < -2s_*\sqrt{\omega}$. In this case we note that $\alpha > 1$. By Lemma 4.2.3, (4.2.8) and (4.2.5), we obtain that

$$\begin{aligned} \|\Phi_{\omega,c}\|_{L^4}^4 &= \frac{4(4\omega - c^2)^{3/2}}{c^2 + \gamma(4\omega - c^2)} \cdot \left[-\frac{2}{\alpha^2 - 1} + \frac{2\alpha}{(\alpha^2 - 1)^{3/2}} \log\left(\alpha + \sqrt{\alpha^2 - 1}\right)\right] \\ &= \frac{8}{\gamma}\sqrt{4\omega - c^2} - \frac{8c}{(-\gamma)^{3/2}} \log\left(\alpha + \sqrt{\alpha^2 - 1}\right) \\ &= \frac{8}{\gamma}\sqrt{4\omega - c^2} - \frac{2c}{-\gamma}M(\Phi_{\omega,c}). \end{aligned} \tag{4.2.27}$$

This is exactly the same as the formula (4.2.25). Hence the momentum has the same formula as the Case 1. □

By the Pohozaev identity, the energy of the solitons is represented by the momentum.

Proposition 4.2.6. *Let γ and (ω, c) satisfy* (4.1.10). *Then we have*

$$E(\phi_{\omega,c}) = -\frac{c}{4}P(\phi_{\omega,c}). \tag{4.2.28}$$

Proof. For $\lambda > 0$, let $u^\lambda(x) = \lambda^{1/2}u(\lambda x)$. It is easily verified that

$$\begin{aligned} S_{\omega,c}(\phi_{\omega,c}^\lambda) &= E(\phi_{\omega,c}^\lambda) + \frac{\omega}{2}M(\phi_{\omega,c}^\lambda) + \frac{c}{2}P(\phi_{\omega,c}^\lambda) \\ &= \lambda^2 E(\phi_{\omega,c}) + \frac{\omega}{2}M(\phi_{\omega,c}) + \lambda \cdot \frac{c}{2}P(\phi_{\omega,c}). \end{aligned} \tag{4.2.29}$$

Since $S'_{\omega,c}(\phi_{\omega,c}) = 0$, we have

$$\frac{d}{d\lambda}S_{\omega,c}(\phi_{\omega,c}^\lambda)\bigg|_{\lambda=1} = \left\langle S'_{\omega,c}(\phi_{\omega,c}), \frac{1}{2}\phi_{\omega,c} + x\phi'_{\omega,c}\right\rangle = 0.$$

From (4.2.29) we deduce that

$$0 = \frac{d}{d\lambda}S_{\omega,c}(\phi_{\omega,c}^\lambda)\bigg|_{\lambda=1} = 2E(\phi_{\omega,c}) + \frac{c}{2}P(\phi_{\omega,c}).$$

Hence the result follows. □

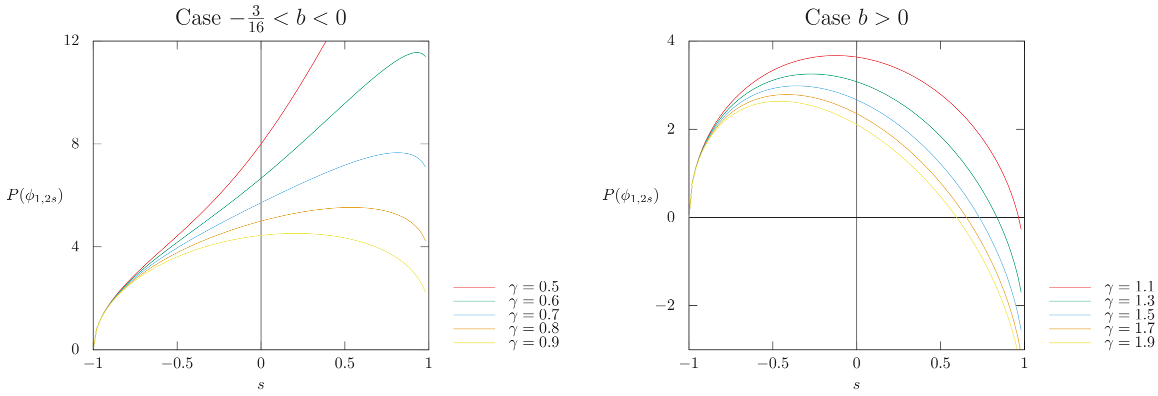

Figure 4.3: The function $s \mapsto P(\phi_{1,2s})$ for several values of $b > -3/16$.

4.2.3 Positivity of the momentum

The effect of the momentum plays an essential role in the arguments on both global existence and orbital stability of the solitons. In this subsection we study the sign of the momentum of the soliton. Let $\omega > 0$ and let s satisfy

$$\begin{aligned} &\text{if } \gamma > 0 \Leftrightarrow b > -3/16, \quad -1 < s \leq 1, \\ &\text{if } \gamma \leq 0 \Leftrightarrow b \leq -3/16, \quad -1 < s < -s_*. \end{aligned} \tag{4.2.30}$$

Since $P(\phi_{\omega,2s\sqrt{\omega}}) = \sqrt{\omega}P(\phi_{1,2s})$ from Proposition 4.2.4, it is enough to check the sign of $P(\phi_{1,2s})$.

Proposition 4.2.7. *Let s satisfy* (4.2.30)*. Then the following properties hold:*

(i) *If $b < 0$, $P(\phi_{1,2s}) > 0$ for any s satisfying* (4.2.30)*.*

(ii) *If $b = 0$, $P(\phi_{1,2s}) > 0$ for $s \in (-1, 1)$ and $P(\phi_{1,2}) = 0$.*

(iii) *If $b > 0$, there exists a unique $s^* = s^*(b) \in (0, 1)$ such that $P(\phi_{1,2s^*}) = 0$. Moreover, we have $P(\phi_{1,2s}) > 0$ for $s \in (-1, s^*)$ and $P(\phi_{1,2s}) < 0$ for $s \in (s^*, 1]$.*

Remark 4.2.8. As in Figure 4.3, the zero point of the function $s \mapsto P(\phi_{1,2s})$ moves to the right and converges to 1 as $b \downarrow 0$. This remark is rigorously proved below.

Proof. First we note that $\phi_{1,-2}$ is the zero solution of the equation (4.1.4) and

$$\lim_{s \to -1+0} P(\phi_{1,2s}) = P(\phi_{1,-2}) = 0, \tag{4.2.31}$$

which follows from Proposition 4.2.4.

(i) If $b = -3/16$, the positivity of the momentum is obvious from the formula (4.2.23). Let us consider the case $-3/16 < b < 0$. First we note that the formula (4.2.22) is rewritten as

$$P(\phi_{1,2s}) = s\left(-1 + \frac{1}{\gamma}\right) M(\phi_{1,2s}) + \frac{4}{\gamma}\sqrt{1 - s^2}. \tag{4.2.32}$$

Since $-1+\frac{1}{\gamma}>0$, $P(\phi_{1,2s})>0$ for $s\in[0,1]$ follows from (4.2.32). It is easily verified that the function $(-1,0)\ni s\mapsto P(\phi_{1,2s})$ is continuous and strictly increasing. Therefore, from (4.2.31) we have

$$0=P(\phi_{1,-2})<P(\phi_{1,2s})$$

for $s\in(-1,0)$. The proof in the case $b<-3/16$ is done similarly.

(ii) This is obvious from the formula (4.2.32).

(iii) Since $-1+\frac{1}{\gamma}<0$ in this case, $P(\phi_{1,2s})>0$ for $s\in(-1,0]$ follows from (4.2.32). We note that

$$\begin{aligned}P(\phi_{1,0})&=\frac{4}{\gamma}>0,\\ P(\phi_{1,2})&=\left(-1+\frac{1}{\gamma}\right)M(\phi_{1,2})=-\frac{4\pi\,(\gamma-1)}{\gamma^{3/2}}<0,\end{aligned}$$

and the function $[0,1]\ni s\mapsto P(\phi_{1,2s})$ is continuous and strictly decreasing. Therefore there exists $s^*\in(0,1)$ such that $P(\phi_{1,2s^*})=0$, $P(\phi_{1,2s})>0$ for $s\in(0,s^*)$ and $P(\phi_{1,2s})<0$ for $s\in(s^*,1]$. This completes the proof. □

4.3 Connection between two types of the solitons

In this section we prove Theorem 4.1.1. It is enough to discuss the convergence of $\phi_{1,2s}$ as $s\to 1$. First we prove the pointwise convergence.

Proposition 4.3.1. *Let $b>-3/16$. For any $x\in\mathbb{R}$ we have*

$$\lim_{s\to 1-0}\phi_{1,2s}(x)=\phi_{1,2}(x). \tag{4.3.1}$$

Proof. Fix any $x\in\mathbb{R}$. First we discuss the convergence of $\Phi_{1,2s}(x)$. From the explicit formula (4.1.8), we have

$$\Phi_{1,2s}^2(x)=\frac{4(1-s^2)}{\sqrt{s^2+\gamma(1-s^2)}\cosh\left(2\sqrt{1-s^2}x\right)-s} \tag{4.3.2}$$

for $s\in(-1,1)$. By the Taylor expansion of cosh, the denominator is rewritten as

$$\sqrt{s^2+\gamma(1-s^2)}\left(1+2(1-s^2)x^2+O\left((1-s^2)^2\right)\right)-s. \tag{4.3.3}$$

By the Taylor expansion of the function $h\mapsto\sqrt{s^2+h}$, we have

$$\begin{aligned}(4.3.3)&=\frac{\gamma}{2s}(1-s^2)+2(1-s^2)\sqrt{s^2+\gamma(1-s^2)}x^2+O\left((1-s^2)^2\right)\\ &=(1-s^2)\left(\frac{\gamma}{2s}+2\sqrt{s^2+\gamma(1-s^2)}x^2+O\left(1-s^2\right)\right).\end{aligned}$$

We note that the numerator and denominator share a common factor $1-s^2$. Hence we deduce that

$$\begin{aligned}\Phi_{1,2s}^2(x) &= \frac{4}{\frac{\gamma}{2s} + 2\sqrt{s^2+\gamma(1-s^2)}x^2 + O\left(1-s^2\right)} \\ &\underset{s\to 1-0}{\longrightarrow} \frac{8}{\gamma+4x^2} = \Phi_{1,2}^2(x).\end{aligned}$$

From this and the formula (4.1.6), the result follows. □

To complete the proof of Theorem 4.1.1, we effectively use the Brézis–Lieb lemma (see Lemma 3.2.10). For convenience we write the statement again.

Lemma 4.3.2 ([11])**.** *Let $1 \le p < \infty$. Let $\{f_n\}$ be a bounded sequence in $L^p(\mathbb{R})$ and $f_n \to f$ a.e. in $\mathbb{R}$ as $n \to \infty$. Then we have*

$$\|f_n\|_{L^p}^p - \|f_n - f\|_{L^p}^p - \|f\|_{L^p}^p \to 0$$

as $n \to \infty$.

Proof of Theorem 4.1.1. From Proposition 4.2.2 and Proposition 4.3.1, we have

$$\begin{aligned}&\lim_{s\to 1-0} \phi_{1,2s}(x) = \phi_{1,2}(x) \text{ for all } x \in \mathbb{R}, \\ &\lim_{s\to 1-0} \|\phi_{1,2s}\|_{L^2}^2 = \|\phi_{1,2}\|_{L^2}^2 .\end{aligned}$$

Applying Lemma 4.3.2, we have

$$\lim_{s\to 1-0} \|\phi_{1,2s} - \phi_{1,2}\|_{L^2}^2 = 0. \tag{4.3.4}$$

In the same way, we also have

$$\lim_{s\to 1-0} \|\Phi_{1,2s} - \Phi_{1,2}\|_{L^2}^2 = 0. \tag{4.3.5}$$

Here we recall that $\Phi_{1,2s}$ is the solution of the equation

$$-\Phi'' + (1-s^2)\Phi + s|\Phi|^2\Phi - \frac{3}{16}\gamma|\Phi|^4\Phi = 0. \tag{4.3.6}$$

We note that

$$\begin{aligned}\|\Phi_{1,2s}\|_{L^\infty}^2 &= \Phi_{1,2s}^2(0) \\ &= \frac{4(1-s^2)}{\sqrt{s^2+\gamma(1-s^2)} - s} \\ &= \frac{4}{\gamma}\left(\sqrt{s^2+\gamma(1-s^2)} + s\right).\end{aligned}$$

This yields that the function $(-1,1) \ni s \mapsto \|\Phi_{1,2s}\|_{L^\infty}$ is strictly increasing and

$$\lim_{s\to 1-0} \|\Phi_{1,2s}\|_{L^\infty}^2 = \frac{8}{\gamma} = \|\Phi_{1,2}\|_{L^\infty}^2.$$

Especially we have

$$\max_{s\in(-1,1]} \|\Phi_{1,2s}\|_{L^\infty} = \|\Phi_{1,2}\|_{L^\infty}. \tag{4.3.7}$$

By Proposition 4.2.2, (4.3.7) and (4.3.5), we obtain

$$\begin{aligned}
\|s\Phi_{1,2s}^3 - \Phi_{1,2}^3\|_{L^2} &\le (1-s)\|\Phi_{1,2s}^3\|_{L^2} + \|\Phi_{1,2s}^3 - \Phi_{1,2}^3\|_{L^2} \\
&\le (1-s)\|\Phi_{1,2}\|_{L^\infty}^2\|\Phi_{1,2}\|_{L^2} + 3\|\Phi_{1,2}\|_{L^\infty}^2\|\Phi_{1,2s} - \Phi_{1,2}\|_{L^2} \\
&\underset{s\to 1-0}{\longrightarrow} 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|\Phi_{1,2s}^5 - \Phi_{1,2}^5\|_{L^2} &\le 4\|\Phi_{1,2}\|_{L^\infty}^4\|\Phi_{1,2s} - \Phi_{1,2}\|_{L^2} \\
&\underset{s\to 1-0}{\longrightarrow} 0.
\end{aligned}$$

Hence, by using the equation (4.3.6), we deduce that

$$\begin{aligned}
\|\Phi_{1,2s}'' - \Phi_{1,2}''\|_{L^2} &\le (1-s^2)\|\Phi_{1,2s}\|_{L^2}^2 + \|s\Phi_{1,2s}^3 - \Phi_{1,2}^3\|_{L^2} \\
&\quad + \frac{3}{16}\gamma\|\Phi_{1,2s}^5 - \Phi_{1,2}^5\|_{L^2} \\
&\underset{s\to 1-0}{\longrightarrow} 0.
\end{aligned}$$

From this and (4.3.5) we have

$$\lim_{s\to 1-0} \|\Phi_{1,2s} - \Phi_{1,2}\|_{H^2} = 0.$$

From the formula (4.1.6), this yields that

$$\lim_{s\to 1-0} \|\phi_{1,2s} - \phi_{1,2}\|_{H^2} = 0.$$

The rest of the proof is done by using the equation (4.1.4) and a standard bootstrap argument. □

4.4 Gauge transformation

The equation (DNLSb) has various equivalent forms under gauge transformation. In this section we discuss the gauge transformations and their application. First we recall the result of local well-posedness for (DNLSb) in the energy space.

Theorem 4.4.1 ([58]). *For every $u_0 \in H^1(\mathbb{R})$, there exist $0 < T_{\min}, T_{\max} \le \infty$ and a unique, maximal solution $u \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R})) \cap L^4((-T_{\min}, T_{\max}), W^{1,\infty}(\mathbb{R}))$ of* (DNLSb) *with $u(0) = u_0$. Furthermore, the following properties hold:*

(i) *If $T_{\max} < \infty$ (resp., if $T_{\min} < \infty$), then $\|\partial_x u(t)\|_{L^2} \to \infty$ as $t \uparrow T_{\max}$ (resp., as $t \downarrow -T_{\min}$).*

(ii) *There is conservation of energy, mass and momentum; i.e., $E(u(t)) = E(u_0)$, $M(u(t)) = M(u_0)$ and $P(u(t)) = P(u_0)$ for all $t \in (-T_{\min}, T_{\max})$.*

(iii) *Continuous dependence is satisfied in the following sense; if $u_{0n} \to u_0$ in $H^1(\mathbb{R})$ and if $I \subset (-T_{\min}(u_0), T_{\max}(u_0))$ is a closed interval, then the maximal solution u_n of* (DNLSb) *with $u_n(0) = u_{0n}$ is defined on I for n large enough and satisfies $u_n \to u$ in $C(I, H^1(\mathbb{R}))$.*

In [58] the proof of Theorem 4.4.1 is done by transforming the equation (DNLSb) into a new system of equations as follows; see also [31, 32, 33]. For the solution u of (DNLSb) we set

$$\varphi(t,x) = \exp\left(\frac{i}{2}\int_{-\infty}^{x} |u(t,y)|^2 dy\right) u(t,x),$$
$$\psi(t,x) = \exp\left(\frac{i}{2}\int_{-\infty}^{x} |u(t,y)|^2 dy\right) \partial u(t,x),$$

then new functions φ and ψ formally satisfy

$$\begin{cases} i\partial_t \varphi + \partial_x^2 \varphi = i\varphi^2 \overline{\psi} + f(\varphi), \\ i\partial_t \psi + \partial_x^2 \psi = -i\psi^2 \overline{\varphi} + \partial_\varphi f(\varphi)\psi + \partial_{\overline{\varphi}} f(\varphi)\overline{\psi}, \end{cases} \tag{4.4.1}$$

where $f(\varphi) = -b|\varphi|^4\varphi$. Since the system (4.4.1) has no loss of derivatives unlike the original equation (DNLSb), one can solve the Cauchy problem by the fixed point argument. Note that in order to construct the solution of (DNLSb) through the system, we need to solve the equation (4.4.1) under the constraint condition

$$\psi = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi,$$

which needs more or less complex calculation; see [33] for details. In Chapter 2 we took a more direct approach without using a system of equations. This approach is also applicable to the equation (DNLSb).

We note that the gauge transformation plays a key role when one transforms the equation (DNLSb) into a system of equations (4.4.1). Here we consider more general gauge transformations as seen in [72]. For $a \in \mathbb{R}$ we define $\mathcal{G}_a : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ by

$$\mathcal{G}_a(u)(t,x) = \exp\left(ia\int_{-\infty}^{x} |u(t,y)|^2 dy\right) u(t,x). \tag{4.4.2}$$

A direct computation shows the following.

Proposition 4.4.2. *Let $a \in \mathbb{R}$, and let $u \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}))$ be a maximal solution of* (DNLSb). *Then $v = \mathcal{G}_a(u) \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}))$, and v satisfies the equation*

$$i\partial_t v + \partial_x^2 v + (-2a+1)i|v|^2\partial_x v - 2aiv^2\partial_x\overline{v} + \left(a^2 + \frac{a}{2} + b\right)|v|^4 v = 0. \tag{4.4.3}$$

Moreover, the equation (4.4.3) *has the following conserved quantities:*

$$\begin{aligned} E_a(v) &= \frac{1}{2}\|\partial_x v\|_{L^2}^2 + \left(a - \frac{1}{4}\right)(i|v|^2\partial_x v, v) + \left(\frac{a^2}{2} - \frac{a}{4} - \frac{b}{6}\right)\|v\|_{L^6}^6, \\ M_a(v) &= \|v\|_{L^2}^2, \\ P_a(v) &= (i\partial_x v, v) + a\|v\|_{L^4}^4. \end{aligned}$$

It is important to choose the suitable parameter $a \in \mathbb{R}$ depending on the situation. If we set $a = 1/2$, the term $i|v|^2\partial v$ is removed in (4.4.3) and it is useful when one treats the Fourier restriction norm (see [66, 18, 19]).

When $a = 1/4$ the interaction term with derivative in the energy is canceled out, which yields the advantage of giving a sufficient condition for global existence of solutions in the energy space (see [32, 72, 73]). In this chapter we apply the gauge transformation in the case $a = 1/4$ for giving the variational characterization of the solitons including the case $b < 0$. By Proposition 4.4.2, $v = \mathcal{G}_{1/4}(u)$ satisfies the equation

$$i\partial_t v + \partial_x^2 v + \frac{i}{2}|v|^2\partial_x v - \frac{i}{2}v^2\partial_x\overline{v} + \frac{3}{16}\gamma|v|^4 v = 0, \tag{DNLSb$'$}$$

where $\gamma = 1 + 16b/3$. The conserved quantities of (DNLSb$'$) are as follows:

$$\mathcal{E}(v) := E_{1/4}(v) = \frac{1}{2}\|\partial_x v\|_{L^2}^2 - \frac{\gamma}{32}\|v\|_{L^6}^6, \tag{Energy}$$

$$\mathcal{M}(v) := M_{1/4}(v) = \|v\|_{L^2}^2, \tag{Mass}$$

$$\mathcal{P}(v) := P_{1/4}(v) = (i\partial_x v, v) + \frac{1}{4}\|v\|_{L^4}^4. \tag{Momentum}$$

We note that the energy functional $\mathcal{E}(v)$ is nonnegative if $\gamma \leq 0$. Hence one can easily prove the following.

Proposition 4.4.3. *Let $b \leq -3/16$. For every $u_0 \in H^1(\mathbb{R})$, the maximal $H^1(\mathbb{R})$-solution u of* (DNLSb) *given by Proposition* 4.4.2 *is global and*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} \leq C(\|u_0\|_{H^1}) < \infty. \tag{4.4.4}$$

Proof. Set $v_0 = \mathcal{G}_{1/4}(u_0)$ and $v = \mathcal{G}_{1/4}(u)$. From Proposition 4.4.2, we have

$$\|\partial_x v(t)\|_{L^2}^2 \leq 2\mathcal{E}(v(t)) = 2\mathcal{E}(v_0) = 2E(u_0)$$

for all $t \in (-T_{\min}, T_{\max})$. This gives that $T_{\min} = T_{\max} = \infty$ and

$$\sup_{t\in\mathbb{R}} \|v(t)\|_{H^1}^2 \leq 2E(u_0) + M(u_0).$$

Since $u = \mathcal{G}_{-1/4}(v)$, we deduce that (4.4.4). □

When $b > -3/16$, if we apply the following sharp Gagliardo–Nirenberg inequality (see [71])

$$\|f\|_{L^6}^6 \leq \frac{4}{\pi^2}\|f\|_{L^2}^4\|\partial_x f\|_{L^2}^2, \tag{4.4.5}$$

we deduce that if the initial data $u_0 \in H^1(\mathbb{R})$ satisfying $\|u_0\|_{L^2}^2 < \frac{2\pi}{\sqrt{\gamma}}$, then the corresponding $H^1(\mathbb{R})$-solution u of (DNLSb) is global. A similar approach was originally taken in [32, 33, 58].

Finally, we discuss the solitons of (DNLSb′). Let (ω, c) satisfy (4.1.10). (DNLSb′) has a two-parameter family of solitons

$$v_{\omega,c}(t,x) = \mathcal{G}_{1/4}(u_{\omega,c})(t,x) = e^{i\omega t}\varphi_{\omega,c}(x-ct), \tag{4.4.6}$$

where $\varphi_{\omega,c}$ is defined by

$$\varphi_{\omega,c}(x) = e^{i\frac{cx}{2}}\Phi_{\omega,c}(x).$$

We note that $\varphi_{\omega,c}$ satisfies the equation

$$-\varphi'' + \omega\varphi + ic\varphi' + \frac{c}{2}|\varphi|^2\varphi - \frac{3}{16}\gamma|\varphi|^4\varphi = 0, \tag{4.4.7}$$

which can be written as $\mathcal{S}'_{\omega,c}(\varphi) = 0$, where

$$\mathcal{S}_{\omega,c}(\varphi) = \mathcal{E}(\varphi) + \frac{\omega}{2}\mathcal{M}(\varphi) + \frac{c}{2}\mathcal{P}(\varphi).$$

Since

$$\mathcal{E}(\mathcal{G}_{1/4}(u)) = E(u),\ \mathcal{M}(\mathcal{G}_{1/4}(u)) = M(u),\ \mathcal{P}(\mathcal{G}_{1/4}(u)) = P(u),$$

we note that

$$\mathcal{S}_{\omega,c}(\varphi_{\omega,c}) = \mathcal{S}_{\omega,c}(\mathcal{G}_{1/4}(\phi_{\omega,c})) = S_{\omega,c}(\phi_{\omega,c}) = d(\omega,c). \tag{4.4.8}$$

4.5 Variational characterization

In this section we give a variational characterization of the soliton $v_{\omega,c}$ defined by (4.4.6). Here we assume that γ and (ω, c) satisfy

$$\begin{aligned} &\text{if } \gamma > 0 \Leftrightarrow b > -3/16, \quad -2\sqrt{\omega} < c \leq 2\sqrt{\omega}, \\ &\text{if } \gamma = 0 \Leftrightarrow b = -3/16, \quad -2\sqrt{\omega} < c < 0. \end{aligned} \tag{4.5.1}$$

We prepare some notations. First we define the functional spaces by

$$\varphi \in X_{\omega,c} \iff \begin{cases} \varphi \in H^1(\mathbb{R}) & \text{if } \omega > c^2/4, \\ e^{-i\frac{cx}{2}}\varphi \in \dot{H}^1(\mathbb{R}) \cap L^4(\mathbb{R}) & \text{if } c = 2\sqrt{\omega}, \end{cases} \tag{4.5.2}$$

$$\|\varphi\|_{X_{c^2/4,c}} := \|e^{-i\frac{c}{2}\cdot}\varphi\|_{\dot{H}^1 \cap L^4}.$$

Note that $H^1(\mathbb{R}) \subset X_{c^2/4,c}$. We define the functional $\mathcal{K}_{\omega,c}$ by

$$\mathcal{K}_{\omega,c}(\varphi) := \|\partial_x\varphi\|_{L^2}^2 + \omega\|\varphi\|_{L^2}^2 + c\,(\partial_x\varphi,\varphi) + \frac{c}{2}\|\varphi\|_{L^4}^4 - \frac{3}{16}\gamma\|\varphi\|_{L^6}^6. \tag{4.5.3}$$

Note that $\mathcal{K}_{\omega,c}(\varphi) = \frac{d}{d\lambda}\,\mathcal{S}_{\omega,c}(\lambda u)|_{\lambda=1}$. We consider the following minimization problem:

$$\mu(\omega,c) := \inf\left\{\mathcal{S}_{\omega,c}(\varphi) : \varphi \in X_{\omega,c}\setminus\{0\}, \mathcal{K}_{\omega,c}(\varphi) = 0\right\}.$$

We introduce the sets $\mathscr{G}_{\omega,c}$ and $\mathscr{M}_{\omega,c}$ defined by

$$\mathscr{G}_{\omega,c} := \left\{\varphi \in X_{\omega,c}\setminus\{0\} : \mathcal{S}'_{\omega,c}(\varphi) = 0\right\},$$
$$\mathscr{M}_{\omega,c} := \left\{\varphi \in X_{\omega,c}\setminus\{0\} : \mathcal{S}_{\omega,c}(\varphi) = \mu(\omega,c), \mathcal{K}_{\omega,c}(\varphi) = 0\right\}.$$

The element of $\mathscr{G}_{\omega,c}$ is called a ground state. $\mathscr{M}_{\omega,c}$ is the set of minimizers of $\mathcal{S}_{\omega,c}$ on the Nehari manifold. The main result in this section is the following.

Proposition 4.5.1. *Let γ and (ω,c) satisfy* (4.5.1). *Then we have*

$$\mathscr{G}_{\omega,c} = \mathscr{M}_{\omega,c} = \left\{e^{i\theta_0}\varphi_{\omega,c}(\cdot - x_0) : \theta_0 \in [0,2\pi), x_0 \in \mathbb{R}\right\}, \tag{4.5.4}$$

and $d(\omega,c) = \mu(\omega,c)$.

Our proof of Proposition 4.5.1 depends on the argument in [17]; see also Chapter 3 for the case $c = 2\sqrt{\omega}$. For convenience of notation, we define

$$\mathcal{L}_{\omega,c}(\varphi) := \|\partial_x\varphi\|_{L^2}^2 + \omega\|\varphi\|_{L^2}^2 + c\,(\partial_x\varphi,\varphi),$$
$$\mathcal{I}_{\omega,c}(\varphi) := \mathcal{S}_{\omega,c}(\varphi) - \frac{1}{4}\mathcal{K}_{\omega,c}(\varphi) = \frac{1}{4}\mathcal{L}_{\omega,c}(\varphi) + \frac{\gamma}{64}\|\varphi\|_{L^6}^6.$$

First we prove the following lemma.

Lemma 4.5.2. *Let γ and (ω,c) satisfy* (4.5.1). *Then the following properties hold:*

(i) *If $\omega > c^2/4$, there exists $C_1 = C_1(\omega,c)$ such that*

$$\mathcal{L}_{\omega,c}(\varphi) \ge C_1\|\varphi\|_{H^1}^2 \text{ for } \varphi \in H^1(\mathbb{R}).$$

(ii) *$\mu(\omega,c) > 0$.*

(iii) *If $\varphi \in X_{\omega,c}$ satisfies $\mathcal{K}_{\omega,c}(\varphi) < 0$, then $\mu(\omega,c) < \mathcal{I}_{\omega,c}(\varphi)$.*

Proof. (i) See Lemma 7 (1) in [17].

(ii) Case 1: $\omega > c^2/4$. Let $\varphi \in H^1(\mathbb{R})\setminus\{0\}$ satisfy $\mathcal{K}_{\omega,c}(\varphi) = 0$. By (i), (4.5.3) and the Sobolev inequality, there exists $C_2 > 0$ such that

$$\begin{aligned} C_1\|\varphi\|_{H^1}^2 \le \mathcal{L}_{\omega,c}(\varphi) &= -\frac{c}{2}\|\varphi\|_{L^4}^4 + \frac{3}{16}\gamma\|\varphi\|_{L^6}^6 \\ &\le \frac{|c|}{2}\|\varphi\|_{L^2}\|\varphi\|_{L^6}^3 + \frac{3}{16}\gamma\|\varphi\|_{L^6}^6 \\ &\le \frac{C_1}{2}\|\varphi\|_{H^1}^2 + C_2\|\varphi\|_{H^1}^6. \end{aligned}$$

This yields that $\|\varphi\|_{H^1}^4 \geq \frac{C_1}{2C_2}$. Hence we have

$$\begin{aligned}\mu(\omega, c) &= \inf\left\{\mathcal{I}_{\omega,c}(\varphi) : \varphi \in H^1(\mathbb{R}) \setminus \{0\}, \mathcal{K}_{\omega,c}(\varphi) = 0\right\} \\ &\geq \frac{1}{4} \inf\left\{\mathcal{L}_{\omega,c}(\varphi) : \varphi \in H^1(\mathbb{R}) \setminus \{0\}, \mathcal{K}_{\omega,c}(\varphi) = 0\right\} \\ &\geq \frac{C_1}{4}\sqrt{\frac{C_1}{2C_2}} > 0.\end{aligned}$$

Case 2: $c = 2\sqrt{\omega}$. In this case we have

$$\mathcal{L}_{\omega,c}(\varphi) = \left\|\partial_x \varphi - \frac{i}{2}c\varphi\right\|_{L^2}^2 + \left(\omega - \frac{c^2}{4}\right)\|\varphi\|_{L^2}^2 = \left\|\partial_x\left(e^{-i\frac{cx}{2}}\varphi\right)\right\|_{L^2}^2 > 0 \tag{4.5.5}$$

for $\varphi \in X_{\omega,c} \setminus \{0\}$. This yields that $\mu(\omega, c) \geq 0$. We prove $\mu(\omega, c) > 0$ by contradiction. Assume that $\mu(\omega, c) = 0$. Then we can take the minimizing sequence $\{\varphi_n\} \subset X_{\omega,c} \setminus \{0\}$ such that

$$\mathcal{S}_{\omega,c}(\varphi_n) \underset{n\to\infty}{\longrightarrow} 0 \text{ and } \mathcal{K}_{\omega,c}(\varphi_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{4.5.6}$$

Since $\mathcal{S}_{\omega,c}$ is rewritten as

$$\mathcal{S}_{\omega,c}(\varphi) = \frac{1}{4}\mathcal{K}_{\omega,c}(\varphi) + \frac{1}{4}\mathcal{L}_{\omega,c}(\varphi) + \frac{\gamma}{64}\|\varphi\|_{L^6}^6, \tag{4.5.7}$$

from (4.5.5) and (4.5.6), we obtain that

$$\left\|\partial_x\left(e^{-i\frac{cx}{2}}\varphi_n\right)\right\|_{L^2},\ \|\varphi_n\|_{L^6} \underset{n\to\infty}{\longrightarrow} 0.$$

By using an elementary interpolation inequality

$$\|f\|_{L^\infty}^4 \leq 4\|f\|_{L^6}^3\|\partial_x f\|_{L^2},$$

we have $\|\varphi_n\|_{L^\infty} \to 0$ as $n \to \infty$. Hence we have

$$\begin{aligned}0 = \mathcal{K}_{\omega,c}(\varphi_n) &= \mathcal{L}_{\omega,c}(\varphi_n) + \frac{c}{2}\|\varphi_n\|_{L^4}^4 - \frac{3}{16}\gamma\|\varphi_n\|_{L^6}^6 \\ &\geq \left(\frac{c}{2} - \frac{3}{16}\gamma\|\varphi_n\|_{L^\infty}^2\right)\|\varphi_n\|_{L^4}^4 > 0\end{aligned}$$

for large $n \in \mathbb{N}$, which contradicts (4.5.6).

(iii) Let $\varphi \in X_{\omega,c} \setminus \{0\}$ satisfy $\mathcal{K}_{\omega,c}(\varphi) < 0$. Then there exists a unique $\lambda_0 \in (0, 1)$ such that $\mathcal{K}_{\omega,c}(\lambda_0\varphi) = 0$. From the definition of $\mu(\omega, c)$, we have

$$\mu(\omega, c) \leq \mathcal{I}_{\omega,c}(\lambda_0\varphi) = \frac{\lambda_0^2}{4}\mathcal{L}_{\omega,c}(\varphi) + \frac{\lambda_0^6\gamma}{64}\|\varphi\|_{L^6}^6 < \mathcal{I}_{\omega,c}(\varphi).$$

This completes the proof. □

By the standard ODE arguments, we have the following lemma.

Lemma 4.5.3. *Let γ and (ω, c) satisfy* (4.1.10). *Then we have*

$$\mathscr{G}_{\omega,c} = \left\{ e^{i\theta_0}\varphi_{\omega,c}(\cdot - x_0) : \theta_0 \in [0, 2\pi), x_0 \in \mathbb{R} \right\}.$$

Next we prove the following result.

Lemma 4.5.4. *Let γ and (ω, c) satisfy* (4.5.1). *Assume that $\mathscr{M}_{\omega,c} \neq \emptyset$. Then we have $\mathscr{G}_{\omega,c} = \mathscr{M}_{\omega,c}$. Moreover we have $d(\omega, c) = \mu(\omega, c)$.*

Proof. First we prove $\mathscr{M}_{\omega,c} \subset \mathscr{G}_{\omega,c}$. Let $\varphi \in \mathscr{M}_{\omega,c}$. Since φ is a minimizer on the Nehari manifold, there exists a Lagrange multiplier $\eta \in \mathbb{R}$ such that $\mathcal{S}'_{\omega,c}(\varphi) = \eta \mathcal{K}'_{\omega,c}(\varphi)$. Thus we have

$$0 = \mathcal{K}_{\omega,c}(\varphi) = \left\langle \mathcal{S}'_{\omega,c}(\varphi), \varphi \right\rangle = \eta \left\langle \mathcal{K}'_{\omega,c}(\varphi), \varphi \right\rangle.$$

By $\mathcal{K}_{\omega,c}(\varphi) = 0$ and $\varphi \neq 0$, we have

$$\begin{aligned} \left\langle \mathcal{K}'_{\omega,c}(\varphi), \varphi \right\rangle &= 2\mathcal{L}_{\omega,c}(\varphi) + 2c\|\varphi\|_{L^4}^4 - \frac{9}{8}\gamma\|\varphi\|_{L^6}^6 \\ &= -2\mathcal{L}_{\omega,c}(\varphi) - \frac{3}{8}\gamma\|\varphi\|_{L^6}^6 < 0. \end{aligned}$$

This yields that $\eta = 0$ and $\varphi \in \mathscr{G}_{\omega,c}$, which implies $\mathscr{M}_{\omega,c} \subset \mathscr{G}_{\omega,c}$. Conversely, let $\varphi \in \mathscr{G}_{\omega,c}$. By Lemma 4.5.3, there exist $\theta_0 \in [0, 2\pi)$ and $x_0 \in \mathbb{R}$ such that $\varphi = e^{i\theta_0}\varphi_{\omega,c}(\cdot - x_0)$. Since $\mathscr{M}_{\omega,c} \neq \emptyset$, we can take some $\psi \in \mathscr{M}_{\omega,c}$. By Lemma 4.5.3 again, there exist $\theta_1 \in [0, 2\pi)$ and $x_1 \in \mathbb{R}$ such that $\psi = e^{i\theta_1}\varphi_{\omega,c}(\cdot - x_1)$. Thus we have

$$\mathcal{S}_{\omega,c}(\varphi) = \mathcal{S}_{\omega,c}(\varphi_{\omega,c}) = \mathcal{S}_{\omega,c}(\psi) = \mu(\omega, c).$$

Since $\mathcal{K}_{\omega,c}(\varphi) = \left\langle \mathcal{S}'_{\omega,c}(\varphi), \varphi \right\rangle$, we deduce that $\varphi \in \mathscr{M}_{\omega,c}$. This completes the proof. □

To complete the proof of Proposition 4.5.1, we need to prove that $\mathscr{M}_{\omega,c} \neq \emptyset$. To this end we use Lieb's concentration compactness (see Lemma 3.2.9). For convenience we write the statement again.

Lemma 4.5.5 ([43, 7])**.** *Let $p \geq 2$. Let $\{f_n\}$ be a bounded sequence in $\dot{H}^1(\mathbb{R}) \cap L^p(\mathbb{R})$. Assume that there exists $q \in (p, \infty)$ such that $\limsup_{n\to\infty} \|f_n\|_{L^q} > 0$. Then, there exist $\{y_n\} \subset \mathbb{R}$ and $f \in \dot{H}^1(\mathbb{R}) \cap L^p(\mathbb{R}) \setminus \{0\}$ such that $\{f_n(\cdot - y_n)\}$ has a subsequence that converges to f weakly in $\dot{H}^1(\mathbb{R}) \cap L^p(\mathbb{R})$.*

The assertion $\mathscr{M}_{\omega,c} \neq \emptyset$ follows from the following proposition.

Proposition 4.5.6. *Let γ and (ω, c) satisfy* (4.5.1). *If a sequence $\{\varphi_n\} \subset X_{\omega,c}$ satisfies*

$$\mathcal{S}_{\omega,c}(\varphi_n) \to \mu(\omega, c) \text{ and } \mathcal{K}_{\omega,c}(\varphi_n) \to 0 \text{ as } n \to \infty, \tag{4.5.8}$$

then there exist a sequence $\{y_n\} \subset \mathbb{R}$ and $v \in \mathscr{M}_{\omega,c}$ such that $\{\varphi_n(\cdot - y_n)\}$ has a subsequence that converges to v strongly in $X_{\omega,c}$.

Remark 4.5.7. If we only prove that $\mathscr{M}_{\omega,c} \neq \emptyset$, we may assume that $\mathcal{K}_{\omega,c}(\varphi_n) = 0$ for all $n \in \mathbb{N}$. However, when one proves orbital stability of the solitons by variational arguments, it is essentially necessary to consider the minimizing sequence $\{\varphi_n\}$ satisfying $\mathcal{K}_{\omega,c}(\varphi_n) \neq 0$; see Section 4.7.

Proof. **Step 1.** $\{\varphi_n\}$ is bounded in $X_{\omega,c}$. If $\omega > c^2/4$, this follows from (4.5.7) and Lemma 4.5.2 (i). If $c = 2\sqrt{\omega}$, from (4.5.5) and (4.5.7) we obtain that

$$\sup_{n\in\mathbb{N}} \|\varphi_n\|_{L^6}^6, \ \sup_{n\in\mathbb{N}} \|\partial_x \left(e^{-i\frac{cx}{2}}\varphi_n\right)\|_{L^2}^2 < \infty.$$

Since we have

$$\mathcal{K}_{\omega,c}(\varphi_n) = \mathcal{L}_{\omega,c}(\varphi_n) + \frac{c}{2}\|\varphi_n\|_{L^4}^4 - \frac{3}{16}\gamma\|\varphi_n\|_{L^6}^6, \tag{4.5.9}$$

we deduce that $\{\varphi_n\}$ is also bounded in $L^4(\mathbb{R})$.
Step 2. $\limsup_{n\to\infty} \|\varphi_n\|_{L^6} > 0$. Suppose that $\lim_{n\to\infty} \|\varphi_n\|_{L^6} = 0$. If $\omega > c^2/4$, by the boundedness of $\{\varphi_n\}$ in $L^2(\mathbb{R})$ we have

$$\|\varphi_n\|_{L^4}^4 \leq \|\varphi_n\|_{L^2}\|\varphi_n\|_{L^6}^3 \underset{n\to\infty}{\longrightarrow} 0.$$

From (4.5.9) we deduce that $\mathcal{L}_{\omega,c}(\varphi_n) \to 0$. By (4.5.7), we have $\mathcal{S}_{\omega,c}(\varphi_n) \to 0$, but this gives a contradiction with $\mu(\omega, c) > 0$. If $c = 2\sqrt{\omega}$, from (4.5.9) we obtain that

$$\mathcal{L}_{\omega,c}(\varphi_n), \ \|\varphi_n\|_{L^4}^4 \underset{n\to\infty}{\longrightarrow} 0,$$

which yields $\mathcal{S}_{\omega,c}(\varphi_n) \to 0$ again and gives a contradiction.
Step 3. By Step 1, Step 2 and Lemma 4.5.5, there exist $\{y_n\} \subset \mathbb{R}$ and $v \in X_{\omega,c} \setminus \{0\}$ such that a subsequence of $\{\varphi(\cdot - y_n)\}$ (we denote it by $\{v_n\}$) converges to v weakly in $X_{\omega,c}$. Taking a subsequence if necessary, we have $v_n \to v$ a.e. in $\mathbb{R}$. By applying Lemma 4.3.2, we have

$$\mathcal{K}_{\omega,c}(v_n) - \mathcal{K}_{\omega,c}(v_n - v) - \mathcal{K}_{\omega,c}(v) \to 0, \tag{4.5.10}$$
$$\mathcal{I}_{\omega,c}(v_n) - \mathcal{I}_{\omega,c}(v_n - v) - \mathcal{I}_{\omega,c}(v) \to 0 \tag{4.5.11}$$

as $n \to \infty$.
Step 4. $\mathcal{K}_{\omega,c}(v) \leq 0$. Suppose that $\mathcal{K}_{\omega,c}(v) > 0$. By $\mathcal{K}_{\omega,c}(v_n) \to 0$ and (4.5.10), we have

$$\mathcal{K}_{\omega,c}(v_n - v) \to \mathcal{K}_{\omega,c}(v) < 0.$$

This implies that $\mathcal{K}_{\omega,c}(v_n - v) < 0$ for large $n \in \mathbb{N}$. Applying Lemma 4.5.2 (iii), we have $\mu(\omega, c) < \mathcal{I}_{\omega,c}(v_n - v)$ for large $n \in \mathbb{N}$. By (4.5.8) and

$$\mathcal{S}_{\omega,c}(\varphi) = \frac{1}{4}\mathcal{K}_{\omega,c}(\varphi) + \mathcal{I}_{\omega,c}(\varphi),$$

we have $\mathcal{I}_{\omega,c}(v_n) \to \mu(\omega, c)$. Combined with (4.5.11), we have

$$\mathcal{I}_{\omega,c}(v) = \lim_{n\to\infty} \{\mathcal{I}_{\omega,c}(v_n) - \mathcal{I}_{\omega,c}(v_n - v)\} \le \mu(\omega, c) - \mu(\omega, c) = 0,$$

which yields that $v = 0$. This is a contradiction.
Step 5. By Step 4, Lemma 4.5.2 (iii), and the weakly lower semicontinuity of $\mathcal{I}_{\omega,c}$, we have

$$\mu(\omega, c) \le \mathcal{I}_{\omega,c}(v) \le \liminf_{n\to\infty} \mathcal{I}_{\omega,c}(v_n) = \mu(\omega, c).$$

Thus we have $\mathcal{I}_{\omega,c}(v) = \mu(\omega, c)$. By Step 4 and Lemma 4.5.2 (iii), we have $\mathcal{K}_{\omega,c}(v) = 0$. Therefore $v \in \mathscr{M}_{\omega,c}$. By (4.5.11) and $\mathcal{I}_{\omega,c}(v) = \mu(\omega, c)$, we have $\mathcal{I}_{\omega,c}(v_n - v) \to 0$, which yields that $v_n \to v$ strongly in $X_{\omega,c}$. This completes the proof. □

4.6 Global existence

In this section we prove Theorem 4.1.3 and Theorem 4.1.11. To this end, we apply the potential theory inspired from the arguments by Payne and Sattinger [60]. Consider the following subset of the energy space:

$$\mathscr{W}_{\omega,c} := \left\{\varphi \in H^1(\mathbb{R}) : \mathcal{S}_{\omega,c}(\varphi) \le d(\omega, c), \mathcal{K}_{\omega,c}(v_0) \ge 0\right\}.$$

By using the variational characterization of the solitons in Section 4.5, we have the following lemma.

Lemma 4.6.1. *Let γ and (ω, c) satisfy* (4.5.1). *If $v_0 \in \mathscr{W}_{\omega,c}$, then the $H^1(\mathbb{R})$-solution v of* (DNLSb′) *with $v(0) = v_0$ exists globally in time and $v(t) \in \mathscr{W}_{\omega,c}$ for all $t \in \mathbb{R}$. Moreover we have*

$$\|\partial_x v\|^2_{L^\infty(\mathbb{R},L^2)} \le 8\mathcal{S}_{\omega,c}(v_0) + \frac{c^2}{2}\mathcal{M}(v_0). \tag{4.6.1}$$

Remark 4.6.2. This lemma yields the following global result;

$$\begin{aligned} &\text{if } b > -3/16, \quad v_0 \in \bigcup_{\substack{-2\sqrt{\omega}<c\le 2\sqrt{\omega}\\ \omega>0}} \mathscr{W}_{\omega,c}, \\ &\text{if } b = -3/16, \quad v_0 \in \bigcup_{\substack{-2\sqrt{\omega}<c<0\\ \omega>0}} \mathscr{W}_{\omega,c}, \end{aligned} \tag{4.6.2}$$

then $H^1(\mathbb{R})$-solution v of (DNLSb′) with $v(0) = v_0$ exists globally in time.

Proof. Let $v \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}))$ be a maximal solution of (DNLSb′) with $v(0) = v_0$. If $\mathcal{K}_{\omega,c}(v_0) = 0$, by Proposition 4.5.1, we have $v_0 = 0$ or $v_0 = e^{i\theta_0}\varphi_{\omega,c}(\cdot - x_0)$ for some $\theta_0, x_0 \in \mathbb{R}$. By uniqueness we have $v(t) = 0$ or $v(t) = e^{i\omega t}e^{i\theta_0}\varphi_{\omega,c}(\cdot - ct - x_0)$

for all $t \in \mathbb{R}$, respectively. This implies that $v(t) \in \mathscr{W}_{\omega,c}$ for all $t \in \mathbb{R}$. Consider the case $\mathcal{K}_{\omega,c}(v_0) > 0$. If there exists $t_* \in (-T_{\min}, T_{\max})$ such that $\mathcal{K}_{\omega,c}(v(t_*)) = 0$, the above argument gives that $\mathcal{K}_{\omega,c}(v(0)) = 0$, which is a contradiction. Since the function $t \mapsto \mathcal{K}_{\omega,c}(v(t))$ is continuous, we deduce that $\mathcal{K}_{\omega,c}(v(t)) > 0$ for all $t \in (-T_{\min}, T_{\max})$. This implies that $v(t) \in \mathscr{W}_{\omega,c}$ for all $t \in (-T_{\min}, T_{\max})$.

Next we prove the solution v exists globally in time. By (4.5.7) and $v(t) \in \mathscr{W}_{\omega,c}$, we obtain that

$$\begin{aligned}
\mathcal{S}_{\omega,c}(v_0) &= \mathcal{S}_{\omega,c}(v(t)) \\
&= \frac{1}{4}\mathcal{K}_{\omega,c}(v(t)) + \frac{1}{4}\mathcal{L}_{\omega,c}(v(t)) + \frac{\gamma}{64}\|v(t)\|_{L^6}^6 \\
&\geq \frac{1}{4}\mathcal{L}_{\omega,c}(v(t)) \\
&\geq \frac{1}{4}\left\|\partial_x\left(e^{-i\frac{cx}{2}}v(t)\right)\right\|_{L^2}^2
\end{aligned}$$

for all $t \in (-T_{\min}, T_{\max})$. This implies that $T_{\min} = T_{\max} = \infty$. More precisely we have

$$\begin{aligned}
\|\partial_x v(t)\|_{L^2}^2 &\leq \left(\left\|\partial_x v(t) - \frac{c}{2}iv(t)\right\|_{L^2} + \frac{|c|}{2}\|v(t)\|_{L^2}\right)^2 \\
&\leq 2\left\|\partial_x\left(e^{-i\frac{cx}{2}}v(t)\right)\right\|_{L^2}^2 + \frac{c^2}{2}\mathcal{M}(v_0) \\
&\leq 8\mathcal{S}_{\omega,c}(v_0) + \frac{c^2}{2}\mathcal{M}(v_0)
\end{aligned}$$

for all $t \in \mathbb{R}$. This completes the proof. □

Next we examine the set $\mathscr{W}_{\omega,c}$ to investigate the initial data satisfying the condition (4.6.2). To this end, we need to calculate the value of $d(\omega, c)$. Here we consider the curve $c = 2s\sqrt{\omega}$, where s satisfies that

$$\text{(4.6.3)} \qquad \begin{aligned}
&\text{if } \gamma > 0 \Leftrightarrow b > -3/16, \quad -1 < s \leq 1, \\
&\text{if } \gamma = 0 \Leftrightarrow b = -3/16, \quad -1 < s < 0.
\end{aligned}$$

We note that $d(\omega, 2s\sqrt{\omega}) = \omega d(1, 2s)$.

Lemma 4.6.3. *Let $\gamma \geq 0$. Then the following properties hold:*

(i) *If $\gamma > 1$, the function $(-1, 1] \ni s \mapsto d(1, 2s)$ is strictly increasing on $(-1, s^*)$ and strictly decreasing on $(s^*, 1]$.*

(ii) *If $0 < \gamma \leq 1$, the function $(-1, 1] \ni s \mapsto d(1, 2s)$ is strictly increasing.*

(iii) *If $\gamma = 0$, the function $(-1, 0) \ni s \mapsto d(1, 2s)$ is strictly increasing.*

Proof. From the definition we have

$$d(1,2s) = S_{1,2s}(\phi_{1,2s}) = E(\phi_{1,2s}) + \frac{1}{2}M(\phi_{1,2s}) + sP(\phi_{1,2s}).$$

Since $S'_{1,2s}(\phi_{1,2s}) = 0$, we have

$$\frac{d}{ds}d(1,2s) = P(\phi_{1,2s}).$$

Hence the result follows from Proposition 4.2.7. □

We are now in a position to complete the proof of Theorem 4.1.3.

Proof of Theorem 4.1.3. Fix the parameter s of the curve $c = 2s\sqrt{\omega}$ which will be determined later. Let $u_0 \in H^1(\mathbb{R})$, and let $u \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}))$ be the maximal solution of (DNLSb) with $u(0) = u_0$. Set $v_0 = \mathcal{G}_{1/4}(u_0)$ and $v = \mathcal{G}_{1/4}(u)$. By Proposition 4.4.2, $v \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}))$, and v satisfies (DNLSb′). We note that

$$M(u_0) = \mathcal{M}(v_0) \text{ and } P(u_0) = \mathcal{P}(v_0).$$

For any $v_0 \in H^1(\mathbb{R})$ we have

$$\begin{aligned}(4.6.4) \qquad \mathcal{K}_{\omega,2s\sqrt{\omega}}(v_0) &= \|\partial_x v_0\|_{L^2}^2 + \omega\|v_0\|_{L^2}^2 \\ &\quad + s\sqrt{\omega}\left(2\left(i\partial_x v_0, v_0\right) + \|v_0\|_{L^4}^4\right) - \frac{3}{16}\gamma\|v_0\|_{L^6}^6 \geq 0\end{aligned}$$

for large $\omega > 0$, where ω depends on s and v_0.

Case 1: $b > 0$. In this case we note that

$$\max_{s\in(-1,1]} d(1,2s) = d(1,2s^*)$$

from Lemma 4.6.3. Hence we set $s = s^*$. By $P(\phi_{1,2s^*}) = 0$ and Proposition 4.2.6, we have

$$\begin{aligned}&\mathcal{S}_{\omega,2s^*\sqrt{\omega}}(v_0) \leq d(\omega, 2s^*\sqrt{\omega}) \\ \Longleftrightarrow\ &\mathcal{E}(v_0) + \frac{\omega}{2}\mathcal{M}(v_0) + s^*\sqrt{\omega}\mathcal{P}(v_0) \leq \frac{\omega}{2}M(\phi_{1,2s^*}) \\ \Longleftrightarrow\ &\mathcal{E}(v_0) + s^*\sqrt{\omega}\mathcal{P}(v_0) \leq \frac{\omega}{2}\big(\mathcal{M}(\varphi_{1,2s^*}) - \mathcal{M}(v_0)\big).\end{aligned}$$

The last inequality holds for large $\omega > 0$ when

$$\mathcal{M}(v_0) < \mathcal{M}(\varphi_{1,2s^*}), \text{ or } \mathcal{M}(v_0) = \mathcal{M}(\varphi_{1,2s^*}) \text{ and } \mathcal{P}(v_0) < 0.$$

Combined with (4.6.4), we deduce that $v_0 \in \mathscr{W}_{\omega,2s^*\sqrt{\omega}}$ for large $\omega > 0$ under the assumption of Theorem 4.1.3. Therefore it follows from Lemma 4.6.1 that $T_{\min} = T_{\max} = \infty$.

Case 2: $-3/16 < b \le 0$. In this case we note that

$$\max_{s\in(-1,1]} d(1,2s) = d(1,2)$$

from Lemma 4.6.3. Hence we set $s=1$. By Proposition 4.2.6, Proposition 4.2.4 and $M(\phi_{1,2}) = \frac{4\pi}{\sqrt{\gamma}}$, we have

$$\begin{aligned}
&\mathcal{S}_{\omega,2\sqrt{\omega}}(v_0) \le d(\omega, 2\sqrt{\omega}) \\
\iff &\mathcal{E}(v_0) + \frac{\omega}{2}\mathcal{M}(v_0) + \sqrt{\omega}\mathcal{P}(v_0) \le \frac{\omega}{2}\left[M(\phi_{1,2}) + P(\phi_{1,2})\right] \\
\iff &\mathcal{E}(v_0) + \sqrt{\omega}\mathcal{P}(v_0) \le \frac{\omega}{2}\left(\frac{4\pi}{\gamma^{3/2}} - \mathcal{M}(v_0)\right).
\end{aligned}$$

The last inequality holds for large $\omega > 0$ when

$$\mathcal{M}(v_0) < \frac{4\pi}{\gamma^{3/2}}, \text{ or } \mathcal{M}(v_0) = \frac{4\pi}{\gamma^{3/2}} \text{ and } \mathcal{P}(v_0) < 0.$$

Combined with (4.6.4), we deduce that $v_0 \in \mathscr{W}_{\omega,2\sqrt{\omega}}$ for large $\omega > 0$ under the assumption of Theorem 4.1.3. Therefore it follows from Lemma 4.6.1 that $T_{\min} = T_{\max} = \infty$. This completes the proof of Theorem 4.1.3. □

We apply the similar strategy to the proof of Theorem 4.1.11.

Proof of Theorem 4.1.11. We consider the curve $c = 2\sqrt{\omega}$. Let u_c be the maximal $H^1(\mathbb{R})$-solution of (DNLSb) with $u_c(0) = u_{0,c}$. Set $v_{0,c} = \mathcal{G}_{1/4}(u_{0,c})$ and $v_c = \mathcal{G}_{1/4}(u_c)$. By Lemma 4.6.1 it is enough to prove that $v_{0,c} \in \mathscr{W}_{c^2/4,c}$ for large $c > 0$. First we note that

$$v_{0,c} = \mathcal{G}_{1/4}(u_{0,c}) = e^{i\frac{cx}{2}}\mathcal{G}_{1/4}(\psi) =: e^{i\frac{cx}{2}}\varphi.$$

From the definition of $\mathcal{S}_{\omega,c}$, we have

$$\begin{aligned}
\mathcal{S}_{c^2/4,c}(v_{0,c}) &= \frac{1}{2}\mathcal{L}_{c^2/4,c}(v_{0,c}) + \frac{c}{8}\|v_{0,c}\|_{L^4}^4 - \frac{\gamma}{32}\|v_{0,c}\|_{L^6}^6 \\
&= \frac{1}{2}\|\partial_x\varphi\|_{L^2}^2 + \frac{c}{8}\|\varphi\|_{L^4}^4 - \frac{\gamma}{32}\|\varphi\|_{L^6}^6.
\end{aligned}$$

Since $d(c^2/4, c) = (c^2/4)d(1,2)$ and $d(1,2) > 0$, we deduce that

$$\mathcal{S}_{c^2/4,c}(v_{0,c}) \le d(c^2/4, c)$$

for large $c > 0$. Similarly we have

$$\begin{aligned}
\mathcal{K}_{c^2/4,c}(v_{0,c}) &= \mathcal{L}_{c^2/4,c}(v_{0,c}) + \frac{c}{2}\|v_{0,c}\|_{L^4}^4 - \frac{3}{16}\gamma\|v_{0,c}\|_{L^6}^6 \\
&= \|\partial_x\varphi\|_{L^2}^2 + \frac{c}{2}\|\varphi\|_{L^4}^4 - \frac{3}{16}\gamma\|\varphi\|_{L^6}^6 \ge 0
\end{aligned}$$

for large $c > 0$. This completes the proof. □

In the case of $b = -3/16$ we have the following Proposition 4.6.4. This gives the new perspective to the global result of Proposition 4.4.3 from the viewpoint of potential theory.

Proposition 4.6.4. *Let $b = -3/16$. Then we have*

$$H^1(\mathbb{R}) = \bigcup_{\substack{-2\sqrt{\omega}<c<0 \\ \omega>0}} \mathscr{W}_{\omega,c}.$$

Proof. From Proposition 4.2.4 and Proposition 4.2.6 we have

$$d(1,2s) = \frac{1-s^2}{3} M(\phi_{1,2s})$$

for $s \in (-1,0)$. It follows from Proposition 4.2.2 that $d(1,2s) \to \infty$ as $s \to 0-$. For $v_0 \in H^1(\mathbb{R})$ we can take $s_0 \in (-1,0)$ such that

$$2\, d(1,2s_0) - \mathcal{M}(v_0) > 0. \tag{4.6.5}$$

We note that

$$\begin{aligned} &\mathcal{S}_{\omega,2s_0\sqrt{\omega}}(v_0) \le d(\omega, 2s_0\sqrt{\omega}) \\ \iff &\mathcal{E}(v_0) + s_0\sqrt{\omega}\mathcal{P}(v_0) \le \frac{\omega}{2}\left(2\, d(1,2s_0) - \mathcal{M}(v_0)\right). \end{aligned}$$

From (4.6.5) the last inequality holds for large $\omega > 0$. Combined with (4.6.4), we deduce that $v_0 \in \mathscr{W}_{\omega,2s_0\sqrt{\omega}}$ for large $\omega > 0$. Hence the result follows. □

4.7 Orbital stability

In this section we study the stability of the solitons. We apply take variational approach for the proof.

4.7.1 The case $b \ge 0$

In this subsection we revisit the stability theory of the solitons in the case $b \ge 0$. First we prepare some notations. We define the functional $K_{\omega,c}$ by

$$\begin{aligned} K_{\omega,c}(u) :=& \frac{d}{d\lambda} S_{\omega,c}(\lambda u)\Big|_{\lambda=1} \\ =& \|\partial_x u\|_{L^2}^2 + \omega\|u\|_{L^2}^2 + c\,(\partial_x u, u) - N(u) - b\|u\|_{L^6}^6, \end{aligned}$$

where $N(u)$ is defined by

$$N(u) := \left(i|u|^2\partial_x u, u\right).$$

We note that

$$S_{\omega,c}(u) = \frac{1}{2}K_{\omega,c}(u) + \frac{1}{4}N(u) + \frac{b}{3}\|u\|_{L^6}^6 =: \frac{1}{2}K_{\omega,c}(u) + J(u). \tag{4.7.1}$$

We define the subsets of the energy space by

$$\begin{aligned}
\mathscr{A}^+_{\omega,c} &:= \left\{u \in H^1(\mathbb{R}) \setminus \{0\} : S_{\omega,c}(u) < d(\omega,c), K_{\omega,c}(u) > 0\right\},\\
\mathscr{B}^+_{\omega,c} &:= \left\{u \in H^1(\mathbb{R}) \setminus \{0\} : S_{\omega,c}(u) < d(\omega,c), J(u) < d(\omega,c)\right\},\\
\mathscr{A}^-_{\omega,c} &:= \left\{u \in H^1(\mathbb{R}) \setminus \{0\} : S_{\omega,c}(u) < d(\omega,c), K_{\omega,c}(u) < 0\right\},\\
\mathscr{B}^-_{\omega,c} &:= \left\{u \in H^1(\mathbb{R}) \setminus \{0\} : S_{\omega,c}(u) < d(\omega,c), J(u) > d(\omega,c)\right\}.
\end{aligned}$$

In the same way as the proof of Proposition 4.5.6, we have the following.

Proposition 4.7.1. *Let $b \geq 0$ and (ω, c) satisfy $-2\sqrt{\omega} < c \leq 2\sqrt{\omega}$. Let $X_{\omega,c}$ be defined by* (4.5.2). *If a sequence $\{u_n\} \subset X_{\omega,c}$ satisfies*

$$S_{\omega,c}(u_n) \to d(\omega,c) \text{ and } K_{\omega,c}(u_n) \to 0 \text{ as } n \to \infty,$$

then there exists a sequence $\{y_n\} \subset \mathbb{R}$ and $\theta_0, y_0 \in \mathbb{R}$ such that $\{u_n(\cdot - y_n)\}$ has a subsequence that converges to $e^{i\theta_0}\phi_{\omega,c}(\cdot - y_0)$ strongly in $X_{\omega,c}$.

Applying the variational characterization of the soliton $\phi_{\omega,c}$, we have the following; see [17, Lemma 11] for details.

Proposition 4.7.2. *Let $b \geq 0$ and (ω, c) satisfy $-2\sqrt{\omega} < c \leq 2\sqrt{\omega}$. Then $\mathscr{A}^+_{\omega,c}$ and $\mathscr{A}^-_{\omega,c}$ are invariant under the flow of* (DNLSb)*, i.e., if u_0 belongs to $\mathscr{A}^+_{\omega,c}$ (resp. $\mathscr{A}^-_{\omega,c}$), then the maximal solution $u(t)$ of* (DNLSb) *with $u(0) = u_0$ belongs to $\mathscr{A}^+_{\omega,c}$ (resp. $\mathscr{A}^-_{\omega,c}$). Moreover, we have $\mathscr{A}^\pm_{\omega,c} = \mathscr{B}^\pm_{\omega,c}$.*

Here we review the stability theory in the papers [17] and [57]. Let (ω_0, c_0) satisfy $\omega_0 > c_0^2/4$. In [17] it was proved that if there exists $\xi \in \mathbb{R}^2$ such that

$$\langle d'(\omega_0, c_0), \xi\rangle \neq 0, \quad \langle d''(\omega_0, c_0)\xi, \xi\rangle > 0, \tag{4.7.2}$$

then the soliton u_{ω_0,c_0} of (DNLSb) is orbitally stable. When $c_0 < 0$, since we have

$$\partial_\omega^2 d(\omega,c)\big|_{(\omega_0,c_0)} = \frac{1}{2}\partial_\omega M(\phi_{\omega,c})\bigg|_{(\omega_0,c_0)} = \frac{-4c_0}{\sqrt{4\omega_0 - c_0^2}\left\{c_0^2 + \gamma(4\omega_0 - c_0^2)\right\}} > 0,$$

(4.7.2) is satisfied by taking $\xi = (1, 0)$. However, in the case $c_0 \geq 0$, we have

$$\begin{aligned}
\partial_\omega^2 d(\omega,c)\big|_{(\omega_0,c_0)} &= \frac{1}{2}\partial_\omega M(\phi_{\omega,c})\bigg|_{(\omega_0,c_0)} \leq 0,\\
\partial_c^2 d(\omega,c)\big|_{(\omega_0,c_0)} &= \frac{1}{2}\partial_c P(\phi_{\omega,c})\bigg|_{(\omega_0,c_0)} \leq 0.
\end{aligned}$$

This means that the calculation as a one-parameter $\omega \mapsto \phi_{\omega,c}$ or $c \mapsto \phi_{\omega,c}$ is not enough to prove the stability of the solitons in the case $c_0 \geq 0$. Instead of that, by computing $d''(\omega, c)$ we have the following (see Lemma 1 in [57]):

$$\det[d''(\omega_0, c_0)] = \frac{-2P(\phi_{\omega_0,c_0})}{\sqrt{4\omega_0 - c_0^2}\,\{c_0^2 + \gamma(4\omega_0 - c_0^2)\}}. \tag{4.7.3}$$

As we have seen in Proposition 4.2.7, $P(\phi_{\omega_0,c_0})$ is positive when (ω_0, c_0) satisfies that

$$\begin{aligned} &\text{if } b > 0, \quad -2\sqrt{\omega_0} < c_0 < 2s^*\sqrt{\omega_0}, \\ &\text{if } b = 0, \quad -2\sqrt{\omega_0} < c_0 < 2\sqrt{\omega_0}. \end{aligned} \tag{4.7.4}$$

Therefore we deduce that $d''(\omega_0, c_0) < 0$ under the condition (4.7.4). This yields the existence of $\xi \in \mathbb{R}^2$ satisfying (4.7.2) since $d''(\omega_0, c_0)$ has one positive eigenvalue.

Our first aim in this section is to provide a simpler approach in the case $c_0 > 0$. Let $c_0 = 2s_0\sqrt{\omega_0}$ where $-1 < s_0 \leq 1$. Set $\mu_0 = \sqrt{\omega_0}$. If we consider the stability problem along the scaling curve $\tau \mapsto \big((\mu_0 + \tau)^2, 2s_0(\mu_0 + \tau)\big)$, we have the following claim.

Proposition 4.7.3. *Let $b \geq 0$ and (ω_0, c_0) satisfy $-2\sqrt{\omega_0} < c_0 \leq 2\sqrt{\omega_0}$. Suppose that $c_0 P(\phi_{\omega_0,c_0}) > 0$. Then, there exists $\varepsilon_0 > 0$, for any $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R})$ satisfies $\|u_0 - \phi_{\omega_0,c_0}\|_{H^1(\mathbb{R})} < \delta$, then the maximal solution $u(t)$ of* (DNLSb) *with $u(0) = u_0$ satisfies*

$$d\big((\mu_0 - \varepsilon)^2, 2s_0(\mu_0 - \varepsilon)\big) < J(u(t)) < d\big((\mu_0 + \varepsilon)^2, 2s_0(\mu_0 + \varepsilon)\big) \tag{4.7.5}$$

for all $t \in (-T_{\min}, T_{\max})$.

Proof. We define the function $g : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ by

$$g(\tau) = d\big((\mu_0 + \tau)^2, 2s_0(\mu_0 + \tau)\big) \quad \text{for } \tau \in (-\varepsilon_0, \varepsilon_0), \tag{4.7.6}$$

where $\varepsilon_0 > 0$ is sufficiently small. Let the function g defined by (4.7.6). We note that

$$\begin{aligned} &g(\tau) = (\mu_0 + \tau)^2 d(1, 2s_0) \quad \text{for } \tau \in (-\varepsilon_0, \varepsilon_0), \\ &g'(0) = 2\mu_0 d(1, 2s_0), \;\; g''(0) = 2d(1, 2s_0). \end{aligned}$$

By Proposition 4.2.6 we have

$$2d(1, 2s_0) = M(\phi_{1,2s_0}) + s_0 P(\phi_{1,2s_0}).$$

Let $\varepsilon \in (0, \varepsilon_0)$. Assume that $u_0 \in H^1(\mathbb{R})$ satisfies $\|u_0 - \phi_{\omega_0,c_0}\|_{H^1} < \delta$, where $\delta > 0$ is determined later. By Proposition 4.7.2 it is enough to prove that

$$u_0 \in \mathscr{B}^+_{(\mu_0+\varepsilon)^2, 2s_0(\mu_0+\varepsilon)} \cap \mathscr{B}^-_{(\mu_0-\varepsilon)^2, 2s_0(\mu_0-\varepsilon)}. \tag{4.7.7}$$

By direct calculations we have

$$\begin{aligned} S_{(\mu_0\pm\varepsilon)^2,2s_0(\mu_0\pm\varepsilon)}(u_0) &= S_{(\mu_0\pm\varepsilon)^2,2s_0(\mu_0\pm\varepsilon)}(\phi_{\mu_0^2,2s_0\mu_0}) + O(\delta) \\ &= E(\phi_{\mu_0^2,2s_0\mu_0}) + \frac{(\mu_0\pm\varepsilon)^2}{2} M(\phi_{\mu_0^2,2s_0\mu_0}) \\ &\qquad + s_0(\mu_0\pm\varepsilon)P(\phi_{\mu_0^2,2s_0\mu_0}) + O(\delta) \\ &= \mu_0^2 d(1,2s_0) \pm \varepsilon\mu_0 \left(M(\phi_{1,2s_0}) + s_0 P(\phi_{1,2s_0})\right) \\ &\qquad + \frac{\varepsilon^2}{2} M(\phi_{1,2s_0}) + O(\delta) \\ &= g(0) \pm \varepsilon g'(0) + \frac{\varepsilon^2}{2} M(\phi_{1,2s_0}) + O(\delta). \end{aligned}$$

By using the Taylor expansion[5], there exists $\tau_1 = \tau_1(\varepsilon) \in (-\varepsilon_0, \varepsilon_0)$ such that

$$g(\pm\varepsilon) = g(0) \pm \varepsilon g'(0) + \frac{\varepsilon^2}{2} g''(\tau_1).$$

Since we have

$$g''(\tau_1) = 2d(1,2s_0) = M(\phi_{1,2s_0}) + s_0 P(\phi_{1,2s_0})$$

and $s_0 P(\phi_{1,2s_0}) > 0$ from the assumption, by taking small $\delta > 0$ we obtain that

$$S_{(\mu_0\pm\varepsilon)^2,2s_0(\mu_0\pm\varepsilon)}(u_0) < g(\pm\varepsilon). \tag{4.7.8}$$

On the other hand, by (4.7.1) and $K_{\omega_0,c_0}(\phi_{\omega_0,c_0}) = 0$, we have

$$g(0) = J(\phi_{\mu_0^2,2s_0\mu_0}) = J(u_0) + O(\delta).$$

Since g is strictly increasing, by taking smaller $\delta > 0$ if necessary, we obtain that

$$g(-\varepsilon) < J(u_0) < g(\varepsilon).$$

Combined with (4.7.8), we deduce that (4.7.7) holds. □

We note that the assumption of Theorem 4.7.3 is satisfied when (ω_0, c_0) satisfies (4.7.4) and $c_0 > 0$. Hence, as a consequence of Theorem 4.7.3, we have the following result.

Corollary 4.7.4. *Let $b \geq 0$. Suppose that (ω_0, c_0) satisfies (4.7.4) and $c_0 > 0$. Then the soliton u_{ω_0,c_0} of (DNLSb) is orbitally stable.*

[5] Actually we do not need to use the Taylor expansion since the function g is the quadratic function with respect to τ.

Proof. For completeness we give the proof. The result is proved by contradiction. Assume that there exist $\varepsilon_1 > 0$, a sequence of the maximal solutions $\{u_n\}$ to (DNLSb) and a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$\|u_n(0) - \phi_{\omega_0,c_0}\|_{H^1} \underset{n\to\infty}{\longrightarrow} 0, \tag{4.7.9}$$

$$\inf_{(\theta,y)\in\mathbb{R}^2} \|u_n(t_n) - e^{i\theta}\phi_{\omega_0,c_0}(\cdot - y)\|_{H^1} \geq \varepsilon_1. \tag{4.7.10}$$

Since $S_{\omega_0,c_0}(u_n(t_n))$ is a conserved quantity, by (4.7.9) we have

$$S_{\omega_0,c_0}(u_n(t_n)) = S_{\omega_0,c_0}(u_n(0)) \underset{n\to\infty}{\longrightarrow} S_{\omega_0,c_0}(\phi_{\omega_0,c_0}) = d(\omega_0, c_0). \tag{4.7.11}$$

By (4.7.9) and Theorem 4.7.3, we obtain that

$$J(u_n(t_n)) \underset{n\to\infty}{\longrightarrow} d(\omega_0, c_0).$$

Combined with (4.7.1), we have

$$K_{\omega_0,c_0}(u_n(t_n)) \underset{n\to\infty}{\longrightarrow} 0. \tag{4.7.12}$$

Hence, by (4.7.11), (4.7.12) and Proposition 4.7.1, there exist a sequence $\{y_n\}$ and $\theta_0, y_0 \in \mathbb{R}$ such that $\{u_n(t_n, \cdot - y_n)\}$ has a subsequence, which we still denote by the same letter, that converges to $e^{i\theta_0}\phi_{\omega_0,c_0}(\cdot - y_0)$ in $H^1(\mathbb{R})$. Therefore we deduce that

$$\inf_{(\theta,y)\in\mathbb{R}^2} \|u_n(t_n) - e^{i\theta}\phi_{\omega_0,c_0}(\cdot - y)\|_{H^1} \underset{n\to\infty}{\longrightarrow} 0,$$

which contradicts (4.7.10). This completes the proof. □

Our approach offers new perspectives to the stability theory of a two-parameter family of solitons. First we note that the estimate (4.7.5) is derived without any calculation of the Hessian matrix $d''(\omega, c)$. The calculation along the scaling curve $c = 2s\sqrt{\omega}$ is much simpler. This indicates that the curve gives not only the scaling but also "good" measure of stability of the solitons. It is also worthwhile to note that positivity of the momentum of the soliton is more directly used in our proof of the stability.

In the end of this subsection we discuss the case $c_0 = 0$. In this case we have

$$\begin{aligned}
\partial_\omega^2 d(\omega, c)\big|_{(\omega_0,0)} &= \frac{1}{2}\partial_\omega M(\phi_{\omega,c})\Big|_{(\omega_0,0)} = 0, \\
\partial_c^2 d(\omega, c)\big|_{(\omega_0,0)} &= \frac{1}{2}\partial_c P(\phi_{\omega,c})\Big|_{(\omega_0,0)} \leq 0, \\
\partial_\omega\partial_c d(\omega, c)\big|_{(\omega_0,0)} &= \partial_c\partial_\omega d(\omega, c)\big|_{(\omega_0,0)} = \frac{1}{2}\partial_c M(\phi_{\omega,c})\Big|_{(\omega_0,0)} = \frac{1}{\gamma\sqrt{\omega_0}} > 0.
\end{aligned}$$

If we set $\xi = (1, \varepsilon)$, we have

$$\langle d''(\omega_0, 0)\xi, \xi\rangle = \frac{2\varepsilon}{\gamma\sqrt{\omega_0}} + \frac{\varepsilon^2}{2}\, \partial_c P(\phi_{\omega,c})|_{(\omega_0,0)}\,,$$

which is positive for small $\varepsilon > 0$. Hence the condition (4.7.2) is satisfied for the vector $\xi = (1, \varepsilon)$ for small $\varepsilon > 0$, which yields that the soliton $u_{\omega_0,0}$ is orbitally stable. We note that $\xi = (1, \varepsilon)$ can be considered as a tangent vector of the curve $c = 2\varepsilon\sqrt{\omega}$ at the point $(1, 2\varepsilon)$.

4.7.2 The defocusing case

In this subsection we study orbital stability of the solitons in the case $b < 0$ by variational approach. To this end we study the stability of the solitons $v_{\omega,c}$ defined by (4.4.6) for (DNLSb′). We note that the functional $\mathcal{S}_{\omega,c}$ is rewritten as

$$\mathcal{S}_{\omega,c}(v) = \frac{1}{2}\mathcal{K}_{\omega,c}(v) - \frac{c}{8}\|v\|_{L^4}^4 + \frac{\gamma}{16}\|v\|_{L^6}^6 =: \frac{1}{2}\mathcal{K}_{\omega,c}(v) + \mathcal{J}_c(v). \tag{4.7.13}$$

In a similar way as before, we define the subsets of the energy space by

$$\begin{aligned}
\mathscr{C}_{\omega,c}^+ &:= \left\{v \in H^1(\mathbb{R}) \setminus \{0\} : \mathcal{S}_{\omega,c}(v) < d(\omega, c), \mathcal{K}_{\omega,c}(v) > 0\right\},\\
\mathscr{D}_{\omega,c}^+ &:= \left\{v \in H^1(\mathbb{R}) \setminus \{0\} : \mathcal{S}_{\omega,c}(v) < d(\omega, c), \mathcal{J}_c(v) < d(\omega, c)\right\},\\
\mathscr{C}_{\omega,c}^- &:= \left\{v \in H^1(\mathbb{R}) \setminus \{0\} : \mathcal{S}_{\omega,c}(v) < d(\omega, c), \mathcal{K}_{\omega,c}(v) < 0\right\},\\
\mathscr{D}_{\omega,c}^- &:= \left\{v \in H^1(\mathbb{R}) \setminus \{0\} : \mathcal{S}_{\omega,c}(v) < d(\omega, c), \mathcal{J}_c(v) > d(\omega, c)\right\}.
\end{aligned}$$

From Proposition 4.5.1 we obtain the following result. The proof is done in the similar way as the one of Proposition 4.7.2.

Proposition 4.7.5. *Let $-3/16 \le b < 0$ and let (ω, c) satisfy* (4.5.1)*. Then $\mathscr{C}_{\omega,c}^+$ and $\mathscr{C}_{\omega,c}^-$ are invariant under the flow of* (DNLSb′)*. Moreover, we have $\mathscr{C}_{\omega,c}^\pm = \mathscr{D}_{\omega,c}^\pm$.*

Proof. For convenience we give the proof. In the same way as the proof of Lemma 4.6.1 one can prove that $\mathscr{C}_{\omega,c}^\pm$ is an invariant set. We only prove that $\mathscr{C}_{\omega,c}^\pm = \mathscr{D}_{\omega,c}^\pm$.

If $v \in \mathscr{C}_{\omega,c}^+$, we have

$$d(\omega, c) > \mathcal{S}_{\omega,c}(v) = \frac{1}{2}\mathcal{K}_{\omega,c}(v) + \mathcal{J}_c(v) > \mathcal{J}_c(v),$$

which implies $v \in \mathscr{D}_{\omega,c}^+$. Conversely, let $v \in \mathscr{D}_{\omega,c}^+$. Assume that $\mathcal{K}_{\omega,c}(v) \le 0$. By Lemma 4.5.2 (iii), we have $d(\omega, c) = \mu(\omega, c) \le \mathcal{I}_{\omega,c}(v)$. From the definition, we have the following relation:

$$\mathcal{J}_c(v) = -\frac{1}{4}\mathcal{K}_{\omega,c}(v) + \mathcal{I}_{\omega,c}(v), \tag{4.7.14}$$

which implies that $\mathcal{J}_c(v) \geq d(\omega, c)$. But, this contradicts $\mathcal{J}_c(v) < d(\omega, c)$. Hence, $\mathcal{K}_{\omega,c}(v) > 0$, which shows $v \in \mathcal{C}^+_{\omega,c}$. This completes the proof of $\mathscr{C}^+_{\omega,c} = \mathscr{D}^+_{\omega,c}$.

Next, we prove that $\mathscr{C}^-_{\omega,c} = \mathscr{D}^-_{\omega,c}$. If $v \in \mathscr{C}^-_{\omega,c}$, by (4.7.14) and Lemma 4.5.2 (iii), we have

$$\mathcal{J}_c(v) = -\frac{1}{4}\mathcal{K}_{\omega,c}(v) + \mathcal{I}_{\omega,c}(v) > \mathcal{I}_{\omega,c}(v) > d(\omega, c),$$

which yields $v \in \mathscr{D}^-_{\omega,c}$. Conversely, if $v \in \mathscr{D}^-_{\omega,c}$, by (4.7.13) we have

$$\frac{1}{2}\mathcal{K}_{\omega,c}(v) = S_{\omega,c}(v) - \mathcal{J}_c(v) < d(\omega, c) - d(\omega, c) = 0$$

which yields $v \in \mathscr{C}^-_{\omega,c}$. This completes the proof of the claim. □

Let $b \geq -3/16$. First we consider the case $-2\sqrt{\omega_0} < c_0 < 0$. By following the approach in [17], we prove the following proposition.

Proposition 4.7.6. *Let $-3/16 \leq b < 0$ and let (ω_0, c_0) satisfy $-2\sqrt{\omega_0} < c_0 < 0$. Then, there exists $\varepsilon_0 > 0$, for any $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta > 0$ such that if $v_0 \in H^1(\mathbb{R})$ satisfies $\|u_0 - \varphi_{\omega_0,c_0}\|_{H^1(\mathbb{R})} < \delta$, then the maximal solution $v(t)$ of* (DNLSb′) *with $v(0) = v_0$ satisfies*

$$d(\omega_0 - \varepsilon, c_0) < \mathcal{J}_{c_0}(v(t)) < d(\omega_0 + \varepsilon, c_0) \tag{4.7.15}$$

for all $t \in (-T_{\min}, T_{\max})$.

Proof. We define the function $h : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ by

$$h(\tau) = d\big(\omega_0 + \tau, c_0\big) \quad \text{for } \tau \in (-\varepsilon_0, \varepsilon_0), \tag{4.7.16}$$

where $\varepsilon_0 > 0$ is sufficiently small. We note that

$$h'(0) = \frac{1}{2}\mathcal{M}(\varphi_{\omega_0,c_0}),\ h''(0) = \frac{1}{2}\partial_\omega \mathcal{M}(\varphi_{\omega,c})\Big|_{(\omega_0,c_0)}. \tag{4.7.17}$$

Assume that $v_0 \in H^1(\mathbb{R})$ satisfies $\|v_0 - \varphi_{\omega_0,c_0}\|_{H^1} < \delta$, where $\delta > 0$ is determined later. By Proposition 4.7.2 it is enough to prove that

$$v_0 \in \mathscr{D}^+_{\omega_0+\varepsilon,c_0} \cap \mathscr{D}^-_{\omega_0-\varepsilon,c_0}. \tag{4.7.18}$$

By direct calculations we have

$$\begin{aligned}
\mathcal{S}_{\omega_0\pm\varepsilon,c_0}(v_0) &= \mathcal{S}_{\omega_0\pm\varepsilon,c_0}(\varphi_{\omega_0,c_0}) + O(\delta)\\
&= \mathcal{E}(\varphi_{\omega_0,c_0}) + \frac{\omega_0 \pm \varepsilon}{2}\mathcal{M}(\varphi_{\omega_0,c_0}) + \frac{c_0}{2}\mathcal{P}(\varphi_{\omega_0,c_0}) + O(\delta)\\
&= d(\omega_0, c_0) \pm \frac{\varepsilon}{2}\mathcal{M}(\varphi_{\omega_0,c_0}) + O(\delta)\\
&= h(0) \pm \varepsilon h'(0) + O(\delta).
\end{aligned}$$

By using the Taylor expansion, there exists $\tau_1 = \tau_1(\varepsilon) \in (-\varepsilon_0, \varepsilon_0)$ such that

$$h(\pm\varepsilon) = h(0) \pm \varepsilon h'(0) + \frac{\varepsilon^2}{2} h''(\tau_1).$$

Since $c_0 < 0$ by the assumption, we have

$$\begin{aligned} h''(\tau_1) &= \frac{1}{2} \partial_\omega M(\phi_{\omega,c}) \bigg|_{(\omega_0+\tau_1, c_0)} \\ &= \frac{-4c_0}{\sqrt{4(\omega_0+\tau_1) - c_0^2}\,\{c_0^2 + \gamma(4(\omega_0+\tau_1) - c_0^2)\}} > 0. \end{aligned}$$

Hence, by taking small $\delta > 0$, we obtain that

$$\mathcal{S}_{\omega_0 \pm \varepsilon, c_0}(v_0) < h(\pm\varepsilon). \tag{4.7.19}$$

On the other hand, by (4.7.13) and $\mathcal{K}_{\omega_0,c_0}(\varphi_{\omega_0,c_0}) = 0$, we have

$$h(0) = \mathcal{J}_{c_0}(\varphi_{\omega_0,c_0}) = \mathcal{J}_{c_0}(v_0) + O(\delta).$$

Since h is strictly increasing, by taking smaller $\delta > 0$ if necessary, we obtain that

$$h(-\varepsilon) < \mathcal{J}_{c_0}(v_0) < h(\varepsilon).$$

Combined with (4.7.19), we deduce that (4.7.18) holds. □

In the similar way to the proof of Corollary 4.7.4, we obtain the following result from Proposition 4.5.6 and Proposition 4.7.6.

Corollary 4.7.7. *Let $-3/16 \le b < 0$ and let (ω_0, c_0) satisfy $-2\sqrt{\omega_0} < c_0 < 0$. Then the soliton v_{ω_0,c_0} of (DNLSb′) is orbitally stable.*

We recall that (DNLSb) and (DNLSb′) are equivalent under the gauge transformation $u \mapsto \mathcal{G}_{1/4}(u)$. Since $v_{\omega_0,c_0} = \mathcal{G}_{1/4}(u_{\omega_0,c_0})$ and the gauge transformation is locally Lipschitz continuous on $H^1(\mathbb{R})$, we have the following result.

Theorem 4.7.8. *Let $-3/16 \le b < 0$ and let (ω_0, c_0) satisfy $-2\sqrt{\omega_0} < c_0 < 0$. Then the soliton u_{ω_0,c_0} of (DNLSb) is orbitally stable.*

Next we study the remaining case $0 \le c_0 \le 2\sqrt{\omega_0}$ when $-3/16 < b < 0$. In this case we need to do calculations more carefully. The main difficulty comes from the lack of the "good" Hamiltonian structure of (DNLSb′). The analysis along the scaling curve provides the following claim.

Proposition 4.7.9. *Let $-3/16 \le b < 0$ and (ω_0, c_0) satisfy $0 \le c_0 \le 2\sqrt{\omega_0}$. Then, for any $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta > 0$ such that if $v_0 \in H^1(\mathbb{R})$ satisfies $\|v_0 - \varphi_{\omega_0,c_0}\|_{H^1} < \delta$, then the maximal solution $v(t)$ of (DNLSb′) with $v(0) = v_0$ satisfies that if $c_0 = 0$,*

$$d(\omega_0, -\varepsilon) - \frac{\varepsilon}{8}\|v(t)\|_{L^4}^4 < \mathcal{J}_0(v(t)) < d(\omega_0, \varepsilon) + \frac{\varepsilon}{8}\|v(t)\|_{L^4}^4 \tag{4.7.20}$$

for all $t \in (-T_{\min}, T_{\max})$, *and if* $0 < c_0 \le 2\sqrt{\omega_0}$,

$$d\big((\mu_0-\varepsilon)^2, 2s_0(\mu_0-\varepsilon)\big) - \frac{s_0\varepsilon}{4}\|v(t)\|_{L^4}^4 < \mathcal{J}_{c_0}(v(t)) < d\big((\mu_0+\varepsilon)^2, 2s_0(\mu_0+\varepsilon)\big) + \frac{s_0\varepsilon}{4}\|v(t)\|_{L^4}^4 \tag{4.7.21}$$

for all $t \in (-T_{\min}, T_{\max})$, *where* $c_0 = 2s_0\sqrt{\omega_0}$ *and* $\mu_0 = \sqrt{\omega_0}$.

Remark 4.7.10. Compared with Proposition 4.7.3 and Proposition 4.7.6, the L^4-norm appears in (4.7.20) or (4.7.21), which comes from the transformed momentum $\mathcal{P}$.

Proof. We only prove the case $0 < c_0 \le 2\sqrt{\omega_0}$. Let the function g defined by (4.7.6) as

$$g(\tau) = d\big((\mu_0+\tau)^2, 2s_0(\mu_0+\tau)\big) \quad \text{for } \tau \in (-\varepsilon_0, \varepsilon_0),$$

where $\varepsilon_0 > 0$ is sufficiently small. We note that

$$\begin{aligned} &g(\tau) = (\mu_0+\tau)^2 d(1, 2s_0) \quad \text{for } \tau \in (-\varepsilon_0, \varepsilon_0),\\ &g'(0) = 2\mu_0 d(1, 2s_0),\ g''(0) = 2d(1, 2s_0). \end{aligned}$$

By Proposition 4.2.6 and the gauge transformation (see Section 4.4), we have

$$2d(1, 2s_0) = \mathcal{M}(\varphi_{1,2s_0}) + s_0\mathcal{P}(\varphi_{1,2s_0}).$$

Let $\varepsilon \in (0, \varepsilon_0)$. Assume that $v_0 \in H^1(\mathbb{R})$ satisfies $\|v_0 - \varphi_{\omega_0,c_0}\|_{H^1} < \delta$, where $\delta > 0$ is determined later. First we prove that

$$v_0 \in \mathscr{D}^+_{(\mu_0+\varepsilon)^2, 2s_0(\mu_0+\varepsilon)} \cap \mathscr{D}^-_{(\mu_0-\varepsilon)^2, 2s_0(\mu_0-\varepsilon)}. \tag{4.7.22}$$

The calculation is done in the similar way as in the proof of Proposition 4.7.3. By direct calculations we have

$$\begin{aligned} \mathcal{S}_{(\mu_0\pm\varepsilon)^2, 2s_0(\mu_0\pm\varepsilon)}(v_0) &= \mathcal{S}_{(\mu_0\pm\varepsilon)^2, 2s_0(\mu_0\pm\varepsilon)}(\varphi_{\mu_0^2, 2s_0\mu_0}) + O(\delta)\\ &= \mathcal{E}(\varphi_{\mu_0^2, 2s_0\mu_0}) + \frac{(\mu_0\pm\varepsilon)^2}{2}\mathcal{M}(\varphi_{\mu_0^2, 2s_0\mu_0})\\ &\qquad + s_0(\mu_0\pm\varepsilon)\mathcal{P}(\varphi_{\mu_0^2, 2s_0\mu_0}) + O(\delta)\\ &= \mu_0^2 d(1, 2s_0) \pm \varepsilon\mu_0\left(\mathcal{M}(\varphi_{1,2s_0}) + s_0\mathcal{P}(\varphi_{1,2s_0})\right)\\ &\qquad + \frac{\varepsilon^2}{2}\mathcal{M}(\varphi_{1,2s_0}) + O(\delta)\\ &= g(0) \pm \varepsilon g'(0) + \frac{\varepsilon^2}{2}\mathcal{M}(\varphi_{1,2s_0}) + O(\delta). \end{aligned}$$

By using the Taylor expansion, we have

$$g(\pm\varepsilon) = g(0) \pm \varepsilon g'(0) + \frac{\varepsilon^2}{2} g''(0).$$

Since we have

$$g''(0) = 2d(1, 2s_0) = \mathcal{M}(\varphi_{1,2s_0}) + s_0\mathcal{P}(\varphi_{1,2s_0})$$

and $s_0\mathcal{P}(\varphi_{1,2s_0}) > 0$ from Proposition 4.2.7, by taking small $\delta > 0$ we obtain that

$$\mathcal{S}_{(\mu_0\pm\varepsilon)^2, 2s_0(\mu_0\pm\varepsilon)}(v_0) < g(\pm\varepsilon). \tag{4.7.23}$$

On the other hand, by (4.7.13) and $\mathcal{K}_{\omega_0,c_0}(\varphi_{\omega_0,c_0}) = 0$, we have

$$\begin{aligned}\mathcal{J}_{c_0+2s_0\varepsilon}(\varphi_{\omega_0,c_0}) &= -\frac{c_0 + 2s_0\varepsilon}{8}\|\varphi_{\omega_0,c_0}\|_{L^4}^4 + \frac{\gamma}{16}\|\varphi_{\omega_0,c_0}\|_{L^6}^6 \\ &< \mathcal{J}_{c_0}(\varphi_{\omega_0,c_0}) = g(0) < g(\varepsilon).\end{aligned}$$

By taking smaller $\delta > 0$ if necessary, we obtain that $\mathcal{J}_{c_0+2s_0\varepsilon}(v_0) < g(\varepsilon)$. Similarly, we have $g(-\varepsilon) < \mathcal{J}_{c_0-2s_0\varepsilon}(v_0)$. Combined with (4.7.23), we deduce that (4.7.22) holds. By Proposition 4.7.5 we have

$$v(t) \in \mathscr{D}^+_{(\mu_0+\varepsilon)^2, 2s_0(\mu_0+\varepsilon)} \cap \mathscr{D}^-_{(\mu_0-\varepsilon)^2, 2s_0(\mu_0-\varepsilon)} \tag{4.7.24}$$

for all $t \in (-T_{\min}, T_{\max})$. Hence we deduce that

$$\begin{aligned}g(\varepsilon) &> \mathcal{J}_{c_0+2s_0\varepsilon}(v(t)) \\ &= -\frac{c_0 + 2s_0\varepsilon}{8}\|v(t)\|_{L^4}^4 + \frac{\gamma}{16}\|v(t)\|_{L^6}^6 \\ &= \mathcal{J}_{c_0}(v(t)) - \frac{s_0\varepsilon}{4}\|v(t)\|_{L^4}^4.\end{aligned}$$

Similarly, we have

$$g(-\varepsilon) < \mathcal{J}_{c_0}(v(t)) + \frac{s_0\varepsilon}{4}\|v(t)\|_{L^4}^4.$$

This completes the proof. □

At last, combined with Proposition 4.5.6, one can prove the following theorem.

Theorem 4.7.11. *Let $-3/16 < b < 0$ and let (ω_0, c_0) satisfy $0 \le c_0 \le 2\sqrt{\omega_0}$. Then the soliton v_{ω_0,c_0} of* (DNLSb′) *is orbitally stable.*

Proof. The result is proved by contradiction. Assume that there exist $\varepsilon_1 > 0$, a sequence of the maximal solutions $\{v_n\}$ to (DNLSb′) and a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$\|v_n(0) - \varphi_{\omega_0,c_0}\|_{H^1} \underset{n\to\infty}{\longrightarrow} 0, \tag{4.7.25}$$

$$\inf_{(\theta,y)\in\mathbb{R}^2} \|v_n(t_n) - e^{i\theta}\varphi_{\omega_0,c_0}(\cdot - y)\|_{H^1} \ge \varepsilon_1. \tag{4.7.26}$$

Since $\mathcal{S}_{\omega_0,c_0}(v_n(t_n))$ is a conserved quantity, by (4.7.25) we have

$$\mathcal{S}_{\omega_0,c_0}(v_n(t_n)) = \mathcal{S}_{\omega_0,c_0}(v_n(0)) \underset{n\to\infty}{\longrightarrow} \mathcal{S}_{\omega_0,c_0}(\varphi_{\omega_0,c_0}) = d(\omega_0, c_0). \tag{4.7.27}$$

By the continuity $t \mapsto v(t) \in H^1(\mathbb{R})$, one can pick up $t_n \in (-T_{\min}, T_{\max})$ (still denoted by the same letter) such that

$$\inf_{(\theta,y)\in\mathbb{R}^2} \|v_n(t_n) - e^{i\theta}\varphi_{\omega_0,c_0}(\cdot - y)\|_{H^1} = \varepsilon_1. \tag{4.7.28}$$

This equality yields the boundedness[6] of $\{v_n(t_n)\}$ in $H^1(\mathbb{R})$, i.e.,

$$\sup_{n\in\mathbb{N}} \|v_n(t_n)\|_{H^1} \le C, \tag{4.7.29}$$

where C only depends on $\|\varphi_{\omega_0,c_0}\|_{H^1}$ and ε_1. From Proposition 4.7.9 and (4.7.29) we obtain that

$$\mathcal{J}_{c_0}(v_n(t_n)) \underset{n\to\infty}{\longrightarrow} d(\omega_0, c_0).$$

Combined with (4.7.13), we have

$$\mathcal{K}_{\omega_0,c_0}(v_n(t_n)) \underset{n\to\infty}{\longrightarrow} 0. \tag{4.7.30}$$

Hence, by (4.7.27), (4.7.30) and Proposition 4.5.6, there exist a sequence $\{y_n\}$ and $\theta_0, y_0 \in \mathbb{R}$ such that $\{v_n(t_n, \cdot - y_n)\}$ has a subsequence, which we still denote by the same letter, that converges to $e^{i\theta_0}\varphi_{\omega_0,c_0}(\cdot - y_0)$ in X_{ω_0,c_0}. When $\omega_0 > c_0^2/4$, this yields that

$$\inf_{(\theta,y)\in\mathbb{R}^2} \|v_n(t_n) - e^{i\theta}\varphi_{\omega_0,c_0}(\cdot - y)\|_{H^1} \underset{n\to\infty}{\longrightarrow} 0, \tag{4.7.31}$$

which contradicts (4.7.28). When $c_0 = 2\sqrt{\omega_0}$, we need to modify the argument slightly. From the definition of $X_{c_0^2/4,c_0}$, we have

$$e^{-i\frac{c_0 x}{2}} v_n(t_n, \cdot - y_n) \underset{n\to\infty}{\longrightarrow} e^{-i\frac{c_0 x}{2}} e^{i\theta_0}\varphi_{\omega_0,c_0}(\cdot - y_0) \text{ in } \dot{H}^1(\mathbb{R}). \tag{4.7.32}$$

By using this convergence one can easily prove that

$$e^{-i\frac{c_0 x}{2}} v_n(t_n, \cdot - y_n) \underset{n\to\infty}{\longrightarrow} e^{-i\frac{c_0 x}{2}} e^{i\theta_0}\varphi_{\omega_0,c_0}(\cdot - y_0) \text{ weakly in } L^2(\mathbb{R}). \tag{4.7.33}$$

From (4.7.25) and mass conservation we obtain that

$$\mathcal{M}(v_n(t_n)) = \mathcal{M}(v_n(0)) \underset{n\to\infty}{\longrightarrow} \mathcal{M}(\varphi_{\omega_0,c_0}). \tag{4.7.34}$$

Hence, it follows from (4.7.33) and (4.7.34) that

$$e^{-i\frac{c_0 x}{2}} v_n(t_n, \cdot - y_n) \underset{n\to\infty}{\longrightarrow} e^{-i\frac{c_0 x}{2}} e^{i\theta_0}\varphi_{\omega_0,c_0}(\cdot - y_0) \text{ in } L^2(\mathbb{R}). \tag{4.7.35}$$

From (4.7.32) and (4.7.35) we deduce that (4.7.31) holds, which contradicts (4.7.28). This completes the proof. □

[6] I did not realize this boundedness first. I thank Noriyoshi Fukaya for pointing out the fact and giving me helpful comments.

By Theorem 4.7.11, $v_{\omega_0,c_0} = \mathcal{G}_{1/4}(u_{\omega_0,c_0})$ and locally Lipschitz continuity of the gauge transformation, we have the following result.

Theorem 4.7.12. *Let* $-3/16 < b < 0$ *and let* (ω_0, c_0) *satisfy* $0 \le c_0 \le 2\sqrt{\omega_0}$. *Then the soliton* u_{ω_0,c_0} *of* (DNLSb) *is orbitally stable.*

The claims of Theorem 4.7.8 and Theorem 4.7.12 are nothing other than Theorem 4.1.13. This completes the proof. of Theorem 4.1.13.

Finally, we give a few remarks about the case $b < -3/16$. If (ω_0, c_0) satisfies (4.1.10) and $\omega_0^2 > c_0^2/4$, by Proposition 4.2.2 and direct computations, the formula (4.7.3) still holds including the case $b < -3/16$, i.e., we have

$$\det[d''(\omega_0, c_0)] = \frac{-2P(\phi_{\omega_0,c_0})}{\sqrt{4\omega_0 - c_0^2}\,\{c_0^2 + \gamma(4\omega_0 - c_0^2)\}}. \tag{4.7.36}$$

By Proposition 4.2.7, the momentum $P(\phi_{\omega_0,c_0})$ is always positive when $b < 0$, which yields that $d''(\omega_0, c_0)$ has one positive eigenvalue. Combined with the calculation of linearized operators[7] (see [45]), by applying the abstract theory of Grillakis, Shatah and Strauss [24, 25], one can prove that the soliton u_{ω_0,c_0} of (DNLSb) is orbitally stable. This argument gives the unified proof for the stability for the solitons in the defocusing case, but it works well in the only case $\omega_0^2 > c_0^2/4$. We note that the case $c_0 = 2\sqrt{\omega_0}$ brings essential difficulties to the proof of the stability by spectral approach due to lack of the coercivity. We will study these problems in more details in our forthcoming paper.

Spectral analysis is a powerful tool to tackle the stability problems and it also works in the case $b \ge 0$, however we need to examine the spectrum of linearized operators. Since the nonlinearity contains the derivative, the calculation of linearized operators for (DNLSb) is complex as can be seen in [45]. We note that our variational approach as in the proofs of Corollary 4.7.4, Theorem 4.7.8 and Theorem 4.7.12 does not need any calculation of linearized operators.

[7] To be specific, we need to examine the spectrum of the operator $S''_{\omega,c}(\phi_{\omega,c})$.

Chapter 5

Long-period limit of periodic traveling wave solutions

5.1 Introduction

5.1.1 Background

The equation (1.1.1) in the periodic setting is also an important problem:

$$i\partial_t \psi + \partial_x^2 \psi + i\partial_x(|\psi|^2\psi) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \tag{5.1.1}$$

where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. Tsutsumi and Fukuda [68] proved well-posedness in $H^s(\mathbb{T})$ for $s > 3/2$ in the same way as the whole line case. To prove well-posedness in $H^1(\mathbb{T})$ one can not directly apply the proof in [31] to the periodic setting since the L^4-Strichartz estimate on a torus holds with a loss of $\varepsilon > 0$ derivatives (see [10]). Herr [35] proved local well-posedness in $H^s(\mathbb{T})$ for $s \geq 1/2$ by using periodic gauge transformation and multilinear estimates in Fourier restriction norm spaces (see also [26]). In [52], by adapting Wu's proof to the periodic setting, it was proved that the $H^1(\mathbb{T})$-solution of (5.1.1) is global if the mass is less than 4π. For global results in $H^s(\mathbb{T})$ with $s < 1$, we refer to [51] and references therein.

In this chapter we study the periodic traveling waves. We consider the equation (DNLS) in the periodic setting:

$$i\partial_t u + \partial_x^2 u + i|u|^2\partial_x u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{T}. \tag{DNLS}$$

The periodic traveling waves of (DNLS) have only been partially studied. As a first mathematical work of this problem, Imamura [36] studied semi-trivial solutions:

$$\phi_\ell^c(x-ct) = \sqrt{c-\ell}e^{i\ell(x-ct)}, \quad \ell \in \mathbb{Z} \setminus \{0\},\ c > \ell,$$

which are 2π-periodic traveling wave solutions of (DNLS). In [36], it was proved that orbital stability of semi-trivial solutions by applying the abstract theory of Grillakis, Shatah, and Strauss [24, 25]. Murai, Sakamoto and Yotsutani [53] discussed the explicit formulae of periodic traveling waves of (DNLS) which are not semi-trivial. If we put the form

$$v(t,x) = e^{i\omega t}U(x-ct)$$

into (DNLS), then U satisfies the equation

$$-U'' + \omega U + icU' - i|U|^2U' = 0, \tag{5.1.2}$$

with periodic boundary conditions. By using polar coordinates $U(x) = r(x)e^{i\theta(x)}$, a direct calculation shows that the functions $r(x)$ and $\theta(x)$ satisfy

$$-r'' + \left(\omega - \frac{c^2}{4} + \frac{b}{2}\right) r + \frac{c}{2}r^3 - \frac{3}{16}r^5 + \frac{b^2}{r^3} = 0, \tag{5.1.3}$$

$$\theta(x) = \frac{c}{2}x - \frac{1}{4}\int_0^x r(y)^2 dy + b\int_0^x \frac{dy}{r(y)^2}, \tag{5.1.4}$$

where b is some constant which comes from integration. If we consider solutions vanishing at infinity, we can take $b=0$. In this case (5.1.3) corresponds to the equation (1.1.7), and (5.1.4) corresponds to the gauge transformation (1.1.5). However, in general b is a non-zero constant in the periodic setting. In [53], they first obtain explicit formulae of all the 2π-periodic solutions of (5.1.3), and then try to find the solutions from among them which satisfy periodic conditions of θ:

$$\theta(0) = 0,\ \theta(2\pi) = 2\pi\ell,$$

which is equivalent

$$2\pi\ell = c\pi - \frac{1}{4}\int_0^{2\pi} r(x)^2 dx + b\int_0^{2\pi} \frac{dx}{r(x)^2}, \tag{5.1.5}$$

where $\ell \in \mathbb{Z}\setminus\{0\}$ is a winding number. Since general solutions of (5.1.3) are complicate as can be seen in [53], it is a quite delicate problem to find the solutions which satisfy the condition (5.1.5). In [53], partial numerical computations are done to confirm the existence of solutions which satisfy special periodic boundary conditions above.

The main difficulty to obtain exact periodic traveling wave solutions of (DNLS) is that the nonlocal problem as (5.1.5) appears. Here, to avoid complex calculation in nonlocal issues, we consider the equation (1.1.14) in the periodic setting; i.e.,

$$i\partial_t u + \partial_x^2 u + \frac{i}{2}|u|^2\partial_x u - \frac{i}{2}u^2\partial_x\overline{u} + \frac{3}{16}|u|^4 u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{T}_{2L}, \tag{5.1.6}$$

where $\mathbb{T}_{2L} = \mathbb{R}/2L \simeq [-L, L]$ is the torus of size $2L$. The energy, mass and momentum of (5.1.6) are given by the followings respectively:

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{T}_{2L}} |\partial_x u|^2 dx - \frac{1}{32}\int_{\mathbb{T}_{2L}} |u|^6 dx,$$

$$\mathcal{M}(u) = \int_{\mathbb{T}_{2L}} |u|^2,$$

$$\mathcal{P}(u) = \mathrm{Re}\int_{\mathbb{T}_{2L}} i\partial_x u\overline{u}dx + \frac{1}{4}\int_{\mathbb{T}_{2L}} |u|^4 dx.$$

Our aim of this chapter is to find exact periodic traveling wave solutions which yield the solitons on the whole line including the massless case in the long-period limit, not to determine the complete structure of periodic traveling wave solutions. Moreover, we study the regularity of the convergence of exact periodic traveling wave solutions in the long-period limit.

In the end of this subsection, we discuss the relation between the equations (5.1.1), (DNLS) and (5.1.6) on $\mathbb{T}_{2L}$. Let us recall the periodic gauge transformation introduced by Herr [35]. For $a \in \mathbb{R}$, let $\mathscr{G}_a : L^2(\mathbb{T}_{2L}) \to L^2(\mathbb{T}_{2L})$ be defined by

$$\mathscr{G}_a(f)(x) = e^{ia\mathcal{J}(f)(x)} f(x), \tag{5.1.7}$$

where $\mathcal{J}(f)$ is defined by

$$\mathcal{J}(f)(x) := \frac{1}{2L} \int_0^{2L} \int_\theta^x \left(|f(y)|^2 - \mu[f] \right) dy d\theta$$

and

$$\mu = \mu[f] := \frac{1}{2L} \|f\|^2_{L^2(\mathbb{T}_{2L})}.$$

We note that $\mathcal{J}(f)$ is the $2L$-periodic primitive of $|f|^2 - \mu(f)$ with mean zero. For the solution v of (DNLS) on $\mathbb{T}_{2L}$, we define the gauge transformed solution by

$$u(t,x) := \mathscr{G}_a(v)(t, x + 2a\mu t).$$

A straightforward calculation shows that u satisfies

$$i\partial_t u + \partial_x^2 u + (1-2a)i|u|^2\partial_x u - 2iau^2\partial_x \overline{u} + a\left(a + \frac{1}{2}\right)|u|^4 u + e_L(u) = 0, \tag{5.1.8}$$

where

$$e_L(u) := \psi(u)u - a\mu|u|^2 u, \tag{5.1.9}$$

$$\psi(u) := \frac{a}{2L}\int_0^{2L} \left(2\mathrm{Im}(\overline{u}\partial_x u)(t,\theta) + \left(\frac{1}{2} - 2a\right)|u|^4(t,\theta)\right) d\theta + a^2\mu^2. \tag{5.1.10}$$

We note that when $a = -1/2$ [resp. $a = 1/4$] the equation (5.1.8) represents (5.1.1) [resp. (1.1.14)] on $\mathbb{T}_{2L}$ with some error term $e_L(u)$. Therefore, three equations (5.1.1), (DNLS) and (5.1.6) on $\mathbb{T}_{2L}$ can be considered to be *almost equivalent* under the suitable periodic gauge transformation. Since e_L formally goes to 0 as $L \to \infty$, it is reasonable to consider that these three equations on $\mathbb{T}_{2L}$ do not have essentially different structure at least when L is sufficiently large. This is compatible with that these three equations on the whole line are gauge equivalent. As can be seen in the proof in [35], the error term $e_L(u)$ does not give any difficulty to prove well-posedness. However, it gives a delicate problem when one tries to obtain exact periodic traveling wave solutions, since the error term $e_L(u)$ is nonlocal. Hence, we consider the equation (5.1.6) as a basic equation.

5.1.2 Main results

We assume that (ω, c) satisfies

$$-2\sqrt{\omega} < c \le 2\sqrt{\omega}. \tag{5.1.11}$$

First, we note that

$$u_{\omega,c}(t,x) = e^{i\omega t}\varphi_{\omega,c}(x-ct), \tag{5.1.12}$$

is a two-parameter family of solitons of the equation (1.1.14) on the whole line, where

$$\varphi_{\omega,c}(x) = e^{i\frac{c}{2}x}\Phi_{\omega,c}(x), \tag{5.1.13}$$

and $\Phi_{\omega,c}$ is defined by (1.1.6). For convenience we write the explicit formulae again;

$$\Phi_{\omega,c}^2(x) = \begin{cases} \dfrac{4\omega - c^2}{\sqrt{\omega}\left(\cosh(\sqrt{4\omega - c^2}x) - \frac{c}{2\sqrt{\omega}}\right)} & \text{if } \omega > c^2/4, \\[2ex] \dfrac{4c}{(cx)^2+1} & \text{if } c = 2\sqrt{\omega}. \end{cases} \tag{5.1.14}$$

Note that $\varphi_{\omega,c}$ satisfies the following equation

$$-\varphi'' + \omega\varphi + ic\varphi' + \frac{c}{2}|\varphi|^2\varphi - \frac{3}{16}|\varphi|^4\varphi = 0. \tag{5.1.15}$$

We consider the elliptic equation (1.1.7) on a torus:

$$-\Phi'' + \left(\omega - \frac{c^2}{4}\right)\Phi + \frac{c}{2}|\Phi|^2\Phi - \frac{3}{16}|\Phi|^4\Phi = 0, \quad x \in \mathbb{T}_{2L}. \tag{5.1.16}$$

To find exact solutions which yield the solitons in the long-period limit, we need to find positive single-bump solutions of (5.1.16). We have the following theorem.

Theorem 5.1.1. *Let $(\omega, c) \in \mathbb{R}^2$ satisfy (5.1.11). Assume that $L > 0$ satisfies*

$$L_0 = L_0(\omega, c) < L < \infty, \tag{5.1.17}$$

where $L_0(\omega, c)$ is a positive constant determined by (ω, c) (see Remark 5.1.3 below). Then, there exists the positive single-bump solution $\Phi^L_{\omega,c}$ of (5.1.16) on $\mathbb{T}_{2L}$ such that $\Phi^L_{\omega,c}(x) \to \Phi_{\omega,c}(x)$ for any $x \in \mathbb{R}$ as $L \to \infty$. Furthermore, $\Phi^L_{\omega,c}$ is explicitly represented as

$$\left(\Phi^L_{\omega,c}(x)\right)^2 = \eta_3 \frac{\mathrm{dn}^2\left(\frac{x}{2g}; k\right)}{1 + \beta^2 \mathrm{sn}^2\left(\frac{x}{2g}; k\right)}, \quad x \in [-L, L] \tag{5.1.18}$$

with parameters η_3, g, k, β depending on (L, ω, c).

Remark 5.1.2. The value η_3 corresponds to the maximal value of $\left(\Phi^L_{\omega,c}\right)^2$. We note that η_3 satisfies

$$\alpha_0 < \eta_3 < \Phi^2_{\omega,c}(0),$$

where α_0 is defined by

$$\alpha_0 := \frac{1}{3}\left(4c + \sqrt{48\omega + 4c^2}\right).$$

It is shown that α_0 is a positive constant when (ω, c) satisfies (5.1.11) (see Lemma 5.3.1).

Remark 5.1.3. L_0 is explicitly represented as

$$L_0 = L_0(\omega, c) := \frac{2\pi}{\sqrt{\alpha_0\sqrt{A(\alpha_0)}}},$$

where $A(x)$ is defined by

$$A(x) := -3x^2 + 8cx + 64\omega.$$

We note that $A(\alpha_0)$ is a positive constant when (ω, c) satisfies (5.1.11) (see Remark 5.3.3). The condition (5.1.17) is optimal in the sense that when $L = L_0$, the constant $\sqrt{\alpha_0}$ is a solution of (5.1.16) and $\Phi^L_{\omega,c}(x) \to \sqrt{\alpha_0}$ for any $x \in [-L_0, L_0]$ as $L \downarrow L_0$. In short, the condition (5.1.17) is optimal in order that $\Phi^L_{\omega,c}$ has a single bump.

The functions dn (dnoidal) and cn (cnoidal) in Theorem 5.1.1 are usual Jacobi's elliptic functions; see Section 5.2 for a precise definition. We note that if we take $c_L \in \frac{2\pi}{L}\mathbb{Z}$, exact periodic traveling waves defined by

$$u^L_{\omega,c_L}(t,x) = e^{i\omega t + i\frac{c_L}{2}(x - c_L t)}\Phi^L_{\omega,c_L}(x - c_L t) =: e^{i\omega t}\varphi^L_{\omega,c_L}(x - c_L t) \tag{5.1.19}$$

satisfy the equation (5.1.6) on $\mathbb{T}_{2L}$. If for each $L > L_0$ we take $c_L \in \frac{2\pi}{L}\mathbb{Z}$ such that $c_L \to c$ as $L \to \infty$, we have

$$\varphi^L_{\omega,c_L}(x) \to \varphi_{\omega,c}(x) \tag{5.1.20}$$

for any $x \in \mathbb{R}$ as $L \to \infty$. This gives the pointwise convergence of periodic traveling waves in the long-period limit.

In the one-parameter case ($\omega > 0$ and $c = 0$), exact solutions defined by (5.1.18) correspond to periodic traveling wave solutions to (1.1.20) and (1.1.22) which are studied in [6]. Construction of solutions in Theorem 5.1.1 is done by a simple quadrature method in the similar way as the one-parameter case. However, derivation of the detailed properties of exact solutions in the two-parameter case is far from being obvious from the result of one-parameter case. For instance, we can show that the modulus of elliptic functions in (5.1.18) has the following long-period limit:

$$k \to \begin{cases} 1 & \text{if } \omega > c^2/4, \\ \dfrac{1}{\sqrt{2}} & \text{if } c = 2\sqrt{\omega}, \end{cases} \tag{5.1.21}$$

as $L \to \infty$ (see Lemma 5.3.7). The difference of long-period limit of modulus is essential in order that exact periodic solutions yield two types[1] of the solitons on the whole line.

To compute the long-period limit, it is often useful to use the maximum value $\sqrt{\eta_3}$ of $\Phi^L_{\omega,c}$ as a parameter instead of the length of torus L. This idea can be seen in [4, 2, 3, 6]. To apply this idea to our setting, we need to prove

$$\sqrt{\eta_3} \to \Phi_{\omega,c}(0) \iff L \to \infty. \tag{5.1.22}$$

The relation (5.1.22) follows from the monotonicity of the functions $\eta_3 \mapsto k$ and $\eta_3 \mapsto T_{\Phi^L_{\omega,c}}$ (see Proposition 5.3.6 and Proposition 5.3.8), where $T_{\Phi^L_{\omega,c}}$ is the fundamental period of $\Phi^L_{\omega,c}$. We note that the proofs of these monotonicity are much more delicate compared with one-parameter case discussed in previous works. Interestingly, the scaling curve $c = 2s\sqrt{\omega}$ to the solitons is useful even in the periodic setting to derive the detailed properties including the monotonicity.

Next, we study the regularity of the convergence of exact periodic traveling wave solutions in the long-period limit. We can improve the pointwise convergence in Theorem 5.1.1 as follows.

Theorem 5.1.4. *Let* $(\omega, c) \in \mathbb{R}^2$ *satisfy* (5.1.11). *If for* $(\omega, c) \in \mathbb{R}^2$ *we take sufficiently large* L *such that* $L_0 < L$, *then* $\Phi^L_{\omega,c}$ *is well-defined by* (5.1.18). *Then, we have*

$$\lim_{L\to\infty} \|\Phi^L_{\omega,c} - \Phi_{\omega,c}\|_{H^m([-L,L])} = 0 \tag{5.1.23}$$

for any $m \in \mathbb{Z}_{\geq 0}$.

Theorem 5.1.5. *Let* $(\omega, c) \in \mathbb{R}^2$ *and* $\Phi^L_{\omega,c}$ *in the same assumption as Theorem* 5.1.4. *Then, we have*

$$\lim_{L\to\infty} \|\Phi^L_{\omega,c} - \Phi_{\omega,c}\|_{C^m([-L,L])} = 0 \tag{5.1.24}$$

for any $m \in \mathbb{Z}_{\geq 0}$.

Remark 5.1.6. We note that Theorem 5.1.5 is not proved directly from Theorem 5.1.4 by using the Sobolev embedding $H^m([-L, L]) \subset C^k([-L, L])$ $(k < m)$, because constants in the Sobolev inequality depend on size of the interval $2L$.

[1] From the explicit formulae (5.1.14), the soliton in the case $\omega > c^2/4$ has exponential decay and the soliton in the massless case has algebraic decay. However, exact periodic solutions are represented by the same formula as (5.1.18) in both two cases.

Remark 5.1.7. We can replace $\Phi^L_{\omega,c}$ [resp. $\Phi_{\omega,c}$] by φ^L_{ω,c_L} [resp. $\varphi_{\omega,c}$] in both (5.1.23) and (5.1.24) if we take $c_L \in \frac{2\pi}{L}\mathbb{Z}$ such that $c_L \to c$ as $L \to \infty$. Especially, we obtain the uniform bound of periodic traveling wave solutions as

$$\sup_{L_0<L<\infty} \|u^L_{\omega,c_L}\|_{L^\infty(\mathbb{R},H^m(\mathbb{T}_{2L}))} < \infty \tag{5.1.25}$$

for any $m \in \mathbb{Z}_{\geq 0}$, where u^L_{ω,c_L} is defined by (5.1.19).

To the best of our knowledge, the regularity results in Theorem 5.1.4 and Theorem 5.1.5 are new even if we restrict the one-parameter case ($\omega > 0$ and $c = 0$). For the proof of Theorem 5.1.4 and Theorem 5.1.5, the L^2-convergence in the long-period limit is the key step. First, we show that the mass of exact periodic solutions is exactly same as the mass of the solitons in the long-period limit (see Theorem 5.4.1). Here again, the difference between the case $\omega > c^2/4$ and the massless case appears. We need to do a delicate calculation of elliptic integrals in this step. Next, by combining with pointwise convergence in Theorem 5.1.1 and the Brézis-Lieb lemma, we obtain L^2-convergence. Since $\Phi^L_{\omega,c}$ and $\Phi_{\omega,c}$ satisfy the same elliptic equation, we can obtain H^2-convergence from L^2-convergence and the equation. Especially, we obtain L^∞-convergence from H^1-convergence. The proof of L^∞-convergence here is related to the proof of the Sobolev inequality, but we need to calculate the dependence of the size L more carefully. The rest of the proof is done by a standard bootstrap argument. We note that the detailed properties of exact periodic solutions are used throughout the proof.

We remark that one can our approach to periodic traveling wave solutions of other type of dispersive equations such that KdV, mKdV and cubic NLS discussed in previous works (see [3] and references therein).

Remark 5.1.8. If we consider the periodic gauge transformed solution

$$v^L_{\omega,c_L} := \mathscr{G}_{-\frac{1}{4}}(u^L_{\omega,c_L})(t, x - \frac{1}{2}\mu t), \tag{5.1.26}$$

then v^L_{ω,c_L} satisfies the following equation:

$$i\partial_t v + \partial_x^2 v + i|v|^2\partial_x v + e_L(v) = 0,$$

where

$$\begin{aligned} e_L(v) &:= \psi(v)v + \frac{1}{4}\mu|v|^2 v, \\ \psi(v) &:= -\frac{1}{8L}\int_0^{2L}\left(2\mathrm{Im}(\overline{v}\partial_x v)(t,\theta) + \frac{1}{8}|v|^4(t,\theta)\right)d\theta + \frac{1}{16}\mu^2, \end{aligned}$$

and $\mu = \frac{1}{2L}\|v\|^2_{L^2(\mathbb{T}_{2L})}$. From the uniform bound (5.1.25) and formula of the error term, we deduce that

$$\|e_L(v^L_{\omega,c})\|_{L^\infty(\mathbb{R},H^m(\mathbb{T}_{2L}))} \to 0 \tag{5.1.27}$$

as $L \to \infty$ for any $m \in \mathbb{Z}_{\geq 0}$. This means that the solution v^L_{ω,c_L} gives the main part of $2L$-periodic traveling wave solutions of (DNLS) which yield the solitons in the long-period limit, at least when L is sufficiently large. One can apply a similar discussion to the equation (5.1.1) on $\mathbb{T}_{2L}$.

5.1.3 Related problems and remarks

Compared with the solitons on the whole line, it is natural to consider that the periodic traveling waves defined by (5.1.19) belong to the ground states, but the rigorous proof has not been obtained yet. Variational characterizations on a torus have different difficulties from the whole line case. Since a torus is compact, the existence of a minimizer for the problems is easily obtained. However, the identification of this minimizer is a delicate problem since the elliptic equation (5.1.16) has rich structure of solutions compared with the one on the whole line. This problem is also related to uniqueness of ground states. Recently, variational characterizations of periodic waves for cubic NLS were obtained in [29], but the problems in our setting are more delicate.

The stability/instability of the periodic traveling waves is a natural problem as a next step. First, we note that φ^L_{ω,c_L} satisfies the equation (5.1.15) on $\mathbb{T}_{2L}$, which is equivalent that

$$\mathcal{S}'_{\omega,c_L}(\varphi^L_{\omega,c_L}) = 0, \tag{5.1.28}$$

where

$$\mathcal{S}_{\omega,c}(\varphi) = \mathcal{E}(\varphi) + \frac{\omega}{2}\mathcal{M}(\varphi) + \frac{c}{2}\mathcal{P}(\varphi).$$

The relation (5.1.28) is important when one considers the problems of both variational characterization and stability. There are several difficulties when one considers the stability/instability problem in our setting. We note that the equation (5.1.6) can not be rewritten as the Hamiltonian form by using the energy functional as (1.1.3). The lack of Hamiltonian structure causes the delicate problems when one considers the stability/instability problem; see [27] for partial results on the stability.[2] To prove stability or instability of solitons, it is important to calculate second derivatives. However, since we only take $c_L \in \frac{2\pi}{L}\mathbb{Z}$ as a discrete value, this gives the difficulty of differential calculation of $\mathcal{S}_{\omega,c_L}(\varphi^L_{\omega,c_L})$. We recall that Colin and Ohta [17] proved orbital stability of solitons (1.1.4) by showing that the matrix $d''(\omega,c)$ has one positive eigenvalue, where $d(\omega,c)$ is defined by

$$d(\omega,c) := S_{\omega,c}(\phi_{\omega,c}). \tag{5.1.29}$$

We note that when $\omega > c^2/4$ and $c > 0$ we have

$$\partial^2_\omega d(\omega,c) = \frac{1}{2}\partial_\omega M(\phi_{\omega,c}) = -\frac{c}{\omega\sqrt{4\omega - c^2}} < 0, \tag{5.1.30}$$

$$\partial^2_c d(\omega,c) = \frac{1}{2}\partial_c P(\phi_{\omega,c}) = -\frac{c}{\sqrt{4\omega - c^2}} < 0. \tag{5.1.31}$$

If one considers the solitons as a one-parameter $\omega \mapsto \phi_{\omega,c}$ or $c \mapsto \phi_{\omega,c}$, (5.1.30) and (5.1.31) seem to indicate that the solitons are unstable, but actually they are stable.

[2] In [27] they consider the stability problem on the whole line in the setting which can not be rewritten as the Hamiltonian form as (1.1.3).

This means that the calculation as a one-parameter is not enough to fix the stability problems, and shows one of the deep structure of a two-parameter family of solitons.

Although there are several difficulties on the stability/instability problems as above, it is important to study these problems in understanding further properties of exact periodic traveling wave solutions and related dynamics. We refer to [2, 3, 4, 5, 6, 21, 22, 29] for the studies on the stability/instability of the periodic profiles. The author hopes that our results in this chapter would provide further insight on the dynamics for the derivative nonlinear Schrödinger equation.

5.1.4 Organization of the chapter

The rest of this chapter is organized as follows. In Section 5.2 we recall the definition and basic properties of elliptic functions and elliptic integrals. In Section 5.3 we discuss construction and fundamental properties of exact periodic traveling wave solutions, and give a proof of Theorem 5.1.1. In Section 5.4, we discuss the regularity of the convergence in the long-period limit and prove Theorem 5.1.4 and Theorem 5.1.5.

5.2 Preliminaries

Here, we recall the definitions and some basic properties of elliptic functions and elliptic integrals. We refer the reader to [12, 42] for more details. Given $k \in (0,1)$, the incomplete elliptic integral of the first kind is defined by

$$u = F(\varphi, k) := \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

The Jacobi elliptic functions are defined through the inverse function of $F(\cdot, k)$ by

$$\mathrm{sn}(u; k) := \sin\varphi, \;\; \mathrm{cn}(u; k) := \cos\varphi, \;\; \mathrm{dn}(u; k) := \sqrt{1 - k^2 \mathrm{sn}^2(u; k)}.$$

The complete elliptic integral of the first kind is defined by

$$K = K(k) := F\left(\frac{\pi}{2}, k\right).$$

The functions sn, cn and dn have a real fundamental period, namely, $4K$, $4K$, and $2K$, respectively. We note that

$$K(k) \to \begin{cases} \frac{\pi}{2} & \text{as } k \to 0, \\ \infty & \text{as } k \to 1. \end{cases} \tag{5.2.1}$$

More specifically, when $k \to 1$, the function $K(k)$ has the following asymptotic behavior:

$$\lim_{k \to 1} \left(K(k) - \log \frac{4}{k'} \right) = 0, \tag{5.2.2}$$

where the complementary modulus k' is defined by

$$k' := \sqrt{1-k^2}.$$

Elliptic functions have the following extremal formulae:

$$\begin{cases} \mathrm{sn}(u;0) = \sin u, \ \mathrm{cn}(u;0) = \cos u, \ \mathrm{dn}(u;0) \equiv 1, \\ \mathrm{sn}(u;1) = \tanh u, \ \mathrm{cn}(u;1) = \mathrm{dn}(u;1) = \mathrm{sech}\, u. \end{cases} \tag{5.2.3}$$

This shows that elliptic functions bridge the gap between trigonometric and hyperbolic functions.

The incomplete elliptic integral of the second kind is defined by

$$E(\varphi, k) := \int_0^{\varphi} \sqrt{1-k^2\sin^2\theta}\, d\theta.$$

The complete elliptic integral of the second kind is defined by

$$E = E(k) := E\left(\frac{\pi}{2}, k\right).$$

We define by

$$\begin{aligned} K' = K'(k) &:= K(k'), \\ E' = E'(k) &:= E(k'). \end{aligned}$$

Then, we have the following Legendre relation

$$EK' + E'K - KK' = \frac{\pi}{2} \quad \text{for all } k \in (0,1). \tag{5.2.4}$$

5.3 Existence of exact periodic traveling waves

5.3.1 Construction of exact solutions

We consider the elliptic equation (5.1.16) on $\mathbb{T}_{2L}$. Set $\psi = \Phi^2$. By multiplying the equation (5.1.16) by Φ' and integrating, ψ satisfies the following equation

$$[\psi']^2 = -\frac{1}{4}\psi^4 + c\psi^3 + 4\left(\omega - \frac{c^2}{4}\right)\psi^2 + 8C_\psi \psi, \tag{5.3.1}$$

where C_ψ is a constant of integration. The formula (5.3.1) can be rewritten as

$$[\psi']^2 = \frac{1}{4}P_\psi(\psi), \tag{5.3.2}$$

where the polynomial P_ψ is defined by

$$\begin{aligned} P_\psi(t) &= -t^4 + 4ct^3 + 16\left(\omega - \frac{c^2}{4}\right)t^2 + 32C_\psi t \\ &= t(t-\eta_1)(t-\eta_2)(\eta_3 - t). \end{aligned}$$

Here, η_1, η_2 and η_3 are roots of the polynomial P_ψ satisfying

$$\begin{cases} \eta_1 + \eta_2 + \eta_3 = 4c, \\ \eta_2\eta_3 + \eta_1\eta_3 + \eta_1\eta_2 = -16\left(\omega - \frac{c^2}{4}\right), \\ \eta_1\eta_2\eta_3 = 32C_\psi. \end{cases} \tag{5.3.3}$$

Since we are interested in the positive solution, we may set $0 < \eta_2 < \eta_3$. We note that η_3 [resp. η_2] is the maximum [resp. minimum] value of ψ by (5.3.2). By (5.3.3) and (5.1.11), η_1 must be negative. By invariance of translations, we may assume that $\psi(0) = \eta_3$ and $\psi'(0) = 0$. From uniqueness of the ordinary differential equation and the equation (5.1.16), ψ is even. Since we want to construct single-bump solutions, we may assume that $\psi(L) = \eta_2$. Therefore, it is enough to consider the equation (5.3.2) on $[0, L]$. Since $\psi'(x) < 0$ when $0 < x < L$, integrating both sides of (5.3.2) over $[0, x]$ yields that

$$-\int_0^x \frac{\psi'(y)}{\sqrt{P_\psi(\psi(y))}}dy = \frac{1}{2}x.$$

Changing variables $t = \psi(x)$ in the integral implies that

$$\int_{\psi(x)}^{\eta_3} \frac{dt}{\sqrt{t(\eta_3 - t)(t-\eta_2)(t-\eta_1)}} = \frac{1}{2}x. \tag{5.3.4}$$

Applying the formula 257.00 in [12], we conclude that

$$\psi(x) = \frac{\eta_3(\eta_2 - \eta_1) + (\eta_3 - \eta_2)\eta_1 \mathrm{sn}^2\left(\frac{x}{2g}; k\right)}{(\eta_2 - \eta_1) + (\eta_3 - \eta_2)\mathrm{sn}^2\left(\frac{x}{2g}; k\right)}, \tag{5.3.5}$$

where

$$k^2 = \frac{-\eta_1(\eta_3 - \eta_2)}{\eta_3(\eta_2 - \eta_1)}, \tag{5.3.6}$$

$$g = \frac{2}{\sqrt{\eta_3(\eta_2 - \eta_1)}}. \tag{5.3.7}$$

We note that $0 < k^2 < 1$ from the inequality $\eta_1 < 0 < \eta_2 < \eta_3$. Using the expression of k, the formula (5.3.5) can be rewritten as

$$\psi(x) = \eta_3 \left[\frac{\mathrm{dn}^2\left(\frac{x}{2g}; k\right)}{1 + \beta^2 \mathrm{sn}^2\left(\frac{x}{2g}; k\right)}\right], \tag{5.3.8}$$

with $\beta^2 = -\eta_3 k^2/\eta_1 > 0$. From the fundamental periods of sn and dn, the fundamental period T_ψ of ψ is given by

$$T_\psi = 4gK(k) = \frac{8}{\sqrt{\eta_3(\eta_2 - \eta_1)}} K(k). \tag{5.3.9}$$

Since we assume that ψ is the single-bump solution, we obtain

$$2L = T_\psi = \frac{8}{\sqrt{\eta_3(\eta_2 - \eta_1)}} K(k). \tag{5.3.10}$$

Substituting the first equation in (5.3.3)

$$\eta_1 = 4c - \eta_2 - \eta_3 \tag{5.3.11}$$

into the second equation in (5.3.3), we obtain

$$\eta_2^2 + \eta_3^2 + \eta_2\eta_3 - 4c(\eta_2 + \eta_3) - 16\left(\omega - \frac{c^2}{4}\right) = 0. \tag{5.3.12}$$

From (5.3.11) and (5.3.12), η_1 and η_2 have expressions as functions of η_3, ω and c as

$$\eta_1 = \frac{-\eta_3 + 4c - \sqrt{A}}{2}, \tag{5.3.13}$$

$$\eta_2 = \frac{-\eta_3 + 4c + \sqrt{A}}{2}, \tag{5.3.14}$$

where A is defined by

$$A = A(\eta_3) := 64\omega - 3\eta_3^2 + 8c\eta_3. \tag{5.3.15}$$

The following two extreme cases can be considered;

(i) $\eta_2 = \eta_3 =: \alpha_0$.

(ii) $\eta_2 = 0$, $\eta_3 =: \alpha_1$.

The case (i) corresponds to the constant solution of (5.1.16). The case (ii) corresponds to the long-period limit as discussed in detail later. From the equation (5.3.12), we obtain that

$$\alpha_0 = \frac{1}{3}\left(4c + \sqrt{48\omega + 4c^2}\right),$$
$$\alpha_1 = 4\sqrt{\omega} + 2c.$$

It is worthwhile to note that $\alpha_1 = \Phi_{\omega,c}^2(0)$, where $\Phi_{\omega,c}$ is defined by (5.1.14).

5.3.2 Fundamental properties of exact solutions

In this subsection, we investigate detailed relation between parameters defined in Section 5.3.1. For the convenience of calculation, we introduce the following notations. When (ω, c) satisfies (5.1.11), we can write

$$c = 2s\sqrt{\omega}$$

for $\omega > 0$ and some $s \in (-1, 1]$. The case $s = 1$ corresponds to massless case. By using this notation, α_0 and α_1 are rewritten as

$$\alpha_0 = \frac{4}{3}\left(2s + \sqrt{3 + s^2}\right)\sqrt{\omega},$$
$$\alpha_1 = 4(1 + s)\sqrt{\omega}.$$

Set $\beta_0 = \frac{\alpha_0}{4\sqrt{\omega}}$ and $\beta_1 = \frac{\alpha_1}{4\sqrt{\omega}}$. We have

$$\beta_0 = \beta_0(s) = \frac{1}{3}\left(2s + \sqrt{3 + s^2}\right), \tag{5.3.16}$$
$$\beta_1 = \beta_1(s) = 1 + s \tag{5.3.17}$$

for $-1 < s \leq 1$. We begin with the following lemma.

Lemma 5.3.1. *Let* (ω, c) *satisfy* (5.1.11). *Then, we have* $0 < \alpha_0 < \alpha_1$.

Proof. From the definition, we note that

$$0 < \alpha_0 < \alpha_1 \iff 0 < \beta_0 < \beta_1.$$

First, we prove $\beta_0 > 0$. This is trivial from the definition (5.3.16) when $0 \leq s \leq 1$. When $s < 0$, we have

$$\begin{aligned} \beta_0 > 0 &\iff -2s < \sqrt{3 + s^2} \\ &\iff 4s^2 < 3 + s^2 \\ &\iff 0 < 3(1 - s^2). \end{aligned}$$

The last inequality holds when $-1 < s < 0$.

Next, we prove $\beta_0 < \beta_1$. When $-1 < s \leq 1$, we have

$$\begin{aligned} \beta_0 < \beta_1 &\iff \frac{1}{3}\left(2s + \sqrt{3 + s^2}\right) < 1 + s \\ &\iff \sqrt{3 + s^2} < 3 + s \\ &\iff 3 + s^2 < (3 + s)^2 \\ &\iff 0 < 6(s + 1). \end{aligned}$$

The last inequality holds when $-1 < s \leq 1$. This completes the proof. □

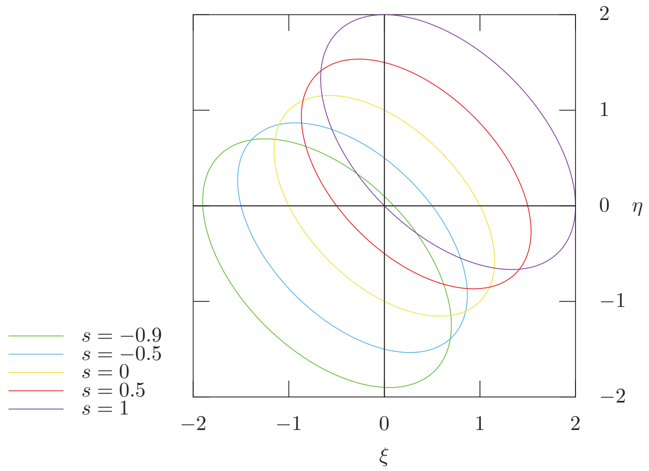

Figure 5.1: The ellipse (5.3.19) for several values of s. Note that the ellipse moves toward upper right when one changes the parameter s from -1 to 1.

We recall that (η_2, η_3) satisfies the constraint condition (5.3.12). Set

$$\xi = \frac{\eta_2}{4\sqrt{\omega}}, \quad \eta = \frac{\eta_3}{4\sqrt{\omega}}. \tag{5.3.18}$$

Substituting (5.3.18) and $c = 2s\sqrt{\omega}$ into (5.3.12), the equation (5.3.12) is equivalent that

$$(\xi - s)^2 + (\eta - s)^2 + \xi\eta = 1 + s^2, \tag{5.3.19}$$

where $-1 < s \leq 1$. The equation (5.3.19) represents the ellipse as in Figure 5.1. Note that (β_0, β_0) corresponds to a intersection point between line $\eta = \xi$ and ellipse (5.3.19), and that $(0, \beta_1)$ corresponds to a intersection point between line $\xi = 0$ and ellipse (5.3.19). Since we assumed that $0 < \eta_2 < \eta_3$ in Section 5.3.1, it follows that

$$\alpha_0 < \eta_3 < \alpha_1, \tag{5.3.20}$$

or equivalently

$$\beta_0 < \eta < \beta_1. \tag{5.3.21}$$

We can prove positivity of A defined by (5.3.15) under the condition (5.3.20).

Lemma 5.3.2. *Let* (ω, c) *satisfy* (5.1.11) *and let* η_3 *satisfy* (5.3.20). *Then, we have* $A = A(\eta_3) > 0$.

Proof. By using $c = 2s\sqrt{\omega}$ and $\eta_3 = 4\sqrt{\omega}\eta$, we can rewrite A as

$$\begin{aligned} A &= 64\omega - 3\eta_3^2 + 8c\eta_3 \\ &= 16\omega(-3\eta^2 + 4s\eta + 4). \end{aligned}$$

We set

$$f_s(\eta) := -3\eta^2 + 4s\eta + 4 \tag{5.3.22}$$

for $-1 < s \le 1$. A positive zero of $f_s(\eta)$ is given by

$$\beta = \frac{2}{3}\left(s + \sqrt{s^2+3}\right). \tag{5.3.23}$$

We obtain $\beta_1 \le \beta$ for $-1 < s \le 1$. Indeed, we have

$$\begin{aligned} \beta_1 \le \beta &\iff 1 + s \le \frac{2}{3}\left(s + \sqrt{s^2+3}\right) \\ &\iff s + 3 \le 2\sqrt{s^2+3} \\ &\iff (s+3)^2 \le 4(s^2+3) \\ &\iff 0 \le 3(s-1)^2. \end{aligned}$$

The last inequality means that $\beta_1 = \beta$ when $s = 1$ and $\beta_1 < \beta$ otherwise. Since $f_s(0) = 4 > 0$ and $0 < \beta_0 < \beta_1$, this implies that $f_s(\eta) > 0$ for $-1 < s \le 1$ and $\beta_0 < \eta < \beta_1$. This completes the proof. □

Remark 5.3.3. From the proof of Lemma 5.3.2 above, we also deduce that $A(\alpha_0)$ is a positive constant depending on (ω, c).

From Figure 5.1, one can observe that η_2 decreases from α_0 to 0 when one changes η_3 from α_0 to α_1. We can prove this claim rigorously.

Lemma 5.3.4. *Let* (ω, c) *satisfy* (5.1.11). *Then, the function* $(\alpha_0, \alpha_1) \ni \eta_3 \mapsto \eta_2 \in (0, \alpha_0)$ *is a strictly decreasing function.*

Proof. It is enough to prove that the function $(\beta_0, \beta_1) \ni \eta \mapsto \xi \in (0, \beta_0)$ is a strictly decreasing function. From (5.3.14) and (5.3.22), we have

$$\xi = \frac{1}{2}\left(-\eta + 2s + \sqrt{f_s(\eta)}\right).$$

For $-1 < s \le 1$ and $\beta_0 < \eta < \beta_1$, we have

$$\begin{aligned} \frac{d\xi}{d\eta} &= -\frac{1}{2} + \frac{1}{4\sqrt{f_s(\eta)}} \cdot \frac{df_s(\eta)}{d\eta} \\ &= -\frac{1}{2} - \frac{1}{4\sqrt{f_s(\eta)}}(6\eta - 4s) \\ &< -\frac{1}{2} - \frac{1}{2\sqrt{f_s(\eta)}}\left(3 \cdot \frac{2s}{3} - 2s\right) = -\frac{1}{2} < 0, \end{aligned}$$

where we used the following inequality:

$$\frac{2s}{3} < \frac{2s + \sqrt{3+s^2}}{3} = \beta_0 < \eta.$$

This completes the proof. □

Next, we discuss the change of the parameters when take the limit $\eta_3 \to \alpha_1$ (or equivalently $\eta \to \beta_1$). We begin with the following lemma.

Lemma 5.3.5. *Let* (ω, c) *satisfy* (5.1.11). *Then, we have*

$$\eta_1 \to -4\sqrt{\omega} + 2c, \ \eta_2 \to 0, \ \frac{1}{2g} \to \frac{\sqrt{4\omega - c^2}}{2} \tag{5.3.24}$$

as $\eta_3 \to \alpha_1$.

Proof. We note that

$$\begin{aligned} A(\alpha_1) &= 64\omega - 3\alpha_1 + 8c\alpha_1 \\ &= (4\sqrt{\omega} - 2c)^2. \end{aligned}$$

From the expressions (5.3.13) and (5.3.14), we have

$$\begin{aligned} \lim_{\eta_3 \to \alpha_1} \eta_1 &= \frac{-\alpha_1 + 4c - \sqrt{A(\alpha_1)}}{2} \\ &= \frac{-4\sqrt{\omega} + 2c - (4\sqrt{\omega} - 2c)}{2} = -4\sqrt{\omega} + 2c, \\ \lim_{\eta_3 \to \alpha_1} \eta_2 &= \frac{-\alpha_1 + 4c + \sqrt{A(\alpha_1)}}{2} \\ &= \frac{-4\sqrt{\omega} + 2c + (4\sqrt{\omega} - 2c)}{2} = 0. \end{aligned}$$

Note that the limit of η_2 compatible with the definition of α_1. From the expression (5.3.7) and the limits of η_2 and η_3, we have

$$\begin{aligned} \lim_{\eta_3 \to \alpha_1} \frac{1}{2g} &= \lim_{\eta_3 \to \alpha_1} \frac{\sqrt{\eta_3(\eta_2 - \eta_1)}}{4} \\ &= \frac{\sqrt{(4\sqrt{\omega} + 2c)(4\sqrt{\omega} - 2c)}}{4} = \frac{\sqrt{4\omega - c^2}}{2}. \end{aligned}$$

This completes the proof. □

It is more delicate to calculate the limit of modulus k of elliptic functions as $\eta_3 \to \alpha_1$. First, we rewrite k^2 defined by (5.3.6) as a function of η. From (5.3.13) and (5.3.14), we have

$$\eta_3(\eta_2 - \eta_1) = 4\sqrt{\omega}\eta \cdot \sqrt{A} = 16\omega\eta\sqrt{f_s(\eta)}.$$

Since

$$\eta_3 - \eta_2 = \frac{3\eta_3 - 4c - \sqrt{A}}{2},$$

we have

$$\begin{aligned}4\left(-\eta_1(\eta_3-\eta_2)\right) &= (3\eta_3-4c-\sqrt{A})(\eta_3-4c+\sqrt{A})\\ &= 3\eta_3^2-16c\eta_3+2\eta_3\sqrt{A}+16c^2-A,\\ &= 32\omega\left(3\eta^2+(\sqrt{f_s(\eta)}-6s)\eta+2(s^2-1)\right).\end{aligned}$$

Hence, we have the expression of k^2 as

$$\begin{aligned}k^2 &= \frac{-\eta_1(\eta_3-\eta_2)}{\eta_3(\eta_2-\eta_1)}\\ &= \frac{3\eta^2+(\sqrt{f_s(\eta)}-6s)\eta+2(s^2-1)}{2\eta\sqrt{f_s(\eta)}}.\end{aligned} \tag{5.3.25}$$

for $\beta_0<\eta<\beta_1$ and $-1<s\le 1$. By using the expression of (5.3.25), we can prove the monotonicity of modulus k of elliptic functions.

Proposition 5.3.6. *Let* (ω,c) *satisfy* (5.1.11). *Then, the function* $(\beta_0,\beta_1)\ni\eta\mapsto k(\eta)\in(0,1)$ *is a strictly increasing function.*

Proof. We define the function b by

$$b=b_s(\eta):=\eta^2f_s(\eta)=-3\eta^4+4s\eta^3+4\eta^2. \tag{5.3.26}$$

Note that by Lemma 5.3.2 $b=b_s(\eta)$ is positive for $-1<s\le 1$ and $\beta_0<\eta<\beta_1$. We differentiate k^2 with respect to η as

$$\begin{aligned}\frac{dk^2}{d\eta} &= \frac{1}{2b}\left[\left(6\eta-6s+\frac{d\sqrt{b}}{d\eta}\right)\sqrt{b}-\left(3\eta^2+\sqrt{b}-6s\eta+2(s^2-1)\right)\frac{d\sqrt{b}}{d\eta}\right]\\ &= \frac{1}{2b}\left[6(\eta-s)\sqrt{b}-\left(3\eta^2-6s\eta+2(s^2-1)\right)\frac{d\sqrt{b}}{d\eta}\right].\end{aligned}$$

A direct computation shows that

$$\frac{dk^2}{d\eta}=\frac{\eta}{b\sqrt{b}}\left(6s\eta g_s(\eta)+4(1-s^2)\right), \tag{5.3.27}$$

where the function g_s is defined by

$$g_s(\eta):=-\left(\eta-(s-1)\right)\left(\eta-(s+1)\right).$$

We note that a positive zero of $g_s(\eta)$ is given by $\beta_1=s+1$. Since $g_s(0)=1-s^2\ge 0$ for $-1<s\le 1$, we have $g_s(\eta)>0$ for $\beta_0<\eta<\beta_1$. Therefore, if $0<s\le 1$, by (5.3.27) we obtain

$$\frac{dk^2}{d\eta}\ge\frac{6s\eta g_s(\eta)}{b\sqrt{b}}>0.$$

If $s = 0$, by (5.3.27) we obtain

$$\frac{dk^2}{d\eta} = \frac{4\eta}{b\sqrt{b}} > 0.$$

Finally, we consider the case $-1 < s < 0$. We note that β_0 gives a positive maximal point of $\eta \mapsto \eta g_s(\eta)$. Therefore, if $-1 < s < 0$, we have

$$6s\eta g_s(\eta) + 4(1 - s^2) > 6s\beta_0 g_s(\beta_0) + 4(1 - s^2) =: h(s). \tag{5.3.28}$$

From the definitions of β_0 and g_s, $h(s)$ is rewritten as

$$h(s) = \frac{4}{9}\left(-s^4 + (3 + s^2)^{3/2}s + 9\right). \tag{5.3.29}$$

We note that $h(0) = 4$, $h(-1) = 0$, and $s \mapsto h(s)$ is strictly increasing on the interval $[-1, 0]$. Hence, from (5.3.27) and (5.3.28), we deduce that

$$\begin{aligned}\frac{dk^2}{d\eta} &= \frac{\eta}{b\sqrt{b}}\big(6s\eta g_s(\eta) + 4(1 - s^2)\big)\\ &> \frac{\eta}{b\sqrt{b}} \cdot h(s) > \frac{\eta}{b\sqrt{b}} \cdot h(-1) = 0.\end{aligned}$$

This completes the proof of Proposition 5.3.6. □

The limits of k and β^2 are given by the following lemma.

Lemma 5.3.7. *Let* (ω, c) *satisfy* (5.1.11). *Then, we have*

$$k \underset{\eta_3\to\alpha_1}{\longrightarrow} \begin{cases} 1 & \text{if } \omega > c^2/4, \\ \dfrac{1}{\sqrt{2}} & \text{if } \omega = c^2/4 \text{ and } c > 0, \end{cases} \tag{5.3.30}$$

$$\beta^2 \underset{\eta_3\to\alpha_1}{\longrightarrow} \begin{cases} \dfrac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c} & \text{if } \omega > c^2/4, \\ \infty & \text{if } \omega = c^2/4 \text{ and } c > 0. \end{cases} \tag{5.3.31}$$

Proof. Case 1: $\omega > c^2/4$. By Lemma 5.3.5 we note that

$$\eta_1 \to -4\sqrt{\omega} + 2c < 0, \quad \eta_2 \to 0$$

as $\eta_3 \to \alpha_1$. From the definitions of k^2 and β, we obtain

$$\begin{aligned}\lim_{\eta_3\to\alpha_1} k^2 &= \lim_{\eta_3\to\alpha_1} \frac{-\eta_1(\eta_3 - \eta_2)}{\eta_3(\eta_2 - \eta_1)}\\ &= \frac{(4\sqrt{\omega} - 2c)\cdot\alpha_1}{\alpha_1(4\sqrt{\omega} - 2c)} = 1,\end{aligned}$$

and

$$\lim_{\eta_3 \to \alpha_1} \beta^2 = \lim_{\eta_3 \to \alpha_1} -\frac{\eta_3 k^2}{\eta_1} = \frac{2\sqrt{\omega} + c}{2\sqrt{\omega} - c}.$$

Case 2: $c = 2\sqrt{\omega}$. Since in this case

$$\eta_1 \to -4\sqrt{\omega} + 2c = 0, \quad \eta_2 \to 0$$

as $\eta_3 \to \alpha_1 = 4c$, the above calculation does not work. In this case $A(\eta_3)$ is rewritten as

$$A(\eta_3) = 16c^2 - 3\eta_3^2 + 8c\eta_3 = (3\eta_3 + 4c)(4c - \eta_3). \tag{5.3.32}$$

By using this identity, η_1 is rewritten as

$$\eta_1 = \frac{-\eta_3 + 4c - \sqrt{A(\eta_3)}}{2} = \frac{\sqrt{4c - \eta_3}}{2}\left(\sqrt{4c - \eta_3} - \sqrt{3\eta_3 + 4c}\right). \tag{5.3.33}$$

Hence, we have

$$k^2 = \frac{-\eta_1(\eta_3 - \eta_2)}{\eta_3(\eta_2 - \eta_1)} = \frac{\eta_3 - \eta_2}{\eta_3} \cdot \frac{-\eta_1}{\sqrt{A(\eta_3)}} = \frac{\eta_3 - \eta_2}{\eta_3} \cdot \frac{\sqrt{3\eta_3 + 4c} - \sqrt{4c - \eta_3}}{2\sqrt{3\eta_3 + 4c}} \longrightarrow \frac{1}{2}$$

as $\eta_3 \to 4c$. For the limit of β^2, since $\eta_1 \to 0$, we have

$$\lim_{\eta_3 \to \alpha_1} \beta^2 = \lim_{\eta_3 \to \alpha_1} -\frac{\eta_3 k^2}{\eta_1} = \infty.$$

This completes the proof. □

The fundamental period T_ψ defined by (5.3.9) is rewritten as

$$T_\psi(\eta_3) = \frac{8}{\sqrt{\eta_3\sqrt{A(\eta_3)}}} K(k(\eta_3))$$

for $\alpha_0 < \eta_3 < \alpha_1$. Combining with Proposition 5.3.6, we can prove the monotonicity of the fundamental period T_ψ.

Proposition 5.3.8. *Let (ω, c) satisfy (5.1.11). Then, the function $(\alpha_0, \alpha_1) \ni \eta_3 \mapsto T_\psi(\eta_3) \in (0, \infty)$ is a strictly increasing function.*

Proof. We recall that the fundamental period T_ψ is defined by

$$T_\psi = \frac{8}{\sqrt{\eta_3(\eta_2 - \eta_1)}} K(k).$$

We note that

$$\eta_3(\eta_2 - \eta_1) = 16\omega\eta\sqrt{f_s(\eta)} = 16\omega\sqrt{b_s(\eta)},$$

where $b_s(\eta)$ is defined by (5.3.26). Hence, T_ψ is rewritten as

$$T_\psi = \frac{2}{\sqrt{\omega} b_s(\eta)^{1/4}} K(k(\eta)), \tag{5.3.34}$$

where $-1 < s \leq 1$ and $\beta_0 < \eta < \beta_1$. We differentiate T_ψ with respect to η as

$$\begin{aligned} \frac{\sqrt{\omega}}{2} \cdot \frac{dT_\psi}{d\eta} &= \frac{dK}{dk}\frac{dk}{d\eta} \cdot \frac{1}{b_s^{1/4}} + K \cdot \left(-\frac{1}{4b_s^{5/4}} \right) \frac{db_s}{d\eta} \\ &= \frac{1}{b_s^{5/4}} \left(\frac{dK}{dk}\frac{dk}{d\eta} \cdot b_s - K\eta a_s(\eta) \right), \end{aligned} \tag{5.3.35}$$

where the function a_s is defined by

$$a_s(\eta) := -3\eta^2 + 3s\eta + 2. \tag{5.3.36}$$

We note that

$$\gamma = \gamma(s) := \frac{3s + \sqrt{9s^2 + 24}}{6} \tag{5.3.37}$$

gives a positive zero of $a_s(\eta)$. Since $a_s(0) = 2$, we have $a_s(\eta) > 0$ for $0 < \eta < \gamma$. When $-1 < s \leq 1$, we have

$$\begin{aligned} \gamma \leq \beta_1 &\iff \frac{3s + \sqrt{9s^2 + 24}}{6} \leq 1 + s \\ &\iff -\frac{1}{3} \leq s. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \beta_0 < \gamma &\iff \frac{1}{3}\left(2s + \sqrt{s^2 + 3}\right) < \frac{1}{6}\left(3s + \sqrt{9s^2 + 24}\right) \\ &\iff s < \sqrt{s^2 + 3}. \end{aligned}$$

Since the last inequality holds for any $s \in \mathbb{R}$, we have $\beta_0 < \gamma$ for $-1 < s \leq 1$. Hence, the following three cases can be considered.

(a) $-\dfrac{1}{3} < s \leq 1$, $\gamma \leq \eta < \beta_1$.

(b) $-\dfrac{1}{3} < s \leq 1$, $\beta_0 < \eta < \gamma$.

(c) $-1 < s \leq -\dfrac{1}{3}$, $\beta_0 < \eta < \beta_1 (\leq \gamma)$.

Since $\dfrac{dK}{dk} > 0$, and $\dfrac{dk}{d\eta} > 0$, we note that the first term on the RHS of (5.3.35) is positive. In the case (a), since $a(\eta) \leq 0$, by (5.3.35) we deduce that

$$\frac{dT_\psi}{d\eta} > 0. \tag{5.3.38}$$

In the latter two cases, since $a_s(\eta) > 0$, we need to calculate a little more carefully. But, by using the formula

$$\frac{dK}{dk} = \frac{1}{kk'^2}(E - k'^2 K)$$

and (5.3.27), one can prove that (5.3.38) holds in these cases. We omit the detail and refer to [2, 6] as similar arguments. □

From the definition (5.3.6) of k, we have

$$k^2(\eta_3) \to 0$$

as $\eta_3 \to \alpha_0$. Since $K(k) \to \frac{\pi}{2}$ as $k \to 0$, we have

$$T_\psi(\eta_3) \to \frac{4\pi}{\sqrt{\alpha_0 \sqrt{A(\alpha_0)}}} =: T_0(\omega, c) \tag{5.3.39}$$

as $\eta_3 \to \alpha_0$. Note that $A(\alpha_0)$ is a positive constant as described in Remark 5.3.3. On the other hand, we have

$$T_\psi(\eta_3) \to \infty \tag{5.3.40}$$

as $\eta_3 \to \alpha_1$. Indeed, when $\omega > c^2/4$, we have $k \to 1$ as $\eta_3 \to \alpha_1$ by Lemma 5.3.7. Since $K(k) \to \infty$ as $k \to 1$, (5.3.40) holds. When $c = 2\sqrt{\omega}$, we have $\eta_1, \eta_2 \to 0$ as $\eta_3 \to \alpha_1$ by Lemma 5.3.5, and hence (5.3.40) holds from the definition (5.3.9) of T_ψ. Therefore, by (5.3.39), (5.3.40) and Proposition 5.3.8 we deduce that

$$\alpha_0 < \eta_3 < \alpha_1 \iff T_0(\omega, c) < T_\psi(\eta_3) < \infty, \tag{5.3.41}$$

and

$$\eta_3 \to \alpha_0 \iff T_\psi(\eta_3) \to T_0(\omega, c), \tag{5.3.42}$$

$$\eta_3 \to \alpha_1 \iff T_\psi(\eta_3) \to \infty. \tag{5.3.43}$$

The relation (5.3.43) means that the limit $\eta_3 \to \alpha_1$ is equivalent to the long-period limit. Since $2L = T_\psi$, L has the following constraint condition:

$$L_0(\omega, c) < L < \infty, \tag{5.3.44}$$

where $L_0(\omega, c)$ is defined by

$$L_0 = L_0(\omega, c) := \frac{T_0(\omega, c)}{2} = \frac{2\pi}{\sqrt{\alpha_0 \sqrt{A(\alpha_0)}}}. \tag{5.3.45}$$

Since by (5.3.43) we have

$$\eta_3 \to \alpha_1 \iff L \to \infty,$$

we can take the limit $\eta_3 \to \alpha_1$ instead of the limit $L \to \infty$.

To clarify the dependence of parameters, we denote the function ψ by $\psi^L_{\omega,c}$. Let $c_L \in \frac{2\pi}{L}\mathbb{Z}$. It is easily verified that the traveling wave

$$u_{\omega,c_L} = e^{i\omega t + i\frac{c_L}{2}(x - c_L t)} (\psi^L_{\omega,c_L})^{\frac{1}{2}}(x - c_L t)$$

is a solution of the equation (5.1.6).

5.3.3 Pointwise convergence in the long-period limit

We complete the proof of Theorem 5.1.1. Fix any $x \in \mathbb{R}$ and consider a large $L > 0$ such that $x \in [-L, L]$. We need to divide two cases to do calculations in the long-period limit.
Case 1: $\omega > c^2/4$. By Lemma 5.3.5, Lemma 5.3.7 and extremal formulae (5.2.3) of elliptic functions, we have

$$\begin{aligned}
\lim_{L\to\infty} \psi^L_{\omega,c}(x) &= \lim_{\eta_3 \to \alpha_1} \eta_3 \left[\frac{\mathrm{dn}^2\left(\frac{x}{2g}; k\right)}{1 + \beta^2 \mathrm{sn}^2\left(\frac{x}{2g}; k\right)} \right] \\
&= (4\sqrt{\omega} + 2c) \left[\frac{\mathrm{sech}^2\left(\frac{\sqrt{4\omega - c^2}}{2} x\right)}{1 + \frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c} \tanh^2\left(\frac{\sqrt{4\omega-c^2}}{2} x\right)} \right] \\
&= \frac{2(4\omega - c^2)}{(2\sqrt{\omega} - c)\cosh^2\left(\frac{\sqrt{4\omega-c^2}}{2} x\right) + (2\sqrt{\omega} + c)\sinh^2\left(\frac{\sqrt{4\omega-c^2}}{2} x\right)} \\
&= \frac{2(4\omega - c^2)}{2\sqrt{\omega}\cosh^2(\sqrt{4\omega - c^2} x) - c} = \Phi^2_{\omega,c}(x).
\end{aligned}$$

Case 2: $c = 2\sqrt{\omega}$. Since in this case $\beta \to \infty$, $\frac{1}{2g} \to 0$ as $\eta_3 \to \alpha_1$, we need to calculate more carefully. We use the following relations

$$\mathrm{dn}(u; k) = 1 + O(u^2), \tag{5.3.46}$$

$$\mathrm{sn}(u; k) = u + O(u^3) \tag{5.3.47}$$

as $u \to 0$ (see, e.g., [42] for the details). From (5.3.46), we have

$$\lim_{\eta_3 \to \alpha_1} \mathrm{dn}\left(\frac{x}{2g}; k\right) = 1. \tag{5.3.48}$$

We note that

$$\begin{aligned} \frac{1}{4g^2} &= \frac{\eta_3(\eta_2 - \eta_1)}{16} \\ &= \frac{\eta_3 \sqrt{A(\eta_3)}}{16} \\ &= \frac{\eta_3}{16} \cdot \sqrt{(4c - \eta_3)(3\eta_3 + 4c)}, \end{aligned} \tag{5.3.49}$$

where in the last equality we used the identity (5.3.32). From (5.3.33), (5.3.49) and (5.3.30), we deduce that

$$\begin{aligned} \frac{\beta^2}{4g^2} &= -\frac{\eta_3}{\eta_1} \cdot \frac{k^2}{4g^2} \\ &= \frac{2\eta_3}{\sqrt{3\eta_3 + 4c} - \sqrt{4c - \eta_3}} \cdot \frac{\eta_3}{16} \sqrt{3\eta_3 + 4c} \cdot k^2 \\ &\longrightarrow \frac{(4c)^2}{8} \cdot \frac{1}{2} = c^2 \end{aligned} \tag{5.3.50}$$

as $\eta_3 \to \alpha_1 = 4c$. Hence, by (5.3.47) and (5.3.50), we obtain that

$$\begin{aligned} \lim_{\eta_3 \to 4c} \beta^2 \mathrm{sn}^2\left(\frac{x}{2g}; k\right) &= \lim_{\eta_3 \to 4c} \beta^2 \left(\frac{x^2}{4g^2} + O\left(4c - \eta_3\right)\right) \\ &= \lim_{\eta_3 \to 4c} \left[\frac{\beta^2}{4g^2} x^2 + O\left(\sqrt{4c - \eta_3}\right)\right] \\ &= (cx)^2, \end{aligned}$$

where we have used the relation

$$\beta^2 O\left(4c - \eta_3\right) = \frac{1}{\eta_1} \cdot O\left(4c - \eta_3\right) = O\left(\sqrt{4c - \eta_3}\right).$$

Combined with (5.3.48), we deduce that

$$\begin{aligned} \lim_{L \to \infty} \psi^L_{c^2/4,c}(x) &= \lim_{\eta_3 \to 4c} \eta_3 \left[\frac{\mathrm{dn}^2\left(\frac{x}{2g}; k\right)}{1 + \beta^2 \mathrm{sn}^2\left(\frac{x}{2g}; k\right)}\right] \\ &= \frac{4\sqrt{\omega} + 2c}{1 + (cx)^2} \\ &= \frac{4c}{1 + (cx)^2} = \Phi^2_{c^2/4,c}(x). \end{aligned}$$

This completes the proof of Theorem 5.1.1. □

5.4 Long-period limit procedure

5.4.1 L^2-convergence

First, we discuss the convergence of the mass $\|\Phi^L_{\omega,c}\|^2_{L^2(\mathbb{T}_{2L})}$ in the long-period limit. We recall that the mass of the soliton on the whole line is given by

$$\|\Phi_{\omega,c}\|^2_{L^2(\mathbb{R})} = 8\tan^{-1}\sqrt{\frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c}}. \tag{5.4.1}$$

Our main purpose in this subsection is to prove the following theorem.

Theorem 5.4.1. *Let* (ω, c) *satisfy* (5.1.11). *Then, we have*

$$\lim_{L\to\infty} \|\Phi^L_{\omega,c}\|^2_{L^2(\mathbb{T}_{2L})} = \|\Phi_{\omega,c}\|^2_{L^2(\mathbb{R})}. \tag{5.4.2}$$

Proof. We calculate the mass of traveling waves on $\mathbb{T}_{2L}$ as

$$\begin{aligned}\|\Phi^L_{\omega,c}\|^2_{L^2(\mathbb{T}_{2L})} &= 2\int_0^L \eta_3 \frac{\mathrm{dn}^2\left(\frac{x}{2g};k\right)}{1+\beta^2\mathrm{sn}^2\left(\frac{x}{2g};k\right)}dx \\ &= 4g\eta_3\int_0^{K(k)} \frac{\mathrm{dn}^2(x;k)}{1+\beta^2\mathrm{sn}^2(x;k)}dx,\end{aligned}$$

where we used $L = 2gK(k)$ in the last equality. Applying formula 410.04 in [12], we have

$$\|\Phi^L_{\omega,c}\|^2_{L^2(\mathbb{T}_{2L})} = 4g\eta_3\sqrt{\frac{k^2+\beta^2}{(1+\beta^2)\beta^2}}G(\mu,k), \tag{5.4.3}$$

where

$$G(\mu,k) := K(k)E(\mu,k') - K(k)F(\mu,k') + E(k)F(\mu,k'), \tag{5.4.4}$$

$$\mu := \sin^{-1}\sqrt{\frac{\beta^2}{\beta^2+k^2}}. \tag{5.4.5}$$

We note that μ is regarded as a function of η_3 and that $0 < \mu < \frac{\pi}{2}$ when $\alpha_0 < \eta_3 < \alpha_1$. We set

$$\mu_1 := \lim_{\eta_3\to\alpha_1}\mu = \lim_{\eta_3\to\alpha_1}\sin^{-1}\sqrt{\frac{\beta^2}{\beta^2+k^2}}. \tag{5.4.6}$$

Case 1: $\omega > c^2/4$. By Lemma 5.3.5 and Lemma 5.3.7, we have

$$\begin{aligned}
\lim_{\eta_3\to\alpha_1} 4g\eta_3\sqrt{\frac{k^2+\beta^2}{(1+\beta^2)\beta^2}} &= \lim_{\eta_3\to\alpha_1}\frac{4g\eta_3}{\beta} \\
&= \frac{8(2\sqrt{\omega}+c)}{\sqrt{(2\sqrt{\omega}+c)(2\sqrt{\omega}-c)}}\sqrt{\frac{2\sqrt{\omega}-c}{2\sqrt{\omega}+c}} \\
&= 8.
\end{aligned} \tag{5.4.7}$$

By the Taylor expansion, we have

$$\frac{1}{\sqrt{1-x}} = 1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^n, \tag{5.4.8}$$

$$\sqrt{1-x} = 1-\sum_{n=1}^{\infty}\frac{1}{2n-1}\cdot\frac{(2n-1)!!}{(2n)!!}x^n \tag{5.4.9}$$

for all $|x|<1$. Let $\tau\in(0,\frac{\pi}{2})$. Applying (5.4.8) and (5.4.9), we have

$$\begin{aligned}
E(\tau,k') &= \int_0^\tau \sqrt{1-k'^2\sin^2\theta}\,d\theta \\
&= \tau - \sum_{n=1}^{\infty}\frac{1}{2n-1}\cdot\frac{(2n-1)!!}{(2n)!!}k'^{2n}\int_0^\tau \sin^{2n}\theta d\theta,
\end{aligned} \tag{5.4.10}$$

$$\begin{aligned}
F(\tau,k') &= \int_0^\tau \frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}} \\
&= \tau + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}k'^{2n}\int_0^\tau \sin^{2n}\theta d\theta.
\end{aligned} \tag{5.4.11}$$

By (5.4.10) and (5.4.11), we have

$$\begin{aligned}
\sup_{0\le\tau\le\frac{\pi}{2}}|E(\tau,k')-F(\tau,k')| &\le \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{2n}{2n-1}\cdot\frac{(2n-1)!!}{(2n)!!}k'^{2n} \\
&\le Ck'^2,
\end{aligned} \tag{5.4.12}$$

where C is independent of k'. By (5.2.2) and (5.4.12), we deduce that

$$\sup_{0\le\tau\le\frac{\pi}{2}}|K(k)(E(\tau,k')-F(\tau,k'))| \le Ck'^2\left(-\log\frac{k'}{4}\right)\to 0 \tag{5.4.13}$$

as $k'\to 0$. Especially, we deduce that

$$\lim_{\eta_3\to\alpha_1} K(k)(E(\mu,k')-F(\mu,k')) = 0. \tag{5.4.14}$$

By (5.4.6) and Lemma 5.3.7, we have

$$\sin \mu_1 = \lim_{\eta_3 \to \alpha_1} \sqrt{\frac{\beta^2}{\beta^2 + k^2}} = \sqrt{\frac{2\sqrt{\omega} + c}{4\sqrt{\omega}}}.$$

Since

$$\sin^2 \mu_1 = \frac{2\sqrt{\omega} + c}{4\sqrt{\omega}}, \quad \cos^2 \mu_1 = \frac{2\sqrt{\omega} - c}{4\sqrt{\omega}}$$

and $\mu_1 \in [0, \frac{\pi}{2}]$, we deduce that

$$\mu_1 = \tan^{-1} \sqrt{\frac{2\sqrt{\omega} + c}{2\sqrt{\omega} - c}}. \tag{5.4.15}$$

By (5.4.3), (5.4.7), (5.4.14) and (5.4.15), we obtain that

$$\begin{aligned} \lim_{L \to \infty} \|\Phi^L_{\omega,c}\|^2_{L^2(\mathbb{T}_{2L})} &= \lim_{\eta_3 \to \alpha_1} 4g\eta_3 \sqrt{\frac{k^2 + \beta^2}{(1 + \beta^2)\beta^2}} G(\mu, k) \\ &= 8E(1)F(\mu_1, 0) \\ &= 8\mu_1 \\ &= 8 \tan^{-1} \sqrt{\frac{2\sqrt{\omega} + c}{2\sqrt{\omega} - c}} = \|\Phi_{\omega,c}\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{5.4.16}$$

Case 2: $c = 2\sqrt{\omega}$. Since

$$\begin{aligned} &k^2 \to \tfrac{1}{2}, \quad k'^2 \to \tfrac{1}{2}, \quad \beta \to \infty, \\ &\eta_1 \to 0, \quad \eta_2 \to 0 \end{aligned} \tag{5.4.17}$$

as $\eta_3 \to \alpha_1$ in this case, we need to modify the previous calculation. By (5.4.17), we have

$$\frac{k^2 + \beta^2}{1 + \beta^2} \to 1 \tag{5.4.18}$$

as $\eta_3 \to \alpha_1$. By using the definition of k, g and β, we have

$$\begin{aligned} \frac{4g\eta_3}{\beta} &= \frac{8\eta_3}{\sqrt{\eta_3(\eta_2 - \eta_1)}} \cdot \sqrt{\frac{-\eta_1}{\eta_3}} \cdot \sqrt{\frac{\eta_3(\eta_2 - \eta_1)}{-\eta_1(\eta_3 - \eta_2)}} \\ &= 8\sqrt{\frac{\eta_3}{\eta_3 - \eta_2}}. \end{aligned} \tag{5.4.19}$$

By (5.4.18) and (5.4.19), we obtain that

$$\begin{aligned}\lim_{\eta_3\to\alpha_1} 4g\eta_3\sqrt{\frac{k^2+\beta^2}{(1+\beta^2)\beta^2}} &= \lim_{\eta_3\to\alpha_1}\frac{4g\eta_3}{\beta}\\ &= \lim_{\eta_3\to\alpha_1} 8\sqrt{\frac{\eta_3}{\eta_3-\eta_2}}\\ &= 8.\end{aligned}\tag{5.4.20}$$

By (5.4.17), we note that

$$\mu_1 = \lim_{\eta_3\to\alpha_1}\sin^{-1}\sqrt{\frac{\beta^2}{\beta^2+k^2}} = \sin^{-1}1 = \frac{\pi}{2}.\tag{5.4.21}$$

Hence, we obtain that

$$\begin{aligned}\lim_{\eta_3\to\alpha_1} G(\mu,k) &= G(\mu_1,\tfrac{1}{\sqrt{2}})\\ &= (KE'-KK'+EK')(\tfrac{1}{\sqrt{2}})\\ &= \frac{\pi}{2},\end{aligned}\tag{5.4.22}$$

where we used the Legendre relation (5.2.4) in the last equality. By (5.4.3), (5.4.20) and (5.4.22), we obtain that

$$\begin{aligned}\lim_{L\to\infty}\|\Phi^L_{c^2/4,c}\|^2_{L^2(\mathbb{T}_{2L})} &= \lim_{\eta_3\to\alpha_1} 4g\eta_3\sqrt{\frac{k^2+\beta^2}{(1+\beta^2)\beta^2}}G(\mu,k)\\ &= 8G(\mu_1,\tfrac{1}{\sqrt{2}})\\ &= 4\pi = \|\Phi_{c^2/4,c}\|^2_{L^2(\mathbb{R})}.\end{aligned}\tag{5.4.23}$$

This completes the proof. □

Next, we prove the following theorem. This is the partial statement of Theorem 5.1.4.

Theorem 5.4.2. *Let* (ω,c) *satisfy* (5.1.11). *Then, we have*

$$\lim_{L\to\infty}\|\Phi^L_{\omega,c}-\Phi_{\omega,c}\|_{H^m([-L,L])} = 0\tag{5.4.24}$$

for all $m=0,1,2$.

To prove Theorem 5.4.2, we recall the Brézis–Lieb lemma.

Lemma 5.4.3 ([11])**.** *Let* $1\le p<\infty$. *Let* $\{f_L\}$ *be a bounded sequence in* $L^p(\mathbb{R})$ *and* $f_L\to f$ *a.e. in* $\mathbb{R}$ *as* $L\to\infty$. *Then we have*

$$\|f_L\|^p_{L^p}-\|f_L-f\|^p_{L^p}-\|f\|^p_{L^p}\to 0$$

as $L\to\infty$.

Proof of Theorem 5.4.2. We consider $\Phi^L_{\omega,c}$ as the function defined on $\mathbb{R}$. More precisely, we extend the function $\Phi^L_{\omega,c}$ as

$$\Phi^L_{\omega,c}(x) = \begin{cases} \Phi^L_{\omega,c}(x) & \text{if } x \in [-L, L], \\ \Phi^L_{\omega,c}(x - 2Lk) & \text{if } x \in [(2k-1)L, (2k+1)L],\ k \in \mathbb{Z} \setminus \{0\}. \end{cases} \tag{5.4.25}$$

We set $f_L = \chi_{[-L,L]}\Phi^L_{\omega,c}$ and $f = \Phi_{\omega,c}$. By Theorem 5.1.1 and Theorem 5.4.1, we have

$$f_L(x) \to f(x) \text{ for all } x \in \mathbb{R},$$
$$\|f_L\|^2_{L^2(\mathbb{R})} \to \|f\|^2_{L^2(\mathbb{R})}$$

as $L \to \infty$. Applying Lemma 5.4.3, we obtain

$$\lim_{L\to\infty} \|f_L - f\|^2_{L^2(\mathbb{R})} = 0. \tag{5.4.26}$$

Since $f \in L^2(\mathbb{R})$, we have

$$\lim_{L\to\infty} \|f\|^2_{L^2(|x|\geq L)} = \lim_{L\to\infty} \|\Phi_{\omega,c}\|^2_{L^2(|x|\geq L)} = 0. \tag{5.4.27}$$

By (5.4.26) and (5.4.27), we deduce that

$$\lim_{L\to\infty} \|\Phi^L_{\omega,c} - \Phi_{\omega,c}\|_{L^2([-L,L])} = \lim_{L\to\infty} \|f_L - f\|_{L^2(|x|\leq L)} = 0. \tag{5.4.28}$$

Next we prove

$$\lim_{L\to\infty} \|\partial^2\Phi^L_{\omega,c} - \partial^2\Phi_{\omega,c}\|_{L^2([-L,L])} = 0. \tag{5.4.29}$$

We note that $\Phi^L_{\omega,c}$ and $\Phi_{\omega,c}$ satisfy the same equation (5.1.16). For each $L > 0$, we have

$$|\Phi^L_{\omega,c}(x)|^2 \leq \eta_3 \text{ for all } x \in [-L, L], \tag{5.4.30}$$

since $\sqrt{\eta_3}$ is maximum value of $\Phi^L_{\omega,c}$. By the explicit formula (5.1.14) of the soliton, we have

$$\|f\|^2_{L^\infty(\mathbb{R})} = \Phi^2_{\omega,c}(0) = 4\sqrt{\omega} + 2c = \alpha_1. \tag{5.4.31}$$

By (5.4.30), (5.4.31) and (5.4.28), we deduce that

$$\begin{aligned} \|f_L^3 - f^3\|_{L^2([-L,L])} &\leq C(\|f_L\|^2_{L^\infty([-L,L])} + \|f\|^2_{L^\infty([-L,L])})\|f_L - f\|_{L^2([-L,L])} \\ &\leq C(\eta_3 + \alpha_1)\|f_L - f\|_{L^2([-L,L])} \\ &\leq 2C\alpha_1\|f_L - f\|_{L^2([-L,L])} \underset{L\to\infty}{\longrightarrow} 0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|f_L^5 - f^5\|_{L^2([-L,L])} &\leq C(\|f_L\|^4_{L^\infty([-L,L])} + \|f\|^4_{L^\infty([-L,L])})\|f_L - f\|_{L^2([-L,L])} \\ &\leq 2C\alpha_1^2\|f_L - f\|_{L^2([-L,L])} \underset{L\to\infty}{\longrightarrow} 0. \end{aligned}$$

Hence, by using the equation (5.1.16), we deduce that

$$\begin{aligned}\|\partial^2\Phi^L_{\omega,c} - \partial^2\Phi_{\omega,c}\|_{L^2([-L,L])} &\leq \left(\omega - \frac{c^2}{4}\right)\|f_L - f\|_{L^2([-L,L])} \\ &\quad + \frac{c}{2}\|f_L^3 - f^3\|_{L^2([-L,L])} + \frac{3}{16}\|f_L^5 - f^5\|_{L^2([-L,L])} \\ &\xrightarrow[L\to\infty]{} 0.\end{aligned}$$

Finally, by integration by parts, we obtain that

$$\begin{aligned}\|\partial\Phi^L_{\omega,c} - \partial\Phi_{\omega,c}\|^2_{L^2([-L,L])} &= -\int_{-L}^{L}\left(\partial^2\Phi^L_{\omega,c} - \partial^2\Phi_{\omega,c}\right)\left(\Phi^L_{\omega,c} - \Phi_{\omega,c}\right)dx \\ &\leq \|\partial^2\Phi^L_{\omega,c} - \partial^2\Phi_{\omega,c}\|_{L^2([-L,L])}\|\Phi^L_{\omega,c} - \Phi_{\omega,c}\|_{L^2([-L,L])} \\ &\xrightarrow[L\to\infty]{} 0.\end{aligned}$$

This completes the proof. □

To prove the estimate (5.4.24) for $m \geq 3$ by using the equation (5.1.16), we need to control L^∞-norm of lower derivative $\partial_x^k\Phi^L_{\omega,c}$ where $k = 1, 2, \cdots, m-1$. To achieve this, we discuss L^∞-convergence of $\Phi^L_{\omega,c}$ in next subsection.

5.4.2 L^∞-convergence

In this subsection, we mainly prove the following proposition.

Proposition 5.4.4. *Let* (ω, c) *satisfy* (5.1.11). *Then, we have*

$$\lim_{L\to\infty}\|\Phi^L_{\omega,c} - \Phi_{\omega,c}\|_{L^\infty([-L,L])} = 0. \tag{5.4.32}$$

Proof. Since $\Phi^L_{\omega,c}$ and $\Phi_{\omega,c}$ are even functions, it is enough to consider the interval $[0, L]$. We use the same notation in the proof of Theorem 5.4.2. By fundamental theorem of calculus, we have

$$\begin{aligned}f_L^2(x) &= f_L(0)^2 + \int_0^x \frac{d}{dy}f_L^2(y)dy, \\ f^2(x) &= f(0)^2 + \int_0^x \frac{d}{dy}f^2(y)dy,\end{aligned}$$

for all $x \in [0, L]$. Since $f_L(0)^2 = \eta_3$ and $f(0)^2 = \alpha_1$, we have

$$f_L^2(x) - f(x)^2 = \eta_3 - \alpha_1 + 2\int_0^x (f_L f_L' - ff')dy \tag{5.4.33}$$

for all $x \in [0, L]$. By Theorem 5.4.1, we note that

$$\sup_{L_0<L<\infty}\|f_L\|_{L^2([0,L])} \leq C = C(\|f\|_{L^2(\mathbb{R})}). \tag{5.4.34}$$

Applying Cauchy-Schwarz's inequality and (5.4.34), we deduce that

$$\int_0^L |f_L f_L' - ff'|dy \leq \|f_L\|_{L^2([0,L])}\|f_L' - f'\|_{L^2([0,L])} + \|f'\|_{L^2([0,L])}\|f_L - f\|_{L^2([0,L])}$$
$$\leq C\|f_L - f\|_{H^1([0,L])}.$$

Combining with (5.4.33), it follows from Theorem 5.4.2 that

$$\begin{aligned}\|f_L^2 - f^2\|_{L^\infty([0,L])} &\leq |\eta_3 - \alpha_1| + 2\int_0^L |f_L f_L' - ff'|dy \\ &\leq |\eta_3 - \alpha_1| + C\|f_L - f\|_{H^1([0,L])} \\ &\underset{L\to\infty}{\longrightarrow} 0.\end{aligned} \tag{5.4.35}$$

By using the elementary inequality

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|} \text{ for all } x, y \geq 0$$

and (5.4.35), we deduce that

$$\begin{aligned}\|f_L - f\|_{L^\infty([0,L])} &\leq \sqrt{\|f_L^2 - f^2\|_{L^\infty([0,L])}} \\ &\underset{L\to\infty}{\longrightarrow} 0.\end{aligned} \tag{5.4.36}$$

This completes the proof. □

Remark 5.4.5. We can also prove Proposition 5.4.4 directly without using the result of Theorem 5.4.2. Given a $\varepsilon > 0$. By the decay of $\Phi_{\omega,c}$ and the pointwise convergence in Theorem 5.1.1, there exists $L_0 > 0$ such that

$$|\Phi_{\omega,c}(L_0)| < \varepsilon, \ |\Phi_{\omega,c}^L(L_0)| < 2\varepsilon \tag{5.4.37}$$

for large $L > L_0 > 0$. Both $\Phi_{\omega,c}^L$ and $\Phi_{\omega,c}$ are radial and decreasing functions, we deduce that

$$\|\Phi_{\omega,c}\|_{L^\infty(L_0\leq|x|\leq L)} < \varepsilon, \ \|\Phi_{\omega,c}^L\|_{L^\infty(L_0\leq|x|\leq L)} < 2\varepsilon \tag{5.4.38}$$

for large $L > 0$. On the other hand, by reviewing the proof of Theorem 5.1.1, it is easily verified that

$$\lim_{L\to\infty} \|\Phi_{\omega,c}^L - \Phi_{\omega,c}\|_{L^\infty([-L_0,L_0])} = 0. \tag{5.4.39}$$

By (5.4.38) and (5.4.39), we obtain

$$\limsup_{L\to\infty} \|\Phi_{\omega,c}^L - \Phi_{\omega,c}\|_{L^\infty([-L,L])} \leq 2\varepsilon.$$

This gives an alternative proof of Proposition 5.4.4.

The following proposition follows from Proposition 5.4.4 and similar discussion on the proof of Theorem 5.4.2.

Proposition 5.4.6. *Let* (ω, c) *satisfy* (5.1.11). *Then, we have*

$$\lim_{L\to\infty} \|\Phi_{\omega,c}^L - \Phi_{\omega,c}\|_{C^m([-L,L])} = 0 \tag{5.4.40}$$

for all $m = 0, 1, 2$.

5.4.3 Proof of Theorem 5.1.4 and Theorem 5.1.5

Proof of Theorem 5.1.5. It is proved by differentiating the equation (5.1.16) and by applying Proposition 5.4.6 and the induction. We omit the detail. □

Proof of Theorem 5.1.4. It is proved similarly as Theorem 5.1.5 by using the induction. We prove only case $m = 3$. By differentiating the equation (5.1.16), we have

$$-\Phi''' + \left(\omega - \frac{c^2}{4}\right)\Phi' + \frac{3}{2}c\Phi^2\Phi' - \frac{15}{16}\Phi^4\Phi' = 0. \tag{5.4.41}$$

By Proposition 5.4.6, we note that

$$\sup_{L_0<L<\infty} \|f_L\|_{W^{1,\infty}([-L,L])} \leq C = C(\|f\|_{W^{1,\infty}(\mathbb{R})}). \tag{5.4.42}$$

By (5.4.42) and Theorem 5.4.2, we have

$$\begin{aligned}\|f_L^2 f_L' - f^2 f'\|_{L^2([-L,L])} &\leq \|f_L'\|_{L^\infty([-L,L])}\|f_L^2 - f^2\|_{L^2([-L,L])} \\ &\quad + \|f^2\|_{L^\infty([-L,L])}\|f_L' - f'\|_{L^2([-L,L])} \\ &\leq C\|f_L - f\|_{H^1([-L,L])} \underset{L\to\infty}{\longrightarrow} 0.\end{aligned}$$

Similarly, we have

$$\lim_{L\to\infty} \|f_L^4 f_L' - f^4 f'\|_{L^2([-L,L])} = 0.$$

By using the equation (5.4.41), we deduce that

$$\lim_{L\to\infty} \|\partial^3\Phi_{\omega,c}^L - \partial^3\Phi_{\omega,c}\|_{L^2([-L,L])}^2 = 0.$$

This completes the proof. □

Bibliography

[1] D.M. Ambrose, G. Simpson, Local existence theory for derivative nonlinear Schrödinger equations with non-integer power nonlinearities, SIAM J. Math. Anal. **47** (2015), 2241–2264.

[2] J. Angulo, Nonlinear stability of periodic traveling wave solutions to the Schrödinger and the modified Korteweg-de Vries equations, J. Differential Equations **235** (2007), 1–30.

[3] J. Angulo, *Nonlinear Dispersive Equations: Existence and Stability of Solitary and Periodic Travelling Wave Solutions*, Math. Surveys Monogr. **156**, Amer. Math. Soc., Providence, 2009.

[4] J. Angulo, J.L. Bona, M. Scialom, Stability of cnoidal waves, Adv. Differential Equations **11**, (2006), 1321–1374.

[5] J. Angulo, F. Natali, Positivity properties of the Fourier transform and the stability of periodic travelling-wave solutions, SIAM J. Math. Anal. **40** (2008), 1123–1151.

[6] J. Angulo, F. Natali, Stability and instability of periodic travelling wave solutions for the critical Korteweg-de Vries and nonlinear Schrödinger equations, Phys. D **238** (2009), 603–621.

[7] J. Bellazzini, R.L. Frank, N. Visciglia, Maximizers for Gagliardo–Nirenberg inequalities and related non-local problems, Math. Ann. **360** (2014), 653–673.

[8] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Rational Mech. Anal. **82** (1983), 313–345.

[9] H. Biagioni, F. Linares, Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations, Trans. Amer. Math. Soc. **353** (2001), 3649–3659.

[10] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations. I. Schrödinger equations, Geom. Fund. Anal. **3** (1993), 107–156.

[11] H. Brézis, E.H. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[12] P.F. Byrd, M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, 2nd edition, Springer-Verlag, New York, 1971.

[13] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Math. vol.**10**, Amer. Math. Soc., 2003.

[14] T. Cazenave, F.B. Weissler, Rapidly decaying solutions of the nonlinear Schrödinger equation, Comm. Math. Phys. **147** (1992), 75–100.

[15] Y. Cher, G. Simpson, C. Sulem, Local structure of singular profiles for a derivative nonlinear Schrödinger equation, SIAM J. Appl. Dyn. Syst. **16** (2016), 514–545.

[16] M. Christ, Illposedness of a Schrödinger equation with derivative nonlinearity, preprint.

[17] M. Colin, M. Ohta, Stability of solitary waves for derivative nonlinear Schrödinger equation, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), 753–764.

[18] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Global well-posedness for Schrödinger equations with derivative, SIAM J. Math. Anal. **33** (2001), 649–669.

[19] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, A refined global well-posedness for Schrödinger equations with derivative, SIAM J. Math. Anal. **34** (2002), 64–86.

[20] N. Fukaya, Instability of solitary waves for a generalized derivative nonlinear Schrödinger equation in a borderline case, Kodai Math. J. **40** 2017, 450–467.

[21] T. Gallay, M. Hărăguş, Stability of small periodic waves for the nonlinear Schrodinger equation, J. Differential Equations **234** (2007), 544–581.

[22] T. Gallay, M. Hărăguş, Orbital stability of periodic waves for the nonlinear Schrödinger equation, J. Dynam. Differential Equations **19** (2007), 825–865.

[23] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*, 7th edition, Elsevier/Academic Press, Amsterdam, 2007.

[24] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry. I, J. Funct. Anal. **74** (1987), 160–197.

[25] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry. II, J. Funct. Anal. **94** (1990), 308–348.

[26] A. Grünrock, S. Herr, Low regularity local well-posedness of the derivative nonlinear Schrödinger equation with periodic initial data, SIAM J. Math. Anal. **39** (2008), 1890–1920.

[27] B. Guo, Y. Wu, Orbital stability of solitary waves for the nonlinear derivative Schrödinger equation, J. Differential Equations **123** (1995), 35–55.

[28] Z. Guo, Y. Wu, Global well-posedness for the derivative nonlinear Schrödinger equation in $H^{\frac{1}{2}}(\mathbb{R})$, Discrete Contin. Dyn. Syst. **37** (2017), 257–264.

[29] S. Gustafson, S. Le Coz, T.-P. Tsai, Stability of periodic waves of 1D cubic nonlinear Schrödinger equations, preprint, arXiv:1606.04215.

[30] C. Hao, Well-posedness for one-dimensional derivative nonlinear Schrödinger equations, Commun. Pure Appl. Anal. **6** (2007), 997–1021.

[31] N. Hayashi, The initial value problem for the derivative nonlinear Schrödinger equation in the energy space, Nonlinear Anal. **20** (1993), 823–833.

[32] N. Hayashi, T. Ozawa, On the derivative nonlinear Schrödinger equation, Phys. D. **55** (1992), 14–36.

[33] N. Hayashi, T. Ozawa, Finite energy solutions of nonlinear Schrödinger equations of derivative type, SIAM J. Math. Anal. **25** (1994), 1488–1503.

[34] N. Hayashi, T. Ozawa, Remarks on nonlinear Schrödinger equations in one space dimension, Differential Integral Equations **7** (1994), 453–461.

[35] S. Herr, On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition, Int. Math. Res. Not. IMRN (2006), 1–33.

[36] K. Imamura, Stability and bifurcation of periodic traveling waves in a derivative non-linear Schrödingier equation, Hiroshima Math. J., **40** (2010), 185–203.

[37] R. Jenkins, J. Liu, P. Perry, C. Sulem, Global existence for the derivative nonlinear Schrödinger equation with arbitrary spectral singularities, preprint, arXiv:1804.01506.

[38] V.I. Judovič, Non-stationary flows of an ideal incompressible fluid, Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 1032–1066.

[39] T. Kato, On nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Phys. Thèor. **46** (1987), 113–129.

[40] D. J. Kaup, A. C. Newell, An exact solution for a derivative nonlinear Schrodinger equation, J. Math. Phys., **9** (1978), 789–801.

[41] S. Kwon, Y. Wu, Orbital stability of solitary waves for derivative nonlinear Schrödinger equation, J. Anal. Math. **135** (2018), 473–486.

[42] D.F. Lawden, *Elliptic Functions and Applications*, Applied Mathematics Series, vol. **80**, Springer, New York, 1989.

[43] E.H. Lieb, On the lowest eigenvalue of the Laplacian for the intersection of two domains, Invent. Math. **74** (1983), 441–448.

[44] X. Liu, G. Simpson, C. Sulem, Focusing singularity in a derivative nonlinear Schrödinger equation, Phys. D: Nonlinear Phenomena **262** (2013), 48–58.

[45] X. Liu, G. Simpson, C. Sulem, Stability of solitary waves for a generalized derivative nonlinear Schrödinger equation, J. Nonlinear Sci. **23** (2013), 557–583.

[46] Y. Martel, F. Merle, Blow up in finite time and dynamics of blow up solutions for the L^2-critical generalized KdV equation, J. Amer. Math. Soc. **15** (2002), 617–664.

[47] F. Merle, Existence of blow-up solutions in the energy space for the critical generalized KdV equation, J. Amer. Math. Soc. **14** (2001), 555–578.

[48] K. Mio, T. Ogino, K. Minami, S. Takeda, Modified nonlinear Schrödinger equation for Alfvén Waves propagating along magneticfield in cold plasma, J. Phys. Soc. **41** (1976), 265–271.

[49] E. Mjølhus, On the modulational instability of hydromagnetic waves parallel to the magnetic field, J. Plasma Phys. **16** (1976), 321–334.

[50] J. Moses, B. Malomed, F. Wise, Self-steepening of ultrashort optical pulses without self-phase-modulation, Phys. Rev. A **76** (2007), 1–4.

[51] R. Mosincat, Global well-posedness of the derivative nonlinear Schrödinger equation with periodic boundary condition in $H^{\frac{1}{2}}$, J. Differential Equations **263** (2017), 4658–4722.

[52] R. Mosincat, T. Oh, A remark on global well-posedness of the derivative nonlinear Schrödinger equation on the circle, C.R. Math. Acad. Sci. Paris **353** (2015), 837–841.

[53] M. Murai, K. Sakamoto, S. Yotsutani, Representation formula for traveling waves to a derivative nonlinear Schrödinger equation with the periodic boundary condition, Discrete Contin. Dyn. Syst. Supplement 2015, AIMS, (2015), 878–900.

[54] C. Ning, M. Ohta, Y. Wu, Instability of solitary wave solutions for derivative nonlinear Schrödinger equation in endpoint case, J. Differential Equations **262** (2017), 1671–1689.

[55] T. Ogawa, A proof of Trudinger's inequality and its application to nonlinear Schrödinger equations, Nonlinear Anal. **14** (1990), 765–769.

[56] T. Ogawa, T. Ozawa, Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrödinger mixed problem, J. Math. Anal. Appl. **155** (1991), 531–540.

[57] M. Ohta, Instability of solitary waves for nonlinear Schrödinger equations of derivative type, SUT J. Math. **50** (2014), 399–415.

[58] T. Ozawa, On the nonlinear Schrödinger equations of derivative type, Indiana Univ. Math. J. **45** (1996), 137–163.

[59] T. Ozawa, Finite energy solutions for Schrödinger equations with quadratic nonlinearity in one space dimension, Funkcial. Ekvac. **41** (1998), 451–468.

[60] L.E. Payne, D.H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, Israel J. Math. **22** (1975), 273–303.

[61] D.E. Pelinovsky, A. Saalmann, Y. Shimabukuro, The derivative NLS equation: global existence with solitons, Dyn. Partial Differ. Equ. **14** (2017), 271–294.

[62] D.E. Pelinovsky, Y. Shimabukuro, Existence of global solutions to the derivative NLS equation with the inverse scattering transform method, Int. Math. Res. Not. (2017), 1–66.

[63] G.N. Santos, Existence and uniqueness of solution for a generalized nonlinear derivative Schrödinger equation, J. Differential Equations **259** (2015), 2030–2060.

[64] J. Shatah, Stable standing waves of nonlinear Klein-Gordon equations, Comm. Math. Phys. **91** (1983), 313–327.

[65] C. Sulem, P.-L. Sulem, *The Nonlinear Schrödinger Equation: Self-focusing and Wave Collapse*, Applied Mathematical Sciences vol. **139**, Springer, 1999.

[66] H. Takaoka, Well-posedness for the one dimensional Schrödinger equation with the derivative nonlinearity, Adv. Diff. Eq. **4** (1999), 561–580.

[67] S.B. Tan, Blow-up solutions for mixed nonlinear Schrödinger equations, Acta Math. Sin. **20** (2004), 115–124.

[68] M. Tsutsumi, I. Fukuda, On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness, Funkcial. Ekvac. **23** (1980), 259–277.

[69] M. Tsutsumi, I. Fukuda, On solutions of the derivative nonlinear Schrödinger equation. II, Funkcial. Ekvac. **24** (1981), 85–94.

[70] M.V. Vladimirov, On the solvability of mixed problem for a nonlinear equation of Schrödinger type, Dokl. Akad. Nauk SSSR **275** (1984), 780–783.

[71] M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. **87** (1982), 567–576.

[72] Y. Wu, Global well-posedness for the nonlinear Schrödinger equation with derivative in energy space, Anal. PDE **6** (2013), 1989–2002.

[73] Y. Wu, Global well-posedness on the derivative nonlinear Schrödinger equation, Anal. PDE **8** (2015), 1101–1112.

List of Papers

[1] M. Hayashi and T. Ozawa, “Well-posedness for a generalized derivative nonlinear Schrödinger equation”, Journal of Differential Equations, vol.261, no.10, pp.5424–5445, 2016.

[2] M. Hayashi and T. Ozawa, “On Landau-Kolmogorov inequalities for dissipative operators”, Proceedings of the American Mathematical Society, vol.145, no.2, pp.847–852, 2017.

[3] N. Fukaya, M. Hayashi and T. Inui, “A sufficient condition for global existence of solutions to a generalized derivative nonlinear Schrödinger equation”, Analysis & PDE, vol.10, no.5, pp.1149–1167, 2017.

[4] M. Hayashi, “A note on the nonlinear Schrödinger equation in a general domain”, Nonlinear Analysis, vol.173, pp.99–122, 2018.

[5] M. Hayashi, “Long-period limit of exact periodic traveling wave solutions for the derivative nonlinear Schrödinger equation”, Annales de l’Institut Henri Poincaré / Analyse Non Linéaire, 2019, https://doi.org/10.1016/j.anihpc.2018.12.003