Waseda University Doctoral Dissertation

Study on Infinite Pure Jump Levy Process under Fuzzy Environment in Asset Pricing Applications

Huiming ZHANG

Graduate School of Information, Production and Systems
Waseda University
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Abstract

The stochastic process is an essential part of modern probability theory; it has been widely used in fields such as physics, biology, chemistry, computer science, communication engineering, management engineering and other scientific engineering fields. It has played a pivotal role in describing random phenomena widely used in various scientific and engineering fields. Among those stochastic processes, the Levy process has been widely and rapidly employed in recent years, which has various mathematical properties, such as independent stationary increments, stochastic continuity, and the ability to fit the characteristics of leptokurtic and fat-tailed distributions. These properties enable to characterize fluctuations in asset prices. Therefore, the Levy process has attracted special attention in the field of asset pricing. Furthermore, as an important part of asset pricing, option pricing is a core issue in asset pricing research. The Levy process includes both a finite jump process and an infinite pure-jump process. However, the finite jump process consider only the finite big jumps in modelling the asset price fluctuations, and it cannot characterize small, high-frequency jumps well. Therefore, a more general jump process has been developed as an infinite pure-jump process. The infinite pure-jump process can be considered a substitute for the finite jump process, as it can better characterize the big jumps and high frequency small jumps simultaneously in the asset price fluctuations, such as bigger jumps that represent market shocks and high-frequency smaller jumps that represent real-time transactions. Since then, the infinite pure-jump process has been the latest research focus and has been widely used in option pricing.

Options are generally used as financial derivatives based on stocks, and play an important part of financial assets. Options give their holders the right to buy or sell underlying assets at an agreed-upon price (i.e., the exercise price). They also provide good risk management and create value in investment transactions. However, after several major financial crises, establishing a more reasonable option pricing model has become an issue of concern for both financial institutions and regulators.

It is worth mentioning that the real financial market environment is full of uncertainties that are not as ideal as the assumptions in a theoretical model. On the one hand, underlying asset yields are not normally distributed but rather exhibit skewed and leptokurtic fat-tailed characteristics in addition to a significant jump phenomenon in asset price fluctuations. These issues are widely recognized in academia. On the other hand, the parameters in the model

are taken as crisp values, but due to many subjective and objective uncertainty factors, such as information asymmetric, individual judgements, different risk preferences, and incomplete information in the real-life financial market, these parameters are often vague and cannot be expressed using crisp values (i.e., co-exist uncertainty of randomness and fuzziness simultaneously). Due to the non-normality of random variables and the jump phenomenon, the infinite pure-jump Levy process can better capture the leptokurtosis and fat-tailed characteristics of the asset yield and the big jumps and high-frequency small jumps in asset price fluctuations. At the same time, the fuzzy set theory is a powerful tool to address uncertainty, and vagueness of social environments; thus, by applying it to option pricing models with an infinite pure-jump Levy process, it can be a useful supplement to option pricing theory. Therefore, to price options more rationally, this thesis introduces the fuzzy set theory and infinite pure-jump Levy process into the option pricing model on the basis of previous studies to further enhance and enrich option pricing theory. In addition, this thesis also discusses the theoretical and practical value of the proposed models through numerical simulations and empirical analyses. The options have mainly two styles to exercise: European options (to be exercised only at the expiration date) and American options (to be exercised before or at the expiration date). Thus, the main contributions of this thesis, which are shown in **Chapter 4** and **Chapter 5**, are summarized based on these two aspects. This thesis consists of 6 chapters.

Chapter 1 introduces the research background, motivation, objective, research position, and structure of this thesis.

Chapter 2 thoroughly reviews the available literature to establish the positioning of our study, examines research on the Levy process and fuzzy set theory for option pricing and combines the Levy process and fuzzy set theory. We summarize and discuss these streams of literature from the two perspectives of European options and American options.

Chapter 3 provides necessary preliminary definitions for this thesis, including the infinite pure-jump Levy process, the Black-Scholes (BS) model, common fuzzy variables, fuzzy random variable, the extension principle and credibility measures.

In **Chapter 4**, for the European option pricing problem, on the basis of the Black-Scholes (BS) model, we make use of the fuzzy set theory to construct a European option pricing model based on the VG (variance gamma) process (which is one of widely used infinite pure-jump Levy processes) in a fuzzy environment, with drift, diffusion, and jump parameters as the trapezoidal fuzzy random variables.

Next, a Monte Carlo simulation algorithm is used to conduct numerical simulations, in which the instrumental variable method is employed to improve the convergence speed of the Monte Carlo algorithm. The numerical simulation experiments, and the empirical analysis which uses Tencent Holding (HK.0700) and its stock options data, are used to compare the pricing results of the Black-Scholes (BS) model in a crisp environment, the variance gamma (VG) process option pricing model in a crisp environment, and the variance gamma (VG) process option pricing model in a fuzzy environment.

The results indicate that the fuzzy VG process option pricing model is more reasonable; the fuzzy interval can cover the market prices of options and the prices that obtained by the crisp VG process option pricing model, moreover, the expectations using fuzzy pricing are closer to the market prices of options than the pricing results obtained by the crisp BS model, the results are more consistent with the real-life market.

According to the evaluation based on the mean absolute percentage error (MAPE), the fuzzy VG process option pricing model achieved 96.68% accuracy rate which is an improvement of 1.33% over the crisp BS model. Furthermore, the variance of the accuracy rate of the proposed fuzzy model is 56.77% of that of the crisp BS model, it is less than the crisp BS model; this shows that the proposed fuzzy model is more stable than the crisp BS model in terms of pricing accuracy rate. The results indicate that the fuzzy VG process option pricing model is feasible and its pricing results are more accurate and stable even when many reality uncertainty factors are included.

In addition, the convergence efficiency of Monte Carlo algorithm can be improved by 50% via the instrumental variable method.

In **Chapter 5**, we further extend our research on the American option pricing problem. Compared with the European option, the American option allows early exercise, which creates an optimal stopping problem; thus, the issue becomes much more complicated.

Taking into account the time-varying, jump and leverage effect (i.e. asymmetric volatility) characteristics of asset price fluctuations, we first obtain the asset return rate model through the GJR-GARCH model (Glosten, Jagannathan and Rundle-generalized autoregressive conditional heteroskedasticity model) and introduce the infinite pure-jump Levy process into the asset return rate model to improve the model's accuracy.

Then, to be more consistent with reality and include more uncertainty factors, we integrate the more generalized parabolic fuzzy variable (which can cover the triangle and trapezoid fuzzy variable) to represent asset price volatility; meanwhile, according to the American option pricing theory, we derive the optimal exercise boundary, the continuation holding region and the stopping holding region for the fuzzy American options. The optimal exercise boundary can provide reasonable investment decision making for the risk managers or investors.

Next, considering more general situations with fuzzy variables with mixed distributions, we apply fuzzy simulation technology to the widely used numerical algorithms (the binomial tree algorithm and the least squares Monte Carlo algorithm) to create fuzzy pricing numerical algorithms, such as the fuzzy binomial tree algorithm and the fuzzy least squares Monte Carlo algorithm. In particular, we apply quasi-random numbers that are produced by the Sobol sequence, and Brownian bridge method, to improve the convergence speed of the least squares Monte Carlo algorithm.

Finally, by using American options data from the Standard & Poor's 100 index, we empirically test our fuzzy pricing model and comparatively analyse the pricing effect of different widely used infinite pure-jump Levy processes (the VG (variance gamma process), NIG (normal inverse Gaussian process) and CGMY (Carr-Geman-Madan-Yor process) under fuzzy and crisp environments with different fuzzy numerical algorithms that are proposed in this chapter.

The main findings are as follows: under a fuzzy environment, the results of the option pricing are more accurate than the results under a crisp environment; the fuzzy least squares Monte Carlo algorithm yields more accurate pricing than the fuzzy binomial tree algorithm, and the pricing effects via different infinite pure-jump Levy processes indicate that the NIG and CGMY models are superior to the VG model.

The fuzzy least squares Monte Carlo-NIG-GJR-GARCH model has the best performance; According to the MAPE evaluation, the model achieved 88.39% accuracy rate which is better improvement by 10.34% than the crisp least squares Monte Carlo-NIG-GJR-GARCH model. Furthermore, the variance of the accuracy rate of the fuzzy least squares Monte Carlo-NIG-GJR-GARCH model is 22.91% of that of the crisp least squares Monte Carlo-NIG-GJR-GARCH model, it is less than the crisp model; this shows that the proposed fuzzy model is more stable than the crisp model in terms of pricing accuracy rate. The results indicate that the proposed fuzzy model is effective and its pricing results are more accurate and stable even with many reality uncertainty factors included.

In addition, the convergence efficiency of the least squares Monte Carlo algorithm can be improved by 60% via the Sobol sequence and Brownian bridge method.

Finally, **Chapter 6** summarizes the thesis and describes future works.

Citation to Our Publications

The main portion of Chapter 3 appears in the following papers:

1. Huiming Zhang, Junzo Watada, " A European Call Options Pricing Model Using the Infinite Pure Jump Levy Process in a Fuzzy Environment," *IEEJ Transactions on Electrical and Electronic Engineering, TEEE C (Electronics, Information and Systems)*, Vol.13, No.10, pp.1-15, October 2018, to be published.

2. Huiming Zhang, Junzo Watada, "Fuzzy Levy-GJR-GARCH American Option Pricing Model Based on an Infinite Pure Jump Process", *IEICE Transactions on Information and Systems*, Vol.E101-D, No.7, pp.1843-1859, July 2018.

The content of Chapter 4 is based on the following papers:

1. Huiming Zhang, Junzo Watada, " A European Call Options Pricing Model Using the Infinite Pure Jump Levy Process in a Fuzzy Environment," *IEEJ Transactions on Electrical and Electronic Engineering, TEEE C (Electronics, Information and Systems)*, Vol.13, No.10, pp.1-15, October 2018, to be published.

2. Huiming Zhang, Junzo Watada, " Building Fuzzy Variance Gamma Option Pricing Models with Jump Levy Process ," *the 9th International Knowledge-Based and Intelligent Engineering Systems Conference on Intelligent Decision Technologies 2017 (KES-IDT 17)*, Springer, Cham, 2017. pp.105-116, Vilamoura, Portugal, 21-23 June 2017.

The content of Chapter 5 is based on the following papers:

1. Huiming Zhang, Junzo Watada, " Fuzzy Levy-GJR-GARCH American Option Pricing Model Based on an Infinite Pure Jump Process," *IEICE Transactions on Information and Systems*, Vol.E101-D, No.7, pp.1843-1859, July 2018.

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Dedication

Dedicated to my beloved family

Dedicated to all my friends who always support me.

"*Even in my darkest moments, I had a sense of who I was and what I could do. More than once I failed, but I refused to give up. I knew in my heart that I was here on earth to achieve good and meaningful things. I never let any failure take that conviction away.*"

Xiaoping Xu (1956–)

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Chapter 1

Introduction

1.1 Background

The stochastic process is an essential part of modern probability theory. It is used as a theoretical tool for studying the stochastic evolution process in the world, which has been widely used in fields such as physics, biology, chemistry, computer science, communication engineering, dynamic reliability, economics, social sciences, management engineering, financial engineering and other scientific engineering fields. In the past three decades, it has pla l role in describing random phenomena widely used in various scientific and engineering fields, and its importance has continuously increased with the rapid development of science and technology. Among those stochastic processes, the Levy process has been widely and rapidly employed in recent years.

Levy process is named after Paul Levy (1886-1971), who was one of the founding fathers of the theory of stochastic processes. As the earliest scholar worked in this field, he and other important scholars, Alexsander Yakovlevich Khintchine (1894-1959), and Kiyoshi Ito (1915-2008), made a massive contribution in the development of stochastic process theory. The Levy process has various mathematical properties, such as independent stationary increments, stochastic continuity, and the ability to fit the characteristics of leptokurtic and fat-tailed distributions. These properties are enable to characterize fluctuations in asset prices. Therefore, the Levy process has attracted more special attention in the field of asset pricing. Furthermore, as an important part of asset pricing, option pricing is a core issue in asset pricing research.

The Levy process includes both of a finite jump process and an infinite pure jump process. For finite jump process, the earliest research on finite jump process was proposed by Merton in 1976, who introduced Levy process into the option pricing and established a jump-diffusion model, where an analytical solution was given to the European option pricing model. The subsequent models, such as the double exponential jump diffusion model proposed by Kou (2004) [1], the hybrid exponential jump model [2] by Ning Cai *et al.*, the exponential jump diffusion process to American option price problem by Levendorskii (2004) [3], are all belong to the finite jumps type. However, finite jump processes consider only the finte big jumps in the asset price fluctuations, they cannot characterize small, high-frequency jumps well. Therefore, the follwing scholars have developed a more general jump process, that is infinite pure jump process, it can better characterize the big jumps and high frequency small jumps simultaneously in the asset price fluctuations. F . (1990) [4] used the Gamma variable to describe the Levy process with a VG (Variance gamma) distribution; although only one parameter was added, the model fits high-order moment characteristics of asset prices well. The NIG (Normal inverse Gaussian) process proposed by Barndorff Nielsen (1997) [5] is one of the most commonly used Levy processes, and this process offers high operating efficiency and the ability to provide accurate characterisation of the tail behaviour of asset prices. The variance gamma model (VG model) [4] proposed by Madan *et al.* (1990), the hyperbolic model proposed by Prause (1999) [6], the NIG model proposed by Barndorff Nielsen (1997) [5] and the CGMY (Carr-Geman-Madan-Yor process) model [7] proposed by Carr Peter *et al.* (2002) are belong to the infinite pure jumps type. The scholars consider that the infinite pure jump process can be a substitute for the finite jump process, such as Carr et al. (2002) [7] and Daal et al. (2005) [8]. Since then, the infinitely jump process has been the latest research focus and has been widely used in the option pricing.

Options are generally used as a financial derivative based on stocks, and play an important part of financial assets. Options give their holders the right to buy or sell underlying assets at an agreed-upon price (i.e., the exercise price). They also provide good risk management and create value in investment transactions. Options trading does not only occur in stock exchanges; rather, financial institutions also engage in a large number of over-the-counter options transactions. Therefore, option pricing has always been the core of asset pricing research. However, after several major financial crises, establishing a more reasonable option pricing model has become an issue of concern for both financial institutions and regulators. With continuous development of the options market, the theory of options pricing is also improving.

It is worth mentioning that the real financial market environment is full of uncertainties that are not as ideal as the assumptions in a theoretical model. On the one hand, the underlying asset yield is not normally distributed but rather exhibits skewed and leptokurtic fat-tailed characteristics in addition to a significant jump phenomenon in the asset price fluctuation. These issues are widely recognized in academia. On the other hand, the parameters in the model are taken as crisp values, but due to many subjective and objective uncertainty factors and incomplete information in the real-life financial market, these parameters are often vague and cannot be expressed using crisp values (i.e., co-exist uncertainty of randomness and fuzziness simultaneously in the real-life financial market).

The concept of fuzzy sets was first proposed by Zadeh in 1965 [9]. Subsequently, S.Nahmias [10] devepIoped fuzzy variable, D.Dubois et.al [11] developed possbility theory, and Liu et al. established a credibility theory with an axiomatic basising in 2002 [12] and 2004 [13]. Based on many scholars' contribution, fuzzy set theory gradually becomes a strong tool to handle incomplete and uncertain situation, which also revealed a new direction for asset pricing theories. Many scholars, including Muzzioli et al. (2001) [14], Carlsson *et al.* (2003) [15], Wu (2004) [16], Thiagarajah *et al.* (2007) [17], Nowak *et al.* (2010) [18], Frank *et al.* (2013) [19], Liu *et al.* (2005) [13] and Wang *et al.* (2014) [20] have successfully applied this theory to option pricing problem.

Owing to the non-normality of random variables and the jump phenomenon of the asset price fluctuation, the infinite pure jump Levy process can better capture the leptokurtosis and fat-tailed characteristics of the assets yield and the big jumps and high-frequency small jumps existing simultaneously in the asset price fluctuation of the real market situation. At the same time, fuzzy set theory is a powerful tool employed to address the uncertainty, and vagueness of the social environment, thus applying it to options pricing models with infinite pure jump Levy process, it can be a useful supplement to the option pricing method and can provide a new theoretical basis for the pricing of options.

1.2 Motivation and Objectives

As the statement above, in asset pricing field, especially in option pricing problem, Levy processes are becoming extremely important tools because they can more accurate to describe the observed reality of financial markets than the models based on Brownian motion. In the “real” market, the asset price fluctuation have jumps or spikes (see Figure 1.1), moreover, the asset yield not a normal distribution but rather exhibits leptokertic and fat tails (see Figure 1.2), these widely recognized issues have to be taken into consideration by the risk managers or investors. Due to the finite jump Levy processes only consider the finite big jumps in the fluctuation of the asset price, ignoring the high-frequency small jumps which is also existing in the fluctuation of asset price, compared with finite jump Levy process, the infinite pure jump Levy processes are more general jump processes which can better capture the big jumps and high frequency small jumps simultaneously in the real market situation, such as bigger jumps that represent market shocks and high-frequency smaller jumps that represent real-time transactions. And in the real financial market, owing to the existence of factors such as information asymmetry, individual judgement, and different risk preferences, the financial market is an incomplete market whose incompleteness is not only randomness but also fuzziness. Thus, introducing the fuzzy set theory can effectively combine randomness and fuzziness to more closer the real market and can provide more reasonable investment decision making for the risk managers or investors. (See Figure 1.3).

Therefore, to price options more rationally, in this thesis, we introduce fuzzy set theory and the infinite pure jump Levy process into the options pricing model on the basis of previous studies to further enhance and enrich option pricing theories. In addition, this thesis also discusses the theoretical and practical values of the options pricing model in a fuzzy environment through numerical simulation and empirical analysis. The style of options mainly includes two types: European options (to be exercised only at the expiration date) and American options (to be exercised before or at the expiration date). Thus, this thesis will study from the following two aspects in options pricing problem (See Figure 1.4, 1.5),

1. **For European option pricing problem**, on the basis of the Black-Scholes (BS) model, we will first base on the VG (variance gamma) process (which is one of widely used infinite pure-jump Levy processes) to construct European option pricing model, then, to be more closer with the real financial market, we will integrate trapezoidal fuzzy random variables to represent drift, diffusion, and jump parameters of VG process.

   Upon obtaining the fuzzy VG process European option pricing model which is proposed in this thesis, the Monte Carlo simulation algorithm is used to conduct numerical simulations, in which the instrumental variable method is employed to improve the convergence speed of the Monte Carlo algorithm.

   The numerical simulation experiments, and the empirical analysis which uses Tencent Holding (HK.0700) and its stock options data, are used to compare the pricing result of the Black-Scholes (BS) model in a crisp environment, the variance gamma (VG) process options pricing model in a crisp environment, and the variance gamma (VG) process options pricing model in a fuzzy environment.

   The result of the analysis indicates that the fuzzy VG options pricing model is more reasonable, the fuzzy interval can cover the market prices of options and the prices that obtained by the crisp VG process option pricing model, and the fuzzy interval narrowing as the option exercise price increases, it is consistent with the real-life market. On the other hand, the fuzzy interval widens as the time to expiration increases. Because the introduction of more uncertainties, the option price obtained under the proposed model is higher than those of other models, it is also consistent with the real-life financial market. The option price under the model is also more sensitive to changes in the jump parameter. As the jump parameter increases, the fuzzy interval narrows. The expectation using fuzzy pricing is closer to the market prices of options than the pricing results obtained by the crisp BS model, the results are more consistent with the real-life market. In addition, the results demonstrate that the instrumental variable method can effectively improve the convergence speed of Monte Carlo algorithm.

2. **For American option pricing problem**. Unlike the European option, American option allow early exercise, there is an optimal stopping problem, thus the issue is much more complicated.

   Therefore, taking into account the characteristics of asset price fluctuation, such as time-varying, jump and leverage effect (i.e. asymmetric volaatility), first we obtain the asset return rate model through GJR-GARCH model (Glosten, Jagannathan and Rundle–Generalized autoregressive conditional heteroskedasticity model), and introduce infinite pure jump Levy process into the asset return rate model for improving the model's accuracy.

   Then, to be more consistent with reality and include more uncertainty factors, we integrate the more generalized parabolic fuzzy variable (which can cover the triangle and trapezoid fuzzy variable) to represent the asset price volatility, meanwhile according to the American option pricing theory, we derive the optimal exercise boundary, the continuation holding region and the stopping holding region for the fuzzy American options. The optimal exercise boundary can provide reasonable investment decision making for the risk managers or investors.

   Upon obtaining the fuzzy Levy-GJR-GARCH American option pricing model which is proposed in this thesis, considering more general situations with the fuzzy variables with mixed distributions, we then apply fuzzy simulation technology to the widely used numerical algorithms (the binomial tree algorithm and the least squares Monte Carlo algorithm) to create fuzzy pricing numerical algorithms, such as fuzzy binomial tree algorithm, fuzzy least squares Monte Carlo algorithm, and we especially apply quasi-random numbers that are produced by Sobol sequence, and Brownian Bridge method, to improve the convergence speed of the least squares Monte Carlo algorithm.

At last, an empirical study will be performed on our fuzzy pricing model using American options data from the Standard & Poor's 100 index, and we comparatively analysed the pricing effect of different widely used infinite pure-jump Levy processes (the VG (variance gamma process), NIG (normal inverse Gaussian process) and CGMY (Carr-Geman-Madan-Yor process)) under fuzzy and crisp environments with different fuzzy numerical algorithms that are proposed in this thesis.

The findings are as follows: under a fuzzy environment, the result of the option pricing is more accurate than the result under a crisp environment; the pricing results of short-term options have higher accuracy than those for medium- and long-term options; the fuzzy least squares Monte Carlo algorithm yields more accurate pricing than the fuzzy binomial tree algorithm, and the pricing effects via different infinite pure-jump Levy processes indicate that the NIG and CGMY models are superior to the VG model. The fuzzy least squares Monte Carlo-NIG-GJR-GARCH model has the best performance. Moreover, the option price increases as the time to expiration of options is extended and the exercise price increases, the membership function curve is asymmetric with an inclined left tendency, and the fuzzy interval narrows as the level set α and the exponent of membership function n increase. In addition, the results demonstrate that the Sobol sequence and Brownian Bridge method can effectively improve the convergence speed of the least squares Monte Carlo algorithm.

1.3 Position of this study

Owing to the style of options mainly includes two types: European option and American option. Thus, the main works of this thesis are summarized based on these two aspects. We show our research position and main works in Fig. 1.6, 1.7.

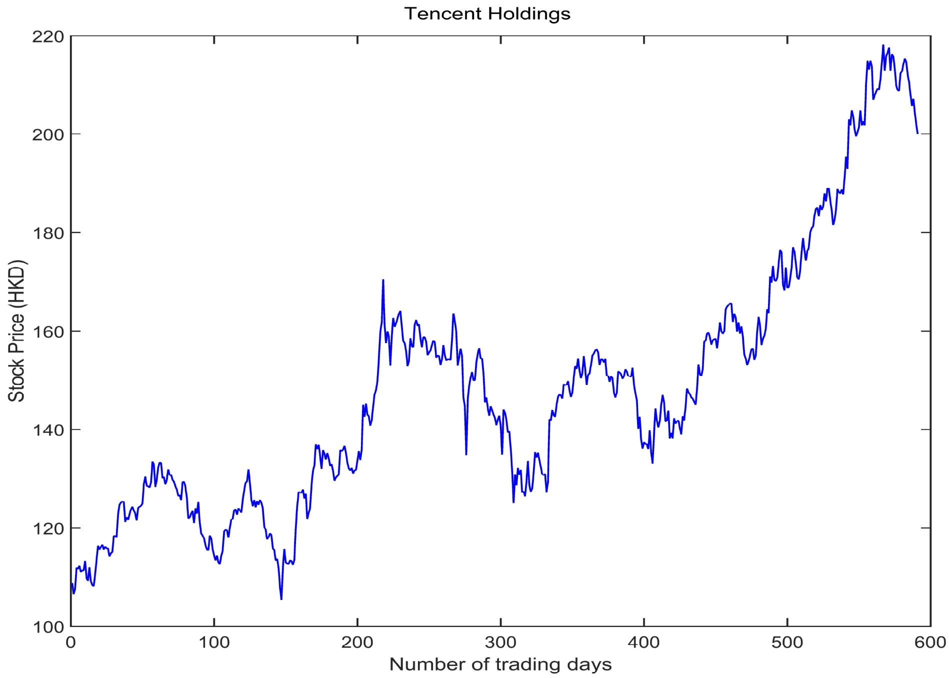


Figure 1.1: Tencent Holding (HK.0700) stock closing price, May 2014 - Nov. 2016

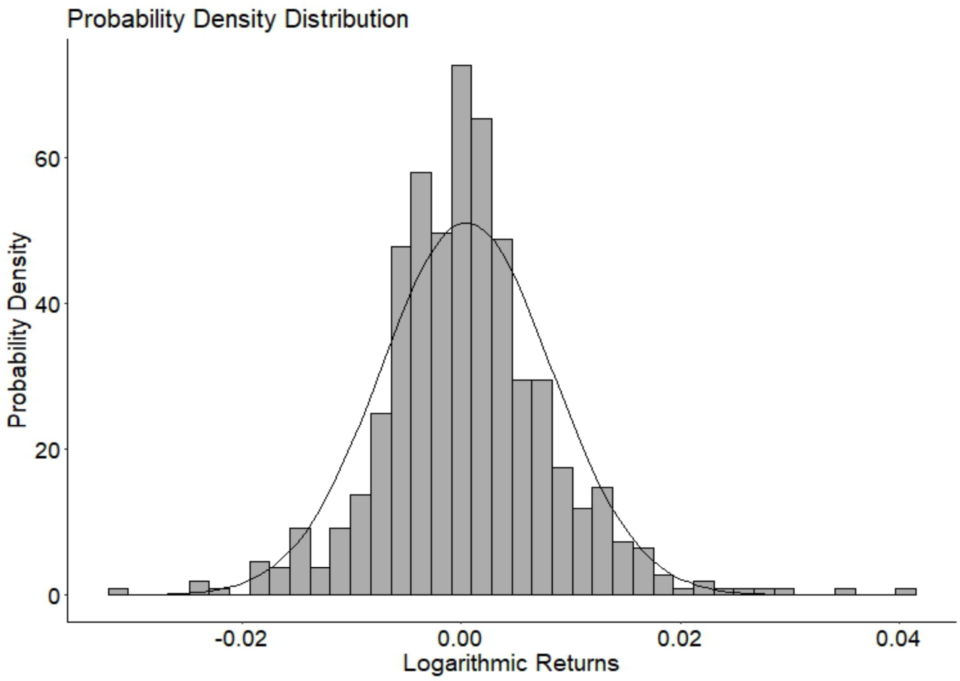


Figure 1.2: Probability Density Function of Daily Returns of Tencent Holding (HK.0700)

1.4 Structure of This Thesis

The rest of this thesis is organized as follows:

In **Chapter 2** thoroughly reviews the available literature to establish the positioning of

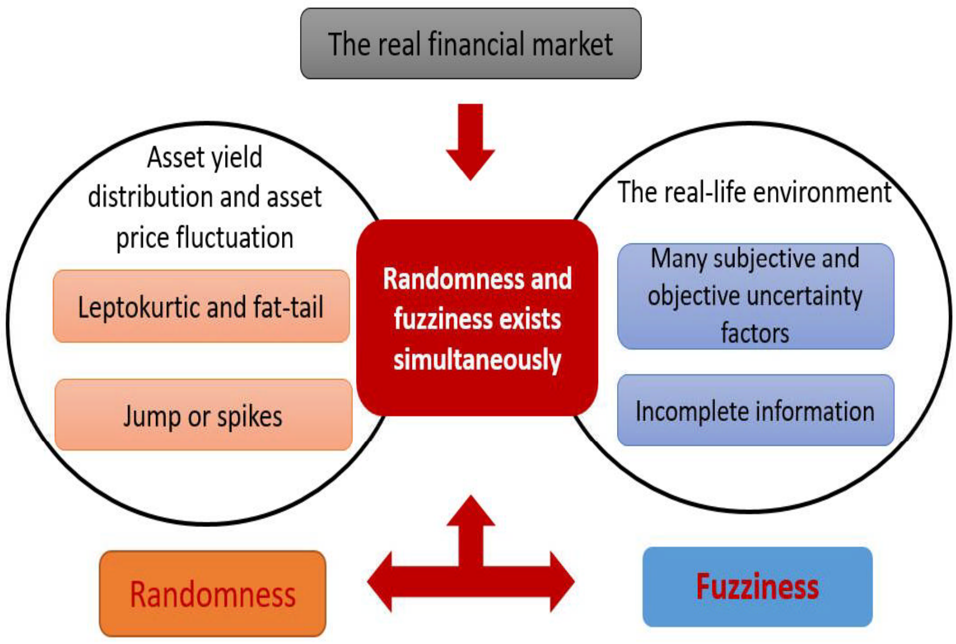


Figure 1.3: The uncertainties of the real financial market environment

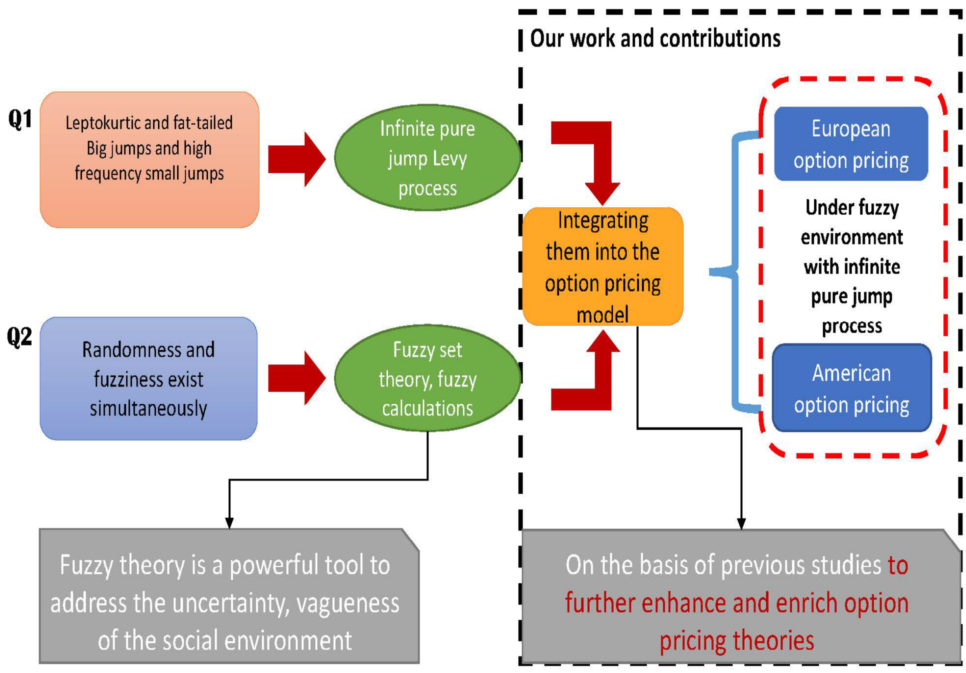


Figure 1.4: Rearch objectives

our study, examines research on the Levy process and fuzzy set theory for option pricing and combines the Levy process and fuzzy set theory. We summarize and discuss these streams of literature from the two perspectives of European options and American options.

In **Chapter 3** provides necessary preliminary definitions for this thesis, including the infinite

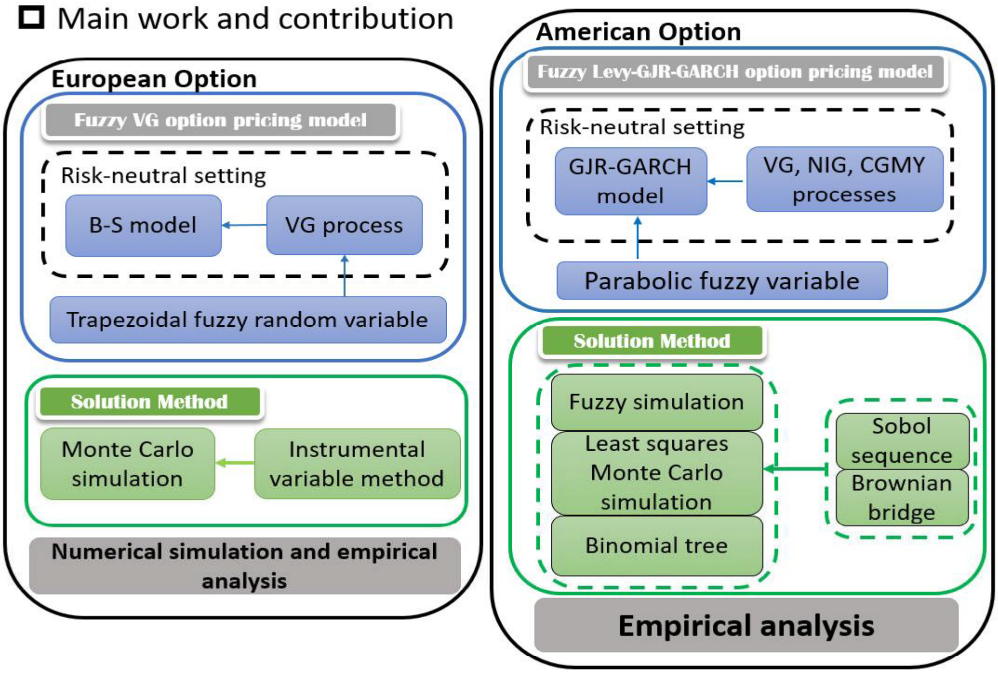


Figure 1.5: Main work and contribution

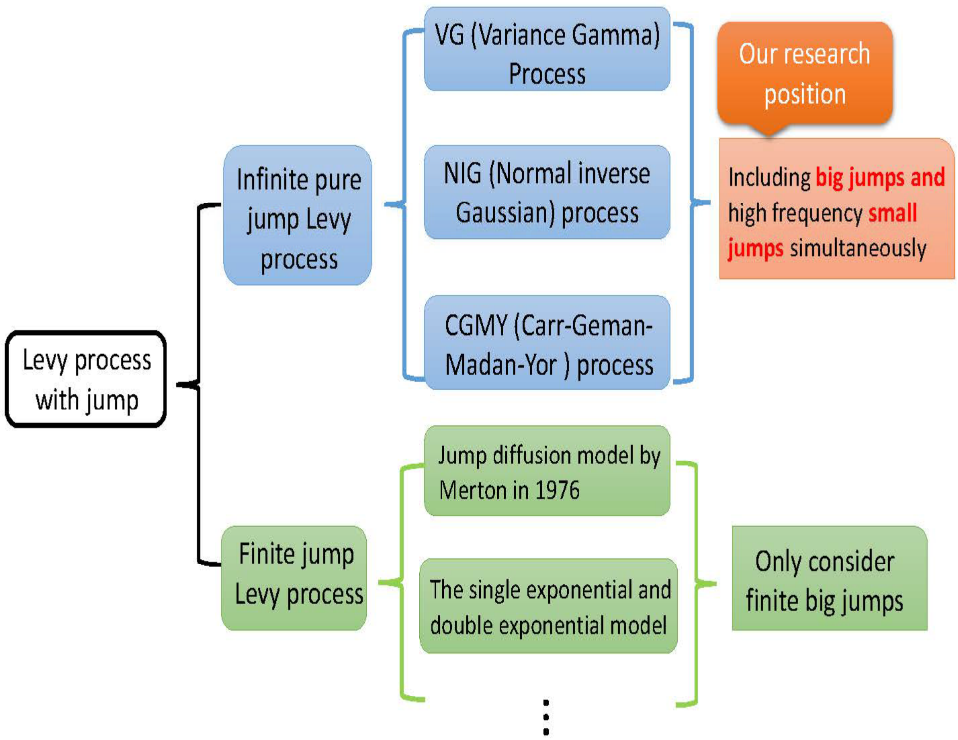


Figure 1.6: Research position

pure-jump Levy process, the Black-Scholes (BS) model, common fuzzy variables, fuzzy random variables, the extension principle and credibility measures.

From **Chapter 4** to **Chapter 5**, we describe the main achievements and contributions of

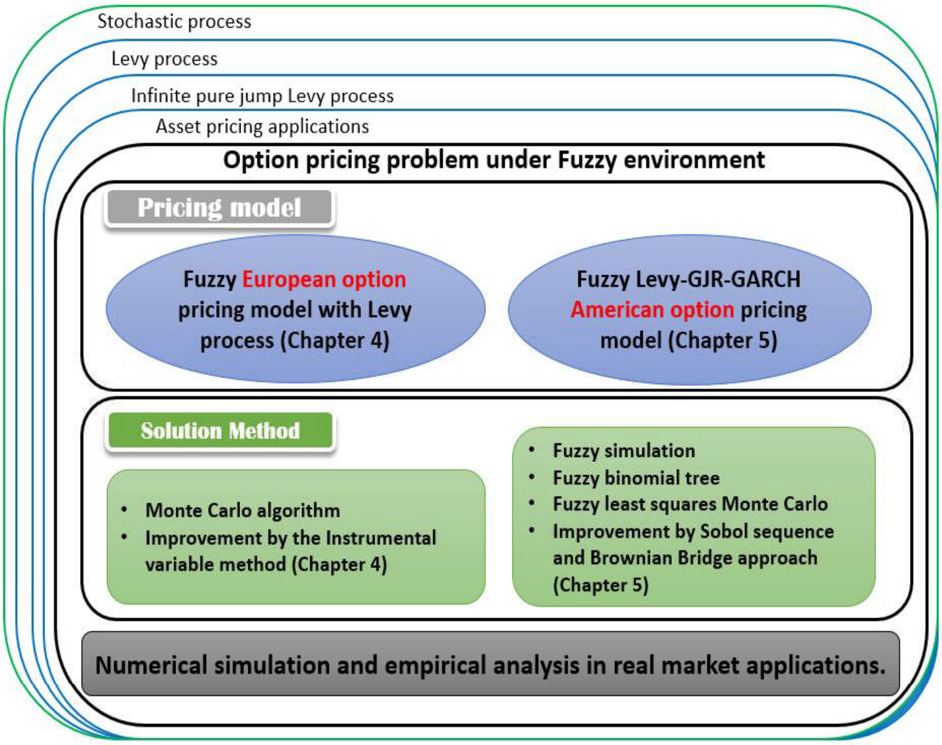


Figure 1.7: Research position and main work.

this thesis, specifically as follows,

In **Chapter 4**, for the European option pricing problem, on the basis of the Black-Scholes (BS) model, we make use of the fuzzy set theory to construct a European option pricing model based on the VG (variance gamma) process (which is one of widely used infinite pure-jump Levy processes) in a fuzzy environment, with drift, diffusion, and jump parameters as the trapezoidal fuzzy random variables.

Following this, the Monte Carlo simulation algorithm is used to conduct numerical simulations, in which the instrumental variable method is employed to improve the convergence speed of the Monte Carlo algorithm.

The numerical simulation experiments, and the empirical analysis which uses Tencent Holding (HK.0700) and its stock options data, are used to compare the pricing results of the Black-Scholes (BS) model in a crisp environment, the variance gamma (VG) option pricing model in a crisp environment, and the variance gamma (VG) option pricing model in a fuzzy environment.

The results indicate that the fuzzy VG option pricing model is more reasonable; the fuzzy interval can cover the market prices of options and the prices that obtained by the crisp VG process option pricing model, moreover, the expectations using fuzzy pricing are closer to the market prices of options than the pricing results obtained by the crisp BS (Black-Scholes) model, the results are more consistent with the real-life market. In addition, the instrumental variable method can effectively improve the convergence speed of Monte Carlo algorithm.

In **Chapter 5**, we further extend our research on the American option pricing problem. Due to the American option allow early exercise, which creates an optimal stopping problem; thus, the issue becomes much more complicated.

Taking into account the time-varying, jump and leverage effect (i.e. asymmetric volatility) characteristics of the asset price fluctuation, we first obtain the asset return rate model through the GJR-GARCH model (Glosten, Jagannathan and Rundle-generalized autoregressive conditional heteroskedasticity model) and introduce the infinite pure-jump Levy process into the asset return rate model to improve the model's accuracy.

Then, to be more consistent with reality and include more uncertainty factors, we integrate the more generalized parabolic fuzzy variable (which can cover the triangle and trapezoid fuzzy variable) to represent the asset price volatility; meanwhile, according to the American option pricing theory, we derive the optimal exercise boundary, the continuation holding region and the stopping holding region for the fuzzy American options. The optimal exercise boundary can provide reasonable investment decision making for the risk managers or investors.

Following this, considering more general situations with the fuzzy variables with mixed distributions, we apply fuzzy simulation technology to the widely used numerical algorithms (the binomial tree algorithm and the least squares Monte Carlo algorithm) to create fuzzy pricing numerical algorithms, such as the fuzzy binomial tree algorithm and fuzzy least squares Monte Carlo algorithm, and we particularly applied quasi-random numbers that are produced by the Sobol sequence, and Brownian bridge method, to improve the convergence speed of the least squares Monte Carlo algorithm.

Finally, by using American options data from the Standard & Poors 100 index, we empirically test our fuzzy pricing model and comparatively analyse the pricing effect of different widely used infinite pure-jump Levy processes (the VG (variance gamma process), NIG (normal inverse Gaussian process) and CGMY (Carr-Geman-Madan-Yor process)) under fuzzy and crisp environments with different fuzzy numerical algorithms that are proposed in this chapter.

The findings are as follows: under a fuzzy environment, the results of the option pricing are more accurate than the results under a crisp environment; the pricing results for short-term options have higher accuracy than those for medium- and long-term options; the fuzzy least squares Monte Carlo algorithm yields more accurate pricing than the fuzzy binomial tree algorithm, and the pricing effects via different infinite pure-jump Levy processes indicate that the NIG and CGMY models are superior to the VG model. The fuzzy least squares Monte Carlo-NIG-GJR-GARCH model has the best performance. In addition, the Sobol sequence and Brownian Bridge method can effectively improve the convergence speed of the least squares Monte Carlo algorithm.

Finally, **Chapter 6** draws conclusions and describes future work based on this thesis.

Chapter 2

Current Research State

In this chapter, we will thoroughly review the available literature to establish the positioning of our studies, examines research on Levy process and fuzzy set theory for option pricing and combines the Levy process and fuzzy set theory. We will summarize and discuss these streams of literature from the two perspectives of European option and American option.

2.1 Literature Review and Remark of European option price model

The Levy process is a stochastic process with good mathematical properties, such as independent stationary increments, stochastic continuity. These properties mean that the process can have plenty of applications. Coupled with its ability to fit leptokurtosis and fat-tailed characteristics, this makes the process play a pivotal role in options pricing models. As early as 1976, Merton introduced the Levy process into options pricing and established the jump-diffusion model, which provides analytic solutions for the European options pricing model [21]. Following this, research works based on the stochastic process with jump can be further divided into two main types: a series of models that measure finite jumps and the other that measures infinite pure jumps. The finite jump series of models include the double exponential jump diffusion model proposed by Kou (2004) [1] and the hybrid exponential jump model [2] by NingCai *et al.*, whereas the infinite pure jump series of models include the variance gamma model (VG model) [4] proposed by Madan *et al.* (1990), the hyperbolic model proposed by Prause (1999) [6], and the CGMY model [7] proposed by Carr Peter *et al.* (2002). In recent years, options pricing based on jump-type Levy processes has also garnered much attention. For example, Song *et al.* (2011) proposed a non-linear regression method to approximate the options pricing problem of pure jump Levy processes [22]. On the other hand, Bollerslev *et al.* (2013) analysed the heterogeneity of stock volatility and found the moment characteristic of the data to be non-normal and that a significant jump tail phenomenon exists. This demonstrated that the usage of normal models to characterise real data is subject to rather significant limitations [23]. Yuji Umezawa *et al.* (2015) assumed that the underlying asset price obeys a time-varying Levy process and studied the pricing of path-dependent discrete derivatives. As a result, they obtained a multivariate characteristic function with a backward recursive relation and semi-analytical pricing formula for discrete options such as the lookback option, the barrier option, and the geometric asian option. [24].

The above-mentioned studies are based on the framework of stochastic processes with jump that reveal the leptokurtosis and fat-tailed characteristics of the underlying asset. Since Zadeh has developed fuzzy set theory, it has been widely used in the field of financial research. Muzzioli et al. (2001) introduced fuzzy numbers into the conventional binomial options pricing model and deduced the fuzzy binomial options pricing formula [14]; Carlsson *et al.* (2003) demonstrated how to price real options in a fuzzy environment and developed a new fuzzy real options pricing model [15]. Wu (2004) designated the parameters of stock price, risk-free rate and volatility in the European options pricing model as fuzzy numbers and applyed sensitivity analysis to find the fuzzy interval of options price in a fuzzy environment to obtain the membership degree of the fuzzy option [16]; Thiagarajah *et al.* (2007) treated the parameters in the options pricing model as adaptive fuzzy numbers and performed numerical experiments using the model [17]. With the continuous development of the measurement model, options pricing models on the basis of fuzzy set therory have also been developed. For example, Nowak *et al.* (2010) used the minimal entropy martingale measure to calculate the fuzzy pricing formula of European call options in a fuzzy environment and used Monte Carlo simulations to conduct numerical experiments [18]; Thavaneswaran *et al.* (2013) f lue of the stock price before multiplying by the original payoff function to obtain the binary option payoff function before incorporating the binary options pricing formula in a fuzzy environment [19]. Liu *et al.* (2005) systematically studied the theory of credibility and fuzzy simulation technology. This provided a good foundation for the application of fuzzy set theory in financial markets [13]. Wang *et al.* (2014) considered both the risk and fuzziness of financial markets before setting the interest rate and volatility as fuzzy numbers to conduct a pricing analysis on the Geske compound option to obtain the fuzzy mean of the option price. Following this, numerical analysis was performed on the fuzzy pricing [20].

Because the Levy process can capture the jump characteristics of the underlying assets well and the pricing model assumptions in a fuzzy environment are more aligned with reality, some scholars have applied both the Levy process and fuzzy set theory to options pricing models. One of the earliest studies performed was by Xu *et al.* in 2009. In the study, fuzzy set theory was applied to the jump diffusion model, and the validity of the model was demonstrated through numerical experiments [25]. Romaniuk *et al.* (2010) proposed the application of stochastic analysis and fuzzy set theory on the basis of options pricing methods. Numerical experiments were performed by means of the Monte Carlo simulation method, and the pricing formula of European call options was analysed in detail [26]. Zhang *et al.* (2012) treated parameters such as the risk-free interest rate, drift rate and jump intensity as fuzzy numbers and studied the double exponential jump diffusion model pricing formula of European options in a fuzzy environment, explaining the rationality of this method [27]. Nowak *et al.* (2014) introduced fuzzy set theory and the geometric Levy process into the European options pricing model and performed numerical experiments [28]. Feng *et al.* (2015) studied the problem of pricing European call options using a time-varying Levy process in a fuzzy environment. Through an empirical study of data from the S&P 500 index, they found the model to be a better fit than using a time-varying Levy process and the B-S model [29]. Table 2.1 and 2.2 summarize the research progress of the European options pricing model based on the Levy process and fuzzy parameters.

2.1.1 Remark of European option price model

In summary, even though scholars have applied fuzzy set theory and the Levy process to perform plenty of theoretical and empirical research about options pricing, these studies mostly focus only on one aspect, whereas studies that consider both aspects of fuzzy set theory and Levy process tend to use finite jump process conditions. Compared with the the finite jump process, the infinite pure jump Levy process can better describe the characteristics of the market such as bigger jumps to represent market shocks, whereas smaller jumps represent real-time transactions. Therefore, to address the shortcomings of existing research, this thesis considers both fuzzy set theory and an infinite pure jump condition and establishes an infinite pure jump European options pricing model in a fuzzy environment.

2.2 Literature Review and Remark of American option price model

Jin-Chuan Duan (1995) [30] was the first to apply the GARCH model (Generalized autoregressive conditional heteroskedasticity model) in European option pricing theory. He also performed a comparative analysis of pricing results obtained using the B-S model under risk conditions. His findings demonstrated that the GARCH model is effective in reducing the systematic error in pricing. Saez Marc (1997) [31] studied the effect of stochastic volatility on option pricing using symmetric and asymmetric GARCH models of Spanish options. He discovered that the IEGARCH (1,2)-M-S model is the most effective model for capturing the stochastic volatility of the return rate of IBEX-35 stocks. Lars Stentoft (2012) [32] applied a GARCH model to the pricing of American options; he used a Monte Carlo approach to form simulation analysis. The results indicate that asset prices with GARCH effect can better reflect the actual conditions. Therefore, we can conclude that by incorporating a GARCH model in an option pricing model, we can improve the pricing accuracy. In addition, many researchers have incorporated Levy processes in option pricing model and studied the non-normality and jumping characteristics of the underlying asset. Levy processes include finite jumps and infinite pure jumps, the ear-

Table 2.1: Research progress of European options pricing model based on the Levy process and fuzzy parameters (1)

Usage method	Formula	Objective
Jump Diffusion Model [21]	$\mathrm{d}P^*/P^* = (\alpha_P^* - \lambda k_P^*)dt + dq_P^*$	Allows jump process in the rate of return, a more general option pricing model that does not depend on investor preferences to be deduced.
Double Exponential Jump Diffusion Model [1]	$\frac{dS(t)}{S(t-)} = (r - \lambda^* \zeta^*)dt + \sigma dW^*(t) + d\left(\sum_{i=1}^{N^*(t)} V_i^* - 1\right)$	Uses the jump model in place of the BS model and applied to American options and path-dependent options pricing to obtain the analytical solution of options pricing.
Hybrid Double Exponential Jump Diffusion Model [2]	$f_Y(y) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i y} 1_{\{y \geq 0\}} + \sum_{j=1}^{n} q_j \theta_j e^{-\theta_j y} 1_{\{y<0\}}$	A flexible, hyper-exponential jump diffusion model based on Laplace transform is proposed, and the non-singularity of the correlated high-dimensional matrices is proved.
Variance Gamma Model (VG Model) [4]	$f(x) = \int_0^\infty \left[e^{-x^2/(2\sigma^2 v)}/(\sigma\sqrt{2\pi v})\right] g(v)dv$	Assumed that the variance of the yield follows the gamma distribution to establish an option pricing model for the pure jump process.
Hyperbolic model [6]	$dP^\vartheta = \exp(\vartheta X_t - t\log M(\vartheta))dP$	Makes use of the hyperbolic Levy process and the inverse Gaussian process to set up an option pricing model with practical applications.
CGMY model [7]	$k_{CGMY}(x) = \begin{cases} C\frac{\exp(-G\|x\|)}{\|x\|^{1+Y}} & \text{for } x < 0 \\ C\frac{\exp(-M\|x\|)}{\|x\|^{1+Y}} & \text{for } x > 0 \end{cases}$	Includes both finite and infinite jumps to more vividly describe financial assets price changes.
Non-linear regression asymptotic pure jump Levy process [22]	$\log S_t^\delta = \log S_0 + \sum_{i=1}^{N_t^\delta} G_i^\delta$	An option pricing method based on the non-linear regression asymptotic of the pure jump Levy process that targets the complexity of option pricing model under the Levy process.
A in-fill asymptotics method to estimate the jump tail parameters of the Levy process [23]	$CV_t^{(j)} = \sum_{i=tn+1}^{tn+n} \|\Delta_i^n p^{(j)}\|^2 1\left(\|\Delta_i^n p^{(j)}\| \leq \alpha_i^{(j)} n^{-\varpi}\right)$	Made use of the in-fill asymptotics method to analyse the dependencies of the jump tail decay parameters and jump parameters in the pricing model of assets with jumps.
Discrete Derivative Pricing Model Based on the Time-varying Levy Process [24]	$G_t := \ln\frac{S_t}{S_0} = (r-q)t + \psi_Y(-i)\tau_t + X_t$	Obtained the multivariate characteristic function of intertemporal joint distribution and obtained the semi–analytical pricing formula of discrete options.
Fuzzy Binomial Option Pricing Model [14]	$P_0^C = \frac{a_3(1-\alpha) - K + a_2\alpha}{(1-\alpha)(a_3 - a_1)} \cdot \left[\frac{P_0^R(1+r) - a_1 - \alpha(a_2 - a_1)}{(1+r)}\right]$	The binomial option pricing model is combined with fuzzy numbers. It has certain advantages and can provide different degrees of information to the market.
Fuzzy Real Option Pricing Model [15]	$FROV = \widetilde{S}_0 e^{-\delta T} N(d_1) - \widetilde{X} e^{-rT} N(d_2)$	The possibilistic mean and variance are set as fuzzy numbers to establish the fuzzy real option pricing model to help determine the best investment opportunity.

Table 2.2: Research progress of European options pricing model based on the Levy process and fuzzy parameters (2)

BS Model in a Fuzzy Environment [16]	$\tilde{c}(\tilde{s},t,K,\tilde{r},\tilde{\sigma}) = (\tilde{s} \otimes \widetilde{N}(\widetilde{d_1})) - (\tilde{1}_{\{K\}} \otimes e^{-\tilde{r}\otimes\tilde{1}_{\{t\}}} \otimes \widetilde{N}(\widetilde{d_2}))$	The interest rate, volatility and stock price are set to fuzzy numbers, and the assumptions on the parameters in the BS model are relaxed.
BS Model Based on Quadratic Adaptive Fuzzy Numbers [17]	$\widehat{FCOV} = \widetilde{S}_0 e^{-\delta\tau} N(d_1) - \widetilde{X} e^{-\tilde{r}\tau} N(d_2)$	Quadratic adaptive fuzzy numbers are applied to the BS model for options pricing.
Pricing European Call Option Based on Fuzzy Theory and Stochastic Analysis [18]	$C_0 = e^{-\kappa_1 T} \sum_{n=0}^{\infty} \frac{(\kappa_1 T)^n}{n!} \Big(S_0 e^{(u_1 - r)T + \frac{\sigma^2 T}{2} + kn} \Phi(d_+^n) - e^{-rT} K \Phi(d_-^n) \Big)$	Considering the jump of the underlying assets and the vagueness of the parameters, an option pricing method based on fuzzy theory and stochastic analysis is established.
Binary Option Pricing Model in a Fuzzy Environment [19]	$p(x,t) = e^{-r(T-t)} E\big[f(xe^X) I\{a(\alpha) \le S_T \le b(\alpha)\}\big]$	By considering the uncertainty in the maturity value of the underlying asset, the binary option pricing model based on fuzzy maturity value is established.
Credibility Theory and Fuzzy Simulation Technology [13]	$\xi_\alpha = \{\xi(\theta) \mid \theta \in \Theta, Pos\{\theta\} \ge \alpha\}$	Systematically explains the uncertainty theory and its simulation method.
Geske Compound Option Pricing Model in a Fuzzy Environment [20]	$C = S N_2(d_1, d_2, \rho) - K_2 e^{-rT_2} N_2(d_3, d_4, \rho) - K_1 e^{-rT_1} N(d_3)$	The Geske compound option pricing model in a fuzzy environment is established by considering the risk and vagueness in the financial environment.
A Jump Diffusion Model Based on Fuzzy Variables [25]	$\widetilde{S}(t) = S(0) e^{(u - \frac{1}{2}\sigma^2)t + \sigma W(t)} \prod_{i=1}^{\widetilde{N}(t)} \widetilde{V}_i$	A jump diffusion model based on fuzzy variables is established, and the weighted possibilistic mean jump diffusion model in a crisp environment is obtained.
Fuzzy Option Pricing Model under the Levy Process [26]	$C_t = \exp\big(-r(T-t) E^Q(f(S) \mid F_t)\big)$	Based on fuzzy numbers and stochastic analysis theory, the fuzzy option pricing formula is established after considering various uncertainties.
Double Exponential Jump Diffusion Model in a Fuzzy Environment [27]	$\begin{aligned}\widetilde{C}_0 =& \tilde{1}_{\{S(0)\}} \otimes \left\{ \kappa \oplus \Phi\left(-\frac{1}{\sqrt{T}} \widetilde{a}_1 \odot \widetilde{\sigma} \right) - \tau \right\} \\ &- \tilde{1}_{\{K\}} \otimes e^{-T\tilde{r}} \left\{ \kappa \oplus \Phi\left(-\frac{1}{\sqrt{T}} \widetilde{a}_2 \odot \widetilde{\sigma} \right) - \tau \right\}\end{aligned}$	It is assumed that the interest rate, drift, volatility and jump parameters are fuzzy numbers, and the double exponential jump diffusion model in a fuzzy environment is established.
European Option Pricing Formula Based on the Martingale Method and Fuzzy Theory [28]	$P_t^E = e^{-\lvert\kappa^E\rvert(T-t)} \cdot \sum_{m=(m_1,m_2,\ldots,m_D)\in N_0^D} \prod_{i=1}^{D} \frac{(\kappa_i^E)^{m_i}}{m_i!} (T-t)^{\lvert m\rvert}$	Applying of fuzzy algorithm and incorporating expert advice or imprecise estimates to establish a European options pricing formula in a fuzzy environment.
Pricing of European Call Options with the Time-changed Levy Process in a Fuzzy Environment [29]	$\widetilde{\phi} S_t = E_0^Q \Big[\exp\Big(-\big[-iu(r-q) + \widetilde{\psi_x}(u) + iu\widetilde{\kappa_x}(1)\big] \Big) T_t \Big]$	This paper studies the high frequency jump, stochastic volatility and stochastic jump in financial markets, and establishes option pricing under the fuzzy time-changed Levy Process.
VG Option Pricing Model in a Fuzzy Environment (**Our model**)	$\begin{aligned}\widetilde{\mathrm{C}} =& S_0 \Psi\left(\tilde{d}\sqrt{\frac{1-\widetilde{c}_1}{\tilde{v}}}, (\alpha+\tilde{s})\sqrt{\frac{\tilde{v}}{1-\widetilde{c}_1}}, \frac{t}{\tilde{v}} \right) \\ &- K\exp(-rt)\Psi\left(\tilde{d}\sqrt{\frac{1-\widetilde{c}_2}{\tilde{v}}}, (\alpha+\tilde{s})\sqrt{\frac{\tilde{v}}{1-\widetilde{c}_2}}, \frac{t}{\tilde{v}} \right)\end{aligned}$	The VG options pricing model in a fuzzy environment is established to provide a better fit for real-life problems.

liest reaserch on finite jumps was proposed by Merton in 1976, subsequently various scholars proposed the single exponential and the double exponential jump diffusion models for option pring, for example, Levendorskii (2004) [3] introduced the exponential jump diffusion process to American option price problem and provided an effective pricing solution; Based on multinomial approximation and exponential jump diffusion process, Maller et al. (2006) [33] studied American option pricing problem, they consider that this scheme is relatively applicable to path-dependent options pricing problem. However, finite jump process can not better characterize high-frequency small jumps, thus Madan et al. (1990) [4] used the Gamma variable to describe the Levy process with a VG (Variance gamma) distribution; although only one parameter was added, the model fits high-order moment characteristics of asset prices well. The NIG (Normal inverse Gaussian) process proposed by Barndorff Nielsen (1997) [5] is one of the most commonly used Levy processes, and this process offers high operating efficiency and the ability to provide accurate characterisation of the tail behaviour of asset prices. And some scholars consider that infinite pure jump process can substitute jump diffusion process, such as Carr et al. (2002) [7] and Daal et al. (2005) [8]. Thus infinite pure jumps offering wider application scope in option pricing, for example, Avramidis et al. (2006) [34] based on VG model studied the Monte-Carlo algorithm for path-dependedent options. Song et al. (2011) [35] based on asymptotic expansion and nonlinear regression method to obtain the approximate option price for the infinite pure jump levy process option pricing problem. As theoretical research advances, Peter Christoffersen et al. (2010) [36] have combined a GARCH model and Levy processes, resulting in the GARCH-Levy option pricing model, which better suits the financial environment. Byun et al. (2013) [37] studied the dynamic volatility and non-normality of underlying asset using the Levy-GARCH model, and based on an empirical analysis of the S&P500, they demonstrated that their model has higher precision in option pricing than previous models.

Regarding the application of fuzzy set theory in option pricing, Cheng-Few Lee et al. (2005) [38] were among the first to incorporate fuzzy decision space in investor decision making, deriving a B-S model under a fuzzy environment. Their research demonstrates that models that fail to incorporate fuzzy numbers tend to underestimate the values of call options. However, empirical research indicates that volatility in asset prices is often time-varying and non normality; consequently, under the framework of a fuzzy system, Leandro Maciel et al. (2015) [39] took into account the time-varying volatility, clustering of volatility and other factors. Using empirical testing, they showed that under a fuzzy environment, the GJR-GARCH model (Glosten, Jagannathan and Rundle-GARCH model) provides better prediction than the traditional GARCH model. Liu Wen-Qiong et al. (2013) [40] used fuzzy set theory to study the European option pricing problem under the condition of jump-diffusion. They treated the interest rate and jump frequency as triangular fuzzy numbers and obtained the option fuzzy price range through empirical analysis. Feng Zhi-Yuan et al. (2015) [41] applied a time-varying Levy process with high-frequency jumps and stochastic volatility for fuzzy pricing of European call option; using option data for the S&P500 Index, they demonstrated that their model better fits the data than state-of-the-art models. Therefore, only by simultaneously studying the fuzziness, time-varying volatility and jump characteristic of the asset prices can we obtain option pricing that fits the actual conditions.

Regarding research works about American options based on numerical approaches, Richard Breen (1991) [42] and Mark Broadie et al. (1994) [43] have each applied the convergence acceleration methodologies and extrapolation methods in the binomial tree option pricing model, which in turn improves the model's convergence speed. Clement et al. (2002) [44] verified the convergence of the least squares Monte Carlo approach; they demonstrated that the asymptotic error obeys a asymptotic Gaussian distribution. Lars Stentoft (2004) [45] demonstrated that among the high-dimensional calculation methods, the least squares Monte Carlo approach is clearly superior to the binomial tree method and finite difference algorithm; the least squares Monte Carlo approach can easily perform ten-dimensional mathematical operation, whereas the finite difference method is no longer valid for calculations with more than five dimensions. Afterwards, in 2008, Lars Stentoft [46] compared the least squares Monte Carlo approach with the American option pricing method proposed by Carriere in 1996, and he found that the least squares Monte Carlo approach provides better results. From financial model research, Jorg Kienitz et al. (2012) [47] performed key analysis regarding how the quasi-random number and Brownian Bridge approach can be used to improve Monte Carlo method, and they applied the improved calculation method in option pricing. However, there are relatively few studies regarding American option pricing theory under a fuzzy environment based on numerical approaches, for example, Yoshida et al. (2006) [48] based on Black-Scholes model constructed an American option pricing model which set the underlying price as fuzzy variable and through numerical simulation to verify the proposed model effectiveness. Rather, most of existing literature provides analysis based on the binomial tree method, such as Silvia Muzzioli et al. (2008) [49] treat the volatility as a fuzzy number and used the multiple-period binomial tree method to obtain risk-neutral valuations of American options. Table 2.3 and 2.4 summarized the research progress of American options pricing model based on the Levy process and fuzzy parameters.

2.2.1 Remark of American option price model

Through review in the existing literature, we observe that there are abundant studies regarding European option, but studies about American option pricing are still limited. Furthermore, the existing studies mainly focus on numerical algorithm improvements, and there is insufficient research intended to improve the theoretical model. Therefore, we have constructed the fuzzy Levy-GJR-GARCH American option pricing model which considered both fuzzy set theory and an infinite pure jump condition, it is more consistent with reality.

2.3 Summary and Remark

This chapter aimed to set up the positioning of our studies through review of the existing literature. Here, we summarized and discussed the development of research in Levy process, fuzzy set theory in option pricing, and a research in combining the Levy process and fuzzy set theory, we concluded them from European option and American option two aspects, for being theoretical supports and explanation better of our research meaning on theoretical and practical values.

Table 2.3: Developments in research regarding American option pricing models based on Levy process, GARCH models and fuzzy parameters (1)

Usage method	Formula	Objective
Symmetric and asymmetric GARCH-type model [31]	$r_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{i=1}^{q} \theta_i \varepsilon_t$	Study the effect of stochastic volatility and stochastic interest rate on option pricing.
American option pricing model based on the GARCH model [32]	$S_1 = S_0 \exp\{m_1(.;\theta_m) - \delta + \sqrt{h_1}\varepsilon_1\}$	Analyse and discuss current American option pricing simulation methods, develop an empirical financial research handbook.
Variance gamma model (VG model) [4]	$f(x) = \int_0^\infty \left[e^{-x^2/(2\sigma^2 v)}/(\sigma\sqrt{2\pi v})\right] g(v)dv$	Establish an pure jump process option pricing model by presuming that the rate of return conforms to gamma distribution.
Normal inverse Gaussian model (NIG model) [5]	$g(\theta) = -\rho\delta(\theta+\beta)\{(\alpha-\theta-\beta)/(\alpha+\theta+\beta)\}^{1/2}$	Use an NIG model to perform theoretical analysis and maximum likelihood estimation.
CGMY model [7]	$k_{CGMY}(x) = \begin{cases} C\dfrac{\exp(-G\lvert x\rvert)}{\lvert x\rvert^{1+Y}} & \text{for } x < 0 \\ C\dfrac{\exp(-M\lvert x\rvert)}{\lvert x\rvert^{1+Y}} & \text{for } x > 0 \end{cases}$	Includes both finite and infinite jumps to more vividly describe financial assets price changes.
Foreign currency option pricing model based on VG process	$S(t) = S(0)\exp[(\mu+\omega)t + X(t;\sigma_J,\nu,\theta)]$	Verify that option pricing model based on Infinite jump process is superior than based on BS model or finite jump process.
Path-dependent pricing model based on VG process	$S(t) = S(0)\exp\{(\omega+r-q)t + X(t)\}$	Verify that combine the gamma bridge sampling with randomized quasiMonte Carlo to reduce the variance can further improve the efficiency of option pricing.
Non-linear regression asymptotic pure jump Levy process [22]	$\log S_t^\delta = \log S_0 + \sum_{i=1}^{N_t^\delta} G_i^\delta$	An option pricing method based on the non-linear regression asymptotic of the pure jump Levy process that targets the complexity of option pricing model under the Levy process.
Discrete time stochastic volatility model (GARCH-levy model) [36]	$d(\ln(S_t)) = (r - 1/2 * \sigma^2)dt + \sigma dz(t)$	Study the effect of stochastic volatility and jump characteristics on option pricing under a risk-neutral condition.
Modified option pricing model based on GARCH and the Levy process [37]	$\ln(S_t/S_{t-1}) = u + \varepsilon_t\sigma_t^2 = w + \beta\sigma_{t-1}^2 + \alpha\varepsilon_{t-1}^2 + \delta I_{t-1}\varepsilon_{t-1}^2$	Study the effect of asset price dynamic volatility and non-normality on option pricing.
Fuzzy GARCH-type model [39]	$\sigma_{i,t}^2 = w_i + \sum_{n=1}^{q} \alpha_{i,n} r_{t-n}^2 + \sum_{n=1}^{p} \beta_{i,n}\sigma_{t-n}^2$	Study the time-varying characteristics, clustering and other characteristics of underlying asset volatility under a fuzzy environment.

Table 2.4: Developments in research regarding American option pricing models based on Levy process, GARCH models and fuzzy parameters (2)

Accelerated binomial tree option pricing model [42]	$C_{t-\Delta t,i} = e^{-r\Delta t}(pC_{t,i+1} + (1-p)C_{t,i-1})$	Apply convergence acceleration technology on the binomial tree option pricing model; improve the precision of pricing and convergence speed.
American option pricing simulation based on the binomial tree calculation method [43]	$C_t^1(S_t) = \hat{\lambda}_1 C_t^l(S_t)$, $C_t^2(S_t) = \hat{\lambda}_2 C_t^l(S_t) + (1-\hat{\lambda}_2)C_t^u(S_t)$	Study the price ceiling and price floor of the dividend-paying American call option and put option.
American option pricing simulation based on the least squares Monte Carlo approach [44]	$F_j(a^m, z, x) = z_j 1_{B_j^c} + \sum_{i=j+1}^{L-1} z_j 1_{B_j \ldots B_{i-1} B_i^c} + z_L 1_{B_j \ldots B_{L-1}}$	Study the convergence of the least squares Monte Carlo approach proposed by Longstaff and Schwartz.
American option pricing simulation based on the least squares Monte Carlo approach [45]	$V(t_k) = \max(Z(t_k), E[V(t_{k+1})\|X(t_k)])$	Comparative analysis of subtle differences between different numeric methods of American option pricing; evaluation of the effectiveness of different methods.
Quasi-Monte Carlo and Brownian Bridge approach [47]	$f_{x\|z} = \frac{f_{x,z}(x, z-x)}{f_z(z)}$	Improve the convergence efficiency of the least squares Monte Carlo approach using a quasi-random number and Brownian Bridge approach.
Fuzzy binomial tree American option pricing model [49]	$v_n(s) = \max\{K-s, \frac{1}{1+r}(p_u v_{n+1}(us) + p_d v_{n+1}(ds))\}$	Study fuzzy binomial tree American option pricing problems when volatility is set as a fuzzy number.
Fuzzy Levy-GJR-GARCH American option pricing model (**Our model**)	$(\widetilde{V})_\alpha = [(\widetilde{V})_\alpha^L, (\widetilde{V})_\alpha^U] == [\min_{\widetilde{S}_\alpha^L \le S \le \widetilde{S}_\alpha^U} V(S,t), \max_{\widetilde{S}_\alpha^L \le S \le \widetilde{S}_\alpha^U} V(S,t)]$	Establish the Levy-GJR-GARCH American option pricing model under a fuzzy environment; improve option pricing results to better match the real-life situation.

Chapter 3

Preliminaries

In this chapter, we provide some necessary basic knowledge used in the studies of this thesis, which including infinite pure jump Levy process, the Black-Scholes (BS) model, common fuzzy variables, fuzzy random variable, the extension principle and credibility measure.

3.1 Infinite Pure Jump Levy processes

A Levy process is a self-adapting process with independent and stable random increments. Its characteristic function is $\Phi_{x_t} = (u|F_t) = E\{\exp(iux_t)\} = \exp(\phi(u))$, where $\phi(u)$ is the characteristic exponent of the characteristic function. $\phi(u)$ is formulated as follows:

$$\phi(u) = i\theta u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (e^{iux} - 1 - iux1_{|x|\leq 1})v(dx) \tag{3.1}$$

The whole expression consists of drift, diffusion and jump elements; θ and σ represent the measure of drift and the measure of diffusion, whereas v is the measure of jump. Hence, (θ, σ, v) represent all information in the Levy process, and it is also known as the three elements of Levy.

Levy processes include two major types of processes: jump-diffusion processes and infinite pure jump processes. The jump-diffusion processes include the Merton model, and double exponential jump model, for example, whereas examples of pure jump processes include the VG (Variance gamma process) model, NIG (Normal inverse Gaussian process) model, and CGMY (Carr-Geman-Madan-Yor process) model. Compared with jump-diffusion processes, there are fewer parameters in the pure jump process, it contains more high-order moment characteristics and offers simpler calculations, and it has broader applications in current research; therefore, our study chose to use a pure jump process for our model analysis.

Our study used more mature Levy processes VG, NIG and CGMY simulation technology to analyse the option pricing and performed comparative analysis of the simulation effect of these three jump processes.

The VG process is a random process driven by Gamma process; it has good mathematical properties and is considered to be a finite variation process. Moreover, the incremental part of the VG process exhibits leptokurtic and fat-tailed characteristics, which can resolve the issue of the "volatility smile".

The NIG process is a random process formed when the IG (Inverse Gaussian) process performs time-varying on Wiener process; this process can provide excellent description of fat-tailed data, and it also possess other good properties, such as the additive property, which makes formula deduction and application easier. Simultaneously, it is easier to achieve measure conversion for this process, which increases its application in option pricing.

The CGMY process is a tempered stable random process based on the VG process with the addition of the Y parameter. The existence of the Y parameter increases the structural complexity of the model, but it also enriches the model's data presentation capability and allows better description of financial data characteristics, such as infinite jumps. The characteristic function of these three Levy processes are expressed as follows:

(1) VG process:

$$
\begin{aligned}
E(e^{iuX_t}) &= \varphi(u;\sigma,v,\theta) \\
&= (1 - iuv\theta + \frac{1}{2}\sigma^2 vu^2)^{-\frac{1}{v}}
\end{aligned}
\tag{3.2}
$$

of which,

$$
\begin{aligned}
C &= \frac{1}{v} > 0 \\
M &= (\sqrt{\frac{1}{4}\theta^2 v^2 + \frac{1}{2}\sigma^2 v} + \frac{1}{2}v\theta)^{-1} > 0 \\
G &= (\sqrt{\frac{1}{4}\theta^2 v^2 + \frac{1}{2}\sigma^2 v} - \frac{1}{2}v\theta)^{-1} > 0
\end{aligned}
$$

where C represent a measure of the overall level of activity, G and M are measure the jump intensity in negative and jump intensity in positive respectively.

(2) NIG process:

$$
\begin{aligned}
E(e^{iuX_t}) &= \varphi(u;\lambda,\eta,\kappa) \\
&= \exp(\eta\sqrt{\lambda^2 - \eta^2} - \kappa\sqrt{\lambda^2 - (\eta + iu)^2})
\end{aligned}
\tag{3.3}
$$

of which, $\lambda > 0$, $\kappa > 0$, $-\lambda < \eta < \lambda$, λ controls the kurtosis, η controls the skewness, κ is the scale parameter.

(3) CGMY process:

$$
\begin{aligned}
E(e^{iuX_t}) &= \varphi(u;C,G,M,Y) \\
&= \exp(\mathrm{C}g(-\mathrm{Y})(M - iu)^Y \\
&\quad + (G + iu)^Y - \mathrm{M}^Y - \mathrm{G}^Y)
\end{aligned}
\tag{3.4}
$$

of which, $C > 0$, $G > 0$, $M > 0$, $Y < 2$, g represents gamma function, C,G,M parameters are the same meaning with VG process, Y describes the statistical characteristics of financial data.

3.2 Black Scholes model

In 1973, Black and Scholes established the famous Black-Scholes (BS) pricing model for European call options, it was certainly a remarkable achievement in options pricing theory, the specific form as follows,

$$C_t = N(d_1)S_0 - N(d_2)Ke^{-rt} \tag{3.5}$$

of which,

$$d_1 = \frac{1}{\sigma\sqrt{t}}[\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})t]$$
$$d_2 = d_1 - \sigma\sqrt{t}$$

where $N(\bullet)$ represents the cumulative distribution function, t represents the time to expiration, S_0 is the underlying asset price at initial time, K is the exercise price, r is the risk-free rate, σ represents the underlying asset price volatility.

3.3 Common Fuzzy Variables

The concept of fuzzy sets was first proposed by Zadeh in 1965 [9]. It gradually developed into a more complete fuzzy theory, which revealed a new direction for asset pricing theories.

Let $\tilde{A}$ be a mapping of the domain X to $[0, 1]$, that is, $\tilde{A}\colon X \to [0, 1]$, $x \to \tilde{A}(x)$ is called a fuzzy set on X. $\tilde{A}(x)$ is called the membership function of the fuzzy set $\tilde{A}$, and the set of all fuzzy sets on X is denoted as $\tilde{F}(X)$. If $\alpha \in [0, 1]$, $\tilde{A}_\alpha = \{x \in X \mid \tilde{A}(x) \geq \alpha\}$, then $\tilde{A}_\alpha$ is called the α-level set of fuzzy set $\tilde{A}$. If $\tilde{a}$ is a regular convex fuzzy set with a upper semi-continuous membership function $\tilde{a}(x)$ and the level set $\tilde{a}_\alpha$ is bounded, i.e., $\alpha \in [0, 1]$, then $\tilde{a}$ is called a fuzzy number. Common fuzzy numbers include triangular fuzzy numbers, trapezoidal fuzzy numbers, parabolic fuzzy numbers and normal fuzzy numbers. The membership function graphs of the respective fuzzy numbers are shown in Figure 3.1. The most commonly used fuzzy numbers are triangular fuzzy numbers and trapezoidal fuzzy numbers, and their respective definitions are as follows:

If the membership function of the fuzzy number $\tilde{A}$ is:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\[2ex] \dfrac{x-a_3}{a_2-a_3}, & a_2 \le x \le a_3 \\[2ex] 0, & \text{others} \end{cases} \tag{3.6}$$

then $\tilde{A}$ is a triangular fuzzy number and is expressed as $\tilde{A} = (a_1, a_2, a_3)$.

If the membership function of the fuzzy number $\tilde{A}$ is:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\[2ex] 1, & a_2 \le x \le a_3 \\[2ex] \dfrac{x-a_4}{a_3-a_4}, & a_3 \le x \le a_4 \\[2ex] 0, & \text{others} \end{cases} \tag{3.7}$$

then $\tilde{A}$ is a trapezoidal fuzzy number and is expressed as $\tilde{A} = (a_1, a_2, a_3, a_4)$. If $a_2 = a_3$, then the abovementioned fuzzy number becomes a triangular fuzzy number; thus, triangular fuzzy numbers are a special case of trapezoidal fuzzy numbers.

If the membership function form of fuzzy number $\tilde{A}$ is:

$$\mu_{\tilde{A}}(x) = \begin{cases} \left(\dfrac{x-a_1}{a_2-a_1}\right)^n, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \left(\dfrac{x-a_4}{a_3-a_4}\right)^n, & a_3 \le x \le a_4 \\ 0, & \text{others} \end{cases} \tag{3.8}$$

where $\tilde{A}$ is a parabolic fuzzy number called $\tilde{A} = (a_1, a_2, a_3, a_4)_n$. If $n = 1$, the above is a trapezoidal fuzzy number; if $n = 1$ and $a_2 = a_3$, the above is a triangular fuzzy number. Therefore, triangular fuzzy numbers and trapezoidal fuzzy numbers are special cases of parabolic fuzzy numbers (See Figure 3.2). At this point, the α level set of $\tilde{A}$ can be expressed as $\widetilde{A}_\alpha = [\widetilde{A}_\alpha^L, \widetilde{A}_\alpha^U] = [a_1 + \sqrt[n]{\alpha}(a_2 - a_1), a_4 - \sqrt[n]{\alpha}(a_4 - a_3)]$, where $\widetilde{A}_\alpha^L$ is the α pessimistic value of fuzzy variable $\widetilde{A}$ and $\widetilde{A}_\alpha^U$ is the α optimistic value of $\widetilde{A}$.

If the membership function form of fuzzy number $\tilde{A}$ is:

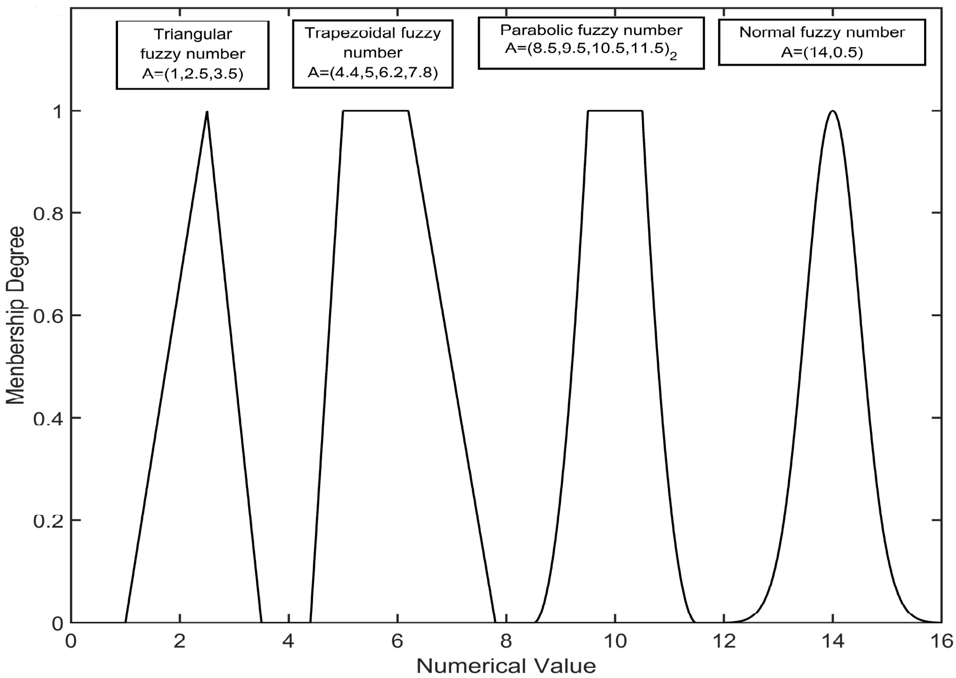
$$\mu_{\tilde{A}}(x) = \exp\left(-\frac{(x-a)^2}{\sigma^2}\right), \quad x, a \in R, \quad \sigma > 0 \tag{3.9}$$

where $\tilde{A}$ is a normal fuzzy number called $\tilde{A} = (a, \sigma)$. The membership function of the normal fuzzy number is shown in Figure 3.1.

3.4 Fuzzy Random Variable

The fuzzy random variable $\tilde{X}$ is a mapping from a probability measure space (Ω, F, P) to a fuzzy class of sets $\tilde{F}_c$, where the mapping $\tilde{X}$ obeys measurability conditions, that is, $\forall \alpha \in [0, 1]$, $\tilde{X}_\alpha(w) = [\tilde{X}_\alpha^L(w), \tilde{X}_\alpha^U(w)]$, and $w \in \Omega$ is a random interval, and therefore, $\tilde{X}_\alpha^L(w)$ and $\tilde{X}_\alpha^U(w)$ are the usual random variables.

Let $f : R^n \to R$ be a measurable function. If $X_1, X_2, ...X_n$ are fuzzy random variables defined in the same probability measure space (Ω_i, F_i, P_i), then $X(w) = f(X_1(w), X_2(w), ...X_n(w))$



Note: Triangular fuzzy number $A = (1, 2.5, 3.5)$, trapezoidal fuzzy number $A = (4.4, 5, 6.2, 7.8)$, parabolic fuzzy number $A = (8.5, 9.5, 10.5, 11.5)_2$, normal fuzzy number $A = (14, 0.5)$.

Figure 3.1: Membership Function Graph of Common Fuzzy Numbers

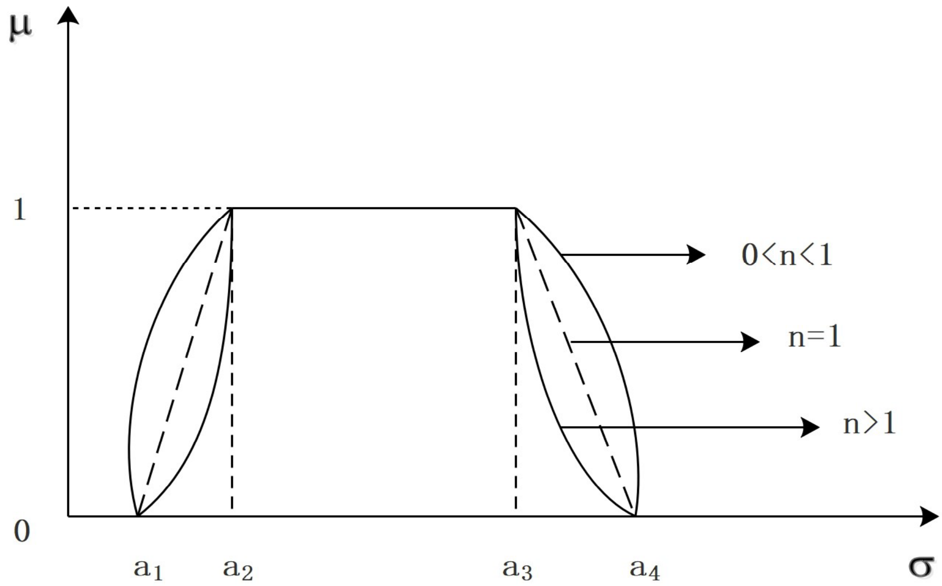


Figure 3.2: Plot of membership function of a parabolic fuzzy number

is the same fuzzy random variable and is defined as

$$X(w) = f(X_1(w), X_2(w), ...X_n(w)),$$
$$\text{of which,} \quad w \in \Omega \tag{3.10}$$

If $X_1, X_2, ...X_n$ are fuzzy random variables not defined in the same probability measure space $(\Omega_i, F_i, P_i)(i = 1, 2...n)$, then $X(w) = f(X_1(w), X_2(w), ...X_n(w))$ is a fuzzy random variable in the product probability measure space $(\Omega_1 \times \Omega_2 \times ...\Omega_n, F_1 \times F_2 \times ... \times F_n, P_1 \times P_2 \times ... \times P_n)$ and is defined as

$$X(w) = f(X_1(w), X_2(w), ...X_n(w)),$$
$$\text{of which, } w \in \Omega_1 \times \Omega_2 \times ...\Omega_n \tag{3.11}$$

3.5 Extension Principle

There is necessary to introduce the extension principle, which is one of most basic concepts that can be used to generalize crisp mathematical concepts to fuzzy sets. Following Zadeh [9], Dubois and Prade [50] and Zimmermann [51], the extension principle defined as follows:

Let X be a Cartesian product of universes, $X = X_1 \times ... \times X_r$, and $\tilde{A}_1, ..., \tilde{A}_r$ be r fuzzy sets in $X_1, ..., X_r$, respectively. f is a mapping from X to a universe Y, $y = f(x_1, ..., x_r)$. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in Y by

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x_1, ...x_r), (x_1, ..., x_r) \in X\} \tag{3.12}$$

where,

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{(x_1,...,x_r) \in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1), ..., \mu_{\tilde{A}_r}(x_r)\}, & f^{-1}(y) \neq \emptyset \\ 0, & \text{others} \end{cases} \tag{3.13}$$

where f^{-1} is the inverse of f

For $r = 1$, the extension principle reduces to

$$\tilde{B} = f(\tilde{A}) = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x), x \in X\} \tag{3.14}$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{others} \end{cases} \tag{3.15}$$

3.6 Credibility Measure

Assuming that ξ is a fuzzy variable, of which lit us define μ_ξ a membership function and g a real number respectively, event $\xi \leq g$ has the credibility in the following:

$$Cr\{\xi \leq g\} = \frac{1}{2}[Nec\{\xi \leq g\} + Pos\{\xi \leq g\}] \tag{3.16}$$

where let us define $Nec\{.\}$ and $Pos\{.\}$ [11] necessity and probability respectively in the following.

$$Nec\{\xi \leq g\} = 1 - \sup_{\eta > g} \mu_\xi(\eta) \tag{3.17}$$

$$Pos\{\xi \leq g\} = \sup_{\eta \leq g} \mu_\xi(\eta) \tag{3.18}$$

A credibility measure have the property of self-duality, thus $Cr\{\xi \leq g\} = 1 - Cr\{\xi > g\}$, and that represents the credibility event of the fuzzy variable. Assuming the fuzzy income of a security is denoted by ξ, then $Cr\{\xi > 4\} = 0.7$ is the credibility of this security with future earnings over 4 of 0.7.

According to formulas (3.17) and (3.18), the following can be obtained.

$$Cr\{\xi \leq g\} = \frac{1}{2}[\sup_{\eta \leq g} \mu_\xi(\eta) + 1 - \sup_{\eta > g} \mu_\xi(\eta)] \tag{3.19}$$

similarly, this is written as follows.

$$Cr\{\xi \geq g\} = \frac{1}{2}[\sup_{\eta \geq g} \mu_\xi(\eta) + 1 - \sup_{\eta < g} \mu_\xi(\eta)] \tag{3.20}$$

with respect to n fuzzy vectors $\xi = \xi_1, \xi_2, ..., \xi_n$, ξ_z is the fuzzy variable, $z = 1, 2, ..., n$ and the membership function of ξ takes the minimum value of individual coordinates, i.e.,

$$\mu_\xi(\eta) = \min\{\mu_{\xi_1}(\eta_1), \mu_{\xi_2}(\eta_2), ..., \mu_{\xi_n}(\eta_n)\} \tag{3.21}$$

where $\eta = (\eta_1, \eta_2, ..., \eta_n)$.

And the expected value of ξ can be written as follows [54]:

$$E[\xi] = \int_0^{+\infty} Cr\{\xi \geq g\}dr - \int_{-\infty}^{0} Cr\{\xi \leq g\}dr \tag{3.22}$$

More information on fuzzy variables and credibility theory are shown in References [52, 53, 12, 10]

3.7 Summary and Remark

This chapter mainly introduced the theoretical tools which we will handle the uncertainty from the real-life financial market, i.e., randomness and fuzziness co-exist at the same time. The asset yield is not a normal distribution but rather exhibits skewed and leptokurtic fat-tail characteristics, in addition to a significant jump phenomenon in the asset price fluctuation. Levy process has the ability to fit characteristics of the leptokurtic and fat-tailed distribution of the asset yield as well as describe the jump phenomenon of the asset price fluctuation. Levy process includes both of finite jump process and infinite pure jump process, however, finite jumps process only consider the finite big jumps, ignoring the high-frequency small jumps which is also existing in the fluctuation of asset price, compared with finite jump Levy process, infinite pure jump Levy process can capture the big jumps and high frequency small jumps simultaneously in the real market situation. Therefore, for the non-normality phenomenon of the underlying asset yield, and the asset price fluctuation including big jumps and high frequency small jumps simultaneously in the real market situation, the infinite pure jump Levy process will be adopted to better characterize them. Meanwhile, fuzzy set theory as a powerful tool employed to address the uncertainty, vagueness from the social environment.

Chapter 4

Fuzzy European Option Pricing Model with Infinite Pure Jump Levy Process

4.1 Introduction

In 1900, Louis Bachelier, a famous French mathematician, first proposed an options pricing model in his doctoral thesis. He also proposed a stock price stochastic model that was based on the random walk hypothesis. This was recognised as a milestone in financial mathematics. However, the model assumes that the stock price is subject to the arithmetic Brownian motion, which may cause the underlying stock price to be negative and is hence not realistic. In 1965, Paul Samuelson, a Nobel laureate in economics, assumed that stock prices obey geometric Brownian motion and established a pricing formula for European call options using partial differential equations. Subsequently, in 1973, Black and Scholes established the famous Black-Scholes (BS) pricing formula [55] for European call options. This formula is independent of investors personal preferences and can be used to provide analytical solutions for options price risk parameters and leverage effects, for example. It was certainly a remarkable achievement in options pricing theory. In the same year, Merton extended the model to include other types of financial transactions. From then on, the BS model has often been referred to as the Black-Scholes-Merton (BSM) model, and it won M. Scholes and R. Merton the Nobel Prize in

1997.

It is worth mentioning that the above theoretical models were established under ideal assumptions, whereas the real financial market environment is full of uncertainties and not as perfect as the assumptions of the theoretical model. Consequently, the drawbacks and shortcomings of the options market based on the BSM pricing system are increasingly obvious. On one hand, the underlying asset yield is not a normal distribution but rather exhibits skewed and leptokurtic fat-tail characteristics in addition to a significant jump phenomenon in the asset price fluctuation. These issues are widely recognised in academia. On the other hand, the parameters in the BSM model are taken as crisp value, but because of the uncertainties and incomplete information in the financial market, these parameters are often vague and cannot be expressed using crisp value (i.e., randomness and fuzziness co-exist simultaneously in the real-life financial market). Owing to the non-normality of random variables, the jump measure of the Levy process can better capture the leptokurtosis and fat-tailed characteristics of the underlying assets and has relatively more applications in options pricing models. At the same time, fuzzy set theory is a powerful tool employed to address the uncertainty, vagueness of the social environment; thus, by applying it to option pricing models with an infinite pure-jump Levy process, it can be a useful supplement to the traditional pricing method and can provide a new theoretical basis for the pricing of options. Therefore, to price options more rationally, this chapter introduces fuzzy set theory and the infinite pure jump Levy process into an European options pricing model on the basis of previous studies to further enhance and enrich options pricing theories. In addition, this chapter also discusses the theoretical and practical values of the options pricing model in a fuzzy environment through numerical simulation and empirical analysis.

In summary, even though scholars have applied fuzzy set theory and the Levy process to perform plenty of theoretical and empirical research about options pricing, these studies mostly focus only on one aspect, whereas studies that consider both aspects of fuzzy theory and Levy process tend to use finite jump process conditions. Compared with the finite jump process, the infinite pure jump Levy process can better describe the characteristics of the market such as bigger jumps representing market shocks, whereas smaller jumps representing real-time transactions. Therefore, to address the shortcomings of existing research, this paper considers both fuzzy set theory and an infinite pure jump condition and establishes an infinite pure jump European option pricing model in a fuzzy environment. The main research contributions include the following: First, theoretically establishing an European option pricing model based on the infinite pure jump Levy process and treating the drift, diffusion and jump terms as the trapezoidal fuzzy random variables. The model assumptions are relaxed to those in a fuzzy environment to better describe real-life problems. Second, Monte Carlo simulations and empirical tests were used to accurately verify the feasibility of the model. Third, a instrumental variable method was applied to improve the Monte Carlo simulation algorithm to achieve better convergence. The structure of the remainder of this chapter is as follows: Section 4.2 is an in-depth theoretical introduction of the European option pricing model that is in accordance with the VG process in a fuzzy environment; Section 4.3 verifies the pricing model by applying the Monte Carlo simulation method to European call options and analyses the sensitivity of the pricing model to the jump parameters, meanwhile the instrumental variable method was chosen to improve the convergence speed of the Monte Carlo algorithm; Section 4.4 presents an empirical test of the pricing model using Tencent Holdings (HK.0700) and its stock options data; Section 4.5 is the summary. Our conceptual framework is illustrated in Figure 4.1.

4.2 Options Pricing Model under the Levy Process in a Fuzzy Environment

We first introduce the notation that is used in the remainder of this chapter:

4.2.1 Pricing Model under the Infinite Pure Jump Levy Process

In the probability space (Ω, F, P) , the adaptation process $X = \{X_t : t \geq 0\}$ is a Levy process. If $X_0 = 0$ and X_t has an independent, stationary random increments, that is $\Delta X_t = X_{t+\Delta t} - X_t$ is independent of any time for $t \geq 0$ and has the same distribution. The characteristic function

Variable	**Description**
$\{X_t\}$	Levy process
θ	Drift rate
σ	Volatility
v	Jump rate
g_t	A random variable that obeys gamma distribution
$W(t)$	Brownian motion
S_t	The general underlying price at time t
T	Time to expiration
K	Exercise price
r	Risk-free interest rate
$C(S_0; K, t)$	Option price at time t
$\tilde{A}$	Fuzzy set
$\tilde{A}_\alpha$	α-level set of fuzzy set $\tilde{A}$
$\mu_{\tilde{A}}(x)$	The membership function of the fuzzy set $\tilde{A}$
$\tilde{C}(S_0; K, t)$	Fuzzy option price at time t

of the Levy process process is $\Phi_{X_t} = (u \mid F_t) = E\{\exp(iuX_t)\} = \exp(t\phi(u))$, of which $\phi(u)$ is the characteristic exponent of the characteristic function and has the following structure:

$$\phi(u) = i\theta u - \tfrac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (e^{iux} - 1 - iux1_{|x|\leq 1})v(dx) \tag{4.1}$$

The entire expression consists of the drift, diffusion and jump. θ and σ are the measures of the drift and diffusion, respectively, whereas v is the measure of the jump; (θ, σ, v) represent all information in the Levy process and are knowns as the three elements in a Levy process, $v(dx)$ represents the arrival rate of a certain jump per unit time, and $v(R)$ represents the sum of probabilities of all jumps. If $v(R) = \int_R v(dx) = \infty$, this means that the stochastic process is an infinite activity rate process, and infinite multiple jumps may occur at any time interval. However, If $v(R) = \lambda < \infty$, then the stochastic process is a finite jump process. The infinite jump process is more general than the finite jump process, and Daal *et al.* (2005) [56]showed that the diffusion part is not necessary and that the underlying asset price can be replaced by pure jump. Consequently, this study is based on the infinite pure jump Levy process. The VG process is a typical infinite pure jump process and is currently the most widely used Levy process. It has very good mathematical properties, belongs to the finite variation process, and has an incremental distribution that exhibits leptokurtosis and fat-tailed characteristics. In

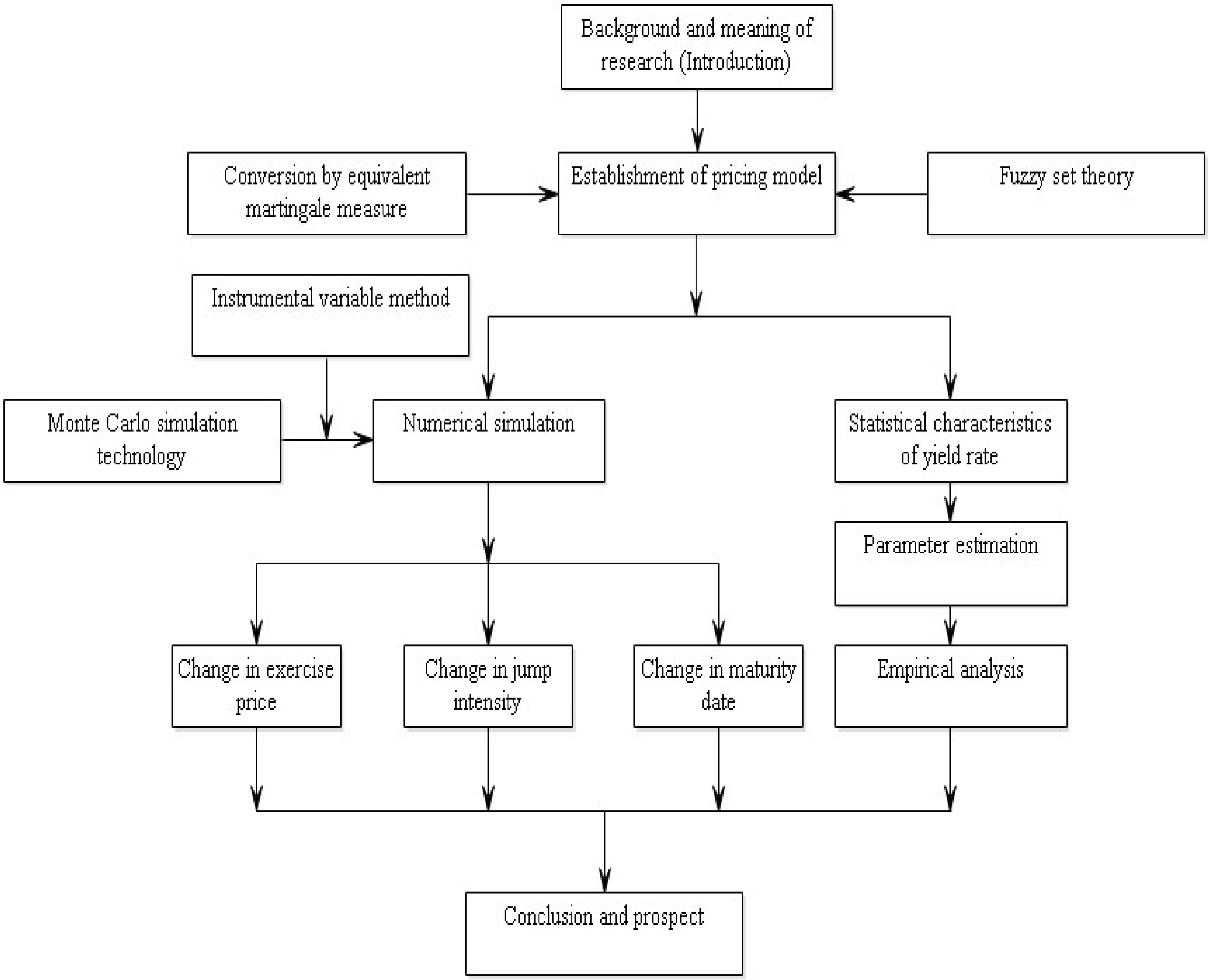


Figure 4.1: Conceptual Framework

addition, an options pricing model based on the VG process can solve the "volatility smile" predicament. Therefore, the infinite pure jump Levy process studied in this paper is analysed using the VG process as an example.

The VG process is formed by changing the variance in the normal distribution into a random variance, that is, assuming that the yield of the underlying asset obeys a normal distribution with a mean of u and variance of $\sigma^2 g$, of which g is a random variable that obeys the gamma distribution. The gamma process is a subordinate process of the VG process, and its density function is

$$f_{Gamma}(g) = \frac{b^a}{\Gamma(g)} g^{a-1} e^{-bg} \tag{4.2}$$

where $\Gamma(\cdot)$ is the gamma function with the boundary conditions $g > 0$, $a > 0$, $b > 0$. The gamma distribution is a very extensively used two-parameter stochastic process, in which a is the shape parameter used to control the shape of the random distribution and b is the scale parameter used to describe its scale. The characteristic function of Gamma distribution is

$$\Phi_{Gamma}(u; a, b) = E(e^{iux}) = \left(1 - \frac{iu}{b}\right)^{-a} \tag{4.3}$$

The VG process is driven by the gamma process. The time randomness of the process is determined by the gamma distribution of $g(t; 1, v)$, that is, $X(t; \sigma, v, \theta) = B(g(t; 1, v); \sigma, v, \theta)$, of which $B(t; \theta, \sigma)$ represents the drift rate, a Gaussian process with θ of σ.

The density function of the VG process is as follows:

$$f_{x_t}(x) = \int_0^{\infty} \frac{g^{\frac{t}{v}-1} e^{-\frac{g}{v}}}{v^{\frac{t}{v}} \Gamma\left(\frac{t}{v}\right)} * \frac{1}{\sigma\sqrt{2\pi g}} \exp\left(-\frac{(x - \theta g)^2}{2\sigma^2 g}\right) dg \tag{4.4}$$

The equation above illustrates the complexity of the VG processs density function. This complexity makes it difficult to estimate the parameters. Therefore, the characteristic function of the pure jump process and the three Levy elements become more important. Their characteristic function is

$$E(e^{iuX_t}) = \varphi(u; \sigma, v, \theta) = e^{iut}(1 - iuv\theta + \tfrac{1}{2}\sigma^2 v u^2)^{-\frac{1}{v}} \tag{4.5}$$

4.2.2 Risk-neutral Model Settings

If the expected return on a particular portfolio in a financial market is greater than the risk-free income, there will be arbitrage space, and this will induce investors to trade to obtain excess returns. If such a portfolio does not exist in the market, the market is said to be arbitrage-free. The no-arbitrage hypothesis lies at the core and is the basis of current asset pricing theories. Therefore, only by converting model parameters under an reality measure into a risk-neutral measure will the pricing of derivatives conform to the no-arbitrage assumption. Generally, the risk-neutral measure conversion is performed in the form of equivalence martingale transformations, which are a form of probability density, or equivalence martingale transformations for characteristic functions. Because the probability density function of the VG model is too complex, the transformation of the model is performed using the equivalence martingale measure conversion method for characteristic functions.

The equivalence martingale measure conversion is performed mainly to convert the assets stochastic process S_t under measure P into the stochastic process $\widehat{S_t}$ under the risk-neutral measure Q. This can be derived through the characteristic function by introducing the risk-free rate of return r to correct the drift. The requirements of the equivalence martingale measure in a risk-neutral environment are

$$E_t^Q[\widehat{S_t} \mid \mathcal{F}_t] = \widehat{S_t} = S_t \exp(-rt) \tag{4.6}$$

The above equation coincides with the form of the characteristic function expression $\varphi_X(u) = E[\exp(iuX_t)]$ of the VG process. Therefore, we revised the risky assets return rate sequence S_t under the P measure to the risk-neutral risky assets return rate $\widehat{S_t}$:

$$\widehat{S_t} = (r - \varphi(-i))t + S_t \tag{4.7}$$

$$E(\widehat{S_t}) = rt - E(\varphi(-i)t) + E(S_t) = rt \tag{4.8}$$

Asset pricing theories normally refer to $r - \varphi(-i)$ as the risk neutral drift rate of S, and it is usually denoted by u^*. We introduce the VG process characteristic function into the $r - \varphi(-i)$ equation to obtain the risk-neutral drift rate for the VG process:

$$u_{VG}^* = r + \frac{\ln\left(1 - v\theta - \frac{\sigma^2 v}{2}\right)}{v} \tag{4.9}$$

The asset pricing model under the VG process then becomes

$$
\begin{aligned}
S_t &= S_0 * \exp[u^*_{VG} * t + X_{VG}] \\
&= S_0 * \exp\left[rt + \frac{\ln\left(1 - v\theta - \frac{\sigma^2 v}{2}\right)}{v} * t + X_{VG}\right] \\
&= S_0 * \exp\left[rt + \frac{\ln\left(1 - v\theta - \frac{\sigma^2 v}{2}\right)}{v} * t + \theta * g_t + \sigma W(g_t)\right]
\end{aligned} \tag{4.10}
$$

where $g_t \sim gamma(a, b)$, $W(g_t)$ represents time-changed Brownian motion.

According to the no-arbitrage pricing method, the price of European call options can be expressed as

$$
C(S(0); K, t) = e^{-rt} E\big[\max(S(t) - K, 0)\big] \tag{4.11}
$$

Under the measure Q, and according to the properties of conditional expectation, we obtain

$$
C(S_0; K, t) = E\Big[e^{-rt} E\big[\max(S(t) - K, 0) \mid g_t = g\big]\Big] \tag{4.12}
$$

such that $c(g) = e^{-rt} E[\max(S(t) - K, 0) \mid g_t = g]$. Madan *et al.* (1991) [57] have proved that:

$$
\begin{aligned}
c(g) = S_0 &\left(1 - \frac{v(\alpha + s)^2}{2}\right)^{\frac{t}{v}} \\
&* \exp\left(\frac{(\alpha + s)^2 g}{2}\right) * N\left(\frac{d}{\sqrt{g}} + (\alpha + s)\sqrt{g}\right) \\
&- K \exp(-rt)\left(1 - \frac{v\alpha^2}{2}\right)^{\frac{t}{v}} \exp\left(\frac{\alpha^2 g}{2}\right) * N\left(\frac{d}{\sqrt{g}} + \alpha\sqrt{g}\right)
\end{aligned} \tag{4.13}
$$

where

$$
\begin{aligned}
s &= \frac{\sigma}{\sqrt{1 + \left(\frac{\theta}{\sigma}\right)^2 \frac{v}{2}}}, \quad c_1 = \frac{v(\alpha + s)^2}{2}, \quad c_2 = \frac{v\alpha^2}{2}, \\
d &= \frac{1}{s}\left[\ln \frac{S_0}{K} + rt + \frac{t}{v} \ln\left(\frac{1 - c_1}{1 - c_2}\right)\right].
\end{aligned}
$$

Consequently, $C(S_0; K, t) = E[c(g) \mid g_t = g]$, that is,

$$C(S_0; K, t) = \int_0^\infty c(g) \frac{g^{\frac{t}{v}-1} e^{-\frac{g}{v}}}{v^{\frac{t}{v}} \Gamma(\frac{t}{v})} dg \tag{4.14}$$

such that $y = \frac{g}{v}$, $\gamma = \frac{t}{v}$. This yields,

$$\begin{aligned} &C(S_0; K, t) \\ &= \int_0^\infty S_0(1 - c_1)^\gamma e^{c_1 y} * N\left(\frac{d/\sqrt{v}}{\sqrt{y}} + (\alpha + s)\sqrt{v}\sqrt{y}\right) \\ &\quad - Ke^{-rt}(1 - c_2)^\gamma e^{c_2 y} * N\left(\frac{d/\sqrt{v}}{\sqrt{y}} + \alpha\sqrt{v}\sqrt{y}\right) \frac{y^{\gamma-1} e^{-y}}{\Gamma(\gamma)} dy \end{aligned} \tag{4.15}$$

Let

$$\Psi(a, b, \gamma) = \int_0^\infty N\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) \frac{u^{\gamma-1} e^{-u}}{\Gamma(\gamma)} du$$

We obtain the pricing model under a risk-neutral measure as

$$\begin{aligned} &C(S_0; K, t) \\ &= S_0 \Psi\left(d\sqrt{\frac{1 - c_1}{v}}, (\alpha + s)\sqrt{\frac{v}{1 - c_1}}, \frac{t}{v}\right) \\ &\quad - K \exp(-rt) \Psi\left(d\sqrt{\frac{1 - c_2}{v}}, (\alpha + s)\sqrt{\frac{v}{1 - c_2}}, \frac{t}{v}\right) \end{aligned} \tag{4.16}$$

4.2.3 Pricing Model with Fuzzy Random Variables

The financial system addresses the issues of asset pricing and market efficiency under risk conditions (mainly) based on microeconomics. Owing to the existence of factors such as information asymmetry, individual judgement, and different risk preferences, the financial market is an incomplete market whose incompleteness is not only random but also fuzzy. Introducing the concept of fuzzy random variables to effectively combine randomness and fuzziness can provide more effective investment decision-making. To establish an options pricing model in a fuzzy environment, consider using the fuzzy yield rate, fuzzy volatility and fuzzy jump arrival rate in place of the yield, volatility and jump arrival rate, that is, these three fuzzy random variables are used instead of the corresponding random variables. The remaining variables are kept fixed, and the final obtained option price is also a fuzzy number. In accordance with the above method, the pricing formula for European call options is as follows:

$$
\begin{aligned}
&\tilde{\mathrm{C}}(S_0; K, t) \\
&\quad = \tilde{\mathrm{C}}(S_0; K, t, \tilde{\theta}, \tilde{\sigma}, \tilde{v}) \\
&\quad = S_0 \Psi\left(\tilde{d}\sqrt{\frac{1-\tilde{c_1}}{\tilde{v}}}, (\alpha+\tilde{s})\sqrt{\frac{\tilde{v}}{1-\tilde{c_1}}}, \frac{t}{\tilde{v}}\right) \\
&\qquad - K\exp(-rt)\Psi\left(\tilde{d}\sqrt{\frac{1-\tilde{c_2}}{\tilde{v}}}, (\alpha+\tilde{s})\sqrt{\frac{\tilde{v}}{1-\tilde{c_2}}}, \frac{t}{\tilde{v}}\right)
\end{aligned} \tag{4.17}
$$

of which,

$$
\begin{aligned}
&\tilde{s} = \frac{\tilde{\sigma}}{\sqrt{1+\left(\frac{\tilde{\theta}}{\tilde{\sigma}}\right)^2 \frac{\tilde{v}}{2}}}, \quad \tilde{c_1} = \frac{\tilde{v}(\alpha+\tilde{s})^2}{2}, \quad \tilde{c_2} = \frac{\tilde{v}\alpha^2}{2}, \\
&\tilde{d} = \frac{1}{\tilde{s}}\left[\ln\frac{S_0}{K} + rt + \frac{t}{\tilde{v}}\ln\left(\frac{1-\tilde{c_1}}{1-\tilde{c_2}}\right)\right].
\end{aligned}
$$

According to the extension principle, the membership function of $\tilde{\mathrm{C}}(S_0; K, t, \tilde{\theta}, \tilde{\sigma}, \tilde{v})$ is

$$
\mu_{\tilde{\mathrm{C}}_t}(c) = \sup_{\{(\theta,\sigma,v):c=C(S_0;K,t,\theta,\sigma,v)\}} \min\{\mu_{\tilde{\theta}}(\theta), \mu_{\tilde{\sigma}}(\sigma), \mu_{\tilde{v}}(v)\} \tag{4.18}
$$

For the fuzzy price of C of $\tilde{\mathrm{C}}(S_0; K, t, \tilde{\theta}, \tilde{\sigma}, \tilde{v})$ at t, the membership λ is required. If the financial analyst is willing to accept the option price C with said membership, it is possible to set the option price at time t as C because he/she is satisfied with the membership. The membership function of $\tilde{\mathrm{C}}$ can also be written as $\mu_{\tilde{\mathrm{C}}_t}(c) = \sup_{0\le\lambda\le1} \lambda \cdot 1_{(\tilde{\mathrm{C}}_t)_\lambda}(c)$ where $(\tilde{\mathrm{C}}_t)_\lambda$ is the λ-level set of $\tilde{\mathrm{C}}_t$. As the λ-level sets of $\tilde{\theta}$, $\tilde{\sigma}$ and $\tilde{v}$ are $\tilde{\theta}_\lambda = [\tilde{\theta}^L_\lambda, \tilde{\theta}^U_\lambda]$, $\tilde{\sigma}_\lambda = [\tilde{\sigma}^L_\lambda, \tilde{\sigma}^U_\lambda]$ and

$\tilde{v}_\lambda = [\tilde{v}^L_\lambda, \tilde{v}^U_\lambda]$, respectively, the λ-level set of $\tilde{\mathrm{C}}_t$ can be expressed as

$$
\begin{aligned}
(\tilde{\mathrm{C}}_t)_\lambda &= [(\tilde{\mathrm{C}}_t)^L_\lambda, (\tilde{\mathrm{C}}_t)^U_\lambda] \\
&= \Bigg[\min_{\tilde{\theta}^L_\lambda \le \theta \le \tilde{\theta}^U_\lambda, \tilde{\sigma}^L_\lambda \le \sigma \le \tilde{\sigma}^U_\lambda, \tilde{v}^L_\lambda \le v \le \tilde{v}^U_\lambda} \mathrm{C}(S_0; K, t, \theta, \sigma, v), \\
&\qquad \max_{\tilde{\theta}^L_\lambda \le \theta \le \tilde{\theta}^U_\lambda, \tilde{\sigma}^L_\lambda \le \sigma \le \tilde{\sigma}^U_\lambda, \tilde{v}^L_\lambda \le v \le \tilde{v}^U_\lambda} \mathrm{C}(S_0; K, t, \theta, \sigma, v) \Bigg]
\end{aligned}
\tag{4.19}
$$

Different investors use different measures to determine the vagueness of the parameters used in pricing options. Therefore, even for the same λ-level, different investors may obtain different price ranges. This is not conducive to unified pricing, so it is necessary to convert the number of intervals into an exact number. The conversion process is called "de-fuzzification". In this study, we chose the general de-f ller and Majlender [58], and obtains a fuzzy expectation with upper and lower weights of the λ-level set of $\tilde{\mathrm{C}}_t$ with the following formula:

$$
\begin{aligned}
M(\tilde{C}) &= \frac{M(\tilde{C})^L + M(\tilde{C})^U}{2} \\
&= \frac{\int_0^1 f(\lambda)\tilde{C}^L_\lambda d\lambda + \int_0^1 f(\lambda)\tilde{C}^U_\lambda d\lambda}{2} \\
&= \int_0^1 \frac{f(\lambda)}{2}(\tilde{C}^L_\lambda + \tilde{C}^U_\lambda) d\lambda
\end{aligned}
\tag{4.20}
$$

4.3 Monte Carlo Numerical Simulation

In this chapter, we utilise numerical simulations and empirical tests to illustrate the applications of the VG model under a fuzzy environment. To highlight the proposed models superiority, the pricing results were compared to those obtained under the BS options pricing model and VG model under a crisp environment, respectively. All experiments were performed on a Dell VOSTRO personal computer which installed MATLAB2015b on Windows 10 and its configuration is i7-2600 CPU 3.40GHz with 16GB RAM. In terms of the pricing method, this paper uses the Monte Carlo simulation method through Matlab programming language to achieve, especially for fuzzy option pricing, the options price interval was divided into 15000 divisions, and then the membership degree of each sample point was obtained to produce the membership function of fuzzy options price. The Monte Carlo simulation method is based on the law of large numbers and is used to calculate the average return of the option by simulating the price paths of the underlying asset to obtain an estimated value of the option. The advantage of this method is that it is still applicable even when the function of the underlying asset is relatively complicated, and the simulation time required increases linearly rather than geometrically with the number of variables.

European call options are the object of study in this chapter. Two simulation experiments were set up. In the first experiment, the time to expiration of the options was kept unchanged while the exercise price was gradually increased. The VG model in a fuzzy environment was studied, in addition to how the options price varies in a crisp environment both under the VG model and the BS model. The sensitivity of options price to changes in the jump parameter, v, was also analysed. In the second experiment, the exercise price was kept unchanged while the time to expiration of options was gradually increased. The VG model in a fuzzy environment was studied, in addition to how the options price varies in a crisp environment both under the VG model and the BS model. At the same time, this study analysed how to improve the convergence rate of the Monte Carlo simulations.

4.3.1 Pricing of European Options with Exercise Price Variations

Before applying the BS and VG methods to the pricing of the options, the initial value of the underlying asset and the model parameters need to be set. In the following numerical analysis, it is assumed that the initial price of the underlying asset is $S_0 = 100$, the annual risk free rate is $r = 0.02$, the exercise price at expiration is $K = 95$, and the time to expiration is half a year, i.e., $T = 0.5$ year. Regarding the BS model parameters, the yield volatility σ is set to 0.2. Regarding the parameters of the VG model in a crisp environment, the drift θ is set to 0.05, the diffusion σ is set to 0.2, and the jump parameter v is 1.5. Regarding the VG model parameters under a fuzzy environment, this paper utilises the most commonly used trapezoidal function as the membership function of the fuzzy random variables $\tilde{\theta}$, $\tilde{\sigma}$ and $\tilde{v}$ such that $\tilde{\theta} = [0.035, 0.04, 0.05, 0.055]$, $\tilde{\sigma} = [0.14, 0.19, 0.25, 0.28]$, $\tilde{v} = [1.25, 1.5, 1.75, 2]$ and the confidence level $\lambda = 0.8$. Similar analysis can also be performed on other types of membership functions. For the Monte Carlo simulation, the time period is divided into M intervals with a total simulation N times, where $M = 10$, and $N = 100000$.

The pricing of options with different exercise prices under the BS and VG models in a crisp environment and VG model in a fuzzy environment is shown in Table 4.1 and Figure 4.2. To unify the pricing, we calculate the expected price of the fuzzy option by using equation (4.20) with $f(\lambda) = 2\lambda$. It can be observed that the higher the exercise price, the lower the option price, which is consistent with the actual situation, because the option price is bullish for European call option. As such, the higher the exercise price, the lower the return and thus the lower the price. The simulation results by VG model in a fuzzy environment are higher than that by VG model in a crisp environment at the exercise prices which except for 82, 83, 88 and 89; The simulation results by VG model in crisp environment are higher than those by BS model in a crisp environment at different exercise prices. To easier observe the simulation results corresponding to the different exercise prices, we shown the part of simulation results by enlarge way in a rectangular frame at exercise price from 89 to 95 in Figure 4.2. This is also consistent with our hypothesis because the BS model is based on Brownian motion and the normal distribution and does not take into account large scale fluctuations in asset prices, whereas the options pricing model under a fuzzy environment takes into account the most random factors and uncertainties. The fuzzy range basically covers all the option prices of the VG model in a crisp environment but not those under the BS model. This result demonstrates that pricing under the BS model pricing is rather different from the actual situation.

Table 4.2 analyses the sensitivity of the jump parameter v under the VG model in both crisp and fuzzy environments. With the exercise price set to $K = 95$, under the VG model in a crisp environment, when the value of v was gradually increased from 1 to 4, the option price dropped from 11.361 to 6.437. For the VG model in fuzzy environment, as the interval of the trapezoidal membership parameter v was increased, the fuzzy interval of the option gradually narrowed, and the fuzzy expectation gradually decreased. The reason for this behaviour is that

Table 4.1: Variation of Option Price with Exercise Price

Sequence	Exercise Price	BS	Crisp VG Model	Fuzzy Expectation	Fuzzy Interval
1	81	19.925	20.625	20.804	[20.561,21.047]
2	82	19.139	19.800	19.650	[19.427,19.873]
3	83	17.977	18.957	18.826	[18.583,19.069]
4	84	17.325	18.006	18.393	[18.081,18.704]
5	85	16.337	16.967	17.209	[16.921,17.496]
6	86	15.532	16.109	16.172	[15.890,16.454]
7	87	14.693	15.335	15.809	[15.449,16.168]
8	88	13.713	14.480	14.035	[13.772,14.298]
9	89	13.097	13.607	13.490	[13.173,13.807]
10	90	12.165	12.728	12.883	[12.518,13.247]
11	91	11.432	11.965	12.248	[11.839,12.656]
12	92	10.656	11.179	11.505	[11.065,11.944]
13	93	10.136	10.544	11.226	[10.697,11.754]
14	94	8.973	9.709	9.945	[9.453,10.437]
15	95	8.725	9.261	9.512	[8.950,10.073]
16	96	8.036	8.333	8.748	[8.157,9.338]
17	97	7.260	7.650	8.674	[7.967,9.380]
18	98	6.913	7.181	7.825	[7.201,8.449]
19	99	6.278	6.596	7.185	[6.616,7.754]
20	100	5.702	5.962	6.381	[5.888,6.873]
21	101	5.420	5.590	6.153	[5.562,6.743]
22	102	4.776	4.848	5.316	[4.549,6.083]
23	103	4.614	4.762	4.906	[4.449,5.363]
24	104	4.202	4.593	4.792	[4.222,5.362]
25	105	3.751	4.034	4.382	[3.607,5.156]
26	106	3.262	3.688	4.287	[3.525,5.049]
27	107	3.097	3.564	3.952	[2.973,4.931]
28	108	2.723	3.479	3.818	[2.890,4.746]
29	109	2.348	3.215	3.434	[2.392,4.476]
30	110	2.289	2.891	3.161	[2.155,4.166]

Note: “Sequence” obtained by sort ascending on the exercise price from 81 to 100, 1-30 represent the sequence number.

in the event of large jumps in market prices, investors in European call options investors will consider the increase in investment risk factors; consequently, their valuation of the options would decrease correspondingly.

With the exercise price fixed to $K = 95$, the Matlab programming language was used to plot the graph of the membership function for options price obtained using the VG model under a fuzzy environment. Firstly, the options price interval was divided into 15000 divisions,

Table 4.2: Option Price under Various Jump Intensities

Crisp VG Model	Jump v value	1.000	1.250	1.500	1.750	2.000	2.250	2.500
	Option price	11.361	9.939	9.028	8.544	7.999	7.809	7.527
	Jump v value	2.750	3.000	3.250	3.500	3.750	4.000	
	Option price	7.184	6.956	6.959	6.717	6.640	6.437	
Fuzzy VG Model	Jump v interval	[1.2,1.45,1.7,1.95]	[1.25,1.5,1.75,2]	[1.3,1.55,1.8,2.05]				
	Fuzzy expectation	9.999	9.986	9.976				
	Fuzzy interval	[9.303,10.694]	[9.364,10.608]	[9.478,10.474]				

and then the membership degree of each sample point was obtained to produce the graph that is shown in Figure 4.3. It can be observed from the figure, which is increasing on the left-hand side and decreasing on the right-hand side. The increasing portion shows the sellers satisfaction increasing as the price rises, whereas the decreasing portion on the right shows the buyers satisfaction decreasing as the price rises. The ranges of values for options at different levels of confidence λ are presented in Table 4.3. When $\lambda = 0.9$, the corresponding option price is in the closed interval $[9.440, 10.517]$, thus indicating that if the investor is satisfied with the confidence level of 0.9, then he/she can choose any number from the closed interval $[9.440, 10.517]$ as the options price. It can be observed more clearly from Figure 4.4, with an increase in the confidence level λ,that the fuzzy interval narrows as the confidence level increases, but the fuzzy expectation basically remains unchanged, thus indicating that the membership function of the option price is a symmetrical one.

4.3.2 How European Options Price Varies with Time to Expiration Changes

Table 4.4 and Figure 4.5. below show the options pricing result under the BS model, VG model in a crisp environment, and VG model in a fuzzy environment for different time to expiration. From the figure, it can be clearly observed that as the time to expiration increases, the option price becomes higher because a longer period means more uncertainties, and the option price will increase as a result. At the same time, the difference between the VG model and BS model increases as the time to expiration increases. The longer the time to expiration, the higher

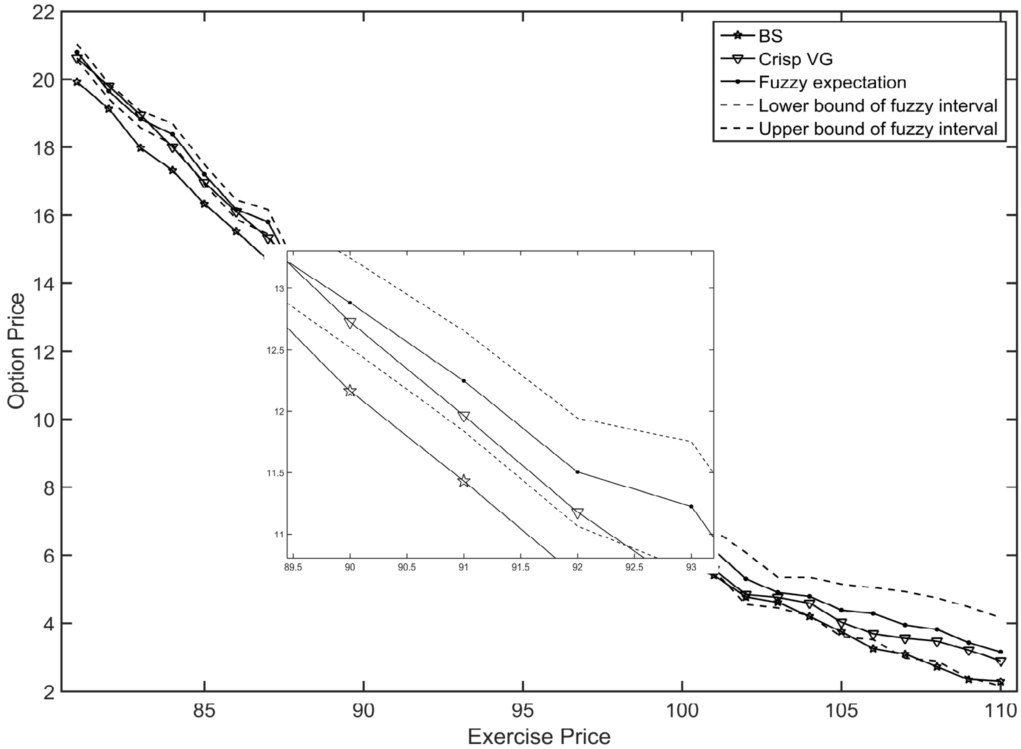


Figure 4.2: Variation of Option Price with Exercise Price

Table 4.3: Option Price at Various Confidence Levels λ

λ	Fuzzy expectation	Fuzzy Interval
0.7	9.992	[9.284,10.699]
0.75	9.989	[9.322,10.656]
0.8	9.986	[9.364,10.608]
0.85	9.979	[9.397,10.560]
0.9	9.979	[9.440,10.517]
0.95	9.976	[9.478,10.474]

the probability of jump(s) and thus the greater the error between the BS model calculation and the true value. Comparatively, the VG model calculation is more accurate. The option prices under the BS model basically fall outside the fuzzy interval, whereas those under the VG model in a crisp environment lie within the fuzzy interval. This further illustrates that there is a relatively large difference between the BS options pricing model and the real situation.

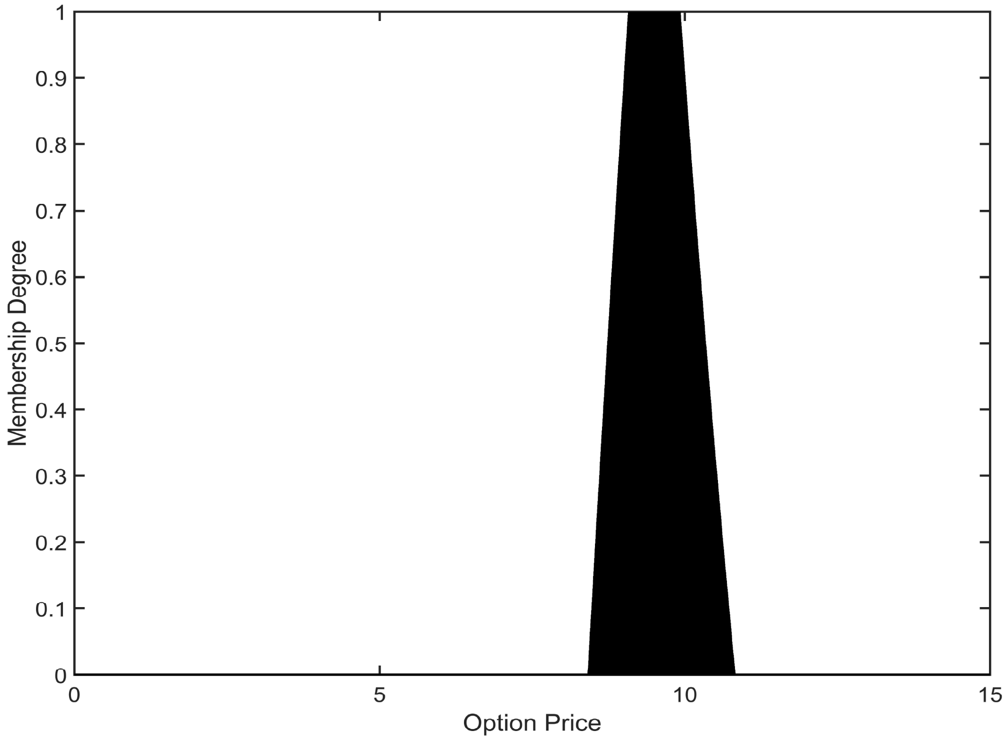


Figure 4.3: Membership Function of the Fuzzy Option Price

4.3.3 The Improvement of Convergence speed for the Monte Carlo Algorithm

The Monte Carlo simulation method is based on the law of large numbers and can be used to estimate the true value by using the sample mean of sufficient random simulations; the estimation error will gradually converge as the number of simulations increases. However, the convergence speed is inversely proportional to the variance between the samples. If the variance between the simulated values can be reduced, for the same simulation time, the Monte Carlo estimation results will be more accurate. At present, the instrumental variable method, antithetic variable method and importance sampling are the most widely used variance reduction techniques. The antithetic variable method mainly reduces the variance by using random numbers that are negatively correlated with the mean. However, the property of gamma random numbers means that negatively correlated random numbers do not exist; consequently, the instrumental variable method was chosen to reduce the variance.

In the method chosen, a instrumental variate Y that is related to the variable X is chosen to construct the synthetic variable. Variance between the simulated samples is reduced while ensuring that the mean value remains unchanged. When using the Monte Carlo algorithm for options pricing, the price of the underlying asset is usually selected as the instrumental variate for analysis. According to the European call options pricing formula in this paper, $C = e^{-rT}(\max((S_T - K), 0))$, $e^{-rT}S_T$ is chosen as the instrumental variate, and the corresponding instrumental variate estimate is

$$\bar{C} = \frac{1}{N}\sum_{i=1}^{N}\big(C_i - a_i(e^{-rt}S_T^i - S_0)\big),$$

$$\text{in which, } a_i = \operatorname{cov}(C_i, S_i)/\operatorname{var}(S_i) \tag{4.21}$$

Figure 5.9 shows the pricing results obtained by combining Monte Carlo simulations with the instrumental variable method under the VG model in a crisp environment. It shows the rate of convergence against the number of simulations. In the case of an ordinary Monte Carlo simulation, at least 6000 simulations are needed before the simulation outcome converges sufficiently accurately to the mean. After using the variance reduction method, the simulation outcome has been controlled within a reliable range with less than 3000 simulations. This shows that in options pricing, the instrumental variable method chosen can effectively reduce the variance and reduce the number of simulations required, thus improving the pricing efficiency. The convergence efficiency improved by 50% via the improved Monte Carlo algorithm.

4.4 Empirical Studies

To test the efficacy of fuzzy options pricing, this chapter chose Tencent Holdings (HK.0700) as the underlying asset to study a set of its European call options traded on the Hong Kong Stock Exchange.

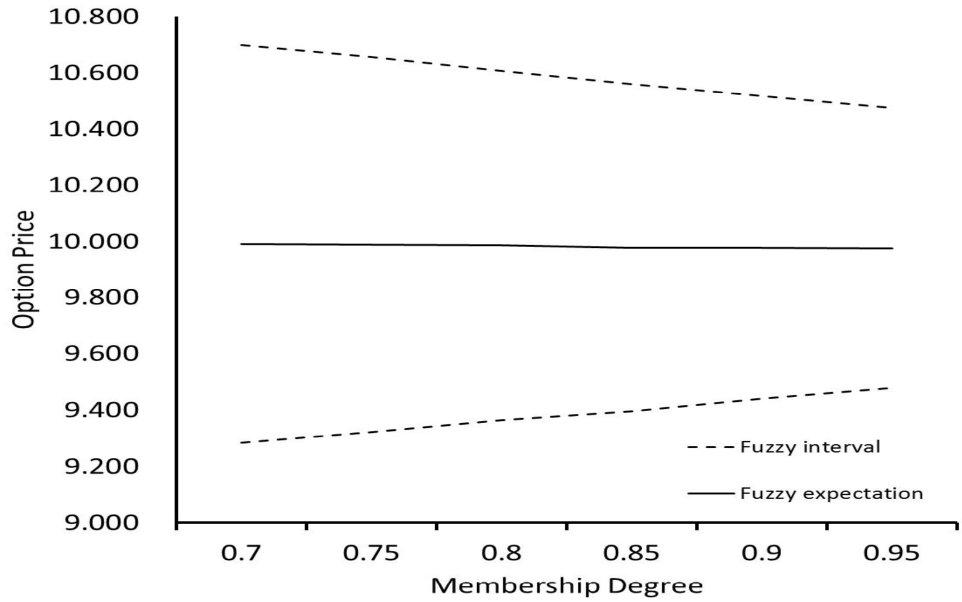


Figure 4.4: Option Price at Various Confidence Levels λ

4.4.1 The Statistical Characteristics of the Rate of Return

The price of Tencent Holdings (HK.0700) at the close of trading (data were obtained from the wind information database) for the period from 15 May 2014 to 4 Nov 2016, a total of 591 trading days, was used as the raw data. From this data, 590 daily logarithmic returns were obtained. Figure 4.7 shows that rate of return is volatile. Descriptive statistics regarding the data are presented in Table 4.5. The skewness of the samples is $0.4579 > 0$; this indicates that the samples are skewed to the right. The kurtosis is $5.7828 > 3$; this indicates that the samples exhibit leptokurtic fat-tailed characteristics.

The probability density distribution of the samples shows that the distribution of daily returns has a higher kurtosis and thicker tail than the normal distribution. At the same time, the Q-Q (Quantile-Quantile) figure shows multiple sample points deviating from the straight line, with the deviation most significant for extreme values. This shows that the daily returns are not normally distributed (as shown in Figure 4.8, 4.9).

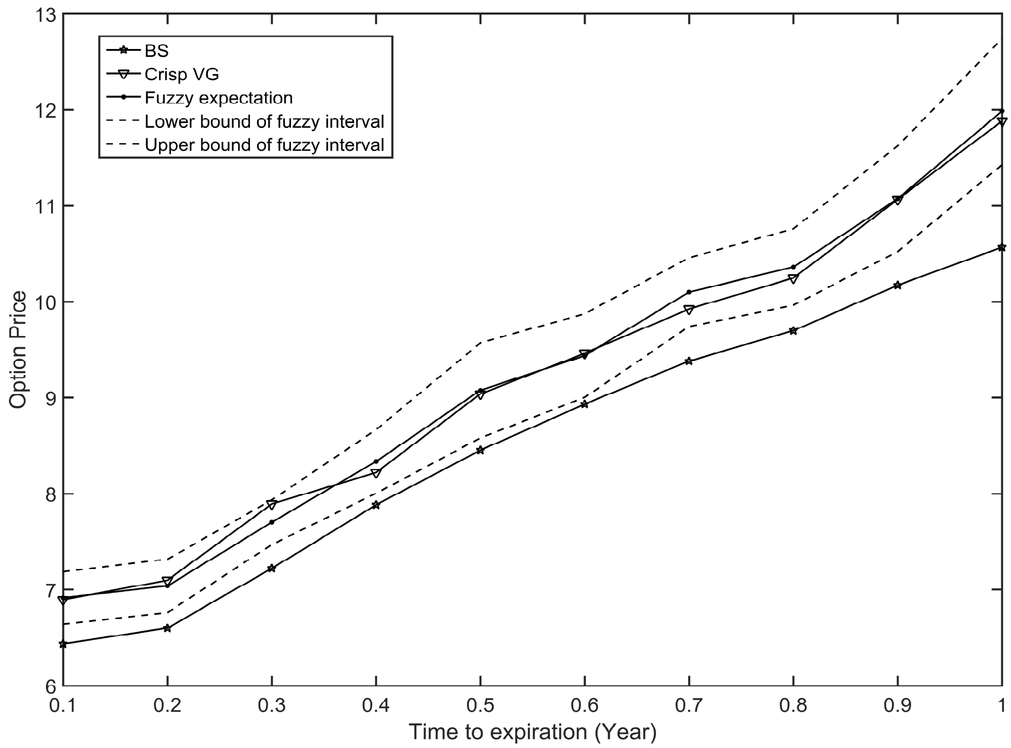


Figure 4.5: Variation of Option Price with Time to Expiration

Table 4.4: Variation of Option Price with Time to Expiration

Sequence	Time to expiration(year)	BS	Crisp VG model	Fuzzy expectation	Fuzzy interval
1	0.1	6.436	6.894	6.915	[6.641,7.188]
2	0.2	6.599	7.100	7.040	[6.761,7.319]
3	0.3	7.223	7.894	7.704	[7.471,7.936]
4	0.4	7.885	8.222	8.337	[8.005,8.668]
5	0.5	8.454	9.039	9.077	[8.582,9.572]
6	0.6	8.933	9.465	9.440	[9.004,9.876]
7	0.7	9.383	9.925	10.102	[9.743,10.461]
8	0.8	9.700	10.249	10.363	[9.964,10.762]
9	0.9	10.173	11.066	11.074	[10.525,11.623]
10	1	10.571	11.883	11.986	[11.228,12.743]

Note: "Sequence" obtained by sort ascending on the time to expiration from 0.1 to 1 year, 1-10 represent the sequence number.

Table 4.5: Descriptive Statistics regarding the Daily Logarithmic Returns of Tencent Holdings (HK.0700)

Sample size	Mean value	Maximum value	Minimum value	Standard deviation	Skewness	Kurtosis
590	0.0045	0.0410	-0.0308	0.0780	0.4579	5.7828

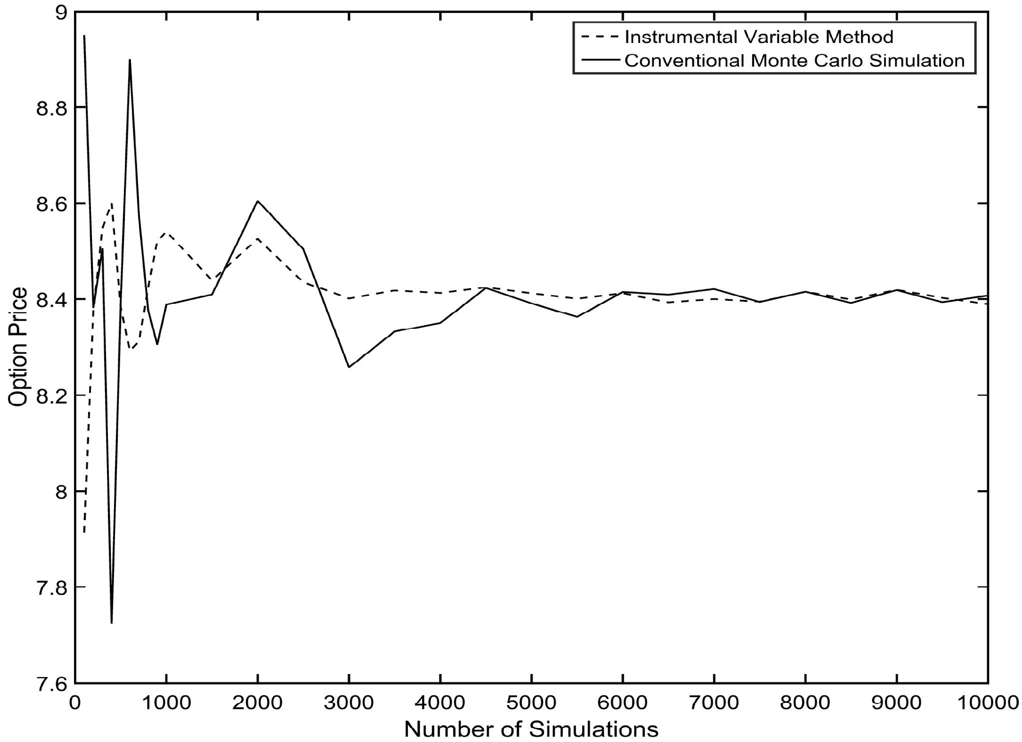


Figure 4.6: Convergence process of Using Instrumental Variable Method in the Monte Carlo Simulation

4.4.2 Parameter Estimation

Before analysing the model, the model parameters θ, σ and v need to be estimated. Under the VG model in a crisp environment, the parameters θ, σ and v were estimated using the moment estimation method on the historical rates of return from the 590 trading days prior to 4 Nov 2016. Because the three parameters, $\tilde{\theta}$, $\tilde{\sigma}$ and $\tilde{v}$ are trapezoidal fuzzy numbers, four parameter values are required for the parameter range of each fuzzy number. The difference between the sample interval and the number of samples will result in different estimation results, thus reflecting different market information. To include more market information, we select the historical rates of return for 120, 240, 360 and 590 trading days before 4 Nov 2016 as the observation samples and obtain the parameter values needed by the trapezoidal fuzzy numbers using the moment estimation method. The parameters estimation results are presented in Table 4.6. The result of the parameters estimation shows that the parameters θ, σ and v are significantly not 0 at the 1% level of significance.

Table 4.6: Model parameters

Parameter		θ	σ	v
Crisp model		0.05(2.3380)***	0.40(7.4339)***	1.74(7.0915)***
Trapezoidal Fuzzy Number	a_1	0.01(3.6094)***	0.38(7.0550)***	1.74(7.0915)***
	a_2	0.02(3.2116)***	0.40(7.4339)***	3.20(6.5430)***
	a_3	0.03(2.7671)***	0.41(11.4326)***	5.64(13.8195)***
	a_4	0.05(2.3380)***	0.53(9.8515)***	7.03(11.0294)***

Note: Bracketed number is the t statistic of the parameter, *, **, and *** represent significance at the 10%, 5%, and 1% level of significance, respectively.

4.4.3 Empirical Analysis

From the derivatives database of the Hong Kong Stock Exchange's official website, the price at the end of the trading day for 20 stock call options of Tencent Holdings (HK.0700) as of 4 Nov 2016 were downloaded. The closing price of said stock was 200 HKD, and the options exercise date was May 2017. Thus, $S_0 = 200$ HKD, the time to expiration $T = 0.417$ year. A one-year-fixed deposit rate was $r = 1.7\%$ was chosen as the risk-free rate. The detailed options pricing results are presented in Table 4.7 and Figure 4.10. The simulation results show that when the options exercise price is less than 170, the model is more consistent with the market price. Specifically, the simulation results of the BS model are all less than the market price, whereas the simulation results of the VG model in a crisp environment and the fuzzy expectation of the VG model in a fuzzy environment are both greater than the market price. The fuzzy expectation is slightly greater than the prices under the VG model in a crisp environment. Nine of the market price (data points) are in the fuzzy interval, with another 2 outside. This indicates that the market price of the option is better covered by the fuzzy price range. At part of different exercise prices, the comparison of the market price and simulated price can be more clearly observed by the enlarge rectangular frame in Figure 4.10. When the exercise price is greater than 170, the simulated price under the model gradually departs from the market price, with the deviation increasing as the exercise price increases. This behaviour means that the volatility information contained in the model does not fully explain fluctuations in the option price, or that in the eyes of the investor, the volatility of the stock price is more than the jump and vagueness of the parameters; rather, it is more greatly influenced by

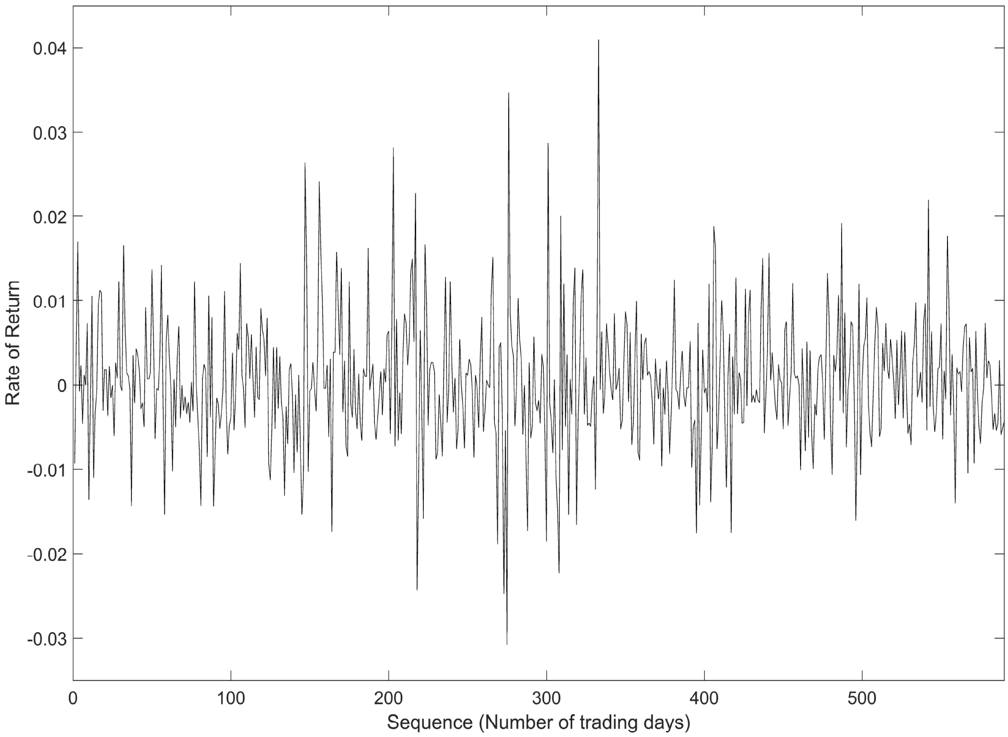


Figure 4.7: Daily Logarithmic Returns of Tencent Holdings (HK.0700).

macroeconomic conditions and the company's health, for example. However, when λ increases, the fuzzy price interval narrows, and more of the market price data points fall outside the price interval. In using the options pricing model in a fuzzy environment, the investor is free to choose an acceptable level set λ to obtain a fuzzy price interval. This can be used to predict the future option price and guide the investor. If the actual price is greater than the upper bound, this means that the options are overvalued, and investors should sell the options; if the actual price is below the lower bound, this means that the options are undervalued, and investors should buy them.

To compare the pricing result of the BS model in a crisp environment, the VG model in a crisp environment, and the VG model in a fuzzy environment. Two statistical indicators, the root mean square error (RMSE) and the average absolute error (AAE), were used to compare the pricing result of the models, and the results are presented in Table 4.8. These two indicators are used to measure the difference between the pricing result and the market price. The smaller

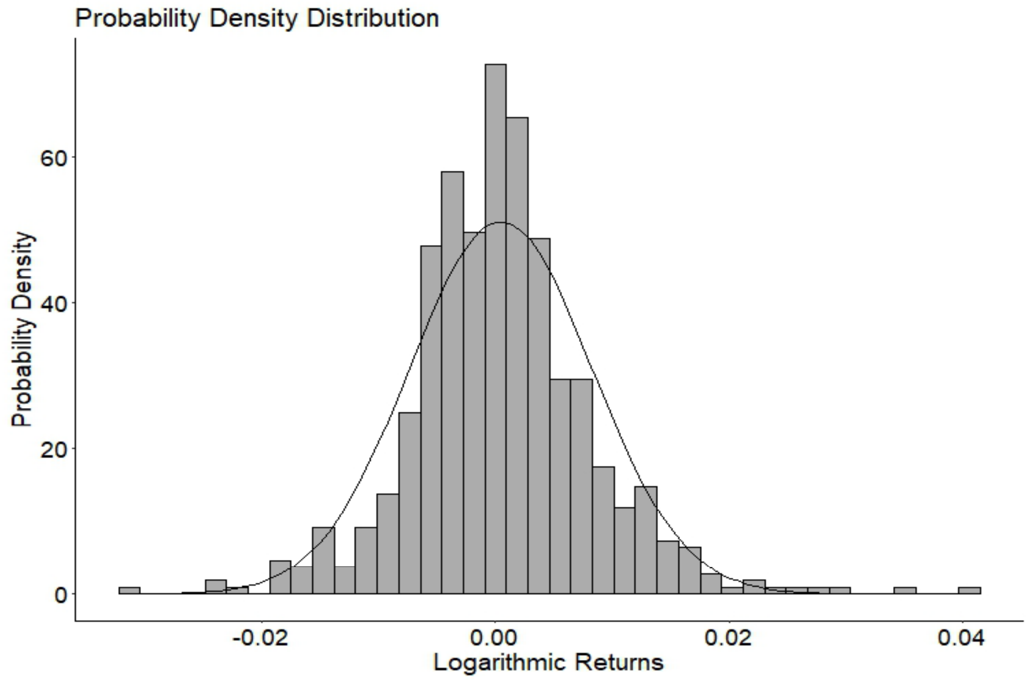


Figure 4.8: Probability Density Function of Daily Returns

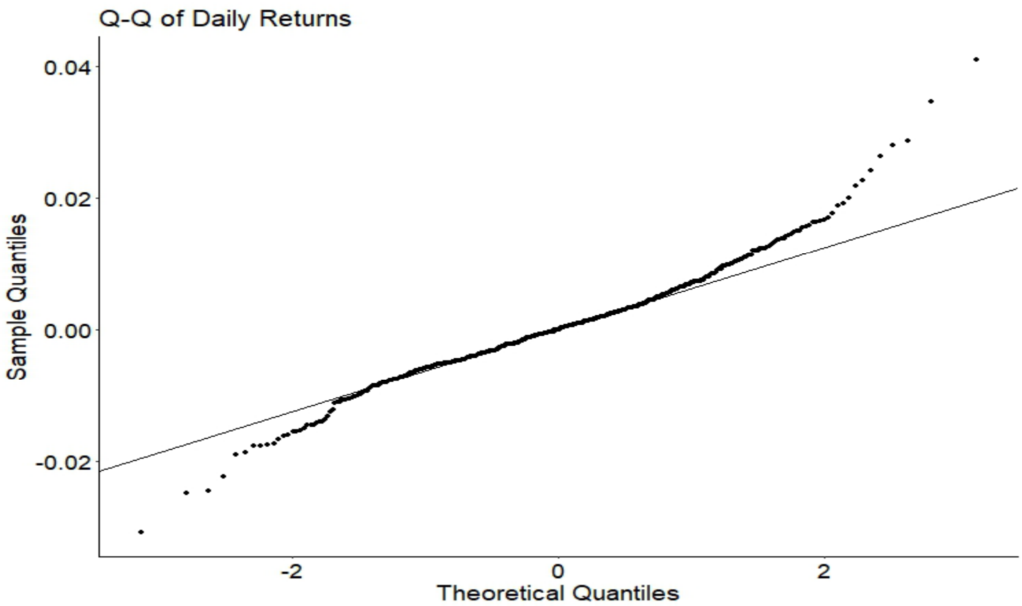


Figure 4.9: Q-Q (Quantile-Quantile) Figure of Daily Returns

the value is, the higher the pricing accuracy. The two indicators are calculated as follows:

$$\mathrm{RM}SE = \sqrt{\sum_{i=1}^{N} \frac{(C_i^{Model} - C_i^{Market})^2}{N}} \tag{4.22}$$

$$AAE = \frac{\sum_{i=1}^{N} |C_i^{Model} - C_i^{Market}|}{N} \tag{4.23}$$

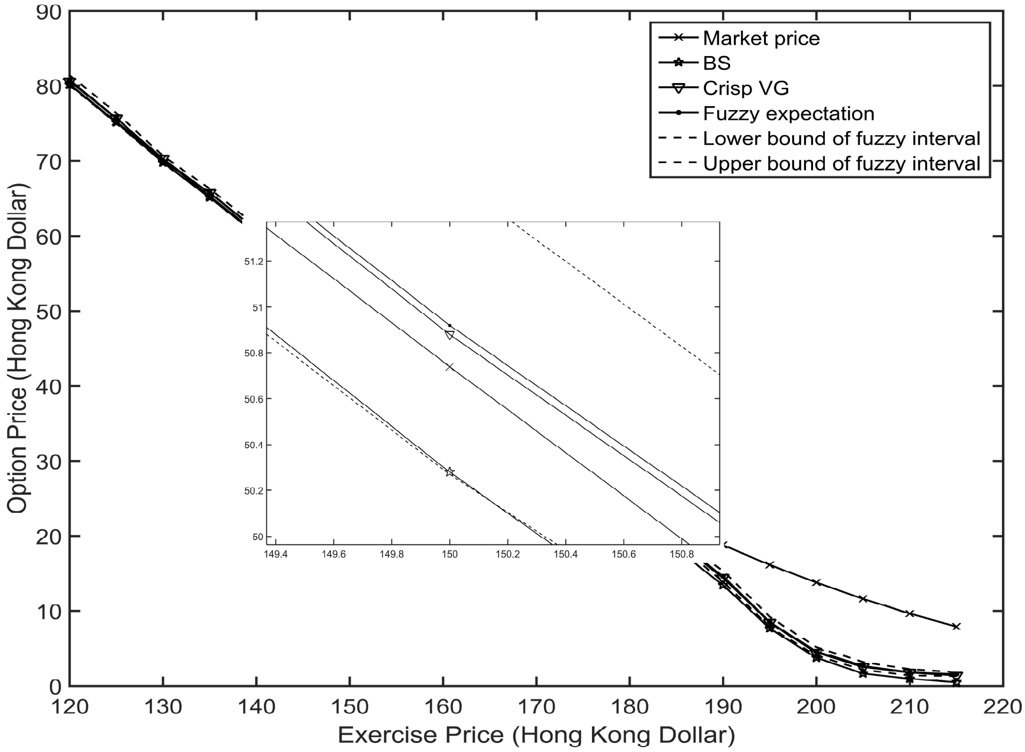


Figure 4.10: Option Pricing Results for May 2017 expiry (Hong Kong Dollar)

It can be observed from Table 4.8 that both indicators indicate that the pricing result of the BS model has the greater error compared with the actual data, whereas the VG model in a fuzzy environment has the smallest error. Regarding the fuzzy expectation, the error is lower than the pricing result of the VG model in a crisp environment. Furthermore, we calculate the accuracy rate by mean absolute percentage error (MAPE) to test our model efficacy, data for exercise price less than 190 (including 190) were chosen, and the results are presented in Table 4.9. It can be observed from Table 4.9 that the fuzzy VG process option pricing model achieved 96.68% accuracy rate which is an improvement of 1.33% over the crisp BS model. The variance of the accuracy rate of the proposed fuzzy model is 56.77% of that of the crisp BS model, it is less than the crisp BS model; this shows that the proposed fuzzy model is more stable than the crisp BS model in terms of pricing accuracy rate. The results indicate that the fuzzy VG process option pricing model is feasible and its pricing results are more accurate and stable even when many reality uncertainty factors are included. From this, it can be observed that the use of the fuzzy VG model for options pricing has better accuracy and stable than the BS model. The accuracy rate and MAPE evaluation are calculated as follows,

Table 4.7: Option Pricing Results (Hong Kong Dollar)

Sequence	Exercise price	Market price	BS	Crisp VG model	Fuzzy expectation	Fuzzy interval
1	120	80.21	80.15	80.60	80.74	[80.05,81.43]*
2	125	75.23	75.14	75.70	75.74	[75.06,76.41]*
3	130	70.26	69.79	70.26	69.99	[69.72,70.66]*
4	135	65.31	65.13	65.73	65.80	[65.26,66.34]*
5	140	60.39	60.21	60.74	60.81	[60.35,61.27]*
6	145	55.53	55.28	55.84	55.80	[55.11,56.49]*
7	150	50.74	50.28	50.88	50.92	[50.27,51.57]*
8	155	46.05	45.82	46.48	46.54	[46.15,46.93]
9	160	41.50	41.13	41.48	41.35	[40.88,41.82]*
10	165	37.11	36.19	37.01	37.19	[36.86,37.52]*
11	170	33.81	32.19	33.07	33.09	[32.43,33.75]
12	175	28.98	27.27	27.61	27.71	[27.03,28.39]
13	180	25.29	23.32	24.20	24.27	[23.85,24.69]
14	185	21.88	18.32	19.27	19.34	[19.06,19.62]
15	190	18.77	13.45	14.44	14.65	[13.92,15.38]
16	195	16.17	7.68	8.44	8.52	[7.83,9.21]
17	200	13.81	3.76	4.44	4.61	[4.04,5.18]
18	205	11.60	1.67	2.55	2.68	[2.17,3.19]
19	210	9.63	0.95	1.79	1.83	[1.40,2.26]
20	215	7.92	0.46	1.40	1.57	[1.31,1.83]

Note: "Sequence" obtained by sort ascending on the exercise price from 120 to 215 Hong Kong Dollars, 1-20 represent the sequence number; * shows that the market price is within the fuzzy interval.

Table 4.8: Comparison of the Pricing Result of the Different Models

Indicator	BS	Crisp VG	Fuzzy Expectation (our model)
RMSE	4.7679	4.2641	4.1946
AAE	3.0996	2.6645	2.6488

$$\text{Accuracy rate } = (1 - \text{ MAPE}) \times 100\%$$

$$\text{MAPE} = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{C_i^{Model} - C_i^{Market}}{C_i^{Market}}\right| \tag{4.24}$$

Table 4.9: Comparison of the accuracy rate of the different models

Indicator	Accuracy rate	Sample Variance
BS	95.35%	62.32
Fuzzy Expectation(our model)	96.68%	35.38
Compare to BS	1.33%(Improved)	56.77%

4.5 Conclusions

In this chapter, on the basis of conventional Black Scholes (BS) model, we incorporates fuzzy set theory to construct an European option pricing model with VG (variance gamma) process (which is one of widely used infinite pure-jump Levy processes) in a fuzzy environment. The drift, diffusion, and jump are treated as the trapezoidal fuzzy random variables in the model.

The Monte Carlo simulation algorithm was then used to provide simulation estimates for the model, and an empirical analysis was performed using Tencent Holdings (HK.0700) and its stock options data. A comparison of the option price under the VG (variance gamma) process model in a fuzzy environment, BS (Black Scholes) model in a crisp environment, and the VG (variance gamma) process model in a crisp environment yielded the following:

1. An analysis of the Monte Carlo numerical simulations and the empirical analysis which uses Tencent Holding (HK.0700) and its stock options data show that treating the drift, diffusion and jump as fuzzy random variables to obtain the options pricing model is more reasonable, the fuzzy interval can cover the market prices of options and the prices that obtained by the crisp VG process option pricing model, moreover, the expectations using fuzzy pricing are closer to the market prices of options than the pricing results obtained by the crisp BS (Black-Scholes) model. According to the evaluation based on the mean absolute percentage error (MAPE), the fuzzy VG process option pricing model achieved 96.68% accuracy rate which is an improvement of 1.33% over the crisp BS model. Furthermore, the variance of the accuracy rate of the proposed fuzzy model is 56.77% of that of the crisp BS model, it is less than the crisp BS model; this shows that the proposed fuzzy model is more stable than the crisp BS model in terms of pricing accuracy rate. The results indicate that the fuzzy VG process option pricing model is feasible and its pricing results are more accurate and stable even when many reality

uncertainty factors are included. The results are more consistent with the real-life market and can provide investors with better investment advice.

2. The results show that expectation obtained through the fuzzy VG model is mostly greater than the pricing results obtained under the crisp VG model and that obtained using the crisp BS model. At the same time, at a confidence level of 0.8, the fuzzy interval basically encompasses the outcomes of the crisp VG model. On the other hand, the option price of the crisp BS model tends to be less than the fuzzy interval. This shows that the greater the number of random factors and uncertainties included in the model, the higher the option price, the results are consistent with the real-life market.

3. Both the VG model under a crisp environment and that under a fuzzy environment are sensitive to variations in the jump parameter. As the jump parameter increases, the option price decreases. At the same time, an increase in the confidence level also causes the fuzzy interval for the pricing model in a fuzzy environment to narrow.

4. The empirical analysis shows that the instrumental variable method can improve the convergence speed faster than the Monte Carlo simulation alone, the convergence efficiency of Monte Carlo algorithm can be improved by 50% via the instrumental variable method.

The object of study in this chapter was the relatively simple European call option, and the pricing of more complex financial asset was not considered. The form of the membership function chosen for the fuzzy random variables was trapezoidal, and other more complicated and reasonable membership function forms were not examined. Consequently, in next chapter, Chapter 4, we further extend our study into the pricing of the complicated options—American options.

Chapter 5

Fuzzy Levy-GJR-GARCH American Option Pricing Model

5.1 Introduction

An option is a kind of fundamental financial derivative, it represents a contract which offer a right to the buyer who can buy (call) or sell (put) a security or other financial assets at a agreed-upon price (the exercise price) without the obligation during a fixed period or on a specific date (exercise date). The buyer should pay the fee for the seller to obtain this right, this fee is called option price. An option includes call option and put option, call option offer a right to the buyer who can buy the underlying asset at a fixed price before or on a specific date, put option offer a right to the buyer who can sell the underlying asset at a fixed price before or on a specific date. In 1973, E. Black and M. Scholes wrote "The Pricing of Options and Corporate Liabilities" and proposed a comprehensive option pricing model that resolves the challenges in option pricing, which has contributed tremendously to the study of option pricing theory. However, the said pricing model primarily addresses pricing problems in European options; hence, it is unsuitable for the pricing of American options, which allow early exercise. Compared with European options, the pricing problems of American options are far more complicated because the holder of an option determines the best time to exercise the option by comparing the value of continuously holding the option at various points of time with the value of immediate option exercise. When the value of immediate option exercise is greater than the value of continuously holding the option, the option holder will choose to exercise the option immediately because that time is the best exercise time; this choice influences the pricing of the American option. A rational investor will choose the best time to exercise the option, which is known as the optimal stopping time problem in mathematics. From the perspective of partial differential equations, the problem is also considered a free-boundary problem. Therefore, the key challenge in the pricing of American options is to determine the best time for investor to exercise an option. This is one of the reasons why American option pricing theory has become a frontier and very active topic in the field of financial research (See Figure 5.1).

The pricing problem of American options is usually solved with either analytical or numerical methods. Earlier studies mainly used analytical methods to determine the price of American options: Johnson (1983) [59] used approximate analysis to determine the value of an American option under the assumption of no dividend; Geske et al. (1984) [60] constructed a model to analyse an American option with a dividend pay-out, but no closed-form solution was obtained. Therefore, numerical methods began discussed to solve American option pricing; these include commonly used binomial tree, finite difference, and least square Monte Carlo methods, among others. Cox et al. (1979) [61] proposed the binomial tree method, which offers simple and effective solutions, and it provides an accurate numerical solution by continuously shrinking the time step; therefore, it is often used as a reference to evaluate the accuracy of other numerical approaches. However, when the model includes multiple random influencing factors, the number of values increases exponentially to calculate in the binomial tree method, which often leads to the curse of dimensionality. The finite difference method mainly converts the asset pricing differential equation into a difference equation, and by obtaining solutions through an iterative method, it mitigates the difficulty in directly solving the differential equation. In 1978, Brennan et al. [62] applied this calculation method in the pricing of American options, but the curse of dimensionality persists when this method is used to solve high-dimensional problems. The Monte Carlo method has the characteristic of forward simulation, so it cannot be applied directly for the pricing of American options, which have a backward iterative search characteristic. Longstaff et al. (2001) [34] modified the Monte Carlo method by using the least squares approach and proposed the least squares Monte Carlo algorithm, which solves the application difficulty of the said method in the pricing of American options; they also provided empirical evidence of the method effectiveness. This method uses the least squares approach to estimate the expected value of continuous holding for each path. By comparing the values to the value associated with immediate exercise, the exercise point of each path is determined. Finally, the value of the American option is obtained by computing the discounted average value of each path's exercise point.

The above calculation methods are effective in pricing American options; however, these studies use the Black-Scholes (B-S) model as their theoretical basis, in which the asset price random process is treated as a geometric Brownian motion, which is unfit for real-life financial markets. Empirical studies have demonstrated that fluctuations in asset price and rate of return are often characterized by non-continuity, clustering and leverage effects (i.e., asymmetric volatility); consequently, we need to construct a more flexible asset pricing model to accurately reflect how asset prices change in reality. Asset price usually jumps in movements, and by adding a Levy process in the pricing model, we can construct a jump model with random jumps of different strengths. Moreover, generalized autoregressive conditional heteroskedasticity (GARCH) models are most frequently used to express the volatility in asset price fluctuations and leverage effects, and such models are highly expandable and more capable of providing accurate descriptions of volatility; therefore, by combining the two models to form the Levy-GARCH model, we can better capture the characteristics of the volatility of the underlying asset. The Levy-GARCH model is widely used in the pricing of European options, but due to the complexity of American options, the model is less frequently applied as a theoretical model for American options. Based on the background described above, jump measure, time-varying volatility and leverage effects are incorporated in this study to construct the Levy-GARCH pricing model for American options proposed by Glosten, Jagannathan and Rundle (Levy-GJR-GARCH) on the basis of an infinite pure jump Levy process and an asymmetric GARCH model. In addition, in real-life financial markets, many subjective and objective uncertainty factors lead to randomness and fuzziness in the price of the option. Therefore, it is necessary to incorporate fuzzy set theory in the pricing model to improve the classic pricing theory. Hence, this study analysed the American option pricing model under a fuzzy environment, incorporated fuzzy simulation technology, and used the least squares Monte Carlo algorithm with higher operational efficiency to analyse the model and compare the operating results with the results computed using the binomial tree algorithm. Lastly, through empirical analysis, we compared the option pricing simulation results of the three infinite pure jump Levy processes (variance gamma (VG), normal inverse Gaussian (NIG), Carr-Geman-Madan-Yor (CGMY)) combined with the GJR-GARCH model, and we verified the convergence efficiency of the modified least squares Monte Carlo algorithm using the quasi-random numbers and Brownian Bridge method.

Through the review of the existing literature, we found abundant studies regarding European option, but studies about American option pricing are still limited. Furthermore, the existing studies mainly focus on numerical algorithm improvements, and insufficient research was pursued to improve the theoretical model. Therefore, we constructed the fuzzy Levy-GJR-GARCH American option pricing model, which is more consistent with reality, and evaluate the model's simulation accuracy by using empirical analysis. The rest of this chapter is structured as follows: Section 5.2 deduces the Levy-GJR-GARCH American option pricing model under a fuzzy environment; Section 5.3 provides a brief introduction of fuzzy simulation technology, then based on it design the algorithms for fuzzy American option pricing model, such as fuzzy binomial tree algorithm, fuzzy least squares Monte Carlo algorithm and especially using quasi-random numbers and Brownian Bridge method to improve the convergence speed of the least squares Monte Carlo algorithm. Section 5.4 combines the Standard & Poor's 100 index (S&P 100 Index) American put option prices to perform empirical testing, followed by a comparative analysis of the fitting precision of different models under fuzzy environments and crisp environments and an examination of the convergence efficiency of least squares Monte Carlo which improved by the quasi-random numbers and Brown Bridge method. Section 5.5 summaries the findings of this study. The theoretical framework is illustrated in Fig. 5.2 below.

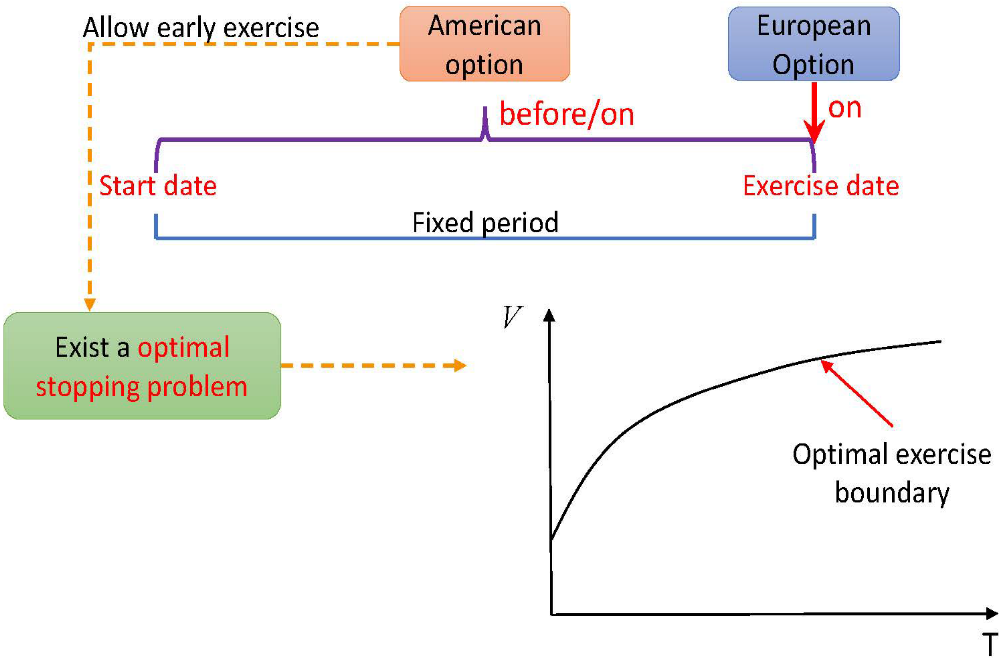


Figure 5.1: American option pricing

5.2 Fuzzy Levy-GJR-GARCH American Option Pricing Model

As the notations used in the remainder of this paper are listed as follows:

Acronyms	Description
VG	Variance gamma process
NIG	Normal inverse Gaussian process
CGMY	Carr-Geman-Madan-Yor process
GARCH	Generalized autoregressive conditional heteroskedasticity model
EGARCH	Exponential GARCH model
TGARCH	Threshold GARCH model
GJR-GARCH	Glosten, Jagannathan and Rundle-GARCH model

5.2.1 The process of the underlying asset price

In this chapter, we assumed the fluctuation of the underlying asset price has the characteristics of time-varying, jump and leverage effect (i.e. asymmetric volatility), thus the sequence of the rate of return of the underlying asset is described using an asymmetric conditional het-

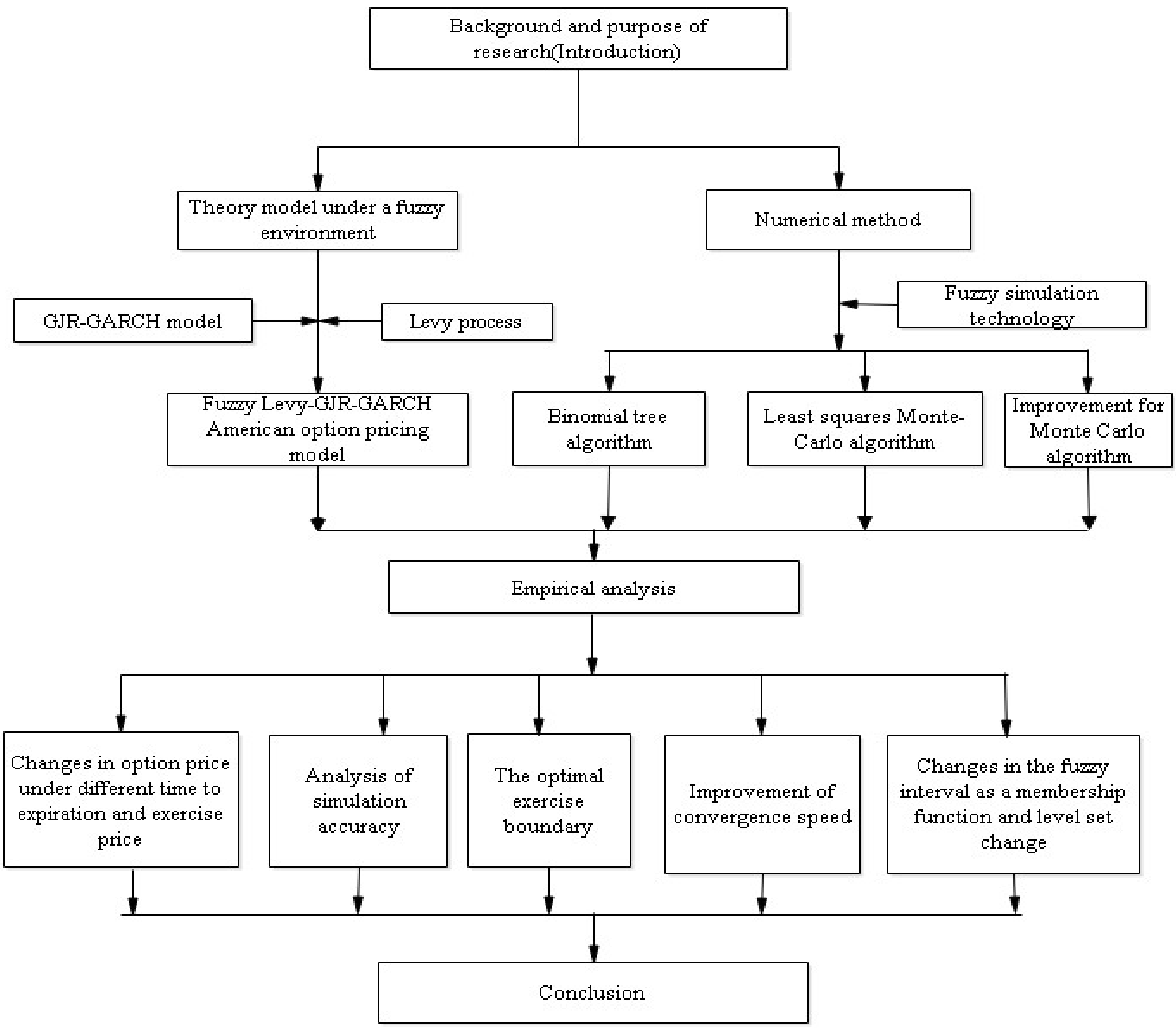


Figure 5.2: Framework Diagram

eroskedasticity model. Among GARCH-type models, models that can express the conditional heteroskedasticity "leverage effect" include the exponential GARCH (EGARCH), threshold GARCH (TGARCH) and GJR(Glosten, Jagannathan and Rundle)-GARCH models, of which the TGARCH and GJR-GARCH models have similar structures and pricing effects. Compared with the EGARCH model, the GJR-GARCH model has better simulation accuracy; therefore, we chose to use GJR-GARCH model proposed by Glosen et al. (1993) [63] as the specific form of the asset return rate model, specifically as follows:

Variable	**Description**
S_0	The underlying asset price at initial time
S_t	The underlying asset price at time t
T	Time to expiration
K	Exercise price
r	Risk-free interest rate
$V(S_t, t)$	Option price at time t
R_t	Logarithmic return rate of the underlying asset price at time t
σ_t	Volatility at time t
X_t	Levy process
θ	Drift rate
v	Jump rate
F_t	Information set at time t
I_t	Indicator Function at time t
w	Intercept
α	The influence coefficient of the variance of previous period to the variance of current period
β	The influence coefficient of the residual of previous period to the residual of current period
δ	Asymmetric effect coefficient
z_t	The innovation of the mean equation at time t
g	Gamma function
Γ	The optimal exercise boundary
$\tilde{A}$	Fuzzy set
$\tilde{A}_\alpha$	α-level set of fuzzy set $\tilde{A}$
$\mu_{\tilde{A}}(x)$	The membership function of the fuzzy set $\tilde{A}$
$\tilde{V}(\tilde{S}_t, t)$	Fuzzy option price at time t

$$\begin{cases} R_t = \ln(\frac{S_t}{S_{t-1}}) = u_t - \gamma_t + \sigma_t z_t \\ \sigma_t^2 = w + \alpha\sigma_{t-1}^2 + \beta\sigma_{t-1}^2 z_{t-1}^2 + \delta I_{t-1}\sigma_{t-1}^2 z_{t-1}^2 \\ I_t = \begin{cases} 1, z_t < 0 \\ 0, z_t \geq 0 \end{cases} \\ z_t | F_{t-1} \sim D(0, 1; \theta_D) \end{cases} \tag{5.1}$$

In the asset return rate model (5.1), R_t is asset's logarithmic return rate, u_t is the expected rate of return under the condition of information set F_{t-1}, γ_t is the mean correction factor, and σ_t^2 is the time-varying variance sequence, I_t represents the indicator function. w represents intercept, α is the influence coefficient of the variance of previous period to the variance of current period, β is the influence coefficient of the residual of previous period to the residual of current period, δ represents asymmetric effect coefficient. z_t represents the innovation of the mean equation, and it follows distribution $D(\bullet)$ with mean value of 0, variance of 1, and parameter θ_D, for which this study will establish several different infinite pure jump Levy processes, such as VG, NIG and CMGY process, which were introduced at chapter 3.

Compared with a standard normal distribution, an infinite pure jump Levy process can describe better high-order moment characteristic of financial data, such as skewness or fat tails. Therefore, it is necessary to incorporate an infinite pure jump Levy process in the GARCH model because the normal random number replaced by the Levy process random number will improve the model's pricing accuracy.

5.2.2 The risk-neutral conversion of the underlying asset pricing

In theory, there should be no arbitrage in the option value; therefore, the asset return rate model (see equation (5.1)) requires risk-neutral conversion to ensure the validity of the no-arbitrage assumption. Under risk-neutral measure Q, $\mathrm{E}^Q(S_t|S_{t-1}) = S_{t-1}e^{r_t}$, where r_t represents the risk-free rate of return. Here, the risk-neutral model is,

$$S_t = S_{t-1}e^{r_t - \varphi^Q(\sigma_t) + \sigma_t \varsigma_t^Q} \tag{5.2}$$

Above, $\varphi^Q(\sigma_t) = \mathrm{E}^Q(e^{\sigma_t \varsigma_t^Q})$ is the mean correction factor, where ς_t^Q is white noise with mean of 0 and variance of 1. Using the Christofersen et al. (2010) [64] method to construct the pricing kernel $\{\varsigma_t\}$, we establish a Radon-Nikodym derivative sequence that can materialise real measurement of risk-neutral measure conversion:

$$\frac{dQ}{dP}|F_{t-1} = \exp(-\sum_{i=1}^{t}(\varsigma_i \sigma_i z_i + \psi(\varsigma_i))) \tag{5.3}$$

Under a non-normal environment, the kernel sequence $\{\varsigma_t\}$ is not the only one that fulfils the following formula:

$$\psi_t(\varsigma_t - 1) - \psi_t(\varsigma_t) + u_t - r_t - \gamma_t = 0 \tag{5.4}$$

Here, $\psi(\bullet)$ represents the exponential part of the moment-generating function. Based on the characteristics of the moment-generating function $\psi_t^{'}(0) = E_{t-1}[\sigma_t z_t]$, $\psi_t^{''}(0) = Var_{t-1}[\sigma_t z_t] = \sigma_t^2$, we obtain the following analytical expression for the kernel sequence $\{\varsigma_t\}$:

$$\varsigma_t \approx \frac{1}{2} + \frac{u_t - r_t - \gamma_t - \psi_t^{'}(0)}{\psi_t^{''}(0)} = \frac{1}{2} + \frac{u_t - r_t - \gamma_t}{\sigma_t^2} \tag{5.5}$$

After obtaining the kernel sequence $\{\varsigma_t\}$, we can perform risk-neutral adjustment on the stochastic item $\varepsilon_t = \sigma_t z_t$ and obtain the following formula:

$$\varepsilon_t^Q = \varepsilon_t - E_{t-1}^Q[\varepsilon_t] = \varepsilon_t - \psi_t^{'}(\varsigma_t) \tag{5.6}$$

Therefore, under the risk-neutral measure, the mean equation can be expressed as follows:

$$R_t^Q = r_t - \psi_{\varepsilon_t^Q}^Q(1) + \varepsilon_t^Q = r_t - \psi_{z_t^Q}^Q(\sigma_t^Q) + \sigma_t^Q z_t^Q \tag{5.7}$$

The conditional variance formula for the risk-neutral asset return rate model can be expressed as

$$(\sigma_t^2)^Q = w^Q + \alpha^Q(\sigma_{t-1}^2)^Q + \beta^Q(\varepsilon_{t-1}^Q + \psi'(\varsigma_{t-1}))^2 \\ + \delta^Q I_{t-1}(\varepsilon_{t-1}^Q + \psi'(\varsigma_{t-1}))^2 \quad (5.8)$$

At this point, we can see that there is some discrepancy between the risk-neutral measure and the real measure of sequence ε_t^Q and $(\sigma_t^2)^Q$; therefore, it is necessary to perform parameter adjustment using kernel sequence $\{\varsigma_t\}$.

5.2.3 American option pricing under a fuzzy environment

Although we can obtain the logarithmic return rate through a heteroskedastic model and obtain the asset price, this method does not provide complete control over future uncertainty factors; therefore, in this study, we assume the asset price volatility σ is a more generalized parabolic fuzzy variable (can cover the triangle and trapezoid fuzzy variable) and analyse the American option pricing model under random and fuzzy environments.

Unlike European options, American options allow early exercise; therefore, American option pricing is a free-boundary problem, in which there is an optimal exercise boundary, and the region $\{0 \leq \mathrm{S_t} \leq \infty, 0 \leq t \leq T\}$ can be segregated into two parts: a region corresponding to continuation option holding and a region corresponding to stopping holding. For a non-dividend-paying American put option, in the continuation holding region $\sum_1$, $V(S_t, t) > (K - S_t)^+$; in the stopping holding region $\sum_2$, $V(S_t, t) = (K - S_t)^+$; K represents exercise price; the optimal exercise boundary $\Gamma : S_t = B(t)$ is located between the two regions. In the above equations, $V(S_t, t)$ and S_t each represents the option value and asset price at time t. Therefore, the following relationship can be obtained:

$$\sum\nolimits_1 = \{(S_t, t)|B(t) \leq S_t < \infty, 0 \leq t \leq T\} \tag{5.9}$$

$$\sum\nolimits_2 = \{(S_t, t)|0 \leq S_t \leq B(t), 0 \leq t \leq T\} \tag{5.10}$$

The relationship for the optimal exercise boundary, Γ, is

$$V(B(t), t) = K - B(t) \tag{5.11}$$

$$\frac{\partial V}{\partial S}(B(t), t) = -1 \tag{5.12}$$

Subsequently, when $S_t \to \infty$, $V(S_t, t) \to 0$, and when $t = T$, $V(S_T, T) = (K - S_T)^+$. Because $B(t)$ is a free boundary, the problem of pricing American put options can be viewed as a parabolic free-boundary problem.

Because the asset price S_t at time t is a function of σ, when σ is a fuzzy number $\tilde{\sigma}$, S_t is also a fuzzy number $\tilde{S_t}$. Similarly, because the option value $V(S_t, t)$ is a function of S_t, when S_t is fuzzy number $\tilde{S_t}$, $V(S_t, t)$ is also a fuzzy number $\tilde{V}(\tilde{\mathrm{S}}_{\mathrm{t}}, t)$. At this point, the α level set of $\tilde{V}(\tilde{\mathrm{S}}_{\mathrm{t}}, t)$ can be expressed as

$$\begin{aligned}(\tilde{V})_\alpha &= [(\tilde{V})^L_\alpha, (\tilde{V})^U_\alpha] \\ &== [\min_{\tilde{S}_\alpha{}^L \leq S \leq \tilde{S}_\alpha{}^U} V(S_t, t), \max_{\tilde{S}_\alpha{}^L \leq S \leq \tilde{S}_\alpha{}^U} V(S_t, t)]\end{aligned} \tag{5.13}$$

Therefore, based on credibility theory, the expected value $E(\tilde{V}(\tilde{\mathrm{S}}_{\mathrm{t}}, t))$ of $\tilde{V}(\tilde{\mathrm{S}}_{\mathrm{t}}, t)$ can be expressed as

$$
\begin{aligned}
E(\widetilde{V}(\widetilde{\mathrm{S}}_{\mathrm{t}},t)) &= \int_0^{+\infty} Cr\{(\widetilde{V}(\widetilde{\mathrm{S}}_{\mathrm{t}},t)) \geq r\}dr \\
&= \frac{1}{2}\int_0^1 ((\widetilde{V})_\alpha^L + (\widetilde{V})_\alpha^U)d\alpha
\end{aligned} \tag{5.14}
$$

At this point, the optimal exercise boundary for fuzzy American options, the continuation holding region and the stopping holding region under fuzzy environment are expressed as follows :

(1) Optimal exercise boundary: $V(B(t),t) = K - B(t), \Gamma : B(t) = E(\widetilde{\mathrm{S}}_{\mathrm{t}})$.

(2) Continuation holding region: $\sum_1 = \{(\widetilde{\mathrm{S}}_{\mathrm{t}},t)|B(t) \leq E(\widetilde{\mathrm{S}}_{\mathrm{t}}) <\infty, 0 \leq t \leq T\}$.

(3) Stopping holding region: $\sum_2 = \{(\widetilde{\mathrm{S}}_{\mathrm{t}},t)|0 \leq E(\widetilde{\mathrm{S}}_{\mathrm{t}}) \leq B(t), 0 \leq t \leq T\}$

5.3 The Algorithms Design for Fuzzy American Option Pricing Model

Upon obtaining the Levy-GARCH model for an American option under a fuzzy environment, considering more general situations where the fuzzy variables with mixed distributions, we combine fuzzy simulation technology [13] and calculation methods frequently used for American options to create a fuzzy pricing method for American options.

5.3.1 Fuzzy simulation technology

Fuzzy simulation technology is used for sampling test of fuzzy models based on probability distributions. This technology only provides a statistical estimate of the model, not the precise result, but it is the only effective method for complex problems for which analytical results are unattainable.

If ξ is a fuzzy variable with probability space $(\Theta, P(\Theta), Pos)$, the function $f(\xi)$ is also a fuzzy variable; at the same time, the membership function of $f(\xi)$ can be obtained using the following simulation method:

Step 1. Randomly and evenly extract a number ξ_k(k=1, 2...N) from the level set of fuzzy variable ξ, calculate ξ_k membership from the membership function of ξ, and denote it as v_k.

Step 2. Based on the formula for function $f(\xi_k)$, calculate the function value$f(\xi_k)$.

Step 3. Repeat Steps 1 through 2 N times.

Step 4. Calculate the expected value $E(f(\xi)) = \frac{1}{N} f(\xi_k)$ of function $f(\xi)$ and draw the membership function of $f(\xi)$ based on $(f(\xi_k),\, v_k)$.

5.3.2 Fuzzy binomial tree algorithm

The binomial tree algorithm assumes that asset prices obey a dispersed time process, where the time $[0, T]$ is divided into n equivalent time steps $\Delta \mathrm{t} = \mathrm{t}_i = T/n$, where $i = 1, 2...n$. There can only be two changes in the asset price S_i at time t_i, whereby the price either increases to u times its original price with probability p or decreases to d times its original price with probability $(1 - p)$, such that $0 < d < 1 < u, ud = 1$; therefore, the asset price at t_{i+1} can only be uS_i or dS_i. When the initial asset price is S_0, there is $i + 1$ probability for asset price S_i at t_i: $S_0 u^j d^{i-j}$, where $j = 0, 1, 2...i$. The exact binomial tree is shown in Fig.5.3.

When using a binomial tree to obtain the price of an American option at each node, pricing is mainly performed using the backward inference method from back to front. Generally, u and d are set as functions of the volatility σ. When σ is a fuzzy number, u and d are also fuzzy numbers; consequently, the asset price S_i at time t_i is also a fuzzy variable. At this point, when the exercise price is K and the time to expiration is T, the value of the American put option is expressed as $V_n(S_{n,j}) = \max\{K - E(S_{n,j}), 0\}$, where $j = 0, 1, 2...n$. Using backward inference, we can obtain the option value at time t_i:

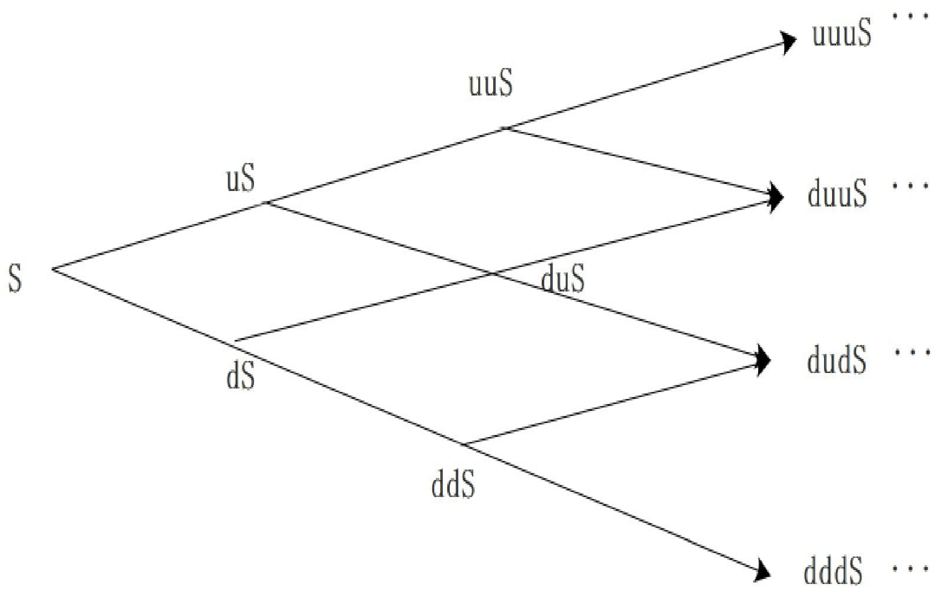


Figure 5.3: Multiple-period binomial tree

$$V_i(S_{i,j}) = \max\{K - E(S_{i,j}), \exp(-r\Delta t)(pV_{i+1,j}(uS_{i,j}) + (1-p)V_{i+1,j}(dS_{i,j}))\} \quad (5.15)$$

The α level set of $V_i(S_{i,j})$ can be expressed as

$$\begin{aligned} \tilde{V}_{i,\alpha}(S_{i,j}) =& [\max\{K - \tilde{S}^U_{i,j}(\alpha), \exp(-r\Delta t)(p\tilde{V}^L_{i+1,j}(uS_{i,j}) \\ & + (1-p)\tilde{V}^L_{i+1,j}(dS_{i,j}))\}, \\ & \max\{K - \tilde{S}^L_{i,j}(\alpha), \exp(-r\Delta t)(p\tilde{V}^U_{i+1,j}(uS_{i,j}) \\ & + (1-p)\tilde{V}^U_{i+1,j}(dS_{i,j}))\}] \end{aligned} \quad (5.16)$$

Therefore, the calculation of option price using fuzzy binomial tree is as follows:

Step 1. Randomly and evenly extract a number σ_k(k=1,2...N) from the α level set of fuzzy variable $\tilde{\sigma}$, calculate σ_k membership from the $\tilde{\sigma}$ membership function, and denote it as v_k.

Step 2. Presume that the asset price upward factor u, downward factor d and probability p are $e^{\sigma_k\sqrt{\Delta t}}$, $e^{-\sigma_k\sqrt{\Delta t}}$ and $\frac{e^{r\Delta t}-d}{u-d}$, respectively; based on these, calculate the asset price $S^k_{i,j}$ at each node of the price tree.

Step 3. Based on option calculation formula, calculate the option value up to the expiry date $V^k_N(S^k_N)$ and use backward inference to obtain the option value at each node $V^k_i(S^k_i)$.

Step 4. Repeat Steps 1 through 3 N times.

Step 5. Calculate the expected value of the option price $E(V_0) = \frac{1}{N}V^k_0$ and draw the membership function diagram of the fuzzy option price according to (V^k_0,υ_k).

5.3.3 Fuzzy least squares Monte Carlo algorithm

The least squares Monte Carlo algorithm mainly compares the exercise value of immediate option exercise and the conditional expected value of continuous option holding to determine the optimal exercise time of an American option. When the value of immediate exercise is greater than or equal to the value of continuous holding, the investor will choose to exercise the option immediately.

Presuming that the number of Monte Carlo algorithm-simulated paths is N and that the time to expiration T is divided into M periods, at time t_i, the exercise value of path-j is $I_{i,j}(S_{i,j}) = \max(K - S_{i,j}, 0)$, where K is the exercise price and $S_{i,j}$ is the asset price on path j during t_i. The conditional expected value of continuous option holding can only obtained using backward inference $E_{i,j}(S_{i,j}) = E[\exp(-r\Delta t)V_{i+1,j}(S_{i+1,j})|S_{i,j}]$. Therefore, the conventional Monte Carlo method is not suitable for numerical simulation of the American option pricing model. The least squares Monte Carlo approach regards the discounted value of the option value at time t_{i+1}, $\exp(-r\Delta t)V_{i+1,j}(S_{i+1,j})$, as the Y variable and $S_{i,j}$ and $S^2_{i,j}$ as X variable, constructing a least squares regression model for Y as a function of X and obtaining regression coefficients a_1, a_2 and a_3. The following formula can yield an approximation for $E_{i,j}(S_{i,j})$ (See Figure 5.4):

$$E_{i,j}(S_{i,j}) \approx a_1 + a_2 S_{i,j} + a_3 S_{i,j}^2 \tag{5.17}$$

Based on the above method, compare the value of continuation holding and the value of exercise at each node of N paths, thereby obtaining the optimal exercise strategy for each path. Discount the option value of each path to the present period and obtain the average of each path's discounted option value; this said average value is the acquired option price (See Figure 5.5).

If the asset price volatility σ is a fuzzy number, the asset price $S_{i,j}$ is also a fuzzy number, whereas the exercise value $I_{i,j}(S_{i,j})$ and value of continuous holding $E_{i,j}(S_{i,j})$ are both functions of $S_{i,j}$; therefore, $I_{i,j}(S_{i,j})$ and $E_{i,j}(S_{i,j})$ are also fuzzy numbers. Their α level set can be expressed as follows:

$$\widetilde{I_{i,j}}^{\alpha}(S_{i,j}) = [\max\{K - \widetilde{S_{i,j}}^{U}(\alpha)\}^{+}, \max\{K - \widetilde{S_{i,j}}^{L}(\alpha)\}^{+}] \tag{5.18}$$

$$\widetilde{E_{i,j}}^{\alpha}(S_{i,j}) = [a_1 + a_2 S_{i,j}^{L} + a_3 {S_{i,j}^{2}}^{L}, a_1 + a_2 S_{i,j}^{U} + a_3 {S_{i,j}^{2}}^{U}] \tag{5.19}$$

Because the asset price and option value are fuzzy variables, when comparing and solving the least squares regression equation, the expected value of fuzzy variable is used in the calculation. The calculation of the option price using the least squares Monte Carlo algorithm is as follows:

Step 1. Randomly and evenly extract a number $\sigma_k(k = 1, 2 \ldots N)$ from the α level set of the fuzzy variable $\widetilde{\sigma}$, calculate the membership degree of σ_k from the membership function of $\widetilde{\sigma}$, and denote it as v_k.

Step 2. Based on the asset price formula, calculate the asset price $S_{k,j}(j = 1, 2 \ldots M)$ at

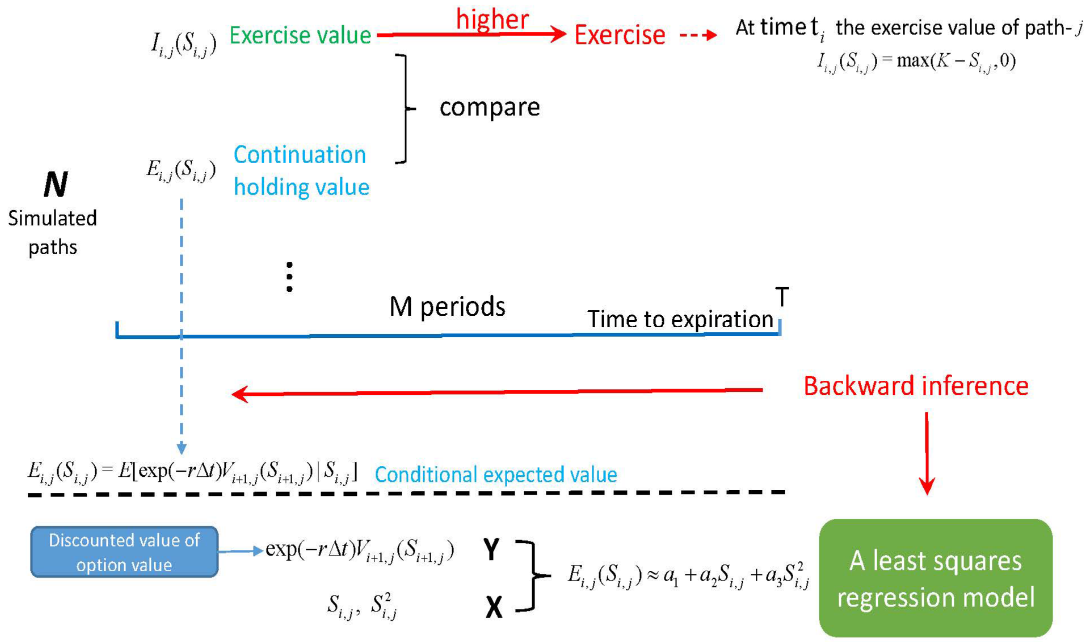


Figure 5.4: The least squares Monte Carlo algorithm (1)

each node of path j.

Step 3. Find the option exercise value $I_{k,j}(S_{k,j})$ at each node of path j, and calculate the value of continuous option holding $E_{k,j}(S_{k,j})$ at each node using the least squares method.

Step 4. Repeat Steps 1 through 3 N times.

Step 5. Calculate the expected option value $E(V_0) = V_{k,0}/N$, and based on $(V_{k,0}, v_k)$, draw the fuzzy option value membership function diagram.

5.3.4 The improvement for Monte Carlo algorithm

There are two key factors that affect the simulation effect of the Monte Carlo algorithm: first, the sampling characteristic of the random sampling determines the skewness of the sample distribution; second, the manner in which the randomly simulated paths are constructed determines whether the simulated paths resemble real paths. Most studies regarding the simulation effectiveness of Monte Carlo method mainly focus on these two factors.

The random numbers generated from Monte Carlo method are pseudo-random numbers.

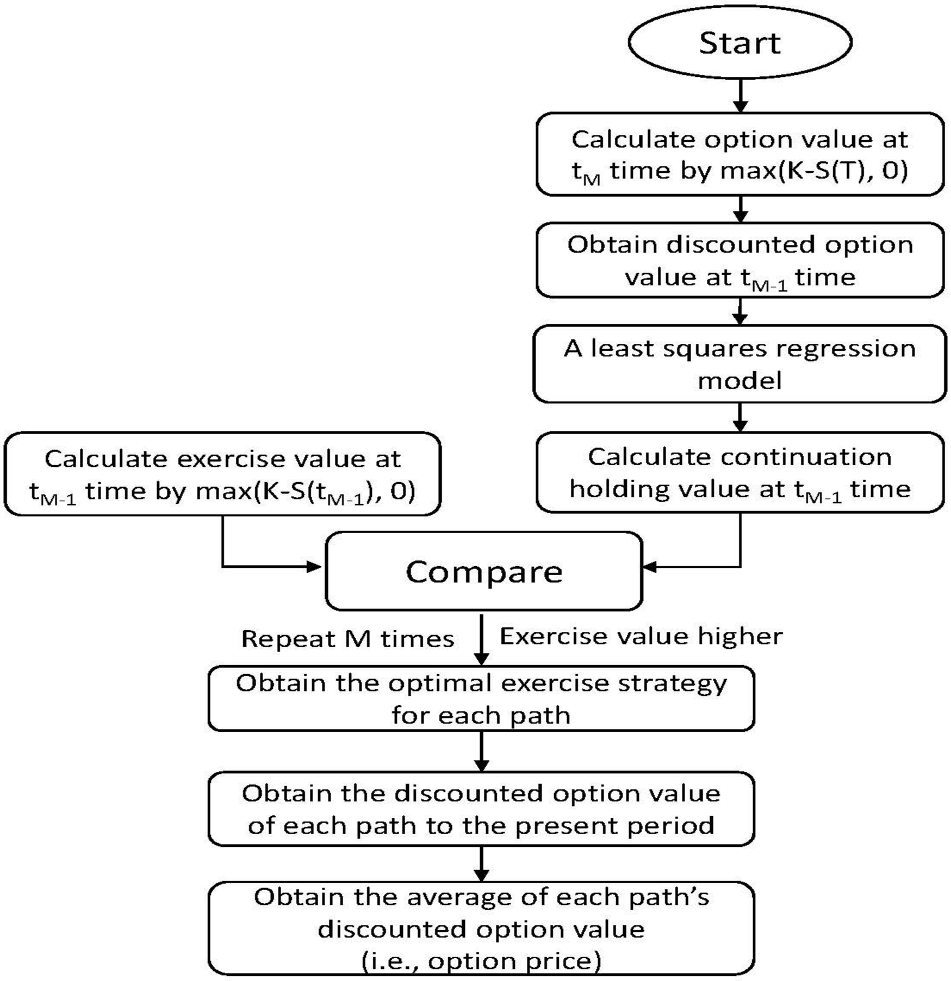


Figure 5.5: The least squares Monte Carlo algorithm (flow chart)(2)

These random points often present clustering or gap problems, causing relatively large deviations in the random number sequence. On the other hand, the quasi-random numbers generated from the quasi-Monte Carlo method incorporate the randomness and the evenness of the sequence distribution in random sequence; hence, using this method reduces the deviation in the random sequence. The Halton, Sobol and Faure sequences are the most common quasi-random sequences; since the Sobol sequence has better evenness and is less time-consuming to generate, this study chose to use Sobol sequence to obtain the quasi-random numbers.

The Sobol sequence [65] is constructed based on a series of "direction numbers" v_i. When q_i is a positive odd number less than 2^i,

$$v_i = \frac{q_i}{2^i} \tag{5.20}$$

The v_i and q_i are obtained by a polynomial with coefficient is 0 or 1, and the form of the polynomial is as follows:

$$f(z) = z^p + a_1 z^{p-1} + ... + a_{p-1} z + a_p \tag{5.21}$$

When $i > p$, the recursion formulas for v_i and q_i are

$$v_i = a_1 v_{i-1} \oplus a_2 v_{i-2} \oplus ... \oplus a_p v_{i-p} \oplus \lfloor v_{i-p}/2^p \rfloor \tag{5.22}$$

$$q_i = 2a_1 q_{i-1} \oplus 2^2 a_2 q_{i-2} \oplus ... \oplus 2^p a_p q_{i-p} \oplus q_{i-p} \tag{5.23}$$

where $\oplus$ indicates the binary bitwise exclusive-OR.

The Brownian Bridge method [28] is a method for constructing a Monte Carlo simulation path. If $X(t)$ is a random process, let $t_1 < t_2$ and the density function of $x \sim F_{X(t_1)}$ and $y \sim F_{X(t_2)}$ be f_{t_1} and f_{t_2} respectively, where $f_{x,y}$ indicates the combined density function of x and y. When the density function of $z = x+y$ is f_z, the density function of $x|z$ can be obtained using the following formula:

$$f_{x|z} = \frac{f_{x,z}(x, z-x)}{f_z(z)} \tag{5.24}$$

For Brownian motion $W(t)$, presuming $t_i < t_j < t_k$ and that $W(t_i)$ and $W(t_k)$ are given, the average value and variance at time t_j satisfy the following Brown Bridge characteristic:

$$E[W(t_j)] = (\frac{t_k - t_j}{t_k - t_i})W(t_i) + (\frac{t_j - t_i}{t_k - t_i})W(t_k) \tag{5.25}$$

$$V[W(t_j)] = \frac{(t_j - t_i)(t_k - t_j)}{t_k - t_i} \tag{5.26}$$

Therefore, the following formula can be used to obtain the sample value of time sequence $\{t_0, t_1, ..., t_n\}$, subsequently obtaining the diffused sample path:

$$f_{x|z}(x) = \frac{1}{\sqrt{2\pi\frac{(t_j - t_i)(t_k - t_j)}{t_k - t_i}}} \exp(-\frac{1}{2}(\frac{x - \frac{t_j - t_i}{t_k - t_i}z}{\sqrt{\frac{(t_j - t_i)(t_k - t_j)}{t_k - t_i}}})^2) \tag{5.27}$$

The Brownian Bridge method mainly increases the low-order coordinate component in the random sequence and reduces the actual dimension of simulation problem to better illustrate the distribution characteristic of quasi-random numbers, hence improving its estimation effect.

5.4 Empirical Analysis

5.4.1 Source of data and descriptive statistics

This study used the S&P 100 Index and American put options acquired from S&P 100 Index as the data for empirical analysis. The S&P 100 Index prices were selected from the closing prices of data of 1526 days dating from March 22, 2011 to March 23, 2017 (data source:

Table 5.1: Descriptive statistics of the data

Indicator	Sample size	Maximum value	Minimum value	Average value	Standard deviation	Skewness	Kurtosis
Rate of return	1525	0.0188	-0.028	0.0002	0.004	-0.4512	7.6373
Option time to expiration	70	0.75	0.0833	0.231	0.184	1.7002	5.1386
Option exercise price	70	1095	990	1041.6	31.2	0.0000	1.8
Option price	70	65.8	0.6	23.4336	17.0523	0.4781	2.1542

Yahoo!Finance), and the American S&P 100 Index put option prices were selected to be the average prices of the final transacted prices for different expiry dates and different exercise prices for put options on March 23, 2017 (data source: Chicago Board of Options Exchange). The data used in this research excluded options with same month expiry, and we categorized options with durations of 1-3 months as short-term options, 4-6 months as medium-term options, and more than 6 months as long-term options. We only took into account American options with exercise prices within the range of 95%-105% of the index prices and eliminated contracts with option values close to 0. Consequently, we obtained 70 data points, of which 22 expire in April, 22 expire in May, 11 expire in June, 5 expire in July, 5 expire in September and 5 expire in December.

Fig.5.6 shows the logarithmic return rate data calculated from S&P 100 Index prices of 1,525 days; it reveals tremendous volatility in the rate of return. Descriptive statistics of the data are listed in Table 5.1. The sampling skewness of the rate of return is $-0.4512 < 0$, which indicates that the sample is skewed to the left. The kurtosis is $7.6373 > 3$, which indicates that the sample is leptokurtic and fat-tailed. Skewness represents the deviation degree of the sample data distribution relative to the symmetrical distribution, when skewness$= 0$, it represents the sample data distribution is symmetrical, when skewness < 0, it represents the sample data is left-skewed distribution, when skewness > 0, it represents the sample data is right-skewed distribution. Kurtosis represents the degree of the sample data distribution more or less peaked than a normal distribution, when kurtosis $= 3$, it represents the sample data is a normal distribution, when kurtosis > 3, it represents the sample data is relatively peaked distribution (leptokertic) and its tail is longer and fatter than a normal distribution, when kurtosis < 3, it represents the sample data is flat-topped distribution (platykurtic) and its tail is shorter and thinner than a normal distribution.

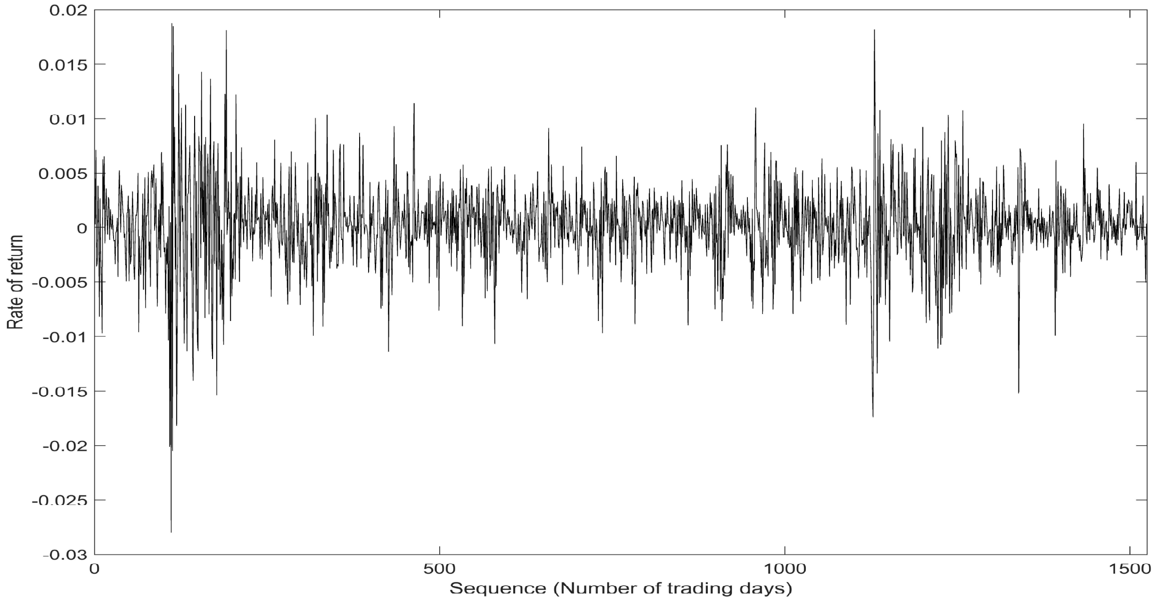


Figure 5.6: Daily logarithmic return rate of the S&P 100 Index

5.4.2 Parameter estimation

Compared with a Gaussian distribution, a Levy process can better characterize the jump behaviour of innovations; however, the form of the Levy process distribution function is complicated. After combining the Levy process distribution function with GJR-GARCH model, there are many parameters to be estimated. If estimation is performed using the maximum likelihood method, the calculation efficiency will be very low. Nevertheless, the form of the Levy process moment condition obtained from the characteristic function is relatively simple, and the parameters of a Levy process can be estimated using the generalised method of moments. Therefore, to reduce the complexity of parameter estimation, this study used a two-step method to estimate the parameters of GJR-GARCH model and Levy process: in step 1, set innovations as Gaussian distribution and use maximum likelihood estimation to estimate the parameters of the GJR-GARCH model; in step 2, based on the innovations data obtained in step 1, use the generalised method of moments to estimate the parameters of the VG, NIG and CGMY models. The results of the parameters estimation are presented in Table 5.2, from which we can see that the "leverage effect" parameter δ of GJR-GARCH model is greater than 0. At the 1% significance level, the significance is not 0, indicating that the changes in volatility are clearly asymmetric, with downward fluctuations stronger than upward fluctuations.

Table 5.2: Estimated results for the Levy-GJR-GARCH model parameters

GJR-GARCH model parameters				VG process parameters			
ω	α	β	δ	θ	σ	υ	-
0.0001***	0.0050	0.7364***	0.4746***	-0.2131	1.0676***	0.3740**	-
(2.9711)	(0.4902)	(40.3428)	(9.9941)	(-1.2262)	(6.1442)	(2.1527)	
NIG process parameters				**CGMY process parameters**			
λ	η	κ	-	C	G	M	Y
1.7806***	-0.2396	0.8883***	-	6.6594***	3.1636***	2.7937***	1.6961***
(5.1771)	(-0.6968)	(2.5827)		(76.5830)	(36.3812)	(32.1268)	(19.5048)

Remark: the numerical values in parentheses correspond to the t-statistics of the parameter values, * indicates significant at the 10% significance level, ** indicates significant at the 5% significance level, and *** indicates significant at the 1% significance level.

Table 5.3: Descriptive statistics regarding volatility and innovations

Indicator	Sample size	Maximum value	Minimum value	Average value	Standard deviation	Skewness	Kurtosis
Time-varying volatility	1525	0.0005	0.0000	0.0000	0.0000	7.7168	88.3883
Innovation	1525	3.4999	-6.6548	0.0753	1.0338	-0.4951	4.9872

Fig. 5.7 and 5.8 show the time-varying volatility sequence and the innovations sequence, and Table 5.3 presents the descriptive statistics of innovations. From the characteristics of time-varying volatility, we can observe the following phenomena: the volatility has a relatively strong clustering characteristic, i.e., major volatility is followed by major volatility, and minor volatility is followed by minor volatility. Therefore, using a skewed GARCH model to describe the volatility data is more consistent with reality. From the characteristics of the innovation data, we can see that volatility in innovations is not white noise; the skewness and kurtosis are $-0.4951 < 0$ and $4.9872 > 3$, respectively, indicating a leptokurtic, fat-tailed distribution. Therefore, a Levy process can provide higher accuracy than a Gaussian distribution.

5.4.3 Empirical result analysis

We downloaded the data for option prices for 70 American options transacted on the S&P 100 Index from the official website of the Chicago Board Options Exchange and performed a

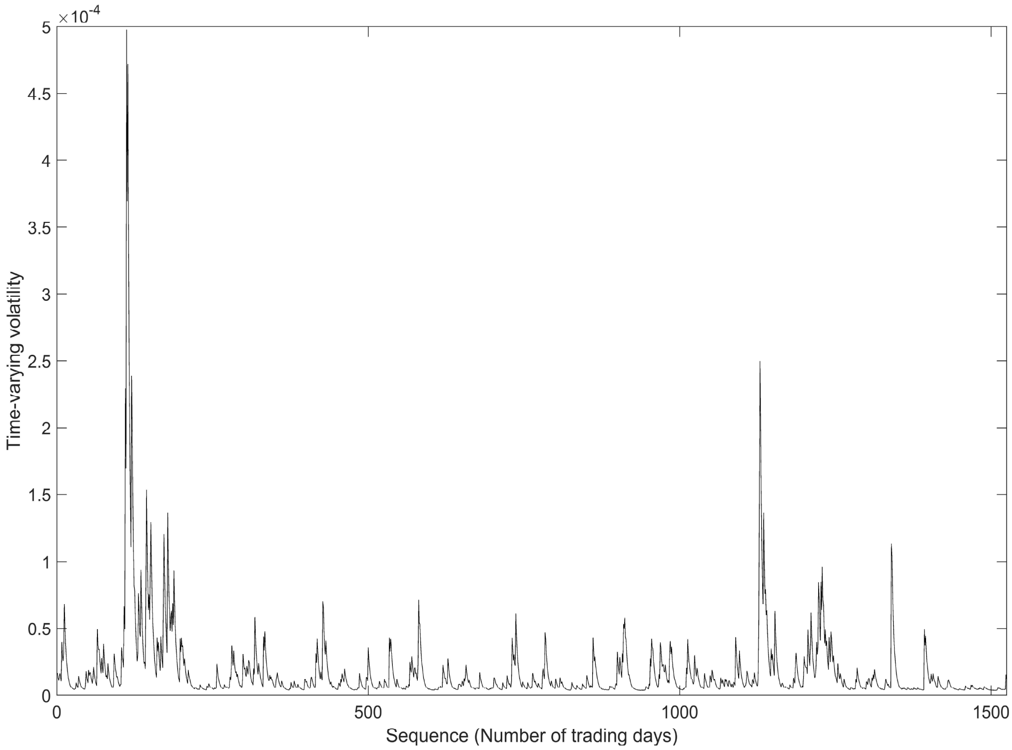


Figure 5.7: Time-varying volatility

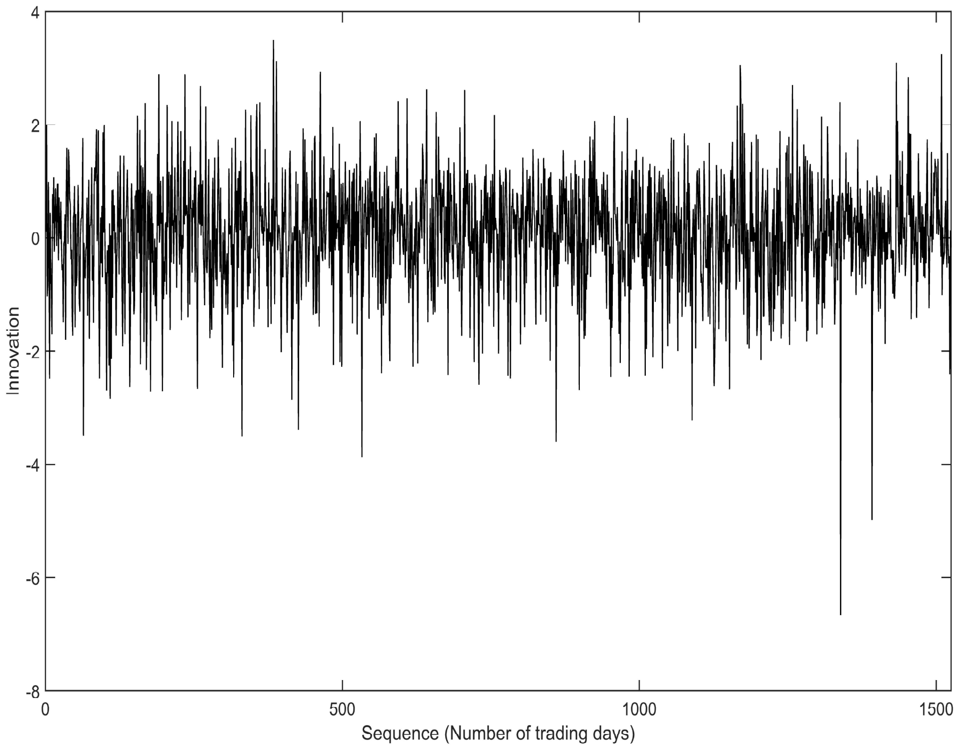


Figure 5.8: Innovations sequence diagram

comparative analysis of different exercise prices and different time to expiration. The multiplier of S&P 100 Index options is 100 USD (i.e. each point represents 100 USD). The closing price of

S&P 100 Index on March 23, 2017 was 1,040, i.e. $S_0 = 1,040$. We used the 10-year T-bond yield as of March 23, 2017 as the risk-free interest rate, $r = 2.4\%$ (data source: official website of US Treasury Department). To examine the pricing effect of the Levy-GJR-GARCH model under a fuzzy environment, we compared the pricing result with that of the Levy-GJR-GARCH model under a crisp environment. Under fuzzy theory, the volatility σ of an asset price is set as a fuzzy variable, whereas in the GARCH model, the volatility σ is set as a time-varying variable. To reduce the complexity of the fuzzy calculation, the membership function of the time-varying volatility $\{\widetilde{\sigma_t}\}$ was set as an equal form of parabolic membership function. Because estimation of four parameter values was required for parameter interval of parabolic fuzzy numbers, the historical rates of return of 1525, 1200, 800 and 400 trading days before March 23, 2017 were selected as observation samples based on different market information reflected by different sampling intervals. The parameter values required for parabolic fuzzy numbers and the results of option pricing are listed in Table 5.4.

The expected values under a fuzzy environment presented in Table 5.4 were obtained from the upper and lower weights of the $\alpha = 0.95$ level set of fuzzy number $\widetilde{\mathrm{V}}_t$, and the exact formula is as follows:

$$
\begin{aligned}
M(\widetilde{V}) &= \frac{M(\widetilde{V})^L + M(\widetilde{V})^U}{2} \\
&= \frac{\int_0^1 f(\alpha)\widetilde{V_\alpha}^L d\alpha + \int_0^1 f(\alpha)\widetilde{V_\alpha}^U d\alpha}{2} \\
&= \int_0^1 \frac{f(\alpha)}{2}(\widetilde{V_\alpha}^L + \widetilde{V_\alpha}^U)d\alpha
\end{aligned} \tag{5.28}
$$

Based on the simulation results presented in Table 5.4, the option prices corresponding to different time to expirations and different exercise prices are shown in Fig. 5.9 and 5.10. From Fig. 5.9, we can see that as the time to expiration of option lengthens, the option price increases gradually; this increase in option price is because the uncertainty increases as time increases. We can see that when the time to expiration is shorter, the model's simulation results are clustered around the market price, whereas the simulation results are more dispersed when

Table 5.4: Option pricing results (Unit: 100 dollars)

Time to expiration (year)	Exercise price	Market price	Least squares Monte Carlo algorithm						Binomial tree algorithm					
			Fuzzy environment (expected value)			Crisp environment			Fuzzy environment (expected value)			Crisp environment		
			VG	NIG	CGMY	VG	NIG	CGMY	VG	NIG	CGMY	VG	NIG	CGMY
0.083	1000	2.78	4.19	2.87	3.21	1.98	1.75	3.59	3.50	1.87	2.89	2.81	1.08	2.72
0.083	1020	5.05	6.81	4.91	6.26	3.68	3.01	7.88	7.05	3.73	3.65	6.94	3.75	3.02
0.083	1040	10.14	10.83	11.63	11.75	10.33	10.69	13.38	15.90	10.87	12.77	14.79	9.92	13.32
0.083	1060	21.24	18.73	25.69	20.17	20.63	25.11	22.86	26.29	23.61	28.40	21.30	25.36	23.55
0.083	1080	39.00	32.26	45.85	32.59	26.65	44.81	30.63	50.63	41.21	45.16	40.41	44.10	43.74
0.167	1000	7.50	4.56	3.02	4.32	5.33	2.62	7.03	9.32	2.52	6.18	5.84	1.55	5.46
0.167	1020	11.10	7.59	5.59	8.34	14.04	7.27	9.04	16.88	6.66	10.17	12.78	6.20	7.32
0.167	1040	17.55	10.45	14.65	13.62	21.79	17.16	16.64	26.10	14.35	17.70	18.48	15.59	15.64
0.167	1060	27.60	18.46	26.63	24.59	26.06	32.83	23.49	37.04	26.30	33.17	22.45	29.08	30.11
0.167	1080	42.50	26.76	46.60	39.76	30.47	51.52	39.68	53.62	42.47	49.47	41.41	45.60	46.10
0.250	1000	11.45	7.67	6.10	6.84	19.33	4.13	9.99	16.14	4.62	10.13	9.84	5.36	15.98
0.250	1020	16.50	11.33	15.59	14.62	21.91	10.52	15.76	24.15	9.59	16.84	17.03	14.95	16.31
0.250	1040	22.80	15.44	25.90	18.31	24.28	21.73	17.78	33.54	17.57	21.52	24.50	26.62	26.63
0.250	1060	32.60	23.76	44.85	28.44	28.24	37.05	26.60	44.37	29.00	36.93	27.05	35.07	38.92
0.250	1080	45.70	32.37	54.81	43.22	33.05	55.54	41.52	56.54	43.94	53.01	43.05	55.01	55.06
0.333	1000	16.55	12.81	14.51	9.38	20.42	6.86	15.98	22.19	6.06	14.58	11.56	8.44	20.98
0.333	1020	21.75	16.20	24.40	18.23	23.03	13.93	18.67	30.51	11.41	23.54	19.18	21.32	24.86
0.333	1040	28.60	24.39	34.73	22.59	25.74	25.76	22.68	35.03	19.48	26.26	25.82	31.26	32.96
0.333	1060	37.70	29.02	43.68	30.67	28.35	41.78	28.24	46.79	30.62	39.63	32.48	43.12	45.45
0.333	1080	49.90	36.02	56.62	44.32	32.73	60.30	43.72	62.73	45.07	55.60	53.03	56.93	63.08
0.500	1000	24.40	15.22	27.14	13.97	25.17	10.69	16.56	30.69	9.53	20.52	17.97	10.92	22.31
0.500	1020	30.10	18.35	37.05	23.26	27.30	19.24	20.68	39.26	15.63	26.81	21.42	24.40	32.63
0.500	1040	37.40	27.26	45.31	28.81	29.73	31.88	26.95	48.84	24.05	29.61	30.67	32.48	35.95
0.500	1060	46.20	33.21	56.29	35.44	32.86	47.85	33.48	54.46	35.07	44.91	41.64	47.62	54.65
0.500	1080	57.40	39.85	68.19	48.93	36.78	66.00	49.31	71.08	48.84	60.72	54.31	62.99	65.80
0.750	1000	35.30	18.24	35.61	17.98	22.86	12.54	19.32	33.78	13.82	26.20	21.29	17.62	27.13
0.750	1020	41.60	24.30	45.46	25.75	24.14	22.35	24.79	42.71	20.51	35.63	28.88	27.24	38.93
0.750	1040	49.20	30.79	53.72	31.66	27.33	35.29	30.99	62.43	29.15	45.47	33.88	38.86	47.85
0.750	1060	57.80	35.77	70.61	41.52	30.41	51.38	35.39	73.03	39.95	60.76	50.33	52.16	59.81
0.750	1080	68.30	41.26	84.45	53.12	33.29	69.45	51.87	84.37	52.94	66.43	61.33	66.83	68.05

the time to expiration is longer. This result indicates that the model has higher simulation accuracy for short-term options than for long-term options. From Fig. 5.10, we can see that the higher the exercise price, the higher the option price, which is consistent with the actual situation. Because for American put options, the option price is bearish, higher exercise prices correspond to higher option profit and thus higher option prices. Moreover, we can clearly observe that the model simulation results for shorter period cluster around the market price, and as the time frame lengthens, the simulated curve becomes more dispersed.

From the above analysis, we know that the theoretical model has greater simulation accuracy for short-term options compared with long-term options; therefore, we selected 22 short-term option pricing results with expiry in April 2017 (see Table 5.5) to further analyse the differences

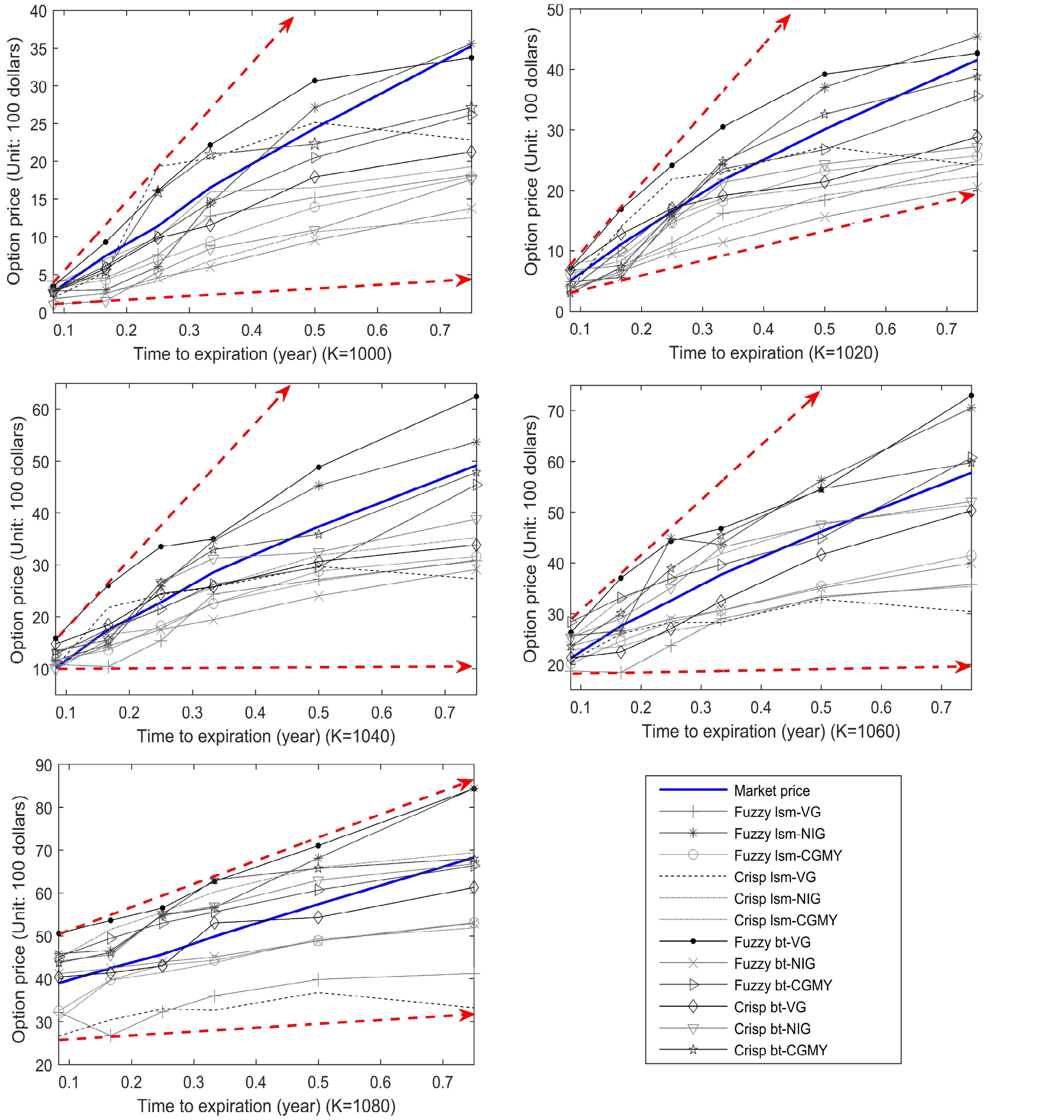


Figure 5.9: Comparison of option prices with different time to expiration

Remark: "Fuzzy" denotes simulation under a fuzzy environment, "Crisp" denotes simulation under a crisp environment,"lsm" denotes the least squares Monte Carlo algorithm, and "bt" denotes the binomial tree algorithm. VG, NIG and CGMY are Levy processes. The red broken line with arrow is the trend line, the blue line is the market price.

in option pricing under fuzzy and crisp environments. The result is shown in Figure 5.11. From the simulation result, we can see that all market prices fall within the fuzzy interval of the VG, NIG and CGMY models under a fuzzy environment, which shows that the market prices of options are better covered when a fuzzy price interval is used. In contrast with the smaller fuzzy interval of the VG model and the greater fuzzy interval of the NIG model, the fuzzy

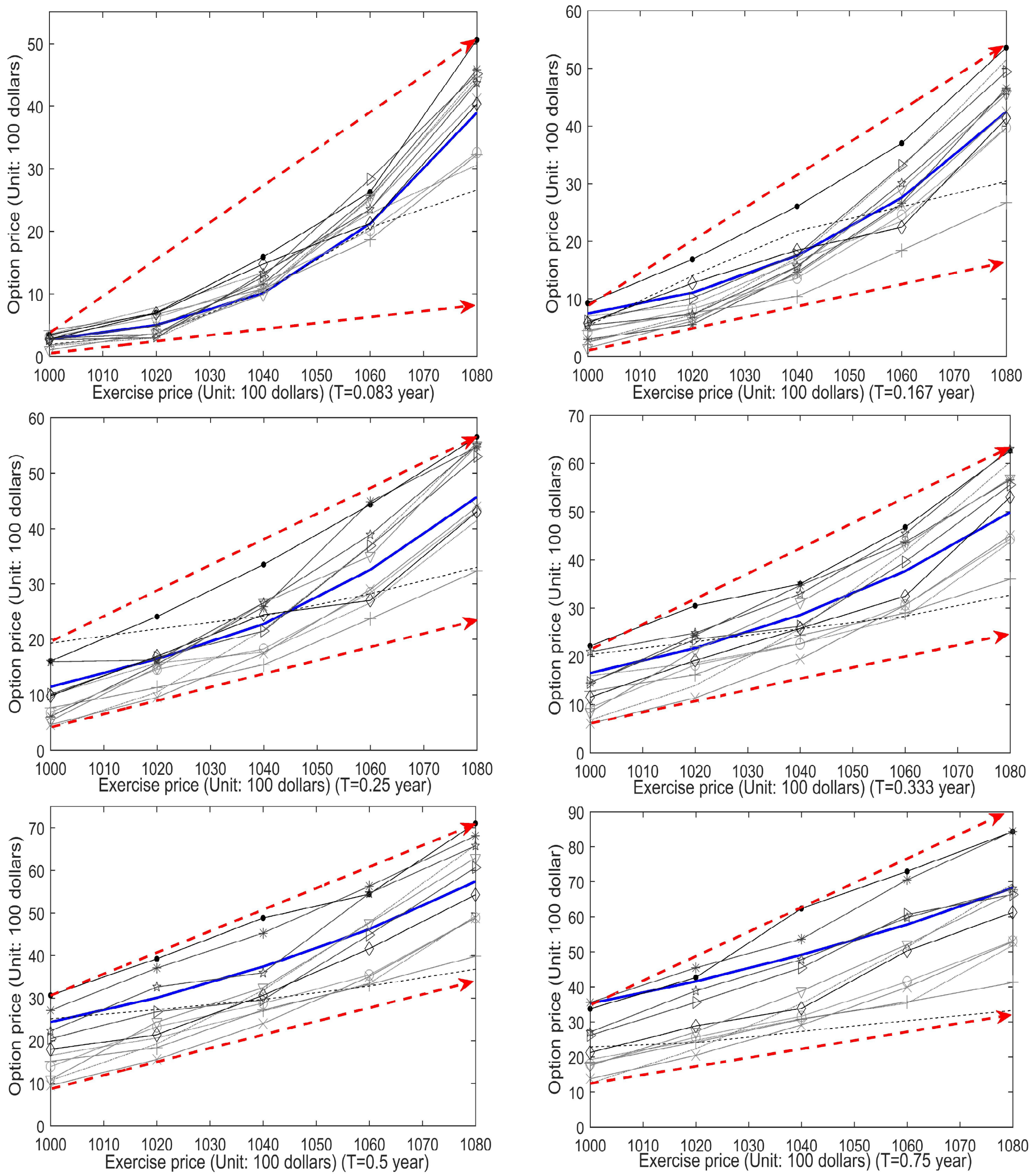


Figure 5.10: Comparison of option prices with different exercise prices

interval of the CGMY model offers better simulation results. Simultaneously, we observe that under a crisp environment, the simulation results of the VG and CGMY models are greater than the market price when the exercise price is lower and less than the market price when the exercise price is higher, whereas the simulation result of the NIG model is less than the market price when the exercise price is lower and greater than the market price when the exercise price is higher.

For exercise price $K = 1,050$, Table 5.6 provides the option pricing result using the NIG-

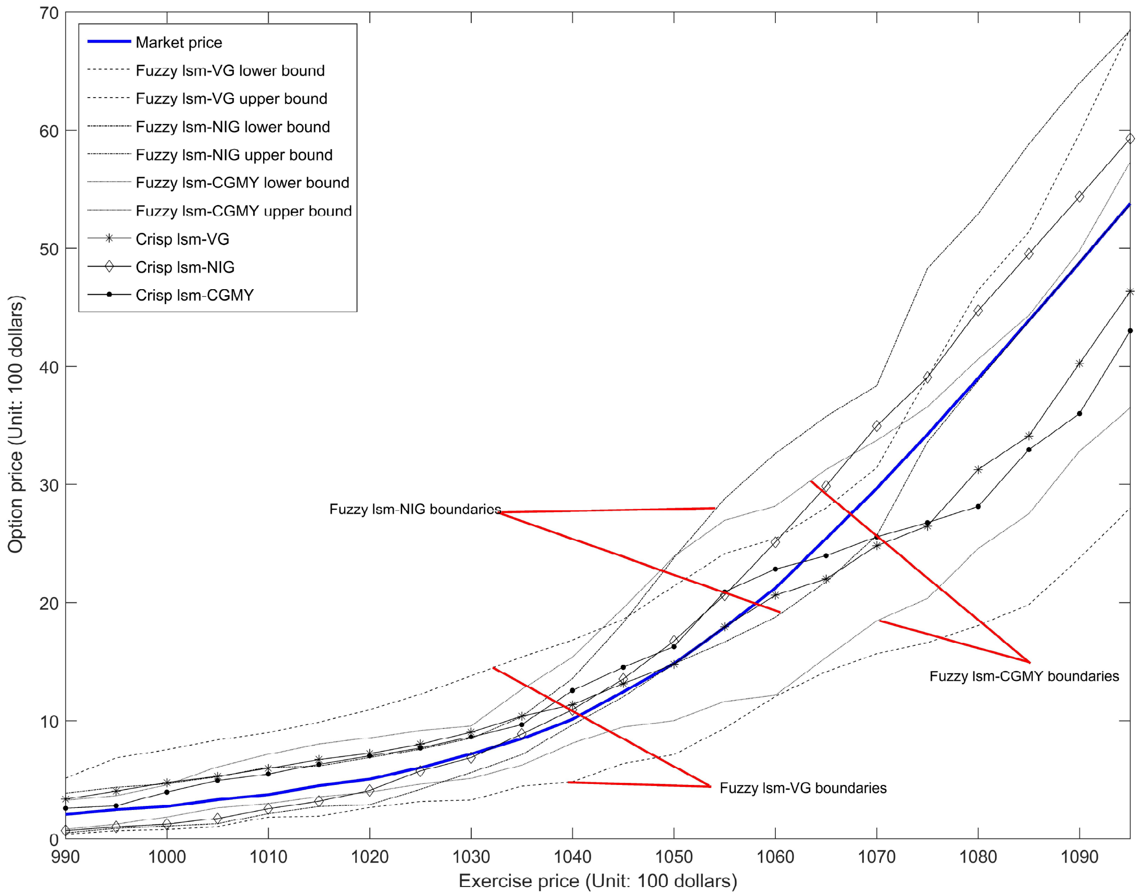


Figure 5.11: Option pricing results for April 2017 expiry

GJR-GARCH model under a fuzzy environment, in addition to the relationship between its membership function and level set. When the exponent of membership function n remains unchanged, the fuzzy interval narrows as the level set α increases while the fuzzy expectation lowers; when the level set α remains unchanged, the fuzzy interval narrows as the membership function exponent n increases, and the fuzzy expectation also decreases. The membership function diagram for different exponent n is shown in Fig. 5.12, from which we can see that when the rate of return of the membership function is parabolic, the option price membership function is also parabolic, and when $n = 1$, the parabolic membership function becomes the trapezoidal membership function. The left half interval of the function is monotonically increasing, whereas the right half interval is monotonically decreasing. The monotonically increasing part reflects that the seller's satisfaction increases as the price increases, whereas the monotonically decreasing part reflects that the buyer's satisfaction reduces as the price falls. Fig. 5.13 shows the relationship between the option price and the level set, and combined with the membership

Table 5.5: Option pricing results for April 2017 expiry (Unit: 100 dollars)

Sequence	Time to expiration (year)	Exercise price	Market price	Least squares Monte Carlo approach					
				Fuzzy environment (expected value)			Crisp environment		
				VG	NIG	CGMY	VG	NIG	CGMY
1	0.083	990	2.09	2.77	2.21	2.06	3.38	0.72	2.61
2	0.083	995	2.50	3.78	2.64	2.46	4.06	1.02	2.83
3	0.083	1000	2.78	4.19	2.87	3.21	4.76	1.25	3.98
4	0.083	1005	3.34	4.70	3.27	4.37	5.29	1.73	4.93
5	0.083	1010	3.76	5.43	4.10	5.10	5.97	2.57	5.48
6	0.083	1015	4.50	5.91	4.47	5.80	6.72	3.21	6.32
7	0.083	1020	5.05	6.81	4.91	6.26	7.25	4.11	7.03
8	0.083	1025	6.03	7.71	5.90	6.91	8.01	5.74	7.70
9	0.083	1030	7.20	8.55	7.06	7.34	9.05	6.88	8.63
10	0.083	1035	8.48	9.94	8.74	9.45	10.39	8.89	9.69
11	0.083	1040	10.14	10.83	11.63	11.75	11.36	10.94	12.55
12	0.083	1045	12.44	12.48	15.16	14.52	13.15	13.54	14.54
13	0.083	1050	14.85	14.28	19.31	16.96	14.79	16.76	16.29
14	0.083	1055	17.90	16.76	22.74	19.31	17.95	20.65	20.93
15	0.083	1060	21.24	18.73	25.69	20.17	20.63	25.11	22.86
16	0.083	1065	25.43	21.09	28.76	23.31	22.02	29.87	23.96
17	0.083	1070	29.68	23.55	32.02	26.10	24.81	34.96	25.58
18	0.083	1075	34.25	27.84	40.94	28.46	26.50	39.07	26.78
19	0.083	1080	39.00	32.26	45.85	32.59	31.27	44.72	28.15
20	0.083	1085	43.88	35.61	51.32	35.92	34.12	49.53	32.95
21	0.083	1090	48.87	41.75	56.41	41.32	40.26	54.38	36.01
22	0.083	1095	53.80	48.28	61.08	46.90	46.35	59.30	43.01

Table 5.6: The relationship between option pricing results and membership function/level set

Level set α	$n=1$		$n=2$		$n=3$	
	Fuzzy interval	Fuzzy expectation	Fuzzy interval	Fuzzy expectation	Fuzzy interval	Fuzzy expectation
0.95	[8.71,24.8]	16.76	[8.81,23.91]	16.36	[8.84,23.61]	16.23
0.90	[8.51,26.6]	17.56	[8.7,24.85]	16.78	[8.77,24.24]	16.51
0.85	[8.31,28.39]	18.35	[8.6,25.81]	17.21	[8.7,24.9]	16.80
0.80	[8.09,30.18]	19.14	[8.48,26.8]	17.64	[8.63,25.58]	17.11
0.75	[7.89,31.97]	19.93	[8.37,27.82]	18.10	[8.54,26.29]	17.42

f ly shows the changes in the option price interval under different level sets. The fuzzy expectation lowers as the level set increases, which shows that the membership function diagram of option prices is asymmetric with an inclined left tendency, which is consistent with the result shown in Fig. 5.12, where n represents the exponent of the parabolic fuzzy variable, when value of n varies, the shape of the membership function of the parabolic fuzzy variable will change accordingly; when $n=1$, the parabolic fuzzy variable converts to a trapezoidal fuzzy variable; when $n=2$, it represents the classical parabolic fuzzy variable.

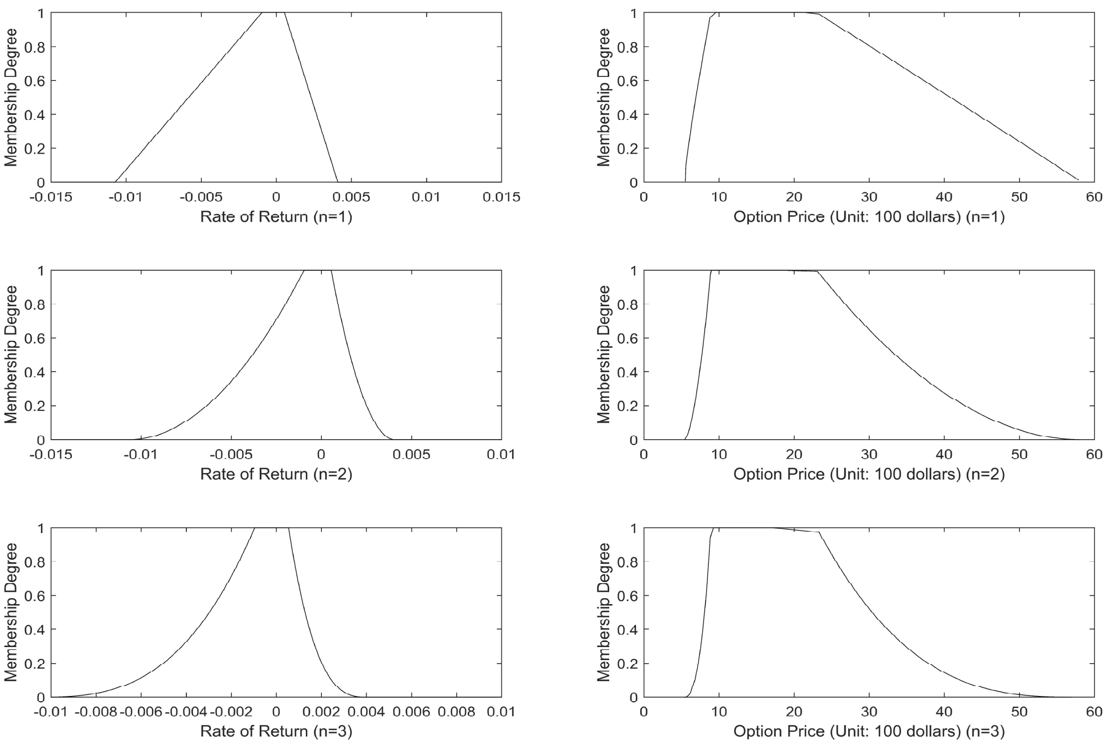


Figure 5.12: Diagrams of membership function

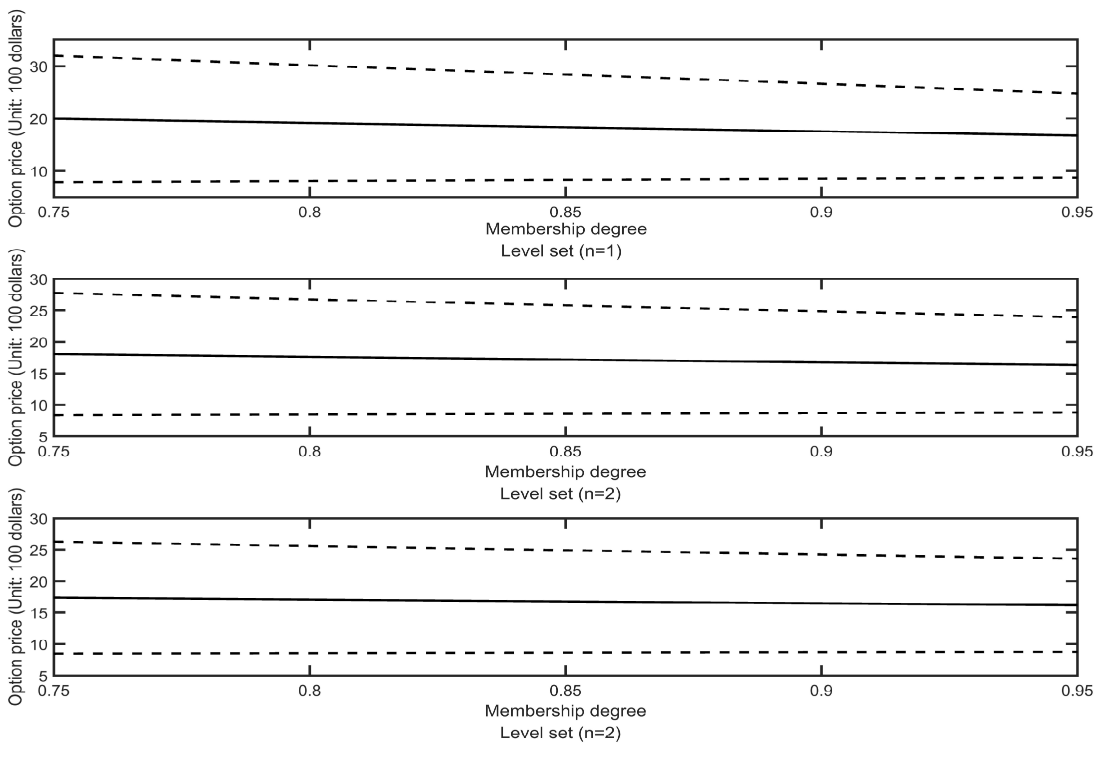


Figure 5.13: Relational chart between option price and level set

Fig. 5.14 shows the optimal option exercise boundaries when the time to expiration $T = 0.5$ year, exercise price $K = 1,060$, and the exponent of membership function n takes on different

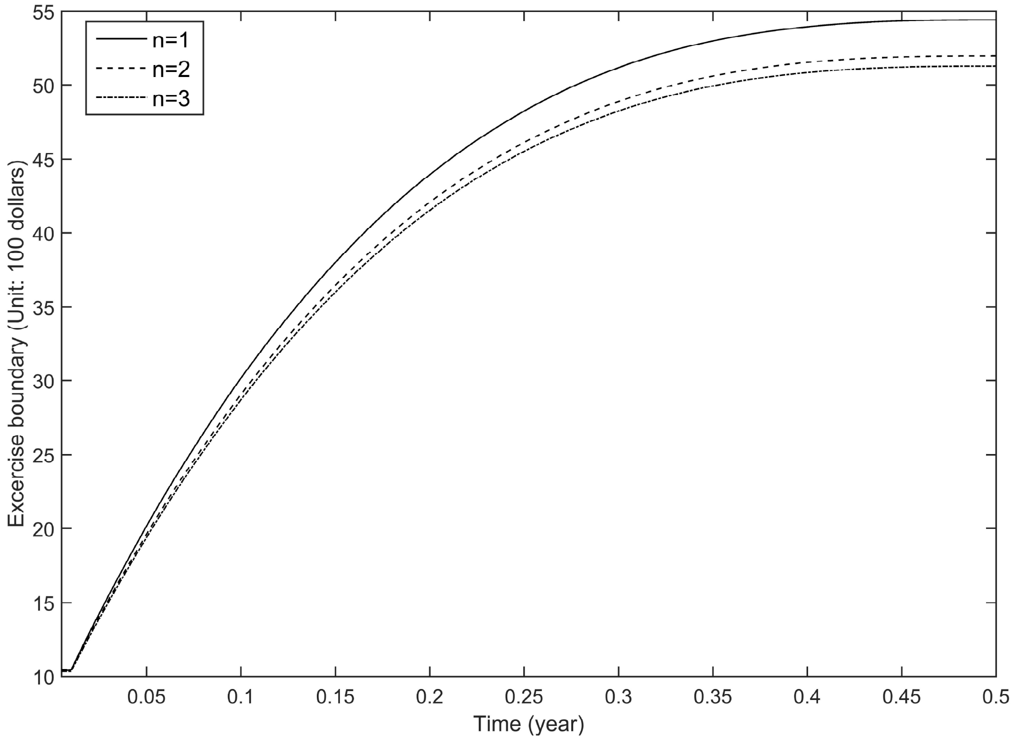


Figure 5.14: The optimal exercise boundary

values. The optimal option exercise boundary shifts higher as time goes. When $n = 1$, the position of the exercise boundary is at its highest, and when $n = 3$, the position of the exercise boundary is at its lowest. Since the price at the optimal exercise boundary satisfies $V(t) = K - S(t)$, as time progresses, $S(t)$ will decrease gradually. At the same time, we can see that as the expiry date approaches, the optimal exercise boundary tends to even out. At this point, if there is any sharp decrease in the option price, it may trigger early option exercise.

For the purpose of a comprehensive comparison of the accuracy of various option pricing models, we used two types of statistical methods – RMSE (root mean square error) and AAE (average absolute error) – to perform an error comparison of the pricing results obtained from different models; the results are presented in Table 5.8. These two indicators quantify the deviation between the pricing result and the market price; the lower the value obtained, the higher the pricing accuracy. The calculation formulas for these indicators are as follows:

$$\text{RMSE} = \sqrt{\sum_{i=1}^{N} \frac{(C_i^{Model} - C_i^{Market})^2}{N}} \tag{5.29}$$

$$\text{AAE} = \frac{\sum_{i=1}^{N} \left| C_i^{Model} - C_i^{Market} \right|}{N} \tag{5.30}$$

The results in Table 5.8 indicate that the least squares Monte Carlo algorithm has a better pricing accuracy than the binomial tree algorithm regardless of whether pricing occurs in a fuzzy or crisp environment. The simulation results of the least squares Monte Carlo algorithm under a fuzzy environment are better than the results under a crisp environment, and among the different models of the least squares Monte Carlo algorithm, the NIG model achieves the most prominent enhancing effect. Comparing the pricing effects of the VG, NIG and CGMY models, the NIG model in the least squares Monte Carlo algorithm has the best effect, followed by the CGMY model, with the VG model exhibiting the poorest effect; in the binomial tree algorithm, the CGMY model has the best effect, followed by the VG and NIG models. The fuzzy least squares Monte Carlo-NIG-GJR-GARCH model has the best performance. Furthermore, we calculate the accuracy rate by mean absolute percentage error (MAPE) to test the efficacy of the fuzzy least squares Monte Carlo-NIG-GJR-GARCH model, data from Table 5.5 were chosen, and the results are presented in Table 5.7. It can be observed from Table 5.7 that the fuzzy least squares Monte Carlo-NIG-GJR-GARCH model achieved 88.39% accuracy rate which is better improvement by 10.34% than the crisp least squares Monte Carlo-NIG-GJR-GARCH model. The variance of the accuracy rate of the proposed fuzzy model is 22.91% of that of the crisp least squares Monte Carlo-NIG-GJR-GARCH model, it is less than the crisp model; this shows that the proposed fuzzy model is more stable than the crisp model in terms of pricing accuracy rate. The results indicate that the proposed fuzzy model is effective and its pricing results are more accurate and stable even with many reality uncertainty factors included. The above outcomes show that fuzzy environment has great significance in the study of option pricing theory. The accuracy rate and MAPE evaluation are calculated as follows,

$$\text{Accuracy rate } = (1 - \text{ MAPE}) \times 100\%$$
$$\text{MAPE} = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{C_i^{Model} - C_i^{Market}}{C_i^{Market}} \right| \tag{5.31}$$

Table 5.7: Comparison of the accuracy rate of the different models

Indicator	Accuracy rate	Sample Variance
Crisp lsm-NIG-GJR-GARCH	78.05%	338.91
Fuzzy lsm-NIG-GJR-GARCH (Fuzzy expectation,our model)	88.39%	77.63
Compare to crisp lsm-NIG-GJR-GARCH	10.34%(Improved)	22.91%

5.4.4 Improvement of the least squares Monte Carlo algorithm

In the evaluation of the improvement effect of quasi-random numbers and Brownian Bridge method on the least squares Monte Carlo algorithm, we are mainly concerned with whether the improved method has increased the convergence speed. Therefore, this section uses the NIG-GJR-GARCH model as an example in the dynamic analysis of the convergence process of the pricing result. The results of a comparison of the convergence speeds of different calculation methods are shown in Fig. 5.15. It appears that under the least squares Monte Carlo simulation, it takes at least 5,000 simulations before the option pricing result can converge accurately to the average value. On the other hand, using the improved method, even within only 2,000 simulations, the pricing result can be kept within the reliable range. This result indicates that for option pricing, the improved method can effectively increase the convergence speed, reduce the number of simulations required, and increase the pricing efficiency. The convergence efficiency of the least squares Monte Carlo algorithm can be improved by 60% via the Sobol sequence and Brownian bridge method.

Table 5.8: Pricing error of the models

Model	RMSE	AAE
Fuzzy lsm-VG	9.622701	7.231907
Fuzzy lsm-NIG	5.876694	4.453933
Fuzzy lsm-CGMY	7.072131	5.227281
Crisp lsm-VG	10.257905	7.839648
Crisp lsm-NIG	7.177533	5.297496
Crisp lsm-CGMY	8.001463	5.800416
Fuzzy bt-VG	9.817527	8.210713
Fuzzy bt-NIG	10.408733	8.196607
Fuzzy bt-CGMY	7.276172	4.463557
Crisp bt-VG	10.974434	8.509217
Crisp bt-NIG	7.715236	8.244162
Crisp bt-CGMY	9.041845	4.018898

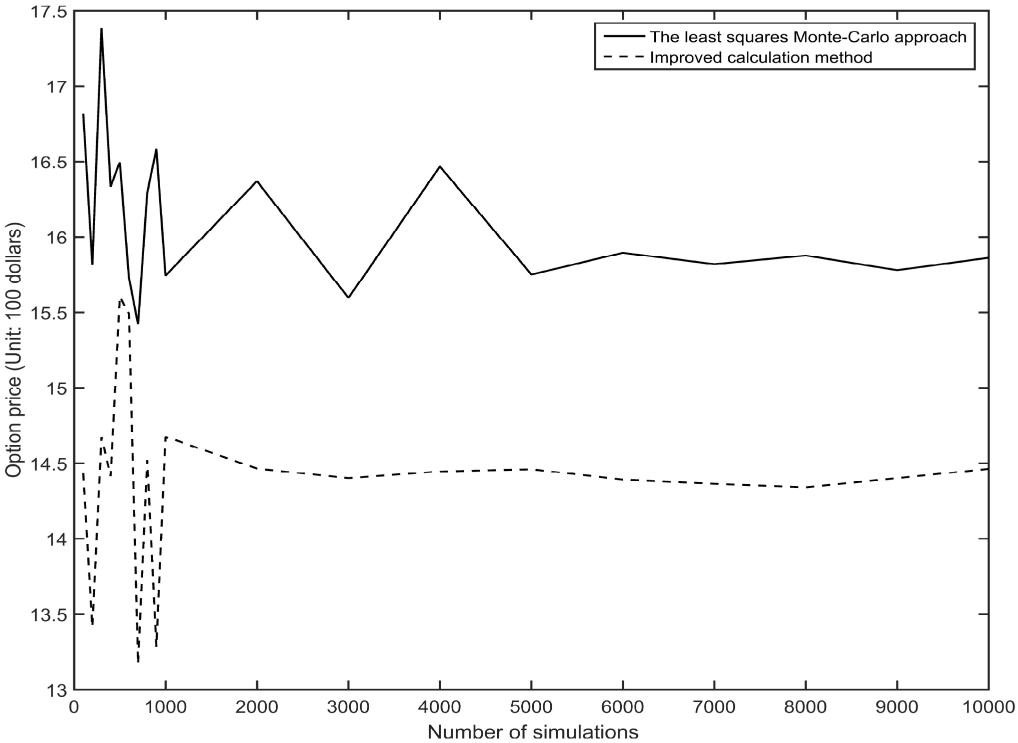


Figure 5.15: Convergence process of the least squares Monte Carlo algorithm and the improved calculation method

5.5 Conclusions

The decision of the optimal stopping time makes American option pricing problems more complicated than the European option pricing problem, and the traditional BS (Black Schloes) model is not capable of deciding American option pricing.

Taking into account the time-varying, jump and leverage effect (i.e. asymmetric volatility) characteristics of the asset price fluctuation, this study built a Levy-GJR-GARCH American option pricing model based on an infinite pure jump process.

Meanwhile we incorporated fuzzy set theory and set the underlying asset price volatility as the more generalized parabolic fuzzy variable (which can cover the triangle and trapezoid fuzzy variable), according to the American option pricing theory we derived the optimal exercise boundary, the continuation holding region and the stopping holding region for fuzzy American options, and considering more general situations with the fuzzy variables with mixed distributions, based on fuzzy simulation technology established fuzzy binomial tree and fuzzy least squares Monte Carlo numerical algorithms for the proposed model, and we particularly applied quasi-random numbers that are produced by the Sobol sequence, and Brownian bridge method, to improve the convergence speed of the least squares Monte Carlo algorithm.

Lastly, using the S&P 100 Index and data for the corresponding American put options, we empirically tested our fuzzy pricing model; comparatively analysed the pricing effect of different widely used infinite pure-jump Levy processes (the VG (variance gamma process), NIG (normal inverse Gaussian process) and CGMY (Carr-Geman-Madan-Yor process)) under fuzzy and crisp environments with different fuzzy numerical algorithms that are proposed in this chapter.

The main conclusions are as follows:

1. There is significant volatility clustering and strong leverage effects and stochastic jump characteristics in the S&P 100 Index.

2. The option price increases as the length of the time to expiration of options increases and as the exercise price increases; in addition, the pricing accuracy for short-term option prices is greater than for medium- and long-term options.

3. Under a fuzzy environment, the market price of short-term options can be better covered by the fuzzy interval of the VG, NIG and CGMY models, and the membership function curve of the option price is asymmetric with an inclined left tendency, whereas the fuzzy interval

narrows as the level set α and the exponent of membership function n increase.

4. The results of the option pricing are more accurate under a fuzzy environment than the results under a crisp environment; the least squares Monte Carlo algorithm yields more accurate pricing than the binomial tree algorithm, whereas among different infinite pure jump Levy processes, the NIG (normal inverse Gaussian process) and CGMY (Carr-Geman-Madan-Yor process) models yield better simulation results than the VG (variance gamma process) model.

5. The fuzzy least squares Monte Carlo-NIG-GJR-GARCH model has the best performance; According to the MAPE evaluation, the model achieved 88.39% accuracy rate which is better improvement by 10.34% than the crisp least squares Monte Carlo-NIG-GJR-GARCH model. Furthermore, the variance of the accuracy rate of the fuzzy least squares Monte Carlo-NIG-GJR-GARCH model is 22.91% of that of the crisp least squares Monte Carlo-NIG-GJR-GARCH model, it is less than the crisp model; this shows that the proposed fuzzy model is more stable than the crisp model in terms of pricing accuracy rate. The results indicate that the proposed fuzzy model is effective and its pricing results are more accurate and stable even with many reality uncertainty factors included. The convergence efficiency is significantly improved by using the Sobol sequence and Brownian Bridge method to improve the least squares Monte Carlo algorithm, the convergence efficiency of the least squares Monte Carlo algorithm can be improved by 60% via the Sobol sequence and Brownian bridge method.

Chapter 6

Conclusion

6.1 Summary of Thesis Achievements

As widely recognised in academia and practical, the real financial market exists complicated uncertainties and it is not as ideal as the assumptions of the theoretical model. One side, the fluctuation of asset price have significant jump phenomenon which including finite big jumps and high frequency small jumps simultaneously, and the asset yield is not a normal distribution but rather exhibits skewed and leptokurtic and fat-tailed characteristics. Another side, because of many subjective and objective uncertain factors and the incomplete information in the real-life financial market, the parameters in the theoretical model are often vagueness and cannot be expressed using crisp values, i.e., randomness and fuzziness co-exist at the same time. Therefore the theoretical asset pricing model should be improve further.

The Levy process is a stochastic process with good mathematical properties, such as independent stationary increments, stochastic continuity. These properties mean that the process can have plenty of applications. Coupled with its ability to fit leptokurtosis and fat-tailed characteristics, this makes the process pla l role in asset pricing field. Furthermore, as an important part of asset pricing, option pricing is a core issue in asset pricing research.

Levy process includes both of finite jump process and infinite pure jump process, however, finite jump process only consider the finite big jumps in the asset price fluctuations, ignoring the high-frequency small jumps which is also existing in the fluctuation of asset price, compared with finite jump Levy process, infinite pure jump Levy process can capture the big jumps and high frequency small jumps simultaneously in the real market situation, such as bigger jumps to represent market shocks, whereas smaller jumps to represent real-time transactions. Therefore, owing to the non-normality phenomenon of the underlying asset yield, and the asset price fluctuation including big jumps and high frequency small jumps simultaneously in the real market situation, the infinite pure jump Levy process was adopted to capture these characteristics of the asset price. At the same time, fuzzy set theory as a powerful tool to address the uncertainty, vagueness of the social environment, thus integrating fuzzy set theory to option pricing models with infinite pure jump Levy process, it can be a useful supplement to the option pricing method and can provide a new theoretical basis for the pricing of options.

To price options more rationally, in this thesis, we introduce the fuzzy set theory and the infinite pure jump Levy process into an options pricing model on the basis of previous studies to further enhance and enrich options pricing theories. In addition, this thesis also discussed the theoretical and practical values of the options pricing model in a fuzzy environment through numerical simulation and empirical analysis. Owing to the style of options mainly includes two types: European options (to be exercised only at the expiration date) and American options (to be exercised before or at the expiration date). Thus, the main contributions of this thesis are summarised from these two aspects, specifically as follows,

In **Chapter 4**, we incorporate fuzzy set theory to construct a European options pricing model based on VG (variance gamma) process (which is one of widely used infinite pure-jump Levy processes) in a fuzzy environment on the basis of Black Schloes (BS) model. The drift, diffusion, and jump parameters are treated as the trapezoidal fuzzy random variables in the model.

The Monte Carlo simulation algorithm was then used to provide simulation estimates for the model, meanwhile the instrumental variable method was introduced to improve the convergence speed of the Monte Carlo algorithm.

At last, the numerical simulation experiments, and an empirical analysis which using Tencent Holdings (HK.0700) and its stock options data, are performed verify the efficiency and accuracy of the proposed model. The following are yielded by the comparison of the option price under the VG model in a fuzzy environment, BS model in a crisp environment, and the VG model in a crisp environment:

1) An analysis of the Monte Carlo numerical simulations and the empirical analysis which uses Tencent Holding (HK.0700) and its stock options data show that treating the drift, diffusion and jump as fuzzy random variables to obtain the options pricing model is more reasonable, the fuzzy interval can cover the market prices of options and the prices obtained by the VG process option pricing model in a crisp environment, and the expectations using fuzzy pricing are closer to the market prices of options than the pricing results obtained by the BS (Black-Scholes) model in a crisp environment. According to the evaluation based on the mean absolute percentage error (MAPE), the fuzzy VG process option pricing model achieved 96.68% accuracy rate which is an improvement of 1.33% over the crisp BS model. Furthermore, the variance of the accuracy rate of the proposed fuzzy model is 56.77% of that of the crisp BS model, it is less than the crisp BS model; this shows that the proposed fuzzy model is more stable than the crisp BS model in terms of pricing accuracy rate. The results indicate that the fuzzy VG process option pricing model is feasible and its pricing results are more accurate and stable even when many reality uncertainty factors are included. The results are more consistent with the real-life market and can provide investors with better investment advice.

2) The results show that expectation obtained through the VG model in a fuzzy environment is mostly greater than the pricing results obtained under the VG model in a crisp environment and that obtained using the BS model in a crisp environment. At the same time, at a confidence level of 0.8, the fuzzy interval basically encompasses the outcomes of the VG model in a crisp environment. On the other hand, the option price of the BS model in a crisp environment tends to be less than the fuzzy interval. This shows that the greater the number of random factors and uncertainties included in the model, the

higher the option price, the results are consistent with the real-life market.

3) Both the VG model under a crisp environment and that under a fuzzy environment are sensitive to variations in the jump parameter. As the jump parameter increases, the option price decreases. At the same time, an increase in the confidence level also causes the fuzzy interval for the pricing model in a fuzzy environment to narrow.

4) The empirical analysis shows that the instrumental variable method can improve the convergence speed faster than the Monte Carlo simulation alone, the convergence efficiency of Monte Carlo algorithm can be improved by 50% via the instrumental variable method.

In **Chapter 5**, we further extended our research in American option pricing problem, due to the existence of the problem of the optimal stopping time, studies regarding American option pricing problems are much more complicated than European option pricing problems, and the traditional BS (Black Schloes) model is not suitable for American option pricing.

Taking into account the time-varying, jump and leverage effect (i.e. asymmetric volatility) characteristics of the asset price fluctuation, this study constructed a fuzzy Levy-GJR-GARCH American option pricing model based on an infinite pure jump process with incorporated fuzzy set theory and set the underlying asset price volatility as the more generalized parabolic fuzzy variable (which can cover the triangle and trapezoid fuzzy variable).

Meanwhile, according to the American option pricing model theory we derived the optimal exercise boundary, the continuation holding region and the stopping holding region for the fuzzy American options, and considering more general situations with the fuzzy variables with mixed distributions, we then applied fuzzy simulation technology to the widely used numerical algorithms (the binomial tree algorithm and the least squares Monte Carlo algorithm) to create fuzzy pricing numerical algorithms for the proposed model, such as fuzzy binomial tree algorithm, fuzzy least squares Monte Carlo algorithm, and we particularly applied quasi-random numbers that are produced by Sobol sequence, and Brownian Bridge method, to improve the convergence speed of the least squares Monte Carlo algorithm.

Finally, by using American option data from the Standard & Poors 100 index, we empirically test our fuzzy pricing model and comparatively analyse the pricing effect of different widely used infinite pure-jump Levy processes (the VG (variance gamma process), NIG (normal inverse Gaussian process) and CGMY (Carr-Geman-Madan-Yor process)) under fuzzy and crisp environments with different fuzzy numerical algorithms that are proposed in this chapter. The main conclusions are as follows:

1) There is significant volatility clustering and strong leverage effects and stochastic jump characteristics in the S&P 100 Index.

2) The option price increases as the length of the time to expiration of options increases and as the exercise price increases; in addition, the pricing accuracy for short-term option prices is greater than for medium- and long-term options.

3) Under a fuzzy environment, the market price of short-term options can be better covered by the fuzzy interval of the VG, NIG and CGMY models, and the membership function curve of the option price is asymmetric with an inclined left tendency, whereas the fuzzy interval narrows as the level set α and the exponent of membership function n increase.

4) The results of the option pricing are more accurate under a fuzzy environment than the results under a crisp environment; the least squares Monte Carlo algorithm yields more accurate pricing than the binomial tree algorithm, whereas among different infinite pure jump Levy processes, the NIG (normal inverse Gaussian process) and CGMY (Carr-Geman-Madan-Yor process) models yield better simulation results than the VG (variance gamma process) model.

5) The fuzzy least squares Monte Carlo-NIG-GJR-GARCH model has the best performance; According to the MAPE evaluation, the model achieved 88.39% accuracy rate which is better improvement by 10.34% than the crisp least squares Monte Carlo-NIG-GJR-GARCH model. Furthermore, the variance of the accuracy rate of the fuzzy least squares Monte Carlo-NIG-GJR-GARCH model is 22.91% of that of the crisp least squares Monte Carlo-NIG-GJR-GARCH model, it is less than the crisp model; this shows that the proposed fuzzy model is more stable than the crisp model in terms of pricing accuracy

rate. The results indicate that the proposed fuzzy model is effective and its pricing results are more accurate and stable even with many reality uncertainty factors included. In addition, the Sobol sequence and Brownian bridge method can effectively improve the convergence speed of the least squares Monte Carlo algorithm, the convergence efficiency of the least squares Monte Carlo algorithm can be improved by 60% via the Sobol sequence and Brownian bridge method.

6.2 Future Work

This thesis incorporated fuzzy set theory and the infinite pure jump Levy process into the options pricing models on the basis of previous studies as a useful supplement to the option pricing theories from European option and American options two aspects. It can provide a new theoretical basis for the pricing of options. However, still room remains to perform in future work.

6.2.1 Future work from option pricing model perspective

The subject of this thesis was relatively straightforward European call options and American put options, without taking into account the pricing of more complex financial derivatives. Therefore, future research may focus on incorporating fuzzy set theory and the infinite pure jump Levy process in the analysis of the pricing of other derivative products.

6.2.2 Future work from fuzzy set theory perspective

In this thesis, the membership functions of fuzzy variables were assumed to have fixed forms, without considering the more complex situation in which the membership degree is also a fuzzy variable. To handle this uncertain situation, Type-2 fuzzy variable plays a key role, it has been widely used as the latest research focus. Therefore, future research may focus on finding a more reasonable calculation method for Type-2 fuzzy variable and other related problems.

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List of Publications

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◯ 1. **Huiming Zhang**, Junzo Watada, " A European Call Options Pricing Model Using the Infinite Pure Jump Levy Process in a Fuzzy Environment ", *IEEJ Transactions on Electrical and Electronic Engineering, TEEE C (Electronics, Information and Systems)*, Vol.13, No.10, pp.1-15, October 2018, to be published.

◯ 2. **Huiming Zhang**, Junzo Watada, "Fuzzy Levy-GJR-GARCH American Option Pricing Model Based on an Infinite Pure Jump Process", *IEICE Transactions on Information and Systems*, Vol.E101-D, No.7, pp.1843-1859, July 2018.

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1. **Huiming Zhang**, Junzo Watada, " A Fuzzy Index Tracking Multi-Objective Approach to Stock Data Analytics ", *the 4th International Conference on computer and information sciences (ICCOINS 2018)*, pp.1-6, Kuala Lumpur Convention Center, Kuala Lumpur, Malaysia, 13-14 August 2018, to be published.

◯ 2. **Huiming Zhang**, Junzo Watada, " Building Fuzzy Variance Gamma Option Pricing

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