Invariant Hilbert schemes and resolutions of singularities of GIT quotients

不変Hilbertスキーム及びGIT商の特異点解消

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Contents

Chapter 1

Introduction

We work over the field of complex numbers $\mathbb{C}$. By a *variety* we mean an irreducible reduced scheme of finite type. Let G be a reductive algebraic group, X an affine G-variety, and $h : \mathrm{Irr}(G) \longrightarrow \mathbb{Z}_{\geq 0}$ a *Hilbert function* that assigns a non-negative integer to each irreducible representation of G. The invariant Hilbert scheme $\mathrm{Hilb}_h^G(X)$ is a moduli space that parametrizes G-stable closed subschemes of X whose coordinate rings have Hilbert function h. It was first introduced by Alexeev and Brion [AB05] for G connected. Later, Brion [Bri13] extended the construction to the case where G is any reductive algebraic group. Therefore, the invariant Hilbert scheme can be considered as a generalization of the G-Hilbert scheme of Ito and Nakamura [IN96] for a finite group G. If we take h to be the Hilbert function of a general fiber of the quotient morphism $\pi : X \longrightarrow X/\!/G := \mathrm{Spec}(\mathbb{C}[X]^G)$, we obtain a morphism

$$\gamma : \mathrm{Hilb}_h^G(X) \longrightarrow X/\!/G, \quad [Z] \mapsto Z/\!/G.$$

The morphism γ is called the *quotient-scheme map*, or the *Hilbert–Chow morphism*. By the choice of the Hilbert function h, it turns out that γ is an isomorphism over a dense open subset Y_0 of $X/\!/G$. Therefore, the Zariski closure $\mathcal{H}^{main} := \overline{\gamma^{-1}(Y_0)}$ equipped with a reduced scheme structure is an irreducible component of the invariant Hilbert scheme $\mathrm{Hilb}_h^G(X)$, which is called the *main component* of $\mathrm{Hilb}_h^G(X)$. Since the restriction of γ to the main component $\mathcal{H}^{main}$ is projective and birational, we can ask the following questions:

Question 1.0.1. Does the restriction $\gamma|_{\mathcal{H}^{main}}$ give a resolution of singularities of the quotient variety $X/\!/G$?

Question 1.0.2. Does the invariant Hilbert scheme $\mathrm{Hilb}_h^G(X)$ coincide with the main component $\mathscr{H}^{main}$? In other words, is $\mathrm{Hilb}_h^G(X)$ irreducible?

When the group G is finite, the G-Hilbert scheme G-Hilb(X) is known to give a crepant resolution of singularities of the quotient variety X/G if X is a smooth variety of dimension less than four and if the G-action is Gorenstein ([IN96, Nak01, BKR01]). It provides a moduli-theoretic perspective to the theory of McKay correspondence and has been actively studied also in connection with representation theory. However, in the case of infinite groups, not many examples of invariant Hilbert schemes are explicitly known except for some cases where G is a classical group and X is a classical representation of G ([JR09], [Bec11], [Ter14a], [Ter14b]). In this thesis, we focus on the fact that any 3-dimensional affine normal quasihomogeneous $SL(2)$-variety can be described as a GIT quotient, and determine the geometric structure of the associated invariant Hilbert scheme. This gives a new family of examples where the corresponding Hilbert–Chow morphism is a resolution of singularities. We also provide a necessary and sufficient condition for the resolution to be minimal. Here we say that a resolution of singularities $f : W \longrightarrow Y$ is *minimal* if $K_W \cdot C \geq 0$ holds for any curve $C \subset W$ that is contracted to a point under f, where K_W denotes the canonical divisor of W.

This thesis consists of six chapters. Below we summarize the content of each chapter.

In chapter 2, we first review general properties of invariant Hilbert schemes and spherical varieties, then overview some known results on quasihomogeneous $SL(2)$-varieties, where a variety with a reductive group action is called *quasihomogeneous* if it contains a dense open orbit. In [Pop73], Popov gives a complete classification of 3-dimensional affine normal quasihomogeneous $SL(2)$-varieties; they are uniquely determined by a pair of numbers $(l, m) \in \{\mathbb{Q} \cap (0, 1]\} \times \mathbb{N}$, where $(0, 1]$ stands for the half-open interval $\{x : 0 < x \leq 1\}$. Popov proves that the variety $E_{l,m}$ corresponding to a pair (l, m) is smooth if and only if $l = 1$; otherwise $E_{l,m}$ contains a unique singular point, which is $SL(2)$-invariant. After the work of Popov, such $SL(2)$-varieties have been extensively studied by Kraft [Kra84], Panyushev [Pan88, Pan91], Gaĭfullin [Gaĭ08], Batyrev and Haddad [BH08], and others. Batyrev and Haddad described $E_{l,m}$ as a GIT quotient of an affine hypersurface H_{q-p} in $\mathbb{C}^5$ modulo an action of $\mathbb{C}^* \times \mu_m$, where p and q are positive coprime integers such that $l = p/q$. By using the quotient description, they show that there exists an equivariant flip

$$\begin{array}{ccc} E_{l,m}^{-} & \dashrightarrow & E_{l,m}^{+} \\ & \searrow \quad \swarrow & \\ & E_{l,m} & \end{array}$$

by variation of GIT quotients. They also show that the varieties $E_{l,m}^{-}$ and $E_{l,m}^{+}$ are dominated by a weighted blow-up $E'_{l,m} := Bl_O^{\omega}(E_{l,m})$ of $E_{l,m}$ with a weight ω, where ω depends on the parameters l and m. Furthermore, they define an additional $\mathbb{C}^*$-action on $E_{l,m}$ and prove that $E_{l,m}$ becomes a spherical $SL(2)\times\mathbb{C}^*$-variety with respect to the Borel subgroup $\widetilde{B} := B\times\mathbb{C}^*$.

In chapters 3,4, and 5, we study the birational geometry of $E_{l,m}$ through the invariant Hilbert scheme $\mathscr{H} := \mathrm{Hilb}_h^{\mathbb{C}^*\times\mu_m}(H_{q-p})$ associated with the triple $(\mathbb{C}^*\times\mu_m, H_{q-p}, h)$, where h is the Hilbert function of a general fiber of the quotient morphism $\pi : H_{q-p} \longrightarrow E_{l,m}$, and by examining the corresponding Hilbert–Chow morphism $\gamma : \mathscr{H} \longrightarrow E_{l,m}$. The main results of this thesis are the following.

Theorem 1.0.3 (Corollaries 3.1.3 and 5.3.3 and Theorem 3.2.4)**.** *The invariant Hilbert scheme $\mathscr{H}$ is irreducible and reduced (therefore, $\mathscr{H}$ coincides with the main component $\mathscr{H}^{main}$), and the Hilbert–Chow morphism γ is an equivariant resolution of singularities of $E_{l,m}$. Moreover, $\mathscr{H}$ is described as follows.*

(i) *If $l = 1$, then $\mathscr{H}$ is isomorphic to $E_{1,m}$.*

(ii) *If $l < 1$ and if $E_{l,m}$ is toric (i.e. if $q-p$ divides m, see Theorem 2.3.3 and Remark 2.3.3.1), then $\mathscr{H}$ is isomorphic to $E'_{l,m}$.*

(iii) *If $l < 1$ and if $E_{l,m}$ is non-toric, then $\mathscr{H}$ is isomorphic to the minimal resolution $\widetilde{E'_{l,m}}$ of $E'_{l,m}$.*

Theorem 1.0.4 (Theorem 5.4.1)**.** *Let $k = g.c.d.(m, q-p)$, $a = m/k$, and $b = (q-p)/k$. Then, the Hilbert–Chow morphism γ is a minimal resolution of $E_{l,m}$ if and only if $1 + b \leq ap$.*

In chapter 3, we describe the minimal resolution $\widetilde{E'_{l,m}}$ in terms of the colored fan by using the combinatorics of the colored cone of $E'_{l,m}$. In chapter 4, we give explicit descriptions of the ideals corresponding to each $SL(2)$-orbit of the invariant Hilbert scheme $\mathscr{H}$. In chapter 5, we first show that the Hilbert–Chow morphism γ factors through $\widetilde{E'_{l,m}}$ building on the results from chapters 3 and 4, and then we complete the proof of Theorem 1.0.3 by calculating the dimension of the Zariski tangent space at each Borel-fixed point. The proof of Theorem 1.0.4 will be given at the end of the chapter.

In chapter 6, we present the following question as a generalization of the framework of the problem we have considered in the previous chapters.

Question 1.0.5. When a singularity is described as a GIT quotient of its Cox ring by the natural action of a quasitorus, does the corresponding invariant Hilbert scheme give a resolution of singularities?

This thesis considers the question for the singularity of the closure of the maximal nilpotent orbit in $\mathfrak{sl}_n$. We will see that, at least if $n = 2, 3$, the corresponding Hilbert–Chow morphism coincides with the Springer's resolution.

Chapter 2

Preliminaries

2.1 The invariant Hilbert scheme

Brion's survey [Bri13] offers a detailed introduction to the invariant Hilbert scheme. In this section, we briefly review some definitions and properties on invariant Hilbert schemes.

2.1.1 The invariant Hilbert scheme and the Hilbert–Chow morphism

Let G be a reductive algebraic group, and denote by $\operatorname{Irr}(G)$ the set of isomorphism classes of irreducible representations of G. For any G-module V, we have its isotypical decomposition:

$$V \cong \bigoplus_{M \in \operatorname{Irr}(G)} \operatorname{Hom}^G(M,V) \otimes M.$$

The dimension of $\operatorname{Hom}^G(M,V)$ is called the *multiplicity* of M in V. If the multiplicity is finite for every $M \in \operatorname{Irr}(G)$, it defines a function

$$h_V : \operatorname{Irr}(G) \longrightarrow \mathbb{Z}_{\geq 0}, \quad M \mapsto h_V(M) := \dim \operatorname{Hom}^G(M,V),$$

which is called the *Hilbert function* of V.

Let X be an affine G-variety, S a Noetherian scheme on which G acts trivially, and Z a closed G-subscheme of $X \times S$. We denote the projection $Z \longrightarrow S$ by f. Then, according to

[Bri13], there is a decomposition of $f_*\mathcal{O}_Z$ as an $\mathcal{O}_S$-G-module

$$f_*\mathcal{O}_Z \cong \bigoplus_{M\in\mathrm{Irr}(G)} \mathcal{F}_M \otimes M,$$

where sheaves of covariants $\mathcal{F}_M := \mathcal{H}om^G_{\mathcal{O}_S}(M\otimes\mathcal{O}_S, f_*\mathcal{O}_Z)$ are sheaves of $\mathcal{O}_S$-modules. Assume that each $\mathcal{F}_M$ is a coherent $\mathcal{O}_S$-module. Then, each of them is locally-free if and only if it is flat over S.

Definition 2.1.2 ([AB05, Definition 1.5])**.** Let $h : \mathrm{Irr}(G) \longrightarrow \mathbb{Z}_{\geq 0}$ be a Hilbert function. For a given triple (G, X, h), the associated functor

$$\mathcal{H}ilb^G_h(X) : (\mathrm{Sch})^{\mathrm{op}} \longrightarrow (\mathrm{Sets})$$

$$S \mapsto \left\{ \begin{array}{c} Z \subset X\times S \\ {\scriptstyle f}\searrow \quad \downarrow \mathrm{pr}_2 \\ S \end{array} \;\middle|\; \begin{array}{l} Z \text{ is a closed } G\text{-subscheme of } X\times S; \\ f \text{ is a flat morphism;} \\ f_*\mathcal{O}_Z \cong \bigoplus_{M\in\mathrm{Irr}(G)} \mathcal{F}_M\otimes M; \\ \mathcal{F}_M \text{ is locally-free of rank } h(M) \text{ over } \mathcal{O}_S \end{array} \right\},$$

is called the *invariant Hilbert functor*.

Theorem 2.1.3 ([Bri13, Theorem 2.11])**.** *The invariant Hilbert functor is represented by a quasiprojective scheme* $\mathrm{Hilb}^G_h(X)$*, the invariant Hilbert scheme associated with the affine G-variety X and the Hilbert function h. We denote by* $\mathrm{Univ}^G_h(X) \subset X\times\mathrm{Hilb}^G_h(X)$ *the universal family over* $\mathrm{Hilb}^G_h(X)$.

We denote by $T_{[Z]}\mathrm{Hilb}^G_h(X)$ the Zariski tangent space to the invariant Hilbert scheme $\mathrm{Hilb}^G_h(X)$ at a closed point $[Z]$. We sometimes represent a closed point of $\mathrm{Hilb}^G_h(X)$ by the defining ideal I_Z of Z if there is no danger of confusion.

Theorem 2.1.4 ([Bri13, Proposition 3.5])**.** *With the above notation, we have*

$$T_{[Z]}\mathrm{Hilb}^G_h(X) \cong \mathrm{Hom}^G_{\mathbb{C}[X]}(I_Z, \mathbb{C}[X]/I_Z).$$

The invariant Hilbert scheme comes with a projective morphism called the *quotient-scheme map*, or the *Hilbert–Chow morphism*. This is a generalization of the Hilbert–Chow morphism from the G-Hilbert scheme G-$\mathrm{Hilb}(X)$ to the quotient variety X/G that sends a

G-cluster to its support. The construction of the quotient-scheme map in a general setting is explained in [Bri13, §3.4]. Here we restrict ourselves to the situation we consider in this thesis. Let

$$\pi : X \longrightarrow X/\!/G := \mathrm{Spec}(\mathbb{C}[X]^G)$$

be the quotient morphism. By the generic flatness theorem, π is flat over a non-empty open subset Y_0 of $X/\!/G$. According to [Bri13, §3.4], every scheme-theoretic fiber of π over the flat locus yields the same Hilbert function. This special function is called the *Hilbert function of a general fiber of* π, and we denote it by h_X. Since $h_X(0) = 1$, where 0 stands for the trivial representation of G, the associated quotient-scheme map is a morphism

$$\gamma : \mathrm{Hilb}^G_{h_X}(X) \longrightarrow X/\!/G, \quad [Z] \mapsto Z/\!/G.$$

Proposition 2.1.5 ([Bri13, Proposition 3.15], [Bud10, Theorem I.1.1])**.** *With the preceding notation, the diagram*

$$\begin{array}{ccc} \mathrm{Univ}^G_{h_X}(X) & \xrightarrow{\mathrm{pr}_1} & X \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Hilb}^G_{h_X}(X) & \xrightarrow[\gamma]{} & X/\!/G \end{array}$$

commutes. Moreover, the pullback of γ to the flat locus Y_0 of π is an isomorphism.

The Zariski closure $\mathscr{H}^{main} := \overline{\gamma^{-1}(Y_0)}$ equipped with a reduced scheme structure is an irreducible component of $\mathrm{Hilb}^G_{h_X}(X)$, which is called the *main component* of $\mathrm{Hilb}^G_{h_X}(X)$ ([Bec11, Definition 2.4], [LT15, Definition 2.3]). Since the restriction

$$\gamma|_{\mathscr{H}^{main}} : \mathscr{H}^{main} \longrightarrow X/\!/G$$

is projective and birational, it is natural to ask whether $\gamma|_{\mathscr{H}^{main}}$ gives a resolution of singularities of the quotient variety $X/\!/G$.

2.1.6 Tools to study the invariant Hilbert scheme

We consider a situation where there is an action on X by another connected reductive algebraic group G'. Suppose that the action of G' on X commutes with that of G. Then, G' acts on $\mathrm{Hilb}^G_{h_X}(X)$, and the quotient-scheme map γ is G'-equivariant ([Bri13, Proposition 3.10]). Let us especially consider the action of a Borel subgroup $B' \subset G'$ on $\mathrm{Hilb}^G_{h_X}(X)$, and denote by $\mathcal{H}^{B'}$ the set of fixed points for the action of B'.

Theorem 2.1.7 ([Ter14a, Lemmas 1.6 and 1.7])**.** *Suppose that $X /\!/ G$ has a unique closed G'-orbit, and that this orbit is a point. Then the following hold.*

(i) *Each G'-stable closed subset of $\mathrm{Hilb}^G_{h_X}(X)$ contains at least one fixed point for the action of the Borel subgroup B'. Moreover, if $\mathrm{Hilb}^G_{h_X}(X)$ has a unique B'-fixed point, then $\mathrm{Hilb}^G_{h_X}(X)$ is connected.*

(ii) *The following are equivalent:*

  (a) $\mathrm{Hilb}^G_{h_X}(X) = \mathcal{H}^{main}$ *and* $\mathrm{Hilb}^G_{h_X}(X)$ *is smooth;*

  (b) $\dim T_{[Z]} \mathrm{Hilb}^G_{h_X}(X) = \dim \mathcal{H}^{main}$ *for any* $[Z] \in \mathcal{H}^{B'}$*, and* $\mathrm{Hilb}^G_{h_X}(X)$ *is connected.*

There is one more useful tool to study the invariant Hilbert scheme. To elaborate, let G, X, h_X, and G' be as above. For any irreducible representation $M \in \mathrm{Irr}(G)$, there is a finite-dimensional G'-module F_M that generates $\mathrm{Hom}^G(M, \mathbb{C}[X])$ as a module over the invariant ring $\mathbb{C}[X]^G$ ([Bec11, Proposition 4.2]). Let $[Z] \in \mathrm{Hilb}^G_{h_X}(X)$, and let

$$f_{M,Z} : F_M \longrightarrow \mathrm{Hom}^G(M, \mathbb{C}[Z])$$

be the composition of the inclusion $F_M \hookrightarrow \mathrm{Hom}^G(M, \mathbb{C}[X])$ and the natural surjection $\mathrm{Hom}^G(M, \mathbb{C}[X]) \longrightarrow \mathrm{Hom}^G(M, \mathbb{C}[Z])$. Then, the quotient vector space $F_M / \mathrm{Ker}\, f_{M,Z}$ defines a point in the Grassmannian $\mathrm{Gr}(h_X(M), F_M^\vee)$. In this way, we obtain a G'-equivariant morphism

$$\eta_M : \mathrm{Hilb}^G_{h_X}(X) \longrightarrow \mathrm{Gr}(h_X(M), F_M^\vee), \quad [Z] \mapsto F_M / \mathrm{Ker}\, f_{M,Z}.$$

Moreover, there is a finite subset $\mathcal{M} \subset \mathrm{Irr}(G)$ such that the morphism

$$\gamma \times \prod_{M \in \mathcal{M}} \eta_M : \mathrm{Hilb}^G_{h_X}(X) \longrightarrow X /\!/ G \times \prod_{M \in \mathcal{M}} \mathrm{Gr}(h_X(M), F_M^\vee)$$

is a closed immersion (see [Bec11, §4.2] for details).

2.2 Spherical varieties

The main references for this section are [Kno91], [Pas17], [Per14], and [Tim11].

2.2.1 Classification

Spherical varieties are classified by combinatorial data called *colored fans*, which are generalization of fans for toric varieties.

Let G be a connected reductive algebraic group, and let H be an algebraic subgroup of G. A normal G-variety X is called *spherical* if it contains a dense open orbit under a Borel subgroup B of G. By a *spherical embedding*, we mean a normal G-variety X together with an equivariant open embedding $G/H \hookrightarrow X$ of a homogeneous spherical variety G/H.

Let X be a spherical embedding of G/H with respect to a Borel subgroup B. We denote by $\mathfrak{X}(B)$ the group of characters of B, and by $\mathbb{C}(G/H)^{(B)}$ the set of rational B-eigenfunctions:

$$\mathbb{C}(G/H)^{(B)} = \left\{ f \in \mathbb{C}(G/H)^* : \exists \chi_f \in \mathfrak{X}(B)\ \forall g \in B\ g \cdot f = \chi_f(g) f \right\}.$$

Consider a homomorphism $\tau : \mathbb{C}(G/H)^{(B)} \longrightarrow \mathfrak{X}(B)$ defined by $f \mapsto \chi_f$, and let $\Gamma \subset \mathfrak{X}(B)$ be its image. Then, Γ is a finitely generated free abelian group. Since G/H contains a dense B-orbit, the kernel of τ consists of constant functions. Therefore, we get the exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathbb{C}(G/H)^{(B)} \longrightarrow \Gamma \longrightarrow 0.$$

We see that any valuation $v : \mathbb{C}(G/H)^* \longrightarrow \mathbb{Q}$ of G/H defines a homomorphism

$$\mathbb{C}(G/H)^{(B)} \longrightarrow \mathbb{Q}, \quad f \mapsto v(f),$$

which factors through Γ. Hence, it induces an element $\rho_v \in Q := \mathrm{Hom}(\Gamma, \mathbb{Q})$, which satisfies $\rho_v(\chi_f) = v(f)$ for any $f \in \Gamma$. A valuation v is called *G-invariant* if $v(g \cdot f) = v(f)$ holds for any $g \in G$. We denote by $\mathcal{V}$ the set of G-invariant valuations. Since it is known that the map $\mathcal{V} \longrightarrow Q$, $v \mapsto \rho_v$ is injective ([LV83, 7.4 Proposition]), we will not distinguish $\mathcal{V}$ and its image in Q. Moreover, the set of G-invariant valuations $\mathcal{V}$ is known to be a finitely generated cone ([Kno91, Corollary 5.3]).

Definition 2.2.2 ([Pas17, Definition 2.8])**.** A primitive element of a ray of the opposite $-\mathcal{V}^\vee$ of the dual in $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ is called a *spherical root* of X.

Let Y be a G-orbit of X, and set $X_{Y,G} := \{x \in X : Y \subset \overline{G \cdot x}\}$. Then, $X_{Y,G}$ is a G-stable open subset of X, and Y is a unique closed G-orbit of $X_{Y,G}$. A spherical variety X is called *simple* if it contains a unique closed G-orbit. It is known that any spherical variety is covered by finitely many simple spherical varieties ([Kno91, §2], [Per14, §3.1]).

Let us denote by $\mathscr{D}(X)$ the set of B-stable prime divisors on X. We simply write $\mathscr{D}$ for $\mathscr{D}(G/H)$, and an element of $\mathscr{D}$ is called a *color*. If a divisor $D \in \mathscr{D}(X)$ non-trivially meets the open orbit G/H, then we have $D \cap (G/H) \in \mathscr{D}$. Otherwise, D is an irreducible component of the complement $X \setminus (G/H)$ and hence is G-stable. Therefore, each G-orbit Y of X determines two sets

$$\mathscr{B}_Y(X) := \{v_D \in \mathscr{V} : D \in \mathscr{D}_Y(X) \text{ is } G\text{-stable}\},$$

where v_D stands for the valuation associated to the divisor D, and

$$\mathscr{F}_Y(X) := \{D \cap (G/H) \in \mathscr{D} : D \in \mathscr{D}_Y(X) \text{ is not } G\text{-stable}\},$$

where $\mathscr{D}_Y(X) := \{D \in \mathscr{D}(X) : Y \subset D\}$.

Remark 2.2.2.1. Let X be a simple spherical variety with a closed orbit Y. Set $X_0 := X \setminus \bigcup_{D \in \mathscr{D}(X) \setminus \mathscr{D}_Y(X)} D$, and set $X_1 := G/H \setminus \bigcup_{D \in \mathscr{D} \setminus \mathscr{F}_Y(X)} D$. Then, X_0 is a B-stable affine open subset, and its coordinate ring is described as follows:

$$\mathbb{C}[X_0] = \{f \in \mathbb{C}[X_1] : v(f) \geq 0 \text{ for all } v \in \mathscr{B}_Y(X)\}.$$

Moreover, we have $X = GX_0$ (see [Kno91, Theorems 2.1 and 2.3]).

Consider a map $\varrho : \mathscr{D} \longrightarrow Q$ given by $D \mapsto \varrho(D) := \rho_{v_D}$.

Definition 2.2.3. A *colored cone* is a pair $(\mathscr{C}, \mathscr{F})$ with $\mathscr{C} \subset Q$ and $\mathscr{F} \subset \mathscr{D}$ that satisfies the following properties: $\mathscr{C}$ is a cone generated by $\varrho(\mathscr{F})$ and finitely many elements of $\mathscr{V}$; and $\mathscr{C}^\circ \cap \mathscr{V} \neq \emptyset$, where $\mathscr{C}^\circ$ stands for the relative interior of $\mathscr{C}$. A colored cone $(\mathscr{C}, \mathscr{F})$ is called *strictly convex* if $\mathscr{C}$ is strictly convex and if $0 \notin \varrho(\mathscr{F})$. A pair $(\mathscr{C}_0, \mathscr{F}_0)$ is called a *face* of a colored cone $(\mathscr{C}, \mathscr{F})$ if $\mathscr{C}_0$ is a face of $\mathscr{C}$, $\mathscr{C}_0^\circ \cap \mathscr{V} \neq \emptyset$, and if $\mathscr{F}_0 = \mathscr{F} \cap \varrho^{-1}(\mathscr{C}_0)$.

For a G-orbit Y of X, $\mathscr{C}_Y(X) \subset Q$ denotes the cone generated by $\varrho(\mathscr{F}_Y(X))$ and $\mathscr{B}_Y(X)$.

Theorem 2.2.4 ([LV83, 8.10 Proposition])**.** *The map $X \mapsto (\mathscr{C}_Y(X), \mathscr{F}_Y(X))$ gives a bijective correspondence between the isomorphism classes of simple spherical varieties X with a closed orbit Y and strictly convex colored cones.*

Theorem 2.2.5 ([Kno91, Lemma 3.2])**.** *Let X be a spherical variety, and let Y be a G-orbit. Then, the map $Z \mapsto (\mathscr{C}_Z(X), \mathscr{F}_Z(X))$ gives a bijective correspondence between G-orbits whose closure contain Y and faces of $(\mathscr{C}_Y(X), \mathscr{F}_Y(X))$.*

Definition 2.2.6. A *colored fan* is a non-empty finite set $\mathfrak{F}$ of colored cones satisfying the following properties: every face of $(\mathscr{C}, \mathscr{F}) \in \mathfrak{F}$ belongs to $\mathfrak{F}$; for every $v \in \mathscr{V}$, there is at most one $(\mathscr{C}, \mathscr{F}) \in \mathfrak{F}$ such that $v \in \mathscr{C}^\circ$. A colored fan $\mathfrak{F}$ is called *strictly convex* if $(0, \emptyset) \in \mathfrak{F}$, namely if all elements of $\mathfrak{F}$ are strictly convex.

For a spherical variety X, set $\mathfrak{F}(X) := \{(\mathscr{C}_Y(X), \mathscr{F}_Y(X)) : Y \subset X \text{ is a } G\text{-orbit}\}$.

Theorem 2.2.7 ([Kno91, Theorem 3.3])**.** *The map $X \mapsto \mathfrak{F}(X)$ gives a bijective correspondence between the isomorphism classes of spherical varieties and strictly convex colored fans.*

Remark 2.2.7.1 ([Kno91, §3]). An order relation can be given to the set of G-orbits of X by the inclusion of closures. Theorems 2.2.4 and 2.2.5 imply that $Y \mapsto (\mathscr{C}_Y(X), \mathscr{F}_Y(X))$ is an order-reversing bijection between the set of G-orbits and $\mathfrak{F}(X)$. In particular, the open orbit corresponds to $(0, \emptyset)$.

Theorem 2.2.8 ([Kno91, Theorem 4.1])**.** *Let X and X' be spherical embeddings of G/H. Then, the following are equivalent.*

(i) *An equivariant birational morphism $X \longrightarrow X'$ exists.*

(ii) *For any $(\mathscr{C}, \mathscr{F}) \in \mathfrak{F}(X)$ there exists $(\mathscr{C}', \mathscr{F}') \in \mathfrak{F}(X')$ such that $\mathscr{C} \subset \mathscr{C}'$ and $\mathscr{F} \subset \mathscr{F}'$.*

Definition 2.2.9. A spherical variety X is called *toroidal* if $\mathscr{F}_Y(X) = \emptyset$ for any G-orbit Y, namely if no $D \in \mathscr{D}$ contains a G-orbit in its closure.

Remark 2.2.9.1. Let X be a toroidal spherical variety whose maximal colored cones are $(\mathscr{C}_1, \emptyset)$, …, $(\mathscr{C}_r, \emptyset)$. A local structure theorem for toroidal spherical varieties ([BP87, 3.4], see also [Tim11, Theorem 29.1] and [Per14, Proposition 3.3.2]) implies that any equivariant resolution of singularities for X can be obtained by subdividing the cones $\mathscr{C}_1$, …, $\mathscr{C}_r$ appropriately, as in the toric case.

2.2.10 Weil divisors on a spherical variety

Let X be a spherical embedding of G/H. According to [Per14, §3.2], any Weil divisor on X is linearly equivalent to a divisor of the form $\delta = \sum_{D\in\mathscr{D}(X)} n_D D$.

Theorem 2.2.11 ([Per14, Theorem 3.2.1])**.** *With the above notation, δ is Cartier if and only if for any G-orbit Y there exists $f_Y \in \mathbb{C}(G/H)^{(B)}$ that satisfies $n_D = v_D(f_Y)$ for any $D \in \mathscr{D}_Y(X)$.*

Definition 2.2.12 ([Per14, Definition 3.2.2])**.** Let X be a spherical variety.

(i) We denote by $\mathscr{C}(X)$ the union of all $\mathscr{C}_Y(X)$, where Y runs over all G-orbits.

(ii) A collection $l = (l_Y)$ indexed by G-orbits Y is called a *piecewise linear function* on the colored fan $\mathfrak{F}(X)$ of X if it satisfies the following conditions:

- for each G-orbit Y, l_Y is the restriction of an element of Γ to $\mathscr{C}_Y(X)$;
- for any G-orbits Y and Z with $Z \subset \overline{Y}$, we have $l_Z|_{\mathscr{C}_Y(X)} = l_Y$.

We denote by $PL(X)$ the abelian group of piecewise linear functions on $\mathfrak{F}(X)$.

Remark 2.2.12.1 ([Per14, Remark 3.2.3]). An element $l \in PL(X)$ depends only on its values on maximal colored cones, namely cones corresponding to closed orbits in the sense of Remark 2.2.7.1.

Let $\mathrm{Car}^B(X)$ be the group of B-stable Cartier divisors on X. Then, we have a homomorphism $\mathrm{Car}^B(X) \longrightarrow PL(X)$, $\delta \mapsto l_\delta$, where $(l_\delta)_Y = f_Y$ with the notation of Theorem 2.2.11. Set $\mathscr{D}_0(X) := \bigcup \mathscr{D}_Y(X)$, where Y runs over all G-orbits.

Theorem 2.2.13 ([Tim11, Theorem 17.18])**.** *For any B-stable Cartier divisor*

$$\delta = \sum_{D\in\mathscr{D}_0(X)} v_D(l_\delta)D + \sum_{D\in\mathscr{D}(X)\setminus\mathscr{D}_0(X)} n_D D$$

on X, the following properties are equivalent.

(i) *The divisor δ is generated by global sections.*

(ii) *For any G-orbit Y, there exists $f_Y \in \mathbb{C}(G/H)^{(B)}$ that satisfies the following conditions:*

- $f_Y|_{\mathscr{C}_Y(X)} = l_\delta|_{\mathscr{C}_Y(X)}$;
- $f_Y|_{\mathscr{C}(X)\setminus\mathscr{C}_Y(X)} \leq l_\delta|_{\mathscr{C}(X)\setminus\mathscr{C}_Y(X)}$;
- $v_D(f_Y) \leq n_D$ *for any* $D \in \mathscr{D}(X) \setminus \mathscr{D}_0(X)$.

Theorem 2.2.14 ([Pas17, Theorem 2.15])**.** *Keep the above notation. Let* $D \in \mathscr{D}$*, and choose a simple root* α *with respect to* B *such that the action of the corresponding parabolic subgroup* P_α *does not preserve the divisor* D*, i.e.,* $P_\alpha \cdot D \neq D$*. Then, one and only one of the following cases occurs: (i)* α *is a spherical root of* G/H*; (ii)* 2α *is a spherical root of* G/H*; (iii) neither* α *nor* 2α *is a spherical root of* G/H*.*

Remark 2.2.14.1 ([Pas17, §2]). The anticanonical divisor of a spherical embedding $G/H \hookrightarrow X$ can be described as

$$-K_X = \sum_{D\in\mathscr{D}(X)\setminus\mathscr{D}} D + \sum_{D\in\mathscr{D}} a_D D,$$

where the coefficient a_D attached to $D \in \mathscr{D}$ is determined according to the type of D classified in Theorem 2.2.14. Denote by $P \subset G$ the stabilizer of the open B-orbit of G/H, and by S_P the set of simple roots α such that $-\alpha$ is not a weight of the Lie algebra of P. Then the integer a_D is given as follows: if D is of type (i) or (ii), $a_D = 1$; and if D is of type (iii), $a_D = \sum_{\alpha\in\mathscr{R}_P^+}\langle\alpha, \alpha^\vee\rangle$, where $\mathscr{R}_P^+$ stands for the set of positive roots with at least one non-zero coefficient for a simple root of S_P.

Remark 2.2.14.2. Keep the notation of Remark 2.2.14.1. For later use, we consider a linear function $h_\mathscr{C}$ associated to a colored cone $(\mathscr{C}, \mathscr{F}) \in \mathfrak{F}(X)$: the function $h_\mathscr{C}$ is defined so that $h_\mathscr{C}(\rho_D) = a_D$ for any $D \in \mathscr{F}$, and that $h_\mathscr{C}(v) = 1$ for any primitive element v of a ray of $\mathscr{C}$ that is not generated by some ρ_D with $D \in \mathscr{F}$ (cf. [Pas17, Proposition 5.2]).

Remark 2.2.14.3. Let X be a $\mathbb{Q}$-Gorenstein spherical G/H-embedding. Given a G-equivariant resolution of singularities $f : Y \longrightarrow X$, one has $K_Y = f^*K_X + \sum_{i\in I} a_i F_i$ for some $a_i \in \mathbb{Q}$, where $\{F_i\ :\ i \in I\}$ is the set of exceptional divisors of f. Let $(\mathscr{C}, \mathscr{F})$ be the colored cone of $\mathfrak{F}(X)$ such that $\rho_{F_i} \in \mathscr{C}$ under the notation of Remark 2.2.14.2. Then, according to the proof of [Pas17, Proposition 5.2], the coefficient a_i attached to F_i is $h_\mathscr{C}(\rho_{F_i}) - 1$.

2.3 Affine normal quasihomogeneous $SL(2)$-varieties

2.3.1 Classification

Popov [Pop73] gives a complete classification of affine normal quasihomogeneous $SL(2)$-varieties. Consult also the book of Kraft [Kra84, III.4].

Theorem 2.3.2 ([Pop73, Corollary of Proposition 9])**.** *Every 3-dimensional affine normal quasihomogeneous $SL(2)$-variety containing more than one orbit is uniquely determined by a pair of numbers $(l,m) \in \{\mathbb{Q} \cap (0,1]\} \times \mathbb{N}$.*

Remark 2.3.2.1. The classification of 3-dimensional affine homogeneous $SL(2)$-varieties can be found in [Pop73, Proposition 6].

We denote by $E_{l,m}$ the variety corresponding to a pair (l,m). The numbers l and m are called the *height* and the *degree* of $E_{l,m}$, respectively. Write $l = p/q$, where $g.c.d.(q,p) = 1$, and set

$$k := g.c.d.(m, q-p), \quad a := \frac{m}{k}, \quad b := \frac{q-p}{k}. \tag{2.1}$$

Theorem 2.3.3 ([Gaĭ08], see also [BH08, Corollary 2.7])**.** *An affine normal quasihomogeneous $SL(2)$-variety $E_{l,m}$ is toric if and only if $q-p$ divides m.*

Remark 2.3.3.1 ([BH08, §3]). Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$. If $q-p$ divides m, i.e., if $m = a(q-p)$, then the toric variety $E_{l,m}$ is defined by the cone $\sum_{i=1}^4 \mathbb{R}_{\geq 0}\mathbf{v}_i$, where $\mathbf{v}_1 = \mathbf{e}_1$, $\mathbf{v}_2 = -\mathbf{e}_1 + aq\mathbf{e}_3$, $\mathbf{v}_3 = \mathbf{e}_2$, and $\mathbf{v}_4 = -\mathbf{e}_2 + ap\mathbf{e}_3$.

We use the following notation for some closed subgroups of $SL(2)$:

$$T := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^* \right\}, \quad B := \left\{ \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^*, u \in \mathbb{C} \right\},$$
$$U_n := \left\{ \begin{pmatrix} \zeta & u \\ 0 & \zeta^{-1} \end{pmatrix} : \zeta^n = 1, u \in \mathbb{C} \right\}, \quad C_n := \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} : \zeta^n = 1 \right\}.$$

Remark 2.3.3.2. An $SL(2)$-variety $E_{l,m}$ is smooth if and only if $l = 1$.

(i) If $l = 1$, then $E_{1,m}$ contains two $SL(2)$-orbits: the open orbit $\mathfrak{U} \cong SL(2)/C_m$ and a 2-dimensional orbit $\mathfrak{D} \cong SL(2)/T$. It is known that

$$E_{1,m} \cong SL(2) \times_T \mathbb{C} := \operatorname{Spec}\left(\mathbb{C}[SL(2) \times \mathbb{C}]^T\right),$$

where T acts on $\mathbb{C}$ by the character $\chi_m : t \mapsto t^m$ (see [Kra84, III.4.5, Beispiel 2] and [Pan91, Proposition 5]).

(ii) If $l < 1$, then $E_{l,m}$ contains three $SL(2)$-orbits: the open orbit $\mathfrak{U} \cong SL(2)/C_m$, a 2-dimensional orbit $\mathfrak{D} \cong SL(2)/U_{a(q+p)}$, and the closed orbit $\{O\}$. The fixed point O is a unique singular point, which is $SL(2)$-invariant.

Remark 2.3.3.3. Let $l \leq 1$. An explicit construction of $E_{l,m}$ reduces to determine a system of generators of the following semigroup ([Kra84, III.4.7, Satz 1], [Pan88]):

$$M^+_{l,m} := \left\{(i,j) \in \mathbb{Z}^2_{\geq 0} \ : \ j \leq li,\ m|(i-j)\right\}.$$

Let (i_1, j_1), …, (i_u, j_u) be a system of generators of $M^+_{l,m}$, and consider a vector

$$v = (X^{i_1}Y^{j_1},\ \ldots,\ X^{i_u}Y^{j_u}) \in V(i_1 + j_1) \oplus \cdots \oplus V(i_u + j_u) = V,$$

where $V(n) := \mathrm{Sym}^n\langle X, Y\rangle$ is the irreducible $SL(2)$-representation of highest weight n. Then, $E_{l,m}$ is isomorphic to the closure $\overline{SL(2)\cdot v} \subset V$.

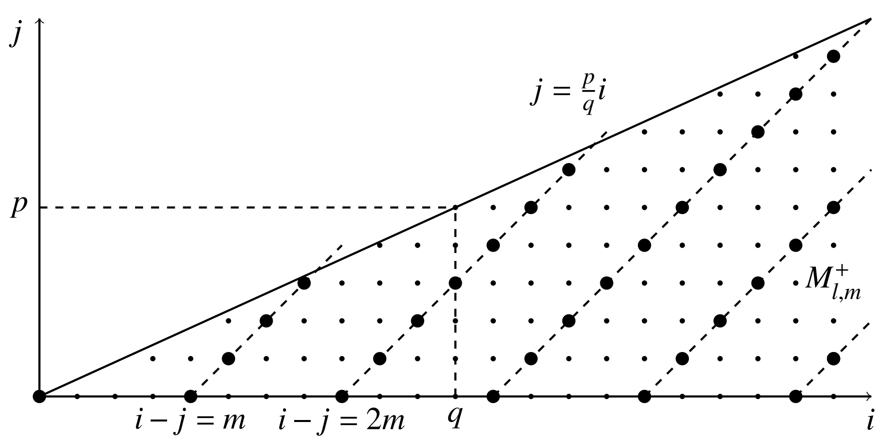


Figure 2.1: The semigroup $M^+_{l,m}$

Remark 2.3.3.4. If $l = 1$, then we see that $M^+_{1,m}$ is minimally generated by $(1,1)$ and $(m,0)$. An algorithm for finding a system of generators of $M^+_{l,m}$ for $l < 1$ can be found in [Pan88]. By applying the algorithm for the case when $m = a(q-p)$, i.e., when $E_{l,m}$ is a toric variety (see Theorem 2.3.3), we see that $M^+_{l,m}$ is minimally generated by $(m,0)$, $(m+1,1)$, …, (aq,ap).

2.3.4 Quotient description

According to [BH08, §1], an affine normal quasihomogeneous $SL(2)$-variety $E_{l,m}$ has a description as a categorical quotient of a hypersurface in $\mathbb{C}^5$. We consider $\mathbb{C}^5$ as the $SL(2)$-module $V(0)\oplus V(1)\oplus V(1)$ with coordinates X_0, X_1, X_2, X_3, X_4, and identify X_1, X_2, X_3, X_4 with the coefficients of the 2×2 matrix $\begin{pmatrix} X_1 & X_3 \\ X_2 & X_4 \end{pmatrix}$ so that $SL(2)$ acts by left multiplication. We consider actions of the following diagonalizable groups on $\mathbb{C}^5$:

$$G_0 := \{\mathrm{diag}(t,\ t^{-p},\ t^{-p},\ t^q,\ t^q) : t \in \mathbb{C}^*\},\ \ G_m := \{\mathrm{diag}(1,\ \zeta^{-1},\ \zeta^{-1},\ \zeta,\ \zeta) : \zeta^m = 1\}.$$

It is easy to see that the $SL(2)$-action on $\mathbb{C}^5$ commutes with the $G_0\times G_m$-action.

Theorem 2.3.5 ([BH08, Theorem 1.6])**.** *Let $E_{l,m}$ be a 3-dimensional affine normal quasihomogeneous $SL(2)$-variety of height $l = p/q \le 1$ and of degree m. Then, $E_{l,m}$ is isomorphic to the categorical quotient of an affine hypersurface $H_{q-p} \subset \mathbb{C}^5$ defined by $X_0^{q-p} = X_1X_4 - X_2X_3$ modulo the action of $G_0\times G_m$.*

Remark 2.3.5.1. According to the proof of [BH08, Theorem 1.6], the dense open orbit $\mathfrak{U}$ of $E_{l,m}$ is isomorphic to the $G_0\times G_m$-quotient of the open subset in H_{q-p} defined by the condition $X_0 \neq 0$. Also, the ring of G_0-invariants of $H_{q-p}\cap\{X_0 \neq 0\}$ is generated by the monomials $X := X_0^pX_1$, $Y := X_0^{-q}X_3$, $Z := X_0^pX_2$, and $W := X_0^{-q}X_4$, which satisfy the equation $\det\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = X_0^{p-q}X_1X_4 - X_0^{p-q}X_2X_3 = 1$.

An $SL(2)$-variety $E_{l,m}$ has another description as an affine categorical quotient. In order to see this, let $H_b \subset \mathbb{C}^5$ be an affine hypersurface defined by the equation $Y_0^b = X_1X_4 - X_2X_3$, and consider the action of the group $G_0'\times G_a$, where

$$G_0' := \{\mathrm{diag}(t^k,\ t^{-p},\ t^{-p},\ t^q,\ t^q) : t \in \mathbb{C}^*\},\ \ G_a := \{\mathrm{diag}(1,\ \zeta^{-1},\ \zeta^{-1},\ \zeta,\ \zeta) : \zeta^a = 1\}.$$

Theorem 2.3.6 ([BH08, Theorem 1.7])**.** *Let $E_{l,m}$ be a 3-dimensional affine normal quasihomogeneous $SL(2)$-variety of height $l = p/q \le 1$ and of degree m. Then, $E_{l,m}$ is isomorphic to the categorical quotient of H_b modulo the action of $G_0'\times G_a$.*

Remark 2.3.6.1. According to the proof of [BH08, Theorem 1.7], $G_0\times G_m$ contains a subgroup isomorphic to $G_k' = \{\mathrm{diag}(\zeta,\ 1,\ 1,\ 1,\ 1) : \zeta^k = 1\}$, and the hypersurface H_b is isomorphic to the G_k'-quotient of H_{q-p}.

Theorem 2.3.7 ([BH08, Corollary 2.6])**.** *For any affine $SL(2)$-variety $E_{l,m}$, the Cox ring $\mathrm{Cox}(E_{l,m})$ of $E_{l,m}$ is isomorphic to the coordinate ring $\mathbb{C}[H_b]$ of H_b.*

Let L^- and L^+ be linearizations of the trivial line bundle over H_b corresponding to non-trivial characters $\chi^- : G'_0 \times G_a \longrightarrow \mathbb{C}^*$, $(t, \zeta) \mapsto t^{k-p+q}$ and $\chi^+ : G'_0 \times G_a \longrightarrow \mathbb{C}^*$, $(t, \zeta) \mapsto t^{-k+p-q}$ of $G'_0 \times G_a$, respectively. Also, consider the Zariski open subsets $U^- := H_b \setminus \{X_3 = X_4 = 0\}$ and $U^+ := H_b \setminus \{X_1 = X_2 = 0\}$ of H_b.

Theorem 2.3.8 ([BH08, Propositions 3.2 and 3.3])**.** *The subsets $H_b^{ss}(L^-)$ and $H_b^{ss}(L^+)$ of semistable points of H_b with respect to the $G'_0 \times G_a$-linearized line bundles L^- and L^+ are U^- and U^+, respectively.*

Theorem 2.3.9 ([BH08, Theorem 3.4])**.** *Set $E^-_{l,m} := H_b^{ss}(L^-)/\!/(G'_0 \times G_a)$, and set $E^+_{l,m} := H_b^{ss}(L^+)/\!/(G'_0 \times G_a)$. Then, the open embeddings $H_b^{ss}(L^-) \subset H_b$ and $H_b^{ss}(L^+) \subset H_b$ define natural birational morphisms $E^-_{l,m} \longrightarrow E_{l,m}$ and $E^+_{l,m} \longrightarrow E_{l,m}$, and the $SL(2)$-equivariant flip*

$$\begin{array}{ccc} E^-_{l,m} & \dashrightarrow & E^+_{l,m} \\ & \searrow \quad \swarrow & \\ & E_{l,m} & \end{array} .$$

Remark 2.3.9.1. Let $E_{l,m} \hookrightarrow V \cong V(i_1 + j_1) \oplus \cdots \oplus V(i_u + j_u)$ be the equivariant closed embedding mentioned in Remark 2.3.3.3, and consider an action of $t \in \mathbb{C}^*$ on V defined by multiplication of $(t^{i_1 - j_1}, \ldots, t^{i_u - j_u})$. Then, since this $\mathbb{C}^*$-action commutes with the $SL(2)$-action, the affine variety $E_{l,m} \subset V$ remains stable under the $\mathbb{C}^*$-action. This enables us to consider $E_{l,m}$ as an $SL(2) \times \mathbb{C}^*$-variety. The same $\mathbb{C}^*$-action on $E_{l,m}$ can be defined in another way: an action of $\mathbb{C}^*$ on H_b defined by the matrices $\{\mathrm{diag}(1, s^{-1}, s^{-1}, s, s) : s \in \mathbb{C}^*\}$ commutes with the $SL(2) \times G'_0 \times G_a$-action, and therefore it descends to $E_{l,m}$ (see [BH08, Remarks 3.12 and 4.2]).

Remark 2.3.9.2 ([BH08, Remark 3.12]). Let $l < 1$, and let $E'_{l,m} := Bl^{\omega}_O(E_{l,m})$ be the weighted blow-up of $E_{l,m}$ with weight ω defined by the $\mathbb{C}^*$-action considered in Remark 2.3.9.1. The exceptional divisor D' of the weighted blow-up $E'_{l,m} \longrightarrow E_{l,m}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and we obtain surjective morphisms $\gamma^- : E'_{l,m} \longrightarrow E^-_{l,m}$ and $\gamma^+ : E'_{l,m} \longrightarrow E^+_{l,m}$ by contracting $\mathbb{P}^1 \times \mathbb{P}^1$ in different directions to $\mathbb{P}^1$, which fit into the following commutative diagram:

$$\begin{array}{ccc} & E'_{l,m} & \\ {}^{\gamma^-}\swarrow & & \searrow^{\gamma^+} \\ E^-_{l,m} & \dashrightarrow & E^+_{l,m} \\ & \searrow \quad \swarrow & \\ & E_{l,m} & \end{array}$$

The exceptional divisor $D' \cong \mathbb{P}^1 \times \mathbb{P}^1$ contains two $SL(2)\times\mathbb{C}^*$-orbits under the diagonal action of $SL(2)$ and under the trivial action of $\mathbb{C}^*$: the closed orbit $C = (SL(2)\times\mathbb{C}^*)\cdot([1:0],[1:0])$ and a 2-dimensional orbit $C' = (SL(2)\times\mathbb{C}^*)\cdot([1:0],[0:1])$.

Theorem 2.3.10 ([BH08, Proposition 3.13])**.** *Let $C^\pm$ be the image of D' under the morphism $\gamma^\pm$. Then the canonical divisor $K_{E^\pm_{l,m}}$ of $E^\pm_{l,m}$ has the following intersection number with $C^\pm$:*

$$K_{E^-_{l,m}} \cdot C^- = -\frac{(1+b)k}{aq^2}, \qquad K_{E^+_{l,m}} \cdot C^+ = \frac{(1+b)k}{ap^2}.$$

Theorem 2.3.11 ([BH08, §3])**.** *The weighted blow-up $E'_{l,m}$ contains a unique closed $SL(2)\times\mathbb{C}^*$-orbit C, which is isomorphic to $\mathbb{P}^1$. Moreover, along the closed orbit C, $E'_{l,m}$ is locally isomorphic to $\mathbb{C}\times\mathbb{C}^2/\mu_b$.*

Remark 2.3.11.1. By Theorem 2.3.3, $E_{l,m}$ is toric if and only if $b = 1$. Therefore, in view of Theorem 2.3.11, $E'_{l,m}$ is smooth if and only if $E_{l,m}$ is toric. Furthermore, if $E_{l,m}$ is toric, then the weight ω is trivial since we have $i_1 - j_1 = \cdots = i_u - j_u = m$ (with the notation of Remark 2.3.9.1) by Remark 2.3.3.4, in which case $E'_{l,m}$ is the usual reduced blow-up of the fixed point O in $E_{l,m}$.

2.3.12 Spherical geometry

Theorem 2.3.13 ([BH08, Proposition 4.1])**.** *An affine $SL(2)\times\mathbb{C}^*$-variety $E_{l,m}$ is spherical with respect to the Borel subgroup $\widetilde{B} := B\times\mathbb{C}^*$.*

Batyrev and Haddad ([BH08, §4]) compute the colored cones of $E_{l,m}$, $E^-_{l,m}$, $E^+_{l,m}$, and $E'_{l,m}$. The lattice Γ of rational $\widetilde{B}$-eigenfunctions on $\mathfrak{U}$ is given as follows (see Remark 2.3.5.1 for the definition of the variables Z and W):

$$\Gamma = \{Z^i W^j \in \mathbb{C}(\mathfrak{U})^* : m|(i-j)\}.$$

The varieties $E_{l,m}$, $E^-_{l,m}$, and $E^+_{l,m}$ contain exactly three $\widetilde{B}$-stable divisors $D = (H_b \cap \{Y_0 = 0\})/\!/G'$, $S^- = (H_b \cap \{X_4 = 0\})/\!/G'$, and $S^+ = (H_b \cap \{X_2 = 0\})/\!/G'$, where $G' = G'_0 \times G_a$, and $E'_{l,m}$ contains an $SL(2)\times\mathbb{C}^*$-stable divisor $D' \cong \mathbb{P}^1\times\mathbb{P}^1$, the exceptional divisor of the weighted blow-up $E'_{l,m} \longrightarrow E_{l,m}$. The divisors D, S^-, S^+, and D' define lattice vectors ρ_{v_D}, $\rho_{v_{S^-}}$, $\rho_{v_{S^+}}$, $\rho_{v_{D'}} \in \Gamma^\vee$ in the dual space $Q = \mathrm{Hom}(\Gamma, \mathbb{Q})$. We can take $\{\rho_{v_{S^-}}, \rho_{v_{S^+}}\}$ as a

$\mathbb{Q}$-basis of Q, and the set $\mathcal{V}$ of $SL(2)\times\mathbb{C}^*$-invariant valuations is given as $\mathcal{V} = \{x\rho_{v_{S^+}} + y\rho_{v_{S^-}} \in Q \ : \ x+y \leq 0\}$. Under the notation of §2.2, the colored cones of $E_{l,m}$, $E^-_{l,m}$, $E^+_{l,m}$, and $E'_{l,m}$ are described as follows:

$$
\begin{aligned}
\mathscr{C} &:= \mathscr{C}(E_{l,m}) = \mathbb{Q}_{\geq 0}\rho_{v_D} + \mathbb{Q}_{\geq 0}\rho_{v_{S^-}}, \qquad & \mathscr{F} &:= \mathscr{F}(E_{l,m}) = \{\rho_{v_{S^+}}, \rho_{v_{S^-}}\};\\
\mathscr{C}^- &:= \mathscr{C}(E^-_{l,m}) = \mathbb{Q}_{\geq 0}\rho_{v_D} + \mathbb{Q}_{\geq 0}\rho_{v_{S^+}}, \qquad & \mathscr{F}^- &:= \mathscr{F}(E^-_{l,m}) = \{\rho_{v_{S^+}}\};\\
\mathscr{C}^+ &:= \mathscr{C}(E^+_{l,m}) = \mathbb{Q}_{\geq 0}\rho_{v_D} + \mathbb{Q}_{\geq 0}\rho_{v_{S^-}}, \qquad & \mathscr{F}^+ &:= \mathscr{F}(E^+_{l,m}) = \{\rho_{v_{S^-}}\};\\
\mathscr{C}' &:= \mathscr{C}(E'_{l,m}) = \mathbb{Q}_{\geq 0}\rho_{v_D} + \mathbb{Q}_{\geq 0}\rho_{v_{D'}}, \qquad & \mathscr{F}' &:= \mathscr{F}(E'_{l,m}) = \emptyset.
\end{aligned}
$$

The weighted blow-up $E'_{l,m}$ is toroidal since $\mathscr{F}' = \emptyset$.

Chapter 3

Invariant Hilbert schemes and resolutions of singularities of affine normal quasihomogeneous $SL(2)$-varieties I: colored fan of the minimal resolution

3.1 Flat locus and the Hilbert function of a general fiber

Let $l \leq 1$, and let

$$\pi : H_{q-p} \longrightarrow H_{q-p}/\!/(G_0 \times G_m) \cong E_{l,m}$$

be the quotient morphism. In this section, we determine the flat locus of π and the Hilbert function $h := h_{H_{q-p}}$ of a general fiber of π. Let $x = (1,1,0,0,1) \in H_{q-p}$. Then, the $SL(2) \times \mathbb{C}^* \times G_0 \times G_m$-orbit of x coincides with the open subset $H_{q-p} \cap \{X_0 \neq 0\}$ of H_{q-p}, and the categorical quotient of $H_{q-p} \cap \{X_0 \neq 0\}$ by $G_0 \times G_m$ is isomorphic to the dense open orbit $\mathfrak{U}$ (see Remark 2.3.5.1). Namely, $\mathfrak{U}$ is the $SL(2) \times \mathbb{C}^*$-orbit of $\pi(x)$. We can verify that $\mathfrak{D}$ is the $SL(2) \times \mathbb{C}^*$-orbit of $\pi(x')$, where $x' = (0,1,0,1,0) \in H_{q-p}$, as follows. Note that $\pi^{-1}(\mathfrak{U}) = H_{q-p} \cap \{X_0 \neq 0\}$. If $l = 1$, then we get $x' \in \pi^{-1}(\mathfrak{D})$ since $\mathfrak{U}$ and $\mathfrak{D}$ are the only orbits of $E_{1,m}$ (see Remark 2.3.3.2). If $l < 1$, and if we assume that $\pi(x') \notin \mathfrak{D}$, then $\pi(x') = O$. But this is a contradiction since $X_1^{aq} X_3^{ap} \in \mathbb{C}[H_{q-p}]^{G_0 \times G_m} \cong \mathbb{C}[E_{l,m}]$, and since the X_1-coordinate and the X_3-coordinate of x' are both 1. Consequently, $\mathfrak{D}$ is the $SL(2) \times \mathbb{C}^*$-orbit of $\pi(x')$.

Proposition 3.1.1. *Let $l \leq 1$. With the above notation, we have the following.*

(i) *For any $g \in SL(2)$, the $G_0 \times G_m$-orbits of $g \cdot x$ and $g \cdot x'$ are closed and isomorphic to $G_0 \times G_m$.*

(ii) *For any $y \in \mathfrak{U} \cup \mathfrak{D}$, the fiber $\pi^{-1}(y)$ is isomorphic to $G_0 \times G_m$.*

(iii) *If $l = 1$, then π is flat. Otherwise, $E_{l,m} \setminus \{O\} = \mathfrak{U} \cup \mathfrak{D}$ is the flat locus of π.*

Proof. We have seen in Remark 2.3.3.2 that $\mathfrak{U} \cup \mathfrak{D}$ is smooth. Therefore, taking into account Remark 2.3.6.1 and Theorem 2.3.7, items (i) and (ii) follow from [ADHL15, Remark 1.6.4.2]. Also, it follows from [ADHL15, Proposition 6.1.3.9] that π is flat over the smooth locus $\mathfrak{U} \cup \mathfrak{D}$ of $E_{l,m}$. Therefore, π is flat if $l = 1$. Suppose that $l < 1$, and let $x'' = (0, 1, 1, 0, 0) \in H_{q-p}$. Then we see that $\pi(x'') = O$, since monomials $X_1^{d_1} X_2^{d_2}$ with $(d_1, d_2) \neq (0, 0)$ are never $G_0 \times G_m$-invariant. The stabilizer of x'' under the $SL(2)$-action is 1-dimensional, which implies that π is not flat at the origin O concerning item (ii). Q.E.D.

Remark 3.1.1.1. Consider the following ideals of $\mathbb{C}[X_0, X_1, X_2, X_3, X_4]$:

$$I_1 := (X_0^{q-p} - X_1 X_4,\ X_2,\ X_3,\ 1 - X_0^{mp} X_1^{m}), \quad J_1 := (X_0^{k},\ X_2,\ X_4,\ 1 - X_1^{aq} X_3^{ap}).$$

By a simple calculation, we see that the underlying topological spaces of the orbits $(G_0 \times G_m) \cdot x$ and $(G_0 \times G_m) \cdot x'$ coincide with the zero sets of I_1 and J_1, respectively. In the case where $l < 1$, we will see in Theorems 4.2.2 and 4.2.3 that the ideals of the scheme-theoretic fibers $\pi^{-1}(\pi(x))$ and $\pi^{-1}(\pi(x'))$ coincide with I_1 and J_1, respectively.

Corollary 3.1.2. *The Hilbert function h of a general fiber of the quotient morphism π coincides with that of the regular representation $\mathbb{C}[G_0 \times G_m]$:*

$$h : \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}_{\geq 0}, \quad (n, d) \mapsto h(n, d) = 1,$$

where we identify $\mathrm{Irr}(G_0 \times G_m)$ *with* $\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$*.*

Let us denote by $\mathscr{H}$ the invariant Hilbert scheme $\mathrm{Hilb}_h^{G_0 \times G_m}(H_{q-p})$ associated to the triple $(G_0 \times G_m, H_{q-p}, h)$, and consider the Hilbert–Chow morphism

$$\gamma : \mathscr{H} \longrightarrow H_{q-p} /\!/ (G_0 \times G_m) \cong E_{l,m}.$$

By Theorem 2.1.5 and Proposition 3.1.1, we see that γ is an isomorphism over $\mathfrak{U} \cup \mathfrak{D}$, and that the restriction of γ to the main component

$$\mathscr{H}^{main} := \overline{\gamma^{-1}(\mathfrak{U} \cup \mathfrak{D})} = \overline{\gamma^{-1}(\mathfrak{U})}$$

is projective and birational. As a consequence of Proposition 3.1.1, we have:

Corollary 3.1.3. *If $l = 1$, then the invariant Hilbert scheme $\mathscr{H}$ is isomorphic to $E_{1,m}$.*

Until the end of chapter 5, we always assume that $l < 1$ taking into account Corollary 3.1.3.

Remark 3.1.3.1. It will be important to have an explicit description of the orbit decomposition of the varieties $E_{l,m}$, $E'_{l,m}$, and the minimal resolution $\widetilde{E'_{l,m}}$ of $E'_{l,m}$. So here we recall that $E_{l,m}$ contains exactly three orbits, and that $E'_{l,m}$ contains exactly four orbits:

- $E_{l,m} = \mathfrak{U} \cup \mathfrak{D} \cup \{O\}$, where $\mathfrak{U}$ is the dense open orbit, $\mathfrak{D}$ is a 2-dimensional orbit, and O is the closed orbit (see Remark 2.3.3.2 (ii)).
- $E'_{l,m} = \mathfrak{U} \cup \mathfrak{D} \cup C' \cup C$, where $C' \cong SL(2)/T$ is a 2-dimensional orbit, and $C \cong SL(2)/B \cong \mathbb{P}^1$ is the closed orbit (see Remark 2.3.9.2).

The orbit decomposition of $\widetilde{E'_{l,m}}$ will be given in Remark 3.2.4.2.

3.2 Minimal resolution of the weighted blow-up

A resolution of singularities $f : \widetilde{X} \longrightarrow X$ is *minimal* if the canonical divisor $K_{\widetilde{X}}$ of $\widetilde{X}$ is f-nef, i.e., $K_{\widetilde{X}} \cdot S \geq 0$ for any curve $S \subset \widetilde{X}$ which is contracted to a point under f.

Recall that the weighted blow-up $E'_{l,m}$ is a simple toroidal spherical $SL(2)\times\mathbb{C}^*$-variety, and that it is locally isomorphic to $\mathbb{C}\times\mathbb{C}^2/\mu_b$ along the closed orbit $C \cong \mathbb{P}^1$ (Theorem 2.3.11). We denote by $\widetilde{E'_{l,m}}$ the minimal resolution of singularities of $E'_{l,m}$ obtained by the minimal resolution of the cyclic quotient singularities $\mathbb{C}^2/\mu_b$. As we have seen in Remark 2.2.9.1 that any equivariant resolution of singularities of a toroidal spherical variety is obtained by subdividing the cones of its colored fan, this applies in particular to the minimal resolution $\widetilde{E'_{l,m}}$ of $E'_{l,m}$ whose we will calculate the colored fan in §3.2.2.

3.2.1 Singularities of the weighted blow-up

We see that the lattice $\Gamma = \{Z^i W^j \in \mathbb{C}(\mathfrak{U})^* : m|(i-j)\}$ of rational $\widetilde{B}$-eigenfunctions on $\mathfrak{U}$ is generated by ZW and Z^m. Since $(t,s) \in T\times\mathbb{C}^* \subset \widetilde{B}$ acts on Z^iW^j via $(t,s)\cdot Z^iW^j =$

$t^{i+j}s^{i-j}Z^iW^j$, the natural homomorphism $f : \Gamma \longrightarrow \mathfrak{X}(\widetilde{B}) \cong \mathbb{Z}^2$ is given by $Z^iW^j \mapsto (i+j, i-j)$. Set $\mathbf{v}_1 := f(ZW) = (2,0)$, and set $\mathbf{v}_2 := f(Z^m) = (m,m)$. We denote the dual basis of $\{\mathbf{v}_1, \mathbf{v}_2\}$ by $\{\mathbf{u}_1, \mathbf{u}_2\}$. By virtue of [Pan91, Theorem 2] and [BH08, Proposition 2.8], we see that the lattice vectors $\rho_{v_D}, \rho_{v_{S^-}}, \rho_{v_{S^+}}, \rho_{v_{D'}} \in \Gamma^\vee$ can be written as follows:

$$\rho_{v_D} = -b\mathbf{u}_1 + ap\mathbf{u}_2, \quad \rho_{v_{S^-}} = \mathbf{u}_1, \quad \rho_{v_{S^+}} = \mathbf{u}_1 + m\mathbf{u}_2, \quad \rho_{v_{D'}} = \mathbf{u}_2.$$

Therefore, $E'_{l,m}$ has singularities of an affine toric surface defined by the following cone (see [BH08, Remark 3.12]):

$$\sigma := \mathbb{Q}_{\geq 0}\mathbf{u}_2 + \mathbb{Q}_{\geq 0}(-b\mathbf{u}_1 + ap\mathbf{u}_2).$$

We denote by X_σ the toric variety of the cone σ. Let α and β be the quotient and the remainder of mp divided by $q-p$, respectively, i.e., $mp = \alpha(q-p) + \beta$, and set

$$t := \frac{q-p-\beta}{k} = (\alpha+1)b - ap. \tag{3.1}$$

We consider the base change

$$\begin{pmatrix} \mathbf{u}'_1 \\ \mathbf{u}'_2 \end{pmatrix} := \begin{pmatrix} -1 & \alpha+1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} \tag{3.2}$$

to make σ into the normal form in the sense of [CLS11, Proposition 10.1.1]:

$$\sigma = \mathbb{Q}_{\geq 0}\mathbf{u}'_2 + \mathbb{Q}_{\geq 0}(b\mathbf{u}'_1 - t\mathbf{u}'_2).$$

It follows that X_σ is a cyclic quotient singularity of type $\frac{1}{b}(1,t)$. Therefore, by Theorem 2.3.3, X_σ is smooth if and only if $E_{l,m}$ is toric. If $E_{l,m}$ is non-toric, then X_σ has a minimal resolution described by the Hirzebruch–Jung continued fraction expansion of b/t (see [CLS11, §10.2], [Ful93, §2.6]).

3.2.2 Colored fan of the minimal resolution

To each $E_{l,m}$, we assign an integer $r = r(E_{l,m})$ as follows.

- If $E_{l,m}$ is toric, then we define $r = 0$.

- If $E_{l,m}$ is non-toric, then we define r to be the *length* of the Hirzebruch–Jung continued fraction expansion of b/t:

$$\frac{b}{t} = [[c_1, \ldots, c_r]] = c_1 - \cfrac{1}{c_2 - \cfrac{1}{\cdots - \cfrac{1}{c_r}}}.$$

Set $P_0 := 0$, set $Q_0 := -1$, set $P_1 := 1$, and set $Q_1 := 0$. For $2 \leq i \leq r+1$ (this only happens if $E_{l,m}$ is non-toric), we recursively define

$$P_i := c_{i-1}P_{i-1} - P_{i-2}, \quad Q_i := c_{i-1}Q_{i-1} - Q_{i-2}. \tag{3.3}$$

Theorem 3.2.3 ([CLS11, Proposition 10.2.2])**.** *The numbers P_i and Q_i $(0 \leq i \leq r+1)$ defined above satisfy the following properties.*

(i) $P_0 < P_1 < \cdots < P_{r+1}$, $Q_0 < Q_1 < \cdots < Q_{r+1}$*;*

(ii) $P_{i-1}Q_i - P_iQ_{i-1} = 1$ *for any* $1 \leq i \leq r+1$*;*

(iii) $\dfrac{b}{t} = \dfrac{P_{r+1}}{Q_{r+1}} < \dfrac{P_r}{Q_r} \cdots < \dfrac{P_2}{Q_2}$.

We define the vectors

$$\rho_i := -P_i\mathbf{u}_1 + \{(\alpha+1)P_i - Q_i\}\mathbf{u}_2 \quad (0 \leq i \leq r+1)$$

in Q and consider the cone spanned by ρ_i and ρ_{i+1}:

$$\mathscr{C}_i := \mathbb{Q}_{\geq 0}\rho_i + \mathbb{Q}_{\geq 0}\rho_{i+1} \quad (0 \leq i \leq r).$$

Let us denote by $\widetilde{E'_{l,m}}$ the toroidal spherical $SL(2) \times \mathbb{C}^*$-variety whose colored fan has $(\mathscr{C}_0, \emptyset)$, $\ldots$, $(\mathscr{C}_r, \emptyset)$ as its maximal colored cones. Then, $\widetilde{E'_{l,m}} = E'_{l,m}$ if $E_{l,m}$ is toric. If $E_{l,m}$ is non-toric, then $\widetilde{E'_{l,m}} \longrightarrow E'_{l,m}$ is the minimal resolution concerning [CLS11, Theorem 10.2.3] and the base change (3.2).

The main result of this thesis is the following:

Theorem 3.2.4. *The main component $\mathscr{H}^{main}$ is isomorphic to $\widetilde{E'_{l,m}}$.*

The proof of Theorem 3.2.4 will be given in §5.3.

Remark 3.2.4.1. Keep the above notation.

(i) We have $\rho_0 = \rho_{v_{D'}}$ and $\rho_{r+1} = \rho_{v_D}$ by definition.

(ii) Let $\widetilde{E_i}$ $(0 \leq i \leq r)$ be the simple spherical subvariety of $\widetilde{E'_{l,m}}$ whose colored cone is $(\mathscr{C}_i, \emptyset)$. Then, $\bigcup_{0 \leq i \leq r} \widetilde{E_i}$ gives an open covering of $\widetilde{E'_{l,m}}$.

Remark 3.2.4.2. In view of Theorem 2.2.5, we can read off information of $SL(2) \times \mathbb{C}^*$-orbits of $\widetilde{E'_{l,m}}$ from its colored fan. First, the colored cones $(0, \emptyset)$, $(\mathbb{Q}_{\geq 0}\rho_{r+1}, \emptyset)$, and $(\mathbb{Q}_{\geq 0}\rho_0, \emptyset)$ correspond to $\mathfrak{U}$, $\mathfrak{D}$, and C', respectively. Next, we denote the closed orbit that corresponds to $(\mathscr{C}_i, \emptyset)$ by Y_i for each $0 \leq i \leq r$. Notice that if $E_{l,m}$ is toric, i.e., if $r = 0$, then Y_0 is nothing but the closed orbit $C \cong \mathbb{P}^1$. In the case where $E_{l,m}$ is non-toric, we denote the orbit corresponding to $(\mathbb{Q}_{\geq 0}\rho_i, \emptyset)$ by O_i for each $1 \leq i \leq r$. Summarizing, the $SL(2) \times \mathbb{C}^*$-orbits of $\widetilde{E'_{l,m}}$ are described as follows.

(i) If $r = 0$, i.e., if $E_{l,m}$ is toric, then $\widetilde{E'_{l,m}} = E'_{l,m}$ contains exactly four orbits: $\mathfrak{U}$, $\mathfrak{D}$, C', and $Y_0 = C$ (see Remark 2.3.9.2, see also Remark 3.1.3.1).

(ii) If $r > 0$, i.e., if $E_{l,m}$ is non-toric, then $\widetilde{E'_{l,m}}$ contains $2r + 4$ orbits: $\mathfrak{U}$, $\mathfrak{D}$, C', Y_i $(0 \leq i \leq r)$, and O_i $(1 \leq i \leq r)$.

Before moving on to the next section, let us define some more notations. Set

$$e_i := (\alpha + 1 + m)P_i - Q_i, \quad l_i := (\alpha + 1)P_i - Q_i, \quad n_i := -pe_i + ql_i$$

for each $0 \leq i \leq r + 1$. Then we get the following lemma as a consequence of Theorem 3.2.3, which will be frequently used in the remaining sections.

Lemma 3.2.5. *Keep the above notation.*

(i) *We have* $n_i = k(tP_i - bQ_i)$ *for any* $0 \leq i \leq r + 1$.

(ii) *If* $E_{l,m}$ *is non-toric, then we have* $n_i = c_{i-1}n_{i-1} - n_{i-2}$ *for any* $2 \leq i \leq r + 1$.

(iii) *We have* $n_i > n_{i+1}$ *for any* $0 \leq i \leq r$.

(iv) *We have* $n_0 = q - p$, $n_r = k$, *and* $n_{r+1} = 0$.

Chapter 4

Invariant Hilbert schemes and resolutions of singularities of affine normal quasihomogeneous $SL(2)$-varieties II: calculation of ideals

4.1 Generators as a module over the invariant ring

Let $r \geq 0$, and let A be the polynomial ring $\mathbb{C}[X_0, X_1, X_2, X_3, X_4]$. We consider the following two families of ideals of A parametrized by $s \in \mathbb{C}$:

$$I_s := (X_0^{q-p} - X_1 X_4,\ X_2,\ X_3,\ s - X_0^{mp} X_1^m), \quad J_s := (X_0^k,\ X_2,\ X_4,\ s - X_1^{aq} X_3^{ap}).$$

Note that the ideals I_1 and J_1 have already appeared in Remark 3.1.1.1. We will see that the closed subschemes of H_{q-p} associated with the ideals I_s and J_s define closed points of $\mathcal{H}^{main}$, and that the $SL(2) \times \mathbb{C}^*$-orbits of the closed points $[I_1]$, $[I_0]$, $[J_1]$, and $[J_0]$ coincide with $\mathfrak{U}$, C', $\mathfrak{D}$, and Y_r under the isomorphism $\mathcal{H}^{main} \cong \widetilde{E'_{l,m}}$, respectively (see Corollary 4.2.5 and the proof of Theorem 3.2.4, which will be given in §5.3). If $E_{l,m}$ is toric, i.e., if $r = 0$, then $\widetilde{E'_{l,m}} = E'_{l,m} = \mathfrak{U} \cup \mathfrak{D} \cup C' \cup Y_0$ by Remark 3.2.4.2. In the case where $r > 0$, i.e., where $E_{l,m}$ is non-toric, we consider r additional families of ideals of A parametrized by $s \in \mathbb{C}$. Let K be the ideal of A generated by monomials of the form

$$X_0^{pu_1 - qu_2} X_1^{u_1} X_3^{u_2}, \quad (u_1, u_2) \in M^+_{l,m} \setminus \{(0,0)\},$$

which are $G_0 \times G_m$-invariant (see Remark 4.1.2.2), and define

$$L_s^i := (X_0^{n_{i-1}},\ X_2,\ X_4,\ sX_0^{n_i} - X_1^{e_i} X_3^{l_i}) + K \subset A$$

for each $1 \leq i \leq r$. We will see that the closed subschemes of H_{q-p} associated with the ideals L_s^i define closed points of $\mathcal{H}^{main}$, and that the $SL(2) \times \mathbb{C}^*$-orbits of $[L_1^i]$ and $[L_0^i]$ coincide with O_i and Y_{i-1} under the isomorphism $\mathcal{H}^{main} \cong \widetilde{E'_{l,m}}$, respectively (refer again to the proof of Theorem 3.2.4 in §5.3). This section is a preparation for the next one, where we calculate the Hilbert functions of the ideals I_s, J_s, and L_s^i (Theorems 4.2.2, 4.2.3, 4.2.7, and 4.2.8).

Remark 4.1.0.1. If $s \in \mathbb{C}^*$, then I_s, J_s, and L_s^i are $SL(2) \times \mathbb{C}^*$-translates of I_1, J_1, and L_1^i, respectively.

Let S be the coordinate ring of H_{q-p}:

$$S := \mathbb{C}[H_{q-p}] \cong A/(X_0^{q-p} - X_1 X_4 + X_2 X_3).$$

Remark 4.1.0.2. For a $G_0 \times G_m$-module V, we denote $\mathrm{Hom}^{G_0 \times G_m}(M_{(n,d)}, V)$ by $V_{(n,d)}$, where $M_{(n,d)}$ stands for the irreducible representation of weight $(n, d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

For any weight $(n, d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, there is a finite-dimensional $SL(2) \times \mathbb{C}^*$-module $F_{n,d}$ that generates the weight space $S_{(n,d)}$ over the invariant ring $S^{G_0 \times G_m}$ (see §2.1.6). In order to calculate the Hilbert function of A/I_s, A/J_s, and A/L_s^i, we need to find an appropriate $F_{n,d}$ for some weights. For each $n \geq 0$, consider the following irreducible $SL(2)$-representations of highest weight n:

$$A(n) := \mathrm{Sym}^n \langle X_1, X_2 \rangle \cong V(n), \quad B(n) := \mathrm{Sym}^n \langle X_3, X_4 \rangle \cong V(n).$$

Also, define $C(n) := \langle X_0^n \rangle \cong V(0)$ for each $n \in \mathbb{Z}$.

Lemma 4.1.1. *With the above notation, we have the following.*

(i) $S_{(-p,-1)} = S^{G_0 \times G_m} X_1 + S^{G_0 \times G_m} X_2$.

(ii) $S_{(q,1)} = S^{G_0 \times G_m} X_3 + S^{G_0 \times G_m} X_4$.

(iii) *We can take* $F_{-p,-1} = A(1)$ *and* $F_{q,1} = B(1)$.

Proof. Since $X_1, X_2 \in S_{(-p,-1)}$, it is clear that $S_{(-p,-1)} \supset S^{G_0\times G_m}X_1 + S^{G_0\times G_m}X_2$. To see the other inclusion, take an arbitrary $f = X_0^{d_0}X_1^{d_1}X_2^{d_2}X_3^{d_3}X_4^{d_4} \in A_{(-p,-1)}$. If either $d_1 > 0$ or $d_2 > 0$ holds, then we clearly have $f \in A^{G_0\times G_m}X_1 + A^{G_0\times G_m}X_2$. Otherwise, f is of the form $f = X_0^{d_0}X_3^{d_3}X_4^{d_4}$. But this contradicts to $f \in A_{(-p,-1)}$, since the G_0-weights of X_0, X_3, and X_4 are all positive. This shows (i). Item (ii) follows in a similar way. Item (iii) is a consequence of items (i) and (ii). Q.E.D.

Remark 4.1.1.1. Let $[I]$ be a closed point of $\mathscr{H}$. By Lemma 4.1.1 (iii), we see that $s_1X_1 + s_2X_2 \in I$ and $s_3X_3 + s_4X_4 \in I$ hold for some $(s_1, s_2) \neq 0$ and $(s_3, s_4) \neq 0$, respectively, concerning $h(-p,-1) = h(q,1) = 1$.

Looking only at the weights $(-p,-1)$ and $(q,1)$ is not enough to calculate the Hilbert function of the ideals I_s, J_s, and L_s^i, and we need to find a suitable $F_{n,d}$ for $(n,d) = (n_i,0)$ $(0 \leq i \leq r)$ as well. The goal of this section is to prove the following

Proposition 4.1.2. *With the above notation, we have the following.*

(i) *We can take* $F_{n_0,0} = A(e_0)\otimes B(l_0)$.

(ii) *Suppose that* $E_{l,m}$ *is non-toric, i.e., that* $r \geq 1$. *Then, we can take* $F_{n_i,0} = A(e_i)\otimes B(l_i) \oplus C(n_i)$ *for any* $1 \leq i \leq r$.

The proof of Proposition 4.1.2 requires intricate combinatorial arguments. In order to simplify the discussion, we introduce new notation and prepare a series of lemmas.

Let $j \in \{3,4\}$, and set

$$R := \mathbb{C}[X_0, X_1, X_j] \subset A. \tag{4.1}$$

For each $c, n \in \mathbb{Z}$, we consider the following vector subspaces of R:

$$R^c := \langle X_0^{d_0}X_1^{d_1}X_j^{d_j} \in R \ : \ d_1 - d_j = c\rangle, \quad R_n := \langle X_0^{d_0}X_1^{d_1}X_j^{d_j} \in R \ : \ d_0 - pd_1 + qd_j = n\rangle.$$

Then we have $R = \bigoplus_{c\in\mathbb{Z}} R^c = \bigoplus_{n\in\mathbb{Z}} R_n$. Define $R_n^c := R^c \cap R_n$. Then, the weight space $R_{(n,d)}$ can be described as $R_{(n,d)} = \bigoplus_{c\equiv d\ (mod\ m)} R_n^c$.

Remark 4.1.2.1. In order to show Proposition 4.1.2, it suffices to determine a subspace of $R_{(n_i,0)}$ that generates $R_{(n_i,0)}$ over the invariant ring $R^{G_0\times G_m}$ concerning that X_1 and X_2 (resp. X_3 and X_4) have the same $SL(2)\times\mathbb{C}^*\times G_0\times G_m$-weight.

Remark 4.1.2.2. The weight space $R_{(0,0)}$ is the invariant ring $R^{G_0 \times G_m}$. By the proof of [BH08, Theorem 1.6], we see that $R^{G_0 \times G_m} = \mathbb{C}[X_0^{pu_1 - qu_2} X_1^{u_1} X_j^{u_2} : (u_1, u_2) \in M^+_{l,m}]$.

Lemma 4.1.3. *If $R_n^c \neq 0$, then we have $c \geq -n/q$. In particular, the minimum $c_{(n,d)} := \min\{c \in \mathbb{Z} : c \equiv d \ (mod\ m),\ R_n^c \neq 0\}$ exists for any $(n, d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.*

Proof. Take an arbitrary $X_0^{d_0} X_1^{d_1} X_j^{d_j} \in R_n^c \setminus \{0\}$. Then, we have $n = d_0 - pd_1 + qd_j = d_0 + (q - p)d_1 - qc$. Since $d_0,\ d_1 \geq 0$, it follows that $c \geq -n/q$. Q.E.D.

Example 4.1.4. If $0 \leq n \leq q - p$, then we have $c_{(n,0)} = 0$. This can be verified as follows. Suppose that $R_n^c \neq 0$ holds for some $c < 0$, and take $X_0^{d_0} X_1^{d_1} X_j^{d_j} \in R_n^c \setminus \{0\}$. Then, $n = d_0 + (q - p)d_1 - qc \geq q > q - p$. Moreover, by a direct calculation, we see that $R_n^0 = \langle X_0^n \rangle$ if $0 \leq n < q - p$, and that $R_n^0 = \langle X_0^{q-p},\ X_1 X_j \rangle$ if $n = q - p$.

Consider a $\mathbb{Z}$-linear map $\mu : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^3$ defined by

$$(d_0, d_1, d_j) \mapsto \mu(d_0, d_1, d_j) := (d_0 - pd_1 + qd_j,\ d_1 - d_j,\ pd_1 - qd_j),$$

which is injective. Let us denote by Λ the image of $\mu|_{\mathbb{Z}^3_{\geq 0}}$, and define

$$R_\lambda := \langle X_0^{d_0} X_1^{d_1} X_j^{d_j} \in R : \mu(d_0, d_1, d_j) = \lambda \rangle$$

for each $\lambda \in \Lambda$. Then we have $R = \bigoplus_{\lambda \in \Lambda} R_\lambda$.

Lemma 4.1.5. *Let $\lambda = (n, c, \omega) \in \Lambda$. Then, R_λ is a 1-dimensional vector space spanned by $f_\lambda := X_0^{n+\omega} X_1^{\frac{qc-\omega}{q-p}} X_j^{\frac{pc-\omega}{q-p}}$. In particular, $n + \omega$, $\frac{qc-\omega}{q-p}$, and $\frac{pc-\omega}{q-p}$ are all non-negative integers.*

Proof. Let $(d_0, d_1, d_j) = \mu^{-1}(\lambda)$. By the definition of μ, one has $n = d_0 - pd_1 + qd_j$, $c = d_1 - d_j$, and $\omega = pd_1 - qd_j$. Therefore, a direct calculation gives $d_0 = n + \omega$, $d_1 = \frac{qc-\omega}{q-p}$, and $d_j = \frac{pc-\omega}{q-p}$. Q.E.D.

Remark 4.1.5.1. Let $\lambda = (n, c, \omega) \in \mathbb{Z}^3$. Lemma 4.1.5 implies that we have $\lambda \in \Lambda$ if and only if all of $n + \omega$, $\frac{qc-\omega}{q-p}$, and $\frac{pc-\omega}{q-p}$ are non-negative integers.

Lemma 4.1.6. *For any λ, $\lambda' \in \Lambda$, we have $f_\lambda f_{\lambda'} = f_{\lambda+\lambda'}$.*

Proof. This follows from the definition of f_λ. Q.E.D.

Remark 4.1.6.1. The polynomial ring $R = \mathbb{C}[X_0, X_1, X_j]$ has a natural $\mathbb{Z}\times\mathbb{Z}/m\mathbb{Z}$-grading defined by the $G_0\times G_m$-action. Although, each graded component $R_{(n,d)}$ with respect to this grading is infinite-dimensional. On the other hand, Lemma 4.1.6 implies that R admits another grading, namely the Λ-grading, such that each graded component R_λ is 1-dimensional. We will see below that this makes it easier to analyze the structure of $R_{(n,d)}$.

Consider the projection $\tilde{\mu} : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2$, $(n, c, \omega) \mapsto (n, c)$ to the first and the second factor, and set $\mu' := \tilde{\mu} \circ \mu$. We denote by Λ' the image of $\mu'|_{\mathbb{Z}^3_{\geq 0}}$. Then, we have $R = \bigoplus_{(n,c)\in\Lambda'} R^c_n$ and $R^c_n = \bigoplus_{\lambda\in\tilde{\mu}^{-1}(n,c)\cap\Lambda} R_\lambda$.

Lemma 4.1.7. *Let $(n, c) \in \Lambda'$. Then, for any $\lambda = (n, c, \omega)$, $\lambda' = (n, c, \omega') \in \tilde{\mu}^{-1}(n, c) \cap \Lambda$, we have $\omega - \omega' \in (q - p)\mathbb{Z}$.*

Proof. Let $\mu^{-1}(\lambda) = (d_0, d_1, d_j)$, and let $\mu^{-1}(\lambda') = (d'_0, d'_1, d'_j)$. By Lemma 4.1.5, we have $d_1 = \frac{qc-\omega}{q-p}$ and $d'_1 = \frac{qc-\omega'}{q-p}$. Therefore, $\omega - \omega' = (q - p)(d'_1 - d_1) \in (q - p)\mathbb{Z}$. Q.E.D.

Let $\lambda = (n, c, \omega)$, and let $(d_0, d_1, d_j) = \mu^{-1}(\lambda)$. By Lemma 4.1.5, we have $\omega = -(q - p)d_1 + qc \geq qc$ and $n = d_0 + (q - p)d_1 - qc$. Combining these, we get $qc \leq \omega \leq n + qc$. Therefore, we see that the maximum $\omega^{\max}_{(n,c)} := \max\left\{\omega \in \mathbb{Z} : (n, c, \omega) \in \tilde{\mu}^{-1}(n, c) \cap \Lambda\right\}$ and the minimum $\omega^{\min}_{(n,c)} := \min\left\{\omega \in \mathbb{Z} : (n, c, \omega) \in \tilde{\mu}^{-1}(n, c) \cap \Lambda\right\}$ exist for any $(n, c) \in \Lambda'$. In particular, the vector space R^c_n is finite-dimensional: we have $R^c_n = \bigoplus_{\omega^{\min}_{(n,c)}\leq\omega\leq\omega^{\max}_{(n,c)}} R_{(n,c,\omega)}$.

Lemma 4.1.8. *Let $(n, c) \in \Lambda'$. If $c < 0$, then $\omega^{\max}_{(n,c)} = qc$. Otherwise, $\omega^{\max}_{(n,c)} = pc$.*

Proof. Let $\mu^{-1}\left(n, c, \omega^{\max}_{(n,c)}\right) = (d_0, d_1, d_j)$. Suppose that d_1, $d_j > 0$, and set $v = (d_0 + q - p, d_1 - 1, d_j - 1)$. Then, we have $v \in \mathbb{Z}^3_{\geq 0}$ and $\mu(v) = \left(n, c, \omega^{\max}_{(n,c)} + q - p\right)$. This implies that $\mu(v) \in \tilde{\mu}^{-1}(n, c) \cap \Lambda$, which contradicts to the maximality of $\omega^{\max}_{(n,c)}$. Thus, either $d_1 = 0$ or $d_j = 0$ holds. If $c < 0$, then we see that $d_1 = 0$, and therefore $\omega^{\max}_{(n,c)} = qc$. Otherwise, we have $d_j = 0$, and hence $\omega^{\max}_{(n,c)} = pc$. Q.E.D.

Lemma 4.1.9. *Let $(n, c, \omega) \in \Lambda$. Then, we have $n + \omega < q - p$ if and only if $\omega = \omega^{\min}_{(n,c)}$.*

Proof. First, suppose that $n+\omega \geq q-p$, and set $v = \left(n + \omega', \frac{qc-\omega}{q-p} + 1, \frac{pc-\omega}{q-p} + 1\right)$, where $\omega' = \omega - (q - p)$. Taking into account Lemma 4.1.5, we see that $v \in \mathbb{Z}^3_{\geq 0}$. Since we get $\mu(v) = (n, c, \omega')$

by a direct calculation, it yields that $(n,c,\omega') \in \tilde{\mu}^{-1}(n,c) \cap \Lambda$. Therefore, one has $\omega > \omega^{\min}_{(n,c)}$. Conversely, suppose that $\omega > \omega^{\min}_{(n,c)}$. Then $\omega - \omega^{\min}_{(n,c)} \geq q-p$ holds by Lemma 4.1.7. Since we have $n+\omega^{\min}_{(n,c)} \geq 0$ by Lemma 4.1.5, it follows that $n+\omega \geq n+\omega^{\min}_{(n,c)}+q-p \geq q-p$. Q.E.D.

Example 4.1.10. Let $0 \leq i \leq r+1$, and set $\lambda = (n_i, mP_i, -n_i) \in \mathbb{Z}^3$. We claim that $\lambda \in \Lambda$. By the equation (3.1) (see §3.2.1) and Lemma 3.2.5 (i), we get the following:

$$pmP_i + n_i = \{\alpha(q-p)+\beta\}P_i + k(tP_i - Q_i) = \{(\alpha+1)P_i - Q_i\}(q-p) = l_i(q-p);$$
$$qmP_i + n_i = (q-p)mP_i + (pmP_i + n_i) = e_i(q-p).$$

By these we obtain $\frac{pmP_i+n_i}{q-p} = l_i$ and $\frac{qmP_i+n_i}{q-p} = e_i$. Since $l_i,\ e_i \geq 0$, we deduce that $\lambda \in \Lambda$ by Remark 4.1.5.1. Note that $f_\lambda = X_1^{e_i} X_j^{l_i}$. Also, one has $pmP_i = \omega^{\max}_{(n_i,mP_i)}$ and $\omega^{\min}_{(n_i,mP_i)} = -n_i$ by Lemmas 4.1.8 and 4.1.9, respectively. Therefore, $\lambda = \left(n_i, mP_i, \omega^{\min}_{(n_i,mP_i)}\right)$.

Lemma 4.1.11. *Let $\lambda = (n,c,\omega) \in \Lambda$. If $\omega > \omega^{\min}_{(n,c)}$, then f_λ is contained in the ideal $(X_0^{n_i}) \subset R$ for any $0 \leq i \leq r$.*

Proof. By Lemmas 3.2.5 and 4.1.9, one has $n+\omega \geq q-p \geq n_i$. Therefore, we see that $f_\lambda \in (X_0^{n_i})$ holds concerning Lemma 4.1.5. Q.E.D.

Definition 4.1.12. For each $(n,d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, we define:

(i) $\Lambda_{(n,d)} := \{(n,c,\omega) \in \Lambda : c \equiv d \ (mod\ m)\}$;

(ii) $\lambda_{(n,d)} := \left(n, c_{(n,d)}, \omega^{\min}_{(n,c_{(n,d)})}\right) \in \Lambda_{(n,d)}$.

Remark 4.1.12.1. We obtain the following three different ways of expressing the weight space $R_{(n,d)}$:

$$R_{(n,d)} = \bigoplus_{\substack{c \equiv d \ (mod\ m) \\ c \geq c_{(n,d)}}} R_n^c = \bigoplus_{\substack{c \equiv d \ (mod\ m) \\ c \geq c_{(n,d)}}} \left(\bigoplus_{\lambda \in \tilde{\mu}^{-1}(n,c) \cap \Lambda} R_\lambda \right) = \bigoplus_{\lambda \in \Lambda_{(n,d)}} R_\lambda.$$

Example 4.1.13. Let $l = p/q = 1/3$, and let $m = 2$. By Remark 2.3.3.4, the semigroup $M^+_{\frac{1}{3},2}$ is minimally generated by $(2,0)$ and $(3,1)$. Therefore, in view of Remark 4.1.2.2, we have

$R_{(0,0)} = R^{G_0 \times G_m} = \mathbb{C}[X_0^2 X_1^2, X_1^3 X_j]$. We can also calculate the following:

$R_0^0 = \mathbb{C}$;

$R_0^2 = R_{(0,2,0)} \oplus R_{(0,2,2)},\ f_{(0,2,0)} = X_1^3 X_j,\ f_{(0,2,2)} = X_0^2 X_1^2$;

$R_1^0 = R_{(1,0,0)},\ f_{(1,0,0)} = X_0$;

$R_1^2 = R_{(1,2,0)} \oplus R_{(1,2,2)},\ f_{(1,2,0)} = X_0 X_1^3 X_j,\ f_{(1,2,2)} = X_0^3 X_1^2$;

$R_2^0 = R_{(2,0,-2)} \oplus R_{(2,0,0)},\ f_{(2,0,-2)} = X_1 X_j,\ f_{(2,0,0)} = X_0^2$;

$R_2^2 = R_{(2,2,-2)} \oplus R_{(2,2,0)} \oplus R_{(2,2,2)},\ f_{(2,2,-2)} = X_1^4 X_j^2,\ f_{(2,2,0)} = X_0^2 X_1^3 X_j,\ f_{(2,2,2)} = X_0^4 X_1^2$.

We see that $\lambda_{(0,0)} = (0,0,0)$, $\lambda_{(1,0)} = (1,0,0)$, and $\lambda_{(2,0)} = (2,0,-2)$.

Lemma 4.1.14. *Let $\lambda = (n,c,\omega)$, $\lambda' = (n,c',\omega') \in \Lambda_{(n,d)}$. Then the following hold.*

(i) *If $c = c'$, then we have $f_\lambda - f_{\lambda'} \in (X_0^{q-p} - X_1 X_j)$.*

(ii) *If $c > c_{(n,d)}$, then we have $f_\lambda \in (X_0^{q-p} - X_1 X_j,\ X_0^{mp} X_1^m)$.*

(iii) *We have $f_\lambda - f_{\lambda'} \in (X_0^{q-p} - X_1 X_j,\ 1 - X_0^{mp} X_1^m)$.*

Proof. By definition, we have $f_\lambda = X_0^{n+\omega} X_1^{\frac{qc-\omega}{q-p}} X_j^{\frac{pc-\omega}{q-p}}$ and $f_{\lambda'} = X_0^{n+\omega'} X_1^{\frac{qc'-\omega'}{q-p}} X_j^{\frac{pc'-\omega'}{q-p}}$. Also, by the definition of $\Lambda_{(n,d)}$, we can write c and c' as $c = c_{(n,d)} + mx$ and $c' = c_{(n,d)} + mx'$ with some $x,\ x' \in \mathbb{Z}_{\geq 0}$.

(i) We may assume that $\omega \geq \omega'$. Then, by Lemma 4.1.7, $\omega - \omega' = y(q-p)$ holds for some $y \geq 0$. Therefore, one obtains $f_\lambda - f_{\lambda'} = X_0^{n+\omega'} X_1^{\frac{qc-\omega}{q-p}} X_j^{\frac{pc-\omega}{q-p}} \{(X_0^{q-p})^y - (X_1 X_j)^y\} \in (X_0^{q-p} - X_1 X_j)$.

(ii) We first remark that $f_{(0,m,mp)} = X_0^{mp} X_1^m \in R^{G_0 \times G_m}$. Hence, we have $f_{(0,xm,xmp)} = (X_0^{mp} X_1^m)^x$ by Lemma 4.1.6. By setting

$$\lambda'' = \lambda_{(n,d)} + (0, xm, xmp) = \left(n, c, \omega_{(n,c_{(n,d)})} + xmp\right),$$

we get $f_\lambda - f_{\lambda''} \in (X_0^{q-p} - X_1 X_j)$ taking into account (i). Since we have $f_{\lambda''} = f_{\lambda_{(n,d)}} (X_0^{mp} X_1^m)^x$ again by Lemma 4.1.6, it follows that $f_\lambda \in (X_0^{q-p} - X_1 X_j,\ X_0^{mp} X_1^m)$.

(iii) By (i), we may assume that $c > c'$. Let λ'' be as in the proof of (ii), and set $\lambda''' = \lambda_{(n,d)} + (0, x'm, x'mp) = \left(n, c', \omega_{(n,c_{(n,d)})} + x'mp\right)$. Then we have

$$f_{\lambda''} - f_{\lambda'''} = f_{\lambda_{(n,d)}} (X_0^{mp} X_1^m)^{x'} \{(X_0^{mp} X_1^m)^{x-x'} - 1\} \in (1 - X_0^{mp} X_1^m).$$

Therefore we get $f_\lambda - f_{\lambda'} = (f_\lambda - f_{\lambda''}) + (f_{\lambda''} - f_{\lambda'''}) + (f_{\lambda'''} - f_{\lambda'}) \in (X_0^{q-p} - X_1X_j,\ 1 - X_0^{mp}X_1^m)$, since we have $f_\lambda - f_{\lambda''},\ f_{\lambda'''} - f_{\lambda'} \in (X_0^{q-p} - X_1X_j)$ by (i). Q.E.D.

Lemma 4.1.15. *Let* $\lambda = \left(n, c, \omega_{(n,c)}^{\min}\right)$, $\lambda' = \left(n, c', \omega_{(n,c')}^{\min}\right) \in \Lambda_{(n,0)}$, *where* $0 \le n < q - p$. *If* $n + \omega_{(n,c)}^{\min} < k$ *and* $n + \omega_{(n,c')}^{\min} < k$ *hold, then the following properties are true.*

(i) *We have* $\omega_{(n,c)}^{\min} = \omega_{(n,c')}^{\min}$ *and* $c - c' \in mb\mathbb{Z}$.

(ii) *If* $c' > c$, *then we have* $f_{\lambda'} \in (f_\lambda X_1^{aq} X_j^{ap})$.

(iii) *We have* $f_{\lambda'} - f_\lambda \in (1 - X_1^{aq} X_j^{ap})$.

Proof. We may assume that $c' > c$. By Example 4.1.4 and the definition of $\Lambda_{(n,0)}$, we can write c' as $c' = c + mx$ with some $x > 0$. Recall that $f_{(0,m,mp)} = X_0^{mp} X_1^m$. Set $\lambda'' = \lambda + x(0, m, mp) = \left(n, c', \omega_{(n,c)}^{\min} + xmp\right)$. Then, $\lambda'' \in \tilde{\mu}^{-1}(n, c') \cap \Lambda$. Therefore, $\omega_{(n,c)}^{\min} + xmp - \omega_{(n,c')}^{\min} \in (q-p)\mathbb{Z}_{\ge 0}$ holds by Lemma 4.1.7. Taking the relations $mp = akp$ and $q - p = bk$ into account (see (2.1) in §2.3.1), we see that $\left(n + \omega_{(n,c)}^{\min}\right) - \left(n + \omega_{(n,c')}^{\min}\right) \in k\mathbb{Z}$. On the other hand, we have $0 \le n + \omega_{(n,c)}^{\min},\ n + \omega_{(n,c')}^{\min} < k$ by Lemma 4.1.5. Therefore, $\omega_{(n,c)}^{\min} = \omega_{(n,c')}^{\min}$. It also follows from Lemma 4.1.5 that $qc - \omega_{(n,c)}^{\min},\ qc' - \omega_{(n,c')}^{\min} \in (q-p)\mathbb{Z}_{\ge 0}$. Thus, $0 = qmx - y(q-p) = k(aqx - by)$ holds for some $y > 0$. Since $g.c.d.(aq, b) = 1$, we have $x = x'b$ with some $x' > 0$. To see items (ii) and (iii), set $\lambda''' = x'(0, aq - ap, 0) = (0, xm, 0)$. Then we have $f_{\lambda'''} = (X_1^{aq} X_j^{ap})^{x'}$ and $\lambda + \lambda''' = \lambda'$. Hence we get $f_{\lambda'} = f_\lambda (X_1^{aq} X_j^{ap})^{x'}$ by Lemma 4.1.6. Q.E.D.

Lemma 4.1.16. *Let* (n, c), $(n', c') \in \Lambda'$. *Then the following properties are true.*

(i) *If* $n = 0$, *then* $0 \le \omega_{(0,c)}^{\min} < q - p$.

(ii) *We have* $\omega_{(n+n',c+c')}^{\min} = \omega_{(n,c)}^{\min} + \omega_{(n',c')}^{\min}$ *if and only if* $\omega_{(n,c)}^{\min} + \omega_{(n',c')}^{\min} + n + n' < q - p$.

Proof. Let $\mu^{-1}\left(n, c, \omega_{(n,c)}^{\min}\right) = (d_0, d_1, d_j)$. If $n = 0$, then we have $0 \le d_0 = \omega_{(0,c)}^{\min} < q - p$ by Lemmas 4.1.5 and 4.1.9. Item (ii) follows from the fact that

$$\left(n + n', c + c', \omega_{(n,c)}^{\min} + \omega_{(n',c')}^{\min}\right) \in \Lambda$$

and Lemma 4.1.9. Q.E.D.

Definition 4.1.17. Let m_1 and m_2 be any positive integers. We denote by $\mathfrak{Q}[m_1, m_2]$ (resp. $\mathfrak{R}[m_1, m_2]$) the quotient (resp. the remainder) of m_1 divided by m_2.

Lemma 4.1.18. *Let $(n, c) \in \Lambda'$. Suppose that $n \geq 0$ and that $c > 0$. Then, we have $\omega^{\min}_{(n,c)} \geq 0$ if and only if $\mathfrak{R}[pc, q-p] + n < q - p$.*

Proof. By Lemmas 4.1.7 and 4.1.8, $pc - \omega^{\min}_{(n,c)} = \omega^{\max}_{(n,c)} - \omega^{\min}_{(n,c)} = x(q-p)$ holds for some $x \geq 0$. Also, we have $n + \omega^{\min}_{(n,c)} < q - p$ by Lemma 4.1.9, and hence $\omega^{\min}_{(n,c)} < q - p$ by $n \geq 0$. Therefore, $\mathfrak{R}[pc, q-p] = \omega^{\min}_{(n,c)}$ if $\omega^{\min}_{(n,c)} \geq 0$. If $\omega^{\min}_{(n,c)} < 0$, then $\mathfrak{R}[pc, q-p] = x'(q-p) + \omega^{\min}_{(n,c)}$ holds for some $x' > 0$. Thus we get $\mathfrak{R}[pc, q-p] + n = x'(q-p) + \omega^{\min}_{(n,c)} + n \geq q - p$, since $\omega^{\min}_{(n,c)} + n \geq 0$ by Lemma 4.1.5. Q.E.D.

The rest of this section is devoted to the proof of the following two propositions.

Proposition 4.1.19. *The weight space $R_{(n_0,0)}$ is generated by $R^0_{n_0}$ as a module over the invariant ring $R^{G_0 \times G_m}$.*

Proposition 4.1.20. *Suppose that $E_{l,m}$ is non-toric, i.e., that $r \geq 1$. Then, for any $1 \leq i \leq r$, the weight space $R_{(n_i,0)}$ is generated by $R_{\left(n_i, 0, \omega^{\min}_{(n_i,0)}\right)}$ and $R_{\left(n_i, mP_i, \omega^{\min}_{(n_i, mP_i)}\right)}$ as a module over the invariant ring $R^{G_0 \times G_m}$.*

Recall that $n_0 = q - p$, and that $e_0 = l_0 = 1$. We have seen in Example 4.1.4 that $R^0_{n_0} = \langle X_0^{n_0} \rangle \oplus \langle X_1^{e_0} X_j^{l_0} \rangle$. Since the $SL(2)$-submodule of $A = \mathbb{C}[X_0, X_1, X_2, X_3, X_4]$ generated by $X_0^{n_0}$ and $X_1^{e_0} X_j^{l_0}$ is $C(n_0) \oplus A(e_0) \otimes B(l_0)$, we see that Proposition 4.1.2 (i) can be obtained as a consequence of Proposition 4.1.19, concerning that the defining ideal of H_{q-p} is $(X_0^{n_0} - X_1 X_4 + X_2 X_3)$. Similarly, since we have $R_{\left(n_i, 0, \omega^{\min}_{(n_i,0)}\right)} = \langle X_0^{n_i} \rangle$ and $R_{\left(n_i, mP_i, \omega^{\min}_{(n_i, mP_i)}\right)} = \langle X_1^{e_i} X_j^{l_i} \rangle$ for any $1 \leq i \leq r$ by Examples 4.1.4 and 4.1.10, we see that Proposition 4.1.2 (ii) follows from Proposition 4.1.20. See also Remark 4.1.2.1.

Proof of Proposition 4.1.19. Recall that $R_{(n_0,0)}$ decomposes as $R_{(n_0,0)} = \bigoplus_{\lambda \in \Lambda_{(n_0,0)}} R_\lambda$. Let $\lambda = (n_0, c, \omega) \in \Lambda_{(n_0,0)}$, and write $f_\lambda = X_0^{d_0} X_1^{d_1} X_j^{d_j}$. It suffices to show that either $f_\lambda \in (X_0^{n_0})$ or $f_\lambda \in (X_1 X_j)$ holds. Indeed, if $f_\lambda \in (X_0^{n_0})$, then we have $f_\lambda = f X_0^{n_0}$ for some $f \in R$. Since both f_λ and $X_0^{n_0}$ are homogeneous elements of $G_0 \times G_m$-weight $(n_0, 0)$, we deduce that $f \in R^{G_0 \times G_m}$. The same holds true if $f_\lambda \in (X_1 X_j)$. Next, suppose that $\omega > \omega^{\min}_{(n_0,c)}$. Then, we have $d_0 = n_0 + \omega \geq n_0$

by Lemmas 4.1.5 and 4.1.9. Therefore, $f_\lambda \in (X_0^{n_0})$. If $\omega = \omega_{(n_0,c)}^{\min}$, then we have $\omega < 0$ again by Lemma 4.1.9. On the other hand, we have $\omega = pd_1 - qd_j$ by Lemma 4.1.5, which tells us that $d_1 > 0$ and $d_j > 0$ hold. Therefore, $f_\lambda \in (X_1 X_j)$. Q.E.D.

We show Proposition 4.1.20 separately for the case $i = 1$ and the case $i > 1$. The former case can be shown by following similar lines as in the proof of Proposition 4.1.19, while the latter case requires more preparatory lemmas.

Proof of Proposition 4.1.20 for $i = 1$. Recall that $P_1 = 1$. Let $\lambda = (n_1, c, \omega) \in \Lambda_{(n_1,0)}$, and write $f_\lambda = X_0^{d_0} X_1^{d_1} X_j^{d_j}$. By the definition of $\Lambda_{(n_1,0)}$, we have $c \in m\mathbb{Z}$. Moreover, since we have $c_{(n_1,0)} = 0$ by Example 4.1.4, we can write $c = mx$ with some $x \geq 0$. As in the proof of Proposition 4.1.19, it suffices to show that either $f_\lambda \in (X_0^{n_1})$ or $f_\lambda \in (X_1^{e_1} X_j^{l_1})$ holds. If $\omega > \omega_{(n_1,c)}^{\min}$, then we have $d_0 > n_1$ by Lemmas 3.2.5, 4.1.5, and 4.1.9. Therefore, $f_\lambda \in (X_0^{n_1})$. Next suppose that $\omega = \omega_{(n_1,c)}^{\min}$. If $c = 0$, then $f_\lambda = X_0^{n_i}$. So let us assume that $c > 0$ and consider an element f of $R_{n_1}^c$ defined by $f = f_{\left(n_1, c, \omega_{(n_1,m)}^{\min} + (x-1)mp\right)} = f_{\left(n_1, m, \omega_{(n_1,m)}^{\min}\right)} f_{(0,m,mp)}^{x-1} = X_1^{e_1} X_j^{l_1} (X_0^{mp} X_1^m)^{x-1}$, where the last equality follows from $f_{\left(n_1, m, \omega_{(n_1,m)}^{\min}\right)} = X_1^{e_1} X_j^{l_1}$ (see Example 4.1.10). Let $\theta = \mathfrak{Q}[(x-1)mp, q-p]$, and let $\Theta = \mathfrak{R}[(x-1)mp, q-p]$. Then, we see that f can be written as $f = X_1^{e_1} X_j^{l_1} (X_0^{q-p})^\theta X_0^\Theta X_1^{(x-1)m}$. We define f' to be the monomial obtained by replacing the factor X_0^{q-p} in f by $X_1 X_j$, i.e., $f' = X_1^{e_1} X_j^{l_1} (X_1 X_j)^\theta X_0^\Theta X_1^{(x-1)m}$. Since both X_0^{q-p} and $X_1 X_j$ are elements of R_{q-p}^0, we see that $f' \in R_{n_1}^c$. Therefore, we can write $f' = f_{\lambda'}$ with some $\lambda' = (n_1, c, \omega') \in \Lambda_{(n_1,0)}$. By Lemma 4.1.5, we have $\Theta = n_1 + \omega'$. Moreover, we have $\Theta < q - p$ by its definition. Thus, we get $\omega' = \omega_{(n_1,c)}^{\min}$ by Lemma 4.1.9. It follows that $\lambda = \lambda'$, and hence $f_\lambda = f_{\lambda'} \in (X_1^{e_1} X_j^{l_1})$. Q.E.D.

Henceforth, we assume that $r > 1$ and prepare some lemmas that we need for the proof of Proposition 4.1.20 for the case $1 < i \leq r$.

Remark 4.1.20.1. Recall that we have considered the Hirzebruch–Jung continued fraction expansion $b/t = [[c_1, \ldots, c_r]]$ in §3.2.2. Set $t_1 := t$. Then we have the following equations that arise from the *modified Euclidean algorithm* (see [CLS11, §10]):

$$b = c_1 t_1 - t_2, \quad t_1 = c_2 t_2 - t_3, \quad \ldots, \quad t_{i-1} = c_i t_i - t_{i+1}, \quad \ldots, \quad t_{r-1} = c_r t_r. \tag{4.2}$$

Since $b = n_0/k$ and $t_1 = n_1/k$, Lemma 3.2.5 and (4.2) yield that $t_i = n_i/k$ holds for any $1 \le i \le r$. Furthermore, the following equation holds for any $1 < i \le r$:

$$b - t_1 = (c_1 - 2)t_1 + (c_2 - 2)t_2 + \cdots + (c_{i-1} - 2)t_{i-1} + t_{i-1} - t_i. \tag{4.3}$$

Let us fix an integer L that satisfies $1 < L \le r$, and let x be any integer such that $0 \le x < P_L - P_{L-1}$. Set $\Theta_x := \mathfrak{R}[t_1(P_{L-1} + x), b]$, and set $\theta_x := \mathfrak{Q}[t_1(P_{L-1} + x), b] - Q_{L-1}$. Then, $\theta_x \ge 0$. Indeed, we have $t_1 P_{L-1} = bQ_{L-1} + t_{L-1}$ with $0 \le t_{L-1} < b$ by Lemma 3.2.5, and this implies that $\mathfrak{Q}[t_1 P_{L-1}, b] = Q_{L-1}$ and $\mathfrak{R}[t_1 P_{L-1}, b] = t_{L-1}$. Therefore, we have $\mathfrak{Q}[t_1(P_{L-1} + x), b] \ge Q_{L-1}$. Also, note that $\Theta_x = t_{L-1} + t_1 x - b\theta_x$ holds.

Remark 4.1.20.2. With the above assumption and notation, we have the following.

(i) We have $\theta_x - \theta_{x-1} = \mathfrak{Q}[t_1(P_{L-1} + x), b] - \mathfrak{Q}[t_1(P_{L-1} + x - 1), b] \in \{0, 1\}$, since $t_1 < b$. Furthermore, the following properties are true.

- We have $\theta_x - \theta_{x-1} = 0$ if and only if $\Theta_{x-1} + t_1 - b < 0$. In this case, $\Theta_x = \Theta_{x-1} + t_1$.
- We have $\theta_x - \theta_{x-1} = 1$ if and only if $\Theta_{x-1} + t_1 - b \ge 0$. In this case, $\Theta_x = \Theta_{x-1} + t_1 - b$.

(ii) Let x' be any integer such that $0 \le x' < P_L - P_{L-1}$. Since $P_L - P_{L-1} < b$, we have $\Theta_x = \Theta_{x'}$ if and only if $x = x'$.

(iii) We have $\theta_0 = 0$ and $\Theta_0 = t_{L-1}$.

(iv) Suppose that $c_s > 2$ holds for some $1 \le s \le L - 1$, and denote by $s_{\max}$ the maximum among them. Then, we have $P_L - P_{L-1} = (c_1 - 1)P_1 + (c_2 - 2)P_2 + \cdots + (c_{s_{\max}} - 2)P_{s_{\max}} = P_{s_{\max}+1} - P_{s_{\max}}$ by Lemma 3.2.5.

Definition 4.1.21. Keep the above notation. For each $1 \le j \le L - 1$, we define $M_j := \max\{\Theta_x : 0 \le x < P_{j+1}\}$ and $N_j := \max\{\Theta_x : 0 \le x < P_{j+1} - P_j\}$.

The next lemma will be the core of the proof of Proposition 4.1.20.

Key Lemma. *Let $1 < L \le r$. Then, $\Theta_x \ge t_{L-1}$ holds for any $0 \le x < P_L - P_{L-1}$. Moreover, we have $\Theta_x = t_{L-1}$ if and only if $x = 0$.*

We need the following lemmas for the proof of Key Lemma.

Lemma 4.1.22. *Let $1 < L \le r$, and let $0 \le x < P_L - P_{L-1}$ as above . Assume that $c_s > 2$ holds for some $1 \le s \le L-1$, and let $s_{\max}$ be as in Remark 4.1.20.2 (iv). If $\Theta_x = t_{L-1} + (c_1 - 2)t_1 + \cdots + (c_{j-1} - 2)t_{j-1} + (c_j - 1)t_j$ holds for some $1 \le j \le s_{\max}$, then we have $\Theta_{x+1} = t_{L-1} + t_{j+1}$.*

Proof. By a direct calculation using (4.3) (see Remark 4.1.20.1), we have $\Theta_x + t_1 - b = t_{L-1} + t_{j+1}$. On the other hand, we have $t_{L-1} + t_{j+1} > 0$ by Lemma 3.2.5, and therefore we obtain $\Theta_{x+1} = \Theta_x + t_1 - b = t_{L-1} + t_{j+1}$ by Remark 4.1.20.2 (i). Q.E.D.

Lemma 4.1.23. *Keep the notation and the assumption of Lemma 4.1.22, and let $1 \le j < s_{\max}$. Then the following properties are true.*

(i) *Suppose that $P_j \le x < P_{j+1}$. Set $\kappa = \mathfrak{Q}[x, P_j]$, and set $\varepsilon = \mathfrak{R}[x, P_j]$, i.e., $x = \kappa P_j + \varepsilon$. Then, we have $\Theta_x = \Theta_\varepsilon + \kappa t_j$. In particular, $\Theta_x > t_{L-1}$ holds.*

(ii) *We have $M_j = \Theta_{P_{j+1} - P_j} = t_{L-1} + b - t_j + t_{j+1}$.*

Remark 4.1.23.1. In Lemma 4.1.23 (i), we see that $1 \le \kappa \le c_j - 1$ holds concerning the relation $P_{j+1} = (c_j - 1)P_j + (P_j - P_{j-1})$. Also, if $\kappa = c_j - 1$, then $0 \le \varepsilon < P_j - P_{j-1}$.

Proof of Lemma 4.1.23. We proceed by induction on j.

Let $j = 1$. Recall that $P_1 = 1$, and that $P_2 = c_1$. By induction on x, we show that $\Theta_x = t_{L-1} + x t_1$ holds for any $P_1 \le x < P_2$. Firstly, by (4.3) and $c_{s_{\max}} > 2$, we have

$$\begin{aligned}\Theta_0 + t_1 - b = t_{L-1} + t_1 - b &= t_L - \{(c_1 - 2)t_1 + \cdots + (c_{L-1} - 2)t_{L-1}\} \\ &= t_L - \{(c_1 - 2)t_1 + \cdots + (c_{s_{\max}} - 2)t_{s_{\max}}\} \le t_L - t_{s_{\max}} < 0.\end{aligned}$$

The last inequality follows from Lemma 3.2.5. By Remark 4.1.20.2, we get $\Theta_1 = t_{i-1} + t_1$. Suppose that $x > 1$. Then, by the induction hypothesis, we have $\Theta_{x-1} = t_{L-1} + (x-1)t_1$. Since $c_1 - x - 1 \ge 0$ and $s_{\max} \ge 2$, we see that the following holds:

$$\Theta_{x-1} + t_1 - b = t_L - \{(c_1 - x - 1)t_1 + (c_2 - 2)t_2 + \cdots + (c_{s_{\max}} - 2)t_{s_{\max}}\} \le t_L - t_{s_{\max}} < 0.$$

Therefore, we get $\Theta_x = t_{L-1} + x t_1$. Furthermore, this yields that $M_1 = \Theta_{c_1 - 1} = \Theta_{P_2 - P_1}$. Also, we see that $t_{L-1} + b - t_1 + t_2 = t_{L-1} + (c_1 - 1)t_1$ holds, since we have $b - t_1 + t_2 = (c_1 - 1)t_1$ by (4.2) (see Remark 4.1.20.1).

Let $j > 1$. We divide the proof into three steps.

Step 1. We show by induction on κ that the following holds for any $1 \le \kappa \le c_j - 1$:

$$\Theta_{\kappa P_j} = t_{L-1} + \kappa t_j. \tag{4.4}$$

Let $\kappa = 1$. The relation $P_j - 1 = (c_{j-1} - 1)P_{j-1} + (P_{j-1} - P_{j-2} - 1)$ implies that $\mathfrak{Q}[P_j - 1, P_{j-1}] = c_{j-1} - 1$, and that $\mathfrak{R}[P_j - 1, P_{j-1}] = P_{j-1} - P_{j-2} - 1$. Taking these into account, it follows from the induction hypothesis for item (i) that $\Theta_{P_j-1} = \Theta_{P_{j-1}-P_{j-2}-1} + (c_{j-1} - 1)t_{j-1}$. By Remark 4.1.20.2, we see that either $\Theta_{P_{j-1}-P_{j-2}} = \Theta_{P_{j-1}-P_{j-2}-1} + t_1$ or $\Theta_{P_{j-1}-P_{j-2}} = \Theta_{P_{j-1}-P_{j-2}-1} + t_1 - b$ holds. If the latter holds, then $\Theta_{P_{j-1}-P_{j-2}} - t_1 < 0$. On the other hand, we have $\Theta_{P_{j-1}-P_{j-2}} = t_{L-1} + b - t_{j-2} + t_{j-1}$ by the induction hypothesis for item (ii), and hence

$$\Theta_{P_{j-1}-P_{j-2}} - t_1 = t_{L-1} + (c_1 - 2)t_1 + \cdots + (c_{j-2} - 2)t_{j-2} > 0$$

by (4.3). This implies that the former holds, i.e., we have $\Theta_{P_{j-1}-P_{j-2}} = \Theta_{P_{j-1}-P_{j-2}-1} + t_1$. Therefore,

$$\begin{aligned}\Theta_{P_j-1} &= \Theta_{P_{j-1}-P_{j-2}-1} + (c_{j-1} - 1)t_{j-1} = \Theta_{P_{j-1}-P_{j-2}} - t_1 + (c_{j-1} - 1)t_{j-1} \\ &= t_{L-1} + (c_1 - 2)t_1 + \cdots + (c_{j-2} - 2)t_{j-2} + (c_{j-1} - 1)t_{j-1}.\end{aligned} \tag{4.5}$$

Hence, we get $\Theta_{P_j} = t_{L-1} + t_j$ by Lemma 4.1.22. Next, let $\kappa > 1$. We first show that

$$\Theta_{(\kappa-1)P_j+\varepsilon} = \Theta_\varepsilon + (\kappa - 1)t_j \tag{4.6}$$

holds for any $1 \le \varepsilon < P_j$. By the induction hypothesis for Step 1, we have $\Theta_{(\kappa-1)P_j} = t_{L-1} + (\kappa - 1)t_j$. Thus, concerning the definition of Θ and Remark 4.1.20.2 (i) and (iii), it suffices to check that $\Theta_\varepsilon + (\kappa-1)t_j < b$ holds. Here, note that $\Theta_\varepsilon \le M_{j-1}$. Since $M_{j-1} = t_{L-1} + b - t_{j-1} + t_j$ holds by the induction hypothesis for item (ii), we have

$$\begin{aligned}b - \{\Theta_\varepsilon + (\kappa - 1)t_j\} &\ge b - \{M_{j-1} + (c_j - 2)t_j\} = t_j - t_{j+1} - t_{L-1} \\ &= (c_{j+1} - 2)t_{j+1} + \cdots + (c_{s_{\max}} - 2)t_{s_{\max}} + (t_{s_{\max}} - t_{s_{\max}+1}) - t_{L-1} \\ &\ge t_{s_{\max}} + (t_{s_{\max}} - t_{s_{\max}+1}) - t_{L-1} > 0.\end{aligned}$$

This shows (4.6). Taking $\varepsilon = P_j - 1$, one obtains $\Theta_{\kappa P_j-1} = \Theta_{P_j-1} + (\kappa-1)t_j$. Therefore, by (4.3) and (4.5), we have $\Theta_{\kappa P_j-1} + t_1 - b = t_{L-1} + \kappa t_j > 0$. Hence we see that $\Theta_{\kappa P_j} = \Theta_{\kappa P_j-1} + t_1 - b = t_{L-1} + \kappa t_j$ by Remark 4.1.20.2. This shows (4.4).

Step 2. In this step, we prove that the following holds for any $0 < \varepsilon < P_j - P_{j-1}$:

$$\Theta_{(c_j-1)P_j+\varepsilon} = \Theta_\varepsilon + (c_j - 1)t_j. \tag{4.7}$$

If $c_1 = \cdots = c_{j-1} = 2$, then we have $P_j - P_{j-1} = 1$, so we may suppose otherwise. Similarly as in the proof of (4.6), it suffices to show that $N_{j-1} + (c_j - 1)t_j < b$ holds. Set $u := \max\{j' : 1 \le j' \le j-1,\ c_{j'} > 2\}$. Then, we have $P_j - P_{j-1} = P_{u+1} - P_u$. In view of this relation, we show that $N_u + (c_j - 1)t_j < b$ holds. Sine $P_{u+1} - P_u = (c_u - 2)P_u + (P_u - P_{u-1})$, we see that N_u coincides with the maximum between $\max\{\Theta_\varepsilon : 0 \le \varepsilon < (c_u - 2)P_u\}$ and $\max\{\Theta_\varepsilon : (c_u-2)P_u \le \varepsilon < P_{u+1} - P_u\}$. Concerning the induction hypothesis for item (i), we see that the relation $(c_u-2)P_u = (c_u-3)P_u + P_u$ implies that $\Theta_\varepsilon = \Theta_{\Re[\varepsilon, P_u]} + (c_u-3)t_u$ and $0 \le \Re[\varepsilon, P_u] < P_u$ hold for any $(c_u - 3)P_u \le \varepsilon < (c_u-2)P_u$. Therefore, we see that $\max\{\Theta_\varepsilon : 0 \le \varepsilon < (c_u-2)P_u\} = (c_u-3)t_u + M_{u-1}$. In a similar manner, we see that $\max\{\Theta_\varepsilon : (c_u-2)P_u \le \varepsilon < P_{u+1} - P_u\} = (c_u-2)t_u + N_{u-1}$. Therefore, $N_u = \max\{(c_u-3)t_u + M_{u-1},\ (c_u-2)t_u + N_{u-1}\}$. By continuing in this way and concerning that $\Theta_0 = t_{L-1}$, one finally obtains

$$N_u = \max\{(c_u-3)t_u + M_{u-1},\ (c_u-2)t_u + \cdots + (c_1-2)t_1 + t_{L-1}\}.$$

Since we have $M_{u-1} = t_{L-1} + b - t_{u-1} + t_u$ by the induction hypothesis for item (ii), we get $(c_u-3)t_u + M_{u-1} = b - t_u + t_L$. This yields that $N_u = (c_u-3) + M_{u-1}$. Therefore,

$$\begin{aligned} b - \{N_u + (c_j-1)t_j\} &= t_u + (c_{u+1}-2)t_{u+1} + \cdots + (c_j-2)t_j \\ &\qquad + \cdots + (c_{s_{\max}}-2)t_{s_{\max}} + t_{s_{\max}} - t_{s_{\max}+1} - t_{L-1} - (c_j-1)t_j \\ &\ge t_u + t_{s_{\max}} + t_{s_{\max}} - t_{s_{\max}+1} - t_j - t_{L-1} > 0. \end{aligned}$$

This completes the proof of (4.7). Since we have $P_{j+1} = (c_j-1)P_j + (P_j - P_{j-1})$, item (i) follows from (4.4), (4.6), and (4.7).

Step 3. In this last step, we complete the proof of item (ii). First, we show that $M_j = t_{L-1} + b - t_j + t_{j+1}$ holds. Set $M_A := \max\{\Theta_x : P_j \le x < (c_j-1)P_j\}$, and set $M_B := \max\{\Theta_x : (c_j-1)P_j \le x < P_{j+1}\}$. Then, M_j is the maximum among M_{j-1}, M_A, and M_B. Following the similar line as in the proof of (4.7), we see that $M_A = (c_j-2)t_j + M_{j-1} = t_{L-1} + b - t_j + t_{j+1}$, and that $M_B = (c_j-1)t_j + N_{j-1}$, which implies that $M_j = \max\{M_A, M_B\}$. If $c_1 = \cdots = c_{j-1} = 2$, then we have $M_B = (c_j-1)t_j + t_{L-1}$, and hence $M_A - M_B = b - t_{j-1} > 0$. Therefore, $M_j = M_A$. Suppose that $c_{j'} > 2$ holds for some $1 \le j' \le j-1$, and let u be as in Step 2. Then, since $M_{u-1} = t_{L-1} + b - t_{u-1} + t_u$, we have $M_B = (c_j-1)t_j + (c_u-3)t_u + M_{u-1} = t_{L-1} + b + t_{u+1} - 2t_u + (c_j-1)t_j$. By using (4.2), we see that $M_A - M_B = (t_u - t_{u+1}) + (t_u - t_{j-1}) > 0$. Therefore, $M_j = M_A = t_{L-1} + b - t_j + t_{j+1}$. Next we show that $t_{L-1} + b - t_j + t_{j+1} = \Theta_{P_{j+1} - P_j}$. By the induction hypothesis, one obtains

$$t_{L-1} + b - t_j + t_{j+1} = (t_{L-1} + b - t_{j-1} + t_j) + (c_j-2)t_j = \Theta_{P_j - P_{j-1}} + (c_j-2)t_j.$$

Since $\Theta_{P_j - P_{j-1}} + (c_j - 2)t_j < b$, it follows that $\Theta_{P_j - P_{j-1}} + (c_j - 2)t_j = \Theta_{P_j - P_{j-1} + (c_j - 2)P_j} = \Theta_{P_{j+1} - P_j}$.

This completes the proof of the lemma. Q.E.D.

Proof of Key Lemma. If $c_1 = \cdots = c_{L-1} = 2$, then $P_L - P_{L-1} = 1$, and we have already verified that $\Theta_0 = t_{L-1}$ in Remark 4.1.20.2. Suppose that $c_s > 2$ holds for some $1 \le s \le L-1$. Let $s_{\max}$ be as in Remark 4.1.20.2, and let x be any integer such that $P_{s_{\max}} \le x < P_{s_{\max}+1} - P_{s_{\max}}$. Set $\kappa = \mathfrak{Q}[x, P_{s_{\max}}]$, and set $\varepsilon = \mathfrak{R}[x, P_{s_{\max}}]$. By following a similar line as in the proof of Lemma 4.1.23, we can check that the following hold:

$$\Theta_\varepsilon + \kappa t_{s_{\max}} \le (c_{s_{\max}} - 3)t_{s_{\max}} + M_{s_{\max}-1} < b \quad (\text{if } 1 \le \kappa \le c_{s_{\max}} - 3);$$
$$\Theta_\varepsilon + \kappa t_{s_{\max}} \le (c_{s_{\max}} - 2)t_{s_{\max}} + N_{s_{\max}-1} < b \quad (\text{if } \kappa = c_{s_{\max}} - 2).$$

These yield that $\Theta_x = \Theta_\varepsilon + \kappa t_{s_{\max}}$. In particular, we have $\Theta_x > t_{L-1}$. Therefore, taking Lemma 4.1.23 into account, we see that $\Theta_x \ge t_{L-1}$ holds for any $0 \le x < P_L - P_{L-1}$, and that the equality is true if and only if $x = 0$. Q.E.D.

Corollary 4.1.24. *Let* $1 \le i \le r$. *Then, we have* $\mathfrak{R}[t_1 P_i, b] = t_i$. *Moreover, if* $i > 1$, *then* $t_{i-1} \le \mathfrak{R}[t_1 x, b] \le b + t_i - t_{i-1}$ *holds for any* $0 < x < P_i$.

Proof. We have already seen that $\mathfrak{R}[t_1 P_i, b] = t_i$. Let $0 < x < P_i$. Then, we see that $P_{L-1} \le x < P_L$ holds for some $1 < L \le i$. By definition, we have $\mathfrak{R}[t_1 x, b] = \Theta_{x - P_{L-1}}$. Therefore, it follows from Lemma 3.2.5 and Key Lemma that $\mathfrak{R}[t_1 x, b] \ge t_{L-1} \ge t_{i-1}$. Let us show that $\mathfrak{R}[t_1 x, b] \le b + t_i - t_{i-1}$. First, suppose that $c_1 = \cdots = c_{L-1} = 2$. Then, we have $\mathfrak{R}[t_1 x, b] = t_{L-1}$ in view of the proof of Key Lemma. Also, we get $b + t_i - t_{i-1} = (c_1 - 1)t_1 + (c_2 - 2)t_2 + \cdots + (c_{i-1} - 2)t_{i-1} \ge t_1 \ge t_{L-1}$ by (4.3). Next, suppose that we have $c_s > 2$ for some $1 \le s \le L-1$, and let $s_{\max}$ be as in Remark 4.1.20.2. Concerning the proof of Key Lemma, we see that $N_{L-1} = (c_{s_{\max}} - 3)t_{s_{\max}} + M_{s_{\max}-1}$ holds, and therefore

$$\begin{aligned} b + t_i - t_{i-1} - \mathfrak{R}[t_1 x, b] &\ge b + t_i - t_{i-1} - \{(c_{s_{\max}} - 3)t_{s_{\max}} + M_{s_{\max}-1}\} \\ &\ge (c_{i-1} - 2)t_{i-1} - (c_{s_{\max}} - 3)t_{s_{\max}} > 0. \end{aligned}$$

Consequently, we get $t_{i-1} \le \mathfrak{R}[t_1 x, b] \le b + t_i - t_{i-1}$. Q.E.D.

Proof of Proposition 4.1.20 for $1 < i \le r$. Let $\lambda = (n_i, c, \omega) \in \Lambda_{(n_i, 0)}$, and write $f_\lambda = X_0^{d_0} X_1^{d_1} X_j^{d_j}$. Then we have $c = xm$ for some $x \ge 0$. As in the proof of Proposition 4.1.20 for the case $i = 1$, we show that either $f_\lambda \in (X_0^{n_i})$ or $f_\lambda \in (X_1^{e_i} X_j^{l_i})$ holds. If $\omega > \omega^{\min}_{(n_i,c)}$, then we have $f_\lambda \in (X_0^{n_i})$ by Lemmas 4.1.5 and 4.1.9. Suppose that $\omega = \omega^{\min}_{(n_i,c)}$. By Example 4.1.4, we have $f_\lambda = X_0^{n_i}$ if $x = 0$. Next, we assume that $0 < x < P_i$ and show that $\omega^{\min}_{(n_i,c)} \geq 0$ holds. First, we have

$$\Re[pc, q-p] + n_i < q-p \Leftrightarrow \Re[pc+n_i, q-p] \geq n_i \Leftrightarrow \Re\left[\frac{pc+n_i}{k}, b\right] \geq t_i. \tag{4.8}$$

By using the equation (3.1) (see §3.2.1), we see that

$$\frac{pc+n_i}{k} = x\{(\alpha+1)b - t_1\} + (t_1 P_i - bQ_i) \equiv t_1(P_i - x) \quad (mod\ b)$$

holds, which yields that

$$\Re\left[\frac{pc+n_i}{k}, b\right] = \Re\left[t_1(P_i - x), b\right]. \tag{4.9}$$

Therefore, it follows from Lemma 4.1.18 and Corollary 4.1.24 that $\omega^{\min}_{(n_i,c)} \geq 0$. Since $d_0 = n_i + \omega^{\min}_{(n_i,c)}$, this implies that $f_\lambda \in (X_0^{n_i})$. If $x = P_i$, then we have $f_\lambda = X_1^{e_i} X_j^{l_i}$ by Example 4.1.10. Therefore, we are left to consider the case where $x > P_i$. We show that $d_1 \geq e_i$ and $d_j \geq l_i$ hold in this case. Set $\omega' = -n_i + q(c - mP_i)$, and set $\omega'' = -n_i + p(c - mP_i)$. Suppose that $d_1 < e_i$. Then we have $qc - \omega^{\min}_{(n_i,c)} < (q-p)e_i = n_i + qmP_i$, and hence $\omega^{\min}_{(n_i,c)} > \omega'$. It follows that $0 \leq pc - \omega^{\min}_{(n_i,c)} < pc - \omega' = n_i + qmP_i - c(q-p)$. Therefore, all of the following are positive integers: $n_i + \omega' = q(c - mP_i)$, $\frac{qc-\omega'}{q-p} = \frac{n_i+qmP_i}{q-p}$, $\frac{pc-\omega'}{q-p} = \frac{n_i+qmP_i}{q-p} - c$. Thus we get $(n_i, c, \omega') \in \tilde{\mu}^{-1}(n_i, c) \cap \Lambda$ by Remark 4.1.5.1. But this contradicts to the minimality of $\omega^{\min}_{(n_i,c)}$. If $d_j < l_i$, then we have $pc - \omega^{\min}_{(n_i,c)} < (q-p)l_i = n_i + pmP_i$, and hence $\omega^{\min}_{(n_i,c)} > \omega''$. In a similar manner, we see that this implies $(n_i, c, \omega'') \in \tilde{\mu}^{-1}(n_i, c) \cap \Lambda$, which is a contradiction. Q.E.D.

Corollary 4.1.25. *Let $1 < i \leq r$. Then, $\Re[pmx + n_i, q-p] = n_i + \Re[pmx, q-p]$ holds for any $0 < x < P_i$.*

Proof. We have seen in the proof of Proposition 4.1.20 that $\Re[pmx + n_i, q-p] \geq n_i$ holds if $0 < x < P_i$. On the other hand, since $n_i < q-p$, we have

$$\Re[pmx + n_i, q-p] = \begin{cases} n_i + \Re[pmx, q-p] & (\text{if } n_i + \Re[pmx, q-p] < q-p) \\ n_i + \Re[pmx, q-p] - q + p & (\text{otherwise}). \end{cases}$$

Therefore we deduce that $\Re[pmx + n_i, q-p] = n_i + \Re[pmx, q-p]$, since otherwise we have $n_i \leq n_i + \Re[pmx, q-p] - q + p < n_i$. Q.E.D.

4.2 Hilbert function of the ideals

4.2.1 Calculation of the Hilbert function I

In this subsection, we show that the Hilbert function of the ideals I_s and J_s coincide with the Hilbert function h of a general fiber of the quotient morphism π (Theorems 4.2.2 and 4.2.3). Recall that h coincides with the Hilbert function of the regular representation of $G_0 \times G_m$ (Corollary 3.1.2).

Theorem 4.2.2. *For any $s \in \mathbb{C}$, the quotient ring A/I_s has Hilbert function h. Namely, $\dim(A/I_s)_{(n,d)} = h(n,d)$ holds for any $G_0 \times G_m$-weight $(n,d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.*

Proof. Taking Remark 4.1.0.1 into account, it suffices to consider the cases where $s = 0, 1$. Let $R = \mathbb{C}[X_0, X_1, X_4]$, i.e., let $j = 4$ in (4.1) (see §4.1), and consider the ideals $\widetilde{I_1} := (X_0^{q-p} - X_1X_4,\ 1 - X_0^{mp}X_1^m)$ and $\widetilde{I_0} := (X_0^{q-p} - X_1X_4,\ X_0^{mp}X_1^m)$ of R. Then, we have $A/I_1 \cong R/\widetilde{I_1}$ and $A/I_0 \cong R/\widetilde{I_0}$.

Case 1: $s = 1$. We first show that $\dim(A/I_1)_{(n,d)} \geq h(n,d)$ holds. Recall that the open orbit $\mathfrak{U} \subset E_{l,m}$ coincides with the $SL(2)$-orbit of $\pi(x)$, where $x = (1,1,0,0,1) \in H_{q-p}$, and that the Hilbert–Chow morphism γ is an isomorphism over $\mathfrak{U} \cup \mathfrak{D}$ (Proposition 3.1.1). Let $[I] = \gamma^{-1}(\pi(x))$. Since $X_0^{mp}X_1^m \in A^{G_0 \times G_m}$ and since the X_0-coordinate and the X_1-coordinate of x are both 1, we have $1 - X_0^{mp}X_1^m \in I$. Similarly, since $X_0^{mp}X_2^m \in A^{G_0 \times G_m}$ and since the X_2-coordinate of x is 0, we have $X_0^{mp}X_2^m \in I$. On the other hand, $s_1X_1 + s_2X_2 \in I$ holds for some $(s_1, s_2) \neq 0$ by Remark 4.1.1.1. Then we see that $s_1 = 0$, since otherwise we have $1 \in I$ by the conditions $1 - X_0^{mp}X_1^m,\ X_0^{mp}X_2^m \in I$. Therefore, we get $X_2 \in I$. Similarly, since we have $X_1^{aq}X_3^{ap},\ X_1^{aq}X_4^{ap} \in A^{G_0 \times G_m}$, it follows that $X_1^{aq}X_3^{ap},\ 1 - X_1^{aq}X_4^{ap} \in I$. This implies that $X_3 \in I$ again by Remark 4.1.1.1. Therefore, we have $I_1 \subset I$, which induces a natural surjection $A/I_1 \longrightarrow A/I$. This yields that $\dim(A/I_1)_{(n,d)} \geq \dim(A/I)_{(n,d)} = h(n,d)$. Next, we show that $\dim(R/\widetilde{I_1})_{(n,d)} \leq h(n,d)$ holds. The weight space $R_{(n,d)}$ decomposes as $R_{(n,d)} = \bigoplus_{\lambda \in \Lambda_{(n,d)}} R_\lambda$. In view of this decomposition, we see that $\dim(R/\widetilde{I_1})_{(n,d)} \leq 1$ holds by Lemma 4.1.14 (iii).

Case 2: $s = 0$. Let $[I'] \in \gamma^{-1}(O)$ be a point such that $\gamma([I']) \in H_{q-p} \cap \{X_2 = X_3 = 0\} /\!/ (G_0 \times G_m)$, where O stands for the origin of $E_{l,m}$ (see Remark 2.3.3.2). Then we see in a similar way as above that $I_0 \subset I'$ holds. Thus, $\dim(A/I_0)_{(n,d)} \geq h(n,d)$. Next, notice that $R_{(n,d)}$ decomposes as $R_{(n,d)} = R_n^{c_{(n,d)}} \oplus R'_{(n,d)}$, where we set $R'_{(n,d)} = \bigoplus_{\substack{c \equiv d\ (mod\ m) \\ c > c_{(n,d)}}} R_n^c$. By Lemma

4.1.14 (ii), we see that $R'_{(n,d)} \subset \widetilde{I_0}$. Hence we get $\dim(R/\widetilde{I_0})_{(n,d)} \leq 1$ by applying Lemma 4.1.14 (i) with $c = c_{(n,d)}$. Q.E.D.

Theorem 4.2.3. *For any $s \in \mathbb{C}$, the quotient ring A/J_s has Hilbert function h.*

Lemma 4.2.4. *For any $s \in \mathbb{C}$ and $(n,d) \in \mathbb{Z}\times\mathbb{Z}/m\mathbb{Z}$, we have* $\dim(A/J_s)_{(n,d)} \geq h(n,d)$.

Proof. We proceed in a similar way to the proof of Theorem 4.2.2.

Case 1: $s = 1$**.** We have seen in §3.1 that $\mathfrak{D}$ coincides with the $SL(2)$-orbit of $\pi(x')$, where $x' = (0,1,0,1,0) \in H_{q-p}$. Let $[J] = \gamma^{-1}(\pi(x'))$. Similarly as in Case 1 of the proof of Theorem 4.2.2, we have $1 - X_1^{aq}X_3^{ap}$, $X_2^{aq}X_3^{ap} \in J$ since $X_1^{aq}X_3^{ap}$, $X_2^{aq}X_3^{ap} \in A^{G_0\times G_m}$ and since the X_1-, X_2-, and X_3-coordinates of x' are $1,0$, and 1, respectively. By Remark 4.1.1.1, it follows that X_2, $X_4 \in J$. Therefore, we have $(X_0^{q-p},\ X_2,\ X_4,\ 1 - X_1^{aq}X_3^{ap}) \subset J$ concerning that the defining ideal of H_{q-q} is $(X_0^{q-p} - X_1X_4 + X_2X_3)$. If $E_{l,m}$ is toric, then we get $J_1 \subset J$, since $k = q-p$. Suppose that $E_{l,m}$ is non-toric. By Proposition 4.1.2, $S_{(k,0)}$ is generated by $\mathrm{Sym}^{e_r}\langle X_1, X_2\rangle \otimes \mathrm{Sym}^{l_r}\langle X_3, X_4\rangle \oplus \langle X_0^k\rangle$ over the invariant ring $S^{G_0\times G_m}$. Therefore, the conditions $\dim(A/J)_{(n,d)} = h(n,d) = 1$ and $X_2, X_4 \in J$ imply that $sX_1^{e_r}X_3^{l_r} + s'X_0^k \in J$ holds for some $(s,s') \neq 0$. If $s \neq 0$, then we deduce from the conditions $e_r \leq e_{r+1} = aq$, $l_r \leq l_{r+1} = ap$, and $k < q-p$ that $1 \in J$. Thus we get $s = 0$, and hence $X_0^k \in J$. Therefore, we see that $J_1 \subset J$ holds in the non-toric case as well. The inclusion $J_1 \subset J$ induces a natural surjection $A/J_1 \longrightarrow A/J$, which yields that $\dim(A/J_1)_{(n,d)} \geq \dim(A/J)_{(n,d)} = h(n,d)$.

Case 2: $s = 0$**.** Let $[J'] \in \gamma^{-1}(O)$ be a point such that $\gamma([J']) \in H_{q-p} \cap \{X_2 = X_4 = 0\}/\!/(G_0 \times G_m)$. Then we can show in a similar way that $J_0 \subset J'$ holds. Therefore, we have $\dim(A/J_0)_{(n,d)} \geq h(n,d)$. Q.E.D.

Proof of Theorem 4.2.3. Let $R = \mathbb{C}[X_0, X_1, X_3]$, i.e., let $j = 3$ in (4.1) (see §4.1).

Case 1: $s = 1$**.** Set $J = (X_0^k,\ 1 - X_1^{aq}X_3^{ap})$. Then, we have $A/J_1 \cong R/J$ and $R_{(n,d)}/J_{(n,d)} \cong \bigoplus_{c\equiv d\ (mod\ m)} R_n^c/(J \cap R_n^c)$. The vector space R_n^c decomposes as $R_n^c = \bigoplus_{\omega \geq \omega_{(n,c)}^{\min}} R_{(n,c,\omega)}$, and we have $\bigoplus_{\omega > \omega_{(n,c)}^{\min}} R_{(n,c,\omega)} \subset J$ by Lemma 4.1.11. Therefore, it suffices to show that $\dim W_{n,d} \leq 1$, where we set $W_{n,d} = \bigoplus_{\lambda = \left(n,c,\omega_{(n,c)}^{\min}\right) \in \Lambda_{(n,d)}} R_\lambda/(J \cap R_\lambda)$. We divide its proof into two steps. Recall that R_λ is a 1-dimensional vector space spanned by f_λ (see Lemma 4.1.5).

Step 1 of Case 1. In this step, we show that $\dim W_{n,d} \leq 1$ holds if $0 \leq n < q-p$ and if $d = 0$. Let us consider the set $C = \left\{c \in \mathbb{Z} : \left(n, c, \omega_{(n,c)}^{\min}\right) \in \Lambda_{(n,0)},\ n + \omega_{(n,c)}^{\min} < k\right\}$. If C is empty, then we see that $f_\lambda \in (X_0^k)$ holds for any $\lambda = \left(n, c, \omega^{\min}_{(n,c)}\right) \in \Lambda_{(n,0)}$ concerning Lemma 4.1.5. This implies that $\dim W_{n,0} = 0$, and hence we get $\dim(R/J)_{(n,0)} = 0$, which contradicts to Lemma 4.2.4. Therefore, C is non-empty. Let $\lambda = \left(n, c, \omega^{\min}_{(n,c)}\right) \in \Lambda_{(n,0)}$. If $c \notin C$, then we have $n + \omega^{\min}_{(n,c)} \geq k$, and hence $f_\lambda \in (X_0^k)$ by Lemma 4.1.5. If $c \in C$, then we have $f_\lambda - f_{\left(n, c_{\min}, \omega^{\min}_{(n,c_{\min})}\right)} \in (1 - X_1^{aq} X_3^{ap})$ by Lemma 4.1.15, where $c_{\min}$ denotes the minimal element of C. Consequently, we get $\dim W_{n,0} \leq 1$.

Step 2 of Case 1. In this step, we show that $\dim W_{n,d} \leq 1$ holds for any (n,d). Let $\lambda = \left(n, c, \omega^{\min}_{(n,c)}\right) \in \Lambda_{(n,d)}$. Set $n' = n + \omega^{\min}_{(n,c_{(n,d)})}$, set $c' = c - c_{(n,d)}$, and set $\lambda' = \left(n', c', \omega^{\min}_{(n',c')}\right) \in \Lambda_{(n',0)}$. Let $\lambda'' \in \Lambda$ be the image of

$$\left(0, \frac{qc_{(n,d)} - \omega^{\min}_{(n,c_{(n,d)})}}{q-p}, \frac{pc_{(n,d)} - \omega^{\min}_{(n,c_{(n,d)})}}{q-p}\right)$$

under the map μ. Then a direct calculation shows $\lambda'' = \left(n - n', c_{(n,d)}, \omega^{\min}_{(n,c_{(n,d)})}\right) \in \Lambda_{(n-n',d)}$. Therefore, $\lambda' + \lambda'' = \left(n, c, \omega^{\min}_{(n',c')} + \omega^{\min}_{(n,c_{(n,d)})}\right)$. Since we have $\omega^{\min}_{(n',c')} + \omega^{\min}_{(n,c_{(n,d)})} + n = \omega^{\min}_{(n',c')} + n' < q - p$ by Lemma 4.1.9, it follows from Lemma 4.1.16 that $\omega^{\min}_{(n',c')} + \omega^{\min}_{(n,c_{(n,d)})} = \omega^{\min}_{(n,c)}$. Thus we get $\lambda = \lambda' + \lambda''$. On the other hand we have $0 \leq n' < q - p$ by Lemmas 4.1.5 and 4.1.9, and therefore $\dim R_{(n',0)}/J_{(n',0)} \leq 1$ by Step 1. This yields that $\dim R_{(n,d)}/J_{(n,d)} \leq 1$, since we have $f_\lambda = f_{\lambda'} f_{\lambda''}$ and $f_{\lambda'} \in R_{(n',0)}$.

Case 2: $s = 0$**.** Set $J' = (X_0^k,\ X_1^{aq} X_3^{ap})$. Then, we have $A/J_0 \cong R/J'$. Concerning Case 1, it suffices to show that $\dim W'_{n,d} \leq 1$ holds if $0 \leq n < q - p$ and if $d = 0$, where we set $W'_{n,d} = \bigoplus_{\lambda = \left(n,c,\omega^{\min}_{(n,c)}\right) \in \Lambda_{(n,d)}} R_\lambda/(J' \cap R_\lambda)$. Let $\lambda = (n, c, \omega) \in \Lambda_{(n,0)}$, and let C be the set defined in Step 1 of Case $s = 1$. If $c \in C \setminus \{c_{\min}\}$, then we have $f_\lambda \in (X_1^{aq} X_3^{ap})$ by Lemma 4.1.15. Otherwise, we get $f_\lambda \in (X_0^k)$. Therefore, we have $f_\lambda \in J'$ whenever $c \neq c_{\min}$, which shows $\dim W'_{n,0} \leq 1$.

This completes the proof of the theorem. Q.E.D.

Corollary 4.2.5. *The subsets $\gamma^{-1}(\mathfrak{U})$ and $\gamma^{-1}(\mathfrak{D})$ of $\mathscr{H} = \mathrm{Hilb}_h^{G_0 \times G_m}(H_{q-p})$ are $SL(2)$-orbits of $[I_1]$ and $[J_1]$, respectively. In particular, the equivariant isomorphism $\gamma|_{\gamma^{-1}(\mathfrak{U} \cup \mathfrak{D})} : \gamma^{-1}(\mathfrak{U} \cup \mathfrak{D}) \longrightarrow \mathfrak{U} \cup \mathfrak{D}$ is given by sending $[I_1]$ and $[J_1]$ to $\pi(x)$ and $\pi(x')$, respectively, where $x = (1,1,0,0,1)$, $x' = (0,1,0,1,0) \in H_{q-p}$.*

Proof. Taking Remark 3.1.1.1 into account, we deduce from Theorems 4.2.2 and 4.2.3 that the defining ideals of $\pi^{-1}(\pi(x))$ and $\pi^{-1}(\pi(x'))$ are I_1 and J_1, respectively. This shows the corollary. Q.E.D.

4.2.6 Calculation of the Hilbert function II

In this subsection, we evaluate the Hilbert function of the ideals L_s^i from above (Theorems 4.2.7 and 4.2.8). Let $R = \mathbb{C}[X_0, X_1, X_3]$, i.e., let $j = 3$ in (4.1) (see §4.1), and let $\widetilde{K}$ be the ideal of R generated by elements of the form $X_0^{pu_1-qu_2}X_1^{u_1}X_3^{u_2}$, where $(u_1,u_2) \in M_{l,m}^+ \setminus (0,0)$. For each $s \in C$ and $1 \leq i \leq r$, we define

$$\widetilde{L_s^i} := (X_0^{n_{i-1}},\ sX_0^{n_i} - X_1^{e_i}X_3^{l_i}) + \widetilde{K} \subset R.$$

Then, $A/L_s^i \cong R/\widetilde{L_s^i}$. The goal of this subsection is to prove the following theorems:

Theorem 4.2.7. *We have* $\dim(R/\widetilde{L_0^i})_{(n,d)} \leq h(n,d)$ *for any* $1 \leq i \leq r$ *and* $(n,d) \in \mathbb{Z}\times\mathbb{Z}/m\mathbb{Z}$.

Theorem 4.2.8. *We have* $\dim(R/\widetilde{L_1^i})_{(n,d)} \leq h(n,d)$ *for any* $1 \leq i \leq r$ *and* $(n,d) \in \mathbb{Z}\times\mathbb{Z}/m\mathbb{Z}$.

We will see in Corollary 5.3.1 that the Hilbert function of L_s^i coincides with h for any $1 \leq i \leq r$ and for any $s \in \mathbb{C}$. The following series of lemmas help to avoid complicated arguments in the proof of Theorems 4.2.7 and 4.2.8.

Lemma 4.2.9. *Let* $\lambda = (n,c,\omega) \in \Lambda_{(n,0)}$. *If* $n,\ \omega \geq 0$ *and* $c > 0$, *then we have* $f_\lambda \in \widetilde{K}$.

Proof. Set $u_1 = \frac{qc-\omega}{q-p}$, and set $u_2 = \frac{pc-\omega}{q-p}$. Then, we have $f_\lambda = X_0^{n+\omega}X_1^{u_1}X_3^{u_2}$ by Lemma 4.1.5. Since we have $u_1 - u_2 \in m\mathbb{Z}_{>0}$ by the conditions $\lambda \in \Lambda_{(n,0)}$ and $c > 0$, we see that $(u_1,u_2) \in M_{l,m}^+ \setminus \{(0,0)\}$. Moreover, one has $pu_1 - qu_2 = \omega$. Therefore, f_λ can be written as $f_\lambda = X_0^n(X_0^{pu_1-qu_2}X_1^{u_1}X_3^{u_3})$, which tells us that $f_\lambda \in \widetilde{K}$. Q.E.D.

Lemma 4.2.10. *Let* $(n,c) \in \Lambda'$. *Assume that* $0 \leq n < q-p$, *and that* $c \geq 0$. *Then:*

(i) *we have* $\omega_{(0,c)}^{\min} + n < q-p$ *if and only if* $\omega_{(0,c)}^{\min} = \omega_{(n,c)}^{\min}$;

(ii) *we have* $\omega_{(0,c)}^{\min} + n \geq q-p$ *if and only if* $\omega_{(0,c)}^{\min} = \omega_{(n,c)}^{\min} + q - p$;

(iii) *we have* $\omega_{(0,c)}^{\min} \geq q-p-\beta$ *if and only if* $\omega_{(0,c+m)}^{\min} = \omega_{(0,c)}^{\min} - q + p + \beta$;

(iv) *we have* $\omega^{\min}_{(0,c)} < q-p-\beta$ *if and only if* $\omega^{\min}_{(0,c+m)} = \omega^{\min}_{(0,c)} + \beta$.

Proof. The if part is easy to check, so we prove the only if part. Note that one has $0 \leq \omega^{\min}_{(0,c)} < q-p$ by Lemma 4.1.16. Since the image of $\left(n+\omega^{\min}_{(0,c)}, \frac{qc-\omega^{\min}_{(0,c)}}{q-p}, \frac{pc-\omega^{\min}_{(0,c)}}{q-p}\right) \in \mathbb{Z}^3_{\geq 0}$ under the map μ is $(n, c, \omega^{\min}_{(0,c)})$, it follows that $(n, c, \omega^{\min}_{(0,c)}) \in \tilde{\mu}^{-1}(n,c) \cap \Lambda$. This implies that $\omega^{\min}_{(0,c)} \geq \omega^{\min}_{(n,c)}$. Concerning this relation, item (i) follows from Lemma 4.1.9. Next, we show (ii). If $n+\omega^{\min}_{(0,c)} \geq q-p$, then we have $\omega^{\min}_{(0,c)} > \omega^{\min}_{(n,c)}$ by Lemma 4.1.9. Thus, $\omega^{\min}_{(0,c)} - \omega^{\min}_{(n,c)} = x(q-p)$ holds for some $x \geq 1$ by Lemma 4.1.7. By Lemma 4.1.5, we have $\omega^{\min}_{(n,c)} + n \geq 0$, and thus $\omega^{\min}_{(n,c)} + q - p > 0$. It follows that $x = 1$, since otherwise one has $\omega^{\min}_{(0,c)} > q-p$, which contradicts to the minimality of $\omega^{\min}_{(0,c)}$. To see (iii), suppose that $\omega^{\min}_{(0,c)} \geq q-p-\beta$, and set $\omega = \omega^{\min}_{(0,c)} - q + p + \beta$. Then, $0 \leq \omega < q-p$. By a direct calculation using (3.1) (see §3.2.1), we get $\frac{q(c+m)-\omega}{q-p} = \frac{qc-\omega^{\min}_{(0,c)}}{q-p} + \alpha + m + 1$ and $\frac{p(c+m)-\omega}{q-p} = \frac{pc-\omega^{\min}_{(0,c)}}{q-p} + \alpha + 1$. Therefore, $\frac{q(c+m)-\omega}{q-p} > 0$ and $\frac{p(c+m)-\omega}{q-p} > 0$ hold, since we have $\frac{qc-\omega^{\min}_{(0,c)}}{q-p}, \frac{pc-\omega^{\min}_{(0,c)}}{q-p} \geq 0$ by Lemma 4.1.5. On the other hand, we see that the image of $\left(\omega, \frac{q(c+m)-\omega}{q-p}, \frac{p(c+m)-\omega}{q-p}\right) \in \mathbb{Z}^3_{\geq 0}$ under μ is $(0, c+m, \omega)$. It follows that $(0, c+m, \omega) \in \tilde{\mu}^{-1}(0, c+m) \cap \Lambda$. Therefore, we have $\omega \geq \omega^{\min}_{(0,c+m)}$. Since $\omega < q-p$, we see that $\omega = \omega^{\min}_{(0,c+m)}$. Finally, we show (iv). Suppose that $\omega^{\min}_{(0,c)} < q-p-\beta$, and set $\omega' = \omega^{\min}_{(0,c)} + \beta$. Then, $0 \leq \omega < q-p$. In a similar way as in the proof of (iii), we see that $(0, c+m, \omega') \in \tilde{\mu}^{-1}(0, c+m) \cap \Lambda$. Therefore, we get $\omega' = \omega^{\min}_{(0,c+m)}$. Q.E.D.

Lemma 4.2.11. *Suppose that* $r > 1$, *and let* $1 < i \leq r$. *Then,* $n_{i-1} - n_i \leq \omega^{\min}_{(0,mx)} \leq q - p - n_{i-1}$ *holds for any* $0 < x < P_i$.

Proof. By the proof of Lemma 4.1.18, we have $\omega^{\min}_{(0,mx)} = \Re[pmx, q-p]$, which coincides with $\Re[pmx + n_i, q-p] - n_i$ by Corollary 4.1.25. Also, we see that $\Re[pmx + n_i, q-p] = k\Re[t_1(P_i - x), b]$ holds by (4.9) in the proof of Proposition 4.1.20. Moreover, we have $t_{i-1} \leq \Re[t_1(P_i - x), b] \leq b + t_i - t_{i-1}$ by Corollary 4.1.24. Therefore, the lemma follows concerning $q - p = kb$, $n_i = kt_i$, and $n_{i-1} = kt_{i-1}$. Q.E.D.

Definition 4.2.12. For each $c \in m\mathbb{Z}_{>0}$, we define

$$\lambda_c := \left(q-p-\omega^{\min}_{(0,c)}, c, \omega^{\min}_{(0,c)} - q + p\right) \in \Lambda_{\left(q-p-\omega^{\min}_{(0,c)}, 0\right)},$$

which coincides with the image of $\left(0, \frac{qc-\omega^{\min}_{(0,c)}}{q-p} + 1, \frac{pc-\omega^{\min}_{(0,c)}}{q-p} + 1\right) \in \mathbb{Z}^3_{\geq 0}$ under the map μ.

By a direct calculation, we have $f_{\lambda_c} = X_1^{\frac{qc-\omega^{\min}_{(0,c)}}{q-p}+1} X_3^{\frac{pc-\omega^{\min}_{(0,c)}}{q-p}+1}$. Also, by applying Lemma 4.2.10 (ii) with $n = q-p-\omega^{\min}_{(0,c)}$, we see that $\omega^{\min}_{(0,c)} - q + p = \omega^{\min}_{\left(q-p-\omega^{\min}_{(0,c)},c\right)}$ holds. Therefore, we have $\lambda_c = \left(q-p-\omega^{\min}_{(0,c)}, c, \omega^{\min}_{(q-p-\omega^{\min}_{(0,c)},c)}\right)$.

Example 4.2.13. By Example 4.1.10, Lemmas 4.1.16 (i), and 4.2.10 (i), (ii), we have $\omega^{\min}_{(0,mP_i)} = \omega^{\min}_{(n_i,mP_i)} + q - p = -n_i + q - p$. Therefore, we see that $\lambda_{mP_i} = \left(n_i, mP_i, \omega^{\min}_{(n_i,mP_i)}\right)$ and $f_{\lambda_{mP_i}} = X_1^{e_i} X_3^{l_i}$ hold.

Lemma 4.2.14. *Let $c, c' \in m\mathbb{Z}_{>0}$. Then we have $f_{\lambda_{c'}} \in (f_{\lambda_c})$ if and only if $c' \geq c$.*

Proof. We may assume that $c' = c + m$. Then, by (3.1) (see §3.2.1) and Lemma 4.2.10, we have $f_{\lambda_{c'}} = X_1^{\alpha+m+1} X_3^{\alpha+1} f_{\lambda_c}$ if $\omega^{\min}_{(0,c)} \geq q - p - \beta$; otherwise, we have $f_{\lambda_{c'}} = X_1^{\alpha+m} X_3^{\alpha} f_{\lambda_c}$. Q.E.D.

Lemma 4.2.15. *Let $\lambda = \left(n, c, \omega^{\min}_{(n,c)}\right) \in \Lambda_{(n,0)}$. Assume that $c > 0$, and that $0 \leq n < q - p$. Then we have the following.*

(i) *If $\omega^{\min}_{(0,c)} + n < q - p$, then we have $f_\lambda \in \widetilde{K}$.*

(ii) *If $\omega^{\min}_{(0,c)} + n \geq q - p$, then we have $f_\lambda = X_0^{n+\omega^{\min}_{(n,c)}} f_{\lambda_c} = X_0^{n+\omega^{\min}_{(0,c)}-q+p} f_{\lambda_c}$.*

Proof. Item (i) follows from Lemmas 4.1.16 (i), 4.2.9, and 4.2.10 (i). Item (ii) is a consequence of Lemma 4.2.10 (ii) and the definition of λ_c. Q.E.D.

Lemma 4.2.16. *Let $0 \leq n < q - p$, and let $\lambda = \left(n, c, \omega^{\min}_{(n,c)}\right) \in \Lambda_{(n,0)}$. Then, c can be written as $c = mx$ with some $x \geq 0$ by Example 4.1.4. Under this assumption and notation, the following properties are true for any $1 \leq i \leq r$.*

(i) *If $0 < x < P_i$ and if $0 \leq n < n_{i-1}$, then we have $f_\lambda \in \widetilde{K}$.*

(ii) *If $x = P_i$ and if $0 \leq n < n_i$, then we have $f_\lambda \in \widetilde{K}$.*

(iii) *If $x = P_i$ and if $n_i \leq n < q - p$, then we have $f_\lambda \in (X_1^{e_i} X_3^{l_i})$.*

(iv) *If $x > P_i$ and if $0 \leq n < q - p$, then we have $f_\lambda \in (X_1^{e_i} X_3^{l_i}) + \widetilde{K}$.*

(v) *If* $x > P_i$ *and if* $0 \leq n < n_i$*, then we have* $f_\lambda \in \widetilde{L_1^i}$.

(vi) *Let* $x > P_i$*, and let* $n_i \leq n < n_{i-1}$.

(vi-1) *If* x *is not a multiple of* P_i*, then we have* $f_\lambda \in \widetilde{L_1^i}$.

(vi-2) *If* x *is a multiple of* P_i*, then we have* $f_\lambda - f_{\left(n, c-mP_i, \omega^{\min}_{(n,c-mP_i)}\right)} \in \widetilde{L_1^i}$.

Proof. Item (i) follows from Lemmas 4.2.11 and 4.2.15 (i). Item (ii) is a consequence of Example 4.2.13 and Lemma 4.2.15 (i). We get item (iii) by applying Lemma 4.2.15 (ii) with $c = mP_i$ and taking into account Example 4.2.13. If $x > P_i$, then we have $f_{\lambda_c} \in (f_{\lambda_{mP_i}})$ by Lemma 4.2.14. Thus, item (iv) follows from Lemma 4.2.15. Next we show (v). If $n + \omega^{\min}_{(0,c)} < q - p$, then we have $f_\lambda \in \widetilde{K}$ by Lemma 4.2.15 (i), and hence $f_\lambda \in \widetilde{L_1^i}$. Suppose that $n + \omega^{\min}_{(0,c)} \geq q - p$, and set $n' = n + \omega^{\min}_{(0,c)} - q + p$. Then, we have $f_\lambda = X_0^{n'} f_{\lambda_c}$. Also, by Lemma 4.2.14, we see that f_{λ_c} can be written as $f_{\lambda_c} = f_{\lambda_{mP_i}} f = X_1^{e_i} X_3^{l_i} f$ with some $f \in R^{c'}_{n-n'-n_i}$, where we set $c' = c - mP_i$. Therefore, we can write f_λ as $f_\lambda = X_0^{n'+n_i} f - X_0^{n'} f (X_0^{n_i} - X_1^{e_i} X_3^{l_i})$. Since $X_0^{n'+n_i} f \in R^{c'}_n$, we have $X_0^{n'+n_i} f = f_{\lambda'}$ with some $\lambda' = (n, c', \omega') \in \Lambda_{(n,0)}$: we have $f_\lambda = f_{\lambda'} - X_0^{n'} f (X_0^{n_i} - X_1^{e_i} X_3^{l_i})$. Therefore, it suffices to show that $f_{\lambda'} \in \widetilde{L_1^i}$. If $\omega' > \omega^{\min}_{(n,c')}$, then we have $f_{\lambda'} \in (X_0^{n_{i-1}})$ by Lemma 4.1.11. Hence we get $f_\lambda \in \widetilde{L_1^i}$. Now we are left to consider the case where $\omega' = \omega^{\min}_{(n,c')}$. If $0 < c' \leq mP_i$, then we have $f_{\lambda'} \in \widetilde{K}$ by (i) and (ii), and hence $f_\lambda \in \widetilde{L_1^i}$. Suppose that $c' > mP_i$. Then, by applying the above discussion to $f_{\lambda'}$ and continuing in this way, one finally obtains $f_\lambda \in \widetilde{L_1^i}$. (vi) is a consequence of the proof of (v). Q.E.D.

Lemma 4.2.17. *Let* $\lambda = \left(n, c, \omega^{\min}_{(n,c)}\right) \in \Lambda_{(n,0)}$ *with* $c = mx$*. Suppose that* $P_j < x < P_i$*,* $n_j \leq n < n_{j-1}$*, and* $n - n_j < n_{i-1}$ *hold for some* $1 \leq j < i \leq r+1$*. Then, we have* $f_\lambda = X_0^{n-n_j} f_{\lambda_{mP_j}} f_{\lambda'}$*, where* $\lambda' = \left(0, c - mP_j, \omega^{\min}_{(0,c-mP_j)}\right) \in \Lambda_{(0,0)}$*. In particular,* $f_\lambda \in \widetilde{K}$.

Proof. Set $\lambda'' = (n - n_j, 0, 0)$. Then, we have $\lambda'' + \lambda_{mP_j} + \lambda' = \left(n, c, \omega^{\min}_{(0,c-mP_j)} - n_j\right)$ and $f_{\lambda''} = X_0^{n-n_j}$. Since $0 < x - P_j < P_i$, we have $n + \omega^{\min}_{(0,c-mP_j)} - n_j < q - p$ by Lemma 4.2.11, and hence $\omega^{\min}_{(0,c-mP_j)} - n_j = \omega^{\min}_{(n,c)}$ by Lemma 4.1.9. It follows that $\lambda'' + \lambda_{mP_j} + \lambda' = \lambda$, and thus we get $f_\lambda = X_0^{n-n_j} f_{\lambda_{mP_j}} f_{\lambda'}$. Taking Remark 4.1.2.2 and the definition of $\widetilde{K}$ into account, we see that $f_{\lambda'} \in \widetilde{K}$. Q.E.D.

Proof of Theorem 4.2.7. Set $L = \widetilde{L_0^i}$. In view of the proof of Theorem 4.2.3, it suffices to show that $\dim V_{n,d} \leq 1$ holds if $0 \leq n < q-p$ and if $d = 0$, where we set $V_{n,d} = \bigoplus_{\lambda=\left(n,c,\omega^{\min}_{(n,c)}\right)} R_\lambda/(L \cap R_\lambda)$. Let $\lambda = \left(n, c, \omega^{\min}_{(n,c)}\right) \in \Lambda_{(n,0)}$. Note that we have $c = mx$ for some $x \geq 0$ by Example 4.1.4.

Case 1. Let $0 \leq n < n_{i-1}$. By Lemma 4.2.16 (i), (ii), (ii), (iv), we see that $f_\lambda \in L$ holds if $x > 0$. This implies $\dim V_{n,0} \leq 1$.

Case 2. Let $n_{i-1} \leq n < q-p$. By Lemma 3.2.5, there is a unique integer j_1 that satisfies $1 \leq j_1 \leq i-1$ and $n_{j_1} \leq n < n_{j_1-1}$. If $n - n_{j_1} \geq n_{i-1}$, then we can take an integer j_2 uniquely to satisfy $1 \leq j_2 \leq i-1$ and $n_{j_2} \leq n - n_{j_1} < n_{j_2-1}$. By continuing in this way, we finally get $n - (n_{j_1} + n_{j_2} + \cdots + n_{j_{u_n-1}} + n_{j_{u_n}}) < n_{i-1}$ for some $1 \leq j_1, j_2, \ldots, j_{u_n} \leq i-1$. Namely, we have

$$\begin{cases} n_{j_1} \leq n < n_{j_1-1} \\ n - n_{j_1} \geq n_{i-1} \end{cases}$$
$$\begin{cases} n_{j_2} \leq n - n_{j_1} < n_{j_2-1} \\ n - (n_{j_1} + n_{j_2}) \geq n_{i-1} \end{cases}$$
$$\cdots\cdots$$
$$\begin{cases} n_{j_{u_n-1}} \leq n - (n_{j_1} + \cdots + n_{j_{u_n-2}}) < n_{j_{u_n-1}-1} \\ n - (n_{j_1} + \cdots + n_{j_{u_n-2}} + n_{j_{u_n-1}}) \geq n_{i-1} \end{cases}$$
$$\begin{cases} n_{j_{u_n}} \leq n - (n_{j_1} + \cdots + n_{j_{u_n-1}}) < n_{j_{u_n}-1} \\ n - (n_{j_1} + \cdots + n_{j_{u_n-1}} + n_{j_{u_n}}) < n_{i-1} \end{cases}$$

In the following, we show $\dim V_{n,0} \leq 1$ by induction on u_n. Set $u = u_n$, and set $P = P_{j_1} + \cdots + P_{j_u}$. First suppose that $u = 1$. Since $j_1 < i$, we have $P < P_i$. We show that $f_\lambda \in L$ holds if $x \neq P$. If $x = 0$, then $f_\lambda = X_0^n$ by Example 4.1.4. Therefore, $f_\lambda \in L$. If $0 < x < P$, then we have $f_\lambda \in \widetilde{K}$ by applying Lemma 4.2.16 (i) with $i = j_1$. If $P < x < P_i$, then by applying Lemma 4.2.17 with $j = j_1$ we see that $f_\lambda \in \widetilde{K}$ holds. If $x \geq P_i$, then we have $f_\lambda \in (X_1^{e_i} X_3^{l_i}) + \widetilde{K}$ by Lemma 4.2.16 (iii), (iv). Therefore, we see that $\dim V_{n,0} \leq 1$ holds if $u = 1$. Next suppose that $u > 1$. If $x = 0$, then we have $f_\lambda \in (X_0^{n_{i-1}})$. Also, we see as above that $f_\lambda \in \widetilde{K}$ holds if $0 < x < P_{j_1}$. Let $x > P_{j_1}$, and set $P' = P - P_{j_1}$, set $n' = n - n_{j_1}$, set $c' = c - mP_{j_1}$, and set $\lambda' = \left(n', c', \omega^{\min}_{(n',c')}\right)$. Since we have $\omega^{\min}_{(n_{j_1}, mP_{j_1})} + \omega^{\min}_{(n',c')} + n_{j_1} + n' = \omega^{\min}_{(n',c')} + n' < q - p$ by Example 4.1.10, it follows from Lemma 4.1.16 that $\omega^{\min}_{(n,c)} = \omega^{\min}_{(n_{j_1}, mP_{j_1})} + \omega^{\min}_{(n',c')}$. Thus we get $\lambda = \lambda_{mP_{j_1}} + \lambda'$, and hence $f_\lambda = f_{\lambda_{mP_{j_1}}} f_{\lambda'}$ by Lemma 4.1.5. Since we have $u_{n'} = u - 1$, it follows from the induction hypothesis and the relation $f_\lambda = f_{\lambda_{mP_{j_1}}} f_{\lambda'}$ that $\dim V_{n,0} \leq 1$ holds. Q.E.D.

Remark 4.2.17.1. Let $0 \leq n < q-p$, and let $\lambda = (n, c, \omega) \in \Lambda_{(n,0)}$. By the proof of Theorem 4.2.7, we deduce the following.

- Let $0 \leq n \leq n_{i-1}$. Then, we have $f_\lambda \in \widetilde{L_0^i}$ if $\lambda \neq \left(n, 0, \omega_{(n,0)}^{\min}\right)$.
- Let $n_{i-1} \leq n < q-p$. Then, we have $f_\lambda \in \widetilde{L_0^i}$ if $\lambda \neq \left(n, mP, \omega_{(n,mP)}^{\min}\right)$, where $P = P_{j_1} + \cdots + P_{j_u}$ as in the proof of Theorem 4.2.7.

Proof of Theorem 4.2.8. Set $L' = \widetilde{L_1^i}$, and set $V'_{n,d} = \bigoplus_{\lambda=\left(n,c,\omega_{(n,c)}^{\min}\right)\in\Lambda_{(n,d)}} R_\lambda/(L' \cap R_\lambda)$. As in the proof of Theorem 4.2.7, it suffices to show that $\dim V'_{n,d} \leq 1$ holds if $0 \leq n < q-p$ and if $d = 0$. Let $\lambda = \left(n, c, \omega_{(n,c)}^{\min}\right) \in \Lambda_{(n,0)}$.

Case 1. Let $0 \leq n < n_i$. By Lemma 4.2.16 (i), (ii), (v), we see that $f_\lambda \in L'$ holds if $c > 0$. Therefore, we get $\dim V'_{n,0} \leq 1$.

Case 2. Let $n_i \leq n < n_{i-1}$. By Lemma 4.2.16 (i), we have $f_\lambda \in L'$ if $0 < x < P_i$. If $x \geq P_i$, then taking $f_{\left(n,mP_i,\omega_{(n,mP_i)}^{\min}\right)} - f_{\left(n,0,\omega_{(n,0)}^{\min}\right)} = X_0^{n-n_i}(X_1^{e_i}X_3^{l_i} - X_0^{n_i}) \in L'$ into account, we deduce from Lemma 4.2.16 (vi) that either of the following holds: $f_\lambda \in L'$; or $f_\lambda - f_{\left(n,c-mP_i,\omega_{(n,c-mP_i)}^{\min}\right)} \in L'$. This implies that $\dim V'_{n,0} \leq 1$.

Case 3. Let $n_{i-1} \leq n < q-p$. We follow similar lines to Case 2 of the proof of Theorem 4.2.7: we define u and P in the same way and proceed by induction on u. Let $u = 1$. In a similar way, we see that $f_\lambda \in L'$ holds if $0 \leq c < mP$. Let $c \geq mP$. Then we can write $f_\lambda = f_{\lambda'} f_{\lambda_{mP}}$, where $\lambda' = \left(n - n_{j_1}, c - mP, \omega_{(n-n_{j_1},c-mP)}^{\min}\right)$. If $0 \leq n - n_{j_1} < n_i$, then we have $f_{\lambda'} \in L'$ by Lemma 4.2.16 (i), (ii), (v), which tells us that $\dim V'_{n,0} \leq 1$ holds in this case. If $n_i \leq n - n_{j_1} < n_{i-1}$, then we can show that $\dim V'_{n,0} \leq 1$ is true by following a similar argument to the one in Case 2. Q.E.D.

Remark 4.2.17.2. Let us define $F_j = f_{\left(0,mj,\omega_{(0,mj)}^{\min}\right)}$ for each $1 \leq j \leq b-1$. We claim that L_0^i coincides with $(X_0^{n_{i-1}}, X_2, X_4, X_1^{e_i}X_3^{l_i}, F_1, \ldots, F_{b-1})$ for any $1 \leq i \leq r$. To see this, let $(u_1, u_2) \in M_{l,m}^+ \setminus \{(0,0)\}$, and set $c = u_1 - u_2$, and set $\omega = pu_1 - qu_2$. Then, we have $X_0^{pu_1-qu_2}X_1^{u_1}X_3^{u_2} = f_{(0,c,\omega)}$. Also, we can write $c = mx$ with some $x > 0$ by the definition of $M_{l,m}^+$. Concerning the definition of L_0^i, it suffices to check that $f_{(0,c,\omega)}$ is contained in the ideal $(X_0^{n_{i-1}}, X_1^{e_i}X_3^{l_i}, F_1, \ldots, F_{b-1}) \subset R$. If $\omega > \omega_{(0,c)}^{\min}$, then we get $f_{(0,c,\omega)} \in (X_0^{n_{i-1}})$ by Lemma 4.1.11. Suppose that $\omega = \omega_{(0,c)}^{\min}$. If $x = b$, then we have $f_{(0,c,\omega)} = X_1^{aq}X_3^{ap} = X_1^{e_{r+1}}X_3^{l_{r+1}} \in (X_1^{e_i}X_3^{l_i})$. If

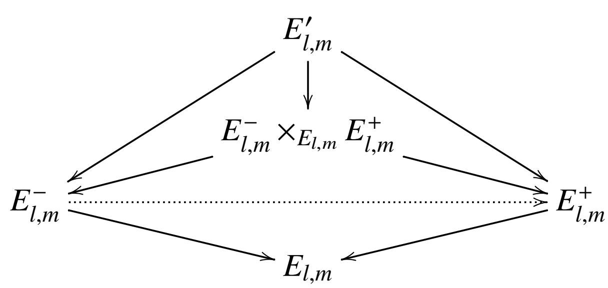
$x > b$, then we see that $f_{(0,c,\omega)} \in (X_1^{aq} X_3^{ap})$ holds. Therefore, the two ideals coincide. In a similar manner, we see that J_0 coincides with $(X_0^{n_r},\ X_2,\ X_4,\ X_1^{e_{r+1}} X_3^{l_{r+1}},\ F_1,\ \ldots,\ F_{b-1})$.

Chapter 5

Invariant Hilbert schemes and resolutions of singularities of affine normal quasihomogeneous $SL(2)$-varieties III: proof of the main results

5.1 Morphism to the fiber product

Let us consider the diagonal $SL(2)\times\mathbb{C}^*$-action on the fiber product $E^-_{l,m}\times_{E_{l,m}} E^+_{l,m}$. Then we have the following equivariant commutative diagram:



In this section, we construct a morphism from the main component $\mathscr{H}^{main} = \overline{\gamma^{-1}(\mathfrak{U})}$ to $E^-_{l,m}\times_{E_{l,m}} E^+_{l,m}$, which is in equivariant bijection with the weighted blow-up $E'_{l,m}$. We have seen in Lemma 4.1.1 that $F_{-p,-1} = \langle X_1, X_2\rangle$ (resp. $F_{q,1} = \langle X_3, X_4\rangle$) generates $S_{(-p,-1)}$ (resp. $S_{(q,1)}$) as a module over the invariant ring $S^{G_0\times G_m}$. Therefore, taking §2.1.6 into account, we can construct the following equivariant morphisms:

$$\eta_{-p,-1} : \mathcal{H} \longrightarrow \mathrm{Gr}(h(-p,-1), F^{\vee}_{-p,-1}) \cong \mathbb{P}^1, \quad \eta_{q,1} : \mathcal{H} \longrightarrow \mathrm{Gr}(h(q,1), F^{\vee}_{q,1}) \cong \mathbb{P}^1,$$

where the isomorphisms $\mathrm{Gr}(h(-p,-1), F^{\vee}_{-p,-1}) \cong \mathbb{P}^1$ and $\mathrm{Gr}(h(q,1), F^{\vee}_{q,1}) \cong \mathbb{P}^1$ are given by $\langle x_1 X_1^{\vee} + x_2 X_2^{\vee} \rangle \mapsto [x_1 : x_2]$ and $\langle x_3 X_3^{\vee} + x_4 X_4^{\vee} \rangle \mapsto [x_3 : x_4]$, respectively. Since X_2, $X_3 \in I_1$, it follows from Lemma 4.1.1 that $(S/I_1)_{(-p,-1)} \cong \langle X_1 \rangle$ and $(S/I_1)_{(q,1)} \cong \langle X_4 \rangle$. Therefore, we have $\eta_{-p,-1}([I_1]) = [1 : 0]$ and $\eta_{q,1}([I_1]) = [0 : 1]$. Set

$$\psi := \gamma \times \eta_{-p,-1} \times \eta_{q,1} : \mathcal{H} \longrightarrow E_{l,m} \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Since $\gamma([I_1]) = \pi(x)$ by Corollary 4.2.5, we see that $\psi([I_1]) = (\pi(x), [1 : 0], [0 : 1])$. Similarly, we have $\psi([J_1]) = (\pi(x'), [1 : 0], [1 : 0])$. In what follows, we show that the image of $\mathcal{H}^{main}$ under ψ is isomorphic to $E^-_{l,m} \times_{E_{l,m}} E^+_{l,m}$.

Lemma 5.1.1. *The equivariant morphism $E'_{l,m} \longrightarrow E^-_{l,m} \times_{E_{l,m}} E^+_{l,m}$ induced by the universal property of the fiber product is bijective and birational.*

Proof. It follows from the description of the surjective morphisms $E'_{l,m} \longrightarrow E^-_{l,m}$ and $E'_{l,m} \longrightarrow E^+_{l,m}$ given in Remark 2.3.9.2 that $E'_{l,m} \longrightarrow E^-_{l,m} \times_{E_{l,m}} E^+_{l,m}$ is bijective. Therefore, the fiber product $E^-_{l,m} \times_{E_{l,m}} E^+_{l,m}$ is irreducible. Since $E^-_{l,m} \times_{E_{l,m}} E^+_{l,m}$ contains an open orbit isomorphic to $\mathfrak{U}$, we deduce that $E'_{l,m} \longrightarrow E^-_{l,m} \times_{E_{l,m}} E^+_{l,m}$ is birational. Q.E.D.

Lemma 5.1.2. *There are $SL(2) \times \mathbb{C}^*$-equivariant embeddings:*

$$E^+_{l,m} \hookrightarrow E_{l,m} \times \mathrm{Gr}(h(-p,-1), F^{\vee}_{-p,-1}) \cong E_{l,m} \times \mathbb{P}^1;$$
$$E^-_{l,m} \hookrightarrow E_{l,m} \times \mathrm{Gr}(h(q,1), F^{\vee}_{q,1}) \cong E_{l,m} \times \mathbb{P}^1.$$

Proof. We have the following equivariant morphism (this morphism was first constructed in the proof of [BH08, Theorem 3.10]):

$$U^+ \longrightarrow \mathrm{Gr}(h(-p,-1), F^{\vee}_{-p,-1}) \cong \mathbb{P}^1, \quad (Y_0, X_1, X_2, X_3, X_4) \mapsto [X_1 : X_2].$$

Also, we have an equivariant morphism $U^+ \longrightarrow E_{l,m}$ as a composition of the inclusion $U^+ \hookrightarrow H_b$ and the quotient morphism $H_b \longrightarrow E_{l,m}$. Therefore, we get a $G'_0 \times G_a$-invariant morphism $U^+ \longrightarrow E_{l,m} \times \mathbb{P}^1$, which factors through $E^+_{l,m}$ by the universal property of the categorical quotient:

$$\begin{array}{ccc} U^+ & \longrightarrow & E_{l,m} \times \mathbb{P}^1 \\ \downarrow & \nearrow_{\alpha^+} & \\ U^+ /\!/ (G'_0 \times G_a) = E^+_{l,m} & & \end{array}$$

Let $[T_1 : T_2]$ be the coordinate of $\mathrm{Gr}(h(-p,-1), F^\vee_{-p,-1}) \cong \mathbb{P}^1$. Then, for each $i \in \{1,2\}$, we have the following commutative diagram:

$$\begin{array}{ccc} U^+ \cap \{X_i \neq 0\} = H_b \cap \{X_i \neq 0\} & \longrightarrow & \mathrm{Spec}\left(\mathbb{C}[E_{l,m}]\left[\frac{T_1}{T_i}, \frac{T_2}{T_i}\right]\right) \\ \downarrow & \nearrow_{\alpha^+|_{\{X_i \neq 0\}}} & \\ (H_b \cap \{X_i \neq 0\}) /\!/ (G'_0 \times G_a) & & \end{array}$$

We see that $(\mathbb{C}[H_b]_{X_i})^{G'_0 \times G_a} = \mathbb{C}[H_b]^{G'_0 \times G_a}\left[\frac{X_1}{X_i}, \frac{X_2}{X_i}\right]$ holds as a subring of $\mathbb{C}[H_b]_{X_i}$. Therefore, α^+ is a closed immersion. Analogously, we have an equivariant morphism $U^- \longrightarrow \mathrm{Gr}(h(q,1), F^\vee_{q,1}) \cong \mathbb{P}^1$, $(Y_0, X_1, X_2, X_3, X_4) \mapsto [X_3 : X_4]$, which induces an equivariant morphism $\alpha^- : E^-_{l,m} \longrightarrow E_{l,m} \times \mathbb{P}^1$. In a similar way, we see that α^- is a closed immersion. Q.E.D.

By Lemmas 5.1.1 and 5.1.2, we get an equivariant closed embedding followed by an equivariant bijection:

$$\varphi : E'_{l,m} \longrightarrow E^-_{l,m} \times_{E_{l,m}} E^+_{l,m} \hookrightarrow E_{l,m} \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Corollary 5.1.3. *We have* $\psi(\mathcal{H}^{main}) = \varphi(E'_{l,m}) \cong E^-_{l,m} \times_{E_{l,m}} E^+_{l,m}$.

Proof. Since $\gamma^{-1}(\mathfrak{U})$ is the $SL(2) \times \mathbb{C}^*$-orbit of $[I_1]$, and since $\psi(\mathcal{H}^{main}) = \overline{\psi(\gamma^{-1}(\mathfrak{U}))}$, we see that $\psi(\mathcal{H}^{main})$ is the $SL(2) \times \mathbb{C}^*$-orbit closure of $\psi([I_1])$. On the other hand, consider the blow-up morphism $f : E'_{l,m} \longrightarrow E_{l,m}$, and let $y \in E'_{l,m}$ be a point such that $f(y) = \pi(x)$. Then, it follows from the construction of φ that $\varphi(y) = \psi([I_1])$. Therefore, one obtains $\psi(\mathcal{H}^{main}) = \varphi(E'_{l,m})$, which is isomorphic to $E^-_{l,m} \times_{E_{l,m}} E^+_{l,m}$. Q.E.D.

In view of Remark 2.3.9.2, $E'_{l,m}$ contains four orbits $\mathfrak{U}$, $\mathfrak{D}$, C, and C', and they are described as follows under $E'_{l,m} \longrightarrow \varphi(E'_{l,m}) \subset E_{l,m} \times \mathbb{P}^1 \times \mathbb{P}^1$:

$$\mathfrak{U} \cong \varphi(\mathfrak{U}) = (SL(2)\times\mathbb{C}^*)\cdot(\pi(x),[1:0],[0:1]),$$
$$\mathfrak{D} \cong \varphi(\mathfrak{D}) = (SL(2)\times\mathbb{C}^*)\cdot(\pi(x'),[1:0],[1:0]),$$
$$\varphi(C) = (SL(2)\times\mathbb{C}^*)\cdot(O,[1:0],[1:0]),$$
$$\varphi(C') = (SL(2)\times\mathbb{C}^*)\cdot(O,[1:0],[0:1]).$$

Lemma 5.1.4. *$\psi|_{\mathcal{H}^{main}} : \mathcal{H}^{main} \longrightarrow \varphi(E'_{l,m})$ is bijective outside $\varphi(C)$.*

Proof. Since $\psi|_{\mathcal{H}^{main}}$ is an isomorphism over $\mathfrak{U}\cup\mathfrak{D}$, it suffices to show the bijectivity of $\psi|_{\mathcal{H}^{main}}$ over $\varphi(C')$. By the construction of ψ, we see that the set-theoretical fiber of $(O,[1:0],[0:1])$ consists of closed points $[I] \in \mathcal{H}^{main}$ that satisfies $X_2, X_3 \in I$ and $\gamma([I]) = O$. In view of the proof of Theorem 4.2.2, we have $I_0 \subset I$. Since A/I_0 has Hilbert function h by Theorem 4.2.2, it follows that $I = I_0$. Therefore, $\psi|_{\mathcal{H}^{main}}$ is bijective over $\varphi(C')$. Q.E.D.

5.2 Morphism to the minimal resolution

The goal of this section is to construct an equivariant morphism $\Psi : \mathcal{H}^{main} \longrightarrow \widetilde{E'_{l,m}}$ in two steps. First, we realize $\widetilde{E'_{l,m}}$ as a closed subscheme of a projective space $E_{l,m}\times\mathbb{P}(V^\vee)$ over $E_{l,m}$. Next, we construct a morphism $\Psi : \mathcal{H} \longrightarrow E_{l,m}\times\mathbb{P}(V^\vee)$ and show that the image $\Psi(\mathcal{H}^{main})$ is isomorphic to $\widetilde{E'_{l,m}}$. In the next section, we will see that $\Psi|_{\mathcal{H}^{main}}$ is an isomorphism, which completes the proof of Theorem 3.2.4.

5.2.1 Equivariant embedding of the minimal resolution

In this subsection, we construct an equivariant morphism $\widetilde{E'_{l,m}} \longrightarrow \mathbb{P}(V^\vee)$ defined by a base-point-free $V \subset \Gamma(\widetilde{E'_{l,m}}, \mathcal{O}(\delta))$, where δ is an $SL(2)\times\mathbb{C}^*$-stable Cartier divisor on $\widetilde{E'_{l,m}}$, and show that the natural morphism $\Phi : \widetilde{E'_{l,m}} \longrightarrow E_{l,m}\times\mathbb{P}(V^\vee)$ is a closed immersion (Proposition 5.2.7). In below, we use notations introduced in §2.2 and in §3.2. Let D_i $(0 \le i \le r+1)$ be an $SL(2)\times\mathbb{C}^*$-stable prime divisor on $\widetilde{E'_{l,m}}$ corresponding to the extremal ray $\mathbb{Q}_{\ge 0}\rho_i$. Note that we have $D_0 = D'$ and $D_{r+1} = D$ (see Remark 3.2.4.1 (i)). Then, the set of $\widetilde{B}$-stable prime divisors on $\widetilde{E'_{l,m}}$ is given as

$$\mathcal{D}(\widetilde{E'_{l,m}}) = \{D_0,\ \ldots,\ D_{r+1},\ \widetilde{S^+},\ \widetilde{S^-}\},$$

where $\widetilde{S^+}$ (resp. $\widetilde{S^-}$) is a non-$SL(2)\times\mathbb{C}^*$-stable prime divisor on $\widetilde{E'_{l,m}}$ such that its image under the resolution of singularities $\widetilde{E'_{l,m}} \longrightarrow E_{l,m}$ is the $\widetilde{B}$-stable divisor S^+ (resp. S^-) on $E_{l,m}$. By definition, we have $v_{D_i}(f) = \rho_{v_{D_i}}(\chi_f) = \rho_i(\chi_f)$ for any $f \in \mathbb{C}(\mathfrak{U})^{\widetilde{B}} \subset \mathbb{C}(X,Y,Z,W)$. For each $0 \le i \le r+1$, we define $\sigma_i,\ f_i \in \mathbb{C}(\mathfrak{U})^{\widetilde{B}}$ to be

$$\sigma_i := Z^{e_i}W^{l_i} = (ZW)^{(\alpha+1)P_i-Q_i}(Z^m)^{P_i}, \quad f_i := \prod_{0\le j\le i} \sigma_j,$$

where $e_i = (\alpha+1+m)P_i - Q_i$ and $l_i = (\alpha+1)P_i - Q_i$ as defined in §3.2.2.

Lemma 5.2.2. *With the preceding notation, the following properties are true.*

(i) *Let* $0 \le i,\ j \le r+1$. *Then we have:* $v_{D_j}(\sigma_i) > 0$ *if* $i > j$*;* $v_{D_j}(\sigma_i) = 0$ *if* $i = j$*; and* $v_{D_j}(\sigma_i) < 0$ *if* $i < j$*. In particular, we have* $v_{D_j}(\sigma_{j+1}) = 1$ *and* $v_{D_j}(\sigma_{j-1}) = -1$.

(ii) *We have* $v_{D_i}(f_i) = v_{D_i}(f_{i-1})$.

Proof. A direct calculation shows $v_{D_j}(\sigma_i) = \rho_j(\chi_{\sigma_i}) = -P_j\{(\alpha+1)P_i - Q_i\} + \{(\alpha+1)P_j - Q_j\}P_i = P_jQ_i - P_iQ_j$. Therefore, we get (i) by Theorem 3.2.3. Item (ii) follows from the definition of f_i and (i). Q.E.D.

Let $\widetilde{E}_i$ $(0 \le i \le r)$ be the simple spherical open subvariety of $\widetilde{E'_{l,m}}$ corresponding to the colored cone $(\mathscr{C}_i,\ \emptyset)$, and let Y_i be the unique closed orbit of $\widetilde{E}_i$. Then we have

$$\mathscr{D}(\widetilde{E}_i) = \{D_i|_{\widetilde{E}_i},\ D_{i+1}|_{\widetilde{E}_i},\ \widetilde{S^+}|_{\widetilde{E}_i},\ \widetilde{S^-}|_{\widetilde{E}_i}\}, \quad \mathscr{D}_{Y_i}(\widetilde{E}_i) = \{D_i|_{\widetilde{E}_i},\ D_{i+1}|_{\widetilde{E}_i}\}.$$

Moreover, let us consider the following $SL(2)\times\mathbb{C}^*$-stable divisor on $\widetilde{E'_{l,m}}$:

$$\delta := \sum_{1\le i\le r+1} v_{D_i}(f_i^{-1})D_i.$$

Though the Cartierness of δ follows immediately from the smoothness of $\widetilde{E'_{l,m}}$, we check the criterion for a Weil divisor to be Cartier given in Theorem 2.2.11 as a preparation for the proof of Lemma 5.2.3: with the notation used in Theorem 2.2.11, we see by Lemma 5.2.2 (ii) and $v_{D_0}(f_0^{-1}) = 0$ that $f_{Y_i} = f_i^{-1}$ $(0 \le i \le r)$ satisfy the required condition.

Lemma 5.2.3. *The Cartier divisor δ is generated by global sections.*

Proof. Taking Theorem 2.2.13 and the fact that the cone $\mathscr{C}_j$ is spanned by ρ_j and ρ_{j+1} into account, it is enough to show the following two conditions:

(C1) $v_{D_j}(f_{Y_i}) \leq v_{D_j}(f_{Y_j})$ and $v_{D_{j+1}}(f_{Y_i}) \leq v_{D_{j+1}}(f_{Y_j})$ hold for any $0 \leq i,\ j \leq r$; and

(C2) $v_{\widetilde{S^+}}(f_{Y_i}) \leq 0$ and $v_{\widetilde{S^-}}(f_{Y_i}) \leq 0$ hold for any $0 \leq i \leq r$.

Condition (C1) follows from Lemma 5.2.2. Moreover, by a direct calculation, we have $v_{\widetilde{S^+}}(f_i) = \sum_{0\leq j\leq i} e_j \geq 0$ and $v_{\widetilde{S^-}}(f_i) = \sum_{0\leq j\leq i} l_j \geq 0$. This shows (C2). Q.E.D.

Remark 5.2.3.1. Since δ is $SL(2)\times\mathbb{C}^*$-stable, there is a linearization of the action of $SL(2)\times\mathbb{C}^*$ with respect to the line bundle $\mathscr{O}(\delta)$ such that the induced action on $\Gamma(\widetilde{E'_{l,m}}, \mathscr{O}(\delta))$ coincides with that on the function field $\mathbb{C}(\widetilde{E'_{l,m}})$ (see [ADHL15]).

We denote by $V(n)$ the irreducible $SL(2)$-representation of highest weight n. Set

$$V := \langle (SL(2)\times\mathbb{C}^*)\cdot f_i \ :\ 1 \leq i \leq r\rangle.$$

Then, we see that V is isomorphic to $\bigoplus_{1\leq i\leq r} V(e_0+e_1+\cdots+e_i)\otimes V(l_0+l_1+\cdots+l_i)$. Also, we can take the following as a basis of V:

$$\mathscr{A} := \left\{ X^{e_0+e_1+\cdots+e_i-e} Z^e Y^{l_0+l_1+\cdots+l_i-l} W^l \in \mathbb{C}(\mathfrak{U}) \ :\ \begin{array}{c} 1 \leq i \leq r; \\ 0 \leq e \leq e_0+e_1+\cdots+e_i; \\ 0 \leq l \leq l_0+l_1+\cdots+l_i \end{array} \right\}.$$

Lemma 5.2.4. *The vector space V is an $SL(2)\times\mathbb{C}^*$-submodule of* $\Gamma(\widetilde{E'_{l,m}},\mathscr{O}(\delta))$.

Proof. Let $1 \leq i \leq r$. For any $0 \leq j \leq r$, we have:

$$\begin{aligned} \mathrm{div}(f_i)|_{\widetilde{E_j}} &= v_{\widetilde{S^+}}(f_i)\widetilde{S^+}|_{\widetilde{E_j}} + v_{\widetilde{S^-}}(f_i)\widetilde{S^-}|_{\widetilde{E_j}} + v_{D_j}(f_i)D_j|_{\widetilde{E_j}} + v_{D_{j+1}}(f_i)D_{j+1}|_{\widetilde{E_j}}; \\ \delta|_{\widetilde{E_j}} &= v_{D_j}(f_{Y_j})D_j|_{\widetilde{E_j}} + v_{D_{j+1}}(f_{Y_{j+1}})D_{j+1}|_{\widetilde{E_j}}. \end{aligned}$$

Therefore, we get $\mathrm{div}(f_i)|_{\widetilde{E_j}} + \delta|_{\widetilde{E_j}} \geq 0$ by comparing each coefficient using the condition (C1) in the proof of Lemma 5.2.3. This shows $f_i \in \Gamma(\widetilde{E'_{l,m}},\mathscr{O}(\delta))$. Q.E.D.

As a consequence, one obtains a natural equivariant morphism

$$\Phi : \widetilde{E'_{l,m}} \longrightarrow E_{l,m}\times\mathbb{P}(V^\vee).$$

We show that Φ is a closed immersion. Recall that $\widetilde{E'_{l,m}}$ is covered by simple open subembeddings $\widetilde{E_0}, \ldots, \widetilde{E_r}$, and that we have $\widetilde{E_i} = (SL(2)\times\mathbb{C}^*)(\widetilde{E_i})_0$, where

$$(\widetilde{E_i})_0 = \widetilde{E_i} \setminus \bigcup_{D\in\mathscr{D}(\widetilde{E_i})\setminus\mathscr{D}_{Y_i}(\widetilde{E_i})} D = \widetilde{E_i} \setminus (\widetilde{S^+}|_{\widetilde{E_i}} \cup \widetilde{S^-}|_{\widetilde{E_i}})$$

with the notation of §2.2. Also, we have $(\widetilde{E_i})_1 = \mathfrak{U} \cap \{ZW \neq 0\}$, $(E_{l,m})_0 = E_{l,m}$, and $(E_{l,m})_1 = \mathfrak{U}$. Therefore, we get the following by Remark 2.2.2.1:

$$\begin{aligned}
\mathbb{C}[(\widetilde{E_i})_0] &= \left\{F \in \mathbb{C}[\mathfrak{U}]_{ZW} : v_{D_{i-1}}(F) \geq 0,\ v_{D_i}(F) \geq 0\right\};\\
\mathbb{C}[E_{l,m}] &= \left\{F \in \mathbb{C}[\mathfrak{U}] : v_{D_{r+1}}(F) \geq 0\right\}.
\end{aligned}$$

Let L be the subring of $\mathbb{C}(\mathfrak{U})$ defined as $L := \{F \in \mathbb{C}[\mathfrak{U}]_{ZW} : v_{D_{r+1}}(F) \geq 0\}$. For each $0 \leq i \leq r$, we consider an open subset $U_i := \operatorname{Spec}\left(L\left[f^\vee/f_i^\vee : f \in \mathscr{A}\right]\right)$ of $E_{l,m}\times\mathbb{P}(V^\vee)$, where $f^\vee$ denotes the dual basis of f. We also consider a homomorphism

$$\Phi_i^{\#} : L\left[f^\vee/f_i^\vee : f \in \mathscr{A}\right] \longrightarrow \mathbb{C}[(\widetilde{E_i})_0] \qquad (0 \leq i \leq r)$$

defined by sending $F\frac{f^\vee}{f_i^\vee}$, where $F \in L$, to $F\frac{f}{f_i}$.

Lemma 5.2.5. *The homomorphism $\Phi_i^{\#}$ is well-defined for any $0 \leq i \leq r$.*

Proof. Let $F \in L$. We may assume that F is of the form $F = \frac{Z^{d_z}W^{d_w}}{(ZW)^d} \in L$, where $d_z, d_w, d \in \mathbb{Z}_{\geq 0}$. Moreover, since F is G_m-invariant, we have $d_z - d_w = cm$ for some $c \in \mathbb{Z}$. Therefore, F can be written as $(ZW)^{d_w-d}(Z^m)^c$. Since $v_{D_{r+1}}(F) \geq 0$, we get $\frac{(\alpha+1)b-t}{b}c \geq d_w - d$ by a direct calculation. This implies that $c \geq 0$, since otherwise we get $c < 0$ and $d_w - d < 0$, which contradicts to $F \in L$. Therefore, we have

$$\begin{aligned}
v_{D_j}(F) &= -P_j(d_w - d) + \{(\alpha+1)P_j - Q_j\}c\\
&\geq \left(\frac{b}{t}P_j - Q_j\right)c = \frac{1}{b}(tP_j - bQ_j)c = \frac{n_j}{bk}c = \frac{n_j}{q-p}c \geq 0 \quad (0 \leq \forall j \leq r+1)
\end{aligned}$$

by Lemma 3.2.5, and this shows $L \subset \mathbb{C}[(\widetilde{E_i})_0]$. Moreover, we have $f_j/f_i \in \mathbb{C}[(\widetilde{E_i})_0]$ by the condition (C1) in the proof of Lemma 5.2.3, and hence $f/f_i \in \mathbb{C}[(\widetilde{E_i})_0]$ holds for any $f \in \mathscr{A}$. Q.E.D.

Lemma 5.2.6. *The homomorphism $\Phi_i^{\#}$ is surjective for any $0 \leq i \leq r$.*

Proof. Let $F \in \mathbb{C}[(\widetilde{E_i})_0]$. Concerning the proof of Lemma 5.2.5, we may assume that $F \notin \mathbb{C}$ and that F is of the form $F = (ZW)^d(Z^m)^c$ for some $d,\ c \in \mathbb{Z}$. Notice that $v_{D_{r+1}}(F) \geq 0$ if and only if $\frac{(\alpha+1)b-t}{b}c \geq d$. If $v_{D_{r+1}}(F) \geq 0$, then we have $F = \Phi_i^{\#}(F)$. Suppose that $v_{D_{r+1}}(F) < 0$, and set $F' = F/\sigma_{i+1}$. Then, as an element of $\mathbb{C}(\mathfrak{U})$, F can be written as $F = F'\frac{f_{i+1}}{f_i}$. We claim that the following two conditions hold: (I) $v_{D_{r+1}}(F') > v_{D_{r+1}}(F)$; (II) $F' \in \mathbb{C}[(\widetilde{E_i})_0]$. Indeed, (I) follows from Lemma 5.2.2. Since we have $v_{D_i}(\sigma_{i+1}) = 1$ and $v_{D_{i+1}}(\sigma_{i+1}) = 0$ by Lemma 5.2.2, it suffices to show that $v_{D_i}(F) \geq 1$ holds to get (II). Suppose that $v_{D_i}(F) < 1$. Since $F \in \mathbb{C}[(\widetilde{E_i})_0]$, this implies that $v_{D_i}(F) = 0$. Namely, we get $0 = -dP_i + \{(\alpha+1)P_i - Q_i\}\,c$. If $i = 0$, then we obtain $c = 0$. It follows that $0 \leq v_{D_1}(F) = -d$, which contradicts to $v_{D_{r+1}}(F) < 0$. Next, let $i \geq 1$. Since we see that $v_{D_{i+1}}(F) = c_i v_{D_i}(F) - v_{D_{i-1}}(F)$ by (3.3), the condition $v_{D_{i+1}}(F) \geq 0$ implies that $0 \geq v_{D_{i-1}}(F)$. Let $i = 1$. Then, we have $0 \geq v_{D_0}(F) = c$ and $0 = v_{D_1}(F) = -d + (\alpha+1)c$. If $c < 0$, then $v_{D_{r+1}}(F) < 0$ implies that $(\alpha+1)b - t > (\alpha+1)b$. Thus, $c = d = 0$, which contradicts to $F \notin \mathbb{C}$. If $i > 1$, then we have $c\frac{Q_i}{P_i} \leq c\frac{Q_{i-1}}{P_{i-1}}$ by $v_{D_i}(F) = 0$ and $0 \geq v_{D_{i-1}}(F)$. It follows that $c = 0$ concerning Theorem 3.2.3. In the same manner, we see that this contradicts to $v_{D_{r+1}}(F) < 0$. Thus, $F' \in \mathbb{C}[(\widetilde{E_i})_0]$. The conditions (I), (II) and $F = F'\frac{f_{i+1}}{f_i}$ yield that there is an $F'' \in \mathbb{C}[(\widetilde{E_i})_0]$ with $v_{D_{r+1}}(F'') \geq 0$ such that $F = F''\left(\frac{f_{i+1}}{f_{i1}}\right)^t$ holds for some $t > 0$. Q.E.D.

Proposition 5.2.7. *The morphism $\Phi : \widetilde{E'_{l,m}} \longrightarrow E_{l,m} \times \mathbb{P}(V^\vee)$ is a closed immersion.*

Proof. This follows from Lemmas 5.2.5 and 5.2.6. Q.E.D.

5.2.8 Morphism to the minimal resolution

In this subsection, we construct an equivariant morphism $\Psi : \mathscr{H} \longrightarrow E_{l,m} \times \mathbb{P}(V^\vee)$ that satisfies $\Psi(\mathscr{H}^{main}) = \Phi(\widetilde{E'_{l,m}}) \cong \widetilde{E'_{l,m}}$ (Proposition 5.2.11). First, by §2.1.6 and Proposition 4.1.2 , we can construct an equivariant morphism $\eta_{n_i,0} : \mathscr{H} \longrightarrow \mathrm{Gr}(h(n_i,0), F^\vee_{n_i,0})$ for each $0 \leq i \leq r$.

Remark 5.2.8.1. If $i = 0$, then $F^\vee_{n_0,0} = \langle (X_2X_4)^\vee, (X_1X_4)^\vee, (X_2X_3)^\vee, (X_1X_3)^\vee \rangle$. For later use, we fix an isomorphism $\mathrm{Gr}(h(n_i,0), F^\vee_{n_0,0}) \cong \mathbb{P}(F^\vee_{n_0,0})$ given by the following:

$$t^{(0)}_{0,0}(X_2X_4)^\vee + t^{(0)}_{e_0,0}(X_1X_4)^\vee + t^{(0)}_{0,l_0}(X_2X_3)^\vee + t^{(0)}_{e_0,l_0}(X_1X_3)^\vee \mapsto [t^{(0)}_{0,0} : t^{(0)}_{e_0,0} : t^{(0)}_{0,l_0} : t^{(0)}_{e_0,l_0}],$$

where $t^{(0)}_{0,0},\ t^{(0)}_{e_0,0},\ t^{(0)}_{0,l_0},\ t^{(0)}_{e_0,l_0} \in \mathbb{C}$. If $i \geq 1$ (this happens only if $E_{l,m}$ is non-toric), then

$$F^\vee_{ni,0} = \langle (X_0^{n_i})^\vee, (X_2^{e_i}X_4^{l_i})^\vee, \ldots, (X_2^{e_i-e}X_1^{e}X_4^{l_i-l}X_3^{l})^\vee, \ldots, (X_1^{e_i}X_3^{l_i})^\vee \rangle.$$

As above, we fix an isomorphism $\mathrm{Gr}(h(n_i,0),F_{n_i,0}^\vee)\cong\mathbb{P}(F_{n_i,0}^\vee)$ given by sending

$$u^{(i)}(X_0^{n_i})^\vee+t_{0,0}^{(i)}(X_2^{e_i}X_4^{l_i})^\vee+\cdots+t_{e,l}^{(i)}(X_2^{e_i-e}X_1^eX_4^{l_i-l}X_3^l)^\vee+\cdots+t_{e_i,l_i}^{(i)}(X_1^{e_i}X_3^{l_i})^\vee$$

to $[u^{(i)}:t_{0,0}^{(i)}:\cdots:t_{e,l}^{(i)}:\cdots:t_{e_i,l_i}^{(i)}]$, where $u^{(i)}$, $t_{0,0}^{(i)}$, …, $t_{e_i,l_i}^{(i)}\in\mathbb{C}$. In below, we denote the composition $\mathscr{H}\longrightarrow\mathrm{Gr}(h(n_i,0),F_{n_i,0}^\vee)\cong\mathbb{P}(F_{n_i,0}^\vee)$ by the same $\eta_{n_i,0}$ $(0\le i\le r)$.

Remark 5.2.8.2. As in Remark 2.3.3.3, we denote by $V(n)=\mathrm{Sym}^n\langle X,Y\rangle$ the irreducible $SL(2)$-representation of highest weight n. For any partition $n=\mu_1+\cdots+\mu_s$, the tensor representation $V(\mu_1)\otimes\cdots\otimes V(\mu_s)$ contains an irreducible representation $V(\mu_1,\ldots,\mu_s)$ isomorphic to $V(n)$ by the Clebsch–Gordan theorem. For each $0\le i\le n$, set

$$\phi_i:=\frac{1}{\binom{n}{i}}\sum_{\substack{i_1+\cdots+i_s=i\\0\le i_1\le\mu_1\\\cdots\\0\le i_s\le\mu_s}}\binom{\mu_1}{i_1}\cdots\binom{\mu_s}{i_s}X^{\mu_1-i_1}Y^{i_1}\otimes\cdots\otimes X^{\mu_s-i_s}Y^{i_s}\in V(\mu_1)\otimes\cdots\otimes V(\mu_s).$$

Then, $\{\phi_0,\ldots,\phi_n\}$ forms a basis of $V(\mu_1,\ldots,\mu_s)$. On the other hand, we can take $\{X^{n-i}Y^i : 0\le i\le n\}$ as a basis of $V(n)$, and the linear map $V(n)\longrightarrow V(\mu_1,\ldots,\mu_s)$ that sends $X^{n-i}Y^i$ to ϕ_i is an $SL(2)$-equivariant isomorphism.

Let $V':=F_{n_0,0}\otimes F_{n_1,0}\otimes\cdots\otimes F_{n_r,0}$. We see that V' coincides with

$$\bigoplus A(e_0)\otimes B(l_0)\otimes A(e_{i_1})\otimes B(l_{i_1})\otimes\ldots A(e_{i_s})\otimes B(l_{i_s})\otimes C(n_{j_1})\otimes\cdots\otimes C(n_{j_u}),$$

where the sum runs over $\{i_1,\ldots,i_s,j_1,\ldots,j_u\}=\{1,\ldots,r\}$ such that $i_1<\cdots<i_s$ and $j_1<\cdots<j_u$. In order to describe a submodule of V' isomorphic to V, let us denote by $A(e_0,e_1,\ldots,e_i)$ the irreducible representation of highest weight $e_0+e_1+\cdots+e_i$ contained in $A(e_0)\otimes A(e_1)\otimes\cdots\otimes A(e_i)$ in the sense of Remark 5.2.8.2. Namely, $A(e_0,e_1,\ldots,e_i)\cong V(e_0,e_1,\ldots,e_i)\cong V(e_o+e_1+\cdots+e_i)$. Likewise, we denote by $B(l_0,l_1,\ldots,l_i)$ the irreducible representation of highest weight $l_0+l_1+\cdots+l_i$ in $B(l_0)\otimes B(l_1)\otimes\cdots\otimes B(l_i)$, i.e., $B(l_0,l_1,\ldots,l_i)\cong V(l_0,l_1,\ldots,l_i)$, which is isomorphic to $V(l_0+l_1+\cdots+l_i)$. Let $\widetilde{V}$ be the submodule of V' defined as follows:

$$\widetilde{V}:=\bigoplus_{1\le i\le r}A(e_0,e_1,\ldots,e_i)\otimes B(l_0,l_1,\ldots,l_i)\otimes C(n_{i+1})\otimes\cdots\otimes C(n_r).$$

Since $V\subset\Gamma(\widetilde{E'_{l,m}},\mathcal{O}(\delta))$ coincides with

$$\bigoplus_{1\le i\le r}A(e_0+e_1+\cdots+e_i)\otimes B(l_0+l_1+\cdots+l_i)\otimes C(-(n_0+n_1+\cdots+n_i)),$$

we see that $V \cong \widetilde{V}$, where the isomorphism

$$C(-(n_0+n_1+\cdots+n_i)) \cong C(n_{i+1}+\cdots+n_r) \cong C(n_{i+1})\otimes\cdots\otimes C(n_r)$$

is given by multiplying $X_0^{n_0+n_1+\cdots+n_r}$.

Example 5.2.9. Let $l = p/q = 1/4$, and let $m = 2$. Then, we have $k = 1$, $a = 2$, $b = 3$, $\alpha = 0$, $\beta = 2$, and $t = 1$. Therefore, the Hirzebruch–Jung continued fraction expansion of b/t is $b/t = c_1 = 3$, and we have $P_0 = 0$, $Q_0 = -1$, $P_1 = 1$, $Q_1 = 0$, $P_2 = c_1 = 3$, and $Q_2 = 1$. Thus, we get $\rho_0 = \mathbf{u}_2$, $\rho_1 = -\mathbf{u}_1+\mathbf{u}_2$, and $\rho_2 = -3\mathbf{u}_1+2\mathbf{u}_2$, and the maximal cones of the colored fan of $\widetilde{E'_{\frac{1}{4},2}}$ are $\mathscr{C}_1 = \mathbb{Q}_{\geq 0}\rho_0+\mathbb{Q}_{\geq 0}\rho_1$ and $\mathscr{C}_2 = \mathbb{Q}_{\geq 0}\rho_1+\mathbb{Q}_{\geq 0}\rho_2$. Also, we have $(e_0,l_0,n_0) = (1,1,3)$, $(e_1,l_1,n_1) = (3,1,1)$, and $(e_2,l_2,n_2) = (8,2,0)$. Thus we get $f_0 = ZW$, $f_1 = Z^4W^2$, and $f_2 = Z^{12}W^4$ by definition, and therefore

$$\begin{aligned} V &= \langle (SL(2)\times\mathbb{C}^*)\cdot ZW\rangle \oplus \langle (SL(2)\times\mathbb{C}^*)\cdot Z^4W^2\rangle \\ &\cong \langle X,Z\rangle\otimes\langle Y,W\rangle \oplus \langle X^4, X^3Z, X^2Z^2, XZ^3, Z^4\rangle\otimes\langle Y^2, YW, W^2\rangle \\ &\cong V(1)\otimes V(1)\oplus V(4)\otimes V(2). \end{aligned}$$

We have $V' = F_{n_0,0}\otimes F_{n_1,0}$, where $F_{n_0,0} = A(1)\otimes B(1) = \langle X_1,X_2\rangle\otimes\langle X_3,X_4\rangle$ and

$$F_{n_1,0} = A(3)\otimes B(1)\oplus C(1) = \langle X_1^3, X_1^2X_2, X_1X_2^2, X_2^3\rangle\otimes\langle X_3,X_4\rangle\oplus\langle X_0\rangle.$$

Furthermore, we have $\widetilde{V} = A(1,3)\otimes B(1,1)\oplus A(1)\otimes B(1)$, where $A(1,3)$ is a subrepresentation of $A(1)\otimes A(3)$ spanned by $X_1\otimes X_1^3$, $\frac{1}{4}(X_2\otimes X_1^3+3X_1\otimes X_1^2X_2)$, $\frac{1}{2}(X_2\otimes X_1^2X_2+X_1\otimes X_1X_2^2)$, $\frac{1}{4}(3X_2\otimes X_1X_2^2+X_1\otimes X_2^3)$, and $X_2\otimes X_2^3$. Also, $B(1,1)$ is a subrepresentation of $B(1)\otimes B(1)$ spanned by $X_3\otimes X_3$, $\frac{1}{2}(X_3\otimes X_4+X_4\otimes X_3)$, and $X_4\otimes X_4$.

We define Ψ' to be the composition of

$$\gamma\times\prod_{0\leq i\leq r}\eta_{n_i,0} : \mathscr{H} \longrightarrow E_{l,m}\times\prod_{0\leq i\leq r}\mathbb{P}(F^\vee_{n_i,0})$$

and $\mathrm{id}_{E_{l,m}}\times\iota$, where ι denotes the Segre embedding $\iota : \prod_{0\leq i\leq r}\mathbb{P}(F^\vee_{n_i,0}) \hookrightarrow \mathbb{P}(V'^\vee)$. Namely,

$$\Psi' : \mathscr{H} \longrightarrow E_{l,m}\times\mathbb{P}(V'^\vee).$$

Let us consider the projection $\mathrm{pr} : E_{l,m}\times\mathbb{P}(V'^\vee) \longrightarrow E_{l,m}\times\mathbb{P}(\widetilde{V}^\vee)$.

Proposition 5.2.10. *The restriction* $\mathrm{pr}|_{\Psi'(\mathcal{H})}$ *of the rational map* pr *to the image of* Ψ' *is a morphism.*

Proof. Suppose that there is a point $[I] \in \mathcal{H}$ such that pr is not defined at $\Psi'([I])$. Let $\eta_{n_0,0}([I]) = [t^{(0)}_{0,0} : t^{(0)}_{e_0,0} : t^{(0)}_{0,l_0} : t^{(0)}_{e_0,l_0}]$, and let $\eta_{n_i,0}([I]) = [u^{(i)} : t^{(i)}_{0,0} : \cdots : t^{(i)}_{e,l} : \cdots : t^{(i)}_{e_i,l_i}]$ $(1 \leq i \leq r)$ following the notation of Remark 5.2.8.1. By Remark 4.1.1.1, we have $s_1X_1 + s_2X_2 \in I$ for some $(s_1, s_2) \neq 0$. Since Ψ' is $SL(2)$-equivariant, we may assume that $X_2 \in I$. Note that the subrepresentation $A(e_0, e_1, \ldots, e_r) \otimes B(l_0, l_1, \ldots, l_r)$ of $\widetilde{V}$ contains $X_1^{e_0} \otimes X_1^{e_1} \otimes \cdots \otimes X_1^{e_r} \otimes X_3^{l_0} \otimes X_3^{l_1} \otimes \cdots \otimes X_3^{l_r}$. Therefore, we have $t^{(0)}_{e_0,l_0} t^{(1)}_{e_1,l_1} \cdots t^{(r)}_{e_r,l_r} = 0$ by the assumption on I. Let $j = \min\{i : t^{(i)}_{e_i,l_i} = 0,\ 0 \leq i \leq r\}$. Then we have $X_1^{e_j} X_3^{l_j} \in I$ by the construction of $\eta_{n_j,0}$, which implies that $X_1^{e_i} X_3^{l_i} \in I$ holds for any $i \geq j$. Next, again by Remark 4.1.1.1, we have $s_3X_3 + s_4X_4 \in I$ for some $(s_3, s_4) \neq 0$. Therefore, one of the following holds: (I) $s_3 \neq 0,\ s_4 \neq 0$; (II) $s_3 = 0,\ s_4 \neq 0$; (III) $s_3 \neq 0,\ s_4 = 0$. Suppose that we are in the case (I). Then, by multiplying $X_1^{e_j} X_3^{l_j-1}$ to $s_3X_3 + s_4X_4$, we get $X_1^{e_j} X_3^{l_j-1} X_4 \in I$. By continuing in this way, we see that $X_1^{e_j-e} X_2^{e} X_3^{l_j-l} X_4^{l} \in I$ holds for any $0 \leq e \leq e_j$ and for any $0 \leq l \leq l_j$ concerning $X_2 \in I$. Lastly, we pay attention to the vector $X_1^{e_0} \otimes X_1^{e_1} \otimes \cdots \otimes X_1^{e_{j-1}} \otimes X_3^{l_0} \otimes X_3^{l_1} \otimes \cdots \otimes X_3^{l_{j-1}} \otimes X_0^{n_j} \otimes \cdots \otimes X_0^{n_r}$ in the subrepresentation $A(e_0, e_1, \ldots, e_{j-1}) \otimes B(l_0, l_1, \ldots, l_{j-1}) \otimes C(n_j) \otimes \cdots \otimes C(n_r)$ of $\widetilde{V}$. Likewise, we have $t^{(0)}_{e_0,l_0} t^{(1)}_{e_1,l_1} \cdots t^{(j-1)}_{e_{j-1},l_{j-1}} u^{(j)} \cdots u^{(r)} = 0$ by the assumption on I. This implies that $u^{(j)} \cdots u^{(r)} = 0$ by the minimality of j. In particular, we have $u^{(j')} = 0$ for some $j \leq j' \leq r$. It follows that $X_0^{n_{j'}} \in I$ by the construction of $\eta_{n_{j'},0}$. Therefore, $X_0^{n_j} \in I$. Consequently, we get $F_{n_j,0} \subset I$, and it follows from Proposition 4.1.2 that $\dim(S/I)_{(n_j,0)} = 0 \neq h(n_j, 0)$, which contradicts to $[I] \in \mathcal{H}$. If we are in the case (II) or (III), we can show that the assumption on the ideal I leads to a contradiction by following a similar line as above. Q.E.D.

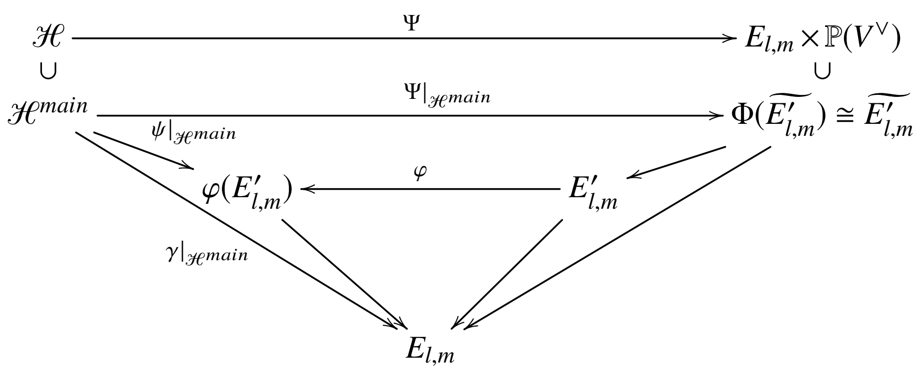
Combining the above discussion, we obtain the following equivariant morphism:

$$\Psi : \mathcal{H} \xrightarrow{\Psi'} E_{l,m} \times \mathbb{P}(V'^{\vee}) \overset{\mathrm{pr}}{\dashrightarrow} E_{l,m} \times \mathbb{P}(\widetilde{V}^{\vee}) \xrightarrow{\sim} E_{l,m} \times \mathbb{P}(V^{\vee}).$$

Proposition 5.2.11. *We have* $\Psi(\mathcal{H}^{main}) = \Phi(\widetilde{E'_{l,m}})$*. In particular,* $\Psi(\mathcal{H}^{main}) \cong \widetilde{E'_{l,m}}$.

Proof. Let $f : \widetilde{E'_{l,m}} \longrightarrow E_{l,m}$ be the resolution of singularities, and let $y \in \widetilde{E'_{l,m}}$ be a point such that $f(y) = \pi(x)$. Then, concerning Remark 2.3.5.1, we have $\Phi(y) = (\pi(x), v)$, where v is a point of $\mathbb{P}(V^{\vee})$ whose coordinates are all 0 except for the ones corresponding to the bases $\{(X^{e_0+e_1+\cdots+e_i} W^{l_0+l_1+\cdots+l_i})^{\vee} : 1 \leq i \leq r\}$. On the other hand, we see by the definition of I_1 and the construction of $\eta_{n_i,0}$ that $\eta_{n_0,0}([I_1]) = \langle (X_1X_4)^\vee \rangle$, and that $\eta_{n_i,0}([I_1]) = \langle (X_0^{n_i})^\vee + (X_1^{e_i}X_4^{l_i})^\vee \rangle$ $(1 \leq i \leq r)$. Taking the relations $X = X_0^p X_1$ and $W = X_0^{-q} X_4$ into account, we deduce that $\Psi([I_1]) = \Phi(y)$. This shows the proposition, since $\Psi(\mathscr{H}^{main})$ and $\Phi(\widetilde{E'_{l,m}})$ are $SL(2) \times \mathbb{C}^*$-orbit closures of $\Psi([I_1])$ and $\Phi(y)$, respectively. Q.E.D.

Summarizing, we get the following equivariant commutative diagram:



5.3 Proof of Theorem 3.2.4

We have seen in Proposition 5.2.11 that $\Psi(\mathscr{H}^{main}) \cong \widetilde{E'_{l,m}}$. Therefore, in order to complete the proof of Theorem 3.2.4, we are left to show that $\Psi|_{\mathscr{H}^{main}}$ is injective. Indeed, it follows from the Zariski's Main Theorem that $\Psi|_{\mathscr{H}^{main}}$ being injective implies $\Psi|_{\mathscr{H}^{main}}$ being a closed immersion.

Proof of Theorem 3.2.4. Let $\tau : \widetilde{E'_{l,m}} \longrightarrow E'_{l,m}$ be the resolution of singularities. We first show the injectivity of $\Psi|_{\Psi^{-1}(\tau^{-1}(C))} : \Psi^{-1}(\tau^{-1}(C)) \longrightarrow \tau^{-1}(C)$ orbit-wise. By Remark 3.2.4.2, we see that $\tau^{-1}(C)$ contains $2r+1$ orbits Y_i $(0 \leq i \leq r)$ and O_i $(1 \leq i \leq r)$. Let us elaborate on this. Recall that we have constructed an equivariant closed immersion $\Phi : \widetilde{E'_{l,m}} \hookrightarrow E_{l,m} \times \mathbb{P}(V^\vee)$. We denote by y_i $(0 \leq i \leq r)$ the point of $\mathbb{P}(V^\vee)$ whose coordinates are all 0 except for the one corresponding to the basis $g \cdot f_i^\vee$, where $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2)$. Then, we have $\Phi(Y_i) = (SL(2) \times \mathbb{C}^*) \cdot v_i$, where we set $v_i = (O, y_i)$. Similarly, we denote by y'_i $(1 \leq i \leq r)$ the point of $\mathbb{P}(V^\vee)$ whose coordinates are all 0 except for the ones corresponding to the bases $g \cdot f_i^\vee$ and $g \cdot f_{i+1}^\vee$. Then, we have $\Phi(O_i) = (SL(2) \times \mathbb{C}^*) \cdot v'_i$, where $v'_i = (O, y'_i)$. In the following, we show that each of the set-theoretical fibers of v_i and v'_i consists of one point. More precisely, we show that $\Psi^{-1}(v_i) = \{[L_0^{i+1}]\}$ and $\Psi^{-1}(v_i') = \{[L_1^i]\}$ hold, where we set $L_0^{r+1} := J_0$ for the sake of convenience.

First, we show that $\Psi^{-1}(v_i) = \{[L_0^{i+1}]\}$ holds for any $0 \le i \le r$. Let $[L] \in \Psi^{-1}(v_i)$. As in the proof of Proposition 5.2.10, write $\eta_{n_0,0}([L]) = [t_{0,0}^{(0)} : t_{e_0,0}^{(0)} : t_{0,l_0}^{(0)} : t_{e_0,l_0}^{(0)}]$, and $\eta_{n_j,0}([L]) = [u^{(j)} : t_{0,0}^{(j)} : \cdots : t_{e_j,l_j}^{(j)}]$ $(1 \le j \le r)$. It follows from $\gamma([L]) = O$ that $K \subset L$, since the ideal $K \subset A$ is generated by $G_0 \times G_m$-invariants. We see that

$$g \cdot f_i = X^{e_0+e_1+\cdots+e_i} Y^{l_0+l_1+\cdots+l_i} = X_0^{-(n_0+n_1+\cdots+n_i)} X_1^{e_0+e_1+\cdots+e_i} X_3^{l_0+l_1+\cdots+l_i}$$

maps to

$$\begin{aligned} X_1^{e_0} \otimes X_1^{e_1} \otimes \cdots \otimes X_1^{e_i} \otimes X_3^{l_0} \otimes X_3^{l_1} \otimes \cdots \otimes X_3^{l_i} \otimes X_0^{n_{i+1}} \otimes \cdots \otimes X_0^{n_r} \\ \in A(e_0, e_1, \ldots, e_i) \otimes B(l_0, l_1, \ldots, l_i) \otimes C(n_{i+1}) \otimes \cdots \otimes C(n_r) \end{aligned}$$

under the isomorphism $V \cong \widetilde{V}$. Therefore, by the definition of y_i,

$$t_{e_0,l_0}^{(0)} t_{e_1,l_1}^{(1)} \cdots t_{e_i,l_i}^{(i)} u^{(i+1)} \cdots u^{(r)} = s \tag{5.1}$$

holds for some $s \in \mathbb{C}^*$. Similarly, by paying attention to $g \cdot f_{i+1}$, we have

$$t_{e_0,l_0}^{(0)} t_{e_1,l_1}^{(1)} \cdots t_{e_i,l_i}^{(i)} t_{e_{i+1},l_{i+1}}^{(i+1)} u^{(i+2)} \cdots u^{(r)} = 0. \tag{5.2}$$

By (5.1) and (5.2), we get $t_{e_{i+1},l_{i+1}}^{(i+1)} = 0$, which implies that $X_1^{e_{i+1}} X_3^{l_{i+1}} \in L$. Next, we see that $Z^{e_0} X^{e_1+\cdots+e_i} W^{l_0} Y^{l_1+\cdots+l_i} = X_0^{-(n_0+n_1+\cdots+n_i)} X_2^{e_0} X_1^{e_1+\cdots+e_i} X_4^{l_0} X_3^{l_1+\cdots+l_i}$ maps to

$$\begin{aligned} X_2^{e_0} \otimes X_4^{l_0} \otimes X_1^{e_1} \otimes X_3^{l_1} \otimes \cdots \otimes X_1^{e_i} \otimes X_3^{l_i} \otimes X_0^{n_{i+1}} \cdots \otimes X_0^{n_r} \\ \in A(e_0) \otimes B(l_0) \otimes A(e_1) \otimes B(l_1) \otimes \cdots \otimes A(e_i) \otimes B(l_i) \otimes C(n_{i+1}) \otimes \cdots \otimes C(n_r) \end{aligned}$$

under $V \cong \widetilde{V} \subset V'$, which yields that $t_{0,0}^{(0)} t_{e_1,l_1}^{(1)} \cdots t_{e_i,l_i}^{(i)} u^{(i+1)} \cdots u^{(r)} = 0$. By comparing this equation with (5.1), we have $t_{0,0}^{(0)} = 0$, which tells us that $X_2^{e_0} X_4^{l_0} = X_2 X_4 \in L$. In a similar way, we see that $X_2 X_3$, $X_1 X_4 \in L$ holds as well. Concerning Remark 4.1.1.1, it follows that $(X_2, X_4) \subset L$. Therefore, we get $(X_0^{q-p}, X_2, X_4, X_1^{e_{i+1}} X_3^{l_{i+1}}) + K \subset L$. If $i = 0$, then we have $L_0^1 \subset L$ since $n_0 = q - p$. It follows that $\dim(A/L_0^1)_{(n,d)} \ge \dim(A/L)_{(n,d)} = h(n,d)$ holds for any weight $(n,d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. On the other hand, we have seen in Theorem 4.2.8 that $\dim(A/L_0^1)_{(n,d)} \le h(n,d)$. Consequently, we get $L_0^1 = L$. Next, suppose that $i > 0$. Since the vector

$$X^{e_0+e_1+\cdots+e_{i-1}} Y^{l_0+l_1+\cdots+l_{i-1}} = X_0^{-(n_0+n_1+\cdots+n_{i-1})} X_1^{e_0+e_1+\cdots+e_{i-1}} X_3^{l_0+l_1+\cdots+l_{i-1}}$$

maps to $X_1^{e_0} \otimes X_3^{l_0} \otimes X_1^{e_1} \otimes X_3^{l_1} \otimes \cdots \otimes X_1^{e_{i-1}} X_3^{l_{i-1}} \otimes X_0^{n_i} \otimes \cdots \otimes X_0^{n_r}$ under the isomorphism $V \cong \widetilde{V} \subset V'$, we see that $t^{(0)}_{e_0,l_0} t^{(1)}_{e_1,l_1} \cdots t^{(i-1)}_{e_{i-1},l_{i-1}} u^{(i)} \cdots u^{(r)} = 0$. Again by comparing this equation with (5.1), we get $u^{(i)} = 0$, which implies that $X_0^{n_i} \in L$. Therefore, one obtains $L_0^{i+1} \subset L$. As above, we deduce that $L_0^{i+1} = L$.

Next, we show that $\Psi^{-1}(v_i') = \{[L_1^i]\}$ holds for any $1 \leq i \leq r$. Let $[L'] \in \Psi^{-1}(v_i')$, and write $\eta_{n_0,0}([L']) = [t^{(0)}_{0,0} : t^{(0)}_{e_0,0} : t^{(0)}_{0,l_0} : t^{(0)}_{e_0,l_0}]$ and $\eta_{n_j,0}([L']) = [u^{(j)} : t^{(j)}_{0,0} : \cdots : t^{(j)}_{e_j,l_j}]$ $(1 \leq j \leq r)$ as above. In a similar manner, we can show that $(X_0^{n_{i-1}}, X_2, X_4) + K \subset L'$. The conditions $X_2, X_4 \in L'$ imply that $t^{(i)}_{0,0} = \cdots = t^{(i)}_{e_i-1,l_i-1} = 0$. Moreover, we see that $t^{(0)}_{e_0,l_0} t^{(1)}_{e_1,l_1} \cdots t^{(i-1)}_{e_{i-1},l_{i-1}} u^{(i)} u^{(i+1)} \cdots u^{(r)} = s$ and $t^{(0)}_{e_0,l_0} t^{(1)}_{e_1,l_1} \cdots t^{(i-1)}_{e_{i-1},l_{i-1}} t^{(i)}_{e_i,l_i} u^{(i+1)} \cdots u^{(r)} = s$ hold for some $s \in \mathbb{C}^*$. Therefore, we get $u^{(i)} = t^{(i)}_{e_i,l_i}$, and hence we have $\eta_{n_i,0}([L']) = [1 : 0 : \cdots : 0 : 1]$. It follows that $X_0^{n_i} - X_1^{e_i} X_3^{l_i} \in L'$ concerning the construction of $\eta_{n_i,0}$ and the fixed isomorphism $\mathrm{Gr}(h(n_i,0), F^\vee_{n_i,0}) \cong \mathbb{P}(F^\vee_{n_i,0})$ (see Remark 5.2.8.1). As a consequence, we get $L_1^i \subset L'$, and hence $L_1^i = L'$.

Lastly, we show that $\Psi|_{\Psi^{-1}(\tau^{-1}(C'))} : \Psi^{-1}(\tau^{-1}(C')) \longrightarrow \tau^{-1}(C') \cong C'$ is bijective. Taking into account Lemma 5.1.4, it suffices to show that $\psi^{-1}(\varphi(C'))$ coincides with $\Psi^{-1}(\tau^{-1}(C'))$ as subsets of $\mathcal{H}^{main}$. By the proof of Lemma 5.1.4, $\psi^{-1}(\varphi(C'))$ is the $SL(2)$-orbit of $[I_0]$, where $I_0 = (X_0^{q-p} - X_1X_4, X_2, X_3, X_0^{mp} X_1^m)$ (see §4.1). With the notation of Remark 5.2.8.1, we have $\eta_{n_0,0}([I_0]) = [0 : 1 : 0 : 0]$. If $1 \leq i \leq r$, we have $(X_0^{q-p} - X_1X_4)^{l_i} X_1^{mP_i} = X_0^{n_i} (X_0^{mp} X_1^m)^{P_i} - X_1^{e_i} X_4^{l_i} = -X_1^{e_i} X_4^{l_i}$ modulo the ideal I_0, since $(q-p)l_i = n_i + p(e_i - l_i)$ and $e_i = l_i + mP_i$ (see §3.2.2). Therefore, $X_1^{e_i} X_4^{l_i} \in I_0$. It follows that $\eta_{n_i,0}([I_0]) = [1 : 0 : \ldots : 0]$, i.e., $u^{(i)} = 1$, $t^{(i)}_{0,0} = \cdots = t^{(i)}_{e_i,l_i} = 0$ with the notation of Remark 5.2.8.1. This shows $\Psi([I_0]) \notin \tau^{-1}(C)$, since there is no $g \in SL(2)$ that translates X_1 to X_1, and X_4 to X_3. Therefore, $\Psi([I_0]) \in \tau^{-1}(C')$. Q.E.D.

Corollary 5.3.1. *For any $s \in \mathbb{C}$ and for any $1 \leq i \leq r$, A/L_s^i has Hilbert function h.*

Proof. By the proof of Theorem 3.2.4, the quotient rings A/L_0^i and A/L_1^i have Hilbert function h for any $1 \leq i \leq r$. Therefore, the corollary follows concerning Remark 4.1.0.1. Q.E.D.

Remark 5.3.1.1. Let $0 \leq n < q-p$, and let $\lambda = (n, c, \omega) \in \Lambda_{(n,0)}$. Taking into account Remark 4.2.17.1 and Corollary 5.3.1, we see that the following properties are true.

- Let $0 \leq n \leq n_{i-1}$. Then, we have $f_\lambda \in \widetilde{L_0^i}$ if and only if $\lambda \neq \left(n, 0, \omega^{\min}_{(n,0)}\right)$.
- Let $n_{i-1} \leq n < q-p$. Then, we have $f_\lambda \in \widetilde{L_0^i}$ if and only if $\lambda \neq \left(n, mP, \omega^{\min}_{(n,mP)}\right)$.

Let us denote by $\mathcal{H}^{\widetilde{B}}$ the set of $\widetilde{B}$-fixed points of $\mathcal{H}$. Recall that we have set $L_0^{r+1} := J_0$ in the proof of Theorem 3.2.4.

Corollary 5.3.2. *We have* $\mathcal{H}^{\widetilde{B}} = \{[L_0^i] : 1 \leq i \leq r+1\}$.

Proof. Let $[L] \in \mathcal{H}^{\widetilde{B}}$. Then, $s_1X_1 + s_2X_2,\ s_3X_3 + s_4X_4 \in L$ hold for some $(s_1, s_2) \neq 0$ and $(s_3, s_4) \neq 0$ by Remark 4.1.1.1. Since L is stable under the action of $\widetilde{B}$, we have $X_2, X_4 \in L$. Therefore, since $\gamma([L]) = O$, we get $(X_2, X_4) + K \subset L$. Concerning the conditions $X_2, X_4 \in L$ and $h(n_j, 0) = 1$, we deduce from Proposition 4.1.2 that either $X_0^{n_j} \in L$ or $X_1^{e_j} X_3^{l_j} \in L$ holds for any $1 \leq j \leq r+1$. Let $i = \min\{j : X_1^{e_j} X_3^{l_j} \in L\}$. Then, we have $(X_0^{n_{i-1}},\ X_1^{e_i} X_3^{l_i}) \subset L$, and hence $L_0^i \subset L$. This implies that $L_0^i = L$, since both L_0^i and L have Hilbert function h. Q.E.D.

Corollary 5.3.3. *The invariant Hilbert scheme $\mathcal{H}$ is irreducible and reduced.*

Proof. By [Ter14a, Lemma 1.6], every closed subset of $\mathcal{H}$ contains at least one fixed point for the action of $\widetilde{B}$. It yields that $\mathcal{H}$ is connected, since Corollary 5.3.2 implies that every $\widetilde{B}$-fixed point is contained in $\mathcal{H}^{main}$. Therefore, we are left to show that $\mathcal{H}$ is smooth. Concerning Theorem 2.1.4 and the proof of [Ter14a, Lemma 1.7], it suffices to show that $\dim \mathrm{Hom}_S^{G_0 \times G_m}(L_0^i, A/L_0^i) = \dim \mathcal{H}^{main} = 3$ holds for any $1 \leq i \leq r+1$, where $S = \mathbb{C}[H_{q-p}]$ as in §4.1. Let R be the subring $\mathbb{C}[X_0, X_1, X_3]$ of A, i.e., let $j = 3$ in (4.1) (see §4.1), and we use notations of §4.1 and §4.2. Recall that we have seen in Remark 4.2.17.2 that $L_0^i = (X_0^{n_{i-1}},\ X_2,\ X_4,\ X_1^{e_i} X_3^{l_i},\ F_1,\ \ldots,\ F_{b-1})$. Let $\phi \in \mathrm{Hom}_S^{G_0 \times G_m}(L_0^i, A/L_0^i)$. Since ϕ is $G_0 \times G_m$-equivariant, we see that $\phi(X_0^{n_{i-1}}) = \alpha_1 X_1^{e_{i-1}} X_3^{l_{i-1}}$, $\phi(X_2) = \alpha_2 X_1$, $\phi(X_4) = \alpha_3 X_3$, $\phi(X_1^{e_i} X_3^{l_i}) = \alpha_4 X_0^{n_i}$, and $\phi(F_j) = \beta_j$ $(1 \leq j \leq b-1)$ hold for some $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \beta_j \in \mathbb{C}$. Moreover, since ϕ is a homomorphism of S-modules, we have

$$0 = \phi(X_0^{q-p} - X_1X_4 + X_2X_3) = \alpha_1 X_0^{q-p-n_{i-1}} X_1^{e_{i-1}} X_3^{l_{i-1}} - \alpha_3 X_1 X_3 + \alpha_2 X_1 X_3.$$

If $i = 1$, then we have $X_0^{q-p-n_{i-1}} X_1^{e_{i-1}} X_3^{l_{i-1}} = X_1 X_3$, and hence $\alpha_1 + \alpha_2 - \alpha_3 = 0$. If $i > 1$, then we have $e_{i-1},\ l_{i-1} \geq 1$, and thus we can write $X_0^{q-p-n_{i-1}} X_1^{e_{i-1}} X_3^{l_{i-1}} = X_1 X_3 f$ with a monomial $f \in A$. On the other hand, we have $X_0^{q-p-n_{i-1}} X_1^{e_{i-1}} X_3^{l_{i-1}} = f_{\left(q-p, mP_{i-1}, \omega^{\min}_{(q-p, mP_{i-1})}\right)}$ and $X_1 X_3 = f_{\left(q-p, 0, \omega^{\min}_{(q-p,0)}\right)}$ by Lemmas 4.1.5 and 4.1.9, which tells us that $f = f_{\left(0, mP_{i-1}, \omega^{\min}_{(0, mP_{i-1})}\right)} = F_{P_{i-1}}$. Therefore, one obtains $\alpha_2 - \alpha_3 = 0$ if $i > 1$.

In the following, we show that $\beta_j = 0$ holds for any $1 \leq j \leq b-1$. Set $d_1 = \frac{qmj - \omega^{\min}_{(0,mj)}}{q-p}$, and set $d_3 = \frac{pmj - \omega^{\min}_{(0,mj)}}{q-p}$. Then, we have $F_j = X_0^{\omega^{\min}_{(0,mj)}} X_1^{d_1} X_3^{d_3}$ and $f_{\lambda_{mj}} = X_1^{d_1+1} X_3^{d_3+1}$ by Lemma

4.1.5 and Definition 4.2.12, respectively. If $j = P_i$, then we have $f_{\lambda_{mP_i}} = X_1^{e_i} X_3^{l_i}$ (see Example 4.2.13).

Case 1. Let $j > P_i$. Then, $d_1 + 1 > e_i$ and $d_3 + 1 > l_i$ hold by Lemma 4.2.14. By setting $f = X_0^{\omega^{\min}_{(0,mj)}} X_1^{d_1+1-e_i} X_3^{d_3+1-l_i}$, we have

$$0 = X_1 X_3 \phi(F_j) - f \phi(X_1^{e_i} X_3^{l_i}) = \beta_j X_1 X_3 - \alpha_4 X_0^{n_i} f.$$

Here we have $X_0^{n_i} f = X_1 X_3 f'$, where $f' = X_0^{\omega^{\min}_{(0,mj)}+n_i} X_1^{d_1-e_i} X_3^{d_3-l_i}$. Since both $X_0^{n_i} f$ and $X_1 X_3$ are homogeneous elements of $G_0 \times G_m$-weight $(q-p, 0)$, it follows that $f' \in K$. Therefore, one has $X_0^{n_i} f \in L_0^i$. On the other hand, we see that $X_1 X_3 \notin L_0^i$, since otherwise we get $\dim(A/L_0^i)_{(q-p,0)} = 0$ by Proposition 4.1.19, which contradicts to Corollary 5.3.1. Therefore, we have $\beta_j = 0$.

Case 2. Let $j = P_i$. Then, we have

$$0 = X_1 X_3 \phi(F_{P_i}) - X_0^{q-p-n_i} \phi(X_1^{e_i} X_3^{l_i}) = \beta_{P_i} X_1 X_3 - \alpha_4 X_0^{q-p}.$$

Since we have $X_1 X_3 \notin L_0^i$ and $X_0^{q-p} \in L_0^i$, it follows that $\beta_{P_i} = 0$.

Case 3. Let $1 \leq j < P_i$. As above, we see that the condition $j < P_i$ implies that $d_1 < e_i$ and $d_3 < l_i$. Then we have

$$0 = X_1^{e_i-d_1} X_3^{l_i-d_3} \phi(F_j) - X_0^{\omega^{\min}_{(0,mj)}} \phi(X_1^{e_i} X_3^{l_i}) = \beta_j f_{\lambda_c} - \alpha_4 X_0^n,$$

where we set $n = \omega^{\min}_{(0,mj)} + n_i$, and $c = m(P_i - j)$. Since we have $n_{i-1} \leq n < q - p$ by Lemma 4.2.11, it follows that $X_0^n \in L_0^i$. Therefore, we are left to show that $f_{\lambda_c} \notin L_0^i$. As in Case 2 of the proof of Theorem 4.2.7, we have $n - (n_{j_1} + \cdots + n_{j_{u_n}}) < n_{i-1}$ for some $1 \leq j_1, \ldots, j_{u_n} \leq i-1$. Set $P = P_{j_1} + \cdots + P_{j_{u_n}}$, and set $\lambda = \left(n, mP, \omega^{\min}_{(n,mP)}\right)$. We show that f_{λ_c} coincides with f_λ. First, we have $f_{\lambda_c} \notin (X_1^{e_i} X_3^{l_i})$ by Lemma 4.2.14. If $f_{\lambda_c} \in K$, then we have $f_{\lambda_c} \in (F_1, \ldots, F_{b-1})$ since $c \leq b-1$. On the other hand, we see that the degree of $F_{j'}$ with respect to X_0 is greater than 0 for any $1 \leq j' \leq b-1$. This is a contradiction since $f_{\lambda_c} \notin (X_0)$. Therefore, we have $f_{\lambda_c} \notin K$, and hence $c = mP$ concerning the proof of Theorem 4.2.7. It follows from Remark 5.3.1.1 that $f_{\lambda_c} \notin L_0^i$, and thus we get $\beta_j = 0$.

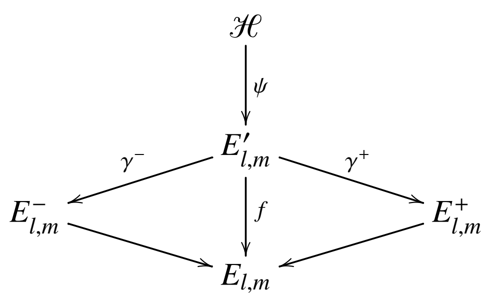
Consequently, we get $\dim \mathrm{Hom}_S^{G_0 \times G_m}(L_0^i, A/L_0^i) \leq 3$, and hence the equality. Q.E.D.

5.4 Minimality of the invariant Hilbert scheme

We have seen that the invariant Hilbert scheme $\mathscr{H} = \mathrm{Hilb}_h^{G_0 \times G_m}(H_{q-p})$ is isomorphic to the minimal resolution of the weighted blow-up $E'_{l,m}$ in the cases where $l = p/q < 1$. It is then natural to ask if $\mathscr{H}$ is minimal over the $SL(2)$-variety $E_{l,m}$; in other words, if the Hilbert–Chow morphism γ is a minimal resolution of singularities. The following gives an answer to this question.

Theorem 5.4.1. *The Hilbert–Chow morphism γ is a minimal resolution of $E_{l,m}$ if and only if $1 + b \leq ap$.*

Recall from the previous sections that the invariant Hilbert scheme $\mathscr{H}$ fits into the following equivariant commutative diagram:



Since ψ is a minimal resolution, it suffices to show that $K_{E'_{l,m}}$ is f-nef if and only if $1 + b \leq ap$, where the former condition is equivalent to $K_{E'_{l,m}}$ being γ^--nef and γ^+-nef. We start by expressing the canonical divisor $K_{E'_{l,m}}$ in two ways with some $\alpha, \beta \in \mathbb{Q}$:

$$K_{E'_{l,m}} = (\gamma^-)^* K_{E^-_{l,m}} + \alpha D' = (\gamma^+)^* K_{E^+_{l,m}} + \beta D'.$$

Lemma 5.4.2. *Let $\widetilde{C^-}$ and $\widetilde{C^+}$ be curves in $E'_{l,m}$ that are contracted to a point under γ^+ and γ^-, respectively. Then the canonical divisor $K_{E'_{l,m}}$ has the following intersection numbers with $\widetilde{C^-}$ and $\widetilde{C^+}$:*

$$K_{E'_{l,m}} \cdot \widetilde{C^-} = \frac{\beta(1+b)k}{(\alpha - \beta)aq^2}, \qquad K_{E'_{l,m}} \cdot \widetilde{C^+} = \frac{\alpha(1+b)k}{(\alpha - \beta)ap^2}.$$

Proof. By the projection formula, we have

$$K_{E'_{l,m}} \cdot \widetilde{C^-} = K_{E^-_{l,m}} \cdot C^- + \alpha D' \cdot \widetilde{C^-} = \beta D' \cdot \widetilde{C^-}$$

and

$$K_{E'_{l,m}} \cdot \widetilde{C^+} = \alpha D' \cdot \widetilde{C^+} = K_{E^+_{l,m}} \cdot C^+ + \beta D' \cdot \widetilde{C^+},$$

so that the lemma follows from Theorem 2.3.10. Q.E.D.

Below we compute the coefficients α and β by using combinatorial data of the colored cones of $E^-_{l,m}$, $E^+_{l,m}$, and $E'_{l,m}$. Denote by $\mathfrak{X}(\widetilde{B})$ the group of characters of $\widetilde{B}$. The lattice $\mathcal{M} = \{Z^i W^j \in \mathbb{C}(\mathfrak{U})^* \,:\, m|(i-j)\}$ of rational $\widetilde{B}$-eigenfunctions on the dense orbit $\mathfrak{U}$ is generated by ZW and Z^m, and the natural homomorphism $f : \mathcal{M} \longrightarrow \mathfrak{X}(\widetilde{B}) \cong \mathbb{Z}^2$ is given by $Z^i W^j \mapsto (i+j, i-j)$. We denote the image of f by Γ. Set $\mathbf{v}_1 := f(ZW) = (2,0)$, and $\mathbf{v}_2 := f(Z^m) = (m,m)$. We remark that $\mathbf{v}_1$ is a simple root of $SL(2)\times\mathbb{C}^*$. If we denote the dual basis of $\{\mathbf{v}_1, \mathbf{v}_2\}$ by $\{\mathbf{u}_1, \mathbf{u}_2\}$, the lattice vectors ρ_{v_D}, $\rho_{v_{S^-}}$, $\rho_{v_{S^+}}$, and $\rho_{v_{D'}}$ in $\Gamma^\vee \subset Q := \mathrm{Hom}(\Gamma, \mathbb{Q})$ defined by the $\widetilde{B}$-stable divisors D, S^-, S^+, and D' are given as follows:

$$\rho_{v_D} = -b\mathbf{u}_1 + ap\mathbf{u}_2, \qquad \rho_{v_{S^-}} = \mathbf{u}_1, \qquad \rho_{v_{S^+}} = \mathbf{u}_1 + m\mathbf{u}_2, \qquad \rho_{v_{D'}} = \mathbf{u}_2.$$

The valuation cone $\mathcal{V} \subset Q$ can be described as $\mathcal{V} = \{x\mathbf{u}_1 + y\mathbf{u}_2 \in Q \,:\, x \le 0\}$, and $-\mathcal{V}^\vee$ is generated by $\mathbf{v}_1$, which turns out that $\mathbf{v}_1$ is a spherical root. Moreover, the colored cones of $E^-_{l,m}$, $E^+_{l,m}$, and $E'_{l,m}$ are descried as follows:

$$\begin{aligned}
\mathcal{C}^- &= \mathcal{C}(E^-_{l,m}) = \mathbb{Q}_{\ge 0}\rho_{v_D} + \mathbb{Q}_{\ge 0}\rho_{v_{S^+}}, & \mathcal{F}^- &= \mathcal{F}(E^-_{l,m}) = \{\rho_{v_{S^+}}\}\\
\mathcal{C}^+ &= \mathcal{C}(E^+_{l,m}) = \mathbb{Q}_{\ge 0}\rho_{v_D} + \mathbb{Q}_{\ge 0}\rho_{v_{S^-}}, & \mathcal{F}^+ &= \mathcal{F}(E^+_{l,m}) = \{\rho_{v_{S^-}}\}\\
\mathcal{C}' &= \mathcal{C}(E'_{l,m}) = \mathbb{Q}_{\ge 0}\rho_{v_D} + \mathbb{Q}_{\ge 0}\rho_{v_{D'}}, & \mathcal{F}' &= \mathcal{F}(E'_{l,m}) = \emptyset.
\end{aligned}$$

Remark 5.4.2.1. Colored cones of $E^-_{l,m}$, $E^+_{l,m}$, and $E'_{l,m}$ have already been given in §2.3.12 and §3.2.1. However, we have included the calculation above to specify the basis of the lattice Γ, which is different from the one chosen in §2.3.12 and more convenient for our later discussion.

Lemma 5.4.3. *Let $h_{\mathcal{C}^-}$ and $h_{\mathcal{C}^+}$ be linear functions corresponding to the colored cones $(\mathcal{C}^-, \mathcal{F}^-)$ and $(\mathcal{C}^+, \mathcal{F}^+)$, respectively, in the sense of Remark 2.2.14.2. Then, one has*

$$h_{\mathcal{C}^-} = \frac{p-k}{q}\mathbf{v}_1 + \frac{1+b}{aq}\mathbf{v}_2, \qquad h_{\mathcal{C}^+} = \mathbf{v}_1 + \frac{1+b}{ap}\mathbf{v}_2.$$

Proof. The anticanonical divisor of $E^-_{l,m}$ (and hence of $E^+_{l,m}$) can be described as $-K_{E^-_{l,m}} = D + a_{S^-}S^- + a_{S^+}S^+$ for some a_{S^-}, $a_{S^+} \in \mathbb{Q}$. Taking into account that the parabolic subgroup corresponding to $\mathbf{v}_1$ is $SL(2)\times\mathbb{C}^*$, and that $(SL(2)\times\mathbb{C}^*)\cdot S^- \ne S^-$ and $(SL(2)\times\mathbb{C}^*)\cdot S^+ \ne S^+$, it follows from Remark 2.2.14.1 that $a_{S^-} = a_{S^+} = 1$. Therefore, we have $h_{\mathscr{C}^-}(\rho_{v_D}) = 1 = h_{\mathscr{C}^-}(\rho_{v_{S^+}})$ and $h_{\mathscr{C}^+}(\rho_{v_D}) = 1 = h_{\mathscr{C}^+}(\rho_{v_{S^-}})$, and hence the lemma. Q.E.D.

Proof of Theorem 5.4.1. By Remark 2.2.14.3, one has

$$\alpha = h_{\mathscr{C}^-}(\rho_{v_{D'}}) - 1 = \frac{1+b}{aq} - 1, \qquad \beta = h_{\mathscr{C}^+}(\rho_{v_{D'}}) - 1 = \frac{1+b}{ap} - 1.$$

In particular, $\alpha - \beta < 0$. Therefore, in view of Lemma 5.4.2, we have $K_{E^-_{l,m}} \cdot \widetilde{C^-} \geq 0$ and $K_{E^+_{l,m}} \cdot \widetilde{C^+} \geq 0$ if and only if $1 + b \leq ap$. Q.E.D.

Remark 5.4.3.1. The existence of the minimal resolution $\mathscr{W}$ of $E_{l,m}$ was proved by Panyushev [Pan88], and he constructed it as the minimal resolution of $E^+_{l,m} \cong SL(2) \times_B S^+$, which is described by the Hirzebruch–Jung continued fraction arising from the cone σ of the toric surface S^+ (see also [BH08]). It follows that the Hilbert–Chow morphism γ factors as

$$\mathscr{H} \longrightarrow \mathscr{W} \longrightarrow E^+_{l,m} \longrightarrow E_{l,m}.$$

Therefore, Theorem 5.4.1 implies that the invariant Hilbert scheme $\mathscr{H}$ and the minimal resolution $\mathscr{W}$ coincide if and only if $1 + b \leq ap$. Consider the subdivision of σ obtained by adding a new ray $\mathbb{R}_{\geq 0} e_1$. This defines the morphism $E'_{l,m} \longrightarrow E^+_{l,m}$. If $1 + b \leq ap$, then the subdivision coincides with the first step of that defined by the Hirzebruch–Jung continued fraction for constructing the minimal resolution $\mathscr{W}$, concerning that the cone σ is in the normal form in the sense of [CLS11, §10.1] if and only if $1 + b \leq ap$.

Chapter 6

Further discussions

6.1 The Cox ring and the associated invariant Hilbert scheme

Given an affine normal variety, there would be several ways to describe it as an affine quotient. One of them, on which we elaborate below, is to consider its *Cox ring* equipped with a natural action of a quasitorus. By definition, a *quasitorus* is an affine algebraic group whose coordinate ring is generated by characters as a $\mathbb{C}$-vector space ([ADHL15, Definition 1.2.1.1]). Below we recall the construction of the *Cox ring* following [BH08, Definition 2.1] (see also [ADHL15, §1.4]).

Let Y be a normal variety with the field of rational functions $\mathbb{C}(Y)$. For a divisor D on Y, we put $\mathscr{L}(D) := \{f \in \mathbb{C}(Y) \,:\, (f) + D \geq 0\}$. Assume that every invertible regular function on Y is constant, and that the divisor class group $\mathrm{Cl}(Y)$ is finitely generated, so that there is an isomorphism $\mathrm{Cl}(Y) \cong \mathbb{Z}^n \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_s\mathbb{Z}$. We fix divisors $E_1, \ldots, E_n$ in Y that generate the free part $\mathbb{Z}^n$. We also fix a divisor W_j $(1 \leq j \leq s)$ in Y whose class generates $\mathbb{Z}/m_j\mathbb{Z}$ and choose a rational function f_j such that $m_j W_j = (f_j)$. For each tuple $k = (k_1, \ldots, k_{n+s}) \in \mathbb{Z}^{n+s}$, we set $D(k) := \sum_{1 \leq i \leq n} k_i E_i + \sum_{1 \leq j \leq s} k_{n+j} W_j$. Then we have the following isomorphism for any $1 \leq j \leq s$:

$$\alpha_j : \mathscr{L}(D(k)) \longrightarrow \mathscr{L}(D(k) + m_j W_j), \qquad f \mapsto \frac{f}{f_j}.$$

We denote by S the $\mathbb{Z}^{n+s}$-graded ring $\bigoplus_{k \in \mathbb{Z}^{n+s}} \mathscr{L}(D(k))$, and consider the ideal I of S generated by $f - \alpha_j(f)$ for every $f \in \mathscr{L}(D(k))$, $k \in \mathbb{Z}^{n+s}$, and $j \in \{1, \ldots, s\}$. Then,

$$\mathrm{Cox}(Y) := S/I$$

is called the *Cox ring* of Y. It is uniquely defined up to isomorphism and does not depend on the choice of E_i, W_j, and f_j. Moreover, it has a natural structure of a $\mathrm{Cl}(Y)$-graded ring:

$$\mathrm{Cox}(Y) = S/I \cong \bigoplus_{[D] \in \mathrm{Cl}(Y)} \Gamma(Y, \mathcal{O}_Y(D)).$$

The $\mathrm{Cl}(Y)$-grading on $\mathrm{Cox}(Y)$ defines an action of the quasitorus $G = \mathrm{Spec}(\mathbb{C}[\mathrm{Cl}(Y)])$ on $X = \mathrm{Spec}(\mathrm{Cox}(Y))$. We consider the quotient $X/\!/G := \mathrm{Spec}(\mathbb{C}[X]^G)$, which comes with the quotient morphism $\pi : X \longrightarrow X/\!/G$.

Theorem 6.1.1 ([ADHL15, Corollary 1.6.3.4])**.** *If Y is affine, then $X/\!/G$ is isomorphic to Y.*

Theorem 6.1.2 ([ADHL15, Remark 1.6.4.2])**.** *Keep the above notation.*

(i) *Let $Y_{\mathrm{reg}} \subset Y$ be the set of smooth points. Then, $\pi^{-1}(Y_{\mathrm{reg}}) \subset X$ is smooth, and G acts freely on $\pi^{-1}(Y_{\mathrm{reg}})$.*

(ii) *For any $x \in \pi^{-1}(Y_{\mathrm{reg}})$, the orbit $G \cdot x$ is closed in X.*

Remark 6.1.2.1. By Theorem 6.1.2, the Hilbert function of a general fiber of π coincides with that of the regular representation $\mathbb{C}[G]$.

We will use the following theorems in the forthcoming sections.

Theorem 6.1.3 ([ADHL15, Lemma 1.5.1.2])**.** *Let Y be as above, and let U be an open subset of Y such that* $\mathrm{codim}(Y \setminus U) \geq 2$. *Then,* $\mathrm{Cox}(Y) \longrightarrow \mathrm{Cox}(U)$, $f \mapsto f|_U$ *is an isomorphism.*

Theorem 6.1.4 ([ADHL15, Theorem 4.5.1.8])**.** *Let H be a connected affine algebraic group with trivial Picard group* $\mathrm{Pic}(H)$ *and trivial character group* $\mathfrak{X}(H)$*, and let $F \subset H$ be a closed subgroup. Define $F_1 := \bigcap_{\chi \in \mathfrak{X}(F)} \mathrm{Ker}(\chi) \subset F$, and $G := \mathrm{Spec}(\mathbb{C}[\mathfrak{X}(F)])$. Then, G is isomorphic to the quasitorus* $\mathrm{Spec}(\mathbb{C}[\mathrm{Cl}(H/F)])$*, and the Cox ring of H/F is given as follows:*

$$\mathrm{Cox}(H/F) \cong \mathbb{C}[H/F_1] \cong \mathbb{C}[H]^{F_1} \cong \bigoplus_{\chi \in \mathfrak{X}(F)} \mathbb{C}[H]^{F_1}_{\chi}.$$

Remark 6.1.4.1. In the situation of Theorem 6.1.4, the H-action on the homogeneous space H/F lifts to the Cox ring $\mathrm{Cox}(H/F)$ via the isomorphism $\mathrm{Cox}(H/F) \cong \mathbb{C}[H/F_1]$. A more general statement about lifting of a given action of an affine algebraic group on Y to $\mathrm{Spec}(\mathrm{Cox}(Y))$ can be found in [ADHL15, Theorem 4.2.3.2]. See also [Gaĭ08, Proposition 2].

Given an affine normal variety Y with only constant invertible regular functions and finitely generated divisor class group, Theorem 6.1.1 tells us that Y can be restored from its Cox ring $\mathrm{Cox}(Y)$. This motivates us to consider the associated invariant Hilbert scheme and ask the following question: if we describe Y as an affine quotient of $X = \mathrm{Spec}(\mathrm{Cox}(Y))$ by the action of $G = \mathrm{Spec}(\mathbb{C}[\mathrm{Cl}(Y)])$ and if we take h to be the Hilbert function of a general fiber of the quotient morphism $\pi : X \longrightarrow Y$, does the associated invariant Hilbert scheme $\mathrm{Hilb}_h^G(X)$ give a resolution of singularities of Y via the Hilbert–Chow morphism $\gamma : \mathrm{Hilb}_h^G(X) \longrightarrow Y$?

In the forthcoming sections, we consider the case where Y is a closure of the maximal nilpotent orbit in the Lie algebra $\mathfrak{sl}_n$ and ask if the corresponding Hilbert–Chow morphism coincide with the Springer's resolution. As we will see below, the case $n = 2$ is classical and known to give a positive answer (Example 6.2.2). The case $n = 3$ will be discussed in the last section.

6.2 Closures of nilpotent orbits in $\mathfrak{sl}_n$

We first review the one to one correspondence between nilpotent orbits in $\mathfrak{sl}_n$ and partitions of n. Given a partition $\mathbf{d} = [d_1, \ldots, d_k]$ of n, i.e., $d_1, \ldots, d_k$ are integers that satisfy $d_1 \geq \cdots \geq d_k > 0$ and $d_1 + \cdots + d_k = n$, we can define a nilpotent element $A_{\mathbf{d}}$ in $\mathfrak{sl}_n$ according to the partition $\mathbf{d}$ as follows:

$$A_{\mathbf{d}} = \begin{pmatrix} J_{d_1} & & \\ & \ddots & \\ & & J_{d_k} \end{pmatrix},$$

where

$$J_{d_i} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

is the Jordan block of size d_i. The orbit of $A_{\mathbf{d}}$ under the conjugate action of $SL(n)$ is denoted by $O_{\mathbf{d}}$. Since any nilpotent element $A \in \mathfrak{sl}_n$ is conjugate to a unique $A_{\mathbf{d}}$ for some partition $\mathbf{d}$ of n, the correspondence $\mathbf{d} \mapsto O_{\mathbf{d}}$ is bijective ([CM93, Proposition 3.1.7]). An order relation is defined on the set of nilpotent orbits in $\mathfrak{sl}_n$ by inclusion of closures. The maximal orbit corresponds to the partition $\mathbf{d} = [n]$. According to [Fu03, FN04], the closure $\overline{O_{[n]}}$ of the maximal nilpotent orbit admits a unique symplectic resolution of singularities, the Springer's resolution:

$$SL(n) \times_B \mathfrak{n} \longrightarrow \overline{O_{[n]}}, \qquad (X, A) \mapsto XAX^{-1},$$

where $\mathfrak{n} = \{A = (a_{i,j}) \in \mathfrak{sl}_n \, : \, a_{i,j} = 0 \, (i \geq j)\}$, and $SL(n) \times_B \mathfrak{n}$ denotes the quotient $(SL(n) \times \mathfrak{n})/B$ under the action of the Borel subgroup $B = \{b = (b_{i,j}) \in SL(n) \, : \, b_{i,j} = 0 \, (i > j)\}$ defined by $b \cdot (X, A) = (Xb^{-1}, bAb^{-1})$.

A natural question that arises is whether the Springer's resolution of $\overline{O_{[n]}}$ can be obtained as the invariant Hilbert scheme associated with the Cox ring $\mathrm{Cox}(\overline{O_{[n]}})$ in the sense of the previous section. We first remark that $\mathrm{codim}(\overline{O_{[n]}} \setminus O_{[n]}) = 2$, so that the Cox ring of $\overline{O_{[n]}}$ is isomorphic to that of $O_{[n]}$ by Theorem 6.1.3. Following the notation of Theorem 6.1.4, F denotes the stabilizer of $A_{[n]}$:

$$F = \{X = (x_{i,j}) \in SL(n) \, : \, x_{i,j} = x_{i+1,j+1} \, (i \leq j), \; x_{i,j} = 0 \, (i > j)\}.$$

Also, we have $F_1 = \{X = (x_{i,j}) \in F \, : \, x_{i,i} = 1\}$. The Cox ring of the maximal orbit $O_{[n]}$ is isomorphic to the invariant ring

$$R := \mathbb{C}[SL(n)]^{F_1} \cong \bigoplus_{\chi \in \mathfrak{X}(F) \cong \mathbb{Z}/n\mathbb{Z}} \mathbb{C}[SL(n)]^{F_1}_{\chi},$$

and the quotient of $\mathrm{Spec}(R)$ by the action of $\mathrm{Spec}(\mathbb{C}[\mathfrak{X}(F)]) \cong \mu_n$ is isomorphic to $\overline{O_{[n]}}$.

We denote by $\mathscr{H}_{[n]}$ the invariant Hilbert scheme $\mathrm{Hilb}^{\mu_n}_h(\mathrm{Spec}(R))$ associated with the triple $(\mu_n, \mathrm{Spec}(R), h)$, where h is the Hilbert function of a general fiber of the quotient morphism $\pi : \mathrm{Spec}(R) \longrightarrow \overline{O_{[n]}}$. As mentioned in Remark 6.1.2.1, h coincides with the Hilbert function of the regular representation $\mathbb{C}[\mu_n]$. The corresponding Hilbert–Chow morphism

$$\gamma : \mathscr{H}_{[n]} \longrightarrow \overline{O_{[n]}}$$

is an isomorphism over the maximal orbit $O_{[n]}$, so that the main component is $\mathscr{H}^{main} = \overline{\gamma^{-1}(O_{[n]})}$. We pose the following question.

Question 6.2.1. Does the Hilbert–Chow morphism γ (or its restriction to the main component $\mathcal{H}^{main}$) give a resolution of singularities of $\overline{O_{[n]}}$? And if it does, does it coincide with the Springer's resolution?

Example 6.2.2. When $n = 2$, we have $F = \left\{ \begin{pmatrix} \xi & a \\ 0 & \xi \end{pmatrix} : \xi^2 = 1,\ a \in \mathbb{C} \right\}$, and the subgroup $F_1 \subset F$ coincides with the unipotent radical of the Borel subgroup $B \subset SL(2)$. The Cox ring of $\overline{O_{[2]}}$ is isomorphic to a polynomial ring in two variables $\mathbb{C}[t_1, t_2]$, and the quasitorus $\operatorname{Spec} \mathbb{C}[\mathfrak{X}(F)] \cong \mu_2$ acts on $\mathbb{C}[t_1, t_2]$ via multiplication on each variable. Therefore, the invariant Hilbert scheme $\mathcal{H}_{[2]}$ is isomorphic to the μ_2-Hilbert scheme μ_2-Hilb$(\mathbb{C}^2)$, which is known to give the minimal (hence crepant) resolution of singularities of $\mathbb{C}^2/\mu_2$ via the Hilbert–Chow morphism ([IN96]). Since the notion of crepant resolutions coincides with symplectic resolutions for symplectic varieties, we see that the case $n = 2$ gives a positive answer to Question 6.2.1.

6.3 The case $n = 3$

In this subsection, we consider the case where $n = 3$ and show the following.

Theorem 6.3.1. *The Hilbert–Chow morphism $\gamma : \mathcal{H}_{[3]} \longrightarrow \overline{O_{[3]}}$ is a resolution of singularities. Moreover, the invariant Hilbert scheme $\mathcal{H}_{[3]}$ is isomorphic to $SL(3) \times_B \mathfrak{n}$, and under this isomorphism, γ coincides with the Springer's resolution.*

Remark 6.3.1.1. The nilpotent matrix corresponding to the partition $\mathbf{d} = [3]$ is

$$A_{[3]} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The closure $\overline{O_{[3]}}$ is a 6-dimensional subvariety of $\mathfrak{sl}_3$, and it has three $SL(3)$-orbits: $O_{[3]}$, $O_{[2,1]}$, and the origin $O = O_{[1^3]}$. Also, the closed subgroups $F_1 \subset F \subset SL(3)$ are given as follows:

$$F = \left\{ \begin{pmatrix} \omega & a & b \\ 0 & \omega & a \\ 0 & 0 & \omega \end{pmatrix} : \omega^3 = 1,\ a, b \in \mathbb{C} \right\} \supset F_1 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

Let $X = (x_{i,j})_{1 \le i,j \le 3}$ be the coordinate of $SL(3)$, and $R = \mathbb{C}[SL(3)]^{F_1}$. The action of F on R gives it a natural structure of a $\mathbb{Z}/3\mathbb{Z}$-graded ring, and for each $0 \le i \le 2$, the component

$R_{(i)}$ of weight i is an $SL(3)$-submodule of R. The F-invariant ring $R^F = R_{(0)} \cong \mathbb{C}[SL(3)/F]$ is generated by $y_{i,j}$ $(1 \leq i, j \leq 3)$, where $y_{i,j}$ denotes the (i, j)-entry of the matrix $Y = XA_{[3]}X^{-1}$. Since $\overline{O_{[3]}}$ is normal and $\operatorname{codim}(\overline{O_{[3]}} \setminus O_{[3]}) = 2$, the regular functions $y_{i,j}$ on $SL(3)/F \cong O_{[3]}$ extend to those on $\overline{O_{[3]}}$, which we denote by the same symbol $y_{i,j}$. Set

$$g_{3,1} = \begin{vmatrix} x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \end{vmatrix}, \qquad g_{3,2} = -\begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{3,1} & x_{3,2} \end{vmatrix}, \qquad g_{3,3} = \begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{vmatrix},$$

and

$$\begin{aligned}
f_1 &= y_{1,1} + y_{2,2} + y_{3,3}, \\
f_2 &= y_{2,2}y_{3,3} - y_{2,3}y_{3,2} + y_{1,1}y_{3,3} - y_{1,3}y_{3,1} + y_{1,1}y_{2,2} - y_{1,2}y_{2,1}, \\
f_3 &= y_{1,1}g_{3,1} + y_{2,1}g_{3,2} + y_{3,1}g_{3,3}, \\
f_4 &= y_{1,2}g_{3,1} + y_{2,2}g_{3,2} + y_{3,2}g_{3,3}, \\
f_5 &= y_{1,3}g_{3,1} + y_{2,3}g_{3,2} + y_{3,3}g_{3,3}, \\
f_6 &= y_{1,1}x_{1,1} + y_{1,2}x_{2,1} + y_{1,3}x_{3,1}, \\
f_7 &= y_{2,1}x_{1,1} + y_{2,2}x_{2,1} + y_{2,3}x_{3,1}, \\
f_8 &= y_{3,1}x_{1,1} + y_{3,2}x_{2,1} + y_{3,3}x_{3,1}, \\
f_9 &= g_{3,1}^2 + y_{2,1}x_{3,1} - y_{3,1}x_{2,1}, \\
f_{10} &= g_{3,1}g_{3,2} + y_{2,2}x_{3,1} - y_{3,2}x_{2,1}, \\
f_{11} &= g_{3,1}g_{3,3} + y_{2,3}x_{3,1} - y_{3,3}x_{2,1}, \\
f_{12} &= g_{3,2}^2 - y_{1,2}x_{3,1} + y_{3,2}x_{1,1}, \\
f_{13} &= g_{3,2}g_{3,3} - y_{1,3}x_{3,1} + y_{3,3}x_{1,1}, \\
f_{14} &= g_{3,3}^2 + y_{1,3}x_{2,1} - y_{2,3}x_{1,1}, \\
f_{15} &= x_{1,1}^2 - y_{1,2}g_{3,3} + y_{1,3}g_{3,2}, \\
f_{16} &= x_{1,1}x_{2,1} - y_{2,2}g_{3,3} + y_{2,3}g_{3,2}, \\
f_{17} &= x_{1,1}x_{3,1} - y_{3,2}g_{3,3} + y_{3,3}g_{3,2}, \\
f_{18} &= x_{2,1}^2 + y_{2,1}g_{3,3} - y_{2,3}g_{3,1}, \\
f_{19} &= x_{2,1}x_{3,1} + y_{3,1}g_{3,3} - y_{3,3}g_{3,1}, \\
f_{20} &= x_{3,1}^2 - y_{3,1}g_{3,2} + y_{3,2}g_{3,1}, \\
f_{21} &= x_{1,1}g_{3,1} - y_{2,2}y_{3,3} + y_{2,3}y_{3,3}, \\
f_{22} &= x_{1,1}g_{3,2} + y_{1,2}y_{3,3} - y_{1,3}y_{3,2}, \\
f_{23} &= x_{1,1}g_{3,3} - y_{1,2}y_{2,3} + y_{1,3}y_{2,2}, \\
f_{24} &= x_{2,1}g_{3,1} - y_{2,1}y_{3,3} + y_{2,3}y_{3,1}, \\
f_{25} &= x_{2,1}g_{3,2} - y_{1,1}y_{3,3} + y_{1,3}y_{3,1}, \\
f_{26} &= x_{2,1}g_{3,3} + y_{1,1}y_{2,3} - y_{1,3}y_{2,1}, \\
f_{27} &= x_{3,1}g_{3,1} - y_{2,1}y_{3,2} + y_{2,2}y_{3,1}, \\
f_{28} &= x_{3,1}g_{3,2} + y_{1,1}y_{3,2} - y_{1,2}y_{3,1}, \\
f_{29} &= x_{3,1}g_{3,3} - y_{1,1}y_{2,2} + y_{1,2}y_{2,1}.
\end{aligned}$$

We can check that the following lemma actually holds, by a brute-force calculation on the computer algebra system Macaulay2 [GS].

Lemma 6.3.2. *The F_1-invariant ring $R = \mathbb{C}[SL(3)]^{F_1}$ is generated by $x_{1,1}$, $x_{2,1}$, $x_{3,1}$, $g_{3,1}$, $g_{3,2}$, $g_{3,3}$, and $y_{i,j}$ $(1 \le i, j \le 3)$, and the ideal of relations among these generators is generated by $f_1, \dots, f_{29}$. Moreover, the weight space $R_{(1)}$ (resp. $R_{(2)}$) is generated by $x_{1,1}$, $x_{2,1}$, and $x_{3,1}$ (resp. $g_{3,1}$, $g_{3,2}$, and $g_{3,3}$) as a module over the F-invariant ring $R^F = R_{(0)}$.*

Remark 6.3.2.1. Let $V_1 = \langle x_{1,1}, x_{2,1}, x_{3,1}\rangle$, and let $V_2 = \langle g_{3,1}, g_{3,2}, g_{3,3}\rangle$. Then, V_1 and V_2 are $SL(3)$-submodules of $R_{(1)}$ and $R_{(2)}$, respectively. Moreover, V_1 is isomorphic to the standard representation V of $SL(3)$, and $x_{1,1}$ is the highest weight vector with respect to the Borel subgroup $B \subset SL(3)$. Also, V_2 is isomorphic to $\bigwedge^2 V \cong V^\vee$, and $g_{3,3}$ is the highest weight vector.

Recall that the Hilbert–Chow morphism $\gamma : \mathscr{H}_{[3]} \longrightarrow \overline{O_{[3]}}$ is an isomorphism over the maximal orbit $O_{[3]}$. The next lemma gives the ideal corresponding to the closed point which is mapped to $A_{[3]}$ under the isomorphism.

Lemma 6.3.3. *Let $I_{[3]}$ be the ideal of R generated by the entries of the matrix*

$$\begin{pmatrix} y_{1,1} & y_{1,2}-1 & y_{1,3} \\ y_{2,1} & y_{2,2} & y_{2,3}-1 \\ y_{3,1} & y_{3,2} & y_{3,3} \end{pmatrix}$$

and $x_{2,1}$, $x_{3,1}$, $g_{3,1}$, $g_{3,2}$, $x_{1,1}^2 - g_{3,3}$, $x_{1,1} - g_{3,3}^2$, $x_{1,1}g_{3,3} - 1$. Then, $I_{[3]}$ defines a point in $\mathscr{H}_{[3]}$ and $\gamma([I_{[3]}]) = A_{[3]}$.

Proof. Let $I \subset R$ be the unique μ_3-stable ideal such that $\gamma([I]) = A_{[3]}$. Since the image of the zero set of I under the quotient morphism $\pi : \operatorname{Spec} R \longrightarrow \overline{O_{[3]}}$ is $A_{[3]}$, we get $(y_{1,1},\ y_{1,2} - 1,\ y_{1,3},\ y_{2,1},\ y_{2,2},\ y_{2,3} - 1,\ y_{3,1},\ y_{3,2},\ y_{3,3}) \subset I$. Taking into account that f_4, f_5, f_6, f_7, f_{14}, f_{15}, $f_{23} \in I$, one obtains $(g_{3,1},\ g_{3,2},\ x_{2,1},\ x_{3,1},\ x_{1,1}^2 - g_{3,3},\ g_{3,3}^2 - x_{1,1},\ x_{1,1}g_{3,3} - 1) \subset I$. Namely, $I_{[3]} \subset I$. On the other hand, we have $R/I_{[3]} \cong \mathbb{C}[x_{1,1}]/(x_{1,1}^3 - 1)$, which implies that the closed subscheme of $\operatorname{Spec} R$ associated with the ideal $I_{[3]}$ defines a point in $\mathscr{H}_{[3]}$. Therefore, $I_{[3]} = I$. Q.E.D.

We consider the set ${\mathscr{H}_{[3]}}^B$ of fixed points in $\mathscr{H}_{[3]}$ for the action of the Borel subgroup B. The lemma below shows that $\mathscr{H}_{[3]}$ contains a unique B-fixed point.

Lemma 6.3.4. *Let I_0 be the ideal of R generated by the entries of the matrix*

$$\begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} \\ y_{2,1} & y_{2,2} & y_{2,3} \\ y_{3,1} & y_{3,2} & y_{3,3} \end{pmatrix}$$

and $x_{2,1}$, $x_{3,1}$, $g_{3,1}$, $g_{3,2}$, $x_{1,1}^2$, $x_{1,1}g_{3,3}$, $g_{3,3}^2$. Then, ${\mathscr{H}_{[3]}}^B = \{[I_0]\}$. In particular, $\mathscr{H}_{[3]}$ is connected.

Proof. Take any $[I] \in \mathscr{H}^B$. Then, we have $\gamma([I]) = O$ since the origin $O \in \overline{O_{[3]}}$ is the unique B-fixed point. Therefore, $y_{i,j} \in I$ for any $1 \le i, j \le 3$. Since $I \subset R$ is B-stable, we see by Remark 6.3.2.1 that the weight spaces $(R/I)_{(1)}$ and $(R/I)_{(2)}$ of μ_3-weight 1 and 2 are spanned by $x_{1,1}$ and $g_{3,3}$, respectively. Therefore, $(x_{2,1},\ x_{3,1},\ g_{3,1},\ g_{3,2}) \subset I$. Moreover, we get $(x_{1,1}^2,\ g_{3,3}^2,\ x_{1,1}g_{3,3}) \subset I$ by the conditions f_{15}, f_{14}, $f_{23} \in I$. Thus, $I_0 \subset I$. Since I_0 is of colength 3, one obtains $I_0 = I$. The connectedness of $\mathscr{H}_{[3]}$ follows from Theorem 2.1.7. Q.E.D.

Lemma 6.3.5. $\dim \mathrm{Hom}_R^{\mu_3}(I_0, R/I_0) = 6$ *($= \dim \mathscr{H}^{main}$).*

Proof. Let $\phi \in \mathrm{Hom}_R^{\mu_3}(I_0, R/I_0)$. Since $R/I_0 \cong \mathbb{C}[x_{1,1}, g_{3,3}]/(x_{1,1}^2,\ g_{3,3}^2,\ x_{1,1}g_{3,3})$ and ϕ preserves the action of μ_3, we have

$$\begin{array}{llll} \phi(x_{2,1}) = \alpha_1 x_{1,1}, & \phi(x_{3,1}) = \alpha_2 x_{1,1}, & \phi(g_{3,1}) = \alpha_3 g_{3,3}, & \phi(g_{3,2}) = \alpha_4 g_{3,3} \\ \phi(x_{1,1}^2) = \alpha_5 g_{3,3}, & \phi(x_{1,1}g_{3,3}) = \alpha_6, & \phi(g_{3,3}^2) = \alpha_7 x_{1,1}, & \phi(y_{i,j}) = \beta_{i,j} \end{array}$$

for some $a_1, \ldots, \alpha_7, \beta_{ij} \in \mathbb{C}$. On the other hand, since ϕ is a homomorphism of R-modules, we get $\beta_{1,1} = \beta_{2,1} = \beta_{3,1} = \beta_{3,2} = \beta_{3,3} = 0$ by $f_3, f_4, f_5, f_6, f_7, f_8 \in I$. Also, by $f_{14}, f_{15} \in I$, we have $\alpha_7 - \beta_{2,3} = \alpha_5 - \beta_{1,2} = 0$. Moreover, we have $\beta_{2,2} = \alpha_6 = 0$ by $f_{16}, f_{23} \in I$. Therefore, $\dim \mathrm{Hom}_R^{\mu_3}(I_0, R/I_0) \le 6$, and hence the equality. Q.E.D.

Corollary 6.3.6. *The invariant Hilbert scheme $\mathscr{H}_{[3]}$ is smooth and coincides with the main component $\mathscr{H}^{main}$.*

Proof. This is an immediate consequence of Theorem 2.1.7. Q.E.D.

Proof of Theorem 6.3.1. Taking into account the discussions above, it remains to show that γ coincides with the Springer's resolution. For each $i \in \{1, 2\}$, we define

$$\eta_i : \mathscr{H}_{[3]} \longrightarrow \mathrm{Gr}(1, V_i^\vee) \cong \mathbb{P}^2, \qquad I \mapsto V_i/\mathrm{Ker}\, f_I,$$

where f_I is the composition of the natural inclusion $F_i \hookrightarrow R_{(i)}$ and the surjection $R_{(i)} \longrightarrow (R/I)_{(i)}$ (see §2.1.6 for details). The isomorphism $\mathrm{Gr}(1, V_1^\vee) \cong \mathbb{P}^2$ is given by sending $s_0 x_{1,1}{}^\vee + s_1 x_{2,1}{}^\vee + s_2 x_{3,1}{}^\vee$ to $[s_0 : s_1 : s_2]$, where $x_{i,1}{}^\vee$ denote the dual basis of $x_{i,1}$ $(1 \leq i \leq 3)$. Similarly, the isomorphism $\mathrm{Gr}(1, V_2^\vee) \cong \mathbb{P}^2$ is given by $t_0 g_{3,1}{}^\vee + t_1 g_{3,2}{}^\vee + t_2 g_{3,3}{}^\vee \mapsto [t_0 : t_1 : t_2]$, where $g_{3,j}{}^\vee$ denotes the dual basis of $g_{3,j}$ $(1 \leq j \leq 3)$. Set $\eta = \eta_1 \times \eta_2 : \mathscr{H}_{[3]} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2$. Then we have $\eta([I_{[3]}]) = ([1:0:0], [0:0:1])$. The stabilizer of $\eta([I_{[3]}])$ is the Borel subgroup B, and hence we get a surjective morphism $\eta : \mathscr{H}_{[3]} \longrightarrow SL(3)/B$. Let $N = \eta^{-1}(1)$, where 1 stands for the identity matrix. Then, we have $\mathscr{H}_{[3]} \cong SL(3) \times_B N$. We claim that $N \cong \mathfrak{n}$, where

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & y_{1,2} & y_{1,3} \\ 0 & 0 & y_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \;:\; y_{1,2},\ y_{1,3},\ y_{2,3} \in \mathbb{C} \right\}.$$

Let $[I] \in N$, and let $\gamma([I]) = (a_{i,j})_{1 \leq i,j \leq 3} \in \overline{O_{[3]}} \subset \mathfrak{sl}_3$. By the construction of η, we have $x_{2,1},\ x_{3,1},\ g_{3,1},\ g_{3,2} \in I$. Then it follows that $a_{1,1} = a_{2,1} = a_{3,1} = a_{2,2} = a_{3,2} = a_{3,3} = 0$ by the conditions $f_3,\ f_4,\ f_5,\ f_6,\ f_7,\ f_8,\ f_{16} \in I$. This implies that $N \cong \mathfrak{n}$. Q.E.D.

Remark 6.3.6.1. One would be able to show that the answers to Question 6.2.1 is also positive for the cases where $n \geq 4$ by using theorems from [Gra92].

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List of Research Achievements

1 Articles

1.1 Published Articles

[1] Ayako Kubota, Invariant Hilbert scheme resolution of Popov's $SL(2)$-varieties, to appear in Transformation Groups.

1.2 Conference Proceedings

[2] 久保田 絢子, Invariant Hilbert scheme resolution of Popov's $SL(2)$-varieties, 「第 11 回数論女性の集まり」報告集, 2018 年 10 月, pp. 44-53.

[3] 久保田 絢子, On minimality of the invariant Hilbert scheme associated to Popov's $SL(2)$-variety, 「第 12 回数論女性の集まり」報告集, 2019 年 10 月, pp. 32-41.

2 Research Presentations

2.1 Oral Presentations

1. 久保田 絢子, 永井 保成, Minimal generating set of a ring of invariants for vectors, linear forms, and matrices, 代数幾何ミニ研究集会, 埼玉大学, 2016 年 3 月 10 日.

2. 久保田 絢子, 不変 Hilbert スキームとアファイン準等質 $SL(2)$-多様体, 第 14 回城崎新人セミナー, 城崎総合支所城崎市民センター大会議室, 2017 年 2 月 15 日.

3. 久保田 絢子, Invariant Hilbert scheme resolution of Popov's $SL(2)$-varieties, 代数幾何ミニ研究集会, 埼玉大学, 2018 年 2 月 28 日.

4. 久保田 絢子, Invariant Hilbert scheme resolution of Popov's $SL(2)$-varieties, 第 23 回代数学若手研究会, 大阪大学吹田キャンパス, 2018 年 3 月 7 日.

5. 久保田 絢子, Invariant Hilbert scheme resolution of Popov's $SL(2)$-varieties, 日本数学会 2018 年度年会, 東京大学駒場キャンパス, 2018 年 3 月 21 日.

6. 久保田 絢子, Invariant Hilbert scheme resolution of Popov's toric $SL(2)$-varieties, 特異点論月曜セミナー, 日本大学文理学部, 2018 年 4 月 16 日.

7. 久保田 絢子, 不変 Hilbert スキームによる 3 次元アファイン正規準等質 $SL(2)$-多様体の特異点解消, 小山高専数学セミナー, 小山工業高等専門学校, 2018 年 6 月 8 日.

8. 久保田 絢子, Invariant Hilbert scheme resolution of Popov's $SL(2)$-varieties, 第 11 回数論女性の集まり, 立教大学池袋キャンパス, 2018 年 6 月 9 日.

9. Ayako Kubota, Invariant Hilbert scheme resolution of Popov's $SL(2)$-varieties, Younger Generations in Algebraic and Complex Geometry V, Hakodate Community Plaza G-SQUARE, August 7, 2018.

10. 久保田 絢子, 準等質 $SL(2)$-多様体について, 第 1 回宇都宮大学代数幾何研究集会, 宇都宮大学峰キャンパス, 2018 年 8 月 22 日.

11. Ayako Kubota, Invariant Hilbert scheme resolution of Popov's $SL(2)$-varieties, Varieties and Group Actions, Mathematics Institute of Polish Academy and Sciences, Warsaw, Poland, September 24, 2018.

12. Ayako Kubota, Invariant Hilbert scheme resolution of Popov's $SL(2)$-varieties, McKay Correspondence and Noncommutative Algebra, Nagoya University, November 17, 2018.

13. 久保田 絢子, Popov の $SL(2)$-多様体に付随する不変 Hilbert スキームの極小性について, 第 12 回数論女性の集まり, 東京理科大学神楽坂キャンパス, 2019 年 5 月 18 日.

14. 久保田 絢子, On minimality of the invariant Hilbert scheme associated with Popov's $SL(2)$-variety, 第 2 回宇都宮大学代数幾何研究集会, 宇都宮大学峰キャンパス, 2019 年 9 月 13 日.

15. 久保田 絢子, On minimality of the invariant Hilbert scheme associated with Popov's $SL(2)$-variety, 日本数学会 2019 年秋季総合分科会, 金沢大学角間キャンパス, 2019 年 9 月 20 日.

2.2 Poster Presentations

16. Ayako Kubota, Invariant Hilbert scheme resolution of Popov's $SL(2)$-varieties, Kinosaki Algebraic Geometry Symposium, Kinosaki International Arts Center, October 25, 2017.

17. Ayako Kubota, Invariant Hilbert scheme resolution of Popov's $SL(2)$-varieties, K3 Surfaces and Related Topics, Nagoya University, December 20, 2017.