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博 士 論 文 概 要
Doctoral Thesis Synopsis

論 文 題 目
Thesis Theme
Well-posedness and ill-posedness of the stationary Navier-Stokes equations in scaling invariant Besov spaces

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In this doctoral thesis, we consider the stationary Navier-Stokes equations, which describe the incompressible viscous fluid independent of the time development, in $\mathbb{R}^n$ with $n \geq 3$ as follows:

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla \Pi = f, & x \in \mathbb{R}^n, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^n, \end{cases} \tag{SNS}$$

Here $u = (u_1(x), \dots, u_n(x))$ and $\Pi = \Pi(x)$ denote the unknown velocity vector field and the unknown pressure of the fluid at the point $x \in \mathbb{R}^n$, respectively, while $f = (f_1(x), \dots, f_n(x))$ is the given external force. In (SNS), $-\Delta u$ denotes the viscosity term, and $u \cdot \nabla u$ denotes the derivative of u in the direction along itself.

In this thesis, we focus on the well-posedness and ill-posedness problems on (SNS). Let us first define the notions of them: Let $(D, \|\cdot\|_D)$ and $(S, \|\cdot\|_S)$ be two Banach spaces (here *D* and *S* indirectly denote the spaces of data (external forces) and of solutions, respectively). We say that (SNS) is well-posed from $(D, \|\cdot\|_D)$ to $(S, \|\cdot\|_S)$ if there exist two constants $\varepsilon > 0$ and $\delta > 0$ such that

(i) For any $f \in B_D(\varepsilon)$, there exist a solution $u \in B_S(\delta)$ of (SNS),

(ii) If there exist two solutions $u_1, u_2 \in B_S(\delta)$ of (SNS) for one external force $f \in B_D(\varepsilon)$, then it holds that $u_1 \equiv u_2$ in S,

(iii) The map $f \in (D, \|\cdot\|_D) \mapsto u \in (S, \|\cdot\|_S)$, which is well defined by (i) and (ii), is continuous,

where $B_D(\varepsilon) \equiv \{f \in D; \|f\|_D < \varepsilon\}$ and $B_S(\delta) \equiv \{f \in S; \|f\|_S < \delta\}$. In addition, (SNS) is ill-posed from *D* to *S* if (SNS) is not well-posed from *D* to *S*. It seems to be an important problem to find more general spaces *D* and *S* where (SNS) is well-posed from *D* to *S*. We now deal with this problem in homogeneous Besov spaces $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$ for $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$.

Recently, the well-posedness of (SNS) in homogeneous Besov spaces was studied by Kaneko-Kozono-Shimizu (2017). They showed that (SNS) is well-posed from $D = \dot{B}^{-3+n/p}_{p,q}$ to $S = \dot{B}^{-1+n/p}_{p,q}$ for all $1 \leq p < n$ and $1 \leq q \leq \infty$. These spaces *D* and *S* are scaling invariant for the external force *f* and the velocity *u* in (SNS) respectively. Indeed, if a triple $\{u, \Pi, f\}$ solves (SNS), so does $\{u_\lambda, \Pi_\lambda, f_\lambda\}$ for every $\lambda > 0$, with $u_\lambda(x) \equiv \lambda u(\lambda x)$, $\Pi_\lambda(x) \equiv \lambda^2 \Pi(\lambda x)$, $f_\lambda(x) \equiv \lambda^3 f(\lambda x)$. Then we see that $\|f_\lambda\|_D \cong \|f\|_D$ and $\|u_\lambda\|_S \cong \|u\|_S$ for all λ. Actually, their study in homogeneous Besov spaces enables us handle a larger class of functions which never belong to the usual Sobolev space, such as the Dirac delta function, which belongs to $\dot{B}^{-n+n/p}_{p,\infty}$ for $1 \leq p \leq \infty$.

In this thesis, we will first review the study by Kaneko-Kozono-Shimizu, and also consider a similar problem in homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$ for comparison, which are also generalization of Sobolev spaces. Actually, even in the case of $D = \dot{F}^{-3+n/p}_{p,q}$ and $S = \dot{F}^{-1+n/p}_{p,q}$, we can prove the well-posedness of (SNS), provided $1 < p < n$ and $1 \leq q \leq \infty$, and provided $p = n$ and $1 \leq q \leq 2$, by similar methods to Kaneko-Kozono-Shimizu. Indeed, we make use of the boundedness of the Riesz transform, the product estimate, and the embedding theorem in homogeneous Besov and Triebel-Lizorkin spaces. Furthermore, in the case of Triebel- Lizorkin space, we can see some advantages in the sense of the regularity of solutions. More precisely, we will prove that if a small external force in the above scaling invariant Triebel-Lizorkin spaces with $1 < p < n$ also belongs to the homogeneous Sobolev space $\dot{H}^{s-2,r}$ with $s > 0$ and $1 < r < \infty$, or with $s = 0$ and $n/(n-1) < r < \infty$, then the solution belongs to $\dot{H}^{s,r}$. Although Kaneko-Kozono-Shimizu showed a similar result on regularity, some additional restrictions for *s* and *r* are required in the case of Besov spaces.

Now our main purpose in this thesis is to show that the well-posedness result by Kaneko-Kozono-Shimizu is almost optimal in the scaling invariant Besov spaces. In other words, we will prove that (SNS) is ill-posed from $D = \dot{B}_{p,q}^{-3+n/p}$ to $S = \dot{B}_{p,q}^{-1+n/p}$ for $n < p \le \infty$ and $1 \le q \le \infty$, and for $p = n,\ 2 < q \le \infty$.

In the second chapter, we will first prove the ill-posedness in the extreme case $p = \infty$, i.e., we will show that (SNS) is ill-posed from $\dot{B}_{\infty,q}^{-3}$ to $\dot{B}_{\infty,q}^{-1}$ for all $1 \le q \le \infty$ in the sense that it occurs a lack of continuity of the solution map $f \mapsto u$. More precisely, we will construct a sequence $\{f_N\}_{N=1}^{\infty}$ of external forces with $f_N \to 0$ in $\dot{B}_{\infty,q}^{-3}$ such that there exists a solution u_N of (SNS) for each f_N, which never converges to zero in $\dot{B}_{\infty,q}^{-1}$. For the proof, we apply the sequence of initial data proposed by Bourgain-Pavlović (2008), which studied the ill-posedness of the non-stationary Navier-Stokes equations, to (SNS) as the external force with some modifications. Actually, we can construct such a sequence by using trigonometric functions. Making use of the method of Sawada (2012) (which may be regarded as a refinement of the original proof by Bourgain-Pavlović), we construct the solution by the successive approximation, and show that the second approximation causes the inflation of the norm. On the other hand, it turns out that the limit of the successive approximation can be constructed as a bounded uniformly smooth function. Based on this fact with the aid of the theorem of termwise differentiation, we can prove that this limit function yields a smooth solution of (SNS) with $\nabla \Pi = 0$.

The above method by Bourgain-Pavlović is, however, not applicable for the case $n \le p < \infty$, since trigonometric functions used above are not in $\dot{B}_{p,q}^{-3+n/p}$ for such p by the lack of integrability in the whole space $\mathbb{R}^n$. Hence, considering the fact that such functions are spacial periodic, we will also discuss (SNS) in the n-dimensional torus space $\mathbb{T}^n \equiv [-\pi, \pi]^n$ for the moment. In fact, it is also useful to deal with the Navier-Stokes equations in $\mathbb{T}^n$. Usually, it is natural to consider the Navier-Stokes equations in $\mathbb{R}^n$ for seeking the general fluid without any boundaries. On the other hand, for instance, in the computational fluid dynamics, we need to discretize the domain periodically to find a numerical solution. In particular, the asymptotic behavior of solutions in $\mathbb{T}_\lambda^n \equiv [-\pi\lambda, \pi\lambda]^n$ as $\lambda \to \infty$ is quite important to investigate the exact solutions in $\mathbb{R}^n$.

Actually, the inhomogeneous toroidal Besov space $B_{p,q}^s(\mathbb{T}^n)$ was defined by Schmeisser-Triebel (1987). They defined such spaces using classical Littlewood-Paley theory and the Fourier series instead of the Fourier transform. Following their idea, we first define the homogeneous space $\dot{B}_{p,q}^s(\mathbb{T}^n)$ so that we can discuss similar problems on (SNS) to Kaneko-Kozono-Shimizu. In addition, we should also define such spaces on $\mathbb{T}_\lambda^n$ for each $\lambda > 0$, since for the functions u, Π, f on $\mathbb{T}^n$, the above scaling ones $u_\lambda, \Pi_\lambda, f_\lambda$ are on $\mathbb{T}_\lambda^n$. In fact, we see that $\dot{B}_{p,q}^s(\mathbb{T}_\lambda^n)$ also has the same properties as $\dot{B}_{p,q}^s(\mathbb{T}^n)$, and that $\|f_\lambda\|_{\dot{B}_{p,q}^{-3+n/p}(\mathbb{T}_\lambda^n)} \cong \|f\|_{\dot{B}_{p,q}^{-3+n/p}(\mathbb{T}^n)}$ and $\|u_\lambda\|_{\dot{B}_{p,q}^{-1+n/p}(\mathbb{T}_\lambda^n)} \cong \|u\|_{\dot{B}_{p,q}^{-1+n/p}(\mathbb{T}^n)}$ for any $\lambda > 0$ and $1 \le p, q \le \infty$.

In the third chapter, we will show the well-posedness of (SNS) from $\dot{B}_{p,q}^{-3+n/p}(\mathbb{T}^n)$ to $\dot{B}_{p,q}^{-1+n/p}(\mathbb{T}^n)$ for $1 \le p < n$ and $1 \le q \le \infty$, by using similar methods to Kaneko-Kozono-Shimizu. Moreover, we show that (SNS) is ill-posed from $\dot{B}_{p,q}^{-3+n/p}(\mathbb{T}^n)$ to $\dot{B}_{p,q}^{-1+n/p}(\mathbb{T}^n)$ if $p = n,\ 2 < q \le \infty$ and $n < p \le \infty,\ \ 1 \le q \le \infty$, by discontinuity of the solution map. According to the same method in the case of $\mathbb{R}^n$, we will also construct a sequence of external forces by using trigonometric functions, which are included in $D = \dot{B}_{p,q}^{-3+n/p}(\mathbb{T}^n)$ even for $p < \infty$. In particular, for the case $p = n$, i.e., $S = \dot{B}_{n,q}^0(\mathbb{T}^n)$, we will multiply such a sequence by the inverse of the harmonic number $\sum_{k=1}^{N} k^{-1}$. This idea is inspired by Yoneda (2010), which advanced the study on the ill-posedness of non-stationary Navier-Stokes equations by Bourgain-Pavlović.

In the fourth chapter, we will return to the problem on the whole space $\mathbb{R}^n$. We will now prove the ill-posedness from $D = \dot{B}_{p,q}^{-3+n/p}$ to $S = \dot{B}_{p,q}^{-1+n/p}$ when $n < p \le \infty$ and $1 \le q \le \infty$, and when $p = n$, $2 < q \le \infty$, using another method by Bejenaru-Tao (2006), which studied on the ill-posedness of the quadratic nonlinear Schrödinger equation. This method is based on the well-posedness of (SNS) from $\dot{B}_{n,q}^{-2}$ to L^n for $1 \le q \le 2$, which can be shown by a similar method as that of Kaneko-Kozono-Shimizu. Actually, we can construct a sequence of external forces which is included in a small ball of $\dot{B}_{n,q}^{-2}$ with $1 \le q \le 2$, and converges to zero in the weaker norm $\dot{B}_{n,\tilde{q}}^{-2}$ for $\tilde{q} > 2$, such that the corresponding sequence of solutions in L^n does not converge to zero even in the weakest norm $\dot{B}_{\infty,\infty}^{-1}$. Although smooth solutions cannot be expected in this method, we can apply a sequence inspired by Bourgain-Pavlović and Yoneda by multiplying some appropriate cut functions. In this method, we have only to check the norm inflation of the second approximation of a solution, while in the Bourgain-Pavlović method, we should also check the norm convergence of all of the other approximations.

From the above studies, it seems that the above ill-posedness results are caused by unboundedness of the bilinear form $(u, v) \mapsto B(u, v) \equiv (-\Delta)^{-1} P(u \cdot \nabla v)$, where P is the Leray projection to the solenoidal vector space. In fact, Kaneko-Kozono-Shimizu showed the boundedness of B on the space $\dot{B}_{p,q}^{-1+n/p}$ when $1 \le p < n$ using the paraproduct estimate by Bony as follows: Let $n \ge 1$, $1 \le p, q \le \infty$, $s > 0$, $\alpha > 0$ and $\beta > 0$. Suppose that $1 \le p_1, p_2, \widetilde{p_1}, \widetilde{p_2} \le \infty$ satisfy $1/p = 1/p_1 + 1/p_2 = 1/\widetilde{p_1} + 1/\widetilde{p_2}$. Then there holds

$$\|fg\|_{\dot{B}_{p,q}^{s}} \le C\left(\|f\|_{\dot{B}_{p_1,q}^{s+\alpha}} \|g\|_{\dot{B}_{p_2,\infty}^{-\alpha}} + \|f\|_{\dot{B}_{\widetilde{p_1},\infty}^{-\beta}} \|g\|_{\dot{B}_{\widetilde{p_2},q}^{s+\beta}}\right),$$

Where $C = C(n, p, q, s, \widetilde{p_1}, \widetilde{p_2})$ is a constant. Indeed, for the well-posedness of (SNS) from $D = \dot{B}_{p,q}^{-3+n/p}$ to $S = \dot{B}_{p,q}^{-1+n/p}$, the restriction of p, $1 \le p < n$, stems from that of s, $s > 0$ in the above estimate (we should note here that $-1 + n/p > 0$ when $1 \le p < n$). On the other hand, we have shown the discontinuity of the solution map $f \mapsto u$ of (SNS) when $p = n$, $2 < q \le \infty$ and $n < p \le \infty$, $1 \le q \le \infty$. Hence, it seems natural to expect that the above estimate should fail necessarily for $s \le 0$.

In the final chapter, we will show that if $s < 0$, then we can construct concrete counter-examples of the above paraproduct estimate. On the other hand, by restricting the ranges of p or q appropriately, we can also find a counter-example when $s = \alpha = \beta = 0$. For construction of such examples, we can apply similar functions as the above sequence of external forces causing the ill-posedness of (SNS). This result can explain not only the ill-posedness of (SNS) above, but also that of the quadratic nonlinear Schrödinger equation in the one-dimensional Sobolev space $H^s(\mathbb{R})$ when $s < -1$, which was showed by Bejenaru-Tao. Indeed, similar negative result of bilinear estimates also holds in Sobolev spaces.

In this way, our study on (SNS) gives a clear borderline between the well-posedness and ill-posedness in Besov spaces, and a new knowledge on the structure of such spaces concerning the product estimate of functions. Moreover, it is expected that our method by mixture of Bourgain-Pavlović and Bejenaru-Tao may be applicable for other stationary or elliptic partial differential equations.

早稲田大学　博士（理学）　学位申請　研究業績書

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論文	1. H. Tsurumi, “Extension criterion via partial components of vorticity on strong solutions to the Navier-Stokes equations in higher dimensions”, J. Differential Equations Vol.263 (2017), 4007-4022. 2. ○ H. Tsurumi, “The stationary Navier-Stokes equations in the scaling invariant Triebel-Lizorkin spaces”, Differ. Integral Equ. Vol.32 (2019), 323-336. 3. ○ H. Tsurumi, “Ill-posedness of the stationary Navier-Stokes equations in Besov spaces”, J. Math.Anal.Appl. Vol.475 (2019), 1732–1743. 4. ○ H. Tsurumi, “Well-posedness and ill-posedness of the stationary Navier-Stokes equations in toroidal Besov spaces”, Nonlinearity Vol.32 (2019), 3798-3819. 5. ○ H. Tsurumi, “Well-poseness and ill-posedness problems of the stationary Navier-Stokes equations in scaling invariant Besov spaces”, Arch. Ration. Mech. Anal. Vol. 234 (2019), 911-923. 6. ○ H. Tsurumi, “Counter-examples of the bilinear estimates of the Holder type inequality in homogeneous Besov spaces”, Tokyo J. Math. (2019) （掲載決定）
講演	1. 鶴見 裕之, “Extension criterion via partial components of vorticity on strong solutions to the Navier-Stokes equations in higher dimensions”, 日本数学会 2017 年度秋季総合分科会, 山形大学, 2017 年 9 月. 2. H. Tsurumi, “Ill-posedness of the stationary Navier-Stokes equations in Besov spaces”, IRTG seminer, ドイツ・ダルムシュタット工科大学, 2017 年 10 月 3. H. Tsurumi, “Well-posedness and ill-posedness of the stationary Navier-Stokes equations in Triebel-Lizorkin spaces”, 第 15 回日独流体数学国際研究集会, 早稲田大学, 2018 年 1 月. 4. 鶴見 裕之, “Ill-posedness of the stationary Navier-Stokes equations in homogeneous Besov spaces”, 若手による流体力学の基礎方程式研究集会, 名古屋大学, 2018 年 1 月. 5. 鶴見 裕之, “Solutions of the stationary Navier-Stokes equations in homogeneous Besov and Triebel-Lizorkin spaces”, RIMS 共同研究(公開型) 『関数空間の深化とその周辺』, 京都大学, 2018 年 2 月. 6. 鶴見 裕之, “Solutions of the stationary Navier-Stokes equations in homogeneous Besov and Triebel-Lizorkin spaces”, 若手のための偏微分方程式と数学解析, 福岡大学セミナーハウス, 2018 年 2 月. 7. 鶴見 裕之, “Well-posedness and ill-posedness of the stationary Navier-Stokes equations in Besov spaces”, Japanese-Indonesian International Workshop on Mathematical Fluid Dynamics, 早稲田大学, 2018 年 3 月.

早稲田大学　博士（理学）　学位申請　研究業績書

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