Well-posedness and ill-posedness of the stationary Navier-Stokes equations in scaling invariant Besov spaces

定常Navier-Stokes方程式のスケール不変なBesov空間における適切性および非適切性

December 2019

Waseda University

Graduate School of Fundamental Science and Engineering
Department of Pure and Applied Mathematics
Research on Functional Analysis and Non-linear
Partial Differential Equations

Hiroyuki TSURUMI

鶴見　裕之

Abstract.

In this doctoral thesis, we study on the stationary incompressible Navier-Stokes equations in the whole space $\mathbb{R}^n$ for $n \geq 3$. In particular, we discuss here the well-posedness problem of that equation, that is, the problem on the uniquely existence of solutions continuously dependent on given small external forces.

We first review the previous result by Kaneko-Kozono-Shimizu [11] on the well-posedness in the scaling invariant Besov space, from the space $\dot{B}_{p,q}^{-3+\frac{n}{p}}(\mathbb{R}^n)$ of given external forces to the solution space $\dot{B}_{p,q}^{-1+\frac{n}{p}}(\mathbb{R}^n)$ with $1 \leq p < n$ and $1 \leq q \leq \infty$. We then show the well-posedness in the homogeneous Triebel-Lizorkin space similarly. Our method is inspired by the Kaneko-Kozono-Shimizu's one, which is based on the boundedness of the Riesz transform, the para-product estimate, and the embedding theorem in homogeneous Besov and Triebel-Lizorkin spaces. Moreover, we can see some advantages for the regularity of solutions in the case of Triebel-Lizorkin spaces compared to Besov spaces.

We next consider the ill-posedness of the stationary Navier-Stokes equations in weaker Besov spaces. It is proved that a sequence of bounded smooth external forces whose $\dot{B}_{\infty,1}^{-3}$ norms converge to zero can produce a sequence of bounded smooth solutions whose $\dot{B}_{\infty,\infty}^{-1}$ norms never converge to zero. Such a discontinuity of the solution map is shown by constructing the sequence of external forces, as similar to those of initial data proposed by Bourgain-Pavlović [5] in the non-stationary problem. This method proves to be applicable for the Besov spaces on the torus $\mathbb{T}^n$ for $n \geq 3$, and we can also show the ill-posedness for the space $\dot{B}_{p,q}^{-3+\frac{n}{p}}(\mathbb{T}^n)$ of external forces when $n < p < \infty$, $1 \leq q \leq \infty$ and $p = n$, $2 < q \leq \infty$.

Finally, we show the ill-posedness for the space $\dot{B}_{p,q}^{-3+\frac{n}{p}}(\mathbb{R}^n)$ of external forces when $n < p < \infty$, $1 \leq q \leq \infty$ and $p = n$, $2 < q \leq \infty$. In this case, we should apply not only the method of Bourgain-Pavlović [5], but also that of Bejenaru-Tao [2] which studied on the ill-posedness of the quadratic Schrödinger equation. In this way, together with the well-posedness result by Kaneko-Kozono-Shimizu [11], our result may be regarded as showing the borderline case between well-posedness and ill-posedness of the stationary Navier-Stokes equations in scaling invariant Besov spaces.

As by-products of our study on the ill-posedness of the stationary Navier-Stokes equations, we can construct counter-examples of the bilinear estimates of the Hölder type inequality in homogeneous Besov spaces showed by Bony [4], which has an important role for the boundedness of the bilinear term in the Navier-Stokes equations. It is proved that if we change the condition of indices denoting differential orders, then we can find examples of functions that never satisfy the bilinear estimates. Such examples can be constructed due to those used in the ill-posedness problem of the stationary Navier-Stokes equations. This existence of counter-examples of this inequality seems to explain not only our ill-posedness results, but also the ill-posedness of other nonlinear equations in similar cases.

Acknowledgements.

This doctoral thesis is a summarization of my Ph.D. studies in Waseda University. Firstly, I am deeply grateful to my supervisor Prof. Hideo Kozono for his continuous and warm support since my undergraduate years. His comments and suggestions were innumerably valuable throughout the course of my studies. I owe all of my submitted papers composing this doctoral thesis to his kind instruction on academic writing. In addition, I would like to thank Mr. Kenta Kaneko, who is a graduate of Waseda University and was my older colleague in Prof. Kozono's laboratory. In fact, my studies on the stationary Navier-Stokes equation are strongly inspired by his master thesis.

Furthermore, I am deeply thankful to my thesis committee, Prof. Tohru Ozawa, Prof. Yoshihiro Shibata, and Prof. Kazunaga Tanaka, for taking time out to review this doctoral thesis and attend the preliminary examination. Their insightful comments and encouragement widened my perspective for study and research. I am also thankful to Prof. Ozawa, who is one of editors of the journal "Differential and Integral Equations", for accepting my paper appearing in Chapter 3 in this thesis.

Throughout of my doctoral course, I was supported by Grant-in-Aid for JSPS Research Fellow Grant Number JP19J11499, Top Global University Project of Waseda University and by the Research Institute for Mathematical Sciences (RIMS) in Kyoto University. I was also helped by Ms. Sachiko Abdelaziz, a secretary of Prof. Kozono, on office procedures relating to many research trips. As for the first study on the ill-posedness in Chapter 4, Prof. Okihiro Sawada (Gifu University) kindly reviewed my paper and encouraged me in my studies. Moreover, as for my paper appearing in Chapter 5, Dr. Patrick Tolksdorf (TU Darmstadt) and Dr. Martin Saal (TU Darmstadt) kindly suggested the study on the torus spaces. I am deeply grateful to those persons and everyone who supported my Ph.D. studies.

Finally, I would also like to express my gratitude to my parents, Hiroko Tsurumi and Masayuki Tsurumi, for their moral support and warm encouragements for many years.

Hiroyuki Tsurumi
December 2019

Contents.

Chapter 1

Introduction

In this doctoral thesis, we consider the stationary Navier-Stokes equations, which describe the incompressible viscous fluid independent of the time development, in the whole n-dimensional Euclid space $\mathbb{R}^n$ with $n \geq 3$;

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla \Pi = f, \\ \text{div } u = 0. \end{cases} \tag{SNS}$$

Here $u = u(x) = (u_1(x), u_2(x), \ldots, u_n(x))$ and $\Pi = \Pi(x)$ denote the unknown velocity vector field and the unknown pressure of the fluid at the point $x \in \mathbb{R}^n$, respectively, while $f = f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ is the given external force. In (SNS), $-\Delta u$ denotes the viscosity term, and $u \cdot \nabla u \equiv \sum_{j=1}^n u_j \frac{\partial u}{\partial x_j}$ denotes the derivative of u in the direction along itself.

For this stationary problem, there have been various studies on existence, uniqueness, and regularity of weak and strong solutions to (SNS). For instance, Leray [15] and Ladyzhenskaya [14] showed the existence of solutions to (SNS), and later on, Heywood [8] constructed the solution of (SNS) as a limit of solutions of the non-stationary Navier-Stokes equations having the same external force. Then Secchi [20] investigated existence and regularity of solutions to (SNS) in $L^n \cap L^p$, $p > n$. Moreover, Chen [6] proved that for every smooth external force which is small in $\dot{H}^{-1,\frac{n}{2}}$ yields a unique solution of (SNS) in $L^n \cap \dot{H}^{1,\frac{n}{2}}$. Here $\dot{H}^{s,r}$ denotes the homogeneous Sobolev space with the norm $\|f\|_{\dot{H}^{s,r}} \equiv \|(-\Delta)^{\frac{s}{2}} f\|_{L^r}$. In this way, it has been important to find more general spaces such that every small external force in these spaces yields a unique solution of (SNS), and to find more regularity of solutions.

In this thesis, we focus on the well-posedness and ill-posedness problems on (SNS). Roughly speaking, the well-posedness means the uniquely existence of solutions to (SNS) continuously depending on given external forces. The precise definition of the well-posedness is as follows:

Definition 1.1. *Let $(D, \|\cdot\|_D)$ and $(S, \|\cdot\|_S)$ be two Banach spaces (here D and S indirectly denote the spaces of data (external forces) and of solutions, respectively). We say that* (SNS) *is well-posed from D to S if there exist two constants $\varepsilon, \delta > 0$ such that*

(i) *For any* $f \in B_D(\varepsilon)$*, there exist a solution* $u \in B_S(\delta)$ *of* (SNS),

(ii) *If there exist two solutions* $u_1, u_2 \in B_S(\delta)$ *of* (SNS) *for one external force* $f \in B_D(\varepsilon)$*, then it holds that* $u_1 \equiv u_2$ *in* S,

(iii) *The solution map* $f \in (B_D(\varepsilon), \|\cdot\|_D) \mapsto u \in (B_S(\delta), \|\cdot\|_S)$*, which is well-defined by* (i) *and* (ii)*, is continuous,*

where $B_D(\varepsilon) \equiv \{f \in D;\ \|f\|_D < \varepsilon\}$ *and* $B_S(\delta) \equiv \{u \in S;\ \|u\|_S < \delta\}$*. In addition,* (SNS) *is ill-posed from* D *to* S *if* (SNS) *is not well-posed from* D *to* S*.*

This notion of well-posedness corresponds to that of the global well-posedness with small initial data in the Cauchy problem of time-evolution partial differential equations, such as the non-stationary Navier-Stokes equations:

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \Pi = 0 & \text{in } x \in \mathbb{R}^n,\ t \in (0,\infty), \\ \operatorname{div} u = 0 & \text{in } x \in \mathbb{R}^n,\ t \in (0,\infty), \\ u|_{t=0} = a, & \text{in } x \in \mathbb{R}^n. \end{cases} \tag{NNS}$$

In this case, D is the space of initial data a and we often let $S \equiv C([0,\infty); D)$. Until now, the global well-posedness of (NNS) has been studied intensively. For example, Koch-Tataru [12] showed the global well-posedness in the case $D = BMO^{-1}$. On the other hand, Bourgain-Pavlović [5] showed the ill-posedness in the case $D = \dot{B}^{-1}_{\infty,\infty}$ (which includes BMO^{-1}). In fact, they proved the ill-posedness by showing the discontinuity of the solution map. Later on, the ill-posedness in the case $D = \dot{B}^{-1}_{\infty,q}$, $1 \le q < \infty$ was also showed by Yoneda [27] ($2 < q < \infty$) and Wang [28] ($1 \le q \le 2$). These spaces play a crucial role since these are scaling invariant for the initial data a in (NNS). In fact, it is easily seen that if (u, Π) is a solution to (NNS) with an initial datum $a = a(x)$, then so is $(u_\lambda, \Pi_\lambda) = (\lambda u(\lambda x, \lambda^2 t), \lambda^2 \Pi(\lambda x, \lambda^2 t))$ with an initial datum $a_\lambda = \lambda a(\lambda x)$ for every $\lambda > 0$. We call the normed space $(X, \|\cdot\|_X)$ scaling invariant for the initial data if $\|a_\lambda\|_X \cong \|a\|_X$. Together with (NNS), it seems to be an important problem to find more general spaces D and S where (SNS) is well-posed.

We now deal with this problem in homogeneous Besov spaces $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$ for $s \in \mathbb{R}$, $1 \le p, q \le \infty$. Actually, in numerous present papers, the Navier-Stokes equations have been handled in such spaces as above. In fact, we see some similarities between $\dot{B}^s_{p,q}$ and homogeneous Sobolev spaces $\dot{H}^{s,p}$. Indeed, s and p denote differentiability and L^p-integrability of functions, respectively. Furthermore, q denotes the interpolation exponent which enlarges the structure of Sobolev spaces. Namely, it holds that $\dot{B}^s_{p,1} \hookrightarrow \dot{H}^{s,p} \hookrightarrow \dot{B}^s_{p,\infty}$ for all $s \in \mathbb{R}$ and $1 \le p \le \infty$. In this way, the study in the Besov spaces is expected to generalize the previous studies in Sobolev spaces.

Recently, the well-posedness of (SNS) in homogeneous Besov spaces was well studied by Kaneko-Kozono-Shimizu [11]. They showed that (SNS) is well-posed from $D = \dot{B}^{-3+\frac{n}{p}}_{p,q}$ to $S = \dot{B}^{-1+\frac{n}{p}}_{p,q}$ for all $1 \le p < n$ and $1 \le q \le \infty$. These spaces D and S are scaling invariant for the external force f and the velocity u in (SNS) respectively.

Indeed, if a triple $\{u, \Pi, f\}$ solves (SNS), so does $\{u_\lambda, \Pi_\lambda, f_\lambda\}$ for every $\lambda > 0$, with $u_\lambda(x) \equiv \lambda u(\lambda x)$, $\Pi_\lambda(x) \equiv \lambda^2 \Pi(\lambda x)$, $f_\lambda(x) \equiv \lambda^3 f(\lambda x)$. Then we see that

$$\|f_\lambda\|_D \cong \|f\|_D, \quad \|u_\lambda\|_S \cong \|u\|_S, \quad \forall \lambda > 0.$$

Actually, their study in homogeneous Besov spaces enables us handle a larger class of functions which never belong to the usual Sobolev space. For instance, in three dimension case, we can solve (SNS) with a singular external force like the Dirac delta function, which belongs to $\dot{B}^{-n+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ for $1 \le p \le \infty$.

In Chapter 3, we will review the study by Kaneko-Kozono-Shimizu [11], and will also consider a similar problem in homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$ for comparison, which are also generalization of Sobolev spaces. Actually, even in the case of $D = \dot{F}^{-3+\frac{n}{p}}_{p,q}$ and $S = \dot{F}^{-1+\frac{n}{p}}_{p,q}$, we can prove the well-posedness of (SNS), provided $1 < p < n$ and $1 \le q \le \infty$, and provided $p = n$ and $1 \le q \le 2$, by similar methods to Kaneko-Kozono-Shimizu [11]. Indeed, we make use of the boundedness of the Riesz transform, the product estimate, and the embedding theorem in homogeneous Besov and Triebel-Lizorkin spaces. Furthermore, in the case of Triebel-Lizorkin space, we can see some advantages in the sense of the regularity of solutions. More precisely, we will prove that if a small external force in the above scaling invariant Triebel-Lizorkin spaces $\dot{F}^{-1+\frac{n}{p}}_{p,q}$ with $1 < p < n$ also belongs to $\dot{H}^{s-2,r}$ with $s > 0$ and $1 < r < \infty$, or with $s = 0$ and $n/(n-1) < r < \infty$, then the solution belongs to $\dot{H}^{s,r}$. Although Kaneko-Kozono-Shimizu [11] showed a similar result, some additional restrictions for s, r are required in the case of Besov spaces. Such difference seems to stem from the facts as follows. First, the Triebel-Lizorkin space can be identified with the usual Sobolev space, namely, $\dot{F}^s_{p,2} = \dot{H}^{s,p}$ $(1 < p < \infty)$, while in the Besov space, it is only known for the inclusion relation, i.e., $\dot{B}^s_{p,1} \subset \dot{H}^{s,p} \subset \dot{B}^s_{p,\infty}$. Second, there holds

$$\dot{F}^{s_1}_{p_1,q} \hookrightarrow \dot{F}^{s_2}_{p_2,r}, \quad 1 \le p_1 < p_2 < \infty,\ 1 \le q, r \le \infty,\ s_1 - n/p_1 = s_2 - n/p_2.$$

We take the above q and r arbitrarily, while in the Besov space, a similar embedding holds only if $q \le r$.

Now our main purpose in this thesis is to show that the well-posedness result by Kaneko-Kozono-Shimizu [11] is almost optimal in the scaling invariant Besov spaces. In other words, we will prove that (SNS) is *ill-posed* from $D = \dot{B}^{-3+\frac{n}{p}}_{p,q}$ to $S = \dot{B}^{-1+\frac{n}{p}}_{p,q}$ for $n < p \le \infty$ and $1 \le q \le \infty$, and for $p = n$, $2 < q \le \infty$.

In Chapter 4, we will first prove the ill-posedness in the extreme case $p = \infty$, i.e., we will show that (SNS) is ill-posed from $\dot{B}^{-3}_{\infty,q}$ to $\dot{B}^{-1}_{\infty,q}$ for all $1 \le q \le \infty$ in the sense that it occurs a lack of continuity of the solution map $f \mapsto u$. More precisely, we will construct a sequence $\{f_N\}_{N\in\mathbb{N}}$ of external forces with $f_N \to 0$ in $\dot{B}^{-3}_{\infty,1}$ such that there exists a solution u_N of (SNS) for each f_N, which never converges to zero in $\dot{B}^{-1}_{\infty,q}$ (and even in $\dot{B}^{-1}_{\infty,\infty}$). For the proof, we apply the sequence of initial data used in the study on the ill-posedness of (NNS) by Bourgain-Pavlović [5], to (SNS) as the external force f with some modifications. Actually, we can construct such a sequence by using trigonometric functions. Making use of the method of Sawada [17] (which may be regarded as a refinement of the original proof by Bourgain-Pavlović [5]), we construct the solution by the successive approximation, and show that the second approximation causes the inflation of the norm $\|u\|_{\dot{B}^{-1}_{\infty,\infty}}$. In fact, such norm inflation is caused by a superposition of waves. For instance, let $s<0$ and

$$w_1^h \equiv \sin(hx_1 + x_2), \quad w_2^h \equiv \cos(hx_1)$$

be two high frequency waves, where h is a large number. In this case, both of $\|w_1^h\|_{\dot{B}^s_{\infty,q}}$ and $\|w_2^h\|_{\dot{B}^s_{\infty,q}}$ converge to zero as h goes to infinity. However, for the product

$$w_3^h \equiv w_1^h \cdot w_2^h = \frac{1}{2}\sin(2hx_1 + x_2) + \frac{1}{2}\sin x_2,$$

the Besov norm $\|w_3^h\|_{\dot{B}^s_{\infty,q}}$ has a positive lower bound, since the second term of w_3^h is independent of h. We apply this fact to the nonlinear term of (SNS), and construct examples causing the ill-posedness. On the other hand, it turns out that the limit of the successive approximation can be constructed as a bounded uniformly smooth function. Based on this fact with the aid of the theorem of termwise differentiation, we can prove that this limit function yields a smooth solution of (SNS) with a pressure Π such that $\nabla\Pi = 0$.

The above method by Bourgain-Pavlović [5] is, however, not applicable for the case $n \le p < \infty$, since trigonometric functions used above are not in $\dot{B}^{-3+\frac{n}{p}}_{p,q}$ for such p by the lack of integrability in the whole space $\mathbb{R}^n$. Hence, considering the fact that such functions are spacial periodic, we will also discuss (SNS) in the n-dimensional torus space $\mathbb{T}^n \equiv [-\pi,\pi]^n$ for the moment. In fact, it is also useful to deal with the Navier-Stokes equations in $\mathbb{T}^n$. Usually, it is natural to consider the Navier-Stokes equations in $\mathbb{R}^n$ for seeking the general fluid without any boundaries. On the other hand, for instance, in the computational fluid dynamics, we need to discretize the domain periodically to find a numerical solution. In particular, the asymptotic behavior of solutions in $\mathbb{T}^n_\lambda \equiv [-\lambda\pi, \lambda\pi]^n$ as $\lambda \to \infty$ is quite important to investigate the exact solutions in $\mathbb{R}^n$.

Actually, the inhomogeneous toroidal Besov space $B^s_{p,q}(\mathbb{T}^n)$ was defined by Schmeisser-Triebel [19]. They defined such spaces using classical Littlewood-Paley theory and the Fourier series instead of the Fourier transform. Following their idea, we first define the homogeneous space $\dot{B}^s_{p,q}(\mathbb{T}^n)$ so that we can discuss similar problems on (SNS) to Kaneko-Kozono-Shimizu [11]. In addition, we should also define such spaces on $\mathbb{T}^n_\lambda \equiv [-\lambda\pi, \lambda\pi]^n$ for each $\lambda > 0$, since for the functions u, Π, f on $\mathbb{T}^n$, the above scaling ones $u_\lambda, \Pi_\lambda, f_\lambda$ are on $\mathbb{T}^n_\lambda$. In fact, we see that $\dot{B}^s_{p,q}(\mathbb{T}^n_\lambda)$ also has the same properties as $\dot{B}^s_{p,q}(\mathbb{T}^n)$, and that

$$\|u_\lambda\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}(\mathbb{T}^n_\lambda)} \cong \|u\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}(\mathbb{T}^n)}, \quad \|f_\lambda\|_{\dot{B}^{-3+\frac{n}{p}}_{p,q}(\mathbb{T}^n_\lambda)} \cong \|f\|_{\dot{B}^{-3+\frac{n}{p}}_{p,q}(\mathbb{T}^n)}$$

for any $\lambda > 0$ and $1 \le p, q \le \infty$.

In Chapter 5, we will first check that the well-posedness of (SNS) from $D = \dot{B}_{p,q}^{-3+\frac{n}{p}}(\mathbb{T}^n)$ to $S = \dot{B}_{p,q}^{-1+\frac{n}{p}}(\mathbb{T}^n)$ also holds for $1 \le p < n$ and $1 \le q \le \infty$, by using similar methods to Kaneko-Kozono-Shimizu [11]. Moreover, we show that (SNS) is ill-posed from $\dot{B}_{p,q}^{-3+\frac{n}{p}}(\mathbb{T}^n)$ to $\dot{B}_{p,q}^{-1+\frac{n}{p}}(\mathbb{T}^n)$ if $p = n$, $2 < q \le \infty$ and $n < p \le \infty$, $1 \le q \le \infty$, by discontinuity of the solution map. According to the same method in the case of $\mathbb{R}^n$, we will also construct a sequence of external forces by using trigonometric functions, which are now included in $\dot{B}_{p,q}^{-3+\frac{n}{p}}(\mathbb{T}^n)$ even for $p < \infty$. In particular, for the case $p = n$, i.e., $S = \dot{B}_{n,q}^{0}(\mathbb{T}^n)$, we will multiply such a sequence by the inverse of the harmonic number $\left(\sum_{k=1}^{N} k^{-1}\right)^{-1}$. This idea is inspired by Yoneda [27], which advanced the study on the ill-posedness of (NNS) by Bourgain-Pavlović [5].

In Chapter 6, we will return to the problem on the whole space $\mathbb{R}^n$. We will now prove the ill-posedness from $D = \dot{B}_{p,q}^{-3+\frac{n}{p}}$ to $S = \dot{B}_{p,q}^{-1+\frac{n}{p}}$ when $p = n$, $2 < q \le \infty$ and $n < p \le \infty$, $1 \le q \le \infty$ using another method proposed by Bejenaru-Tao [2], which studied on the ill-posedness of the quadratic nonlinear Schrödinger equation. This method is based on the well-posedness of (SNS) from $\dot{B}_{n,q}^{-2}$ to L^n for $1 \le q \le 2$, which can be shown by a similar method as that of Kaneko-Kozono-Shimizu [11]. Actually, we can construct a sequence of external forces which is included in a small ball of $\dot{B}_{n,q}^{-2}$ with $1 \le q \le 2$ and converges to zero in the weaker norm $\dot{B}_{\infty,\tilde{q}}^{-2}$ for $\tilde{q} > 2$, such that the corresponding sequence of solutions in L^n does not converge to zero even in the weakest norm $\dot{B}_{\infty,\infty}^{-1}$. Although smooth solutions cannot be expected in this method, we can apply a sequence inspired by Bourgain-Pavlović [5] and Yoneda [27] by multiplying some appropriate cut functions. In this method, we have only to check the norm inflation of the second approximation of a solution, while in the Bourgain-Pavlović method, we should also check the norm convergence of all of the other approximations.

From the above studies, it seems that the above ill-posedness results are caused by unboundedness of the bilinear form $(u, v) \mapsto B(u, v) \equiv (-\Delta)^{-1} P(u \cdot \nabla v)$, where P is the Leray projection to the solenoidal vector space. In fact, Kaneko-Kozono-Shimizu [11] showed the boundedness of B on the space $\dot{B}_{p,q}^{-1+\frac{n}{p}}$ when $1 \le p < n$ using the paraproduct estimate by Bony [4] as follows:

Proposition 1.2. *Let $n \ge 1$, $1 \le p, q \le \infty$, $s > 0$, $\alpha > 0$ and $\beta > 0$. Suppose that $1 \le p_1, p_2, \tilde{p}_1, \tilde{p}_2 \le \infty$ satisfy $1/p = 1/p_1 + 1/p_2 = 1/\tilde{p}_1 + 1/\tilde{p}_2$. Then for every $f \in \dot{B}_{\tilde{p}_1,q}^{s+\alpha} \cap \dot{B}_{\tilde{p}_1,\infty}^{-\beta}$ and $g \in \dot{B}_{p_2,\infty}^{-\alpha} \cap \dot{B}_{\tilde{p}_2,q}^{s+\beta}$, it holds that $f \cdot g \in \dot{B}_{p,q}^{s}$ with the estimate*

$$\|f \cdot g\|_{\dot{B}_{p,q}^{s}} \le C \left(\|f\|_{\dot{B}_{p_1,q}^{s+\alpha}} \|g\|_{\dot{B}_{p_2,\infty}^{-\alpha}} + \|f\|_{\dot{B}_{\tilde{p}_1,\infty}^{-\beta}} \|g\|_{\dot{B}_{\tilde{p}_2,q}^{s+\beta}} \right),$$

where $C = C(n, p, q, s, \tilde{p}_1, \tilde{p}_2)$ is a constant.

Indeed, for the well-posedness of (SNS) from $D = \dot{B}_{p,q}^{-3+\frac{n}{p}}$ to $S = \dot{B}_{p,q}^{-1+\frac{n}{p}}$, the restriction of p, $1 \le p < n$, stems from that of s, $s > 0$ in Proposition 1.2 (we should note here that $-1+n/p > 0$ when $1 \leq p < n$). On the other hand, as seen in Chapter 4-6, we can show the discontinuity of the solution map $f \mapsto u$ of (SNS) when $p = n$, $2 < q \leq \infty$ and $n < p \leq \infty$, $1 \leq q \leq \infty$. Hence, it seems natural to expect that Proposition 1.2 should fail necessarily for $s \leq 0$.

In Chapter 7, we will show that if $s < 0$, then we can construct concrete counter-examples of the above paraproduct estimate. On the other hand, by restricting the ranges of p or q appropriately, we can also find a counter-example when $s = \alpha = \beta = 0$. For construction of such examples, we can apply similar functions as the above sequence of external forces causing the ill-posedness of (SNS). This result can explain not only the ill-posedness of (SNS) above, but also that of the quadratic nonlinear Schrödinger equation in $H^s(\mathbb{R})$ when $s < -1$, which was showed by Bejenaru-Tao [2]. Indeed, similar negative result of bilinear estimates also holds in Sobolev spaces.

In this way, our study on (SNS) gives a clear borderline between the well-posedness and ill-posedness in Besov spaces, and a new knowledge on the structure of such spaces concerning the product estimate of functions. Moreover, it is expected that our method by mixture of Bourgain-Pavlović [5] and Bejenaru-Tao [2] may be applicable for other stationary equations.

Chapter 2

Preliminary

In this chapter, we prepare some theories on harmonic analysis and partial differential equations required for our studies.

2.1 Definitions and properties of function spaces

First of all, let us define some spaces of functions and distributions. Before starting discussion, we review here some fundamental notation on multi-indices: If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index and f is a function, then

$$\partial^\alpha f \equiv \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}, \quad x^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

where $|\alpha| \equiv \alpha_1 + \alpha_2 + \cdots + \alpha_n$. In what follows, we shall denote by C the constants which may change from line to line.

2.1.1 Smooth function spaces and distributions

We denote by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing functions on $\mathbb{R}^n$, which is usually called the Schwartz class. More precisely, a function f belongs to $\mathcal{S}$ if f is infinitely differentiable ($f \in C^\infty$) and satisfies

$$\rho_{\alpha,\beta} \equiv \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty$$

for any two indices $\alpha, \beta \in \mathbb{N}^n$. It is known that this space $\mathcal{S}$ is complete and metrizable with a family $\{\rho_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}^n}$ of semi-norms. Clearly, the space $C_0^\infty = C_0^\infty(\mathbb{R}^n)$ of compact supported smooth functions is densely embedded in $\mathcal{S}$. Furthermore, the space $\mathcal{S}$ is densely included in the Lebesgue space L^p for $1 \le p < \infty$, and hence we can define the Fourier transform on $\mathcal{S}$. On the other hand, $f(x) = e^{-|x|^2}$, which is often used as a normal distribution in statistics, is a well-known example in $\mathcal{S} \backslash C_0^\infty$.

Moreover, we denote by $\mathcal{S}'$ the dual space (the space of bounded linear functionals) on $\mathcal{S}$, which is called the space of tempered distributions. For example, for every

$1 \le p \le \infty$, the space L^p is included in $\mathcal{S}'$ (to see this, we identify $f \in L^p$ with a functional

$$\phi \to \langle f, \phi \rangle \equiv \int_{\mathbb{R}^n} f(x)\phi(x)dx$$

for $\phi \in \mathcal{S}$). On the other hand, the Dirac delta function $\phi \mapsto \delta(\phi) \equiv \phi(0)$ is a well-known distribution which is never written as a usual function.

In addition, we define $\mathcal{S}_0 = \mathcal{S}_0(\mathbb{R}^n)$ to be the space of all $\varphi \in \mathcal{S}$ such that

$$\int_{\mathbb{R}^n} x^\alpha \varphi(x)dx = 0, \quad \text{for any } \alpha \in \mathbb{N}^n, \tag{2.1}$$

and define $\mathcal{S}_0'$ as the dual space of $\mathcal{S}_0$. It is known that $\mathcal{S}_0$ is a closed subspace of $\mathcal{S}$, and that there holds the topological isomorphism

$$\mathcal{S}_0' \cong \mathcal{S}'/\mathcal{P},$$

where $\mathcal{S}'/\mathcal{P}$ denotes the quotient space with the polynomials space $\mathcal{P}$. (Here we omit the proof of the isomorphism. However, it is directly seen from (2.1) that all constants and polynomials are regarded as zero in $\mathcal{S}_0'$. For the detail, see Grafakos [7, Proposition 1.1.3])

For $f \in \mathcal{S}$, we define the Fourier transform $f \mapsto \mathcal{F}f$ from $\mathcal{S}$ onto itself and its inverse $\mathcal{F}^{-1}$ as

$$\mathcal{F}f(\xi) \equiv \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx,$$

$$\mathcal{F}^{-1}f(x) \equiv \int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}d\xi,$$

and for a distribution $f \in \mathcal{S}'$, we define $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ and its inverse $\mathcal{F}^{-1}$ as

$$\langle \mathcal{F}f, \varphi \rangle \equiv \langle f, \mathcal{F}\varphi \rangle, \quad \varphi \in \mathcal{S},$$

$$\left\langle \mathcal{F}^{-1}f, \varphi \right\rangle \equiv \left\langle f, \mathcal{F}^{-1}\varphi \right\rangle, \quad \varphi \in \mathcal{S}.$$

It is known that each Fourier transform as above is isomorphic. For example, we see the boundedness of $\mathcal{F}$ on $\mathcal{S}$ from

$$|\xi^\alpha D^\beta \mathcal{F}f(\xi)| = C|\mathcal{F}[D^\alpha x^\beta f](\xi)| \le C\|D^\alpha x^\beta f\|_{L^1}$$

and the embedding $\mathcal{S} \hookrightarrow L^1$.

2.1.2 Riesz potentials and homogeneous Sobolev spaces

Next, let us define the Riesz potential $(-\Delta)^{\frac{s}{2}}$, which has an important role for our discussion on the stationary Navier-Stokes equations and homogeneous Besov spaces later. For $f \in \mathcal{S}'/\mathcal{P}$ and $s \in \mathbb{R}$, we define

$$(-\Delta)^{\frac{s}{2}} f \equiv \mathcal{F}^{-1}\left[|\xi|^s \mathcal{F}f\right],$$

where

$$\langle |\xi|^s \mathcal{F} f, \varphi\rangle \equiv \langle f, |\xi|^s \mathcal{F}\varphi\rangle\,, \quad \varphi \in \mathcal{S}_0.$$

Actually, this operator is well-defined even for $s < 0$, since the singularity at the origin is negligible as for $f \in \mathcal{S}_0'$. More precisely, we can roughly show it as follows: Let us take $\varphi \in \mathcal{S}_0$. We should notice that (2.1) is equivalent to the condition as

$$D^\alpha(\mathcal{F}\varphi)(0) = 0, \quad \text{for any } \alpha \in \mathbb{N}^n. \tag{2.2}$$

Now we take an integer $N > 1 - s$. Considering the Taylor expansion of $\mathcal{F}\varphi$ to degree N and (2.2), we see $|\mathcal{F}\varphi(\xi)| \le C|\xi|^N$ with some constant $C = C(N)$. Hence there holds the boundedness of $|\xi|^s \mathcal{F}\varphi$ at the origin as

$$|\xi|^s |\mathcal{F}\varphi(\xi)| \le C|\xi|^{s+N} \le C \quad \text{on } \{\xi \in \mathbb{R}^n;\ |\xi| \le 1\},$$

which yields that $|\xi|^s \mathcal{F}\varphi$ is a rapidly decreasing function. This shows that we can define $(-\Delta)^{\frac{s}{2}}\varphi$ for every $\varphi \in \mathcal{S}_0$ and $s \in \mathbb{R}$. By a similar discussion, we can show that we also define $(-\Delta)^{\frac{s}{2}} f$ even for $f \in \mathcal{S}_0' = \mathcal{S}'/\mathcal{P}$. For the detail of the proof, see Grafakos [7, page 3-4].

Then we define the homogeneous Sobolev space $\dot{H}^{s,r} = \dot{H}^{s,r}(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and $1 < p < \infty$ as

$$\dot{H}^{s,r} \equiv \left\{ f \in \mathcal{S}'/\mathcal{P};\ \|f\|_{\dot{H}^{s,p}} \equiv \|(-\Delta)^{\frac{s}{2}} f\|_{L^p} < \infty \right\}.$$

It is known that this is complete as a subspace of $\mathcal{S}'/\mathcal{P}$.

2.1.3 Homogeneous Besov and Triebel-Lizorkin spaces

We next introduce the Littlewood-Paley decomposition. First, we take $\phi \in \mathcal{S}$ such that

$$0 \le \phi \le 1, \quad \operatorname{supp}\phi = \left\{\xi \in \mathbb{R}^n;\ \frac{1}{2} \le |\xi| \le 2\right\}, \quad \sum_{j\in\mathbb{Z}} \phi(2^{-j}\xi) = 1\ (\xi \ne 0). \tag{2.3}$$

We can actually choose such a function ϕ. For example, by taking a non-negative smooth function $\psi \in C^\infty(\mathbb{R}^N)$ such that

$$\psi(\xi_1) = \psi(\xi_2) \ \text{ if } \ |\xi_1| = |\xi_2|, \quad \psi(\xi) = \begin{cases} 0, & |\xi| \le \frac{1}{2}, \\ 1, & |\xi| \ge 1, \end{cases}$$

we can construct ϕ satisfying (2.3) as

$$\phi(\xi) \equiv \psi(\xi) - \psi\left(\frac{\xi}{2}\right).$$

Then, we define a family $\{\varphi_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}$ of functions as

$$\mathcal{F}\varphi_j(\xi) = \phi(2^{-j}\xi), \quad j \in \mathbb{Z}. \tag{2.4}$$

By (2.3), (2.4), and boundedness of $\mathcal{F}$ and $\mathcal{F}^{-1}$ in $\mathcal{S}'$, we see that every $f \in \mathcal{S}'$ can be decomposed in $\mathcal{S}'/\mathcal{P}$ as

$$f = \sum_{j\in\mathbb{Z}} \varphi_j * f.$$

We should note here that this decomposition is not necessary valid in $\mathcal{S}'$. For example, if we take $f \equiv 1$, then we see for any $j \in \mathbb{Z}$ that

$$\begin{aligned} \varphi_j * f(x) &= \int_{\mathbb{R}^n} \varphi_j(x)dx \\ &= \mathcal{F}\varphi_j(0) = 0, \end{aligned}$$

which is implied by $0 \notin \mathrm{supp}\ \phi$. Hence we have $\sum_{j\in\mathbb{Z}} \varphi_j * f = 0$. We can justify this decomposition if we regard constants and polynomials as zero.

Associated with $\{\varphi_j\}_{j\in\mathbb{Z}}$ above, we define the homogeneous Besov spaces $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$ by

$$\dot{B}^s_{p,q} \equiv \left\{ f \in \mathcal{S}'/\mathcal{P};\ \|f\|_{\dot{B}^s_{p,q}} < \infty \right\}$$

for $s \in \mathbb{R}$, $1 \le p, q \le \infty$ with the norms

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \begin{cases} \left(\sum_{j\in\mathbb{Z}} (2^{sj}\|\varphi_j * f\|_{L^p})^q \right)^{\frac{1}{q}}, & q < \infty, \\ \sup_{j\in\mathbb{Z}} (2^{sj}\|\varphi_j * f\|_{L^p}), & q = \infty. \end{cases}$$

It is known that each homogeneous Besov space is complete. Moreover, this definition is independent of choice of a function ϕ satisfying (2.3). Indeed, we take such two functions ϕ, ϕ', and according to (2.4), we define $\{\varphi_j\}_{j\in\mathbb{Z}}$ and $\{\varphi'_j\}_{j\in\mathbb{Z}}$ respectively. Since it holds from (2.3) and (2.4) that

$$\mathrm{supp}\ \mathcal{F}\varphi_j \subset \{\xi \in \mathbb{R}^n; 2^{j-1} \le |\xi| \le 2^{j+1}\},$$

we see for every $j \in \mathbb{Z}$ that

$$\mathrm{supp}\ \mathcal{F}\varphi_j \cap \mathrm{supp}\ \mathcal{F}\varphi_k = \emptyset \quad \forall k \text{ s.t. } |j-k| \ge 2,$$

which yields

$$\varphi'_j * f = \sum_{k\in\mathbb{Z}} \varphi_k * (\varphi'_j * f) = \tilde{\varphi}_j * \varphi'_j * f, \quad \text{where } \tilde{\varphi}_j \equiv \varphi_{j-1} + \varphi_j + \varphi_{j+1} \tag{2.5}$$

for any $f \in \mathcal{S}'/\mathcal{P}$. Moreover, it holds for every $j \in \mathbb{Z}$ that

$$\begin{aligned} \|\varphi'_j\|_{L^1} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \phi'(2^{-j}\xi)e^{ix\cdot\xi}d\xi \right| dx \\ &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \phi'(\eta)e^{iy\cdot\eta}d\eta \right| dy \le C \end{aligned}$$

with some constant $C = C(n, \phi')$. Therefore, we see from the Young inequality that

$$\begin{aligned}\|\varphi_j' * f\|_{L^p} &= \|\tilde{\varphi}_j * \varphi_j' * f\|_{L^p} \\ &\leq \|\varphi_j'\|_{L^1}\|\tilde{\varphi}_j * f\|_{L^p} \\ &\leq C\|\varphi_j * f\|_{L^p}. \end{aligned} \tag{2.6}$$

It is easily seen that (2.6) also holds conversely. Considering that the last constant C in (2.6) does not depend on j, we have the equivalence between the Besov norms from ϕ and those from ϕ'.

Next, we define the homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$ by

$$\dot{F}^s_{p,q} \equiv \left\{ f \in \mathcal{S}'/\mathcal{P};\ \|f\|_{\dot{F}^s_{p,q}} < \infty \right\}$$

for $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ with the norms

$$\|f\|_{\dot{F}^s_{p,q}} \equiv \begin{cases} \left\| \left\{ \sum_{j=1}^{\infty} (2^{sj}|\varphi_j * f(\cdot)|)^q \right\}^{\frac{1}{q}} \right\|_{L^p}, & 1 \leq p, q < \infty, \\ \left\| \sup_{j \in \mathbb{Z}} 2^{js}|\varphi_j * f(\cdot)| \right\|_{L^p}, & 1 \leq p \leq \infty,\ q = \infty, \\ \sup_{Q \in \mathcal{Q}} \left\{ \frac{1}{|Q|} \int_Q \sum_{j=-[\log_2 l(Q)]}^{\infty} (2^{sj}|\varphi_j * f(x)|)^q dx \right\}^{\frac{1}{q}}, & p = \infty,\ 1 \leq q < \infty, \end{cases}$$

where

$$\mathcal{Q} \equiv \bigcup_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \left\{ x \in \mathbb{R}^n; 2^{-k}m_i \leq x_i \leq 2^{-k}(m_i + 1), i = 1, 2, \ldots, n \right\}$$

denotes a family of dyadic cube, and $|Q|$ and $l(Q)$ denote volume and side length of Q, respectively. These are also Banach spaces, and this definition is also independent of choice of ϕ. To show the equivalence of norms with different ϕ, we should use the fact (2.5), and apply the vector valued multiplier theorem (see Sawano [18, Theorem 1.5.3], for example).

2.1.4 Properties of homogeneous Besov and Triebel-Lizorkin spaces

The above function spaces, homogeneous Sobolev, Besov, and Triebel-Lizorkin spaces, have properties with regard to the embedding and the boundedness of important operators. Here we review some of them required for our studies:

Proposition 2.1. (1) *Let $s \in \mathbb{R}$, and let $1 \leq p \leq \infty$, $1 \leq q_1 \leq q_2 \leq \infty$. Then there holds*

$$\dot{B}^s_{p,q_1} \hookrightarrow \dot{B}^s_{p,q_2},\ \ \dot{F}^s_{p,q_1} \hookrightarrow \dot{F}^s_{p,q_2}. \tag{2.7}$$

(2) *Let $s_1 > s_2$, and let $1 \le p_1 < p_2 < \infty$, $1 \le q, r \le \infty$. If $s_1 - n/p_1 = s_2 - n/p_2$, then there holds*

$$\dot{B}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,q}, \;\; \dot{F}^{s_1}_{p_1,q} \hookrightarrow \dot{F}^{s_2}_{p_2,r}. \tag{2.8}$$

(3) *Let $s \in \mathbb{R}$, and let $1 < p < \infty$. Then there holds*

$$\dot{F}^s_{p,2} \cong \dot{H}^{s,p}. \tag{2.9}$$

On the other hand, there hold $\dot{F}^0_{1,2} \cong \mathcal{H}^1$ (Hardy space) and $\dot{F}^0_{\infty,2} \cong BMO$.

(4) *Let $s \in \mathbb{R}$, and let $1 \le p, q \le \infty$. Then there holds*

$$\dot{B}^s_{p,\min(p,q)} \hookrightarrow \dot{F}^s_{p,q} \hookrightarrow \dot{B}^s_{p,\max(p,q)}. \tag{2.10}$$

(5) *Let $s, s_0 \in \mathbb{R}$, and let $1 \le p, q \le \infty$. Then the Riesz potential $(-\Delta)^{\frac{s}{2}}$ is isomorphic from $\dot{B}^{s_0}_{p,q}$ onto $\dot{B}^{s_0-s}_{p,q}$, and from $\dot{F}^{s_0}_{p,q}$ onto $\dot{F}^{s_0-s}_{p,q}$.*

(6) *Let $s \in \mathbb{R}$, and let $1 \le p, q \le \infty$. Moreover, we define the dilation of a function v as $v_\lambda(x) \equiv v(\lambda x)$ with $\lambda > 0$. If v belongs to $\dot{B}^s_{p,q}$, then so does v_λ for every $\lambda > 0$ and there holds*

$$\|v_\lambda\|_{\dot{B}^s_{p,q}} \cong \lambda^{s-\frac{n}{p}} \|v\|_{\dot{B}^s_{p,q}}. \tag{2.11}$$

This claim also holds similarly for $\dot{F}^s_{p,q}$.

Remark 2.2. (i) In Proposition 2.1, (1) means that for fixed s and p, the spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ become wider if q becomes larger.

(ii) The claim (2) is similar one to the Sobolev embedding theorem. We should note in (2) that in the case of Triebel-Lizorkin spaces, we can take q and r independently (for example, we can take $q = \infty$ and $r = 1$). Moreover, as seen in (3), Triebel-Lizorkin spaces have strong relationship with Sobolev spaces, while for Besov spaces, it is only known for the inclusion relation, i.e., $\dot{B}^s_{p,1} \subset \dot{H}^{s,p} \subset \dot{B}^s_{p,\infty}$ (see also (4)).

(iii) From the claim (5), the Riesz potential $(-\Delta)^{\frac{s}{2}}$ is also called as a lift operator in Besov and Triebel-Lizorkin spaces. In the later discussion, we will often calculate the $\dot{B}^0_{p,q}$ norm of $(-\Delta)^{\frac{s}{2}} f$ instead of $\|f\|_{\dot{B}^s_{p,q}}$.

(iv) The claim (6) shows the reason why the spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ are called "homogeneous". Moreover, the equivalence (2.11) explains the scaling invariance of the spaces of external forces and solutions to the equation (SNS), which is mentioned later.

Outline of the proof of Proposition 2.1. Here we do not consider the space $\dot{F}^s_{\infty,q}$ for the simplicity. For the detail of each proof, we refer to Triebel [21], Jawerth [10], and Sawano [18].

(1) We can easily see the claim by the embedding of sequence spaces

$$l^{q_1} \hookrightarrow l^{q_2}, \quad \text{if } 1 \le q_1 \le q_2 \le \infty,$$

where $l^q \equiv \{\{a_n\}_{n \in \mathbb{R}};\ \sum_{n=1}^{\infty} |a_n|^q < \infty\}$.

(2), (5) First, we consider in Besov spaces. Since

$$\begin{aligned}(-\Delta)^{\frac{s}{2}}\varphi_j &= \mathcal{F}^{-1}\left\{\mathcal{F}\left((-\Delta)^{\frac{s}{2}}\varphi_j\right)\right\}\\ &= \int_{\mathbb{R}^n}|\xi|^s\phi(2^{-j}\xi)e^{ix\cdot\xi}d\xi\\ &= 2^{(s+n)j}\int_{\mathbb{R}^n}|\xi|^s\phi(\xi)e^{2^j ix\cdot\xi}d\xi,\end{aligned}$$

it holds for $1 \le r < \infty$ that

$$\begin{aligned}\|(-\Delta)^{\frac{s}{2}}\varphi_j\|_{L^r} &= 2^{(s+n)j}\left\{\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}|\xi|^s\phi(\xi)e^{2^j ix\cdot\xi}d\xi\right|^r dx\right\}^{\frac{1}{r}}\\ &\le C2^{\left\{s+n\left(1-\frac{1}{r}\right)\right\}j},\end{aligned}$$

which is also valid for $r=\infty$. From this estimate, (2.5), and the Hausdorff-Young inequality, we obtain important estimates for $1 \le p \le q \le \infty$ such that

$$\begin{aligned}\|\varphi_j * (-\Delta)^{\frac{s}{2}}f\|_{L^q} &= \|(-\Delta)^{\frac{s}{2}}\tilde{\varphi}_j * \varphi_j * f\|_{L^q}\\ &\le \|(-\Delta)^{\frac{s}{2}}\tilde{\varphi}_j\|_{L^r}\|\varphi_j * f\|_{L^p} \quad (1/q+1=1/r+1/p)\\ &\le C2^{\left\{s+n\left(\frac{1}{p}-\frac{1}{q}\right)\right\}j}\|\varphi_j * f\|_{L^p}.\end{aligned}\tag{2.12}$$

Hence it holds from (2.12) with $p=q$ that

$$\|(-\Delta)^{\frac{s}{2}}f\|^q_{\dot{B}^{s_0-s}_{p,q}} \le C\sum_{j\in\mathbb{Z}}2^{j(s_0-s)q}2^{jsq}\|\varphi_j * f\|^q_{L^p} = C\|f\|^q_{\dot{B}^{s_0}_{p,q}},$$

which yields the boundedness of $(-\Delta)^{\frac{s}{2}}$ from $\dot{B}^{s_0}_{p,q}$ to $\dot{B}^{s_0-s}_{p,q}$. By considering the inverse $(-\Delta)^{-\frac{s}{2}}$ similarly, we obtain the isomorphism (5) in Besov spaces. Moreover, under the assumption of (2), we also see from (2.12) that

$$\|\varphi_j * (-\Delta)^{\frac{s_2}{2}}f\|_{L^{p_2}} \le C2^{\left\{s_2+n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\right\}j}\|\varphi_j * f\|_{L^{p_1}} = C2^{s_1 j}\|\varphi_j * f\|_{L^{p_1}}.$$

Hence from this and (5), we have

$$\begin{aligned}\|f\|^q_{\dot{B}^{s_2}_{p_2,q}} &\cong \|(-\Delta)^{\frac{s_2}{2}}f\|^q_{\dot{B}^0_{p_2,q}}\\ &= \sum_{j\in\mathbb{Z}}\|\varphi_j * (-\Delta)^{\frac{s_2}{2}}f\|^q_{L^{p_2}}\\ &\le C\sum_{j\in\mathbb{Z}}2^{s_1 jq}\|\varphi_j * f\|^q_{L^{p_1}} = \|f\|^q_{\dot{B}^{s_2}_{p_2,q}},\end{aligned}$$

which yields (2) in Besov spaces.

We can also show (5) in Triebel-Lizorkin spaces by the equality

$$\begin{aligned}\varphi_j * (-\Delta)^{\frac{s}{2}}f &= (-\Delta)^{\frac{s}{2}}\tilde{\varphi}_j * \varphi_j * f\\ &= 2^{sj}\cdot 2^{nj}\mathcal{F}^{-1}[|\xi|^s\phi](2^j\cdot) * \varphi_j * f\end{aligned}$$

and the vector-valued multiplier theorem (see Sawano [18, Theorem 1.5.3], for example). Here let us show (2) in Triebel-Lizorkin spaces according to Jawerth [10]. From (1) and (5), it suffices to show

$$\dot{F}^0_{p_1,\infty} \hookrightarrow \dot{F}^s_{p_2,1} \tag{2.13}$$

under the assumption

$$1 \le p_1 < p_2 < \infty, \quad s = -n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) < 0.$$

Let $f \in \dot{F}^0_{p_1,\infty}$. We can assume that $\|f\|_{\dot{F}^0_{p_1,\infty}} = 1$. From (2.12) with $s = 0$, $p = p_1$ and $q = \infty$, we see that

$$\begin{aligned} \|\varphi_j * f\|_{L^\infty} &\le C2^{\frac{n}{p_1}j}\|\varphi_j * f\|_{L^{p_1}} \\ &\le C2^{\frac{n}{p_1}j}\|f\|_{\dot{F}^0_{p_1,\infty}} = C2^{\frac{n}{p_1}j}, \end{aligned}$$

Therefore, for any integer $M \in \mathbb{Z}$, we have

$$\sum_{j=-\infty}^{M} 2^{sj}|(\varphi_j * f(x))| \le C\sum_{j=-\infty}^{M} 2^{\frac{n}{p_2}j} \le C2^{\frac{n}{p_2}M} \equiv Ct_M,$$

where $t_M = 2^{\frac{M}{p_2}j}$. On the other hand, there holds

$$\begin{aligned} \sum_{j=M}^{\infty} 2^{sj}|(\varphi_j * f(x))| &\le C2^{sM}\sup_{j\in\mathbb{Z}}|(\varphi_j * f)(x)| \\ &\le Ct_M^{1-\frac{p_2}{p_1}}\sup_{j\in\mathbb{Z}}|(\varphi_j * f)(x)|, \end{aligned}$$

which is implied by $s < 0$. Hence, considering the equality

$$\begin{aligned} \|g\|_{L^p}^p &= \int_{\mathbb{R}^n}\left\{\int_0^{|g(x)|} pt^{p-1}dt\right\}dx \\ &= p\int_0^\infty t^{p-1}\left\{\int_{\mathbb{R}^n}\chi_{[0,|g(x)|]}(t)dx\right\}dt \\ &= p\int_0^\infty t^{p-1}|\{x \in \mathbb{R}^n; |g(x)| > t\}|dt \end{aligned}$$

and taking $t \sim t_M$, we obtain

$$\begin{aligned} \|f\|_{F^s_{p_2,1}}^{p_2} &= p_2\int_0^\infty t^{p_2-1}\left|\left\{x \in \mathbb{R}^n; \sum_{j\in\mathbb{Z}} 2^{sj}|\varphi_j * f(x)| > t\right\}\right|dt \\ &\le p_2\int_0^\infty t^{p_2-1}\left|\left\{x \in \mathbb{R}^n; \sup_{j\in\mathbb{Z}}|\varphi_j * f(x)| > Ct^{\frac{p_2}{p_1}}\right\}\right|dt \\ &\le C\int_0^\infty t^{p_1-1}\left|\left\{x \in \mathbb{R}^n; \sup_{j\in\mathbb{Z}}|\varphi_j * f(x)| > t\right\}\right|dt = C\|f\|_{\dot{F}^0_{p_1,\infty}}^{p_2}, \end{aligned}$$

which implies (2.13).

(3) We see the isomorphism (2.9) directly by (5) and the Littlewood-Paley theorem as

$$\|f\|_{L^p} \cong \left\| \left(\sum_{j\in\mathbb{Z}} |\varphi_j * f(\cdot)|^2 \right)^{\frac{1}{2}} \right\|_{L^p}, \quad 1 < p < \infty.$$

The proof of this theorem is so complicated that we omit it here (see Triebel [21, section 2.5.8], for example).

(4) Since it is clear from the definition that $\dot{B}^s_{p,p} \cong \dot{F}^s_{p,p}$, it suffices to show the followings:

$$\dot{B}^s_{p,q} \hookrightarrow \dot{F}^s_{p,q}, \quad \text{if } 1 \le q \le p \le \infty, \tag{2.14}$$

$$\dot{F}^s_{p,q} \hookrightarrow \dot{B}^s_{p,q}, \quad \text{if } 1 \le p < q \le \infty. \tag{2.15}$$

For the case $q \le p$, we use the Minkowski inequality on $L^{\frac{p}{q}}$ and see that

$$\begin{aligned}
\|f\|^q_{\dot{F}^s_{p,q}} &= \left\{ \int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} 2^{sjq} |\varphi_j * f(x)|^q \right)^{\frac{p}{q}} dx \right\}^{\frac{q}{p}} \\
&\le \sum_{j\in\mathbb{Z}} 2^{sjq} \left\{ \int_{\mathbb{R}^n} |\varphi_j * f(x)|^p dx \right\}^{\frac{q}{p}} = \|f\|^q_{\dot{B}^s_{p,q}},
\end{aligned}$$

which yields (2.14). On the other hand, if $p < q$, then it holds from the reverse Minkowski inequality on $L^{\frac{p}{q}}$ that

$$\begin{aligned}
\|f\|^q_{\dot{B}^s_{p,q}} &= \sum_{j\in\mathbb{Z}} 2^{sjq} \left\{ \int_{\mathbb{R}^n} |\varphi_j * f(x)|^p dx \right\}^{\frac{q}{p}} \\
&= \sum_{j\in\mathbb{Z}} \left\{ \int_{\mathbb{R}^n} \left(2^{sjq} |\varphi_j * f(x)|^q \right)^{\frac{p}{q}} dx \right\}^{\frac{q}{p}} \\
&\le \left\{ \int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} 2^{sjq} |\varphi_j * f(x)|^q \right)^{\frac{p}{q}} dx \right\}^{\frac{q}{p}} = \|f\|^q_{\dot{F}^s_{p,q}},
\end{aligned}$$

which yields (2.15).

(6) Here we consider only in Besov spaces. Let $\lambda > 0$ and $v_\lambda(x) \equiv v(\lambda x)$. Since $\mathcal{F}v_\lambda(\xi) = \lambda^{-n}\mathcal{F}v(\lambda^{-1}\xi)$, it is seen for every $j \in \mathbb{Z}$ that

$$\begin{aligned}
(\varphi_j * v_\lambda)(x) &= \lambda^{-n} \int_{\mathbb{R}^n} \phi(2^{-j}\xi)\mathcal{F}v(\lambda^{-1}\xi) e^{ix\cdot\xi} d\xi \\
&= \int_{\mathbb{R}^n} \phi(2^{-j}\lambda\eta)\mathcal{F}v(\eta) e^{ix\cdot\lambda\eta} d\eta \\
&= (\varphi_{j-\log_2\lambda} * v)(\lambda x),
\end{aligned}$$

where $\varphi_{j-\log_2\lambda} \in \mathcal{S}_0$ is a function satisfying

$$\mathcal{F}\varphi_{j-\log_2\lambda}(\xi) = \phi(2^{-j+\log_2\lambda}\xi) = \phi(2^{-j}\lambda\xi).$$

Hence there holds

$$\begin{aligned} 2^{sj}\|\varphi_j * v_\lambda\|_{L^p} &= 2^{sj}\left\{\int_{\mathbb{R}^n} |(\varphi_{j-\log_2\lambda} * v)(\lambda x)|^p dx\right\}^{\frac{1}{p}} \\ &= 2^{sj}\lambda^{-\frac{n}{p}}\|\varphi_{j-\log_2\lambda} * v\|_{L^p} \\ &= \lambda^{s-\frac{n}{p}}2^{s(j-\log_2\lambda)}\|\varphi_{j-\log_2\lambda} * v\|_{L^p}. \end{aligned}$$

Therefore, it suffices to show that

$$\sum_{j\in\mathbb{Z}} \left(2^{s(j-\log_2\lambda)}\|\varphi_{j-\log_2\lambda} * v\|_{L^p}\right)^q \cong \sum_{j\in\mathbb{Z}} \left(2^{sj}\|\varphi_j * v\|_{L^p}\right)^q. \tag{2.16}$$

To see this, we let $\log_2\lambda = [\log_2\lambda] + \alpha(\lambda)$, where $[\log_2\lambda]$ and $\alpha(\lambda)$ denote the integer and fractional parts of $\log_2\lambda$, respectively. In a similar way to (2.5), we see that

$$\varphi_{j-\log_2\lambda} * v = \tilde{\varphi}_{j-[\log_2\lambda]} * \varphi_{j-\log_2\lambda} * v,$$

where we should take $\tilde{\varphi}_{j-[\log_2\lambda]}$ as

$$\tilde{\varphi}_{j-[\log_2\lambda]} = \sum_{k=-2}^{2} \varphi_{j-[\log_2\lambda]+k}.$$

Hence, by a similar discussion to (2.6), we obtain the equivalence

$$2^{s(j-\log_2\lambda)}\|\varphi_{j-\log_2\lambda} * v\|_{L^p} \cong 2^{s(j-[\log_2\lambda])}\|\varphi_{j-[\log_2\lambda]} * v\|_{L^p}$$

By summing up both sides of the above on $j \in \mathbb{Z}$ (or taking the supremum if $q = \infty$), we obtain (2.16). $\square$

Proposition 2.3. *Let $s, s_0 \in \mathbb{R}$, and let $1 \le p, q \le \infty$. Then for each $j = 1, 2, \ldots, n$, the Riesz transform $R_j \equiv \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$ is bounded from $\dot{B}^s_{p,q}$ to itself. This boundedness also holds similarly for $\dot{F}^s_{p,q}$.*

Proposition 2.3 was showed by Kaneko-Kozono-Shimizu [11] (in Besov spaces) and Iwabuchi-Nakamura [9] (in Triebel-Lizorkin spaces). Here let us prove Proposition 2.3 in Besov spaces according to [11].

Proof of Proposition 2.3 in Besov spaces. By (2.5), we have

$$\varphi_j * R_k f = R_k\tilde{\varphi}_j * \varphi_j * f$$

for every $k = 1, 2, \dots, n$. Moreover, each $R_k\varphi_j$ can be expressed as

$$\begin{aligned} R_k\varphi_j(x) &= \int_{\mathbb{R}^n} \frac{i\xi_k}{|\xi|}\phi(2^{-j}\xi)e^{ix\cdot\xi}d\xi \\ &= 2^{nj}\int_{\mathbb{R}^n} \frac{i\eta_k}{|\eta|}\phi(\eta)e^{ix\cdot 2^j\eta}d\eta, \end{aligned}$$

and we have

$$\begin{aligned} \|R_k\varphi_j\|_{L^1} &= 2^{nj}\int_{\mathbb{R}^n_x}\left|\int_{\mathbb{R}^n_\eta} \frac{i\eta_k}{|\eta|}\phi(\eta)e^{ix\cdot 2^j\eta}d\eta\right| dx \\ &= \int_{\mathbb{R}^n_y}\left|\int_{\frac{1}{2}\le|\eta|\le 2} \frac{i\eta_k}{|\eta|}\phi(\eta)e^{iy\cdot\eta}d\eta\right| dy, \end{aligned}$$

which has a global upper bound independent of j and k. Hence by the Young inequality, we obtain the estimate

$$\begin{aligned} \|\varphi_j * R_kf\|_{L^p} &\le \|R_k\tilde{\varphi}_j\|_{L^1}\|\varphi_j * f\|_{L^p} \\ &\le C\|\varphi_j * f\|_{L^p} \end{aligned}$$

for every $1 \le p \le \infty$, where C denotes a constant independent of j and k. From this estimate and the definition of homogeneous Besov spaces, we have

$$\|R_kf\|_{\dot{B}^s_{p,q}} \le C\|f\|_{\dot{B}^s_{p,q}}$$

for every $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, which proves Proposition 2.3. □

2.2 Definitions of well-posedness and ill-posedness

In this subsection, we define the concept of the well-posedness of abstract equations. Let $(D, \|\cdot\|_D)$ and $(S, \|\cdot\|_S)$ be two Banach spaces, the space of given data and of solutions, respectively. In addition, we let $L : D \to S$ be a densely defined linear operator, and let $B : S \times S \to S$ be a densely defined bilinear form. Then we consider an abstract equation

$$u = Lf + B(u, u), \tag{E}$$

where $f \in D$ is a given data, and $u \in S$ is an unknown solution.

Remark 2.4. Actually, the abstract equation (E) appears in various integral equations. For example, the Cauchy problem of non-stationary Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + u\cdot\nabla u + \nabla\Pi = 0 & \text{in } x \in \mathbb{R}^n,\ t \in (0, T), \\ \text{div } u = 0 & \text{in } x \in \mathbb{R}^n,\ t \in (0, T), \\ u|_{t=0} = a, & \text{in } x \in \mathbb{R}^n \end{cases} \tag{NNS}$$

can be rewritten as (E), where we assume

$$La(t) \equiv e^{-At}a, \quad B(u,v)(t) \equiv -\int_0^t e^{-A(t-s)}P(u\cdot\nabla v)(s)ds, \quad 0<t<T$$

with the Leray projection $P : L^p \to L^p_\sigma \equiv \overline{\{f \in C_0^\infty;\ \mathrm{div}\ f = 0\}}^{\|\cdot\|_{L^p}}$ and the Stokes operator $A \equiv -P\Delta$.

Now we define the well-posedness and ill-posedness of (E) as follows:

Definition 2.5. *We call that the equation* (E) *is well-posed from* $(D, \|\cdot\|_D)$ *to* $(S, \|\cdot\|_S)$ *if there exist two constants* $\varepsilon, \delta > 0$ *such that*

(i) *For any* $f \in B_D(\varepsilon)$*, there exist a solution* $u \in B_S(\delta)$ *of* (E)*,*

(ii) *If there exist two solutions* $u_1, u_2 \in B_S(\delta)$ *of* (E) *for one external force* $f \in B_D(\varepsilon)$*, then it holds that* $u_1 \equiv u_2$ *in* S*,*

(iii) *The map* $f \in (B_D(\varepsilon), \|\cdot\|_D) \mapsto u \in (B_S(\delta), \|\cdot\|_S)$*, which is well-defined by* (i) *and* (ii)*, is continuous with regard to each topology,*

where $B_D(\varepsilon) \equiv \{f \in D;\ \|f\|_D < \varepsilon\}$ *and* $B_S(\delta) \equiv \{u \in S;\ \|u\|_S < \delta\}$*. In addition,* (E) *is ill-posed from* D *to* S *if* (E) *is not well-posed from* D *to* S*.*

Furthermore, we define the *quantitatively* well-posedness of (E) as follows:

Definition 2.6. *We call that the equation* (E) *is quantitatively well-posed from the data space* $(D, \|\cdot\|_D)$ *to the solution space* $(S, \|\cdot\|_S)$ *if there hold two estimates as follows*:

$$\|Lf\|_S \le C_1\|f\|_D, \ \forall f \in D, \tag{2.17}$$

$$\|B(u,v)\|_S \le C_2\|u\|_S\|v\|_S, \ \forall u, v \in S, \tag{2.18}$$

where C_1 *and* C_2 *are two positive constants depending only on* D *and* S*.*

Now let us show that the quantitatively well-posedness in Definition 2.6 is stronger than the well-posedness in Definition 2.5:

Proposition 2.7. *Suppose that the equation* (E) *is quantitatively well-posed from* $(D, \|\cdot\|_D)$ *to* $(S, \|\cdot\|_S)$*. Then* (E) *is well-posed from* $(D, \|\cdot\|_D)$ *to* $(S, \|\cdot\|_S)$ *in the sense of Definition 2.5.*

Proof of Proposition 2.7. We define the approximative sequence $\{u_j\}_{j\in\mathbb{N}}$ to the solution of (E) as

$$\begin{cases} u_1 \equiv Lf, \\ u_{j+1} \equiv u_1 + B(u_j, u_j), \quad j \ge 1. \end{cases} \tag{2.19}$$

By (2.17), we see that $u_1 \in S$ for any $f \in D$. Moreover, if $u_j \in S$, then $u_{j+1} \in S$ with the estimate

$$\|u_{j+1}\|_S \leq C_1\|f\|_D + C_2\|u_j\|_S^2, \tag{2.20}$$

which is implied by (2.17) and (2.18). Hence $u_j \in S$ for all $j \geq 1$ by induction. We should notice from this estimate that if

$$\|f\|_D < \varepsilon \equiv \frac{1}{4C_1C_2}, \tag{2.21}$$

then the following quadratic equation

$$\lambda = C_1\|f\|_D + C_2\lambda^2$$

has a real solution as

$$\lambda = \delta_1 \equiv \frac{1 \pm \sqrt{1 - 4C_1C_2\|f\|_D}}{2C_2}.$$

Under such a condition , we see from (2.17) and (2.20) that

$$\begin{aligned} \|u_1\|_S = \|Lf\|_S &\leq C_1\|f\|_D \\ &\leq C_1\|f\|_D + C_2\delta_1^2 \\ &= \delta_1, \end{aligned}$$

and if $\|u_j\|_S \leq \delta_1$ for some $j \in \mathbb{Z}$, then

$$\begin{aligned} \|u_{j+1}\|_S &\leq C_1\|f\|_D + C_2\|u_j\|_S^2 \\ &\leq C_1\|f\|_D + C_2\delta_1^2 \\ &= \delta_1. \end{aligned}$$

Therefore, by induction, we see that the sequence $\{\|u_j\|_S\}_{j\in\mathbb{N}}$ is bounded with the estimate

$$\|u_j\|_S \leq \delta_1 \equiv \frac{1 - \sqrt{1 - 4C_1C_2\|f\|_D}}{2C_2}, \quad j \geq 1, \tag{2.22}$$

provided (2.21) holds. On the other hand, there holds

$$\begin{aligned} u_{j+1} - u_j &= B(u_j, u_j) - B(u_{j-1}, u_{j-1}) \\ &= B(u_j, u_j - u_{j-1}) + B(u_j - u_{j-1}, u_{j-1}), \quad j \geq 2. \end{aligned}$$

Hence, if f satisfies (2.21), we have by (2.17), (2.18) and (2.22) that

$$\begin{aligned} \|u_{j+1} - u_j\|_S &\leq 2C_2\delta_1\|u_j - u_{j-1}\|_S \\ &\leq (2C_2\delta_1)^{j-1}\|u_2 - u_1\|_S \\ &= (2C_2\delta_1)^{j-1}\|B(u_1, u_1)\|_S \\ &\leq (2C_2\delta_1)^{j-1} \cdot C_2C_1^2\|f\|_D^2 \end{aligned}$$

for all $j \geq 2$. Since $2C_2\delta_1 < 1$ by (2.22), we have

$$\sum_{j=1}^{\infty} \|u_{j+1} - u_j\|_S < \infty, \tag{2.23}$$

which means that $\{u_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in S. Therefore, by completeness, u_j converges to some $u^* \in S$ under the condition (2.21). This limit u^* satisfies $\|u^*\|_S \leq \delta_1$ by (2.22) and there holds

$$\|B(u^*, u^*) - B(u_j, u_j)\|_S \leq 2C_2M\|u^* - u_j\|_S \to 0 \quad \text{as} \quad j \to \infty.$$

Therefore, letting $j \to \infty$ in (2.19), we see that u^* is a solution of (E).

We next consider the uniqueness. Let $u, v \in S$ be two solutions of (E) for the same external force f satisfying (2.21). By (2.18), we have

$$\begin{aligned}
\|u - v\|_S &= \|B(u, u) - B(v, v)\|_S \\
&= \|B(u, u) - B(u, v) + B(u, v) - B(v, v)\|_S \\
&= \|B(u, u - v) + B(u - v, v)\|_S \\
&\leq C_2(\|u\|_S + \|v\|_S)\|u - v\|_S.
\end{aligned}$$

Hence, if

$$\|u\|_S, \|v\|_S < \delta_2 \equiv \frac{1}{2C_2},$$

then

$$C_2(\|u\|_S + \|v\|_S) < 1,$$

which yields that $u = v$ in S. Hence we obtain (i) and (ii) in Definition 2.5 by taking ε as (2.21) and δ such that $\delta_1 < \delta < \delta_2$.

Finally, we prove the continuity of the solution map $f \in (B_D(\varepsilon), \|\cdot\|_D)$ to $u \in (B_S(\delta), \|\cdot\|_S)$. Take an arbitrary sequence $\{g_N\}_{N\in\mathbb{N}} \subset B_S(\delta)$ of data which converges to $g_0 \in B_S(\delta)$, and let $v_j \in B_S(\delta)$, $j \in \mathbb{N}$, be an unique solution of (E) with a datum g_j, and v_0 be a solution with g_0. Then we have

$$\begin{aligned}
\|v_j - v_0\|_S &\leq \|L(g_j - g_0)\|_S + \|B(v_j, v_j) - B(v_0, v_0)\|_S \\
&\leq C_1\|g_j - g_0\|_D + C_2(\|v_0\|_S + \|v_j\|_S)\|v_0 - v_j\|_S.
\end{aligned}$$

Since $C_2(\|v_0\|_S + \|v_j\|_S) < 1$, we see that v_j converges v_0. This completes the proof of Proposition 2.7. □

Next we rewrite the stationary Navier-Stokes equations (SNS) to the generalized form like (E) so that we can apply the above discussion. First of all, we apply the Leray projection P, which is abstractly defined by

$$Pv \equiv v + \nabla(-\Delta)^{-1}\text{div } v.$$

for a vector-valued function v. As can be seen from this form, we see that

$$\mathrm{div}\ (Pv) = 0, \quad P(\nabla v) = 0$$

and if div $v = 0$, then $Pv = v$. In $\mathbb{R}^n$, the Leray projection P is defined as a matrix-valued operator $P = (P_{jk})_{1\le j,k\le n}$ with $P_{jk} \equiv \delta_{jk} + R_j R_k$, where

$$\delta_{jk} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k \end{cases}$$

denotes the Kronecker delta, and $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$, $j = 1, 2, \ldots, n$ denotes the Riesz transform. Indeed, by using the Fourier transform, we see

$$\mathcal{F}[R_j f](\xi) = \frac{i\xi_j}{|\xi|}\mathcal{F}[f](\xi)$$

and hence we have

$$\begin{aligned}
\mathcal{F}[Pv](\xi) &= \mathcal{F}[v](\xi) + \mathcal{F}[\nabla(-\Delta)^{-1}\mathrm{div}\ v](\xi) \\
&= \mathcal{F}[v](\xi) - i\mathcal{F}[(-\Delta)^{-1}\mathrm{div}\ v](\xi)\xi \\
&= \mathcal{F}[v](\xi) - \frac{i}{|\xi|^2}\mathcal{F}[\mathrm{div}\ v](\xi)\xi \\
&= \mathcal{F}[v](\xi) - \frac{1}{|\xi|^2}(\xi_j\xi_k)_{1\le j,k\le n}\mathcal{F}[v](\xi) \\
&= \mathcal{F}[(\delta_{jk} + R_j R_k)_{1\le j,k\le n} v])(\xi),
\end{aligned}$$

where $(a_{jk})_{1\le j,k\le n}$ denotes a n-th square matrix whose (j,k) component is a_{jk}. Applying P to (SNS), we obtain

$$-\Delta u + P(u\cdot\nabla u) = Pf,$$

implied by $P(\nabla\Pi) = 0$ and $Pu = u$, since div $u = 0$. Hence, the solution u of (SNS) can be expressed as

$$\begin{aligned}
u &= (-\Delta)^{-1}Pf - (-\Delta)^{-1}P(u\cdot\nabla u) \\
&\equiv Lf + B(u,u). \qquad \text{(rSNS)}
\end{aligned}$$

Here and in what follows entirely, we let

$$Lf \equiv (-\Delta)^{-1}Pf, \;\; B(u,v) \equiv -(-\Delta)^{-1}P(u\cdot\nabla v),$$

which are linear and bilinear operators, respectively. We should note here that for any vectors u and v with div $u = 0$, there holds

$$\begin{aligned}
u\cdot\nabla v &= \sum_{i=1}^{n} u_i \frac{\partial v}{\partial x_i} \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(u_i v) \equiv \nabla\cdot(u\otimes v),
\end{aligned}$$

where $u \otimes v$ denotes the tensor product with $(u \otimes v)_{ij} \equiv u_i v_j$, $1 \leq i, j \leq n$. Hence under our condition, the above bilinear form B can be also written as

$$B(u, v) = -(-\Delta)^{-1} P \nabla \cdot (u \otimes v) \equiv K(u \otimes v). \tag{2.24}$$

From the next chapter, we will consider the (quantitatively) well-posedness and ill-posedness of this abstract equation (rSNS) with a solenoidal solution space S.

Chapter 3

Well-posedness in Besov and Triebel-Lizorkin spaces

3.1 Well-posedness in Besov spaces

In the beginning of our discussion, we review the previous well-posedness result shown by Kaneko-Kozono-Shimizu [11].

Proposition 3.1. (Kaneko-Kozono-Shimizu [11]) *Let $n \geq 3$, and let $1 \leq p < n$ and $1 \leq q \leq \infty$. Then there hold the followings* (1) *and* (2) :

(1) (rSNS) *is quantitatively well-posed from $D = \dot{B}^{-3+\frac{n}{p}}_{p,q}$ to $S = P\dot{B}^{-1+\frac{n}{p}}_{p,q}$, where*

$$P\dot{B}^{-1+\frac{n}{p}}_{p,q} \equiv \left\{ Pv;\ v \in \dot{B}^{-1+\frac{n}{p}}_{p,q} \right\}.$$

(2) *Let D and S be as* (1), *and let $\varepsilon > 0$ be a constant in Definition 2.5 which guarantees the well-posedness of* (rSNS) *from D to S. Suppose that $1 < r < \infty$ and $s \geq 0$ satisfy*

$$q \leq r < \infty,\ \ \frac{n}{r} - n + 1 < s < \min\left\{\frac{n}{p}, \frac{n}{r}\right\}.$$

Then there exists a positive constant $\varepsilon' = \varepsilon'(n, p, q, r, s) \leq \varepsilon$ such that for every $f \in B_D(\varepsilon') \cap \dot{H}^{s-2,r}$, the solution u obtained by (1) *has an additional regularity such as $u \in S \cap \dot{H}^{s,r}$.*

Remark 3.2. From the condition (iii) of continuity of the solution map in Definition 2.5 and the estimates in Definition 2.6, We see that not only the spaces D and S but also those norms (topologies) $\|\cdot\|_D$ and $\|\cdot\|_S$ have an important role for the well-posedness and quantitatively well-posedness. Hence when we state the (quantitatively) well-posedness of equations, it is desirable to write not only concerned spaces but also norms, such as "$(D, \|\cdot\|_D)$ to $(S, \|\cdot\|_S)$". However, only in the case that the concerned norms directly define the spaces, we omit such a norm in what follows, like the above.

Remark 3.3. The above space $P\dot{B}^{-1+\frac{n}{p}}_{p,q}$ is well-defined. Indeed, since the projection P can be written as a matrix $P=(P_{jk})_{1\le j,k\le n}$ with $P_{jk}\equiv\delta_{jk}+R_jR_k$ (see Chapter 2), we see the boundedness of P from $\dot{B}^{-1+\frac{n}{p}}_{p,q}$ to itself by Proposition 2.3.

Remark 3.4. We should note here that the space $\dot{B}^{-3+\frac{n}{p}}_{p,q}$ $(1\le p,q\le\infty)$ for the external force f and the space $\dot{B}^{-1+\frac{n}{p}}_{p,q}$ for the solution u are both scaling invariant with respect to (SNS), respectively. Indeed, if a triple $\{u,\Pi,f\}$ satisfies (SNS), then for any $\lambda>0$, so does $\{u_\lambda,\Pi_\lambda,f_\lambda\}$ defined as

$$u_\lambda(x)\equiv\lambda u(\lambda x),\ \ \Pi_\lambda(x)\equiv\lambda^2\Pi(\lambda x),\ \ f_\lambda(x)\equiv\lambda^3 f(\lambda x).$$

Hence by Proposition 2.1 (6), we see that

$$\|f_\lambda\|_{\dot{B}^{-3+\frac{n}{p}}_{p,q}}=\|f\|_{\dot{B}^{-3+\frac{n}{p}}_{p,q}},\quad \|u_\lambda\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}}=\|u\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}}.$$

Moreover, it is seen from Proposition 2.1 (2) that

$$\dot{B}^{-3+\frac{n}{p_1}}_{p_1,q}\hookrightarrow\dot{B}^{-3+\frac{n}{p_2}}_{p_2,q},\quad \dot{B}^{-1+\frac{n}{p_1}}_{p_1,q}\hookrightarrow\dot{B}^{-1+\frac{n}{p_2}}_{p_2,q}$$

for any $1\le p_1\le p_2\le\infty$.

Remark 3.5. Proposition 3.1 (1) enables us handle a larger class of functions which never belong to the usual Sobolev space. For example, in three dimension case, we may solve (SNS) with a singular external force like the Dirac delta function δ. Indeed, since

$$\begin{aligned}\varphi_j(x)&=\int_{\mathbb{R}^n}\phi(2^{-j}\xi)e^{ix\cdot\xi}d\xi\\&=2^{nj}\int_{\mathbb{R}^n}\phi(\eta)e^{ix\cdot 2^j\eta}d\eta=2^{nj}\mathcal{F}^{-1}\phi(2^jx),\end{aligned}\tag{3.1}$$

we see for each $j\in\mathbb{R}^n$ that

$$\begin{aligned}2^{sj}\|\varphi_j*\delta\|_{L^p}&=2^{sj}\|\varphi_j\|_{L^p}\\&=2^{s+nj}\|\mathcal{F}^{-1}\phi(2^j\cdot)\|_{L^p}\\&=2^{\left(s+n-\frac{n}{p}\right)j}\|\mathcal{F}^{-1}\phi\|_{L^p},\end{aligned}$$

which yields $\delta\in\dot{B}^{-n+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ for $1\le p\le\infty$. Hence we solve (SNS) for an external force $f=\delta a$ with a sufficiently small constant vector $a\in\mathbb{R}^n$ if $n=3$.

Remark 3.6. In Proposition 3.1 (2), we need smallness of f only on the scaling invariant norm, i.e., in $D=\dot{B}^{-3+\frac{n}{p}}_{p,q}$. Hence, a smallness assumption for f in the Sobolev norm $\dot{H}^{s-2,r}$ is not required. Previously, Chen [6] proved that for every smooth external force which is small in $\dot{H}^{-1,\frac{n}{2}}$, there exists a unique solution of (SNS) in $L^n\cap\dot{H}^{1,\frac{n}{2}}$

which is small in $\dot{H}^{1,\frac{n}{2}}$. Using the embedding $\dot{H}^{-1,\frac{n}{2}} \hookrightarrow \dot{B}^{-3+\frac{n}{p}}_{p,\frac{n}{2}}$ for $n/2 < p < n$, we can see that Proposition 3.1 includes the result by Chen [6], by taking $n/2 < p < n$, $q = r = n/2$, and $s = 1$.

In the next subsection, we will show the quantitatively well-posedness of (rSNS) in Triebel-Lizorkin spaces using a similar method to Kaneko-Kozono-Shimizu [11]. Hence we state here only the outline of the proof of Proposition 3.1.

Outline of the proof of Proposition 3.1. As for (1), it suffices to show (2.17) and (2.18) for

$$Lf \equiv (-\Delta)^{-1}Pf, \;\; B(u,v) \equiv -(-\Delta)^{-1}P(u\cdot\nabla v).$$

In fact, we can easily see from Proposition 2.3 that the estimate (2.17) of L holds for any $1 \le p, q \le \infty$. On the other hand, we can show the estimate (2.18) of B by using the properties in Proposition 2.1 and the following proposition on Hölder type estimates in homogeneous Besov spaces as follows:

Proposition 3.7. *Let $n \ge 1$, $s > 0$, $\alpha > 0$, $\beta > 0$, and $1 \le p, q \le \infty$. Suppose that the exponents $1 \le p_1, p_2, p_3, p_4 \le \infty$ satisfy $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$.*

(1) *There exists a constant $C = C(n, s, \alpha, \beta, p, q, p_1, p_2, p_3, p_4)$ such that for every $f \in \dot{B}^{s+\alpha}_{p_1,q} \cap \dot{B}^{-\beta}_{p_3,\infty}$ and $g \in \dot{B}^{-\alpha}_{p_2,\infty} \cap \dot{B}^{s+\beta}_{p_4,q}$, there holds $fg \in \dot{B}^s_{p,q}$ with the estimate*

$$\|fg\|_{\dot{B}^s_{p,q}} \le C\left(\|f\|_{\dot{B}^{s+\alpha}_{p_1,q}}\|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{B}^{-\beta}_{p_3,\infty}}\|g\|_{\dot{B}^{s+\beta}_{p_4,q}}\right). \tag{3.2}$$

(2) *There exists a constant $C = C(n, s, p, q, p_1, p_2, p_3, p_4)$ such that for every $f \in \dot{B}^s_{p_1,q} \cap L^{p_3}$ and $g \in L^{p_2} \cap \dot{B}^s_{p_4,q}$, there holds $fg \in \dot{B}^s_{p,q}$ with the estimate*

$$\|fg\|_{\dot{B}^s_{p,q}} \le C\left(\|f\|_{\dot{B}^s_{p_1,q}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_3}}\|g\|_{\dot{B}^s_{p_4,q}}\right). \tag{3.3}$$

For the proof of this estimates, see Appendix A.

In order to show Proposition 3.1 (2), we use the lemma as follows, which can be also proved by Proposition 2.1 and Proposition 3.7:

Lemma 3.8. *Let n, p, q, r and s be as assumption of Proposition 3.1, and let $D' \equiv \dot{H}^{s-2,r}$, $S' \equiv \dot{H}^{s,r}$. Then there hold*

$$\|Lf\|_{S'} \le C_1'\|f\|_{D'}, \;\; \forall f \in D' \tag{3.4}$$

and

$$\|B(u,v)\|_{S'} \le C_2'\left(\|u\|_S\|v\|_{S'} + \|u\|_{S'}\|v\|_S\right), \;\; \forall u, v \in S \cap S', \tag{3.5}$$

where $C_1' = C_1'(n, s, r)$, $C_2' = C_2'(n, s, p, q, r)$ are positive constants.

Let us return the abstract discussion in Chapter 2 and consider again the approximative sequence $\{u_j\}_{j\in\mathbb{N}}$ defined by (2.19). By (2.17) and (3.4), we see that $u_1 \in S \cap S'$ for any $f \in D \cap D'$. Moreover, if $u_j \in S \cap S'$, then $u_{j+1} \in S \cap S'$ with the estimates (2.20) and

$$\|u_{j+1}\|_{S'} \leq C_1'\|f\|_{D'} + 2C_2'\delta_1\|u_j\|_{S'}, \tag{3.6}$$

implied by (3.4), (3.5) and the boundedness of $\{u_j\}_{j\in\mathbb{N}}$ in S as (2.22). Hence $u_j \in S \cap S'$ for all $j \geq 1$ by induction. We see from this estimate that if

$$\delta_1 < \frac{1}{2C_2'}, \tag{3.7}$$

then there holds

$$\|u_j\|_{S'} \leq \delta_1' \equiv \frac{C_1'\|f\|_{D'}}{1 - 2C_2'\delta_1}, \quad j \geq 1. \tag{3.8}$$

Since it is seen from (2.22) that $\delta_1 \to 0$ as $\|f\|_D \to 0$, there is a constant $0 < \varepsilon' < \varepsilon$ such that if $\|f\|_D < \varepsilon'$, then (3.8) holds. Under this condition, it holds by (2.22), (3.5), and (3.8) that

$$\begin{aligned}
\|u_{j+1} - u_j\|_{S'} &= \|B(u_j, u_j - u_{j-1}) + B(u_j - u_{j-1}, u_{j-1})\|_{S'} \\
&\leq C_2'\left(\|u_j\|_S\|u_j - u_{j-1}\|_{S'} + \|u_j\|_{S'}\|u_j - u_{j-1}\|_S\right) \\
&\quad + C_2'\left(\|u_j - u_{j-1}\|_S\|u_{j-1}\|_{S'} + \|u_j - u_{j-1}\|_{S'}\|u_{j-1}\|_S\right) \\
&\leq 2C_2'\delta_1\|u_j - u_{j-1}\|_{S'} + 2C_2'\delta_1'\|u_j - u_{j-1}\|_S
\end{aligned}$$

for all $j \geq 2$. Since $2C_2'\delta_1 < 1$ by (3.7) and since (2.23) holds, we have

$$\sum_{j=1}^{\infty} \|u_{j+1} - u_j\|_{S'} < \infty.$$

Hence there holds $u_j \to u$ in S', which proves (2) of Proposition 3.1.

3.2 Well-posedness in Triebel-Lizorkin spaces

Using a similar method to Kaneko-Kozono-Shimizu, we can show the well-posedness of (rSNS) in Triebel-Lizorkin spaces as follows:

Theorem 3.9. (Tsurumi [24]) (1) *Let $n \geq 3$, and suppose that the exponents p and q satisfy the following either* (i) *or* (ii);

(i) $1 < p < n$, $1 \leq q \leq \infty$,

(ii) $p = n$, $1 \leq q \leq 2$.

Then (rSNS) *is quantitatively well-posed from* $D = \dot{F}_{p,q}^{-3+\frac{n}{p}}$ *to* $S = P\dot{F}_{p,q}^{-1+\frac{n}{p}}$*, where*

$$S = P\dot{F}_{p,q}^{-1+\frac{n}{p}} \equiv \left\{ Pv;\ v \in \dot{F}_{p,q}^{-1+\frac{n}{p}} \right\},$$

which is well-defined by Proposition 2.3.

(2) *Let* $n \geq 3$*, and suppose that the exponents* p, q, r*, and* s *satisfy the following either* (i)*,* (ii)*, or* (iii)*;*

(i) $s > 0$*,* $1 < r < \infty$*,* p *and* q *satisfy either* (i) *or* (ii) *of* (1)*,*

(ii) $s = 0$*,* $n/(n-1) < r < \infty$*,* p *and* q *satisfy* (i) *of* (1)*,*

(iii) $s = 0$*,* $r = n$*,* p *and* q *satisfy* (ii) *of* (1)*.*

Moreover, let D *and* S *be as* (1) *and let* $\varepsilon > 0$ *be in Definition 2.5 which guarantees the well-posedness of* (rSNS) *from* D *to* S*. Then there exists a positive constant* $\varepsilon' = \varepsilon'(n, p, q, r, s) \leq \varepsilon$ *such that for every* $f \in B_D(\varepsilon') \cap \dot{H}^{s-2,r}$*, the solution* u *obtained by Theorem 3.9 has an additional regularity such as* $u \in S \cap \dot{H}^{s,r}$*.*

Remark 3.10. In Theorem 3.9, the spaces $\dot{F}_{p,q}^{-1+\frac{n}{p}}$ for solutions u and $\dot{F}_{p,q}^{-3+\frac{n}{p}}$ for external forces f are both scaling invariant with respect to (SNS).

Remark 3.11. Theorem 3.9 (2) means that a smooth external force whose scaling invariant Triebel-Lizorkin norm is small enough yields a smooth solution of (rSNS). We should note that the $\dot{H}^{s-2,r}$ norm of an external force do not have to be small. Moreover, in the case (i) of (2), we can take $s \geq 0$ arbitrary large, while in Besov spaces, there is a restriction on the exponent s (compare with Proposition 3.1).

In particular, it is seen from Theorem 3.9 (1) with $p = n, q = 2$ that a small external force f in $\dot{H}^{-2,n} \cong \dot{F}_{n,2}^{-2}$ yields an unique solution $u \in L^n \cong \dot{F}_{n,2}^{0}$ of (E). Moreover, if this f also belongs to L^n, then it holds from (2) with $s = 2$ and $r = n$ that u also belongs to $\dot{H}^{2,n}$. Hence u belongs to the inhomogeneous Sobolev space $H^{2,n} = L^n \cap \dot{H}^{2,n}$, which implies that u satisfies the original equation (SNS) almost everywhere in $\mathbb{R}^n$.

Remark 3.12. If we let $p > n/2$ and $1 \leq q \leq \infty$, then we have $\dot{H}^{-1,\frac{n}{2}} \hookrightarrow \dot{F}_{p,q}^{-3+\frac{n}{p}}$. Therefore, Theorem 3.9 includes the result by Chen [6], provided $p > n/2$, $1 \leq q \leq \infty$, $s = 1$, and $r = n/2$.

Proof of Theorem 3.9. For the proof of our main theorems, it suffices to show four lemmata as follows.

Lemma 3.13. *Let* $n \geq 2$*,* $s \in \mathbb{R}$ *and let* $1 \leq p < \infty$*,* $1 \leq q \leq \infty$*. Then the operator* $L \equiv (-\Delta)^{-1}P$ *is bounded from* $\dot{F}_{p,q}^{s-2}$ *to* $P\dot{F}_{p,q}^{s}$ *with the estimate*

$$\|Lf\|_{\dot{F}_{p,q}^{s}} \leq C\|f\|_{\dot{F}_{p,q}^{s-2}},$$

where $C = C(n, s, p, q)$ is a constant.

Lemma 3.14. *Let $n \geq 2$, $s \in \mathbb{R}$, and let $1 \leq p < \infty$, $1 \leq q \leq \infty$. Then the operator $K \equiv -(-\Delta)^{-1}P\nabla\cdot$ (see (2.24)) is bounded from $\dot{F}^{s-1}_{p,q}$ to $P\dot{F}^{s}_{p,q}$ with the estimate*

$$\|Kg\|_{\dot{F}^s_{p,q}} \leq C\|g\|_{\dot{F}^{s-1}_{p,q}},$$

where $C = C(n, s, p, q)$ is a constant.

Lemma 3.15. *Let $n \geq 3$, and let $1 < p < n$, $1 \leq q, \tilde{q} \leq \infty$. Then for $u, v \in \dot{F}^{-1+\frac{n}{p}}_{p,q}$, we have $u \otimes v \in \dot{F}^{-2+\frac{n}{p}}_{p,\tilde{q}}$ with the estimate*

$$\|u \otimes v\|_{\dot{F}^{-2+\frac{n}{p}}_{p,\tilde{q}}} \leq C\|u\|_{\dot{F}^{-1+\frac{n}{p}}_{p,q}}\|v\|_{\dot{F}^{-1+\frac{n}{p}}_{p,q}},$$

where $C = C(n, p, q, \tilde{q})$ is a constant. Moreover, this claim is true if $p = n$, $1 \leq q \leq 2$ and $1 \leq \tilde{q} \leq \infty$.

Lemma 3.16. *Let $n \geq 2$, and suppose that p, q, r, and s satisfy either* (i), (ii), *or* (iii) *of Theorem 3.9 (2). Then for $u, v \in \dot{F}^{-1+\frac{n}{p}}_{p,q} \cap \dot{H}^{s,r}$, we have $u \otimes v \in \dot{H}^{s-1,r}$ with the estimate*

$$\|u \otimes v\|_{\dot{H}^{s-1,r}} \leq C\left(\|u\|_{\dot{F}^{-1+\frac{n}{p}}_{p,q}}\|v\|_{\dot{H}^{s,r}} + \|u\|_{\dot{H}^{s,r}}\|v\|_{\dot{F}^{-1+\frac{n}{p}}_{p,q}}\right),$$

where $C = C(n, s, p, q, r)$ is a constant.

For the moment, let us assume these lemmata. Under the assumption of Theorem 3.9 (1), let $D \equiv \dot{F}^{-3+\frac{n}{p}}_{p,q}$ and $S \equiv P\dot{F}^{-1+\frac{n}{p}}_{p,q}$. By Lemma 3.13, we see that $Lf \in S$ for any $f \in D$ with the estimate

$$\|Lf\|_S \leq C_1\|f\|_D.$$

Moreover, by Lemma 3.14-3.15 and (2.24), $B(u, v) \equiv K(u \otimes v) \in S$ for any $u, v \in S$ with the estimate

$$\begin{aligned} \|B(u,v)\|_S &\leq C_2'\|u \otimes v\|_{\dot{F}^{-2+\frac{n}{p}}_{p,q}} \\ &\leq C_2\|u\|_S\|v\|_S, \end{aligned}$$

where C_1, C_2' and C_2 are constants depending only on n, p, and q. Therefore, we see that (rSNS) is quantitative well-posed from D to S, which proves (1) of Theorem 3.9. On the other hand, under the assumption of Theorem 3.9 (2), let $D' \equiv \dot{H}^{s-2,r}$ and $S' \equiv \dot{H}^{s,r}$. By Proposition 2.1 (3) and Lemma 3.13, there holds

$$\|Lf\|_{S'} = \|Lf\|_{\dot{F}^s_{r,2}} \leq C_3\|f\|_{\dot{F}^{s-2}_{r,2}} = C_3\|f\|_{D'}$$

for any $f \in D'$. Moreover, by Lemma 3.14 and Lemma 3.16, it holds that

$$\begin{aligned}\|B(u,v)\|_{S'} &= \|K(u \otimes v)\|_{\dot{F}^s_{r,2}} \\ &\leq C_4' \|u \otimes v\|_{\dot{F}^{s-1}_{r,2}} \\ &= C_4' \|u \otimes v\|_{\dot{H}^{s-1,r}} \\ &\leq C_4 \left(\|u\|_S \|v\|_{S'} + \|u\|_{S'} \|v\|_S\right)\end{aligned}$$

for any $u, v \in S \cap S'$. Here $C_3 = C_3(n,s,r)$, $C_4' = C_4'(n,s,r)$, and $C_4 = C_4(n,s,p,q,r)$ are constants. Hence by the same discussion as the case of Besov spaces (see Lemma 3.8 and the proof of Proposition 3.1 (2)), we obtain (2) of Theorem 3.9.

Now let us show Lemmata 3.13-3.16.

Proof of Lemma 3.13. Since the projection P is defined as a matrix-valued operator $P = (P_{jk})_{1\leq j,k\leq n}$ with $P_{jk} \equiv \delta_{jk} + R_j R_k$, P has the same boundedness as that of Riesz transforms in Proposition 2.2. Together with Proposition 2.1 (5), we can see that

$$\begin{aligned}\|(-\Delta)^{-1} P f\|_{\dot{F}^s_{p,q}} &\leq C \|Pf\|_{\dot{F}^{s-2}_{p,q}} \\ &\leq C \|f\|_{\dot{F}^{s-2}_{p,q}},\end{aligned}$$

for every $s \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $f \in \dot{F}^{s-2}_{p,q}$.

Proof of Lemma 3.14. Let $g = (g_{ij})_{1\leq i,j\leq n}$ be a matrix-valued function. By commutativity of P and $(-\Delta)^{\frac{s}{2}}$, $Kg = ((Kg)_1, (Kg)_2, \ldots, (Kg)_n)$ can be written as

$$\begin{aligned}(Kg)_j &= -(-\Delta)^{-1} P \sum_{i=1}^n \frac{\partial}{\partial x_i} g_{ij} \\ &= -(-\Delta)^{-\frac{1}{2}} P \sum_{i=1}^n (-\Delta)^{-\frac{1}{2}} \frac{\partial}{\partial x_i} g_{ij} \\ &= -(-\Delta)^{-\frac{1}{2}} P \sum_{i=1}^n R_i g_{ij}. \qquad (3.9)\end{aligned}$$

Hence we see from Proposition 2.1 (5) and Proposition 2.2 that there holds

$$\|Kg\|_{\dot{F}^s_{p,q}} \leq C \|g\|_{\dot{F}^{s-1}_{p,q}}$$

for every $s \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $g \in \dot{F}^{s-1}_{p,q}$.

Proof of Lemma 3.15. We first consider the case $p = n$, $1 \leq q \leq 2$ and $1 \leq \tilde{q} \leq \infty$. Since it is seen from Proposition 2.1 (2) and (3) that there holds $L^{\frac{n}{2}} \cong \dot{F}^0_{\frac{n}{2},2} \hookrightarrow \dot{F}^{-1}_{n,\tilde{q}}$, we see by Hölder inequality that

$$\begin{aligned}\|u\otimes v\|_{\dot F^{-1}_{n,\tilde q}} &\le C\|u\otimes v\|_{L^{\frac{n}{2}}}\\ &\le C\|u\|_{L^n}\|v\|_{L^n}\\ &= C\|u\|_{\dot F^0_{n,2}}\|v\|_{\dot F^0_{n,2}}\\ &\le C\|u\|_{\dot F^0_{n,q}}\|v\|_{\dot F^0_{n,q}}.\end{aligned}$$

We next consider the case $1 < p < n$ and $1 \le q, \tilde q \le \infty$. Here we use the following proposition, which is an alternative to Proposition 3.7:

Proposition 3.17. (Kozono-Shimada [13]) *Let $s, \alpha > 0$, $1 < p < \infty$, and let us take $1 < p_1, p_2 < \infty$ so that $1/p = 1/p_1 + 1/p_2$. Then there is a constant $C = C(s, \alpha, p, p_1, p_2)$ such that for every $f, g \in \dot F^{s+\alpha}_{p_1,\infty} \cap \dot F^{-\alpha}_{p_2,\infty}$, there holds $f\cdot g \in \dot F^s_{p,\infty}$ with the estimate*

$$\|f\cdot g\|_{\dot F^s_{p,\infty}} \le C\left(\|f\|_{\dot F^{s+\alpha}_{p_1,\infty}}\|g\|_{\dot F^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot F^{-\alpha}_{p_2,\infty}}\|g\|_{\dot F^{s+\alpha}_{p_1,\infty}}\right).$$

Since $n \ge 3$, we can take p_0 satisfying $1 < p_0 < \min\{p, n/2\}$. Moreover, since $p < n$, we can choose p_1 and p_2 as

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}, \quad p_1 > p, \quad p_2 > n$$

by choosing p_0 properly. Indeed, if $p < n/2$, then we should let $p_0 = p/(1+\varepsilon)$ with small $0 < \varepsilon < p/n$ so that $p_0 > 1$ and

$$\frac{1}{p_0} = \frac{1}{p} + \frac{\varepsilon}{p} < \frac{1}{p} + \frac{1}{n}.$$

On the other hand, if $n/2 \le p$, then we should let $p_0 = n/(2+\varepsilon)$ with small $0 < \varepsilon < (n/p) - 1$ so that $p_0 > 1$ and

$$\frac{1}{p_0} = \frac{1+\varepsilon}{n} + \frac{1}{n} < \frac{1}{p} + \frac{1}{n}.$$

From Proposition 2.1 (2) and Proposition 3.17 with $s = -2 + (n/p_0) > 0$ and $\alpha = 1 - (n/p_2) > 0$, we have

$$\begin{aligned}\|u\otimes v\|_{\dot F^{-2+\frac{n}{p}}_{p,\tilde q}} &\le C\|u\otimes v\|_{\dot F^{-2+\frac{n}{p_0}}_{p_0,\infty}}\\ &\le C\left(\|u\|_{\dot F^{-2+\frac{n}{p_0}+1-\frac{n}{p_2}}_{p_1,\infty}}\|v\|_{\dot F^{-1+\frac{n}{p_2}}_{p_2,\infty}} + \|u\|_{\dot F^{-1+\frac{n}{p_2}}_{p_2,\infty}}\|v\|_{\dot F^{-2+\frac{n}{p_0}+1-\frac{n}{p_2}}_{p_1,\infty}}\right)\\ &\le C\|u\|_{\dot F^{-1+\frac{n}{p}}_{p,q}}\|v\|_{\dot F^{-1+\frac{n}{p}}_{p,q}},\end{aligned}$$

which proves Lemma 3.15.

Proof of Lemma 3.16.

For the case of the condition (iii) in Theorem 3.9 (2), we can show this lemma in a similar way to the first paragraph in the proof of Lemma 3.15. Indeed, it is seen that

$$\begin{aligned}\|u\otimes v\|_{\dot H^{-1,n}} &\le C\|u\otimes v\|_{L^{\frac{n}{2}}}\\ &\le C\left(\|u\|_{L^n}\|v\|_{L^n}+\|u\|_{L^n}\|v\|_{L^n}\right)\\ &\le C\left(\|u\|_{\dot F^0_{n,q}}\|v\|_{\dot H^{0,n}}+\|u\|_{\dot H^{0,n}}\|v\|_{\dot F^0_{n,q}}\right).\end{aligned}$$

In what follows, we consider the cases (i) and (ii) in Theorem 3.9 (2).

Case 1: Under the condition (i) *in Theorem 3.9 (2).*

Since $s>0$ and $r>1$, we can choose $s_0>0$, $\alpha>0$ as

$$\max\left\{0, 1-n+\frac{n}{r}\right\}<\alpha=s-s_0<1.$$

We next choose r_0, $\tilde r$ such that

$$\frac{1}{r_0}=\frac{1}{r}+\frac{1}{\tilde r},\quad \tilde r=\frac{n}{1-\alpha}.$$

By this definition, we have

$$s_0>s-1,\quad r_0<r,\quad s_0-\frac{n}{r_0}=s-1-\frac{n}{r},$$

and

$$-1+\frac{n}{p}>-\alpha,\quad p\le n<\tilde r,\quad -1+\frac{n}{p}-\frac{n}{p}=-\alpha-\frac{n}{\tilde r}.$$

Hence, by Proposition 2.1 (2) and (3), we see that

$$\dot F^{s_0}_{r_0,\infty}\hookrightarrow \dot F^{s-1}_{r,2}\cong \dot H^{s-1,r} \tag{3.10}$$

and

$$\dot F^{-1+\frac{n}{p}}_{p,q}\hookrightarrow \dot F^{-\alpha}_{\tilde r,\infty}. \tag{3.11}$$

On the other hand, there holds $r_0>1$, since

$$\frac{1}{r_0}=\frac{1}{r}+\frac{1-\alpha}{n}<\frac{1}{r}+\frac{1-\left(1-n+\frac{n}{r}\right)}{n}=1.$$

Therefore, by (3.10) and Proposition 3.17, we obtain

$$\begin{aligned}\|u\otimes v\|_{\dot H^{s-1,r}} &\le C\|u\otimes v\|_{\dot F^{s_0}_{r_0,\infty}}\\ &\le C\left(\|u\|_{\dot F^{s_0+\alpha}_{r,\infty}}\|v\|_{\dot F^{-\alpha}_{\tilde r,\infty}}+\|u\|_{\dot F^{-\alpha}_{\tilde r,\infty}}\|v\|_{\dot F^{s_0+\alpha}_{r,\infty}}\right).\end{aligned}$$

Moreover, by (3.11) and the embedding

$$\dot{H}^{s,r} \cong \dot{F}^{s}_{r,2} \hookrightarrow \dot{F}^{s}_{r,\infty} = \dot{F}^{s_0+\alpha}_{r,\infty},$$

we obtain the estimate

$$\|u \otimes v\|_{\dot{H}^{s-1,r}} \le C \left(\|u\|_{\dot{F}^{-1+\frac{n}{p}}_{p,q}} \|v\|_{\dot{H}^{s,r}} + \|u\|_{\dot{H}^{s,r}} \|v\|_{\dot{F}^{-1+\frac{n}{p}}_{p,q}} \right).$$

Case 2: Under the condition (ii) *in Theorem 3.9 (2).*

Since $p < n$, we can take $s_0 > 0$ and $\alpha > 0$ satisfying

$$s_0 + \alpha = -1 + \frac{n}{p}, \quad 0 < \alpha < \frac{n}{r}, \tag{3.12}$$

whose specific values will be decided later on. Then we define r_0, r_1 and r_2 as

$$\frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2}, \quad r_1 = p, \quad \frac{1}{r_2} = \frac{1}{r} - \frac{\alpha}{n}. \tag{3.13}$$

By taking appropriate $\alpha > 0$, we have $r_0 > 1$. Indeed, if

$$\frac{n}{n-1} < r < \frac{np}{n-p}, \quad \left(-1 + \frac{n}{p} < \frac{n}{r} \right)$$

then we take $\gamma > 1$ such that $n/(n-1) < n/(n-\gamma) < r$, and decide $\alpha > 0$ as $\alpha = -1 + (n/p) - (\gamma - 1)$ so that

$$\begin{aligned} \frac{1}{r_0} &= \frac{1}{p} + \frac{1}{r} - \frac{1}{n}\left\{ -1 + \frac{n}{p} - (\gamma - 1) \right\} \\ &< \frac{n-\gamma}{n} + \frac{1}{n} + \frac{\gamma - 1}{n} = 1. \end{aligned}$$

On the other hand, if

$$\frac{np}{n-p} \le r < \infty, \quad \left(\frac{n}{r} \le -1 + \frac{n}{p} \right)$$

then we take $\varepsilon > 0$ such that $(1/p) + (\varepsilon/n) < 1$, and decide $\alpha > 0$ as $\alpha = (n/r) - \varepsilon$ so that

$$\frac{1}{r_0} = \frac{1}{p} + \frac{1}{r} - \frac{1}{n}\left(\frac{n}{r} - \varepsilon \right) = \frac{1}{p} + \frac{\varepsilon}{n} < 1.$$

Therefore, we can choose s_0, α, r_0, r_1, and r_2 satisfying (3.12), (3.13), and $r_0 > 1$.

Since $s_0 > -1$, $r_0 < r$ (because of $\alpha < n/p$), and

$$s_0 - \frac{n}{r_0} = s_0 - \frac{n}{p} - n\left(\frac{1}{r} - \frac{\alpha}{n} \right) = -1 - \frac{n}{r},$$

it is seen by Proposition 2.1 (2) and (3) that $\dot{F}^{s_0}_{r_0,\infty} \hookrightarrow \dot{F}^{-1}_{r,2} \cong \dot{H}^{-1,r}$. Therefore, by Proposition 3.17, we obtain

$$\begin{aligned}
\|u \otimes v\|_{\dot{H}^{-1,r}} &\leq C\|u \otimes v\|_{\dot{F}^{s_0}_{r_0,\infty}} \\
&\leq C\left(\|u\|_{\dot{F}^{s_0+\alpha}_{r_1,\infty}}\|v\|_{\dot{F}^{-\alpha}_{r_2,\infty}} + \|u\|_{\dot{F}^{-\alpha}_{r_2,\infty}}\|v\|_{\dot{F}^{s_0+\alpha}_{r_1,\infty}}\right).
\end{aligned}$$

Moreover, since $\dot{F}^{-1+\frac{n}{p}}_{p,q} = \dot{F}^{s_0+\alpha}_{r_1,q} \hookrightarrow \dot{F}^{s_0+\alpha}_{r_1,\infty}$, and since $\dot{H}^{0,r} \cong \dot{F}^0_{r,2} \hookrightarrow \dot{F}^0_{r,\infty} \hookrightarrow \dot{F}^{-\alpha}_{r_2,\infty}$ because of $-n/r = -\alpha - (n/r_2)$, we obtain

$$\|u \otimes v\|_{\dot{H}^{-1,r}} \leq C\left(\|u\|_{\dot{F}^{-1+\frac{n}{p}}_{p,q}}\|v\|_{\dot{H}^{0,r}} + \|u\|_{\dot{H}^{0,r}}\|v\|_{\dot{F}^{-1+\frac{n}{p}}_{p,q}}\right).$$

This proves Lemma 3.16.

By the foregoing, we have proved Lemmata 3.13-3.16, which completes the proof of Theorem 3.9. □

As can be seen in the above proof, the restrictions on the dimension n and the integral exponent p such as $n \geq 3$ and $1 \leq p < n$ seem to be due to the validity of Proposition 3.17 (in Besov spaces, of Proposition 3.7), which has an important role for the boundedness of the bilinear form B. This problem is also true of the case in Besov spaces as well. From the next section, we will treat the case $n \leq p \leq \infty$ from the *negative* approach, that is, we will show that in such a case, (rSNS) is *ill-posed* from $D = \dot{B}^{-3+\frac{n}{p}}_{p,q}$ to $S = P\dot{B}^{-1+\frac{n}{p}}_{p,q}$.

Chapter 4

Ill-posedness by the Bourgain-Pavlović method

In this chapter, we show the ill-posedness of (rSNS) from $\dot{B}_{p,q}^{-3+\frac{n}{p}}$ to $P\dot{B}_{p,q}^{-1+\frac{n}{p}}$ when $p=\infty$. Our claim is as follows:

Theorem 4.1. (Tsurumi [23]) *Let $n \geq 3$. For any $\delta > 0$, there exists a sequence $\{f_N\}_{N\in\mathbb{N}}$ of external forces in $BUC^2 \cap \dot{B}_{\infty,1}^{-3}$ with* div $f_N = 0$ *such that*

(i) $\|f_N\|_{\dot{B}_{\infty,1}^{-3}} \to 0$ *as* $N \to \infty$,

(ii) *For each f_N, there exists a solution u_N of* (rSNS) *in PL^∞ and $P\dot{B}_{\infty,1}^{-1}$ senses. Moreover, each u_N satisfies*

$$\begin{cases} -\Delta u_N(x) + (u_N \cdot \nabla u_N)(x) + \nabla \Pi(x) = f_N(x), \\ \mathrm{div}\ u_N(x) = 0 \end{cases}$$

for all $x \in \mathbb{R}^n$ with a constant pressure Π, *i.e.*, $\nabla\Pi = 0$.

(iii) *There exists another constant $0 < \kappa < 1$ independent of δ such that*

$$\kappa\delta < \|u_N\|_{\dot{B}_{\infty,\infty}^{-1}} \leq \|u_N\|_{\dot{B}_{\infty,1}^{-1}} < \delta$$

for any $N \in \mathbb{N}$.

Here BUC^2 denotes the space of bounded uniformly continuous functions up to the second order derivatives.

Remark 4.2. This result shows the ill-posedness of (SNS) from $\dot{B}_{\infty,q}^{-3}$ to $P\dot{B}_{\infty,q}^{-1}$ for all $1 \leq q \leq \infty$ (see also Proposition 2.1 (1)). Indeed, Theorem 4.1 means that for any constants ε and δ, the solution map $f \in B_{\dot{B}_{\infty,q}^{-3}}(\varepsilon) \mapsto u \in B_{\dot{B}_{\infty,q}^{-1}}(\delta)$ is, even if it is well-defined, not continuous at zero in each norm. We should note here that each solution u_N above is a strong solution of the original equation (SNS) with a constant pressure Π and is not necessarily unique one.

It is also seen from Theorem 4.1 that there is a external force which is arbitrary small in $\dot{B}^{-3}_{\infty,1}$ can admit a solution which is arbitrary large in $\dot{B}^{-1}_{\infty,\infty}$. In order to see this phenomenon, take a huge $\delta > 0$, choose a sequence of external forces in Theorem 4.1, and fix a number N sufficiently large.

Remark 4.3. Theorem 4.1 also holds for the homogeneous Triebel-Lizorkin spaces with the same exponents. In fact, since it holds from Proposition 2.1 (4) that

$$\dot{B}^{-3}_{\infty,q} \hookrightarrow \dot{F}^{-3}_{\infty,q}, \quad \dot{B}^{-1}_{\infty,\infty} \cong \dot{F}^{-1}_{\infty,\infty},$$

we can show the ill-posedness by using the same sequence $\{f_N\}_{N\in\mathbb{N}}$ of external forces.

In the proof of our result, trigonometric functions and their linear sum will appear frequently. Indeed, Bourgain-Pavlović [5] showed the ill-posedness of non-stationary Navier-Stokes equations by using a sequence of initial data composed of trigonometric functions. Therefore, in order to apply their method to stationary equations, we note here some important properties of such functions in harmonic analysis. In what follows, we write $\frac{\partial}{\partial x_i}$ as ∂_i, $i = 1, 2, \ldots, n$ for simplicity.

We now take a trigonometric function

$$g(x) \equiv \cos(a \cdot x), \quad x \in \mathbb{R}^n$$

with a constant vector $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \backslash \{0\}$. Since

$$g(x) = \frac{e^{ia\cdot x} + e^{-ia\cdot x}}{2}$$

by the Euler's formula, we can see that

$$\begin{aligned}
\mathcal{F}g(\xi) &= \frac{1}{2}\left\{\int_{\mathbb{R}^n} e^{ia\cdot x} \cdot e^{-ix\cdot\xi} dx + \int_{\mathbb{R}^n} e^{-ia\cdot x} \cdot e^{-ix\cdot\xi} dx\right\} \\
&= \frac{1}{2}(\mathcal{F}[1](\xi - a) + \mathcal{F}[1](\xi + a)) \\
&= \frac{1}{2}(\delta(\xi - a) + \delta(\xi + a)),
\end{aligned}$$

where δ denotes the Dirac measure on $\mathbb{R}^n$ having a unit mass at the origin. Therefore,

$$\begin{aligned}
(-\Delta)^{\frac{s}{2}} g(x) &= \mathcal{F}^{-1}[|\xi|^s \mathcal{F}g(\xi)](x) \\
&= \mathcal{F}^{-1}\left[\frac{1}{2}|\xi|^s(\delta(\xi - a) + \delta(\xi + a))\right] \\
&= |a|^s \cos(a \cdot x).
\end{aligned}$$

We should note that if $s = 2$, we have

$$-\Delta g(x) = \left(-\sum_{i=1}^{n} \partial_i^2\right) g(x)$$

and

$$\begin{aligned}(-\Delta)(-\Delta)^{-1}g(x) &= (-\Delta)^{-1}(-\Delta)g(x)\\ &= g(x).\end{aligned}$$

Moreover, by assuming that ϕ in the definition of Littlewood-Paley decomposition, (2.3), is spherical symmetric, it holds that

$$\begin{aligned}(\varphi_j * g)(x) &= \mathcal{F}^{-1}[\mathcal{F}\varphi_j(\xi)\mathcal{F}g(\xi)](x)\\ &= \mathcal{F}^{-1}\left[\frac{1}{2}\phi(2^{-j}\xi)(\delta(\xi-a)+\delta(\xi+a))\right]\\ &= \phi(2^{-j}a)\cos(a\cdot x), \quad \forall j\in\mathbb{Z}. \end{aligned}\tag{4.1}$$

Hence, based on (2.3), (2.4), and the fact $\|\cos(a\cdot)\|_{L^\infty}=1$, we obtain the following two key estimates for our main result;

$$\begin{aligned}\|g\|_{\dot{B}^{-m}_{\infty,1}} &\le C_\#^m\|(-\Delta)^{-\frac{m}{2}}g\|_{\dot{B}^0_{\infty,1}}\\ &= \left(\frac{C_\#}{|a|}\right)^m\sum_{j\in\mathbb{Z}}\phi(2^{-j}a)\\ &= \left(\frac{C_\#}{|a|}\right)^m, \quad m\in\mathbb{N}, \end{aligned}\tag{4.2}$$

$$\begin{aligned}\|g\|_{\dot{B}^{-1}_{\infty,\infty}} &\ge \frac{1}{C_\#|a|}\sup_{j\in\mathbb{Z}}\phi(2^{-j}a)\\ &\ge \frac{1}{2C_\#|a|}, \end{aligned}\tag{4.3}$$

where $C_\#$ denotes a constant dependent only on the dimension n satisfying

$$\frac{1}{C_\#}\|(-\Delta)^{-\frac{1}{2}}f\|_{\dot{B}^{s+1}_{p,q}} \le \|f\|_{\dot{B}^s_{p,q}} \le C_\#\|(-\Delta)^{-\frac{1}{2}}f\|_{\dot{B}^{s+1}_{p,q}} \tag{4.4}$$

for any $s\in\mathbb{R}$ and $1\le p,q\le\infty$ (see Proposition 2.1 (5)). In what follows, we suppose that every constant $C_\#$ appearing below denotes this constant entirely. In addition, let us prepare the L^∞ estimate as follows for the sake of the proof of the main theorem:

$$\begin{aligned}\|(-\Delta)^{-1}g\|_{L^\infty} &\le \|\partial_i(-\Delta)^{-1}g\|_{L^\infty}\\ &\le \|\partial_i\partial_j(-\Delta)^{-1}g\|_{L^\infty}\\ &\le \|g\|_{L^\infty} \quad \text{for } 1\le i,j\le n, \quad \text{if } |a_i|,|a_j|\ge 1. \end{aligned}\tag{4.5}$$

It can be easily seen that the above estimates are valid as it is to the case $g(x)=\sin(a\cdot x)$.

Proof of Theorem 4.1. Here we take the parametrized external force as

$$f_{Q,r}(x) \equiv Qr^2 \left\{ e_2 \cos(rx_1) + e_3 \cos(rx_1 - x_2) \right\}, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n,$$

where $e_2 \equiv (0,1,0,0,\dots,0)$ and $e_3 \equiv (0,0,1,0,\dots,0)$ are unit vectors in $\mathbb{R}^n$, while $Q > 0$ and $r \in \mathbb{N}$ are parameters. Actually, this external force is inspired by the study of Bourgain-Pavlović [5], who showed the ill-posedness of the non-stationary Navier-Stokes equations in $\dot{B}^{-1}_{\infty,\infty}$. In fact, they used the parametrized initial data as

$$u_0(x) = \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s \left\{ e_2 \cos(h_s x_1) + e_3 \cos(h_s x_1 - x_2) \right\},$$

where $h_s \equiv 2^{\frac{s(s-1)}{2}} \gamma^{s-1} \eta$ and Q, r, s, γ, and η are parameters (see also Sawada [17], who refined the study by Bourgain-Pavlović [5]).

We now define the approximative sequence $\{u_j\}_{j\in\mathbb{N}}$ to the solution u of (rSNS) such as (2.19):

$$\begin{cases} u_1 \equiv L f_{Q,r}, \\ u_j \equiv u_1 + B(u_{j-1}, u_{j-1}), \quad j \geq 2. \end{cases}$$

Moreover, we rewrite these u_j as forms of series in accordance with Sawada [17]. Let

$$\begin{cases} v_1 \equiv u_1, \\ v_2 \equiv B(u_1, u_1) = B(v_1, v_1), \\ v_k \equiv B(u_{k-1}, u_{k-1}) - B(u_{k-2}, u_{k-2}), \quad k \geq 3. \end{cases} \tag{4.6}$$

Obviously, it holds

$$u_j = \sum_{k=1}^{j} v_k, \quad j \geq 1. \tag{4.7}$$

As for $f_{Q,r}$ and $\{v_k\}_{k\in\mathbb{N}}$, we can show the following key lemma.

Lemma 4.4. *Let $\{v_k\}_{k\in\mathbb{N}}$ be as (4.6). Then it holds that $v_k \in BUC^2 \cap \dot{B}^{-1}_{\infty,1}$ and* div $v_k = 0$ *for all $k \geq 1$. Moreover, we have the following estimates*;

$$\|f_{Q,r}\|_{\dot{B}^{-3}_{\infty,1}} \leq 2C_\#^3 \frac{Q}{r}, \quad \|v_1\|_{\dot{B}^{-1}_{\infty,1}} \leq 2C_\# \frac{Q}{r} \quad \textit{for all } Q > 0,\ r \in \mathbb{N}, \tag{4.8}$$

$$\frac{Q^2}{16C_\#} \leq \|v_2\|_{\dot{B}^{-1}_{\infty,\infty}} \leq \|v_2\|_{\dot{B}^{-1}_{\infty,1}} \leq C_\# Q^2, \quad \textit{if} \quad r > C_\#, \tag{4.9}$$

$$\|v_k\|_{L^\infty} \leq Q^2 \left(\frac{Q}{r}\right)^{k-2}, \ \|v_k\|_{\dot{B}^{-1}_{\infty,1}} \leq C_\# Q^2 \left(\frac{Q}{r}\right)^{k-2} \quad \textit{for all } k \geq 3,\ \textit{if } r > Q. \tag{4.10}$$

For the moment, let us assume this lemma. Once we fix Q and r with $Q/r < 1$, then there hold

$$\sum_{k=1}^{\infty} \|v_k\|_{L^\infty} < \infty, \quad \sum_{k=1}^{\infty} \|v_k\|_{\dot{B}^{-1}_{\infty,1}} < \infty.$$

Hence there exist $u^*_{Q,r} \in BUC$ and $\tilde{u}^*_{Q,r} \in \dot{B}^{-1}_{\infty,1}$ such that

$$\begin{aligned} u^*_{Q,r} &= \lim_{j\to\infty} u_j \ \text{ in } L^\infty, \qquad \|u^*_{Q,r}\|_{L^\infty} \le \sum_{k=1}^{\infty} \|v_k\|_{L^\infty}, \\ \tilde{u}^*_{Q,r} &= \lim_{j\to\infty} u_j \ \text{ in } \dot{B}^{-1}_{\infty,1}, \qquad \|\tilde{u}^*_{Q,r}\|_{\dot{B}^{-1}_{\infty,1}} \le \sum_{k=1}^{\infty} \|v_k\|_{\dot{B}^{-1}_{\infty,1}}. \end{aligned} \tag{4.11}$$

Actually, $u^*_{Q,r} = \tilde{u}^*_{Q,r}$ in the sense of $\mathcal{S}_0'$. Indeed, since $u^*_{Q,r}$ is a tempered distribution, it holds that

$$\langle u_j, \varphi\rangle \to \left\langle u^*_{Q,r}, \varphi\right\rangle \quad \text{as } j\to\infty \ \text{ for all } \varphi \in \mathcal{S}_0.$$

On the other hand, since $u_j \to \tilde{u}^*_{Q,r}$ as $j \to \infty$ in $\dot{B}^{-1}_{\infty,2}$ in particular, there holds

$$\langle u_j, \varphi\rangle \to \left\langle \tilde{u}^*_{Q,r}, \varphi\right\rangle \quad \text{as } j\to\infty \ \text{ for all } \varphi \in \dot{B}^1_{1,2},$$

because the dual space of $\dot{B}^1_{1,2}$ is $\dot{B}^{-1}_{\infty,2}$ (see Triebel [21, Theorem 2.11.2 and Section 5.2.5], for example). From this convergence and the embedding $\mathcal{S}_0 \subset \dot{B}^1_{1,2}$, we have $u^*_{Q,r} = \tilde{u}^*_{Q,r}$ in $\mathcal{S}_0'$. Therefore, $u^*_{Q,r}$ also belongs to $\dot{B}^{-1}_{\infty,1}$ with the estimate

$$\|u^*_{Q,r}\|_{\dot{B}^{-1}_{\infty,1}} = \|\tilde{u}^*_{Q,r}\|_{\dot{B}^{-1}_{\infty,1}} \le \sum_{k=1}^{\infty} \|v_k\|_{\dot{B}^{-1}_{\infty,1}}. \tag{4.12}$$

Considering the convergence of the bilinear form $B(u_j, u_j)$, we have the following lemma, which will be shown later.

Lemma 4.5. *Let $Q < r$ and let $u^*_{Q,r}$ be a function defined by* (4.11)*. Then we have that $u^*_{Q,r} \in BUC^2 \cap \dot{B}^{-1}_{\infty,1}$,* div $u^*_{Q,r} = 0$*, and it holds that*

$$\lim_{j\to\infty} B(u_j, u_j) = B(u^*_{Q,r}, u^*_{Q,r}) \ \text{ in } L^\infty \text{ and } \dot{B}^{-1}_{\infty,1}. \tag{4.13}$$

Moreover, $u = u^*_{Q,r}$ satisfies (SNS) *for all $x \in \mathbb{R}^n$ in the pointwise sense, with $f = f_{Q,r}$ and $\nabla\Pi = 0$.*

Hence, $u^*_{Q,r}$ is a solution not only of (rSNS) in L^∞ or $\dot{B}^{-1}_{\infty,1}$ sense, but also (SNS) with respect to $f = f_{Q,r}$ in pointwise sense.

Proof of Theorem 4.1. We first take $Q_0 > 0$ so that

$$\frac{\delta}{4C_\#} < Q_0^2 < \frac{\delta}{2C_\#},$$

and let $r_0 \in \mathbb{N}$ be such that

$$\frac{Q_0}{r_0} < \min\left\{\frac{\delta}{256C_\#^3}, \frac{1}{2}\right\}.$$

Then by Lemma 4.4, it holds for every $r \geq r_0$ that

$$\begin{aligned}\|u^*_{Q_0,r}\|_{\dot{B}^{-1}_{\infty,1}} &\leq \sum_{k=1}^{\infty} \|v_k\|_{\dot{B}^{-1}_{\infty,1}} \\ &\leq 2C_\# \frac{Q_0}{r} + \frac{C_\# Q_0^2}{1-\frac{Q_0}{r}} < \delta.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|u^*_{Q_0,r}\|_{\dot{B}^{-1}_{\infty,\infty}} &\geq \|v_2\|_{\dot{B}^{-1}_{\infty,\infty}} - \|v_1\|_{\dot{B}^{-1}_{\infty,1}} - \sum_{k=3}^{\infty} \|v_k\|_{\dot{B}^{-1}_{\infty,1}} \\ &\geq \frac{Q_0^2}{16C_\#} - 2C_\# \frac{Q_0}{r} - C_\# Q_0^2 \frac{\frac{Q_0}{r}}{1-\frac{Q_0}{r}} \\ &> \frac{\delta}{64C_\#^2} - \frac{\delta}{128C_\#^2} - \frac{\delta}{256C_\#^2} \\ &= \frac{\delta}{256C_\#^2}\end{aligned}$$

for every $r \geq r_0$. Moreover, we can easily see from (4.8) that $\|f_{Q_0,r}\|_{\dot{B}^{-3}_{\infty,1}} \to 0$ as $r \to \infty$. Hence, by the above argument from Lemma 4.4 to Lemma 4.5, we obtain the claim of Theorem 4.1 by taking a sequence $\{f_N\}_{N\in\mathbb{N}}$ of external forces as $f_N \equiv f_{Q_0,r_0+N}$ and letting $\kappa \equiv 1/256C_0^2$. □

Now let us prove Lemma 4.4 and Lemma 4.5.

Proof of Lemma 4.4. Using (4.2), we have

$$\|f_{Q,r}\|_{\dot{B}^{-3}_{\infty,1}} \leq Qr^2 \cdot 2\left(\frac{C_\#}{r}\right)^3 = 2C_\#^3 \frac{Q}{r},$$

Since div $f_{Q,r} = 0$, we obtain v_1 as

$$\begin{aligned}v_1(x) &= (-\Delta)^{-1} P f_{Q,r}(x) \\ &= (-\Delta)^{-1} f_{Q,r}(x) \\ &= Q\left\{e_2 \cos(rx_1) + e_3 \frac{r^2}{r^2+1} \cos(rx_1 - x_2)\right\},\end{aligned}$$

which also satisfies div $v_1 = 0$ and

$$\|v_1\|_{\dot{B}^{-1}_{\infty,1}} \leq Q \cdot 2\frac{C_\#}{r} = 2C_\# \frac{Q}{r}.$$

Next we deal with v_2. Let us write the i-th component of v as $v^{(i)}$, and we have

$$\begin{aligned}(v_1 \cdot \nabla v_1)(x) &= \sum_{m=1}^{n} v_1^{(m)} \partial_m v_1 \\ &= v_1^{(2)} \partial_2 v_1 \\ &= Q\cos(rx_1) \cdot e_3 Q \frac{r^2}{r^2+1} \sin(rx_1 - x_2) \\ &= e_3 \frac{1}{2} Q^2 \frac{r^2}{r^2+1} \left\{-\sin x_2 + \sin(2rx_1 - x_2)\right\}.\end{aligned}$$

Hence, it holds that

$$\begin{aligned}v_2(x) &= -(-\Delta)^{-1}(v_1 \cdot \nabla v_1)(x) \\ &= e_3 \frac{1}{2} Q^2 \frac{r^2}{r^2+1} \sin x_2 - e_3 \frac{1}{2} Q^2 \frac{r^2}{r^2+1} \frac{1}{4r^2+1} \sin(2rx_1 - x_2) \\ &\equiv N_1 + N_2.\end{aligned}$$

By virtue of (4.2) and (4.3), we have

$$\|N_1\|_{\dot{B}^{-1}_{\infty,1}} \leq \frac{1}{2} C_\# Q^2$$

and

$$\|N_1\|_{\dot{B}^{-1}_{\infty,\infty}} \geq \frac{1}{2} Q^2 \cdot \frac{1}{2} \cdot \frac{1}{2C_\#} = \frac{Q^2}{8C_\#},$$

while

$$\|N_2\|_{\dot{B}^{-1}_{\infty,1}} \leq C_\# \frac{Q^2}{16r^3}.$$

Therefore, if $r > C_\#$, there holds

$$\|v_2\|_{\dot{B}^{-1}_{\infty,1}} \leq C_\# Q^2$$

and

$$\|v_2\|_{\dot{B}^{-1}_{\infty,\infty}} \geq \|N_1\|_{\dot{B}^{-1}_{\infty,\infty}} - \|N_2\|_{\dot{B}^{-1}_{\infty,1}} > \frac{Q^2}{16C_\#}.$$

For the estimate of v_k with $k \geq 3$, we need the following proposition.

Proposition 4.6. *For $k \geq 3$, v_k has an explicit representation as*

$$v_k = B(v_1, v_{k-1}) = -(-\Delta)^{-1}(v_1^{(2)}\partial_2 v_{k-1})(x) = -e_3 v_k^{(3)}(x_1, x_2). \tag{4.14}$$

Moreover, there holds div $v_k = 0$ *for all* $k \geq 1$.

Proof of Proposition 4.6. First, notice that (4.14) is valid for $k = 2$. Suppose that this is true in the cases $2 \leq k \leq l$ for some $l \geq 2$. Then we have

$$\begin{aligned} v_{l+1} &= B(u_l, u_l) - B(u_{l-1}, u_{l-1}) \\ &= B\left(\sum_{k=1}^{l} v_k, \sum_{k=1}^{l} v_k\right) - \left(\sum_{k=1}^{l-1} v_k, \sum_{k=1}^{l-1} v_k\right) \\ &= B(v_1, v_l) + B\left(\sum_{k=2}^{l-1} v_k, v_l\right) + B\left(v_l, \sum_{k=1}^{l} v_k\right). \end{aligned}$$

Since $v_k = e_3 v_k^{(3)}(x_1, x_2)$ for all $2 \leq k \leq l$ and since v_k does not depend on x_3 for all $1 \leq k \leq l$, we see that

$$\left(\sum_{k=2}^{l-1} v_k\right) \cdot \nabla v_l = 0, \qquad v_l \cdot \nabla \left(\sum_{k=1}^{l} v_k\right) = 0.$$

Moreover, it holds that

$$\begin{aligned} (v_1 \cdot \nabla v_l)(x) &= v_1^{(2)}(x_1) \cdot \partial_2(e_3 v_l^{(3)}(x_1, x_2)) \\ &= e_3 \left\{ v_1^{(2)}(x_1) \cdot \partial_2 v_l^{(3)}(x_1, x_2) \right\}. \end{aligned}$$

Hence, we obtain (4.14) for $k = l + 1$. By induction, we see that (4.14) holds for all $k \geq 3$. This proves Proposition 4.6.

Let us return to the proof of Lemma 4.4. According to Proposition 4.6, we may identify v_k with $v_k^{(3)}$ for $k \geq 2$. Moreover, we rewrite v_1 and v_2 as follows:

$$v_1(x) = Q\left[e_2 \cos(rx_1) + e_3 M_1(r^0) \cos(rx_1 - x_2)\right], \tag{4.15}$$

$$v_2(x) = \frac{1}{2} Q^2 \left[M_1(r^0) \sin x_2 - M_2(r^{-2}) \sin(2rx_1 - x_2)\right]. \tag{4.16}$$

Here and in what follows, for $j \in \mathbb{N}$, we denote by $M_j(r^\alpha)$ the positive functions of r which may change from line to line, and satisfy the estimate

$$M_j(r^\alpha) \leq r^\alpha \quad \text{for all } r > 1.$$

Let us handle v_3 and v_4. Since

$$\begin{aligned} (v_1 \cdot \nabla v_2)(x) &= \frac{1}{2} Q^3 \cos(rx_1) \left[M_1(r^0) \cos x_2 + M_2(r^{-2}) \cos(2rx_1 - x_2)\right] \\ &= \frac{1}{4} Q^3 \Big[M_1(r^0) \left\{\cos(rx_1 + x_2) + \cos(rx_1 - x_2)\right\} \\ &\qquad + M_2(r^{-2}) \{\cos(3rx_1 - x_2) + \cos(-rx_1 + x_2)\}\Big], \end{aligned}$$

we see that v_3 is expressed as

$$\begin{aligned} v_3(x) &= -(-\Delta)^{-1}(v_1 \cdot \nabla v_2)(x) \\ &= -\frac{1}{4}Q^3\Big[M_1(r^{-2})\{\cos(rx_1+x_2)+\cos(rx_1-x_2)\} \\ &\qquad +M_2(r^{-4})\cos(3rx_1-x_2)+M_3(r^{-4})\cos(-rx_1+x_2)\Big]. \end{aligned}$$

Moreover, since

$$\begin{aligned} (v_1 \cdot \nabla v_3)(x) &= \frac{1}{4}Q^4\cos(rx_1) \\ &\quad \cdot\Big[M_1(r^{-2})\{\sin(rx_1+x_2)-\sin(rx_1-x_2)\} \\ &\qquad -M_2(r^{-4})\sin(3rx_1-x_2)+M_3(r^{-4})\sin(-rx_1+x_2)\Big] \\ &= \frac{1}{8}Q^4\Big[M_1(r^{-2})\{2\sin x_2+\sin(2rx_1+x_2)-\sin(2rx_1-x_2)\} \\ &\qquad -M_2(r^{-4})\{\sin(4rx_1-x_2)+\sin(2rx_1-x_2)\} \\ &\qquad +M_3(r^{-4})\{\sin x_2+\sin(2rx_1-x_2)\}\Big], \end{aligned}$$

we see that v_3 has an expression as

$$\begin{aligned} v_4(x) &= -(-\Delta)^{-1}(v_1 \cdot \nabla v_3)(x) \\ &= \frac{1}{8}Q^4\Big[-2M_1(r^{-2})\sin x_2 \\ &\qquad +M_2(r^{-4})\{-\sin(2rx_1+x_2)+\sin(2rx_1-x_2)\} \\ &\qquad +M_3(r^{-6})\{\sin(4rx_1-x_2)+\sin(2rx_1-x_2)\} \\ &\qquad -M_4(r^{-4})\sin x_2-M_5(r^{-6})\sin(2rx_1-x_2)\Big], \end{aligned}$$

Repeating such a procedure, we see that

$$\begin{aligned} v_{2l+1}(x) = \frac{Q^{2l+1}}{2^{2l}}\Big[&-M_1(r^{-2l})\cdot(-2)^{l-1}\{\cos(rx_1+x_2)+\cos(rx_1-x_2)\} \\ &+\sum_{i=1}^{2^{2l}-2^l} M_{i+1}(r^{-2})\cos(a_{l,i}\cdot x)\Big], \end{aligned} \tag{4.17}$$

$$v_{2l+2}(x) = \frac{Q^{2l+2}}{2^{2l+1}}\Big[M_1(r^{-2l})\cdot(-2)^l\sin x_2+\sum_{i=1}^{2^{2l+1}-2^l} M_{i+1}(r^{-2})\sin(b_{l,i}\cdot x)\Big] \tag{4.18}$$

for $l \geq 1$, where $a_{l,i} = (a_{l,i}^{(1)}, a_{l,i}^{(2)}, \ldots, a_{l,i}^{(n)})$, $b_{l,i} = (b_{l,i}^{(1)}, b_{l,i}^{(2)}, \ldots, b_{l,i}^{(n)}) \in \mathbb{R}^n$ are vectors depending only on r with

$$|a_{l,i}^{(2)}|,\ |b_{l,i}^{(2)}| = 1, \quad a_{l,i}^{(j)} = b_{l,i}^{(j)} = 0 \ \text{ for } \ 3 \leq j \leq n.$$

Hence we obtain the following estimates.

$$\|v_{2l+1}\|_{L^\infty} \le \frac{Q^{2l+1}}{2^{2l}}\frac{1}{r^{2l}}\cdot 2^{2l} = Q\left(\frac{Q}{r}\right)^{2l}, \qquad \|v_{2l+1}\|_{\dot{B}^{-1}_{\infty,1}} \le C_\# Q\left(\frac{Q}{r}\right)^{2l},$$

$$\|v_{2l+2}\|_{L^\infty} \le \frac{Q^{2l+2}}{2^{2l+1}}\frac{1}{r^{2l}}\cdot 2^{2l+1} = Q^2\left(\frac{Q}{r}\right)^{2l}, \qquad \|v_{2l+2}\|_{\dot{B}^{-1}_{\infty,1}} \le C_\# Q^2\left(\frac{Q}{r}\right)^{2l}.$$

This completes the proof of Lemma 4.4.

Proof of Lemma 4.5. Let us first show smoothness of $u^*_{Q,r}$. It is easily seen that each v_k, $k \ge 1$, is twice continuously differentiable, and that each of their partial derivatives is uniformly bounded. In particular, since each second component of $a_{k,i}$ in (4.17) and (4.18) is 1 or -1, we can obtain the same estimates of $\|\partial_2 v_k\|_{L^\infty}$ and $\|\partial_2^2 v_k\|_{L^\infty}$ as those of $\|v_k\|_{L^\infty}$ in (4.10) in the same as in the proof of Lemma 4.4. Moreover, since $\|v_1^{(2)}\|_{L^\infty} = Q$, we have by (4.5) and (4.14) that

$$\|\partial_1 v_k\|_{L^\infty},\ \|\partial_1^2 v_k\|_{L^\infty},\ \|\partial_1\partial_2 v_k\|_{L^\infty} \le Q^3\left(\frac{Q}{r}\right)^{k-3}, \qquad k \ge 3.$$

Therefore, by the theorem of termwise differentiation, we see that $u^*_{Q,r} = \sum_{k=1}^\infty v_k$ belongs to C^2 and is termwise differentiable provided $Q < r$.

Since the series $\sum_{k=1}^\infty v_k$ is termwise differentiable, and since the identity

$$\sum_{k=1}^\infty v_1^{(2)}\partial_2 v_k = -\Delta\left(\sum_{k=1}^\infty (-\Delta)^{-1} v_1^{(2)}\partial_2 v_k\right) \tag{4.19}$$

holds, we have by (4.14) that

$$\begin{aligned}
\|B(u^*_{Q,r}, u^*)_{Q,r} - B(u_j,u_j)\|_{L^\infty} &= \left\|-(-\Delta)^{-1} v_1^{(2)}\partial_2\left(\sum_{k=j+1}^\infty v_k\right)\right\|_{L^\infty} \\
&= \left\|\sum_{k=j+1}^\infty \left\{-(-\Delta)^{-1}\left(v_1^{(2)}\partial_2 v_k\right)\right\}\right\|_{L^\infty} \\
&\le \sum_{k=j+1}^\infty \|v_{k+1}\|_{L^\infty} \to 0, \quad \text{as } j\to\infty.
\end{aligned}$$

In the same way, it is also easily shown that $B(u_j,u_j) \to B(u^*_{Q,r}, u^*_{Q,r})$ in $\dot{B}^{-1}_{\infty,1}$ as $j \to \infty$.

Finally, let us show that $u = u^*_{Q,r} = \sum_{k=1}^\infty v_k$ actually satisfies (SNS) with $f = f_{Q,r}$ and $\nabla\Pi = 0$ for all $x \in \mathbb{R}^n$. Indeed, by termwise differentiation, we see from (4.14) and

(4.19) that div $v^*_{Q,r}(x) = 0$ and

$$\begin{aligned} -\Delta u^*_{Q,r}(x) &= -\Delta v_1(x) - \Delta \left(\sum_{k=2}^{\infty} -(-\Delta)^{-1} v_1(x) \partial_2 v_{k-1}(x) \right) \\ &= f_{Q,r}(x) - \sum_{k=1}^{\infty} v_1(x) \partial_2 v_k(x), \\ (u^*_{Q,r} \cdot \nabla u^*_{Q,r})(x) &= v_1(x) \cdot \partial_2 \sum_{k=1}^{\infty} v_k(x) = \sum_{k=1}^{\infty} v_1(x) \partial_2 v_k(x). \end{aligned}$$

Hence we obtain

$$-\Delta u^*_{Q,r}(x) + (u^* \cdot \nabla u^*_{Q,r})(x) = f_{Q,r}(x), \qquad x \in \mathbb{R}^n.$$

This completes the proof of Lemma 4.5, and we have proved Theorem 4.1. □

Until now, we have seen that from $D = \dot{B}^{-3+\frac{n}{p}}_{p,q}$ to $S = P\dot{B}^{-1+\frac{n}{p}}_{p,q}$, (rSNS) is well-posed when $1 \le p < n$ and $1 \le q \le \infty$, and ill-posed when $p = \infty$ and $1 \le q \le \infty$. Therefore, the rest of case is when $n \le p < \infty$. Actually, in the case $n \le p < \infty$, we cannot apply the method above. Indeed, for every trigonometric function $g(x) \equiv \cos(a \cdot x)$, $\varphi_j * g$ is not integrable in the whole space $\mathbb{R}^n$, which means that g is not included in $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$ for $1 \le p < \infty$. Hence, in the next chapter, we reconsider (rSNS) in the torus $\mathbb{T}^n$, taking into account the fact that a function g is spacial periodic.

Chapter 5

The study in toroidal Besov spaces

In this chapter, we consider the well-posedness and ill-posedness problems of (rSNS) on the n-dimensional torus $\mathbb{T}^n \equiv [-\pi,\pi]^n$. Before stating our main results, we should define some function spaces on tori.

5.1 Definition and properties of toroidal Besov spaces

We denote by $\mathbb{T}^n_\lambda \equiv [-\lambda\pi,\lambda\pi]^n$ the n-dimensional $2\pi\lambda$-periodic torus for $\lambda > 0$, and let $\mathbb{T}^n \equiv \mathbb{T}^n_1 = [-\pi,\pi]^n$ in particular. We define the spaces $\mathcal{D}(\mathbb{T}^n_\lambda)$, $\mathcal{D}_0(\mathbb{T}^n_\lambda)$ and $\mathcal{S}_\lambda(\mathbb{Z}^n)$ by

$$\mathcal{D}(\mathbb{T}^n_\lambda) \equiv \left\{ f \in C^\infty(\mathbb{R}^n); f \text{ is } 2\pi\lambda\text{−periodic on each component } x_1,\dots,x_n \right\},$$

$$\mathcal{D}_0(\mathbb{T}^n_\lambda) \equiv \left\{ f \in \mathcal{D}(\mathbb{T}^n_\lambda); \int_{[-\pi\lambda,\pi\lambda]^n} f(x)dx = 0 \right\},$$

$$\mathcal{S}_\lambda(\mathbb{Z}^n) \equiv \left\{ g : \mathbb{Z}^n \to \mathbb{R}^n; \forall s \geq 0, \exists c = c(g,s) > 0 \ s.t. \ \sup_{m\in\mathbb{Z}^n} (1+|\lambda^{-1}m|^2)^{\frac{s}{2}} |g(m)| < c \right\},$$

and let $\mathcal{D}'(\mathbb{T}^n_\lambda)$, $\mathcal{D}'_0(\mathbb{T}^n_\lambda)$ and $\mathcal{S}'_\lambda(\mathbb{Z}^n)$ be dual spaces of $\mathcal{D}(\mathbb{T}^n_\lambda)$ $\mathcal{D}_0(\mathbb{T}^n_\lambda)$ and $\mathcal{S}_\lambda(\mathbb{Z}^n)$, respectively. We define the toroidal Fourier transform (the Fourier series) $\mathcal{F}_{\mathbb{T}^n_\lambda} : \mathcal{D}(\mathbb{T}^n_\lambda) \to \mathcal{S}_\lambda(\mathbb{Z}^n)$ by

$$\mathcal{F}_{\mathbb{T}^n_\lambda} f(m) \equiv \frac{1}{(2\pi\lambda)^n} \int_{[-\pi\lambda,\pi\lambda]^n} f(x) e^{-i\lambda^{-1}m\cdot x} dx, \quad f \in \mathcal{D}(\mathbb{T}^n_\lambda), \quad m \in \mathbb{Z}^n,$$

and the inversion $\mathcal{F}^{-1}_{\mathbb{T}^n_\lambda} : \mathcal{S}_\lambda(\mathbb{Z}^n) \to \mathcal{D}(\mathbb{T}^n_\lambda)$ by

$$\mathcal{F}^{-1}_{\mathbb{T}^n_\lambda} g(x) \equiv \sum_{m\in\mathbb{Z}^n} g(m) e^{i\lambda^{-1}m\cdot x}, \quad g \in \mathcal{S}_\lambda(\mathbb{Z}^n), \quad x \in \mathbb{T}^n_\lambda.$$

We can also define these transforms in dual spaces, $\mathcal{F}_{\mathbb{T}^n_\lambda} : \mathcal{D}'_0(\mathbb{T}^n_\lambda) \to \mathcal{S}'_\lambda(\mathbb{Z}^n)$ and $\mathcal{F}^{-1}_{\mathbb{T}^n_\lambda} : \mathcal{S}'_\lambda(\mathbb{Z}^n) \to \mathcal{D}'(\mathbb{T}^n_\lambda)$, by

$$\left\langle \mathcal{F}_{\mathbb{T}^n_\lambda} f, \varphi \right\rangle \equiv \left\langle f, \mathcal{F}^{-1}_{\mathbb{T}^n_\lambda} \varphi(-\cdot) \right\rangle, \quad f \in \mathcal{D}'_0(\mathbb{T}^n_\lambda), \ \varphi \in \mathcal{S}_\lambda(\mathbb{Z}^n),$$

$$\left\langle \mathcal{F}_{\mathbb{T}_\lambda^n}^{-1} g, \psi \right\rangle \equiv \left\langle g, \mathcal{F}_{\mathbb{T}_\lambda^n} \psi(-\cdot) \right\rangle, \quad g \in \mathcal{S}'_\lambda(\mathbb{Z}^n), \quad \psi \in \mathcal{D}(\mathbb{T}_\lambda^n).$$

In addition, we define the convolution of $f, g \in \mathcal{D}(\mathbb{T}_\lambda^n)$ by

$$(f * g)(x) \equiv \int_{[-\pi\lambda, \pi\lambda]^n} f(x-y) g(y) dy, \quad x \in \mathbb{T}_\lambda^n.$$

We also define the convolution of $(h, f) \in \mathcal{D}(\mathbb{T}_\lambda^n) \times \mathcal{D}'(\mathbb{T}_\lambda^n)$ by

$$\langle h * f, \varphi \rangle \equiv \langle f, h(-\cdot) * \varphi \rangle = \int_{[-\pi\lambda, \pi\lambda]^n} \langle f, h(y - \cdot) \rangle \varphi(y) dy, \quad \varphi \in \mathcal{D}(\mathbb{T}_\lambda^n),$$

and it is seen that $h * f$ is actually in $\mathcal{D}(\mathbb{T}_\lambda^n)$. We can also define that of $(h, f) \in \mathcal{D}_0(\mathbb{T}_\lambda^n) \times \mathcal{D}'_0(\mathbb{T}_\lambda^n)$ by a similar way.

Now let us define some important operators and spaces related to $\mathcal{D}_0(\mathbb{T}_\lambda^n)$ and $\mathcal{D}'_0(\mathbb{T}_\lambda^n)$. Since $\mathcal{F}_{\mathbb{T}_\lambda^n} f(0) = 0$ for every $f \in \mathcal{D}_0(\mathbb{T}_\lambda^n)$, we can define the toroidal Riesz potential I_λ^s with $s \in \mathbb{R}$ and Riesz transform R_k with $k = 1, \ldots, n$ on $\mathcal{D}'_0(\mathbb{T}_\lambda^n)$ by

$$I_\lambda^s f \equiv \mathcal{F}_{\mathbb{T}_\lambda^n}^{-1} \left[|\lambda^{-1} m|^s \mathcal{F}_{\mathbb{T}_\lambda^n} f(m) \right],$$

$$R_k f \equiv \mathcal{F}_{\mathbb{T}_\lambda^n}^{-1} \left[i m_k |m|^{-1} \mathcal{F}_{\mathbb{T}_\lambda^n} f(m) \right].$$

Secondly, we define the homogeneous toroidal Besov spaces. We take a non-negative smooth function $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$0 \le \phi \le 1, \ \ \mathrm{supp}\, \phi \subset \left\{ x \in \mathbb{R}^n; \frac{1}{2} < |\xi| < 2 \right\}, \quad \sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi) = 1 \ \ \forall \xi \in \mathbb{R}^n \backslash \{0\}. \tag{5.1}$$

Then we define

$$\phi_j(\xi) \equiv \phi(2^{-j}\xi), \quad \varphi_{\lambda,j} \equiv \mathcal{F}_{\mathbb{T}_\lambda^n}^{-1} \left[\phi_j(\lambda^{-1} \cdot) \right], \quad j \in \mathbb{Z}. \tag{5.2}$$

We can easily see that each $\varphi_{\lambda,j}$ belongs to $\mathcal{D}_0(\mathbb{T}_\lambda^n)$. Moreover, since $\mathcal{F}_{\mathbb{T}_\lambda^n} f(0) = 0$ for every $f \in \mathcal{D}_0(\mathbb{T}_\lambda^n)$, it is seen from (5.1) and (5.2) that

$$\sum_{j=-\infty}^{\infty} \varphi_{\lambda,j} * f = f, \quad \forall f \in \mathcal{D}'_0(\mathbb{T}_\lambda^n).$$

According to the above family $\{\varphi_{\lambda,j}\}_{j \in \mathbb{Z}}$, we define the homogeneous toroidal Besov space $\dot{B}_{p,q}^s(\mathbb{T}_\lambda^n)$ for $s \in \mathbb{R}$, $1 \le p, q \le \infty$ by

$$\dot{B}_{p,q}^s(\mathbb{T}_\lambda^n) \equiv \left\{ f \in \mathcal{D}'_0(\mathbb{T}_\lambda^n); \ \|f\|_{\dot{B}_{p,q}^s(\mathbb{T}_\lambda^n)} < \infty \right\}$$

with the norm

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{T}_\lambda^n)} \equiv \begin{cases} \left\{ \sum\limits_{j=-\infty}^{\infty} (2^{sj} \|\varphi_{\lambda,j} * f\|_{L^p(\mathbb{T}_\lambda^n)})^q \right\}^{\frac{1}{q}}, & 1 \le q < \infty, \\ \sup\limits_{j \in \mathbb{Z}} 2^{sj} \|\varphi_{\lambda,j} * f\|_{L^p(\mathbb{T}_\lambda^n)}, & q = \infty, \end{cases}$$

where $L^p(\mathbb{T}^n_\lambda)$ is the space of $2\pi\lambda$-periodic measurable functions with the norm

$$\|f\|_{L^p(\mathbb{T}^n_\lambda)} \equiv \begin{cases} \left(\int_{[-\pi\lambda,\pi\lambda]^n} |f(x)|^p dx\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname{esssup}_{x\in\mathbb{T}^n_\lambda} |f(x)|, & p = \infty. \end{cases}$$

Remark 5.1. For classical concepts and definitions of (non-homogeneous) toroidal Besov spaces $B^s_{p,q}(\mathbb{T}^n)$, we refer to Schmeisser-Triebel [19] and Xiong-Xu-Yin [29] for example. By following similar methods to [19], [29], and classical theories of homogeneous Besov spaces on $\mathbb{R}^n$, we can see that the above homogeneous space are also complete (for the completeness, we should define the space in $\mathcal{D}_0'(\mathbb{T}^n_\lambda)$), and that a definition of homogeneous Besov spaces is independent of the choice of ϕ.

Now we prepare some important properties of toroidal Besov spaces to prove our results. Here we only consider the case $\lambda = 1$ for simplicity. In what follows, $\sigma[T]$ denotes formally the multiplier on $\mathcal{D}_0'(\mathbb{T}^n)$ with a symbol $T : \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{Z}^n \to \mathbb{R}$) defined by

$$\sigma[T]f \equiv \mathcal{F}^{-1}_{\mathbb{T}^n}\left[T\mathcal{F}_{\mathbb{T}^n} f\right] = \sum_{m\in\mathbb{Z}^n} T(m)\mathcal{F}_{\mathbb{T}^n} f(m) e^{im\cdot x}.$$

Let $\varphi_j \equiv \varphi_{1.j}$ implied by (5.2). Since supp $\phi_j \subset \{2^{j-1} < |\xi| < 2^{j+1}\}$ by (5.1), we see that

$$\begin{aligned} \varphi_j * f &= \sigma\left[(\phi_{j-1} + \phi_j + \phi_{j+1})\,\phi_j\right] f \\ &= \tilde{\varphi}_j * \varphi_j * f, \end{aligned} \tag{5.3}$$

where $\tilde{\varphi}_j \equiv \varphi_{j-1} + \varphi_j + \varphi_{j+1}$. Moreover, for $T \in \mathcal{F}_{\mathbb{R}^n} L^1(\mathbb{R}^n)$ ($\mathcal{F}_{\mathbb{R}^n}$ denotes the Fourier transform in $\mathbb{R}^n$), there holds for any $f \in \mathcal{D}(\mathbb{T}^n)$ that

$$\begin{aligned} \sigma[T]f(x) &= \sum_{m\in\mathbb{Z}^n} T(m)\mathcal{F}_{\mathbb{T}^n} f(m) e^{im\cdot x} \\ &= \sum_{m\in\mathbb{Z}^n} \left(\int_{\mathbb{R}^n} \mathcal{F}^{-1}_{\mathbb{R}^n} T(y) e^{-im\cdot y} dy\right) \mathcal{F}_{\mathbb{T}^n} f(m) e^{im\cdot x} \\ &= \int_{\mathbb{R}^n} \mathcal{F}^{-1}_{\mathbb{R}^n} T(y) \left(\sum_{m\in\mathbb{Z}^n} \mathcal{F}_{\mathbb{T}^n} f(m) e^{im\cdot(x-y)}\right) dy \\ &= \int_{\mathbb{R}^n} \mathcal{F}^{-1}_{\mathbb{R}^n} T(y) f(x-y) dy, \quad x \in \mathbb{T}^n. \end{aligned}$$

Hence, it is seen from the Young inequality (and its proof) that

$$\|\sigma[T]f\|_{L^p(\mathbb{T}^n)} \le \|\mathcal{F}^{-1}_{\mathbb{R}^n} T\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{T}^n)}, \quad f \in \mathcal{D}(\mathbb{T}^n), \quad 1 \le p \le \infty. \tag{5.4}$$

On the other hand, since $\varphi_j * f$ is a trigonometric polynomial of degree 2^{j+1}, it is seen from the Nikolski's inequality that

$$\|\varphi_j * f\|_{L^p(\mathbb{T}^n)} \leq C 2^{jn\left(\frac{1}{q}-\frac{1}{p}\right)} \|\varphi_j * f\|_{L^q(\mathbb{T}^n)}, \quad 1 \leq q \leq p \leq \infty, \tag{5.5}$$

where C denotes a constant dependent only on p, q, and n. The above (5.3), (5.4), and (5.5) yield the followings which are alternative to Proposition 2.1 and Proposition 3.7:

Proposition 5.2. (1) *Let $s \in \mathbb{R}$, and let $1 \leq p \leq \infty$, $1 \leq q_1 \leq q_2 \leq \infty$. Then there holds*

$$\dot{B}^s_{p,q_1}(\mathbb{T}^n) \hookrightarrow \dot{B}^s_{p,q_2}(\mathbb{T}^n). \tag{5.6}$$

(2) *Let $s_1 > s_2$, and let $1 \leq p_1 < p_2 < \infty$, $1 \leq q, r \leq \infty$. If $s_1 - n/p_1 = s_2 - n/p_2$, then there holds*

$$\dot{B}^{s_1}_{p_1,q}(\mathbb{T}^n) \hookrightarrow \dot{B}^{s_2}_{p_2,q}(\mathbb{T}^n). \tag{5.7}$$

(3) *Let $s, s_0 \in \mathbb{R}$, and let $1 \leq p, q \leq \infty$. Then the Riesz potential $(-\Delta)^{\frac{s}{2}}$ is isomorphic from $\dot{B}^{s_0}_{p,q}(\mathbb{T}^n)$ onto $\dot{B}^{s_0-s}_{p,q}(\mathbb{T}^n)$.*

(4) *Let $s \in \mathbb{R}$ and let $1 \leq p, q \leq \infty$. Then for each $j = 1, 2, \ldots, n$, the Riesz transform $R_j \equiv \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$ is bounded from $\dot{B}^s_{p,q}(\mathbb{T}^n)$ to itself.*

Proposition 5.3. (Para-product estimate) *Let $n \geq 2$, $1 \leq p, q \leq \infty$, $s > 0$, $\alpha > 0$ and $\beta > 0$. Suppose that $1 \leq p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leq \infty$ satisfy $1/p = 1/p_1 + 1/p_2 = 1/\tilde{p}_1 + 1/\tilde{p}_2$. Then for every $f \in \dot{B}^{s+\alpha}_{\tilde{p}_1,q}(\mathbb{T}^n) \cap \dot{B}^{-\beta}_{\tilde{p}_1,\infty}(\mathbb{T}^n)$ and $g \in \dot{B}^{-\alpha}_{p_2,\infty}(\mathbb{T}^n) \cap \dot{B}^{s+\beta}_{\tilde{p}_2,q}(\mathbb{T}^n)$, it holds that $f \cdot g \in \dot{B}^s_{p,q}(\mathbb{T}^n)$ with the estimate*

$$\|f \cdot g\|_{\dot{B}^s_{p,q}(\mathbb{T}^n)} \leq C \left(\|f\|_{\dot{B}^{s+\alpha}_{\tilde{p}_1,q}(\mathbb{T}^n)} \|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}(\mathbb{T}^n)} + \|f\|_{\dot{B}^{-\beta}_{\tilde{p}_1,\infty}(\mathbb{T}^n)} \|g\|_{\dot{B}^{s+\beta}_{\tilde{p}_2,q}(\mathbb{T}^n)} \right), \tag{5.8}$$

where $C = C(n, p, q, s, \tilde{p}_1, \tilde{p}_2)$ is a constant.

Proof of Proposition 5.2. (1) is easily seen from the well-known embedding of sequence spaces $l^{q_1} \hookrightarrow l^{q_2}$ for $1 \leq q_1 \leq q_2 \leq \infty$. On the other hand, it is seen from (5.3) that

$$\begin{aligned} \varphi_j * (-\Delta)^{\frac{s}{2}} f &= \sigma\left[\tilde{\phi}_j(m)\phi_j(m)|m|^s\right] f \\ &= \sigma\left[|m|^s \tilde{\phi}_j(m)\right](\varphi_j * f), \end{aligned}$$

where $\tilde{\phi}_j \equiv \phi_{j-1} + \phi_j + \phi_{j-1}$. Since $0 \notin \operatorname{supp} \phi$, it holds that

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} |\xi|^s \phi_j(\xi) e^{i\xi \cdot x} d\xi \right| dx &= 2^{js} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} |\xi|^s \phi(\xi) e^{i\xi \cdot x} d\xi \right| dx \\ &\leq C 2^{js}, \end{aligned}$$

with a constant $C > 0$ which does not depend on $j \in \mathbb{Z}$. Hence, it is found by (5.4) that

$$2^{j(s_0-s)} \|\varphi_j * (-\Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{T}^n)} \leq C 2^{js_0} \|\varphi_j * f\|_{L^p(\mathbb{T}^n)},$$

which yields the boundedness $(-\Delta)^{\frac{s}{2}} : \dot{B}^{s_0}_{p,q}(\mathbb{T}^n) \to \dot{B}^{s_0-s}_{p,q}(\mathbb{T}^n)$. The boundedness of the inverse $(-\Delta)^{-\frac{s}{2}}$ can be seen by the same way. Hence we obtain the isomorphism (3). By using this morphism and (5.5), we can also show (5.7). Furthermore, since $\varphi_j * R_k f = \sigma\left[\frac{im_k}{|m|}\tilde{\phi}_j(m)\right](\varphi_j * f)$ and since

$$\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\frac{i\xi_k}{|\xi|}\phi_j(\xi)e^{i\xi\cdot x}d\xi\right|dx = \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\frac{i\xi_k}{|\xi|}\phi(\xi)e^{i\xi\cdot x}d\xi\right|dx \le C, \quad \forall j \in \mathbb{Z},$$

(4) also holds by (5.4). This completes the proof of Proposition 5.2. □

On the other hand, we can prove Proposition 5.3 by a similar way to Proposition 3.7 (see also Kaneko-Kozono-Shimizu [11]), using (5.3) and (5.4).

5.2 Well-posedness and ill-posedness

Our main theorems in this chapter now read as follows. First, we state the well-posedness of (rSNS) for $1 \le p \le \infty$.

Theorem 5.4. (Tsurumi [25]) *Let $n \ge 3$, $1 \le p < n$, $1 \le q \le \infty$. Then* (rSNS) *is quantitatively well-posed from $D = \dot{B}^{-3+\frac{n}{p}}_{p,q}(\mathbb{T}^n)$ to $S = P\dot{B}^{-1+\frac{n}{p}}_{p,q}(\mathbb{T}^n)$.*

We should note here that $\{u, \Pi, f\}$ satisfy the equation (SNS) on $\mathbb{T}^n$, then $\{u_\lambda, \Pi_\lambda, f_\lambda\}$ with $u_\lambda(x) \equiv \lambda^{-1}u(\lambda^{-1}x)$, $\Pi_\lambda(x) \equiv \lambda^{-2}\Pi(\lambda^{-1}x)$, $f_\lambda(x) \equiv \lambda^{-3}(\lambda^{-1}x)$ $(\lambda > 0)$ also satisfy (SNS) on $\mathbb{T}^n_\lambda$. On the other hand, we can see that

$$\|u\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}(\mathbb{T}^n)} = \|u_\lambda\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}(\mathbb{T}^n_\lambda)}, \quad \|f\|_{\dot{B}^{-3+\frac{n}{p}}_{p,q}(\mathbb{T}^n)} = \|f_\lambda\|_{\dot{B}^{-3+\frac{n}{p}}_{p,q}(\mathbb{T}^n_\lambda)}$$

for any $\lambda > 0$ and $1 \le p, q \le \infty$. This fact can be regarded as alternative to the concept of scaling invariant with respect to the scaling transform $\{u, f\} \mapsto \{u_\lambda, f_\lambda\}$.

On the other hand, the following result on the ill-posedness holds.

Theorem 5.5. (Tsurumi [25]) *Let $n \ge 3$. Suppose that p and q satisfy either of following conditions*:

(1) $p = n, \quad 2 < q \le \infty,$

(2) $n < p \le \infty, \quad 1 \le q \le \infty.$

Let $D = \dot{B}^{-3+\frac{n}{p}}_{p,q}(\mathbb{T}^n)$ and $S = P\dot{B}^{-1+\frac{n}{p}}_{p,q}(\mathbb{T}^n)$. Then for every $\delta > 0$, there exists a sequence $\{f_N\}_{N=1}^\infty \subset \mathcal{D}_0(\mathbb{T}^n)$ of external forces satisfying the following (i), (ii), *and* (iii) *as follows*:

(i) $\|f_N\|_D \to 0$ *as* $N \to \infty$,

(ii) *For each* f_N*, there exists a solution* u_N *of* (rSNS) *in* $\mathcal{D}_0(\mathbb{T}^n) \cap S$*. Moreover, each* u_N *satisfies*

$$\begin{cases} -\Delta u_N(x) + (u_N \cdot \nabla u_N)(x) + \nabla \Pi(x) = f_N(x), \\ \operatorname{div}\, u_N(x) = 0 \end{cases}$$

for all $x \in \mathbb{R}^n$ *with a constant pressure* Π*, i.e.,* $\nabla \Pi = 0$.

(iii) *There exists another constant* $0 < \kappa < 1$ *independent of* δ *such that*

$$\kappa\delta < \|u_N\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{T}^n)} \le \|u_N\|_S < \delta.$$

for any $N \in \mathbb{N}$.

Remark 5.6. Since $\dot{B}^{-1+\frac{n}{p}}_{p,q}(\mathbb{T}^n) \hookrightarrow \dot{B}^{-1}_{\infty,\infty}(\mathbb{T}^n)$ for any $1 \le p, q \le \infty$, it is seen that (SNS) is ill-posed from D to S with such exponents p and q as in (1) and (2) of Theorem 5.5 by the lack of continuity of the solution map. Moreover, since it holds that $\|f\|_{L^\infty(\mathbb{T}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)}$ for any $f \in \mathcal{D}(\mathbb{T}^n)$, Theorem 5.5 also holds on $\mathbb{R}^n$ instead of $\mathbb{T}^n$ provided $p = \infty$.

Remark 5.7. It is still unknown whether (SNS) is well-posed or ill-posed from $\dot{B}^{-2}_{n,q}(\mathbb{T}^n)$ to $P\dot{B}^0_{n,q}(\mathbb{T}^n)$ when $1 \le q \le 2$. However, since $\dot{B}^0_{n,2}(\mathbb{T}^n) \hookrightarrow L^n(\mathbb{T}^n)$ (see Xiong-Xu-Yin[29]), we can show the well-posedness of (SNS) from $\dot{B}^{-2}_{n,q}(\mathbb{T}^n)$ to $PL^n(\mathbb{T}^n)$, using Hölder inequality instead of Proposition 5.3 in the proof of Theorem 5.4.

Only in this chapter, we let $\mathcal{D}_0 \equiv \mathcal{D}_0(\mathbb{T}^n)$ and $\dot{B}^s_{p,q} \equiv \dot{B}^s_{p,q}(\mathbb{T}^n)$ for simplicity .

5.2.1 Proof of the well-posedness

By the second section in this thesis, it suffices to show the lemma as follows in order to prove Theorem 5.4.

Lemma 5.8. (1) *Let* $n \ge 2$*,* $s \in \mathbb{R}$ *and let* $1 \le p, q \le \infty$*. Then the operator* $L \equiv (-\Delta)^{-1} P$ *is bounded from* $\dot{B}^{s-2}_{p,q}$ *onto* $P\dot{B}^s_{p,q}$ *with the estimate*

$$\|Lf\|_{\dot{B}^s_{p,q}} \le C\|f\|_{\dot{B}^{s-2}_{p,q}},$$

where $C = C(n, s, p, q)$ *is a constant.*

(2) *Let* $n \ge 2$*,* $s \in \mathbb{R}$*, and let* $1 \le p, q \le \infty$*. Then the operator* $K \equiv -(-\Delta)^{-1} P \nabla \cdot$ *is bounded from* $\dot{B}^{s-1}_{p,q}$ *onto* $P\dot{B}^s_{p,q}$ *with the estimate*

$$\|Kg\|_{\dot{B}^s_{p,q}} \le C\|g\|_{\dot{B}^{s-1}_{p,q}},$$

where $C = C(n, s, p, q)$ is a constant.

(3) Let $n \geq 3$, and let $1 \leq p < n$, $1 \leq q \leq \infty$. Then for $u, v \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$, we have $u \otimes v \in \dot{B}_{p,q}^{-2+\frac{n}{p}}$ with the estimate

$$\|u \otimes v\|_{\dot{B}_{p,q}^{-2+\frac{n}{p}}} \leq C\|u\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}\|v\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}},$$

where $C = C(n, p, q)$ is a constant.

Proof of Lemma 5.8. Since the projection P is defined as a matrix-valued operator $P = (P_{jk})_{1\leq j,k\leq n}$ with $P_{jk} \equiv \delta_{jk} + R_j R_k$ and the bilinear form K can be written as (3.9), we can prove (1) and (2) of Lemma 5.8 by the boundedness of the Riesz potential and Riesz transforms. Now let us show (3).

Since $n \geq 3$, we can take p_0 satisfying $1 < p_0 < \min\{p, n/2\}$. Moreover, since $p < n$, we can choose p_1 and p_2 as

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}, \quad p_1 > p, \quad p_2 > n$$

by choosing p_0 properly. Indeed, if $p < n/2$, then we should let $p_0 = p/(1 + \varepsilon)$ with small $0 < \varepsilon < p/n$ so that $p_0 > 1$ and

$$\frac{1}{p_0} = \frac{1}{p} + \frac{\varepsilon}{p} < \frac{1}{p} + \frac{1}{n}.$$

On the other hand, if $n/2 \leq p$, then we should let $p_0 = n/(2 + \varepsilon)$ with small $0 < \varepsilon < (n/p) - 1$ so that $p_0 > 1$ and

$$\frac{1}{p_0} = \frac{1+\varepsilon}{n} + \frac{1}{n} < \frac{1}{p} + \frac{1}{n}.$$

From Proposition 5.2, Proposition 5.3 with $s = -2+(n/p_0) > 0$ and $\alpha = 1-(n/p_2) > 0$, and (5.6), we have

$$\begin{aligned}
\|u \otimes v\|_{\dot{B}_{p,q}^{-2+\frac{n}{p}}} &\leq C\|u \otimes v\|_{\dot{B}_{p_0,q}^{-2+\frac{n}{p_0}}} \\
&\leq C\left(\|u\|_{\dot{B}_{p_1,q}^{-2+\frac{n}{p}+1-\frac{n}{p_2}}}\|v\|_{\dot{B}_{p_2,\infty}^{-1+\frac{n}{p_2}}} + \|u\|_{\dot{B}_{p_2,\infty}^{-1+\frac{n}{p_2}}}\|v\|_{\dot{B}_{p_1,q}^{-2+\frac{n}{p}+1-\frac{n}{p_2}}}\right) \\
&\leq C\|u\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}\|v\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}},
\end{aligned}$$

which proves (3) of Lemma 5.8. □

Remark 5.9. We can show that Theorem 5.4 also holds for $D = \dot{B}_{n,q}^{-2}$ and $S = PL^n$ when $1 \leq q \leq 2$. Indeed, since $\dot{B}_{n,q}^{0} \hookrightarrow L^n$ (see Xiong-Xu-Yin[29] for example), we have

$$\begin{aligned}
\|Lf\|_{L^n} &\leq C\|(-\Delta)^{-1}Pf\|_{\dot{B}_{n,q}^{0}} \\
&\leq C\|f\|_{\dot{B}_{n,q}^{-2}}.
\end{aligned}$$

Moreover, by Hölder inequality, we obtain

$$\begin{aligned}\|K(u\otimes v)\|_{L^n} &\leq C\|u\otimes v\|_{\dot{H}^{-1,n}}\\ &\leq C\|u\otimes v\|_{L^{\frac{n}{2}}}\\ &\leq C\|u\|_{L^n}\|v\|_{L^n},\end{aligned}$$

where

$$\dot{H}^{s,p}=\dot{H}^{s,p}(\mathbb{T}^n)\equiv\{f\in\mathcal{D}_0';\|f\|_{H^{s,p}}\equiv\|(-\Delta)^{\frac{s}{2}}\|_{L^p(\mathbb{T}^n)}<\infty\}$$

denotes the homogeneous toroidal potential space and we have used the boundedness of the Riesz transform on there.

5.2.2 Proof of the ill-posedness

For the proof of Theorem 5.5, we use a parameterized function defined by

$$f_{Q,r}(x)\equiv\frac{Q}{\sqrt{\Gamma(r)}}\sum_{s=1}^{r}s^{-\frac{1}{2}}h_s^2\left\{\cos(h_sx_1)e_2+\cos(h_sx_1-x_2)e_3\right\},\quad x=(x_1,\dots,x_n)\in\mathbb{T}^n,$$

where $Q>0$ and $r\in\mathbb{N}$ are parameters, $h_s\equiv 2^{s^2}$, and

$$\Gamma(r)\equiv\sum_{s=1}^{r}s^{-1}$$

denotes a partial harmonic series. We should note here that $\Gamma(r)\to\infty$ as $r\to\infty$. This function $f_{Q,r}$ is similar to the parameterized initial data proposed by Yoneda [27] on the topic of ill-posedness of non-stationary Navier-Stokes equations in $\mathbb{R}^n$.

Now let us construct a solution of (rSNS) with an external force $f=f_{Q,r}$. As similar to (2.19), we define again the approximative sequence $\{u_j\}_{j\in\mathbb{N}}$ to the solution of (rSNS) with $f=f_{Q,r}$ as

$$\begin{cases}u_1\equiv Lf_{Q,r},\\ u_j\equiv u_1+B(u_{j-1},u_{j-1}),\quad j\geq 2.\end{cases}$$

Moreover, we rewrite these u_j as forms of series in accordance with Sawada [17]. Let

$$\begin{cases}v_1\equiv u_1,\\ v_2\equiv B(u_1,u_1)=B(v_1,v_1),\\ v_k\equiv B(u_{k-1},u_{k-1})-B(u_{k-2},u_{k-2}),\quad k\geq 3.\end{cases}\tag{5.9}$$

Obviously, it holds

$$u_j=\sum_{k=1}^{j}v_k,\quad j\geq 1.\tag{5.10}$$

As for $f_{Q,r}$ and $\{v_k\}_{k\in\mathbb{N}}$, we can show the following key lemma.

Lemma 5.10. *Let* $\{v_k\}_{k\in\mathbb{N}}$ *be as* (5.9), *and suppose that* $n < \tilde{p} \le \infty$ *and* $2 < \tilde{q} \le \infty$. *Then it holds that* $\operatorname{div} v_k = 0$ *for all* $k \ge 1$. *In addition, we have the following estimates.*

$$\|f_{Q,r}\|_{\dot{B}^{-2}_{n,\tilde{q}}}, \|f_{Q,r}\|_{\dot{B}^{-3+\frac{n}{\tilde{p}}}_{\tilde{p},1}} \le C\frac{Q}{\sqrt{\Gamma(r)}}, \quad \text{for all } r, Q > 1, \tag{5.11}$$

$$\|v_1\|_{\dot{B}^{0}_{n,\tilde{q}}}, \|v_1\|_{\dot{B}^{-1+\frac{n}{\tilde{p}}}_{\tilde{p},1}} \le C\frac{Q}{\sqrt{\Gamma(r)}} \quad \text{for all } r, Q > 1, \tag{5.12}$$

$$C^{-1}Q^2 \le \|v_2\|_{\dot{B}^{-1}_{\infty,\infty}} \le \|v_2\|_{\dot{B}^{0}_{n,1}} \le CQ^2 \quad \text{if} \quad r \gg Q, \tag{5.13}$$

$$\|v_k\|_{\dot{B}^{0}_{n,1}}, \|v_k\|_{L^\infty} \le CQ^2\left(\frac{Q}{\sqrt{\Gamma(r)}}\right)^{k-2} \quad \text{for all } k \ge 3, \text{ if } r \gg Q, \tag{5.14}$$

where $C > 0$ *denotes a global constant which depends only on* n, $\tilde{p}$, *and* $\tilde{q}$.

For the moment, let us assume this lemma. Once we fix Q and r with

$$\frac{Q}{\sqrt{\Gamma(r)}} < 1,$$

then by (5.14), there hold

$$\sum_{k=1}^{\infty}\|v_k\|_{L^\infty} < \infty, \quad \sum_{k=1}^{\infty}\|v_k\|_{\dot{B}^{0}_{n,1}} < \infty.$$

Hence, there exists $u^*_{Q,r} \in C(\mathbb{T}^n) \cap \dot{B}^0_{n,1}$ such that

$$u^*_{Q,r} = \lim_{j\to\infty} u_j = \sum_{k=1}^{\infty} v_k \quad \text{in } L^\infty \text{ and } \dot{B}^0_{n,1} \tag{5.15}$$

and $\operatorname{div} u^*_{Q,r} = 0$. Actually, this function $u^*_{Q,r}$ becomes a solution of (rSNS), which is implied by the convergence of the bilinear form $B(u_j, u_j)$ as the following lemma.

Lemma 5.11. *Let* $r \gg Q$ *and let* u^* *be a function defined by* (5.15). *Then it holds that*

$$\lim_{j\to\infty} B(u_j, u_j) = B(u^*_{Q,r}, u^*_{Q,r}) \quad \text{in } L^\infty \text{ and } \dot{B}^0_{n,1}. \tag{5.16}$$

Moreover, $u = u^*_{Q,r}$ *satisfies the original equation* (SNS) *for all* $x \in \mathbb{R}^n$ *in the pointwise sense, with* $f = f_{Q,r}$ *and a constant pressure* Π.

By the above two lemmata, we can easily show Theorem 5.5 by the same method as Theorem 4.1 in the last chapter. Here we should note that by (5.6), it suffices to show Theorem 5.5 in the cases

$$(D, S) = (\dot{B}^{-2}_{n,\tilde{q}}, P\dot{B}^{0}_{n,\tilde{q}}), (\dot{B}^{-3+\frac{n}{\tilde{p}}}_{\tilde{p},1}, P\dot{B}^{-1+\frac{n}{\tilde{p}}}_{\tilde{p},1}), \quad n < \tilde{p} \le \infty, \quad 2 < \tilde{q} \le \infty.$$

Now let us show Lemma 5.10-5.11. First of all, we prepare some properties about the Riesz potential and the toroidal Besov norm of trigonometric functions as follows.

Proposition 5.12. (1) *Let* $a \in \mathbb{Z}^n \backslash \{0\}$ *and let* $g_1(x) \equiv \cos(a \cdot x)$, $x \in \mathbb{T}^n$. *Then*

$$(-\Delta)^{\frac{s}{2}} g_1 = |a|^s g_1, \quad s \in \mathbb{R}, \tag{5.17}$$

and for any $1 \le p, q \le \infty$, *we have*

$$\|g_1\|_{\dot{B}^0_{p,q}} \le C, \tag{5.18}$$

where $C = C(p, q, n)$ *is a constant.*

(2) *Let* $r \in \mathbb{N}$, $\alpha_1, \dots, \alpha_r \in \mathbb{R}$, $a_1, \dots, a_r \in \mathbb{Z}^n \backslash \{0\}$, *and let*

$$g_2(x) \equiv \sum_{s=1}^{r} \alpha_s \cos(a_s \cdot x), \quad x \in \mathbb{T}^n.$$

Suppose that the vectors $a_1, \dots, a_r$ *satisfy*

$$|a_s| \ne |a_t|, \quad \frac{|a_s|}{|a_t|} \notin \{x \in \mathbb{R}; 2^{-2} < |x| < 2^2\}, \quad \text{if} \quad s \ne t. \tag{5.19}$$

Then for any $1 \le p, q \le \infty$, *we have*

$$\|g_2\|_{\dot{B}^0_{p,q}} \le \begin{cases} C \left\{ \sum\limits_{s=1}^{r} |\alpha_s|^q \right\}^{\frac{1}{q}}, & 1 \le q < \infty, \\ C \max\limits_{1 \le s \le r} |\alpha_s|, & q = \infty, \end{cases}$$

where $C = C(p, q, n)$ *is a constant.*

Proof of Proposition 5.12. (1) Since $g(x) = (e^{ia \cdot x} + e^{-ia \cdot x})/2$, it holds that

$$\mathcal{F}_{\mathbb{T}^n} g_1(m) = \begin{cases} \frac{1}{2}, & m = \pm a, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, considering the definition of the Riesz potential $(-\Delta)^{\frac{s}{2}}$, we see

$$\begin{aligned} (-\Delta)^{\frac{s}{2}} g_1(x) &= \sum_{m \in \mathbb{Z}^n} |m|^s \mathcal{F}_{\mathbb{T}^n} g_1(m) e^{im \cdot x} \\ &= |a|^s \cos(a \cdot x), \end{aligned}$$

which implies (5.17). On the other hand, we can assume that ϕ in the definition of $\dot{B}^s_{p,q}$ (see (5.1)) is a radial function. Then we have

$$\begin{aligned} (\varphi_j * g_1)(x) &= \sum_{m \in \mathbb{Z}^n} \phi_j(m) \mathcal{F}_{\mathbb{T}^n} g_1(m) e^{im \cdot x} \\ &= \phi_j(a) \cos(a \cdot x), \ j \in \mathbb{Z}. \end{aligned}$$

Since supp $\phi_j = \{x \in \mathbb{R}^n; 2^{j-1} < |x| < 2^{j+1}\}$, there exist at most two indices $j \in \mathbb{Z}$ such that $\phi_j(a) \not\equiv 0$. Let j_1 and j_2 be such indices. Then since $|\phi_j| \le 1$ uniformly, we see that

$$\|g_1\|_{\dot{B}^0_{p,q}} = (\|\phi_{j_1}(a)\cos(a\cdot))\|^q_{L^p(\mathbb{T}^n)} + \|\phi_{j_2}(a)\cos(a\cdot))\|^q_{L^p(\mathbb{T}^n)})^{\frac{1}{q}} < C.$$

(2) For each $s = 1, \ldots, r$, there exist at most two indices $j \in \mathbb{Z}$ such that

$$(\varphi_j * \cos(a_s\cdot)(x) = \phi_j(a_s)\cos(a_s \cdot x) \not\equiv 0.$$

Let $j(s)$ and $j'(s)$ be such indices. On the other hand, it is seen from the assumption (5.19) that there exist at most one vector a_s for each $j \in \mathbb{Z}$ such that $\phi_j(a_s) \not\equiv 0$. Hence, since $|\phi_j| \le 1$ uniformly, we can see that

$$\begin{aligned}
\|g_2\|^q_{\dot{B}^0_{p,q}} &= \sum_{j\in\mathbb{Z}} \|\varphi_j * g_2\|^q_{L^p(\mathbb{T}^n)} \\
&= \sum_{s=1}^{r} \Big(|\alpha_s \phi_{j(s)}|^q \|\cos(a_s\cdot)\|^q_{L^p(\mathbb{T}^n)} + |\alpha_s \phi_{j'(s)}|^q \|\cos(a_s\cdot)\|^q_{L^p(\mathbb{T}^n)} \Big) \\
&\le C \sum_{s=1}^{r} |\alpha_s|^q
\end{aligned}$$

for any $1 \le q < \infty$, and that

$$\begin{aligned}
\|g_2\|_{\dot{B}^0_{p,\infty}} &= \sup_{j\in\mathbb{Z}} \|\varphi_j * g_2\|_{L^\infty(\mathbb{T}^n)} \\
&\le \max_{1\le s\le r} |\alpha_s|.
\end{aligned}$$

This completes the proof of Proposition 5.12. □

We should note here that the above proposition also holds for $\sin(\cdot)$ instead of $\cos(\cdot)$.

Proof of Lemma 5.10. We prove Lemma 5.10 by three steps.

Step 1. Estimates of $f_{Q,r}$ and v_1. It is clear that $f_{Q,r} \in \mathcal{D}_0$ and div $f_{Q,r} = 0$. Hence, it is seen from Proposition 5.12 (1) that

$$\begin{aligned}
v_1 &= (-\Delta)^{-1} f_{Q,r} \\
&= \frac{Q}{\sqrt{\Gamma(r)}} \sum_{s=1}^{r} s^{-\frac{1}{2}} \left\{\cos(h_s x_1)e_2 + k_s \cos(h_s x_1 - x_2)e_3\right\},
\end{aligned}$$

where $k_s \equiv h_s^2/(h_s^2+1)$, and that div $v_1 = 0$. Since $h_s > h_t$ and $h_s/h_t \ge 2^2$ for any

$s > t$, we obtain the estimates from Proposition 5.2 (3) and Proposition 5.12 (2) as

$$\begin{aligned}\|f_{Q,r}\|_{\dot{B}^{-2}_{n,\tilde{q}}} &= \|v_1\|_{\dot{B}^{0}_{n,\tilde{q}}} \\ &\leq C\frac{Q}{\sqrt{\Gamma(r)}}\left(\sum_{s=1}^{r} s^{-\frac{\tilde{q}}{2}}\right)^{\frac{1}{q}} \\ &\leq C\frac{Q}{\sqrt{\Gamma(r)}}\end{aligned}$$

and

$$\begin{aligned}\|f_{Q,r}\|_{\dot{B}^{-3+\frac{n}{\tilde{p}}}_{\tilde{p},1}} = \|v_1\|_{\dot{B}^{-1+\frac{n}{\tilde{p}}}_{\tilde{p},1}} &= \|(-\Delta)^{\frac{1}{2}(-1+\frac{n}{\tilde{p}})}v_1\|_{\dot{B}^{0}_{\tilde{p},1}} \\ &\leq C\frac{Q}{\sqrt{\Gamma(r)}}\sum_{s=1}^{r}|h_s|^{\frac{1}{2}(-1+\frac{n}{\tilde{p}})} \\ &\leq C\frac{Q}{\sqrt{\Gamma(r)}},\end{aligned}$$

which prove (5.11) and (5.12).

Step 2. Estimates of v_2. In what follows, let $v_k^{(i)}$ be the i-th component of v_k. It is easily seen that

$$\begin{aligned}v_1\cdot\nabla v_1 &= v_1^{(2)}\cdot\partial_2 v_1 \\ &= \left(\frac{Q}{\sqrt{\Gamma(r)}}\sum_{s=1}^{r}s^{-\frac{1}{2}}\cos(h_s x_1)\right)\frac{Q}{\sqrt{\Gamma(r)}}\sum_{t=1}^{r}t^{-\frac{1}{2}}k_t\sin(h_t x_1 - x_2)e_3 \\ &= \frac{Q^2}{\Gamma(r)}\sum_{s=1}^{r}s^{-1}k_s\left(-\frac{1}{2}\sin x_2\right)e_3 \\ &\quad + \frac{Q^2}{\Gamma(r)}\sum_{s=1}^{r}s^{-1}k_s\left(\frac{1}{2}\sin(2h_s x_1 - x_2)\right)e_3 \\ &\quad + \frac{Q^2}{\Gamma(r)}\sum_{\substack{s,t=1\\ s\neq t}}^{r}s^{-\frac{1}{2}}t^{-\frac{1}{2}}k_t\frac{1}{2}\left\{\sin((h_s+h_t)x_1 - x_2) - \sin((h_s-h_t)x_1 + x_2)\right\}e_3,\end{aligned}$$

Therefore, $\nabla \cdot (v_1 \cdot \nabla v_1) = 0$, and

$$\begin{aligned}
v_2 &= -(-\Delta)^{-1}(v_1 \cdot \nabla v_1) \\
&= \frac{Q^2}{\Gamma(r)} \sum_{s=1}^{r} s^{-1} k_s \left(\frac{1}{2} \sin x_2 \right) e_3 \\
&\quad + \frac{Q^2}{\Gamma(r)} \sum_{s=1}^{r} s^{-1} k_s \frac{1}{4h_s^2+1} \left(-\frac{1}{2} \sin(2h_s x_1 - x_2) \right) e_3 \\
&\quad + \frac{Q^2}{\Gamma(r)} \sum_{\substack{s,t=1 \\ s \neq t}}^{r} s^{-\frac{1}{2}} t^{-\frac{1}{2}} k_t \frac{1}{2} \left\{ -l_{s,t} \sin((h_s+h_t)x_1 - x_2) + \tilde{l}_{s,t} \sin((h_s-h_t)x_1 + x_2) \right\} e_3 \\
&\equiv N_1 + N_2 + N_3,
\end{aligned}$$

where

$$l_{s,t} \equiv \frac{1}{(h_s+h_t)^2+1}, \quad \tilde{l}_{s,t} \equiv \frac{1}{(h_s-h_t)^2+1}.$$

Since $1/2 < k_s < 1$ for every $s = 1, \ldots, r$, we have

$$\begin{aligned}
\|N_1\|_{\dot{B}^0_{n,1}} &\leq \frac{1}{2} \frac{Q^2}{\Gamma(r)} \left(\sum_{s=1}^{r} s^{-1} \right) \|\phi(e_2) \sin x_2\|_{L^n(\mathbb{T}^n)} \\
&\leq CQ^2,
\end{aligned}$$

while

$$\begin{aligned}
\|N_1\|_{\dot{B}^{-1}_{\infty,\infty}} &\geq \frac{1}{4} \frac{Q^2}{\Gamma(r)} \left(\sum_{s=1}^{r} s^{-1} \right) \|\phi(e_2) \sin x_2\|_{L^\infty(\mathbb{T}^n)} \\
&\geq CQ^2.
\end{aligned}$$

Moreover, by Proposition 5.12 (1), we have

$$\begin{aligned}
\|N_2\|_{\dot{B}^0_{n,1}} &\leq \frac{1}{2} \frac{Q^2}{\Gamma(r)} \sum_{s=1}^{r} (4h_s^2+1)^{-1} \\
&\leq C \frac{Q^2}{\Gamma(r)}.
\end{aligned}$$

In order to estimate N_3, it suffices to estimate

$$\tilde{N}_3 \equiv \frac{Q^2}{\Gamma(r)} \sum_{s=2}^{r} \sum_{t=1}^{s-1} s^{-\frac{1}{2}} t^{-\frac{1}{2}} \frac{1}{2} \left\{ l_{s,t} \sin((h_s+h_t)x_1 - x_2) - \tilde{l}_{s,t} \sin((h_s-h_t)x_1 + x_2) \right\} e_3$$

by the symmetry of s and t. Since

$$l_{s,t} \leq h_s^{-2}, \ \tilde{l}_{s,t} \leq \frac{1}{h_s(h_s - 2h_t)} \leq 2h_s^{-2}, \quad \forall t < s,$$

we can see from Proposition 5.12 (1) that

$$\begin{aligned}\|\tilde{N}_3\|_{\dot{B}^0_{n,1}} &\leq C\frac{Q^2}{\Gamma(r)}\sum_{s=2}^{r} s^{-\frac{1}{2}}h_s^{-2}(s-1)\\ &\leq C\frac{Q^2}{\Gamma(r)}.\end{aligned}$$

Therefore, if $r \gg Q$, then there hold from the estimates of N_1, N_2, and N_3 that

$$\begin{aligned}\|v_2\|_{\dot{B}^0_{n,1}} &\leq \sum_{i=1,2,3}\|N_i\|_{\dot{B}^0_{n,1}}\\ &\leq CQ^2,\end{aligned}$$

and that

$$\begin{aligned}\|v_2\|_{\dot{B}^{-1}_{\infty,\infty}} &\geq \|v_2\|_{\dot{B}^{-1}_{\infty,\infty}} - \sum_{i=2,3}\|N_i\|_{\dot{B}^0_{n,1}}\\ &\geq CQ^2,\end{aligned}$$

which implies (5.13).

Step 3. Estimates of v_k, $k \geq 3$. By the induction, we can see that

$$v_k \in \mathcal{D}_0, \quad \text{div } v_k = 0, \quad \forall k \geq 1, \tag{5.20}$$

$$v_k = B(v_1, v_{k-1}) = \theta_k e_3, \quad \forall k \geq 2, \tag{5.21}$$

where

$$\theta_k \equiv -(-\Delta)^{-1}\left(v_1^{(2)}\partial_2 v_{k-1}^{(3)}\right) \tag{5.22}$$

denotes a scalar-valued function depend only on x_1 and x_2. Indeed, we can see all of the above (5.20)-(5.22) in a similar way to Proposition 4.6.

Now let us estimate v_3. By (5.21), It suffices to estimate θ_3. Since

$$\begin{aligned}v_1^{(2)}\partial_2 N_1^{(3)} &= \left(\frac{Q}{\sqrt{\Gamma(r)}}\right)^3\left(\sum_{s=1}^{r} s^{-\frac{1}{2}}\cos(h_s x_1)\right)\sum_{t=1}^{r} t^{-1}k_t\left(-\frac{1}{2}\cos x_2\right)\\ &= -\frac{1}{4}\left(\frac{Q}{\sqrt{\Gamma(r)}}\right)^3\left(\sum_{t=1}^{r} t^{-1}k_t\right)\sum_{s=1}^{r} s^{-\frac{1}{2}}\left\{\cos(h_s x_1 + x_2) + \cos(h_s x_1 - x_2)\right\},\end{aligned}$$

we have

$$\begin{aligned}\theta_3 &= -(-\Delta)^{-1}\left(v_1^{(2)}\partial_2(N_1^{(3)} + N_2^{(3)} + N_3^{(3)})\right)\\ &= \frac{1}{4}\left(\frac{Q}{\sqrt{\Gamma(r)}}\right)^3\left(\sum_{t=1}^{r} t^{-1}k_t\right)\sum_{s=1}^{r} s^{-\frac{1}{2}}\frac{1}{h_s^2+1}\left\{\cos(h_s x_1 + x_2) + \cos(h_s x_1 - x_2)\right\} + R_3\\ &\equiv M_3 + R_3,\end{aligned}$$

where $R_3 \equiv -(-\Delta)^{-1}\left(v_1^{(2)}\partial_2(N_2^{(3)}+N_3^{(3)})\right)$ is a reminder term. For M_3, it is seen from Proposition 5.12 (1) that

$$\begin{aligned}\|M_3\|_{\dot{B}^0_{n,1}} &\le \frac{1}{4}C\frac{Q^3}{\sqrt{\Gamma(r)}}\sum_{s=1}^{r}h_s^{-2}\\ &\le \frac{1}{4}C\frac{Q^3}{\sqrt{\Gamma(r)}}\end{aligned}$$

and

$$\|M_3\|_{L^\infty} \le \frac{1}{4}C\frac{Q^3}{\sqrt{\Gamma(r)}}.$$

Moreover, it is easily seen by a similar calculation that R_3 is small compared with the main term M_3. Hence, we have the estimate (5.14) for $k=3$.

We next estimate v_4 (θ_4). It holds that

$$\begin{aligned}v_1^{(2)}\partial_2 M_3 &= \frac{1}{4}\left(\frac{Q}{\sqrt{\Gamma(r)}}\right)^4\left(\sum_{s=1}^{r}s^{-\frac{1}{2}}\cos(h_s x_1)\right)\\ &\quad\times\left(\sum_{l=1}^{r}l^{-1}k_l\right)\sum_{t=1}^{r}t^{-\frac{1}{2}}\frac{1}{h_t^2+1}\left\{-\sin(h_t x_1+x_2)+\sin(h_t x_1-x_2)\right\}\\ &= -\frac{1}{4}\left(\frac{Q}{\sqrt{\Gamma(r)}}\right)^4\left(\sum_{l=1}^{r}l^{-1}k_l\right)\sum_{s=1}^{r}s^{-1}\frac{1}{h_s^2+1}\sin x_2\\ &\quad-\frac{1}{8}\left(\frac{Q}{\sqrt{\Gamma(r)}}\right)^4\left(\sum_{l=1}^{r}l^{-1}k_l\right)\sum_{s=1}^{r}s^{-1}\frac{1}{h_s^2+1}\left\{\sin(2h_s x_1+x_2)-\sin(2h_s x_1-x_2)\right\}\\ &\quad+R_4',\end{aligned}$$

where R_4' is a reminder. Hence we have

$$\begin{aligned}\theta_4 &= -(-\Delta)^{-1}\left(v_1^{(2)}\partial_2(M_3+R_3)\right)\\ &= \frac{1}{4}\left(\frac{Q}{\sqrt{\Gamma(r)}}\right)^4\left(\sum_{l=1}^{r}l^{-1}k_l\right)\sum_{s=1}^{r}s^{-1}\frac{1}{h_s^2+1}\sin x_2\\ &\quad-\frac{1}{8}\left(\frac{Q}{\sqrt{\Gamma(r)}}\right)^4\left(\sum_{l=1}^{r}l^{-1}k_l\right)\sum_{s=1}^{r}s^{-1}\frac{1}{h_s^2+1}\frac{1}{4h_s^2+1}\left\{\sin(2h_s x_1+x_2)-\sin(2h_s x_1-x_2)\right\}\\ &\quad+R_4\\ &\equiv M_4+R_4,\end{aligned}$$

where $R_4 \equiv -(-\Delta)^{-1}(R_4'+v_1^{(2)}\partial_2 R_3)$ is another reminder. Therefore, by a similar calculation on v_3, the norms of M_4 are estimated as

$$\|M_4\|_{\dot{B}^0_{n,1}} \le \frac{1}{4}C\frac{Q^4}{\Gamma(r)^2},\quad \|M_4\|_{L^\infty} \le \frac{1}{4}C\frac{Q^4}{\Gamma(r)^2},$$

and so are that of R_4. These estimates show (5.14) for $k = 4$.

From (5.17), (5.21), (5.22), and a similar calculation on trigonometric functions as above, we see by induction that

$$\theta_k = \left(\frac{1}{2}\right)^{\left[\frac{k+1}{2}\right]} \left(\frac{Q}{\sqrt{\Gamma(r)}}\right)^k \left(\sum_{l=1}^{r} l^{-1} k_l\right) \tilde{M}_k(x) + R_k(x), \quad k \geq 3,$$

where $[q] \equiv \max\{m \in \mathbb{N}; m \leq q\}$, each R_k denotes a small reminder, and $\tilde{M}_k$ has a form as

$$\tilde{M}_k(x) = \begin{cases} \sum_{s\in\mathbb{N}:finite} \gamma_s^k \cos(H_s^k \cdot x) & k \text{ is odd}, \\ \sum_{s\in\mathbb{N}:finite} {\gamma'}_s^k \sin x_2 & k \text{ is even}, \end{cases}$$

with some coefficients γ_s^k, ${\gamma'}_s^k$ and vectors H_s^k such that

$$H_\sigma^k = l_k h_s e_1 + \sigma_k e_2, \quad l_k \in \mathbb{N},\ \sigma_k \in \{-1, 1\},$$
$$|\gamma_s^k|,\ |{\gamma'}_s^k| \leq \frac{1}{h_s^2}, \quad \forall k \geq 3.$$

Hence we see that (5.14) holds for any $k \geq 3$. This completes the proof of Lemma 5.10. □

Proof of Lemma 5.11. From the calculation in the proof of Lemma 5.1, it is seen that $\partial_2^m v_k$ can be estimated the same as (5.14) for any order $m = 1, 2, \ldots$. Moreover, it is also seen that for every order $m = 1, 2, \ldots$, $\partial_1^m v_k$ can be estimated as (5.14) if k is large enough. Therefore, we see that $u^*_{Q,r} = \sum_{k=1}^\infty v_k$ belongs to $\mathcal{D}_0$ and is termwise differentiable provided $r \gg Q$. In addition, from the identity

$$\begin{aligned} \sum_{k=1}^\infty v_1^{(2)} \partial_2 v_k &= -(-\Delta)\left(\sum_{k=1}^\infty -(-\Delta)^{-1} v_1^{(2)} \partial_2 v_k\right) \\ &= (-\Delta) \sum_{k=1}^\infty v_{k+1}, \end{aligned}$$

there holds

$$\begin{aligned} \|B(u^*_{Q,r}, u^*_{Q,r}) - B(u_j, u_j)\|_{L^\infty} &= \left\| B\left(v_1, \sum_{k=j+1}^\infty v_k\right)\right\|_{L^\infty} \\ &= \left\| -(-\Delta)^{-1} v_1^{(2)} \partial_2 \left(\sum_{k=j+1}^\infty v_k\right)\right\|_{L^\infty} \\ &= \left\| \sum_{k=j+1}^\infty \left\{-(-\Delta)^{-1}\left(v_1^{(2)} \partial_2 v_k\right)\right\}\right\|_{L^\infty} \\ &\leq \sum_{k=j+1}^\infty \|v_{k+1}\|_{L^\infty} \to 0, \quad \text{as } j \to \infty. \end{aligned}$$

In the same way, it is also easily shown that $B(u_j, u_j) \to B(u^*_{Q,r}, u^*_{Q,r})$ in $\dot{B}^{-1}_{n,1}$ as $j \to \infty$.

Finally, let us show that $u = u^*_{Q,r} = \sum_{k=1}^{\infty} v_k$ actually satisfies (SNS) with $f = f_{Q,r}$ and $\nabla \Pi = 0$ for all $x \in \mathbb{T}^n$. Indeed, by termwise differentiation, we see that

$$\begin{aligned} -\Delta u^*_{Q,r}(x) &= -\Delta v_1(x) - \Delta \left(\sum_{k=2}^{\infty} -(-\Delta)^{-1} v_1(x) \partial_2 v_{k-1}(x) \right) \\ &= f_{Q,r}(x) - \sum_{k=1}^{\infty} v_1(x) \partial_2 v_k(x), \\ (u^*_{Q,r} \cdot \nabla u^*_{Q,r})(x) &= v_1(x) \cdot \partial_2 \sum_{k=1}^{\infty} v_k(x) \\ &= \sum_{k=1}^{\infty} v_1(x) \partial_2 v_k(x). \end{aligned}$$

Hence we obtain

$$-\Delta u^*_{Q,r}(x) + (u^*_{Q,r} \cdot \nabla u^*_{Q,r})(x) = f_{Q,r}(x), \qquad x \in \mathbb{T}^n.$$

This completes the proof of Lemma 5.11. □

Chapter 6

Ill-posedness by Bejenaru-Tao method

From this chapter, we return to the problem on the well-posedness of (SNS) from $D = \dot{B}_{p,q}^{-3+\frac{n}{p}} = \dot{B}_{p,q}^{-3+\frac{n}{p}}(\mathbb{R}^n)$ to $S = \dot{B}_{p,q}^{-1+\frac{n}{p}} = \dot{B}_{p,q}^{-1+\frac{n}{p}}(\mathbb{R}^n)$ in the whole space. As seen in Chapter 4, the Bourgain-Pavlović method for the ill-posedness is applicable only when $p = \infty$. Hence in this chapter, we approach this problem in the case $n \le p < \infty$ using another method proposed by Bejenaru-Tao [2].

6.1 The important proposition by Bejenaru-Tao

Bejenaru-Tao [2] showed the following important proposition. Here we return to the abstract problem on the equation (E) in Chapter 2.

Proposition 6.1. (Bejenaru-Tao[2]) *Suppose that* (E) *is quantitatively well-posed from* $(D, \|\cdot\|_D)$ *to* $(S, \|\cdot\|_S)$*. We define the nonlinear maps* $A_m : D \to S$ *for* $m \in \mathbb{N}$ *by*

$$\begin{cases} A_1 f \equiv Lf, \\ A_m f \equiv \displaystyle\sum_{k,l \ge 1, k+l=m} B(A_k f, A_l f), \quad n \ge 2. \end{cases}$$

(1) *Each* $A_m f$ *belongs to* S *and there exists a constant* $C > 0$ *such that*

$$\|A_m f\|_S \le C^m \|f\|_D^m, \quad \forall m \in \mathbb{N}.$$

Moreover, a solution $u \in B_S(\delta)$ *of* (E) *for* $f \in B_D(\varepsilon)$*, which is obtained by the well-posedness defined in Definition 2.5, is expressed as*

$$u = u(f) = \sum_{m=1}^{\infty} A_m f \quad \text{in } S.$$

(2) *Suppose that* D *and* S *are given other norms* $\|\cdot\|_{\tilde{D}}$ *and* $\|\cdot\|_{\tilde{S}}$*, respectively, which are weaker than* D *and* S *in the sense that*

$$\|f\|_{\tilde{D}} \le C\|f\|_D, \quad \|u\|_{\tilde{S}} \le C\|u\|_S.$$

Assume that the solution map $f \mapsto u$ of (E) *given by Definition 2.5 is continuous from* $(B_D(\varepsilon), \|\cdot\|_{\tilde{D}})$ *to* $(B_S(\delta), \|\cdot\|_{\tilde{S}})$. *Then for every $m \in \mathbb{N}$, $A_m : D \to S$ is also continuous from* $(B_D(\varepsilon), \|\cdot\|_{\tilde{D}})$ *to* $(B_S(\delta), \|\cdot\|_{\tilde{S}})$.

In the above, for example, $(B_D(\varepsilon), \|\cdot\|_{\tilde{D}})$ denotes the ball $\{f \in D; \|f\|_D < \delta\}$ equipped with the weak norm $\|\cdot\|_{\tilde{D}}$.

Remark 6.2 Suppose that the equation (E) is quantitatively well-posed from D to S, and $(\tilde{D}, \|\cdot\|_{\tilde{D}})$ and $(\tilde{S}, \|\cdot\|_{\tilde{S}})$ are Banach spaces such that there hold the embeddings $D \hookrightarrow \tilde{D}$ and $S \hookrightarrow \tilde{S}$. Then Proposition 6.1 means that if at least one of A_m is discontinuous from $\tilde{D}$ to $\tilde{S}$ for any small $\varepsilon > 0$, then the solution map of (E) becomes discontinuous in such weaker spaces, which implies that the equation (E) is ill-posed from $\tilde{D}$ to $\tilde{S}$. For the proof of Proposition 6.1, see Appendix B.

In order to apply this proposition for showing the ill-posedness of (rSNS), we require appropriate spaces D and S which guarantee the quantitatively well-posedness of (rSNS). Actually, the well-posedness result stated in Proposition 3.1 by Kaneko-Kozono-Shimizu [11] does not seem to be satisfactory for showing the ill-posedness in the case $n \le p < \infty$ by Proposition 6.1. Hence, before stating our new ill-posedness result, we should show the following:

Proposition 6.3. (Quantitatively well-posedness when $p = n$ and $1 \le q \le 2$) *Let $n \ge 3$. Then* (rSNS) *is quantitatively well-posed from $\dot{H}^{-2,n}$ to PL^n, and in particular, from $\dot{B}^{-2}_{n,q}$ to PL^n if $1 \le q \le 2$.*

Indeed, by the boundedness of P (or Riesz transforms) in homogeneous Sobolev spaces, we have

$$\|(-\Delta)^{-1}Pf\|_{L^n} = \|Pf\|_{\dot{H}^{-2,n}} \le C\|f\|_{\dot{H}^{-2,n}}.$$

for all $f \in \dot{H}^{-2,n}$. Moreover, by the embedding $L^{\frac{n}{2}} \hookrightarrow \dot{H}^{-1,n}$ and the Hölder inequality, it holds that

$$\begin{aligned}\|(-\Delta)^{-1}P\nabla\cdot(u \otimes v)\|_{L^n} &\le C\|u \otimes v\|_{\dot{H}^{-1,n}} \\ &\le C\|u\|_{L^n}\|v\|_{L^n}\end{aligned}$$

for all $u, v \in PL^n$, which completes the proof of Proposition 6.3.

Remark 6.4 Actually, there are other previous results on the well-posedness in the case $p = n$. For example, Bjorland-Brandolese-Iftimie-Schonbek [3] showed the well-posedness with more general space of external forces. In fact, they proved that there are constants $\varepsilon, \delta > 0$ such that if $f \in \mathcal{S}'$ satisfies $\|(-\Delta)^{-1}f\|_{L^{n,\infty}} < \varepsilon$ ($L^{n,\infty}$ denotes the weak L^n space), then there exists a unique solution $u \in B_{PL^{n,\infty}}(\delta)$ to (rSNS), which belongs to L^n if and only if $Pf \in \dot{H}^{-2,n}$. In addition, Phan-Phuc [16] showed the well-posedness in the largest critical space of external forces including $\dot{H}^{-2,n}$. However, in this thesis, it suffices to consider Proposition 6.3 for our main purpose.

6.2 Ill-posedness in the remaining cases

Our result now reads as follows:

Theorem 6.5. (Tsurumi [26]) *Let $n \geq 3$. Suppose that D and $\tilde{D}$ are two Banach spaces with $D \hookrightarrow \tilde{D}$ as either* (1) *or* (2):

(1) $D = \dot{B}^{-2}_{n,1}$, $\tilde{D} = \dot{B}^{-3+\frac{n}{p}}_{p,q}$ *with $n < p \leq \infty$ and $1 \leq q \leq \infty$,*

(2) $D = \dot{B}^{-2}_{n,2}$, $\tilde{D} = \dot{B}^{-2}_{n,q}$ *with $2 < q \leq \infty$.*

Let $\varepsilon, \delta > 0$ be constants appearing in Definition 2.5 which guarantee the well-posedness of (rSNS) *from D to PL^n (see Proposition 6.3), and take $0 < \eta < \varepsilon$ arbitrarily. Then the solution map*

$$f \in (B_D(\eta), \|\cdot\|_{\tilde{D}}) \mapsto u \in (B_{PL^n}(\delta), \|\cdot\|_{\dot{B}^{-1}_{\infty,\infty}})$$

is discontinuous, where $(B_D(\eta), \|\cdot\|_{\tilde{D}})$ and $(B_{PL^n}(\delta), \|\cdot\|_{\dot{B}^{-1}_{\infty,\infty}})$ denote the ball $B_D(\eta)$ equipped with the $\tilde{D}$ topology and $B_{PL^n}(\delta)$ with the $\dot{B}^{-1}_{\infty,\infty}$ topology, respectively. In other words, (rSNS) *is ill-posed from $\tilde{D}$ to $P\dot{B}^{-1}_{\infty,\infty}$.*

Remark 6.6. Suppose that D and $\tilde{D}$ are as the above theorem. We now arbitrarily choose a sequence $\{g_N\}_{N\in\mathbb{N}}$ such that $\sup_{N\in\mathbb{N}} \|g_N\|_D < \varepsilon$. Then by Proposition 6.3, there exists a unique solution $v_N \in PL^n$ for each g_N. In addition, if $g_N \to 0$ in D, then we see $v_N \to 0$ in PL^n by the well-posedness (continuity of the solution map). Theorem 6.5 means, however, that the weaker convergence $g_N \to 0$ in $\tilde{D}$ cannot sufficiently guarantee $v_N \to 0$ even in the weakest scaling invariant norm $\dot{B}^{-1}_{\infty,\infty}$.

Remark 6.7. Actually, as seen in Lemma 6.9, we will show Theorem 6.5 by constructing a sequence $\{f_N\}_{N\in\mathbb{N}}$ of external forces with $\|f_N\|_{\tilde{D}} \to 0$ such that the corresponding sequence $\{u_N\}_{N\in\mathbb{N}}$ of solutions does not converges to zero in $\dot{B}^{-1}_{\infty,\infty}$. Hence we can easily see by Proposition 2.1 (4) that we can also show Theorem 6.5 for the homogeneous Triebel-Lizorkin spaces with the same exponents. Indeed, in the case (1), we consider the embedding

$$\dot{B}^{-3+\frac{n}{p}}_{p,1} \hookrightarrow \dot{F}^{-3+\frac{n}{p}}_{p,q}$$

for $n < p \leq \infty$ and $1 \leq q \leq \infty$, while in the case (2) with $2 < q \leq \infty$, take $2 < r < \min\{n, q\}$ and consider the embedding

$$\dot{B}^{-2}_{n,r} \hookrightarrow \dot{F}^{-2}_{n,q}.$$

Then together with the isomorphism $\dot{B}^{-1}_{\infty,\infty} \cong \dot{F}^{-1}_{\infty,\infty}$, we see that there holds Theorem 6.5 with Besov spaces replaced by Triebel-Lizorkin ones.

Remark 6.8. It is still unknown whether or not Theorem 6.5 would hold in the case $n = 2$. Actually, in $\mathbb{R}^2$, even Proposition 6.3 has not been proved for any indices p and q. Indeed, for the well-posedness, it is hard to show the bilinear estimate

$$\|(-\Delta)^{-1}P\nabla\cdot(u\otimes v)\|_{\dot{B}^{-1+\frac{2}{p}}_{p,q}(\mathbb{R}^2)} \le C\|u\|_{\dot{B}^{-1+\frac{2}{p}}_{p,q}(\mathbb{R}^2)}\|v\|_{\dot{B}^{-1+\frac{2}{p}}_{p,q}(\mathbb{R}^2)}.$$

On the other hand, to show the ill-posedness, we require external forces having at least three components (see the proof in the next section).

Proof of Theorem 6.5. By Proposition 6.1 and Proposition 6.3, it suffices to show the following lemma in order to prove Theorem 6.5.

Lemma 6.9. *Let $n \ge 3$. Suppose that D and $\tilde{D}$ are two spaces with $D \hookrightarrow \tilde{D}$ as either* (1) *or* (2) *of Theorem 6.5, and $\eta > 0$ is a constant given in that theorem. Then there exists a sequence $\{f_N\}_{N\in\mathbb{N}}$ of external forces and a constant $C = C(\eta) > 0$ satisfying the following* (i), (ii) *and* (iii):

(i) $\sup_{N\in\mathbb{N}}\|f_N\|_D < \eta$,

(ii) $\|f_N\|_{\tilde{D}} \to 0$ *as* $N \to \infty$,

(iii) $\inf_{N\in\mathbb{N}}\|A_2(f_N)\|_{\dot{B}^{-1}_{\infty,\infty}} = \inf_{N\in\mathbb{N}}\|B(Lf_N, Lf_N)\|_{\dot{B}^{-1}_{\infty,\infty}} > C$.

Proof of Lemma 6.9. We first take $\psi \in \mathcal{S}$ as

$$\operatorname{supp}(\mathcal{F}\psi) = \{\xi\in\mathbb{R}^n; |\xi|\le 1\}, \quad \mathcal{F}\psi(\xi) > 0 \text{ in } \{\xi\in\mathbb{R}^n; |\xi| < 1\}, \tag{6.1}$$

and we define

$$\Psi^{(j)}_m \equiv (-\Delta)\left\{\psi_{x_j}\cos(mx_1)\right\}, \quad j = 2, 3, \ m\in\mathbb{N},$$

where $\psi_{x_j} \equiv \frac{\partial\psi}{\partial x_j}$. Using this function, we construct $\{f_N\}_{N\in\mathbb{N}}$ differently in the case (1) and (2) of Theorem 6.5.

Step 1. The case (1) : $D = \dot{B}^{-2}_{n,1}$, $\tilde{D} = \dot{B}^{-3+\frac{n}{p}}_{p,q}$ *with $n < p \le \infty$ and $1 \le q \le \infty$.* We define a parametrized vector-valued function as

$$g_{\lambda,M} \equiv \lambda\{e_2\Psi^{(3)}_M - e_3\Psi^{(2)}_M\}, \quad \lambda > 0, \ M \ge 100,$$

This function is inspired by a initial data sequence proposed by Bourgain-Pavlović [5]. It is clearly seen that div $g_{\lambda,M} = 0$ and hence $Pg_{\lambda,M} = g_{\lambda,M}$. Therefore, we have

$$\begin{aligned} Lg_{\lambda,M} &= (-\Delta)^{-1}g_{\lambda,M} \\ &= \lambda\cos(Mx_1)\{e_2\psi_{x_3}(x) - e_3\psi_{x_2}(x)\}. \end{aligned}$$

Now let us consider the estimate of $g_{\lambda,M}$. We recall $\{\varphi_j\}_{j\in\mathbb{Z}}$ in the definition of Besov spaces (see (2.3)-(2.4)). Since

$$\mathcal{F}[\psi_{x_j}\cos(Mx_1)](\xi) = -\frac{1}{2}i\xi_j\{\mathcal{F}\psi(\xi - Me_1) + \mathcal{F}\psi(\xi + Me_1)\}, \quad j = 2, 3,$$

we see that there exist at most three indices $j \in \mathbb{Z}$ such that $\varphi_j * Lg_{\lambda,M} \not\equiv 0$. Indeed, such indices must satisfy

$$\{\xi \in \mathbb{R}^n; 2^{j-1} \le |\xi| \le 2^{j+1}\} \cap \{\xi \in \mathbb{R}^n; M - 1 \le |\xi| \le M + 1\} \ne \emptyset,$$

that is,

$$\frac{M-1}{2} \le 2^j \le 2(M+1).$$

Therefore, we obtain the estimates

$$\begin{aligned} \|g_{\lambda,M}\|_D &= \|g_{\lambda,M}\|_{\dot{B}^{-2}_{n,1}} \\ &= \|(-\Delta)^{-1}g_{\lambda,M}\|_{\dot{B}^{0}_{n,1}} \\ &= \sum_{j\in\mathbb{Z}} \|\varphi_j * Lg_{\lambda,M}\|_{L^n} \\ &\le C\lambda \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} \|g_{\lambda,M}\|_{\tilde{D}} &= \|g_{\lambda,M}\|_{\dot{B}^{-3+\frac{n}{p}}_{p,q}} \\ &\le \|(-\Delta)^{-1}g_{\lambda,M}\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \\ &= \sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p})}\|\varphi_j * Lg_{\lambda,M}\|_{L^p} \\ &\le C\lambda M^{-1+\frac{n}{p}} \to 0 \;\; as \;\; M \to \infty \end{aligned} \tag{6.3}$$

for any $M \ge 100$, implied by $-1 + n/p < 0$. Here we have used the Young inequality, the equality

$$\begin{aligned} \|\varphi_j\|_{L^1} &= \|2^{nj}\varphi_0(2^j\cdot)\|_{L^1} \\ &= \|\varphi_0\|_{L^1}, \quad \forall j \in \mathbb{Z}, \end{aligned} \tag{6.4}$$

and the estimate

$$\|Lg_{\lambda,M}\|_{L^p} \le C\|\nabla\psi\|_{L^p}, \quad \forall\lambda > 0,\ \forall M \ge 100,\ 1 \le \forall p \le \infty. \tag{6.5}$$

We next calculate $B(Lg_{\lambda,M}, Lg_{\lambda,M})$. Since $Lg_{\lambda,M}$ has only two non-trivial components, it is seen that

$$\begin{aligned} (Lg_{\lambda,M})\cdot\nabla(Lg_{\lambda,M}) &= (Lg_{\lambda,M})_2\frac{\partial}{\partial x_2}(Lg_{\lambda,M}) + (Lg_{\lambda,M})_3\frac{\partial}{\partial x_3}(Lg_{\lambda,M}) \\ &= \lambda^2\cos^2(Mx_1)\{e_2(\psi_{x_3}\psi_{x_2x_3} - \psi_{x_2}\psi_{x_3^2}) + e_3(-\psi_{x_3}\psi_{x_2^2} + \psi_{x_2}\psi_{x_2x_3})\} \\ &\equiv \frac{1}{2}\lambda^2(e_2\Phi_1 + e_3\Phi_2) + \frac{1}{2}\lambda^2(e_2\Phi_1\cos(2Mx_1) + e_3\Phi_2\cos(2Mx_1)) \\ &\equiv I_1 + I_2. \end{aligned}$$

Here $\psi_{x_2^\alpha x_3^\beta} \equiv \frac{\partial^{(\alpha+\beta)}}{\partial x_2^\alpha x_3^\beta}\psi$ and

$$\Phi_1 \equiv \psi_{x_3}\psi_{x_2 x_3} - \psi_{x_2}\psi_{x_3^2}, \quad \Phi_2 \equiv -\psi_{x_3}\psi_{x_2^2} + \psi_{x_2}\psi_{x_2 x_3}. \tag{6.6}$$

We note here that $B(Lg_{\lambda,M}, Lg_{\lambda,M}) = -(-\Delta)^{-1}P(I_1+I_2)$ belongs to PL^n (in particular, to $P\dot{B}^{-1}_{\infty,\infty}$) by Proposition 6.3. We can show that $PI_1 \not\equiv 0$ (see Appendix C), and that $(-\Delta)^{-1}PI_1$ is not constant. Furthermore, since

$$\begin{aligned} \operatorname{supp}(\mathcal{F}I_1) &\subset \operatorname{supp}(\mathcal{F}\psi * \mathcal{F}\psi) \\ &\subset \{\xi \in \mathbb{R}^n; |\xi| \le 2\}, \end{aligned} \tag{6.7}$$

we have

$$|((-\Delta)^{-1}PI_1)(x)| = \left| \int_{\mathbb{R}^n} \frac{1}{|\xi|^2} \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right)_{1 \le j,k \le n} \mathcal{F}I_1(\xi) e^{-ix\cdot\xi} d\xi \right| \le C, \quad \forall x \in \mathbb{R}^n$$

for some constant $C > 0$ (where δ_{jk} denotes the Kronecker delta), which yields that $(-\Delta)^{-1}PI_1$ does not belong to the polynomial space $\mathcal{P}$. From this fact and (6.7), we see that

$$\begin{aligned} \|(-\Delta)^{-1}PI_1\|_{\dot{B}^{-1}_{\infty,\infty}} &= \sup_{j\in\mathbb{Z}, j\le 2} 2^{-j} \|\varphi_j * (-\Delta)^{-1}PI_1\|_{L^\infty} \\ &\ge C\lambda^2 > 0 \end{aligned}$$

for some constant $C > 0$. On the other hand, it is seen that

$$\begin{aligned} \operatorname{supp}(\mathcal{F}I_2) &\subset \operatorname{supp}((\mathcal{F}\psi * \mathcal{F}\psi)(\cdot \pm 2Me_1)) \\ &\subset \{\xi \in \mathbb{R}^n; 2M - 2 \le |\xi| \le 2M + 2\}, \end{aligned}$$

which yields $\varphi_j * ((-\Delta)^{-1}PI_2) \equiv 0$ for any $j \ge 2$. Therefore, we obtain the estimate that

$$\begin{aligned} \|B(Lg_{\lambda,M}, Lg_{\lambda,M})\|_{B^{-1}_{\infty,\infty}} &= \sup_{j\in\mathbb{Z}} 2^{-j} \|\varphi_j * (-\Delta)^{-1}P(I_1+I_2)\|_{L^\infty} \\ &\ge \sup_{j\in\mathbb{Z}, j\le 2} 2^{-j} \|\varphi_j * (-\Delta)^{-1}PI_1\|_{L^\infty} \\ &\ge C\lambda^2 \end{aligned} \tag{6.8}$$

for any $M \ge 100$.

Now for given $\eta > 0$, we can fix $\lambda = \lambda_0$ so that

$$\sup_{M\ge 100} \|g_{\lambda_0,M}\|_D < \eta$$

from (6.2). In addition, from (6.3) and (6.8), we see that a sequence $\{f_N\}_{N\in\mathbb{N}}$ defined by

$$f_N \equiv g_{\lambda_0, N+100}, \quad N = 1, 2, 3, \ldots$$

satisfies (i), (ii), and (iii) of Lemma 6.9. This proves Lemma 6.9 in the case (1) of Theorem 6.5.

Step 2. The case (2) : $D = \dot{B}^{-2}_{n,2}$, $\tilde{D} = \dot{B}^{-2}_{n,q}$ *with* $2 < q \le \infty$. We define another parametrized vector-valued function as

$$h_{\lambda,M} \equiv \frac{\lambda}{\sqrt{\Gamma(M)}} \sum_{k=10}^{M} k^{-\frac{1}{2}} \{ e_2 \Psi^{(3)}_{2^{k^2}} - e_3 \Psi^{(2)}_{2^{k^2}} \}, \quad \lambda > 0, \ M \ge 100,$$

where

$$\Gamma(M) \equiv \sum_{k=10}^{M} k^{-1}.$$

This function is inspired by a initial data sequence proposed by Yoneda [27]. As similar to $g_{\lambda,M}$, we see that div $h_{\lambda,M} = 0$ and

$$\begin{aligned} Lh_{\lambda,M} &= (-\Delta)^{-1} h_{\lambda,M} \\ &= \frac{\lambda}{\sqrt{\Gamma(M)}} \sum_{k=10}^{M} k^{-\frac{1}{2}} \cos(2^{k^2} x_1) \{ e_2 \psi_{x_3}(x) - e_3 \psi_{x_2}(x) \} \end{aligned}$$

Let us consider the estimate of $h_{\lambda,M}$. By a similar way to Step 1, we see that for each k, there exist at most three indices $j \in \mathbb{Z}$ such that $\varphi_j * (\psi_{x_l} \cos(2^{k^2} x_1)) \not\equiv 0$ $(l = 2, 3)$, which must satisfy

$$\frac{2^{k^2} - 1}{2} \le 2^j \le 2(2^{k^2} + 1).$$

Moreover, the set $\{2^{k^2}\}_{k \ge 10}$ is so discrete that we see

$$\{ j \in \mathbb{Z}; \varphi_j * (\psi_{x_l} \cos(2^{k_1^2} x_1)) \not\equiv 0 \} \cap \{ j \in \mathbb{Z}; \varphi_j * (\psi_{x_l} \cos(2^{k_2^2} x_1)) \not\equiv 0 \} = \emptyset$$

for any $k_1, k_2 \ge 10$ with $k_1 \ne k_2$. Hence we obtain the estimate

$$\begin{aligned} \|h_{\lambda,M}\|_{\dot{B}^{-2}_{n,q}} &= \|(-\Delta)^{-1} h_{\lambda,M}\|_{\dot{B}^{0}_{n,q}} \\ &= \left\{ \sum_{j \in \mathbb{Z}} \|\varphi_j * (-\Delta)^{-1} h_{\lambda,M}\|^q_{L^n} \right\}^{\frac{1}{q}} \\ &\le \frac{C\lambda}{\sqrt{\Gamma(M)}} \left\{ \sum_{k=10}^{M} k^{-\frac{q}{2}} \right\}^{\frac{1}{q}} \\ &\le \begin{cases} C\lambda, & q = 2, \\ \frac{C\lambda}{\sqrt{\Gamma(M)}}, & 2 < q \le \infty. \end{cases} \end{aligned} \tag{6.9}$$

Here we have used the Young inequality, (6.4), and (6.5), and should notice that

$$\lim_{M \to \infty} \sum_{k=10}^{M} k^{-\frac{q}{2}} < \infty, \quad \text{if } 2 < q \le \infty.$$

Since $\Gamma(M) \to \infty$ as $M \to \infty$, we see from (6.9) that

$$\|h_{\lambda,M}\|_{\dot{B}^{-2}_{n,q}} \to 0 \quad \text{as } M \to \infty, \quad \text{if } 2 < q \leq \infty. \tag{6.10}$$

We next calculate $B(Lh_{\lambda,M}, Lh_{\lambda,M})$. It is seen that

$$\begin{aligned}
&(Lh_{\lambda,M}) \cdot \nabla(Lh_{\lambda,M}) \\
&= (Lh_{\lambda,M})_2 \frac{\partial}{\partial x_2}(Lh_{\lambda,M}) + (Lh_{\lambda,M})_3 \frac{\partial}{\partial x_3}(Lh_{\lambda,M}) \\
&= \frac{\lambda^2}{\Gamma(M)}(e_2\Phi_1 + e_3\Phi_2) \sum_{k,l=10}^{N} k^{-\frac{1}{2}} l^{-\frac{1}{2}} \cos(2^{k^2}x_1)\cos(2^{l^2}x_1) \\
&= \frac{\lambda^2}{\Gamma(M)}(e_2\Phi_1 + e_3\Phi_2) \left\{ \sum_{k=10}^{M} k^{-1}\cos^2(2^{k^2}x_1) + \sum_{\substack{10 \leq k,l \leq M \\ k \neq l}} k^{-\frac{1}{2}} l^{-\frac{1}{2}} \cos(2^{k^2}x_1)\cos(2^{l^2}x_1) \right\} \\
&= \frac{\lambda^2}{2}(e_2\Phi_1 + e_3\Phi_2) \\
&\quad + \frac{\lambda^2}{2\Gamma(M)}(e_2\Phi_1 + e_3\Phi_2) \sum_{k=10}^{M} k^{-1}\cos(2^{k^2+1}x_1) \\
&\quad + \frac{\lambda^2}{2\Gamma(M)}(e_2\Phi_1 + e_3\Phi_2) \left\{ \sum_{\substack{10 \leq k,l \leq M \\ k \neq l}} k^{-\frac{1}{2}} l^{-\frac{1}{2}} \cos((2^{k^2} + 2^{l^2})x_1) + \cos((2^{k^2} - 2^{l^2})x_1) \right\} \\
&\equiv J_1 + J_2 + J_3,
\end{aligned}$$

where Φ_1 and Φ_2 are as (6.6). Since the above coefficients 2^{k^2+1}, $2^{k^2} + 2^{l^2}$ and $|2^{k^2} - 2^{l^2}|$ are large enough, we see

$$\varphi_j * (-\Delta)^{-1}P(J_1 + J_2) \equiv \varphi_j * (-\Delta)^{-1}PJ_1, \quad \forall j \leq 2.$$

Hence, by a similar way to the argument on I_1 and I_2 in Step 1, we obtain

$$\|B(Lh_{\lambda,M}, Lh_{\lambda,M})\|_{\dot{B}^{-1}_{\infty,\infty}} \geq \|(-\Delta)^{-1}PJ_1\|_{\dot{B}^{-1}_{\infty,\infty}} \geq C\lambda^2 > 0. \tag{6.11}$$

Now for given $\eta > 0$, we can fix $\lambda = \lambda_0$ so that

$$\sup_{M \geq 100} \|h_{\lambda_0,M}\|_{\dot{B}^0_{n,2}} < \eta$$

from (6.9). In addition, from (6.10) and (6.11), we see that a sequence $\{f_N\}_{N\in\mathbb{N}}$ defined by

$$f_N \equiv h_{\lambda_0,N+100}, \quad N = 1, 2, 3, \ldots$$

satisfies

$$\sup_{N\in\mathbb{N}} \|f_N\|_{\dot{B}^0_{n,2}} < \eta, \quad \lim_{N\to\infty} \|f_N\|_{\dot{B}^0_{n,q}} = 0 \quad \text{if } 2 < q \le \infty,$$

and

$$\inf_{N\in\mathbb{N}} \|B(Lf_N, Lf_N)\|_{\dot{B}^{-1}_{\infty,\infty}} \ge C\lambda_0^2.$$

This proves Lemma 6.9 in the case (2) of Theorem 6.5, and hence the proof of Lemma 6.9 is completed.

By the foregoing, the whole proof of Theorem 6.5 has been completed. □

Chapter 7

Counter-example for the product estimate

In this chapter, we treat some by-products produced by our studies in Chapter 2-6.

Here we reconsider the factor causing the ill-posedness of (rSNS). Let $D = \dot{B}_{p,q}^{-3+\frac{n}{p}}$ and $S = P\dot{B}_{p,q}^{-1+\frac{n}{p}}$ again. As seen in Chapter 2, to see the well-posedness, it suffices to show (2.17) and (2.18) with

$$Lf \equiv (-\Delta)^{-1}Pf,\ \ B(u,v) \equiv -(-\Delta)^{-1}P(u \cdot \nabla v).$$

Among these, the estimate (2.17) of $L : D \to S$ always holds for any $1 \le p, q \le \infty$ by Proposition 2.1. On the other hand, in order to show (2.18), we should use the paraproduct estimate in Proposition 3.7. In fact, for the well-posedness, the restriction of p, $1 \le p < n$ stems from that of s, $s > 0$ in Proposition 3.7 (we should note here that $-1 + n/p > 0$ when $1 \le p < n$). On the other hand, we have showed the discontinuity of the solution map $f \in \dot{B}_{p,q}^{-3+\frac{n}{p}} \mapsto u \in P\dot{B}_{p,q}^{-1+\frac{n}{p}}$ of (SNS) when $p = n$, $2 < q \le \infty$ and $n < p \le \infty$, $1 \le q \le \infty$ in the last chapter, which implies that (2.18) does not necessary hold in such conditions of p and q. Hence, it seems natural to expect that Proposition 3.7 should fail necessarily for $s \le 0$.

Our result on counter-examples of the paraproduct estimate is as follows:

Theorem 7.1. (Tsurumi [22]) *Let $n \ge 1$, $s \in \mathbb{R}$, and $1 \le p, p_1, p_2, p_3, p_4 \le \infty$.*

(1) *Suppose that the exponents $s_1, s_2, s_3, s_4 \in \mathbb{R}$ satisfy $s_1 + s_2 < 0$ and $s_3 + s_4 < 0$. Then for any $M > 0$, there exist functions $f, g \in \mathcal{S}_0$ such that $fg \in \mathcal{S}_0(\mathbb{R}^n)$ and*

$$\|fg\|_{\dot{B}^s_{p,\infty}} \ge M \left(\|f\|_{\dot{B}^{s_1}_{p_1,1}} \|g\|_{\dot{B}^{s_2}_{p_2,1}} + \|f\|_{\dot{B}^{s_3}_{p_3,1}} \|g\|_{\dot{B}^{s_4}_{p_4,1}} \right). \tag{7.1}$$

(2) *Suppose that the exponents $1 \le q_1, q_2, q_3, q_4 \le \infty$ satisfy*

$$2 \le q_1, q_2, q_3, q_4 \le \infty, \quad \max\{q_1, q_2\} > 2, \quad \max\{q_3, q_4\} > 2.$$

Then for any $M > 0$, there exist functions $f, g \in \mathcal{S}_0$ such that $fg \in \mathcal{S}_0(\mathbb{R}^n)$ and

$$\|fg\|_{\dot{B}^s_{p,\infty}} \ge M \left(\|f\|_{\dot{B}^{0}_{p_1,q_1}} \|g\|_{\dot{B}^{0}_{p_2,q_2}} + \|f\|_{\dot{B}^{0}_{p_3,q_3}} \|g\|_{\dot{B}^{0}_{p_4,q_4}} \right). \tag{7.2}$$

As can be seen in the assumption of Theorem 7.1, we can choose $s \in \mathbb{R}$ and $1 \leq p, p_1, p_2, p_3, p_4 \leq \infty$ independently. In particular, those indices do not have to satisfy $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$.

Corollary 7.2. (1) *The inequality* (3.2) *is invalid in the case* (i) *or* (ii) *as follows.*

(i) $s < 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $1 \leq p, q \leq \infty$,

(ii) $s = \alpha = \beta = 0$, $1 \leq p \leq \infty$, $2 \leq q \leq \infty$.

(2) *The inequality* (3.3) *is invalid in the case* (i) *or* (ii) *as follows.*

(i) $s < 0$, $1 \leq p, q \leq \infty$,

(ii) $s = 0$, $2 \leq p < \infty$, $2 < q \leq \infty$.

Proof of Corollary 7.2 from Theorem 7.1. (1) In Theorem 7.1, take

$$s_1 = s + \alpha, \ s_2 = -\alpha, \ s_3 = -\beta, \ s_4 = s + \beta,$$

and take p_1, p_2, p_3, p_4 as $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. Then by using the embedding in Proposition 2.1, we have the claim in the case (i) from (1) of Theorem 7.1, and in the case (ii) from (2) of Theorem 7.1.

(2) In (1) of Theorem 7.1, take

$$s_1 = s_4 = s, \quad s_2 = s_3 = 0,$$

and p_1, p_2, p_3, p_4 as $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. Then by using the embedding in Proposition 2.1, we have the claim in the case (i) from (1) of Theorem 7.1. To see (2) of Corollary 7.2 in the case (ii), we should let

$$q_1 = q_4 = q > 2, \quad q_2 = q_3 = 2,$$

and let p_1, p_2, p_3, p_4 as above in Theorem 7.1 (2). Since $p_2, p_3 \geq p \geq 2$, it is seen from Proposition 2.1 that

$$\dot{B}^0_{p_2,2} \hookrightarrow L^{p_2}, \quad \dot{B}^0_{p_3,2} \hookrightarrow L^{p_3},$$

which yields (2) of Corollary 7.2 in the case (ii). □

Remark 6.3. Our result can be applied to the bilinear estimates in homogeneous Triebel-Lizorkin spaces. In fact, under the condition $s, \alpha, \beta > 0$, Kozono-Shimada [13] showed the estimates

$$\|fg\|_{\dot{F}^s_{p,q}} \leq C \left(\|f\|_{\dot{F}^{s+\alpha}_{p_1,q}} \|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{p_3,\infty}} \|g\|_{\dot{F}^{s+\beta}_{p_4,q}} \right) \tag{7.3}$$

when $1 < p, q < \infty$, $1 < p_1, p_4 < \infty$, and $1 < p_2, p_3 \le \infty$ so that $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. Moreover, Iwabuchi-Nakamura [9] also showed (7.3) in the case $p = \infty$.

By Proposition 2.1 (4), Theorem 7.1 (1) gives counter examples of (7.3) when $s < 0$ and $\alpha, \beta \in \mathbb{R}$. Moreover, we see that Theorem 7.1 (2) also gives counter examples of (7.3) when $s = \alpha = 0$ provided $2 \le q \le p \le \infty$, which is implied by $\dot{B}^s_{p,q} \hookrightarrow \dot{F}^s_{p,q}$ if $q \le p$. In addition, by the isomorphism

$$\dot{H}^{s,p} \simeq \dot{F}^s_{p,2}, \quad s \in \mathbb{R},\ 1 < p < \infty,$$

our negative result can be applied to bilinear estimates in the homogeneous Sobolev spaces $\dot{H}^{s,p} = \{f \in \mathcal{S}'/\mathcal{P};\ \|f\|_{\dot{H}^{s,p}} \equiv \|(-\Delta)^{\frac{s}{2}} f\|_{L^p} < \infty\}$.

Proof of Theorem 7.1. First of all, we choose $\Phi_1, \Phi_2 \in \mathcal{S}_0$ so that

$$\mathrm{supp}(\mathcal{F}\Phi_1) = \{\xi \in \mathbb{R}^n; 1 \le |\xi| \le 2\}, \quad \mathcal{F}\Phi_1 > 0 \ \text{ in } \{\xi \in \mathbb{R}^n; 1 < |\xi| < 2\},$$

$$\mathrm{supp}(\mathcal{F}\Phi_2) = \{\xi \in \mathbb{R}^n; 3 \le |\xi| \le 4\}, \quad \mathcal{F}\Phi_2 > 0 \ \text{ in } \{\xi \in \mathbb{R}^n; 3 < |\xi| < 4\}.$$

We define

$$\Phi_3(x) \equiv \Phi_1(x)\Phi_2(x).$$

Since

$$\begin{aligned} \mathrm{supp}(\mathcal{F}\Phi_1(\eta - \cdot)) &= \{\xi \in \mathbb{R}^n; 1 \le |\eta - \xi| \le 2\} \\ &\subset \{\xi \in \mathbb{R}^n; |\eta| - 2 \le |\xi| \le |\eta| + 2\}, \end{aligned}$$

for each $\eta \in \mathbb{R}^n$, it is seen that $\mathcal{F}\Phi_1(\eta - \xi)\mathcal{F}\Phi_2(\xi) = 0$ for all $\xi \in \mathbb{R}^n$ if η satisfies

$$\{\xi \in \mathbb{R}^n; |\eta| - 2 \le |\xi| \le |\eta| + 2\} \cap \{\xi \in \mathbb{R}^n; 3 \le |\xi| \le 4\} = \emptyset,$$

that is, $0 \le |\eta| < 1$ or $|\eta| > 6$. Hence we see that $\Phi_3 \in \mathcal{S}_0$ and

$$\begin{aligned} \mathrm{supp}(\mathcal{F}\Phi_3) &= \mathrm{supp}(\mathcal{F}\Phi_1 * \mathcal{F}\Phi_2) \\ &\subset \{\xi \in \mathbb{R}^n; 1 \le |\xi| \le 6\}. \end{aligned}$$

Let $s \in \mathbb{R}$, and $1 \le p, p_1, p_2, p_3, p_4 \le \infty$. We prove Theorem 7.1 by two steps.

Step 1. (1) *of Theorem 7.1.* We choose parametrized functions f_r, g_r as

$$f_r(x) \equiv \Phi_1(x)\cos(rx_1), \quad g_r(x) \equiv \Phi_2(x)\cos(rx_1)$$

for $r > 100$. We should note again here that the Fourier transform of $c_v(x) \equiv \cos(v \cdot x)$ $(v \in \mathbb{R}^n \backslash \{0\})$ can be written as

$$\mathcal{F}c_v(\xi) = \frac{1}{2}\left\{\delta(\xi - v) + \delta(\xi + v)\right\},$$

where δ denotes the Dirac measure at the origin. Hence, we have

$$\begin{aligned} \mathcal{F}f_r(\xi) &= (\mathcal{F}\Phi_1 * \mathcal{F}c_{re_1})(\xi) \\ &= \frac{1}{2}\left\{\mathcal{F}\Phi_1(\xi - re_1) + \mathcal{F}\Phi_1(\xi + re_1)\right\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}g_r(\xi) &= (\mathcal{F}\Phi_2 * \mathcal{F}c_{re_1})(\xi) \\ &= \frac{1}{2}\left\{\mathcal{F}\Phi_2(\xi - re_1) + \mathcal{F}\Phi_2(\xi + re_1)\right\}, \end{aligned}$$

where $e_1 \equiv (1, 0, \ldots, 0)$. Since

$$\mathrm{supp}(\mathcal{F}\Phi_1(\cdot \pm re_1)) \subset \{\xi \in \mathbb{R}^n; r - 2 \le |\xi| \le r + 2\},$$

there are at most 3 indices $j \in \mathbb{Z}$ such that $\varphi_j * f_r \not\equiv 0$, where φ_j are defined in (2.4). Such indices should satisfy

$$\left\{\xi \in \mathbb{R}^n; 2^{j-1} \le |\xi| \le 2^{j+1}\right\} \cap \{\xi \in \mathbb{R}^n; r - 2 \le |\xi| \le r + 2\} \neq \emptyset$$

(recall also (2.3). Roughly, there should be $2^j \sim r$). Moreover, by the Hausdorff-Young inequality, it holds that

$$\begin{aligned} \|\varphi_j * f_r\|_{L^p} &\le \|\varphi_j\|_{L^1}\|f_r\|_{L^p} \\ &\le \|\varphi_0\|_{L^1}\|\Phi_1\|_{L^p} \end{aligned}$$

for every $1 \le p \le \infty$ and $j \in \mathbb{Z}$. Here we have used the fact that $\|\varphi_j\|_{L^1} = \|\varphi_0\|_{L^1}$ for every $j \in \mathbb{Z}$, which is implied by $\varphi_j(x) = 2^{nj}\varphi_0(2^j x)$. Therefore, we have

$$\begin{aligned} \|f_r\|_{\dot{B}^{s_i}_{p_i,1}} &= \sum_{j\in\mathbb{Z}} 2^{js_i}\|\varphi_j * f_r\|_{L^{p_i}} \\ &\le Cr^{s_i}, \quad i = 1, 3 \end{aligned}$$

with some constant $C = C(n, s, p_i)$. By a similar way, we can also see that

$$\|g_r\|_{\dot{B}^{s_i}_{p_i,1}} \le Cr^{s_i}, \quad i = 2, 4.$$

Hence, we obtain the estimate

$$\|f_r\|_{\dot{B}^{s_1}_{p_1,1}}\|g_r\|_{\dot{B}^{s_2}_{p_2,1}} + \|f_r\|_{\dot{B}^{s_3}_{p_3,1}}\|g_r\|_{\dot{B}^{s_4}_{p_4,1}} \le C(r^{s_1+s_2} + r^{s_3+s_4}). \tag{7.4}$$

On the other hand, we have

$$\begin{aligned} f_r(x)g_r(x) &= \Phi_3(x)\cos^2(rx_1) \\ &= \frac{1}{2}\Phi_3(x) + \frac{1}{2}\Phi_3(x)\cos(2rx_1) \\ &\equiv \frac{1}{2}\Phi_3(x) + \frac{1}{2}R(x), \end{aligned}$$

where $R(x) \equiv \Phi_3(x)\cos(2rx_1)$. From the similar argument as above, it is seen that

$$\operatorname{supp}(\mathcal{F}R) \subset \{\xi \in \mathbb{R}^n; 2r-6 \le |\xi| \le 2r+6\},$$

which implies $\varphi_0 * R \equiv 0$ for $r > 100$. Therefore, we have

$$\begin{aligned} \|f_r g_r\|_{\dot{B}^s_{p,\infty}} &= \sup_{j\in\mathbb{Z}} 2^{sj}\|\varphi_j * f_r g_r\|_{L^p} \\ &\ge \frac{1}{2}\|\varphi_0 * \Phi_3\|_{L^p}. \end{aligned} \tag{7.5}$$

We should note here that $\|\varphi_0 * \Phi_3\|_{L^p} > 0$. Indeed, let $\xi_0 \equiv \frac{3}{2}e_1$. Then $\mathcal{F}\Phi_1(\xi_0 - \eta)\mathcal{F}\Phi_2(\eta) > 0$ in some neighborhood of $\eta = \frac{13}{4}e_1$, which yields $\mathcal{F}\Phi_3(\xi_0) > 0$. Therefore, $\phi(\xi)\mathcal{F}\Phi_3(\xi) > 0$ in some neighborhood of $\xi = \xi_0$, which yields $\varphi_0 * \Phi_3 \not\equiv 0$.

Since $s_1 + s_2 < 0$ and $s_3 + s_4 < 0$, it is seen from (7.4) and (7.5) that for any $M > 0$, f_r and g_r satisfy (7.1) if we take r large enough so that the right hand side becomes smaller than $\frac{1}{2}\|\varphi_0 * \Phi_3\|_{L^p}$. This proves (1) of Theorem 7.1.

Step 2. (2) *of Theorem 7.1.* We choose parametrized functions $\tilde{f}_r$, $\tilde{g}_r$ as

$$\tilde{f}_r(x) \equiv \Phi_1(x)K_r(x), \quad \tilde{g}_r(x) \equiv \Phi_2(x)K_r(x)$$

for $r > 100$, where

$$K_r(x) \equiv \frac{1}{\sqrt{\Gamma(r)}}\sum_{l=10}^{r} l^{-\frac{1}{2}}\cos(2^{l^2}x_1), \quad \Gamma(r) = \sum_{l=10}^{r} l^{-1}.$$

Let

$$\Psi_l(x) \equiv \Phi_1(x)\cos(2^{l^2}x_1).$$

By a similar argument to that in Step 1, there are at most 3 indices $j \in \mathbb{Z}$ such that $\varphi_j * \Psi_l \not\equiv 0$ for each l, which satisfy $j \sim 2^{l^2}$. Moreover, since

$$\operatorname{supp}(\mathcal{F}\Psi_l) = \{\xi \in \mathbb{R}^n; 2^{l^2} - 2 \le |\xi| \le 2^{l^2} + 2\},$$

it holds that

$$\{j \in \mathbb{Z}; \varphi_j * \Psi_l \not\equiv 0\} \cap \{j \in \mathbb{Z}; \varphi_j * \Psi_k \not\equiv 0\} = \emptyset, \quad \text{if } l \ne k.$$

Hence, we have

$$\begin{aligned} \|\tilde{f}_r\|_{\dot{B}^0_{p_i,q_i}} &= \frac{1}{\sqrt{\Gamma(r)}}\left\{\sum_{j\in\mathbb{Z}}\left\|\sum_{l=10}^{r} l^{-\frac{1}{2}}(\varphi_j * \Psi_l)\right\|_{L^{p_i}}^{q_i}\right\}^{\frac{1}{q_i}} \\ &\le C\frac{1}{\sqrt{\Gamma(r)}}\left\{\sum_{l=10}^{r} l^{-\frac{q_i}{2}}\right\}^{\frac{1}{q_i}} \\ &\le \begin{cases} C, & \text{if } q_i = 2, \\ C\frac{1}{\sqrt{\Gamma(r)}}, & \text{if } 2 < q_i \le \infty, \end{cases} \qquad i = 1, 3, \end{aligned}$$

which is implied by

$$\|\varphi_j * \Psi_l\|_{L^{p_i}} \le \|\varphi_0\|_{L^1}\|\Phi_1\|_{L^{p_i}}$$

for all $j \in \mathbb{Z}$ and $10 \le l \le r$. In a similar way, we also have

$$\|\tilde{g}_r\|_{\dot{B}^0_{p_i,q_i}} \le \begin{cases} C, & \text{if } q_i = 2, \\ C\frac{1}{\sqrt{\Gamma(r)}}, & \text{if } 2 < q_i \le \infty, \end{cases} \qquad i = 2, 4.$$

Since $\max\{q_1, q_2\} > 2$, $\max\{q_3, q_4\} > 2$, it is seen that

$$\|\tilde{f}_r\|_{\dot{B}^0_{p_1,q_1}}\|\tilde{g}_r\|_{\dot{B}^0_{p_2,q_2}} + \|\tilde{f}_r\|_{\dot{B}^0_{p_3,q_3}}\|\tilde{g}_r\|_{\dot{B}^0_{p_4,q_4}} \le C\frac{1}{\sqrt{\Gamma(r)}}. \tag{7.6}$$

We should note here that

$$\lim_{r\to\infty} \frac{1}{\sqrt{\Gamma(r)}} = 0.$$

On the other hand, we have

$$\begin{aligned} \tilde{f}_r(x)\tilde{g}_r(x) &= \Phi_3(x)K_r^2(x) \\ &= \frac{1}{\Gamma(r)}\Phi_3(x)\left\{\sum_{l=10}^{r} l^{-1}\cos^2(2^{l^2}x_1) + \sum_{\substack{10\le l,k\le r \\ l\ne k}} l^{-\frac{1}{2}}k^{-\frac{1}{2}}\cos(2^{l^2}x_1)\cos(2^{k^2}x_1)\right\} \\ &= \frac{1}{2}\Phi_3(x) \\ &\quad +\frac{1}{2}\Phi_3(x)\sum_{l=10}^{r} l^{-1}\cos(2^{l^2+1}x_1) \\ &\quad +\frac{1}{2}\Phi_3(x)\sum_{\substack{10\le l,k\le r \\ l\ne k}} l^{-\frac{1}{2}}k^{-\frac{1}{2}}\left\{\cos((2^{l^2}+2^{k^2})x_1) + \cos((2^{l^2}-2^{k^2})x_1)\right\} \\ &\equiv \frac{1}{2}\Phi_3(x) + \tilde{R}(x). \end{aligned}$$

Since $2^{l^2+1}, 2^{l^2}+2^{k^2}$ and $|2^{l^2}-2^{k^2}|$ are large enough, there holds $\varphi_0 * \tilde{R}(x) \equiv 0$. Hence, we have

$$\begin{aligned} \|\tilde{f}_r\tilde{g}_r\|_{\dot{B}^s_{p,\infty}} &= \sup_{j\in\mathbb{Z}} 2^{sj}\|\varphi_j * \tilde{f}_r\tilde{g}_r\|_{L^p} \\ &\ge \frac{1}{2}\|\varphi_0 * \Phi_3\|_{L^p} > 0. \end{aligned} \tag{7.7}$$

Therefore, it is seen from (7.6) and (7.7) that for any $M > 0$, $\tilde{f}_r$ and $\tilde{g}_r$ satisfy (7.2) if we take r large enough. This proves (2) of Theorem 7.1. □

Chapter 8

Appendix

For self-containment of this thesis, we prove here some propositions which we have admitted in the main parts without proves for simplicity.

Appendix A

Here we state the proof of Proposition 3.7, the paraproduct estimates in homogeneous Besov spaces, according to Kaneko-Kozono-Shimizu [11].

Proof. (1) By method by Bony [4], a product of functions in $\mathcal{S}'/\mathcal{P}$ can be decomposed as

$$\begin{aligned} f \cdot g &= \sum_{k\in\mathbb{Z}} (\varphi_k * f)(P_k g) + \sum_{k\in\mathbb{Z}} (P_k f)(\varphi_k * g) + \sum_{k\in\mathbb{Z}} \sum_{|l-k|\le 2} (\varphi_k * f)(\varphi_l * g) \\ &\equiv h_1 + h_2 + h_3, \end{aligned} \tag{A.1}$$

where $P_k g \equiv \sum_{l=-\infty}^{k-3} \varphi_l * g$. First, we consider the case $1 \le q < \infty$. From (2.3), (2.4), and

$$\operatorname{supp} \mathcal{F}[(\varphi_k * f)(P_k g)] \subset \{\xi \in \mathbb{R}^n;\ 2^{k-2} \le |\xi| \le 2^{k+2}\},$$

we have an equality that

$$\begin{aligned} \|h_1\|_{\dot{B}^s_{p,q}} &= \left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj} \|\varphi_j * h_1\|_{L^{p_1}}\right)^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj} \left\| \sum_{k\in\mathbb{Z}} \varphi_j * ((\varphi_k * f)(P_k g)) \right\|_{L^{p_1}} \right)^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj} \left\| \sum_{|k-j|\le 2} \varphi_j * ((\varphi_k * f)(P_k g)) \right\|_{L^{p}} \right)^q \right\}^{\frac{1}{q}}. \end{aligned}$$

From (3.1), the Young inequality, and the Hölder inequality we see that

$$\|\varphi_j * ((\varphi_k * f)(P_k g))\|_{L^p} \le \|\mathcal{F}^{-1}\phi\|_{L^1} \|\varphi_k * f\|_{L^{p_1}} \|P_k g\|_{L^{p_2}}$$

for every $j, k \in \mathbb{Z}$. Therefore, by using the Minkowski inequality, we obtain that

$$\begin{aligned}
\|h_1\|_{\dot{B}^s_{p,q}} &\le C \left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj} \sum_{|k-j|\le 2} \|\varphi_k * f\|_{L^{p_1}} \|P_k g\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\
&= C \left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj} \sum_{|l|\le 2} \|\varphi_{j+l} * f\|_{L^{p_1}} \|P_{j+l} g\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\
&\le C \sum_{|l|\le 2} \left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj} \|\varphi_{j+l} * f\|_{L^{p_1}} \|P_{j+l} g\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\
&= C \sum_{|l|\le 2} \left\{ \sum_{m\in\mathbb{Z}} \left(2^{sm} 2^{-sl} \|\varphi_m * f\|_{L^{p_1}} \|P_m g\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\
&= C \sum_{|l|\le 2} 2^{-sl} \left\{ \sum_{m\in\mathbb{Z}} \left(2^{(s+\alpha)m} \|\varphi_m * f\|_{L^{p_1}} 2^{-\alpha m} \left\| \sum_{k=-\infty}^{m-3} \varphi_k * g \right\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\
&\le C \left\{ \sum_{m\in\mathbb{Z}} \left(2^{(s+\alpha)m} \|\varphi_m * f\|_{L^{p_1}} \sum_{k=-\infty}^{m-3} 2^{-\alpha k} \|\varphi_k * g\|_{L^{p_2}} 2^{-\alpha(m-k)} \right)^q \right\}^{\frac{1}{q}} \\
&\le C \sup_{k\in\mathbb{Z}} 2^{-\alpha k} \|\varphi_k * g\|_{L^{p_2}} \left\{ \sum_{m\in\mathbb{Z}} \left(2^{(s+\alpha)m} \|\varphi_m * f\|_{L^{p_1}} \sum_{l=3}^{\infty} 2^{-\alpha l} \right)^q \right\}^{\frac{1}{q}} \\
&\le C \|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}} \|f\|_{\dot{B}^{s+\alpha}_{p_1,q}}, \qquad \text{(A.2)}
\end{aligned}$$

where $C = C(n, p, p_1, p_2, q, s, \alpha)$. Here we should note that the final estimate in (A.2) is valid since $\alpha > 0$. For the case $q = \infty$, we see by a similar calculation to (A.2) that

$$\begin{aligned}
\|h_1\|_{\dot{B}^s_{p,\infty}} &\le C \sup_{k\in\mathbb{Z}} 2^{-\alpha k} \|\varphi_k * g\|_{L^{p_2}} \sup_{m\in\mathbb{Z}} 2^{(s+\alpha)m} \|\varphi_m * f\|_{L^{p_1}} \sum_{l=3}^{\infty} 2^{-\alpha l} \\
&\le C \|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}} \|f\|_{\dot{B}^{s+\alpha}_{p_1,\infty}},
\end{aligned}$$

with a constant $C = C(n, p, p_1, p_2, s, \alpha)$. Hence we have

$$\|h_1\|_{\dot{B}^s_{p,q}} \le C \|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}} \|f\|_{\dot{B}^{s+\alpha}_{p_1,q}} \qquad \text{(A.3)}$$

for every $1 \le q \le \infty$ with $C = C(n, p, p_1, p_2, q, s, \alpha)$. Moreover, by the symmetry with regard to f and g, we also have

$$\|h_2\|_{\dot{B}^s_{p,q}} \le C \|f\|_{\dot{B}^{-\beta}_{p_3,\infty}} \|g\|_{\dot{B}^{s+\beta}_{p_4,q}} \qquad \text{(A.4)}$$

for every $1 \le q \le \infty$ with $C = C(n, p, p_3, p_4, q, s, \beta)$.

Next, we estimate h_3 in $\dot{B}^s_{p,q}$. First we consider the case $1 \le q < \infty$. Since

$$\text{supp}\ \mathcal{F}((\varphi_k * f)(\varphi_l * g)) \subset \left\{\xi \in \mathbb{R}^n; |\xi| \le 2^{\max\{k,l\}+2}\right\},$$

we have

$$\begin{aligned}
\|h_3\|_{\dot{B}^s_{p,q}} &= \left\{\sum_{j\in\mathbb{Z}} \left(2^{sj}\left\|\varphi_j * h_3\right\|_{L^p}\right)^q\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{j\in\mathbb{Z}} \left(2^{sj}\left\|\sum_{k\in\mathbb{Z}}\sum_{|l-k|\le 2}\varphi_j * ((\varphi_k * f)(\varphi_l * g))\right\|_{L^p}\right)^q\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{j\in\mathbb{Z}} \left(2^{sj}\left\|\sum_{\max\{k,l\}\ge j-2}\sum_{|l-k|\le 2}\varphi_j * ((\varphi_k * f)(\varphi_l * g))\right\|_{L^p}\right)^q\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{j\in\mathbb{Z}} \left(2^{sj}\left\|\sum_{r\ge -4}\sum_{|t|\le 2}\varphi_j * ((\varphi_{j+r} * f)(\varphi_{j+r+t} * g))\right\|_{L^p}\right)^q\right\}^{\frac{1}{q}} \\
&\le \left\{\sum_{j\in\mathbb{Z}} \left(2^{sj}\sum_{r\ge -4}\sum_{|t|\le 2}\left\|\varphi_j * ((\varphi_{j+r} * f)(\varphi_{j+r+t} * g))\right\|_{L^p}\right)^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

From (3.1), the Young inequality, and the Hölder inequality we see that

$$\|\varphi_j * ((\varphi_{j+r} * f)(\varphi_{j+r+t} * g))\|_{L^p} \le \|\mathcal{F}^{-1}\phi\|_{L^1}\|\varphi_{j+r} * f\|_{L^{p_1}}\|\varphi_{j+r+t} * g\|_{L^{p_2}}$$

for every $j, r, t \in \mathbb{Z}$. Therefore, by using the Minkowski inequality, we obtain that

$$\begin{aligned}
\|h_3\|_{\dot{B}^s_{p,q}} &\le C\left\{\sum_{j\in\mathbb{Z}}\left(2^{sj}\sum_{r\ge -4}\sum_{|t|\le 2}\|\varphi_{j+r} * f\|_{L^{p_1}}\|\varphi_{j+r+t} * g\|_{L^{p_2}}\right)^q\right\}^{\frac{1}{q}} \\
&\le C\sum_{r\ge -4}\sum_{|t|\le 2}\left\{\sum_{j\in\mathbb{Z}}\left(2^{sj}\|\varphi_{j+r} * f\|_{L^{p_1}}\|\varphi_{j+r+t} * g\|_{L^{p_2}}\right)^q\right\}^{\frac{1}{q}} \\
&= C\sum_{r\ge -4}2^{-sr}\sum_{|t|\le 2}2^{\alpha t}\left\{\sum_{j\in\mathbb{Z}}\left(2^{(s+\alpha)(j+r)}\|\varphi_{j+r} * f\|_{L^{p_1}}2^{-\alpha(j+r+t)}\|\varphi_{j+r+t} * g\|_{L^{p_2}}\right)^q\right\}^{\frac{1}{q}} \\
&\le C\sup_{l\in\mathbb{Z}}2^{-\alpha l}\|\varphi_l * g\|_{L^{p_2}}\sum_{r\ge -4}2^{-sr}\sum_{|t|\le 2}2^{\alpha t}\left\{\sum_{k\in\mathbb{Z}}\left(2^{(s+\alpha)k}\|\varphi_k * f\|_{L^{p_1}}\right)^q\right\}^{\frac{1}{q}} \\
&\le C\|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}}\|f\|_{\dot{B}^{s+\alpha}_{p_1,q}}, \qquad \text{(A.5)}
\end{aligned}$$

where $C = C(n, p, p_1, p_2, q, s, \alpha)$. Here we should note that the final estimate in (A.5) is valid since $s > 0$. For the case $q = \infty$, we see by a similar calculation to (A.5) that

$$\begin{aligned} \|h_3\|_{\dot{B}^s_{p,\infty}} &\leq C \sup_{l\in\mathbb{Z}} 2^{-\alpha l}\|\varphi_l * g\|_{L^{p_2}} \sum_{r\geq -4} 2^{-sr} \sum_{|t|\leq 2} 2^{\alpha t} \sup_{k\in\mathbb{Z}} 2^{(s+\alpha)k}\|\varphi_k * f\|_{L^{p_1}} \\ &\leq C\|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}}\|f\|_{\dot{B}^{s+\alpha}_{p_1,\infty}}, \end{aligned}$$

with a constant $C = C(n, p, p_1, p_2, s, \alpha)$. Hence we have

$$\|h_3\|_{\dot{B}^s_{p,q}} \leq C\|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}}\|f\|_{\dot{B}^{s+\alpha}_{p_1,q}} \tag{A.6}$$

for every $1 \leq q \leq \infty$ with $C = C(n, p, p_1, p_2, q, s, \alpha)$.

From (A.3), (A.4), and (A.6), we obtain (1) of Proposition 3.7.

(2) We also use the paraproduct formula (A.1). We first consider the case $1 \leq q < \infty$. By a similar calculation in (A.2), we obtain

$$\begin{aligned} \|h_1\|_{\dot{B}^s_{p,q}} &\leq C \sum_{|l|\leq 2} 2^{-sl} \left\{ \sum_{m\in\mathbb{Z}} (2^{sm}\|\varphi_m * f\|_{L^{p_1}}\|P_m g\|_{L^{p_2}})^q \right\}^{\frac{1}{q}} \\ &\leq C \sup_{m\in\mathbb{Z}} \|P_m g\|_{L^{p_2}}\|f\|_{\dot{B}^s_{p_1,q}}. \end{aligned} \tag{A.7}$$

Let ψ be as $\mathcal{F}\psi(\xi) = 1 - \sum_{j=1}^{\infty} \phi(2^{-j}\xi)$. We should note here that there holds

$$\sum_{l=\infty}^{k} \varphi_l(x) = 2^{kn}\psi(2^k x) \equiv \psi_{2^{-k}}(x)$$

for every $k \in \mathbb{Z}$, where $\psi_\lambda \equiv \lambda^{-n}\psi(\lambda^{-1}\cdot)$ for $\lambda > 0$. Therefore, we see

$$\left\| \sum_{l=-\infty}^{k} \varphi_l \right\|_{L^1} = \|\psi\|_{L^1}$$

and hence

$$\begin{aligned} \|P_m g\|_{L^{p_2}} &= \left\| \sum_{l=-\infty}^{m-3} \varphi_l * g \right\|_{L^{p_2}} \\ &= \|\psi_{2^{m-3}} * g\|_{L^{p_2}} \\ &\leq \|\psi\|_{L^1}\|g\|_{L^{p_2}} \end{aligned}$$

for every $m \in \mathbb{Z}$. From this estimate and (A.7), we obtain the estimate

$$\|h_1\|_{\dot{B}^s_{p,q}} \leq C\|g\|_{L^{p_2}}\|f\|_{\dot{B}^s_{p_1,q}}, \tag{A.8}$$

where $C = C(n, p, p_1, p_2, q, s)$. We can easily see that this is also true for the case $q = \infty$. Moreover, by the symmetry, we have

$$\|h_2\|_{\dot{B}^s_{p,q}} \le C\|f\|_{L^{p_3}}\|g\|_{\dot{B}^s_{p_4,q}}, \tag{A.9}$$

where $C = C(n, p, p_3, p_4, q, s)$.

For the estimate of h_3 in $\dot{B}^s_{p,q}$ with $1 \le q < \infty$, we have by a similar calculation to (A.5) that

$$\begin{aligned}
\|h_3\|_{\dot{B}^s_{p,q}} &\le C \sum_{r\ge -4} 2^{-sr} \sum_{|t|\le 2} \left\{ \sum_{j\in\mathbb{Z}} \left(2^{s(j+r)}\|\varphi_{j+r} * f\|_{L^{p_1}}\|\varphi_{j+r+t} * g\|_{L^{p_2}}\right)^q \right\}^{\frac{1}{q}} \\
&\le C \sup_{l\in\mathbb{Z}} \|\varphi_l * g\|_{L^{p_2}} \sum_{r\ge -4} 2^{-sr} \left\{ \sum_{k\in\mathbb{Z}} (2^{sk}\|\varphi_k * f\|_{L^{p_1}})^q \right\}^{\frac{1}{q}} \\
&\le C\|g\|_{L^{p_2}}\|f\|_{p_1,q}
\end{aligned} \tag{A.10}$$

where $C = C(n, p, p_1, p_2, q, s)$, which is also valid for $q = \infty$.

From (A.8)-(A.10), we obtain (2) of Proposition 3.7. □

Appendix B

Here we sketch the proof of Proposition 6.1 (2) by Bejenaru-Tao [2]. First of all, we should show the following estimate:

$$\|A_m f - A_m g\|_S \leq \|f-g\|_D C^n(\|f\|_D + \|g\|_D)^{m-1}, \quad \forall f, g \in D,\ \forall n \geq 1, \tag{B.1}$$

where $C > 0$ is a constant independent of f, g and m.

Proof of (B.1). By the symmetry and the equality

$$A_m(\lambda f) = \lambda^m A_m f \quad \forall f \in D,\ \forall \lambda \in \mathbb{R} \tag{B.2}$$

which is seen by induction, we can assume

$$f \neq g, \quad \|f\|_D \leq \|g\|_D \leq 1.$$

Indeed, if (B.1) holds under such an assumption, then for general f and g with $\|f\|_D \leq \|g\|_D$, we see that

$$\begin{aligned} \|g\|_D^{-m}\|A_m f - A_m g\|_S &= \left\| A_m\left(\frac{f}{\|g\|_D}\right) - A_m\left(\frac{g}{\|g\|_D}\right)\right\|_S \\ &\leq C^m \left\| \frac{f}{\|g\|_D} - \frac{g}{\|g\|_D}\right\|_D \\ &\leq C^m \|g\|_D^{-1}\|f-g\|_D \end{aligned}$$

and hence we have (B.1) for f and g by considering $\|g\|_D \leq \|f\|_D + \|g\|_D$. Moreover, let $t \equiv \|f-g\|_D$. Then we see that

$$f = g + th, \quad \text{where } h \equiv \frac{f-g}{\|f-g\|_D}, \quad 0 < t \leq 2.$$

Hence, it suffices to show

$$\|A_m(g+th) - A_m(g)\|_S \leq tC^m. \tag{B.3}$$

Now let us fix g and h. Then the function $s \mapsto A_m(g+sh) - A_m(g)$ is a polynomial of degree at most m having no constant. Therefore, this can be written as

$$A_m(g+sh) - A_m(g) = \sum_{j=1}^{m} F_j s^j \tag{B.4}$$

with some $F_1, F_2, \ldots, F_n \in S$. By the estimate in (1) of Proposition 6.1 (which is shown by induction), we have

$$\begin{aligned} \|A_m(g+th) - A_m(g)\|_S &\leq C^m(\|g\|_D + s\|h\|_D)^m \\ &\leq (4C)^m \end{aligned}$$

for every $0 < s \leq 2$. Together with (B.4), it is seen that

$$\left\| \sum_{j=1}^{m} F_j s^j \right\|_S \leq (4C)^m. \tag{B.5}$$

Since each F_j can be written as linear summation of $\{\sum_{j=1}^{m} F_j s_k^j\}_{k=1}^{m}$ ($\{s_k\}_{k=1}^{m} \subset (0,2]$ are m-different data), we see that

$$\|F_j\|_S \leq C^m, \quad \forall j = 1, 2, \ldots, m.$$

From this and (B.4), we conclude that

$$\|A_m(g + sh) - A_m(g)\|_S \leq s \left\| \sum_{j=1}^{m} F_j s^{j-1} \right\|_S \leq sC^m,$$

which yields (B.3). □

Proof of Proposition 6.1 (2). Let us prove the claim by induction on m. More precisely, we assume that $A_{m'} : D \to S$ is continuous from $(B_D(\varepsilon), \|\cdot\|_{\tilde{D}})$ to $(B_S(\delta), \|\cdot\|_{\tilde{S}})$ for every $1 \leq m' \leq m - 1$.

We take a sequence $\{f_k\}_{k=1}^{\infty} \subset B_D(\varepsilon)$ such that $f_k \to 0$ in the norm $\|\cdot\|_{\tilde{D}}$ and a parameter $0 < \lambda \leq 1$ arbitrarily. Considering that the solution u can be written as $u = u(f) = \sum_{m=1}^{\infty} A_m f$, we see that

$$\lim_{k\to\infty} \|u(\lambda f_k) - u(\lambda f)\|_{\tilde{S}} = 0,$$

and hence

$$\lim_{k\to\infty} \left\| \sum_{m'=1}^{\infty} \lambda^{m'} (A_{m'} f_k - A_{m'} f) \right\|_{\tilde{S}} = 0$$

by the equality (B.2). On the other hand, by the assumption of induction, there holds

$$\lim_{k\to\infty} \left\| \sum_{m'=1}^{m-1} \lambda^{m'} (A_{m'} f_k - A_{m'} f) \right\|_{\tilde{S}} = 0.$$

Therefore, we see that

$$\lim_{k\to\infty} \left\| \sum_{m'=m}^{\infty} \lambda^{m'-m} (A_{m'} f_k - A_{m'} f) \right\|_{\tilde{S}} = 0.$$

Then by the triangle inequality and (B.1), we have

$$\begin{aligned}
\overline{\lim_{k\to\infty}} \|A_m f_k - A_m f\|_{\tilde{S}} &\le \sum_{m'=m+1}^{\infty} \lambda^{m'-m} \sup_{k\in\mathbb{N}} \|A_{m'} f_k - A_{m'} f\|_{\tilde{S}} \\
&\le \sum_{m'=m+1}^{\infty} \lambda^{m'-m} C^{m'} \sup_{k\in\mathbb{N}} (\|f_k\|_D + \|f\|_D)^{m'} \\
&\le \sum_{m'=m+1}^{\infty} \lambda^{m'-m} (2C\varepsilon)^{m'}
\end{aligned}$$

Taking $\lambda \to 0$, we obtain

$$\overline{\lim_{k\to\infty}} \|A_m f_k - A_m f\|_{\tilde{S}} = 0,$$

which yields a continuity of A_m from $(B_D(\varepsilon), \|\cdot\|_{\tilde{D}})$ to $(B_S(\delta), \|\cdot\|_{\tilde{S}})$. □

Appendix C

Here let us prove $PI_1 \not\equiv 0$ in Step 1 of the proof of Lemma 6.9, that is,

$$P(e_2\Phi_1 + e_3\Phi_2) \not\equiv 0 \tag{C.1}$$

with Φ_1 and Φ_2 in (6.6), and ψ characterized by (6.1). We can assume here that ψ is radial symmetric, i.e.,

$$\psi(x) = \psi(y), \quad x, y \in \mathbb{R}^n \text{ with } |x| = |y|,$$

which implies that we can write $\psi = \psi(r)$, $r = r(x) = |x|$.

Let us show (C.1) with a proof by contradiction. Suppose that

$$P(e_2\Phi_1 + e_3\Phi_2) \equiv 0, \tag{C.2}$$

which yields that there exists some distribution F such that

$$e_2\Phi_1 + e_3\Phi_2 = \nabla F.$$

Hence it must hold that

$$\nabla \times (e_2\Phi_1 + e_3\Phi_2) \equiv 0.$$

Therefore, all of its components should vanish, and especially we have

$$\frac{\partial \Phi_1}{\partial x_1} \equiv 0.$$

Actually, this yields $\Phi_1 \equiv 0$, i.e.,

$$\psi_{x_3}\psi_{x_2x_3} \equiv \psi_{x_2}\psi_{x_3^2} \tag{C.3}$$

Indeed, if not, there exists a point $x^* \in \mathbb{R}^n$ such that $\Phi_1(x^*) \neq 0$ Then by $\frac{\partial \Phi_1}{\partial x_1} \equiv 0$, we see that $\Phi_1(x) = \Phi_1(x^*)$ on the line $\{x \in \mathbb{R}^n; x_j = x_j^*, \ \forall j = 2, 3, \ldots, n\}$, which contradicts $\Phi_1 \in \mathcal{S}$. Now we let $\psi_{r^\alpha} \equiv \frac{\partial^\alpha \psi}{\partial r^\alpha}$ and $r_{x_2^\alpha x_3^\beta} \equiv \frac{\partial^{(\alpha+\beta)}}{\partial x_2^\alpha x_3^\beta} r$. By a chain rule of differentiation, we can rewrite (C.3) as

$$\psi_r r_{x_3}(\psi_{r^2} r_{x_2} r_{x_3} + \psi_r r_{x_2x_3}) \equiv \psi_r r_{x_2}(\psi_{r^2} r_{x_2} r_{x_3} + \psi_r r_{x_3^3}).$$

Since $\psi \in \mathcal{S}\backslash\{0\}$, we have

$$r_{x_3} r_{x_2x_3} \equiv r_{x_2} r_{x_3^3}.$$

On the other hand,

$$r_{x_3} r_{x_2x_3} = \frac{x_3}{r}\left(-\frac{x_2x_3}{r^3}\right), \quad r_{x_2} r_{x_3^3} = \frac{x_2}{r}\left(\frac{1}{r} - \frac{x_3^2}{r^3}\right),$$

which yields $\frac{x_2}{r^2} = 0$ for any $x \in \mathbb{R}^n$ and it is not true. Therefore, we see that the assumption (C.2) is false and hence (C.1) holds.

References

[1] J. Bergh, J. Löfström, *Interpolation spaces, An introduction*, Berlin-New York-Heidelberg, Springer-Verlag, 1976.

[2] I. Bejenaru, T. Tao, Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation, J. Funct. Anal. **233** (2006), 228-259.

[3] C. Bjorland, L. Brandolese, D. Iftimie, M. E. Schonbek, L^p-solutions of the steady-state NavierStokes equations with rough external forces, Comm. Part. Diff. Equ. **36** (2011), 216-246.

[4] J. M. Bony, Calcul symbolique et propagation des singulartiṕour les équations aux dérivées partielles nonlinéaries. Ann. Sci. École Norm. Sup. (4) **14** (1981), 209-246.

[5] J. Bourgain, N. Pavlović, Ill-posedness of the Navier-Stokes equations in a critical space in 3D, J. Funct. Anal. **255** (2008), 2233-2247.

[6] Z. Chen, L^n solutions of the stationary and nonstationary Navier-Stokes equations in $\mathbb{R}^n$, Pacific J. Math. **158** (1993), 293-303.

[7] L. Grafakos, *Modern Fourier Analysis*, Springer, 2014.

[8] J. G. Heywood, On stationary solutions of the Navier-Stokes equations as limits of non-stationary solutions, Arch. Ration. Mech. Anal. **37** (1970), 48-60.

[9] T. Iwabuchi and M. Nakamura, Small solutions for nonlinear heat equations, the Navier-Stokes equation, and the Keller-Segel system in Besov and Triebel-Lizorkin spaces, Adv. Differ. Equ. **18** (2013), 687-736.

[10] B. Jawerth, Some observations on Besov and Lizorkin-Triebel spaces, Math. Scand. **40** (1977), 94-104.

[11] K. Kaneko, H. Kozono, S. Shimizu, Stationary solution to the Navier-Stokes equations in the scaling invariant Besov space and its regularity, Indiana Univ. Math. J. **68** (2019), 857-880.

[12] H. Koch, D. Tataru: Well-posedness for the Navier-Stokes equations. Adv. Math. **157** (2001), 22-35.

[13] H. Kozono, Y. Shimada, Bilinear estimates in homogeneous Triebel-Lizorkin spaces and the Navier-Stokes equations, Math. Nachr. **276** (2004), 63-74.

[14] O. A. Ladyzhenskaya, Investigation of the Navier-Stokes equation for stationary motion of an incompressible fluid, Uspehi Mat. Nauk **14** (1959), 75-97.

[15] J. Leray, Etude de diverses équations integrales non linéaries et de quelques proléms que pose l'hydrodynamique, J. Math. Pures Appl. **12** (1933), 1-82.

[16] T. V. Phan, N. C. Phuc, Stationary NavierStokes equations with critically singular external forces: Existence and stability results, Adv. Math. **241** (2013), 137-161.

[17] O. Sawada, A description of Bourgain-Pavlović's ill-posedness theorem of the Navier-Stokes equations in the critical Besov space (English summary), Harmonic analysis and nonlinear partial differential equations, RIMS Kôkyûroku Bessatsu, Res. Inst. Math. Sci. (RIMS) **B33** (2012), 59-85.

[18] Y. Sawano, *Theory of Besov spaces*, Developments in Mathematics, Springer, 2018.

[19] H. J. Schmeisser, H. Triebel, Topics in Fourier analysis and function spaces, Mathematik und ihre Anwendungen in Physik und Technik [Mathematics and its Applications in Physics and Technology], **42**, Akademische Verlagsgesellschaft Geest & Portig K.G., Leipzig, 1987.

[20] P. Secchi, On the stationary and nonstationary Navier-Stokes equations in $\mathbb{R}^n$, Ann. Mat. Pure. Appl. **153** (1988), 293-305.

[21] H. Triebel, *Theory of Function Spaces*, Birkhauser, Basel-Boston-Stuttgart, 1983.

[22] H. Tsurumi, Counter-examples of the bilinear estimates of the Hölder type inequality in homogeneous Besov spaces, to appear in Tokyo J. Math.

[23] H. Tsurumi, Ill-posedness of the stationary Navier-Stokes equations in Besov spaces, J. Math. Anal. Appl. **475** (2019), 1732-1743.

[24] H. Tsurumi, The stationary Navier-Stokes equations in the scaling invariant Triebel-Lizorkin spaces, Differ. Integral Equ. **32** (2019), 323-336.

[25] H. Tsurumi, Well-posedness and ill-posedness of the stationary Navier-Stokes equations in toroidal Besov spaces, Nonlinearity **32** (2019), 3798-3819.

[26] H. Tsurumi, Well-posedness and ill-posedness problems of the stationary Navier-Stokes equations in scaling invariant Besov spaces, Arch. Ration. Mech. Anal. **234** (2019), 911-923.

[27] T. Yoneda, Ill-posedness of the 3D-Navier-Stokes equations in a generalized Besov space near BMO^{-1}, J. Funct. Anal. **258** (2010), 3376-3387.

[28] B. Wang: Ill-posedness for the Navier-Stokes equations in critical Besov spaces $\dot{B}^{-1}_{\infty,q}$. Adv. Math. **268** (2015), 350-372.

[29] X. Xiong, Q. Xu, Z. Yin: Sobolev, Besov and Triebel-Lizorkin spaces on quantum tori. Memoirs of the American Mathematical Society, **252/1203**, American Mathematical Society, 2018.

List of original papers

The papers composing this doctoral thesis are as follows.

1. H. Tsurumi, The stationary Navier-Stokes equations in the scaling invariant Triebel-Lizorkin spaces, Differ. Integral Equ. **32** (2019), 323-336.
   (Chapter 3, [24])

2. H. Tsurumi, Ill-posedness of the stationary Navier-Stokes equations in Besov spaces, J. Math. Anal. Appl. **475** (2019), 1732-1743.
   (Chapter 4, [23])

3. H. Tsurumi, Well-posedness and ill-posedness of the stationary Navier-Stokes equations in toroidal Besov spaces, Nonlinearity **32** (2019), 3798-3819.
   (Chapter 5, [25])

4. H. Tsurumi, Well-posedness and ill-posedness problems of the stationary Navier-Stokes equations in scaling invariant Besov spaces, Arch. Ration. Mech. Anal. **234** (2019), 911-923.
   (Chapter 6, [26])

5. H. Tsurumi, Counter-examples of the bilinear estimates of the Hölder type inequality in homogeneous Besov spaces, to appear in Tokyo J. Math.
   (Chapter 7, [22])