Graduate School of Fundamental Science and Engineering
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博 士 論 文 概 要
Doctoral Thesis Synopsis

論 文 題 目
Thesis Theme

On a class number problem of the cyclotomic Z_2-extension of $\mathrm{Q}(\sqrt{5})$
$\mathrm{Q}(\sqrt{5})$ の円分的 Z_2-拡大の類数問題について

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The applicant studies on algebraic number theory. In algebraic number theory, the class number of an algebraic number field is one of the most important objects. An algebraic number field is an extension field of the rational number field Q with finite degree. Every algebraic number field K has the special subring O_K, which is called the ring of algebraic integer of K. The class number h_K of K is the cardinal of the quotient group I_K/P_K, where I_K is the group of all the fractional ideals of O_K and P_K is the group of all the principal fractional ideals of O_K. It is well known that h_K is finite for any algebraic number field K. Moreover, O_K is a unique factorization domain if and only if the class number of K is equal to 1.

More than two hundred years ago, Gauss conjectured that there are infinitely many real quadratic fields with class number 1. This conjecture, however, is still open. Moreover, it is not known whether there are infinitely many algebraic number fields with class number 1.

In order to approach this problem, we study "Weber's class number problem". Let p be a prime number and n a non-negative integer. We denote by ζ_n a primitive n-th root of unity in the complex number field. Then the cyclotomic field $Q(\zeta_{2p^{n+1}})$ has the unique real subfield $B_{p,n}$ whose Galois group over Q is isomorphic to Z/p^nZ. Since $B_{p,n} \subseteq B_{p,n+1}$, the set $B_{p,\infty} = \cup_{n=1}^{\infty} B_{p,n}$ is also a field. This field is called the cyclotomic Z_p-extension of Q. $B_{p,n}$ is called the n-th layer of the cyclotomic Z_p-extension of Q. We denote by $h_{p,n}$ the class number of $B_{p,n}$.

In terms of $h_{p,n}$, Weber showed that $h_{2,1} = h_{2,2} = h_{2,3} = 1$. Moreover, he proved that $h_{2,n}$ is odd for all positive integer n. Later, $h_{2,4} = 1$ was shown by Chon, Bauer and Masley. Van der Linden showed $h_{2,5} = 1$. Recently, J. C. Miller proved that $h_{2,6} = 1$. He also proved that $h_{2,7} = 1$ holds under the assumption of the generalized Riemann hypothesis. Moreover, it is known that $h_{p,n} = 1$ for $(p,n) = (3,1), (3,2), (3,3), (5,1),$ $(5,2), (5,3), (7,1), (11,1), (13,1)$. Then we consider the following problem:

Weber's class number problem. Is the class number $h_{p,n}$ equal to 1 for each prime number p and positive integer n?

However, it is very difficult to calculate the class numbers directly. To approach Weber's class number problem, we discuss on the indivisibility of $h_{p,n}$ by a prime number l:

Problem(l-divisibility)**.** Does there exist a prime number l dividing some $h_{p,n}$?

For the case of $p = l$, Weber and Iwasawa showed that p does not divide $h_{p,n}$ for any positive integer n. For the case of p$\neq l$, on the other hand, Washington proved that the l-part of $h_{p,n}$ is bounded as n tends to ∞. Regarding the l-indivisibility of $h_{p,n}$, K. Horie and K. Horie-M. Horie gave various results. The case of $p = 2$ has been studied very deeply by Fukuda-Komatsu, Okazaki and Morisawa-Okazaki. The case of $p = 3$ is also studied very deeply by Morisawa.

The applicant, on the other hand, studied the cyclotomic Z_2-extension of the quadratic field $K_0 := Q(\sqrt{5})$. For an arbitrary number field K, the cyclotomic Z_p-extension $K_{p,\infty}$ of K is defined by

$$K_{p,\infty} := K \cdot B_{p,\infty}.$$

The reason why the applicant treated $K_0 = Q(\sqrt{5})$ is because K_0 has the minimal discriminant among all the real quadratic fields. The n-th layer K_n of the cyclotomic Z_2-extension of K_0 is defined by

$$K_n = B_{2,n}(\sqrt{5}).$$

We denote by h_n the class number of K_n. Applying the result of Iwasawa, we have that h_n is odd for any positive integer n. Moreover, van der Linden showed that $h_1 = h_2 = h_3 = 1$. The aim of this thesis is to discuss the following problem:

Problem (divisibility for h_n)**.** Does there exist an odd prime number dividing some h_n?

The above problem is a natural generalization of Weber's class number problem for the cases of real quadratic fields.

This problem is also related to Coates' conjecture. An algebraic number field K is called totally real if all the embeddings of K into the complex number field are real. For a totally real field K, we denote by K^{cyc} the composite of $K_{p,\infty}$ for all prime number p. By Galois theory, there is a unique intermediate field $K(m)$ of K^{cyc}/K with degree m over K for all positive integer m. We denote by $h(m)$ the class number of $K(m)$. Then Coates asked the following problem:

Problem (Coates). Does there exist a positive constant $C(K)$, which is not depending on m, such that $h(m)$ is at most $C(K)$ for all positive integer m?

The applicant's thesis consists of 6 chapters. In chapter 1, we recall fundamental facts of algebraic number fields, that is, the ideal class groups, the discriminants, the root discriminants and the cyclotomic Z_p-extensions. In this chapter, we explain how to calculate the discriminant of K_n because the explicit value of the discriminant plays an important role in chapter 5.

In chapter 2, we give an explicit upper bound of Washington's theorem for the cyclotomic Z_2-extension of $Q(\sqrt{5})$.

Our result in this chapter is as follows:

Theorem 1. For an odd prime number l, let δ_l be 0 or 1 according as $l \equiv 1\ (mod\ 4)$ or not and c_l the exact power of 2 dividing $l^{\delta_l+1} - 1$. We denote by $\lfloor x \rfloor$ the maximal rational integer not exceeding a real number x. We put

$$m_l = 2c_l + \lfloor log_2(5l-1) \rfloor - \delta_l - 2$$

for $l \neq 5$ and

$$m_5 = 2c_5 + \lfloor log_2(5-1) \rfloor - \delta_5 - 2 = 4$$

for $l = 5$. Then l does not divide h_n/h_{m_l} for any $n \geq m_l$.

We show theorem 1 by calculating the l-parts of the generalized Bernoulli numbers and using the reflection theorem. We need to deal with the case of $l = 5$ separately because the prime number 5 divides the discriminant of K_0. For $l = 5$, we cannot obtain the same group isomorphism as the case of $l \neq 5$, which plays an important role in our argument. To obtain a similar group isomorphism for $l = 5$, we use the fact that the 5-part of $h_{2,n}$ is trivial for all positive integer n. This is an original method of this thesis.

In chapter 3, we give a numerical result as follows:

Theorem 2. Let l be an odd prime number. If l is less than 60000, then l does not divide h_n for any positive integer n.

Theorem 2 is obtained by using theorem 1 and a computer. In this chapter, we explain how to verify that l does not divide h_{m_l}. For this purpose, we treat 4 cases and give 4 criteria. For l with large c_l, however, it takes too much time to verify that l does not divide h_{m_l}. In order to speed up the computer calculation, we also explain the logarithmic algorithm.

In chapter 4, we explain how to establish an upper bound for the class number of a totally real number field K by following to Miller's method. This method is very important when we treat a totally real field with the root discriminant larger than 60.839. The root discriminants of K_4 and K_5 exceed this value.

We define C^{∞}-function F_c on the real number field by

$$F_c(x) = exp(-(x/c)^2)/cosh(x/2)$$

for a positive constant c. F_c is an important function to construct an upper bound of the class number. We denote by $S(K)$ the set of all prime numbers each of which splits completely into a product of principal prime ideals of K. By using Poitou version of Weil's explicit formula and class field theory, we have that if we can choose a suitable c and construct an enough subset T of $S(K)$, then we can establish an upper bound of the class number of K.

In chapter 5, we establish an upper bound of h_5, the class number of K_5 by using the method introduced in chapter 4. The upper bound we obtained is as follows:

Theorem 3. The class number of K_5 is at most 133.

To obtain theorem 3, we need to verify many small prime numbers are in $S(K_5)$. A prime number q is in $S(K_5)$ if and only if there is an algebraic integer of K_5 whose absolute norm has absolute value q. For this purpose, we use a special integral basis of K_5, some of which are Galois conjugate. Owing to this property, we can find algebraic integers whose absolute norms have small absolute values. Then we can construct a subset T_0 of $S(K_5)$, which consists of 741,766 elements. This T_0 and $c = 210$ are used to obtain theorem 3.

Theorems 2 and 3 imply that the class number of K_5 is 1. Moreover, since h_{n-1} divides h_n for all positive integer n, we also have the following theorem in chapter 5:

Theorem 4. The class numbers of K_4 and K_5 are 1.

In chapter 5, we also show that the class number of K_4 is at most 518 without the knowledge of theorem 3.

In chapter 6, we describe perspectives of the research by comparing to known results on Weber's class number problem. It is not known by this thesis whether there exist infinitely many prime numbers each of which does not divide h_n for any positive integer n. To approach this question, it is natural to consider the following question:

Problem 1. We fix a prime number l. Can we provide an explicit lower bound for l-indivisibility of K_∞/K?

As we mentioned, this thesis is related to Coates' conjecture. To approach this conjecture, it is very important to discuss the following problem:

Problem 2. We fix a prime number l. Let $p_1, p_2, \ldots, p_t$ be distinct prime numbers. Does there exist an intermediate field of $Z_{p_1} \times Z_{p_2} \times \cdots \times Z_{p_t}$-extension of K_0 whose class number is divided by l ?

早稲田大学　博士（理学）　学位申請　研究業績書

(List of research achievements for application of doctorate (Dr. of Science), Waseda University)

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Academic papers	(1) A Class Number Problem for the Cyclotomic Z_2-extension of $\mathrm{Q}(\sqrt{5})$, Tokyo J. Math. 39, No.1 (2016), 69-81, Takuya Aoki. (2) A Class Number Calculation of the 5^{th} layer of the Cyclotomic Z_2-extension of $\mathrm{Q}(\sqrt{5})$, to appear in Funct. Approx. Commet. Math, Takuya Aoki.
Lectures	(1) $\mathrm{Q}(\sqrt{5})$ の円分的 Z_2-拡大の類数問題, 早稲田整数論セミナー, 早稲田大学, 2015 年 1 月 23 日. (2) A Class Number Problem for the Cyclotomic Z_2-extension of $\mathrm{Q}(\sqrt{5})$, 早稲田大学整数論研究集会, 早稲田大学,2016 年 3 月 23 日. (3) Odlyzko bound と総実代数体の類数評価について, 早稲田大学整数論セミナー, 早稲田大学, 2017 年 12 月 22 日. (4) Odlyzko bound と総実代数体の類数評価について, 野々市代数的整数論 2018, 金沢工業大学, 2018 年 2 月 24 日. (5) Odlyzko bound と総実代数体の類数評価について, 早稲田大学整数論研究集会, 早稲田大学, 2018 年 3 月 15 日. (6) $\mathrm{Q}(\sqrt{5})$ の円分的 Z_2-拡大の中間体の類数の計算について, 津田塾大学整数論ワークショップ 2019, 津田塾大学, 2019 年 11 月 24 日.