Doctoral thesis

# Type IIB Effective Theories from the Viewpoint of the Matrix Model
# 行列模型に基づく IIB型超弦理論の有効理論の解析

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**Abstract**

In this thesis, we study the features of effective theories of type IIB superstring theory from the perturbative and non-perturbative points of view. We mainly focus on coupling constants in magnetized compactifications. The magnetized compactifications are constructed from super Yang-Mill theory that is an effective theory of D-branes in type IIB superstring theory. In the previous works, two features of the coupling constants have been shown in several magnetized compactifications: the coupling constants are determined by the product property of the space of the zero modes, higher order coupling constants are decomposed by the three-point coupling constants. We revealed that these are common features to general compact spin manifolds. This means that the origin of those features is not the property of just a specific function but the property of a Dirac operator on such general manifolds. On the other hand, the above is the result in the context of perturbative superstring theory. Hence, we also study those features of the coupling constants based on the IKKT matrix model that is known as a non-perturbative formulation of type IIB superstring theory. We show that those features are essentially valid even if we consider magnetized noncommutative torus and magnetized fuzzy sphere constructed based on the IKKT matrix model.

# Contents

# Chapter 1

# Introduction

The standard model (SM) is a successful model to explain the high-energy experiments. However, the SM has several problems to be solved, e.g., the origin of the generations and the chiral structure and the difficulties in incorporating gravity. Recent cosmological observations also imply the existence of additional structures for the SM. These facts lead us to consider the concept *beyond* the SM. As a possibility to overcome such problems, superstring theory has energetically been studied. Traditional quantum field theory is constructed on the basis of a point particle. On the other hand, superstring theory is constructed from an open string and a closed string. According to the spectral analysis based on the canonical quantization, the open string creates a gauge field and the closed string creates a gravitational field. Therefore, superstring theory is regarded as a promising candidate of the unified theory of all forces in nature.

In addition, superstring theory is rich in variety so called D-branes, which are various dimensional objects. Classically, the D-branes are defined through a boundary condition to consider the equation of motion with respect to an open string. For the open string, we can select the Dirichlet boundary conditions with respect to the two endpoints of the open string. The endpoints cannot move along the directions that are orthogonal to the directions constrained by the Dirichlet boundary conditions. Therefore, it seems that the open string can move only in a specific spatial area. The classical concept of the D-branes is such an area. However, the modern interpretation of the D-branes is a various dimensional object that couples naturally to a tensor fields defined in Type II superstring theory.

Type II superstring theory is constructed from a closed string. There are two types based on the property of the supersymmetry: type IIA and type IIB. The physical states of a closed string are classified as the left-mover and the right-mover, and the whole theory is defined by combining them. As we mentioned, the spectrum of the closed string includes a gravitational field. Actually, there are several massless tensor fields in the spectrum. The effective action with respect to these massless fields is described by supergravity. In the type II supergravity actions, the D-branes can be described as solutions of equations of motion. On the other hand, an effective action of the D-branes is described by the super Yang-Mills action. As we mentioned, the spectrum of the open string includes a gauge field. If we consider an open string between several parallel D-branes, we have a degree of freedom to select which D-branes the end points attach. If those D-branes coincide each other, the open string state can be a massless state. Therefore, such a state can be identified as a non-Abelian gauge field. Because of the supersymmetry, the effective action is described by super Yang-Mills theory.

On the other hand, a feature of superstring theory is the requirement of a ten-dimensional spacetime because of the consistency, e.g., physical state conditions based on the Virasoro algebra. Our world is four-dimensional spacetime on the basis of the experiments and the observations so far. Therefore, the problem is how to do with an additional six-dimensional space so called an extra dimensional space. We have several ways to consider an extra dimensional space. In this thesis, we focus on *compactifications*. In general, the *compactifications* is a method to construct a lower dimensional effective field theory (low-energy effective field theory) from a higher dimensional field theory by assuming the space associated to such a difference of the dimensions is very *small*. From the viewpoint of superstring theory, considering a ten-dimensional field theory is natural. However, in practice, we may consider higher dimensions lower than ten dimensions (e.g., six dimensions) as effective field theories of superstring theory. In this thesis, we refer to four dimensions as lower dimensions and $D > 4$ spacetime dimensions as higher dimensions. In this case, the *smallness* is restricted from the experiments and the observations. According to Ref. [1], extra dimensional spaces wider than TeV-size are excluded. In general, low-energy effective field theories derived from compactifications have moduli fields as degrees of freedom of an extra dimensional space and topological information. In practice, analytical information that is mentioned later is also necessary.

In compactifications of superstring theory, moduli fields appear as scalar fields in low-energy effective field theories. In this context, moduli fields parameterize possible string backgrounds. More precisely, expectation values of the moduli fields play roles of parameters governing coupling constants (e.g., the string coupling) and the shape of an extra dimensional space. Typical examples are complex structure moduli fields and Kähler moduli fields associated with the metric of an extra dimensional space. In general, coupling constants in low-energy effective field theories depend on these moduli fields. From the viewpoint of phenomenology, the value of a coupling constant should be fixed uniquely. To fix, we have to understand the mechanism to obtain expectation values of moduli fields. The processes that determine the expectation values of moduli fields through a certain mechanism are referred to as the moduli stabilization. On the other hand, such moduli fields have shift symmetries. The shift symmetry corresponds to the invariance of an action (up to total derivatives) under a constant shift of the value of a field. In the context of superstring theory, such a field is called a stringy axion. There are several works on applications of stringy axions to, e.g., dark matters, inflations and the CP problem. In practice, the decay constants of stringy axions are important. For example, the decay constant should be at around the Planck scale for the natural inflation from the Planck data [2, 3], which is one of the typical inflation scenarios within the framework of superstring theory.

On the other hand, typical topological information is the index based on the Index theorem. The index counts the difference of the numbers of chiral and anti-chiral zero modes of the Dirac operator on an extra dimensional space. In compactifications, we typically consider the Kaluza-Klein expansion. This is the separation of varieties by eigenfunctions of an operator on an extra dimensional space. In low-energy effective field theories applied the Kaluza-Klein expansion, the eigenvalues play the role of mass parameters. In the SM, the fermions as the matter fields are massless fields before the spontaneous symmetry breaking. Therefore, zero modes in the Kaluza-Klein expansion based on a Dirac operator are identified with the matter fields in the SM. Hence, a non-zero index means a realization of the chiral structure in a low-energy effective field theory. The chiral structure is required from the SM. Therefore, we have to consider setups that can lead to a non-zero index.

The analytical information mean wavefunctions and the integral on an extra dimensional space. Coupling constants in low-energy effective field theories are important to compare the theory with the experiments and the observations. To compute such a coupling constant, information of the wavefunctions and the integral on an extra dimensional space are necessary. According to the Kaluza-Klein expansion, the fields in a higher dimensional field theory are decomposed by the eigenfunctions in principle. Accordingly, each term in the action is decomposed into two parts: the four-dimensional part and the extra dimensional part. The four-dimensional part consists of an expected form from the SM, e.g., four-dimensional kinetic terms and Yukawa interactions. Therefore, the extra dimensional parts play the role of coupling constants because they are constants from the viewpoint of the four-dimensional field theory. Such an extra dimensional part is constructed by a product of wavefunctions as the eigenfunctions and the integral over the extra dimensional space. Therefore, the analytical information is necessary to obtain explicit values of coupling constants in low-energy effective field theories.

As we mentioned, coupling constants in low-energy effective field theories depend on moduli fields in general. This is because they are determined by the wavefunctions and the integral on an extra dimensional space. Typically, we perform the Kaluza-Klein expansion based on the Dirac operator or the Laplacian on an extra dimensional space. Such an operator is defined by the metric. The metric, in general, depends on moduli fields, e.g., complex structure moduli mentioned above. Thus, the operators defined with the metric and their eigenfunctions depend naturally on moduli fields. Therefore, moduli fields and matter fields cannot be considered separately. In conclusion, we have to investigate the dependence of moduli fields and topological and analytical information to obtain a low-energy effective field theory and consider its phenomenological implications.

One of interesting string-inspired models is a magnetized compactification model. Typically, we start from super Yang-Mills theory as a low-energy effective field theory of the D-branes. In particular, such a model can be realized in type IIB superstring theory. Therefore, in this thesis, we mainly focus on low-energy effective field theories of type IIB superstring theory. In magnetized compactification models, the magnetic flux obtained from a background gauge field can induce the non-trivial index. Therefore, the chiral structure is ensured by the magnetic flux. Although it is difficult generally to obtain the explicit forms of wavefunctions and compute the integral on an extra dimensional space, there are several works that have obtained and computed them.

In Ref. [4], the authors considered the magnetized toroidal compactifications and obtained the explicit zero modes, which are given by the Jacobi-theta function. In addition, they proved that the lightest mode of the Laplacian has the same functional form with the zero modes. Moreover, they computed the Yukawa couplings constructed from those eigenfunctions. In Ref. [4], it is shown that the product of two zero modes can be described by the linear combination of zero modes. Namely, the space of the zero modes is closed under the multiplication in the sense of a standard function space. The Yukawa couplings are nothing but the coefficients of the linear combination. In Ref. [6], the authors showed an interesting feature of the coupling constants in the magnetized toroidal compactifications by using the product property of the Jacobi-theta function. According to Ref. [6], generic multi-point coupling constants in the magnetized toroidal compactifications can be decomposed by the product of the three-point coupling constants. In the decomposition, we can observe the structure that is similar to the conformal blocks in conformal field theory. In Ref. [5], the authors generalized the results of Refs. [4, 6] to include the massive modes. In Ref. [75], the authors considered the magnetized spherical compactifications and obtain the explicit zero modes of the Dirac operator by rational functions whose numerators are monomials. In addition, they proved that the lightest mode of the Laplacian has the same functional form with the zero modes. It is obvious that the Yukawa couplings (three-point coupling constants) are important in comparison with the SM. The generic multi-point coupling constants are also important to consider higher dimensional operators, for example, the dimension 5 operator for the neutrino masses from the viewpoint of models beyond the SM.

So far, we discussed the features of low-energy effective field theories based on superstring theory, especially type IIB superstring theory. However, superstring theory is established perturbatively at present. In the traditional quantum field theory, we first define an action at the beginning, and the rule of the Feynman diagrams is derived on the basis of the action to compute amplitudes perturbatively. On the other hand, in superstring theory, we have only the rule of Feynman diagrams for string amplitudes based on superconformal field theory, but the origin of the rule is indefinite. Because the discussions so far are based on the perturbative superstring theory. There is a possibility that the above features, especially for coupling constants, will be affected by non-perturbative effects. Hence, it is important to reconsider the above discussion from the viewpoint of non-perturbative perspectives.

The purpose of this thesis is to investigate features of coupling constants in magnetized compactifications models as effective theories of type IIB superstring from perturbative and non-perturbative points of view.

As a perturbative point of view, we discuss the generalization of the features of the coupling constants in the magnetized toroidal and spherical compacifications to magnetized compacifications of general compact spin manifolds. In the previous works, the features of the coupling constants are derived from the product property of the Jacobi-theta function or the rational functions. Actually, we will prove that the features of the coupling constants can be attributed to two general reasons: the space of the eigenfunctions of the Dirac operator configures the complete orthonormal system, and the action of the Dirac operator satisfies the Leibniz rule.

As a non-perturbative point of view, we consider matrix models as non-perturbative formulations of superstring theory. In this thesis, we mainly consider Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) matrix model. There are two reasons why the IKKT matrix model is considered as a non-perturbative formulations of superstring theory: we can define the action of the IKKT matrix model without Feynman diagrams, and type IIB string field theory in the light-cone gauge can be derived from the IKKT matrix model. The action of the IKKT matrix model can be defined by super Yang-Mills action. As we mentioned, magnetized compactification models are described by super Yang-Mills theory. Therefore, we are naturally lead to consider magnetized compactifications in the IKKT matrix model to investigate non-perturbative effects for the perturbative results.

The organization of this thesis is as follows. Chapter 1 contains the introduction and the structure of this thesis. In Chapter 2, we introduce superstring theory, especially type IIB superstring theory. In Chapter 3, we review the results of the magnetized toroidal and spherical compactifications as low-energy effective field theories of type IIB superstring theory on the basis of Refs. [4–6, 75]. In Chapter 4, we review the IKKT matrix model. We define the action of the IKKT matrix model from the matrix regularization of the Green-Schwarz action and the large-$N$ reduction of the super Yang-Mills action. Although the momentum cut-off is introduced effectively by the matrix regularization or the large-$N$ reduction, the gauge symmetry is preserved. The noncommutative torus in the IKKT matrix model is defined on the basis of the gauge symmetry. Therefore, we introduce the noncommutative torus in this Chapter. In addition, we introduce a modified IKKT matrix model to describe the fuzzy sphere. In Chapter 5, we discuss the generalization of the results reviewed in Chapter 3 to the case with more general manifold. We will prove the results are hold on magnetized compactifications of general compact spin manifolds. In Chapter 6, we reconsider the result of Chapter 5 from a non-perturbative perspective. We will consider the magnetized compactification models on the noncommutative torus and fuzzy sphere based on the IKKT matrix model. The quantization condition of the magnetic flux on the noncommutative torus is deformed by the noncommutative parameter. We will derive zero modes of the Dirac operator. In addition, we compute Yukawa couplings as examples to confirm the features of the coupling constants. We will observe that the result based on the IKKT matrix model is almost the same with magnetized toroidal and spherical compactifications in Chapter 5. Chapter 7 contains the summary of this thesis. We take the natural unit convention $\hbar = c = 1$, where $\hbar$ is the Dirac constant and $c$ is the speed of light. The Minkowski metric is set as $\eta_{MN} = \mathrm{diag}(-, +, \cdots, +)$.

# Chapter 2

# Type IIB superstring theory

In this thesis, we mainly focus on type IIB superstring theory and its low-energy effective field theories. Hence, in this chapter, we survey superstring theory, especially type IIB superstring theory on the basis of Refs. [8–11].

## 2.1 Superstring theory

There are two approaches to establish superstring theory: the *Ramond-Neveu-Schwarz* (RNS) formalism, and the *Green-Schwarz* (GS) formalism. The RNS formalism is constructed on the basis of the supersymmetry on the string world-sheet. On the other hand, the GS formalism handles ten-dimensional Minkowski space-time with the supersymmetry. In this section, we review superstring theory based on the RNS formalism.

### 2.1.1 RNS formalism

Let us start from the bosonic string theory. We consider a surface so called the world-sheet since the trajectory of a string describes a surface. Therefore, we can characterize the world-sheet by two-dimensional coordinates $(\tau, \sigma)$. We also use $\sigma^\alpha$ $(\sigma^0 = \tau, \sigma^1 = \sigma)$. The dynamics of the string in $D$ dimensions can be described $D$ functions $X^M(\tau, \sigma)$ $(M = 0, \dots, D-1)$. In addition, we define the inner product of $X^M$ as $X \cdot X := \eta_{MN} X^M X^N$ where $\eta_{MN} = \mathrm{diag}(-, +, \cdots, +)$. The well-known action of the bosonic string is the Nambu-Goto action [12, 13]

$$S_{\mathrm{NG}} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{\left(\dot{X} \cdot X'\right)^2 - (\dot{X})^2 (X')^2}, \tag{2.1}$$

where $\dot{X}^M := dX^M/d\tau$, $X^{M\prime} := dX^M/d\sigma$, and $\alpha'$, which is called the Regge slope, has units of length-squared. However, the square root is an obstacle when we consider quantization. To perform quantization of the string, we introduce an auxiliary world-sheet metric $h_{\alpha\beta}(\tau, \sigma)$ whose signature is $(-, +)$. By using the metric $h_{\alpha\beta}$, we can define the action that is equivalent to the Nambu-Goto action (2.1) as

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X, \tag{2.2}$$

where $h := \det h_{\alpha\beta}$. The action (2.2) is called Brink-Di Cecchia-Howe-Deser-Zumino action [14, 15] or Polyakov action [16].

The action (2.2) possesses the following symmetries:

1. Poincaré transformations (global)

$$\delta X^M = a^M{}_N X^N + b^M, \quad \delta h_{\alpha\beta} = 0,$$

   where $a_{MN} = -a_{NM}$ is the Lorentz transformation parameter and $b^M$ is the translation parameter in $D$ dimensions.

2. Diffeomorphisms (local)

$$\sigma^\alpha \to f^\alpha(\sigma, \tau) = \sigma'^\alpha, \quad h_{\alpha\beta}(\tau, \sigma) = \frac{\partial f^\gamma}{\partial \sigma^\alpha}\frac{\partial f^\delta}{\partial \sigma^\beta} h_{\gamma\delta}(\tau', \sigma').$$

   This is a reparameterization of the world-sheet coordinate.

3. Weyl transformations (local)

$$h_{\alpha\beta} \to e^{\phi(\tau,\sigma)} h_{\alpha\beta}, \quad \delta X^M = 0.$$

   This local symmetry induces the vanishing of the energy-momentum tensor.

We can select a gauge of the auxiliary metric since diffeomorphisms and Weyl transformations are local symmetries. On a two-dimensional space, the metric has three degrees of freedom because of the symmetric condition. On the other hand, diffeomorphisms and Weyl transformations have two and one degrees of freedom, respectively. Therefore, we can fix a gauge of the auxiliary metric completely. For convenience, we select the following gauge:

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, the action (2.2) takes

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left( (\dot{X})^2 - (X')^2 \right). \tag{2.3}$$

In the following, we set the unit $\alpha' = 1/2$. To establish superstring action, we generalize the action (2.3) to the action containing fermions on the world-sheet. The expected action is obtained by

$$S = -\frac{1}{2\pi} \int d^2\sigma \left( \partial_\alpha X_M \partial^\alpha X^M + \bar{\psi}^M \Gamma^\alpha \partial_\alpha \psi_M \right) \tag{2.4}$$

where $\Gamma^\alpha$ is the two-dimensional Gamma matrix, e.g.,

$$\Gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The spinors in the action (2.4) must be Grassmann numbers classically. We will impose the anti-commutation relation. We label the two components of the spinors $\psi^M$ as

$$\psi^M = \begin{pmatrix} \psi_-^M \\ \psi_+^M \end{pmatrix}. \tag{2.5}$$

On the other hand, the Dirac conjugate is defined by

$$\bar{\psi}^M := \psi^{M\dagger}\mathcal{C}, \quad \mathcal{C} := i\Gamma^0.$$

Since the spinors included in the action (2.4) should be Majorana spinors and the Majorana spinors are defined by $\psi^T\mathcal{C}$, the fermionic part of the action can be written as

$$S_{\mathrm{f}} = \frac{i}{\pi}\int d^2\sigma \left(\psi_-^M\partial_+\psi_{M,-} + \psi_+^M\partial_{M,-}\psi_+\right), \tag{2.6}$$

where we introduce the light-cone coordinates, i.e.,

$$\sigma^{\pm} = \tau \pm \sigma, \quad \partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma).$$

The equations of motion are

$$\partial_+\partial_- X^M = 0, \tag{2.7}$$
$$\partial_\pm \psi_\mp^M = 0. \tag{2.8}$$

We have to select the boundary conditions. For the bosonic part, there are three types of boundary conditions: the Dirichlet boundary condition, the Neumann boundary condition, and closed string. The first two conditions are for the open string.

- Neumann boundary condition

$$X^{M\prime}|_{\sigma=0,\pi} = 0. \tag{2.9}$$

- Dirichlet boundary condition

$$X^M|_{\sigma=0} = X_0^M, \quad X^M|_{\sigma=\pi} = X_\pi^M, \tag{2.10}$$

Actually, we can consider mixed boundary conditions, e.g., $X^I$ $(I = 1, \ldots, D-p-1)$ Dirichlet boundary condition and $X^J$ $(J = D-p, \ldots, D-1)$ Neumann boundary condition. We will consider such a case in the context of D-branes.

- Closed string

$$X^M(\tau,\sigma) = X^M(\tau, \sigma+\pi). \tag{2.11}$$

For the fermionic part, there are two types of boundary conditions if we fix the overall sign between $\psi_+^M$ and $\psi_-^M$: the Ramond boundary condition, and the Neveu-Schwarz boundary condition.

- Ramond boundary condition

$$\psi_+^M|_{\sigma=\pi} = \psi_-^M|_{\sigma=\pi} \quad \text{for open string,} \tag{2.12}$$
$$\psi_\pm^M(\tau,\sigma) = \psi_\pm^M(\tau,\sigma+\pi) \quad \text{for closed string} \tag{2.13}$$

- Neveu-Schwarz boundary condition

$$\psi_+^M|_{\sigma=\pi} = -\psi_-^M|_{\sigma=\pi} \quad \text{for open string,} \tag{2.14}$$
$$\psi_\pm^M(\tau,\sigma) = -\psi_\pm^M(\tau,\sigma+\pi) \quad \text{for closed string} \tag{2.15}$$

To consider canonical quantization, the harmonic oscillator representation is convenient. In this thesis, we are interested in type IIB superstring theory, and type IIB superstring theory is defined by the closed string only. Hence, we obtain only the closed string solution in the harmonic oscillator representation.

- Bosonic closed string

$$X_R^M = \frac{1}{2}x^M + \frac{1}{2}p^M(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^M e^{-2in(\tau-\sigma)} \quad \text{for right movers,}$$
$$X_L^M = \frac{1}{2}x^M + \frac{1}{2}p^M(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^M e^{-2in(\tau+\sigma)} \quad \text{for left movers.}$$

- Fermionic closed string: Ramond sector (R-sector)

$$\psi_-^M = \sum_{n\in\mathbb{Z}} d_n^M e^{-2in(\tau-\sigma)} \quad \text{for right movers,}$$
$$\psi_+^M = \sum_{n\in\mathbb{Z}} \tilde{d}_n^M e^{-2in(\tau+\sigma)} \quad \text{for left movers.}$$

- Fermionic closed string: Neveu-Schwarz sector (NS-sector)

$$\psi_-^M = \sum_{r\in\mathbb{Z}+1/2} b_r^M e^{-2ir(\tau-\sigma)} \quad \text{for right movers,}$$
$$\psi_+^M = \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^M e^{-2ir(\tau+\sigma)} \quad \text{for left movers.}$$

For canonical quantization, we introduce the canonical commutation relations and the canonical anti-commutation relations for the bosonic part and the fermionic part, respectively. We define the canonical momentum by

$$P^M := \frac{\delta S}{\delta \dot{X}_M} \tag{2.16}$$

Hence,

$$\left[P^M(\tau,\sigma), P^N(\tau',\sigma')\right] = \left[X^M(\tau,\sigma), X^N(\tau',\sigma')\right] = 0, \tag{2.17}$$

$$\left[P^M(\tau,\sigma), X^N(\tau',\sigma')\right] = -i\eta^{MN}\delta(\sigma-\sigma'), \tag{2.18}$$

$$\left\{\psi_A^M(\tau,\sigma), \psi_B^N(\tau',\sigma')\right\} = \pi\eta^{MN}\delta_{AB}\delta(\sigma-\sigma'), \tag{2.19}$$

where $A, B = \pm$. The conditions (2.17), (2.18), (2.19) induce the following relations of each mode operators defined through the Fourier coefficients

$$[\alpha_m^M, \alpha_n^N] = [\tilde{\alpha}_m^M, \tilde{\alpha}_n^N] = m\eta^{MN}\delta_{m+n=0}, \quad [\alpha_m^M, \tilde{\alpha}_n^N] = 0, \tag{2.20}$$

$$\{b_r^M, b_s^N\} = \{\tilde{b}_r^M, \tilde{b}_s^N\} = \eta^{MN}\delta_{r+s=0}, \quad \{d_m^M, d_n^N\} = \{\tilde{d}_m^M, \tilde{d}_n^N\} = \eta^{MN}\delta_{m+n=0}. \tag{2.21}$$

In the harmonic oscillator representation, we can introduce annihilation operators and creation operators. From the relations (2.20) and (2.21), we can identify $\alpha_m^M, \tilde{\alpha}_m^M, d_m^M, \tilde{d}_m^M, b_r^M, \tilde{b}_r^M$ $(m, r > 0)$ as the annihilation operators. Therefore, the ground state is defined by

$$\alpha_m^M |0\rangle = \tilde{\alpha}_m^M |0\rangle = d_m^M |0\rangle = \tilde{d}_m^M |0\rangle = 0,$$

or

$$\alpha_m^M |0\rangle = \tilde{\alpha}_m^M |0\rangle = b_r^M |0\rangle = \tilde{b}_r^M |0\rangle = 0.$$

The ground state with respect to the R-sector is degenerate since there is $d_0^\mu$. The number operator with respect to the R-sector is defined by

$$N_R = \sum_{m=1}^{\infty} m\left(d_{-m}\cdot d_m + \tilde{d}_{-m}\cdot\tilde{d}_m\right) \tag{2.22}$$

The number operator (2.22) commutes with the operator $d_0^\mu$ and $\tilde{d}_0^\mu$. This induces the degeneracy of the ground state of the R-sector. On the other hand, the operator $d_0^\mu$ satisfies

$$\{d_0^M, d_0^N\} = \eta^{MN}. \tag{2.23}$$

If we set $\sqrt{2}d_0^M = \Gamma^M$, the relation (2.23) is nothing but the Dirac algebra. Hence, the degenerated ground states can be labeled by the spinor index. Namely,

$$d_0^M |a\rangle = \Gamma_{ab}^M |b\rangle,$$

where $a, b$ are the D-dimensional spinor indices. Therefore, all states of the R-sector are spece-time fermions.

We note that the set of the states include negative-norm states since the space-time metric appears in the right-hand side in eqs. (2.20) and (2.21). To exclude the negative-norm states, we introduce the Virasoro algebra. Let us start from the bosonic part as a demonstration.

The energy-momentum tensor $T_{\alpha\beta}$ is defined by the variation of the action with respect to the metric

$$T_{\alpha\beta} := -2\pi \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h^{\alpha\beta}}. \tag{2.24}$$

On the other hand, eq. (2.24) is nothing but the equation of motion with respect to the world-sheet metric. Therefore, the energy-momentum tensor must vanish, i.e., $T_{\alpha\beta} = 0$. By using the light-cone coordinates of the world-sheet, the vanishing of the energy-momentum tensor can be written by

$$\begin{aligned} T_{++} &= \partial_+ X^M \partial_+ X_M = 0, \\ T_{--} &= \partial_- X^M \partial_{+-} X_M = 0. \end{aligned}$$

while obviously $T_{+-} = T_{-+} = 0$. For example, we obtain the energy-momentum tensor based on the closed string by

$$T_{--} = 2 \sum_{m \in \mathbb{Z}} L_m e^{-2im(\tau - \sigma)}, \quad T_{++} = 2 \sum_{m \in \mathbb{Z}} \tilde{L}_m e^{-2im(\tau + \sigma)},$$

where the coefficients

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n, \quad \tilde{L}_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n$$

are called the Virasoro generators. Therefore, the vanishing the energy-momentum tensor in the classical sense is identified by the vanishing of the Virasoro generators, i.e.,

$$L_m = \tilde{L}_m = 0 \quad \text{for} \quad m = 0, \pm 1, \dots$$

Let us go back to the superstring case. The energy-momentum tensor derived from the superstring action also vanished. This means that

$$\begin{aligned} T_{++} &= \partial_+ X^M \partial_+ X_M + \frac{i}{2} \psi_+^M \partial_+ \psi_{+M} = 0, \\ T_{--} &= \partial_- X^M \partial_{+-} X_M + \frac{i}{2} \psi_-^M \partial_- \psi_{-M} = 0. \end{aligned}$$

In addition, we have other currents called the supercurrents since the action possesses the supersymmetry, i.e.,

$$J_+ = \psi_+^M \partial_+ X_M, \quad J_- = \psi_-^M \partial_- X_M.$$

We can also introduce the Virasoro generators, which is called the super Virasoro generators in this case. They are defined by the integration of the above currents along the space-like direction of the world-sheet. Namely,

$$L_m = \frac{1}{\pi}\int_{-\pi}^{\pi} d\sigma T_{++} e^{im\sigma} = L_m^{(B)} + L_m^{(F)},$$

$$L_m^{(B)} = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \alpha_{-n}\alpha_{m+n} : \quad m \in \mathbb{Z},$$

$$L_m^{(F,NS)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+1/2}(r+\frac{m}{2}) : b_{-r}b_{m+r} : \quad \text{or} \quad L_m^{(F,R)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}(n+\frac{m}{2}) : d_{-n}d_{m+n} : \quad m \in \mathbb{Z}.$$

We introduce the normal ordering to consider quantization. There is an ambiguity according to this ordering. We will discuss the resolution of this ambiguity. On the other hand, the supercurrents are obtained by

$$G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma J_+ e^{ir\sigma} = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot b_{r+n} \quad r \in \mathbb{Z}+\frac{1}{2},$$
$$F_m = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma J_- e^{im\sigma} = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot d_{m+n} \quad m \in \mathbb{Z}.$$

From the general discussion, the algebraic structure of the super Virasoro algebra is obtained as follows,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m+n=0},$$
$$[L_m, F_n] = \left(\frac{m}{2}-n\right)F_{m+n},$$
$$\{F_m, F_n\} = 2L_{m+n} + \frac{D}{2}m^2\delta_{m+n=0},$$

for the NS-sector, and

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}m(m^2-1)\delta_{m+n=0},$$
$$[L_m, G_r] = \left(\frac{m}{2}-r\right)G_{m+r},$$
$$\{G_r, G_s\} = 2L_{m+n} + \frac{D}{2}\left(r^2-\frac{1}{4}\right)\delta_{r+s=0},$$

for the R-sector. In addition, the physical states must satisfy the following conditions to avoid the negative-norm states

$$\begin{aligned} G_r \left|\phi\right\rangle &= 0 \quad r > 0, \\ L_m \left|\phi\right\rangle &= 0 \quad m > 0, \\ (L_0 - a_{\mathrm{NS}}) \left|\phi\right\rangle &= 0 \end{aligned}$$

in the NS-sector, and

$$\begin{aligned} F_m \left|\phi\right\rangle &= 0 \quad m \geq 0, \\ L_m \left|\phi\right\rangle &= 0 \quad m > 0, \\ (L_0 - a_{\mathrm{R}}) \left|\phi\right\rangle &= 0 \end{aligned}$$

in the R-sector. The constants $a_{\mathrm{R}}$ and $a_{\mathrm{NS}}$ are introduced from the normal-ordering ambiguity as we mentioned.

The constant $a_{\mathrm{R}}$ must be zero because of the super Virasoro algebra with respect to $L_0$ and $F_0$. On the other hand, we can determine the constant $a_{\mathrm{NS}}$ by the following way.

Let us start from the NS-sector physical state

$$\left|\psi\right\rangle := G_{-1/2} \left|\chi\right\rangle,$$

where the state $\left|\chi\right\rangle$ satisfies

$$G_{1/2} \left|\chi\right\rangle = G_{3/2} \left|\chi\right\rangle = \left(L_0 - a_{\mathrm{NS}} + \frac{1}{2}\right) \left|\chi\right\rangle = 0.$$

From the definition of the states $\left|\psi\right\rangle$ and $\left|\chi\right\rangle$, $a_{\mathrm{NS}} = 1/2$ since

$$0 = G_{1/2} \left|\psi\right\rangle = (2a_{\mathrm{NS}} - 1) \left|\chi\right\rangle.$$

We can determine the space-time dimension $D$ on the basis of the similar way for the determination of $a_{\mathrm{NS}}$. To confirm, we define the Neveu-Schwarz physical state

$$\left|\psi\right\rangle := (G_{-3/2} + \lambda G_{-1/2} L_{-1}) \left|\chi\right\rangle,$$

where the state $\left|\chi\right\rangle$ satisfies

$$G_{1/2} \left|\chi\right\rangle = G_{3/2} \left|\chi\right\rangle = (L_0 + 1) \left|\chi\right\rangle = 0.$$

We have already derived the result, $a_{\mathrm{NS}} = 1/2$. Therefore, we can obtain the following equations

$$0 = G_{1/2}|\psi\rangle = (2-\lambda)L_{-1}|\chi\rangle\,,$$
$$0 = G_{3/2}|\psi\rangle = (D-2-4\lambda)|\chi\rangle\,.$$

From the conditions, the space-time dimension $D$ must be 10 as we mentioned in the Chapter 1 since the parameter $\lambda$ is 2. We can obtain the same result on the basis of the R-sector.

Before proceeding to the discussion about the supergravity action, we mention the classical notion of the D-branes. Let us consider open strings with the Dirichlet boundary condition. For simplicity, we assume that we have two options ($k, l = 1, 2$) for the boundary condition. Namely,

$$X^I|_{\sigma=0} = x^I_{k,0}, \quad X^I|_{\sigma=\pi} = x^I_{l,\pi}, \tag{2.25}$$

where $I = 1, \ldots, D-p-1$ and $X^I_{k,0}$ and $X^I_{l,\pi}$ are constants.

Comparing with the Neumann boundary condition, the solution is given by

$$X^I = X^I_{k,0} + (X^I_{l,\pi} - X^I_{k,0})\frac{\sigma}{\pi} + \sum_{n\neq 0}\frac{1}{n}\alpha^I_n e^{-in\tau}\sin n\sigma. \tag{2.26}$$

We have already defined the canonical momentum by eq. (2.16). The mass-squared operator $M^2$ is defined as $M^2 = -P^2$. In this case, the mass-squared operator has $(X^I_{l,\pi} - X^I_{k,0})^2$ that is similar to the potential energy of a spring. In other words, all states of open strings described by eq. (2.26) have the masses depending on the square of the distance $|X^I_{l,\pi} - x^I_{k,0}|$. From the set, all states have two labels to distinguish two options at the endpoints. If two labels are different from each other, such a state has the masses depending on the square of the distance. However, such a state can be a massless state if two D-branes coincide. In that case, the massless states have the two types of the label: the spacetime vector, and the coincident D-branes. Therefore, we can identify these states as a $U(2)$ non-Abelian gauge field. Obviously, we can consider $N$ coincident D-branes. Hence, the effective field theory can be described by $U(N)$ super Yang-Mills theory, and we will discuss it below.

## 2.2 Type IIB superstring theory and supergravity action

In the following, we introduce the physical states in superstring theory by imposing the Virasoro condition. According to Section 2.1, we can select the boundary conditions for each left- and right-mover of the fermionic part: the NS-sector, the R-sector. Therefore, we have four types of physical states: the NS-NS sector, the NS-R sector, the R-NS sector, the R-R sector. Actually, the whole states are described by the combinations of the states from the bosonic and the fermionic parts.

Let us start from the NS-NS sector. The massless states of the NS-NS sector can be described by

$$|0\rangle_{(B)} \otimes \tilde{b}^i_{-1/2}|0\rangle_L \otimes b^j_{-1/2}|0\rangle_R\,, \tag{2.27}$$

where $|0\rangle_{(B)}$, $|0\rangle_L$,and $|0\rangle_R$ are the ground states of the bosonic part and the left-movers and right-movers of the fermionic part. These states (2.27) have two indices corresponding to the spacetime vector after the light-cone gauge, hence, $i = 1, 2, \ldots, 8$. In other words, we can identify these states as the 2-rank tensor field. More precisely, the rank-2 tensor can be decomposed into the symmetric traceless part, the trace part and the anti-symmetric part. They correspond to the gravitational field, and the dilaton field and the Kalb-Ramond field.

Next, we consider the NS-R sector and the R-NS sector. As we mentioned, the R-sector has the label of the spacetime fermion. Hence, both of the sectors have the labels: the spacetime fermion, the spacetime vector. Namely,

$$|0\rangle_{(B)} \otimes \tilde{b}^i_{-1/2} |0\rangle_L \otimes |a\rangle_R \,, \tag{2.28}$$
$$|0\rangle_{(B)} \otimes |a\rangle_L \otimes b^i_{-1/2} |0\rangle_R \,. \tag{2.29}$$

Let us consider a Dirac fermion in a ten-dimensional spacetime. Originally, such a fermion has 32 components. However, due to the equation of motion, the number of the physical degree of freedom is 16. This equals to that of the spinor in an eight-dimensional spacetime. Additionally, the massless states (2.28) and (2.29) have the index of the spacetime vector. Hence, the massless states can be identified as a Rarita-Schwinger field and a Majorana fermion.

Next, we consider the R-R sector. The massless states of the R-R sector can be described by

$$|0\rangle_{(B)} \otimes |a\rangle_L \otimes |b\rangle_R \,. \tag{2.30}$$

In general, the field with two spinor indices can be decomposed by the Gamma matrices and their anti-symmetric combinations. In other words,

$$\left(\not{C}\right)_{ab} = C + C_i(\Gamma^i)_{ab} + \cdots + \frac{1}{8!} C_{i_1\cdots i_8}(\Gamma^{i_1\cdots i_8})_{ab}, \tag{2.31}$$

where the slash means the Feynman slash and $\Gamma^i$ $(i = 1, \ldots, 8)$ is the Gamma matrix in eight dimensions. Hence, we can obtain anti-symmetric tensors from 0-rank to 8-rank.

From the above, the number of the bosonic fields is 320. On the other hand, we can find 256 fermionic fields. This means that the ten-dimensional field theory of these massless fields does not have supersymmetry. Actually, we have to consider the restricted Fock spaces because of the modular invariance. They can be realized by the GSO projection [17]. The GSO projection is described by the action of the fermionic number operator $(-1)^{\mathrm{F_L}}$ and $(-1)^{\mathrm{F_R}}$. The action of these operators for the ground states is defined as

$$\begin{aligned}
(-1)^{F_L} |0\rangle_L &= - |0\rangle_L \,, \\
(-1)^{F_L} |a\rangle_L &= -\gamma_{ab}\xi \, |b\rangle_L \,, && (2.32) \\
(-1)^{F_R} |0\rangle_R &= - |0\rangle_R \,, \\
(-1)^{F_R} |a\rangle_R &= -\gamma_{ab}\tilde{\xi} \, |b\rangle_R \,, && (2.33) \\
& && (2.34)
\end{aligned}$$

where $\xi, \tilde{\xi} = \pm 1$. The ambiguity of the sign assignment corresponds to the existence of two supersymmetric theories: type IIA superstring theory, and type IIB superstring theory. The difference lies in the R-R sector. The GSO projection imposes the chirality condition with respect to the R-R sector. It induces that type IIA superstring theory and type IIB superstring theory can have the anti-symmetric tensor fields with even number indices and odd number indices, respectively. In addition, the fifth-rank anti-symmetric tensor field must satisfy the anti-self-dual condition. In the following, we focus on type IIB superstring theory.

The effective action of these massless fields is described by supergravity so called type IIB supergravity. In type IIB supergravity, the contents are dilaton $\Phi$, graviton $G_{MN}$, Kalb-Ramond field $B_2$ (field strength $H_3$), R-R scalar field $C_0$, R-R 2-form field $C_2$ and R-R 4-form field $C_4$. We denote the field strength of $p$-form fields by $F_{p+1}$. The action is given by

$$\begin{aligned}
S_{\mathrm{IIB}} &= S_{\mathrm{NS}} + S_{\mathrm{RR}} + S_{\mathrm{CS}}, \\
S_{\mathrm{NS}} &= \int d^{10}x(-G)^{1/2} e^{-2\Phi} \left( R + 4\partial_M \Phi \partial^M \Phi - \frac{1}{2}|H_3|^2 \right), \\
S_{\mathrm{RR}} &= -\frac{1}{2} \int d^{10}x(-G)^{1/2} e^{-2\Phi} \left( |F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2 \right), \\
S_{\mathrm{CS}} &= -\frac{1}{2} \int C_4 \wedge H_3 \wedge F_3,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{F}_3 &:= F_3 - C_0 \wedge H_3, \\
\tilde{F}_5 &:= F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3.
\end{aligned}$$

In addition, we have to impose the anti-self dual condition to $F_5 = *F_5$, where $*$ is the Hodge dual in a ten-dimensional spacetime.

## 2.3 D-branes

Classically, the D-branes are various dimensional objects where the endpoints of the open strings attach. As we discussed, type IIB superstring theory is constructed on the basis of the closed strings. Therefore, it seems that type IIB superstring theory does not contain the D-branes. However, the massless spectrum of type IIB superstring theory contains the higher form anti-symmetric tensor fields like the Maxwell gauge field. This fact implies the existence of the D-branes in type IIB superstring theory.

Let us consider the Maxwell equations in four dimensions with electric and magnetic sources

$$dF = *J_{\mathrm{mag}}, \tag{2.35}$$

$$d * F = *J_{\mathrm{elec}}. \tag{2.36}$$

The point-like electric charge density is described by the delta function such that

$$\rho = e\delta^{(3)}(\vec{r}) \tag{2.37}$$

For the point-like magnetic charge density is described in a similar way by replacing $e$ to $e_m$. The electric and magnetic charges are defined by the integration of the field strength

$$e = \int_{S^2} *F, \quad e_m = \int_{S^2} F \tag{2.38}$$

The magnetic field depending on the magnetic charge, for example

$$\vec{B} = \frac{e_m}{r^2}\vec{r} \tag{2.39}$$

can satisfy the Maxwell equation (2.36). This magnetic field induces a gauge potential depending on the magnetic charge, for example

$$A_\phi = e_m(1 - \cos\theta) \tag{2.40}$$

in the polar coordinate. On the other hand, the existence of the gauge field induces the covariant derivatives in the Schrödinger equations. The covariant derivatives have the gauge field dependence such that $eA_\mu$. Therefore, the gauge transformations carry both of charges. Hence, if we consider the wavefunctions of an electrically charged particle in a magnetic monopole background, the electric charge $e$ and the magnetic charge $e_m$ satisfy Dirac's quantization condition [18]

$$ee_m \in 2\pi\mathbb{Z}. \tag{2.41}$$

We can generalize the above result to the $(p+1)$-form gauge fields in $D$ dimensions [19,20]. The interaction with $(p+1)$-form fields can be described by

$$\int A_{p+1} = \frac{1}{(p+1)!}\int d^{p+1}\sigma A_{M_1\cdots M_{p+1}}\frac{\partial x^{M_1}}{\partial\sigma^0}\cdots\frac{\partial x^{M_{p+1}}}{\partial\sigma^p}, \tag{2.42}$$

where $\sigma^0, \cdots, \sigma^p$ are the coordinates of $(p+1)$ dimensions. The electric charge $e_p$ is also defined by the integration of the field strength on the sphere $S^{D-p-2}$

$$e_p = \int_{S^{D-p-2}} *F_{p+2}. \tag{2.43}$$

We can interpret eq. (2.42) as the coupling between the anti-symmetric tensor field and an extended charged object. Let us consider the following source in $(p+1)$ dimensions

$$J^{M_1\dots M_{p+1}} = e_p\int d^{p+1}\sigma\delta^{(D)}(y-x)\frac{\partial x^{M_1}}{\partial\sigma^0}\cdots\frac{\partial x^{M_{p+1}}}{\partial\sigma^p}.$$

Then, eq. (2.42) can be

$$\int A_{p+1} = \frac{1}{(p+1)!} \int d^{p+1}\sigma A_{M_1 \cdots M_{p+1}} J^{M_1 \ldots M_{p+1}}.$$

On the other hand, the magnetic charge $e_{p,m}$ is also defined by the integration of the field strength on the sphere $S^{p+2}$

$$e_{m,p} = \int_{S^{p+2}} F_{p+2}. \tag{2.44}$$

For the Maxwell gauge field (the 1-form gauge field), we consider the integral on a loop $S^1$. For the $(p+1)$-form gauge field in $D$ dimensions, we have to consider the integral on a sphere $S^{p+1}$. By the Stokes' theorem, the integral converts to the integration of the field strength on a $(p+2)$-dimensional hypersurface in $D$ dimensions. If we consider two $(p+2)$-dimensional hypersurfaces, the difference is topologically the sphere $S^{D-p-2}$. Originally, the $(p+1)$-form gauge fields carry the charge $e_p$. Therefore, we come to the generalization of Dirac's quantization condition

$$e_p e_{m,6-p} \in 2\pi\mathbb{Z} \tag{2.45}$$

if $D = 10$. In Ref. [21], the author derived the condition (2.45) by comparing the amplitude of the closed string channel and the effective potential of the higher form gauge fields.

# Chapter 3

# Magnetized extra dimensions

## 3.1 Super Yang-Mills theory

We discussed the D-branes in type IIB superstring theory. In addition, the world-volume theory of $N$ coincident supersymmetric D-branes is described by super Yang-Mills theory as we mentioned in Section 2.1. In this section, we consider super Yang-Mills theory as an effective field theory of type IIB superstring theory.

We consider the $\mathcal{N}=1$ super Yang-Mills action of the group $G$ in $D$ dimensions. The action is given by

$$S=\int d^D w\left[-\frac{1}{4g^2}\operatorname{Tr}\left(F^{MN}F_{MN}\right)+\frac{i}{2g^2}\operatorname{Tr}\left(\bar{\lambda}\Gamma^M D_M\lambda\right)\right], \tag{3.1}$$

where $M,N=0,\ldots,D-1$. The action has the gauge symmetry. The infinitesimal gauge transformations are described by

$$A_M \to A_M+\partial_M\theta+i[\theta,A_M], \tag{3.2}$$

$$\lambda \to \lambda+i[\theta,\lambda], \tag{3.3}$$

where $\theta$ is a $G$-valued arbitrary function as a gauge transformation parameter. In the following, we consider the group $G=U(N)$ for concreteness. In this case, the bases of the Lie algebra can be $(U_a)_{kl}=\delta_{ak}\delta_{al}$ and $(e_{ab})_{kl}=\delta_{ak}\delta_{bl}$. Hence, we can expand the gauge field and the fermions in terms of the bases, i.e.,

$$A_M=B_M^a U_a+W_M^{ab}e_{ab}, \quad \lambda=\chi^a U_a+\Psi^{ab}e_{ab}. \tag{3.4}$$

By inserting these, the Lagrangian can be rewritten as

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_B + \mathcal{L}_F && (3.5)\\
\mathcal{L}_B &= -\frac{1}{2g^2}\,\mathrm{Tr}\left[D_M W_N D^M W^N - D_M W_N D^N W^M - iG_{MN}[W^M, W^N]\right]\\
&\quad + \frac{1}{4g^2}\,\mathrm{Tr}\left[[W_M, W_N][W^M, W^N]\right]\\
&= \frac{i}{2g^2}\,\mathrm{Tr}\left[(D_M W_N - D_N W_M)[W^M, W^N] - \frac{1}{2i} G_{MN} G^{MN}\right], && (3.6)\\
\mathcal{L}_F &= \frac{i}{2g^2}\,\mathrm{Tr}\left(\bar{\Psi}\Gamma^M \partial_M \Psi - i\bar{\Psi}\Gamma^M [B_M, \Psi]\right)\\
&\quad + \frac{i}{2g^2}\,\mathrm{Tr}\left(\bar{\Psi}\Gamma^M [W_M, \Psi]\right)\\
&\quad + \frac{i}{2g^2}\,\mathrm{Tr}\left(\bar{\chi}\Gamma^M \partial_M \chi - i\bar{\chi}\Gamma^M [W_M, \Psi] - i\bar{\Psi}\Gamma^M [W_M, \Psi]\right). && (3.7)
\end{aligned}
$$

where

$$
\begin{aligned}
G_{MN} &= \partial_M B_N - \partial_N B_M,\\
D_M W_N &= \partial_M W_N - i[B_M, W_N].
\end{aligned}
$$

Let us consider the compactifications by a $D-4$ dimensional manifold with expectation values of the components of the gauge fields since we are interested in magnetized compactifications. We use indices $\mu$ for $0, \ldots, 3$ and $i$ for $4, \ldots, D-1$. To preserve the four-dimensional Poincaré invariance, we introduce the non-vanishing expectation values for $B_i^a$ and $W_i^{ab}$ only

$$
B_i^a(w) =< B_i^a > (y) + C_i^a(w), \quad W_i^{ab}(w) =< W_i^{ab} > (y) + \Phi_i^{ab}(w), \tag{3.8}
$$

where $w^M$ is the coordinate of the whole spacetime, $y^i$ is that of the extra dimensional space and $x^\mu$ is that of the four-dimensional Minkowski spacetime. For simplicity, we consider $< W_i^{ab} >= 0$. Inserting the expansions around the above expectation values, we can rewrite the action as

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_B^{(2)} + \mathcal{L}_F^{(2)} + \mathcal{L}_Y + \tilde{\mathcal{L}}, && (3.9)\\
\mathcal{L}_B^{(2)} &= \frac{i}{2g^2}\left(G_{ij}^a - G_{ij}^b\right)\left((\Phi_{ab}^i)^* \Phi_{ab}^j - (\Phi_{ab}^j)^* \Phi_{ab}^i\right) - \frac{1}{2g^2}\left[(D_\mu \Phi_{i,ab})^* (D^\mu \Phi_{ab}^i)\right.\\
&\left. + (\tilde{D}_i \Phi_{j,ab})^* (\tilde{D}^i \Phi_{ab}^j) - (D_\mu \Phi_{i,ba})(\tilde{D}^i W_{ab}^\mu) - (\tilde{D}_i \Phi_{j,ab})^* (\tilde{D}^j \Phi_{ab}^i)\right],\\
\mathcal{L}_F^{(2)} &= \frac{i}{2g^2}\bar{\Psi}_{ba}\Gamma^\mu D_\mu \Psi_{ab} + \frac{i}{2g^2}\bar{\Psi}_{ba}\Gamma^i \tilde{D}_i \Psi_{ab},\\
\mathcal{L}_Y &= \frac{1}{2g^2}\left(\bar{\Psi}_{ab}\Gamma^i \Phi_{i,bd}\Psi_{da} - \bar{\Psi}_{ab}\Gamma^i \Phi_{i,ca}\Psi_{bc}\right),
\end{aligned}
$$

where $\tilde{D}_i = \partial_i - ig < B_i >$, and $\tilde{D}_i \Phi_{j,ab} = \partial_i \Phi_{j,ab} - i < B_i^a > W_{j,ab} + i < B_i^b > W_{j,ab}$. The part $\tilde{\mathcal{L}}$ contains the irrelevant terms for the following discussions.

Let us remark on the gauge-fixing. The quadratic bosonic part $\mathcal{L}_B^{(2)}$ contains the term $-(D_\mu \Phi_{i,ba})(\tilde{D}^i W^\mu_{ab})$. By integration by parts, this term becomes $(\tilde{D}^i \Phi_{i,ba})(D_\mu W^\mu_{ab})$. Hence, it is convenient to consider the gauge-fixing condition $\tilde{D}^i \Phi_{i,ba} = 0$ since we can avoid complicated mass terms.

We can read the Dirac operator and the Laplacian from the action (3.9)

$$i\tilde{\not{D}}_{D-4}\psi_{ab,n} = i\Gamma^i \tilde{D}_i \psi_{ab,n}, \tag{3.10}$$
$$\Delta_{D-4}\phi_{i,ab,n} = -\tilde{D}_j \tilde{D}^j \phi_{i,ab,n}. \tag{3.11}$$

Then, the Kaluza-Klein expansions with respect to these operators are given as

$$\Psi_{ab}(w) = \sum_n \chi_{ab,n}(x) \otimes \psi_{ab,n}(y)$$
$$\Phi_{i,ab}(w) = \sum_n \varphi i, ab, n(x) \otimes \phi_{ab,i,n}(y),$$

where $\psi_{ab,n}(y)$ and $\phi_{ab,i,n}(y)$ are eigenfunctions of the Dirac operator (3.10) and the Laplacian (3.11), respectively. In the following parts, we will focus on coupling constants in four-dimensional effective field theories. Based on the action (3.9), The Yukawa couplings can be described by

$$\begin{aligned} S \supset \frac{1}{2g^2} \sum_{I,J,K} \left[ \int d^{D-4}y \psi^\dagger_{ab,I} \phi_{i,bd,J} \Gamma^i \psi_{da,K} \right] \int d^4x \bar{\chi}_{ab,I} \varphi_{i,bd,J,i} \chi_{da,K} \\ - \left[ \int d^{D-4}y \psi^\dagger_{ab,I} \phi_{i,ca,J} \Gamma^i \psi_{bc,K} \right] \int d^4x \bar{\chi}_{ab,I} \varphi_{i,ca,J} \chi_{bc,K}. \end{aligned} \tag{3.12}$$

The integration of the type

$$\int d^{D-4}y \psi^\dagger_{ab,I} \phi_{i,ca,J} \Gamma^i \psi_{bc,K}$$

is called an overlap integral.

## 3.2 Magnetized toroidal compactifications

From eq. (3.12), we have to derive the lightest modes of the operators (3.10) and (3.11)to compute the product of those modes and perform the integration on the extra dimensional space. For concreteness, let us consider magnetized toroidal compactifications on the basis of Refs. [4, 7]. For simplicity, we consider a two-dimensional torus as an extra dimensional space. Hence, we ignore the directions from $y^6$ to $y^{D-1}$.

For convenience, we adopt the real-coordinate description. Let us denote $v_4 = (2\pi R_4, 0)$ and $v_5 = (0, 2\pi R_5)$ as vectors of $\mathbb{R}^2$. We define the lattice space $\Lambda \simeq \mathbb{Z}^2$ generated by

$v_i$ $(i = 4, 5)$. The torus can be defined as the quotient space such that $T^2 \simeq \mathbb{R}^2/\Lambda$. The metric[1] is obtained from the line element

$$ds^2_{T^2} = dy^4 dy^4 + dy^5 dy^5.$$

We are interested in magnetized compactifications. Hence, we introduce an Abelian gauge field on $T^2$. Let us recall the gauge transformations of the gauge field and its field strength. Their gauge transformations are given by

$$A_i \to A_i + \partial_i \Omega, \quad F_{ij} \to F_{ij},$$

where $\Omega$ is an arbitrary function corresponding to a degree of freedom of the gauge transformations. The field strength must be well-defined on the torus because of the gauge invariance. On the other hand, the gauge field should be well-defined up to gauge transformations, i.e.,

$$A_i(y + v_j) = A_i(y) + \partial_i \Omega_j(y), \tag{3.13}$$

where $\Omega_j(y)$ is an arbitrary function on $T^2$ corresponding to a degree of freedom of the gauge transformations that are associated with the translation along the $v_j$-direction. This induces the transformation of the wavefunctions on the torus such that

$$\psi(y + v_j) = e^{i\Omega_j(y)} \psi(y). \tag{3.14}$$

If the gauge transformation function $\Omega_j(y)$ is zero, the gauge transformation (3.14) is nothing but the periodic boundary condition. Therefore, eq. (3.14) is interpreted as a kind of boundary condition. The boundary condition (3.14) is called the twisted boundary condition.

The torus has the loop that goes around the fundamental domain along the edges. We evaluate at the same point before and after the parallel transport along the loop. Since the wavefunctions must be single-valued functions, the gauge transformation functions must satisfy the following equation

$$\Omega_i(y + v_j) - \Omega_i(y) - \Omega_j(y + v_i) + \Omega_j(y) \in 2\pi\mathbb{Z}. \tag{3.15}$$

For concreteness, we consider the axial gauge defined by

$$A_4 = 0, \quad A_5 = F y^4,$$

The gauge transformation functions in eq. (3.13) can be obtained by

$$\Omega_i(y) = 2\pi R_4 F y^5 \delta^4_i + \text{const.}, \tag{3.16}$$

[1] We can introduce the complex structure modulus at this level. However, the dependence of the modulus is irrelevant to the following discussions. Hence, we consider the real coordinate of the torus for simplicity as mentioned.

where the constant is called the Wilson line, which we ignore, because it is irrelevant to the following discussion. The magnetic flux $F$ in eq. (3.16) must satisfy the following quantization condition

$$F \cdot \frac{A}{2\pi} \in \mathbb{Z} \to F = \frac{2\pi}{A} N\nu, \tag{3.17}$$

where $A = (2\pi)^2 R_4 R_5$, $N \in \mathbb{N}$, and $\nu = \pm 1$.

Let us start from a fermion in the fundamental representation. The zero mode equation is obtained by

$$\not{D}\psi = \begin{pmatrix} 0 & \partial_4 - i\partial_5 - \frac{2\pi}{A} N\nu y^4 \\ \partial_4 + i\partial_5 + \frac{2\pi}{A} N\nu y^4 & 0 \end{pmatrix} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = 0. \tag{3.18}$$

According to Refs. [4, 7], the zero modes can be described by the Jacobi-theta function

$$\psi_I^s = \left(2\pi R_5 \sqrt{\frac{\pi}{|F|}}\right)^{-1/2} \exp\left[-\frac{1}{2} F \cdot s (y^4)^2\right] \vartheta \begin{bmatrix} \frac{I \cdot s \cdot \nu}{N} \\ 0 \end{bmatrix} \left(\frac{N \cdot s \cdot \nu}{2\pi i R_5}(y^4 + isy^5), i\frac{R_4}{R_5} N \cdot s \cdot \nu\right), \tag{3.19}$$

where $s = \pm$ is the label of the chirality, $I = 0, \ldots, N-1$ is the label of the degeneracy, and the Jacobi-theta function is defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\delta, \tau) := \sum_{l \in \mathbb{Z}} e^{\pi i (a+l)^2 \tau + 2\pi i (a+l)(\delta + b)},$$

where $\mathrm{Im}\,\tau$ must satisfy $\mathrm{Im}\,\tau > 0$. Because of the normalization condition and the requirement from the definition of the Jacobi-theta function, the chirality $s$ and the sign of the magnetic flux $\nu$ must satisfy $s \cdot \nu = +1$. In addition, we can confirm the orthogonality of them. The zero modes can be written as

$$\psi_I^s = \left(2\pi R_5 \sqrt{\frac{\pi}{|F|}}\right)^{-1/2} \exp\left[-\frac{1}{2} |F| (y^4)^2\right] \vartheta \begin{bmatrix} \frac{I}{N} \\ 0 \end{bmatrix} \left(\frac{N}{2\pi i R_5}(y^4 + isy^5), i\frac{R_4}{R_5} N\right), \tag{3.20}$$

where $s \cdot \nu = +1$, which has already been incorporated.

Consequently, the chirality is determined by the sign of the magnetic flux and the generations as the degeneracy is determined by the absolute value of the magnetic flux. This means the realization of the chiral structure and the generations in the low energy effective field theory.

On the other hand, we are interested in a bifundamental fermion. For simplicity, we consider the $U(2)$ adjoint representation and the gauge background such that

$$A_4 = 0, \quad A_5 = \begin{pmatrix} F_1 y^4 & 0 \\ 0 & F_2 y^4 \end{pmatrix}, \tag{3.21}$$

where $F_m = 2\pi N_m \nu_m / A$ $(m = 1, 2)$. The magnetic flux (3.21) breaks $U(2)$ down to $U(1) \times U(1)$. Then, we can realize the bifundamental fermions from the off-diagonal component of adjoint fermions, i.e.,

$$\psi^{\pm} = \begin{pmatrix} \psi^{\pm}_{11} & \psi^{\pm}_{12} \\ \psi^{\pm}_{21} & \psi^{\pm}_{22} \end{pmatrix}, \tag{3.22}$$

From the above, the zero mode equation for each chirality is obtained by

$$(\partial_4 + is\partial_5) \begin{pmatrix} \psi^{\pm}_{11} & \psi^{\pm}_{12} \\ \psi^{\pm}_{21} & \psi^{\pm}_{22} \end{pmatrix} + s \begin{pmatrix} 0 & (F_1 - F_2) y^4 \psi^{\pm}_{12} \\ (F_2 - F_1) y^5 \psi^{\pm}_{21} & 0 \end{pmatrix} = 0 \quad (s = \pm). \tag{3.23}$$

The diagonal components are not affected by the magnetic fluxes. Therefore, the boundary condition becomes the double periodic boundary condition. Then, the zero mode must be a constant function. On the other hand, the off-diagonal components are affected by the magnetic fluxes. However, the net effect is the same as the fundamental representation case (3.18). Therefore, when $F_{12} = F_1 - F_2 > 0$ $(N_{12} := N_1 - N_2 > 0)$, the zero modes can be also described by the Jacobi-theta function

$$\psi^{+}_{12,I} = \left(2\pi R_5 \sqrt{\frac{\pi}{F_{12}}}\right)^{-1/2} \exp\left[-\frac{1}{2} F_{12} (y^4)^2\right] \vartheta \begin{bmatrix} \frac{I}{N_{12}} \\ 0 \end{bmatrix} \left(\frac{N_{12}}{2\pi i R_5}(y^4 + isy^5), i\frac{R_4}{R_5} N_{12}\right), \tag{3.24}$$

where $I = 0, \ldots, N_{12} - 1$.

Comparing with the Yukawa couplings (3.12), we have to consider the Kaluza-Klein expansion with respect to the Laplacian (3.11). To obtain the eigenfunctions of the Laplacian (3.11), let us compute the commutation relation of the Dirac operators, i.e.,

$$D := \partial_4 - i\partial_5 - i[A_4 - iA_5, \cdot\,], \quad D' = \partial_4 + i\partial_5 - i[A_4 + iA_5, \cdot\,], \tag{3.25}$$

$$[D, D'] = 2F_{12}, \tag{3.26}$$

Therefore, we can identify $D$ and $D'$ as an annihilation operator and a creation operator, respectively. If $\mathrm{sign}(F_{12}) = -1$, it is enough to interchange the role of the annihilation and the creation operators each other. On the other hand, the Laplacian (3.11) can be rewritten as

$$\not{D}^2 = \Delta + \begin{pmatrix} -F_{12} & 0 \\ 0 & F_{12} \end{pmatrix}.$$

Therefore, the zero modes of the Dirac operator are the lightest modes of the Laplacian. The generalization to $U(N)$ gauge group is straightforward. Consequently, it is enough to compute the product of the Jacobi-theta functions for the Yukawa couplings (3.12). According to Ref. [22], the Jacobi-theta functions satisfy the following product property

$$\vartheta\begin{bmatrix}\frac{I}{N_1}\\0\end{bmatrix}(z_1,\tau N_1)\cdot\vartheta\begin{bmatrix}\frac{J}{N_2}\\0\end{bmatrix}(z_2,\tau N_2)=\sum_{K\in\mathbb{Z}_{N_1+N_2}}\vartheta\begin{bmatrix}\frac{I+J+N_1K}{N_1+N_2}\\0\end{bmatrix}(z_1+z_2,\tau(N_1+N_2))$$
$$\times\vartheta\begin{bmatrix}\frac{N_2I-N_1J+N_1N_2K}{N_1N_2(N_1+N_2)}\\0\end{bmatrix}(z_1N_2-z_2N_1,\tau N_1N_2(N_1+N_2)). \tag{3.27}$$

It is important that the product of the Jacobi-theta functions is written by the linear combination of the Jacobi-theta functions. This indicates that the zero mode space is closed under the multiplicity in the sense of the usual multiplication of functions. Therefore, we can compute the Yukawa couplings from the coefficients of the linear combinations (3.27). To compute eq. (3.12), we need to apply the formula (3.27) to $\psi^I_{bd}\cdot\psi^J_{da}$. The result is

$$\psi_{bd,I}\cdot\psi_{da,J}=\frac{(2N_{bd})^{1/4}(2N_{da})^{1/4}}{(2N_{ba})^{1/4}}\sum_{K\in\mathbb{Z}_{N_{ba}}}\psi_{ba,I+J+N_{bd}K}\times\vartheta\begin{bmatrix}\frac{M_{da}I-M_{bd}J+M_{bd}M_{da}K}{M_{bd}M_{da}M_{ba}}\\0\end{bmatrix}(0,iM_{bd}M_{da}M_{ba}).$$

Consequently, the Yukawa coupling constants are obtained by

$$Y_{IJL}=\frac{(2N_{bd})^{1/4}(2N_{da})^{1/4}}{(2N_{ba})^{1/4}}\vartheta\begin{bmatrix}\frac{M_{da}I-M_{bd}J+M_{bd}M_{da}K}{M_{bd}M_{da}M_{ba}}\\0\end{bmatrix}(0,iM_{bd}M_{da}M_{ba}), \tag{3.28}$$

where $L=I+J+N_{bd}K$.

We showed that the Yukawa couplings in magnetized toroidal compactifications can be computed from the product property of the Jacobi-theta function. The Yukawa couplings are three-point coupling constants. According to Ref. [6], we generalize from three-point coupling constants to higher order coupling constants in magnetized toroidal compactifications.

- Four-point coupling constants

Because of the Lorentz invariance, the number of the fermions in general higher order coupling constants must be even. Therefore, it is enough to consider just boson-boson-boson-boson and boson-boson-fermion-fermion for four-point coupling constants. On the other hand, the lightest modes of the Laplacian are the same as the zero modes of the Dirac operator. Therefore, the computations for the two cases are essentially the same. This is similar for general higher order coupling constants in magnetized toroidal compactifications. Hence, it is enough to consider the following four-point coupling constants

$$Y_{IJKL}=\int d^2y\psi^{M_1}_I\psi^{M_2}_J\psi^{M_3}_K\left(\psi^{M_4}_L\right)^*. \tag{3.29}$$

where we denote the magnetic flux by $M_a$ for simplicity. From the gauge symmetry, $M_1+M_2+M_3=M_4$. In addition, we assume $\gcd(M_1,M_2,M_3)=1$ for simplicity. We have shown the product property of the Jacobi-theta functions. Therefore, we can evaluate the four-point coupling constants (3.29) by using the product property (3.27). The result is obtained by

$$Y_{IJKL} = \sum_{P \in \mathbb{Z}_{M_1+M_2}} \vartheta \begin{bmatrix} \frac{M_2 I - M_1 J + M_1 M_2 P}{M_1 M_2 M} \\ 0 \end{bmatrix} (0, iM_1 M_2 M) \times \vartheta \begin{bmatrix} \frac{M_3 L - M_4 K + M_3 M_4 P'}{M_3 M_4 M} \\ 0 \end{bmatrix} (0, iM_3 M_4 M), \tag{3.30}$$

where $M = M_1 + M_2 = -M_3 + M_4$ and $I + J + K + M_1 P + (M_1 + M_2) P'' = L + M_4 P'$ for the constraint of $P''$. On the other hand, according to Ref. [6], the result (3.30) can be interpreted as the insertion of the completeness relation. It is well-known that the eigensystem of the Dirac operator on a compact manifold forms the complete orthonormal system. Therefore, the entire eigenfunctions satisfy the completeness relation with respect to an appropriate inner product. In this case, we define the inner product by

$$(\Phi, \Psi) := \int_{T^2} \mathrm{Tr} \left( \Phi^\dagger \cdot \Psi \right),$$

where the dot $\cdot$ means the product with respect to the gauge group and the spinors if $\Phi$ and $\Psi$ are spinors. If we consider the splitting of eq. (3.29) by the delta function, we can insert the completeness relation. Namely,

$$Y_{IJKL} = \int d^2y d^2y' \psi_I^{M_1}(y) \psi_J^{M_2}(y) \delta^{(2)}(y - y') \psi_K^{M_3}(y') \left( \psi_L^{M_4}(y') \right)^*. \tag{3.31}$$

The contributions including the massive modes vanish because of the orthogonality between the massless modes and the massive modes. Hence, each of the two separated integrals is a three-point coupling constant (3.28), and the result is the same with the direct computation (3.30).

On the other hand, we can consider other decomposition of eq. (3.29). In eqs. (3.30) and (3.31), we first contracted $\psi_I^{M_1}$ and $\psi_J^{M_2}$. Actually, we can contract $\psi_I^{M_1}$ and $\psi_K^{M_3}$ or $\psi_J^{M_2}$ and $\psi_K^{M_3}$ at first. In that case, the decomposition process is as follows

$$Y_{IJKL} = \int d^2y d^2y' \psi_I^{M_1}(y) \psi_K^{M_3}(y) \delta^{(2)}(y - y') \psi_J^{M_2}(y') \left( \psi_L^{M_4}(y') \right)^*. \tag{3.32}$$

$$Y_{IJKL} = \int d^2y d^2y' \psi_I^{M_1}(y) \left( \psi_L^{M_4}(y) \right)^* \delta^{(2)}(y - y') \psi_J^{M_2}(y') \psi_K^{M_3}(y'). \tag{3.33}$$

In general, we have to consider the spin statistics for the interchange of the positions. However, we ignore the extra minus sign since it is irrelevant to the following discussion. The left-hand sides of eqs. (3.32) and (3.33) are the same as that of eq. (3.31). Therefore, these decomposition must be the same with each other. This structure is similar to the conformal block in conformal field theory.

- $n$-point coupling constants

The generalization to $n$-point coupling constants is straightforward. The $n$-point coupling constants are described as

$$Y_{I_1 \dots I_n} = \int d^2y \prod_{j=1}^{n} \psi_{I_j}^{M_j}(y), \tag{3.34}$$

Because of the Lorentz invariance and the gauge invariance, we have a selection rule

$$\sum_{j=1}^{n} M_j = 0.$$

By the insertion of the completeness relation, we can obtain a decomposition as an example, $M_n < 0$

$$\begin{aligned} Y_{I_1 \dots I_n} &= \int d^2y d^2(y') \prod_{j=1}^{n-2} \psi_{I_j}^{M_j}(y) \delta^{(2)}(y-y') \psi_{I_{n-1}}^{M_{n-1}}(y') \psi_{I_n}^{M_n}(y') \\ &= \sum_I Y_{I_1 \dots I_{n-2} I} Y_{I I_{n-1} I_n} \end{aligned} \tag{3.35}$$

## 3.3 Magnetized spherical compactifications

We have shown the product property of the zero modes in magnetized toroidal compactifications. In this section, we confirm that magnetized spherical compactifications have the same structure with respect to the zero modes.

In spherical compactifications, the difference from toroidal compactifications is the contribution of the curvature. Therefore, we have to derive the zweibein. Let us consider a complex projective space $\mathbb{P}^1$ as a complexified Riemann sphere. The metric, the Fubini-Study metric, is given by

$$ds^2 := g_{z\bar{z}} dz d\bar{z} = 4R^2 \frac{1}{(1+|z|^2)^2} dz d\bar{z}, \tag{3.36}$$

where $R$ is the radius of the sphere. The zweibein is defined as an object that connects a curved space and the flat space since several mathematical objects are defined only on a flat space, e.g., the Gamma matrix. The explicit definition of the zweibein ${e^\alpha}_i$ ($\alpha = 1, 2, i = z, \bar{z}$) is

$$g_{ij} = \eta_{\alpha\beta} {e^\alpha}_i {e^\beta}_j,$$

where $\eta_{\alpha\beta}$ is the Euclidean metric on a flat space. The metric (3.36) implies the following

$${e^1}_z = {e^1}_{\bar{z}} = \frac{R}{1+|z|^2}, \quad {e^2}_z = -{e^2}_{\bar{z}} = \frac{iR}{1+|z|^2}.$$

To define the Dirac operator, we have to introduce a spin connection. We can define the spin connection $w^{\alpha\beta}{}_i$ by the zweibein

$$w^{\alpha\beta}{}_i = \frac{1}{2}\left[e^{\alpha j}\left(\partial_i e^\beta{}_j - \partial_j e^\beta{}_i\right) - e^{\beta j}\left(\partial_i e^\alpha{}_j - \partial_j e^\alpha{}_i\right) - e^{j\alpha}e^{k\beta}\left(\partial_j e_{k\rho} - \partial_k e_{j\rho}\right)e^\rho{}_i\right].$$

In this case, its non-zero components are

$$w^{12}{}_z = \frac{iz}{1+|z|^2}, \quad w^{12}{}_{\bar{z}} = \frac{-i\bar{z}}{1+|z|^2}.$$

We define the gamma matrix as follows

$$\Gamma^z := e^z{}_\alpha \Gamma^\alpha, \quad \Gamma^{\bar{z}} := e^{\bar{z}}{}_\alpha \Gamma^\alpha,$$

where $\Gamma^\alpha$ is the gamma matrix on a flat space

$$\Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In Section 3.2, we introduced the background gauge field that induced the chirality and the degeneracy of the zero modes. In this case, we also introduce a background gauge field

$$A_z = \frac{iM}{2}\frac{\bar{z}}{1+|z|^2}, \quad A_{\bar{z}} = \frac{-iM}{2}\frac{z}{1+|z|^2}, \tag{3.37}$$

where $M \in \mathbb{Z}$. The field strength satisfies $\int_{S^2} F = -2\pi M$.

The Dirac operator and the zero mode equation are defined by

$$\not{D} := \Gamma^i\left[\partial_i + \frac{1}{4}w_{i\alpha\beta}\Gamma^{\alpha\beta} - iA_i\right], \quad \not{D}\psi = 0.$$

Explicitly,

$$\frac{1}{R}\begin{pmatrix} 0 & (1+|z|^2)\partial_{\bar{z}} - z\left(\frac{M+1}{2}\right) \\ (1+|z|^2)\partial_z - \bar{z}\left(\frac{-M+1}{2}\right) & 0 \end{pmatrix}\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = 0.$$

We can easily set the following ansatz

$$\psi^+ = f^+(\bar{z})(1+|z|^2)^{(1-M)/2}, \quad \psi^- = f^-(z)(1+|z|^2)^{(1+M)/2},$$

where $f^+$ and $f^-$ are arbitrary anti-holomorphic and holomorphic functions, respectively. The concrete forms of $f^\pm$ are determined by the normalization condition. Since we have to consider normalizable zero modes on $\mathbb{P}^1$, the normalization condition is the square integrability on $\mathbb{P}^1$, i.e.,

$$\int_{\mathbb{P}^1} d^2z\sqrt{g}\psi^\dagger \cdot \psi,$$

where $g = \det(g_{ij})$. In magnetized toroidal compactifications, the chirality and the degeneracy are determined completely by the magnetic flux. The square-integrable condition only determines the normalization condition. On the other hand, in magnetized spherical compactifications, we have to consider the square-integrable condition to determine the explicit form of the zero modes. By inserting the ansatz, we have, for example,

$$\int_{\mathbb{P}^1} d^2z\sqrt{g}\psi^\dagger \cdot \psi = 2\int_{\mathbb{P}^1} d^2z \frac{|f^+(\bar{z})|^2}{(1+|z|^2)^{(M+1)}}.$$

The complex projective space $\mathbb{P}^1$ is the one-point compactification of the complex plane. Therefore, if we convert from the complex coordinates to the polar coordinates, the convergence at the infinity is important. Obviously, the convergence is determined by the sign of $M$ and the polynomial power of $f^+$. Actually, if $M \leq 0$, there is no normalizable solution. On the other hand, if $M > 0$, the power of $f^+$ can be $M-1$ or less. Similarly, $\psi^-$ is normalizable if $M < 0$ and the polynomial power of $f^-$ can be $|M|-1$ or less.

Comparing with the Yukawa couplings (3.12), we have to consider the Kaluza-Klein expansion with respect to the Laplacian (3.11). We discuss the spectrum of scalar fields and vector fields separately since there is a curvature contribution.

Let us start from the scalar fields. In this case, there is no contribution of the curvature. Hence, it is sufficient to consider the Laplacian (3.11) without the curvature contribution. Namely,

$$-g^{ij}D_iD_j\phi = -(g^{z\bar{z}}D_zD_{\bar{z}} + g^{\bar{z}z}D_{\bar{z}}D_z)\phi = -m^2\phi, \tag{3.38}$$

where $D_z$ and $D_{\bar{z}}$ are the gauge covariant derivatives

$$D_z := \partial_z - iA_z = \partial_z + \frac{M}{2}\frac{\bar{z}}{1+|z|^2},$$
$$D_{\bar{z}} := \partial_{\bar{z}} - iA_{\bar{z}} = \partial_{\bar{z}} - \frac{M}{2}\frac{z}{1+|z|^2}.$$

In magnetized toroidal compactifications, we confirmed that the lightest modes of the Laplacian are the same functions as the zero modes of the Dirac operator. In this case, we can observe the same situation. To see it, we arrange the Laplacian (3.38) as follows,

$$\begin{aligned}-g^{ij}D_iD_j\phi &= -2g^{z\bar{z}}D_{\bar{z}}D_z\phi - g^{\bar{z}z}[D_z, D_{\bar{z}}]\phi \\ &= -2g^{z\bar{z}}D_{\bar{z}}D_z\phi + \frac{M}{2R^2}\phi.\end{aligned}$$

Since the spectrum of the operator $-2g^{z\bar{z}}D_{\bar{z}}D_z$ is positive semi-definite, the lightest modes are the solutions of $D_z\phi = 0$. On the other hand, the operator $D_z$ is nothing but the component of the Dirac operator for $\psi^+$. Therefore, the solution of $D_z\phi = 0$ has the same function as the zero modes of $\psi^+$. Accordingly, the condition of the magnetic flux for normalizable solutions is $M > 0$, and it induces $M + 1$ degeneracies of the solution.

Similarly, it is sufficient to consider the Laplacian (3.11) for the vector fields. For the sphere with the metric (3.36), the curvature contribution is given by

$$[\nabla_i, \nabla_j]\phi^{i,ab} = g^{\bar{z}z}[\nabla_{\bar{z}}, \nabla_z]\phi_z^{ab} = \frac{1}{R^2}\phi_z^{ab}.$$

Similarly, we can consider the case of $\phi_{\bar{z}}^{ab}$. More precisely, according to Ref. [75], the lightest modes are the tachyonic, but the zero modes can exist if $|M| \geq 2$ due to the normalizability. Consequently, it is sufficient for these bosonic fields to consider the equation, $D_z\phi = 0$.

In the following, we assume that the magnetic flux is positive, i.e., $M > 0$. The eigenfunctions are given by,

$$\psi^M_{(F),I} = \frac{1}{\mathcal{N}^M_{(F),I}} \cdot \frac{\bar{z}^I}{(1+|z|^2)^{\frac{M-1}{2}}}, \quad \psi^M_{(B),I} = \frac{1}{\mathcal{N}^M_{(B),I}} \cdot \frac{\bar{z}^I}{(1+|z|^2)^{\frac{M}{2}}} \tag{3.39}$$

where $\mathcal{N}^M_{(F),I}$ and $\mathcal{N}^M_{(B),I}$ are the normalization constants such that

$$|\mathcal{N}^M_{(F),I}|^2 = 4\pi R^2 \frac{\Gamma(I+1)\Gamma(M-I)}{\Gamma(M+1)}, \quad |\mathcal{N}^M_{(B),I}|^2 = 4\pi R^2 \frac{\Gamma(I+1)\Gamma(M+1-I)}{\Gamma(M+2)}.$$

In magnetized toroidal compactifications, the zero mode fermions cannot be distinguished from the lightest mode bosons. On the other hand, we can distinguish them in this case. Therefore, we have to consider three types of the products based on the spin statistics,

$$\begin{aligned}
\psi^M_{(B),I} \cdot \psi^{M'}_{(B),J} &= \frac{1}{\mathcal{N}^M_{(B),I} \cdot \mathcal{N}^{M'}_{(B),J}} \frac{\bar{z}^{I+J}}{(1+|z|^2)^{\frac{M+M'}{2}}} = \frac{\mathcal{N}^{M+M'}_{(B),I+J}}{\mathcal{N}^M_{(B),I} \cdot \mathcal{N}^{M'}_{(B),J}} \cdot \psi^{M+M'}_{(B),I+J}, \\
\psi^M_{(B),I} \cdot \psi^{M'}_{(F),J} &= \frac{1}{\mathcal{N}^M_{(B),I} \cdot \mathcal{N}^{M'}_{(F),J}} \frac{\bar{z}^{I+J}}{(1+|z|^2)^{\frac{M+M'-1}{2}}} = \frac{\mathcal{N}^{M+M'}_{(F),I+J}}{\mathcal{N}^M_{(B),I} \cdot \mathcal{N}^{M'}_{(B),J}} \cdot \psi^{M+M'}_{(F),I+J}, \\
\psi^M_{(F),I} \cdot \psi^{M'}_{(F),J} &= \frac{1}{\mathcal{N}^M_{(B),I} \cdot \mathcal{N}^{M'}_{(B),J}} \frac{\bar{z}^{I+J}}{(1+|z|^2)^{\frac{M+M'-2}{2}}} = \frac{\mathcal{N}^{M+M'-2}_{(B),I+J}}{\mathcal{N}^M_{(B),I} \cdot \mathcal{N}^{M'}_{(B),J}} \cdot \psi^{M+M'-2}_{(B),I+J}.
\end{aligned}$$

Since Yukawa couplings contain two fermions and single boson and one of the fermions is Hermitian conjugate, the Yukawa coupling constants can be obtained by

$$Y_{IJK} = \frac{\mathcal{N}^{M+M'}_{(F),I+J}}{\mathcal{N}^M_{(B),I} \cdot \mathcal{N}^{M'}_{(B),J}},$$

where $I+J=K$.

We can easily extend the results of the three-point couplings to higher order coupling constants. For example, the four-boson coupling constant can be obtained by

$$\begin{aligned}
Y_{IJKL} &= \int d^2z \psi^{M_1}_{(B),I}\psi^{M_2}_{(B),J}\psi^{M_3}_{(B),K}\left(\psi^{M_4}_{(B),L}\right)^* \\
&= \frac{\mathcal{N}^{M_1+M_2}_{(B),I+J}}{\mathcal{N}^{M_1}_{(B),I}\cdot\mathcal{N}^{M_2}_{(B),J}}\cdot\frac{\mathcal{N}^{M_1+M_2+M_3}_{(B),I+J+K}}{\mathcal{N}^{M_1+M_2}_{(B),I+J}\cdot\mathcal{N}^{M_3}_{(B),K}} \\
&= \frac{\mathcal{N}^{M_1+M_2+M_3}_{(B),I+J+K}}{\mathcal{N}^{M_1}_{(B),I}\cdot\mathcal{N}^{M_2}_{(B),J}\cdot\mathcal{N}^{M_3}_{(B),K}},
\end{aligned} \tag{3.40}$$

where $M_1+M_2+M_3=M_4$ and $I+J+K=L$. We mentioned that there are other decompositions, for example,

$$\begin{aligned}
Y_{IJKL} = \int d^2z \psi^{M_1}_{(B),I}\psi^{M_2}_{(B),J}\psi^{M_3}_{(B),K}\left(\psi^{M_4}_{(B),L}\right)^* &= \frac{\mathcal{N}^{M_1+M_3}_{(B),I+K}}{\mathcal{N}^{M_1}_{(B),I}\cdot\mathcal{N}^{M_3}_{(B),K}}\cdot\frac{\mathcal{N}^{M_1+M_2+M_3}_{(B),I+J+K}}{\mathcal{N}^{M_1+M_3}_{(B),I+K}\cdot\mathcal{N}^{M_2}_{(B),J}} \\
= \frac{\mathcal{N}^{M_1+M_2+M_3}_{(B),I+J+K}}{\mathcal{N}^{M_1}_{(B),I}\cdot\mathcal{N}^{M_2}_{(B),J}\cdot\mathcal{N}^{M_3}_{(B),K}}.
\end{aligned} \tag{3.41}$$

As we expected, eq. (3.40) is the same as eq. (3.41).

In conclusion, we can find two features of the coupling constants in the magnetized toroidal compactifications: the three-point coupling constants are determined by the expansion coefficients of the zero mode product, and any higher order coupling constants can be decomposed by the three-point coupling constants. In addition, the gauge symmetry constrains the value of the expansion coefficients, equivalently the coupling constants. For example, the gauge symmetry implies the vanishing of the coefficients of the terms violating the gauge symmetry. Moreover, it can be seen that a hierarchical structure of the three-point coupling constants is inherited to a higher order coupling constants. In Ref. [23], the hierarchical structure of the quark masses is realized in the magnetized toroidal compactifications.

On the other hand, it seems that the reason of the existence of the product property is the Jacobi-theta functions or the rational functions have a good property under a usual multiplication for functions. However, we can obtain strong suggestions for model buildings from a top-down approach if the similar result can be proved for general magnetized compactifications models. We will discuss this point in Chapter 5.

# Chapter 4

# Matrix Model

## 4.1 Ishibashi-Kawai-Kitazawa-Tsuchiya matrix model

Matrix models are proposed as nonperturbative formulations of superstring theory. In this thesis, we focus on Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) matrix model [24].

### 4.1.1 Green-Schwarz action in Schild gauge

The action of the IKKT matrix model is derived in two ways: the matrix regularization of Green-Schwarz action [25], and the large-$n$ reduction of super Yang-Mills theory [27,31,41–43]. In the following, we show both derivations.

The Green-Schwarz action is the action of a worldsheet theory for type IIB superstring theory. To derive the action of the IKKT matrix model, let us start from the covariant form of the Green-Schwarz action

$$S = -T \int d^2\sigma \left( \sqrt{-\frac{1}{2}\Sigma^2} + i\epsilon^{\alpha\beta}\partial_\alpha X^M \left(\bar{\theta}^1\Gamma_M\partial_\beta\theta^1 + \bar{\theta}^2\Gamma_M\partial_\beta\theta^2\right) + \epsilon^{\alpha\beta}\bar{\theta}^1\Gamma^M\partial_\alpha\theta^1\bar{\theta}^2\Gamma_M\partial_\beta\theta^2 \right),$$

where $\alpha, \beta = 1, 2,\ M = 0, \ldots, 9,\ T$ is the string tension, $\theta^1$ and $\theta^2$ are ten-dimensional Majorana spinors, and $\Gamma^M$ are Gamma matrices in 10 dimensions. $\Sigma^{MN}$ and ${\Pi^M}_\alpha$ are defined by

$$\begin{aligned}
\Sigma^{MN} &= \epsilon^{\alpha\beta}\Pi^M_\alpha\Pi^N_\beta, \\
\Pi^M_\alpha &= \partial_\alpha X^M - i\bar{\theta}^1\Gamma^M\partial_\alpha\theta^1 + i\bar{\theta}^2\Gamma^M\partial_\alpha\theta^2.
\end{aligned}$$

This action has $\mathcal{N} = 2$ SUSY of a target space and $\kappa$-symmetry as the gauge symmetry on the worldsheet.

- $\mathcal{N} = 2$ supersymmetry

$$\begin{aligned}
\delta\theta^1 &= \epsilon^1, \\
\delta\theta^2 &= \epsilon^2, \\
\delta X^M &= i\bar{\epsilon}^1\Gamma^M\theta^1 - i\bar{\epsilon}^2\Gamma^M\theta^2
\end{aligned}$$

- $\kappa$-symmetry

$$\begin{aligned}\delta\theta^1 &= \alpha^1,\\ \delta\theta^2 &= \alpha^2,\\ \delta X^M &= i\bar{\theta}^1\Gamma^M\alpha^1 - i\bar{\theta}^2\Gamma^M\alpha^2,\end{aligned}$$

where

$$\begin{aligned}\alpha^1 &= (1+\tilde{\Gamma})\kappa^1,\\ \alpha^2 &= (1-\tilde{\Gamma})\kappa^2,\\ \tilde{\Gamma} &= \frac{1}{2\sqrt{-\frac{1}{2}(\Sigma)^2}}\Sigma_{MN}\Gamma^{MN},\end{aligned}$$

where $\Gamma^{MN} = [\Gamma^M,\Gamma^N]/2$ and $\kappa^i$ $(i = 1,2)$ are local Majorana spinors. The $\kappa$-symmetry implies that a half of the degrees of freedom of $\theta^1$ and $\theta^2$ are redundant because of $\tilde{\Gamma}^2 = 1$. We fix the $\kappa$-symmetry by imposing $\theta^1 = \theta^2 \equiv \psi$. Then, the gauge-fixed action is given by

$$S = -T\int d^2\sigma\sqrt{-\frac{1}{2}\sigma^{MN}\sigma_{MN}} + 2i\epsilon^{\alpha\beta}\partial_\alpha X^M\bar{\psi}\Gamma_M\partial_\beta\psi. \tag{4.1}$$

where $\sigma^{MN} = \epsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N$. The action (4.1) still has $\mathcal{N} = 2$ SUSY, which is provided by mixing the original $\mathcal{N} = 2$ SUSY and the $\kappa$-symmetry under the gauge fixing condition.

To obtain the Schild gauge action, we have to consider the Euclidean action. We consider the Wick rotation of the target space and the worldsheet. Such a Wick rotation can be obtained by the following replacements

$$\begin{aligned}X^0 &\to -iX^0_{(E)},\\ \sigma^0 &\to -i\sigma^0_{(E)},\\ \Gamma^0 &\to -i\Gamma^0_{(E)},\\ \epsilon^{ab} &\to -i\epsilon^{ab}_{(E)}.\end{aligned}$$

We will omit the subscript $(E)$. We note that the Dirac conjugation is defined by $\bar{\psi} := \psi^t C$, where $C$ is the charge conjugate matrix.

The Schild gauge action is defined by introducing a scalar density as an independent variable. We introduce the scalar density of the worldsheet $\sqrt{g}$ and define the action as

$$S = \frac{1}{2}\int d^2\sigma\left(\frac{\xi}{\sqrt{g}}\det(\partial_\alpha X^M\partial_\beta X_M) + \rho\sqrt{g}\right) + \xi\int d^2\sigma\frac{i}{2}\epsilon^{\alpha\beta}\partial_\alpha X^M\bar{\psi}\Gamma_M\partial_\beta\psi, \tag{4.2}$$

where $\xi$ and $\rho$ are arbitrary constants. The action (4.2) is equivalent to the action (4.1) by eliminating the scalar density $\sqrt{g}$ and setting $T = \sqrt{\xi\cdot\rho}$.

To consider the matrix regularization, the symmetry with respect to the worldsheet should be kept as only the area preserving diffeomorphism. To realize it, let us consider a gauge fixing of $\sqrt{g}$. The diffeomorphism of the worldsheet has two degrees of freedom. By using one of these degrees of freedom, we can fix $\sqrt{g}$ as

$$\sqrt{g} = \eta\omega(\sigma),$$

where $\eta$ is an arbitrary constant and $\omega(\sigma)$ is a fixed scalar density. Then the action (4.2) can be rewritten as

$$S_{\text{Schild}} = \xi \int d^2\sigma \eta\omega(\sigma) \left( \frac{1}{4}\{X^M, X^N\}^2 + \frac{i}{2}\bar{\psi}\Gamma_M\{X^M, \psi\} \right) + \frac{1}{2}\rho \int d^2\sigma\eta\omega(\sigma), \tag{4.3}$$

where the Poisson bracket is defined by

$$\{f, g\} := \frac{1}{\eta\omega(\sigma)} \epsilon^{\alpha\beta}\partial_\alpha f \partial_\beta g.$$

### 4.1.2 IKKT matrix model action from matrix regularization

We cannot assure that the path integral

$$Z = \int \mathcal{D}\sqrt{g}\mathcal{D}X\mathcal{D}\psi e^{-S_{\text{Schild}}}$$

has no divergence or defines a controllable quantum theory. The principle of matrix models is to obtain a controllable action by the matrix regularization. The matrix regularization is an approximation of a Poisson algebra on a manifold by a Hermitian matrix algebra. In this case, we consider the matrix regularization with respect to a worldsheet $\Sigma$. More precisely, the matrix regularization is defined by the following conditions (c.f., Ref. [26]) with respect to linear maps $\{T_n\}_{n=1}^{\infty}$ from $f, g \in C^\infty(\Sigma, \mathbb{R})$ to $n \times n$ Hermitian matrices with the parameter $h_n$ such that $\lim_{n\to\infty} nh_n < \infty$:

$$\begin{aligned}
&\lim_{n\to\infty} ||T_n(f)|| < \infty, \\
&\lim_{n\to\infty} ||T_n(fg) - T_n(f)T_n(g)|| = 0, \\
&\lim_{n\to\infty} ||\partial_n^{f_1} \cdots \partial_n^{f_l}(T_n(f)) - T_n(\partial^{f_1} \cdots \partial^{f_l} f)|| = 0, \\
&\lim_{n\to\infty} 2\pi h_n \operatorname{Tr}(T_n(f)) = \int_\Sigma f
\end{aligned}$$

where $\partial^{f_1}(f) := \{f_1, f\}$ and $\partial_n^{f_1}(T_n(f)) := [T_n(f), T_n(f_1)]/ih_n$.

The key concept of the matrix regularization is to replace Poisson brackets by commutators. This is similar to the relationship between analytical mechanics and quantum mechanics. In addition, the degree of freedom is no longer infinite, and the theory becomes statistical mechanics. This fact opens the possibility to obtain a controllable quantum theory of superstring theory.

The prescription of the matrix regularization is

$$\{\cdot,\cdot\} \to -i\frac{n}{2}[\cdot,\cdot], \quad \int_\Sigma \to \frac{4\pi}{n}\operatorname{Tr}. \tag{4.4}$$

We can apply the prescription to the action (4.3), which becomes

$$S = \xi' \operatorname{Tr}\left(-\frac{1}{4}[X^M, X^N]^2 + \frac{1}{2}\bar{\psi}\Gamma_M[X^M,\psi]\right) + \rho' \operatorname{Tr}(\mathbf{1}), \tag{4.5}$$

where $\eta = n/2$, $\xi' = 2\pi\xi$, and $\rho' = \pi\rho$. The dimension of matrices is $n \times n$. On the other hand, according to Ref. [24], the IKKT (type IIB) matrix model is defined by

$$S = \xi' \operatorname{Tr}\left(-\frac{1}{4}[X^M, X^N]^2 + \frac{1}{2}\bar{\psi}\Gamma_M[X^M,\psi]\right) \tag{4.6}$$

The matrices in the action (4.6) have a block matrix structure. By considering the integral out of the off-diagonal part, the term that is proportional to $\operatorname{Tr}\mathbf{1}$ in the action (4.5) can be derived. Therefore, we can identify the action (4.6) as the fundamental action of the IKKT matrix model.

### 4.1.3 IKKT matrix model from Large-$n$ reduction

In the previous section, we derived the action of the IKKT matrix model from the Green-Schwarz action. Hence, the connection with type IIB superstring theory is natural. On the other hand, we have to discover some non-perturbative aspects like D-branes. In Section 3.1, we considered super Yang-Mills theory as an effective theory for D-branes. The large-$n$ reduction connects super Yang-Mills theory to the IKKT matrix model.

#### Large-$n$ reduction in lattice gauge theory

Originally, the large-$n$ reduction is proposed in the context of lattice field theory by Eguchi and Kawai [27]. Let us start from the partition function of $U(n)$ gauge theory in lattice field theory, e.g.,

$$Z = \int \Pi_{x,M}\mathcal{D}U_{x,M}e^{-S}, \quad S = \beta\sum_x\sum_{M\neq N}^{D}\left(1 - \frac{1}{n}\operatorname{Re}\operatorname{Tr}\left(U_{x,M}U_{x+M,N}{U^\dagger}_{x+N,M}{U^\dagger}_{x,N}\right)\right), \tag{4.7}$$

where $U_{x,M}$ is a link variable in $D$-dimensional spacetime and $\beta = 2n/g^2$ is the normalization factor. This is the standard action of $U(n)$ gauge theory in lattice field theory. According to Ref. [27], $U(n)$ gauge theory described by eq. (4.7) can be replaced effectively in the limit $n \to \infty$ with fixed $\lambda := ng^2$ by

$$Z = \int \Pi_M \mathcal{D}U_M e^{-S}, \quad S = \beta \sum_{M \neq N}^{D} \left( 1 - \frac{1}{n} \mathrm{Tr}\left( U_M U_N U^\dagger{}_M U^\dagger{}_N \right) \right), \tag{4.8}$$

where we impose the periodic boundary condition to define the path integral. The action (4.8) does not contain the information of spacetime points, c.f., the label $x$ in the action (4.7). Therefore, we can identify the theory described by the action (4.8) as a 0-dimensional field theory. In the following, we call a theory describe by eq. (4.8) the reduced model.

To prove the equivalence, we focus on Wilson loop amplitudes since we are interested in gauge invariant quantities. Based on the action (4.7), a standard Wilson loop amplitude along a contour $C$ is written by

$$< W(C) >=< \mathrm{Tr}\left( U_{x,M_1} U_{x+M_1,M_2} \cdots U_{x-M_3,M_3} \right) >, \tag{4.9}$$

where the bracket $< \cdot >$ means the expectation value defined by the partition function (4.7). We can write the corresponding Wilson loop amplitude in the reduced model

$$< W_r(C) >=< \mathrm{Tr}\left( U_{M_1} U_{M_2} \cdots U_{M_3} \right) >, \tag{4.10}$$

where the bracket $< \cdot >$ means the expectation value defined by the partition function (4.8).

In Ref. [27], the authors proved the equivalence by confirming that the Wilson loop amplitudes (4.9),(4.10) obey the same Schwinger-Dyson equations in the limit $n \to \infty$.

Let us consider the loop $C$ includes the link variable $U_{x,M}$, which is only encountered once in this loop. We denote $C'$ as the contour without the link $(x, x+M)$ in $C$. Hence, we can write the Wilson loop amplitude $W(C)$ as

$$< W(C) >=< \mathrm{Tr}\left( U_C \right) >=< \mathrm{Tr}\left( U_{C'} U_{x,M} \right) >,$$

where $U_C$ means the appropriate combination of the link variable along the contour $C$.

In the following, we derive the Schwinger-Dyson equation of $< \mathrm{Tr}\left( U_{C'} \tau^a U_{x,M} \right) >$ as an example, where $\tau^a$ is the generator of the $U(n)$ gauge group. The Schwinger-Dyson equation can be derived by imposing the invariance of the expectation value under the infinitesimal transformation of $U_{x,M}$. Namely,

$$U_{x,M} \to (1 + i\epsilon\tau^a) U_{x,M},$$

where $\epsilon$ is a real-valued infinitesimal parameter. If we collect the first order corrections with respect to $\epsilon$, we can obtain the identity

$$\begin{aligned}
&< \mathrm{Tr}\left( U_{C'} \tau^a U_{x,M} \right) > \\
&= \frac{\beta}{2n} < \mathrm{Tr}\left( U_{C'} \tau^a U_{x,M} \right) \times \\
&\sum_{N \neq M} \Big( \mathrm{Tr}\left[ U_{x,N} U_{x+N,M} U^\dagger{}_{x+M,N} U^\dagger{}_{x,M} \right] + \mathrm{Tr}\left[ U^\dagger{}_{x-N,N} U_{x-N,M} U_{x+M-N,N} U^\dagger{}_{x,M} \tau^a \right] \\
&- \mathrm{Tr}\left[ \tau^a U_{x,M} U_{x+M,N} U^\dagger{}_{x+N,M} U^\dagger{}_{x,N} \right] - \mathrm{Tr}\left[ \tau^a U_{x,M} U^\dagger{}_{x+M-N,N} U^\dagger{}_{x-N,M} U_{x-N,N} \right] \Big) > .
\end{aligned}$$

By applying the formula

$$\sum_{a=1}^{N^2} \tau_{ij}\tau_{kl} = \frac{1}{2}\delta_{il}\delta_{jk},$$

we can obtain the Schwinger-Dyson equation as

$$< W(C) >= \frac{n}{\lambda}\sum_{N\neq M}\left(< W(D_{1,+}) > + < W(D_{1,-}) > - < W(D_{2,+}) > - < W(D_{2,-}) >\right),$$

where the contours are defined by

$$\begin{aligned}
D_{1,+} &= (y,\dots,x,x+N,x+N+M,x+M,\dots y),\\
D_{1,-} &= (y,\dots,x,x-N,x-N+M,x+M,\dots y),\\
D_{2,+} &= (y,\dots,x,x+M,x+N+M,x+N,x,x+M,\dots,y),\\
D_{2,-} &= (y,\dots,x,x+M,x-N+M,x-N,x,x+M,\dots,y).
\end{aligned}$$

Let us consider the corresponding computation in the reduced model. We can identify the contour $C_r$ in the reduced model from the contour $C = (x, x+M_1, x+M_1+M_2, \dots, x+M_1+M_2+\cdots+M_3)$ in the original theory, i.e., $C_r = (M_1, M_2, \dots, M_3)$.

The difference in the Schwinger-Dyson equations is the contribution of the contour $C_r$ corresponding to $C = (x,\dots,y,y+M,\dots,z,z+M,\dots,x)$, where $y \neq z$. Consequently, we have to consider the contribution $< W(L_{1,r})W(L_{2,r}) >$, where $L_1 = (y, y+M,\dots,z)$ and $L_2 = (z, z+M,\dots,y)$ are the open contours in the original theory.

Originally, the large-$n$ limit induces the simplification of the perturbative expansion of the theory [28]. In other words, only planar diagrams survive in the limit. In addition, the following factorization is proved at large-$n$ [29, 30]

$$< \hat{A}\hat{B} >=< \hat{A} >< \hat{B} > + \frac{1}{n}.$$

By considering this factorization, the different contribution can be written as

$$< W(L_{1,r})W(L_{2,r}) >=< W(L_{1,r}) >< W(L_{2,r}) > + \frac{1}{n}.$$

On the other hand, the action (4.8) has the global symmetry called the center symmetry such that

$$U_M \to e^{i\theta}U_M, \quad \theta \in \{0, \frac{2\pi}{n}, \cdots, \frac{2(N-1)\pi}{n}\}.$$

Therefore,

$$< W(L_{1,r}) >=< W(L_{2,r}) >= 0,$$

if we impose that the center symmetry is not broken spontaneously. Consequently, all different contributions vanish at the limit $n \to \infty$, and the equivalence is proved.

We have to remark about the center symmetry. Actually, the center symmetry is spontaneously broken at high $\beta$ for $D > 2$. This can be seen in the Monte Carlo simulations [31–33] and in perturbative computations [31, 34]. Hence, there are several works to care about the center symmetry, e.g., twisted Eguchi-Kawai model [35–40].

**Large-n reduction in gauge theory**

In the previous subsection, we introduced large-$n$ reduction in lattice field theory. Actually, this reduction holdsin continuous field theory. To describe large-$n$ reduction in continuous field theory, the following prescription for a matrix field $\phi(x)$ is proposed in Refs [41–43],

$$\begin{aligned}
&\phi(x) \to e^{iP\cdot x}\phi e^{-P\cdot x}, \\
&i\partial_M \to P_M,
\end{aligned}$$

where $(P_M)_{ab} := p^a_M \delta_{ab}$ is a set of diagonal matrices of quenched momenta. We assume that the quenched momenta lie in the hypercube whose edges have the length $\Lambda$. We can confirm that the length $\Lambda$ corresponds to the ultraviolet cutoff.

Let us consider pure $U(n)$ gauge theory with a Euclidean action

$$S = \frac{1}{4}\int d^D x \, \mathrm{Tr}(F_{MN})^2, \quad F_{MN} = \partial_M A_N - \partial_N A_M + i[A_M, A_N].$$

By applying the above prescription, we can obtain the 0-dimensional $U(n)$ gauge theory as

$$\begin{aligned}
&S = \left(\frac{2\pi}{\Lambda}\right)^D \frac{1}{4}\,\mathrm{Tr}\left(\tilde{F}_{MN}\right)^2, \qquad (4.11)\\
&F_{MN} = i[\mathcal{D}_M, \mathcal{D}_N], \quad \mathcal{D}_M = P_M + A_M.
\end{aligned}$$

In the definition of the $D_M$, $A_M$ can be absorbed into $P_M$ by the redefinition. Therefore, the action (4.11) becomes

$$S = -\left(\frac{2\pi}{\Lambda}\right)^D \frac{1}{4}\,\mathrm{Tr}\left([P_M, P_N]\right)^2 .$$

This is nothing but the bosonic part of the action of the IKKT matrix model (4.6). More precisely, we can obtain the action (4.6) from super Yang-Mills action by applying the above prescription.

In summary, the IKKT matrix model is defined by the matrix regularization of the Green-Schwarz action or the large-$n$ reduction of super Yang-Mills theory. The action is given by

$$S=\frac{1}{g^2}\,\mathrm{Tr}\left(-\frac{1}{4}[X^M,X^N]^2+\frac{1}{2}\bar{\psi}\Gamma_M[X^M,\psi]\right), \tag{4.12}$$

where we introduced $g$ as a scale factor for convenience. This factor is not essential since the factor $g$ can be absorbed into $X^M$ and $\psi$. In the following, we consider the action (4.12).

## 4.2 Symmetry and equations of motion

In the above, we defined the action of IKKT matrix model (4.12). Clearly, the action (4.12) has the 10D Lorentz invariance if $X^M$ and $\psi$ transform as 10D vectors and 10D spinors, respectively. Additionally, the action (4.12) has three symmetries:

1. Translations

$$\delta X^M = C^M\cdot\mathbf{1},\quad \delta\psi=0, \tag{4.13}$$

2. Gauge transformations

$$\delta X^M = i[\lambda, X^M],\quad \delta\psi = i[\lambda,\psi], \tag{4.14}$$

3. $\mathcal{N}=2$ SUSY

$$\delta^{(1)}X_M = i\bar{\epsilon}_1\Gamma_M\psi,\quad \delta^{(1)}\psi=\frac{i}{2}\Gamma^{MN}[X_M,X_N]\epsilon_1, \tag{4.15}$$

$$\delta^{(2)}X_M=0,\quad \delta^{(2)}\psi=\epsilon_1\mathbf{1}. \tag{4.16}$$

If the eigenvalues of $X^\mu$ can be interpreted as space-time points, we can obtain "$\mathcal{N}=2$ SUSY on 10D space-time" by mixing the $\mathcal{N}=2$ SUSY (4.15),

$$Q^{(1)'}:=Q^{(1)}+Q^{(2)},\quad Q^{(2)'}:=i(Q^{(1)}-Q^{(2)}),$$

where $Q^{(1)}$ and $Q^{(2)}$ are the generators of the supersymmetry (4.15). In addition, we denote $P_M$ as the generator of the translations (4.13). Then

$$[\bar{\epsilon}_1 Q^{(i)'},\bar{\epsilon}_2 Q^{(j)'}]=-2\delta^{ij}\bar{\epsilon}_1\Gamma^M\epsilon_2 P_M.$$

where $i,j=1,2$. On the other hand, eq. (4.14) is infinitesimal form of the gauge transformation. More precisely, finite forms can be written as unitary transformations by a unitary matrix $U$

$$X^M\to UX^MU^\dagger,\quad \psi\to U\psi U^\dagger, \tag{4.17}$$

if the trace in the action (4.12) satisfies the cyclic property. In general, the trace should be well-defined in the definition of the IKKT matrix model action (4.12). However, we have to confirm it when we consider a concrete model. We will discuss this point for the case of noncommutative torus in Chapter 6.

On the other hand, "equations of motion" in the IKKT matrix model mean variational problems since the IKKT matrix model has no space-time by definition as mentioned above. By considering the variations of $X^M$ and $\psi$, the equations of motion are given by

$$\begin{aligned} &[[X^M, X^N], X_N] - \frac{1}{2}(C\Gamma^M)_{ab}\{\psi^a, \psi^b\} = 0, \\ &\Gamma^M_{ab}[X_M, \psi^b] = 0. \end{aligned}$$

Let us consider $\psi = 0$ to discuss classical solutions. The simplest solution is given by a completely simultaneously diagonalizable case, i.e.,

$$[X^M, X^N] = 0 \quad \text{for all} \quad M, N = 0, \cdots 9.$$

In this case, however, there is an attractive force between the eigenvalues from the one-loop potential. Then, the eigenvalues cannot have enough spread, and the correspondence to an original theory will not be valid.

The second simplest solution is

$$[X^M, X^N] = i\theta^{MN}, \tag{4.18}$$

where $\theta^{MN}$ is an arbitrary constant. We omitted the identity matrix in the right-hand side of eq. (4.18). This solution is interpreted as a D-brane (BPS) configuration as follows. The transformation $\delta^{(1)}$ can be canceled by the other transformation $\delta^{(2)}$ since the transformation $\delta^{(1)}$ is proportional to the identity due to the solution (4.18). More precisely, by setting $\epsilon_2 = \pm\frac{1}{2}\theta^{MN}\Gamma_{MN}\epsilon_1$, then

$$(\delta^{(1)} \pm \delta^{(2)})X^M = 0, \quad (\delta^{(1)} \pm \delta^{(2)})\psi = 0.$$

Therefore, the solution (4.18) preserves $\mathcal{N} = 1$ SUSY. In addition, the 1-loop potential around the solution (4.18) reproduces the D-brane interactions through closed strings [24].

On the other hand, the bosonic matrices may have degree of freedom of space-time. Therefore, the solution (4.18) may represent the noncommutativity of space-time. For example, super Yang-Mills theory with the star-product can be derived from IKKT matrix model [44].

## 4.3 Noncommutative torus

We considered magnetized toroidal compactifications in Section 3.2. In that case, the torus is defined by the identifications. The twisted boundary condition assures the compatibility between the gauge transformations and such identifications. It is important that the shifts to define the torus are interpreted as the gauge transformations since there is no change in physics.

In the IKKT matrix model, we can consider the similar concept by combining the translations (4.13) and the unitary transformations (4.17). According to Ref [45], noncommutative torus is realized by this unitary transformation. To define noncommutative torus, we restrict ourselves to the subspace of contents $(X^M, \psi)$ belonging to the same gauge class before and after the shifts of $(X^M, \psi)$, i.e.,

$$
\begin{aligned}
&U_4 Y^4 U_4^{-1} = Y^4 + 2\pi R_4, \quad U_5 Y^5 U_5^{-1} = Y^5 + 2\pi R_5, \\
&U_i Y^j U_i^{-1} = Y^j \quad \text{if} \quad i \neq j \ (i,j = 4,5), \\
&U_i X^M U_i^{-1} = X^M \quad \text{if} \quad X^M \neq Y^i \\
&U_i \psi U_i^{-1} = \psi,
\end{aligned}
\tag{4.19}
$$

where $Y^i$ $(i = 4,5)$ denotes as the direction of the noncommutative torus. The above conditions are compatible with the equations of motions (4.18). Then, in the remaining part of this chapter and Chapter 6, we adopt the solutions of (4.18) when we consider noncommutative torus. On the other hand, the conditions (4.19) cannot be satisfied by any finite dimensional representations since the trace on both hands are not equal. Therefore, we should interpret $X^M$ and $\psi$ as operators on an infinite dimensional Hilbert space. From this fact, the cyclic property of the trace in the action (4.12) is not ensured as mentioned above. We will construct the trace satisfying the cyclic property in Chapter 6.

The conditions (4.19) imply that $U_4 U_5 U_4^{-1} U_5^{-1}$ commutes with $X^M$ and $\psi$. Therefore, we can assume that $U_4 U_5 U_4^{-1} U_5^{-1}$ is a scalar operator, i.e.,

$$
U_4 U_5 = e^{2\pi i \theta} U_5 U_4, \tag{4.20}
$$

where $\theta$ is a real number that is called a noncommutative parameter since $U_4$ and $U_5$ can correspond to the basis of the Fourier expansion, which are usual functions and commutative with each other, for double periodic functions. This real num as a noncommutative parameter. More precisely, the parameter $\theta$ can be restricted to $\theta \in [0, 1/2)$ because of some isomorphisms. Equation (4.20) is called the algebra of noncommutative torus.

The mathematical structure of the algebra of noncommutative torus depends on whether the parameter $\theta$ is a rational number or irrational number. When the parameter $\theta$ is a rational number $\theta = p/N$, where $p$ and $N$ are natural numbers, the concrete representation matrices can be obtained by the following $N \times N$ matrices

$$
V_{\text{shift}} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & 0 & 1 & \cdots \\ & \vdots & \vdots & 1 \\ 1 & \cdots & \cdots & 0 \end{pmatrix}, \quad V_{\text{clock}} = \begin{pmatrix} 1 & & & \\ & w_N & & \\ & & \cdots & \\ & & & w_N^{N-1} \end{pmatrix}, \tag{4.21}
$$

where $w_N = e^{2\pi i/N}$. The above matrices are called the shift matrix and clock matrix, respectively. By considering the determinant on both hands of eq. (4.20), the selection $\theta = p/N$ is compatible. On the other hand, the case of irrational parameter $\theta \in \mathbb{R} \backslash \mathbb{Q}$ cannot be realized by any finite dimensional representation since the trace on both hands of eq. (4.20) are not equal. In such a case, the trace, which is restricted to $X^M$, does not have the cyclic property. Then, the algebra of noncommutative torus is consistent.

In the remaining part of this chapter and Chapter 6, we consider the algebra of noncommutative torus with irrational parameter $\theta \in \mathbb{R}\backslash\mathbb{Q}$. We use a "hat" to indicate an operator. A concrete infinite dimensional representation can be realized as follows. Let us start from the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^m$, where $m \in \mathbb{N}$ ($m$ corresponds to $\mathbb{Z}_m$ in the constructions of Refs. [45–47]). We define $\hat{U}_4$ and $\hat{U}_5$ as operators on $\mathcal{H}$

$$\hat{U}_4 \left| x : j \right\rangle = \left| x - \frac{n}{m} + \theta : j - 1 \right\rangle, \quad \hat{U}_5 \left| x : j \right\rangle = \exp\left[ -2\pi i \left( x - \frac{nj}{m} \right) \right] \left| x : j \right\rangle,$$

where $n$ is some integer. For simplicity, we set $m$ and $n$ are positive and co-prime or $m = 1$ and $n = 0$. From the above, we define $\hat{Y}^4$ and $\hat{Y}^5$ as

$$\hat{Y}^4 := \frac{2\pi m}{n - m\theta} R_4 \hat{x} \otimes \mathbf{1}_m, \quad \hat{Y}^5 := R_5 \hat{p} \otimes \mathbf{1}_m,$$

where $\hat{x}$ and $\hat{p}$ are the position and the momentum operators in one-dimensional quantum mechanics (where $\hbar = 1$). We can confirm that the above operators satisfy the conditions (4.19) and (4.20) with $\theta^{45} = \frac{2\pi m}{n - m\theta} R_4 R_5$. In the following, we omit the identity on $\mathbb{C}^m$. From an analogy of quantum mechanics,

$$\langle Y^4 | Y^5 \rangle = \frac{1}{A} \exp\left[ i \frac{Y^4 Y^5}{\theta^{45}} \right], \quad A = (2\pi)^2 R_4 R_5,$$

where $\hat{Y}^4 \left| Y^5 \right\rangle = Y^4 \left| Y^4 \right\rangle$ and $\hat{Y}^5 \left| Y^5 \right\rangle = Y^5 \left| Y^5 \right\rangle$. We can restrict spectra of $\hat{Y}^4$ and $\hat{Y}^5$ to $0 \le Y^4 \le 2\pi R_4$ and $0 \le Y^5 \le 2\pi R_5$ since the periods correspond to the unitary equivalence by $\hat{U}_4$ and $\hat{U}_5$.

We introduced noncommutative torus as a classical solution of the IKKT matrix model. To compare with the result of Section 5, we have to introduce the notion of gauge fields. We will discuss this point in Chapter 6.

## 4.4 Modified IKKT matrix model

In the previous section, we derived the action of the IKKT matrix model by applying the matrix regularization or the large-$n$ reduction. These methods can be applied to other theories. Actually, the authors of Ref. [48] proposed the modified IKKT matrix model by applying the above methods to super Yang-Mills theory on three dimensions with the Chern-Simons and Majorana mass terms,

$$S = \frac{1}{g^2} \operatorname{Tr} \left( -\frac{1}{4} [X^m, X^n]^2 + \frac{2}{3} i \alpha \epsilon_{lmn} X^l X^m X^n + \frac{1}{2} \bar{\psi} \Gamma_m [X^m, \psi] + \alpha \bar{\psi} \psi \right), \tag{4.22}$$

where $l, m, n = 1, 2, 3$ and $\alpha$ is a real-valued parameter. The action (4.22) is proposed to introduce various noncommutative or fuzzy geometry[1]. In this thesis, we consider three of

[1] We use *noncommutative* to describe infinite dimensional representations or operator algebras and *fuzzy* to describe finite dimensional representations or approximations.

the ten dimensions described by the IKKT matrix model (4.12) to be replaced by the modified action (4.22).

The action (4.22) possesses $SO(3)$ symmetry and the following symmetries:

1. Translations

$$\delta X^m = C^m \cdot \mathbf{1}, \quad \delta\psi = 0, \tag{4.23}$$

2. Gauge transformations

$$\delta X^m = i[\lambda, X^m], \quad \delta\psi = i[\lambda, \psi], \tag{4.24}$$

3. $\mathcal{N} = 1$ SUSY

$$\delta X_m = i\bar{\epsilon}\Gamma_m\psi, \quad \delta\psi = \frac{i}{2}\Gamma^{mn}[X_m, X_n]\epsilon. \tag{4.25}$$

Compared with the action (4.6), the action (4.22) does not possess the translation symmetry of $\psi$.

### 4.4.1 Fuzzy sphere

Let us consider the equation of motion based on the action (4.22) with $\psi = 0$,

$$[X_m, [X_m, X_n]] = -i\alpha\epsilon_{nkl}\,[X_k, X_l]\,. \tag{4.26}$$

The simplest solution is again given by completely simultaneously diagonalizable one. We are interested in the following solution

$$[X_m, X_n] = i\alpha\epsilon_{mnl}X_l. \tag{4.27}$$

This solution is called the algebra of fuzzy sphere. If we define $X_m = \alpha L_m$, the algebra (4.27) becomes

$$[L_m, L_n] = i\epsilon_{mnl}L_l. \tag{4.28}$$

Therefore, we can identify $L_m$ as the generator of $su(2)$ algebra. Hence, we can consider the quadratic Casimir operator

$$\sum_{i=1}^{3}(X_i)^2 = \alpha^2\sum_{i=1}^{3}(\hat{L}_i)^2 = \alpha^2 L(L+1). \tag{4.29}$$

In this case, the dimension of the representation is controlled by $L$ in eq. (4.29). Therefore, we are interested in a finite $L$ in the sense of the matrix regularization.

Let us consider the limit $L \to \infty$ with $\rho := \alpha L$ is fixed. To fix the value of $\rho$, the parameter $\alpha$ should be $\alpha \to 0$. Accordingly, the algebra of fuzzy sphere (4.27) becomes trivial, i.e.,

$$[X_m, X_m] = 0.$$

Therefore, we can identify $X_m$ as a c-number satisfying

$$\sum_{m=1}^{3} (X_m)^2 = \rho^2. \tag{4.30}$$

Eq. (4.30) is the defining equation of the sphere in three-dimensional Euclidean space. Hence, the degree of freedom described by the set $\{X_m\}_{m=1}^{3}$ at a finite $L$ is called fuzzy sphere (more precisely, the coordinate of the fuzzy sphere). The fuzzy sphere is studied in several context: membrane theory in M(atrix)-theory (c.f., Refs. [49–55]), noncommutative gauge theory in superstring theory (c.f., Refs. [56–66]).

On the other hand, the $su(2)$ algebra (4.28) holdseven if the algebra (4.28) is induced from the algebra of fuzzy sphere (4.27). At the limit $L \to \infty$, the generator $L_m$ should be in the infinite representation, especially the differential representation. Actually, the generator of the $su(2)$ algebra in the differential representation is

$$-i\epsilon_{lmn} x^m \partial_n. \tag{4.31}$$

In the algebra of fuzzy sphere,

$$[L_l, X_k] = \alpha [L_l, L_k] = -i\epsilon_{lmk} X_m,$$

then the representation (4.31) can be recovered by $[L_l, \cdot]$ at the limit $L \to \infty$.

Consequently, we can introduce noncommutative torus and fuzzy sphere as the solutions of the equations of motion derived from the IKKT matrix model and the modified IKKT matrix model, respectively. This means that such a solution is realized dynamically. In this thesis, we will consider these geometries by hands. However, it is believed that the IKKT matrix model contains a mechanism called emergent geometry that determines the spacetime structure dynamically without a spacetime structure at the beginning of the theory. Several approaches imply the existence of such a structure, c.f., Refs. [67–73].

# Chapter 5

# Higher order coupling constants in magnetized compactifications

In this chapter, we discuss the structure of higher order coupling constants in 4D effective theories. In 4D effective theories, matter contents and their coupling constants are important for phenomenological applications. In type IIB effective theory, magnetized compactifications are promising models. Such models are described by super Yang-Mills theory with a background gauge field. Advantages of magnetized compactifications are seen in the realization of

1. Chiral structure, and
2. Generations.

These advantages are consequences of the Index theorem. Moreover, we showed the two features of the coupling constants in magnetized toroidal and spherical compactifications in Chapter 3. In this chapter, we generalize the two features of coupling constants to $d$-dimensional manifolds and its magnetized compactifications on the basis of Ref. [74].

## 5.1 Setup

In this section, we consider $D = (4 + d)$-dimensional field theories with an arbitrary gauge group. As an extra dimensional space, we admit an arbitrary $d$-dimensional compact spin manifold $\mathcal{M}_d$. From the viewpoint of extra dimensional models, the Kaluza-Klein expansion is important. For a fermion $\Psi$ and a scalar $\Phi$, the formal Kaluza-Klein expansions can be written as

$$\Psi(w) = \sum_n \chi_n(x) \otimes \psi_n(y),$$
$$\Phi(w) = \sum_n \varphi_n(x) \otimes \phi_n(y),$$

where the fields $\psi_n(y)$ and $\phi_n(y)$ satisfy the following eigenvalue problems with respect to a Dirac operator and a Laplacian on $\mathcal{M}_d$

$$
\begin{aligned}
i\not{D}_d\psi_n &= m_n\psi_n,\\
\Delta_d\phi_n &= M_n^2\phi_n.
\end{aligned}
\tag{5.1}
$$

$D$-dimensilnal Lorentz invariance is broken by the compactifications. As we mentioned, the lightest modes of the eigenvalue problems (5.1) are important since they are identified with the SM matter contents. We will omit the subscript $n=0$ since we often consider the lightest modes labeled by $n=0$.

In this section, we assume that the eigenvalue problem for the scalar can be written as

$$
-g^{ij}D_iD_j\phi_n = M_n^2\phi_n, \tag{5.2}
$$

where $i,j=4,\dots,(4+d)-1$, $g^{ij}$ is the inverse of the metric on $\mathcal{M}_d$, and $D_i=\nabla_i - i[A_i,\cdot]$ is a gauge covariant derivative, c.f., Subsection 3.1. Generally, the difference between $-g^{ij}D_iD_j$ and the Laplacian is a factor depending on the field strength. In magnetized compactifications, such a contribution is a constant factor in the previous works. Actually, such a situation is realized on the torus [4], complex projective spaces [75], and internal four-cycle in the conifold [76]. Therefore, we assume that the difference between $-g^{ij}D_iD_j$ and the Laplacian is a constant factor. Hence, it is sufficient to consider $D_i\phi=0$ for the lightest modes since $-g^{ij}D_iD_j$ is semi-positive definite.

In addition, we assume that the equation of motion of an internal vector field is essentially the same as that of a scalar field. According to Refs. [4, 75], the equation of motion of an internal vector field $\Phi_i^{ab}$ is given by

$$
D_iD^i\Phi_j^{ab} + 2iF_j^{ab,i}\Phi_i^{ab} - [\nabla^i,\nabla_j]\Phi_i^{ab} = -m^2\Phi_j^{ab}.
$$

If the contribution from $2iF_j^{ab,i}\Phi_i^{ab} - [\nabla^i,\nabla_j]\Phi_i^{ab}$ is a constant, the eigenvalue problem reduces to the part $D_iD^i\Phi_j^{ab}$. This part is nothing but the eigenvalue problem of a scalar field. We assume the realization of this situation. Actually, such a situation is also realized on the torus [4], complex projective spaces [75], and internal four-cycle in the conifold [76]. More generally, if we can introduce the magnetic flux along the symplectic form of the compactification manifold (c.f., the symplectic gauge), our expected situation will be realized since the background gauge potential plays a role like the spin connection. As a consequence, we will focus mainly on scalar fields and fermions in the remaining of this chapter.

## 5.2 Product property of the zero modes

We denote flat indices and curved indices by Roman characters and Greek characters, respectively. Let us start from two Dirac-type equations of adjoint fields in magnetized compactifications of $\mathcal{M}_d$

$$
i\not{D}^{(A)}\psi = i\Gamma^i\left[\left(\partial_i + \frac{1}{4}\omega_{i\alpha\beta}\Gamma^{\alpha\beta}\right)\psi - i[A_i,\psi]\right] = 0, \tag{5.3}
$$

$$
D_i^{(A)}\phi = \partial_i\phi - i[A_i,\phi] = 0, \tag{5.4}
$$

where $\Gamma^\alpha$ is a Gamma matrix in $d$ dimensions, $\Gamma^{\alpha\beta} := [\Gamma^\alpha, \Gamma^\beta]/2$, and $\omega_i^{\alpha\beta}$ is a spin connection. The superscript $(A)$ describes the symbol of the background gauge field, c.f., Section 3. The fields $\psi$ and $\phi$ are $N \times N$ matrices since they are in the adjoint representations of a gauge group whose dimension is $N$. We call the solutions of eqs (5.3) and (5.4) zero modes although the solutions of eq. (5.4) correspond to the lightest modes of the Laplacian.

We denote degenerated solutions of eq. (5.3) by $\psi_I^{(A)}$ and $\phi_I^{(A)}$, where $I$ denotes the label of the degeneracy. For example, we have shown such a degenerated chiral zero mode in magnetized toroidal compactifications in Subsection 3.2. We apply the same notation for solutions of eq. (5.4). In addition, we assume that the solutions satisfy the appropriate boundary conditions and the orthogonality with each other.

We have to mention the normalization of the zero modes. The normalization condition is defined as

$$\delta_{IJ} = \int_{\mathcal{M}_d} d^d y \,\mathrm{Tr}\left[\phi_I^{(A)} \cdot \phi_J^{(A)\dagger}\right] \equiv \mathrm{Tr}\left[\mathbf{B}_{IJ}^{(A)}\right], \tag{5.5}$$

$$\delta_{IJ} = \int_{\mathcal{M}_d} d^d y \,\mathrm{Tr}\left[\psi_I^{(A)} \cdot \psi_J^{(A)\dagger}\right] \equiv \mathrm{Tr}\left[\mathbf{F}_{IJ}^{(A)}\right]. \tag{5.6}$$

where $\mathbf{B}_{IJ}^{(A)}$ and $\mathbf{F}_{IJ}^{(A)}$ are matrix-valued constants

$$\begin{aligned}
\mathbf{B}_{IJ}^{(A)} &= \int_{\mathcal{M}_d} d^d y \phi_I^{(A)} \cdot \phi_J^{(A)\dagger}, \\
\mathbf{F}_{IJ}^{(A)} &= \int_{\mathcal{M}_d} d^d y \psi_I^{(A)} \cdot \psi_J^{(A)\dagger}.
\end{aligned}$$

In general, the eigenfunctions of a Dirac-type operator on a compact manifold form the complete orthonormal system since the Dirac-type operator is an elliptic essentially self-adjoint operator. Therefore, we have the completeness relations

$$\begin{aligned}
&\sum_{n,I} \psi_{n,I;ab;s}^{(A)\dagger}(y) \cdot \psi_{n,I;cd;s'}^{(A)}(y') = \delta_{ad}\delta_{bc}\delta_{ss'}\delta^d(y-y'), \\
&\sum_{n,I} \phi_{n,I;ab}^{(A)\dagger}(y) \cdot \phi_{n,I;cd}^{(A)}(y') = \delta_{ad}\delta_{bc}\delta^d(y-y'),
\end{aligned}$$

where $a, b, c, d$ describe the matrix indices and $s$ describes the spinor indices. We note that the Kronecker deltas in the right-hand sides transform under gauge transformations appropriately. From the completeness relations, we can confirm $\mathbf{B}_{IJ}^{(A)} = \mathbf{F}_{IJ}^{(A)} = N\delta_{IJ} \cdot \mathbf{1}$. In fact,

$$\begin{aligned}
&\sum_{n,I,a,b,d} \int d^d y d^d y' \phi_{J,a'a}^{(A)}(y) \phi_{n,I;ab}^{(A)\dagger}(y) \cdot \phi_{n,I;bd}^{(A)}(y') \phi_{J,dd'}^{(A)\dagger}(y') = \delta_{IJ}(\mathbf{B}_{JJ}^2)_{a'd'}, \\
&\sum_{a,b,d} \int d^d y d^d y' \phi_{J,a'a}^{(A)}(y) \delta_{ad}\delta_{bb}\delta^d(y-y') \phi_{J,dd'}^{(A)\dagger}(y') = N \cdot (\mathbf{B}_{JJ})_{a'd'},
\end{aligned}$$

where we have used the orthogonality among different and degenerate modes. Therefore, the normalization conditions (5.5) and (5.6) can be realized by the following redefinition: $\phi_I^{(A)} \to \phi_I^{(A)}/\sqrt{N}$, and $\psi_I^{(A)} \to \psi_I^A/\sqrt{N}$. In the following, we use $\phi_I^{(A)}$ and $\psi_I^{(A)}$ as degenerated solutions after the redefinition.

To investigate the structure of higher order coupling constants, the products of the zero modes are important. Let us start from the product of $\psi_I^{(A)}$ and $\phi_J^{(A)}$. Actually, this quantity is a zero mode of $i\not{D}^{(A)}$,

$$
\begin{aligned}
& i\Gamma^i D_i^{(A)}(\psi_I^{(A)} \cdot \phi_J^{(A)}) \\
&= i\Gamma^i \left[\left(\partial_i + \frac{1}{4}\omega_{i\alpha\beta}\Gamma^{\alpha\beta}\right)\psi_I^{(A)} - i[A_i, \psi_I^{(A)}]\right] \cdot \phi_J^{(A)} + i\Gamma^i \psi_I^{(A)} \cdot \left[\partial_i \phi_J^{(A)} - i[A_i, \phi_J^{(A)}]\right] \\
&= 0.
\end{aligned}
$$

Therefore, the product $\psi_I^{(A)} \cdot \phi_J^{(A)}$ can be written by a linear combination of the zero mode solutions of eq. (5.3) since they constitute a complete orthonormal system. The expansion is

$$
\psi_I^{(A)} \cdot \phi_J^{(A)} = \sum_K \mathbf{s}_{IJK}^{(A)} \cdot \psi_K^{(A)}, \tag{5.7}
$$

where $\mathbf{s}_{IJK}^{(A)}$ is a matrix-valued constant coefficient.

Similarly, the product $\phi_I^A \cdot \phi_J^A$ can be expanded by $\phi_K^A$ since the product $\phi_I^A \cdot \phi_J^A$ satisfies eq. (5.4). Namely,

$$
\phi_I^{(A)} \cdot \phi_J^{(A)} = \sum_K \mathbf{t}_{IJK}^{(A)} \cdot \phi_K^{(A)}, \tag{5.8}
$$

where $\mathbf{t}_{IJK}^{(A)}$ is a matrix-valued constant coefficient.

On the other hand, the product $\psi_I^{(A)} \cdot \psi_J^{(A)}$ cannot be expanded by $\phi_K^{(A)}$ in general since eq. (5.4) does not include the Gamma matrix. Hence, eq. (5.4) is not satisfied. As a special case, we can expand the product $\psi_I^{(A)} \cdot \psi_J^{(A)}$ by $\phi_K^{(A)}$ if we consider a two-dimensional manifold and chiral or anti-chiral zero mode $\psi_I^A$. To investigate higher order coupling constants, only $\mathbf{s}_{IJK}^{(A)}$ and $\mathbf{t}_{IJK}^{(A)}$ are important.

Let us remark about the case of a fundamental representation. In this case, the Dirac-type equations are defined by

$$
i\not{D}^{(A)}\psi = i\Gamma^i \left[\left(\partial_i + \frac{1}{4}\omega_{i\alpha\beta}\Gamma^{\alpha\beta}\right)\psi - iA_i\psi\right] = 0, \tag{5.9}
$$

$$
D_i^{(A)}\phi = \partial_i\phi - iA_i\phi = 0. \tag{5.10}
$$

Although the commutator satisfies the Leibniz rule, the gauge interaction parts of eqs. (5.9) and (5.10) do not have such a property. This induces the modification of the product property of zero modes. Actually, the product of $\psi_I^{(A)}$ and $\phi_I^{(A)}$ is not a zero mode of $i\not{D}^A$ but $i\not{D}^{(2A)}$,

$$
\begin{aligned}
& i\Gamma^i D_i^{(2A)}(\psi_I^{(A)}\cdot\phi_J^{(A)}) \\
&= i\Gamma^i\left[\left(\partial_i+\frac{1}{4}\omega_{i\alpha\beta}\Gamma^{\alpha\beta}\right)\psi_I^{(A)}-iA_i\psi_I^{(A)}\right]\cdot\phi_J^{(A)}+i\Gamma^i\psi_I^{(A)}\cdot\left[\partial_i\phi_J^{(A)}-iA_i\phi_J^{(A)}\right] \\
&= 0.
\end{aligned}
\tag{5.11}
$$

In general, the product of $\psi_I^{(A)}$ and $\phi_I^{(A')}$ is a zero mode of $i\not{D}^{(A+A')}$ and that of $\phi_I^{(A)}$ and $\phi_I^{(A')}$ is a zero mode of $D_i^{(A+A')}$. However, we have to replace the positions of $A_i'$ and $\psi_I^{(A)}$ or $\phi^{(A)}$ e.g., from the first line to the second line in eq. (5.11). Therefore, the product property for a fundamental representation is valid on $U(1)$ gauge theory only.

## 5.3 Generic $n$-point coupling constants

### 5.3.1 Three-point coupling constants

In this subsection, we consider three-point coupling constants constructed from the zero mode fermions and bosons. In the following, we compute boson-boson-boson and boson-fermion-fermion couplings since the four-dimensional Lorentz symmetry requires even number of fermions.

- boson-boson-boson

$$
\begin{aligned}
b_{IJK}^{(A)} &:= \int_{\mathcal{M}_d} d^d y \mathrm{Tr}\left[\phi_I^{(A)}(y)\cdot\phi_J^{(A)}(y)\cdot\phi_K^{(A),\dagger}(y)\right] \\
&= \int_{\mathcal{M}_d} d^d y \mathrm{Tr}\left[\sum_L \mathbf{t}_{IJL}^{(A)}\cdot\phi_L^{(A)}(y)\cdot\phi_K^{(A),\dagger}(y)\right] \\
&= \sum_L \mathrm{Tr}\left[\mathbf{t}_{IJL}^{(A)}\cdot\mathbf{B}_{LK}^{(A)}\right] \\
&= \mathrm{Tr}\left[\mathbf{t}_{IJK}^{(A)}\right],
\end{aligned}
\tag{5.12}
$$

- boson-fermion-fermion

$$
\begin{aligned}
y_{IJK}^{(A)} &:= \int_{\mathcal{M}_d} d^d y \mathrm{Tr}\left[\phi_I^{(A)}(y)\cdot\psi_J^{(A)}(y)\cdot\psi_{\bar{K}}^{(A),\dagger}(y)\right] \\
&= \int_{\mathcal{M}_d} d^d y \mathrm{Tr}\left[\sum_L \mathbf{s}_{IJL}^{(A)}\cdot\psi_L^{(A)}(y)\cdot\psi_{\bar{K}}^{(A),\dagger}(y)\right] \\
&= \sum_L \mathrm{Tr}\left[\mathbf{s}_{IJL}^{(A)}\cdot\mathbf{F}_{L\bar{K}}^{(A)}\right] \\
&= \mathrm{Tr}\left[\mathbf{s}_{IJ\bar{K}}^{(A)}\right].
\end{aligned}
\tag{5.13}
$$

In both cases, the coupling constants are obtained by the expansion coefficients in eqs. (5.7) and (5.8).

### 5.3.2 Four-point coupling constants

In this subsection, we compute four-point coupling constants and will show the decomposition by the three-point coupling constants (5.12) and (5.13). There are three kinds of four-point coupling constants because of the four-dimensional Lorentz symmetry: boson-boson-boson-boson, boson-boson-fermion-fermion and fermion-fermion-fermion-fermion.

- boson-boson-boson-boson

$$\begin{aligned}
&b^{(A)}_{HIJK}\\
&:= \int_{\mathcal{M}_d} d^d y \mathrm{Tr}\left[\phi^{(A)}_H(y)\cdot\phi^{(A)}_I(y)\cdot\phi^{(A)}_J(y)\cdot\phi^{(A),\dagger}_K(y)\right]\\
&= \int_{\mathcal{M}_d} d^d y \sum_{a,b,c,d}\left[\phi^{(A)}_{H;ab}(y)\cdot\phi^{(A)}_{I;bc}(y)\cdot\phi^{(A)}_{J;cd}(y)\cdot\phi^{(A),\dagger}_{K;da}(y)\right]\\
&= \int_{\mathcal{M}_d} d^d y \int_{\mathcal{M}_d} d^d y' \sum_{a,b,c,d,s,t}\delta_{cs}\delta_{at}\delta^d(y-y')\left[\phi^{(A)}_{H;ab}(y)\cdot\phi^{(A)}_{I;bc}(y)\cdot\phi^{(A)}_{J;sd}(y')\cdot\phi^{(A),\dagger}_{K;dt}(y')\right]\\
&= \int_{\mathcal{M}_d} d^d y \int_{\mathcal{M}_d} d^d y' \sum_{a,b,c,d,s,t}\sum_{n=0,S=1}\phi^{(A),\dagger}_{n,S;ca}(y)\phi^{(A)}_{n,S;ts}(y')\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\left[\phi^{(A)}_{H;ab}(y)\cdot\phi^{(A)}_{I;bc}(y)\cdot\phi^{(A)}_{J;sd}(z)\cdot\phi^{(A),\dagger}_{\bar{K};dt}(z)\right]\\
&= \sum_{n=0,S=1}\left[\int_{\mathcal{M}_d} d^d y \mathrm{Tr}\left(\phi^{(A)}_H(y)\cdot\phi^{(A)}_I(y)\cdot\phi^{(A),\dagger}_{n,S}(y)\right)\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\left[\int_{\mathcal{M}_d} d^d y' \mathrm{Tr}\left(\phi^{(A)}_{n,S}(y')\cdot\phi^{(A)}_J(y')\cdot\phi^{(A),\dagger}_K(y')\right)\right]\\
&= \sum_S \mathrm{Tr}\left[\mathbf{t}^{(A)}_{HIS}\right]\times\mathrm{Tr}\left[\mathbf{t}^{(A)}_{SJK}\right], \qquad (5.14)
\end{aligned}$$

The insertion of the completeness relation is important for the above decomposition. However, we can alter the decomposition (5.14) since the representation of the completeness relation is not unique. For example,

- boson-boson-boson-boson

$$\begin{aligned}
b^{(A)}_{HIJK} &= \int_{\mathcal{M}_d} d^d y \int_{\mathcal{M}_d} d^d y' \sum_{a,b,c,d,s,t}\delta_{ds}\delta_{bt}\delta^d(y-y')\left[\phi^{(A)}_{I;bc}(y)\cdot\phi^{(A)}_{J;sd}(y)\cdot\phi^{(A)}_{H;at}(y')\cdot\phi^{(A),\dagger}_{K;st}(y')\right]\\
&= \sum_T \mathrm{Tr}\left[\mathbf{t}^{(A)}_{IJT}\right]\times\mathrm{Tr}\left[\mathbf{t}^{(A)}_{HTK}\right], \qquad (5.15)
\end{aligned}$$

where we contracted $\phi^{(A)}_I$ and $\phi^{(A)}_J$ first by inserting the completeness relation.

Needless to say, eq. (5.14) must be equal to eq. (5.15), i.e.,

$$b^{(A)}_{HIJK} = \sum_S \mathrm{Tr}\left[\mathbf{t}^{(A)}_{HIS}\right]\times\mathrm{Tr}\left[\mathbf{t}^{(A)}_{SJK}\right] = \sum_T \mathrm{Tr}\left[\mathbf{t}^{(A)}_{IJT}\right]\times\mathrm{Tr}\left[\mathbf{t}^{(A)}_{HTK}\right]$$

This structure is similar to the conformal block or degenerations of Riemann surfaces in conformal field theory. In the language of scattering processes, this alteration can be identified, for example, with the crossing symmetry between s-channel and t-channel. In scattering theory, we have another channel called u-channel. In a fundamental representation, we can obtain another decomposition (i.e., first contraction $\phi_I^{(A)}$ and $\phi_K^{(A)\dagger}$) corresponding to u-channel. However, in the adjoint representation, the non-Abelian structure is an obstacle to alter to u-channel.

- boson-boson-fermion-fermion

$$\begin{aligned}
y_{HIJK}^{(A)} &= \int_{\mathcal{M}_d} d^d y \mathrm{Tr}\left[\phi_H^{(A)}(y)\cdot\phi_I^{(A)}(y)\cdot\psi_J^{(A)}(y)\cdot\psi_K^{(A),\dagger}(y)\right] \\
&= \int_{\mathcal{M}_d} d^d y \sum_{a,b,c,d}\left[\phi_{H;ab}^{(A)}(y)\cdot\phi_{I;bc}^{(A)}(y)\cdot\psi_{J;cd}^{(A)}(y)\cdot\psi_{K;da}^{(A),\dagger}(y)\right] \\
&= \int_{\mathcal{M}_d} d^d y \int_{\mathcal{M}_d} d^d y' \sum_{a,b,c,d,s,t} \delta_{cs}\delta_{at}\delta^d(y-y')\left[\phi_{H;ab}^{(A)}(y)\cdot\phi_{I;bc}^{(A)}(y)\cdot\psi_{J;sd}^{(A)}(y')\cdot\psi_{K;dt}^{(A),\dagger}(y')\right] \\
&= \int_{\mathcal{M}_d} d^d y \int_{\mathcal{M}_d} d^d y' \sum_{a,b,c,d,s,t}\sum_{n=0,S=1} \phi_{n,S;ca}^{(A),\dagger}(y)\phi_{n,S;ts}^{(A)}(y') \\
&\qquad\qquad\qquad\qquad\qquad \times\left[\phi_{H;ab}^{(A)}(y)\cdot\phi_{I;bc}^{(A)}(y)\cdot\psi_{J;sd}^{(A)}(y')\cdot\psi_{K;dt}^{(A),\dagger}(y')\right] \\
&= \sum_S \mathrm{Tr}\left[\mathbf{t}_{HIS}^{(A)}\right]\times\mathrm{Tr}\left[\mathbf{s}_{SJK}^{(A)}\right].
\end{aligned} \tag{5.16}$$

In this case, we can also find other decomposition

$$\begin{aligned}
y_{HIJK}^{(A)} &= \int_{\mathcal{M}_d} d^d y \int_{\mathcal{M}_d} d^d y' \sum_{a,b,c,d,s,s',t,u} \delta_{dt}\delta_{bu}\delta_{ss'}\delta^d(y-y') \\
&\qquad\qquad \times\left[\phi_{H;au}^{(A)}(y')\cdot\phi_{I;bc}^{(A)}(y)\cdot\psi_{J;cd;s}^{(A)}(y)\cdot\psi_{K;ta;s'}^{(A),\dagger}(y')\right] \\
&= \int_{\mathcal{M}_d} d^d y \int_{\mathcal{M}_d} d^d y' \sum_{a,b,c,d,s,s',t,u}\sum_{n=0,T=1} \psi_{n,T;db;s}^{(A),\dagger}(y)\psi_{n,T;ut;s'}^{(A)}(y') \\
&\qquad\qquad \times\left[\phi_{H;au}^{(A)}(y')\cdot\phi_{I;bc}^{(A)}(y)\cdot\psi_{J;cd;s}^{(A)}(y)\cdot\psi_{K;ta;s'}^{(A),\dagger}(y')\right] \\
&= \sum_T \mathrm{Tr}\left[\mathbf{s}_{IJT}^{(A)}\right]\times\mathrm{Tr}\left[\mathbf{s}_{HTK}^{(A)}\right].
\end{aligned} \tag{5.17}$$

- fermion-fermion-fermion-fermion

$$
\begin{aligned}
f^{(A)}_{HIJK} &:= \int_{\mathcal{M}_d} d^d y \mathrm{Tr}\left[\left(\psi^{(A)}_H(y)\cdot\psi^{(A),\dagger}_I(y)\right)\cdot\left(\psi^{(A)}_J(y)\cdot\psi^{(A),\dagger}_K(y)\right)\right] \\
&= \int_{\mathcal{M}_d} d^d y \sum_{a,b,c,d}\left[\left(\psi^{(A)}_{H,ab}(y)\cdot\psi^{(A),\dagger}_{I,bc}(y)\right)\cdot\left(\psi^{(A)}_{J,cd}(y)\cdot\psi^{(A),\dagger}_{K,da}(y)\right)\right] \\
&= \int_{\mathcal{M}_d} d^d y \int_{\mathcal{M}_d} d^d y' \sum_{a,b,c,d,s,t} \delta_{sc}\delta_{at}\delta^d(y-y') \\
&\qquad\qquad \left[\left(\psi^{(A)}_{H,ab}(y)\cdot\psi^{(A),\dagger}_{I,bc}(y)\right)\cdot\left(\psi^{(A)}_{J,cd}(y)\cdot\psi^{(A),\dagger}_{\bar{K},da}(y)\right)\right] \\
&= \sum_{n=0,L=1}\int_{\mathcal{M}_d} d^d y \mathrm{Tr}\left[\psi^{(A)}_H(y)\cdot\psi^{(A),\dagger}_{\bar{I}}(y)\cdot\phi^{(A),\dagger}_{n,\bar{L}}(y)\right] \\
&\qquad\qquad \times\int_{\mathcal{M}_d} d^d y' \mathrm{Tr}\left[\phi^{(A)}_{n,L}(y')\cdot\psi^{(A)}_J(y')\cdot\psi^{(A),\dagger}_K(y')\right].
\end{aligned}
\tag{5.18}
$$

As we proved, the product of $\psi^{(A)}_I$ and $\phi^{(A)}_J$ is in the space spanned by $\{\psi^{(A)}_I\}_I$. In general, however, the product of the zero mode $\psi^{(A)}_I$ and a fixed excited mode $\phi^{(A)}_{n,J}$ in not in the specific space but the whole space of the eigenmodes of the Dirac operator $\not{D}^{(A)}$. For convenience, by introducing the three-point coupling constants including higher mode scalars

$$
\mathbf{s}^{(A)}_{n,IJK} := \int_{\mathcal{M}_d} d^d y \,\mathrm{Tr}\left[\phi^{(A)}_{n,I}(y)\cdot\psi^{(A)}_J(y)\cdot\psi^{(A)\dagger}_K(y)\right] \quad (n\geq 0),
\tag{5.19}
$$

then

$$
f^{(A)}_{HIJK} = \sum_{n,L}\mathrm{Tr}\left[\mathbf{s}^{(A),n,\dagger}_{LIH}\right]\times\mathrm{Tr}\left[\mathbf{s}^{(A),n}_{LJK}\right].
$$

In magnetized toroidal compactifications, the concrete form of three-point coupling constants (5.19) are studied in Ref. [5].

### 5.3.3 $n$-point coupling constants

An $n$-point coupling is given by

$$
Y^{(A)}_{M_1\ldots M_i N_1\ldots N_j N_{j+1}\ldots N_{2j}} := \int_{\mathcal{M}_d} d^d y \mathrm{Tr}\left[\prod_{k=1}^{i}\left(\phi^{(A)}_{M_k}(y)\right)\prod_{l=1}^{j}\left[\psi^{(A)}_{N_l}(y)\left(\psi^{(A),\dagger}_{N_{j+l}}(y)\right)\right]\right],
\tag{5.20}
$$

where $n = i+2j$. By the similar computations as above, the n-point coupling (5.20) can be reduced to $-1+i+2j$-point coupling, e.g.,

$$
\begin{aligned}
&Y^{(A)}_{M_1\ldots M_i N_1\ldots N_j N_{j+1}\ldots N_{2j}} \\
&= \int_{\mathcal{M}_d} d^d y \phi^{(A)}_{M_1,ab}(y)\phi^{(A)}_{M_2,bc}(y) \\
&\qquad\qquad \times \int_{\mathcal{M}_d} d^d y' \delta_{cs}\delta_{at}\delta^d(y-y')\left[\prod_{k=3}^{i}\left(\phi^{(A)}_{M_k}(y')\right)\prod_{l=1}^{j}\left[\psi^{(A)}_{N_l}(y')\left(\psi^{(A),\dagger}_{N_{j+l}}(y')\right)\right]\right]_{st} \\
&= \sum_{n=0,L=1}\int_{\mathcal{M}_d} d^d y \mathrm{Tr}\left[\phi^{(A)}_{M_1}(y)\phi^{(A)}_{M_2}(y)\phi^{(A),\dagger}_{n,L}(y)\right] \\
&\qquad\qquad\qquad \times \int_{\mathcal{M}_d} d^d y' \mathrm{Tr}\left[\phi^{(A)}_{n,L}(y')\prod_{k=3}^{i}\left(\phi^{(A)}_{M_k}(y')\right)\prod_{l=1}^{j}\left[\psi^{(A)}_{N_l}(y')\left(\psi^{(A),\dagger}_{N_{j+l}}(y')\right)\right]\right] \\
&= \sum_{L=1}\frac{1}{N}\mathrm{Tr}\left[\mathbf{t}^{(A)}_{M_1M_2L}\right]\cdot Y^{(A)}_{L,M_3\ldots M_i N_1\ldots N_j N_{j+1}\ldots N_{2j}}.
\end{aligned} \tag{5.21}
$$

We can obtain other decomposition that must be the same as the coupling constant (5.21). In addition, we can continue the decomposition process, e.g., for the $-1+i+2j$-coupling $Y^{(A)}_{L,M_3\ldots M_i N_1\ldots N_j N_{j+1}\ldots N_{2j}}$ into a $-2+i+2j$-coupling constant.

As a summary, we define the selection rule for higher-order coupling constants among the zero modes: *Higher-order coupling constants can be decomposed into three-point coupling constants determined by the product property of the zero modes.* We note that an actual $n$-point coupling constant is obtained by the multiplication of the above computations and a coupling constant of a gauge group and a sign coming from spin statistics.

# Chapter 6

# Higher order coupling constants in IKKT matrix model

In this chapter, we reconsider the results of Section 5 from the viewpoint of the IKKT matrix model.

## 6.1 Conjecture

Let us fix a classical solution $X_0^M$ satisfying eq. (4.18), and we consider fluctuation fields $A_M$, i.e.,

$$X^M = X_0^M + \theta^{MN} A_N \tag{6.1}$$

On the other hand, the fermionic part of the IKKT matrix model is

$$S_f = \frac{1}{2} \operatorname{Tr} \left( \bar{\psi} \Gamma_M [X^M, \psi] \right) . \tag{6.2}$$

By substituting the eq. (6.1) into the eq. (6.2), the fermionic part of the IKKT matrix model is written by

$$S_f = \frac{1}{2} \operatorname{Tr} \left( \bar{\psi} \Gamma_M [X_0^M + \theta^{MN} A_N, \psi] \right) \tag{6.3}$$

Therefore, if the fluctuation fields have an expectation value as a background, $< A_N >$, we can define a Dirac operator in the IKKT matrix model as

$$D := -i\Gamma^M (\theta^{-1})_{MN} \left[ X_0^N + \theta^{NL} < A_L >, \cdot \right] . \tag{6.4}$$

This Dirac operator is similar to the gauge interaction part of the Dirac operator (5.3). The essence of the selection rule obtained in Chapter 5 is that Dirac-type operators satisfy the Leibniz rule. From this point, the same result with the Chapter 5 will be achieved since the

Dirac operator (6.4) satisfy the Leibniz rule. However, there is no guarantee for the existence of the complete orthonormal system on the noncommutative or fuzzy manifold. In the case of a compact manifold, it is guaranteed by the general result with respect to essentially self-adjoint operators on compact manifolds. Therefore, at this level, we conjecture that the selection rule proposed in Chapter 5 is valid even if we consider a noncommutative or fuzzy manifold.

In the following, we confirm our conjecture by considering the magnetized noncommutative torus and the magnetized fuzzy sphere on the basis of Ref. [77, 78].

## 6.2 Magnetized noncommutative torus

### 6.2.1 Dirac operator on noncommutative torus

In Subsection 4.3, noncommutative torus is defined by the algebraic structure. Therefore, differential operators on noncommutative torus are defined as linear maps on an algebra.

Let us denote a $C^*$-algebra over $\mathbb{C}$ as $\mathcal{A}$. A map $d : \mathcal{A} \to \mathcal{A}$ is called a derivation on $\mathcal{A}$ if $d$ satisfies the Leibniz rule

$$d(ab) = (da)b + a(db), d(\lambda a) = \lambda(da),$$

for all $a, b \in \mathcal{A}, \lambda \in \mathbb{C}$.

In this thesis, noncommutative torus is defined as a $C^*$-algebra generated by $\hat{U}_4$ and $\hat{U}_5$ as we mentioned in Section 4.3[1]. Therefore, derivations can be defined by the action to each generator. We define the derivations $\hat{\delta}^i$ $(i = 4, 5)$, which are called the basic derivations, as

$$\delta_4 \hat{U}_4 := \frac{i}{R_4} \hat{U}_4, \quad \delta_5 \hat{U}_5 := \frac{i}{R_5} \hat{U}_5, \quad \delta_4 \hat{U}_5 = \delta_5 \hat{U}_4 = 0,$$

and extend them to satisfy the linearity and the Leibniz rule. The factors $\frac{1}{R_4}$ and $\frac{1}{R_5}$ are convenient factors for the commutative torus whose periods are $2\pi R_4$ and $2\pi R_5$.

The Dirac operator on noncommutative torus is defined by the basic derivations

$$\not{D} := i \sum_{i=4,5} \Gamma^i \delta_i, \tag{6.5}$$

where $\Gamma^4$ and $\Gamma^5$ are the Pauli matrices $\sigma_1$ and $\sigma_2$, respectively[2].

In this thesis, we consider noncommutative torus based on the IKKT matrix model. Therefore, the above differential operators should be realized in the framework of the IKKT matrix model. In addition, we are interested in a magnetized model. Hence, we have to construct, for example, a gauge covariant derivative. For this purpose, let us start from noncommutative super Yang-Mills theory based on the IKKT matrix model.

---

[1] In the following, we consider the infinite dimensional representation of the noncommutative torus. Hence, we use “hat” for operators to distinguish them from finite dimensional matrices.

[2] We keep in mind that the noncommutative torus corresponds to an extra dimensional space. Hence, the notation is unified as before.

To obtain noncommutative super Yang-Mills theory, we expand the action (4.6) around the classical solution, i.e., eq. (6.1). Then, the bosonic part of the action (4.6) is written as

$$S_b = -\frac{1}{4g^2}\,\mathrm{Tr}\left(\eta_{IK}\eta_{JL}[\hat{X}_0^I + \theta^{IM}\hat{A}_M, \hat{X}_0^J + \theta^{JM}\hat{A}_M][\hat{X}_0^K + \theta^{KN}\hat{A}_N, \hat{X}_0^L + \theta^{LN}\hat{A}_N]\right). \quad (6.6)$$

In the following, we consider a $U(N)$ gauge group since we focus on the type IIB effective theory. The basic expansion process is the same as the reductionof super Yang-Mills theory in Section 3.1. The difference is whether the degrees of freedom are fields or matrices. Therefore, we can follow the same process as in Section 3.1 by replacing fields by matrices if the cyclic property of the trace is valid. More precisely, we have to define four following items.

- Partial derivatives

$$\partial_M := -i(\theta^{-1})_{MN}[\hat{X}_0^N, \cdot],$$

- Field strength of the $U(1)$ gauge group

$$\hat{F}_{MN} := \partial_M \hat{B}_N - \partial_N \hat{B}_M - i[\hat{B}_M, \hat{B}_N]$$

  The commutation relation does not vanish because of the noncommutativity of the matrix algebra.

- Covariant derivatives with respect to the $U(1)$ gauge group

$$D_M \hat{W}_N = \partial_M \hat{W}_N - i[\hat{B}_M, \hat{W}_N]$$

- Effective metric

$$G^{MN} = \theta^{MN}\theta^{NJ}\eta_{IJ}$$

By inserting these items, the action (6.6) can be written as

$$\begin{aligned} S_b = &\frac{1}{4g^2}\,\mathrm{Tr}\left(\left(\hat{F}_{MN} - (\theta^{-1})_{MN}\right)\left(\hat{F}^{MN} - (\theta^{-1})^{MN}\right) - [\hat{W}_M, \hat{W}_N][\hat{W}^M, \hat{W}^N]\right) \\ &+ \frac{1}{2g^2}\mathrm{Tr}\left(D_M\hat{W}_N D^M\hat{W}^N - D_M\hat{W}_N D^N\hat{W}^M - i(\hat{F}_{MN} - (\theta^{-1})_{MN})[\hat{W}^M, \hat{W}^N])\right), \end{aligned} \quad (6.7)$$

where the indices are contracted by the effective metric $G^{MN}$.

We are interested in a magnetic flux. We introduce an Abelian magnetic flux on the extra dimensional space. By arranging on the basis of the prescription in Subsection 3.1, we can obtain the action after turning on the magnetic flux

$$S_b = -\frac{1}{2g^2}\,\mathrm{Tr}\left(\hat{\Phi}_i D_\mu D^\mu \hat{\Phi}^i + \hat{\Phi}_j \tilde{D}_i \tilde{D}^i \hat{\Phi}^j\right) - \frac{i}{2g^2}\left(< \hat{F}^a_{ij} > - < \hat{F}^b_{ij} >\right)\hat{\Phi}^{i,ab}\hat{\Phi}^{j,ba} + S_{b,\mathrm{other}}, \quad (6.8)$$

where

$$\tilde{D}_i \hat{\Phi}_j^{ab} := \partial_i \hat{\Phi}_j^{ab} - i < \hat{B}_i^a > \hat{\Phi}_j^{ab} + i \hat{\Phi}_j^{ab} < \hat{B}_i^b >,$$

and $S_{b,\text{other}}$ is irrelevant part to the following discussion.

Similarly, the fermionic part of the IKKT matrix model after turning on the magnetic flux becomes

$$S_f = -\frac{1}{2g^2} \mathrm{Tr} \left( i \bar{\hat{\Psi}} \tilde{\Gamma}^\mu D_\mu \hat{\Psi} + i \bar{\hat{\Psi}} \tilde{\Gamma}^i \tilde{D}_i \hat{\Psi} + \bar{\hat{\Psi}} \tilde{\Gamma}^i [\hat{\Phi}_i, \hat{\Psi}] \right) + S_{f,\text{other}}, \tag{6.9}$$

where the indices are contracted by the Minkowski metric and $\tilde{\Gamma}^M := \theta^{MN} \Gamma_N$, and $S_{f,\text{other}}$ is also an irrelevant part.

From the action (6.8), we can define the Dirac operator and the Laplacian

$$D := i \tilde{\Gamma}^M [\hat{X}_M, \cdot], \tag{6.10}$$

$$\Delta := -G^{MN} \tilde{D}_M \tilde{D}_N = [\hat{X}_M, [\hat{X}^M, \cdot]], \tag{6.11}$$

where $\hat{X}^M$ has the same expression as eq. (6.1). In this thesis, we are interested in the noncommutative torus in an infinite dimensional representation. Therefore, the Dirac operator (6.10) and the Laplacian (6.11) are operators act on operators written by $\hat{Y}^4$ and $\hat{Y}^5$.

In the following, we consider the gauge background on noncommutative torus corresponding to Section 3.2, i.e.,

$$\hat{A}_4 = 0, \quad \hat{A}_5 = \mathcal{F} \hat{Y}^4, \tag{6.12}$$

where the field strength is $\hat{F}_{45} = \mathcal{F}$.

We have to consider a boundary condition corresponding to the twisted boundary condition in Section 3.2. Because of the noncommutativity, the gauge transformation is obtained by

$$\hat{A}_M \to \hat{A}'_M = \hat{\Omega} \hat{A}_M \hat{\Omega}^{-1} + i \hat{\Omega} \partial_M \hat{\Omega}^{-1}, \tag{6.13}$$

where $\hat{\Omega}$ is a unitary operator. From the construction of noncommutative torus, the background gauge field translates as

$$\begin{aligned}
&\hat{A}_4(\hat{Y}^4 + 2\pi R_4, \hat{Y}^5) = \hat{A}_4(\hat{Y}^4, \hat{Y}^5 + 2\pi R_5) = 0, \\
&\hat{A}_5(\hat{Y}^4 + 2\pi R_4, \hat{Y}^5) = \hat{A}_5(\hat{Y}^4, \hat{Y}^5) + 2\pi R_5 \mathcal{F}, \\
&\hat{A}_5(\hat{Y}^4, \hat{Y}^5 + 2\pi R_5) = \hat{A}_5(\hat{Y}^4, \hat{Y}^5).
\end{aligned} \tag{6.14}$$

We require that the above translations can be realized as gauge transformations. In fact, the unitary operator $\Omega$ in eq. (6.14) can be constructed by

$$\hat{\Omega}_4(\hat{Y}^4, \hat{Y}^5) \propto \exp\left[\frac{2\pi i R_5}{1+\theta^{45}\mathcal{F}} \cdot \mathcal{F}\hat{Y}^5\right], \quad \hat{\Omega}_5(\hat{Y}^4, \hat{Y}^5) \propto \hat{1}, \tag{6.15}$$

where $\propto$ represents an action on $\mathbb{C}^m$ part. This part is not important for the following discussion since the $\mathbb{C}^m$ part of $\hat{Y}^4$ and $\hat{Y}^5$ is the identity matrix. Therefore, in the following, we assume this part is the identity matrix and omit it.

In addition, we have to consider the consistency condition, i.e.,

$$\hat{\Omega}_5(\hat{Y}^4 + 2\pi R_4, \hat{Y}^5)\hat{\Omega}_4(\hat{Y}^4, \hat{Y}^5) = \hat{\Omega}_4(\hat{Y}^4, \hat{Y}^5 + 2\pi R_5)\hat{\Omega}_5(\hat{Y}^4, \hat{Y}^5). \tag{6.16}$$

Consequently, eq. (6.16) requires that the magnetic flux $\mathcal{F}$ must be quantized in such a way that

$$\frac{\mathcal{F}}{1+\theta^{12}\mathcal{F}} \cdot \frac{A}{2\pi} \equiv \mathcal{N} \in \mathbb{Z}. \tag{6.17}$$

We have to note about the gauge choice. The choice (6.12) is called the axial gauge. The following discussion in this chapter, e.g., explicit derivation of zero modes, depends on this gauge choice. Therefore, we should confirm the gauge invariance of our discussion.

In the commutative case, another gauge commonly used is the symmetric gauge

$$A_4 = -\frac{1}{2}Fy^5, \quad A_5 = \frac{1}{2}Fy^4.$$

More generally, we can consider the following gauge

$$A_4 = -tFy^5, \quad A_5 = (1-t)Fy^4, \tag{6.18}$$

where $t \in [0, 1]$. We can realize the gauge transformations from ${}^\forall t_1$ to ${}^\forall t_2$ by the unitary element $U = \exp[iF(t_2 - t_1)y^4 y^5]$.

Let us consider an analogy in the noncommutative case. We can introduce the similar background gauge field

$$\hat{A}_4 = -t\mathcal{F}\hat{Y}^5, \quad \hat{A}_5 = (1-t)\mathcal{F}\hat{Y}^4.$$

We expect that we can find the gauge transformations from ${}^\forall t_1$ to ${}^\forall t_2$. To find a unitary element, let us consider a unitary element $\hat{U} = \exp\left[i\alpha(\hat{Y}^4\hat{Y}^5 + \hat{Y}^5\hat{Y}^4)\right]$, where $\alpha \in \mathbb{R}$. This unitary element is the analogy of $U = \exp[iF(t_2 - t_1)y^1 y^2]$. Based on the transformation by $\hat{U}$, we can find the following identity,

$$(1 - t_1\theta^{45}\mathcal{F})(1 + (t_1 - 1)\theta^{45}\mathcal{F}) = (1 - t_2\theta^{45}\mathcal{F})(1 + (t_2 - 1)\theta^{45}\mathcal{F}). \tag{6.19}$$

Eq. (6.19) is a quadratic equation with respect to $t_2$, and its solution are $t_1$ and $1 - t_1$. In other words, non-trivial gauge transformation is uniquely determined as $t_1 \to 1 - t_1$. This may impose an additional condition to the gauge space compared to the commutative case. However, we restricted ourselves to $\hat{U}$ in the above discussion. Therefore, it may be possible that the gauge space recovers by other unitary elements.

### 6.2.2 Zero modes of the Dirac operator

In the previous subsection, we introduced the gauge background (6.12). In the following, we obtain the explicit Dirac operator and derive zero modes.

As a first step, let us consider a zero mode equation for a fundamental representation for convenience. We can define the covariant derivative for the fundamental representation

$$D_M := \partial_M - i\hat{A}_M \tag{6.20}$$

By replacing the basic derivations to the covariant derivative (6.20), a zero mode equation for each chirality is given by

$$(\partial_4 + is\partial_5 + s\mathcal{F}\hat{Y}^4)\hat{\psi}^s = 0 \quad (s = \pm 1). \tag{6.21}$$

with the twisted boundary condition

$$\begin{aligned}
\hat{\psi}^s(\hat{Y}^4 + 2\pi R_4, \hat{Y}^5) &= \exp\left[\frac{2\pi i R_4}{1+\theta^{45}\mathcal{F}} \cdot \mathcal{F}\hat{Y}^5\right]\hat{\psi}^s(\hat{Y}^4, \hat{Y}^5) \\
\hat{\psi}^s(\hat{Y}^4, \hat{Y}^5 + 2\pi R_5) &= \hat{\psi}^s(\hat{Y}^4, \hat{Y}^5).
\end{aligned}$$

We expect that the zero modes of eq. (6.21) can be constructed from those of the commutative case since eq. (6.21) is similar to the commutative case.

Here, we consider the viewpoint of Fourier transformation. The Fourier transformation of the whole zero mode of the commutative case is written as

$$\psi^s(y^4, y^5) = \int \frac{dk}{2\pi} \sum_{n=|N|p+q\in\mathbb{Z}} C\exp\left[-\frac{k^2}{2|F|} - \frac{ikn}{R_5 F}\right]\exp\left[iky^4\right]\exp\left[i\frac{n}{R_5}y^5\right], \tag{6.22}$$

where $p \in \mathbb{Z}$, $q = 0, \dots, |N| - 1$, $C$ is the normalization constant, and the magnetic flux $F$ satisfies the quantization condition (3.17). We expect that the whole zero mode of the eq. (6.21) is obtained by

$$\hat{\psi}^s(\hat{Y}^4, \hat{Y}^5) = \int \frac{dk}{2\pi} \sum_{n=|N|p+q\in\mathbb{Z}} C\exp\left[-\frac{k^2}{2|F|} - \frac{ikn}{R_5 F}\right]\exp\left[ik\hat{Y}^4\right]\exp\left[i\frac{n}{R_5}\hat{Y}^5\right] \tag{6.23}$$

However, this is not a zero mode solution because of the difference in the quantization condition with respect to the magnetic flux.

To obtain zero mode solutions of eq. (6.21), let us focus on the decomposition of the label of the summation. In eq.(6.22), the label of the summation $n$ is restricted as $n = |N|p + q$. However, $p$ and $q$ do not appear in eq. (6.22). Therefore, we can consider other decomposition like $n = |\mathcal{N}|p + q$, where $p \in \mathbb{Z}$ and $q = 0, \dots, |\mathcal{N}| - 1$. In addition, the magnetic flux $F$ should be replaced by the magnetic flux $\mathcal{F}$ satisfying eq. (6.17). From these manipulations, the zero mode solution of eq. (6.21) is obtained by

$$\hat{\psi}^s_I(\hat{Y}^4,\hat{Y}^5) = \int \frac{dk}{2\pi}\sum_{p\in\mathbb{Z}} C\exp\left[-\frac{k^2}{2|\mathcal{F}|} - \frac{ik}{R_5\mathcal{F}}(|\mathcal{N}|p+q)\right]\exp\left[ik\hat{Y}^4\right]\exp\left[i\frac{|\mathcal{N}|p+q}{R_5}\hat{Y}^5\right], \tag{6.24}$$

where $I = 0,\dots,\mathcal{N}-1$. Actually, we can confirm that eq. (6.24) satisfies eq. (6.21) if $\mathcal{F}s > 0$. We postpone the computation of the normalization constant to the next section.

In Section 3.2, the zero mode solutions are written by the Jacobi-theta function. To discuss the property of the zero mode space, the Jacobi-theta function is useful. We can write eq. (6.24) by operator-valued Jacobi-theta function

$$\hat{\psi}^s_I(\hat{Y}^4,\hat{Y}^5) = C\sqrt{\frac{|\mathcal{F}|}{2\pi}}\exp\left[-\frac{|\mathcal{F}|}{2}(\hat{Y}^4)^2\right]\vartheta\begin{bmatrix} I/|\mathcal{N}| \\ 0\end{bmatrix}\left(\frac{s|\mathcal{N}|}{2\pi i R_5}(\hat{Y}^4+is\hat{Y}^5), i\frac{R_4}{R_5}s\mathcal{N}\right).$$

If we consider the eigenstate of $\hat{Y}^4 + is\hat{Y}^5$, the factor $s\mathcal{N}$ should be positive because of the convergence. Therefore, in the following, we require $s\mathcal{N} > 0$[3]. Accordingly, $|\mathcal{N}| = s\mathcal{N}$ and

$$\hat{\psi}^s_I(\hat{Y}^4,\hat{Y}^5) = C\sqrt{\frac{|\mathcal{F}|}{2\pi}}\exp\left[-\frac{|\mathcal{F}|}{2}(\hat{Y}^4)^2\right]\vartheta\begin{bmatrix} I/|\mathcal{N}| \\ 0\end{bmatrix}\left(\frac{s|\mathcal{N}|}{2\pi i R_5}(\hat{Y}^4+is\hat{Y}^5), i\frac{R_4}{R_5}|\mathcal{N}|\right).$$

Let us go back to the zero mode equation of a bifundamental fermion. As a demonstration, we consider the magnetic fluxes that break the gauge group $U(2) \to U(1)\times U(1)$. Such a symmetry breaking can be realized by the following gauge background

$$\hat{A}_4(\hat{Y}^4,\hat{Y}^5) = 0, \quad \hat{A}_5(\hat{Y}^4,\hat{Y}^5) = \begin{pmatrix} \mathcal{F}_1\hat{Y}^4 & 0 \\ 0 & \mathcal{F}_2\hat{Y}^5\end{pmatrix},$$

where $\mathcal{F}_i$ $(i=1,2)$ satisfies the quantization condition (6.17).

A fermion in $U(2)$ adjoint representation is described as a $2\times 2$ matrix such that

$$\hat{\psi} = \begin{pmatrix}\hat{\psi}^+ \\ \hat{\psi}^-\end{pmatrix}, \quad \hat{\psi}^s = \begin{pmatrix}\hat{\psi}^s_{11} & \hat{\psi}^s_{12} \\ \hat{\psi}^s_{21} & \hat{\psi}^s_{22}\end{pmatrix} \quad (s=\pm).$$

The zero mode equation for each chirality is obtained by

$$(\partial_4 + is\partial_5)\begin{pmatrix}\hat{\psi}^s_{11} & \hat{\psi}^s_{12} \\ \hat{\psi}^s_{21} & \hat{\psi}^s_{22}\end{pmatrix} + s\begin{pmatrix}\mathcal{F}_1[\hat{Y}^4,\hat{\psi}^s_{11}] & \mathcal{F}_1\hat{Y}^4\hat{\psi}^s_{12} - \mathcal{F}_2\hat{\psi}^s_{12}\hat{Y}^5 \\ \mathcal{F}_2\hat{Y}^4\hat{\psi}^s_{21} - \mathcal{F}_1\hat{\psi}^s_{21}\hat{Y}^4 & \mathcal{F}_2[\hat{Y}^4,\hat{\psi}^s_{22}]\end{pmatrix} = 0 \quad (s=\pm). \tag{6.25}$$

Because of the noncommutativity, we must maintain the ordering in eq. (6.25).

First, let us start from the diagonal components. The zero mode equation of the diagonal component is written as

[3] The condition $2\pi - \theta^{45}\cdot\mathcal{N} > 0$ is necessary for both $s\mathcal{F} > 0$ and $s\mathcal{N} > 0$ to be compatible.

$$(\partial_4 + is\partial_5)\hat{\psi}^s_{11} + s\mathcal{F}_1[\hat{Y}^4, \hat{\psi}^s_{11}] = 0. \tag{6.26}$$

A unitary operator representing a gauge transformation is also described by a $2 \times 2$ matrix and act on a fermion in $U(2)$ adjoint representation such that

$$\hat{\Omega} = \begin{pmatrix} \hat{\Omega}(\mathcal{F}_1) & 0 \\ 0 & \hat{\Omega}(\mathcal{F}_2) \end{pmatrix}, \quad \hat{\psi}^s \to \hat{\Omega}\hat{\psi}^s\hat{\Omega}^\dagger$$

Therefore, the twisted boundary condition of the diagonal component is obtained by

$$\begin{aligned} \hat{\psi}^s_{11}(\hat{Y}^4 + 2\pi R_4, \hat{Y}^5) &= \exp\left[\frac{2\pi i R_4}{1+\theta^{45}\mathcal{F}_1} \cdot \mathcal{F}_1 \hat{Y}^5\right] \hat{\psi}^s_{11}(\hat{Y}^4, \hat{Y}^5) \exp\left[-\frac{2\pi i R_4}{1+\theta^{45}\mathcal{F}_1} \cdot \mathcal{F}_1 \hat{Y}^5\right], \\ \hat{\psi}^s_{11}(\hat{Y}^4, \hat{Y}^4 + 2\pi R_4) &= \hat{\psi}^s_{11}(\hat{Y}^4, \hat{Y}^5). \end{aligned} \tag{6.27}$$

In this case, the right hand side of the first condition induces the shift of $\hat{Y}^4$ such that $\hat{Y}^4 \to \hat{Y}^4 + \frac{\mathcal{N}_1\theta^{45}}{R_5}$. This shift induces the contradiction: $\frac{\mathcal{F}_1\theta^{45}}{1+\theta^{45}\mathcal{F}_1} = 1$. Therefore, $\hat{\psi}^s_{11}$ must be a scalar operator, i.e., $\hat{\psi}^s_{11} = \text{const.} \times \mathbf{1}$.

Next, we consider the top right component. The zero mode equation is

$$(\partial_4 + is\partial_5)\hat{\psi}^s_{12} + s(\mathcal{F}_1\hat{Y}^4\hat{\psi}^s_{12} - \mathcal{F}_2\hat{\psi}^s_{12}\hat{Y}^4) = 0 \tag{6.28}$$

with the twisted boundary condition

$$\begin{aligned} \hat{\psi}^s_{12}(\hat{Y}^4 + 2\pi R_4, \hat{Y}^5) &= \exp\left[\frac{2\pi i R_4}{1+\theta^{45}\mathcal{F}_1} \cdot \mathcal{F}_1 \hat{Y}^5\right] \hat{\psi}^s_{12}(\hat{Y}^4, \hat{Y}^5) \exp\left[-\frac{2\pi i R_1}{1+\theta^{45}\mathcal{F}_2} \cdot \mathcal{F}_2 \hat{Y}^5\right] \\ \hat{\psi}^s_{12}(\hat{Y}^4, \hat{Y}^5 + 2\pi R_5) &= \hat{\psi}^s_{12}(\hat{Y}^4, \hat{Y}^5) \end{aligned}$$

The third and fourth terms of eq. (6.28) represent the noncommutativity as we mentioned.

In the case of the fundamental representation, the factor $\exp\left[-|\mathcal{F}|(\hat{Y}^4)^2/2\right]$ comes from the term $\mathcal{F}\hat{Y}^4$ in eq. (6.21). Therefore, we expect that zero mode solutions can be written by the operator-valued Jacobi-theta function. Actually, the zero mode solutions are obtained by

$$\begin{aligned} \hat{\psi}^s_{12,I}(\hat{Y}^4, \hat{Y}^5) = C_{12}\sqrt{\frac{|\mathcal{F}_{12}|}{2\pi}} \exp\left[-\frac{s\mathcal{F}_1}{2}(\hat{Y}^4)^2\right] \\ \times \vartheta\begin{bmatrix} I/|\mathcal{N}_{12}| \\ 0 \end{bmatrix}\left(\frac{s|\mathcal{N}_{12}|}{2\pi i R_5}(\hat{Y}^4 + is\hat{Y}^5), i\frac{R_4}{R_5}|\mathcal{N}_{12}|\right) \exp\left[\frac{s\mathcal{F}_2}{2}(\hat{Y}^4)^2\right] \end{aligned} \tag{6.29}$$

where $\mathcal{F}_{12} = \mathcal{F}_1 - \mathcal{F}_2$, $\mathcal{N}_{12} = \mathcal{N}_1 - \mathcal{N}_2$, and $I = 0, \ldots, \mathcal{N}_{12} - 1$. In addition, we restrict ourselves to $s\mathcal{N}_{12} > 0$. We can confirm that eq. (6.29) satisfies the zero mode equation (6.28) and the periodic boundary conditions if $s\mathcal{F}_{12} > 0$[4].

Obviously, the above result can be generalized to $U(N)$ gauge group and symmetry breaking by magnetic fluxes.

[4]The condition $2\pi - \theta^{45} \cdot \mathcal{N}_{12} > 0$ is necessary for both $s\mathcal{F}_{12} > 0$ and $s\mathcal{N}_{12} > 0$ to be compatible.

### 6.2.3 Laplacian in the IKKT matrix model

In Chapter 5, we discussed the product of a scalar field and a fermion field. Therefore, to compare with Chapter 5, we have to consider the Laplacian and its eigenvalue problem.

We have already defined the Laplacian in the IKKT matrix model in eq. (6.11). The Laplacian has the relationship with the square of the Dirac operator even if we consider the noncommutative case. To see this relationship, let us denote the Dirac operator as

$$\not{D} = \begin{pmatrix} 0 & -D \\ D' & 0 \end{pmatrix}.$$

The square of $\not{D}$ can be arranged as

$$\not{D}^2 = \begin{pmatrix} -DD' & 0 \\ 0 & -D'D \end{pmatrix} = \Delta + \begin{pmatrix} -\frac{(\theta^{45})^2}{2}[D, D'] & 0 \\ 0 & -\frac{(\theta^{45})^2}{2}[D', D] \end{pmatrix}. \tag{6.30}$$

If we consider a fermion in an adjoint representation, the action of the operator (6.30) on the $(1, 2)$ component of the fermion can be written as

$$\not{D}^2 \hat{\psi}^s_{12} = \Delta \hat{\psi}^s_{12} - s(\theta^{45})^2 \mathcal{F}_{12} \hat{\psi}^s_{12}. \tag{6.31}$$

The relation (6.31) indicates that the zero modes of $D$ ($D'$) are also the lightest mode solutions of the Laplacian.

In addition, we can construct the eigenmodes of the Laplacian by the similar way in Section 3.2. Actually, the commutation relation between $D$ and $D'$ is

$$[D, D']\hat{\psi}^s_{12} = 2(\theta^{45})^2 \mathcal{F}_{12} \hat{\psi}^s_{12}.$$

If we set the magnetic flux $\mathcal{F}_{12} > 0$, the operators $D$ and $-D'$ can be interpreted as a creation operator and an annihilation operator, respectively. If we set the magnetic flux $\mathcal{F}_{12} < 0$, it is sufficient to exchange the roles of $D$ and $-D'$. This interpretation allows us to consider the analogy of the harmonic oscillator. Namely,

$$N := -DD', \quad \Delta = N + (\theta^{45})^2 \mathcal{F}_{12},$$

Therefore, the eigenmodes of the Laplacian can be obtained by

$$\begin{aligned} &\Delta \hat{\psi}^{+,n}_{12} = \lambda_{\theta^{45},n} \hat{\psi}^{+,n}_{12}, \\ &\hat{\psi}^{+,n}_{12} := D^n \hat{\psi}^{+}_{12}, \quad \lambda_{\theta^{45},n} = (\theta^{45})^2 \mathcal{F}_{12}(2n+1). \end{aligned}$$

The difference from the commutative case lies in the factor $(\theta^{45})^2$. In addition, it seems that all eigenvalues $\lambda_{\theta^{45},n}$ vanish in the limit $\theta^{45} \to 0$. Let us consider the mass dimension. Based on the principle of the IKKT matrix model, we assume the mass dimensions $[\hat{X}^M_0] = M^{-1}$

and $[\hat{A}_i] = M$. This assumption implies that the mass dimension of the scale factor $g$ is -2. In addition, the noncommutative parameter $\theta^{45}$ and the field strength have the mass dimensions -2 and +2, respectively. Therefore, the combination $\lambda_{\theta^{45},n}/g^2$ has the mass dimension +2. On the other hand, the trace is dimensionless. We expect that the trace with respect to four dimensions becomes the integral on a four-dimensional space at an appropriate limit. From the dimensionless of the trace, the expected correspondence is $\mathrm{Tr}_{4\mathrm{D}} \sim \int d^4x/\det(\theta^{MN})$, and $\det(\theta^{MN})$ includes the factor $(\theta^{45})^2$. Therefore, the factor $(\theta^{45})^2$ is natural.

### 6.2.4 Normalizations

In this subsection, we define the trace and compute the normalization constant of the zero mode solutions. We have to assure the cyclic property with respect to a gauge transformation realized by a unitary operator. In addition, we should define the trace without facing an inconsistency with the translations along $\hat{Y}^4$ and $\hat{Y}^5$ directions.

We define the trace by

$$\mathrm{Tr}(\cdot) = \int_0^{2\pi R_4} dY^4 \left\langle Y^4 \right| \cdot \left| Y^4 \right\rangle .$$

First, let us confirm the equivalence with the case of $Y^4$ and that of $Y^5$

$$\begin{aligned}
\int_0^{2\pi R_4} dY^4 \left\langle Y^4 \right| \hat{f}(\hat{Y}^4, \hat{Y}^5) \left| Y^4 \right\rangle &= \int_0^{2\pi R_4} dY^4 \int_0^{2\pi R_5} dY^5 \left\langle Y^4 \right| \hat{f}(\hat{Y}^4, \hat{Y}^5) \left| Y^5 \right\rangle \left\langle Y^5 \,\middle|\, Y^4 \right\rangle \\
&= \int_0^{2\pi R_4} dY^4 \int_0^{2\pi R_5} dY^5 \left\langle Y^5 \,\middle|\, Y^4 \right\rangle \left\langle Y^4 \right| \hat{f}(\hat{Y}^4, \hat{Y}^5) \left| Y^5 \right\rangle \\
&= \int_0^{2\pi R_5} dY^5 \left\langle Y^5 \right| \hat{f}(\hat{Y}^4, \hat{Y}^5) \left| Y^5 \right\rangle \\
&= \mathrm{Tr}\left( \hat{f}(\hat{Y}^4, \hat{Y}^5) \right),
\end{aligned}$$

where we have used

$$\mathbf{1} = \int_0^{2\pi R_4} dY^4 \left| Y^4 \right\rangle \left\langle Y^4 \right| , \quad \mathbf{1} = \int_0^{2\pi R_5} dY^5 \left| Y^5 \right\rangle \left\langle Y^5 \right|$$

on the Hilbert space defined in Section 4.3. Next, we confirm the consistency with the compactification condition.

$$\begin{aligned}\int_{2\pi R_4}^{4\pi R_4} dY^4 \left\langle Y^4 \right| \hat{f}(\hat{Y}^4,\hat{Y}^5) \left| Y^4 \right\rangle &= \int_0^{2\pi R_4} dY^4 \left\langle Y^4+2\pi R_4 \right| \hat{f}(\hat{Y}^4,\hat{Y}^5) \left| Y^4+2\pi R_4 \right\rangle \\ &= \int_0^{2\pi R_4} dY^4 \left\langle Y^4 \right| \hat{U}_4\hat{f}(\hat{Y}^4,\hat{Y}^5)\hat{U}_4^{-1} \left| Y^4 \right\rangle \\ &= \int_0^{2\pi R_4} dY^4 \left\langle Y^4 \right| \hat{\Omega}_4\hat{f}(\hat{Y}^4,\hat{Y}^5)\hat{\Omega}_4^{-1} \left| Y^4 \right\rangle \\ &= \int_0^{2\pi R_5} dY^5 \left\langle Y^5 \right| \hat{\Omega}_4\hat{f}(\hat{Y}^4,\hat{Y}^5)\hat{\Omega}_4^{-1} \left| Y^5 \right\rangle \\ &= \mathrm{Tr}\left(\hat{f}(\hat{Y}^4,\hat{Y}^5)\right),\end{aligned}$$

$$\begin{aligned}\int_{2\pi R_5}^{4\pi R_5} dY^5 \left\langle Y^5 \right| \hat{f}(\hat{Y}^4,\hat{Y}^5) \left| Y^5 \right\rangle &= \int_0^{2\pi R_5} dY^5 \left\langle Y^5+2\pi R_5 \right| \hat{f}(\hat{Y}^4,\hat{Y}^5) \left| Y^5+2\pi R_5 \right\rangle \\ &= \int_0^{2\pi R_5} dY^5 \left\langle Y^5 \right| \hat{U}_5\hat{f}(\hat{Y}^4,\hat{Y}^5)\hat{U}_5^{-1} \left| Y^5 \right\rangle \\ &= \int_0^{2\pi R_5} dY^5 \left\langle Y^5 \right| \hat{f}(\hat{Y}^4,\hat{Y}^5+2\pi R_5) \left| Y^5 \right\rangle \\ &= \int_0^{2\pi R_5} dY^5 \left\langle Y^5 \right| \hat{f}(\hat{Y}^4,\hat{Y}^5) \left| Y^5 \right\rangle \\ &= \mathrm{Tr}\left(\hat{f}(\hat{Y}^4,\hat{Y}^5)\right).\end{aligned}$$

where we used the equivalence between $Y^4$ and $Y^5$ in the fourth line.

For general gauge transformations, we can show the equivalence of the trace between before and after transformations if we assume the existence of the completeness relation of the unitary operator as a gauge transformation. This ensures the cyclic property with respect to the whole gauge transformations, and this is physically sufficient.

Then, we can compute the normalization constant with the above definition of the trace,

$$\begin{aligned}1 = \mathrm{Tr}\left(\hat{\psi}^{s,\dagger}_{12,I}(\hat{Y}^4,\hat{Y}^5)\hat{\psi}^{s}_{12,J}(\hat{Y}^4,\hat{Y}^5)\right) &:= \int_0^{2\pi R_4} dY^4 \left\langle Y^4 \right| \hat{\psi}^{s,\dagger}_{12,I}(\hat{Y}^4,\hat{Y}^5)\hat{\psi}^{s}_{12,J}(\hat{Y}^4,\hat{Y}^5) \left| Y^5 \right\rangle \\ &= \frac{1}{A}\int_0^{2\pi R_4} dY^4 \int_0^{2\pi R_5} dY^5 \hat{\psi}^{s,\dagger}_{12,I}(Y^4,Y^5)\hat{\psi}^{s}_{12,J}(Y^4,Y^5) \\ &= \delta_{IJ}|C_{12}|^2\frac{R_5|\mathcal{F}_{12}|}{A}\cdot I_{12}(\theta^{45})\end{aligned}$$

where

$$I_{12}(\theta^{45}) := \sum_{p\in\mathbb{Z}} \int_{\frac{|\mathcal{N}_{12}|p+q}{R_5\mathcal{F}_{12}}(1+\theta^{45}\mathcal{F}_2)}^{\frac{|\mathcal{N}_{12}|p+q}{R_5\mathcal{F}_{12}}(1+\theta^{45}\mathcal{F}_2)+2\pi R_4} \exp\left[-|\mathcal{F}_{12}|x^2\right] dx$$

If we take the limit $\theta^{45} \to 0$, the integral becomes the Gaussian integral and its value is $I_{12}(0) = \sqrt{\pi/|\mathcal{F}_{12}|}$. However, for $\theta^{45} \neq 0$, we need to use numerical approach.

Therefore, the normalization constant $C_{12}$ is obtained by

$$C_{12} = \left( \frac{R_5 |\mathcal{F}_{12}|}{A} \cdot I_{12}(\theta^{45}) \right)^{-1/2} .$$

Then, the zero modes can be written by

$$\hat{\psi}^s_{12,I}(\hat{Y}^4, \hat{Y}^5) = \left( \frac{2\pi R_5}{A} \cdot I_{12}(\theta^{45}) \right)^{-1/2} \exp\left[ -\frac{s\mathcal{F}_1}{2} (\hat{Y}^4)^2 \right] \\ \times \vartheta \begin{bmatrix} I/|\mathcal{N}_{12}| \\ 0 \end{bmatrix} \left( \frac{s|\mathcal{N}_{12}|}{2\pi i R_5} (\hat{Y}^4 + is\hat{Y}^5), i\frac{R_4}{R_5} |\mathcal{N}_{12}| \right) \exp\left[ \frac{s\mathcal{F}_2}{2} (\hat{Y}^4)^2 \right]$$

If we consider the limit $\theta^{45} \to 0$, there is the difference in the overall factor, explicitly $A^{-1/2}$, between the commutative case in Section 3.2 and the noncommutative case. However, this difference is due to the normalization of the integral on the torus. Hence, the same result can be obtained if the normalization is unified.

### 6.2.5 Three-point and higher order coupling constants

In Subsection 6.2.4, we defined the trace and showed the orthogonality of the zero modes. The purpose of this section is to compare the product property with Section 3.2.

Let us consider a product of zero modes (6.29). A key observation is that the Jacobi-theta function part can be arranged as a c-number if two zero modes have the same chirality and we consider their product. Based on these observations, the product of two zero modes (6.29) is obtained as follows.

$$\begin{aligned} &\hat{\psi}_{ba,I} \cdot \hat{\psi}_{ac,J} \\ &= \sqrt{\frac{2\pi |\mathcal{F}_{ba}\mathcal{F}_{ac}|}{|\mathcal{F}_{bc}|}} \frac{C_{ba}C_{ac}}{C_{bc}} \\ &\quad \times \sum_{K \in \mathbb{Z}_{|\mathcal{N}_{ba}|+|\mathcal{N}_{ac}|}} \hat{\psi}_{bc, I+J+|\mathcal{N}_{ba}|K} \times \vartheta \begin{pmatrix} \frac{|\mathcal{N}_{ac}|I - |\mathcal{N}_{ba}|J + |\mathcal{N}_{ba}||\mathcal{N}_{ac}|K}{|\mathcal{N}_{ba}\mathcal{N}_{bc}\mathcal{N}_{ac}|} \\ 0 \end{pmatrix} \left( 0, i\frac{R_4}{R_5} |\mathcal{N}_{ba}\mathcal{N}_{bc}\mathcal{N}_{ac}| \right) . \end{aligned} \tag{6.32}$$

We have used the formula for the Jacobi-theta function as we showed in Section 3.2. Equation (6.32) indicates that our conjecture mentioned at the beginning of this chapter is realized and this is the same result as in Chapter 5.

Based on eq. (6.32), we compute Yukawa couplings. As a demonstration, we consider the $U(N)$ gauge group and its breaking $U(N) \to \prod_{a=1}^{3} U(N_a)$, where $\sum_{a=1}^{3} N_a = N$. To realized such a symmetry breaking, we introduce the background gauge field as

$$\hat{A}_4(\hat{Y}^4,\hat{Y}^5)=0,\quad \hat{A}_5(\hat{Y}^4,\hat{Y}^5)=\begin{pmatrix}\mathcal{F}_1\hat{Y}^4\mathbf{1}_{N_1} & 0 & 0\\ 0 & \mathcal{F}_2\hat{Y}^4\mathbf{1}_{N_2} & 0\\ 0 & 0 & \mathcal{F}_3\hat{Y}^4\mathbf{1}_{N_3}\end{pmatrix},$$

We have to consider a sign assignment with respect to the magnetic fluxes. Obviously, the following identity is valid

$$\mathcal{F}_{12}+\mathcal{F}_{23}+\mathcal{F}_{31}=0$$

This identity implies that only one of the three has a different sign. In the following, we select $\mathcal{F}_{23}>0$ and $\mathcal{F}_{12},\mathcal{F}_{31}<0$ to compare the following result with Section 3.2. This selection also implies $\mathcal{N}_{23}>0,\mathcal{N}_{12},\mathcal{N}_{31}<0$.

A fermion has a $3\times 3$ block matrix structure since we consider $U(N)$ gauge group and its breaking $U(N)\to\prod_{a=1}^{3}U(N_a)$. Base on Subsection 6.2.2, the component of the zero mode fermion can be written as

$$\hat{\psi}=\begin{pmatrix}\hat{\psi}^+\\ \hat{\psi}^-\end{pmatrix},\quad \hat{\psi}^+=\begin{pmatrix}\text{const.} & 0 & \hat{\psi}^+_{13,I}\\ \hat{\psi}^+_{21,J} & \text{const.} & \hat{\psi}^+_{23,K}\\ 0 & 0 & \text{const.}\end{pmatrix},\quad \hat{\psi}^-=\hat{\psi}^{+,\dagger},$$

where const. $=\mathbf{1}$ and $I,J,K$ denote the degeneracies. The lightest mode scalars have the same matrix structure as the zero mode fermion. In addition, we showed that the lightest mode scalars have the same functional form as the zero mode fermions. Therefore, we denote $\hat{\Phi}$ to describe both the lightest scalars and the zero mode fermions.

Let us denote by $\hat{\Phi}_{ab,I}$ $(a,b=1,2,3)$ the $(a,b)$ block component. Based on the action (6.8), the Yukawa couplings are obtained by

$$Y_{IJL}:=\mathrm{Tr}\left(\hat{\Phi}^\dagger_{23,L}\cdot\hat{\Phi}_{21,I}\cdot\hat{\Phi}_{13,J}\right). \tag{6.33}$$

The ordering in the Yukawa couplings (6.33) is important since the factor $\exp\left[\frac{s\mathcal{F}_1}{2}(\hat{Y}^4)^2\right]$ of $\hat{\Phi}_{21,I}$ can be cancelled by the factor $\exp\left[-\frac{\mathcal{F}_1}{2}(\hat{Y}^4)^2\right]$ of $\hat{\Phi}_{13,J}$. In addition, $\mathcal{F}_{21},\mathcal{F}>0$ from our sign assignment. This situation is the same as that in eq. (6.32). Therefore, we can rewrite it by the linear combination of the Jacobi-theta function, and the result is as,

$$\begin{aligned}&\hat{\Phi}_{21,I}\cdot\hat{\Phi}_{13,J}\\ &=\sqrt{\frac{2\pi|\mathcal{F}_{21}\mathcal{F}_{13}|}{|\mathcal{F}_{23}|}}\frac{C_{21}C_{13}}{C_{23}}\\ &\quad\times\sum_{K\in\mathbb{Z}_{|\mathcal{N}_{21}|+|\mathcal{N}_{13}|}}\hat{\Phi}_{23,I+J+|\mathcal{N}_{21}|K}\times\vartheta\begin{pmatrix}\frac{|\mathcal{N}_{13}|I-|\mathcal{N}_{21}|J+|\mathcal{N}_{21}||\mathcal{N}_{13}|K}{|\mathcal{N}_{21}\mathcal{N}_{23}\mathcal{N}_{13}|}\\ 0\end{pmatrix}\left(0,i\frac{R_4}{R_5}|\mathcal{N}_{21}\mathcal{N}_{23}\mathcal{N}_{13}|\right),\end{aligned}$$

On the other hand, we have already proved the orthogonality. Consequently, the Yukawa coupling (6.33) is obtained by

$$Y_{IJL} = \sqrt{\frac{2\pi A}{R_5} \cdot \frac{I_{23}}{I_{21}I_{13}}}\vartheta \begin{pmatrix} \frac{1}{|\mathcal{N}_{21}|}\left(\frac{L}{|\mathcal{N}_{23}|} - \frac{J}{|\mathcal{N}_{13}|}\right) \\ 0 \end{pmatrix} \left(0, i\frac{R_4}{R_5}|\mathcal{N}_{21}\mathcal{N}_{23}\mathcal{N}_{13}|\right), \tag{6.34}$$

where we assume ${}^\exists K \in \mathbb{Z}_{|\mathcal{N}_{21}|+|\mathcal{N}_{13}|}$ such that $L = I + J + |\mathcal{N}_{21}|K$.

The difference from the commutative case is the presence of an overall factor if we fix each generation number $\mathcal{N}_{ab}$ to the same value with the commutative case. The Yukawa coupling (6.34) goes back to the commutative case in the limit $\theta^{45} \to 0$ since the normalization constant $I_{ab}$ and the generation number $\mathcal{N}_{ab}$ go back to the commutative case completely.

On the other hand, the product property of the zero modes is preserved (c.f., eq. (6.32)) even if the completeness relation is obscure since there is no assurance that the Dirac operator (6.25) has the complete orthonormal system as we mentioned. Hence, we cannot admit the interpretation as an insertion of the delta function (e.g., eq. (5.14)) at this level. However, the product property is the most important for the decomposition. Therefore, we can conclude that the selection rule that we proposed in Section 3.2 is completed. In other words, we can obtain the decomposition of higher order coupling constants in the IKKT matrix model by the three-point coupling constants (6.34).

## 6.3 Magnetized fuzzy sphere

### 6.3.1 Ginsparg-Wilson algebra and Dirac operator

In the previous section, we defined the Dirac operator on the basis of the fermionic part of the action of the IKKT matrix model (4.6). We are interested in the magnetized fuzzy sphere to compare the product property with the result in Section 3.3. However, the Dirac operator for magnetized fuzzy sphere cannot be defined by the fermionic part of the action (4.22) straightforwardly. On the other hand, in Ref. [79], the authors proposed a method to construct a Dirac operator on any fuzzy manifolds on the basis of the Ginsparg-Wilson (GW) algebra. Therefore, we introduce the GW algebra in this subsection and define the Dirac operator for magnetized fuzzy sphere in the next subsection.

The GW algebra $\mathcal{A}_{\mathrm{GW}}$ is defined as a unital $*$-algebra over $\mathbb{C}$ generated by $\Gamma$ and $\Gamma'$ such as

$$\mathcal{A}_{\mathrm{GW}} := \left\langle \Gamma, \Gamma' : \Gamma^2 = \Gamma'^2 = \mathbf{1}, \Gamma^* = \Gamma, \Gamma'^* = \Gamma' \right\rangle, \tag{6.35}$$

where $*$ is a conjugate, typically the Hermitian conjugate. We will define $\Gamma$ and $\Gamma'$ as matrices since the fuzzy sphere is constructed from the finite dimensional representation of the $su(2)$ algebra.

In general, a GW Dirac operator $D_{\mathrm{GW}}$ is defined as an element of $\mathcal{A}_{\mathrm{GW}}$ [79] such that

$$f(a, \Gamma) D_{\mathrm{GW}} = 1 - \Gamma\Gamma', \tag{6.36}$$

where $a$ is a parameter which corresponds to a lattice spacing and $f(a, \Gamma)$ is an arbitrary function depending on $a$ and $\Gamma$. In addition, we assume that $f(a, \Gamma)$ has an inverse element.

Since the GW Dirac operator satisfies

$$\Gamma D_{\mathrm{GW}} + D_{\mathrm{GW}} \Gamma^{'} = 0, \tag{6.37}$$

we can prove the index theorem

$$\mathrm{Index}\,(D_{\mathrm{GW}}) = \frac{1}{2}\,\mathrm{Tr}\left(\Gamma + \Gamma^{'}\right).$$

Originally, the relation (6.37), which is a kind of the GW relation, is introduced to avoid the fermion doubling problem in lattice field theory. Therefore, we do not suffer from the fermion doubling problem if we consider zero modes of the GW Dirac operator.

In the following, we consider

$$D_{\mathrm{GW}} = \frac{1}{a}\left(\Gamma - \Gamma^{'}\right), \quad S = \mathrm{Tr}\left(\bar{\Psi} D_{\mathrm{GW}} \Psi\right).$$

### 6.3.2 Dirac operator of magnetized fuzzy sphere

In Section 3.3, the magnetic flux is introduced as the 't Hooft-Polyakov monopole background that is a geometrical object. However, the fuzzy sphere is defined as an algebraic object. Therefore, we have to introduce the concept of the vector bundle as an algebraic object.

To define a vector bundle as an algebraic object, the Serre-Swan theorem or Swan's theorem [80,81] are important. These theorems argue that the algebraic meaning of vector bundles can be defined as projective modules.

Let us start from the case of the sphere. Each point of a vector bundle is composed of a point of the manifold and an element of a vector space. Based on the Serre-Swam theorem or Swan's theorem, any vector bundles are direct summands of trivial bundles. In other words, we can construct a non-trivial vector bundle by considering of a projection on a trivial bundle. The projection for the 't Hooft-Polyakov monopole bundle with the charge $\pm N$ on the sphere can be defined by

$$\mathcal{P}^{\pm N} = \prod_{i=1}^{N} \frac{1 \pm \vec{\sigma}^{(i)} \cdot \vec{n}}{2}, \tag{6.38}$$

$$\vec{\sigma}^{(i)} := \underbrace{\mathbf{1}_2 \otimes \cdots \otimes \overset{i}{\vec{\sigma}} \otimes \cdots \otimes \mathbf{1}_2}_{N}, \tag{6.39}$$

where $n_i$ $(i = 1, 2, 3)$ is a unit vector on the sphere.

From the projection (6.38), we expect naturally that the following operator is an appropriate projection for the 't Hooft-Polyakov monopole with the charge $\pm 1$ on the fuzzy sphere,

$$p^{(\pm 1)} = \frac{1 \pm \vec{\sigma} \cdot \vec{\hat{n}}}{2}, \tag{6.40}$$

where $\hat{n}_i$ $(i = 1, 2, 3)$ is the normalized generator of the $su(2)$ algebra satisfying $\sum_{i=1}^{3} (\hat{n}_i)^2 = 1$. In the following, we use the hat to describe the representation matrix of the $su(2)$ algebra.

However, the operator (6.40) cannot play a role of a projection operator because of the noncommutativity of $\hat{n}_i$. To define a projective operator, it is necessary to introduce another generator $\hat{T}_i$ of the $su(2)$ algebra with the Casimir operator eigenvalue $T(T+1)$ [82, 83]. Namely,

$$p^{(+1)} := \frac{1+\gamma_\chi}{2}, \quad \gamma_\chi := \frac{\vec{\sigma}\cdot\vec{\hat{T}}+1/2}{T+1/2}. \tag{6.41}$$

This is an idempotent operator, which is remarked in Refs. [60, 61], hence we can admit this idempotent operator as a projection operator. On the other hand, the projection operator (6.41) can be also interpreted as the projection operator of the angular momentum $\hat{K}_i^{(+1)} := \hat{T}_i + \frac{1}{2}\sigma_i$ with the Casimir operator eigenvalue $K(K+1)$, $(K = T\pm\frac{1}{2})$ into $K = T+\frac{1}{2}$ subspace,

$$p^{(+1)} = \frac{\sum_{i=1}^{3}\left(\hat{K}_i^{(+1)}\right)^2 - (T-1/2)(T+1/2)}{(T+1/2)(T+3/2)-(T-1/2)(T+1/2)}.$$

Hence, based on this interpretation, we can define the projection operator for the 't Hooft-Polyakov monopole with the charge $\pm N$ on the fuzzy sphere as

$$p^{(+N)} := \frac{\prod_{K\neq K_{\max}}[\sum_{i=1}^{3}\left(K_i^{(+N)}\right)^2 - K(K+1)]}{\prod_{K\neq K_{\max}}[K_{\max}(K_{\max}+1)-K(K+1)]},$$

where $\hat{K}_i^{(+N)} := \hat{T}_i + \sum_{i=1}^{N}\frac{1}{2}\sigma_i^{(i)}$ and $K_{\max} = K + \frac{N}{2}$. Similarly, the projection operator for the monopole charge $-N$ can be constructed from the minimum value $K_{\min} = T - \frac{N}{2}$.

As a result, according to Refs. [82, 83], the GW algebra for magnetized fuzzy sphere can be defined as

$$\Gamma^{\pm} := \frac{\vec{\sigma}(\vec{\hat{L}}^L+\vec{\hat{T}})+1/2}{L\pm T+1/2}, \quad \Gamma' = -\frac{-\vec{\sigma}\cdot\vec{\hat{L}}^R+1/2}{L+1/2}, \quad a = \frac{1}{\sqrt{(L+1/2)(L\pm T+1/2)}},$$

where $\hat{T}_i$ with the Casimir operator spectrum $T(T+1)$ corresponds to the extra angular momentum operator. The superscripts $L$ and $R$ mean the left-action and the right-action, respectively. The IKKT matrix model is constructed by the commutator. Hence, we have to consider the left-action and the right-action naturally. In addition, we have to restrict ourselves into $\hat{L}_i^L + \hat{T}_i$ with the Casimir operator eigenvalue $(L\pm T)(L\pm T+1)$. From the definition, the square of $D_{\mathrm{GW}}$ is given by

$$(D_{\mathrm{GW}})^2 = \left(\vec{\hat{L}}^L - \vec{\hat{L}}^R + \vec{\hat{T}} + \frac{1}{2}\vec{\sigma}\right)^2 + \frac{1}{4} - T^2.$$

If we define the generator $J_i = \hat{L}_i^L - \hat{L}_i^R + \hat{T}_i + \frac{1}{2}\sigma_i$ with the Casimir operator eigenvalue $J(J+1)$, then $J = T - \frac{1}{2}$ corresponds to the zero mode states with the degeneracy $2T$. On the other hand, the topological charge or the index of this monopole background is $2T$, and this value is the same as the continuous limit [83]. Therefore, this result is consistent with Section 3.3.

### 6.3.3 Laplacian of magnetized fuzzy sphere

As we discussed in Section 3.3, the zero mode fermions are distinguished from the lightest mode bosons because of the curvature contribution. Therefore, we have to consider the lightest modes of the Laplacian on magnetized fuzzy sphere. In the following, we focus on states corresponding to the scalar fields since the eigenvalue problem of the vector fields is essentially the same as that of the scalar fields.

Let us start from the standard action of the scalar sector on the fuzzy sphere without the monopole,

$$S = \mathrm{Tr}\left([\hat{L}_i, \Phi][\hat{L}_i, \Phi]\right) = \mathrm{Tr}\left(-\Phi[\hat{L}_i, [\hat{L}_i, \Phi]]\right) = \mathrm{Tr}\left(-\Phi(\hat{\mathcal{L}}_i)^2\Phi\right),$$

where $\Phi$ is a Hermitian matrix and $\hat{\mathcal{L}}_i = [\hat{L}_i, \cdot] = \hat{L}_i^L - \hat{L}_i^R$ is the $su(2)$ generator with the Casimir operator eigenvalue $\mathcal{L}(\mathcal{L}+1), \mathcal{L} \in \{0, \dots, 2L\}$.

As we discussed, the 't Hooft-Polyakov monopole can be introduced as an extra angular momentum. Therefore, the action of the scalar sector on the fuzzy sphere with the 't Hooft-Polyakov monopole is defined naturally

$$S = \mathrm{Tr}\left(-\Phi(\hat{\mathcal{K}}_i)^2\Phi\right) = \mathrm{Tr}\left(-\mathcal{K}(\mathcal{K}+1)\Phi\Phi\right),$$

where $\hat{\mathcal{K}}_i := \hat{\mathcal{L}}_i + \hat{T}_i$.

As we mentioned in Subsection 6.3.2, we need to consider the projection into $\hat{L}_i^L + \hat{T}_i$ whose Casimir operator eigenvalue is $(L \pm T)(L \pm T + 1)$. This implies that the Casimir operator eigenvalue of $\hat{\mathcal{K}}_i$ is $\mathcal{K}(\mathcal{K}+1)$, where $\mathcal{K}$ runs from $T$ to $2L \pm T$. Hence, the lightest eigenvalue is $T^2 + T$. However, the lightest eigenvalue we expect is $T$ if the corresponding radius is $R = 1$. Therefore, the natural extension of the standard action of the scalar sector is not an appropriate action.

To construct an appropriate action, we consider the correspondence between wavefunctions and states by using the coherent spin state. The coherent spin state is introduced as an analogy of the coherent state of the harmonic oscillator in quantum mechanics [84].

Let us start from the generator $\hat{S}_i$ of the $su(2)$ algebra with the Casimir operator eigenvalue $S(S+1)$. The ground state is the highest spin state,

$$\hat{S}_z |S, S\rangle = S |S, S\rangle .$$

The other states can be constructed by acting the lowering operator, i.e.,

$$(\hat{S}_-)^I |S, S\rangle = \left(\frac{I!(2S)!}{(2S-I)!}\right)^{1/2} |S, S-I\rangle \quad (0 \le I \le 2S),$$

where $\hat{S}_- := \hat{S}_x - i\hat{S}_y$. In Ref. [84], the coherent spin state is defined by

$$|z\rangle := \frac{1}{N^{1/2}} \exp\left(z\hat{S}_-\right) |S, S\rangle = \frac{1}{N^{1/2}} \sum_{I=0}^{2S} \left(\frac{(2S)!}{p!(2S-I)!}\right)^{1/2} z^I |S, S-I\rangle , \tag{6.42}$$

where $N$ is a normalization factor. Since the inner product of the spin coherent state should be finite, i.e.,

$$\langle z|z\rangle = \frac{1}{N}\sum_{I=0}^{2S}\frac{(2S)!}{I!(2S-I)!}|z|^{2I} = \frac{1}{N}(1+|z|^2)^{2S} < \infty, \tag{6.43}$$

then $N$ should be proportional to $(1+|z|^2)^S$.

On the other hand, the coherent spin state satisfies the overcompleteness relation since the set $\{|S : S-I\rangle\}_{I=0}^{2S}$ satisfies the completeness relation. In other words,

$$\int d^2z\ m(|z|^2)\,|z\rangle\,\langle z| = \sum_{I=0}^{2S}|S,S-I\rangle\,\langle S,S-I| = \mathbf{1} \tag{6.44}$$

determines the integral measure $m(|z|^2) \propto \frac{2S+1}{\pi}\frac{1}{(1+|z|^2)^2}$. The proportionality constant depends on the normalization condition (6.43).

Let us consider the coherent spin state for the zero modes of the Dirac operator. The set of the zero mode states is $\{|T-1/2 : T-1/2-I\rangle\}_{I=0}^{2T}$. Therefore, the coherent spin state is defined by

$$|z\rangle := \frac{1}{N^{1/2}}\sum_{I=0}^{2T-1}\left(\frac{(2T-1)!}{I!(2T-I-1)!}\right)^{1/2} z^I\,|T-1/2,T-1/2-I\rangle\,.$$

We require the normalization factor $N$ satisfies

$$\langle z|z\rangle = \frac{1}{N}(1+|z|^2)^{2T-1} \equiv \frac{2T}{4\pi R^2}$$

since there are $2T$ zero mode states in the area $4\pi R^2$. We can confirm the consistency of this requirement from the integral measure. This requirement implies that the integral of the overcompleteness relation must be the following form

$$4R^2\int d^2z\frac{1}{(1+|z|^2)^2}. \tag{6.45}$$

Equation (6.45) is nothing but the integral over the sphere with the metric (3.36).

The normalized state is written as

$$|z\rangle := \frac{1}{(1+|z|^2)^{\frac{2T-1}{2}}}\sum_{I=0}^{2T-1}\left(\frac{1}{4\pi R^2}\cdot\frac{(2T)!}{I!(2T-I-1)!}\right)^{1/2} z^I\,|T-1/2,T-1/2-I\rangle\,.$$

Then, we can derive the wavefunctions (c.f., Ref. [75], Section 3.3) as the inner product. Namely,

$$\langle T-1/2,T-1/2-I|z\rangle = \left(\frac{1}{4\pi R^2}\cdot\frac{(2T)!}{I!(2T-I-1)!}\right)^{1/2}\frac{z^I}{(1+|z|^2)^{\frac{2T-1}{2}}}. \tag{6.46}$$

The right-hand side of eq. (6.46) is the same as the zero mode fermions $\psi^{2T}_{(F),I}$.

On the other hand, the lightest mode bosons can be written as zero-mode fermions with the shift of the magnetic flux (3.39). Since the magnetic flux for fermions is $2T$, the lightest-mode bosons should have the magnetic flux $2T+1$ and the corresponding states can be written as $\{|T, T-I\rangle\}_{I=0}^{2T}$. This difference comes from the curvature contribution. We can expect that an appropriate Laplacian can be constructed on the basis of the GW algebra without the contribution of the curvature.

Let us consider

$$\Gamma_i^{\pm} := \frac{\hat{L}_i^L + \hat{T}_i}{L \pm T}, \quad \Gamma_i^{'} = -\frac{-\hat{L}_i^R}{L}, \quad a = \frac{1}{\sqrt{L(L \pm T)}},$$

where we also restrict ourselves into $\hat{L}_i^L + \hat{T}_i$ with the Casimir operator eigenvalue $(L \pm T)(L \pm T + 1)$. These are $\Gamma^{\pm}$ and $\Gamma^{'}$ without the spinor and the curvature contributions ($\sigma_i$ and $1/2$ factor, respectively). Based on the spectral analysis, we can confirm that the following $R^2\Delta_{\mathrm{GW}}$ can be an appropriate Laplacian

$$R^2\Delta_{\mathrm{GW}} = \sum_{i=1}^{3} (D_{\mathrm{GW},i})^2, \quad D_{\mathrm{GW},i} := \frac{1}{a}\left(\Gamma_i^{\pm} - \Gamma_i^{'}\right).$$

### 6.3.4 Three-point and higher order coupling constants

We obtained the zero modes of the Dirac operator and the lightest modes of the Laplacian on the magnetized fuzzy sphere. Those modes are states in a representation of the $su(2)$ algebra. We summarize the results as follows,

$$\begin{cases} |T-1/2, T-1/2-I\rangle \quad (I = 0, \ldots, 2T-1) & \text{zero modes of the Dirac operator} \\ |T, T-I\rangle \quad (I = 0, \ldots, 2T) & \text{the lightest modes of the Laplacian} \end{cases}$$

In Section 3.3, we computed the products of wavefunctions corresponding to these modes. On the other hand, we confirmed that these modes correspond to the wavefunctions in Subsection 6.3.3. Therefore, we expect naturally that we can obtain the product property in Section 3.3 in the language of the states.

To confirm our expectation, let us consider the tensor product and Racah's formula. The zero modes and the lightest modes are the eigenstates of the angular momenta. Therefore, the Clebsch-Gordan coefficients appear in the tensor product. Hence, we have to compute the Clebsch-Gordan coefficients explicitly. The Racah's formula [85] is the formula to compute the general Clebsch-Gordan coefficients. The explicit form of Racah's formula is

$$
\begin{aligned}
&\langle j_1, m_1 : j_2, m_2 \,|\, j, m\rangle \\
&= \delta_{m_1+m_2,m} \left[\frac{(2j+1)(j_1+j_2-j)!(j_1-j_2+j)!(-j_1+j_2+j)!}{(j_1+j_2+j+1)!}\right]^{1/2} \\
&\times \left[(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j+m)!(j-m)!\right]^{1/2} \\
&\sum_n (-1)^n \left[\frac{1}{n!(j_1+j_2-j-n)!(j_1-m_1-n)!(j_2+m_2-n)!(j-j_2+m_1+n)!(j-j_1-m_2-n)!}\right]^{-1},
\end{aligned}
\tag{6.47}
$$

where $n$ is an integer, and the sum over $n$ is taken as long as the factorial numbers are not negative.

Let us start from the tensor product of the lightest-mode bosons. These states are written as the coherent spin states such that

$$
\begin{aligned}
|z_1\rangle_{(B)} &:= \frac{1}{(1+|z|^2)^{\frac{2T_1}{2}}} \sum_{I_1=0}^{2T_1} \left(\frac{1}{4\pi R^2} \cdot \frac{(2T_1+1)!}{I_1!(2T_1-I_1)!}\right)^{1/2} z^{I_1} \, |T_1, T_1 - I_1\rangle = \sum_{I_1=0}^{2T_1} \psi^{2T_1}_{(B),I_1} \, |T_1, T_1 - I_1\rangle \,, \\
|z_2\rangle_{(B)} &:= \frac{1}{(1+|z|^2)^{\frac{2T_2}{2}}} \sum_{I_2=0}^{2T_2} \left(\frac{1}{4\pi R^2} \cdot \frac{(2T_2)!}{I_2!(2T_2-I_2)!}\right)^{1/2} z^{I_2} \, |T_2, T_2 - I_2\rangle = \sum_{I_2=0}^{2T_2} \psi^{2T_2}_{(B),I_2} \, |T_2, T_2 - I_2\rangle \,.
\end{aligned}
$$

Since we need to consider the projection to the maximum or minimum spin state, we have to compute the Clebsch-Gordan coefficient $\langle T_1, T_1 - I_1; T_2, T_2 - I_2 \,|\, T_3, T_3 - I_3\rangle$, where $T_3 = T_1 + T_2$ and $I_1 + I_2 = I_3$. According to Racah's formula, the Clebsch-Gordan coefficient is

$$
\begin{aligned}
&\langle T_1, T_1 - I_1 : T_2, T_2 - I_2 \,|\, T_3, T_3 - I_3\rangle \\
&= \left[\frac{\Gamma(2T_1+1)}{\Gamma(2T_1-I_1+1)\Gamma(I_1+1)} \cdot \frac{\Gamma(2T_2+1)}{\Gamma(2T_2-I_2+1)\Gamma(I_2+1)} \cdot \frac{\Gamma(2T_3-I_3+1)\Gamma(I_3+1)}{\Gamma(2T_3+1)}\right]^{1/2}.
\end{aligned}
$$

In addition, we have to adjust the normalization factor. We are interested in the tensor product $|z_3\rangle_{(B)} \propto |z_1\rangle_{(B)} \otimes |z_2\rangle_{(B)}$. The normalization condition of each states is

$$
\langle z_i \,|\, z_i\rangle = \frac{2T_i+1}{4\pi R^2} \quad (i = 1, 2, 3).
$$

Therefore, we can read the product property in Section 3.3 from

$$
|z_3\rangle_{(B)} = \left[\frac{4\pi R^2}{2T_1+1} \cdot \frac{4\pi R^2}{2T_2+1} \cdot \frac{2T_3+1}{4\pi R^2}\right]^{1/2} |z_1\rangle_{(B)} \otimes |z_2\rangle_{(B)} \,.
$$

Actually,

$$
\begin{aligned}
&|z_3\rangle_{(B)} \\
&= \sum_{I_1,I_2} \left[ \frac{1}{4\pi R^2} \frac{\Gamma(2T_1+1)}{\Gamma(2T_1-I_1+1)\Gamma(I_1+1)} \cdot \frac{1}{4\pi R^2} \frac{\Gamma(2T_2+1)}{\Gamma(2T_2-I_2+1)\Gamma(I_2+1)} \cdot \frac{2T_3+1}{4\pi R^2} \right]^{1/2} \\
&\times \frac{z^{I_1+I_2}}{(1+|z|^2)^{\frac{2(T_1+T_2)}{2}}} \left| T_1, T_1-I_1 : T_2, T_2-I_2 \right\rangle \\
&= \sum_{I_1,I_2} \psi^{2(T_1+T_2)}_{(B),I_1+I_2} \left| T_1, T_1-I_1 : T_2, T_2-I_2 \right\rangle \\
&\times \left[ \frac{\Gamma(2T_1+1)}{\Gamma(2T_1-I_1+1)\Gamma(I_1+1)} \cdot \frac{\Gamma(2T_2+1)}{\Gamma(2T_2-I_2+1)\Gamma(I_2+1)} \cdot \frac{\Gamma(2T_3-I_1+1)\Gamma(I_3+1)}{\Gamma(2T_3+1)} \right]^{1/2} \\
&= \sum_{I_1,I_2} \psi^{2(T_1+T_2)}_{(B),I_1+I_2} \left| T_1, T_1-I_1 : T_2, T_2-I_2 \right\rangle \left\langle T_1, T_1-I_1 : T_2, T_2-I_2 \,\middle|\, T_3, T_3-I_3 \right\rangle |_{T_3=T_1+T_2, I_3=I_1+I_2} \\
&= \sum_{I_1,I_2,I_3} \psi^{2(T_1+T_2)}_{(B),I_3} \left| T_1, T_1-I_1 : T_2, T_2-I_2 \right\rangle \left\langle T_1, T_1-I_1 : T_2, T_2-I_2 \,\middle|\, T_3, T_3-I_3 \right\rangle |_{T_3=T_1+T_2} \\
&= \sum_{I_1,I_2,I_3} \psi^{2(T_1+T_2)}_{(B),I_3} \left| T_3, T_3-I_3 \right\rangle ,
\end{aligned}
$$

where we used the completeness relation in the fourth line since the Kronecker delta $\delta_{I_1+I_2=I_3}$ is included naturally in the Clebsch-Gordan coefficient.

Therefore, we can find the product property

$$
\begin{aligned}
\psi^{2T_1}_{(B),I_1} \cdot \psi^{2T_2}_{(B),I_2} &= \left[ \frac{4\pi R^2}{2T_1+1} \cdot \frac{4\pi R^2}{2T_2+1} \cdot \frac{2T_3+1}{4\pi R^2} \right]^{-1} \left\langle T_1, T_1-I_1 : T_2, T_2-I_2 \,\middle|\, T_1+T_2, I_3 \right\rangle \psi^{2(T_1+T_2)}_{(B),I_3} \\
&= \frac{\mathcal{N}^{2(T_1+T_2)}_{(B),I_3}}{\mathcal{N}^{2T_1}_{(B),I_1} \cdot \mathcal{N}^{2T_2}_{(B),I_2}} \psi^{2(T_1+T_2)}_{(B),I_3},
\end{aligned}
$$

then the result in Section 3.3 is recovered completely.

Similarly, we can confirm for the remaining combinations. The tensor products should be $|z_3\rangle_{(F)} \propto |z_1\rangle_{(F)} \otimes |z_2\rangle_{(B)}$ and $|z_3\rangle_{(B)} \propto |z_1\rangle_{(F)} \otimes |z_2\rangle_{(F)}$, where

$$
\begin{aligned}
|z_i\rangle_{(F)} &:= \frac{1}{(1+|z|^2)^{\frac{2T_i-1}{2}}} \sum_{I_i=0}^{2T_i-1} \left( \frac{1}{4\pi R^2} \cdot \frac{(2T_i)!}{I_i!(2T_i-I_i-1)!} \right)^{1/2} z^{I_i} \left| T_i - 1/2, T_i - 1/2 - I_i \right\rangle \\
&= \sum_{I_i=0}^{2T_i-1} \psi^{2T_i}_{(F),I_i} \left| T_i - 1/2, T_i - 1/2 - I_i \right\rangle , \\
|z_2\rangle_{(B)} &:= \frac{1}{(1+|z|^2)^{\frac{2T_2}{2}}} \sum_{I_2=0}^{2T_2} \left( \frac{1}{4\pi R^2} \cdot \frac{(2T_2)!}{I_2!(2T_2-I_2)!} \right)^{1/2} z^{I_2} \left| T_2, T_2-I_2 \right\rangle \\
&= \sum_{I_2=0}^{2T_2} \psi^{2T_2}_{(B),I_2} \left| T_2, T_2-I_2 \right\rangle ,
\end{aligned}
$$

where $i = 1, 2$. The necessary Clebsch-Gordan coefficients are obtained by

$$
\begin{aligned}
&\langle T_1 - 1/2, T_1 - 1/2 - I_1 : T_2, T_2 - I_2 \,|\, T_3, T_3 - I_3 \rangle \,|_{T_3 = T_1 - 1/2 + T_2, m_3 = m_1 + m_2} \\
&= \left[ \frac{\Gamma(2T_1)}{\Gamma(2T_1 - I_1)\Gamma(I_1 + 1)} \cdot \frac{\Gamma(2T_2 + 1)}{\Gamma(2T_2 - I_2 + 1)\Gamma(I_2 + 1)} \cdot \frac{\Gamma(2T_3 - I_3 + 1)\Gamma(I_3 + 1)}{\Gamma(2T_3 + 1)} \right]^{1/2}, \\
&\langle T_1 - 1/2, T_1 - 1/2 - I_1 : T_2 - 1/2, T_2 - 1/2 - I_2 \,|\, T_3, T_3 - I_3 \rangle \,|_{T_3 = T_1 - 1/2 + T_2 - 1/2, m_3 = m_1 + m_2} \\
&= \left[ \frac{\Gamma(2T_1)}{\Gamma(2T_1 - I_1)\Gamma(I_1 + 1)} \cdot \frac{\Gamma(2T_2)}{\Gamma(2T_2 - I_2)\Gamma(I_2 + 1)} \cdot \frac{\Gamma(2T_3 - I_3 + 1)\Gamma(I_3 + 1)}{\Gamma(2T_3 + 1)} \right]^{1/2}.
\end{aligned}
$$

In addition, the normalization conditions of each states are

$$
\langle z_i \,|\, z_i \rangle_{(F)} = \frac{2T_i}{4\pi R^2} \quad (i = 1, 2, 3), \quad \langle z_j \,|\, z_j \rangle_{(B)} = \frac{2T_i + 1}{4\pi R^2} \quad (j = 2, 3).
$$

Then, the adjustments for $|z_3\rangle_{(F)}$ and $|z_3\rangle_{(B)}$ are

$$
\begin{aligned}
|z_3\rangle_{(F)} &= \left[ \frac{4\pi R^2}{2T_1} \cdot \frac{4\pi R^2}{2T_2 + 1} \cdot \frac{2T_3 + 1}{4\pi R^2} \right]^{1/2} |z_1\rangle_{(F)} \otimes |z_2\rangle_{(B)} \\
|z_3\rangle_{(B)} &= \left[ \frac{4\pi R^2}{2T_1} \cdot \frac{4\pi R^2}{2T_2} \cdot \frac{2T_3 + 1}{4\pi R^2} \right]^{1/2} |z_1\rangle_{(F)} \otimes |z_2\rangle_{(F)}.
\end{aligned}
$$

If we require

$$
|z_3\rangle_{(F)} = \sum_{I_3} \psi^{2(T_1 + T_2)}_{(F), I_3} \, |T_3, T_3 - I_3\rangle, \qquad |z_3\rangle_{(B)} = \sum_{I_3} \psi^{2(T_1 + T_2 - 1)}_{(B), I_3} \, |T_3, T_3 - I_3\rangle,
$$

we can find the product property

$$
\begin{aligned}
\psi^{2T_1}_{(F), I_1} \cdot \psi^{2T_2}_{(B), I_2} &= \frac{\mathcal{N}^{2(T_1 + T_2)}_{(F), I_1 + I_2}}{\mathcal{N}^{2T_1}_{(F), I_1} \cdot \mathcal{N}^{T_2}_{(B), I_2}} \cdot \psi^{2(T_1 + T_2)}_{(F), I_3}, \\
\psi^{2T_1}_{(F), I_1} \cdot \psi^{2T_2}_{(F), I_2} &= \frac{\mathcal{N}^{2(T_1 + T_2 - 1)}_{(B), I_1 + I_2}}{\mathcal{N}^{2T_1}_{(F), I_1} \cdot \mathcal{N}^{2T_2}_{(F), I_2}} \cdot \psi^{2(T_1 + T_2 - 1)}_{(B), I_3}.
\end{aligned}
$$

In conclusion, the product property in Section 3.3 can be recovered completely from magnetized fuzzy sphere. By similar computations, we can obtain the same result for higher order coupling constants as the result we derived in Section 3.3.

Let us remark about the product property. In Section 6.2, we considered the Dirac operator like eq. (6.4). Equation (6.4) is essentially the same as the case of the adjoint representation in Chapter 5. In Chapter 5, we mentioned the case of the fundamental representation, and we proved that the product property is valid for $U(1)$ gauge theory only. In the case of noncommutative torus, even $U(1)$ gauge theory does not hold the product property because the degrees of freedom are described as operators. Normally, this situation is the same as for the case of fuzzy sphere in the IKKT matrix model. However, we converted the Dirac operator from the type like eq. (6.4) to the Ginsparg-Wilson type. Accordingly, the product of the zero modes was defined by the tensor product. Therefore, the commutativity between the gauge field and the zero mode recovers, and the product property holds as we proved.

# Chapter 7

# Summary

In this thesis, we studied type IIB effective theories from the perturbative and non-perturbative points of view. In Chapter 3, we discussed the features of coupling constants in magnetized toroidal and spherical compactifications. The framework of magnetized compactifications are given by super Yang-Mills theory that is an effective theory of D-branes in type IIB perturbative superstring theory. The coupling constants can be obtained by the overlap integrals. We confirmed that higher order coupling constants can be decomposed by the three-point coupling constants due to the product property of the eigenfunctions.

In Chapter 5, we generalized the features of the coupling constants in Chapter 3 to any compact spin manifolds and its magnetized compactifications. We proved that the origin of the features of coupling constants resides in the fact that the Dirac-type operator has the complete orthonormal system. We confirmed the product property of the zero modes by direct computations. In magnetized toroidal and spherical compactifications, the decomposition of higher order coupling constants has the variations like s-channel and t-channel in the language of scattering theory. We found that the two of them are allowed in general magnetized compactifications because of the non-Abelian structure.

In Chapter 6, we reconsider the features of coupling constants based on the IKKT matrix model as a non-perturbative formulation of superstring theory. We conjectured that the product property of the zero modes holds since the Dirac operator in the IKKT model has the same action structure as Dirac-type operators (c.f., eqs (5.3) and (5.4).). We confirmed this conjecture by considering magnetized noncommutative toroidal and fuzzy spherical compactifications of the IKKT matrix model. In magnetized noncommutative toroidal compactifications, we found that the quantization condition of the magnetic flux is deformed by the noncommutative parameter . However, the zero modes can also be obtained by the (operator-valued) Jacobi-theta function, and the product property of the zero modes is essentially the same as the formula we showed in Chapter 3. In magnetized fuzzy spherical compactifications, the product feature in Section 3.3 can be reproduced as the Clebsch-Gordan coefficients. Therefore, we concluded that the selection rule we proposed in Chapter 5 holds even if we consider non-perturbative formulation in the range of our discussion.

The coupling constants are fundamental objects since we can observe them through several experiments and observations. The features of coupling constants we found are important for top-down approaches since we discussed in general set-ups. In addition, we open the possibility to compute physical quantity like coupling constants from the IKKT matrix model. We expect future progress of the IKKT matrix model as it provides a promising non-perturbative formulation of superstring theory and investigations of its phenomenological aspects are just getting started.

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No.1

# 早稲田大学　博士（理学）　学位申請　研究業績書

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| ○論文 | “Matrix model and Yukawa couplings on the noncommutative torus”, Journal of High Energy Physics 04 (2019) 079, 2019 年 4 月発行, Masaki Honda. |
| 論文 | “Wilson-line Scalar as a Nambu-Goldstone Boson in Flux Compactifications and Higher-loop Corrections”, Journal of High Energy Physics 03 (2020) 031, 2020 年 3 月発行, Masaki Honda and Toshihide Shibasaki. |
| 論文 | “String Backgrounds in String Geometry”, International Journal of Modern Physics A Vol.35 No.27 2050176 (2020), Masaki Honda and Matsuo Sato. |
| 講演（口頭） | 本多正樹, 及川茜, 大塚啓, “II 型超弦理論のおける小さいアクシオン崩壊定数”, 日本物理学会 2016 年秋季大会, 宮崎大学, 2016 年 9 月. |
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| 講演（口頭） | 本多正樹, 佐藤松夫, 杉本裕司, “弦幾何と弦の背景”, 日本物理学会 2018 年秋季大会, 信州大学, 2018 年 9 月. |
| 講演（口頭） | 本多正樹, “行列模型における背景磁場付き非可換トーラス上の湯川結合”, 日本物理学会 2018 年秋季大会, 信州大学, 2018 年 9 月. |
| 講演（口頭） | 本多正樹, 佐藤松夫, 杉本裕司, “弦幾何理論におけるニュートン極限”, 日本物理学会 第 74 回年次大会 (2019 年), 九州大学, 2019 年 3 月. |

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| 講演<br>（口頭） | 本多正樹，佐藤松夫，杉本裕司，“弦幾何理論の摂動的真空とニュートン極限”，日本物理学会 2019年秋季大会，山形大学，2019年9月. |
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| 講演（セミナー） | 本多正樹，“IKKT-inspired model and magnetized extradimensional model”，京都大学（オンライン開催），2020年6月. |
| 講演（セミナー） | 本多正樹，“Towards computing Yukawa couplings from magnetized Riemann surface of higher genus”，北海道大学（オンライン開催），2021年1月. |